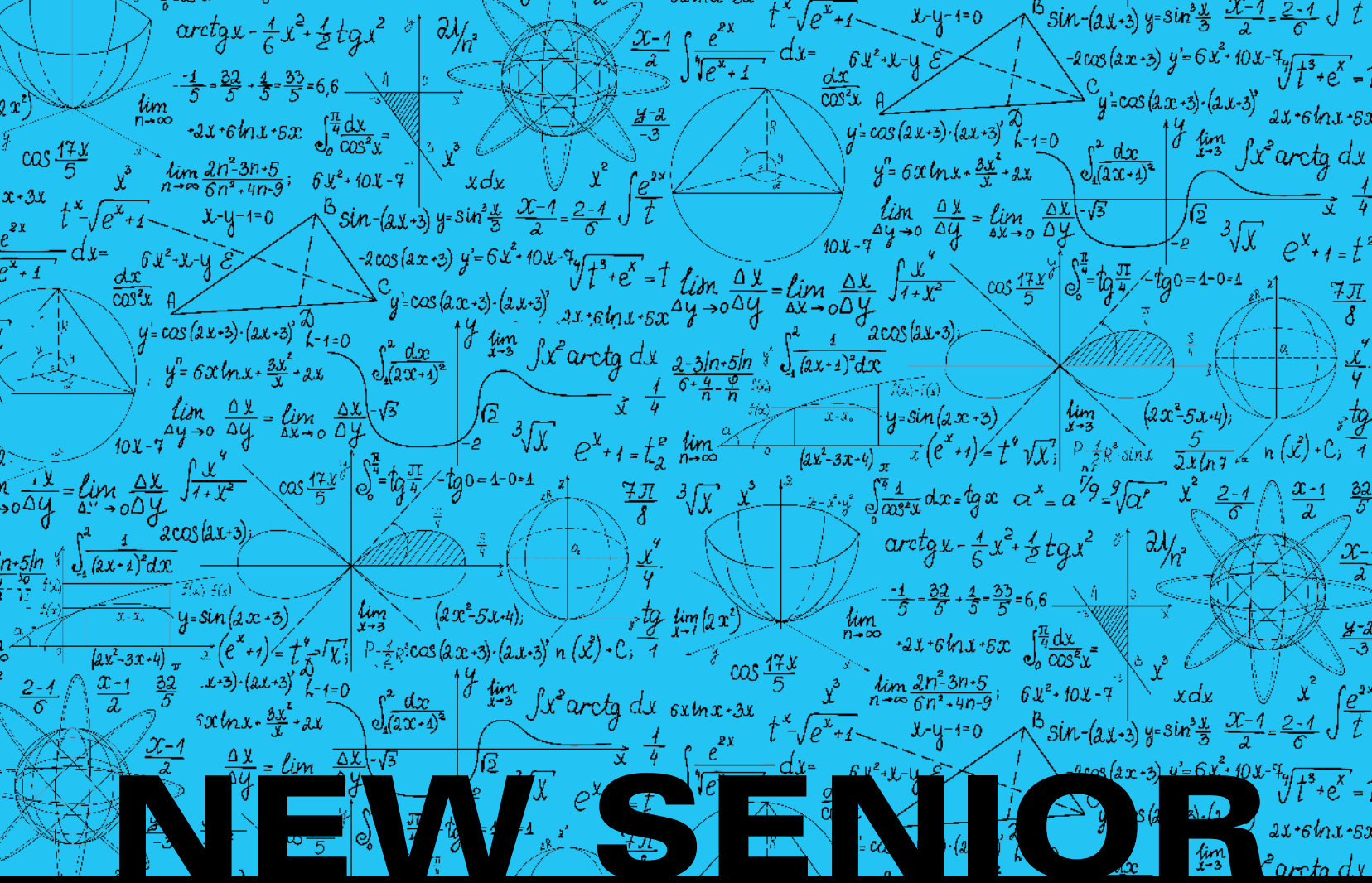

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EXTENSION 1

FOR YEARS 11 & 12

THIRD EDITION

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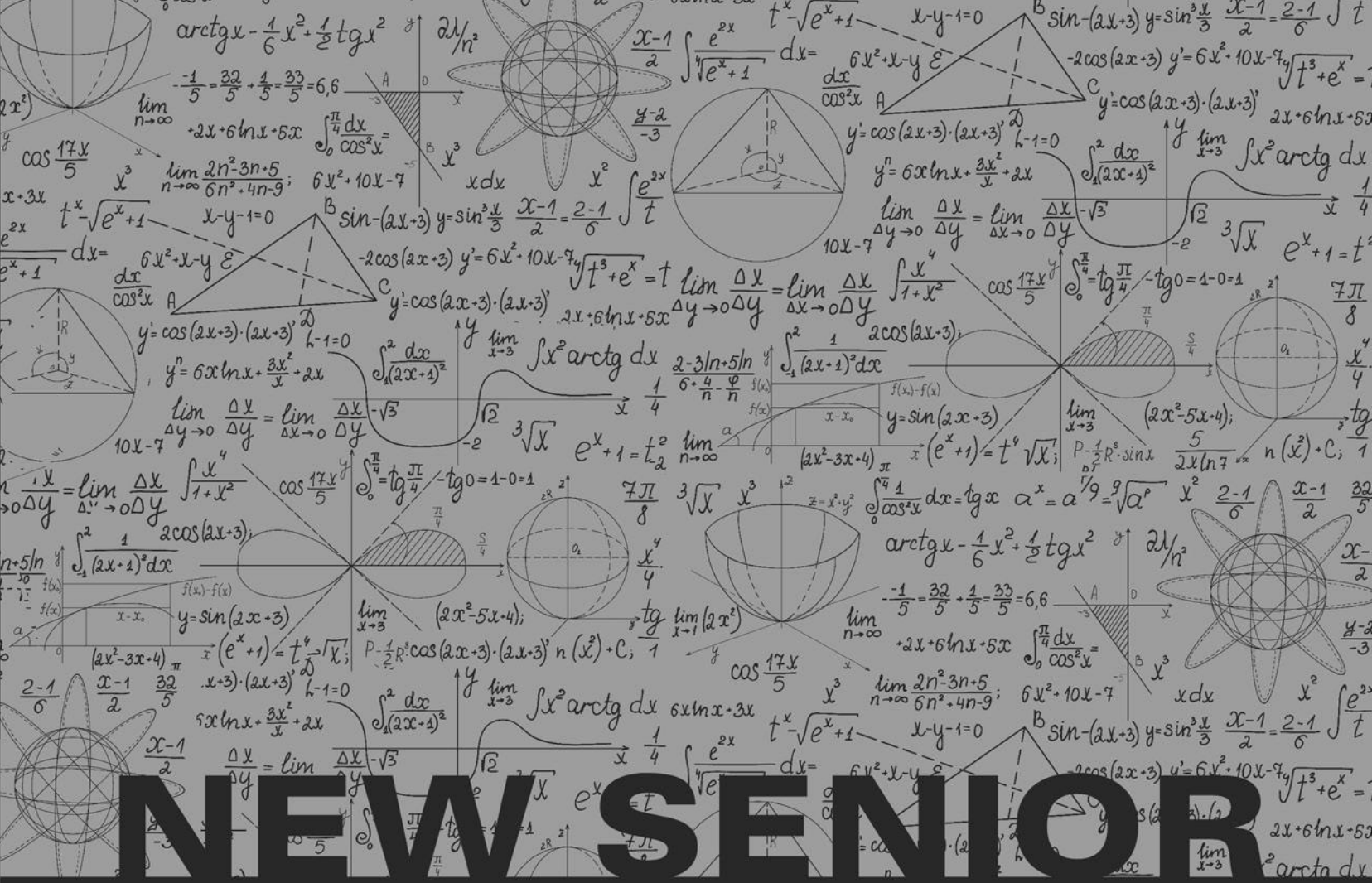

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INTRODUCTION AND DEDICATION

J.B. Fitzpatrick

It is interesting to wonder whether J. B. Fitzpatrick ('Bernie') realised in 1983 just how popular his book *New Senior Mathematics* would be. That first edition of *New Senior Mathematics* was to remain in print for almost 30 years. It has stood the test of time thanks to the quality, rigour and variety of its questions, its accuracy and its high mathematical standards.

As Fitzpatrick wrote in 1983: 'Mathematics, like many other things, is best learnt by doing. A student begins to appreciate the power of mathematics when he or she has achieved a mastery of basic techniques, not after reading lengthy explanations … The emphasis throughout the book is on the understanding of mathematical concepts' (Introduction, *New Senior Mathematics* 1984).

J. B. Fitzpatrick passed away in 2008. Fitzpatrick was a respected author, teacher and figurehead of mathematics education.

Bob Aus

Bob Aus taught in New South Wales high schools for 40 years, retiring in 2007. During that time Bob taught all courses from Years 7 to 12 up to Level 1 / 4-unit / Extension 2. He has marked HSC examination papers and has been involved in the standards setting process as judge and chief judge for the three Calculus-based courses over four years. He has also completed review work for the NSW Board of Studies and represented NSW at a week-long review and standards setting of the upper level course from each state prior to the development of the Australian National Curriculum for senior students.

Bob spent time as Regional Vocational Education Consultant in the North Coast region and was a Mathematics consultant in the Hunter region. When he retired he was Head Teacher Mathematics at Merewether High School and enjoyed teaching an Extension 2 class with 24 students.

Bob's first publication was in 1983 and he has been involved with writing a range of textbooks and study guides since then, including revising and updating the *New Senior Mathematics* series 2nd edition in 2013.

Bob has presented talks on the three Calculus-based courses throughout the state. He has co-written the Years 6–9 Mathematics syllabus for the Abu Dhabi Education Authority, as well as managing the writing project for support material for this course. He also wrote the Years 10–12 syllabus for their Calculus-based course.

This third edition of *New Senior Mathematics* updates it for the new Stage 6 HSC courses in NSW to be implemented in Year 11, 2019.

NEW SENIOR MATHEMATICS

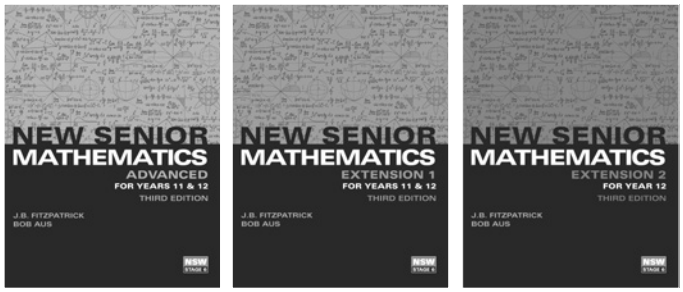
THIRD EDITION

New Senior Mathematics Extension 1 for Years 11 & 12 is part of a new edition of the well-known Mathematics series for New South Wales. The series has been updated to address all requirements of the new Stage 6 syllabus. We have maintained our focus on mathematical rigour and challenging student questions, while providing new opportunities for students to consolidate their understanding of concepts and ideas with the aid of digital resources and activities.

Student Book

The first three chapters of the first student book contain revision material that provides the necessary foundation for the development of senior mathematics concepts. In the new edition you'll also find:

- content built on a rigorous, academic approach that promotes excellence and prepares students for higher education
- a simple, convenient approach with Year 11 and 12 content in one book for Advanced and Extension 1, with colour coding to distinguish year levels
- digital technology activities that promote a deeper understanding, allowing students to make connections, and visualise and manipulate data in real time.



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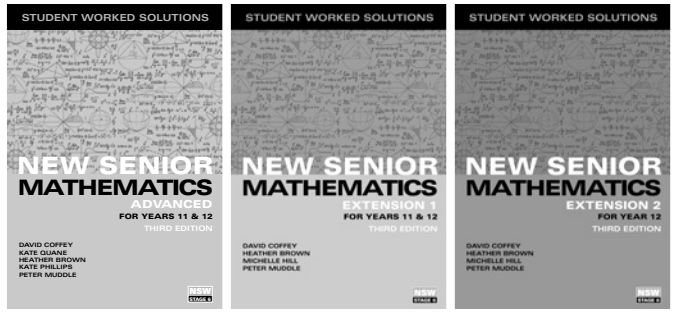

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Reader+, our next generation eBook, features content and digital activities, with technology such as graphing software and spreadsheets, to help students engage on their devices.

There are also teacher support materials, such as practice exams, question banks, investigation assignments, and fully worked solutions to cover all internal and external assessment items and save you time.

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FEATURES OF THE 3RD EDITION STUDENT BOOK/READER+

YEAR LEVELS

Year levels are indicated on each page for easy identification of Year 11 and 12 content.

MAKING CONNECTIONS

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their conceptual understanding.

EXPLORING FURTHER

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skill base. These activities should take approximately 5–15 minutes to complete.

CHAPTER REVIEW

Each chapter contains a comprehensive review of chapter content.

SUMMARY PAGES

A comprehensive course summary is provided at the end of the book.

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CHAPTER 1
Further work with functions

1.1 QUADRATIC INEQUALITIES

Here are some of the main properties of inequalities. In particular, you should note the operations that reverse the inequality sign.

- Adding or subtracting the same number from both sides does not alter an inequality.
- Multiplication by a positive number does not alter the direction of the inequality.
- Multiplying both sides of an inequality by a negative number reverses the direction of the inequality:
  e.g. $3 > 2$ but $-6 < -4$ after multiplying by -2.
- Taking the reciprocal of both sides of an inequality reverses its direction when both sides have the same sign, but *not* if the signs are different:
  e.g. $5 < 7$ but $\frac{1}{5} > \frac{1}{7}$
  $3 > -4$ and $\frac{1}{3} > -\frac{1}{4}$
  It is interesting to prove this result, as follows:

  Let $a > b$, where a, b have the same sign.
  Divide both sides by ab, which must be positive as a, b have the same sign:
  $$\frac{a}{ab} > \frac{b}{ab}$$
  $$\frac{1}{b} > \frac{1}{a}$$
  $$\therefore \frac{1}{a} < \frac{1}{b}$$

  Let $a > b$, where a, b have different signs.
  Divide both sides by ab, which must be negative as a, b have different signs:
  $$\frac{a}{ab} < \frac{b}{ab} \quad \text{(inequality changes)}$$
  $$\frac{1}{b} < \frac{1}{a}$$
  $$\therefore \frac{1}{a} > \frac{1}{b}$$

- Squaring both sides of an inequality produces a positive number on both sides, so the direction of the inequality $a^2 \lessgtr b^2$ depends on which of $|a|$ and $|b|$ is larger.

  Thus, if a and b are positive and $a > b$ then $a^2 > b^2$ (i.e. you can square both sides of an inequality without changing direction if both sides are known to be positive).

This can be proved as follows:

Let $a > b$ and a, b both be positive.

Multiply the original inequality by a, which is positive: $a^2 > ab$

Multiply the original inequality by b, which is positive: $ab > b^2$

Link the two results together: $a^2 > ab > b^2$

$\therefore$ $a^2 > b^2$

- Square root of both sides of an inequality is only defined if both sides of the inequality are non-negative. The direction of the inequality does not change (i.e. if $a > b$ then $\sqrt{a} > \sqrt{b}$, provided a is positive and b is positive or zero).

Some methods for solving harder inequalities will now be looked at.

Quadratic inequalities

Example 1

Solve $x^2 - 4x > 0$.

Solution

To solve a quadratic inequality you must *not* simply factorise it like a quadratic equation, as this is wrong:

$x(x-4) > 0$

$x > 0,\ x - 4 > 0$

$x > 0,\ x > 4$

If you now substitute any negative number for x in the original inequality, you will see that it is a solution, so $x > 0$, $x > 4$ must not be the complete answer. Similarly, $x = 1$ is not a solution even though it is included in $x > 0$, $x > 4$, so this method must be wrong. What has happened? Where is the error?

The answer is that for $x(x-4) > 0$ to be true (i.e. for the product of two factors to be positive), both x and $x - 4$ must be positive *or* both factors must be negative.

The solution can proceed as follows:

$x(x-4) > 0$

$x > 0$	and	$x - 4 > 0$	**or**	$x < 0$	and	$x - 4 < 0$
$x > 0$	and	$x > 4$		$x < 0$	and	$x < 4$

For both $x > 0$ and $x > 4$ to be true: $x > 4$

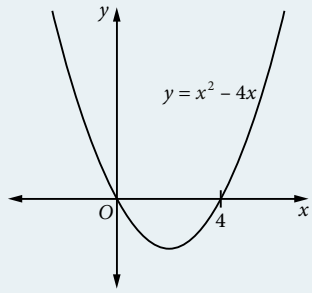
For both $x < 0$ and $x < 4$ to be true: $x < 0$

Hence the correct solution is $x < 0$, $x > 4$ (which is read as '$x < 0$ or $x > 4$').

Alternatively, a graphical method can be used:

- Sketch the parabola $y = x^2 - 4x$ (concave up, cuts the x-axis at 0 and 4).
- Identify the x values for which the parabola is above the x-axis (because you are looking for the places where $y > 0$). These values are the solution.

The graph shows that the solution is $x < 0$ or $x > 4$.



The graphical method in Example **1** has the advantage of providing a visual picture. It is also an easy method for higher-degree inequalities.

Polynomial inequalities

- Use the factored form of the polynomial to sketch the graph of the function.
- Use the graph to identify the x values for which the graph is above (or below) the x-axis (depending on the inequality). These values are the solution.
- If the inequality is $\leq$ or $\geq$, be careful to include the values where the graph cuts or touches the x-axis.

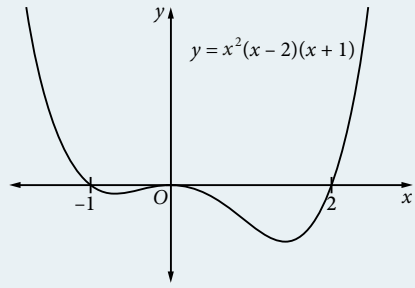
Example 2

Solve $x^2(x-2)(x+1) \geq 0$.

Solution

Sketch $y = x^2(x-2)(x+1)$, noting the double zero at $x = 0$ (a turning point on the x-axis), and zeros at $x = 2$ and $x = -1$.

Find where the graph is on or above the x-axis. The solution is $x \leq -1$, $x = 0$, $x \geq 2$.



EXPLORING FURTHER

Quadratic inequalities

Use technology to solve quadratic inequalities graphically.

EXERCISE 1.1 QUADRATIC INEQUALITIES

For questions **1** to **11**, solve the following inequalities.

1 $x(x-1) \leq 0$ **2** $x^2 - 2x - 15 \leq 0$ **3** $4x^2 - 12x + 10 > 0$ **4** $x(x-1) \leq 6$

5 $(x-1)(x+3)(x-2) < 0$ **6** $-3x^2 + 10x + 8 \leq 0$ **7** $x^2(x-1) \leq 0$ **8** $(2-x)(x-5)(x+1) > 0$

9 $2^{2x} - 5(2^x) + 4 \leq 0$ **10** $2^{2x} - 2(2^x) \leq -1$ **11** $1 - x < 2x + 1 < x + 4$

1.2 RATIONAL FUNCTION INEQUALITIES (x IN DENOMINATOR)

Example 3

Solve $\frac{1}{x-2} \geq -1$.

Solution

There are two important things to note here:

- $x \neq 2$, as the function $f(x) = \frac{1}{x-2}$ does not exist where $x = 2$
- if you multiply both sides by $(x-2)$, you don't know whether $(x-2)$ is positive or negative, so you don't know whether to reverse the inequality or not.

There are several possible methods to solve an inequality like this (see below). You should memorise the one or two methods that you find easiest and most useful, but you should be aware of the other methods too.

Method 1 (algebraic method requiring consideration of all possible cases)

$\frac{1}{x-2} \geq -1$

If $x > 2$, multiply by $(x-2)$, which is positive:

$1 \geq -(x-2)$
$1 \geq -x + 2$
$-1 \geq -x$
$1 \leq x$ (multiplying by -1)
$x \geq 1$

Both $x > 2$ and $x \geq 1$ must be true.

This requires $x > 2$.

If $x < 2$, multiply by $(x-2)$, which is negative:

$1 \leq -(x-2)$ (note change of inequality)
$1 \leq -x + 2$
$-1 \leq -x$
$1 \geq x$ (multiplying by -1)
$x \leq 1$

Both $x < 2$ and $x \leq 1$ must be true.

This requires $x \leq 1$.

Thus the complete solution is $x > 2$, $x \leq 1$.

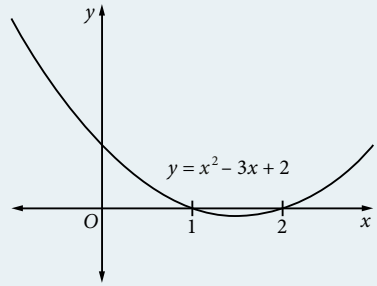
Method 2 (algebraic method avoiding the need to consider different cases)

$\frac{1}{x-2} \geq -1$ Note that $x \neq 2$.

Multiply both sides by $(x-2)^2$, which is known to be positive (so the inequality does not change):

$x - 2 \geq -(x-2)^2$
$x - 2 \geq -x^2 + 4x - 4$
$x^2 - 3x + 2 \geq 0$

You can now solve this quadratic inequality using the graphical method (as shown in Example **1**).



The graph (at right) is on or above the x-axis for $x \leq 1$, $x \geq 2$. However, $x \neq 2$, so the solution is $x \leq 1$, $x > 2$.

Method 3 (variation on Method 2, producing an inequality in factorised form)

$$\frac{1}{x-2} \geq -1 \quad \text{Note that } x \neq 2.$$

Make the right-hand side become zero, then simplify the resulting left-hand side:

$$\frac{1}{x-2} + 1 \geq 0$$

$$\frac{1+(x-2)}{x-2} \geq 0$$

$$\frac{x-1}{x-2} \geq 0$$

Multiply both sides by $(x-2)^2$:

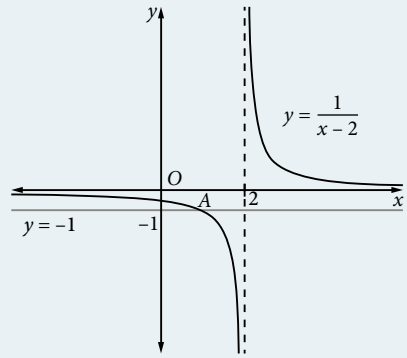
$$(x-1)(x-2) \geq 0$$

This is the factorised form of the quadratic inequality obtained in Method 2. From this point on, the solution is identical to Method 2.

Method 4 (graphical method)

$$\frac{1}{x-2} \geq -1$$

Sketch a graph of $y = \frac{1}{x-2}$: it is a hyperbola with vertical asymptote at $x = 2$ and horizontal asymptote at the x-axis.



Also graph the horizontal line $y = -1$.

Use the graph to find the x values for which the hyperbola is on or above the horizontal line $y = -1$. You can see that this happens for all x values from point A back to the left, including point A, and for all x values to the right of the asymptote.

To find the x-coordinate of the point A, solve $\frac{1}{x-2} = -1$: the solution is $x = 1$. Thus the solution is $x \leq 1$, $x > 2$.

Method 5 (like the graphical method, but without drawing the graph)

$$\frac{1}{x-2} \geq -1$$

List all the x values that are excluded because they make the denominator become zero:

$x \neq 2$ (the vertical asymptote)

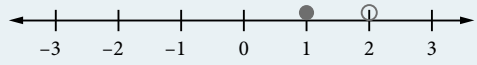
Solve the inequality's corresponding equation:

$$\frac{1}{x-2} = -1$$

$$1 = -x + 2$$

$x = 1$ This is the x-coordinate of the point of intersection of $y = \frac{1}{x-2}$ and $y = -1$.

The x values found by these two steps are 'critical values', to be marked on a number line. Use open circles to mark the excluded (asymptotic) values. If the inequality is $<$ or $>$, use open circles to mark the x values from solving the equation; if the inequality is $\leq$ or $\geq$, use filled circles.



Now choose representative x values from each section of the number line and test to see whether the x values satisfy the original inequality:

Test $x = 0$: Is $\frac{1}{(0-2)} \geq -1$? Yes

Test $x = 1\frac{1}{2}$: Is $\frac{1}{1\frac{1}{2}-2} \geq -1$? No

Test $x = 3$: Is $\frac{1}{(3-2)} \geq -1$? Yes

This finds where the graph of $y = \frac{1}{x-2}$ is above the line $y = -1$.
Thus the solution is $x \leq 1$, $x > 2$.

Example 4

Solve $\frac{1}{x^2 - x} > \frac{1}{x^2 - 1}$.

Solution

Using **Method 3** (from Example **3**):

$$\frac{1}{x(x-1)} > \frac{1}{(x-1)(x+1)}, \text{ so } x \neq 0, 1, -1$$

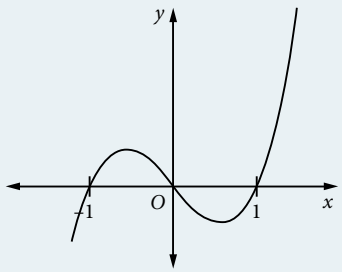
$$\frac{1}{x(x-1)} - \frac{1}{(x-1)(x+1)} > 0$$

$$\frac{(x+1)-x}{x(x-1)(x+1)} > 0$$

$$\frac{1}{x(x-1)(x+1)} > 0$$

Multiply by $x^2(x-1)^2(x+1)^2$:

$$x(x-1)(x+1) > 0$$



The graph is above the x-axis for $-1 < x < 0$, $x > 1$, which is the required solution.

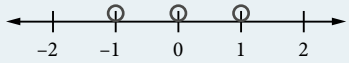
Using **Method 5** (from Example **3**):

$x \neq 0, 1, -1$

Solve $\frac{1}{x^2 - x} = \frac{1}{x^2 - 1}$: $x^2 - 1 = x^2 - x$

$x = 1$

Critical values are 0, 1, −1; all must be not included (i.e. open circles on number line)



Test $x = -2$:

Is $\frac{1}{\left((-2)^2 - (-2)\right)} > \frac{1}{\left((-2)^2 - 1\right)}$? No

Test $x = -\frac{1}{2}$:

Is $\frac{1}{\left((-\frac{1}{2})^2 - (-\frac{1}{2})\right)} > \frac{1}{\left((-\frac{1}{2})^2 - 1\right)}$? Yes

Test $x = \frac{1}{2}$:

Is $\frac{1}{\left((\frac{1}{2})^2 - (\frac{1}{2})\right)} > \frac{1}{\left((\frac{1}{2})^2 - 1\right)}$? No

Test $x = 2$:

Is $\frac{1}{\left((2)^2 - (2)\right)} > \frac{1}{\left((2)^2 - 1\right)}$? Yes

The solution is $-1 < x < 0$, $x > 1$.

EXERCISE 1.2 RATIONAL FUNCTION INEQUALITIES (x IN DENOMINATOR)

1 When asked to solve $\frac{2}{x-1} \leq \frac{1}{2}$, four students began their solutions as follows:

Student A: $4 \leq x - 1$ Student B: $4(x-1) \leq (x-1)^2$

Student C: $\frac{x-1}{2} \geq 2$ Student D: $4(x-1) \leq (x-1)^2$ noting that $x \neq 1$

(a) Which student has started their solution correctly?

(b) Explain the errors made by the other students.

For questions **2** to **11**, solve the following inequalities.

2 $\frac{x-3}{x+1} > 0$ **3** $\frac{x-2}{x+3} > -2$ **4** $\frac{4x-3}{2x+1} \leq 3$ **5** $\frac{2}{1-x} > -1$

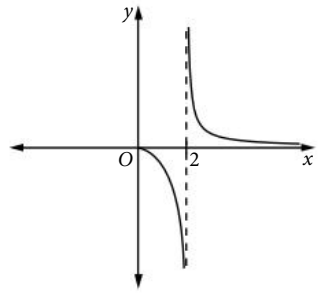
6 $\frac{2x-3}{4x-5} + 2 < 0$ **7** $\frac{1}{(x-1)(x-3)} \leq -1$ **8** $\frac{7}{(3-x)(x+3)} > -1$ **9** $\frac{x}{x^2-1} < 0$

10 $\frac{2x-4}{x+3} > \frac{x+2}{2x+6}$

11 **(a)** Show that $f(x) = \frac{x}{x^2-4}$ is an odd function.

(b) Copy and complete the graph of $y = \frac{x}{x^2-4}$ (as shown at right).

(c) Hence solve $\frac{x}{x^2-4} < \frac{1}{3}$.



1.3 INEQUALITIES INVOLVING ABSOLUTE VALUE AND SQUARE ROOTS

Example 5

Solve $|x-2| < \frac{x}{2}$.

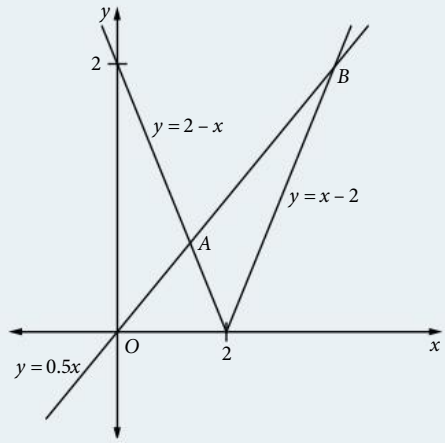
Solution

From the definition of absolute value, you know that $|x-2|$ is simply equal to $x-2$ where $x \geq 2$, but it is equal to $-(x-2)$ where $x < 2$.

Using graphical method (**Method 4** from Example **3**):

Graph $y = |x-2|$ and $y = \frac{x}{2}$.

Note that the 'V'-shaped absolute value graph consists of the ray $y = 2 - x$ for $x < 2$ and the ray $y = x - 2$ for $x \geq 2$.



The solution of the inequality $|x-2| < \frac{x}{2}$ is shown on the graph where the 'V'-shaped absolute value graph is below the straight line $y = \frac{x}{2}$. You need to find the x-coordinates of the points of intersection A and B.

- At A, $y = 2 - x$ and $y = \frac{x}{2}$ intersect. Solving these: $x = \frac{4}{3}$
- At B, $y = x - 2$ and $y = \frac{x}{2}$ intersect. Solving these: $x = 4$

Thus the solution is: $\frac{4}{3} < x < 4$

Using analytical method (**Method 1** from Example **3**):

For $x \geq 2$: $x - 2 < \frac{x}{2}$

$\frac{x}{2} < 2$

$x < 4$

Both $x \geq 2$ and $x < 4$ must be true.

$\therefore 2 \leq x < 4$

For $x < 2$: $2 - x < \frac{x}{2}$

$2 < \frac{3x}{2}$

$x > \frac{4}{3}$

Both $x < 2$ and $x > \frac{4}{3}$ must be true.

$\therefore \frac{4}{3} < x < 2$

Thus the complete solution is $\frac{4}{3} < x < 4$.

Example 6

Solve $|x-4|+|x+2|>7$.

Solution

Although it is possible to solve this analytically, a graphical approach is easier.

- First sketch $y=|x-4|$ and $y=|x+2|$ on the same axes, writing the equations of their rays.
  Remember that $y=|x-4|$ is made up of the ray $y=x-4$ (with positive gradient) and $y=4-x$.
  Similarly, $y=|x+2|$ is made up of $y=x+2$ and $y=-x-2$.

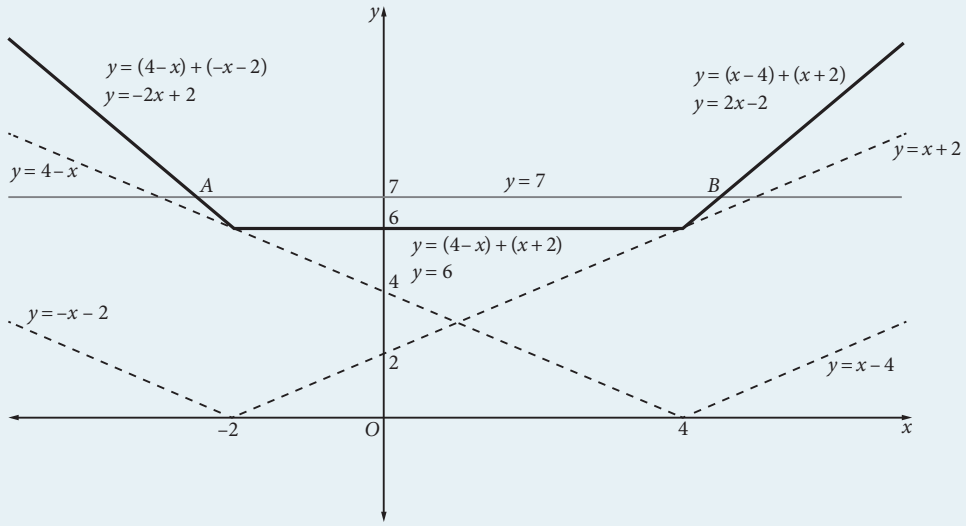
  Note that the graphs have the bases of their 'V'-shapes at $x=4$ and $x=-2$.

Next, sketch $y=|x-4|+|x+2|$ by determining the equations of each of its three parts:

- For $x<-2$, find the equation by adding the ordinates (y values) that apply. Over this domain the two rays that apply are $y=-x-2$ and $y=4-x$, so the equation is $y=(-x-2)+(4-x)$, i.e. $y=-2x+2$. This can easily be sketched. At $x=-2$, $y=-2(-2)+2=6$, so the ray starts at $(-2,6)$ and is drawn back to the left with a gradient of -2 (which is steeper than the existing rays on the diagram).
- For $-2<x<4$, the two rays that apply are $y=x+2$ and $y=4-x$, so the equation is $y=(x+2)+(4-x)$, i.e. $y=6$. Add this horizontal interval to the graph.
- For $x>4$, the two rays that apply are $y=x+2$ and $y=x-4$, so the equation is $y=(x+2)+(x-4)$, i.e. $y=2x-2$.
  Draw this ray on the graph. It starts at $(4,6)$ and proceeds to the right with a gradient of 2.

You have now sketched the 'trough'-shaped graph of $y=|x-4|+|x+2|$ and found the equations of each of its three parts.

- Draw the horizontal line $y=7$ on the graph and label points A and B where $y=|x-4|+|x+2|$ and $y=7$ intersect.



To solve $|x-4|+|x+2|>7$, you need to find the x values for which the graph is above the horizontal line $y=7$, which means you need to find the x-coordinates of A and B.

- To find A, solve simultaneously $y=-2x+2$ and $y=7$: $x=-\frac{5}{2}$
- To find B, solve simultaneously $y=2x-2$ and $y=7$: $x=\frac{9}{2}$

Hence the solution is $x<-\frac{5}{2}$, $x>\frac{9}{2}$.

Inequalities involving square roots

Example 7

(a) Sketch $y = \sqrt{1-x^2}$ and state the domain of this function.

(b) On the same diagram sketch $y = |x-1|$.

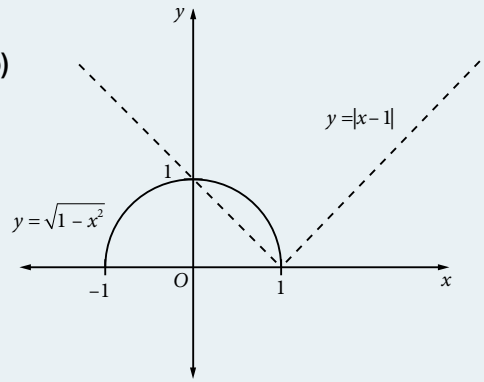
(c) Hence, or otherwise, solve $|x-1| \geq \sqrt{1-x^2}$.

(d) Solve $x - 1 \geq \sqrt{1-x^2}$.

Solution

(a) The sketch is shown at right.
The domain is $-1 \leq x \leq 1$.

(b)



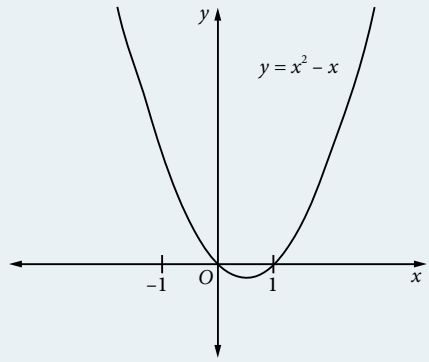
(c) To solve $|x-1| \geq \sqrt{1-x^2}$ you need to state the x values for which the 'V'-shaped absolute graph is on or above the semicircle. The graphs intersect at $x = 0$ and $x = 1$. Also remember the domain from **(a)**, i.e. that the only applicable x values are between -1 and 1. Thus the solution is $-1 \leq x \leq 0$, $x = 1$.

Alternatively:

In this inequality, both sides are known to be non-negative (as $\sqrt{1-x^2}$ means the positive square root), so you can square both sides and know that the inequality will not change.

$$\begin{aligned} \left(|x-1|\right)^2 &\geq \left(\sqrt{1-x^2}\right)^2 \\ x^2 - 2x + 1 &\geq 1 - x^2 \\ 2x^2 - 2x &\geq 0 \\ x^2 - x &\geq 0 \end{aligned}$$

Graphing $y = x^2 - x$:
The parabola is on or above the x-axis for $x \leq 0$, $x \geq 1$.
But the domain is restricted to $-1 \leq x \leq 1$, so the solution is $-1 \leq x \leq 0$, $x = 1$.

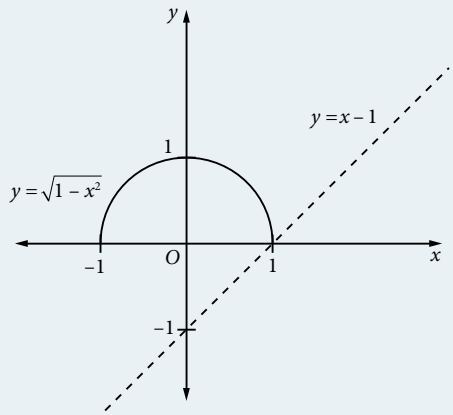


(d) Graphing $y = \sqrt{1-x^2}$ and $y = x - 1$:

The only point of intersection is at $x = 1$.

You now find the x values for which the straight line is on or above the semicircle, keeping in mind that x values must be between -1 and 1.

Thus the solution is $x = 1$.



EXERCISE 1.3 INEQUALITIES INVOLVING ABSOLUTE VALUE AND SQUARE ROOTS

1 $\dfrac{1}{|x|} < 3$ (Hint: The denominator is known to be non-negative.)

2 **(a)** On the same axes sketch $y = 1 - |x|$ and $y = 3x$. **(b)** Hence solve $|x| + 3x > 1$.

3 **(a)** On the same axes sketch $y = |x-2|$ and $y = 2x$. **(b)** Hence solve $|x-2| < 2x$.

4 Solve $x^2 - |x| > 0$.

5 **(a)** On the same axes sketch $y = |x-2|$ and $y = \dfrac{1}{x}$. **(b)** Hence solve $|x-2| > \dfrac{1}{x}$.

6 Solve $\dfrac{x}{x+1} < 1$. (*Note:* This looks like a standard problem, but in fact requires some analysis, depending on which method you use.)

7 Solve $\left|\dfrac{1-x}{2x+1}\right| \geq 1$. (Hint: $|2x+1|$ is known to be non-negative.)

8 Solve $|x+1| - |x-2| \geq 2$.

9 **(a)** On the same axes sketch $y = |x+1|$ and $y = |x-5|$. **(b)** Hence graph $y = |x+1| + |x-5|$.
(c) Solve $|x+1| + |x-5| > 7$. **(d)** Solve $|x+1| + |x-5| = 6$.

1.4 CIRCULAR AND SIMULTANEOUS INEQUALITIES

A circle divides the number plane into two regions, a finite region called its **interior** and an infinite region called its **exterior**, as well as the set of points that make up the circle.

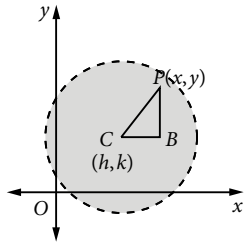
A parabola, cubic, quartic or hyperbola curve divides the number plane into two infinite regions, as well as the set of points that make up the curve.

Points inside and outside a circle

A circle divides the number plane into three sets of points: the sets of points **on** the circle, **inside** the circle and **outside** the circle. The set of points on a circle of centre $C(h, k)$ and radius r is given by the equation $(x-h)^2 + (y-k)^2 = r^2$.

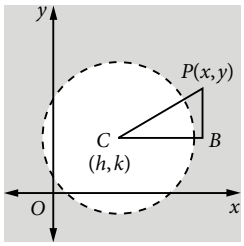
A point $P(x, y)$ lies on this circle if $CP = r$.

If $CP < r$, the point P is inside the circle.



The graph of $(x-h)^2 + (y-k)^2 < r^2$ gives the interior of the circle.

If $CP > r$, the point P is outside the circle.



The graph of $(x-h)^2 + (y-k)^2 > r^2$ gives the exterior of the circle.

Regions involving simultaneous inequalities

Example 8

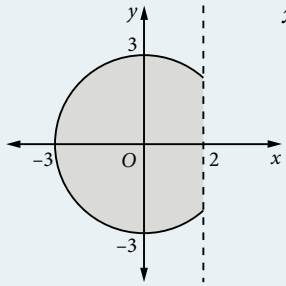
For the circle with centre $(0, 0)$ and radius 3 units, sketch the region of the Cartesian plane that includes all points on or inside the circle that are also:

(a) to the left of the line $x = 2$ **(b)** on or above the line $x + y = 3$.

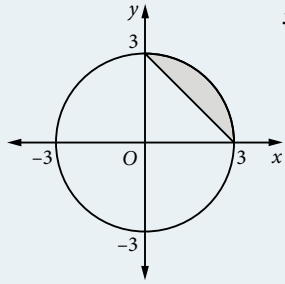
In each case give the inequalities that define the region.

Solution

(a) $x^2 + y^2 \leq 9, x < 2$



(b) $x^2 + y^2 \leq 9, x + y \geq 3$



Example 9

Sketch the region defined by $y \geq x^2$ and $y \leq 2x + 3$. Describe this region in words.

Solution

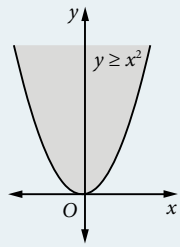
To find the points of intersection, solve simultaneously the equations $y = x^2$ and $y = 2x + 3$.

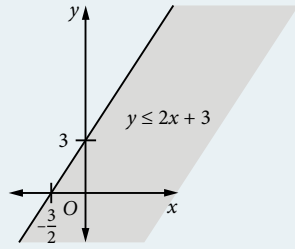
This gives: $x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

Factorise: $(x + 1)(x - 3) = 0$

Solve: $x = -1, 3$

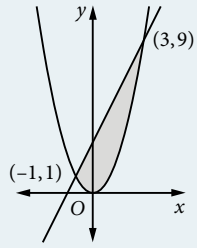




Substitute into $y = 2x + 3$: $x = -1, y = 1$; $x = 3, y = 9$

Hence the points of intersection are $(-1, 1)$ and $(3, 9)$.

The shaded region is the points on and above the parabola $y = x^2$ that are also on or below the line $y = 2x + 3$.



Example 10

Describe the region of the x–y plane whose points satisfy the inequalities $y < 2 + x - x^2$ and $y + 2x \leq 2$.

Solution

The graph of $y = 2 + x - x^2$ can be obtained by completing a table of values and then plotting points. It can also be obtained by completing the square for x and then graphing according the shape and properties of $y = x^2$: $2 + x - x^2 = 2 - \left(x^2 - x\right)$

$$= 2 + \tfrac{1}{4} - \left(x^2 - x + \tfrac{1}{4}\right)$$

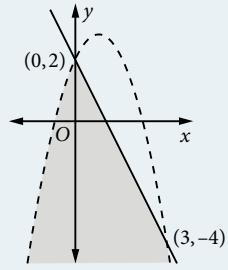
$$= 2\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2$$

Hence you can graph $y = 2\frac{1}{4} - \left(x - \frac{1}{2}\right)^2$, which is the graph of $y = x^2$ turned upside down, moved 0.5 units to the right and moved 2.25 units up:

$y < 2 + x - x^2$ is the region below this curve.

$y + 2x \leq 2$ is the region on or below the line $y + 2x = 2$.

As shown in the diagram, the required region is the region on or below the line $y + 2x = 2$ that is contained between the 'arms' of $y = 2 + x - x^2$.



MAKING CONNECTIONS

Simultaneous inequalities

Use technology to solve simultaneous inequalities graphically.

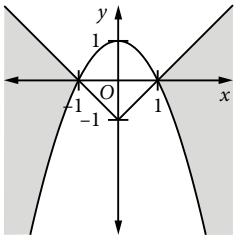
EXERCISE 1.4 CIRCULAR AND SIMULTANEOUS INEQUALITIES

1 Sketch the region defined by each inequality.

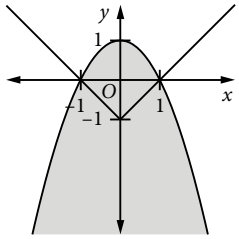
(a) $x^2 + y^2 \geq 16$ **(b)** $x^2 + y^2 < 4$ **(c)** $(x-1)^2 + y^2 > 9$

2 Which diagram shows the region satisfying $y \leq 1 - x^2$ and $y \geq |x| - 1$?

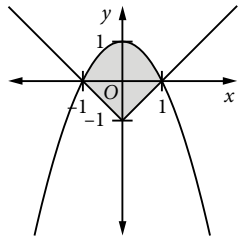
A



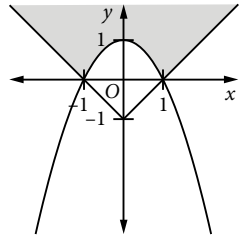
B



C



D



3 Sketch the region defined by each inequality.

(a) $(x+3)^2 + y^2 < 1$ **(b)** $y \leq x^2 + 1$ **(c)** $y \geq 9 - x^2$

(d) $y \geq |x|$ **(e)** $y < 2x + 4$ **(f)** $y < |2x + 4|$

4 Sketch the region defined by the given inequalities.

(a) $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$ **(b)** $(x-1)^2 + (y-1)^2 < 1, x > 0, y > 0$

(c) $y < 4 - x^2, y \geq 0$ **(d)** $y < |x|, y > 0, 1 < x < 2$

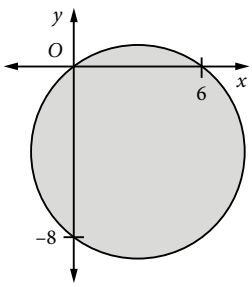
(e) $x^2 + y^2 \leq 4, y > x + 2$ **(f)** $y \geq x^2 - 4, x + y < 2$

(g) $y > |x - 2|, y > 3$ **(h)** $y \leq 1 - x^2, y \geq 0$

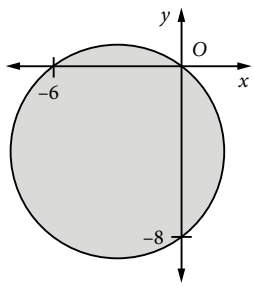
(i) $x^2 + y^2 \leq 1, y \leq 2x, x \geq 0$ **(j)** $(x-1)^2 + y^2 \geq 1, x \geq 0, y \leq 1$

5 Which graph represents the region defined by $(x-3)^2 + (y+4)^2 \leq 25$?

A



B



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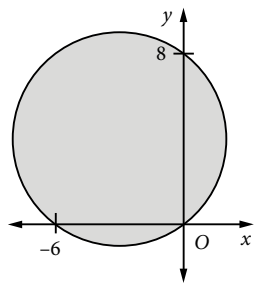
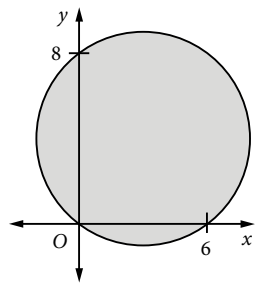


D



6 Sketch the region of the Cartesian plane whose boundary consists of:

(a) the curve $y = x^2$, the ordinates at $x = 1$ and $x = 2$ and the x-axis

(b) the lines $y = 1$ and $y = 3$ and the circle with centre $(0, 2)$ and radius 2 units

(c) the circle $(x-3)^2 + (y-4)^2 = 25$ and the y-axis

(d) the arc of the parabola $y = 4 - x^2$ between $x = 2$ and $x = -2$ and the x-axis

(e) the graph of $y = |x - 1|$, the x-axis and the y-axis

(f) the circle $x^2 + y^2 = 36$ and the lines $y = 6$ and $x = 6$

(g) the circle centred at $(-2, 0)$, radius 2 units, and the circle centred at $(-2, 0)$, radius 1 unit

(h) the parabola $y = x^2 - 2$ and the line $y = x$

(i) the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$

(j) the semicircle $y = \sqrt{1 - x^2}$ and the x-axis.

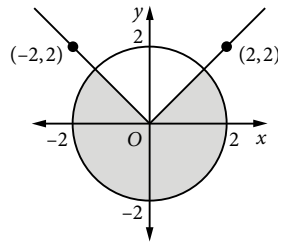
7 For the shaded region in the diagram, state whether each statement is correct or incorrect.

(a) The shaded region is defined by $y \le |x|$ and $x^2 + y^2 \le 4$.

(b) The shaded region is the part of the interior of the circle of centre $(0, 0)$ and radius 2 that is below the lines given by $y = |x|$.

(c) The shaded region is defined by $y \ge |x|$ and $x^2 + y^2 \le 4$.

(d) The shaded region is the part of the circle with centre $(0, 0)$ and radius 2, and its interior, that is on or below the lines given by $y = |x|$.



CHAPTER REVIEW 1

1 The diagram shows the graphs of $y = |x + 2|$ and $y = \sqrt{4 - x^2}$.

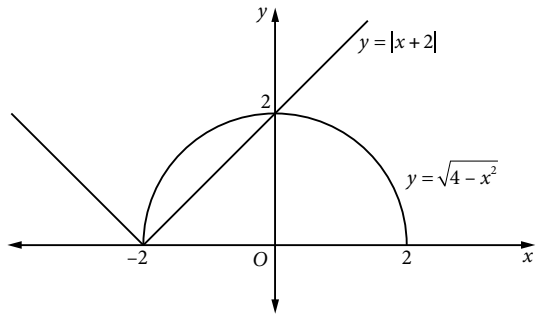
The solution of $\sqrt{4 - x^2} \le |x + 2|$ is:

A $0 \le x \le 2$

B $-2 \le x \le 0$

C $x = -2, 0 \le x \le 2$

D $x \ge 0$



2 Solve the following inequalities.

(a) $\dfrac{2}{1-x} > 1$ **(b)** $\dfrac{1}{x+3} \le \dfrac{2}{x}$ **(c)** $\dfrac{1}{|x-1|} > \dfrac{1}{|x+1|}$ **(d)** $\dfrac{x}{x^2-1} \le 0$

(e) $\dfrac{4x}{x^2-x-6} \le 1$ **(f)** $\dfrac{x}{x-1} < 0$ **(g)** $|x-1| + |x+3| > 6$ **(h)** $x^2 \le |2x-1|$

3 Sketch the region of the Cartesian plane bounded by curves $y = \dfrac{1}{x}$, $x = 1$, $x = 3$ and the x-axis.

4 Sketch the region of the Cartesian plane that satisfies $y \ge x^2 - 1$ and $y \le 1 - |x|$.

5 Sketch the region of the Cartesian plane bounded by the curves $y \ge x^2 - 4$ and $y \le 4 - x^2$.

6 Show that the straight lines $2x + y = 20$ and $x + y = 14$ intersect at $(6, 8)$. Hence sketch the region of the Cartesian plane for which $y \ge 20 - 2x$, $y \le 14 - x$ and $y \ge 0$ are all true.

7 Sketch the region in the number plane defined by $(x - 1)^2 + (y - 1)^2 < 1$ and $x > 1$.

8 Sketch the region of the Cartesian plane bounded by the curves $y = x^2 - 4$ and $y = |x| + 1$.

CHAPTER 2
Polynomials

2.1 POLYNOMIALS

A real polynomial $P(x)$ is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $n, n-1, \ldots$ are all positive integers and $a_n, a_{n-1}, \ldots$ are the coefficients, which for convenience will usually be chosen as real integers.

The term $a_n x^n$ is the **leading term**. If $a_n \neq 0$, the polynomial is said to be of the n-th degree, or 'degree n'.

e.g. $P(x) = 4x^5 - 2x^4 + x^2 + 5x - 6$ is a polynomial of degree 5.

A real polynomial $P(x)$ is defined for all real x and is a continuous and differentiable function of x.

$P(x) = 0$ is a polynomial equation of degree n. Real numbers x that satisfy this polynomial equation are called the real **roots** of the equation or the real **zeros** of the corresponding polynomial.

In the Mathematics Advanced course (see *New Senior Mathematics Advanced for Years 11 & 12*) you have already investigated the roots of the quadratic equation $ax^2 + bx + c = 0$ and found that there are at most two real roots, but there may be only one or none.

Similarly, any polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ will have at most n real roots. (Finding these real roots will become easier after you study the factor theorem.)

- $2x^3 + 4x^2 - 3x - 5$ is a polynomial of the 3rd degree in x, because the highest power of x is 3.
- $x^4 - 2x + 7$ is a polynomial of the 4th degree in x.
- $x^{\frac{1}{2}} + 2 + x$ and $\frac{1}{x} + 2x^{-3}$ are not polynomials, because they include powers of x that are not positive integers.

When the leading coefficient $a_n = 1$, the polynomial is said to be **monic**. $x^4 - 2x + 7$ is a monic polynomial.
A polynomial can also be described according to the subset of real numbers that contain the coefficients $a_n, a_{n-1}, \ldots, a_0$.

- $2x^2 - 7x + 1$ is a 2nd-degree polynomial over the integers.
- $\frac{1}{4}x^5 + 2x + \frac{3}{8}$ is a 5th-degree polynomial over the rational numbers (or 'the rational number field').
- $\sqrt{3}x - \sqrt{6}$ is a 1st-degree polynomial over the irrational numbers.
- $4x^2 + \pi x - \frac{1}{2}$ is a 2nd-degree polynomial over the real numbers.

Example 1

Express the polynomial $P(x) = x^3 - x^4 + 6x^2$ in standard form. Then write:

(a) its degree **(b)** the constant term **(c)** the coefficient of x^2 **(d)** the leading term

(e) the greatest number of real zeros possible. **(f)** Hence solve the equation $P(x) = 0$.

Solution

$P(x) = -x^4 + x^3 + 6x^2$. In standard form (fully expanded): $P(x) = -x^4 + x^3 + 6x^2 + 0x + 0$

(a) degree = 4 **(b)** constant term = 0 **(c)** coefficient of $x^2 = 6$ **(d)** leading term $= -x^4$

(e) The polynomial is of degree 4, so there are at most four zeros.

(f) $-x^4 + x^3 + 6x^2 = 0$
$-x^2(x^2 - x - 6) = 0$
$x^2(x-3)(x+2) = 0$
$\therefore\ x = 0, 3, -2$

The equation has three real roots. $x = 0$ is a double root.

A graph of $y = x^3 - x^4 + 6x^2$ shows that at $x = 0$, the curve touches the x-axis.

Operations with polynomials

You have added, subtracted and multiplied simple algebraic expressions before. With more complicated polynomials the only difference is that more terms are involved, so you must be more careful that you don't miss any terms.

Example 2

If $A(x) = x^2 + 2x + 3$, $B(x) = 2x - 5$, $C(x) = 3x^4 - 3x^2 + 5x + 6$ and $D(x) = x^5 - 3x^2 + 1$, simplify:

(a) $A(x) + C(x)$ **(b)** $B(x) \times D(x)$ **(c)** $D(x) - C(x)$ **(d)** $A(x) \times D(x)$ **(e)** $A(x) + 2C(x) - 3B(x)$

Solution

(a) $A(x) + C(x) = x^2 + 2x + 3 + 3x^4 - 3x^2 + 5x + 6$
$= 3x^4 - 2x^2 + 7x + 9$

(b) $B(x) \times D(x) = (2x - 5)(x^5 - 3x^2 + 1)$
$= 2x^6 - 6x^3 + 2x - 5x^5 + 15x^2 - 5$
$= 2x^6 - 5x^5 - 6x^3 + 15x^2 + 2x - 5$

(c) $D(x) - C(x) = x^5 - 3x^2 + 1 - (3x^4 - 3x^2 + 5x + 6)$
$= x^5 - 3x^2 + 1 - 3x^4 + 3x^2 - 5x - 6$
$= x^5 - 3x^4 - 5x - 5$

(d) $A(x) \times D(x) = (x^2 + 2x + 3)(x^5 - 3x^2 + 1)$
$= x^7 - 3x^4 + x^2 + 2x^6 - 6x^3 + 2x + 3x^5 - 9x^2 + 3$
$= x^7 + 2x^6 + 3x^5 - 3x^4 - 6x^3 - 8x^2 + 2x + 3$

(e) $A(x) + 2C(x) - 3B(x) = x^2 + 2x + 3 + 2(3x^4 - 3x^2 + 5x + 6) - 3(2x - 5)$
$= x^2 + 2x + 3 + 6x^4 - 6x^2 + 10x + 12 - 6x + 15$
$= 6x^4 - 5x^2 + 6x + 30$

EXERCISE 2.1 POLYNOMIALS

1 For the polynomial $P(x) = 3x^4 + 2x^3 + 7$, which statement is correct?

A degree = 3 **B** leading term = 3 **C** leading coefficient = 3 **D** constant term = 3

2 Express the polynomial $P(x) = x^2 - x^3 + 6x$ in standard form. Then write:

(a) its degree **(b)** the constant term **(c)** the coefficient of x^2 **(d)** the leading term
(e) the greatest number of real zeros possible. **(f)** Hence solve the equation $P(x) = 0$.

3 Write the following polynomials in standard form and then state:

(i) the degree **(ii)** the constant term **(iii)** the coefficient of x^2
(iv) whether or not it is monic **(v)** the greatest number of real zeros possible.

(a) $x^2 + 5x^3 + 7 - 6x$ **(b)** $27 - x^3$ **(c)** $ax^3 + bx + cx^2 - d$
(d) $x(5x^3 - 3x^2 + 2)$ **(e)** $(3x + 1)(2x^3 - 5)$ **(f)** $(3x^2 - 2)^2$

4 State whether each expression is a polynomial or not. If it is not a polynomial, explain why.

(a) $x^2 - 6x + 3$ **(b)** $x + 4$ **(c)** $\sqrt{3}x - 4$ **(d)** $x^9 + 1$ **(e)** $4 - \frac{1}{x}$
(f) $\frac{6x + 2}{3}$ **(g)** $x^2 + 3x^{\frac{1}{2}} - 4x^{-1}$ **(h)** $\frac{3x + 2}{3x - 1}$ **(i)** $2^x + 3x - 5$

5 If $A(x) = x^2 - 5x + 1$ and $B(x) = 3x^4 - 2x^2 + 5x + 3$, then $B(x) - A(x) = \ldots$

A $-3x^4 + 3x^2 - 10x - 2$ **B** $3x^4 - 3x^2 + 2$ **C** $3x^4 - 3x^2 - 10x + 2$ **D** $3x^4 - 3x^2 + 10x + 2$

6 If $A(x) = 3x^2 - 2x + 1$, $B(x) = 5x - 2$, $C(x) = 2x^4 - 5x^2 + 3x + 4$ and $D(x) = 2x^5 - 4x^2 - 3$, simplify:

(a) $A(x) + C(x)$ **(b)** $B(x) \times D(x)$ **(c)** $D(x) - C(x)$
(d) $A(x) \times B(x)$ **(e)** $A(x) - 3C(x) + 2B(x)$

7 If $E(x) = x^2 - 3$, $F(x) = 3x + 2$, $G(x) = x^2 + 2x + 1$ and $H(x) = x^2 - 3x + 2$, find the polynomial for:

(a) $E(x) \times F(x)$ **(b)** $F(x) \times G(x)$ **(c)** $3G(x) - 4H(x)$
(d) $(x - 3)G(x)$ **(e)** $[F(x)]^2$ **(f)** $E(x) \times G(x) + F(x) \times H(x)$

2.2 DIVISION OF POLYNOMIALS AND THE REMAINDER THEOREM

If $P(x) = 3x^2 + 7x + 4$, then $P(1)$ is the value of $P(x)$ when $x = 1$, which is found by substituting $x = 1$ into the polynomial. For example:

$P(1) = 3 + 7 + 4 = 14$

$P(-1) = 3 \times (-1)^2 + 7 \times (-1) + 4 = 0$

$P(a) = 3a^2 + 7a + 4$

You already know how to perform long division using integers (e.g. $532 \div 19 = 28$). You can also perform long division of a polynomial by a linear expression or by another polynomial. For example:

$\frac{x^2 + 7x + 12}{x + 3}, x \neq -3$

$(x^2 + 7x + 12) \div (x + 3), x \neq -3$

The two ways of writing the division above are equivalent.

Example 3

Find $\frac{x^2 + 7x + 12}{x + 3}, x \neq -3$.

Solution

The condition $x \neq -3$ exists because it cannot be divided by zero.

Divide the leading term of the dividend (x^2) by the leading term of the divisor (x) and write the answer (x) above the x^2.

Multiply the divisor $(x + 3)$ by the result of $x^2 \div x$ (i.e. x) to obtain $x^2 + 3x$.

Subtract $(x^2 + 3x)$ from $(x^2 + 7x)$ to get $4x$, then bring down the $+12$ from above.

Divide this new leading term ($4x$) by x and write the answer ($+4$) above $7x$.

Multiply $(x + 3)$ by 4 to obtain $4x + 12$.

Subtract the last two lines to get a remainder of 0, completing the division.

$$\begin{array}{r} x + 4 \\ x+3 \overline{)\, x^2 + 7x + 12} \\ \underline{x^2 + 3x} \\ 4x + 12 \\ \underline{4x + 12} \\ 0 \end{array}$$

Hence you have $\frac{x^2 + 7x + 12}{x + 3} = x + 4$, with no remainder.

You could also write $x^2 + 7x + 4 = (x + 3)(x + 4) + 0$:

Dividend = Divisor × Quotient + Remainder

Example 4

If $P(x) = 3x^3 - 7x^2 + 2x + 4$, divide $P(x)$ by $x - 3$.

Solution

Divide $3x^3$ by x and write the answer ($3x^2$) above the $3x^3$.

Multiply $(x - 3)$ by $3x^2$ to obtain $3x^3 - 9x^2$ and write it under $3x^3 - 7x^2$.

Subtract $(3x^3 - 9x^2)$ from $(3x^3 - 7x^2)$ to get $2x^2$, then bring down the $+2x$ from above.

Divide $(2x^2 + 2x)$ by x and write the answer ($+2x$) above $-7x^2$.

Multiply $(x - 3)$ by $2x$ to obtain $2x^2 - 6x$ and write it under $2x^2 + 2x$.

Subtract $(2x^2 - 6x)$ from $(2x^2 + 2x)$ to get $8x$, then bring down the $+4$ from above.

Divide $8x$ by x and write the answer ($+8$) above $+2x$.

Multiply $(x - 3)$ by 8 to obtain $8x - 24$ and write it under $8x + 4$.

Subtract $(8x - 24)$ from $(8x + 4)$ to get 28.

$$\begin{array}{r} 3x^2 + 2x + 8 \\ x-3 \overline{)\, 3x^3 - 7x^2 + 2x + 4} \\ \underline{3x^3 - 9x^2} \\ 2x^2 + 2x \\ \underline{2x^2 - 6x} \\ 8x + 4 \\ \underline{8x - 24} \\ 28 \end{array}$$

The number 28 is a constant with a degree less than the divisor, so it is not divisible by $(x-3)$.
Thus the remainder is 28.

Hence: $\dfrac{3x^3-7x^2+2x+4}{x-3} = 3x^2+2x+8+\dfrac{28}{x-3}$

or $3x^3-7x^2+2x+4 = (x-3)(3x^2+2x+8)+28$

i.e. $P(x) = (x-3)Q(x)+R$

where $Q(x)$ is the quotient and R is the remainder (i.e. in this case, 28). *Note:*

The degree of the remainder < the degree of the divisor.
The degree of the quotient < the degree of the dividend.

When the divisor is a linear function (first degree), the remainder will be a constant.

Note also that $P(x) = (x-3)(3x^2+2x+8)+28$, so:

$$P(3) = (3-3)\times Q(3)+28 = 0+28 = 28$$

This suggests that the remainder of division by $(x-3)$ could be found just by finding the value of $P(3)$, without doing the long division.

Example 5

Divide $4x^3-19x+9$ by $2x-3$.

Solution

When writing out the dividend, include all missing terms by writing them with zero coefficients.

$$\begin{array}{r} 2x^2+3x-5 \\ 2x-3\overline{)\,4x^3+0x^2-19x+9} \\ \underline{4x^3-6x^2} \\ 6x^2-19x \\ \underline{6x^2-9x} \\ -10x+9 \\ \underline{-10x+15} \\ -6 \end{array}$$

The quotient is $2x^2+3x-5$ and the remainder is -6.

$$\frac{4x^3-19x+9}{2x-3} = 2x^2+3x-5-\frac{6}{2x-3}$$

or $(4x^3-19x+9) = (2x-3)(2x^2+3x-5)-6$

$$P(x) = (2x-3)\times Q(x)-6$$

If $2x-3=0$ then $x=\dfrac{3}{2}$.

$$P\left(\frac{3}{2}\right) = 0\times Q\left(\frac{3}{2}\right)-6 = -6$$

This again suggests a simpler method for finding the remainder.

Example 6

Divide $x^4+x^3-7x^2-x+6$ by x^2-1.

Solution

$$\begin{array}{r} x^2+x-6 \\ x^2-1\overline{)\,x^4+x^3-7x^2-x+6} \\ \underline{x^4\quad -x^2} \\ x^3-6x^2-x \\ \underline{x^3\quad -x} \\ -6x^2\quad +6 \\ \underline{-6x^2\quad +6} \\ 0 \end{array}$$

The remainder is zero, so this means that (x^2-1) is a factor of $x^4+x^3-7x^2-x+6$. This then means that $(x+1)$ and $(x-1)$ are also factors of $x^4+x^3-7x^2-x+6$.

The remainder theorem

If a polynomial $P(x)$ is divided by $(x-a)$ until the remainder R does not contain x, then $R = P(a)$.

For any polynomial, $P(x) = (x-a)Q(x) + R$, where $Q(x)$ is a polynomial.

Thus $P(a) = (a-a)Q(a) + R$

i.e. $P(a) = 0 \times Q(a) + R$

$\therefore \quad R = P(a)$

Important considerations:

1. If $P(x)$ is divided by $x+a$, as $x+a = x-(-a)$, then $R = P(-a)$.
2. If $P(x)$ is divided by $ax-b$, as $ax-b = a\left(x-\frac{b}{a}\right)$, then $R = P\left(\frac{b}{a}\right)$.

Example 7

Find the remainder when $P(x) = 2x^3 - 6x^2 + 4x + 3$ is divided by the following.

(a) $x-2$ **(b)** $x+3$ **(c)** $2x-1$ **(d)** $3x+2$

Solution

(a) $R = P(2) = 2\times 2^3 - 6\times 2^2 + 4\times 2 + 3 = 16 - 24 + 8 + 3 = 3$

(b) $R = P(-3) = 2\times(-3)^3 - 6\times(-3)^2 + 4\times(-3) + 3 = -54 - 54 - 12 + 3 = -117$

(c) $R = P\left(\frac{1}{2}\right) = 2\times\left(\frac{1}{2}\right)^3 - 6\times\left(\frac{1}{2}\right)^2 + 4\times\frac{1}{2} + 3 = \frac{1}{4} - \frac{3}{2} + 2 + 3 = 3\frac{3}{4}$

(d) $R = P\left(-\frac{2}{3}\right) = 2\times\left(-\frac{2}{3}\right)^3 - 6\times\left(-\frac{2}{3}\right)^2 + 4\times\left(-\frac{2}{3}\right) + 3 = -\frac{16}{27} - \frac{8}{3} - \frac{8}{3} + 3 = -\frac{79}{27}$

EXERCISE 2.2 DIVISION OF POLYNOMIALS AND THE REMAINDER THEOREM

1 Perform the following long divisions.

(a) $(3x^2 - 2x + 5) \div (x-2)$
(b) $(x^3 - x^2 + x - 1) \div (x-1)$
(c) $(4x^3 - 8x^2 + 8x - 1) \div (2x+1)$
(d) $(2x^4 - 3x^3 + 6x^2 - 5x - 4) \div (2x-3)$
(e) $(x^3 - 27) \div (x-3)$
(f) $(x^4 + 3x^3 + 5x^2 - 6x - 8) \div (x+2)$
(g) $(-2x^3 + 7x^2 - 7x + 1) \div (2x+3)$
(h) $(x^4 - 3x^2 + 2) \div (x-1)$
(i) $(x^3 + 3x^2 - 2x + 1) \div (x^2 - 1)$
(j) $(x^4 + 2x^3 - 3x^2 + x) \div (x^2 + 1)$
(k) $(x^3 - 4x^2 + 2x + 3) \div (x^2 - x + 1)$
(l) $(x^4 + 5x^3 - x - 5) \div (x^2 + x + 1)$

2 If $P(x) = 2x^3 - 4x^2 + 2x - 1$, find $P(1)$, $P(-3)$, $P(a)$, $P(-2)$.

3 If $P(x) = x^4 - 2x^3 + x^2 - 4$, find $P(2)$, $P(-1)$, $P(a)$.

4 If $P(x) = x^3 - 3x^2 + kx - 7$ and $P(2) = 0$, then $k = \ldots$

A 5.5 **B** 0 **C** −5.5 **D** −6.5

5 $P(x) = x^4 - x^3 + px^2 - 4x + q$. Find p and q if $P(0) = 3$ and $P(-1) = 11$.

6 $P(x) = ax^3 - 2x^2 + bx + c$. Find a, b and c if $P(0) = 12$, $P(-1) = 3$ and $P(2) = 36$.

7 Using the remainder theorem, find the remainder when:

(a) $x^3 + 3x^2 + 2x - 7$ is divided by $(x+2)$
(b) $3x^2 + 7x - 2$ is divided by $(x-4)$
(c) $2y^3 + y^2 - 2y + 3$ is divided by $(y+1)$
(d) $x^4 - 3x^3 + 5x^2 - 2$ is divided by $(x-4)$
(e) $x^4 + 2x^2 - 13x - 60$ is divided by $(2x+3)$
(f) $x^4 - 3x^3 + x + 1$ is divided by $(2x-1)$.

8 When $x^3 + 3x^2 + 7x - b$ is divided by $(x-2)$ the remainder is 20. Find b.

9 When $x^4 + 5x^3 - ax + b$ and $ax^2 + bx - 1$ are each divided by $(x + 1)$, the remainders are 7 and -6 respectively. Find a and b.

10 When the polynomial $x^3 + 3x^2 - mx + n$ is divided by $(x + 2)$ the remainder is 9; when divided by $(x - 3)$ the remainder is 49. Find m and n.

11 If $x^3 + 2x^2 + 3x + p$ and $x^3 + x^2 + 9$ have the same remainder when divided by $x + 2$, find p.

12 When $3x^3 - ax^2 - bx + 1$ is divided by $(x - 2)$ the remainder is 5; when divided by $(x - 1)$ there is no remainder. Find a and b.

13 When $2x^3 + 7x^2 + ax + b$ is divided by $(x - 3)$ the remainder is 120; when divided by $(x + 1)$ the remainder is -8. Find the values of a and b.

14 Let $P(x) = x^3 - ax^2 + x$ be a polynomial, where a is a real number. When $P(x)$ is divided by $(x - 2)$ the remainder is -2. Find the remainder when $P(x)$ is divided by $(x + 1)$.

2.3 THE FACTOR THEOREM

When the remainder of a division is zero, the divisor is a factor of the expression being divided. For example, 2 is a factor of 14 because $14 = 2 \times 7 + 0$. Similarly, $(x + 3)$ is a factor of $x^2 + 4x + 3$ because $x^2 + 4x + 3 = (x + 3)(x + 1) + 0$.

The **factor theorem** states:

For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

The converse of this result is also true:

If $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

By finding the zeros of a polynomial (i.e. values of a, such that $P(a) = 0$), you can **factorise** the polynomial. The first factor can usually be found by trial and error, and then by long division you can find the quotient, which will have a degree of one less than the original polynomial. You may then be able to find factors of the quotient, which must also be factors of the original polynomial.

For example, if the original polynomial is cubic, then the quotient will be a quadratic polynomial, which you can then factorise into real factors (if they exist). For higher degree polynomials, further trial and error may be needed after long division until you can reach a factorisable quotient.

Example 8

$P(x) = x^3 - x^2 - 14x + 24$. Find one zero of the polynomial and then express $P(x)$ as a product of linear factors.

Solution

$P(x) = x^3 - x^2 - 14x + 24$ is monic, so the factors of 24 can be regarded as the only possible zeros, namely $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$:

$P(1) = 1 - 1 - 14 + 24 = 10 \neq 0$, $(x - 1)$ is not a factor

$P(-1) = -1 - 1 + 14 + 24 = 36 \neq 0$, $(x + 1)$ is not a factor

$P(2) = 8 - 4 - 28 + 24 = 0$, $(x - 2)$ is a factor

Hence $P(x) = (x - 2)(x^2 + x - 12)$

$= (x - 2)(x + 4)(x - 3)$

The factor theorem could also have been used to find the zeros 3 and -4. You should use the method that you find easiest and quickest in each case.

$$
\begin{array}{r}
x^2 + x - 12 \\
x-2 \overline{)\, x^3 - x^2 - 14x + 24} \\
\underline{x^3 - 2x^2} \\
x^2 - 14x \\
\underline{x^2 - 2x} \\
-12x + 24 \\
\underline{-12x + 24} \\
0
\end{array}
$$

Example 9

Find the linear factors of $x^4 + x^3 - 7x^2 - x + 6$.

Solution

$P(x) = x^4 + x^3 - 7x^2 - x + 6$

$P(1) = 1 + 1 - 7 - 1 + 6 = 0$ Hence $(x - 1)$ is a factor.

$P(-1) = 1 - 1 - 7 + 1 + 6 = 0$ Hence $(x + 1)$ is a factor.

You now know that $(x - 1)(x + 1) = x^2 - 1$ is a factor, so you can use this as the divisor in the long division.

$$x^2 - 1 \overline{)\, x^4 + x^3 - 7x^2 - x + 6}$$

$$\begin{array}{r} x^2 + x - 6 \\ x^4 \quad - x^2 \\ \hline x^3 - 6x^2 - x \\ x^3 \quad - x \\ \hline -6x^2 + 6 \\ -6x^2 + 6 \\ \hline 0 \end{array}$$

Hence $P(x) = x^4 + x^3 - 7x^2 - x + 6 = (x - 1)(x + 1)(x^2 + x - 6)$

$= (x - 1)(x + 1)(x + 3)(x - 2)$

EXERCISE 2.3 THE FACTOR THEOREM

1 If $P(x) = x^4 + x^2 - 2$, indicate whether each statement is correct or incorrect.

(a) $P(1) = 0$ **(b)** $P(2) = 0$ **(c)** $P(-1) = 0$ **(d)** $P(-2) = 0$

2 Use the factor theorem to factorise each polynomial over the rational numbers.

(a) $x^3 + 4x^2 - 7x - 10$ **(b)** $x^3 + 2x^2 - 41x - 42$ **(c)** $2x^3 + x^2 - 22x + 24$

(d) $6x^3 - 5x^2 - 12x - 4$ **(e)** $3x^3 + 10x^2 + 9x + 2$ **(f)** $2x^3 + 7x^2 - 10x - 24$

3 Use the factor theorem to find the linear factors of each polynomial over the rational number field.

(a) $5 - 6x - 29x^2 + 6x^3$ **(b)** $4x^3 + 12x^2 - x - 3$ **(c)** $6x^3 + 35x^2 + 34x - 40$

(d) $6x^3 - 29x^2 - 40x - 12$ **(e)** $x^3 - 5x^2 + 6x$ **(f)** $x^3 + 2x^2 - 4x - 8$

4 Use the factor theorem to find the linear factors of each polynomial over the rational number field.

(a) $6x^3 - 5x^2 - 2x + 1$ **(b)** $3x + 7x^2 - 6x^3$ **(c)** $2x^3 - 7x^2 + 7x - 2$

(d) $2x^3 + 3x^2 - 9x$ **(e)** $6 + 5x - 2x^2 - x^3$ **(f)** $x^3 - x^2 - 10x - 8$

5 The only linear factor of $x^3 - 8$ is: **A** $x - 1$ **B** $x + 4$ **C** $x + 2$ **D** $x - 2$

6 Find the values of a and b that make $x^4 + 4x^3 + ax^2 - b$ divisible by $(x - 1)$ and $(x + 2)$.

7 Given that $(x - 3)$ and $(x + 2)$ are factors of $x^3 - 6x^2 + px + q$, find the values of p and q.

8 If $5x^2 - 6x - 56$ and $3x^2 - 14x + a$ have a common factor $(x + b)$, find the values of a and b.

9 Given that $(x - 1)$ and $(x + 2)$ are factors of $6x^4 + ax^3 - 17x^2 + bx - 4$, find a and b and the other two factors.

10 Find the values of a and b that make $2x^3 + ax^2 - 13x + b$ exactly divisible by $x^2 - x - 6$.

11 Find the linear factors over the real number field of $x^3 - 3x^2 - 8x + 4$.

12 Let $P(x) = (x - 1)(x + 3)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $(x + 3)$. When $P(x)$ is divided by $(x - 1)$ the remainder is 8.

(a) Find the values of a and b. **(b)** Find the remainder when $P(x)$ is divided by $(x - 1)(x + 3)$.

13 The polynomial $P(x)$ is given by $P(x) = ax^3 + 12x^2 + cx - 60$, where a and c are constants. The three zeros of $P(x)$ are 2, -3 and β. Find the value of β.

14 When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$. What is the remainder when $P(x)$ is divided by $x - 1$?

2.4 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

Quadratic equations

The general quadratic equation is: $ax^2 + bx + c = 0,\ a \neq 0$

Dividing both sides by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ [1]

Let the roots of this equation be α and β.

Hence: $(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ [2]

i.e. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha + \beta = -\frac{b}{a}$ [3]

Product of roots: $\alpha\beta = \frac{c}{a}$ [4]

The equations [3] and [4] show the relationships between the roots α, β and the coefficients a, b, c of a quadratic equation.

Example 10

Write the quadratic equation with roots that are the squares of the roots of $2x^2 + 3x + 5 = 0$.

Solution

This can be answered without solving the original equation.

Let the roots of $2x^2 + 3x + 5 = 0$ be α and β.

Now, $a = 2$, $b = 3$, $c = 5$: $\alpha + \beta = -\frac{3}{2}$ and $\alpha\beta = \frac{5}{2}$

The new equation is of the form: $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

But $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$, so $\alpha^2 + \beta^2 = \left(-\frac{3}{2}\right)^2 - 2 \times \frac{5}{2} = \frac{9}{4} - 5 = -\frac{11}{4}$

and $\alpha^2\beta^2 = \frac{25}{4}$

Hence the required equation is: $x^2 + \frac{11}{4}x + \frac{25}{4} = 0$

i.e. $4x^2 + 11x + 25 = 0$

Two useful identities:

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

Cubic equations

The general cubic equation is: $ax^3 + bx^2 + cx + d = 0,\ a \neq 0$

Dividing both sides by a: $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ [1]

Let the roots of this equation be α, β and γ.

Hence: $(x - \alpha)(x - \beta)(x - \gamma) = 0$

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$ [2]

i.e. $x^3 - (\text{sum of roots})x^2 + (\text{sum of products of each pair of roots})x - (\text{product of roots}) = 0$

Equating the coefficients of like powers of x in [1] and [2]:

$$\text{Sum of roots: } \alpha + \beta + \gamma = -\frac{b}{a} \qquad [3]$$

$$\text{Sum of products of each pair of roots: } \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \qquad [4]$$

$$\text{Product of roots: } \alpha\beta\gamma = -\frac{d}{a} \qquad [5]$$

The equations [3], [4] and [5] show the relationships between the roots α, β, γ and the coefficients a, b, c, d of a cubic equation.

Note: These relationships between the roots and the coefficients are not enough to find the roots of an equation without some additional information.

Example 11

Show that $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$.

Solution

$$\begin{aligned}\text{RHS} &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \beta^2 + \gamma^2 = \text{LHS}\end{aligned}$$

Example 12

If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, find the value of the following.

(a) $\alpha + \beta + \gamma$ **(b)** $\alpha\beta + \alpha\gamma + \beta\gamma$ **(c)** $\alpha\beta\gamma$ **(d)** $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(f)** $\alpha^2 + \beta^2 + \gamma^2$ **(g)** $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

Solution

(a) Sum of roots $= \alpha + \beta + \gamma = -\frac{b}{a} = -2$

(b) Sum of products of pairs of roots $= \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 3$

(c) Product of roots $= \alpha\beta\gamma = -\frac{d}{a} = -4$

(d) $$\begin{aligned}(\alpha - 1)(\beta - 1)(\gamma - 1) &= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1\\ &= -4 - 3 - 2 - 1\\ &= -10\end{aligned}$$

(e) $$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\\ &= -\frac{3}{4}\end{aligned}$$

(f) $$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= (-2)^2 - 2 \times 3\\ &= -2\end{aligned}$$

(g) $$\begin{aligned}\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} &= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}\\ &= \frac{-2}{-4}\\ &= \frac{1}{2}\end{aligned}$$

Another useful identity:

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

Example 13

Solve the equation $2x^3 - 7x^2 - 12x + 45 = 0$, given that two of its roots are equal.

Solution

Let the roots be α, α, β.

Sum of roots: $2\alpha + \beta = \frac{7}{2}$ [1]

Sum of products of pairs of roots: $\alpha^2 + 2\alpha\beta = -6$ [2]

Product of roots: $\alpha^2\beta = -\frac{45}{2}$ [3]

From [1]: $\beta = \frac{7}{2} - 2\alpha$

Substitute into [2]:

$$\alpha^2 + 2\alpha\left(\frac{7}{2} - 2\right) = -6$$
$$\alpha^2 + 7\alpha - 4\alpha^2 = -6$$
$$3\alpha^2 - 7\alpha - 6 = 0$$
$$(3\alpha + 2)(\alpha - 3) = 0$$
$$\alpha = 3 \quad \textbf{or} \quad -\frac{2}{3}$$

Substitute into [1]: $\beta = -\frac{5}{2}$ **or** $\frac{29}{6}$

Substitute $\alpha = 3$, $\beta = -\frac{5}{2}$ into [3]: $\text{LHS} = 3^2 \times \left(-\frac{5}{2}\right) = -\frac{45}{2} = \text{RHS}$

Substitute $\alpha = -\frac{2}{3}$, $\beta = \frac{29}{6}$ into [3]: $\text{LHS} = \left(-\frac{2}{3}\right)^2 \times \frac{29}{6} = -\frac{58}{27} \neq \text{RHS}$

Therefore, the roots of the equation are 3, 3, –2.5.

Example 14

If the roots of the equation $x^3 + px^2 + qx + r = 0$ are consecutive terms of an arithmetic series, find an equation linking p, q and r.

Solution

Let the roots be $\alpha - \beta$, α, $\alpha + \beta$ (i.e. three terms with the common difference β).

Sum of roots: $3\alpha = -p$ [1]

Sum of products of pairs of roots: $\alpha(\alpha - \beta) + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = q$

$3\alpha^2 - \beta^2 = q$ [2]

Product of roots: $\alpha(\alpha^2 - \beta^2) = -r$ [3]

From [1]: $\alpha = -\frac{p}{3}$

Substitute into [2]:

$$\frac{p^2}{3} - \beta^2 = q$$
$$\beta^2 = \frac{p^2}{3} - q$$

Substitute into [3]: $-\frac{p}{3}\left(\frac{p^2}{9}-\left(\frac{p^2}{3}-q\right)\right)=-r$

$$p\left(\frac{p^2}{9}-\frac{p^2}{3}+q\right)=3r$$

$$p(p^2-3p^2+9q)=27r$$

$$p(9q-2p^2)=27r$$

Quartic equations (4th degree)

The general quartic equation is: $ax^4+bx^3+cx^2+dx+e=0,\ a\neq 0$

Dividing both sides by a: $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$ [1]

Let the roots of this equation be α, β, γ and δ.

Hence: $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$

$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x+\alpha\beta\gamma\delta=0$ [2]

i.e. x^4-(sum of roots)x^3+(sum of products of each pair of roots)x^2
$-$(sum of products of each triplet of roots)$x+$(product of roots) $=0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha+\beta+\gamma+\delta=-\frac{b}{a}$ [3]

Sum of products of pairs of roots: $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$ [4]

Sum of products of triplets of roots: $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$ [5]

Product of roots: $\alpha\beta\gamma\delta=\frac{e}{a}$ [6]

EXERCISE 2.4 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

1 If α and β are roots of the equation $x^2+8x-5=0$, find the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

2 If α and β are roots of the equation $x^2+4x+1=0$, find the value of:

(a) $\alpha+\beta$ **(b)** $\alpha\beta$ **(c)** $\alpha^2+\beta^2$ **(d)** $\alpha^3+\beta^3$

3 If α and β are roots of the equation $x^2+mx+n=0$, find the roots of $nx^2+(2n-m^2)x+n=0$ in terms of α and β.

4 If α and β are roots of the equation $px^2+qx+r=0$, find the following in terms of p, q and r.

(a) $\frac{1}{\alpha}+\frac{1}{\beta}$ **(b)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

5 If α and β are roots of the equation $x^2+5x+7=0$, the quadratic equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$ is:

A $x^2+22x+75=0$ **B** $x^2+22x-75=0$ **C** $x^2-22x-75=0$ **D** $x^2-22x+75=0$

6 Solve the equation $x^3-3x^2-4x+12=0$, given that the sum of two of its roots is zero.

7 Solve the equation $2x^3-21x^2+42x-16=0$, given that the roots are consecutive terms of a geometric series. (Hint: Let the roots be $\frac{\alpha}{\beta}$, α, $\alpha\beta$.)

8 Solve the equation $4x^3+32x^2+79x+60=0$, given that one root is equal to the sum of the other two roots.

9 Solve the equation $3x^3-17x^2-8x+12=0$, given that the product of two of the roots is 4.

10 If the roots of the equation $x^3 + px^2 + qx + r = 0$ are consecutive terms of a geometric series, prove that $q^3 = p^3 r$. Show that this condition is satisfied for the equation $8x^3 - 100x^2 + 250x - 125 = 0$ and solve this equation.

11 Find two values of m, such that the roots of the equation $x^3 + 2x^2 + mx - 16 = 0$ are $\alpha, \beta, \alpha\beta$. Using these values of m, find α and β.

12 If the roots of the equation $24x^4 - 52x^3 + 18x^2 + 13x - 6 = 0$ are $\alpha, -\alpha, \beta, \frac{1}{\beta}$, find the values of α and β.

13 If two of the roots of the equation $x^3 + qx + r = 0$ are equal, show that $4q^3 + 27r^2 = 0$.

14 If the roots of the equation $x^3 + 3x^2 - 2x + 1 = 0$ are α, β, γ, find the value of:

(a) $\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)$ **(b)** $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

15 Solve the equation $4x^3 - 12x^2 + 9x - 2 = 0$, given that two of its roots are equal.

16 Solve the equation $6x^4 - 11x^3 - 26x^2 + 22x + 24 = 0$, given that the product of two of the roots is equal to the product of the other two roots.

17 Solve the equation $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, given that the product of two of the roots is 2.

18 Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$, given that the roots are consecutive terms of an arithmetic series.

19 Solve the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, given that the sum of two of its roots is zero.

20 If α, β, γ are the roots of $3x^3 + 8x^2 - 1 = 0$, find the value of: $\left(\beta + \frac{1}{\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right)\left(\alpha + \frac{1}{\beta}\right)$.

21 Solve the equation $x^4 - 2x^3 - 3x^2 + 4x + 2 = 0$, given that the sum of two of its roots is zero.

22 If the roots of the equation $x^4 - 6x^3 + 3x^2 + 12x + 4 = 0$ are $\alpha, 2\alpha, \beta, 2\beta$, find the values of α and β.

2.5 MULTIPLE ROOTS OF A POLYNOMIAL EQUATION

A polynomial of degree n has n zeros, but they are not necessarily all different. You say that c is a zero of multiplicity r $(r > 1)$ when the factor $(x - c)$ occurs r times.

For example, if $P(x) = (x-1)^3(x-5)^2(x-6)$, then 1 is a zero of multiplicity three, 5 is a zero of multiplicity two and 6 is a zero of multiplicity one.

Furthermore, if $x = c$ is a zero of multiplicity r of the real polynomial $P(x)$, then $x = c$ is also a zero of multiplicity $(r-1)$ of the derived polynomial $P'(x)$, a zero of multiplicity $(r-2)$ of the second derived polynomial $P''(x)$, and so on. The proof of this result follows.

If $P(x) = (x-c)^r S(x)$, where $r > 0$, $S(c) \neq 0$

$$\begin{aligned} \text{then } P'(x) &= r(x-c)^{r-1} S(x) + (x-c)^r S'(x) \\ &= (x-c)^{r-1}[rS(x) + (x-c)S'(x)] \\ &= (x-c)^{r-1} Q(x) \quad [1] \end{aligned}$$

i.e. the polynomial $P'(x)$, has a zero $x = c$ of multiplicity $(r-1)$.

Applying the product rule to $P'(x)$ in [1] produces the polynomial $P''(x)$ with a zero $x = c$ of multiplicity $(r-2)$.

If $P(x)$ is a polynomial of degree n, then $P'(x)$ must be a polynomial of degree $(n-1)$, $P''(x)$ a polynomial of degree $(n-2)$, and so on. This property allows us to use calculus techniques to solve equations that are known to have multiple roots.

Example 15

Solve $x^3 - 4x^2 - 3x + 18 = 0$, given that it has a root of multiplicity 2.

Solution

Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$

Differentiate: $P'(x) = 3x^2 - 8x - 3$

Factorise: $P'(x) = (3x + 1)(x - 3)$

Solve: $P'(x) = 0$: $x = -\frac{1}{3}, 3$

As $P(x) = 0$ has a root of multiplicity 2 (i.e. a double root), the solution must be either $x = -\frac{1}{3}$ *or* $x = 3$, but not both.

Evaluate: $P\left(\frac{1}{3}\right) = -\frac{1}{27} - \frac{4}{9} + 1 + 18 = 17\frac{14}{27} \neq 0$

$P(3) = 27 - 36 - 9 + 18 = 0$. Thus $x = 3$ is a double zero.

$\therefore P(x) = (x - 3)(x - 3)(x - a)$

As $P(0) = 18$: $(-3) \times (-3) \times (-a) = 18$

$a = -2$

Thus $P(x) = (x - 3)^2(x + 2)$

The roots of the equation are $x = -2, 3, 3$.

EXERCISE 2.5 MULTIPLE ROOTS OF A POLYNOMIAL EQUATION

1 The polynomial $x^3 - x^2 - 5x - 3$ has a double root at $x = \alpha$. What is the value of α?

A $-\frac{5}{3}$ **B** -1 **C** 1 **D** $\frac{5}{3}$

2 If $P(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$, then:

(a) show that $x = 1$ is a zero of multiplicity 4 **(b)** fully factorise $P(x)$.

3 If $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, then:

(a) show that $P(x)$ has a zero of multiplicity 3 **(b)** fully factorise $P(x)$ **(c)** solve the equation $P(x) = 0$.

4 If $P(x) = x^3 - x^2 - 8x + 12$, then:

(a) show that $P(x)$ has a zero of multiplicity 2 **(b)** fully factorise $P(x)$ **(c)** solve the equation $P(x) = 0$.

5 If $P(x) = 3x^4 - 8x^3 + 6x^2 - 24x + 40$, then:

(a) show that $P(x)$ has a zero of multiplicity 2 **(b)** fully factorise $P(x)$ **(c)** solve the equation $P(x) = 0$.

6 The polynomial $P(x) = ax^4 + bx^3 + cx^2 + e$ has remainder -4 when divided by $(x + 1)$. The polynomial has a double root at $x = 1$.

(a) Show that $2a + c = -3$.

(b) Hence, or otherwise, find the slope of the tangent to the graph $y = P(x)$ where $x = -1$.

7 Let $P(x) = x^3 + ax^2 + bx - 4$, where a and b are real numbers. If $(x - 2)^2$ is a factor of $P(x)$, find the values of a and b.

8 The polynomial $P(x) = ax^3 + bx + c$ has a multiple zero at -1 and has remainder 8 when divided by $(x - 1)$. If a, b and c are real, find their values.

9 Solve each equation using the properties of polynomials.

(a) $4x^3 - 8x^2 + 5x - 1 = 0$, given that it has a root of multiplicity 2.

(b) $x^4 + 4x^3 - 16x - 16 = 0$, given that it has a root of multiplicity 3.

10 Solve each equation using the properties of polynomials.

(a) $9x^3 + 12x^2 - 11x + 2 = 0$, given that it has a root of multiplicity 2.

(b) $2x^4 + 5x^3 + 3x^2 - x - 1 = 0$, given that it has a root of multiplicity 3.

2.6 POLYNOMIAL FUNCTIONS

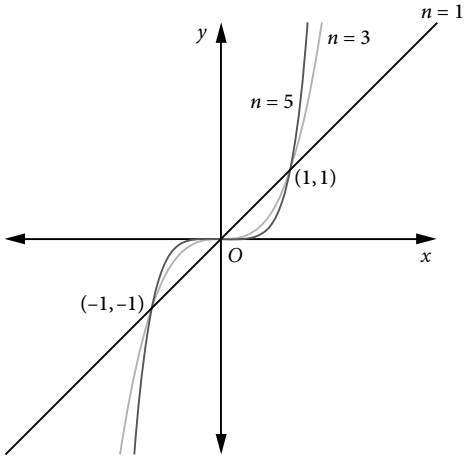
The function f, where $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \ \ (a_n \neq 0)$, is called the **general polynomial function**. This function is defined for all real values of x and is continuous and differentiable. Following the earlier notation, $P(x)$ is often written instead of $f(x)$ when defining rules for polynomial functions. For $a \neq 0$, there is:

- $f(x) = ax + b$ (general linear function)
- $f(x) = ax^2 + bx + c$ (general quadratic function)
- $f(x) = ax^3 + bx^2 + cx + d$ (general cubic function)

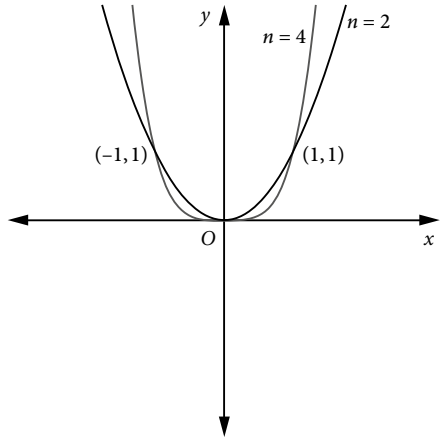
If the coefficient of the highest power of x is unity (1), the polynomial is said to be monic.

Graphs of polynomial functions

The simplest polynomial function of degree n is $f(x) = x^n$. Graphs of these basic polynomials for $n = 1, 2, 3, 4, 5$ are shown below (with $n = 1, 3, 5$ on the left and $n = 2, 4$ on the right).



Odd functions $f(-x) = -f(x)$



Even functions $f(-x) = f(x)$

Important features of $f(x) = x^n$

1. The x-axis is a tangent to each graph at the origin ($n \neq 1$).
2. $f(x) = x^n$ for even values of n defines **even functions** (see above right).
   For even functions, $f(-x) = f(x)$, so their graphs are symmetrical about the y-axis.
   For example, if $f(x) = x^4$, $f(-x) = (-x)^4 = x^4 = f(x)$.
3. $f(x) = x^n$ for odd values of n defines **odd functions** (see above left).
   For odd functions, $f(-x) = -f(x)$. Because $f(-x)$ and $f(x)$ are opposite in sign, the graph of f for $x \leq 0$ can be obtained by rotating the graph of f for $x \geq 0$ through 180° about the origin.
   For example, if $f(x) = x^3$, $f(-x) = (-x)^3 = -x^3 = -f(x)$.
4. Recognising that a function is odd or even means that you only need to draw half of the graph in detail. The other half can then be drawn using the symmetry properties.
5. Odd functions will have an inverse function, although some may require a restriction on the domain.
6. Even functions will not have a single inverse function, but can be split into two parts (by restricting the domain), so that each part has an inverse function. (Inverse functions are covered in Chapter 5.)
7. Note that most functions are neither even nor odd, e.g. $f(x) = x^2 + x$, $f(x) = e^x$.

Graphs and graphing software

It is a good idea to use graphing software to check your sketches. Software-generated graphs will also allow you to zoom in and to easily find the coordinates of important points.

Example 16

Sketch the graph of the following.

(a) $f(x) = x^3$ **(b)** $f(x) = (x-2)^3$ **(c)** $f(x) = (x-2)^3 + 1$

Solution

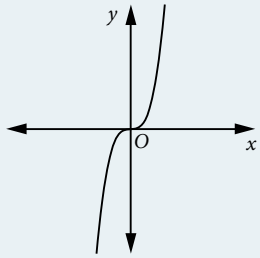
(a) $f(x) = x^3 = 0$ for $x = 0$

For all $x < 0, f(x) < 0$

For all $x > 0, f(x) > 0$

$f(-a) = (-a)^3, f(a) = a^3$

$\therefore$ $f(-x) = -f(x)$, so the function is odd.



(b) $f(x) = (x-2)^3$

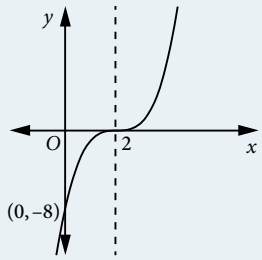
$f(x) = 0$ at $x = 2$

The graph of $f(x) = (x-2)^3$ can be obtained from the graph of $f(x) = x^3$ by a translation of 2 units to the right, parallel to the x-axis.

For all $x < 2, f(x) < 0$

For all $x > 2, f(x) > 0$

The function does not pass through the origin, so it cannot be odd.

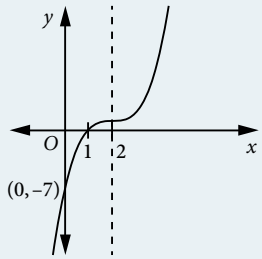


(c) $f(x) = (x-2)^3 + 1$

$f(x) = 0$ at $x = 1$

The graph of $f(x) = (x-2)^3 + 1$ can be obtained from the graph of $f(x) = (x-2)^3$ by a translation of 1 unit upwards, parallel to the y-axis.

In general, the graph of $f(x) = (x+b)^n + c$ will have the same general shape as the graph of $f(x) = x^n$.



MAKING CONNECTIONS

Graphing polynomial functions

Use graphing software to construct graphs of polynomial functions.

Cubic functions

A general cubic function is a polynomial function f of the 3rd degree, defined by $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$. Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$, where α is a real number. Factors like this can be found using the factor theorem.

Example 17

Sketch the graph of f, where $f(x) = (x+1)(x-2)(x-3)$.

Solution

At the points where the graph of f crosses the x-axis, $f(x) = 0$:

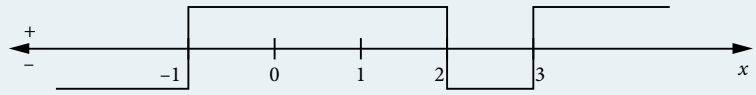
i.e. $(x+1)(x-2)(x-3) = 0$

$x = -1, 2, 3$

The function value will change sign from positive to negative or from negative to positive at these points ($x = -1, x = 2, x = 3$).

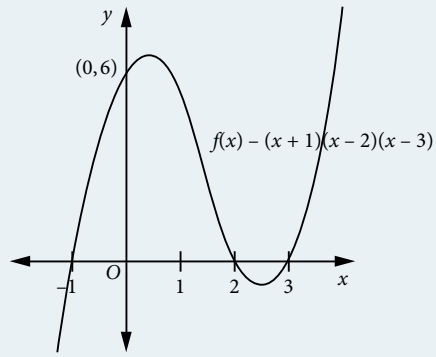
- For all $x < -1$, each of the three factors is negative, so $f(x) < 0$. e.g. $x = -2$: $f(-2) = -20 < 0$
- For $-1 < x < 2$, the factors $(x+1) > 0, (x-2) < 0, (x-3) < 0$, so $f(x) > 0$. e.g. $x = 0$: $f(0) = 6 > 0$
- For $2 < x < 3$, the factors $(x+1) > 0, (x-2) > 0$ and $(x-3) < 0$, so $f(x) < 0$. e.g. $x = 2.5$: $f(2.5) = -0.875 < 0$
- For all $x > 3$, each of the factors is positive, so $f(x) > 0$. e.g. $x = 4$: $f(4) = 20 > 0$

This information can be summarised in a sign diagram:



At the point where the graph of f crosses the y-axis, $x = 0$,
so $f(0) = 1 \times (-2) \times (-3) = 6$. Thus the graph crosses the y-axis at $(0, 6)$.

Note: this cubic equation has a positive coefficient of x^3, so that as x increases, y also increases (except in the domain between the two turning points). For $x > 3$, $f(x) > 0$ for all values of x.



Example 18

Sketch the graph of f, where $f(x) = 2x^3 - x^2 - 13x - 6$, and find the values of x for which $2x^3 - x^2 - 13x - 6 > 0$.

Solution

It is useful to find the linear factors of $2x^3 - x^2 - 13x - 6$. (Remember there will always be at least one.)

$f(1) = 2 - 1 - 13 - 6 = -18 \neq 0$ so $(x - 1)$ is not a factor

$f(-1) = -2 - 1 + 13 - 6 = 4 \neq 0$ so $(x + 1)$ is not a factor

$f(2) = 16 - 4 - 26 - 6 = -20 \neq 0$ so $(x - 2)$ is not a factor

$f(-2) = -16 - 4 + 26 - 6 = 0$ so $(x + 2)$ is a factor

To get the other linear factors (if any), you can divide $2x^3 - x^2 - 13x - 6$ by $(x + 2)$ and then factorise the quotient.

$$
\begin{array}{r}
2x^2 - 5x - 3 \\
x+2 \overline{)\,2x^3 - x^2 - 13x - 6} \\
\underline{2x^3 + 4x^2} \\
-5x^2 - 13x \\
\underline{-5x^2 - 10x} \\
-3x - 6 \\
\underline{-3x - 6} \\
0
\end{array}
$$

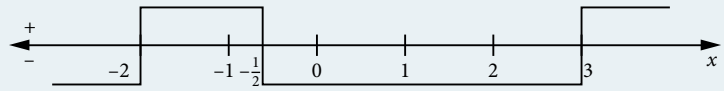
The factors of $2x^2 - 5x - 3$ are $(2x + 1)$ and $(x - 3)$:

$\therefore \quad f(x) = (x + 2)(2x + 1)(x - 3)$

$f(x) = 0$ at $x = -2, -\frac{1}{2}, 3$

- For all $x < -2$, $f(x) < 0$ e.g. $x = -3$: $f(-3) = -30 < 0$
- For $-2 < x < -\frac{1}{2}$, $f(x) > 0$ e.g. $x = -1$: $f(-1) = 4 > 0$
- For $-\frac{1}{2} < x < 3$, $f(x) < 0$ e.g. $x = 0$: $f(0) = -6 < 0$
- For all $x > 3$, $f(x) > 0$ e.g. $x = 4$: $f(4) = 54 > 0$

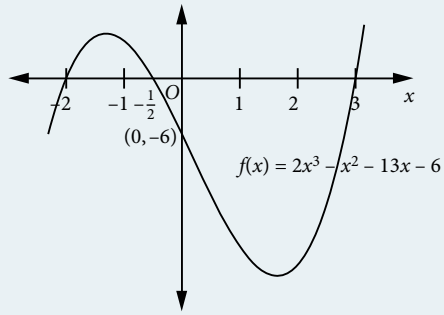
This information is summarised in the following sign diagram.



Hence $f(x) > 0$ for $-2 < x < -\frac{1}{2}$ and for $x > 3$.

At $x = 0$, $f(0) = -6$, so the graph crosses the y-axis at $(0, -6)$.

Note that this cubic equation has a positive coefficient of x^3, so that as x increases, y also increases (except in the domain between the two turning points). For $x > 3$, $f(x) > 0$ for all values of x.



Example 19

Without showing too much detail, sketch graphs of the polynomial functions defined by the following rules.

(a) $y = (x+2)(x-1)^2(x+1)$ **(b)** $y = x^2(x-3)(x+1)^3$

Solution

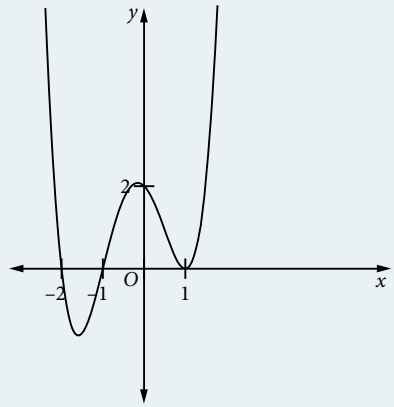
(a) $y = (x+2)(x-1)^2(x+1)$. When $|x|$ is large, y behaves like x^4.

$(x-1)^2 \geq 0$, so the sign of y is determined by $(x+2)(x+1)$.

Hence $y \geq 0$ for $x < -2$ and for $x > -1$ (equality at $x = 1$);
$y < 0$ for $-2 < x < -1$.

Near $x = -2$, y behaves like a multiple of $(x+2)$ and cuts the x-axis like a straight line at $x = -2$. Similarly at $x = -1$.

Near $x = 1$, y behaves like a multiple of $(x-1)^2$, so that $x = 1$ is a stationary point (minimum turning point) and the x-axis is a tangent to the curve at $x = 1$.



(b) $y = x^2(x-3)(x+1)^3$. When $|x|$ is large, y behaves like x^5.

$x^2 \geq 0$, $(x+1)^2 \geq 0$, so the sign of y is determined by $(x-3)(x+1)$.

Hence $y \geq 0$ for $x < -1$ and for $x > 3$ (equality at $x = -1$ and $x = 3$);
$y \leq 0$ for $-1 < x < 3$ (equality at $x = 0$).

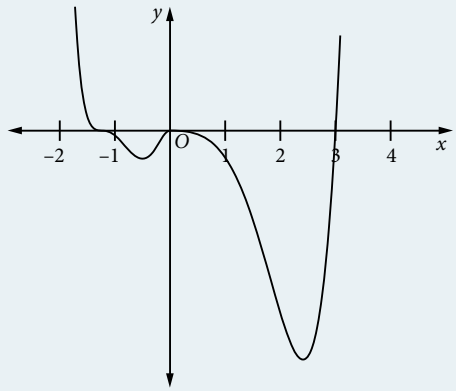
Near $x = 3$, y behaves like a multiple of $(x-3)$ and cuts the x-axis like a straight line at $x = 3$.

Near $x = 0$, y behaves like a multiple of x^2 and the graph touches the x-axis at $x = 0$.

Near $x = -1$, y behaves like a multiple of $(x+1)^3$ and the graph cuts the x-axis at $x = -1$ like a basic cubic curve, i.e. with a horizontal point of inflection.

This sketch is **not** drawn to scale. The minimum between $x = -1$ and $x = 0$ cannot be shown at a reasonable scale without the other minimum being below the page.

You can use graphing software to draw this graph. As you zoom out to find the minimum turning point between $x = 0$ and $x = 3$, the curve between $x = -1$ and $x = 0$ appears to become a straight line.



Summary of polynomial functions

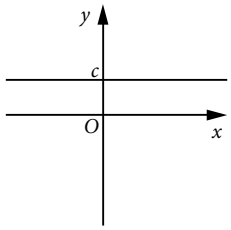
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, then:

1. For very large $|x|$, $P(x) \approx a_n x^n$.
2. A polynomial of odd degree always has at least one real zero (i.e. its graph cuts the x-axis at least once).
3. At least one maximum or minimum value of P occurs between any two distinct real zeros.
4. For a polynomial of odd degree, the ends of the graph go in opposite directions.
5. For a polynomial of even degree, the ends of the graph go in the same direction.
6. When the graph of a polynomial function meets the x-axis, it may cut it (single zero), touch it (double zero) or cut it at a point of inflection (triple zero).

Polynomial graphs, standard forms

Degree 0 (constant)

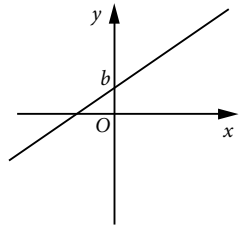
$P(x) = c$



Does not cut
No zeros

Degree 1 (linear)

$P(x) = mx + b$

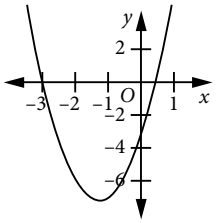


Cuts once
One zero

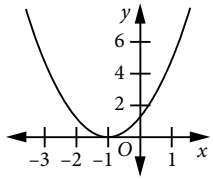
Degree 2 (quadratic)

$P(x) = ax^2 + bx + c$

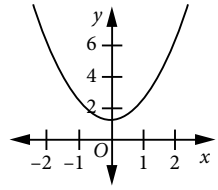
***a* > 0:**



Cuts twice
$\Delta > 0$
Two zeros

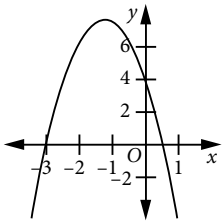


Touches
$\Delta = 0$
Double zero

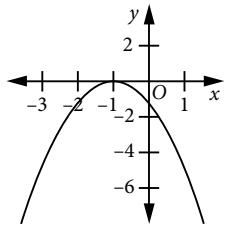


Does not cut
$\Delta < 0$
No real zeros

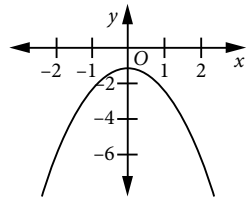
***a* < 0:**



Cuts twice
$\Delta > 0$
Two zeros



Touches
$\Delta = 0$
Double zero



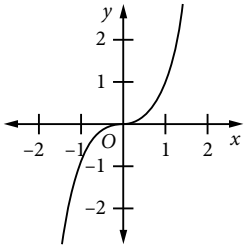
Does not cut
$\Delta < 0$
No real zeros

A quadratic polynomial may have two, one or no real zeros.

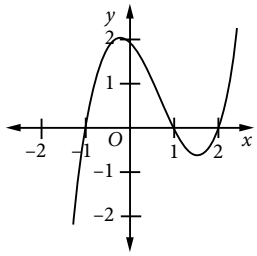
Degree 3 (cubic)

$P(x) = ax^3 + bx^2 + cx + d$

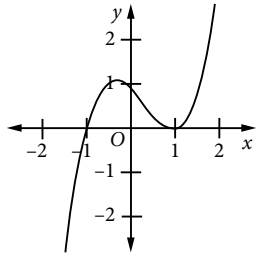
***a* > 0:**



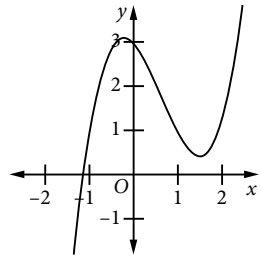
Cuts once
Triple zero



Cuts three times
Three zeros

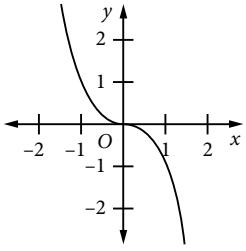


Cuts once and touches
One zero, one double zero

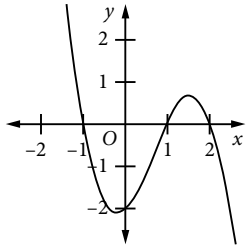


Cuts once
One zero

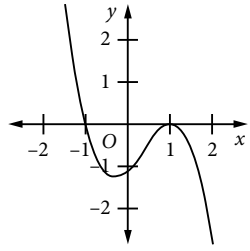
***a* < 0:**



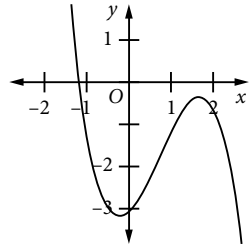
Cuts once
Triple zero



Cuts three times
Three zeros



Cuts once and touches
One zero, one double zero



Cuts once
One zero

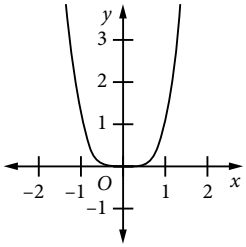
A cubic polynomial may have one, two or three real zeros.

A cubic polynomial always has at least one real zero.

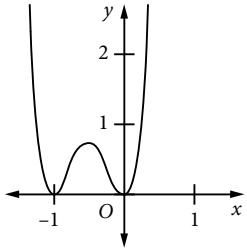
Degree 4 (quartic)

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

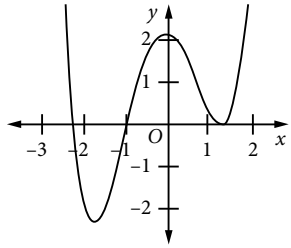
***a* > 0:**



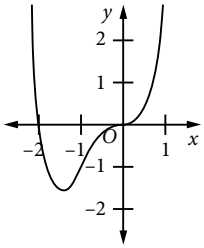
Touches
Quadruple zero



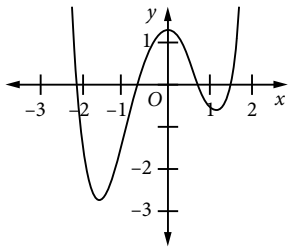
Touches twice
Two double zeros



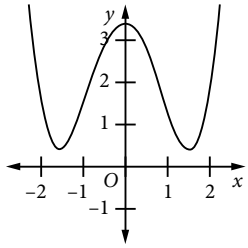
Touches once, cuts twice
Two zeros, one double zero



Cuts twice
One zero, one triple zero



Cuts four times
Four zeros



Does not cut
No real zeros

The addition of an appropriate constant to each equation can create a polynomial with no real zeros.

***a* < 0:**

A negative *a* inverts each of the six graphs above so that they open downwards, with the properties of their zeros the same.

Summary of quartic polynomials

1. A quartic polynomial may have four, three, two, one or no real zeros.
2. If a quartic polynomial has only one real zero, then it must be a quadruple zero.
3. If a quartic polynomial has only two distinct real zeros, then they are either a triple zero and a single zero or they are both double zeros.
4. If a quartic polynomial has only three distinct real zeros, then they are a double zero and two single zeros.
5. If a quartic polynomial has four distinct real zeros, then it can be factorised into four real linear factors.
6. If a quartic polynomial has no real zeros, then it cannot be factorised into any real linear factors.

EXERCISE 2.6 POLYNOMIAL FUNCTIONS

1 Sketch the graph of each function and find the values of x for which: **(i)** $f(x) = 0$ **(ii)** $f(x) > 0$

(a) $f(x) = x(x-2)(x+3)$ **(b)** $f(x) = 2(x-\frac{1}{2})(x+1)(2x+3)$ **(c)** $f(x) = x(x-2)^2$

2 Sketch the graph of f where $f(x)$ is the polynomial given. Draw a sign diagram and label your sketch showing the points of intersection with the axes. Find the values of x for which $f(x) \geq 0$.

(a) $f(x) = x^3 - 4x$ **(b)** $f(x) = x^3 + 4x^2 + 4x$ **(c)** $f(x) = 2x^2 - x^3$ **(d)** $f(x) = x^3 - x^2 - 10x - 8$

3 **(a)** Find the linear factors of $6 + 5x - 2x^2 - x^3$.

(b) Find the values of x for which: **(i)** $6 + 5x - 2x^2 - x^3 = 0$ **(ii)** $6 + 5x - 2x^2 - x^3 > 0$

(c) Sketch the graph of f where $f(x) = 6 + 5x - 2x^2 - x^3$.

4 **(a)** Find the linear factors of $x^3 - 5x^2 + 8x - 4$.

(b) Find the values of x for which: **(i)** $x^3 - 5x^2 + 8x - 4 = 0$ **(ii)** $x^3 - 5x^2 + 8x - 4 > 0$

(c) Sketch the graph of f where $f(x) = x^3 - 5x^2 + 8x - 4$.

5 **(a)** Find the linear factors of $-x^3 + 2x^2 + x - 2$.
(b) Draw a sign diagram to find the values of x for which $-x^3 + 2x^2 + x - 2 > 0$.
(c) Sketch the graph of f where $f(x) = -x^3 + 2x^2 + x - 2$.

6 The number of zeros of the polynomial $f(x) = x^3 - x$ is:
A 0 **B** 1 **C** 2 **D** 3

7 Describe how the graph of $f(x) = x^3$ can be transformed into the graph of the following.
(a) $f(x) = (x + 3)^3$ **(b)** $f(x) = (x + 3)^3 - 2$

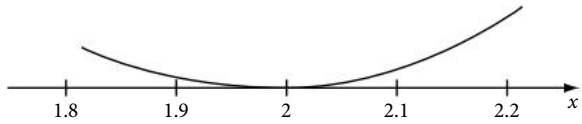
8 Show that the graph of f, where $f(x) = x^3 - 8$, cuts the x-axis at one point only.

9 Show that the graph of f, where $f(x) = x^3 - x^2 - 8x + 12$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

10 Show that the graph of f, where $f(x) = x^3 - 4x^2 + 8x - 8$, cuts the x-axis at one point only.

11 Sketch graphs to show the general form of the following functions.
(a) $y = (x + 1)(x - 2)(x + 3)$ **(b)** $y = x(x - 1)(x - 2)(x - 3)$ **(c)** $y = x(x - 1)^2$ **(d)** $y = x^2(x - 1)$
(e) $y = x(x + 3)^2(2 - x)$ **(f)** $y = (x - 1)(2x - 3)(x + 4)^2$ **(g)** $y = x(2x - 1)(x - 1)^3$
(h) $y = x(x + 2)(x - 1)^4$ **(i)** $y = (3x - 5)^3(x - 1)^2$ **(j)** $y = x(x - 1)^2(2 - x)^3$

12 Part of the graph of $y = P(x)$ is shown, where $P(x)$ is a polynomial of degree four:
Which of the following could be the polynomial $P(x)$?
A $P(x) = x^2(x + 2)^2$ **B** $P(x) = (x - 2)^4$
C $P(x) = x(x - 2)^3$ **D** $P(x) = (x - 1)^2(x + 2)^2$



13 Sketch graphs to show the basic features of the following functions.
(a) $y = x(x - 2)^3(x + 1)^4$ **(b)** $y = x(x^2 - 1)(x - 1)^2$
(c) $y = (3 - x)(1 - x)^4(x + 2)$ **(d)** $y = (2 - x)^3(1 + x)^2(x - 3)$

CHAPTER REVIEW 2

1 Perform the following polynomial divisions.
(a) $(x^3 + 2x^2 - 3x + 4) \div (x - 1)$ **(b)** $(4x^4 - 6x^2 + 10x - 40) \div (x + 3)$

2 Use the remainder theorem to find the remainder of the following.
(a) $x^3 - 4x^2 + 3x - 5$ divided by $(x - 2)$ **(b)** $x^4 + x^3 - 5x^2 + 4x - 2$ divided by $(x + 1)$

3 Use the factor theorem to find the linear factors (over the rational number field) of each polynomial.
(a) $x^3 - 2x^2 - 5x + 6$ **(b)** $x^3 + 7x^2 + 14x + 8$ **(c)** $x^3 + 5x^2 - x - 5$ **(d)** $x^4 - x^3 - 16x^2 + 4x + 48$

4 Let $P(x) = (x - 1)(x + 2)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $x + 2$. When $P(x)$ is divided by $x - 1$ the remainder is 6.
(a) Find the values of a and b.
(b) Find the remainder when $P(x)$ is divided by $(x - 1)(x + 2)$.

5 Let $P(x) = x^3 + ax^2 - x + 1$ be a polynomial where a is a real number. When $P(x)$ is divided by $x - 2$ the remainder is 15. Find the remainder when $P(x)$ is divided by $x + 3$.

6 The polynomial $P(x) = x^3 + ax - b$ has a remainder of 5 when divided by $(x + 1)$ and a remainder of 2 when divided by $(x - 2)$. Find the values of a and b and hence find the remainder when $P(x)$ is divided by $(x - 3)$.

7 The polynomial $P(x)$ is given by $P(x) = ax^3 + 15x^2 + cx - 72$, where a and c are constants. The three zeros of $P(x)$ are -3, 2 and α. Find the value of α.

8 The cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ (where b, c, d are real numbers) has three real zeros: -1, α and $-\alpha$.

(a) Find the value of b. **(b)** Find the value of $c - d$.

9 The polynomial $P(x) = x^3 - 4x^2 + kx + 12$ has zeros α, β, γ.

(a) Find the value of $\alpha + \beta + \gamma$. **(b)** Find the value of $\alpha\beta\gamma$.

(c) Two of the three zeros are equal in magnitude but opposite in sign. Find the third zero and hence find the value of k.

10 Sketch graphs of each function. For what values of x is each function positive?

(a) $y = (x - 1)(x + 2)(x - 3)$ **(b)** $y = (x - 2)(x + 2)^2$ **(c)** $y = x(x^2 - 1)(x + 2)$ **(d)** $y = x^2(x - 2)^2$

CHAPTER 3
Graphing functions

In the Mathematics Advanced course, you looked at the graphs of linear, quadratic, cubic and quartic polynomials. In this section you are going to look at the effects of applying other function rules to linear, quadratic and cubic polynomials and drawing their graphs.

3.1 RECIPROCAL FUNCTIONS

Given the graph of $y = f(x)$, the graph of $y = \dfrac{1}{f(x)}$ is the **reciprocal function** of $f(x)$.

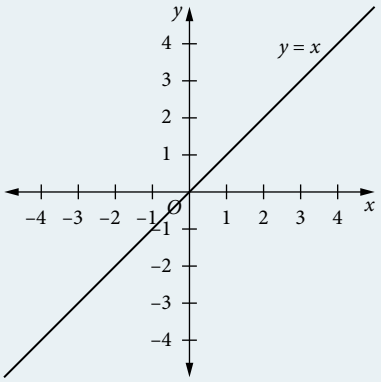
When graphing reciprocal functions, it is important to find where $f(x) = 0$, as these x values will give vertical asymptotes for the reciprocal function.

It is also important to note that where $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0$.

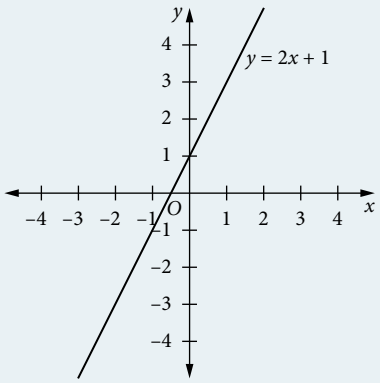
Example 1

In each part, use the graph of the given function to draw the graph of $y = \dfrac{1}{f(x)}$.

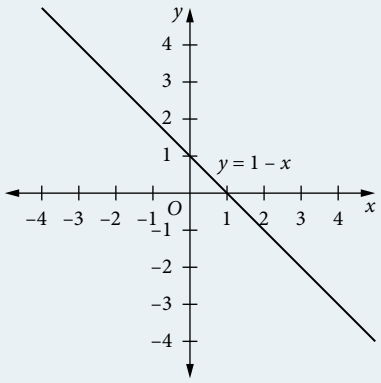
(a) Given $y = x$, draw $y = \dfrac{1}{x}$.



(b) Given $y = 2x + 1$, draw $y = \dfrac{1}{2x+1}$.

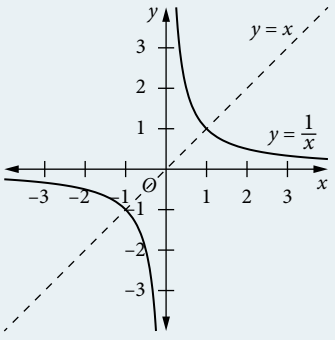


(c) Given $y = 1 - x$, draw $y = \dfrac{1}{1-x}$.

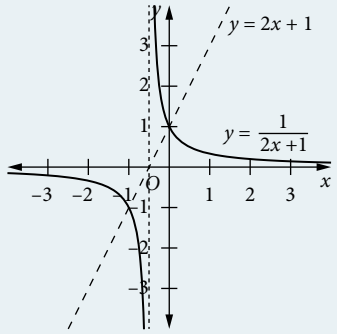


Solution

(a) The graph of $y = \dfrac{1}{x}$ is undefined at $x = 0$, so $x = 0$ is a vertical asymptote.
The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$y = 0$ is the horizontal asymptote.
$x = \dfrac{1}{x}$ where $x = \pm 1$, hence the curves intersect at $(-1, -1)$ and $(1, 1)$.

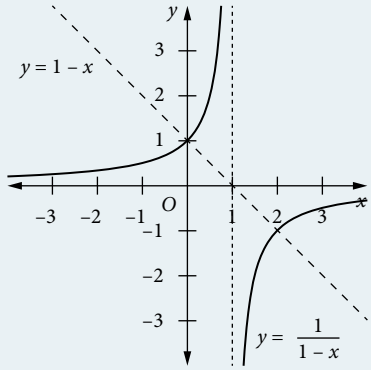


(b) The graph of $y = \frac{1}{2x+1}$ is undefined at $x = -\frac{1}{2}$, so $x = -\frac{1}{2}$ is a vertical asymptote.
The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$y = 0$ is the horizontal asymptote.
$2x + 1 = \frac{1}{2x+1}$ where $x = -1, 0$. Hence the curves intersect at $(-1, -1)$ and $(0, 1)$.



(c) The graph of $y = \frac{1}{1-x}$ is undefined at $x = 1$.

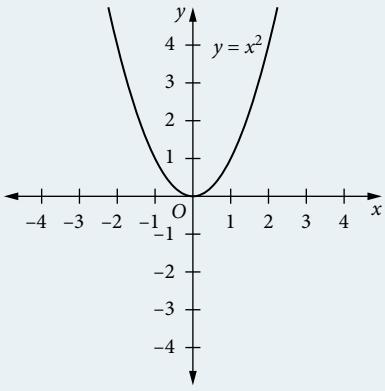
The graph approaches $y = 0$ from below as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
$1 - x = \frac{1}{1-x}$ where $x = 0, 2$. The curves intersect at $(0, 1)$ and $(2, -1)$.
$x = 1$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.



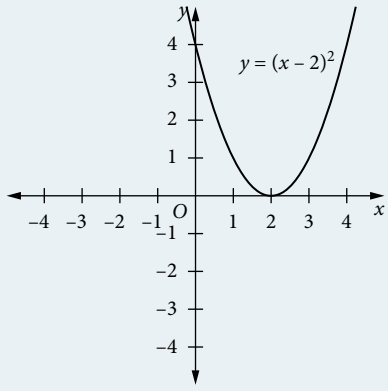
All these reciprocal functions are rectangular hyperbolas.

Example 2

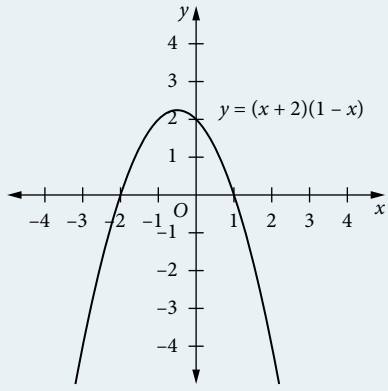
(a) Given the graph of $y = x^2$, draw $y = \frac{1}{x^2}$.



(b) Given the graph of $y = (x-2)^2$, draw $y = \frac{1}{(x-2)^2}$.

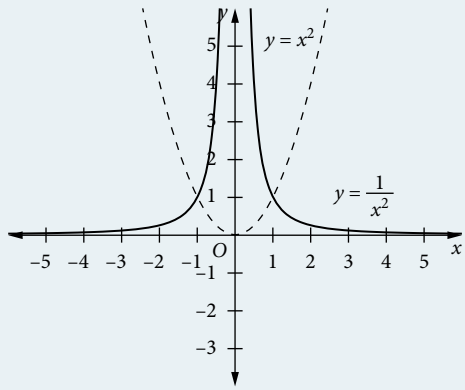


(c) Given the graph of $y = (x+2)(1-x)$, draw $y = \frac{1}{(x+2)(1-x)}$.



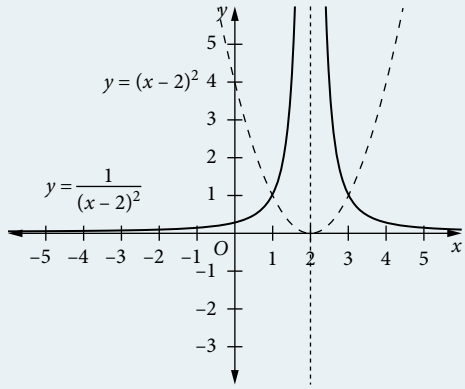
Solution

(a) The graph of $y = \frac{1}{x^2}$ is undefined at $x = 0$.
The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
The function is never negative.
$x^2 = \frac{1}{x^2}$ where $x = \pm 1$. The curves intersect at $(-1, 1)$ and $(1, 1)$.
$x = 0$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.



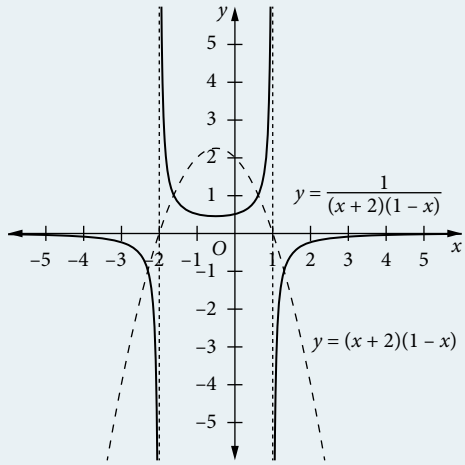
(b) The graph of $y = \dfrac{1}{(x-2)^2}$ is undefined at $x = 2$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
The function is never negative.
$(x-2)^2 = \dfrac{1}{(x-2)^2}$ where $x = 1, 3$. The curves intersect at $(1, 1)$ and $(3, 1)$.
$x = 2$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.
$x = 2$ is an axis of symmetry.



(c) The graph of $y = \dfrac{1}{(x+2)(1-x)}$ is undefined at $x = -2, 1$.

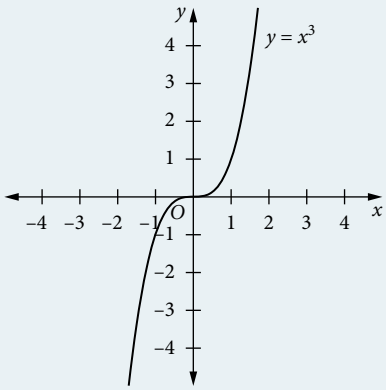
The graph approaches $y = 0$ from below as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
The maximum value of $(x+2)(1-x)$ is $\dfrac{9}{4}$ and occurs at $x = -\dfrac{1}{2}$.
Thus the least positive value of $\dfrac{1}{(x+2)(1-x)}$ is $\dfrac{4}{9}$ and occurs at $x = -\dfrac{1}{2}$.
$x < -2, \dfrac{1}{(x+2)(1-x)} < 0; -2 < x < 1, \dfrac{1}{(x+2)(1-x)} > \dfrac{4}{9}$;
$x > 1, \dfrac{1}{(x+2)(1-x)} < 0$.
$x = -2$ and $x = 1$ are the vertical asymptotes, $y = 0$ is the horizontal asymptote.
$x = -\dfrac{1}{2}$ is an axis of symmetry.



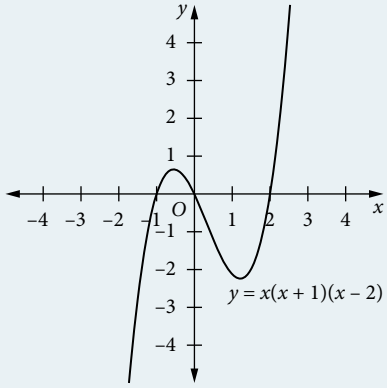
Each of these graphs has a vertical axis of symmetry.

Example 3

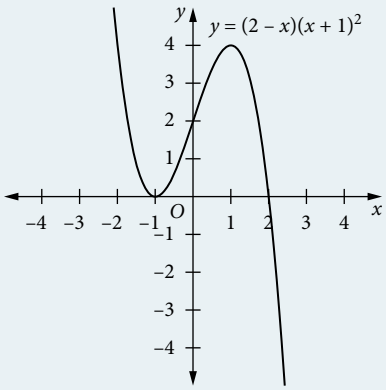
(a) Given the graph of $y = x^3$, draw $y = \dfrac{1}{x^3}$.



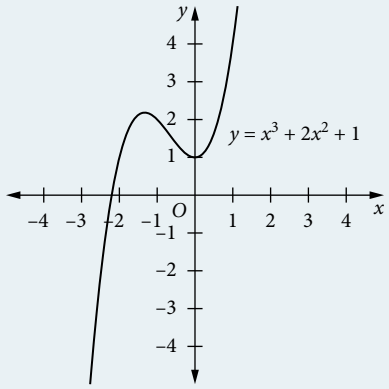
(b) Given the graph of $y = x(x+1)(x-2)$, draw the graph of $y = \dfrac{1}{x(x+1)(x-2)}$.



(c) Given the graph of $y = (2-x)(x+1)^2$, draw the graph of $y = \dfrac{1}{(2-x)(x+1)^2}$.



(d) Given the graph of $y = x^3 + 2x^2 + 1$, draw the graph of $y = \dfrac{1}{x^3 + 2x^2 + 1}$.



Solution

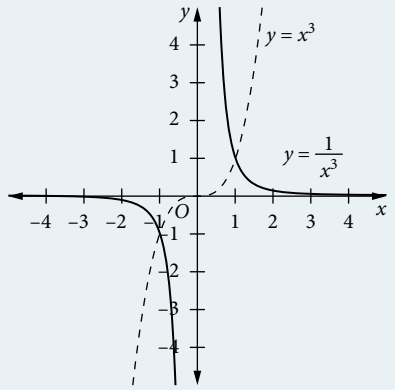
(a) The graph of $y = \dfrac{1}{x^3}$ is undefined at $x = 0$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.

$x^3 = \dfrac{1}{x^3}$ where $x = \pm 1$. The curves intersect at $(-1, -1)$ and $(1, 1)$.

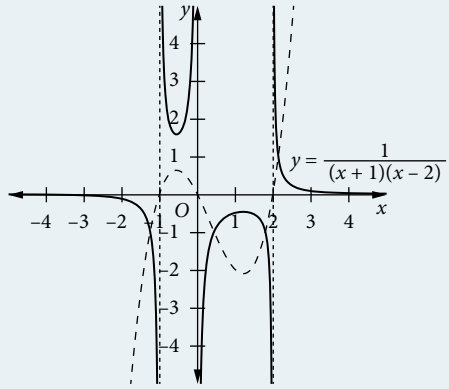
$x = 0$ is the vertical asymptote, $y = 0$ is the horizontal asymptote.

The curve does not have an axis of symmetry, but has rotational (point) symmetry about the origin.



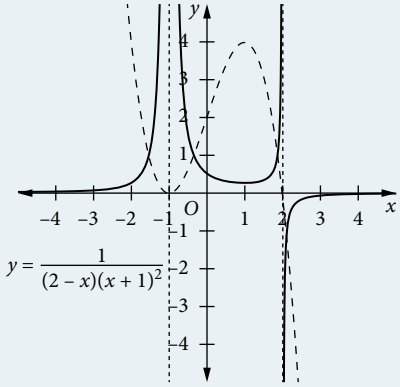
(b) The graph of $y = \dfrac{1}{x(x+1)(x-2)}$ is undefined at $x = -1, 0, 2$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$x = -1, 0, 2$ are the vertical asymptotes, $y = 0$ is the horizontal asymptote.
The curve does not have an axis of symmetry.



(c) The graph of $y = \dfrac{1}{(2-x)(x+1)^2}$ is undefined at $x = -1, 2$.

The graph approaches $y = 0$ from below as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
$x = -1, 2$ are the vertical asymptotes, $y = 0$ is the horizontal asymptote.
It looks as though $y = (2-x)(x+1)^2$ has a local maximum value of 4 at $x = 1$. (This can be shown using calculus.)
Hence $y = \dfrac{1}{(2-x)(x+1)^2}$ will have a local minimum value of $\dfrac{1}{4}$ at $x = 1$.



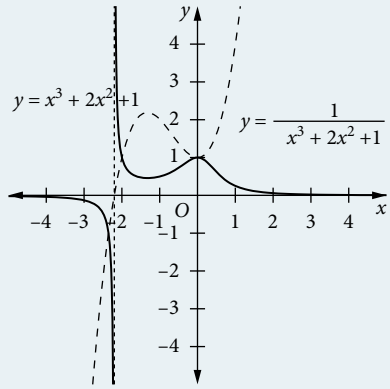
(d) The graph of $y = \dfrac{1}{x^3 + 2x^2 + 1}$ is undefined at $x \approx -2.2$.

The graph approaches $y = 0$ from above as $x \to \infty$.

The graph approaches $y = 0$ from below as $x \to -\infty$.

$x = -2.2$ is the vertical asymptote, $y = 0$ is the horizontal asymptote.

The curves touch at (0, 1), a local minimum of the original function becomes a local maximum of the reciprocal function.



As the Examples above show, a maximum turning point on the original function becomes a minimum turning point on the reciprocal function (or equivalent asymptote). A minimum turning point on the original function becomes a maximum turning point on the reciprocal function (or equivalent asymptote).

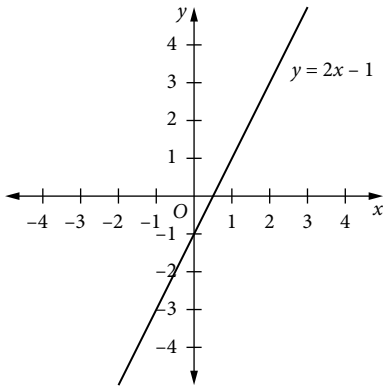
EXPLORING FURTHER

Graphing reciprocal functions

Use graphing software to explore graphs of reciprocal functions.

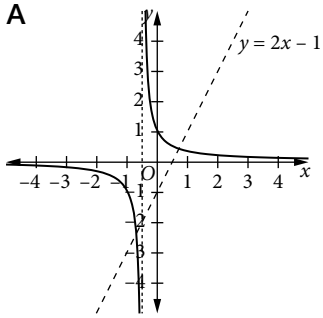
EXERCISE 3.1 RECIPROCAL FUNCTIONS

1 The graph of $y = 2x - 1$ is shown.

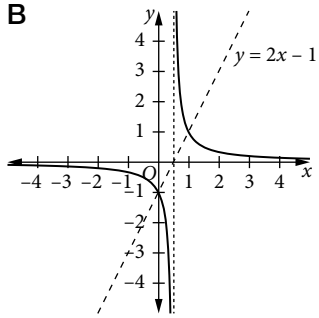


Which of the following represents the graph of $y = \dfrac{1}{2x - 1}$?

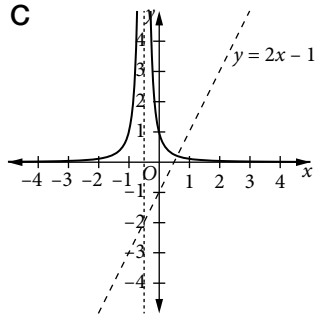
A



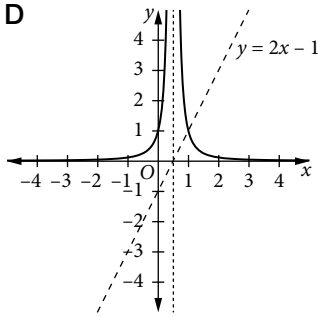
B



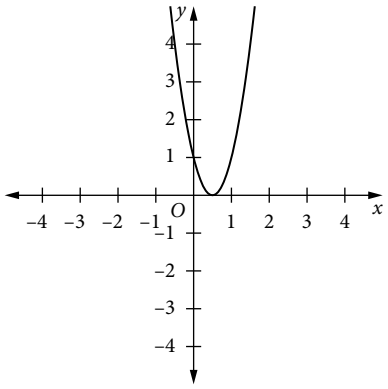
C



D

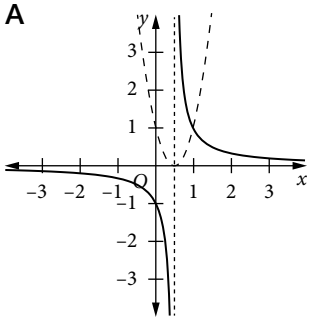


2 The graph of $y = (2x - 1)^2$ is shown.

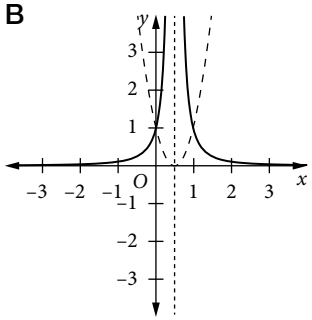


Which of the following represents the graph of $y = \dfrac{1}{(2x-1)^2}$?

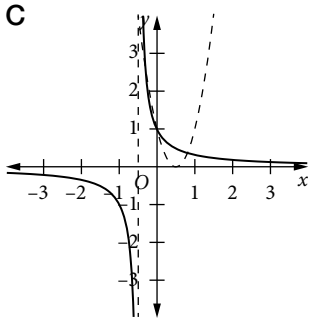
A



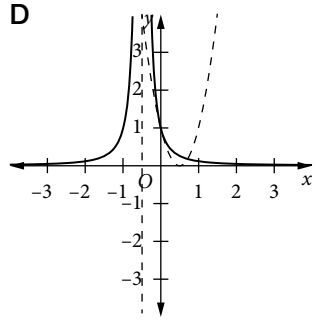
B



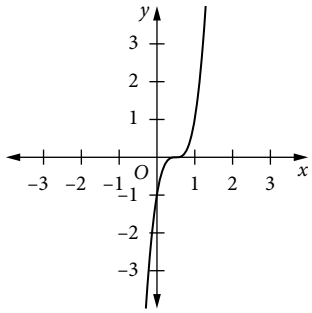
C



D

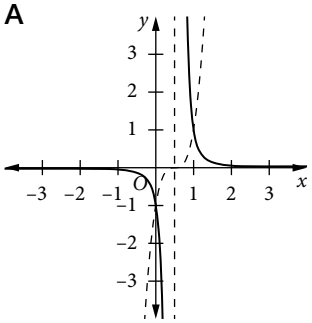


3 The graph of $y = (2x - 1)^3$ is shown.

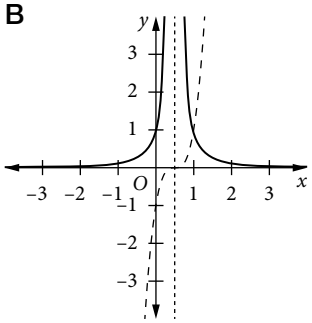


Which of the following represents the graph of $y = \dfrac{1}{(2x-1)^3}$?

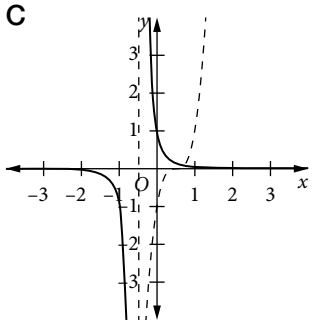
A



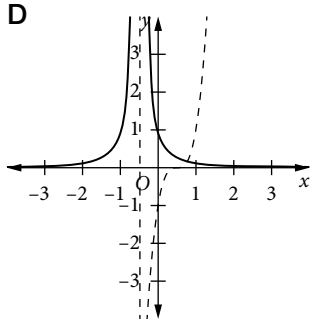
B



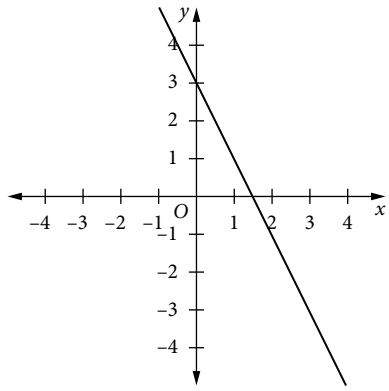
C



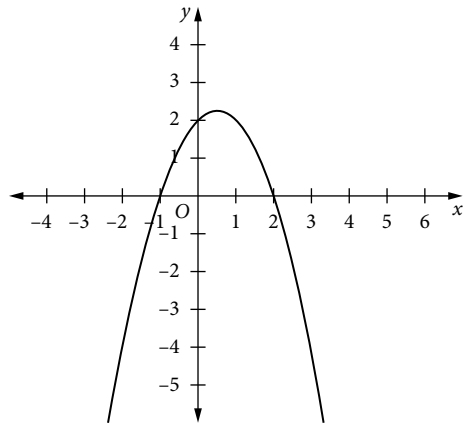
D



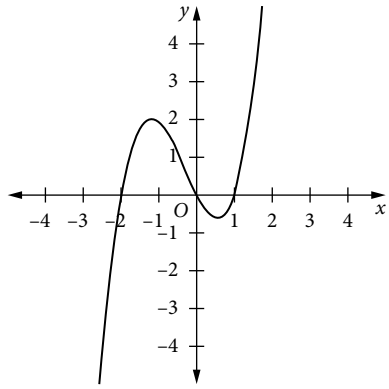
4 Given the graph of $y = 3 - 2x$, draw the graph of $y = \dfrac{1}{3-2x}$.



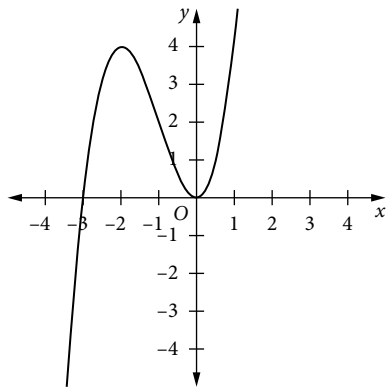
5 Given the graph of $y = (x+1)(2-x)$, draw the graph of $y = \dfrac{1}{(x+1)(2-x)}$.



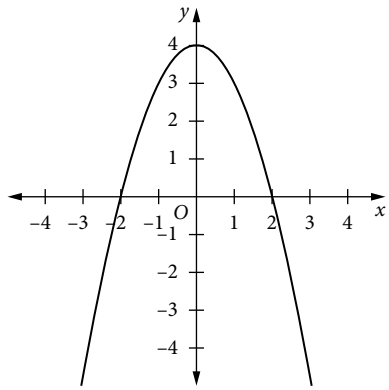
6 Given the graph of $y = x(x-1)(x+2)$, draw the graph of $y = \dfrac{1}{x(x-1)(x+2)}$.



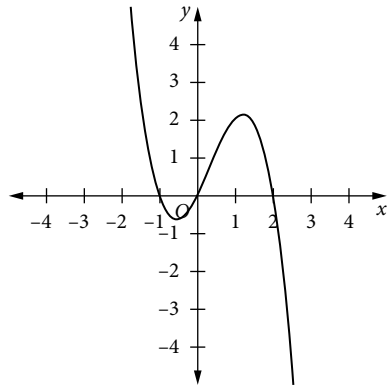
7 Given the graph of $y = x^3 + 3x^2$, draw the graph of $y = \dfrac{1}{x^3+3x^2}$.



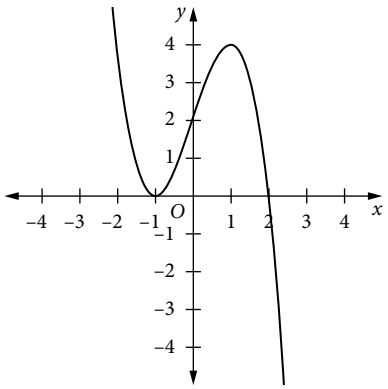
8 Given the graph of $y = 4 - x^2$, draw the graph of $y = \dfrac{1}{4-x^2}$.



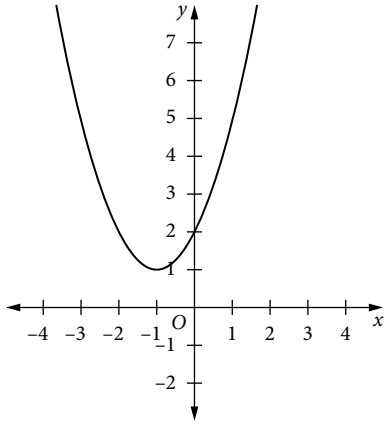
9 Given the graph of $y = x(x+1)(2-x)$, draw the graph of $y = \dfrac{1}{x(x+1)(2-x)}$.



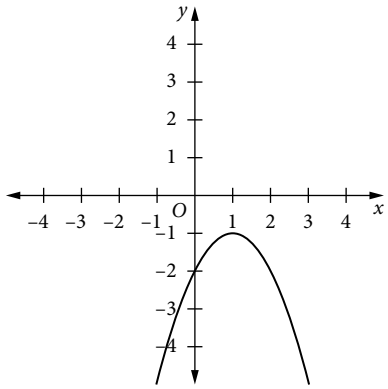
10 Given the graph of $y = 2 + 3x - x^3$, draw the graph of $y = \dfrac{1}{2 + 3x - x^3}$.



11 Given the graph of $y = x^2 + 2x + 2$, draw the graph of $y = \dfrac{1}{x^2 + 2x + 2}$.



12 Given the graph of $y = -x^2 + 2x - 2$, draw the graph of $y = \dfrac{1}{-x^2 + 2x - 2}$.



3.2 SQUARE ROOT FUNCTIONS

Given the graph of $y = f(x)$, it is often useful or necessary to draw the graph of $y = \sqrt{f(x)}$ and the graph of $y^2 = f(x)$.

To graph these functions, it is important to find out where $f(x) < 0$, as $\sqrt{f(x)}$ is not defined for these values of x (because the square root of a negative number is not a real number). Similarly, $y^2 = f(x)$ will not be defined for these values of x, because y^2 cannot be negative.

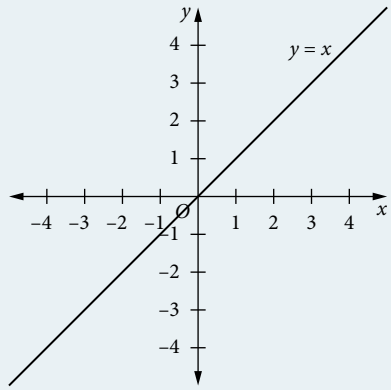
Remember that $\sqrt{0} = 0$ and $\sqrt{1} = 1$. Therefore, for $0 < f(x) < 1$, $\sqrt{f(x)} > f(x)$; for $f(x) > 1$, $f(x) > \sqrt{f(x)}$.

Graphically this means that for $0 < f(x) < 1$, the graph of $y = \sqrt{f(x)}$ is above $y = f(x)$, and for $f(x) > 1$, the graph of $y = \sqrt{f(x)}$ is below $y = f(x)$.

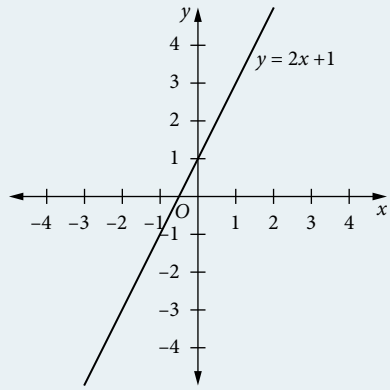
Example 4

In each part, use the graph of the given function to draw the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$.

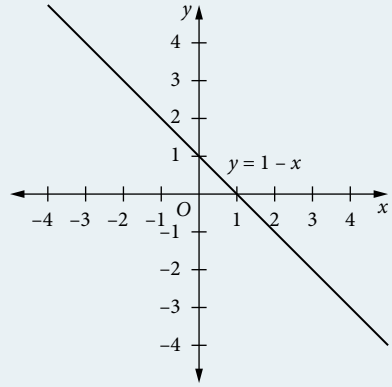
(a) Given $y = x$, draw $y = \sqrt{x}$ and $y^2 = x$.



(b) Given $y = 2x + 1$, draw $y = \sqrt{2x+1}$ and $y^2 = 2x + 1$.

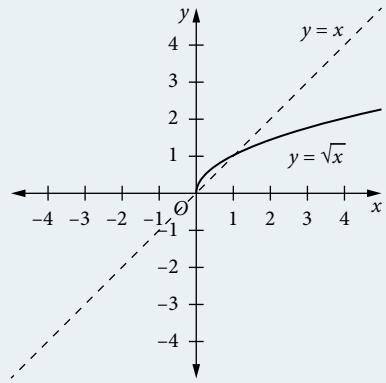


(c) Given $y = 1 - x$, draw $y = \sqrt{1-x}$ and $y^2 = 1 - x$.

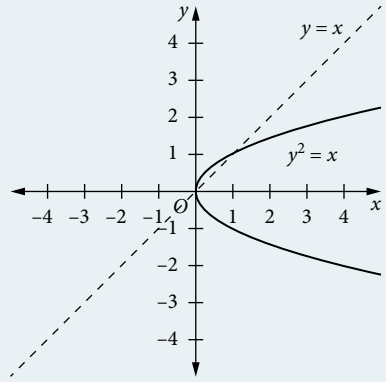


Solution

(a) The graph of $y = \sqrt{x}$ is undefined for $x < 0$.
$x = \sqrt{x}$ at $x = 0, 1$. Graphs intersect at (0, 0) and (1, 1).
For $0 < x < 1$, the graph of $y = \sqrt{x}$ is above the graph of $y = x$.



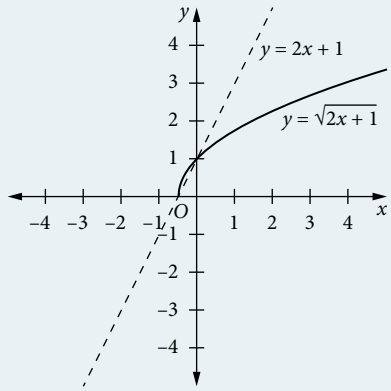
The graph of $y^2 = x$ is undefined for $x < 0$ as y^2 cannot be negative.
It has two branches, $y = \sqrt{x}$ and $y = -\sqrt{x}$.
$y = x$ and $y^2 = x$ intersect at (0, 0) and (1, 1).



(b) The graph of $y = \sqrt{2x+1}$ is undefined for $x < -\frac{1}{2}$.

$2x + 1 = \sqrt{2x+1}$ at $x = -\frac{1}{2}, 0$. Graphs intersect at $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$.

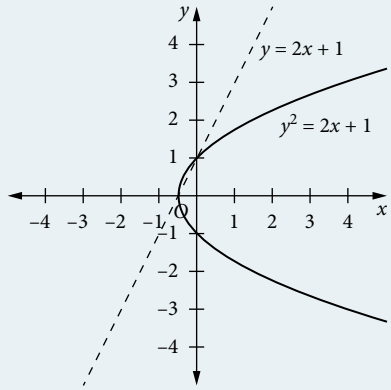
For $-\frac{1}{2} < x < 0$, the graph of $y = \sqrt{2x+1}$ is above the graph of $y = 2x + 1$.



The graph of $y^2 = 2x + 1$ is undefined for $x < -\frac{1}{2}$.

It has two branches, $y = \sqrt{2x+1}$ and $y = -\sqrt{2x+1}$.

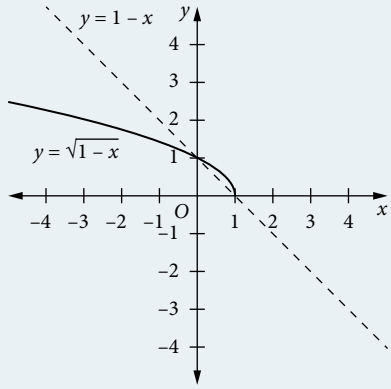
Graphs intersect at $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$.



(c) The graph of $y = \sqrt{1-x}$ is undefined for $x > 1$.

$1 - x = \sqrt{1-x}$ at $x = 0, 1$. Graphs intersect at $(0, 1)$ and $(1, 0)$.

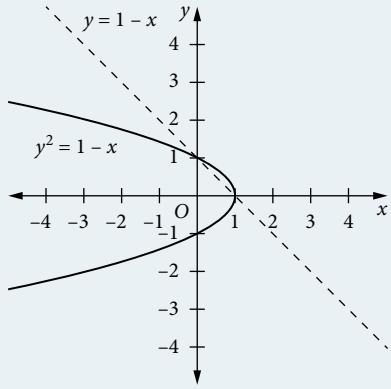
For $0 < x < 1$, the graph of $y = \sqrt{1-x}$ is above the graph of $y = 1 - x$.



The graph of $y^2 = 1 - x$ is undefined for $x > 1$.

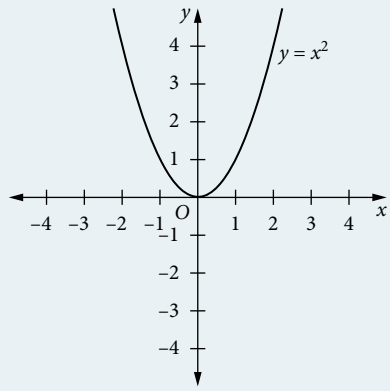
It has two branches, $y = \sqrt{1-x}$ and $y = -\sqrt{1-x}$.

Graphs intersect at $(0, 1)$ and $(1, 0)$.

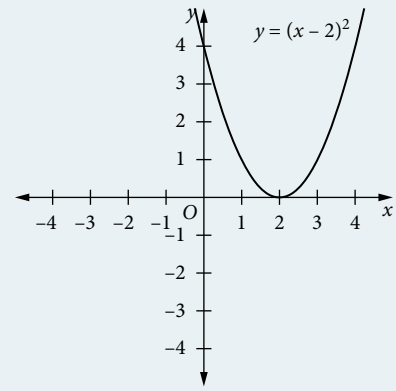


Example 5

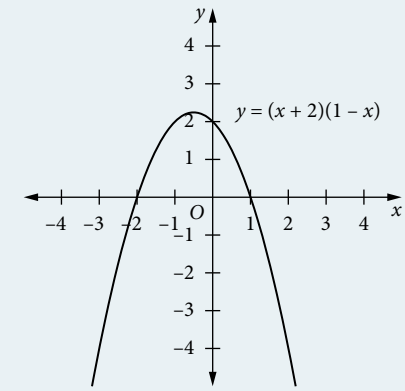
(a) Given $y = x^2$, draw $y = \sqrt{x^2}$ and $y^2 = x^2$.



(b) Given $y = (x-2)^2$, draw $y = \sqrt{(x-2)^2}$ and $y^2 = (x-2)^2$.

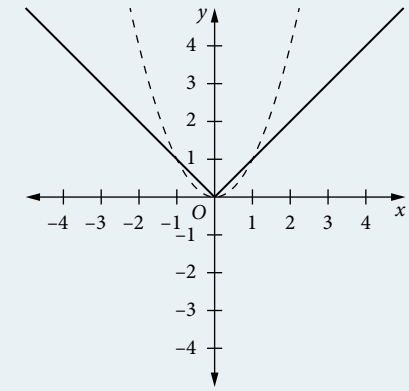


(c) Given $y = (x+2)(1-x)$, draw $y = \sqrt{(x+2)(1-x)}$ and $y^2 = (x+2)(1-x)$.

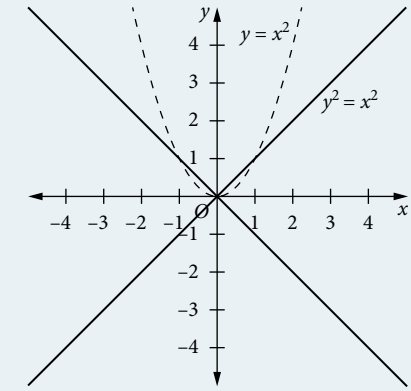


Solution

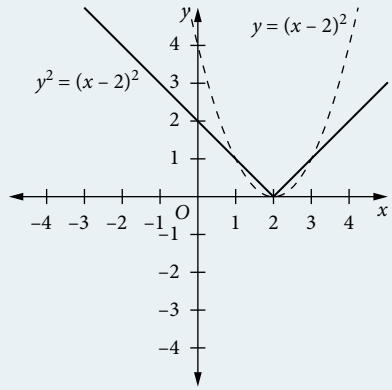
(a) The graph of $y = \sqrt{x^2}$ is defined for all x.
The function is never negative.
The graphs intersect at $(-1, 1)$, $(0, 0)$ and $(1, 1)$.
The resulting graph is the same as $y = |x|$.



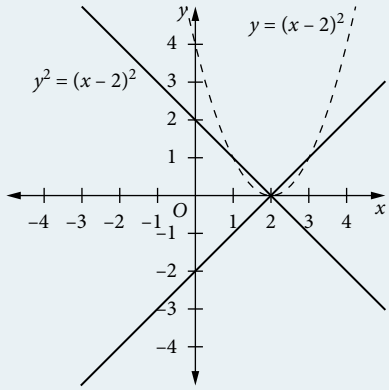
The graph of $y^2 = x^2$ is defined for all x.
The graphs intersect at $(-1, 1)$, $(0, 0)$ and $(1, 1)$.
The resulting graph is the same as the graph of $y = \pm x$.



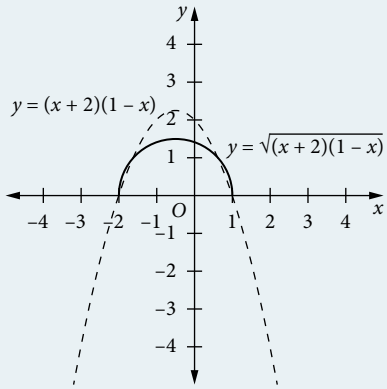
(b) The graph of $y = \sqrt{(x-2)^2}$ is defined for all x.
The function is never negative.
The graphs intersect at (1, 1), (2, 0) and (3, 1).
The resulting graph is the same as the graph of $y = |x-2|$.



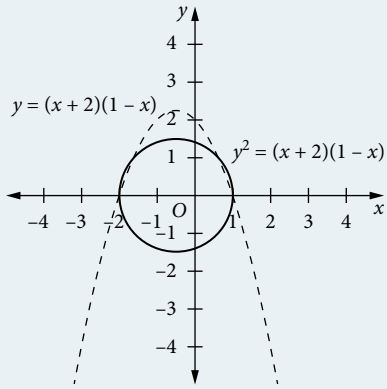
The graph of $y^2 = (x-2)^2$ is defined for all x.
The graphs intersect at (1, 1), (2, 0) and (3, 1).
The resulting graph is the same as the graph of $y = \pm(x-2)$.



(c) The graph of $y = \sqrt{(x+2)(1-x)}$ is undefined for $x < -2$ and $x > 1$.
The maximum value of $(x+2)(1-x)$ is $\frac{9}{4}$ and occurs at $x = -\frac{1}{2}$.
The greatest value of $\sqrt{(x+2)(1-x)}$ is $\frac{3}{2}$ and occurs at $x = -\frac{1}{2}$.
$-2 < x < 1$: $0 \le y \le 1.5$.
Further algebraic investigation shows that the graph is a semicircle in the upper half plane, centre (–0.5, 0), radius 1.5.

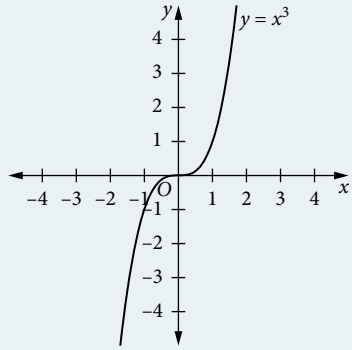


The graph of $y^2 = (x+2)(1-x)$ is undefined for $x < -2$ and $x > 1$.
$-2 < x < 1$: $-1.5 \le y \le 1.5$.
Further algebraic investigation shows that the graph is a circle, centre (–0.5, 0), radius 1.5.

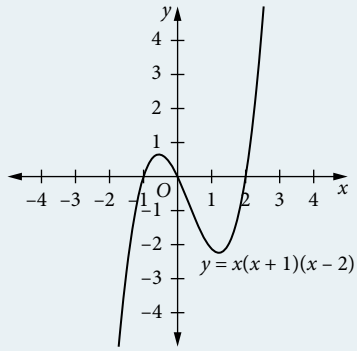


Example 6

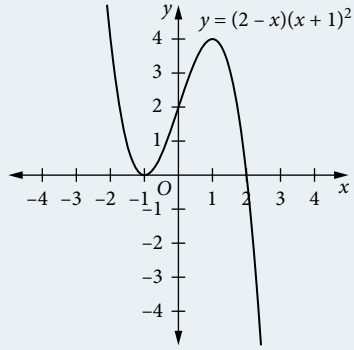
(a) Given $y = x^3$, draw $y = \sqrt{x^3}$ and $y^2 = x^3$.



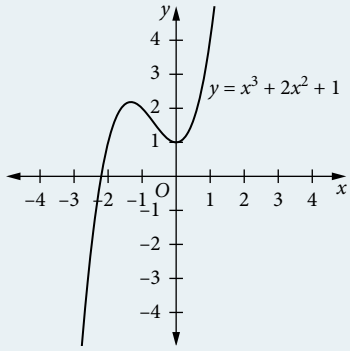
(b) Given $y = x(x+1)(x-2)$, draw $y = \sqrt{x(x+1)(x-2)}$ and $y^2 = x(x+1)(x-2)$.



(c) Given $y = (2-x)(x+1)^2$, draw $y = \sqrt{(2-x)(x+1)^2}$ and $y^2 = (2-x)(x+1)^2$.



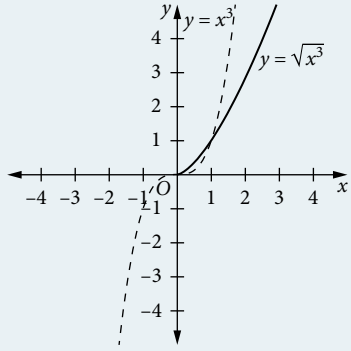
(d) Given $y = x^3 + 2x^2 + 1$, draw $y = \sqrt{x^3 + 2x^2 + 1}$ and $y^2 = x^3 + 2x^2 + 1$.



Solution

(a) The graph of $y = \sqrt{x^3}$ is undefined for $x < 0$.

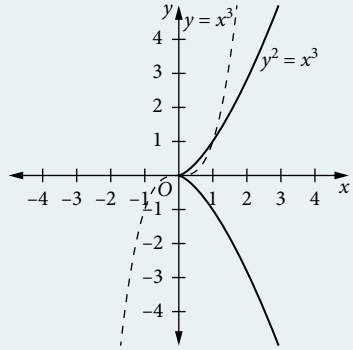
$x^3 = \sqrt{x^3}$ at $x = 0, 1$. The curves intersect at (0, 0) and (1, 1).



The graph of $y^2 = x^3$ is undefined for $x < 0$.

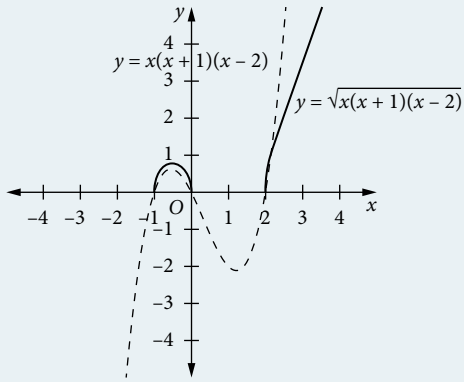
The curves intersect at (0, 0) and (1, 1).

$y^2 = x^3$ is equivalent to $y = \pm\sqrt{x^3}$.

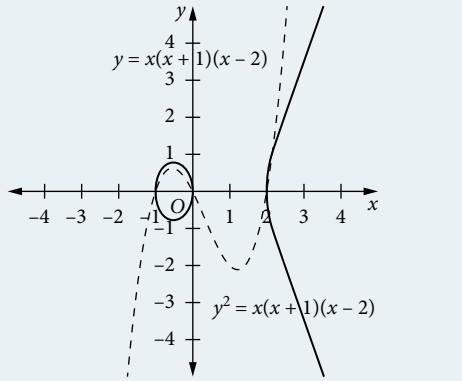


(b) The graph of $y = \sqrt{x(x+1)(x-2)}$ is undefined for $x < -1$, $1 < x < 2$.
The graphs meet at $(-1, 0)$, $(0, 0)$, $(2, 0)$.

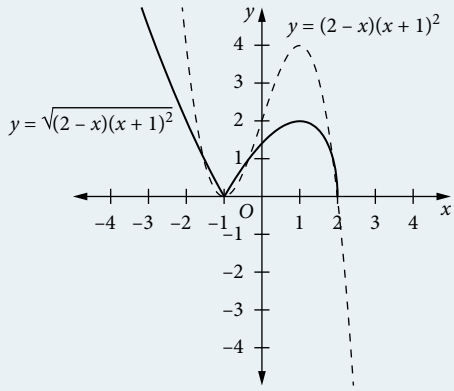
For $x > 2$, $\sqrt{x(x+1)(x-2)} > x(x+1)(x-2)$ until the RHS becomes greater than 1.



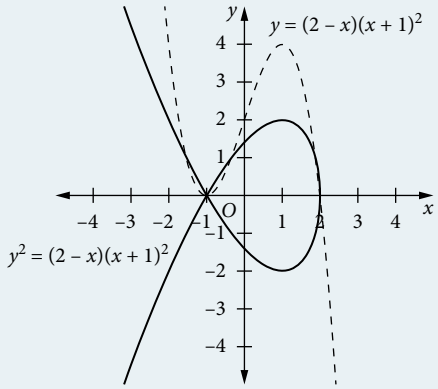
The graph of $y^2 = x(x+1)(x-2)$ is undefined for $x < -1$, $1 < x < 2$.
The graphs meet at $(-1, 0)$, $(0, 0)$, $(2, 0)$.



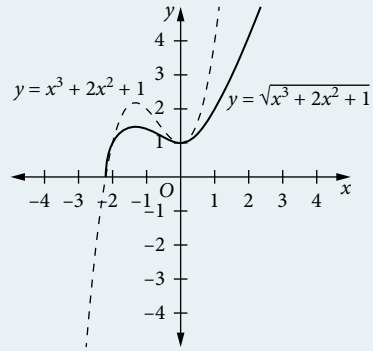
(c) The graph of $y = \sqrt{(2-x)(x+1)^2}$ is undefined for $x > 2$.
The maximum turning point of $y = (2-x)(x+1)^2$ is $(1, 4)$, so the maximum turning point of $y = \sqrt{(2-x)(x+1)^2}$ is $(1, 2)$.
$(-1, 0)$ is not called a turning point of $y = \sqrt{(2-x)(x+1)^2}$ because at this point the curve changes sharply, not smoothly. Instead this point is called a **cusp**.



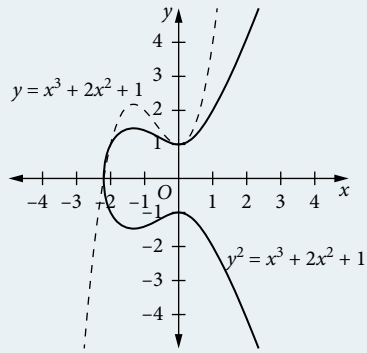
The graph of $y^2 = (2-x)(x+1)^2$ is undefined for $x > 2$.
The maximum turning point of $y = (2-x)(x+1)^2$ is $(1, 4)$, so the maximum turning point of $y^2 = (2-x)(x+1)^2$ is $(1, 2)$ and the minimum turning point is $(1, -2)$.



(d) The graph of $y = \sqrt{x^3 + 2x^2 + 1}$ is undefined for $x < -2.2$
The graphs intersect at $(-2.206, 0)$, $(-2, 1)$ and $(0, 1)$.

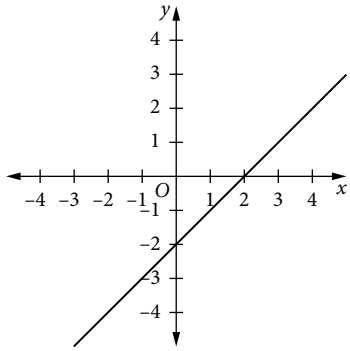


The graph of $y^2 = x^3 + 2x^2 + 1$ is undefined for $x < -2.2$.
The graphs intersect at $(-2.206, 0)$, $(-2, 1)$ and $(0, 1)$.



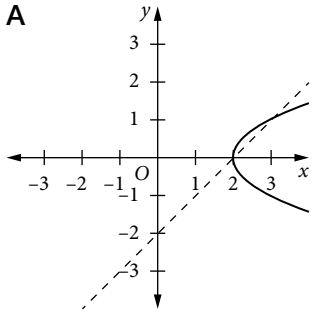
EXERCISE 3.2 SQUARE ROOT FUNCTIONS

1 The graph of $y = x - 2$ is shown.

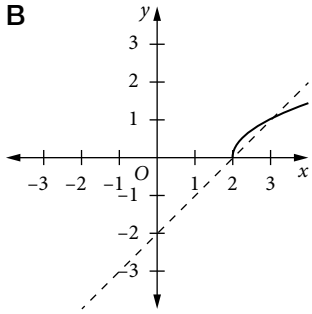


Which of the following represents the graph of $y = \sqrt{x - 2}$?

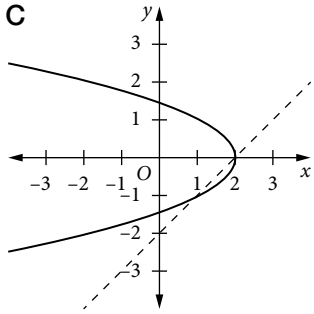
A



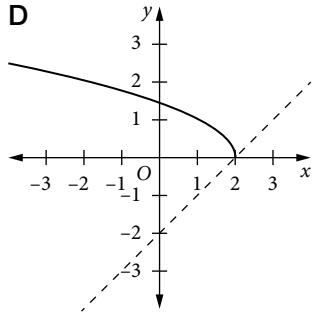
B



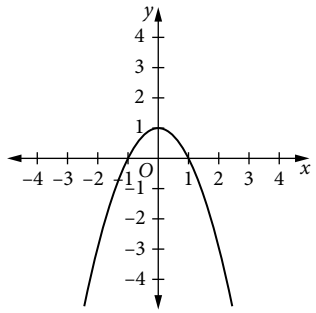
C



D

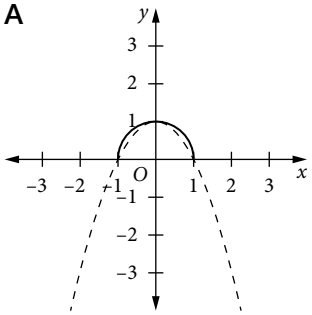


2 The graph of $y = 1 - x^2$ is shown.

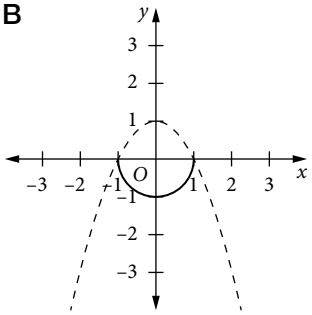


Which of the following represents the graph of $y^2 = 1 - x^2$?

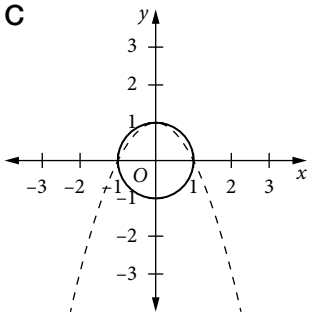
A



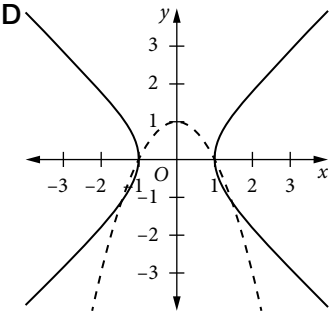
B



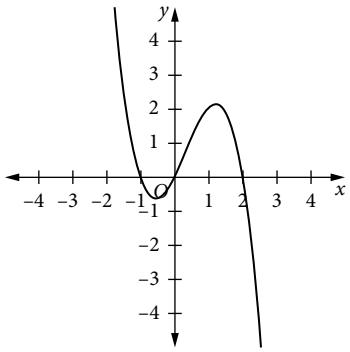
C



D

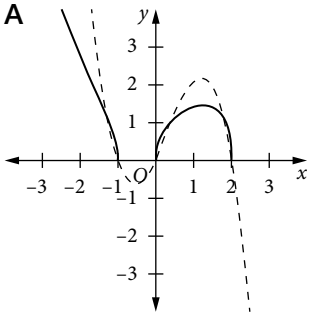


3 The graph of $y = x(2 - x)(x + 1)$ is shown.

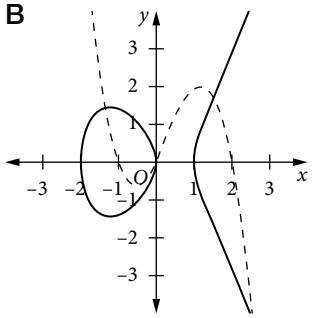


Which of the following represents the graph of $y^2 = x(2 - x)(x + 1)$?

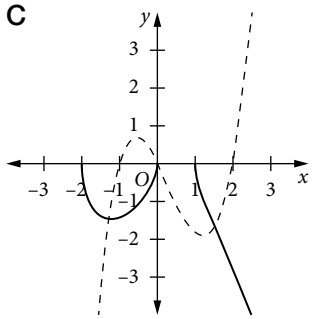
A



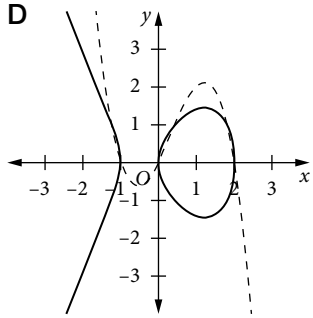
B



C

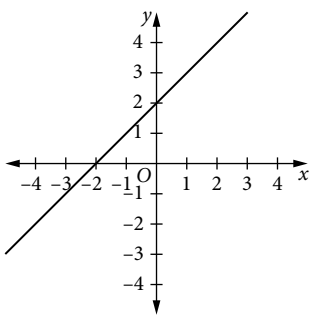


D



4 Given the graph of $y = x + 2$, draw:

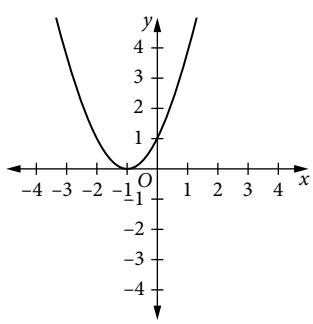
(a) $y = \sqrt{x + 2}$



(b) $y^2 = x + 2$

5 Given the graph of $y = (x + 1)^2$, draw:

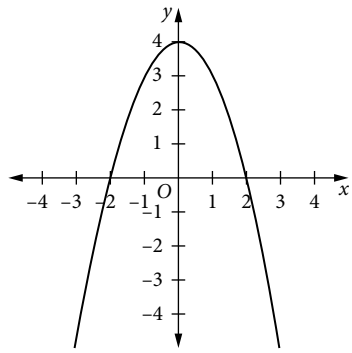
(a) $y = \sqrt{(x + 1)^2}$



(b) $y^2 = x + 1$

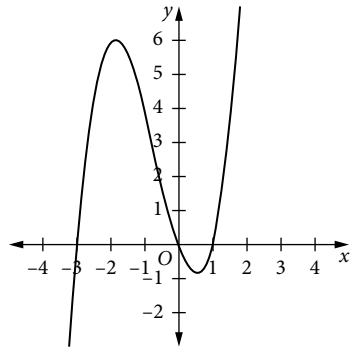
6 Given the graph of $y = (x+2)(2-x)$, draw:

(a) $y = \sqrt{(x+2)(2-x)}$ **(b)** $y^2 = (x+2)(2-x)$



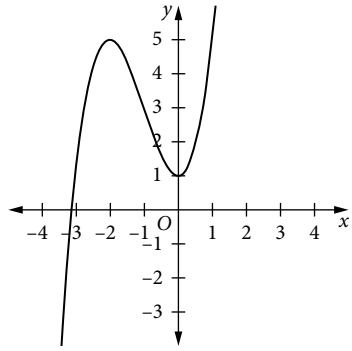
7 Given the graph of $y = x(x-1)(x+3)$, draw:

(a) $y = \sqrt{x(x-1)(x+3)}$ **(b)** $y^2 = x(x-1)(x+3)$



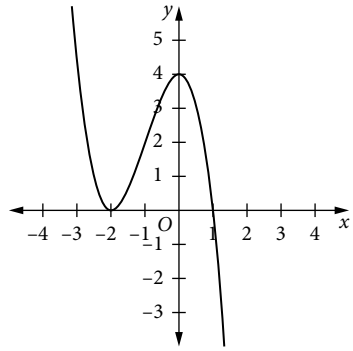
8 Given the graph of $y = x^3 + 3x^2 + 1$, draw:

(a) $y = \sqrt{x^3 + 3x^2 + 1}$ **(b)** $y^2 = x^3 + 3x^2 + 1$



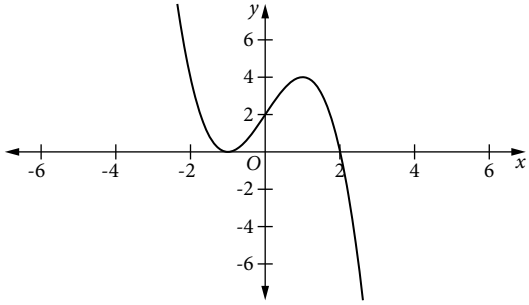
9 Given the graph of $y = (1-x)(x+2)^2$, draw:

(a) $y = \sqrt{(1-x)(x+2)^2}$ **(b)** $y^2 = (1-x)(x+2)^2$



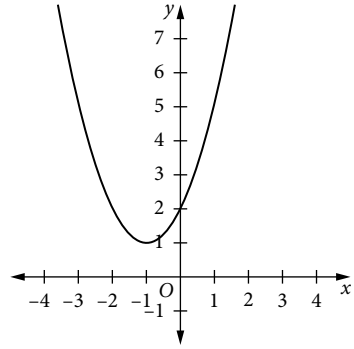
10 Given the graph of $y = 2 + 3x - x^3$, draw:

(a) $y = \sqrt{2 + 3x - x^3}$ **(b)** $y^2 = 2 + 3x - x^3$



11 Given the graph of $y = x^2 + 2x + 2$, draw:

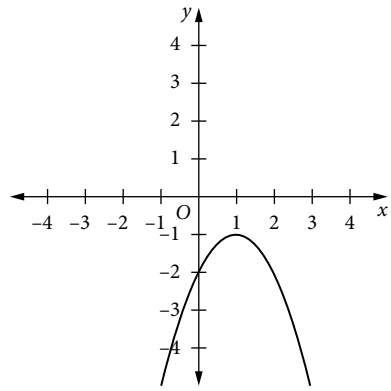
(a) $y = \sqrt{x^2 + 2x + 2}$ **(b)** $y^2 = x^2 + 2x + 2$



12 Given the graph of $y = -x^2 + 2x - 2$, draw:

(a) $y = \sqrt{-x^2 + 2x - 2}$

(b) $y = -x^2 + 2x - 2$



3.3 ABSOLUTE VALUE FUNCTIONS

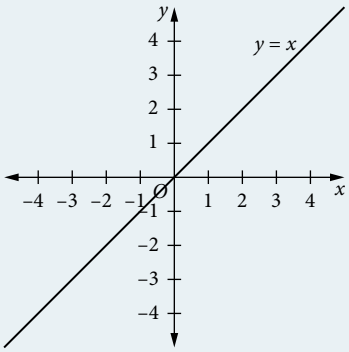
Given the graph of $y = f(x)$, it is often useful or necessary to draw the graphs of the absolute value functions $y = |f(x)|$ and $y = f(|x|)$. Sometimes these graphs will be the same, but not always.

$y = |f(x)|$ is defined wherever $f(x)$ exists. $|f(x)| \geq 0$ if $-\infty < f(x) < \infty$. $|f(x)| > 0$ wherever $f(x) < 0$.
$y = f(|x|)$ is the same as $y = f(x)$ for $x \geq 0$, but different for $x < 0$.

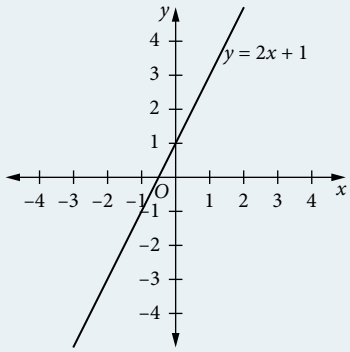
Example 7

In each part, use the graph of the given function to draw the graph of $y = |f(x)|$ and the graph of $y = f(|x|)$.

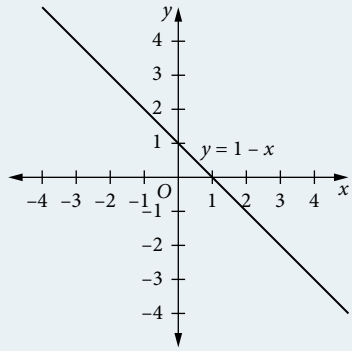
(a) Given the graph of $y = x$, draw $y = |x|$.



(b) Given the graph of $y = 2x + 1$, draw $y = |2x + 1|$ and $y = 2|x| + 1$.

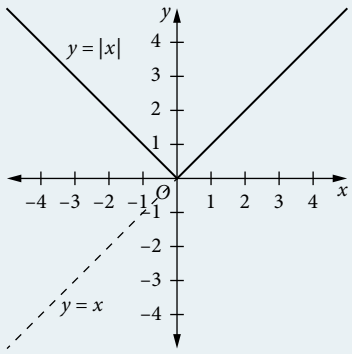


(c) Given the graph of $y = 1 - x$, draw $y = |1 - x|$ and $y = 1 - |x|$.



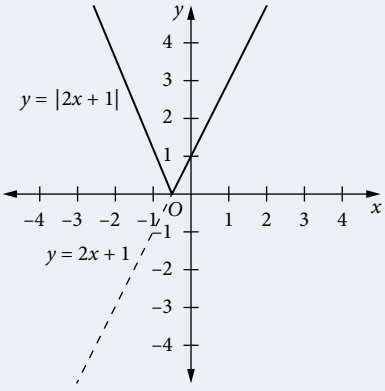
Solution

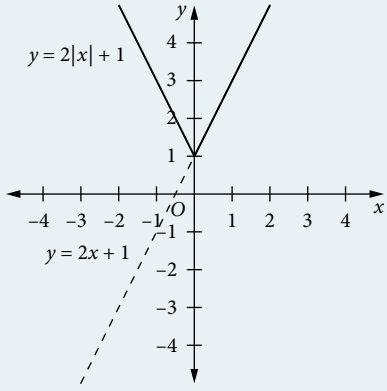
(a) The graphs are the same for $x \geq 0$.
$|x| \geq 0$ for all x.



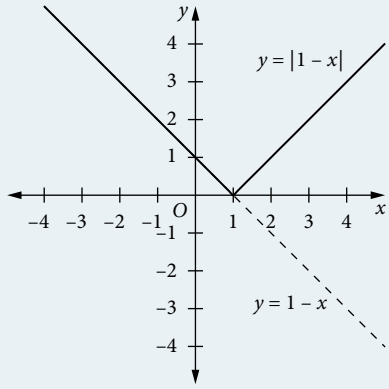
(b) The graphs of $y = 2x + 1$ and $y = |2x + 1|$ are the same for $x \geq -\frac{1}{2}$.
$|2x + 1| \geq 0$ for all x.

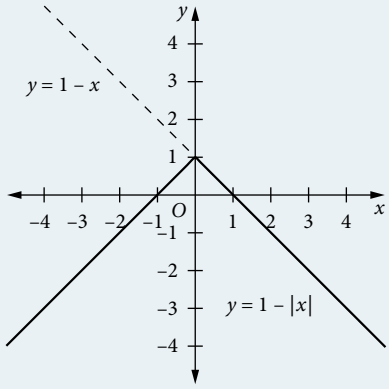
The graphs of $y = 2x + 1$ and $y = 2|x| + 1$ are the same for $x \geq 0$.
$|2x + 1| \geq 1$ for all x.





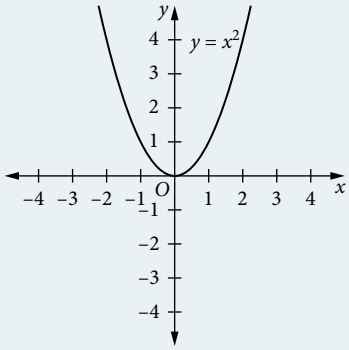
(c) The graphs of $y = 1 - x$ and $y = |1 - x|$ are the same for $x \leq 1$.
$|1 - x| \geq 0$ for all x.
The graphs of $y = 1 - x$ and $y = 1 - |x|$ are the same for $x \geq 0$.
$1 - |x| \leq 1$



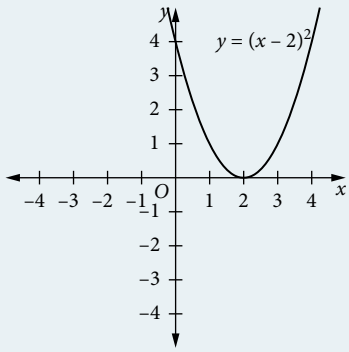


Example 8

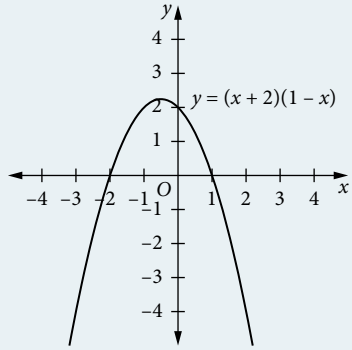
(a) Given the graph of $y = x^2$, draw $y = |x^2|$ and $y = |x|^2$.



(b) Given the graph of $y = (x - 2)^2$, draw $y = |(x - 2)^2|$ and $y = (|x| - 2)^2$.

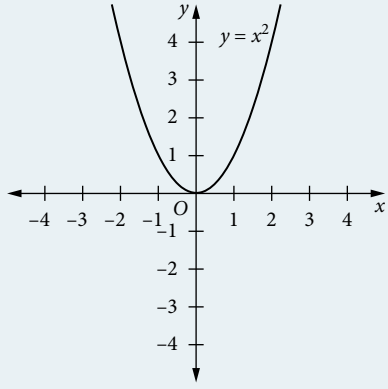


(c) Given the graph of $y = (x + 2)(1 - x)$, draw $y = |(x + 2)(1 - x)|$ and $y = (|x| + 2)(1 - |x|)$.

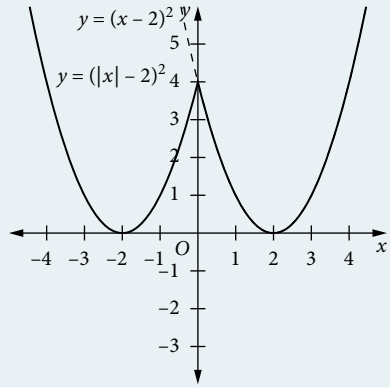


Solution

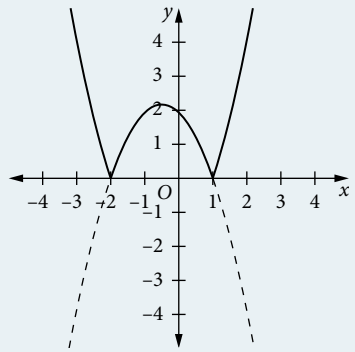
(a) The graphs of $y = |x^2|$ and $y = |x|^2$ are the same as the original graph $y = x^2$.
By the definition, $|x| = \sqrt{x^2}$ so $|x|^2 = x^2$.



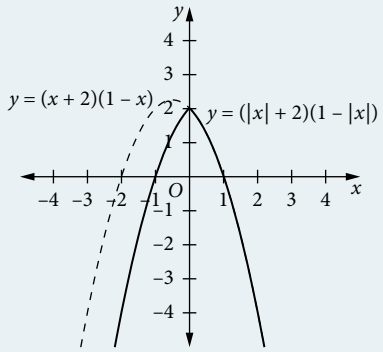
(b) The graphs of $y = (x - 2)^2$ and $y = |(x - 2)^2|$ are the same.
The graphs of $y = (x - 2)^2$ and $y = (|x| - 2)^2$ are the same for $x \geq 0$.
There is a cusp at $(0, 4)$. $x = 0$ is an axis of symmetry.
$(|x| - 2)^2 \geq 0$



(c) The graphs of $y = (x+2)(1-x)$ and $y = \left|(x+2)(1-x)\right|$ are the same for $-2 \le x \le 1$.
There is a cusp at $(-2, 0)$ and $(1, 0)$.
$x = -\frac{1}{2}$ is an axis of symmetry.

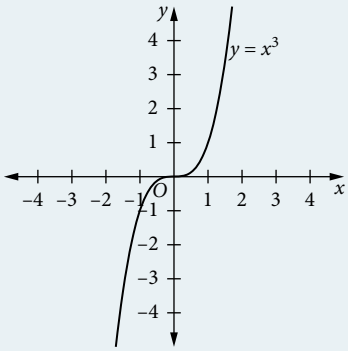


The graphs of $y = (x+2)(1-x)$ and $y = (|x|+2)(1-|x|)$ are the same for $x \ge 0$.
$(|x|+2)\,(1-|x|) \le 2$. $(0, 2)$ is not a turning point.
$x = 0$ is an axis of symmetry.

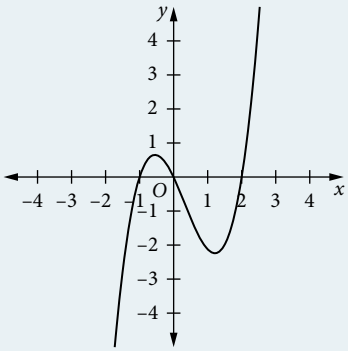


Example 9

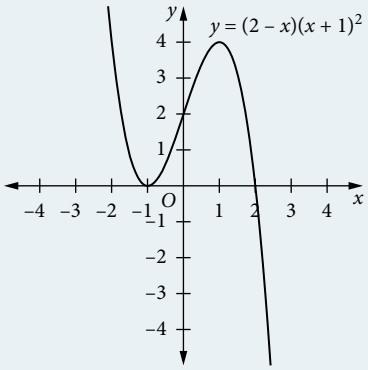
(a) Given the graph of $y = x^3$, draw $y = \left|x^3\right|$ and $y = |x|^3$.



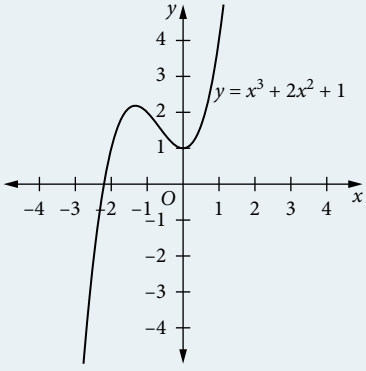
(b) Given the graph of $y = x(x+1)(x-2)$, draw $y = \left|x(x+1)(x-2)\right|$ and $y = |x|\left(|x|+1\right)\left(|x|-2\right)$.



(c) Given the graph of $y = (2-x)(x+1)^2$, draw $y = \left|(2-x)(x+1)^2\right|$ and $y = \left(2-|x|\right)\left(|x|+1\right)^2$.



(d) Given the graph of $y = x^3 + 2x^2 + 1$, draw $y = \left|x^3 + 2x^2 + 1\right|$ and $y = |x|^3 + 2|x|^2 + 1$.

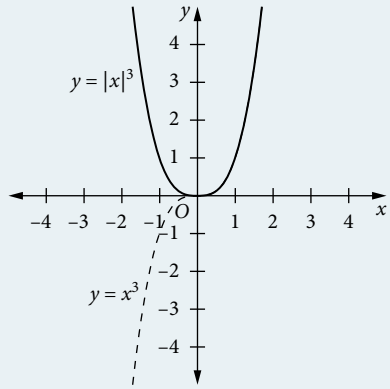


Solution

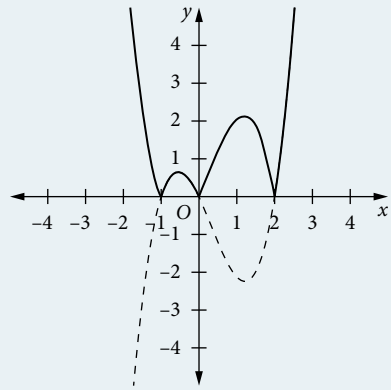
(a) $x^3 = |x^3|$ for $x \geq 0$, i.e. the curves are identical for $x \geq 0$.

Also $x^3 = |x|^3$ for $x \geq 0$, i.e. the curves are identical for $x \geq 0$.

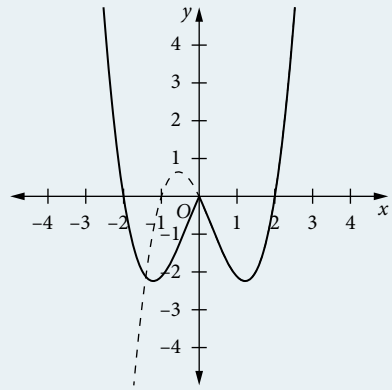
The graph of $y = |x|^3$ is the same as the graph of $y = |x^3|$.
They are both the solid line in the graph.



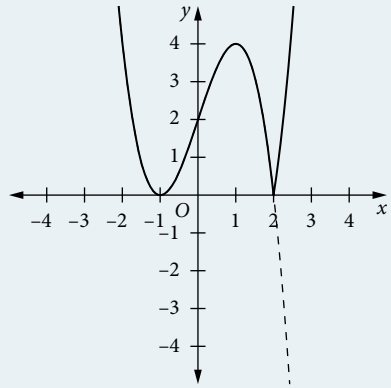
(b) For $y = |x(x+1)(x-2)|$:
The graphs meet at $(-1, 0)$, $(0, 0)$, $(2, 0)$.
The graphs are the same for $-1 \leq x \leq 0$ and $x \geq 2$.
$|x(x+1)(x-2)| \geq 0$ for all x.



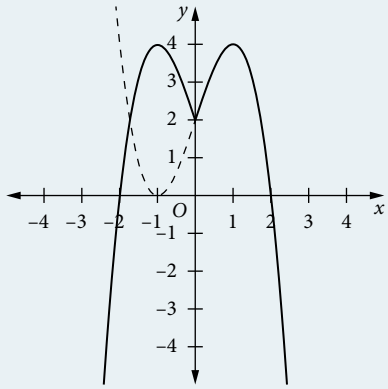
For $y = |x|(|x|+1)(|x|-2)$:
The graphs meet at $(0, 0)$, $(2, 0)$ and $(-\sqrt{2}, -2)$. This last point can be solved using graphing software.
The graphs are the same for $x \geq 0$. There is a cusp at $(0, 0)$.
$|x|(|x|+1)(|x|-2) \geq -2$.



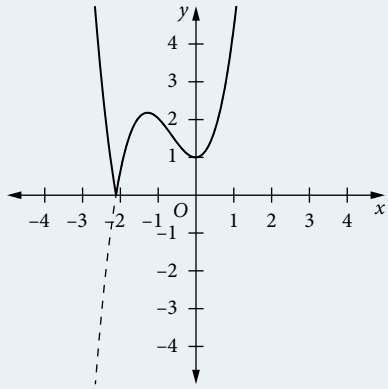
(c) For $y = |(2-x)(x+1)^2|$:
The graphs are the same for $x \leq 2$.
$|(2-x)(x+1)^2| \geq 0$. $(2, 0)$ is a cusp.
$(-1, 0)$ is a minimum turning point, $(1, 4)$ is a maximum turning point.



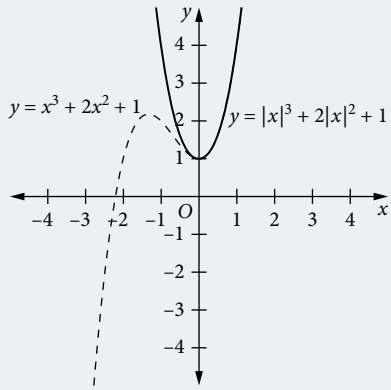
For $y = \left(2 - |x|\right)\left(|x| + 1\right)^2$:
The graphs are the same for $x \geq 0$.
The curves also intersect at $\left(-\sqrt{3}, 2\right)$.
(0, 2) is a cusp. (−1, 4) and (1, 4) are maximum turning points.
$\left(2 - |x|\right)\left(|x| + 1\right)^2 \leq 4$



(d) For $y = \left|x^3 + 2x^2 + 1\right|$:
The graphs are the same for about $x \geq -2.2$.
(−2.2, 0) is a cusp. (0, 1) is a minimum turning point. (−1.33, 2.19) is a maximum turning point.

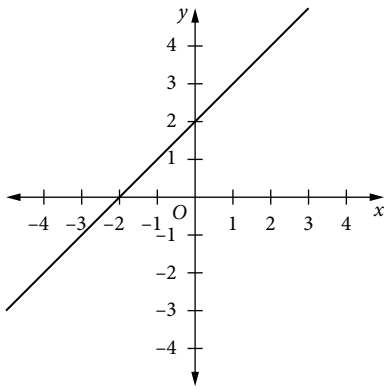


For $y = |x|^3 + 2|x|^2 + 1$:
The graphs are the same for $x \geq 0$.
$|x|^3 + 2|x|^2 + 1 \geq 1$
(0, 1) is a minimum turning point.

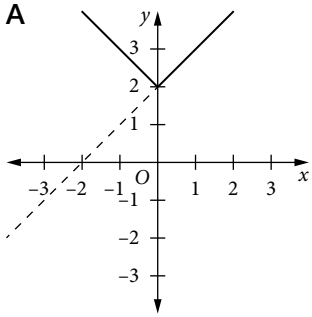


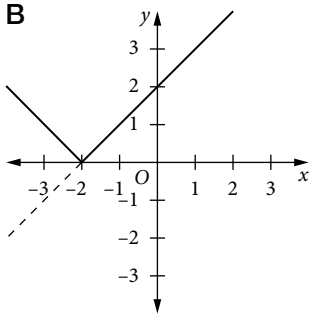
EXERCISE 3.3 ABSOLUTE VALUE FUNCTIONS

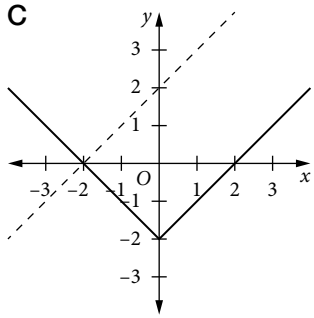
1 The graph of $y = x + 2$ is shown.

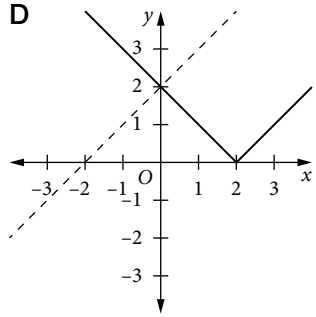


Which of the following represents the graph of $y = |x + 2|$?

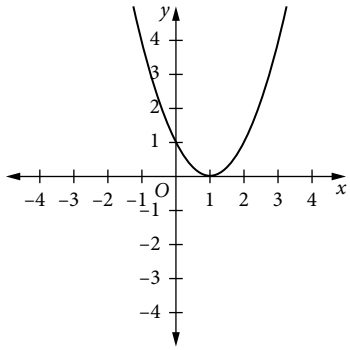
A 

B 

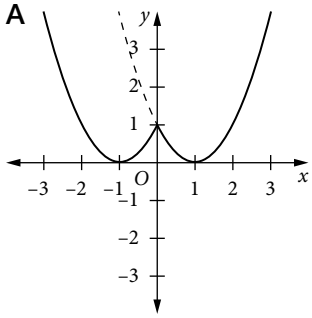
C 

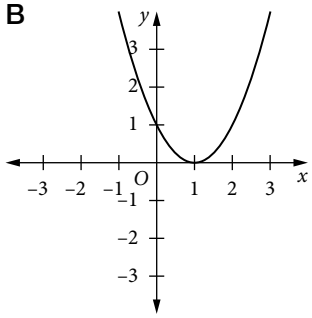
D 

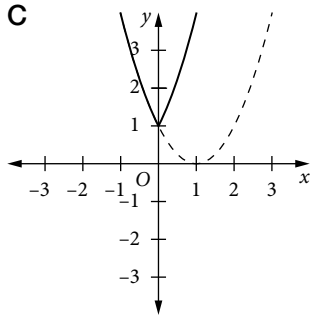
2 The graph of $y = (x - 1)^2$ is shown.

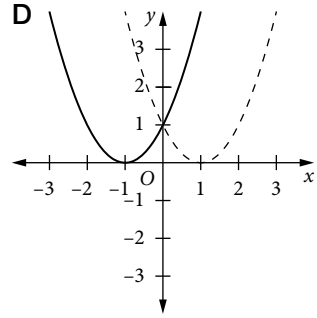


Which of the following represents the graph of $y = (|x| - 1)^2$?

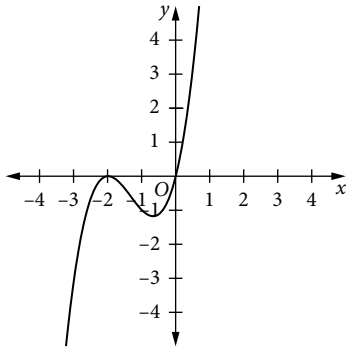
A 

B 

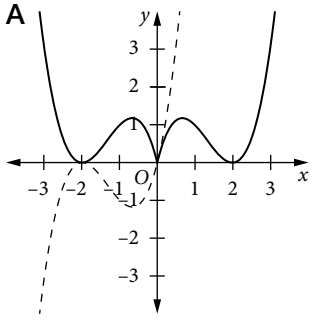
C 

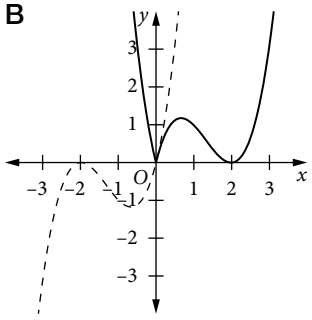
D 

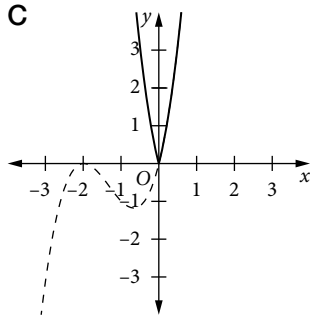
3 The graph of $y = x(x + 2)^2$ is shown.

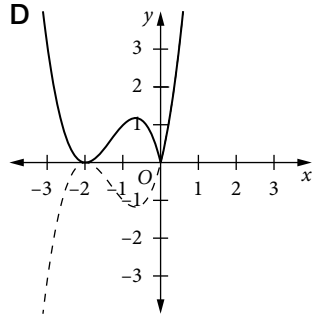


Which of the following represents the graph of $y = |x|(|x| + 2)^2$?

A 

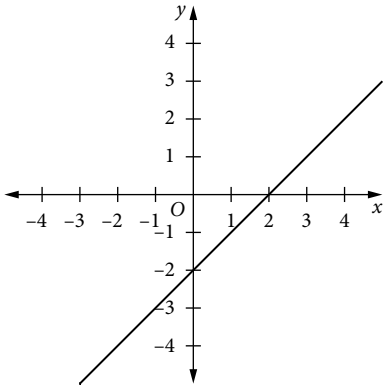
B 

C 

D 

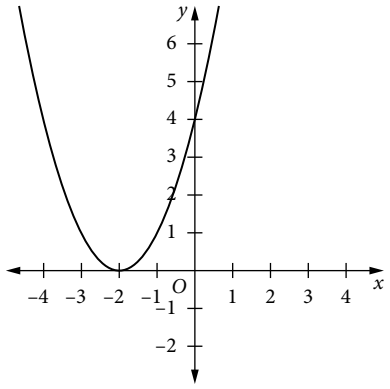
4 Given the graph of $y = x - 2$, draw:

(a) $y = |x - 2|$ **(b)** $y = |x| - 2$



5 Given the graph of $y = (x + 2)^2$, draw:

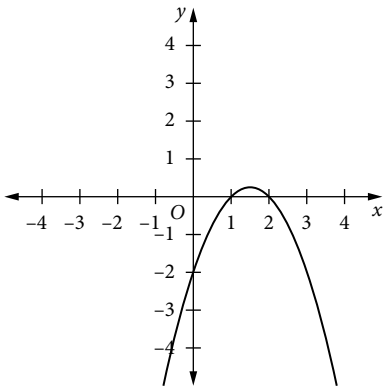
(a) $y = \left|(x + 2)^2\right|$ **(b)** $y = \left(|x| + 2\right)^2$



6 Given the graph of $y = (x - 1)(2 - x)$, draw:

(a) $y = \left|(x - 1)(2 - x)\right|$

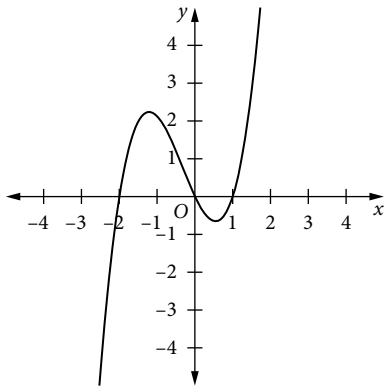
(b) $y = (|x| - 1)(2 - |x|)$



7 Given the graph of $y = x(x - 1)(x + 2)$, draw:

(a) $y = \left|x(x - 1)(x + 2)\right|$

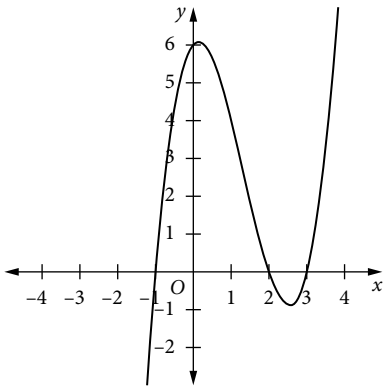
(b) $y = |x|(|x| - 1)(|x| + 2)$



8 Given the graph of $y = (x - 3)(x - 2)(x + 1)$, draw:

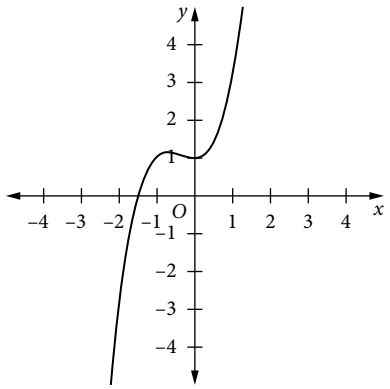
(a) $y = \left|(x - 3)(x - 2)(x + 1)\right|$

(b) $y = (|x| - 3)(|x| - 2)(|x + 1)$



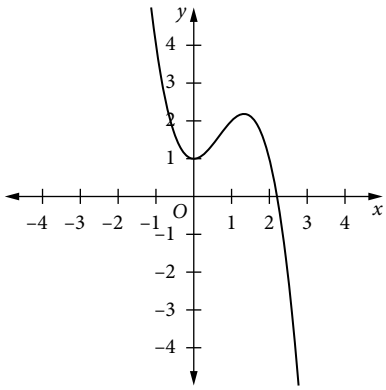
9 Given the graph of $y = x^3 + x^2 + 1$, draw:

(a) $y = \left|x^3 + x^2 + 1\right|$ **(b)** $y = |x|^3 + |x|^2 + 1$



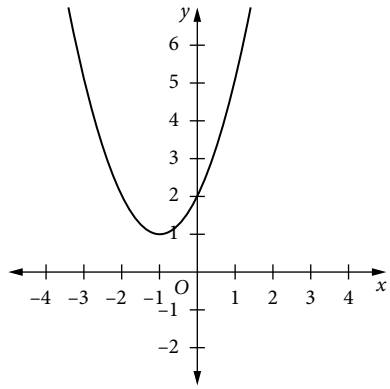
10 Given the graph of $y = -x^3 + 2x^2 + 1$, draw:

(a) $y = \left|-x^3 + 2x^2 + 1\right|$ **(b)** $y = -|x|^3 + 2|x|^2 + 1$



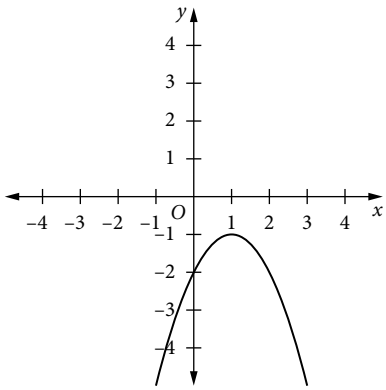
11 Given the graph of $y = x^2 + 2x + 2$, draw:

(a) $y = \left|x^2 + 2x + 2\right|$ **(b)** $y = |x|^2 + 2|x| + 2$



12 Given the graph of $y = -x^2 + 2x - 2$, draw:

(a) $y = \left|-x^2 + 2x - 2\right|$ **(b)** $y = -|x|^2 + 2|x| - 2$



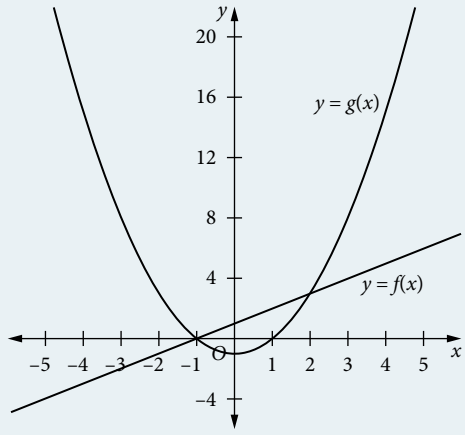
3.4 GRAPHING POLYNOMIALS BY ADDING ORDINATES

Given the graphs of two polynomial functions $y = f(x)$ and $y = g(x)$, the graph of a new function $y = f(x) + g(x)$ can be obtained by a process of adding the ordinates for each x value. This process is demonstrated in the following examples.

Example 10

The graphs of $y = f(x)$ and $y = g(x)$ are shown.

By drawing vertical lines and adding ordinates, draw the graph of $y = f(x) + g(x)$. Comment on the new curve.



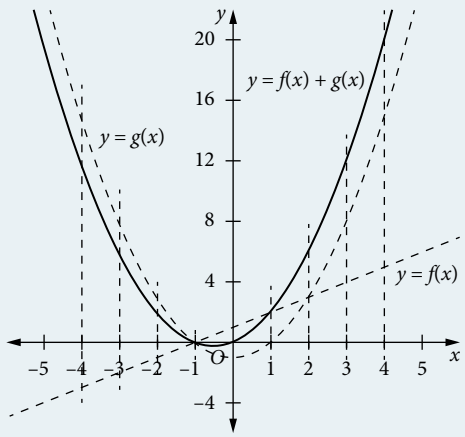
Solution

On the diagram, vertical lines are drawn through important points such as turning points, points where a curve cuts the axes, and points where the curves intersect.

On each vertical line, the intercepts of the two curves are added to find the position of a new point, which is marked on the line. These new points are then joined to obtain $y = f(x) + g(x)$.

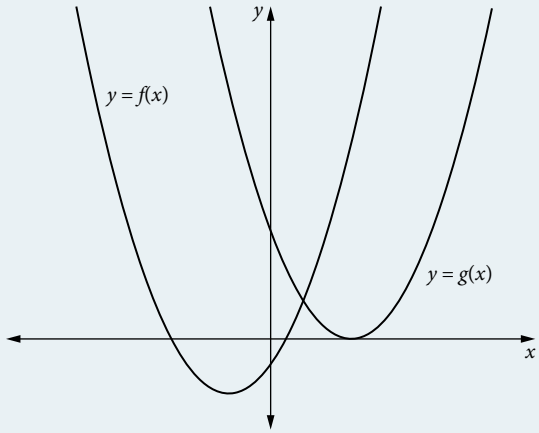
The solid curve is the graph of $y = f(x) + g(x)$.

This curve is above $y = g(x)$ to the right of $x = -1$, and below $y = g(x)$ to the left of $x = -1$. They intersect at $x = -1$. $y = f(x) + g(x)$ cuts the x-axis at $(-1, 0)$ and $(0, 0)$.



Example 11

The graphs of $y = f(x)$ and $y = g(x)$ are shown.
By drawing vertical lines and adding ordinates, draw the graph of $y = f(x) + g(x)$. Comment on the new curve.



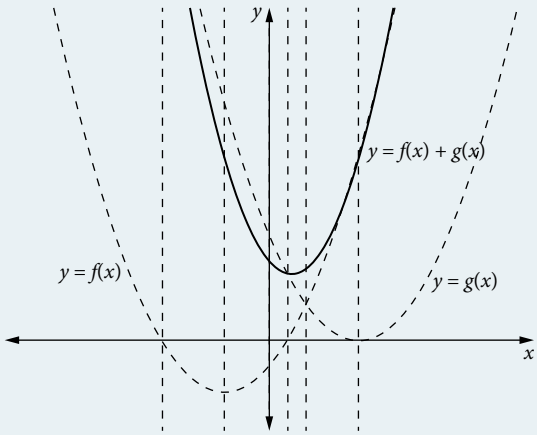
Solution

On the diagram, vertical lines are drawn through important points such as turning points, points where a curve cuts the axes and points where the curves intersect.

On each vertical line, the intercepts of the two curves are added to find the position of a new point, which is marked on the line. These new points are then joined to obtain $y = f(x) + g(x)$.

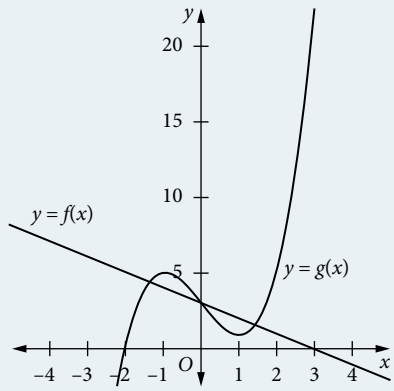
The solid curve is the graph of $y = f(x) + g(x)$.

This curve is above $y = f(x)$ to the right of its turning point, and below $y = g(x)$ to the left of its turning point.



Example 12

The graphs of $y = f(x)$ and $y = g(x)$ are shown.
By drawing vertical lines and adding ordinates, draw the graph of $y = f(x) + g(x)$. Comment on the new curve.



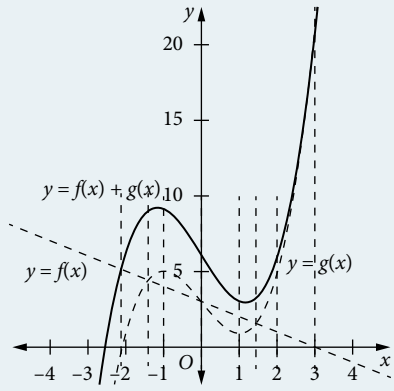
Solution

On the diagram, vertical lines are drawn through important points, such as turning points, points where a curve cuts the axes and points where the curves intersect.

On each vertical line, the intercepts of the two curves are added to find the position of a new point, which is marked on the line. These new points are then joined to obtain $y = f(x) + g(x)$.

The solid curve is the graph of $y = f(x) + g(x)$.

The new curve is above $y = g(x)$ for $x < 4$, passing through $(0, 6)$. They intersect at $x = 4$, and $y = f(x) + g(x)$ is just below $y = g(x)$ for $x > 4$.



EXPLORING FURTHER

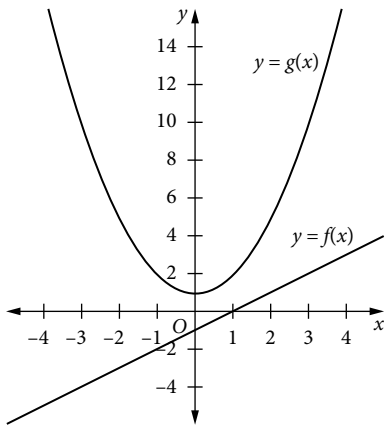
Graphing polynomials by adding ordinates

Use graphing software to verify the addition of ordinates for combined polynomial graphs.

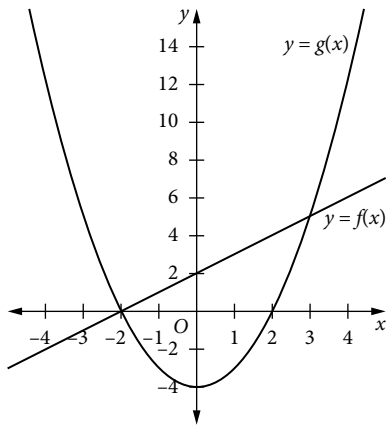
EXERCISE 3.4 GRAPHING POLYNOMIALS BY ADDING ORDINATES

1 The graphs of $y = f(x)$ and $y = g(x)$ are shown. By drawing vertical lines and adding ordinates, draw the graph of $y = f(x) + g(x)$. Comment on the new curve.

(a)



(b)



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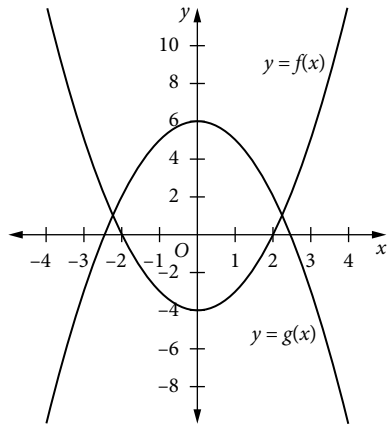
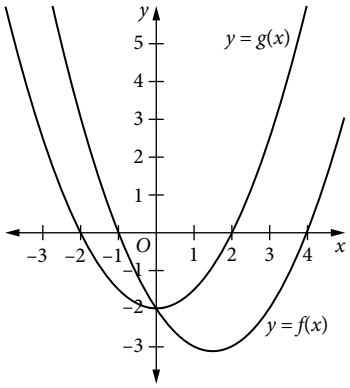
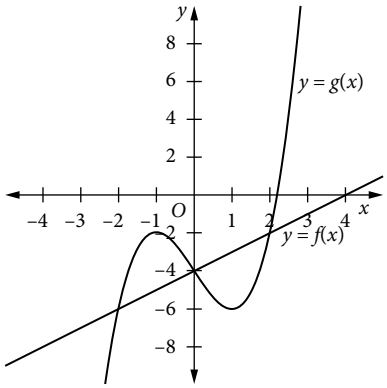


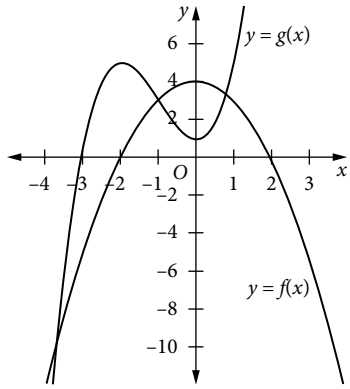
(d)



(e)



(f)

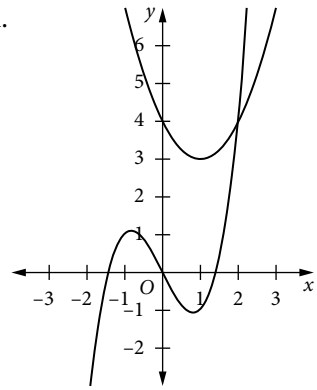


2 On the same diagram, sketch the graphs of $y = x^2 + 2x$ and $y = 1 - 2x$. Use these graphs to sketch $y = x^2 + 1$.

3 On the same diagram, sketch the graphs of $y = x^2 - 2x$ and $y = x^3$. Use these graphs to sketch $y = x^3 + x^2 - 2x$.

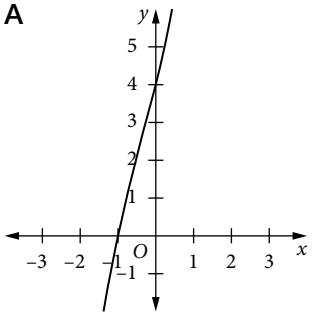
4 On the same diagram, sketch the graphs of $y = x^2 - 3x$ and $y = x^3 - 3x$. Use these graphs to sketch $y = x^3 + x^2 - 6x$.

5 The graphs of $y = x^2 - 2x + 4$ and $y = x^3 - 2x$ are shown.

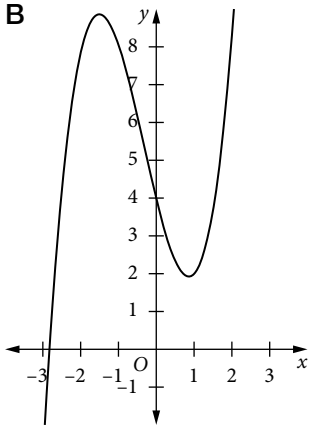


Which diagram represents the graph of $y = x^3 + x^2 - 4x + 4$?

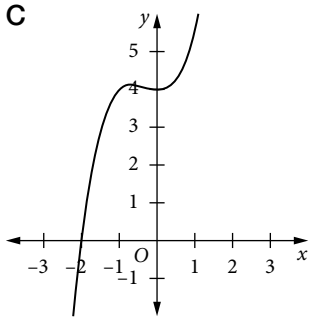
A



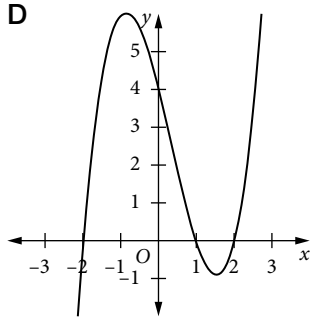
B



C



D

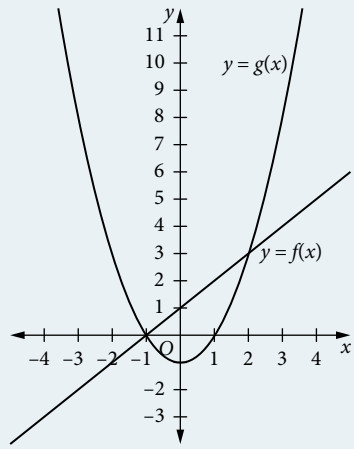


3.5 GRAPHING POLYNOMIALS BY MULTIPLYING ORDINATES

Given the graphs of two polynomial functions, $y = f(x)$ and $y = g(x)$, the graph of a new function $y = f(x)g(x)$ can be obtained by a process of multiplying the ordinates for each x value. This process is demonstrated in the following examples.

Example 13

The graphs of $y = f(x)$ and $y = g(x)$ are shown. By drawing vertical lines and multiplying ordinates, draw the graph of $y = f(x)g(x)$. Comment on the new curve.



Solution

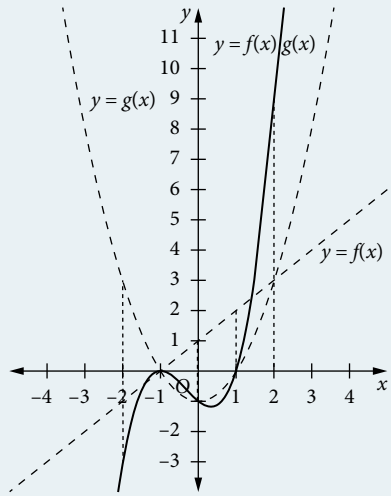
On the diagram, vertical lines are drawn through important points, such as turning points, points where a curve cuts the axes, points where the curves intersect, and points where the function value is 1 or −1.

On each vertical line, the intercepts of the two curves are multiplied to find the position of a new point, which is marked on the line.

- $x = -2$: Ordinates are 3 and −1, so $3 \times (-1) = -3$. The point on $f(x)g(x)$ is $(-2, -3)$.
- $x = -1$: Ordinates are 0 and 0, so the point on $f(x)g(x)$ is $(-1, 0)$.
- $x = 0$: Ordinates are −1 and 1, so $(-1) \times 1 = -1$. The point on $f(x)g(x)$ is $(0, -1)$.

This makes $(-1, 0)$ a local maximum turning point.

- $x = 1$: Ordinates are 0 and 2, so the point on $f(x)g(x)$ is $(1, 0)$.
- $x = 2$: Ordinates are 3 and 3, so $3 \times 3 = 9$. The point on $f(x)g(x)$ is $(2, 9)$.
- $x = 0.5$: Ordinates are approximately −0.8 and 1.5 $(-0.8) \times 1.5 = -1.2$.
  A point on $f(x)g(x)$ is $(-0.5, -1.2)$. The minimum turning point of $y = f(x)\ g(x)$ will be near this point.



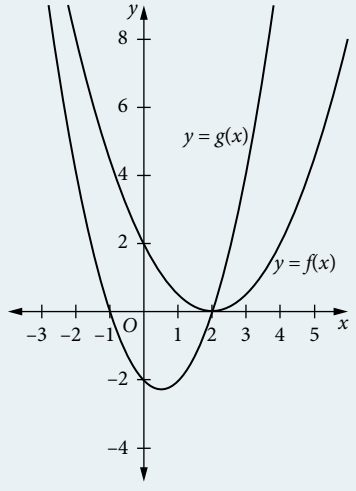
The new points are joined to obtain $y = f(x)g(x)$.

The solid curve is the graph of $y = f(x)g(x)$.

As $x \to \infty$, $f(x)g(x) \to \infty$. As $x \to -\infty$, $f(x)g(x) \to -\infty$.

Example 14

The graphs of $y = f(x)$ and $y = g(x)$ are shown. By drawing vertical lines and multiplying ordinates, draw the graph of $y = f(x)g(x)$. Comment on the new curve.



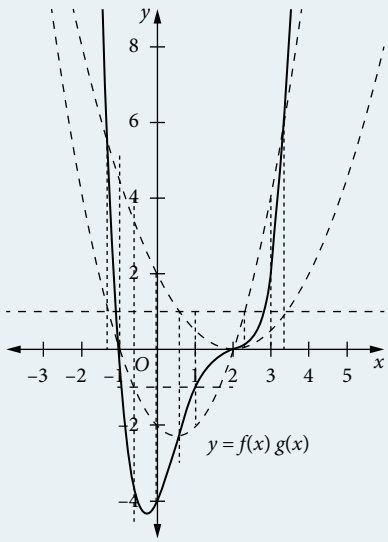
Solution

On the diagram, draw vertical lines through important points, such as turning points, points where a curve cuts the axes, where the curves intersect, and points where the function value is 1 or −1.

Multiply the intercepts on these vertical lines of the two curves and mark a new point.

Where the horizontal line $y = 1$ intersects a curve it shows the x value for which the product of the two functions is the same as the other function value.

Where the horizontal line $y = -1$ intersects a curve it shows the x value for which the product of the two functions has the opposite sign to the other function value.



- $x = -1$: Ordinates are 0 and 4.5, so the point on $f(x)g(x)$ is $(-1, 0)$.
- $x = 0$: Ordinates are −2 and 2, so $(-2) \times 2 = -4$. The point on $f(x)g(x)$ is $(0, -4)$.
- $x = 1$: Ordinates are −2 and 0.5, so $(-2) \times 0.5 = -1.5$. The point on $f(x)g(x)$ is $(1, -1.5)$.
- $x = 2$: Ordinates are 0 and 0, so the point on $f(x)\ g(x)$ is $(2, 0)$.
- $x = 3$: Ordinates are 4 and 0.5, so $4 \times 0.5 = 2$. The point on $f(x)g(x)$ is $(3, 2)$.

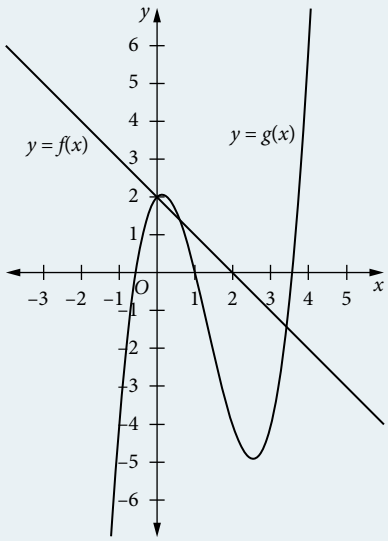
Join the points to obtain $y = f(x)g(x)$.

The solid curve is the graph of $y = f(x)g(x)$.

The new curve is a quartic, the product of two quadratic functions. It is positive where the original functions both have the same sign, zero where at least one of them is zero, and negative where only one of the functions is negative. At $(2, 0)$, where both $f(x)$ and $g(x)$ are zero, $f(x)\ g(x)$ has a horizontal point of inflection.

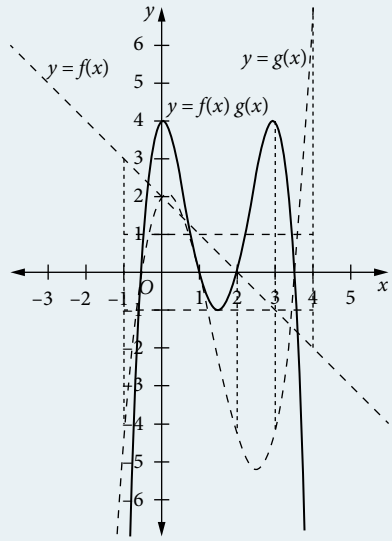
Example 15

The graphs of $y = f(x)$ and $y = g(x)$ are shown. By drawing vertical lines and multiplying ordinates, draw the graph of $y = f(x)\ g(x)$. Comment on the new curve.



Solution

On the diagram, draw vertical lines through important points, such as turning points, points where a curve cuts the axes, where the curves intersect, and points where the function value is 1 or −1.



Multiply the intercepts on these vertical lines of the two curves and mark a new point.

Where the horizontal line $y = 1$ intersects a curve it shows the x value for which the product of the two functions is the same as the other function value.

Where the horizontal line $y = -1$ intersects a curve it shows the x value for which the product of the two functions has the opposite sign to the other function value.

- $x = -1$: Ordinates are 3 and −4, so the point on $f(x)g(x)$ is $(-1,-12)$.
- $x = 0$: Ordinates are 2 and 2, so $2 \times 2 = 4$. The point on $f(x)g(x)$ is $(0, 4)$.
- $x = 1$: $g(1) = 0$, so $f(x)g(x) = 0$ and cuts the x-axis at $(1, 0)$.
- $x = 2$: $f(2) = 0$, so $f(x)g(x) = 0$ and cuts the x-axis at $(2, 0)$.
- $x = 3$: Ordinates are (-1) and (-4), so $(-1) \times (-4) = 4$. The point on $f(x)g(x)$ is $(3, 4)$.
- $x = 4$: Ordinates are (-2) and 6, so $(-2) \times 6 = -12$. The point on $f(x)g(x)$ is $(4, -12)$.

$g(x)$ cuts the x-axis again at 0.6 and 3.6, so $f(x)g(x) = 0$ and cuts the x-axis at $(-0.6, 0)$ and $(3.6, 0)$.

Drawing $x = 1.5$ will help locate the turning point which is at approximately $(1.5, -1)$.

Join the points to obtain $y = f(x)g(x)$.

The solid curve is the graph of $y = f(x)g(x)$.

The new curve is a quartic, the product of a linear and a cubic function. It is positive where the original functions both have the same sign, zero where one of them is zero, and negative where only one of the functions is negative. As $x \to \pm\infty$, $f(x)g(x) \to -\infty$. It has a greatest value of 4, which occurs at $x = 0$ and $x = 3$.

EXPLORING FURTHER

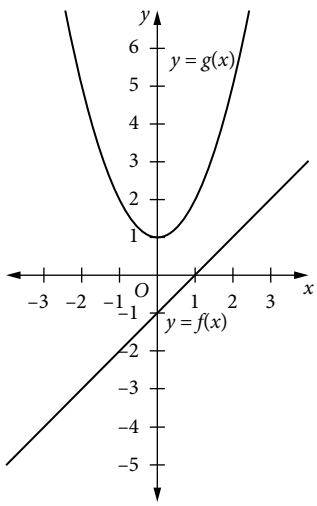
Graphing polynomials by multiplying ordinates

Use graphing software to verify the multiplication of ordinates for combined polynomial graphs.

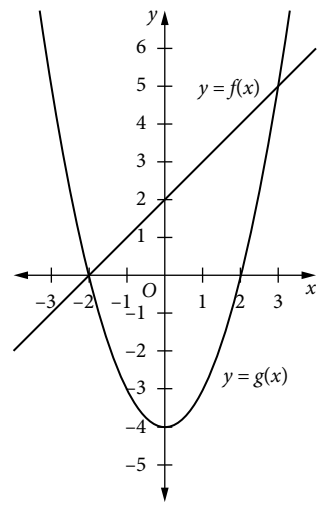
EXERCISE 3.5 GRAPHING POLYNOMIALS BY MULTIPLYING ORDINATES

1 The graphs of $y = f(x)$ and $y = g(x)$ are shown. By drawing vertical lines and multiplying ordinates, draw the graph of $y = f(x)\,g(x)$. Comment on the new curve.

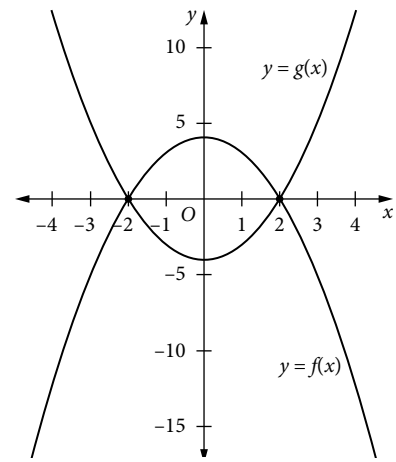
(a)



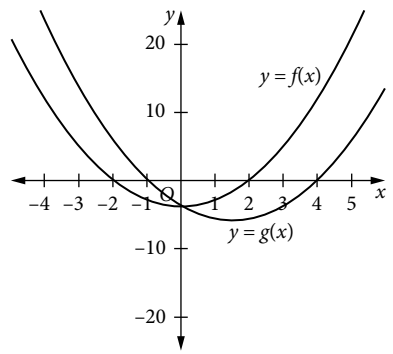
(b)



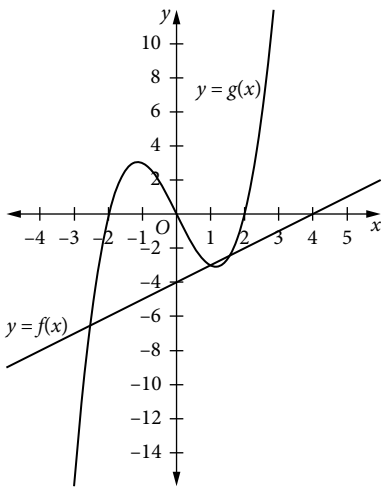
(c)



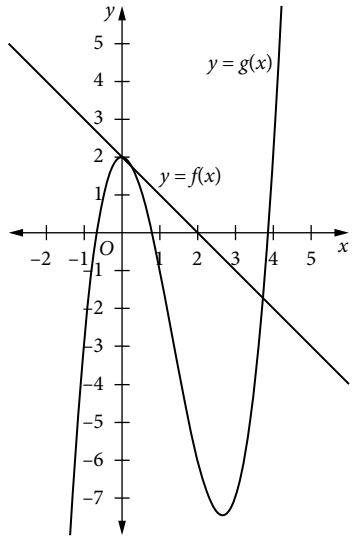
(d)



(e)



(f)

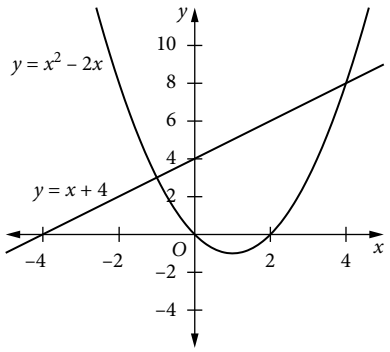


2 On the same diagram, sketch the graphs of $y = x^2$ and $y = 1 - x$. Use these graphs to sketch $y = x^2 - x^3$.

3 On the same diagram, sketch the graphs of $y = x - 2$ and $y = x^3$. Use these graphs to sketch $y = x^4 - 2x^3$.

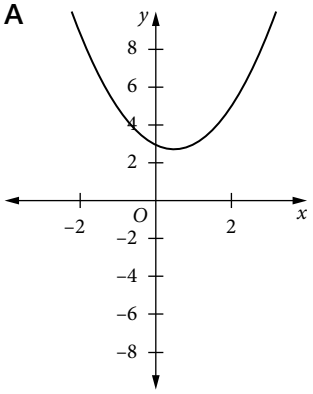
4 On the same diagram, sketch the graphs of $y = x^2 - 3x$ and $y = x^2 - 1$. Use these graphs to sketch $y = (x^2 - 3x)(x^2 - 1)$.

5 The graphs of $y = x + 4$ and $y = x^2 - 2x$ are shown.

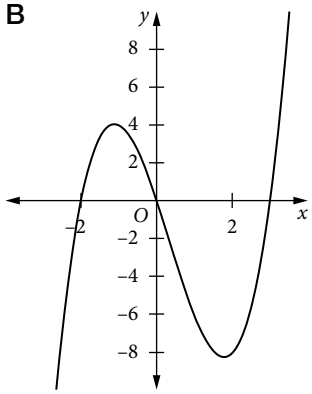


Which diagram represents the graph of $y = (x + 4)(x^2 - 2x)$?

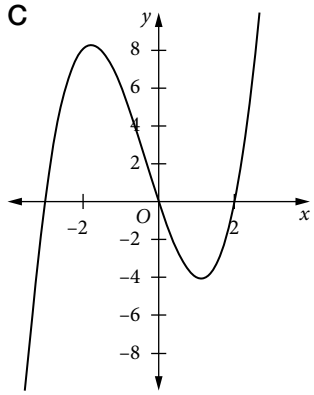
A



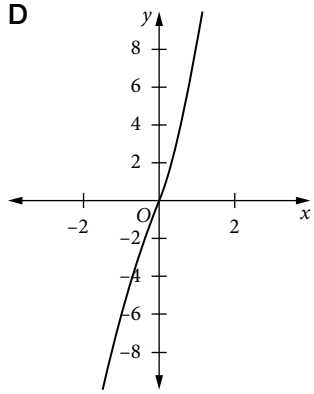
B



C



D



3.6 PARAMETRIC FORM OF A FUNCTION OR RELATION

It is often useful in mathematics to express two related variables (e.g. x and y) in terms of a third variable (e.g. t or θ), so that, for example: $x = f(t)$, $y = g(t)$ or $x = f(\theta)$, $y = g(\theta)$

Equations like these are called **parametric equations** and the third variable (e.g. t or θ) is called the **parameter**.

For example, recall that the functions cosine and sine can be defined as the x- and y-coordinates respectively of a point on the unit circle $x^2 + y^2 = 1$. Thus the unit circle can be represented by the parametric equations:

$x = \cos\theta, \quad y = \sin\theta$

where θ is the parameter. When the unit circle is described by the equation $x^2 + y^2 = 1$, it is said to be in **Cartesian** form.

Example 16

Find the Cartesian equation of the curve and describe it in words, given the parametric equations:

(a) $x = t, y = t + 1$ **(b)** $x = 2t - 1, y = 3t + 2$.

Solution

(a) $x = t, y = t + 1$

Make t the subject of the equation in x: $t = x$

Substitute in the equation for y: $y = x + 1$

The parametric equations represent a straight line with gradient 1 and y-intercept 1.

(b) $x = 2t - 1, y = 3t + 2$

Make t the subject of the equation in x: $2t = x + 1$

$$t = \frac{x+1}{2}$$

Substitute in the equation for y: $y = 3 \times \frac{x+1}{2} + 2$

$$2y = 3x + 3 + 4$$

$$3x - 2y + 7 = 0$$

The parametric equations represent a straight line with gradient 1.5 and y-intercept 3.5.

Example 17

Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t$, $y = t^2$.

Solution

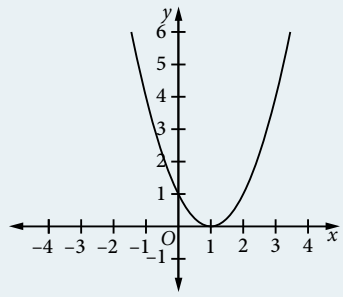
$x = 1 + t$ [1]

$y = t^2$ [2]

From [1]: $t = x - 1$

Substitute into [2]: $y = (x - 1)^2$

Hence the Cartesian equation is $y = (x - 1)^2$ and the graph is the parabola shown.



Example 18

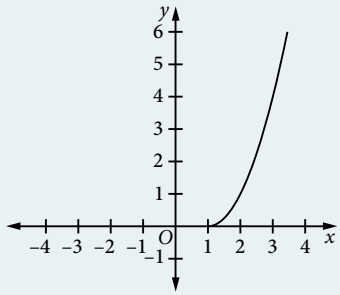
Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t$, $y = t^2$, $t \geq 0$.

Solution

As in Example 17, these parametric equations give the Cartesian equation $y = (x-1)^2$, but there is now also the condition $t \geq 0$.

$x = 1 + t$ and $t \geq 0$, so the condition is equivalent to $x \geq 1$.

Hence the Cartesian equation is $y = (x-1)^2$ with the domain restricted to $x \geq 1$, as shown.



Example 19

Find the Cartesian equation of the curve whose parametric equations are given by $x = 2\sin\theta$, $y = 2\cos\theta$. Describe the curve in words and sketch its graph.

Solution

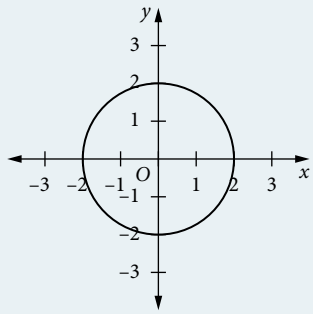
Recall the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.

$\sin\theta = \frac{x}{2}$, so $\sin^2\theta = \frac{x^2}{4}$. $\cos\theta = \frac{y}{2}$, so $\cos^2\theta = \frac{y^2}{4}$.

Hence, using the identity: $\frac{x^2}{4} + \frac{y^2}{4} = 1$

or $x^2 + y^2 = 4$

The curve is a circle with centre at the origin and radius 2.



Example 20

Write each Cartesian equation in parametric form.

(a) $3x + y - 3 = 0$ **(b)** $x^2 = 4(y-3)$ **(c)** $(x-1)^2 + (y+2)^2 = 9$

Solution

There may be more that one set of parametric equations for each Cartesian equation, depending on how the parameter is defined for x and y.

(a) **Method 1**

Rewrite the equation: $y = 3 - 3x$

Let $t = x$: $y = 3 - 3t$

The parametric equations are $x = t$ and $y = 3 - 3t$.

Method 2

Rewrite the equation: $y = 3(1-x)$

Let $t = 1 - x$: $y = 3t$

The parametric equations are $x = 1 - t$ and $y = 3t$.

(b) Rewrite the equation: $\left(\frac{x}{2}\right)^2 = y - 3$

Method 1

Let $t = \frac{x}{2}$: $t^2 = y - 3$

$x = 2t$ $y = t^2 + 3$

The parametric equations are $x = 2t$ and $y = t^2 + 3$.

Method 2

Rewrite the equation: $x^2 = 4(y-3)$

Let $t = x$: $t^2 = 4(y-3)$

$y = \frac{t^2}{4} + 3$

The parametric equations are $x = t$ and $y = \frac{t^2}{4} + 3$.

(c) The equation is the sum of two squares, which suggests that the identity $\sin^2\theta + \cos^2\theta = 1$ may be useful.

Rewrite the equation: $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{9} = 1$

Method 1

Let $\dfrac{x-1}{3} = \sin\theta$ and $\dfrac{y+2}{3} = \cos\theta$

$x = 3\sin\theta + 1$ $\quad$ $y = 3\cos\theta - 2$

The parametric equations are $x = 3\sin\theta + 1$ and $y = 3\cos\theta - 2$.

Method 2

Swapping the position of $\sin\theta$ and $\cos\theta$ would give different parametric equations.

Example 21

Sketch the graph of each curve from its parametric equations.

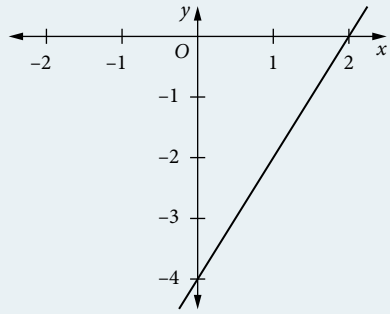
(a) $x = t + 2, y = 2t$ **(b)** $x = 2t, y = 2t^2$ **(c)** $x = 2\sin\theta, y = 2\cos\theta$

Solution

Either use graphing software or draw up a table of values and plot points.

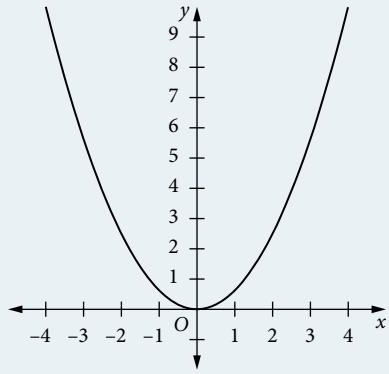
(a)

t	-2	-1	0
x	0	1	2
y	-4	-2	0



(b)

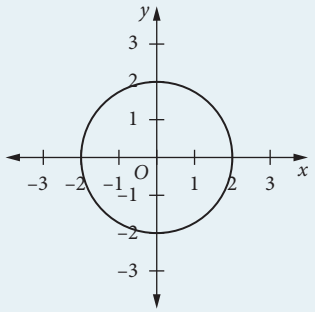
t	-2	-1	0	1	2
x	-4	-2	0	2	4
y	8	2	0	2	8



(c)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$x = 2\sin\theta$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	0.5	0
$y = 2\cos\theta$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-1

This table gives the right half of the graph of the relation. By changing the signs on values as the quadrant for θ changes, the left half may be graphed.



Parametric equations of the parabola

The parabola $x^2 = 4ay$ can be represented by the parametric equations: $x = 2at, y = at^2$

This can be verified by eliminating the parameter: $x = 2at$ [1]

$y = at^2$ [2]

From [1]: $t = \dfrac{x}{2a}$

Substitute into [2]: $y = a\left(\dfrac{x}{2a}\right)^2$

$x^2 = 4ay$

The point $P(2at, at^2)$ on the parabola is the variable point that depends on the value of t, so it is frequently called 'the point t'.

EXERCISE 3.6 PARAMETRIC FORM OF A FUNCTION OR RELATION

For questions **1** to **14**, find the Cartesian equation of the curves with the parametric equations given.

1 $x = 2t, y = t + 2$ **2** $x = t, y = t^2$ **3** $x = t, y = \dfrac{1}{t}$ **4** $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le 2\pi$

5 $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le \pi$ **6** $x = t + 3, y = t^2 - 5, t \ge 0$ **7** $x = 2u - 2, y = 3u + 1, 1 \le u \le 3$

8 $x = v^3, y = 1 - v^2, -1 \le v \le 1$ **9** $x = t + 2, y = t^2 - 1$ **10** $x = \cos t, y = \cos t, 0 \le t \le 2\pi$ **11** $x = 2t^2, y = 4t$

12 $x = 2\cos\theta, y = \sqrt{3}\sin\theta, 0 \le \theta \le 2\pi$ **13** $x = 2\cos t, y = \sin t, 0 \le t \le \pi$ **14** $x = \dfrac{2t}{1+t^2}, y = \dfrac{1-t^2}{1+t^2}$

15 Two boats on a lake start sailing at the same time. Boat A moves on a course given by $x = \dfrac{t}{2}, y = t + 1$, while boat B moves on a course given by $x = t - 2, y = -2t + 9$, where t is the time elapsed in hours.

(a) Find the Cartesian equation for the course of each boat. Show that the courses intersect at the point $(1, 3)$.

(b) Do the boats collide? Justify your answer.

16 Find the Cartesian equation for the parametric equations $x = \sec\theta, y = \tan\theta, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

17 Find the Cartesian equation for each of the following parabolas.

(a) $x = 2t, y = 2t^2$ **(b)** $x = 12t, y = 6t^2$ **(c)** $x = t + 2, y = t^2$

(d) $x = 4t, y = -2t^2$ **(e)** $x = \sqrt{t}, y = t + 1$

18 Sketch the graph of each curve from its parametric equations.

(a) $x = t + 1, y = 2t - 1$ **(b)** $x = t, y = 2t^2$ **(c)** $x = \dfrac{t}{2}, y = t^2$

(d) $x = 4\sin\theta, y = 4\cos\theta$ **(e)** $x = \dfrac{\sin\theta}{2}, y = \dfrac{\cos\theta}{2}$

19 Find the Cartesian equations of the tangent and the normal for each of the following parabolas.

(a) $x = 2t, y = t^2$ **(b)** $x = 12t, y = 6t^2$ **(c)** $x = t + 2, y = t^2$ **(d)** $x = 4t, y = -2t^2$

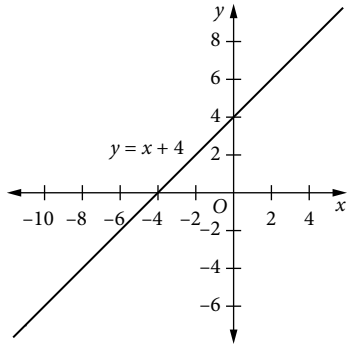
EXPLORING FURTHER

Conchoid of Nicomedes

An interesting family of curves are the *conchoids* of Nicomedes. The parametric equations of these curves are $x = a + \cos t$ and $y = a\tan t + \sin t$. Observe the change in shape for changing values of a.

CHAPTER REVIEW 3

1 Given the graph of $y = x + 4$, draw:



(a) $y = \dfrac{1}{x+4}$

(b) $y = \sqrt{x+4}$

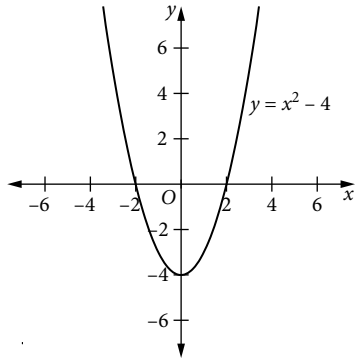
(c) $y = |x+4|$

(d) $y = \dfrac{1}{(x+4)^2}$

(e) $y^2 = x + 4$

(f) $y = |x| + 4$

2 Given the graph of $y = x^2 - 4$, draw:



(a) $y = \dfrac{1}{x^2-4}$

(b) $y = \sqrt{x^2-4}$

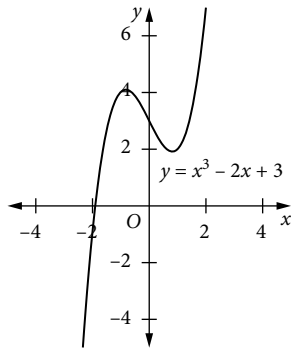
(c) $y = |x^2 - 4|$

(d) $y = \dfrac{1}{(x^2-4)^2}$

(e) $y^2 = x^2 - 4$

(f) $y = |x^2| - 4$

3 Given the graph of $y = x^3 - 2x + 3$, draw:



(a) $y = \dfrac{1}{x^3-2x+3}$

(b) $y = \sqrt{x^3-2x+3}$

(c) $y = |x^3 - 2x + 3|$

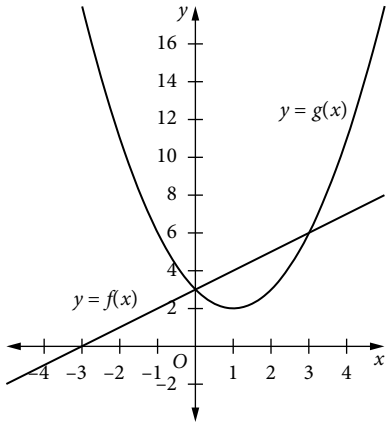
(d) $y^2 = x^3 - 2x + 3$

(e) $y = |x^3| - 2|x| + 3$

4 Given the graph of $y = f(x)$ and $y = g(x)$, draw:

(a) $y = f(x) + g(x)$

(b) $y = f(x)\, g(x)$



5 For each set of parametric equations given, **(i)** sketch the graph of the curve; **(ii)** find the Cartesian equation of the curve.

(a) $x = 2 - t, y = t + 2$ **(b)** $x = 3t, y = 3t^2$ **(c)** $x = 5\sin\theta, y = 5\cos\theta$

6 The point $P(t + 3, t^2 - 5)$ is a variable point on a parabola. Find the Cartesian equation of this parabola.

7 Match the parametric equations to their Cartesian equations. (The Cartesian equations **A B C D** may be used more than once.)

(a) $x = 2ap, y = -ap^2$ **(b)** $x = ap^2, y = 2ap$ **(c)** $x = -2ap, y = -ap^2$ **(d)** $x = 2ap, y = ap^2$

A $y^2 = 4ax$ **B** $y^2 = -4ax$ **C** $x^2 = 4ay$ **D** $x^2 = -4ay$

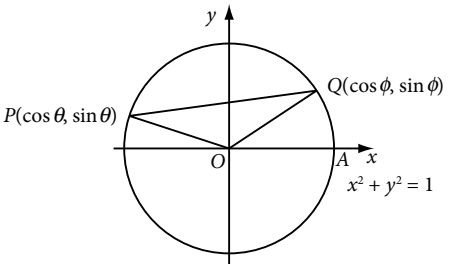
8 $x = t - 2, y = 2t^2 - 1$ are the parametric equations for a parabola. Find the Cartesian equation.

CHAPTER 4
Further trigonometric identities

4.1 SUM AND DIFFERENCE OF TWO ANGLES

You can derive formulae for the trigonometric functions of sums and differences of angles, i.e. formulae for $\cos(\theta \pm \phi)$, $\sin(\theta \pm \phi)$ and $\tan(\theta \pm \phi)$, in terms of trigonometric functions of the angles θ and ϕ.

First, obtain a formula for $\cos(\theta - \phi)$, then deduce formulae for the others from it.



Starting from A, mark a unit circle to show $\angle AOP = \theta$ and $\angle AOQ = \phi$ (where $\theta > \phi$ for convenience), as shown. The coordinates of P and Q are thus $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ respectively, and $\angle POQ = \theta - \phi$.

Using the distance formula: $PQ^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2$

$= 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$ [a]

Using the cosine rule in ΔPOQ: $PQ^2 = 1 + 1 - 2\cos(\angle POQ)$

$= 2 - 2\cos(\theta - \phi)$ [b]

Equating [a] and [b]: $2 - 2\cos(\theta - \phi) = 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$

$\therefore$ $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ [1]

This formula [1] is true for all values of θ and ϕ.

By writing $(-\phi)$ in place of ϕ, remembering that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, you also have:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \quad [2]$$

Similarly, with $(90° - \theta)$ in place of θ in [1]:

$$\cos[90° - (\theta + \phi)] = \cos(90° - \theta)\cos\phi + \sin(90° - \theta)\sin\phi$$

And using the complementary angles formulae, i.e. $\cos(90° - \alpha) = \sin\alpha$ and $\sin(90° - \alpha) = \cos\alpha$:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \quad [3]$$

Writing $(-\phi)$ in place of ϕ in [3]: $\sin(\theta - \phi) = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \quad [4]$$

Combining these for the tangent functions:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \quad [5]$$

And for $(\theta - \phi)$:

$$\tan(\theta - \phi) = \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \qquad [6]$$

Example 1

Find the expansion for each expression, simplifying where possible.

(a) $\sin(3x + 2y)$ **(b)** $\cos(2\alpha + \beta)$ **(c)** $\tan(A + 45°)$

Solution

(a) $\sin(3x + 2y) = \sin 3x\cos 2y + \cos 3x\sin 2y$

(b) $\cos(2\alpha + \beta) = \cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta$

(c) $\tan(A + 45°) = \dfrac{\tan A + \tan 45°}{1 - \tan A\ \tan 45°} = \dfrac{1 + \tan A}{1 - \tan A}$

Example 2

Simplify each expression.

(a) $\sin(2\alpha + \beta)\cos\beta - \cos(2\alpha + \beta)\sin\beta$ **(b)** $\cos(2\theta - 3\alpha)\cos 2\theta + \sin(2\theta - 3\alpha)\sin 2\theta$

Solution

By recognising the form of the equation, the two-angle expansion can be used in reverse:

(a) $\sin(2\alpha + \beta)\cos\beta - \cos(2\alpha + \beta)\sin\beta = \sin[(2\alpha + \beta) - \beta] = \sin 2\alpha$

(b) $\cos(2\theta - 3\alpha)\cos 2\theta + \sin(2\theta - 3\alpha)\sin 2\theta = \cos[(2\theta - 3\alpha) - 2\theta]$
$= \cos(-3\alpha) = \cos 3\alpha$

Alternatively, the expressions on the LHS can be expanded and like terms collected. For example, the solution to **(a)** becomes:

$$\begin{aligned}\sin(2\alpha + \beta)\cos\beta - \cos(2\alpha + \beta)\sin\beta &= [\sin 2\alpha\cos\beta + \cos 2\alpha\sin\beta]\cos\beta - [\cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta]\sin\beta \\ &= \sin 2\alpha\cos^2\beta + \cos 2\alpha\sin\beta\cos\beta - \cos 2\alpha\cos\beta\sin\beta + \sin 2\alpha\sin^2\beta \\ &= \sin 2\alpha[\cos^2\beta + \sin^2\beta] = \sin 2\alpha\end{aligned}$$

Example 3

If θ and ϕ are acute angles and $\sin\theta = \frac{3}{5}$ and $\tan\phi = \frac{24}{7}$, find, without using a calculator, the exact value of the following expressions:

(a) $\sin(\theta + \phi)$ **(b)** $\cos(\theta - \phi)$ **(c)** $\tan(\theta - \phi)$

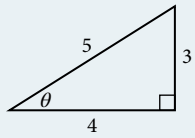
Solution

Draw right-angled triangles for each ratio and use Pythagoras' theorem to find the third side.

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

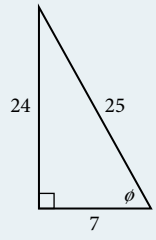
$\tan\theta = \frac{3}{4}$



$\tan\phi = \frac{24}{7}$

$\sin\phi = \frac{24}{25}$

$\cos\phi = \frac{7}{25}$



(a) $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$= \frac{3}{5}\times\frac{7}{25} + \frac{4}{5}\times\frac{24}{25}$

$= \frac{117}{125}$

(b) $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$= \frac{4}{5}\times\frac{7}{25} + \frac{3}{5}\times\frac{24}{25}$

$= \frac{100}{125}$

$= \frac{4}{5}$

(c) $\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

$= \frac{\frac{3}{4} - \frac{24}{7}}{1+\frac{3}{4}\times\frac{24}{7}}$

$= \frac{21-96}{28+72}$

$= -\frac{75}{100}$

$= -\frac{3}{4}$

EXERCISE 4.1 SUM AND DIFFERENCE OF TWO ANGLES

1 Expand:

(a) $\sin(A+2B)$ **(b)** $\sin(2x-y)$ **(c)** $\cos(2x-3y)$

(d) $\cos(2\theta+60°)$ **(e)** $\tan(\theta+\alpha)$ **(f)** $\tan(A-135°)$

2 Simplify:

(a) $\sin A\cos(A-B) + \cos A\sin(A-B)$ **(b)** $\cos(\theta+\alpha)\cos(\theta-\alpha) + \sin(\theta+\alpha)\sin(\theta-\alpha)$

(c) $\sin 2A\cos A - \cos 2A\sin A$ **(d)** $\cos 60°\cos 30° - \sin 60°\sin 30°$

(e) $\frac{\tan\theta - \tan 20°}{1+\tan 20°\tan\theta}$ **(f)** $\frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha\tan\alpha}$ **(g)** $\sin(2A+B)\cos(A+B) - \cos(2A+B)\sin(A+B)$

(h) $\cos(3\theta+\alpha)\cos(2\theta+\alpha) + \sin(3\theta+\alpha)\sin(\theta+\alpha)$ **(i)** $\frac{\tan 3x - \tan x}{1+\tan 3x\tan x}$

3 The expression $\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$ simplifies to:

A $\tan A + \tan B + \tan C$ **B** $\tan(A-B+C)$ **C** $\tan(A+B-C)$ **D** $\tan(A+B+C)$

4 **(a)** Find the exact value of $\sin 38°\cos 22° + \cos 38°\sin 22°$.

(b) Find the exact value of $\frac{\tan 119° + \tan 16°}{1-\tan 119°\tan 16°}$.

(c) Find the exact value of $\cos 165°$. **(d)** Expand and simplify $\sin(x+40°) + \sin(x-40°)$.

5 Write the expansion of $\cos(A+B)$. From this, deduce the expansion of $\cos(A-B)$.

6 Write the expansion of $\cos(\theta-\phi)$. Write $(90° - \theta)$ in place of θ to deduce the expansion of $\sin(\theta+\phi)$.

7 If θ and ϕ are angles between 0° and 90°, $\sin\theta = \frac{3}{5}$, $\tan\phi = \frac{7}{24}$, find the following without using a calculator.

(a) $\sin(\theta-\phi)$ **(b)** $\cos(\theta+\phi)$ **(c)** $\tan(\theta-\phi)$

8 If $\tan A = 4$, $\tan B = \frac{3}{5}$, and A and B are acute angles, then $A - B = \ldots$

A 45° **B** 30° **C** 60° **D** 135°

9 If $\tan\alpha = \frac{4}{3}$ and $\cos\beta = \frac{12}{13}$, where $0 < \beta < \alpha < 90°$, evaluate the following without using a calculator.

(a) $\sin 2\alpha$ **(b)** $\tan 2\alpha$ **(c)** $\cos(\alpha-\beta)$

10 **(a)** Using the expansion of $\sin(A+B)$, prove that $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) Using the expansion of $\tan(A+B)$, prove that $\tan 75° = 2+\sqrt{3}$.

11 Find the value (in simplest surd form) of:

(a) $\cos 75°$ **(b)** $\tan 15°$ **(c)** $\cos 15°$

12 Use the expansion of $\sin(A + B)$ to evaluate $\sin 195°$.

4.2 DOUBLE ANGLE FORMULAE

Using the sum-and-difference-of-two-angles formula [2] (see previous section), writing θ in place of ϕ:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \qquad [2]$$

$$\therefore \quad \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

From $\sin^2\theta + \cos^2\theta = 1$:

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad [7]$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Similarly, using formula [3] and writing θ in place of ϕ:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \qquad [3]$$

$$\therefore \quad \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad [8]$$

And using formula [5], writing θ in place of ϕ:

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \qquad [5]$$

$$\therefore \quad \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}, \ \tan\theta \neq \pm 1 \qquad [9]$$

The double angle formulae can be used in many different ways.

For example, as $4\theta = 2 \times 2\theta$, therefore $\sin 4\theta = 2\sin 2\theta\cos 2\theta$.

This result can be further simplified by again using the double angle formulae to obtain an expression in terms of θ:

$\sin 4\theta = 2 \times 2\sin\theta\cos\theta \times (\cos^2\theta - \sin^2\theta) = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$.

Also, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, so it is possible to express a function of an angle in terms of half the angle.

Example 4

If $\tan A = -\frac{3}{4}$, $90° < A < 180°$, and $\cos B = \frac{5}{13}$, $0° < B < 90°$, write the exact value of the following.

(a) $\sin(A - B)$ **(b)** $\cos 2A$ **(c)** $\tan(A + B)$ **(d)** $\sin 2B$

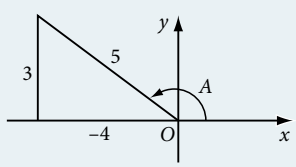
Solution

Draw diagrams to show the given ratio for each angle, then use the diagrams to find the other ratios for the angle.

$\sin A = \frac{3}{5}$

$\cos A = -\frac{4}{5}$

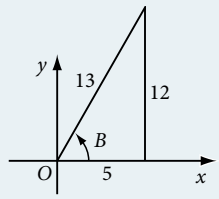
$\tan A = -\frac{3}{4}$



$\sin B = \frac{12}{13}$

$\cos B = \frac{5}{13}$

$\tan B = \frac{12}{5}$



(a) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= \frac{3}{5}\times\frac{5}{13} + \frac{4}{5}\times\frac{12}{13} = \frac{63}{65}$

(b) $\cos 2A = \cos^2 A - \sin^2 A$

$= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

(c) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$= \frac{-\frac{3}{4} + \frac{12}{5}}{1 + \frac{3}{4}\times\frac{12}{5}} = \frac{33}{56}$

(d) $\sin 2B = \sin B\cos B$

$= 2\times\frac{12}{13}\times\frac{5}{13}$

$= \frac{120}{169}$

Example 5

(a) Prove that $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$.

(b) Prove that $\frac{\sin 2\alpha + \sin\alpha}{1+\cos 2\alpha + \cos\alpha} = \tan\alpha$.

Solution

(a) $\text{LHS} = \frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta}$

$= \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$

$= \frac{\cos(3\theta - \theta)}{\frac{1}{2}\sin 2\theta}$

$= \frac{2\cos 2\theta}{\sin 2\theta}$

$= 2\cot 2\theta = \text{RHS}$

(b) **Method 1**

$\text{LHS} = \frac{\sin 2\alpha + \sin\alpha}{1+\cos 2\alpha + \cos\alpha}$

$= \frac{2\sin\alpha\cos\alpha + \sin\alpha}{1 + 2\cos^2\alpha - 1 + \cos\alpha}$

$= \frac{\sin\alpha(2\cos\alpha + 1)}{\cos\alpha(2\cos\alpha + 1)}$

$= \frac{\sin\alpha}{\cos\alpha}$ if $2\cos\alpha + 1 \neq 0$

$= \tan\alpha = \text{RHS}$

Method 2

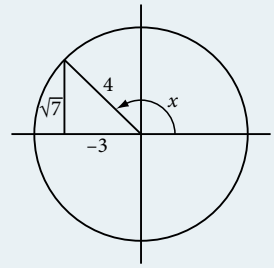
In the denominator, use formula [7] to directly replace $1 + \cos 2\alpha$ with $2\cos^2\alpha$.

Example 6

If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} \leq x \leq \pi$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$

Solution

From the diagram (drawn to show the given ratio for the angle):



(a) $\sin x = \frac{\sqrt{7}}{4}$

(b) $\sin 2x = 2\sin x\cos x$

$= 2\times\frac{\sqrt{7}}{4}\times\left(-\frac{3}{4}\right)$

$= -\frac{3\sqrt{7}}{8}$

Example 7

Simplify:

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6}$ **(b)** $\sin\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}-\phi\right)$ **(c)** $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

Solution

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6}=\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$ (using formula [2] from page 75)

$=\cos\frac{\pi}{2}=0$

(b) $\sin\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}-\phi\right)=\cos\theta+\cos\phi$

(c) $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)=\frac{1}{2}\times 2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

$=\frac{1}{2}\sin 2\left(\frac{\pi}{4}-x\right)$ (using double-angle formula [8] from page 75)

$=\frac{1}{2}\sin\left(\frac{\pi}{2}-2x\right)=\frac{1}{2}\cos 2x$

EXERCISE 4.2 DOUBLE ANGLE FORMULAE

1 **(a)** By writing $\sin 3\theta$ as $\sin(2\theta+\theta)$, write $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence write $\cos 3\theta$ in terms of $\cos\theta$. **(c)** Hence write $\tan 3\theta$ in terms of θ.

2 If $\sin\theta=\frac{3}{4}$, $90°<\theta<180°$, evaluate (in surd form):

(a) $\sin 2\theta$ **(b)** $\cos 2\theta$ **(c)** $\tan 2\theta$. **(d)** In which quadrant is 2θ?

3 Simplify:

(a) $\frac{\sin 2A}{1+\cos 2A}$ **(b)** $\frac{1}{2}\sin 2\theta\tan\theta$ **(c)** $\cos^2 2\theta-\sin^2 2\theta$ **(d)** $\cos^2 30°-\sin^2 30°$

(e) $\sin 4x\cos 4x$ **(f)** $1+\cos(180°+2\theta)$ **(g)** $\sin x\cos x\cos 2x$ **(h)** $2\sin 2x\cos 2x$

(i) $(\sin\theta+\cos\theta)^2$ **(j)** $(\sin A-\cos A)^2$ **(k)** $\frac{2\tan\theta}{1-\tan^2\theta}$ for $\theta=22.5°$ **(l)** $\sin^2 50°+\sin^2 40°$

(m) $\sin(45°-x)\cos(45°-x)$ **(n)** $\frac{1-\cos 2\theta}{1+\cos 2\theta}$ **(o)** $2\cos^2 3x-1$

4 If $\sin\theta=\frac{3}{5}$, $\frac{\pi}{2}\le\theta\le\pi$ and $\tan\phi=\frac{7}{24}$, $0\le\phi\le\frac{\pi}{2}$, find the value of:

(a) $\sin(\theta-\phi)$ **(b)** $\cos(\theta-\phi)$ **(c)** $\tan(\theta-\phi)$

5 Simplify:

(a) $1+\tan^2\left(\frac{\pi}{2}-\alpha\right)$ **(b)** $1-\cos^2(\pi+\theta)$ **(c)** $\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)+\cos\theta\sin\left(\frac{\pi}{2}-\theta\right)$

(d) $2\cos^2\frac{\pi}{6}-1$ **(e)** $1-\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)$ **(f)** $\sin(\pi-\theta)\cos\phi-\cos(\pi-\theta)\sin\phi$

4.3 HALF-ANGLE FORMULAE—THE *t* FORMULAE

The double-angle formulae has been established: $\sin 2A = 2\sin A\cos A$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A},\ \tan A \neq \pm 1$$

Just as A is the half-angle of $2A$, similarly $\frac{A}{2}$ is the half-angle of A, so you can use these results to obtain:

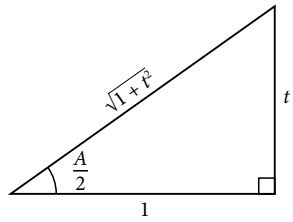
$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}},\ \tan\frac{A}{2} \neq \pm 1$$

If you let $t = \tan\frac{A}{2}$, then: $\tan A = \frac{2t}{1-t^2}$

Using the right-angled triangle, in which $\tan\frac{A}{2} = \frac{t}{1}$, you can obtain the ratios:



$$\sin\frac{A}{2} = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos\frac{A}{2} = \frac{1}{\sqrt{1+t^2}}$$

Hence: $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

$$= 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}$$

$$= \frac{2t}{1+t^2}$$

and: $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

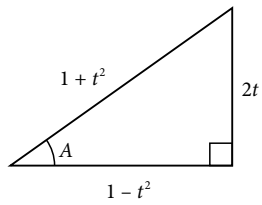
These results, known as the '*t* formulae', can be summarised as follows:

If $t = \tan\frac{A}{2}$, then:

$$\tan A = \frac{2t}{1-t^2}$$

$$\sin A = \frac{2t}{1+t^2}$$

$$\cos A = \frac{1-t^2}{1+t^2}$$



Example 8

Given that $\tan A = \frac{5}{12}$, find the exact values of $\sin 2A$ and $\cos 2A$.

Solution

$\tan A = \frac{5}{12}$ so $t = \frac{5}{12}$: $\sin 2A = \frac{2t}{1+t^2}$

$$= \frac{2\times\frac{5}{12}}{1+\left(\frac{5}{12}\right)^2}$$

$$= \frac{120}{169}$$

$\cos 2A = \frac{1-t^2}{1+t^2}$

$$= \frac{1-\left(\frac{5}{12}\right)^2}{1+\left(\frac{5}{12}\right)^2}$$

$$= \frac{119}{169}$$

Note: If a calculator is used to find intermediate values then the answer will be only approximate. For example, $\tan A = \frac{5}{12}$ so $A = 22°\,37'$, $2A = 45°\,14'$, $\sin 2A \approx 0.709\,98$, while $\frac{120}{169} \approx 0.710\,06$ (both to five d.p.)—close, but not the same.

Example 9

Use the t formulae to prove that: $\dfrac{\cos\theta+\sin\theta-1}{\cos\theta-\sin\theta+1}=\tan\dfrac{\theta}{2}$

Solution

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta+\sin\theta-1}{\cos\theta-\sin\theta+1} \\
&= \frac{\left(\dfrac{1-t^2}{1+t^2}+\dfrac{2t}{1+t^2}-1\right)}{\left(\dfrac{1-t^2}{1+t^2}-\dfrac{2t}{1+t^2}+1\right)} \\
&= \left(\frac{1-t^2+2t-(1+t^2)}{1+t^2}\right)\div\left(\frac{1-t^2-2t+(1+t^2)}{1+t^2}\right) \\
&= \frac{1-t^2+2t-1-t^2}{1-t^2-2t+1+t^2} \\
&= \frac{2t-2t^2}{2-2t} \\
&= \frac{2t(1-t)}{2(1-t)} \\
&= t = \tan\frac{\theta}{2} = \text{RHS}
\end{aligned}$$

EXERCISE 4.3 HALF-ANGLE FORMULAE—THE t FORMULAE

1 If $t=\tan\dfrac{A}{2}$, then $\sin A+\cos A=\ldots$

A $\dfrac{1+2t-t^2}{1+t^2}$ **B** $\dfrac{t^2-2t+1}{1+t^2}$ **C** $\dfrac{(1-t)^2}{1+t^2}$ **D** $\dfrac{(1+t)^2}{1+t^2}$

2 Simplify:

(a) $\dfrac{2\tan 9^\circ}{1-\tan^2 9^\circ}$ **(b)** $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$ **(c)** $\dfrac{1+\tan^2 22.5^\circ}{2\tan 22.5^\circ}$

3 If $t=\tan\dfrac{A}{2}$, express each of the following in terms of t:

(a) $\sin A-\cos A$ **(b)** $3\sin A+4\cos A$ **(c)** $2\cos A-\sin A$ **(d)** $\cot A$

(e) $\cot A-\tan A$ **(f)** $\dfrac{\cot A-\tan A}{\cot A+\tan A}$ **(g)** $1-\dfrac{1}{2}\sin A\tan\dfrac{A}{2}$ **(h)** $1+\tan A\tan\dfrac{A}{2}$

(i) $\dfrac{\tan A-\tan\frac{A}{2}}{\cot\frac{A}{2}+\tan A}$ **(j)** $\cot\dfrac{A}{2}-2\cot A$ **(k)** $\dfrac{1+\sin A+\cos A}{1+\sin A-\cos A}$ **(l)** $\dfrac{\sin A+\sin\frac{A}{2}}{1+\cos A+\cos\frac{A}{2}}$

4 If $t=\tan\dfrac{\theta}{2}$, then $\dfrac{1-\cos\theta}{\sin\theta}=\ldots$

A 0 **B** $\dfrac{1+t^2}{t}$ **C** $\dfrac{1-2t+t^2}{2t}$ **D** t

5 If $t=\tan\dfrac{\theta}{2}$, solve for t the equation $\tan\theta=d$.

6 If $t=\tan\dfrac{A}{2}$, solve for t the equation $12\tan A=5$, $180^\circ<A<270^\circ$.

7 If $\tan\dfrac{\theta}{2}=\dfrac{a}{b}$, prove that $b\cos\theta+a\sin\theta=b$.

8 If $\sec\theta-\tan\theta=x$, prove that $x=\dfrac{1-t}{1+t}$.

4.4 USING IDENTITIES TO SIMPLIFY EXPRESSIONS AND PROVE RESULTS

When working with some trigonometric equations, you will also need to use the Pythagorean identities learnt in the Mathematics Advanced course:

$\sin^2\theta+\cos^2\theta=1$	$\sin^2\theta=1-\cos^2\theta$	$\cos^2\theta=1-\sin^2\theta$
$\sec^2\theta-\tan^2\theta=1$	$\sec^2\theta=1+\tan^2\theta$	$\tan^2\theta=\sec^2\theta-1$
$\operatorname{cosec}^2\theta-\cot^2\theta=1$	$\operatorname{cosec}^2\theta=1+\cot^2\theta$	$\cot^2\theta=\operatorname{cosec}^2\theta-1$

Example 10

Prove that $\dfrac{\cos 2\theta+\sin 2\theta-1}{\cos 2\theta-\sin 2\theta+1}=\tan\theta$.

Solution

$$\begin{aligned}\text{LHS} &= \frac{\cos 2\theta+\sin 2\theta-1}{\cos 2\theta-\sin 2\theta+1}\\ &= \frac{1-2\sin^2\theta+2\sin\theta\cos\theta-1}{2\cos^2\theta-1-2\sin\theta\cos\theta+1}\\ &= \frac{2\sin\theta(\cos\theta-\sin\theta)}{2\cos\theta(\cos\theta-\sin\theta)}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta = \text{RHS}\end{aligned}$$

Notice how two different expansions for $\cos 2\theta$ are used in Example **10** above. To decide which expansion is the best to use in each part you must consider the -1 in the numerator and the $+1$ in the denominator. The aim is to remove these constants by using the appropriate form.

Using $\cos 2\theta=\cos^2\theta-\sin^2\theta$ would have made the question more complicated. Try this to see for yourself.

EXERCISE 4.4 USING IDENTITIES TO SIMPLIFY EXPRESSIONS AND PROVE RESULTS

Prove the following identities (questions **1** to **21**):

1 $\dfrac{\sin A+\cos A\tan B}{\cos A-\sin A\tan B}=\tan(A+B)$

2 $\dfrac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\cos 2\theta\cos\theta+\sin 2\theta\sin\theta}=\tan\theta$

3 $\dfrac{\tan A-\tan B}{\tan A+\tan B}=\dfrac{\sin(A-B)}{\sin(A+B)}$

4 $\sin(\theta+\alpha)\sin(\theta-\alpha)=\sin^2\theta-\sin^2\alpha$

5 $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}=2$

6 $\tan(45^\circ+A)+\tan(45^\circ-A)=\dfrac{2}{\cos 2A}$

7 $\cos 4x=8\cos^4 x-8\cos^2 x+1$

8 $\dfrac{\sin 2\theta+1}{\cos 2\theta}=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

9 $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=2\sec 2\theta$

10 $\dfrac{1-\cos x}{\sin x}=\tan\dfrac{x}{2}$

11 $\dfrac{\sin A+\sin(90^\circ-A)+1}{\sin A-\sin(90^\circ-A)+1}=\cot\dfrac{A}{2}$

12 $\dfrac{\sin x+1-\cos x}{\sin x-1+\cos x}=\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

13 $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B-\sin A\sin B\sin C$

What is the resulting identity if C is replaced by $(90^\circ-B)$?

14 $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin C\sin A-\cos C\sin A\sin B$

What is the resulting identity if B is replaced by $(90^\circ-C)$?

15 $\tan(\theta+\alpha)\tan(\theta-\alpha)=\dfrac{\tan^2\theta-\tan^2\alpha}{1-\tan^2\theta\tan^2\alpha}$

16 $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\dfrac{1}{2}\sin 2\theta$

17 $\dfrac{2\cos\frac{\theta}{2}-1-\cos\theta}{2\cos\frac{\theta}{2}+1+\cos\theta}=\dfrac{1-\cos\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}$

18 $\cot(x+y)=\dfrac{\cot x\cot y-1}{\cot x+\cot y}$

19 $\dfrac{\tan 2\theta-\tan\theta}{\tan 2\theta+\cot\theta}=\tan^2\theta$

20 $\tan(\theta+45°)\tan(\theta-45°)=-1$

21 $\dfrac{1-\tan\theta\tan 2\theta}{1+\tan\theta\tan 2\theta}=4\cos^2\theta-3$

22 If $\tan A=\dfrac{p}{q}$, express the following in terms of p and q.

(a) $q\sin A\cos A+p\sin^2 A$ **(b)** $p\sin 2A+q\cos 2A$

23 If A, B and C are the angles of a triangle, prove that $\cos A\cos B-\sin A\sin B+\cos C=0$.

24 Given that $\sin 18°=\dfrac{1}{4}\left(\sqrt{5}-1\right)$, find $\cos 36°$ in surd form.

25 Find $\tan x$ in terms of $\tan\theta$ if $\tan\theta=\dfrac{\cos(\theta+x)}{\cos(\theta-x)}$.

26 Three points P, Q, R are in a horizontal plane. Angles RPQ and RQP are α and β respectively. If PQ is x units in length, show that the perpendicular distance y from R to PQ is given by $y=\dfrac{x\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$.

27 Using the double-angle formula for $\tan\theta$, find $\tan 22.5°$ in simplest surd form.

28 $\cos^2\left(\dfrac{\pi}{4}-x\right)-\sin^2\left(\dfrac{\pi}{4}-x\right)$ simplifies to:

A $\cos 2x$ **B** $\sin 2x$ **C** $\cos x$ **D** $\sin x$

29 If $\tan\theta=\dfrac{3}{5}$ and $\pi<\theta<\dfrac{3\pi}{2}$, find the value of: **(a)** $\sin\theta$ **(b)** $\cos\theta$ **(c)** $\cos 2\theta$

30 If $\dfrac{\pi}{2}\le x\le\pi$ and $\cos x=-\dfrac{5}{6}$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$ **(c)** $\tan 2x$

31 If $\operatorname{cosec}\alpha=-\dfrac{17}{8}$ and $\pi<\alpha<\dfrac{3\pi}{2}$, find the value of: **(a)** $\cot\alpha$ **(b)** $\tan 2\alpha$

32 Prove the following.

(a) $2\cos\left(\dfrac{\pi}{4}+x\right)\cos\left(\dfrac{\pi}{4}-x\right)=\cos 2x$ **(b)** $\tan\left(\theta+\dfrac{\pi}{4}\right)\tan\left(\theta-\dfrac{\pi}{4}\right)=-1$

(c) $\left(\sec^2\theta-1\right)\tan\left(\dfrac{\pi}{2}-\theta\right)=\tan\theta$ **(d)** $\tan\left(x+\dfrac{3\pi}{4}\right)=\dfrac{\tan x-1}{\tan x+1}$

(e) $\dfrac{1-\sin\left(\frac{\pi}{2}-2x\right)}{\sin 2x}=\tan x$ **(f)** $\tan\left(\dfrac{\pi}{4}+A\right)+\tan\left(\dfrac{\pi}{4}-A\right)=2\sec 2A$

33 If $\tan x=\dfrac{5}{4}$, $\tan y=\dfrac{1}{9}$ and $0<y<x<\dfrac{\pi}{2}$, prove that $x-y=\dfrac{\pi}{4}$.

34 Simplify: **(a)** $\dfrac{2\tan\theta}{1-\tan^2\theta}$ where $\theta=\dfrac{7\pi}{8}$ **(b)** $2\cos^2 3x-1$ where $x=\dfrac{2\pi}{9}$

35 By expanding each term on the left-hand side, prove that $\sin\left(\theta+\dfrac{\pi}{6}\right)\sin\left(\theta-\dfrac{\pi}{6}\right)=\sin^2\theta-\dfrac{1}{4}$.

36 If $0\le\theta\le\dfrac{\pi}{2}$, prove that $\tan\theta=\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$. Hence show that the exact value of $\tan\dfrac{\pi}{8}$ is $\sqrt{2}-1$.

37 **(a)** By writing expansions for $\sin(A+B)$ and $\sin(A-B)$, find a simplified expression for $\sin(A+B)+\sin(A-B)$.

(b) By writing $\theta=A+B$ and $\phi=A-B$, find an expression for $\sin\theta+\sin\phi$ as the product of two trigonometric functions.

38 If $\sec\theta-\tan\theta=\dfrac{3}{5}$, show that $\sin\theta=\dfrac{8}{17}$. (Hint: Use t formulae.)

39 If $4\tan(\alpha-\beta)=3\tan\alpha$, prove that $\tan\beta=\dfrac{\sin 2\alpha}{7+\cos 2\alpha}$.

40 Use the factors of $x^3 - y^3$ to show that $\cos^6\theta - \sin^6\theta = \left(1 - \frac{1}{4}\sin^2 2\theta\right)\cos 2\theta$.

41 If $\tan\theta = t$, express $\sin 2\theta$ and $\cos 2\theta$ in terms of t. Find the values of t for which $(k+1)\sin 2\theta + (k-1)\cos 2\theta = k+1$.

42 If A, B and C are successive terms of an arithmetic series, prove that $\sin A + \sin C = 2\sin B\cos(B - A)$.

43 If $\cos\theta = \frac{l^2 - m^2}{l^2 + m^2}$ and $0 < \theta < \frac{\pi}{2}$, express $\tan\theta$ and $\sin 2\theta$ in terms of l and m.

44 If $\tan\alpha = k\tan\beta$, show that $(k-1)\sin(\alpha + \beta) = (k+1)\sin(\alpha - \beta)$.

45 Show that $4\sin\theta\sin\left(\theta - \frac{\pi}{3}\right)\sin\left(\theta - \frac{2\pi}{3}\right) = \sin 3\theta$.

4.5 TRIGONOMETRIC PRODUCTS AS SUMS OR DIFFERENCES

In section 4.1 you established the formulas for the sum and difference of trigonometric functions (here rewritten using the variables A and B):

$\sin(A + B) = \sin A\cos B + \cos A\sin B$ [1]

$\sin(A - B) = \sin A\cos B - \cos A\sin B$ [2]

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ [3]

$\cos(A - B) = \cos A\cos B + \sin A\sin B$ [4]

Add [1] and [2]: $2\sin A\cos B = \sin(A + B) + \sin(A - B)$ [a]

$= \sin(\text{sum}) + \sin(\text{difference})$

Subtract [2] from [1]: $2\cos A\sin B = \sin(A + B) - \sin(A - B)$ [b]

$= \sin(\text{sum}) - \sin(\text{difference})$

Add [3] and [4]: $2\cos A\cos B = \cos(A + B) + \cos(A - B)$ [c]

$= \cos(\text{sum}) + \cos(\text{difference})$

Subtract [3] from [4]: $2\sin A\sin B = \cos(A - B) - \cos(A + B)$ [d]

$= \cos(\text{difference}) - \cos(\text{sum})$

These results are summarised below.

$\sin A\cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$ $\quad$ $\cos A\sin B = \frac{1}{2}(\sin(A + B) - \sin(A - B))$

$\cos A\cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$ $\quad$ $\sin A\sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$

Example 11

Express each product as a sum or difference of trigonometric functions:

(a) $2\cos 5x\sin x$ **(b)** $2\sin 4A\sin A$ **(c)** $\cos 3\theta\cos 5\theta$ **(d)** $\sin 3\theta\cos\theta$

Solution

(a) $2\cos 5x\sin x = \sin(5x + x) - \sin(5x - x)$

$= \sin 6x - \sin 4x$

(b) $2\sin 4A\sin A = \cos(4A - A) - \cos(4A + A)$

$= \cos 3A - \cos 5A$

(c) $\cos 3\theta\cos 5\theta = \frac{1}{2}(\cos(3\theta + 5\theta) + \cos(3\theta - 5\theta))$

$= \frac{1}{2}(\cos 8\theta + \cos(-2\theta))$

$= \frac{1}{2}(\cos 8\theta + \cos 2\theta)$

(d) $\sin 3\theta\cos\theta = \frac{1}{2}(\sin(3\theta + \theta) + \sin(3\theta - \theta))$

$= \frac{1}{2}(\sin 4\theta + \sin 2\theta)$

Sums or differences as products

Let $\theta = A + B$ and $\phi = A - B$, giving $\theta + \phi = 2A$ and $\theta - \phi = 2B$.

Hence $A = \dfrac{\theta + \phi}{2}$ and $B = \dfrac{\theta - \phi}{2}$.

Substitute in [a]: $\sin\theta + \sin\phi = 2\sin\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$

Substitute in [b]: $\sin\theta - \sin\phi = 2\cos\left(\dfrac{\theta+\phi}{2}\right)\sin\left(\dfrac{\theta-\phi}{2}\right)$

Substitute in [c]: $\cos\theta + \cos\phi = 2\cos\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$

Substitute in [d]: $\cos\phi - \cos\theta = 2\sin\left(\dfrac{\theta+\phi}{2}\right)\sin\left(\dfrac{\theta-\phi}{2}\right)$

or $\cos\theta - \cos\phi = -2\sin\left(\dfrac{\theta+\phi}{2}\right)\sin\left(\dfrac{\theta-\phi}{2}\right)$

Example 12

Convert the following sums or differences into products:

(a) $\sin 6x - \sin 4x$ **(b)** $\cos 3A - \cos 5A$ **(c)** $\cos 8\theta + \cos 2\theta$ **(d)** $\sin 3x + \sin x$

Solution

(a) $\sin 6x - \sin 4x = 2\cos\left(\dfrac{6x+4x}{2}\right)\sin\left(\dfrac{6x-4x}{2}\right)$

$= 2\cos 5x \sin x$

(b) $\cos 3A - \cos 5A = -2\sin\left(\dfrac{3A+5A}{2}\right)\sin\left(\dfrac{3A-5A}{2}\right)$

$= -2\sin 4A \sin(-A)$

$= 2\sin 4A \sin A$

(c) $\cos 8\theta + \cos 2\theta = 2\cos\left(\dfrac{8\theta+2\theta}{2}\right)\cos\left(\dfrac{8\theta-2\theta}{2}\right)$

$= 2\cos 5\theta \cos 3\theta$

(d) $\sin 3x + \sin x = 2\sin\left(\dfrac{3x+x}{2}\right)\cos\left(\dfrac{3x-x}{2}\right)$

$= 2\sin 2x \cos x$

Example 13

Show that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$.

Solution

$$\text{LHS} = \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$$

$$= \frac{(\sin 5\theta + \sin\theta) + \sin 3\theta}{(\cos 5\theta + \cos\theta) + \cos 3\theta}$$

$$= \frac{2\sin 3\theta\cos 2\theta + \sin 3\theta}{2\cos 3\theta\cos 2\theta + \cos 3\theta}$$

$$= \frac{\sin 3\theta(2\cos 2\theta + 1)}{\cos 3\theta(2\cos 2\theta + 1)}$$

$$= \frac{\sin 3\theta}{\cos 3\theta}$$

$$= \tan 3\theta = \text{RHS}$$

EXERCISE 4.5 TRIGONOMETRIC PRODUCTS AS SUMS OR DIFFERENCES

1 $2\cos 6\theta \sin 2\theta$ can be written as:

A $\cos 4\theta - \cos 8\theta$ **B** $\cos 8\theta + \cos 4\theta$ **C** $\sin 8\theta - \sin 4\theta$ **D** $\sin 8\theta + \sin 4\theta$

Express each of the following as sums or differences:

2 $2\sin 4\theta \cos 2\theta$

3 $2\cos 3A \cos 5A$

4 $\cos 4A \sin 2A$

5 $\sin(\theta + \alpha)\cos(\theta - \alpha)$

6 $2\cos(45° + A)\sin(45° - A)$

7 $\cos(2\theta + \alpha)\cos(2\theta - \alpha)$

8 $2\cos(3\alpha + 2\beta)\sin(\alpha - \beta)$

9 $2\sin\theta\sin 2\theta$

10 $\sin 4\theta \sin(-2\theta)$

11 $\sin\left(\frac{3x+y}{2}\right)\cos\left(\frac{3x-y}{2}\right)$

12 $2\sin 75°\cos 15°$

13 $\sin(A - B)\sin(A + B)$

14 $\cos 75°\cos 45°$

15 $2\sin(\theta + \phi)\cos(\theta - \phi)$

16 $\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

17 $\sin 100°\sin 130°$

18 $2\sin 3\theta\cos\theta$

19 $\cos(\theta + 2\phi)\sin(2\theta + \phi)$

20 $2\sin 2A\cos 4A$

21 $2\cos 3A\sin 7A$

22 $\cos 5x + \cos x$ may be written as:

A $2\sin 3x\cos 2x$ **B** $2\cos 3x\sin 2x$ **C** $-2\sin 3x\sin 2x$ **D** $2\cos 3x\cos 2x$

Express the following as products:

23 $\sin 3x - \sin x$

24 $\sin(x + \alpha) - \sin x$

25 $\cos(x + h) - \cos x$

26 $\sin(\theta + \alpha) + \sin(\theta - \alpha)$

27 $\cos\left(\frac{\theta+\alpha}{2}\right) + \cos\left(\frac{\theta-\alpha}{2}\right)$

28 $\cos(A + B + C) - \cos(A - B + C)$

29 $\cos(2x + y) + \cos(x + 2y)$

30 $\sin(90° - A) + \sin(90° - B)$

31 $\sin(90° - A) + \cos 3A$

32 $\cos x + \cos(x + 120°) + \cos(x + 240°)$

33 $\cos(2A + B) - \cos(A + 2B)$

34 $\cos(A + B + C) + \cos(A - B - C)$

35 $\sin 2x + \sin 3x$

36 $\sin(2A + 2B) - \sin(2A - 2B)$

37 $\sin 165° - \sin 105°$

38 $\sin\theta - \cos\phi$

39 $\cos 75° - \cos 45°$

40 $\sin 50° + \cos 20°$

41 $\sin(A - B) - \sin A$

42 $\cos 80° - \sin 50°$

Prove the following results.

43 $\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan\theta$

44 $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x+y}{2}\right)$

45 $\frac{\cos x - \cos 3x}{\sin x - \sin 3x} = -\tan 2x$

46 $\frac{\sin 2A - \sin 2B}{\sin 2A + \sin 2B} = \frac{\tan(A-B)}{\tan(A+B)}$

47 $\frac{\cos(\theta+\alpha) - \cos(\theta-\alpha)}{\sin(\theta+\alpha) + \sin(\theta-\alpha)} = -\tan\alpha$

48 $\frac{\sin x + \sin(x+y) + \sin(x+2y)}{\cos x + \cos(x+y) + \cos(x+2y)} = \tan(x+y)$

49 $\sin 5\theta + \sin 3\theta - 2\sin 2\theta\cos\theta = 2\sin 2\theta\cos 3\theta$

50 $\sin 2x + \sin 4x + \sin 6x = 4\cos x\cos 2x\sin 3x$

51 $\sin^2 5\theta - \sin^2 3\theta = \sin 8\theta\sin 2\theta$

52 $\sin 35° - \sin 25° = \sqrt{3}\sin 5°$

53 $\cos 35° + \cos 45° + \cos 75° + \cos 85° = 2\cos 5°\cos 20°$

54 $2\cos 37.5°\sin 7.5° = \frac{\sqrt{2}-1}{2}$

55 $\sin 25°\sin 35° - \sin 20°\sin 10° = \frac{\sqrt{3}-1}{4}$

56 $\sin 10° + \cos 40° = \sin 70°$

57 $\frac{\sin 48° + \sin 12°}{\cos 48° + \cos 12°} = \frac{\sqrt{3}}{3}$

58 $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

59 $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

60 $\dfrac{\sin\theta + \sin 7\theta}{\sin 3\theta + \sin 5\theta} = 2\cos 2\theta - 1$

61 $\dfrac{\sin(\theta+\phi) - \sin(\theta-\phi)}{\cos(\theta+\phi) - \cos(\theta-\phi)} = -\cot\theta$

62 $\dfrac{\cos 75° + \cos 15°}{\sin 75° - \sin 15°} = \sqrt{3}$

63 $\dfrac{\sin A + \sin(A+B) + \sin(A+2B)}{\cos A + \cos(A+B) + \cos(A+2B)} = \tan(A+B)$

64 $\dfrac{\sin(n+1)\theta + 2\sin n\theta + \sin(n-1)\theta}{\cos(n-1)\theta - \cos(n+1)\theta} = \cot\dfrac{\theta}{2}$

65 If $\alpha+\beta+\gamma=\pi$, show that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

4.6 OVERVIEW OF TRIGONOMETRIC EQUATIONS

Equations of the form $\sin\theta = \sin\alpha$

Example 14

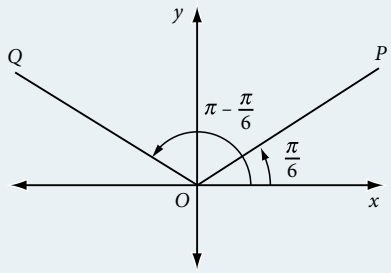
Find all values of θ for which $\sin\theta = \frac{1}{2}$.

Solution

$$\sin\theta = \frac{1}{2}$$

$$\therefore \quad \sin\theta = \sin\frac{\pi}{6}, \sin\left(\pi - \frac{\pi}{6}\right), \sin\left(2\pi + \frac{\pi}{6}\right), \ldots$$

Consider a coordinate diagram.



$$\angle XOP = \frac{\pi}{6}$$

$$\angle XOQ = \pi - \frac{\pi}{6}$$

The ray OP defines an infinite number of angles in the first quadrant. If you rotate OP about the origin (either clockwise or anticlockwise), then during each revolution it is along the original ray OP once.
Each full rotation increases the angle by 2π, so you find that OP is the terminal ray defining the angles:

- $\frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, \ldots$ for anticlockwise rotation
- $-2\pi + \frac{\pi}{6}, -4\pi + \frac{\pi}{6}, -6\pi + \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi + \frac{\pi}{6}$ where $n = 0, \pm 2, \pm 4, \ldots$ [1]

or: $n \times 180° + 30°$ (in degrees)

Similarly, the terminal ray OQ defines an infinite number of angles:

$\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 5\pi - \frac{\pi}{6}, \ldots$ for anticlockwise rotation

$-\pi - \frac{\pi}{6}, -3\pi - \frac{\pi}{6}, -5\pi - \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi - \frac{\pi}{6}$ where $n = \pm 1, \pm 3, \pm 5, \ldots$ [2]

or: $n \times 180° - 30°$ (in degrees)

Statements [1] and [2] can be written together as:

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

or: $\theta = n \times 180° + (-1)^n \times 30°$ (in degrees)

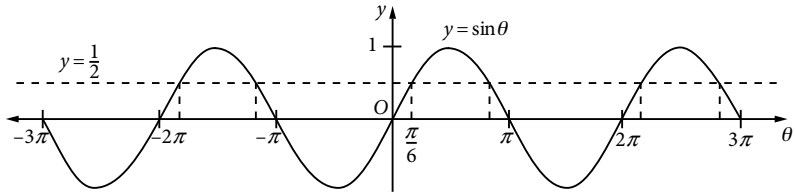
Note: $(-1)^n$ is 1 when n is zero or even, and is -1 when n is odd.

Following Example **14** above, the general solution to the equation $\sin\theta = \sin\alpha$ can be written as:

$\theta = n\pi + (-1)^n\alpha$ (in radians)
$\theta = n \times 180° + (-1)^n\alpha°$ (in degrees) for any integer n.

The general solution of trigonometric equations has not been included in this course. This material is included to show why you have to be very careful to consider all possible results in the given domain when solving trigonometric equations.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \sin\theta$ and $y = \frac{1}{2}$ (from Example **14**), as shown in the following diagram.



From symmetry, you can observe that the line $y = \frac{1}{2}$ intersects the sine curve at values of θ that are $\frac{\pi}{6}$ units to the right of $n\pi$ where $n = 0, \pm2, \pm4, \ldots$ and that are $\frac{\pi}{6}$ units to the left of $n\pi$ when $n = \pm1, \pm3, \ldots$

Both of these solutions are contained in the statement: $\theta = n\pi + (-1)^n \frac{\pi}{6}$

$$\text{or: } \theta = \begin{cases} n\pi + \frac{\pi}{6}, & n \text{ even or zero} \\ n\pi - \frac{\pi}{6}, & n \text{ odd} \end{cases}$$

Example 15

Solve $\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ for $0 \le \theta \le 2\pi$.

Solution

$$\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$\sin\theta < 0$ in third and fourth quadrants: $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}, \sin\frac{7\pi}{4}$

$$\theta + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\theta = \pi, \frac{3\pi}{2}$$

Equations of the form $\cos\theta = \cos\alpha$

Example 16

Find all angles θ for which $\cos\theta = \frac{1}{2}$.

Solution

$$\cos\theta = \frac{1}{2}$$

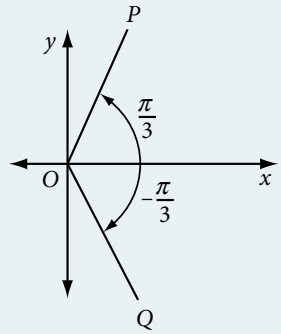
i.e. $\cos\theta = \cos\frac{\pi}{3}, \cos\left(2\pi - \frac{\pi}{3}\right), \cos\left(2\pi + \frac{\pi}{3}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, \ldots$

and OQ defines the angles: $-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, \ldots$

These results can be summarised as: $\theta = 2n\pi \pm \frac{\pi}{3}$ where n is any integer

or: $\theta = n \times 360° \pm 60°$ (in degrees)

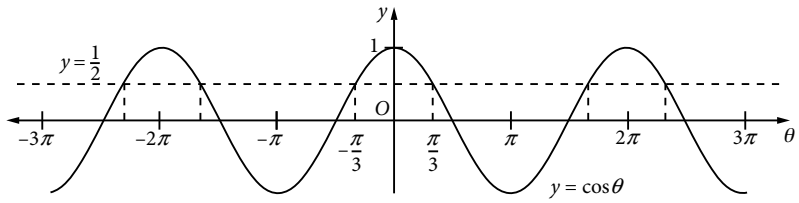


Following Example **16**, above, the general solution to the equation $\cos\theta = \cos\alpha$ can be written as:

$\theta = 2n\pi \pm \alpha$ (in radians)

$\theta = n \times 360° \pm \alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \cos\theta$ and $y = \frac{1}{2}$ (from Example **16**), as shown in the following diagram.



From symmetry, you can observe that the line $y = \frac{1}{2}$ intersects the cosine curve at values of θ that are $\frac{\pi}{3}$ units to the left and right of $2n\pi$ where $n = 0, \pm2, \pm4, \ldots$ i.e. $\theta = 2n\pi \pm \frac{\pi}{3}$.

Example 17

Solve $2\cos(3x + 30°) + \sqrt{3} = 0$ for $0° \le x \le 360°$.

Solution

$2\cos(3x + 30°) = -\sqrt{3}$

$\cos(3x + 30°) = -\frac{\sqrt{3}}{2}$

$\cos\theta < 0$ in second and third quadrants: $\cos(3x + 30°) = \cos 150°, \cos 210°, \ldots$

As $0° \le x \le 360°$, thus $0° \le 3x \le 3 \times 360°$, so two more revolutions are needed.

$3x + 30° = 150°, 210°, 510°, 570°, 870°, 930°$

$3x = 120°, 180°, 480°, 540°, 840°, 900°$

$x = 40°, 60°, 160°, 180°, 280°, 300°$

Equations of the form $\tan\theta = \tan\alpha$

Example 18

Find all the angles for which $\tan\theta = 1$, where θ is in radians.

Solution

$\tan\theta = 1$

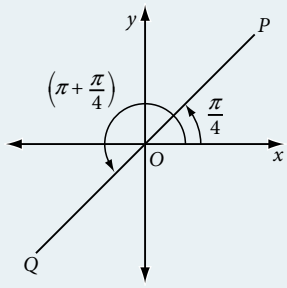
i.e. $\tan\theta = \tan\frac{\pi}{4}, \tan\left(\pi + \frac{\pi}{4}\right), \tan\left(2\pi + \frac{\pi}{4}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, \ldots$

and OQ defines the angles: $\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}, \ldots$

These results can be summarised as: $\theta = n\pi + \frac{\pi}{4}$ where n is any integer

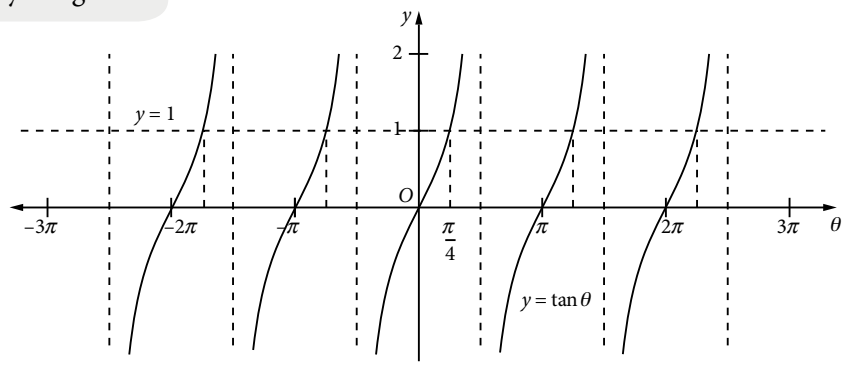
or: $\theta = n \times 180° + 45°$ (in degrees)



Following Example **18**, the general solution to the equation $\tan\theta = \tan\alpha$ can be written as:

$\theta = n\pi + \alpha$ (in radians)
$\theta = n \times 180° + \alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \tan\theta$ and $y = 1$ (from Example **17**), as shown in the diagram at right.



From symmetry, you can observe that the line $y = 1$ intersects the tangent curve at values of θ, which are $\frac{\pi}{4}$ units to the right of $n\pi$ where $n = 0, \pm1, \pm2, \ldots$

Example 19

Solve $\tan x = 3\cot x$ for $-\pi \le x \le \pi$.

Solution

$\tan x = 3\cot x$

$\tan x = \dfrac{3}{\tan x}$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Solution is in all four quadrants: $x = -\pi + \frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

EXERCISE 4.6 OVERVIEW OF TRIGONOMETRIC EQUATIONS

1 Solve for $0 \le x \le 2\pi$.

(a) $\sin x = 1$ **(b)** $\cos x = 0$ **(c)** $\tan x = -1$ **(d)** $\sqrt{3}\operatorname{cosec} x = 2$ **(e)** $\sec x = -2$

(f) $\cot x = \sqrt{3}$ **(g)** $2\sin\left(x - \frac{\pi}{6}\right) + 1 = 0$ **(h)** $\cos\frac{x}{2} = 1$ **(i)** $2\sin^2 x = 1$ **(j)** $\sin x = 0.3894$

2 The solution to $4\cos^2 x - 1 = 0$ for $0 \le x \le 2\pi$ is:

A $x = \frac{\pi}{3}, \frac{2\pi}{3}$ **B** $x = \frac{\pi}{6}, \frac{5\pi}{6}$ **C** $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **D** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3 Solve for $-\pi \le x \le \pi$.

(a) $\cos^2 x - 2\cos x + 1 = 0$ **(b)** $\sin^2 x = \sin x$ **(c)** $\cos 2x = \sin x$

(d) $\sin^2 x = 1 - \cos x$ **(e)** $\cos 2x = 2 + \cos x$ **(f)** $\tan 2x = \cot x$

(g) $\cos 2x = \cos x$ **(h)** $2\sin x = \sec x$ **(i)** $\tan^2 x = \tan x$

4 The solution to $2\cos(2x - 60°) = \sqrt{3}$ for $0° \le x \le 180°$ is:

A $x = 45°$ **B** $x = 145°$ **C** $x = 30°, 330°$ **D** $x = 15°, 145°$

5 Solve for $0 \le \theta \le 2\pi$.

(a) $\sqrt{2}\sin 2\theta + 1 = 0$ **(b)** $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ **(c)** $\cos 2\theta\cos\frac{\pi}{6} - \sin 2\theta\sin\frac{\pi}{6} = 0.5$

(d) $\tan\theta = \sin 2\theta$ **(e)** $\tan\theta = \cot\theta$ **(f)** $\sin 3\theta + \sin\theta = 0$ **(g)** $\sin 4\theta - \sin 2\theta = 0$

6 Solve for $-\pi \le \theta \le \pi$.

(a) $\cos 3\theta = \cos\theta$ **(b)** $2\cos 2\theta = 4\cos\theta - 3$ **(c)** $3\tan 2\theta = 2\tan\theta$

(d) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ **(e)** $2\cos\left(2\theta - \frac{\pi}{3}\right) = \sqrt{3}$ **(f)** $2\sin^2\theta + \cos\theta = 1$

7 Solve for $0° \le x \le 360°$.

(a) $4 + \sin x = 6\cos^2 x$ **(b)** $\sin x = \cos x$ **(c)** $1 + 2\cos^2 x = 5\sin x$

8 Solve for $0 \le \theta \le 2\pi$.

(a) $\tan^3\theta - \tan\theta = 0$ **(b)** $\tan\theta = \sin\theta$ **(c)** $\sec 2\theta = \operatorname{cosec} 2\theta$

(d) $\sin 2\theta = \tan\theta$ **(e)** $\sin 3\theta = \sin 2\theta$

4.7 SIMPLE TRIGONOMETRIC EQUATIONS

There is no general method for solving trigonometric equations and inequalities, but there are certain standard procedures and types, as illustrated in the examples below.

Example 20

Solve the equation $3\sin 2\theta = 1.5$, $0 \le \theta \le \pi$.

Solution

$$3\sin 2\theta = 1.5$$
$$\therefore \quad \sin 2\theta = 0.5$$

$0 \le \theta \le \pi$ means that $0 \le 2\theta \le 2\pi$:

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Hence: $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

This solution can be checked graphically by using graphing software to find the intersection of $y = 3\sin 2\theta$ and $y = 1.5$. Over the domain $0 \le \theta \le \pi$ these functions intersect only twice. If the domain is increased, there will be two more intersections for each domain increase of π units.

Example 21

Solve the equation $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

Solution

Let $\theta = 2x - \frac{\pi}{6}$: $\cos\theta = \frac{\sqrt{3}}{2}$

Hence: $\theta = \ldots -\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{11\pi}{6}, \ldots$

From the limits: $0 \le x \le 2\pi$

$$0 \le 2x \le 4\pi$$

But $2x = \theta + \frac{\pi}{6}$: $0 \le \theta + \frac{\pi}{6} \le 4\pi$

$$-\frac{\pi}{6} \le \theta \le 4\pi - \frac{\pi}{6}$$

$$\therefore \quad \theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{11\pi}{6}$$

$$\therefore \quad 2x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{11\pi}{6}$$

$$2x = 0, \frac{\pi}{3}, 2\pi, \frac{7\pi}{3}, 4\pi$$

$$x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$$

Example 22

Find the values of x for which $\cos 2x \le \frac{1}{\sqrt{2}}$, $0 \le x \le 2\pi$.

Solution

First solve the equation, then solve the inequality graphically.

Solve the equation: $\cos 2x = \frac{1}{\sqrt{2}}$ $\quad (0 \le x \le 2\pi)$

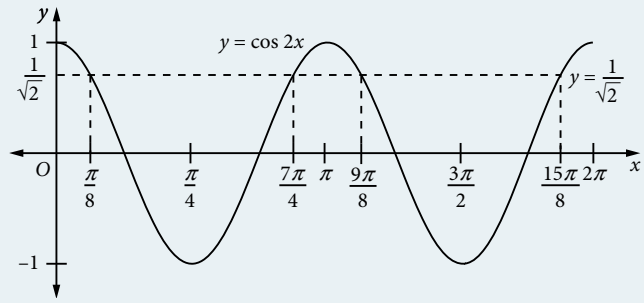
$2x = \frac{\pi}{4}, \frac{7\pi}{4}, 2\pi + \frac{\pi}{4}, 2\pi + \frac{7\pi}{4}$ $\quad$ (as $0 \le 2x \le 4\pi$, around the circle twice)

$x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

Now sketch the graph of $y = \cos 2x$ for $0 \le x \le 2\pi$, showing the line $y = \frac{1}{\sqrt{2}}$ also:

Use the diagram to find where the graph of $y = \cos 2x$ is on or below the line $y = \frac{1}{\sqrt{2}}$.

Hence $\cos 2x \le \frac{1}{\sqrt{2}}$ for $\frac{\pi}{8} \le x \le \frac{7\pi}{8}$ and for $\frac{9\pi}{8} \le x \le \frac{15\pi}{8}$.



EXERCISE 4.7 SIMPLE TRIGONOMETRIC EQUATIONS

1 Solve for values of θ and x between 0 and 2π inclusive:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ **(b)** $\tan x = -1$ **(c)** $\cos x = -0.5$ **(d)** $\sqrt{3}\tan\theta = 1$

(e) $\sin 2\theta = -\frac{1}{2}$ **(f)** $\operatorname{cosec}\theta = -2$ **(g)** $\cot 2x = \sqrt{3}$ **(h)** $\sec 2\theta = \sqrt{2}$

2 Solve between 0° and 360°: **(a)** $\cos x = 0.4$ **(b)** $4\tan 2\theta + 3 = 0$

3 The solution to $\sqrt{2}\sin 2\theta + 1 = 0$ for $0 \le \theta \le 2\pi$ is:

A $\frac{5\pi}{4}, \frac{7\pi}{4}$ **B** $\frac{5\pi}{8}, \frac{7\pi}{8}$ **C** $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ **D** $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$

4 Solve for $-\pi \le x \le \pi$: **(a)** $2\cos 2x + 1 = 0$ **(b)** $\sqrt{2}\sin 2x - 1 = 0$

5 Solve between 0 and 2π inclusive:

(a) $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ **(b)** $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ **(c)** $\cos\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ **(d)** $2\sin\left(2x - \frac{\pi}{6}\right) = 1$

(e) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ **(f)** $2\cos\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$ **(g)** $\sin 2\theta = -\cos\frac{7\pi}{4}$ **(h)** $\sin x = \cos x$

6 Solve for $0° < x < 360°$: **(a)** $\cos(2x + 60°) = 0.7242$ **(b)** $5\sin(2x - 70°) + 4 = 0$

7 If $0 \le x \le 2\pi$, the solution to $\sin x \le \frac{\sqrt{3}}{2}$ is:

A $x \le \frac{\pi}{3}$ **B** $x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **C** $0 \le x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **D** $0 \le x \le \frac{\pi}{3}$ or $\frac{2\pi}{3} \le x \le 2\pi$

8 If $0 \le x \le 2\pi$, solve: **(a)** $\sin x \ge \frac{1}{2}$ **(b)** $\cos x < \frac{1}{2}$ **(c)** $\sin x > 0$ **(d)** $\cos x > \frac{\sqrt{3}}{2}$

(e) $\sin x < 1$ **(f)** $\cos x > 0$ **(g)** $\tan x > 1$ **(h)** $\sqrt{2}\cos x > -1$

9 Solve for $0 < x < \pi$: **(a)** $\sin 2x \ge \frac{1}{2}$ **(b)** $\cos 2x \le 0$

10 Solve for $-\pi < x < \pi$: **(a)** $2\sin 2x \le \sqrt{3}$ **(b)** $2\cos 2x > -1$

4.8 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

You have already solved simple trigonometric equations, you can now use the angle formulae found in this chapter to solve harder equations.

Example 23

Solve the equation $\sin 2x = 3\cos x$, $0 \le x \le 2\pi$.

Solution

As $\sin 2x = 2\sin x \cos x$: $2\sin x \cos x = 3\cos x$

$\cos x(2\sin x - 3) = 0$

$\therefore \cos x = 0$ or $\sin x = 1.5$

Because $|\sin x| \le 1$, the only solution is $\cos x = 0$.

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

Example 24

Solve the equation $\cos 2x \cos\alpha - \sin 2x \sin\alpha = -0.5$, $0 \le x \le 2\pi$, where $\alpha = \frac{\pi}{6}$.

Solution

Use the expansion of $\cos(A + B)$ to simplify the LHS:

$\cos 2x \cos\alpha - \sin 2x \sin\alpha = \cos(2x + \alpha)$

Hence, as $\alpha = \frac{\pi}{6}$: $\cos\left(2x + \frac{\pi}{6}\right) = -0.5$

$$2x + \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}$$

$$2x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{5\pi}{2}, \frac{19\pi}{6}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$$

Example 25

Solve for $0 \le \theta \le \pi$, the equation $\sin 2\theta \cos\theta = \sin 3\theta \cos 2\theta$.

Solution

$\sin 2\theta \cos\theta = \sin 3\theta \cos 2\theta$.

Convert each product to a sum: $\frac{1}{2}(\sin(2\theta + \theta) + \sin(2\theta - \theta)) = \frac{1}{2}(\sin(3\theta + 2\theta) + \sin(3\theta - 2\theta))$

$$\sin 3\theta + \sin\theta = \sin 5\theta + \sin\theta$$

Simplify: $\sin 3\theta = \sin 5\theta$

Rewrite: $\sin 5\theta = \sin 3\theta$

Solve: $5\theta = 3\theta, \pi - 3\theta, 2\pi + 3\theta, 3\pi - 3\theta, 4\pi + 3\theta, 5\pi - 3\theta, 6\pi + 3\theta, 7\pi - 3\theta$.

$0 \le \theta \le \pi$: $2\theta = 0, 2\pi, 4\pi, \ldots$

$\theta = 0, \pi$

and $8\theta = \pi, 3\pi, 5\pi, 7\pi$.

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

The complete solution is: $\theta = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$.

Example 26

Solve for $0 \le \theta \le \pi$, the equation $\cos 3\theta + \cos\theta = \cos 2\theta$.

Solution

$\cos 3\theta + \cos\theta = \cos 2\theta$

Convert the LHS to a product: $2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) = \cos 2\theta$

$$2\cos 2\theta \cos\theta = \cos 2\theta$$

Rearrange: $\cos 2\theta\,(2\cos\theta - 1) = 0$

Solve: $\cos 2\theta = 0$ or $\cos\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = \frac{\pi}{3}$.

The complete solution is: $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}$.

EXERCISE 4.8 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

1 Solve: **(a)** $\cos 2\theta = \cos\theta$, $0 \le \theta \le 2\pi$ **(b)** $2\cos 2\theta = 4\cos\theta - 3$, $0 \le \theta \le 2\pi$

(c) $3\tan 2\theta = 2\tan\theta$, $0 \le \theta \le 2\pi$ **(d)** $\tan\theta + 2\cot\theta = 3$, $0° \le \theta \le 360°$

2 The solution to $5\sin x = 2\sec x$ for $0° \le x \le 180°$ is:

A $x = 11°47'$ or $78°13'$ **B** $x = 23°34'$ or $156°26'$

C $x = 26°34'$ or $63°26'$ **D** $x = 53°8'$ or $126°52'$

3 Solve: **(a)** $\cos 2x \cos\frac{\pi}{6} - \sin 2x \sin\frac{\pi}{6} = \frac{1}{2}$, $0 \le x \le 2\pi$ **(b)** $\sin 2x \cos\frac{\pi}{3} + \cos 2x \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $-\pi \le x \le \pi$

4 The solution to $\sin\theta = \cos 2\theta$ for $0 \le \theta \le 2\pi$ is:

A $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ **B** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **C** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **D** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

5 Solve $\tan\theta = \sin 2\theta$, $0 \le \theta \le 2\pi$.

6 Solve $\cos^2\theta = 2\cos^2\frac{\theta}{2}$, $0° \le \theta \le 360°$.

7 Solve $\sin 3x \cos x - \cos 3x \sin x = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

8 Solve $\tan 2\theta = 2\tan\theta$, $0 \le \theta \le 2\pi$.

9 Solve for $0 \le \theta \le \pi$, the equations:

(a) $\sin 4\theta \cos\theta = \sin 3\theta \cos 2\theta$ **(b)** $\sin 3\theta \cos 2\theta = \sin 5\theta \cos 4\theta$ **(c)** $\cos 2\theta \sin\theta = \cos 3\theta \sin 2\theta$

10 Solve for $0 \le x \le 2\pi$, the equations:

(a) $2\cos\left(x + \frac{\pi}{3}\right)\cos x = 1$ **(b)** $2\sin 3x \sin x = 1$

(c) $\sin x \cos\left(x - \frac{\pi}{3}\right) = 0.3$ **(d)** $\cos 2x \cos x = \cos 3x$

11 Solve for $0 \le \theta \le \pi$, the equations:

(a) $\sin 4\theta - \sin 2\theta = \cos 3\theta$ **(b)** $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

(c) $\sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta + \frac{\pi}{12}\right) = 1$ **(d)** $\sin 4\theta + \sin 2\theta = \sin 3\theta + \sin\theta$

12 Solve for $0 \le x \le 2\pi$, the equations:

(a) $\sin 2x - \sin x = \cos 2x - \cos x$ **(b)** $\sin x + \sin 2x + \sin 3x = 0$

(c) $\cos 2x - \cos x + 1 = 0$ **(d)** $\cos 3x - \cos 2x + \cos x = 1$

CHAPTER REVIEW 4

1 Simplify: **(a)** $\dfrac{1-t^2}{1+t^2}$, where $t=\tan\dfrac{\theta}{2}$ **(b)** $\dfrac{\tan\theta-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{6}\tan\theta}$ **(c)** $\dfrac{\sin 2\theta-\sin\theta}{\cos 2\theta-\cos\theta+1}$

2 Solve $2\tan 2x-1=0$ for $0°<x<360°$.

3 Simplify:

(a) $\sin(\theta+\phi)\cos\phi-\cos(\theta+\phi)\sin\phi$ **(b)** $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$ **(c)** $\sin x\cos x\cos 2x\cos 4x$

4 **(a)** Show that $\cos(A+B)=\cos A\cos B(1-\tan A\tan B)$.

(b) Suppose that $0<A<\dfrac{\pi}{2}$ and $0<B<\dfrac{\pi}{2}$. Show by deduction that if $\tan A\tan B=1$ then $A+B=\dfrac{\pi}{2}$.

5 Show that: **(a)** $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$ **(b)** $\tan^2\theta=\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$, given that $\cos 2\theta\neq -1$.

6 Use the expansion of $\tan 2A$ to show that the exact value of $\tan 22.5°=\sqrt{2}-1$. Hence find the exact value of $\tan 11.25°$.

7 Solve the following equations for $0\leq x\leq\pi$.

(a) $\cos 3x=\cos 2x\cos x$ **(b)** $\cos 3x+\cos 5x+\cos 7x=0$

8 Solve for $-\pi\leq x\leq\pi$.

(a) $\cos x-\sin x=1$ **(b)** $\sin 4x-\sin 2x=0$ **(c)** $\cos x-\sqrt{3}\sin x=1$

CHAPTER 5
Inverse functions

5.1 INVERSE FUNCTIONS

Perform the following activities, either using graphing software or by hand. Be sure to adjust the sizing of your graphs so that both axes have the same scale.

Reflection in the line $y = x$

On a Cartesian plane, sketch the graph $y = x$ and plot the points (1, 3) and (−2, 4). Reflect the points in the line $y = x$ and consider how the reflection changes each of the coordinates.

You should see that when a point is reflected in the line $y = x$, 'what was x becomes y and what was y becomes x'.

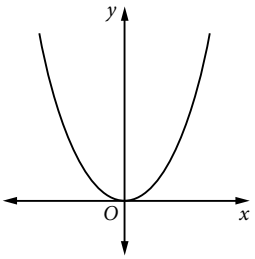
In general, the point (a, b) becomes the point (b, a) on reflection in $y = x$. The midpoint of these two points is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, which is on the line $y = x$.

MAKING CONNECTIONS

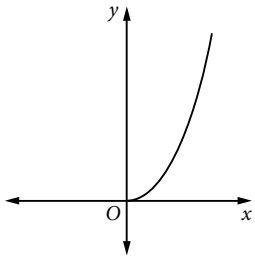
Reflection in the line $y = x$

Use technology to explore the reflection of points in the line $y = x$.

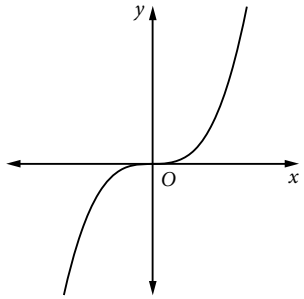
One-to-one functions



$f(x) = x^2$



$g(x) = x^2, x \geq 0$



$h(x) = x^3$

A **one-to-one** function is a function that has no ordered pairs with the same y-coordinate. Any line parallel to the x-axis will intersect the graph of a one-to-one function no more than once. It is said that such graphs pass the **horizontal line test**. This means that in the diagrams above, $g(x)$ and $h(x)$ are examples of one-to-one functions.

Note that $f(x)$ is not one-to-one, because a straight line parallel to the x-axis could intersect the graph of $f(x)$ at two points, e.g. $(-a, a^2)$ and (a, a^2). This means that the graph of $f(x)$ is a **two-to-one** function.

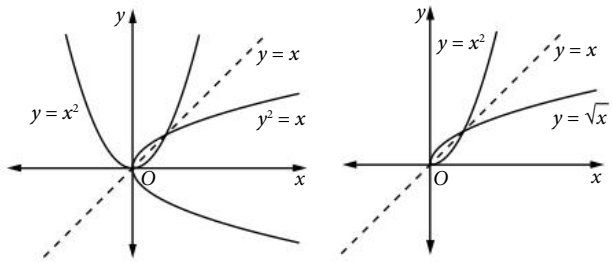
An example of a **many-to-one** function is $s(x) = \sin x$. But a function $t(x) = \sin x$, $0 \leq x \leq \frac{\pi}{2}$ is a one-to-one restriction of $s(x)$.

One-to-one functions are significant because of their properties when reflected in the line $y = x$. This is important in the development of inverse functions. For example, the reflection of the graph of $y = x^2$ in the line $y = x$ is the parabola $y^2 = x$, which is the graph of a relation and not a function (see the left diagram below).

The reflection of the graph of $y = x^2$ in the line $y = x$ is the positive branch only of the parabola $y^2 = x$, i.e. $y = +\sqrt{x}$. This (the diagram on the right) is the graph of a function.

The graph of a function will pass the vertical line test (i.e. no vertical line can cut the graph more than once).

The graph of a one-to-one function will also pass the horizontal line test (i.e. no horizontal line can cut the graph more than once).



If $f(x)$ is a one-to-one function, then the reflection of its graph in the line $y = x$ is the graph of a function, i.e. if a graph passes the horizontal line test then its reflection in the line $y = x$ will pass the vertical line test.

Example 1

Define a restriction on the domain of f, where $f(x) = x^2 - 4x + 5$, so that f is one-to-one.

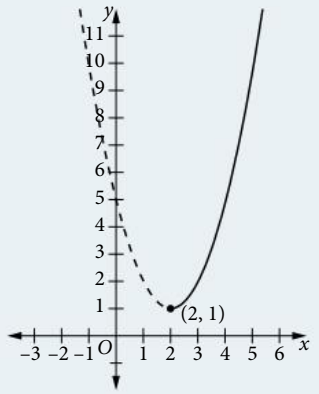
Solution

$f(x) = (x-2)^2 + 1$: the graph of f is a parabola (concave up) with vertex at $(2, 1)$.

f is thus one-to-one for $x \geq 2$ or for $x \leq 2$.

Thus: $f_1(x) = x^2 - 4x + 5, x \geq 2$ and $f_2(x) = x^2 - 4x + 5, x \leq 2$ are both one-to-one restrictions of f.

These restrictions give the largest possible domains for which f is one-to-one. (Of course, any narrower restriction, such as $f_3(x) = x^2 - 4x + 5, x \geq 5$, would also give a one-to-one function.)

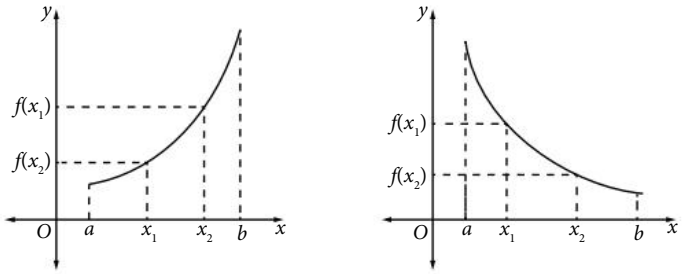


The dashed graph represents $f(x) = x^2 - 4x + 5$ for $x \leq 2$.

The solid graph represents $f(x) = x^2 - 4x + 5$ for $x \geq 2$.

Increasing and decreasing functions

A function f defined on the interval $a \leq x \leq b$ is said to be **strictly monotonic increasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. This means that the curve always slopes upwards to the right, as shown in the left diagram below.



Similarly, a function is said to be **strictly monotonic decreasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. This means that the curve slopes downwards to the right, as shown in the right diagram above.

Functions that are strictly monotonic increasing or decreasing are clearly one-to-one.

Inverse functions

Consider two operations such that each is the 'reverse' of the other. For example, the **cube** of 2 is 8 and the **cube root** of 8 is 2. The cube function defined by $f(x) = x^3$, $x \in R$ (i.e. 'x is a member of the set of real numbers'), and the cube root function, defined by $g(x) = \sqrt[3]{x}$, $x \in R$, are said to be a pair of **inverse** functions: $f(2) = 8$ and $g(8) = 2$.

If you cube any real number x and then take the cube root of the answer, the result is x again. Likewise, if you take the cube root of any number x and then cube the answer, the result is x.

$f(x)=x^3$ $\qquad$ $g(x)=\sqrt[3]{x}$

$f(g(x))=(g(x))^3$ $\qquad$ $g(f(x))=\sqrt[3]{f(x)}$

$=\left(\sqrt[3]{x}\right)^3$ $\qquad$ $=\sqrt[3]{x^3}$

$=x$ $\qquad$ $=x$

Thus you can define, for any pair of inverse functions:

$$f(g(x))=g(f(x))=x$$

If you denote $g(x)$ as $f^{-1}(x)$ (meaning 'the inverse function of f'):

$$f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x$$

The **square** of 2 is 4, but in the reverse operation 4 has two **square roots**, ± 2. If you wish to return to 2 in the square root operation, you must take the positive square root.

This means that the square function, defined by $f(x)=x^2$, $x\geq 0$ and the positive square root function, defined by $g(x)=\sqrt{x}$, $x\geq 0$, are a pair of inverse functions. For the inverse to define a function, the functions must be one-to-one.

$f(x)=x^2 \quad (x\geq 0)$ $\qquad$ $g(x)=\sqrt{x} \quad (x\geq 0)$

$f(g(x))=(g(x))^2$ $\qquad$ $g(f(x))=\sqrt{f(x)}$

$=(\sqrt{x})^2$ $\qquad$ $=\sqrt{x^2}$

$=x$ $\qquad$ $=x$ if $x\geq 0$

Thus $f(g(x))=g(f(x))=x$

i.e. $f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x$ where $g(x)=f^{-1}(x)$

Example 2

Find the inverse function f^{-1} for the function f where $f(x)=2x+4$.

Solution

You require f^{-1}, such that: $f(f^{-1}(x))=x=f^{-1}(f(x))$

$$\therefore\ 2f^{-1}(x)+4=x$$

i.e. $$f^{-1}(x)=\frac{1}{2}(x-4)$$

It is now easy to show that: $f^{-1}(f(x))=\frac{1}{2}[(2x+4)-4]$

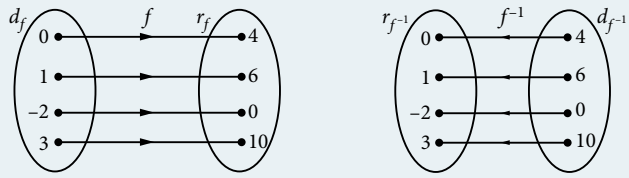
$$=\frac{1}{2}(2x)$$

$$=x$$

Hence $f^{-1}(x)=\frac{1}{2}(x-4)$ is the rule defining this inverse function. You now need to find the domain and range of f^{-1}.

Let d_f and r_f denote the domain and range respectively of f, and $d_{f^{-1}}$ and $r_{f^{-1}}$ denote the domain and range respectively of f^{-1}.

Consider a few typical elements of d_f, e.g. 0, 1, −2, 3. Under the rule $f(x)=2x+4$:



This correspondence can be represented by the ordered pairs $(0,4)$, $(1,6)$, $(-2,0)$, $(3,10)$.

Every real number in d_f has a corresponding real number image in r_f.

The inverse mapping is obtained by operating f^{-1} on the elements of r_f. This means that these elements can now be regarded as belonging to $d_{f^{-1}}$, i.e. $r_f=d_{f^{-1}}$.

For example: $f^{-1}(4)=\frac{1}{2}(4-4)=0$ $f^{-1}(6)=\frac{1}{2}(6-4)=1$

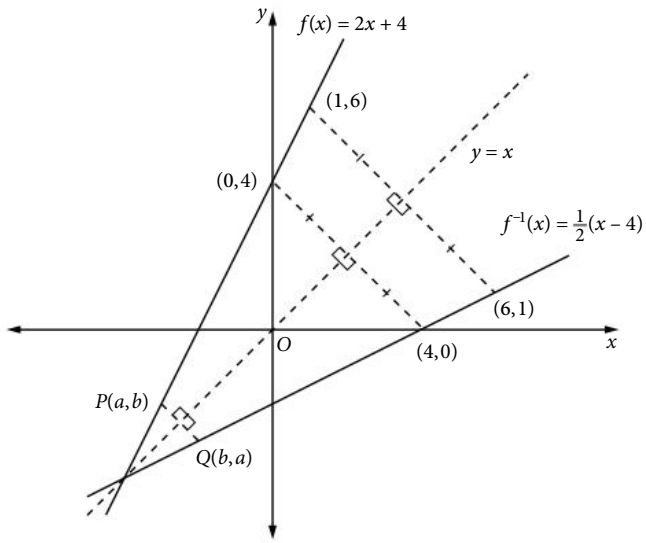
$f^{-1}(0)=-2$ $f^{-1}(10)=3$

This inverse correspondence can be represented by $(4,0)$, $(6,1)$, $(0,-2)$, $(10,3)$.
The elements 0, 1, −2, 3 that belong to the range of f^{-1} can be identified as the elements of the domain of f, and so $r_{f^{-1}}=d_f$.

The two diagrams in Example **2**, considered together, show that an element $x\in d_f$ is mapped back to itself by the composite function $f(f^{-1}(x))$. This is as required by the general defining property $f(f^{-1}(x))=x$.
Example **2** illustrates several important general properties of inverse functions.

1. $d_f=r_{f^{-1}}$ and $r_f=d_{f^{-1}}$, so the domain and range are interchanged between a function and its inverse.
2. If (a,b) is a point on the graph of f, then (b,a) is a point on the graph of f^{-1}. On the graphs these are called **corresponding points**.
3. Representing $P(a,b)$ and $Q(b,a)$ as corresponding points on the graphs of f and f^{-1}, the coordinates of the midpoint of PQ are $\left[\frac{1}{2}(a+b),\frac{1}{2}(b+a)\right]$, which is a point on the graph of $y=x$ for any a and b. Hence $y=x$ bisects the line segment PQ so that P and Q are mirror images of each other in this line.
4. The graph of f^{-1} is thus seen to be a reflection of the graph of f in the line $y=x$. This is very useful when sketching graphs. It also follows that any intersection between the graphs of f and f^{-1} must be on $y=x$.

The diagram shows the graphs of f and f^{-1} for Example **2**.



The property that (x,y) and (y,x) are on the graphs of f and f^{-1} respectively provides an alternative means of finding f^{-1}.
For example, the function f of Example **2** is defined by:

$y=f(x)$ for $x\in d_f$ and $y\in r_f$

i.e. $y=2x+4$ for $x\in R$ and $y\in R$

Hence f^{-1} is defined by:

$x=2y+4$ for $y\in R$ and $x\in R$

(interchanging x and y)

i.e. $y=\frac{1}{2}(x-4)$ for $x\in R$, $y\in R$

Thus: $f^{-1}(x)=\frac{1}{2}(x-4)$ as before.

Existence of inverse functions

Not every function has an inverse function. The graphs of a function and its inverse must be reflections of each other in the line $y=x$. When the graph of a one-to-one function is reflected in $y=x$, the reflection is itself the graph of a function. But when the graph of a many-to-one function is reflected in $y=x$, the reflection is not the graph of a function—it is the graph of a relation (which would fail the vertical line test), so it cannot be called f^{-1}.

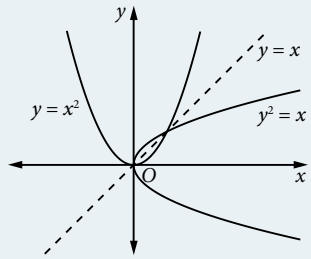
A function f will have an inverse function f^{-1} if and only if f is a one-to-one function.

Example 3

Investigate the existence of an inverse function where f is defined by $f(x) = x^2$.

Solution

f is not a one-to-one function. Its graph, the parabola $y = x^2$, produces the parabola with equation $y^2 = x$ when reflected in the line $y = x$. This latter parabola consists of an upper branch $y = \sqrt{x}$ and a lower branch $y = -\sqrt{x}$: it is the graph of a relation with two values of y for each value of x in the domain (except $x = 0$). Therefore, f^{-1} does not exist.



Example 4

Investigate the existence of an inverse function where f is defined by $f(x) = x^2$, $x \geq 0$.

Solution

This is a restriction of the function in Example **3**. In this case there is only the right-hand branch of the parabola $y = x^2$, which is the graph of a one-to-one function. Hence f^{-1} exists and can be determined as follows.

Method 1 (using algebraic definition)

You require f^{-1} such that $f^{-1}(f(x)) = x$.

$\therefore \left(f^{-1}(x)\right)^2 = x$

$\therefore \quad f^{-1}(x) = \pm\sqrt{x}$ Which one?

The range of f^{-1} consists of non-negative numbers, so take the positive square root.

$\therefore \quad f^{-1}(x) = \sqrt{x}$

Method 2 (using reflection property)

Let $y = x^2$.

Interchange x and y: $x = y^2$

Make y the subject: $y = \pm\sqrt{x}$ Which one?

The range of f^{-1} consists of non-negative numbers, so take the positive square root.

$\therefore \; f^{-1}(x) = \sqrt{x}$

Example 5

Consider the function $y = -\dfrac{1}{\sqrt{x}}$.

(a) Explain why the largest possible domain of the function is $x > 0$.

(b) Use calculus to show that the function has no stationary points and is strictly monotonic increasing for all x in its domain.

(c) Show that the graph of the function does not meet the x-axis.

(d) Find y when x is a very small number and also when x is a very large number. Hence state the range of the function.

(e) Explain why an inverse function exists.

(f) Sketch the graph of $y = -\dfrac{1}{\sqrt{x}}$ and its inverse on the same axes.

(g) Find the equation of the inverse function, carefully specifying any restrictions on its domain.

Solution

(a) $x > 0$ because you can't take the square root of a negative number and you can't divide by 0.

(b) $y = -x^{-\frac{1}{2}} \qquad \therefore \dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{3}{2}}$ i.e. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x^3}}$

$\dfrac{dy}{dx} \neq 0$, so there are no stationary points.

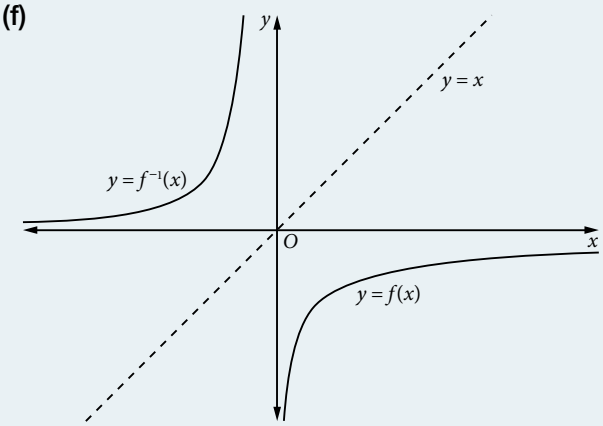
For all x in the domain, $\dfrac{dy}{dx} > 0$, so the function is increasing.

(c) There are no solutions to $-\frac{1}{\sqrt{x}}=0$, hence no x-intercepts.

(d) As $x \to 0$, $y \to -\infty$; as $x \to \infty$, $y \to 0$; range is $y < 0$

(e) Inverse function exists because f is strictly monotonic increasing (one-to-one).

(f)



(g) f is $y=-\frac{1}{\sqrt{x}}$ with domain $x > 0$ and range $y < 0$

$\therefore$ f^{-1} has domain $x < 0$ and range $y > 0$

To find the equation of f^{-1}, interchange x and y:

$x=-\frac{1}{\sqrt{y}}$

Make y the subject: $y=\frac{1}{x^2}$.

Remember the restriction on the domain of f^{-1}, i.e. $x < 0$.

Hence the inverse function is $f^{-1}(x)=\frac{1}{x^2}$, $x < 0$.

Example 6

$f(x)$ is defined by $f(x) = x^2 - 2x - 1$, $x \leq 1$.

(a) Sketch $y = f(x)$ and the inverse function $y = f^{-1}(x)$ on the same diagram.

(b) State the domain and range of f^{-1}.

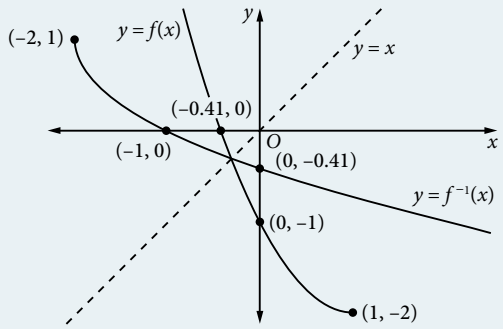
(c) Find the equation of the inverse function.

(d) Show that f and f^{-1} intersect where $x=\frac{3-\sqrt{13}}{2}$.

(e) Show that the gradient of the tangent to $y = f(x)$ at the point $(-1, 2)$ is -4. Hence find the coordinates of the point on $y = f^{-1}(x)$ where the gradient is -0.25.

Solution

(a) $y = x^2 - 2x - 1$ is a concave-up parabola with vertex at $(1, -2)$.
It cuts the y-axis at $(0, -1)$ and cuts the x-axis at $(1 + \sqrt{2}, 0)$ and $(1 - \sqrt{2}, 0)$.
As the domain of f is restricted to $x \leq 1$, its graph is the left half of the parabola.
The graph of f^{-1} is then obtained by reflection in the line $y = x$.



(b) For f^{-1} the domain is $x \geq -2$ and the range is $y \leq 1$.

(c) f is $y = x^2 - 2x - 1$
Interchange x and y: $x = y^2 - 2y - 1$

Make y the subject:

Method 1 (complete the square)

$$y^2 - 2y = x + 1$$
$$y^2 - 2y + 1 = x + 2$$
$$(y-1)^2 = x + 2$$
$$y - 1 = \pm\sqrt{x+2}$$
$$y = 1 \pm \sqrt{x+2}$$

Which one?

The range of f^{-1} is $y \le 1$, so it must be:

$$y = 1 - \sqrt{x+2}$$

Method 2 (quadratic formula)

$y^2 - 2y - x - 1 = 0$

This is a quadratic in y with $a = 1$, $b = -2$ and $c = -(x+1)$.

$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-[x+1])}}{2}$$
$$y = \frac{2 \pm \sqrt{4 + 4x + 4}}{2}$$
$$y = \frac{2\left(1 \pm \sqrt{x+2}\right)}{2}$$
$$y = 1 \pm \sqrt{x+2}$$

Which one?

The range of f^{-1} is $y \le 1$, so it must be:

$$y = 1 - \sqrt{x+2}$$

(d) To find the point of intersection, you would usually solve simultaneously the equations of the two curves. However, for these two equations you need to solve $x^2 - 2x - 1 = 1 - \sqrt{x+2}$, which is a formidable task.

A better way is to remember that a function and its inverse intersect on the line $y = x$.
Now you can find the solution of $x^2 - 2x - 1 = x$, i.e. $x^2 - 3x - 1 = 0$.

The solution is $x = \dfrac{3 \pm \sqrt{13}}{2}$.

However, f and f^{-1} intersect at one point only (as shown in the diagram on p. 99). The second solution is a different point of intersection that would exist if the entire parabola were intersected with the line $y = x$ (instead of only half the parabola).

From the diagram, you can see that the required value in this case is the smaller of the two solutions, so the point of intersection is at $x = \dfrac{3 - \sqrt{13}}{2}$.

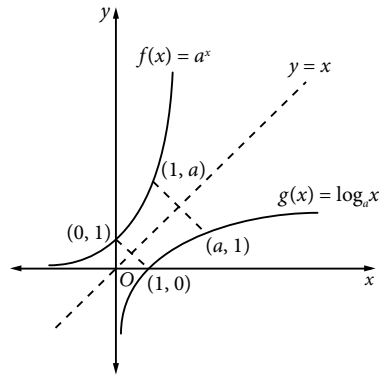
(e) $y = x^2 - 2x - 1 \quad \therefore \dfrac{dy}{dx} = 2x - 2$

At $x = -1$, the gradient of the tangent is -4.
At corresponding points in the reflection in $y = x$, the product of the gradients of a function and its inverse must be 1, so f^{-1} has a gradient of -0.25 at the point $(2, -1)$.

Exponential and logarithmic functions

Exponential and logarithmic functions, which you have met in the Mathematics Advanced course, provide a very good example of inverse functions.

An exponential function has the form $f(x) = a^x$. The base a is any positive real number except 1; the domain is R and the range is R^+ (the set of positive real numbers). It is a strictly monotonic increasing function, i.e. a one-to-one function, without having to restrict its domain.



Let $y = f(x) = a^x$

Interchanging x and y: $x = a^y$

$\therefore \quad y = \log_a x$ (definition of a logarithm)

The function g, where $g(x) = \log_a x$, is called a logarithmic function. Its domain is R^+ (the range of f) and its range is R (the domain of f). It is also a strictly monotonic increasing function.

Hence $\log_a a^x = x$ for all real x (i.e. for all x in the domain of the exponential function), and $a^{\log_a x} = x$ for all $x > 0$ (i.e. for all x in the domain of the logarithmic function).

It should be noted that for any pair of inverse functions f and g, f is the inverse of g and g is the inverse of f.

MAKING CONNECTIONS

Inverse functions

Use technology to explore the graphs of inverse functions and their properties.

EXERCISE 5.1 INVERSE FUNCTIONS

1 Which of the following are one-to-one functions?

A $f(x) = x - 2$ **B** $f(x) = x^2 - 2x + 1, x \geq 1$ **C** $f(x) = \sqrt{4 - x^2}$

D $f(x) = 9 - x$ **E** $f(x) = \dfrac{1}{9 - x}$ **F** $f(x) = |9 - x|$ **G** $f(x) = 9 - x^2$

H $f(x) = \cos x, 0 \leq x \leq \dfrac{\pi}{2}$ **I** $f(x) = \cos x$ **J** $f(x) = x^3 - 4x, -2 \leq x \leq 2$

2 Find the largest possible domain for which the following are one-to-one increasing functions.

(a) $f(x) = \sqrt{4 - x^2}$ **(b)** $f(x) = \sqrt{x^2 - 4}$ **(c)** $f(x) = -\dfrac{1}{x + 2}$

(d) $f(x) = 3x - x^2$ **(e)** $f(x) = x^2 + 6x + 8$

3 For each of the following, find the inverse function and state the domain and range of the inverse.

(a) $f(x) = 2x - 4$ **(b)** $f(x) = x^2 - 1, x \geq 0$ **(c)** $g(x) = \sqrt{x - 3}$ **(d)** $f(x) = \sqrt{9 - x^2}, -3 \leq x \leq 0$

(e) $f(x) = x^3$ **(f)** $f(x) = (x + 2)^2, x \leq -2$ **(g)** $f(x) = x^2 + 2x, x \geq 0$ **(h)** $f(x) = \log_e(x + 1)$

(i) $f(x) = 2 - \sqrt{x - 2}$ **(j)** $g(x) = \sqrt{(5 - x)} - 1$ **(k)** $f(x) = 2^{-x}, x > 0$ **(l)** $h(x) = \dfrac{1}{x + 1}, x > -1$

4 **(a)** Sketch the graph of $y = \dfrac{1}{1 + x^2}$.

(b) State two domains for which the function $f(x) = \dfrac{1}{1 + x^2}$ is one-to-one. (State the largest possible domain in each case.)

(c) Find and graph the two functions that are the inverse of the functions defined in **(b)**.

5 Explain why the following functions do not have an inverse function. Suggest suitable restrictions to their domain so that the restricted functions will have an inverse.

(a) $f(x) = \sqrt{a^2 - x^2}$ **(b)** $f(x) = 4 - x^2$ **(c)** $f(x) = \dfrac{1}{x^2}$

6 Show that the following pairs of functions are inverses by showing that $f(g(x)) = g(f(x)) = x$.

(a) $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}(x + 1)$ **(b)** $f(x) = \dfrac{e^x}{2}$ and $g(x) = \log_e 2x$

(c) $f(x) = \sqrt{16 - x^2}, -4 \leq x \leq 0$ and $g(x) = -\sqrt{16 - x^2}, 0 \leq x \leq 4$

(d) $f(x) = 2x - x^2, x \geq 1$ and $g(x) = 1 + \sqrt{1 - x}, x \leq 1$ **(e)** $f(x) = \dfrac{1}{2x - 1}, x > \frac{1}{2}$ and $g(x) = \dfrac{x + 1}{2x}, x > 0$

7 $f(x) = \sqrt{4 - x^2}, -2 \leq x \leq 0$. The inverse function is given by:

A $x^2 + y^2 = 4$ **B** $y = \sqrt{4 - x^2}, -2 \leq x \leq 2$ **C** $y = \sqrt{4 - x^2}, 0 \leq x \leq 2$ **D** $y = -\sqrt{4 - x^2}, 0 \leq x \leq 2$

8 **(a)** Show that $y = x^3 - 9x + 6\sqrt{3}$ has turning points at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 12\sqrt{3})$.

(b) Use the 'sum of roots' (covered in section 2.4) to find the other point where the graph cuts the x-axis.

(c) Let $f(x) = x^3 - 9x + 6\sqrt{3}$ for the restricted domain $-\sqrt{3} \le x \le \sqrt{3}$. Sketch the graph of $y = f(x)$ and explain why the inverse function $f^{-1}(x)$ exists.

(d) State the domain and range of $f^{-1}(x)$.

(e) Sketch $y = f^{-1}(x)$.

(f) Find the equation of the tangent to the curve $y = f^{-1}(x)$ at the point where the curve cuts the x-axis.

9 **(a)** Show that $y = e^x - 1$ is a monotonic increasing function.

(b) Show that the graph of $y = e^x - 1$ has a gradient of 1 at the point $(0, 0)$.

(c) Carefully sketch $y = e^x - 1$, $y = x$ and the inverse function on the same diagram.

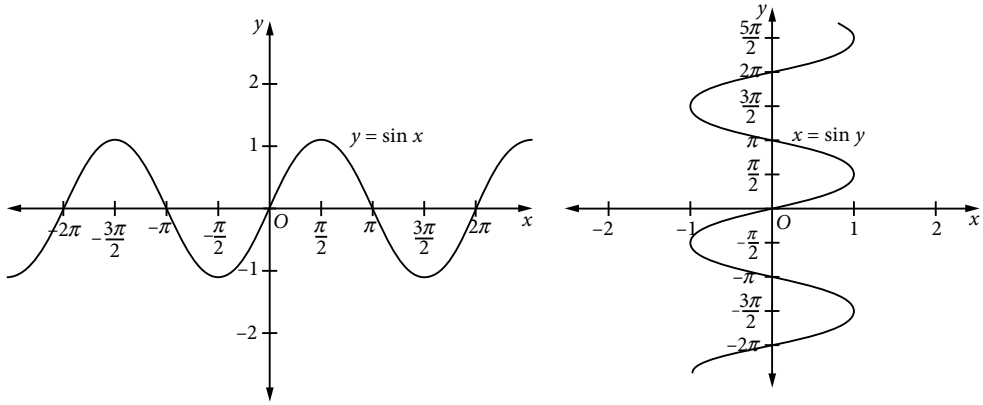
(d) Show that the inverse function is $y = \log_e(x + 1)$.

(e) Solve $\log_e(x + 1) < x$.

5.2 INVERSE TRIGONOMETRIC FUNCTIONS

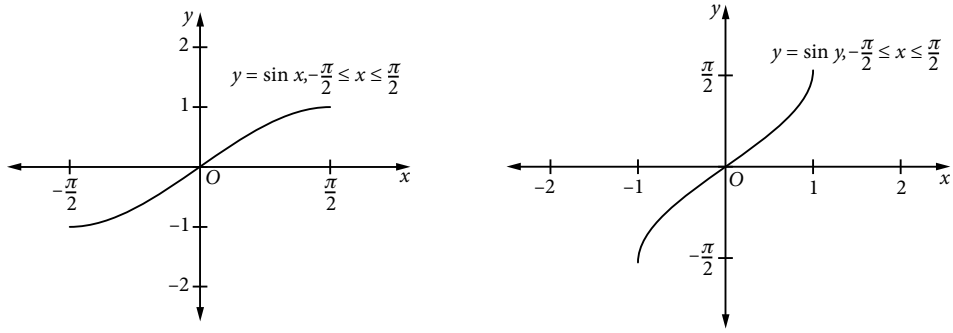
The inverse sine function

The trigonometric function $y = \sin x$ is a many-to-one function (see graph below left). The reflection of $y = \sin x$ in the line $y = x$ is shown below right. It has the equation $x = \sin y$, but clearly it is not a function.



There are infinitely many ways to restrict the domain of $y = \sin x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so you will now consider the function $f(x) = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The graph of this is shown below left, and its reflection can be seen in the line $y = x$ is shown below right.

Note that the reflection is that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

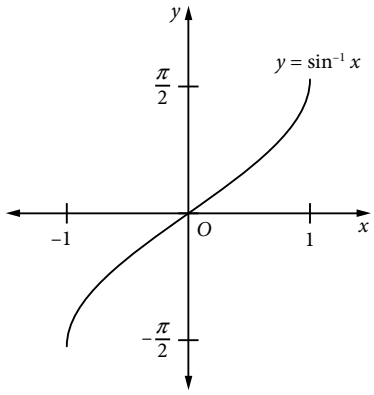


By restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, a one-to-one function is formed, whose inverse is that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Our final concern is how to make y the subject of this inverse function. The answer is to say that $y = \sin^{-1} x$, where $y = \sin^{-1} x$ means 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$'.

Note that this raised number notation $^{-1}$ here is different to $y = \sin^2 x$, where the ^{2} means $y = (\sin x)^2$.
Here $y = \sin^{-1} x$ does *not* mean $y = (\sin x)^{-1}$, as that would be $y = \operatorname{cosec} x$.
Instead, there is a new function, the **inverse sine function**, $y = \sin^{-1} x$ or $y = \arcsin x$.

Note the following features of the inverse sine function:

- increasing function
- domain is $-1 \le x \le 1$
  Remember: 'you can only take $\sin^{-1}$ of values from -1 to 1 inclusive'.
- range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
  Remember: '$\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive'
- vertical tangents at the endpoints
- gradient of 1 at (0, 0)
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function.
  Hence $\sin^{-1}(-x) = -\sin^{-1} x$.



Graph the functions $y = \sin(\sin^{-1} x)$ and $y = \sin^{-1}(\sin x)$ for $-2\pi \le x \le 2\pi$ using graphing software or by completing a table of values.

Note:

- $y = \sin(\sin^{-1} x)$ or $y = \sin(\arcsin x)$ exists only for values of x between -1 and 1 inclusive;
  $y = \sin(\sin^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \sin^{-1}(\sin x)$ or $y = \arcsin(\sin x)$ exists for all real x, but it is equivalent to $y = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
  Its graph has a 'sawtooth' shape.

MAKING CONNECTIONS

Inverse sine functions

Use technology to explore the graphs of the sine function and its inverse.

Example 7

Find the exact values of the following. **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **(c)** $\arcsin 1.2$ **(d)** $\sin^{-1}(\sin 1.2)$ **(e)** $\arcsin\left(\sin\frac{\pi}{4}\right)$ **(f)** $\sin^{-1}(\sin\pi)$

Solution

(a) As $-1 \le \frac{\sqrt{3}}{2} \le 1$, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ exists. It is the number y or angle y^c (i.e. in radians), such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and whose sine is $\frac{\sqrt{3}}{2}$. Hence $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

(b) Similarly, $\sin^{-1}\left(-\frac{1}{2}\right)$ can be evaluated as a number y or angle y^c, such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, whose sine is $-\frac{1}{2}$. Hence $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Alternatively: $\sin^{-1} x$ is an odd function, so $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

(c) 1.2 is not within the domain $-1 \le x \le 1$, so $\arcsin 1.2$ does not exist.

(d) 1.2 is within the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\sin^{-1}(\sin 1.2) = 1.2$.

(e) $\frac{\pi}{4}$ is within the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$.

(f) π is outside the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\sin^{-1}(\sin\pi) \ne \pi$. Instead, $\sin^{-1}(\sin\pi) = \sin^{-1} 0 = 0$.

Example 8

By first considering the domain and range, sketch $y = 3\sin^{-1}(2x-1)$.

Solution

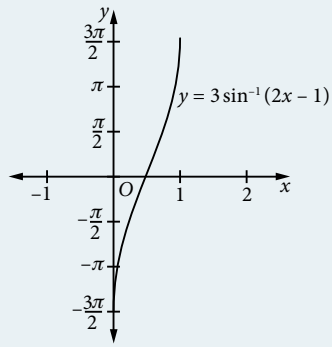
Remember, 'you can only take $\sin^{-1}$ of values from -1 to 1 inclusive', so, $-1 \le 2x-1 \le 1$. Solving this inequality:

$0 \le x \le 1$ is the required domain.

Similarly, '$\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive', so the range of $3\sin^{-1}(2x-1)$ is $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ (i.e. 3 times the range of $y = \sin^{-1}x$).

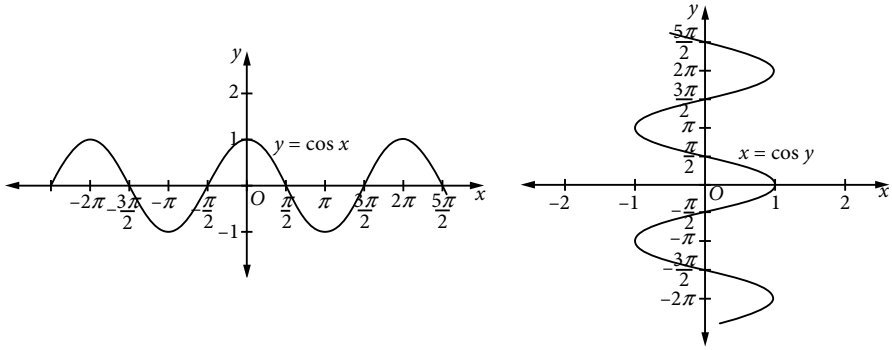
In other words: $-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$

Multiplying by 3: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



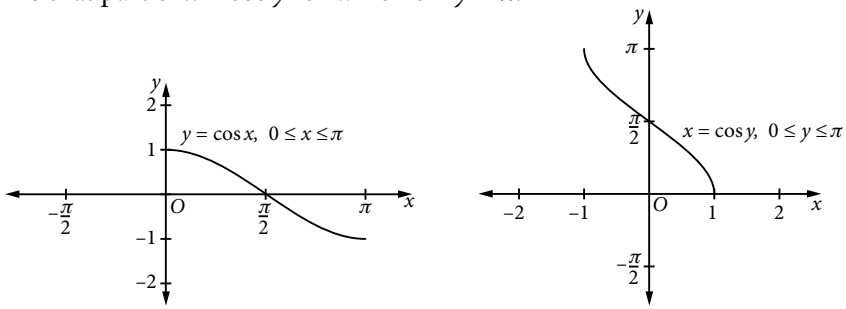
The inverse cosine function

The trigonometric function $y = \cos x$ is a many-to-one function (see graph below left). The reflection of $y = \cos x$ in the line $y = x$ is shown below right. It has the equation $x = \cos y$, but clearly it is not a function.



There are infinitely many ways to restrict the domain of $y = \cos x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $0 \le x \le \pi$, so you will now consider the function $f(x) = \cos x$, $0 \le x \le \pi$. The graph of this is shown below left, and its reflection can be seen in the line $y = x$ is shown below right.

Note that the reflection is that part of $x = \cos y$ for which $0 \le y \le \pi$.

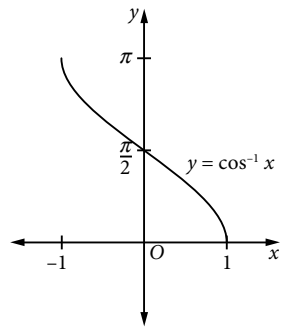


By restricting the domain of $y = \cos x$ to $0 \le x \le \pi$, a one-to-one function is formed, whose inverse is that part of $x = \cos y$ for which $0 \le y \le \pi$.

This function is the **inverse cosine function**, $y = \cos^{-1}x$ or $y = \arccos x$, where $y = \cos^{-1}x$ means 'that part of $x = \cos y$ for which $0 \le y \le \pi$'.

Note the following features of the inverse cosine function:

- decreasing function
- domain is $-1 \le x \le 1$
  Remember: 'you can only take $\cos^{-1}$ of values from -1 to 1 inclusive'
- range is $0 \le y \le \pi$
  Remember: '$\cos^{-1}$ gives values from 0 to π inclusive'
- vertical tangents at the endpoints



- gradient of -1 at $\left(0, \frac{\pi}{2}\right)$
- the function is neither even nor odd, but it does have rotational symmetry about its y-intercept.
- In particular: for any x in the domain, the sum of the function's heights at x and $-x$ will always be π:
  $\cos^{-1}(-x) + \cos^{-1} x = \pi$
  $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Graph the functions $y = \cos(\cos^{-1} x)$ or $y = \cos(\arccos x)$ and $y = \cos^{-1}(\cos x)$ or $y = \arccos(\cos x)$ for $-2\pi \le x \le 2\pi$ using graphing software or by completing a table of values.

Note:

- $y = \cos(\cos^{-1} x)$ exists only for values of x between -1 and 1 inclusive;
  $y = \cos(\cos^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \cos^{-1}(\cos x)$ exists for all real x, but it is equivalent to $y = x$ only for $0 \le x \le \pi$.
  Its graph has a 'sawtooth' shape.

MAKING CONNECTIONS

Inverse cosine functions

Use technology to explore the graphs of the cosine function and its inverse.

Example 9

Find the exact values of the following. **(a)** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **(b)** $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$ **(c)** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(d)** $\arccos\left(\cos\frac{5\pi}{3}\right)$ **(e)** $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ **(f)** $\tan\left(\arccos\left(-\frac{2}{3}\right)\right)$

Solution

(a) **Method 1**

Let $y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Then $\cos y = -\frac{1}{\sqrt{2}}$ and $0 \le y \le \pi$

$\therefore y = \frac{3\pi}{4}$

$\therefore \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Method 2

$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

(b) $\cos(\arccos x) = x$ for $-1 \le x \le 1$, so $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

(c) $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$, so $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

(d) $\frac{5\pi}{3}$ is not in the domain $0 \le x \le \pi$, so $\arccos\left(\cos\frac{5\pi}{3}\right) \ne \frac{5\pi}{3}$

The solution is: $\arccos\left(\cos\frac{5\pi}{3}\right) = \arccos\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$

(e) **Method 1**

$\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin\frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2}$

Method 2

$\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\pi - \cos^{-1}\frac{1}{2}\right)$

$= \sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin\frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2}$

Method 2 here shows a good approach. When you have to take an inverse trigonometric function of a negative value, use the symmetry properties of the inverse trigonometric functions:

$\sin^{-1}(-x) = -\sin^{-1}x \qquad \cos^{-1}(-x) = \pi - \cos^{-1}x \qquad \tan^{-1}(-x) = -\tan^{-1}x$

This process ensures that the function is evaluated with a first quadrant angle.

(f) **Method 1**

Let $\arccos\left(-\frac{2}{3}\right) = \theta$

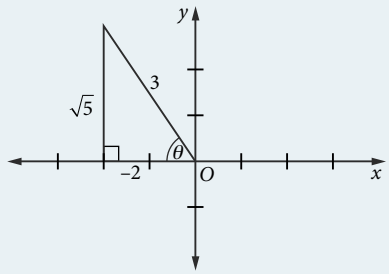
Then $\cos\theta = -\frac{2}{3}$ and $0 \le \theta \le \pi$

So θ is a second quadrant angle.

Need to evaluate:

$\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = \tan\theta$

The graph below shows this:



θ is in the second quadrant, $\cos\theta = -\frac{2}{3}$

Need to find the value of $\tan\theta$.

$\therefore \tan\left(\arccos\left(-\frac{2}{3}\right)\right) = \tan\theta = -\frac{\sqrt{5}}{2}$

Method 2

Use the symmetry properties:

$\arccos\left(-\frac{2}{3}\right) = \pi - \arccos^{-1}\left(\frac{2}{3}\right)$ and $\tan(\pi - \theta) = -\tan\theta$

$\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = \tan\left(\pi - \arccos\frac{2}{3}\right)$

$= -\tan\left(\arccos\frac{2}{3}\right)$

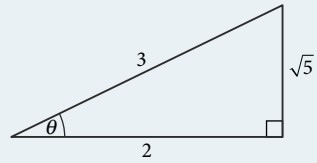
Now find the exact value of $\tan\left(\arccos\frac{2}{3}\right)$.

Let $\theta = \arccos\frac{2}{3}$

$\therefore \cos\theta = \frac{2}{3}$ (where θ is acute)

Now evaluate $\tan\left(\arccos\frac{2}{3}\right) = \tan\theta$.

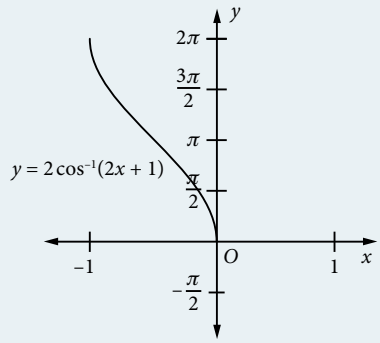
The diagram below shows that if $\cos\theta = \frac{2}{3}$ then $\tan\theta = \frac{\sqrt{5}}{2}$:



Hence $\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{5}}{2}$.

Example 10

State the domain and range of the function $y = 2\cos^{-1}(2x+1)$.
Hence sketch the graph of the function.



Solution

It is only possible to take the inverse cosine of values between −1 and 1 inclusive, so $-1 \le 2x+1 \le 1$.

Solving this, the domain is $-1 \le x \le 0$.

As the inverse cosine gives values between 0 and π inclusive:

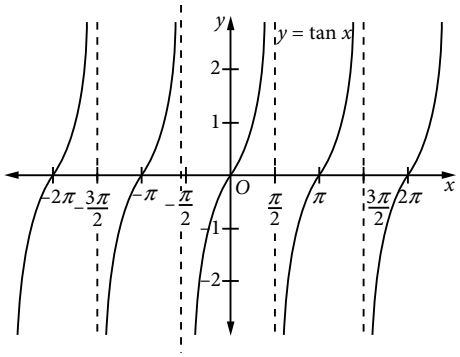
$0 \le \cos^{-1}(2x+1) \le \pi$

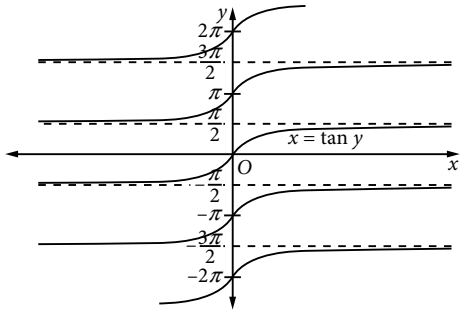
$0 \le 2\cos^{-1}(2x+1) \le 2\pi$

$\therefore$ range is $0 \le y \le 2\pi$

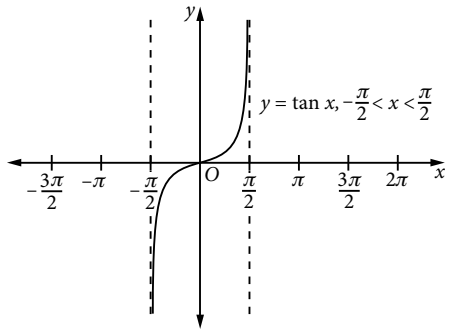
The inverse tangent function

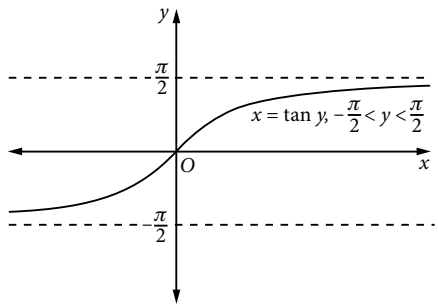
The trigonometric function $y = \tan x$ is a many-to-one function (see graph below left). The reflection of $y = \tan x$ in the line $y = x$ is shown below right. It has the equation $x = \tan y$, but clearly it is not a function.





There are infinitely many ways to restrict the domain of $y = \tan x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so you will now consider the function $f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
The graph of this is shown below left, and its reflection in the line $y = x$ is shown below right.
Note that the reflection is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

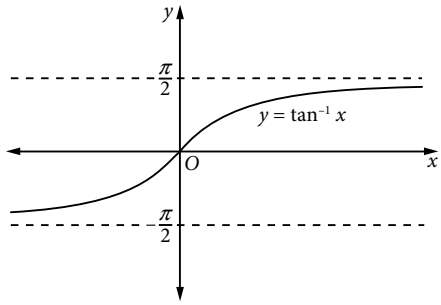




By restricting the domain of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, a one-to-one function is formed, whose inverse is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
This function is the **inverse tangent function**, $y = \tan^{-1} x$, or $y = \arctan x$, where $y = \tan^{-1} x$ means 'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

Note the following features of the inverse tangent function:

- increasing function
- domain is all real x
  Remember: 'you can take $\tan^{-1}$ of any real value'
- range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$
  Remember: '$\tan^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ not inclusive'
- horizontal asymptotes at $y = \pm\frac{\pi}{2}$
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function.
  Hence $\tan^{-1}(-x) = -\tan^{-1} x$.



Graph the functions $y = \tan(\tan^{-1} x)$ and $y = \tan^{-1}(\tan x)$ for $-2\pi \le x \le 2\pi$ using graphing software or by completing a table of values.

Note:

- $y = \tan(\tan^{-1} x)$ or $y = \tan(\arctan x)$ is equivalent to $y = x$ for all real x.
- $y = \tan^{-1}(\tan x)$ or $y = \arctan(\tan x)$ exists for all real x except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
  It is equivalent to $y = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Its graph looks like an infinite set of parallel intervals with open circles on each end.

MAKING CONNECTIONS

Inverse tangent functions

Use technology to explore the graphs of the tangent function and its inverse.

Example 11

Find the exact values of the following.

(a) $\arctan\left(\frac{1}{\sqrt{3}}\right)$ **(b)** $\tan(\tan^{-1} 1)$ **(c)** $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$ **(d)** $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$

Solution

(a) $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. It is the value of θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive), for which $\tan\theta = \frac{1}{\sqrt{3}}$.

(b) $\tan(\tan^{-1} 1) = 1$ **(c)** $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, as $\frac{\pi}{3}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive).

(d) $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$ is not equal to $-\frac{4\pi}{3}$, because $-\frac{4\pi}{3}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Using the symmetry properties that $\tan\theta$ and $\tan^{-1} x$ are odd functions, and that $\tan(\pi + \theta) = \tan\theta$:

$$\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right) = \tan^{-1}\left(-\tan\left(\frac{4\pi}{3}\right)\right) = -\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right) = -\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$$

Example 12

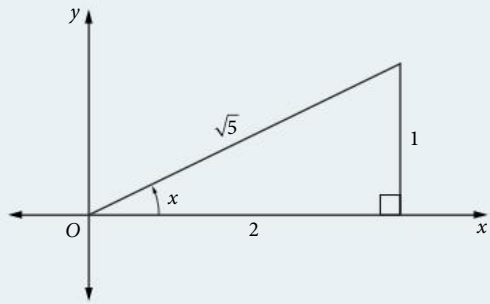
Find the exact value of $\sin\left(2\tan^{-1}\frac{1}{2}\right)$.

Solution

Let $\tan^{-1}\frac{1}{2} = x$

Thus $\tan x = \frac{1}{2}$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Hence x can be represented as a first quadrant angle.



Then: $\sin\left(2\tan^{-1}\frac{1}{2}\right) = \sin 2x$

$= 2\sin x \cos x$

$= 2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$

$= \frac{4}{5}$

Example 13

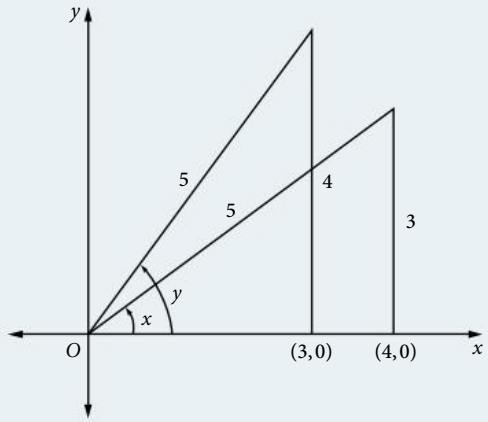
Find $\sin\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\left(-\frac{4}{3}\right)\right]$.

Solution

$$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]=\sin\left[\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{4}{3}\right]$$

$$=\sin(x-y) \text{ where } x=\cos^{-1}\frac{4}{5} \text{ and } y=\tan^{-1}\frac{4}{3}$$

$$=\sin\left(\cos^{-1}\frac{4}{5}\right)\cos\left(\tan^{-1}\frac{4}{3}\right)-\cos\left(\cos^{-1}\frac{4}{5}\right)\sin\left(\tan^{-1}\frac{4}{3}\right)$$



Using expansion of $\sin(x-y)$:

$\cos^{-1}\frac{4}{5}=x$, so $\cos x=\frac{4}{5}$ and $0\le x\le\pi$

$\tan^{-1}\frac{4}{3}=y$, so $\tan y=\frac{4}{3}$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Hence both x and y can be represented as first quadrant angles:

$$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]=\sin x\cos y-\cos x\sin y$$

$$=\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)$$

$$=-\frac{7}{25}$$

Example 14

Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$.

Solution

Let $\alpha=\sin^{-1}x \quad \therefore \sin\alpha=x$ where $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$

Recall that $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$, so $\cos\left(\frac{\pi}{2}-\alpha\right)=x$

Also, as $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, thus $0\le\frac{\pi}{2}-\alpha\le\pi$

$\therefore \frac{\pi}{2}-\alpha=\cos^{-1}x$ (noting that $\theta=\cos^{-1}x$ only when $\cos\theta=x$ and $0\le\theta\le\pi$)

$\therefore \frac{\pi}{2}-\sin^{-1}x=\cos^{-1}x$ so $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

You should remember this result: $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$

This can be proved by an alternative method involving differentiation which will be looked at later in section 11.7 (page 252).

You could also verify it by graphing $y=\sin^{-1}x+\cos^{-1}x$.

Example 15

Solve $2\sin^{-1}x+\cos^{-1}x=\frac{\pi}{3}$.

Solution

Using the result of Example **14**, the equation becomes: $2\sin^{-1}x+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{3}$

$$\sin^{-1}x=-\frac{\pi}{6}$$

$$\therefore \quad x=\sin\left(-\frac{\pi}{6}\right)$$

$$\therefore \quad x=-\frac{1}{2}$$

EXERCISE 5.2 INVERSE TRIGONOMETRIC FUNCTIONS

1 $\cos^{-1}x - \cos^{-1}(-x) = \ldots$

A π **B** $2\cos^{-1}x - \pi$ **C** $2\cos^{-1}x$ **D** 0

2 Find the exact values of the following (if they exist).

(a) $\arcsin 1$ **(b)** $\arcsin 0$ **(c)** $\arcsin(-1)$ **(d)** $\arcsin\frac{\sqrt{3}}{2}$ **(e)** $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ **(f)** $\arccos 1$

(g) $\arccos 0$ **(h)** $\arccos(-1)$ **(i)** $\arccos\frac{1}{2}$ **(j)** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **(k)** $\arctan 1$ **(l)** $\arctan(-1)$

(m) $\tan^{-1}\sqrt{3}$ **(n)** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **(o)** $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$ **(p)** $\sin^{-1}(\sin 80°)$ **(q)** $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(r) $\sin^{-1}\left(\sin\left(-\frac{3\pi}{4}\right)\right)$ **(s)** $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$ **(t)** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(u)** $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ **(v)** $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

3 Evaluate the following.

(a) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ **(b)** $\tan\left(\tan^{-1}\left(-\frac{5}{13}\right)\right)$ **(c)** $\tan^{-1}(\tan 245°)$ **(d)** $\cos^{-1}(\cos 540°)$

(e) $\cos\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$ **(f)** $\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right)$ **(g)** $\cos\left(2\cos^{-1}\frac{5}{13}\right)$ **(h)** $\sec\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$

4 Show that:

(a) $\tan^{-1}4 - \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$

(b) $\sin^{-1}\frac{3}{5} + \sin^{-1}\left(-\frac{3}{5}\right) = 0$

(c) $\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13} = \frac{\pi}{2}$

(d) $\cos^{-1}\frac{3}{5} - \tan^{-1}\left(-\frac{3}{4}\right) = \frac{\pi}{2}$

(e) $2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25}$

(f) $\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{3}{5}$

(g) $\sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63} = \cos^{-1}\frac{4}{5}$

5 Find the exact values of the following.

(a) $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right] + \sin\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$

(b) $\sin\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\left(-\frac{3}{5}\right)\right]$

(c) $\cos\left[\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}\right]$

(d) $\sin\left(2\tan^{-1}\frac{4}{3}\right)$

(e) $\cos\left[\tan^{-1}\frac{4}{3} - \cos^{-1}\frac{5}{13}\right]$

(f) $\sin\left[\cos^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{3}{4}\right)\right]$

(g) $\tan\left[\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{12}{13}\right]$

6 Prove that: **(a)** $\sin^{-1}(x) = \cos^{-1}\sqrt{1-x^2}$, $0 \le x \le 1$ **(b)** $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, $0 \le x \le 1$

7 Sketch graphs of each of the following, stating their domain and range.

(a) $y = 2\sin^{-1}x$ **(b)** $y = 2\sin^{-1}3x$ **(c)** $y = 3\sin^{-1}(2x+1)$ **(d)** $y = 3\sin^{-1}2x + 1$

(e) $y = 2\tan^{-1}x$ **(f)** $y = 2\tan^{-1}(x-1)$ **(g)** $y = 2\cos^{-1}\left(\frac{x}{2}\right)$

(h) $y = 2\sin^{-1}(-x)$ **(i)** $y = \sin(\cos^{-1}x)$, by first simplifying $\sin(\cos^{-1}x)$

8 Consider the functions $f(x) = 2\sin^{-1}(2x-1)$ and $g(x) = 2\sin^{-1}(1-2x)$.

(a) Find the domain and range of each function.
(b) Show that $g(x) = -f(x)$.
(c) Sketch the graphs of the two functions.

9 Solve the following equations. For part **(c)**, answer correct to two decimal places and check your solution by substituting values back into the original equation.

(a) $\sin^{-1}x\cos^{-1}x = 0$ **(b)** $\sin^{-1}(1-x) + 2\cos^{-1}(x-1) = \frac{\pi}{2}$ **(c)** $\sin^{-1}x\cos^{-1}x = -1$

10 Solve the following pairs of simultaneous equations.

(a) $2\sin^{-1}x+\cos^{-1}y=-\frac{\pi}{12}$ and $\sin^{-1}x-2\cos^{-1}y=-\frac{2\pi}{3}$

(b) $\sin^{-1}x+\cos^{-1}y=-\frac{\pi}{12}$ and $\cos^{-1}x-\sin^{-1}y=\frac{5\pi}{12}$

11 **(a)** Show that $\sin(\sin^{-1}x-\cos^{-1}x)=2x^2-1$ for $-1\le x\le 1$.

(b) Show that if $-1\le x\le 1$ then $-1\le 2x^2-1\le 1$.

(c) Hence solve $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(5x-4)$.

12 Prove that $f(x)=\tan(\cos^{-1}x)$ is an odd function.

13 If $f(x)=3\cos^{-1}\left(\frac{x}{2}\right)$, determine the inverse function f^{-1} and specify the domain and range of f^{-1}.

14 **(a)** Sketch the graph of $y=\sin^{-1}x+\cos^{-1}x$.

(b) Hence find $\int_{-1}^{1}\left(\sin^{-1}x+\cos^{-1}x\right)dx$.

CHAPTER REVIEW 5

1 **(a)** Find the vertex and intercepts of the parabola $y=x^2-2x-8$.

(b) State the restricted domain (containing $x=0$) for which $f(x)=x^2-2x-8$ has an inverse function.

(c) Find the equation of the inverse function $f^{-1}(x)$.

(d) Sketch $y=f(x)$ and $y=f^{-1}(x)$ on the same graph.

(e) Show that $y=f(x)$ and $y=f^{-1}(x)$ intersect where $x^2-3x-8=0$. Find the exact value of α, the x-coordinate of the point of intersection T.

(f) Find the gradient of $f^{-1}(x)$ at T.

2 Find the exact value of the following: **(a)** $\tan^{-1}\sqrt{3}$ **(b)** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **(c)** $\sin(\sin^{-1}1)$

(d) $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$ **(e)** $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ **(f)** $\sin\left(2\tan^{-1}\frac{1}{3}\right)$ **(g)** $\tan(2\cos^{-1}(-0.4))$

3 Show that: **(a)** $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}2$ **(b)** $\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{1}{2}\right)=\sin^{-1}\left(\frac{2\sqrt{5}}{25}\right)$

4 For each of the following, state the domain and range and sketch the graph.

(a) $y=2\cos^{-1}\frac{x}{2}$ **(b)** $y=\frac{1}{2}\sin^{-1}(2x-1)$ **(c)** $y=\frac{1}{2}\sin^{-1}(1-2x)$ (Hint: $1-2x=-[2x-1]$)

(d) $y=\frac{\pi}{2}-\sin^{-1}2x$ **(e)** $y=\cos(\sin^{-1}x)$

5 Find the equation of the normal to the curve $y=\tan^{-1}(2x+1)$ at the point on the curve where $y=-\frac{\pi}{4}$.

6 Sketch the graph of the inverse function of $y=3\cos^{-1}\frac{x}{2}$.

7 **(a)** Sketch both $y=\sin^{-1}x$ and $y=\cos^{-1}x$ on the same number plane.

Hence determine the domain of the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

(b) Find the intercepts of the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ with the coordinate axes.

(c) Show that the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ is increasing for all x in its domain.

(d) Sketch the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

CHAPTER 6
Permutations and combinations

6.1 FUNDAMENTAL COUNTING PRINCIPLE

If an outcome can happen in m different ways, and a second outcome can happen in n different ways, then the total number of ways in which the two outcomes can happen together is $m \times n$.

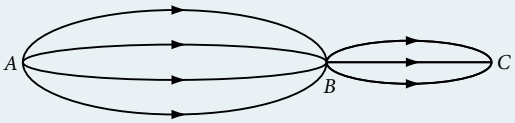
Total number of ways $= m \times n$ where m = number of ways first outcome can happen
n = number of ways second outcome can happen

For example, this simply means that if you have 5 different shirts and 7 different hats, then the total number of different selections of a shirt and hat is $5 \times 7 = 35$.

This is known as the **multiplication principle**.

Example 1

There are four roads from town A to town B, and three roads from town B to town C.
How many different ways are there to travel by road from A to B to C?



Solution

Consider the first road from A to B. After this road there are three ways to travel from B to C.

Similarly, after taking the second road from A to B there are then three ways to go from B to C.

As there are 4 different ways from A to B, and after each of these there are 3 different ways to go from B to C, there are in total $4 \times 3 = 12$ ways to go from A to B to C (i.e. from A to C via B).

Example 2

The old New South Wales black-and-yellow number plates showed three letters and three digits. How many different number plates could be made?

Solution

There are 26 letters of the alphabet and 10 digits to choose from.

Number of different number plates $= 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3 = 17\,576\,000$

Example 3

In how many different ways can six people A, B, C, D, E, F be arranged:

(a) in a row of 3 **(b)** in a row of 6?

Solution

(a) There is a choice of 6 people for the first place.

There is a choice of 5 people for the second place, as one person already occupies the first place.

There is a choice of 4 people for the third place, as two people have already been used.

Hence there are $6 \times 5 \times 4 = 120$ different ways to fill the three places in the row.

This problem could also be considered using a diagram to represent the places:

6	5	4

Each number shows how many ways there are to fill each place.

Using the fundamental counting principle, this gives $6 \times 5 \times 4 = 120$ different ways.

(b) The same method as in part **(a)**, continued for six places:

6	5	4	3	2	1

Number of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Factorial notation

Factorial notation uses an exclamation mark **!** as a short way to write the product of consecutive positive integers. The answer to Example **3(b)** above can be written as 6!, which means the product of the consecutive integers from 6 to 1.

6! is read as '6 factorial' or 'factorial 6'. In general, the product of n consecutive positive integers from n to 1 is written $n!$ and read as 'n factorial'.

$$n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$$

Make sure you know how to use the $n!$ key on your calculator.

Example 4

A group of 6 boys and 5 girls decide to go to the movies.

(a) In how many ways can the boys sit together in a row?

(b) In how many ways can the girls sit together in a row?

(c) In how many ways can the whole group sit together in a row?

(d) Why is the answer to part **(c)** not the product of the answers in parts **(a)** and **(b)**?

Solution

(a) The 6 boys can sit together in 6! ways, i.e. 720 ways.

(b) The 5 girls can sit together in 5! ways, i.e. 120 ways.

(c) The group of 11 people can sit together in 11! ways, i.e. 39 916 800 ways.
(*Note:* This answer is best written simply as '11!' rather than as '39 916 800'.)

(d) In part **(c)** there is no grouping by gender, so 11 different people are being arranged in one big group. But multiplying the answers to parts **(a)** and **(b)** would keep the two groups separate, only finding half the number of ways a group of 6 people can sit in a row with a group of 5 people.

EXERCISE 6.1 FUNDAMENTAL COUNTING PRINCIPLE

1 There are five roads from town A to town B, and two roads from town B to town C. In how many different ways can you travel by road from A to B to C?

2 A man has three pairs of shoes, four suits and six ties. How many different sets of shoes, suits and ties can he wear?

3 In how many ways can seven books be arranged in a row?

4 A Mathematics test contains 20 multiple-choice questions. Each question has four possible answers, *A*, *B*, *C* and *D*. If a student guesses every answer, in how many different ways can the answers be given?

5 A restaurant menu has three choices of soup, five choices of main course and three choices of dessert. How many different meals of soup, main course and dessert are possible?

6 There are 10 candidates for school captain and vice-captain. The number of different ways they might be selected is: **A** 90 **B** 45 **C** 10 **D** 9

7 New South Wales black-and-white number plates consist of three letters, two digits and one letter. How many different number plates can be made?

8 How many arrangements of the letters of the word PENCIL are possible?

9 In how many different ways can *A*, *B*, *C*, *D*, *E* be arranged:

(a) in a row of three **(b)** in a row all together?

10 The newer New South Wales black-and-yellow number plates consist of two letters, two digits and two letters.

(a) How many different number plates can be made?

(b) What is the reason for changing the number plates from three letters and three digits to two letters, two digits, two letters?

11 The Olympic teams from eight countries are swimming in the 200-metre relay final. How many different finishing results are possible? (Assume no tied results.)

12 The digits 0 to 9 are used to make 10-digit numbers (not beginning with zero). How many different numbers are possible if:

(a) each digit can be used only once **(b)** each digit can be used any number of times?

6.2 PIGEONHOLE PRINCIPLE

The **pigeonhole principle** is a way of understanding how a number of items can be placed into a number of containers. Traditionally, mathematicians have illustrated this concept by using the example of placing pigeons into pigeonholes. For example, if you have five pigeons and four pigeonholes, then one of the pigeonholes must contain at least two pigeons.

If you place one pigeon in each pigeonhole, then the only way that you can place the remaining pigeon is to put it in with one of the pigeons already in a pigeonhole.

If $(n + 1)$ items occupy n containers, then at least one of the containers must contain at least two items.

Proof

If each container contains at most one item, then n containers will contain n items.

Thus any additional items must go into a container already containing an item.

Hence the statement is true: at least one of the containers must contain at least two items.

This result may be extended to:

If n items are sitting in k containers, where $n > k$, then there is at least one container with at least $\frac{n}{k}$ items in it.

Proof

Assume there are no containers that have at least $\frac{n}{k}$ items. Hence every container has fewer than $\frac{n}{k}$ items.

Total number of items $< \frac{n}{k} \times k = n$, as there are k containers.

This says that the number of items is less than n, which contradicts the initial statement that there were n items.

Hence the assumption 'assume there are no containers that have at least $\frac{n}{k}$ items' must be false.

Thus, there is at least one container with at least $\frac{n}{k}$ items in it.

Another example of this principle is that if you pick up four gloves, then you must have either at least two left-hand gloves or at least two right-hand gloves: $\frac{4}{2} = 2$. The actual set of possible outcomes is {4L, 3L1R, 2L2R, 1L3R, 4R}, which verifies the statement.

Example 5

(a) You have a drawer full of red and blue socks that have not been arranged in pairs. Socks can be worn on either foot. What is the least number of socks that you must take from the drawer to be sure that you have one pair of the same colour?

(b) Explain why any 27-word sequence in this book must have at least two words that start with the same letter.

Solution

(a) In this example the socks are the items and the colour of the socks are the containers. Thus there are two containers, so you need three items to be sure that you have two items in the same container.

Hence you must select three socks from the drawer to be sure that you have one pair of the same colour. You would have either three socks of the same colour or two socks of the same colour and the other sock the remaining colour.

(b) There are 27 words (items) that can start with one of the 26 different letters of the alphabet (containers). By the pigeonhole principle, two of the words must start with the same letter.

Example 6

There are seven pigeons sitting in three pigeonholes.

Explain why one of the pigeonholes must contain at least three pigeons.

Solution

$n = 7$, $k = 3$, so $\frac{n}{k} = \frac{7}{3} = 2\frac{1}{3}$

Hence one pigeonhole must contain at least $2\frac{1}{3}$ pigeons, but since part-pigeons are impossible, it must contain three pigeons.

EXERCISE 6.2 PIGEONHOLE PRINCIPLE

1 How many cards must be selected from a standard pack of 52 playing cards to make sure that you have two cards of the same suit?

2 If you select five cards from a standard pack of 52 playing cards, at least how many must be of the same suit?

3 There are three pairs of socks in a drawer, coloured brown, grey and black. They are not paired up. How many socks must be selected from the drawer to be sure that you have a pair of the same colour?

4 Six pairs of shoes, of different colours and styles, have been thrown into the bottom of my wardrobe. It is dark. What is the minimum number of shoes I need to take from the wardrobe to be sure I have a matching pair?

5 Twenty-five students attend a class reunion and shake hands with each other. If no students shakes hands with the same person twice, explain why two students will each have shaken the same number of hands.

6 There are 400 students attending a Senior College. Explain why at least two of them will celebrate their birthday on the same day.

7 **(a)** When taking pairs of numbers from the integers 1 to 8, list all the pairs of integers that add up to 9.
(b) How many numbers do you need to select from the integers 1 to 8 to be sure that you have a pair of numbers that add up to 9?

8 Prove that in a group of 13 people there are two people who were born in the same month.

6.3 PERMUTATIONS

A **permutation** is an ordered selection or arrangement of all or part of a set of objects.

It is very important to remember that permutations are used *when the **order** is important!*

Example **3(b)** above is an example of the number of arrangements of a set of six objects. Example **3(a)** is an example of the number of arrangements of three members of a set of six objects. Note that in general:

- the number of arrangements of n different objects is $n!$
- the number of arrangements of three objects taken from a set of n different objects is $n(n-1)(n-2)$.

Consider the following statement: $n(n-1)(n-2) = \dfrac{n(n-1)(n-2)(n-3)(n-4)\times\ldots\times 3\times 2\times 1}{(n-3)(n-4)\times\ldots\times 3\times 2\times 1} = \dfrac{n!}{(n-3)!}$

Example 7

From a standard pack of 52 playing cards, the set of hearts (♥) are taken. In how many ways:

(a) can the 13 cards be arranged in a row

(b) can 6 of the 13 cards be arranged in a row?

Solution

(a) 13 cards can be arranged in 13! ways.

(b) 6 out of 13 cards can be arranged in $13\times 12\times 11\times 10\times 9\times 8 = 1\,235\,520$ ways.

This could be written as $\dfrac{13!}{(13-6)!} = \dfrac{13!}{7!} = 1\,235\,520$ ways.

Number of permutations ${}^nP_r$

The symbol ${}^nP_r$ means the number of arrangements (i.e. ordered subsets, permutations) of any n different objects taken r at a time (or in other words, the arrangements of r objects taken from a group of n objects). This denotes the number of ways of filling r places with the n different objects that are available ($r \leq n$).

Using the diagram notation from Example **3** above:

${}^nP_r =$	n	$n-1$	$n-2$	$\ldots$	$n-r+1$

The first place can be filled in n ways, because any one of the n objects can occupy this place. When the first place has been filled in any one of these ways, there remain $(n-1)$ objects, any one of which can occupy the second place. By the fundamental counting principle, each way of filling the first place can be associated with each way of filling the second place, so the number of ways of filling the first two places is $n(n-1)$.

Similarly, after the first two places have been filled in any one of these ways there remain $(n-2)$ objects, any one of which can occupy the third place. Hence the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing this pattern, the number of ways of filling r places is given by:

$${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!} \qquad [1]$$

$$\therefore \ {}^nP_r = \frac{n!}{(n-r)!} \qquad [2]$$

Similarly, note that the symbol ${}^nP_n$ would denote the number of ways of filling n places with n objects available:

From [1]: ${}^nP_n = n(n-1)(n-2)\ldots(n-n+1)$

$= n(n-1)(n-2)\times\ldots\times 3\times 2\times 1 = n!$

But from [2]: ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$

Hence you have: $n! = \frac{n!}{0!}$

For consistency, 0! is therefore defined to be equal to 1: $0! = 1$

Example 8

(a) In how many ways can a first, second and third prize be awarded in a class of 10 students?

(b) In how many ways can a Mathematics prize, a Physics prize and a Chemistry prize be awarded in a class of 10 students?

Solution

(a) This question is asking us to find how many different arrangements of three students can be selected from 10 students.

Number of ways $= {}^{10}P_3 = 10\times 9\times 8 = 720$

(b) In this question each student could win any or all of the prizes, so permutations are not used.

Number of ways $= 10\times 10\times 10 = 1000$

EXPLORING FURTHER

Permutations

Use technology to explore the calculation of numbers of permutations.

Arrangements with restrictions

Example 9

In how many ways can six students and two teachers be arranged in a row if:

(a) the two teachers are together **(b)** the two teachers are not together

(c) there are at least three students separating the teachers?

Solution

(a) If the teachers are together they can be regarded as one 'unit'. Thus there are seven 'units' (six students plus the unit of two teachers) to be arranged in a row. This can be done in 7! ways.

However, the two teachers (label them A and B) can be arranged among themselves in 2! (i.e. two) ways, AB or BA.

$\therefore$ Number of arrangements $= 2!\times 7! = 10\,080$

(b) Number of arrangements without restriction = 8!

Number of arrangements with the teachers together = 2! × 7!

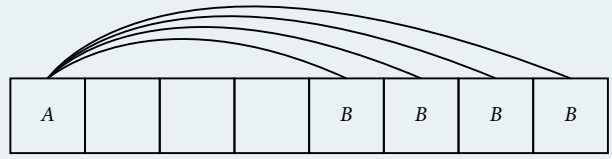
∴ Number of arrangements with the teachers not together = 8! − 2! × 7!

= 8 × 7! − 2 × 7!

= 6 × 7! = 30 240

(c) There is a restriction on the places to be occupied by the teachers, so it is easiest to arrange the teachers first.

If teacher *A* is in the first position then teacher *B* can occupy the fifth, sixth, seventh or eighth position, as shown:



Similarly, for all the possible first positions of *A* there are:

- 4 arrangements of *A* and *B* with *A* in first position
- 3 arrangements of *A* and *B* with *A* in second position
- 2 arrangements of *A* and *B* with *A* in third position
- 1 arrangement of *A* and *B* with *A* in fourth position
- 1 arrangement of *A* and *B* with *A* in fifth position
- 2 arrangements of *A* and *B* with *A* in sixth position
- 3 arrangements of *A* and *B* with *A* in seventh position
- 4 arrangements of *A* and *B* with *A* in eighth position.

∴ There are 20 different ways of arranging *A* and *B* (i.e. the sum of the above arrangements).

For each of these 20 arrangements there are six places left, which can be filled by the students in 6! ways.

∴ Number of arrangements (total) = 20 × 6! = 14 400

Example 10

How many different arrangements of the letters of the word TUESDAY are possible if:

(a) the three vowels are together **(b)** the first and last places are consonants

(c) the consonants and vowels occupy alternate positions?

Solution

The vowels are A, E, U; the consonants are D, S, T, Y.

(a) As the three vowels are together they can be regarded as one unit.

Thus there are five units (the four consonants plus the group of vowels) to be arranged in a row. This can be done in 5! ways.

Also, the three vowels can be arranged in 3! ways.

∴ Total number of arrangements = 3! × 5! = 720

(b) There are restrictions on the first and last places, so fill them first.

- There are four consonants, so the first place can be filled in 4 ways.
- After the first place is filled, the last place can be filled in 3 ways.
- The remaining five places can be filled without restriction in 5! ways.

4	5	4	3	2	1	3

∴ Total number of arrangements = 4 × 3 × 5! = 1440

(c) The consonants (C) and vowels (V) occupy alternate positions, as shown in the table on the right. There are more consonants than vowels, so the consonants must occupy the end places.

C	V	C	V	C	V	C

The four consonants can fill their places in 4! ways. The three vowels can fill their places in 3! ways.

$\therefore$ Total number of arrangements $= 4! \times 3! = 144$

Arrangements in a circle

The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, can be shown to be:

$$\frac{n!}{n} = (n-1)!$$ (This is shown in Example **11** below.)

In a circular arrangement there is no start or finish, so to count the arrangements one object needs to be fixed as the 'starting' object.

Example 11

In how many ways can the four people, Alexia, Bronwyn, Chanda and Divya, be arranged in a circle?

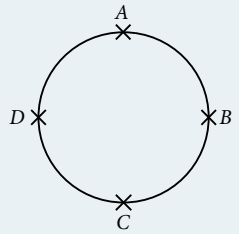
Solution

Method 1

There is no first place to fill, so any one person can be fixed as the 'starting' person (or the 'fixed position') and the other three people arranged around her. This can be done in 3! ways.

Method 2

The diagram at right shows an arrangement of the four people (labelled with the first letter of each name, A, B, C, D).



If A, B, C, D are kept in the same position relative to each other and then moved all one place clockwise, they still have the same arrangement—the arrangement has not changed.

If A, B, C, D were arranged in a straight line and they were all moved along one position to the right (so that the fourth position moved to the first position), then the arrangement would have changed.

A	B	C	D

D	A	B	C

C	D	A	B

B	C	D	A

You should be able to see that for every one arrangement of the group in a circle, there are four different arrangements of the same group in a straight line.

$\therefore$ Number of circular arrangements = Number of linear arrangements ÷ Number of elements

$$= \frac{4!}{4} = 3! = 6$$

Mutually exclusive operations—the addition principle

In Example **9(c)** above there are two teachers A and B seated in a row and separated by at least three students. A could occupy the first position and B the fifth position *or* A the first and B the sixth *or* A the first and B the seventh, and so on. Arrangements like these are said to be **mutually exclusive**, because they cannot both be true at the same time. For example, 'A is first and B is fifth' excludes the possibility that 'A is first and B is sixth'—these two descriptions cannot both be true at the same time.

Example 12

How many numbers greater than 6000 can be formed using the digits 3, 4, 6, 8, 9 if no digit can be used more than once per number?

Solution

Numbers containing either four digits *or* five digits can be formed.

For four-digit numbers, the first place can be filled in three ways by either 6, 8 or 9.

After the first place is filled (in any one of these three ways), four digits remain, any one of which can occupy second place. Following the multiplication principle, the second place can be filled in four ways, the third place in three ways and the fourth place in two ways:

∴ Number of arrangements $= 3 \times 4 \times 3 \times 2 = 72$

3	4	3	2

or Number of arrangements $= 3 \times {}^4P_3 = 3 \times \dfrac{4!}{(4-3)!} = 72$

For five-digit numbers the first place can be filled in five ways, the second in four ways, and so on.

∴ Number of arrangements $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

5	4	3	2	1

or Number of arrangements $= {}^5P_5 = \dfrac{5!}{(5-5)!} = 120$

You cannot form a four-digit number and a five-digit number at the same time. These two operations are mutually exclusive.

∴ Total number of arrangements = 4-digit arrangements + 5-digit arrangements $= 72 + 120 = 192$

EXERCISE 6.3 PERMUTATIONS

1 Simplify: **(a)** ${}^5P_3$ **(b)** ${}^4P_2$ **(c)** ${}^6P_5$ **(d)** ${}^8P_4$ **(e)** ${}^9P_1$ **(f)** ${}^{10}P_{10}$ **(g)** ${}^nP_2$ **(h)** ${}^nP_4$

2 Simplify: **(a)** 5! **(b)** 4! **(c)** 8! **(d)** 7! **(e)** $\dfrac{5!}{2!}$ **(f)** $\dfrac{7!}{6!}$ **(g)** $\dfrac{10!}{8!}$ **(h)** $\dfrac{n!}{(n-2)!}$

3 How many different arrangements can be made using three of the letters of the word SUNDAY?

4 In how many different ways can five people be arranged in a row?

A 20 **B** 60 **C** 120 **D** 720

5 In how many ways can a first, second and third prize be awarded in a class of eight students?

6 How many different arrangements of the letters of the word MINOR are possible if:

(a) the two vowels are next to each other **(b)** the first and last letters are consonants?

7 In how many ways can four different consonants and four different vowels be arranged in a row, if:

(a) the four vowels are together **(b)** the first and last letters are consonants

(c) the vowels and consonants occupy alternate positions?

8 In how many ways can four girls and three boys be arranged in a row, so that:

(a) the boys are always together **(b)** the girls and boys occupy alternate places?

9 How many five-digit numbers can be formed using the digits 2, 3, 5, 6, 8, 9 if no digit can be used more than once per number? How many even numbers can be formed?

10 Five different magazines and four different books are arranged in a row with the books standing together. Indicate whether each statement below is a correct or incorrect step in the calculation of the total number of arrangements.

(a) $5! \times 5!$ **(b)** $6! \times 4!$ **(c)** 9! **(d)** 17 280

11 Five swimmers and five runners are to be seated alternately in a row (i.e. alternating so that no two swimmers or two runners sit next to each other). In how many different ways can this be done?

12 How many four-digit even numbers can be formed using the digits 3, 4, 7, 8 if:

(a) no digit is repeated **(b)** digits may be repeated?

13 How many numbers greater than 4000 can be formed using the digits 3, 5, 7, 8, 9 if repetition is not allowed?

14 How many numbers of seven digits can be formed using the digits 1, 2, 3, 4, 5, 6, 7 if:

(a) the numbers all begin with 1 and end with 2 **(b)** the numbers are all odd?

15 In how many ways can five different Mathematics books, four different Physics books and two different Chemistry books be arranged on a shelf if the books in each subject must be together?

16 In how many ways can three doctors, three nurses and three patients be arranged in a row if the three patients must be together?

17 Find the number of possible arrangements of the letters in the word PENCILS if:

(a) 'E' is next to 'I' **(b)** 'E' comes before 'I' **(c)** there are three letters between 'E' and 'I'.

18 How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5 if the same digit can be used more than once per number and each number is:

(a) divisible by 5 **(b)** an even number **(c)** an odd number **(d)** divisible by 4?

19 A car holds three people in the front seat and four in the back seat. In how many ways can seven people be seated in the car if two particular people must sit in the back seat and one particular person is the driver?

20 If ${}^6P_r = 120$, find the value of r.

21 In how many ways can five lawyers and three judges be arranged in a circle if the judges are always to stand together?

22 In how many ways can six people be arranged in a circle if two particular people are always:

(a) together **(b)** separated?

23 A father, a mother and six children stand in a ring. In how many ways can they be arranged if the father and the mother do not stand together?

24 Four actors and four comedians are seated alternately at a round table (i.e. alternating so that no two actors or two comedians sit next to each other). In how many ways can this be done?

25 The ratio of the number of arrangements of $(2n + 2)$ different objects taken n at a time to the number of arrangements of $2n$ different objects taken n at a time is 14 : 5. Find the value of n.

26 If ${}^{2n}P_n = 8 \times {}^{2n-1}P_{n-1}$, find the value of n.

27 Prove from the formula for ${}^nP_r$ that: ${}^{n+1}P_r = {}^nP_r + r \times {}^nP_{r-1}$

28 Show that: ${}^nP_r = {}^{n-2}P_r + 2r \times {}^{n-2}P_{r-1} + r(r-1) \times {}^{n-2}P_{r-2}$

29 In how many ways can five writers and five artists be arranged in a circle so that the writers are separated? In how many ways can this be done if two particular artists must not sit next to a particular writer?

6.4 ARRANGEMENT OF *n* OBJECTS WHEN SOME ARE IDENTICAL

The number of ways of arranging n objects in a row when p of the objects are identical, and q of the objects are identical (but different to the others), and so on, is not $n!$ but instead:

$$\frac{n!}{p!q!\ldots}$$

(This is shown in Example **13** below.)

Example 13

In how many ways can the six letters of the word MAMMAL be arranged in a line?

Solution

There are six letters, of which three are 'M' and two are 'A'.

$$\therefore \text{ Number of arrangements} = \frac{6!}{3! \times 2!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = 60$$

One of the 60 arrangements is MMMAAL. If the letters 'M' were different, e.g. M_1, M_2, M_3, then this arrangement could be six different arrangements (3!):

$M_1M_2M_3AAL$ $M_1M_3M_2AAL$ $M_2M_1M_3AAL$ $M_2M_3M_1AAL$ $M_3M_1M_2AAL$ $M_3M_2M_1AAL$

Without the subscripts, all these arrangements are the same. This is why 6! is divided by 3! in the calculation of identical object arrangements.

Similarly, the two letters 'A' have 2! arrangements the same, so the result is also divided by 2!.

EXERCISE 6.4 ARRANGEMENT OF *n* OBJECTS WHEN SOME ARE IDENTICAL

1 In how many ways can the letters of MOTOR be arranged in a line?

2 In how many ways can the letters of NEWCASTLE be arranged in a line?

A 4! **B** 8! **C** 9! **D** $\frac{9!}{2!}$

3 The letters of PRINCIPLE are arranged in a line. In how many ways:

(a) can this be done **(b)** can the letters 'P' be next to each other?

4 In how many ways can the letters of PERMUTE be arranged if:

(a) consonants occupy the first and the last places

(b) the vowels and the consonants occupy alternate places?

5 **(a)** In how many ways can the letters of PRECISION be arranged?

(b) In how many of these arrangements do the vowels occupy all the 'even' places (second, fourth, sixth, eighth)?

6 How many arrangements can be made of the letters of DEFINITION if:

(a) the letters 'I' do not occupy the first or last place **(b)** the letters 'I' are together?

7 How many arrangements of the letters of TOMATO are possible if the letters 'O' are never next to each other? Indicate whether each statement is correct or incorrect.

(a) $\frac{6!}{2! \times 2}$ **(b)** $\frac{5!}{2!}$ **(c)** $\frac{6!}{2! \times 2!} - \frac{5!}{2!}$ **(d)** 120

8 How many arrangements of the letters of PARRAMATTA are possible?

9 Seven cubes, identical except that four are red and three are black, are arranged in a row. How many different arrangements are possible?

10 Three blue, three white and three red balls are placed in a row.

(a) How many different arrangements are possible?

(b) In how many of these arrangements are the red balls together?

11 How many seven-digit numbers can be formed that contain all the digits 2, 3, 3, 3, 4, 5, 6?

12 Find the number of different ways in which n students can stand in a row when two are boys, the rest are girls, and the boys all stand together.

13 Five drummers and five singers sit together in a row.

(a) In how many ways can this be done?

(b) In how many ways can this be done if a particular drummer must not sit between two particular singers?

14 In how many ways can four people be accommodated at a hotel if there are four rooms available? (Assume that each room has enough beds for everyone.)

6.5 COMBINATIONS

A **combination** (or 'selection') is an unordered permutation of all or part of a set of objects.

Note: For combinations, the order is not important.

Consider a set of four people, A, B, C and D. In groups of two people at a time, they can be arranged as follows: AB, AC, AD, BC, BD, BA, CD, CA, CB, DA, DB, DC.

There are ${}^4P_2 = 12$ arrangements.

However, the number of combinations (selections) of two people at a time is only six. For example, AB and BA are two arrangements, but only one combination—the order in which they are combined does not matter. A and B can be arranged together in 2! ways, but combined in only one way.

The six different combinations (selections) are as follows: AB, AC, AD, BC, BD, CD.

Thus there are $\frac{{}^4P_2}{2!} = \frac{12}{2} = 6$ combinations.

The number of combinations of four people taken two at a time is represented by the symbols ${}^4C_2$ or $\binom{4}{2}$, and you can write: ${}^4C_2 = \frac{{}^4P_2}{2!} = \frac{12}{2} = 6$.

Example 14

How many groups of three can be selected from the four people A, B, C and D?

Solution

The selections are: ABC, ABD, ACD, BCD.

Number of combinations $= {}^4C_3 = \frac{{}^4P_3}{3!} = \frac{4 \times 3 \times 2}{3 \times 2} = 4$

Number of combinations ${}^nC_r$ or $\binom{n}{r}$

The symbol ${}^nC_r$ or $\binom{n}{r}$ denotes the number of combinations of n different objects taken r at a time. Each combination consists of a group of r different elements that can be ordered in $r!$ ways.

Note the difference between ordered and unordered sets of objects:

- An ordered sample of r objects taken from n different objects can be chosen in ${}^nP_r$ ways.
- An unordered sample of r objects taken from n different objects can be chosen in ${}^nC_r$ ways.

An unordered sample can be ordered in $r!$ ways.

Hence $r! \times {}^nC_r = {}^nP_r$

$$\therefore \quad {}^nC_r = \binom{n}{r} = \frac{{}^nP_r}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

This result can be written differently using factorial notation:

$$^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)\times(n-r)!}{r!\times(n-r)!}$$

$$\therefore \quad ^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Hence, for example: $^7C_3 = \frac{7\times(7-1)\times\ldots\times(7-3+1)}{3!} = \frac{7\times6\times5}{3\times2\times1} = 35$ **or** $^7C_3 = \frac{7!}{3!(7-3)!} = \frac{7\times6\times5}{3\times2\times1} = 35$

Example 15

In how many ways can a group of four people be selected from ten people if:

(a) there are no restrictions

(b) the oldest person is included in the group

(c) the oldest person is excluded from the group?

(d) What proportion of all possible groups contain the oldest person?

Solution

(a) Number of ways $= {}^{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 210$

(b) Automatically including the oldest, the selection is now for three out of nine people.

Number of ways $= {}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7}{3\times2\times1} = 84$

(c) Excluding the oldest, the selection is now for four out of nine people.

Number of ways $= {}^9C_4 = \frac{9!}{4!(9-4)!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$

(d) 84 out of the 210 possible sets contain the oldest person.

Proportion $= \frac{84}{210} = \frac{2}{5} = 40\%$

You can use the ^nC_r key or function on your calculator to check your answers to this work.

It is a good idea to write the numerical expression for the ^nC_r result and to practise the numerical simplification. This practice will help you to simplify the algebraic expressions that can occur in this work.

MAKING CONNECTIONS

Combinations

Use technology to calculate the number of combinations of n different objects taken r at a time.

Important result

From the definition for ^nC_r:

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \text{ and } ^nC_{n-r} = \binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Thus: $^nC_r = {}^nC_{n-r}$ **or** $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects, taken r at a time, is equal to the number of combinations of n objects taken $(n-r)$ at a time.

For each set of r objects selected, there is a set containing $(n-r)$ objects is left behind. Thus there must be the same number of sets containing $(n-r)$ objects as there are containing r objects.

For example, if a group of six is selected from a set of nine people, then a group containing three people is left behind:

$${}^9C_6 = \frac{9!}{6!(9-6)!} = \frac{9\times8\times7\times6!}{6!\times3!} = \frac{9\times8\times7}{3!} = {}^9C_3$$

This useful result allows you to simplify your calculations by calculating smaller factorials:

$${}^9C_6 = {}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7}{3!} = 84$$

Example 16

A class consists of 15 students, of whom five are prefects. How many committees of eight students can be formed if:

(a) there are no restrictions **(b)** each committee contains exactly two prefects

(c) each committee contains at least two prefects?

Solution

(a) Number of committees $= \binom{15}{8} = \binom{15}{7} = \frac{15!}{7!(15-7)!} = \frac{15\times14\times13\times12\times11\times10\times9}{7!} = 6435$

(b) The class contains five prefects and ten non-prefects. You have to select two prefects and six non-prefects.

Number of ways of selecting two prefects $= \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\times4}{2} = 10$

Number of ways of selecting six non-prefects $= \binom{10}{6} = \binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10\times9\times8\times7}{4\times3\times2} = 210$

Number of committees $= \binom{5}{2} \times \binom{10}{6} = 10 \times 210 = 2100$

(c) The committees may consist of:

- two prefects and six non-prefects
- three prefects and five non-prefects
- four prefects and four non-prefects
- five prefects and three non-prefects

These results are mutually exclusive, so the results are added.

$$\text{Number of committees} = \binom{5}{2}\times\binom{10}{6} + \binom{5}{3}\times\binom{10}{5} + \binom{5}{4}\times\binom{10}{4} + \binom{5}{5}\times\binom{10}{3}$$

$$= \frac{5\times4}{2\times1}\times\frac{10\times9\times8\times7}{4\times3\times2\times1} + \frac{5\times4\times3}{3\times2\times1}\times\frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} + \frac{5}{1}\times\frac{10\times9\times8\times7}{4\times3\times2\times1} + 1\times\frac{10\times9\times8}{3\times2\times1}$$

$$= 2100 + 2520 + 1050 + 120 = 5790$$

Doing the numerical calculations by showing this working-out will help you to develop the skills you will need to simplify expressions involving $\binom{n}{r}$. It's tempting to just write the first line of working and then use your calculator, but you shouldn't do that!

Note: Part **(c)** could also be calculated by finding how many committees contain only one or zero prefects, then subtracting this from the total number of committees without restrictions.

Number of committees $= \binom{15}{8} - \binom{10}{8} - \binom{10}{7}\times\binom{5}{1} = 5790$

Example 17

In how many ways can four runners and three swimmers be arranged in a row if there are eight runners and five swimmers to select from?

Solution

First select the total group of seven runners and swimmers, then arrange the group in a row:

- four runners and three swimmers can be selected in ${}^8C_4 \times {}^5C_3$ ways.
- The group of seven can be arranged in 7! ways.

Number of arrangements $= {}^8C_4 \times {}^5C_3 \times 7!$

$$= \frac{8\times7\times6\times5}{4\times3\times2\times1} \times \frac{5\times4}{2\times1} \times 7! = 3\,528\,000$$

This answer is not ${}^8P_4 \times {}^5P_3$, because the group has to be selected before it can be arranged. When selecting the group the order is not important, so the number of ways the group of seven can be selected is ${}^8C_4 \times {}^5C_3$. After the group is selected they can then be arranged in a row.

Example 18

Prove that: ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^mC_2\,{}^nC_{r-2} + {}^mC_1\,{}^nC_{r-1} + {}^nC_r$

Solution

The symbol ${}^{m+n}C_r$ denotes the number of selections, each of r objects, when you select from two sets containing m and n objects respectively.

The r objects can be selected as follows (assuming r is not greater than either m or n):

r objects from m and no objects from n in ${}^mC_r$ ways

or $(r-1)$ objects from m and one object from n in ${}^mC_{r-1}\,{}^nC_1$ ways

or $(r-2)$ objects from m and two objects from n in ${}^mC_{r-2}\,{}^nC_2$ ways

and so on until:

no objects from m and r objects from n in ${}^nC_r$ ways.

These selections are mutually exclusive, so they are added to obtain the final result:

$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^mC_2\,{}^nC_{r-2} + {}^mC_1\,{}^nC_{r-1} + {}^nC_r$$

Example 19

Prove that: ${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$

Solution

Method 1

The symbol ${}^{n+1}C_k$ denotes the number of selections, each of k objects, when you select from two sets containing n objects and 1 object respectively.

The k objects can be selected as follows (assuming $k \leq n$):

- k objects from the set of n and none from the set of one in ${}^nC_k\,{}^1C_0 = {}^nC_k$ ways
- $(k-1)$ objects from the set of n and the other one object in ${}^nC_{k-1}\,{}^1C_1 = {}^nC_{k-1}$ ways.

Note that ${}^1C_0 = 1$ and ${}^1C_1 = 1$.

These two selections are mutually exclusive, so they are added to obtain the final result:

$${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$$

Method 2

Use the result for ${}^nC_r$ in factorial notation:

$$
\begin{aligned}
{}^nC_k + {}^nC_{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= n!\left(\frac{1}{k!(n-k)!} + \frac{1}{(k-1)!(n-k+1)!}\right) \\
&= n!\left(\frac{n-k+1}{k(k-1)!(n-k+1)!} + \frac{k}{k(k-1)!(n-k+1)!}\right) \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= {}^{n+1}C_k
\end{aligned}
$$

from $k! = k(k-1)!$
and $(n-k+1)! = (n-k+1) \times (n-k)!$

Note: $(n-k+1) = (n+1-k)$

Combinations of any number of objects

The symbol ${}^nC_r$ can also be described as representing the number of r-subsets in a given set, where $0 \leq r \leq n$.

If no specific value of r is stated, r can assume any value between 0 and n inclusive. Hence the number of subsets that each contain at least one element, i.e. the number of non-empty subsets in an n-set, is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r$.

From a set of n objects, each object can be dealt with in two ways: it can be included or it can be excluded. The number of ways of dealing with the n objects is thus $2 \times 2 \times 2 \times \ldots$ to n factors, or 2^n. But this includes the case where all n objects are excluded, so the number of combinations of at least one object is $2^n - 1$.

$\therefore\ {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r = 2^n - 1$

Example 20

How many selections can be made from five different books, taking any number of books at a time?

Solution

The books can be selected one at a time, two at a time, or up to five at a time.

$$
\begin{aligned}
\text{Number of selections} &= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \\
&= 5 + 10 + 10 + 5 + 1 \\
&= 31 = 2^5 - 1
\end{aligned}
$$

Example 21

In how many ways can four people be divided into:

(a) an A pair and then a B pair **(b)** any pairs?

Solution

(a) Number of ways the A pair can be selected $= {}^4C_2 = 6$

Number of ways the B pair can now be selected $= 1$

$\therefore$ Number of ways an A pair and a B pair can be selected $= 6 \times 1 = 6$

After the first pair is selected, there is only one pair left for the B pair.

In this question you are distinguishing between the two pairs selected by calling them A and B.

(b) Number of ways of selecting two people from four $= {}^4C_2 = 6$

Let the people be W, X, Y, Z, so the six possible pairs are WX, WY, WZ, XY, XZ, YZ.

Unlike in part **(a)**, the order of selecting the two pairs now does not matter, so WX and YZ is not considered different to YZ and WX.

Thus the number of ways of dividing four people into two pairs $= \frac{{}^4C_2}{2!} = \frac{6}{2} = 3$

Example 22

In how many ways can six people be placed into three groups containing:

(a) three, two and one people respectively **(b)** two people in each group?

Solution

(a) Three people can be selected from the six people in ${}^6C_3$ ways.

From the remaining three people, two people can be selected in ${}^3C_2$ ways.

This leaves one person for the last group.

Number of ways $= {}^6C_3 \times {}^3C_2 \times 1$

$$= \frac{6!}{3!(6-3)!} \times \frac{3!}{2!(3-2)!} \times 1 = \frac{6 \times 5 \times 4}{2} = 60$$

(b) Two people can be selected from six people in ${}^6C_2$ ways.

From the remaining four people, two people can be selected in ${}^4C_2$ ways.

This leaves two people for the last group.

You might distinguish between the groups, e.g. by calling them X, Y, Z.

The number of ways people can be put into three distinct groups of 2 $= {}^6C_2 \times {}^4C_2 \times 1 = 90$

If the groups are indistinguishable (i.e. do not have names), then you have to divide the total number of arrangements by 3! (the number of arrangements of the three groups).

Number of ways people can be put into any three groups of two $= \frac{{}^6C_2 \times {}^4C_2 \times 1}{3!} = \frac{90}{6} = 15$

EXERCISE 6.5 COMBINATIONS

1 Simplify, without using your calculator:

(a) ${}^7C_2$ **(b)** ${}^9C_1$ **(c)** ${}^8C_4$ **(d)** ${}^{12}C_{10}$ **(e)** $\binom{12}{4}$ **(f)** $\binom{15}{1}$

(g) $\binom{9}{9}$ **(h)** $\binom{6}{4}$ **(i)** ${}^nC_2$ **(j)** $\binom{n}{4}$ **(k)** ${}^nC_n$ **(l)** $\binom{n}{k}$

2 Evaluate:

(a) ${}^{10}C_5$ **(b)** ${}^{20}C_9$ **(c)** ${}^{25}C_{10}$ **(d)** ${}^nC_5$

(e) $\binom{15}{10}$ **(f)** $\binom{18}{11}$ **(g)** $\binom{16}{8}$ **(h)** $\binom{48}{45}$

3 In how many ways can three books be selected from eight different books?

4 How many different hands of five cards can be dealt from a standard pack of 52 playing cards?

5 In how many ways can a jury of 12 people be chosen from 20 people?

6 The number of ways of picking six numbers from 45 numbers is:

A ${}^{45}C_6$ **B** ${}^{45}P_6$ **C** $6 \times {}^{45}C_1$ **D** $6 \times {}^{45}P_1$

7 In how many ways can a set of two cooks and three waiters be selected from five cooks and four waiters?

8 How many different selections can be made by taking three of the digits 4, 5, 6, 7, 8, 9?

9 In how many ways can a committee of four teachers and five parents be formed from eight teachers and seven parents?

10 From eight lawyers, seven clerks and five judges, how many different groups could be formed that contain five lawyers, four clerks and three judges? Indicate whether each answer is correct or incorrect.

(a) ${}^{20}C_{12}$ **(b)** ${}^{8}P_5 \times {}^{7}P_4 \times {}^{5}P_3$ **(c)** ${}^{8}C_5 \times {}^{7}C_4 \times {}^{5}C_3$ **(d)** 19 600

11 In how many ways can eight different rabbits be divided into a group of five and a group of three?

12 A committee of six is selected from ten people, of whom A and B are two. How many committees can be formed:

(a) containing both A and B **(b)** excluding A if B is included?

13 In how many ways can three cards be selected from a standard pack of 52 playing cards if:

(a) at least one selected card is an Ace **(b)** no more than one selected card is an Ace?

14 In how many ways can a team of three runners and four hurdlers be chosen from six runners and seven hurdlers?

15 A committee of seven politicians is chosen from ten Liberal members, eight Labor members and five independents. In how many ways can this be done to include exactly one independent, at least three Liberal members and at least one Labor member?

16 A team of 11 is chosen from 15 cricketers. Five of the 15 cricketers are bowlers only, two are wicketkeepers only and the rest are batters only. How many possible teams can be chosen that contain:

(a) four bowlers, one wicketkeeper and six batters

(b) at least four bowlers and at least one wicketkeeper?

17 From seven teachers and five students, a committee of seven is formed. How many different committees can be selected if teachers and students are both represented and the teachers are in a majority?

18 From four oranges, three bananas and two apples, how many selections of five pieces of fruit can be made, taking at least one of each kind?

19 In horse racing, a 'trifecta' is when the first three horses are picked in the correct order, while a 'quinella' is when the first two horses are picked in any order. In a ten-horse race, what is the number of possible:

(a) trifecta combinations **(b)** quinella combinations?

20 In how many ways can a jury of twelve people be chosen from ten women and seven men so that there are at least six women and not more than four men?

21 In how many ways can a group of three or more be selected from nine?

22 In how many ways can a committee of three women and four girls be chosen from seven women and six girls so that if the eldest woman is serving on the committee then the youngest girl is not?

23 How many **(a)** selections or **(b)** arrangements, consisting of three consonants and two vowels, can be made from eight different consonants and four different vowels?

24 In how many ways can four Physics books and three Mathematics books be arranged on a shelf if a selection is made from six different Physics books and five different Mathematics books? In how many of these arrangements are all the Physics books next to each other?

25 In how many ways can three glasses and two plates be arranged in a row if a selection is made from five different glasses and four different plates? In how many of these arrangements does a glass occupy the middle position?

26 How many words (i.e. any arrangements of letters) containing three consonants and two vowels can be formed from the letters of the word PROMISE?

27 If, ${}^nC_6 = {}^nC_4$, find the value of n.

28 In how many ways can nine books be distributed among a teacher, a parent and a child, if the teacher receives four, the parent three and the child two?

29 In how many ways can eight different toys be divided into two unequal groups?

30 In how many ways can eight basketball players be divided into four groups of two?

31 In how many ways can n objects be shared between two people? ('Shared' means that each person gets at least one.)

32 From the definition of ${}^nC_r$, prove each of the following:

(a) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

(b) ${}^nC_k + 2 \times {}^nC_{k-1} + {}^nC_{k-2} = {}^{n+2}C_k$

(c) ${}^nC_r = \dfrac{n-r+1}{r} \times {}^nC_{r-1}$

(d) ${}^{m+n}C_3 = {}^mC_3 + {}^mC_2\,{}^nC_1 + {}^mC_1\,{}^nC_2 + {}^nC_3$

(e) ${}^{n+3}C_r = {}^nC_r + 3 \times {}^nC_{r-1} + 3 \times {}^nC_{r-2} + {}^nC_{r-3}$

33 The ratio of the number of combinations of $(2n + 2)$ different objects taken n at a time to the number of combinations of $(2n - 2)$ different objects taken n at a time is 99 : 7. Find the value of n.

6.6 COUNTING TECHNIQUES IN PROBABILITY

This section builds on the counting techniques developed earlier in the chapter. A brief summary of the necessary knowledge follows.

Factorial notation

$n! = n(n-1)(n-2)\ldots \times 3 \times 2 \times 1 \qquad 0! = 1$

Permutations

A permutation is an *ordered* selection or arrangement of all or part of a set of objects.

- The number of arrangements of n different objects is $n!$
- The number of arrangements of r objects selected from n different objects is ${}^nP_r = \dfrac{n!}{(n-r)!}$
- The number of ways of arranging n objects when there are p identical objects of one kind, q identical objects of another kind, etc. is $\dfrac{n!}{p!q!\ldots}$

Combinations

A combination is an *unordered* selection of all or part of a set of objects.
A combination (or 'selection') is an unordered permutation (or 'arrangement').

- The number of combinations of r objects selected from n different objects is ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Useful results

${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ and ${}^nC_{n-r} = \binom{n}{n-r} = \dfrac{n!}{(n-r)!r!}$

Thus: ${}^nC_r = {}^nC_{n-r}$ **or** $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

Probability

$$P(A)=\frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}},\ 0\le P(A)\le 1 \qquad P\left(\overline{A}\right)=1-P(A)$$

Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$

Not mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

Example 23

PINs (Personal Identification Numbers) are short number codes used as a security device.

(a) How many four-digit PINs can be made using the digits 0, 1, 2, … 9?

(b) What is the probability that someone can randomly guess the four-digit PIN?

(c) As a further security measure, some financial institutions require a four-symbol code that is made using the 26 letters of the alphabet and/or the digits 0–9. How many different four-symbol codes can be created?

(d) What is the probability of someone correctly guessing one of these four-symbol codes?

(e) What is the probability of someone correctly guessing the four-digit PIN and the correct four-symbol code?

Solution

(a) Number of PINs $= 10^4 = 10\,000$

(b) $P(\text{guessing PIN})=\frac{1}{10^4}=\frac{1}{10\,000}$

(c) There are 26 letters of the alphabet and 10 digits. Number of codes $= 36^4 = 1\,679\,616$

(d) $P(\text{guessing symbol code})=\frac{1}{36^4}=\frac{1}{1\,679\,616}$

(e) $P(\text{guessing both PIN and symbol code})=\frac{1}{10^4}\times\frac{1}{36^4}=\frac{1}{16\,796\,160\,000}$

Example 24

In horse racing, a 'quinella' is when the first two horses are picked correctly (in any order). In a twelve-horse race, what is the probability of picking the quinella, assuming that each horse is equally likely to win?

Solution

12 horses total, two picked for a quinella: Number of quinella combinations $= {}^{12}C_2 = 66$

Only one of these combinations wins: $P(\text{winning quinella})=\frac{1}{66}$

Example 25

Eight people in total, including two people A and B, randomly arrange themselves in a straight line. What is the probability that:

(a) A and B are next to each other

(b) A and B are not next to each other

(c) A and B occupy the end positions

(d) there are at least three people between A and B?

Solution

(a) Eight people can be arranged in a row in 8! ways = 40 320
As A and B are next to each other, consider them as one unit.
This unit and the other six people can be arranged in 7! ways.
A and B can be arranged in two ways (AB or BA).
Total ways of arranging the people = $2 \times 7!$

$$P(A \text{ and } B \text{ next to each other}) = \frac{2 \times 7!}{8!} = \frac{2 \times 7!}{8 \times 7!} = \frac{1}{4}$$

(b) $P(A \text{ and } B \text{ not next to each other}) = 1 - P(A \text{ and } B \text{ next to each other}) = 1 - \frac{1}{4} = \frac{3}{4}$

(c) A and B can be placed at the ends in two ways (A at the beginning and B at the end, or B at the beginning and A at the end). The remaining six places can be filled in 6! ways.
Total number of ways with A and B at the ends = $2 \times 6!$

$$P(A \text{ and } B \text{ occupy end positions}) = \frac{2 \times 6!}{8!} = \frac{2 \times 6!}{8 \times 7 \times 6!} = \frac{1}{28}$$

(d) A diagram helps to visualise the possibilities:

$A___B___$, $A____B__$, $A_____B_$, $A_____B$.
$_A___B__$, $_A____B_$, $_A_____B$.
$__A___B_$, $__A____B$.
$___A___B$.

A and B can be swapped, so this gives $10 \times 2 = 20$ arrangements for A and B with 6! ways of filling the remaining six places.
Total number of favourable arrangements = $20 \times 6!$

$$P(\text{at least three people between } A \text{ and } B) = \frac{20 \times 6!}{8!} = \frac{20 \times 6!}{8 \times 7 \times 6!} = \frac{5}{14}$$

Example 26

A bag contains nine cubes: three that are white and six that are black. Two cubes are drawn at random without replacement. Calculate the probability that both cubes are black.

Solution

Method 1

Number of ways two cubes can be selected from nine cubes without replacement = ${}^9C_2$
Number of ways the two cubes can be black = ${}^6C_2$

$$P(\text{both cubes are black}) = \frac{{}^6C_2}{{}^9C_2} = \frac{6 \times 5}{2 \times 1} \times \frac{2 \times 1}{9 \times 8} = \frac{5}{12}$$

Method 2

Let A be the event 'black cube in the first draw' and B be the event 'black cube in the second draw'.
Each cube is equally likely to be selected, so $P(A) = \frac{6}{9}$.
The outcome of the second draw is dependent on the outcome of the first draw:
if event A has happened then there are eight cubes left, of which five are black.

Hence $P(B|A) = \frac{5}{8}$
Now $P(AB) = P(A) \times P(B|A) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$

Example 27

A group of nine people contains three teachers and six students. A random sample of five people is selected. What is the probability that it contains: **(a)** exactly two teachers **(b)** not more than two teachers?

Solution

(a) Exactly two teachers means the sample contains two teachers and three students.

Number of ways of selecting five people from nine people $= \binom{9}{5}$

Number of ways of selecting two teachers from three teachers $= \binom{3}{2}$

Number of ways of selecting three students from six students $= \binom{6}{3}$

Number of ways of selecting two teachers and three students $= \binom{3}{2} \times \binom{6}{3}$

Hence: $P(\text{two teachers and three students}) = \dfrac{\binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}}$

Remember $\binom{9}{5} = \binom{9}{4}$: $P(\text{two teachers and three students}) = \dfrac{\binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}} = 3 \times \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \dfrac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7 \times 6} = \dfrac{10}{21}$

(b) Not more than two teachers means either zero teachers and five students, one teacher and four students or two teachers and three students.

Zero teachers and five students can be selected in $\binom{6}{5}$ ways.

One teacher and four students can be selected in $\binom{3}{1} \times \binom{6}{4}$ ways.

Two teachers and three students can be selected in $\binom{3}{2} \times \binom{6}{3}$ ways.

Hence: $P(\text{not more than two teachers}) = \dfrac{\binom{6}{5} + \binom{3}{1} \times \binom{6}{4} + \binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}} = \dfrac{6 + 45 + 60}{126} = \dfrac{37}{42}$

Example 28

From a set of five cards numbered 1, 2, 3, 4, 5, two cards are selected at random without replacement.

(a) What is the probability that both cards are odd-numbered?

(b) If the cards are used to make a two-digit number, what is the probability that this number is odd?

(c) If the cards are used to make a two-digit number, what is the probability that this number has one odd digit and one even digit?

Solution

(a) Number of different two-card selections $= {}^5C_2 = 10$
Number of two-card selections where both are odd $= {}^3C_2 \times {}^2C_0 = 3$
$P(\text{both cards are odd}) = \frac{3}{10}$

(b) **Method 1**
The second digit must be odd, so the order is important.
Number of odd numbers formed $= {}^3P_1 \times {}^4P_1 = 12$
Number of two-digit numbers formed $= {}^5P_2 = 20$
$P(\text{odd two-digit number formed}) = \frac{12}{20} = \frac{3}{5}$

Method 2
To be an odd number, the cards must be odd–odd or even–odd.
$P(\text{odd number}) = P(\text{odd, odd}) + P(\text{even, odd})$

$$= \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4}$$
$$= \frac{12}{20}$$
$$= \frac{3}{5}$$

(c) One even digit and one odd digit: number of ways $= 2 \times 3 = 6$
Number of different two-card selections $= {}^5C_2 = 10$
$P(\text{one even digit and one odd digit}) = \frac{6}{10} = \frac{3}{5}$
If you consider the order: $P(\text{one even digit and one odd digit}) = \frac{12}{20} = \frac{3}{5}$ (i.e. the same answer)

Example 29

A box contains 10 pairs of headphones, two of which are defective. A sample of three pairs of headphones is drawn at random from the box without replacement. Find the probability that not more than one pair of headphones is defective.

Solution

Number of possible selections $= {}^{10}C_3 = 120$

Number of favourable selections = [0 defective] + [1 defective]

'0 defective' means that 0 (none) of the two defectives are selected and three of the eight non-defectives are selected,
i.e. ${}^2C_0 \times {}^8C_3$ ways.

'1 defective' means that one of the two defectives is selected and two of the eight non-defectives are selected,
i.e. ${}^2C_1 \times {}^8C_2$ ways.

Number of favourable selections $= {}^2C_0 \times {}^8C_3 + {}^2C_1 \times {}^8C_2 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} + 2 \times \frac{8 \times 7}{2 \times 1} = 56 + 56 = 112$

$P(\text{not more than 1 defective}) = \frac{112}{120} = \frac{14}{15}$

Example 30

A scientific study of penguins uses the 'capture–recapture' technique. In the first stage of the study, 36 penguins are caught, tagged and then released. Later, in the second stage of the study, some penguins are again captured from the same area. Of these penguins, 27 of them are found to be tagged, which is 30% of the total number captured in this second stage.

(a) In the second stage of the study, how many penguins are captured in total?

(b) Calculate the estimate for the total population of penguins in this area.

Solution

(a) 27 is 30% of total captured, N: $27 = 0.3N$

$$N = \frac{27}{0.3} = 90$$ Hence 90 penguins were captured.

(b) P is the total population of penguins in the area.

Use the ratio: $\frac{27}{90} = \frac{36}{P}$

$$P = \frac{36 \times 90}{27} = 120$$

or: $0.3 = \frac{36}{P}$ The penguin population is about 120.

EXERCISE 6.6 COUNTING TECHNIQUES IN PROBABILITY

1 Five cards are drawn at random from a standard pack of 52 playing cards. What is the probability that:

(a) they are all from the same suit **(b)** they include four Aces **(c)** they include three 10s and two 9s?

2 A bag contains five red balls and four white balls. Three balls are withdrawn without replacement. The probability of drawing at least two red balls is:

A $\frac{1}{21}$ **B** $\frac{5}{42}$ **C** $\frac{20}{42}$ **D** $\frac{25}{42}$

3 A student writes a random three-digit number using the digits 1–9. What is the probability that the three digits in the number are all the same?

4 In horse racing, a 'trifecta' is when the first three horses are picked in the correct order. In a 10-horse race, what is the probability of picking the trifecta, assuming that each horse is equally likely to win?

5 A bag contains three white balls, four red balls and five black balls. Three balls are drawn at random without replacement. What is the probability that they are: **(a)** different colours **(b)** the same colour?

6 From a group of seven teachers and five students, a random selection of seven people is made. What is the probability that the selection contains at least four teachers?

7 A committee of three judges and four lawyers is to be chosen from six judges and seven lawyers. What is the probability that it contains a particular judge and a particular lawyer?

8 Five cards are drawn from a standard pack of 52 playing cards (without replacement). What is the probability of drawing at least three Aces?

9 A person correctly picks the first and second horses in a race of 10 horses. If each horse was equally likely to win, what is the probability of this?

10 The letters of the word PROMISE are arranged in a row. What is the probability that there are three letters between 'P' and 'R'?

11 Six giraffes randomly arrange themselves in a line. What is the probability that the tallest giraffe and the shortest giraffe are together?

12 Four different trucks and three different cars are parked in a row. What is the probability that the trucks and the cars occupy alternate positions?

13 A car's number plates contain three letters of the alphabet followed by three numerals. How many number plates like this can be made? What proportion of these would contain three letters the same and three numerals the same?

14 The letters of the word INDEPENDENCE are arranged in a row. What is the probability of all the letters 'E' being together? Indicate whether each statement below is a correct or incorrect step in solving this problem.

(a) Number of arrangements of the letters $= \frac{12!}{4!3!2!}$

(b) Number of ways that the letters 'E' are together $= \frac{9!}{4!3!2!}$

(c) Number of ways that the letters 'E' are together $= \frac{9!}{3!2!}$

(d) Probability of the letters 'E' being together $= \frac{1}{55}$

15 A carton contains a dozen eggs, of which three have a double yolk. If three eggs are taken to make a cake, find the probability that the three eggs all have double yolks.

16 A team of six is selected from 10 people. What is the probability that the youngest and oldest people are on the team?

17 A box contains five red cubes and four white cubes. Three cubes are drawn in succession without replacement. What is the probability that:

(a) the first two cubes are red and the third cube is white

(b) any two cubes are red and one cube is white?

18 An angler has caught 15 fish, of which three are undersized. A random sample of three fish is drawn without replacement by an inspector. The angler is fined if one or more of the fish in the sample is undersized. What is the probability that the angler is fined?

19 Eight different jackets are divided into two piles. What is the probability that there will be four in each pile?

20 A party of 12 people, including A and B, are arranged at random in a straight line. What is the probability that A and B are not next to one another?

21 A box contains 15 memory cards, of which 5 are defective. If a random sample of 6 memory cards is drawn from the box (without replacement), determine the probability of there being 0, 1, 2, 3, 4, 5 defective memory cards in the sample.

22 An urn contains 12 distinguishable cubes of which five are red and the remainder black. If a random sample of six cubes is drawn without replacement, calculate the probabilities of 0, 1, 2, 3, 4, 5 red cubes in the sample.

23 Two boxes each contain eight balls. In box A there are three black and five white balls; in box B there are one black and seven white balls. For each box, find the probability that two balls chosen at random without replacement will both be white.

24 A sample of three coins is selected without replacement from a handful of eight coins that consists of four 10c coins and four 20c coins. What is the probability that the sample contains at least two 10c coins?

25 A hand of five cards is dealt from a standard pack of 52 playing cards. What is the probability that it contains at least one Ace?

26 From a group of 12 people, of whom eight are painters and four are carpenters, a sample of four is selected at random. What is the probability that the sample contains at least two carpenters?

27 Box A contains six white and four black balls. Box B contains two white and two black balls. From box A, two balls are selected at random and placed in box B. Two balls are then selected at random from box B. What is the probability that exactly one of these two balls is white?

28 From a set of 10 cards numbered 1–10, two cards are drawn without replacement. What is the probability that:

(a) both numbers are even

(b) one is even and the other is odd

(c) the sum of the two numbers is 12

(d) both numbers are even and the sum of the two numbers is 12?

29 The letters of the word TOMATO are arranged in a row. What is the probability that:
(a) the two letters 'O' are together **(b)** the two letters 'O' are not together?

30 The letters of the word TUESDAY are arranged at random in a row. What is the probability that:
(a) the vowels and consonants occupy alternate positions **(b)** the vowels are together
(c) the vowels are together and the letter 'T' occupies the first place?

31 Four girls and four boys arrange themselves at random in a row. What is the probability that the girls and the boys occupy alternate positions?

32 Six different robots, including A and B, arrange themselves at random in a row. What is the probability that:
(a) A and B occupy the end positions **(b)** A and B are not next to each other
(c) there are at least three robots between A and B?

33 The digits 1, 2, 3, 4, 5, 6 are used to form numbers that contain two or more digits. The same digit cannot be used more than once in a number. What proportion of the numbers formed are even numbers?

34 Five cards are selected with replacement from a standard pack of 52 playing cards. What is the probability that the selected cards include:
(a) exactly three hearts **(b)** four Aces **(c)** no hearts?

35 A game involves choosing six numbers from a set of 44 numbers.
(a) What is the probability of winning (i.e. choosing the six numbers correctly) in a single game?
(b) If you play four games, choosing a different set of numbers for each game, what is the probability that one of your games will win?

36 A scientific study uses the 'capture–recapture' technique. In the first stage of the study, 48 possums are caught, tagged and then released. Later, in the second stage of the study, some possums are again captured from the same area. Of these possums, 24 of them are found to be tagged, which is 40% of the total captured in this second stage.
(a) In the second stage of the study, how many possums are captured in total?
(b) Calculate the estimate for the total population of possums in this area.

An interesting problem—who shares your birthday?

Example 31

In a random group of people, what is the probability that at least two people in the group have the same birthday (i.e. the same day and month, but not necessarily the same year)? This example investigates this problem.

(*Note:* 29 February and leap years will be ignored, assuming instead 1 year = 365 days.)

(a) In a group of four people, what is the probability that there is at least one birthday shared?
(b) In a group of 10 people, what is the probability that there is at least one birthday shared?
(c) In a group of n people, what is the probability that there is at least one birthday shared?
(d) For what value of n is there a 50% chance that at least two people in the group have the same birthday?

Solution

(a) If two people have their birthday on different days, then there are 364 days on which the second person could have been born.

In a group of four people, the probability that no two people are born on the same day $= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$

Probability of at least one birthday in common $= 1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \approx 0.0164$

(b) In a group of 10 people:

Probability that no two people are born on the same day $= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{355}{365} = \frac{{}^{364}P_{10}}{365^{10}}$

Probability of at least one birthday in common $= 1 - \frac{{}^{364}P_{10}}{365^{10}} \approx 0.1411$

(c) In a group of n people:

Probability that no two people are born on the same day $= \frac{{}^{364}P_{n}}{365^{n}}$

Probability of at least one birthday in common $= 1 - \frac{{}^{364}P_{n}}{365^{n}}$

(d) To find a 50% chance you must solve the equation: $1 - \frac{{}^{364}P_{n}}{365^{n}} = 0.5$

That is, solve for n the equation: $\frac{{}^{364}P_{n}}{365^{n}} = 0.5$

This cannot be done analytically, but by substituting values for n you can find the answer by trial and error.

From **(b)** you know $n > 10$. Trying $n = 20$: $\frac{{}^{364}P_{20}}{365^{20}} \approx 0.556$

Trying $n = 25$: $\frac{{}^{364}P_{25}}{365^{25}} \approx 0.4012$

Trying $n = 22$: $\frac{{}^{364}P_{22}}{365^{22}} \approx 0.5234$

Trying $n = 23$: $\frac{{}^{364}P_{23}}{365^{23}} \approx 0.4917$

Hence the closest value of n is 23.

An interesting problem similar to the birthday investigation above is the 'Hermit Problem', which you may wish to research.

6.7 EXPANSION OF $(1 + x)^n$, PASCAL'S TRIANGLE

By multiplying to expand the brackets, it can be shown that $(1 + x)^n$ is equal to the following:

$n = 0$: $(1 + x)^0 = 1$

$n = 1$: $(1 + x)^1 = 1 + x$

$n = 2$: $(1 + x)^2 = 1 + 2x + x^2$

$n = 3$: $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$

$n = 4$: $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

$n = 5$: $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$n = 6$: $(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

You can see that in each case, the expansion of $(1 + x)^n$ for $n = 1, 2, 3, \ldots$ is a polynomial of degree n in the variable x.

The coefficients of the successive powers of the expansion of $1 + x$ can be arranged in a triangular pattern called **Pascal's triangle**, as shown on the right.

$(1 + x)^0$ 1

$(1 + x)^1$ 1 1

$(1 + x)^2$ 1 2 1

$(1 + x)^3$ 1 3 3 1

$(1 + x)^4$ 1 4 6 4 1

$(1 + x)^5$ 1 5 10 10 5 1

$(1 + x)^6$ 1 6 15 20 15 6 1

MAKING CONNECTIONS

Pascal's triangle

Create and explore the construction of Pascal's triangle.

Properties of Pascal's triangle

1. The first and last number in each row is 1.
2. Every number not at the end of a row is the sum of the two numbers in the row above it to the left and right (e.g. $10 = 4 + 6$, $15 = 5 + 10$, $20 = 10 + 10$).
3. The number of terms in each row is one more than the value of n.
4. The second and second-last terms in each row are the value of n.

These properties enable the expansion of $(1 + x)^n$ to be written for any integer value of n. For example:

$n = 7$: $(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$ (eight terms)

$n = 8$: $(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$ (nine terms)

Unfortunately, using these properties of Pascal's triangle to find the coefficients of any particular expansion requires all the previous rows. For example, to expand $(1 + x)^{12}$ in this way you would first need all the rows up to $n = 11$ to find the 12th row of Pascal's triangle.

However, there is a more efficient way to find the coefficients. You can establish a relationship for the coefficients for the expansion of the general case $(1 + x)^n$.

To do this, you can use the notation ${}^nC_r$ in place of the coefficient, where the right subscript r is the power of x and the left superscript n is the power n. Previously defined as the number of combinations of n different objects taken r at a time, ${}^nC_r$, is also by definition the coefficient of x^r in $(1 + x)^n$.

Thus you can write the general expansion of $(1 + x)^n$ as:

$$\begin{aligned}(1+x)^n &= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n \\ &= \sum_{r=0}^{n} {}^nC_rx^r\end{aligned}$$

If you write $\frac{b}{a}$ in place of x and multiply both sides by a^n, you obtain:

$$\begin{aligned}(a+b)^n &= a^n\left(1+\frac{b}{a}\right)^n \\ &= a^n\left({}^nC_0 + {}^nC_1\frac{b}{a} + {}^nC_2\left(\frac{b}{a}\right)^2 + \ldots + {}^nC_r\left(\frac{b}{a}\right)^r + \ldots + {}^nC_n\left(\frac{b}{a}\right)^n\right) \\ &= {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n \\ &= \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r\end{aligned}$$

This is the **binomial theorem**. Notice that in each term the sum of the powers in $a^{n-r}b^r$ is $n - r + r = n$. As the power of a decreases with each term, the power of b increases by the same amount.

Example 32

Write the expansion of $(1 + 2x)^4$, using Pascal's triangle to obtain the coefficients.

Solution

Write: $(1+2x)^4 = {}^4C_0 + {}^4C_1(2x)^1 + {}^4C_2(2x)^2 + {}^4C_3(2x)^3 + {}^4C_4(2x)^4$

From Pascal's triangle: ${}^4C_0 = 1, {}^4C_1 = 4, {}^4C_2 = 6, {}^4C_3 = 4, {}^4C_4 = 1$

$\therefore \quad (1+2x)^4 = 1 + 4 \times 2x + 6 \times 4x^2 + 4 \times 8x^3 + 1 \times 16x^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Example 33

Write the expansion of $(1-x)^3$.

Solution

Write: $(1-x)^3 = {}^3C_0 + {}^3C_1(-x)^1 + {}^3C_2(-x)^2 + {}^3C_3(-x)^3$

From Pascal's triangle: ${}^3C_0 = 1, {}^3C_1 = 3, {}^3C_2 = 3, {}^3C_3 = 1$

$\therefore \quad (1-x)^3 = 1 + 3 \times (-x) + 3 \times x^2 + 1 \times (-x^3)$

$= 1 - 3x + 3x^2 - x^3$

Example 34

Write the simplified expansion of $\left(1+\sqrt{3}\right)^6$.

Solution

$$\left(1+\sqrt{3}\right)^6 = {}^6C_0 + {}^6C_1 \times \sqrt{3} + {}^6C_2 \times \left(\sqrt{3}\right)^2 + {}^6C_3 \times \left(\sqrt{3}\right)^3 + {}^6C_4 \times \left(\sqrt{3}\right)^4 + {}^6C_5 \times \left(\sqrt{3}\right)^5 + {}^6C_6 \times \left(\sqrt{3}\right)^6$$

$$= 1 + 6 \times \sqrt{3} + 15 \times \left(\sqrt{3}\right)^2 + 20 \times \left(\sqrt{3}\right)^3 + 15 \times \left(\sqrt{3}\right)^4 + 6 \times \left(\sqrt{3}\right)^5 + \left(\sqrt{3}\right)^6$$

$$= 1 + 6\sqrt{3} + 45 + 60\sqrt{3} + 135 + 54\sqrt{3} + 27$$

$$= 208 + 120\sqrt{3}$$

Example 35

Find the coefficient of a^4 in $(1+5a)^5$.

Solution

Method 1

$(1+5a)^5 = 1 + 5 \times 5a + 10 \times (5a)^2 + 10 \times (5a)^3 + 5 \times (5a)^4 + \ldots$

After the required term is found, there is no need to find succeeding terms.

$(1+5a)^5 = 1 + 25a + 250a^2 + 1250a^3 + 3125a^4 + \ldots$

The coefficient of a^4 is 3125.

Method 2

For the expansion, you can write $T_1 = {}^5C_0$, $T_2 = {}^5C_1(5a)^1$ and so on, leading to the general term $T_{r+1} = {}^5C_r(5a)^r$.

For the term in a^4, $r = 4$: $T_5 = {}^5C_4(5a)^4 = 5 \times 5^4 a^4 = 3125a^4$

Hence the coefficient of a^4 is 3125.

Example 36

Use the expansion of $\left(1-\frac{1}{x}\right)^5$ to find an approximation for $\left(\frac{99}{100}\right)^5$ correct to four decimal places.

Solution

This can be written as an expansion: $\left(\frac{99}{100}\right)^5 = \left(1-\frac{1}{100}\right)^5$

$$= 1 - 5\times\frac{1}{100} + 10\times\frac{1}{100^2} - 10\times\frac{1}{100^3} + \ldots$$
$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} + \ldots$$
$$= 1 - 0.05 + 0.001 - 0.00001 + \ldots$$
$$\approx 0.9510$$

As the accuracy is only to four decimal places, the terms smaller than 0.000 01 are insignificant and can be ignored.

$\therefore \left(\frac{99}{100}\right)^5 \approx 0.9510$ correct to four decimal places.

EXERCISE 6.7 EXPANSION OF $(1 + x)^n$, PASCAL'S TRIANGLE

1 Use Pascal's triangle to find the expansion of each of the following.

(a) $(1+x)^6$ **(b)** $(1+x)^8$ **(c)** $(1+b)^7$ **(d)** $(1+2x)^3$ **(e)** $(1+4x)^5$

(f) $(1+3x)^6$ **(g)** $(1-x)^3$ **(h)** $(1-a)^8$ **(i)** $(1-2x)^4$ **(j)** $\left(1+\frac{x}{2}\right)^5$

(k) $\left(1-\frac{x}{3}\right)^6$ **(l)** $\left(1+\frac{2x}{5}\right)^5$ **(m)** $(1+x^2)^3$ **(n)** $(1-x^2)^4$ **(o)** $\left(x+\frac{1}{x}\right)^5$

2 Find the coefficient of x^2 in the expansion of:

(a) $(1+x)^4$ **(b)** $(1+x)^7$ **(c)** $(1-x)^5$ **(d)** $(1+3x)^3$ **(e)** $(1+2x)^5$ **(f)** $(1-3x)^4$

3 Find the coefficient of b^3 in the expansion of: **(a)** $(1+2b)^4$ **(b)** $(1-4b)^3$ **(c)** $\left(1+\frac{2b}{3}\right)^5$

4 The coefficient of x^4 in the expansion of $\left(x-\sqrt{2}\right)^6$ is:

A –30 **B** 30 **C** –60 **D** 60

5 Find the fourth term in the expansion of: **(a)** $(1+x)^5$ **(b)** $(1-2a)^7$ **(c)** $\left(1+\frac{3x}{4}\right)^6$

6 Write the simplified expansion for each of the following. **(a)** $\left(1+\sqrt{2}\right)^4$ **(b)** $\left(1-\sqrt{3}\right)^3$

(c) $\left(1+\sqrt{5}\right)^6$ **(d)** $\left(1+2\sqrt{3}\right)^3$ **(e)** $\left(1-3\sqrt{2}\right)^4$ **(f)** $\left(1+\frac{\sqrt{3}}{2}\right)^5$

7 Use the expansion of $\left(1-\frac{1}{x}\right)^n$ to find the value of each expression correct to four decimal places.

(a) $\left(\frac{9}{10}\right)^6$ **(b)** $\left(\frac{19}{20}\right)^5$ **(c)** $\left(\frac{99}{100}\right)^4$

6.8 MORE PASCAL'S TRIANGLE EXPANSIONS

You will now consider expansions of the type:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

where the ${}^nC_r$ are the numbers in the $(n+1)$-th row of Pascal's triangle.

Example 37

Write the expansion of $(a+x)^5$.

Solution

$$(a+x)^5 = {}^5C_0a^5x^0 + {}^5C_1a^4x^1 + {}^5C_2a^3x^2 + {}^5C_3a^2x^3 + {}^5C_4a^1x^4 + {}^5C_5a^0x^5$$

where the ${}^5C_r$ are the numbers in the sixth row of Pascal's triangle.

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5a^1x^4 + x^5$$

After writing the expansion, you can check your result by adding the powers on the variables in each term to check that the sum is 5.

Example 38

Write the expansion of $(2+x)^4$.

Solution

Coefficients are the numbers in the fifth row of Pascal's triangle:

$$\begin{aligned}(2+x)^4 &= 1\times 2^4x^0 + 4\times 2^3x^1 + 6\times 2^2x^2 + 4\times 2^1x^3 + 1\times 2^0x^4\\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

Example 39

Write the expansion of $(2a-b)^5$.

Solution

Coefficients are the numbers in the sixth row of Pascal's triangle:

$$\begin{aligned}(2a-b)^5 &= (2a)^5 + 5\times(2a)^4\times(-b)^1 + 10\times(2a)^3\times(-b)^2 + 10\times(2a)^2\times(-b)^3 + 5\times(2a)\times(-b)^4 + (-b)^5\\ &= 32a^5 - 5\times 16a^4b + 10\times 8a^3b^2 - 10\times 4a^2b^3 + 10ab^4 - b^5\\ &= 32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 10ab^4 - b^5\end{aligned}$$

Example 40

Write the expansion of $\left(x+\frac{1}{x}\right)^4$.

Solution

Coefficients are the numbers in the fifth row of Pascal's triangle:

$$\begin{aligned}\left(x+\frac{1}{x}\right)^4 &= x^4 + 4x^3\times\frac{1}{x} + 6x^2\times\frac{1}{x^2} + 4x\times\frac{1}{x^3} + \frac{1}{x^4}\\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\end{aligned}$$

In this example, two terms that were functions of x create a constant term in the expansion. You can say that 'the term independent of x' (i.e. the term without an x) is 6.

EXERCISE 6.8 MORE PASCAL'S TRIANGLE EXPANSIONS

1 Use Pascal's triangle to find the expansion of the following.

(a) $(2+x)^3$ **(b)** $(3+x)^5$ **(c)** $(b+2)^4$ **(d)** $(a+b)^7$ **(e)** $(x-y)^8$

(f) $(3-x)^3$ **(g)** $(2-a)^8$ **(h)** $(3y-2x)^4$ **(i)** $\left(x-\frac{1}{x}\right)^4$ **(j)** $\left(x^2+\frac{1}{x}\right)^6$

2 Write the simplified expansion for each of the following. **(a)** $\left(2+\sqrt{2}\right)^4$ **(b)** $\left(4-\sqrt{3}\right)^3$

(c) $\left(\sqrt{2}+\sqrt{5}\right)^6$ **(d)** $\left(\sqrt{5}+\sqrt{3}\right)^5$ **(e)** $\left(2\sqrt{3}-3\sqrt{2}\right)^4$ **(f)** $\left(\sqrt{6}+2\sqrt{3}\right)^5$

3 Write the coefficient of x^5 in the expansion of: **(a)** $(3+x)^6$ **(b)** $(2-x)^7$ **(c)** $\left(x+\frac{1}{x}\right)^7$

4 The coefficient of x^5 in the expansion of $(2x+5)^8$ is:

A 32 000 **B** 4000 **C** 224 000 **D** 1792

5 Find the term independent of x in each expansion. **(a)** $(2x+3)^4$ **(b)** $\left(x+\frac{1}{x}\right)^6$ **(c)** $\left(x+\frac{1}{x^2}\right)^6$

6.9 PASCAL'S TRIANGLE RELATIONS AND THE BINOMIAL THEOREM

There are two separate Pascal's triangle relations: one for the outer coefficients and another for the other (inner) coefficients.

Pascal's triangle relation—outer coefficients

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n$$

When $a=1$ and $b=0$: $(1+0)^n = {}^nC_0(1)^n + 0 + 0 + \ldots$

$$\therefore \quad {}^nC_0 = 1$$

When $a=0$ and $b=1$: $(0+1)^n = 0 + 0 + \ldots + {}^nC_n(1)^n$

$$\therefore \quad {}^nC_n = 1$$

Pascal's triangle relation—inner coefficients

$$(1+x)^{n-1} = {}^{n-1}C_0 + {}^{n-1}C_1x + {}^{n-1}C_2x^2 + \ldots + {}^{n-1}C_rx^r + \ldots + {}^{n-1}C_{n-1}x^{n-1}$$

$$x(1+x)^{n-1} = \quad {}^{n-1}C_0x + {}^{n-1}C_1x^2 + \ldots + {}^{n-1}C_{r-1}x^r + \ldots + {}^{n-1}C_{n-2}x^{n-1} + {}^{n-1}C_{n-1}x^n$$

Adding the above two lines, the left-hand side becomes: $(1+x)^{n-1} + x(1+x)^{n-1} = (1+x)(1+x)^{n-1} = (1+x)^n$

Hence:

$$(1+x)^n = {}^{n-1}C_0 + \left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \left({}^{n-1}C_1 + {}^{n-1}C_2\right)x^2 + \ldots + \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)x^r + \ldots + \left({}^{n-1}C_{n-2} + {}^{n-1}C_{n-1}\right)x^{n-1} + {}^{n-1}C_{n-1}x^n$$

Comparing these two expansions for $(1+x)^n$, you now have:

$$(1+x)^n = {}^nC_0 + {}^nC_1x + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n \qquad [1]$$

$$(1+x)^n = {}^{n-1}C_0 + \left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \ldots + \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)x^r + \ldots + {}^{n-1}C_{n-1}x^n \qquad [2]$$

Equating the coefficients of x^r: $\quad {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{for } 1 \le r \le n-1$

In terms of Pascal's triangle, this result is equivalent to saying that a term in one row is the sum of the two terms in the row above to the left and to the right (e.g. ${}^5C_3 = {}^4C_2 + {}^4C_3$).

The formula for ${}^nC_r$

The formula for ${}^nC_r$ can be proved by mathematical induction. (For a discussion of induction, see Chapter 9.)

When $n = 4$, $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

The ratios of successive coefficients are:

- $\frac{4}{1} = \frac{n}{1}$
- $\frac{6}{4} = \frac{3}{2} = \frac{n-1}{2}$
- $\frac{4}{6} = \frac{2}{3} = \frac{n-2}{3}$
- $\frac{1}{4} = \frac{n-3}{4}$

Thus, the coefficients in this expansion from left to right are:

- 1
- $\frac{n}{1}$
- $\frac{n}{1} \times \frac{n-1}{2}$
- $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$

This may lead you to guess that perhaps ${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times r}$, $1 \leq r \leq n$.

Call this statement $S(n)$. The coefficient of ${}^nC_0$ has already been shown to be 1.

Step 1 Prove that statement $S(n)$ is true for $n = 2$.

$(1+ x)^2 = 1 + 2x + x^2 = {}^2C_0x^0 + {}^2C_1x^1 + {}^2C_2x^2$

$\therefore {}^2C_0 = 1, {}^2C_1 = \frac{2}{1}, {}^2C_2 = \frac{2}{1} \times \frac{2-1}{2} = 1$

$\therefore S(2)$ is true.

Step 2 Assume $S(n-1)$ is true (to show that if $S(n-1)$ is true, then $S(n)$ is true).

i.e. assume that ${}^{n-1}C_r = \frac{(n-1)(n-2)(n-3)\ldots(n-r)}{1 \times 2 \times 3 \times \ldots \times r}$ is true.

First, for $r = 1$: assume that ${}^{n-1}C_1 = \frac{n-1}{1}$ from the statement $S(n-1)$.

It has previously been shown that ${}^{n-1}C_0 = 1$.

Adding these results using the Pascal's triangle relation (also shown previously):

${}^nC_1 = {}^{n-1}C_0 + {}^{n-1}C_1$

${}^nC_1 = 1 + \frac{n-1}{1} = \frac{n}{1}$, which agrees with the statement $S(n)$.

Now, for $r \geq 2$, i.e. $r - 1 \geq 1$: assume that ${}^{n-1}C_{r-1} = \frac{(n-1)(n-2)(n-3)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times (r-1)}$ from the statement $S(n-1)$. Again using the Pascal's triangle relation:

$$
\begin{aligned}
{}^nC_r &= {}^{n-1}C_{r-1} + {}^{n-1}C_r \\
&= \frac{(n-1)(n-2)(n-3)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times (r-1)} + \frac{(n-1)(n-2)(n-3)\ldots(n-r)}{1 \times 2 \times 3 \times \ldots \times r} \\
&= \frac{(n-1)(n-2)(n-3)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times (r-1)}\left[1 + \frac{n-r}{r}\right] \\
&= \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times r} \text{ as } 1 + \frac{n-r}{r} = \frac{n}{r}
\end{aligned}
$$

Assuming $S(n-1)$ is true, this now proves the statement $S(n)$ for $r \geq 2$ (which was also already proved for $r = 1$ above).

Step 3 Conclusion

$S(n)$ is true for $r \geq 1$ if $S(n-1)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for $n \geq 2$, $r \geq 1$

Note: This formula for ${}^nC_r$ can now be used to obtain binomial expansions without having to construct the rows of Pascal's triangle.

Another useful form of this formula is: $${}^nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Binomial theorem—alternative approach

The expansion of $(1+x)^n$ can be thought of as the product of n factors, each of which is $(1+x)$. The term containing x^r must be found by selecting x out of any r of the n factors and 1 out of the remaining $(r-1)$ factors. The coefficient of x^r is therefore equal to the number of ways of selecting r lots of x from a total of n lots of x. Hence by definition the coefficient of x^r is ${}^nC_r$. By making $r = 0, 1, 2, 3, \ldots, n$ in succession you can obtain the coefficient of each term. Thus:

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + x^n = \sum_{r=0}^{n} {}^nC_rx^r$$

The coefficients ${}^nC_0$ and ${}^nC_n$ are both equal to 1.

Making $x = \frac{b}{a}$ and multiplying both sides by a^n gives the expansion for $(a+b)^n$:

$$(a+b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + b^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

Note:

- The expansion of $(a+b)^n$ contains $(n+1)$ terms. ${}^nC_ra^{n-r}b^r$ is the $(r+1)$-th term and is called the **general term**.
- If you write $-b$ in place of b, the expansion becomes:
  $$(a-b)^n = a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 - \ldots + {}^nC_ra^{n-r}(-b)^r + \ldots + (-b)^n = \sum_{r=0}^{n} (-1)^r {}^nC_ra^{n-r}b^r$$
  The terms are alternately positive and negative.
- Because $(a+b)^n = (b+a)^n$ and ${}^nC_r = {}^nC_{n-r}$, any term from one end of the expansion of $(a+b)^n$ is the same term counted from the opposite end of the expansion of $(b+a)^n$.

Example 41

Write the expansion of $(2x-y)^5$.

Solution

$$\begin{aligned}(2x-y)^5 &= (2x)^5 - {}^5C_1(2x)^4y + {}^5C_2(2x)^3y^2 - {}^5C_3(2x)^2y^3 + {}^5C_4(2x)y^4 - y^5\\ &= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5\end{aligned}$$

Example 42

Find the coefficient of x^4 in each expansion. **(a)** $(1+x)^6$ **(b)** $(3x-2)^7$

Solution

(The alternative notation $\binom{n}{r} = {}^nC_r$ is used below. Either notation may be used.)

Method 1

(a) $(1+x)^6 = 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \ldots$

The expansion does not need to be written out in full; only far enough to find the required term.

Coefficient of x^4: $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6\times5\times4!}{4!\times2} = \frac{6\times5}{2} = 15$

(Recall from earlier in the chapter that $\binom{6}{4} = \binom{6}{6-4} = \binom{6}{2} = \frac{6\times5}{2\times1} = 15$, which can simplify the calculation.)

(b) $(3x-2)^7 = (3x)^7 - {}^7C_1(3x)^6\times2 + {}^7C_2(3x)^5\times2^2 - {}^7C_3(3x)^4\times2^3 + \ldots$

Term in x^4: $(-1)\times{}^7C_3\times3^4\times2^3x^4 = -\frac{7\times6\times5}{3\times2\times1}\times81\times8x^4$

Coefficient of x^4: $-22\,680$

Method 2

(a) In the expansion of $(1+x)^6$, $T_{r+1} = \binom{6}{r}x^r$

Term in x^4: $T_5 = \binom{6}{4}x^4 = \frac{6\times5}{2\times1}x^4 = 15x^4$

Coefficient of x^4: 15

(b) In the expansion of $(3x-2)^7$, $T_{r+1} = \binom{7}{r}(3x)^{7-r}(-2)^r$

The term for x^4 requires $7-r=4$, so $r=3$.

Term in x^4: $T_4 = \binom{7}{3}(3x)^4(-2)^3 = -\frac{7\times6\times5}{3\times2\times1}\times81\times8x^4 = -22\,680x^4$

Coefficient of x^4: $-22\,680$

Example 43

Write the sixth term in the expansion of $(2x-3y)^8$.

Solution

The general term is ${}^8C_r(2x)^r(-3y)^{8-r}$. This is the $(r+1)$-th term: $T_{r+1} = {}^8C_r(2x)^r(-3y)^{8-r}$

For the sixth term, $r+1=6$, so $r=5$: $T_6 = {}^8C_5(2x)^5(-3y)^3$ (Remember ${}^8C_5 = {}^8C_3$)

$= (-1)^3\frac{8\times7\times6}{3\times2\times1}\times2^5\times3^3x^5y^3$

$= -48\,384x^5y^3$

EXERCISE 6.9 PASCAL'S TRIANGLE RELATIONS AND THE BINOMIAL THEOREM

1 Without using a calculator, find the value of:

(a) ${}^5C_2$ **(b)** ${}^8C_4$ **(c)** ${}^{10}C_6$ **(d)** $\binom{7}{3}$ **(e)** $\binom{9}{4}$ **(f)** $\binom{15}{12}$

2 Write the expansion of: **(a)** $(1+x)^6$ **(b)** $(2x+3y)^4$ **(c)** $(x^2+a)^5$ **(d)** $\left(a+\frac{1}{a}\right)^6$

(e) $(x^2-3y^2)^4$ **(f)** $\left(\frac{2a}{3}-\frac{3}{2b}\right)^5$ **(g)** $\left(2y-\frac{3x^2}{2}\right)^4$ **(h)** $(x^3-2x^{-2})^6$ **(i)** $\left(\frac{a^2}{2}-3b\right)^5$

3 Find the coefficient of x^3 for each expansion.

(a) $(x+2)^5$ **(b)** $(x-3)^4$ **(c)** $(4-x)^3$ **(d)** $(3+x)^6$ **(e)** $(2x+1)^7$ **(f)** $(3x-2)^8$

4 Write the fourth term in the expansion of:

(a) $(x+2)^5$ **(b)** $(x-3)^4$ **(c)** $(4-x)^3$ **(d)** $(3y+x)^6$ **(e)** $(2x+y)^7$ **(f)** $(3x-2z)^8$

5 The fourth term in the expansion of $\left(\frac{a}{b}-\frac{b}{a}\right)^6$ is:

A -20 **B** 20 **C** $\frac{15a^2}{b^2}$ **D** $-\frac{15b^2}{a^2}$

6 Write and simplify the required term for each.

(a) the fourth term of $\left(\frac{m}{2}+3n\right)^8$ **(b)** the seventh term of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^{11}$

(c) the fifth term of $\left(a-\frac{2b}{3}\right)^7$ **(d)** the fourth term of $\left(x-\frac{1}{x}\right)^n$

7 Find the coefficient of:

(a) x^2 in $\left(x+\frac{1}{x}\right)^8$ **(b)** x^{17} in $(2x^2-3x)^{12}$ **(c)** a^5b^4 in $\left(3a-\frac{b}{3}\right)^9$ **(d)** a^9 in $\left(2a-\frac{3}{a}\right)^{13}$

8 Write and simplify the $(k+2)$-th term of $(a+b)^{2n}$.

CHAPTER REVIEW 6

1 A coin is tossed, a die is rolled, and a card is selected from a standard pack of 52 playing cards and its suit noted. Which would be the best form to display the possible outcomes of this experiment: table, list or tree diagram?

2 A bag contains four discs, each of a different colour. How many different arrangements can be made of four discs if:

(a) they are selected at random one at a time and placed in a row until the bag is empty

(b) one disc is selected at random, its colour is noted and it is put back in the bag before the next disc is selected?

3 How many different arrangements can be made of the five vowels *a*, *e*, *i*, *o*, *u*?

4 How many different four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, 7, 8 if:

(a) none of the digits are repeated **(b)** the digits may be repeated?

5 How many four-digit or five-digit numbers greater than 4000 can be formed using 0, 2, 4, 5, 7 if:

(a) none of the digits are repeated

(b) the digits may be repeated

(c) none of the digits are repeated and the numbers formed are even?

6 **(a)** How many arrangements of the letters in the word ISOSCELES are possible?

(b) How many arrangements of the letters in the word ISOSCELES are possible if the consonants and vowels must alternate?

7 In how many ways can a cycling team of four riders be selected from a squad of ten riders?

8 A committee of six is chosen from 14 people. In how many ways can this be done?

9 A mixed volleyball team of eight players is selected from eight males and nine females. In how many ways can this be done if the team must have an equal number of male and female players?

10 A committee of five is to be selected from a class of 30 students.

(a) In how many ways can this be done?

(b) After the committee is selected, one person is elected chairperson and another is elected secretary. In how many ways can these positions be filled?

11 In how many ways can five people be seated in a five-seater car if only two people have a licence to legally occupy the driver's seat?

12 **(a)** In how many ways can a committee of three parents and four teachers be chosen from six parents and eight teachers?

(b) The members of the committee are seated in a row of seven chairs on a stage. In how many different ways can they be seated?

(c) Find the number of ways of seating this committee if the parents refuse to sit at the ends of the row.

(d) What is the probability that for a chosen committee, the parents do not sit at the ends of the row?

13 A box contains four marbles labelled 2, 3, 4, 6. If two marbles are drawn at random from the box, what is the probability that their sum is greater than 7?

14 The letters of the word CONSIDER are rearranged. What is the probability that the rearranged set of letters will begin with 'SI'?

15 Three-digit numbers are formed using the digits 1, 2, 3, 4, 5, 6, 7. If none of the digits are repeated, what is the probability of forming an even number?

16 The letters of the word ARRANGEMENT are arranged in a row.

(a) In how many ways can this be done using all the letters?

(b) What is the probability that the vowels are all together?

17 Three cards are selected from a standard pack of 52 playing cards. What is the probability that at least one of the cards will be a King?

18 If a coin is tossed six times, what is the probability of obtaining exactly three heads?

19 Seven-digit numbers are formed using all the digits 2, 2, 3, 4, 6, 7, 7.

(a) How many different numbers can be formed?

(b) What is the probability that the number formed is odd?

(c) What is the probability that the number formed is greater than 4 000 000?

(d) What is the probability that the number formed is less than 7 000 000?

20 A test consists of six multiple-choice questions. Each question has four possible answers: one that is correct and three that are incorrect. Amanda randomly selects an answer to each of the six questions.

(a) What is the probability that Amanda selects four correct and two incorrect answers?

(b) What is the probability that Amanda selects four or more correct answers?

(c) What is the probability that Amanda selects at least one incorrect answer?

21 Four ordinary six-sided dice are rolled. What is the probability that exactly three of the dice roll a 5? Leave your answer in unsimplified form.

22 A four-person team is chosen at random from seven women and nine men.

(a) In how many ways can the team be chosen?

(b) What is the probability that the team will consist of four men?

23 Mr and Mrs Zeno and their four children go to the theatre. They are randomly allocated six adjacent seats in a single row. What is the probability that Mr and Mrs Zeno are allocated seats next to each other?

CHAPTER 7
Rates of change and their application

7.1 RATES OF CHANGE WITH RESPECT TO TIME

The gradient as a rate of change was introduced in the Mathematics Advanced course. This chapter will extend that topic.

There are many practical situations in which the change in a physical quantity can be described mathematically. The speed of a car depends on the rate at which the distance it travels changes with respect to time. The rate of change in the population of Australia depends on the number of people alive at a given time, together with the birth rate, the death rate and the immigration/emigration rate.

You have seen that the rate of change of a function is the derivative of that function. Hence if you have information about a real-life situation expressed as a function of time, then the rate of change of this can be calculated by finding the derivative of this function.

Example 1

The volume of water in a cylindrical tank of constant cross-section is given by $V = 25\pi(100 - t)\,\text{m}^3$, where t is the time in minutes.

(a) How much water is in the tank initially?

(b) If $\frac{dV}{dt}$ gives the rate at which water flows out from the bottom of the tank when a tap is opened, find $\frac{dV}{dt}$ and comment on your answer.

(c) How much time does it take for the tank to empty?

Solution

(a) $t = 0$: $V = 25\pi \times 100 = 2500\,\pi\,\text{m}^3$

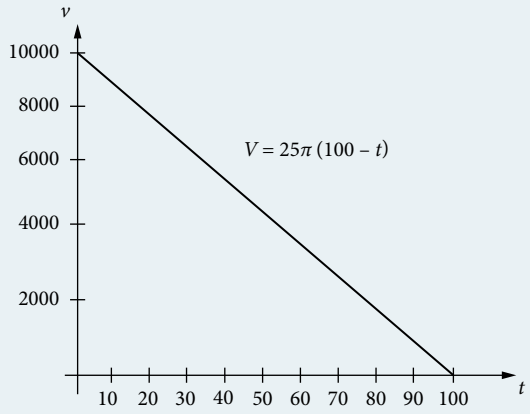
(b) $\frac{dV}{dt} = -25\pi$

The water is flowing out of the tank at a constant rate.

(c) The tank is empty when $V = 0$: $25\pi(100 - t) = 0$

$t = 100$ minutes

This could also be solved by considering where the graph of $V = 25\pi(100 - t)$ cuts the t-axis.



The graph cuts the t-axis (that is, reaches $V = 0$) at $t = 100$, so the tank is empty after 100 minutes.

Example 2

A large cube of ice has edges of length 10 cm. As it melts its volume decreases at a constant rate of 25 cm^3 per hour.

(a) What is the initial volume of the cube of ice?

(b) What is the volume of ice remaining after 2 hours?

(c) Obtain a formula for the volume $V\,cm^3$ remaining after t hours.

(d) How much time will it take for the ice to melt completely?

Solution

(a) Volume of cube of ice $= 10^3 = 1000\,cm^3$.

(b) After 2 hours, the volume melted $= 25 \times 2 = 50\,cm^3$.

Volume remaining $= 1000 - 50 = 950\,cm^3$.

(c) After t hours, the volume melted $= 25 \times t = 25t\,cm^3$.

$V = 1000 - 25t$.

(d) $V = 0$: $1000 - 25t = 0$, so $t = 40$ hours.

It takes 40 hours for the cube of ice to melt completely.

EXERCISE 7.1 RATES OF CHANGE WITH RESPECT TO TIME

1 When concentrated chemical solutions are allowed to evaporate slowly, crystals are formed. The surface area of a particular crystal is given by $A = 0.8t^2$, where A is mm^2 and t is days of evaporation. The rate at which the surface area is increasing after 4 days is:

A 0.8 $mm^2\,day^{-1}$ **B** 1.6 $mm^2\,day^{-1}$ **C** 6.4 $mm^2\,day^{-1}$ **D** 12.8 $mm^2\,day^{-1}$

2 The length of the sides of a square, x cm, is given by $x = 4t + 1$ where t is in seconds.

(a) At what rate is the length of the side of the square increasing at t seconds?

(b) At what rate is the length of the side of the square increasing when $t = 5$ seconds?

(c) Write an expression for the area A cm^2 of the square as a function of t.

(d) At what rate is the area of the square increasing when $t = 5$ seconds?

3 A petrol pump delivers petrol at the rate of 4 litres per minute.

(a) How much petrol is delivered in 5 minutes?

(b) If V litres of petrol are delivered in t minutes at this rate, find an expression for V in terms of t.

(c) How much time will it take to deliver 45 litres of petrol?

4 A tank holds 50 000 litres of water. The water drains from the tank in 40 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{40}\right)^2$, where V is measured in litres.

(a) Find the rate at which the water is draining from the tank after: **(i)** 5 minutes **(ii)** 10 minutes **(iii)** 20 minutes.

(b) How much water remains in the tank after 20 minutes?

(c) How much time will it take until only half the initial volume of water remains in the tank?

5 The volume of water in a tank is given by $V = 1000 - 2t + \frac{t^2}{1000}$, where V is measured in litres and t is in minutes.

(a) How much water is in the tank initially, assuming it was full?

(b) Find an expression for $\frac{dV}{dt}$ as a function of t.

(c) At what rate is the water flowing out of the tank at 25 minutes?

(d) How much time will it take to empty the tank at this rate?

6 The rate at which solvent in a nail polish evaporates is given by $\frac{dV}{dt} = \frac{1}{500}\left(1 - \frac{t}{60}\right)$, where V mL is the volume of solvent present and t is in seconds.

(a) What is the initial rate of evaporation of the solvent?

(b) When does the evaporation of the solvent stop?

7 An electric current exists whenever electric charges move through a surface. The quantity of charge Q in coulombs (C) that has passed through a surface after time t, measured in seconds, is given by $Q = t^3 - 6t^2 + 12t + 5$. The current, in amperes (A), where 1 A = 1 C s^{-1} is given by $\frac{dQ}{dt}$. Find the current when:

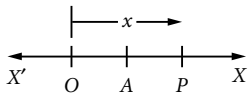
(a) $t = 0.5$ s **(b)** $t = 1$ s.

(c) What is the initial charge? **(d)** When does the current stop flowing?

7.2 VELOCITY AND ACCELERATION AS RATES OF CHANGE

Particle is the term used for a body that behaves as if all forces acting on the body are acting through a single point. This means that the body can be represented as a single point, regardless of its actual size and shape. This definition of a particle means that quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement

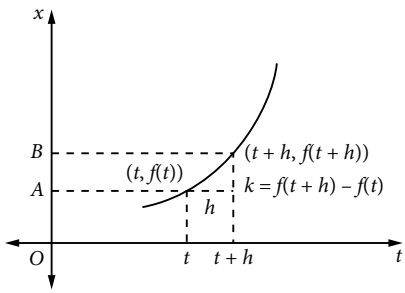


Consider a particle, represented by a point P, moving in a straight line $X'OX$.

The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O. The origin of the motion is not necessarily at O, so when $t = 0$, P may be (for example) at the point A.

Velocity

Consider the equation $x = f(t)$, which gives the position coordinate x of a particle moving in a straight line at time t.



At time t, the particle is at A, and at time $(t + h)$ the particle is at B, as shown in the diagram. Thus in the small time interval h the particle has changed its position by an amount $k = f(t + h) - f(t)$.

The average **velocity** in this time interval $= \frac{k}{h} = \frac{f(t+h) - f(t)}{h}$, $h \neq 0$.

The instantaneous velocity of the particle at time t is defined by $\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$. This may be denoted by $v(t)$, $f'(t)$, $\frac{dx}{dt}$ or $\dot{x}$:

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.
Velocity can be positive or negative, depending on the direction of travel.
Speed is the magnitude of the velocity and is always positive.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time.
Acceleration, like velocity, can be positive or negative.
Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing (which is often called deceleration or retardation).

(*Note:* 'increasing velocity' is not necessarily 'faster speed'; it only means acceleration in the direction of positive displacement.)

If velocity is denoted by $v(t)$, then the average acceleration over the time interval from t to $(t+h)$ is $\frac{v(t+h)-v(t)}{h}$.

The instantaneous acceleration at time t is defined by $\lim\limits_{h\to 0}\frac{v(t+h)-v(t)}{h}$. This may be denoted by $v'(t)$, $a(t)$, $f''(t)$, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$ or $\ddot{x}$:

$$a(t)=v'(t)=\frac{d^2x}{dt^2}=\ddot{x}=\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$$

Summary of important motion terms

'initially':	$t=0$	'at the origin':	$x=0$
'at rest':	$v=0$	'velocity is constant':	$a=0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
Constant acceleration due to gravity on Earth can be assumed to be $9.8\,\text{m s}^{-2}$ ($\approx 10\,\text{m s}^{-2}$).

Example 3

A particle is moving in a straight line, so that its displacement x metres from a fixed point O on the line at time t seconds ($t \geq 0$) is given by $x = t^3 - 2t^2 - 4t$.

(a) Determine expressions for the velocity and acceleration of the particle.

(b) Find when the particle is at rest.

(c) Find when the acceleration is negative.

(d) By drawing the graph of the velocity function, comment on the velocity when the acceleration is negative.

(e) When is the acceleration positive? What does this mean for the velocity of the particle?

Solution

(a) $x = t^3 - 2t^2 - 4t$

Differentiate: $\dfrac{dx}{dt} = 3t^2 - 4t - 4$

$\therefore v = 3t^2 - 4t - 4$

Differentiate again: $\dfrac{dv}{dt} = 6t - 4$

$\therefore a = 6t - 4$

(b) At rest when $v = 0$: $3t^2 - 4t - 4 = 0$

$(3t + 2)(t - 2) = 0$

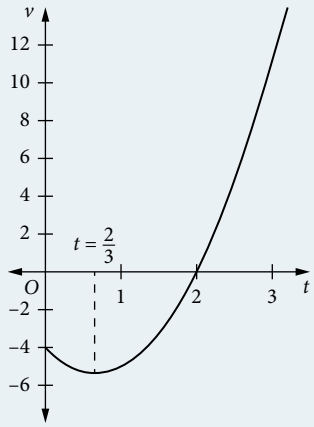
$t = -\dfrac{2}{3}, 2$

As $t \geq 0$, the particle is at rest after 2 seconds.

(c) $a < 0$: $6t - 4 < 0$

$t < \dfrac{2}{3}$

(d)



When $t < \dfrac{2}{3}$ the velocity function is a decreasing function and the velocity is negative. It obtains its least value when $t = \dfrac{2}{3}$.

(e) The acceleration is positive when $t > \dfrac{2}{3}$. This means the velocity is increasing when $t > \dfrac{2}{3}$, and since the graph of the velocity is getting steeper it means the velocity is increasing at an increasing rate.

EXPLORING FURTHER

Displacement, velocity, acceleration

Use technology to explore the relationship between the displacement, velocity and acceleration of a particle.

EXERCISE 7.2 VELOCITY AND ACCELERATION AS RATES OF CHANGE

1 A particle is moving in a straight line, so that its displacement x metres from a fixed point O on the line at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 - 4t$.

(a) Find the velocity and acceleration of the particle at any time t.
(b) Find the initial velocity and acceleration.
(c) When is the particle at rest?
(d) When is the acceleration zero? What is the velocity and displacement at this time?

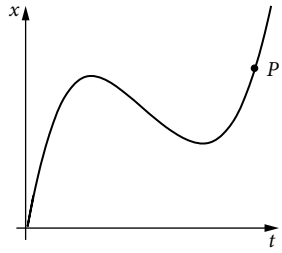
2 A particle is moving in a straight line so that its displacement x metres from a fixed point O on the line at time t seconds ($t \geq 0$) is given by $x = t^3 - 3.5t^2 - 6t$.

(a) Find expressions for the velocity and acceleration of the particle.
(b) Find when the particle is at rest.
(c) Find when the acceleration is negative.
(d) By drawing the graph of the velocity function, comment on the velocity when the acceleration is negative.
(e) When is the acceleration positive? What does this mean for the velocity of the particle?

3 A particle moves in a straight line. Its velocity v ms^{-1} at time t is given by $v = 5 - \dfrac{10}{t+1}$.

(a) Find the initial velocity.
(b) Find the acceleration of the particle when the particle is at rest.
(c) Sketch the graph of v for $t \geq 0$, showing any intercepts and asymptotes.

4 The graph shows the displacement x of a particle moving along a straight line as a function of time t.



Which statement best describes the motion of the particle at the point P?

A The velocity is negative and the acceleration is positive.
B The velocity is negative and the acceleration is negative.
C The velocity is positive and the acceleration is positive.
D The velocity is positive and the acceleration is negative.

5 The displacement of a particle moving along the x-axis is given by $x = 2t - \dfrac{1}{t+1}$, where x is the displacement from the origin in metres, t is in seconds and $t \geq 0$.

(a) Find the expression for the velocity v and draw the graph of v against t.
(b) What value does the velocity approach as t increases indefinitely?
(c) Find the expression for the acceleration a and draw the graph of a against t.
(d) Show that the acceleration of the particle is always negative.

6 A particle is moving along the x-axis. The displacement of the particle at time t is x metres. At a certain time, $v = -4$ m s^{-1} and $a = 3$ m s^{-2}.

Which statement describes the motion of the particle at that time?

A The particle is moving to the left with decreasing speed.
B The particle is moving to the left with increasing speed.
C The particle is moving to the right with decreasing speed.
D The particle is moving to the right with increasing speed.

7 A driver takes 3 hours to travel the distance between two points A and B on a country road. At time t hours after passing A, the driver's speed v km h^{-1} is given by $v = 60 + 40e^{-t}$.

(a) Calculate the speeds when the driver passes points A and B.
(b) Write the acceleration in terms of: **(i)** t **(ii)** v
(c) Sketch the velocity–time curve and comment on the motion for large values of t.

8 A particle moves in a straight line so that its displacement x from a fixed origin at any time t is given by $x(t) = 2(1 - e^{-t})$.

(a) Find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$. **(b)** Sketch the graph of $x(t)$. **(c)** Find t when $x(t) = 1$.

9 A body starts from O and moves in a straight line. At any time t its velocity is given by $\dot{x} = 6t - 4$. Indicate whether each statement below is correct or incorrect.

(a) $x = 3t^2 - 4t + C$ **(b)** $x = 3t^2 - 4t$ **(c)** $\ddot{x} = 3t^2 - 4t$ **(d)** $\ddot{x} = 6$

10 A body starts from O and moves in a straight line. At any time t, its velocity is $t^2 - 4t^3$. Find, in terms of t:

(a) the displacement x **(b)** the acceleration.

11 The velocity $v\,\text{m s}^{-1}$ at time t seconds ($t \geq 0$) of a body moving in a straight line is given by $v = 6t^2 + 6t - 12$. Find the acceleration at any time t.

12 A particle is projected vertically upwards from a point O with an velocity of $25\,\text{m s}^{-1}$ and a downward acceleration of $10\,\text{m s}^{-2}$.

(a) Find its velocity and height above O at any time t.
(b) What maximum height does the particle reach?
(c) At what time has its velocity been reduced to half the velocity of projection?

13 A body is projected vertically upwards with an initial velocity of $30\,\text{m s}^{-1}$. It rises with a deceleration of $10\,\text{m s}^{-2}$.

(a) Find its velocity at any time t. **(b)** Find its height h m above the point of projection at any time t.
(c) Find the greatest height reached. **(d)** Find the time taken to return to the point of projection.

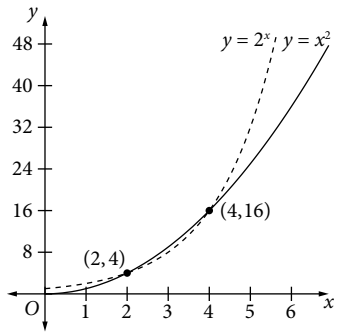
14 The velocity $v\,\text{m s}^{-1}$ of a body moving in a straight line is given by $v = 3t^2 - 2t - 1$. Find the acceleration at any time t.

7.3 EXPONENTIAL GROWTH AND DECAY

A quantity has **exponential growth** when it increases by a constant percentage of its whole over a given time period. This means that the larger the quantity at the start, the bigger the increase will be. For example: if a country of 100 million people and a country of 10 million people are each growing their populations at an annual rate of 10%, then the first country's population increases by 10 million next year while the second country's population increases by only 1 million.

The following table shows values of x^2 (quadratic growth) and 2^x (exponential growth) for selected values of x. Corresponding graphs for $x \geq 0$ are shown at right. Note how exponential growth far exceeds quadratic growth as x increases beyond the value $x = 4$.

x	0	1	2	3	4	5	6	10	100
x^2	0	1	4	9	16	25	36	100	10^4
2^x	1	2	4	8	16	32	64	1024	$\approx 10^{30}$



In 1972, a political think-tank called the Club of Rome published *The Limits to Growth*, a controversial but influential study of the possible interactions between various kinds of linear and exponential growth and decline in global systems of population, pollution, natural resources, food and other factors.

The general principle behind their models of exponential growth can be appreciated if you imagine the vertical axis of the exponential graph above to represent the growth of a quantity (e.g. population or pollution) and the horizontal axis to represent time. An increase of 1 unit of x ('time') from $x = 1$ to $x = 2$ causes only a small absolute increase in y, from $y = 2$ to $y = 4$. You could imagine this might represent world population growth in past centuries. However, an increase of one unit of 'time' from $x = 10$ to $x = 11$ causes y to increase by more than 1000.

This illustrates the enormous increases that can occur when an exponentially growing variable becomes large enough to reach the steeply rising part of the curve. (Of course, global population growth is more complicated than a simple exponential curve, but the general principle of exponential change is the important point to remember.) Exponential decline can be equally dramatic, which is one reason for concern about the use of non-renewable resources.

There are many examples of exponential growth and decay in the physical world, including population growth, the growth of bacteria, radioactive decay, rates of heating and cooling, the decrease in atmospheric pressure at higher altitudes, the decrease in light intensity through water or glass, and the fading away of sound vibrations. In each case, the quantities follow the exponential law (exactly or approximately):

- $y = Ae^{kx}$ for exponential growth
- $y = Ae^{-kx}$ for exponential decay (or 'exponential decline').

For $y = Ae^{kx}$, you have: $\frac{dy}{dx} = kAe^{kx}$
$= ky$

$\therefore \frac{dy}{dx} = ky$ is the exponential rate of change.

This means that exponential growth occurs when the rate of change of a quantity y with respect to another quantity x is proportional to y. For example:

- The rate of growth of a colony of bacteria is proportional to the number of bacteria N present at any time, i.e. $\frac{dN}{dt} = kN$.
- The rate of decay of a radioactive isotope is proportional to the mass of that isotope present at any time, i.e. $\frac{dM}{dt} = -kM$. (The negative sign indicates decay, meaning the amount is reducing over time.)
- The rate of cooling of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, i.e. $\frac{d\theta}{dt} = -k\theta$, where θ is the temperature difference at any time. (This is Newton's 'law of cooling'.)
- The rate of decrease of atmospheric pressure with respect to height above sea level is proportional to the pressure at that height, i.e. $\frac{dP}{dh} = -kP$.
- Light passing through a transparent medium loses its intensity. The rate of loss of light intensity with respect to the distance is proportional to the light's intensity at the distance, i.e. $\frac{dI}{dx} = -kI$.

In each case there is a **differential equation** of the form $\frac{dy}{dx} = ky$. A differential equation is an equation that involves derivatives.

It is important to understand the role played by the constants A and k in the equation $y = Ae^{kx}$.

- At $x = 0$, $y = Ae^0 = A$, so if $x \geq 0$ then A is the initial value of y.
- Because $\frac{dy}{dx} = ky$, k is the **growth rate** and influences the slope of the curve.

For $y = Ae^{kx}$: • A is the value of y when $x = 0$ • k is the growth rate.

Example 4

The annual growth rate of the population of two towns P and Q are 10% and 5% respectively of their populations at any time. If the initial population of P is 20 000 and of Q is 10 000, find their populations 3 years later.

Solution

Let N be the population at any time t years.

For P: $\frac{dN}{dt} = 0.1N$

$\therefore N = Ae^{0.1t}$

At $t = 0$: $N = 20\,000$, so $A = 20\,000$

So: $N = 20\,000e^{0.1t}$

At $t = 3$: $N = 20\,000e^{0.3}$

$N \approx 27\,000$

For Q: $\frac{dN}{dt} = 0.05N$

$\therefore N = Ae^{0.05t}$

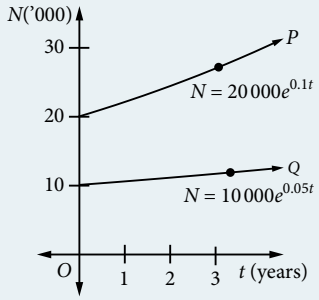
At $t = 0$: $N = 10\,000$, so $A = 10\,000$

So: $N = 10\,000e^{0.05t}$

At $t = 3$: $N = 10\,000e^{0.15}$

$N \approx 12\,000$

The answers are given to the nearest thousand.



The graphs indicate the roles played by A and k. Because A is the value of N when $t = 0$, it is the point where the curve crosses the N-axis.

The graph for P is steeper than the graph for Q. This indicates the role of k, the growth rate. The population of P grows more rapidly than the population of Q.

Note that exponential growth or decay is continuous, so k indicates the **instantaneous** rate of change. Compare this to the growth of money invested at compound interest: interest is calculated periodically, so change only happens at fixed intervals (e.g. daily, monthly, quarterly, yearly).

Example 5

A vessel containing water is being emptied. The volume $V(t)$ cubic metres of water remaining in the vessel after t minutes is given by $V(t) = Ae^{-kt}$.

(a) If $V(0) = 100$, find the value of A. **(b)** If $V(5) = 90$, find the value of k. **(c)** Find $V(20)$.

Solution

(a) When $t = 0$, $V = 100$, so: $A = 100$

(b) When $t = 5$, $V = 90$, so: $90 = 100e^{-5k}$

$0.9 = e^{-5k}$

$\therefore -5k = \log_e 0.9$

$k = -0.2 \log_e 0.9 \approx 0.02$

(c) $V(t) \approx 100e^{-0.02t}$

$V(20) \approx 100e^{-0.4} \approx 67$ (Note that without rounding, this answer is closer to 66.)

Example 6

The pressure of the atmosphere, measured as P kilopascals (kPa), decreases with the altitude h km above sea level approximately according to $P = 101e^{-0.2h}$. Find the rate at which the air pressure falls with respect to height above sea level when: **(a)** $h = 5$ **(b)** $P = 20$

Solution

$$P = 101e^{-0.2h}$$

$$\frac{dP}{dh} = -0.2 \times 101e^{-0.2h}$$

$$\frac{dP}{dh} = -20.2e^{-0.2h} \qquad [1]$$

or: $$\frac{dP}{dh} = -0.2P \qquad [2]$$

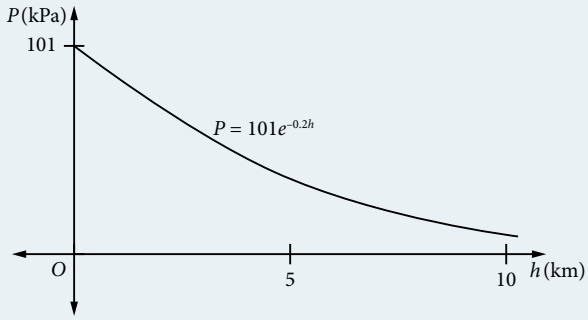
Here [1] gives $\frac{dP}{dh}$ as a function of h, while [2] gives $\frac{dP}{dh}$ as a function of P.

(a) $h = 5$: $\frac{dP}{dh} = -20.2e^{-1} = -7.43$

The pressure falls at a rate of 7.43 kPa/km when $h = 5$.

(b) $P = 20$: $\frac{dP}{dh} = -0.2 \times 20 = -4$

The pressure falls at a rate of 4 kPa/km when $P = 20$.



Example 7

The mass M of a radioactive substance is initially 10 grams. Twenty years later the mass of remaining radioactive substance is 9.6 grams.

(a) Find the annual decay rate, given that the rate of decay of a radioactive substance is proportional to the mass of the substance present at any time.

(b) In how many years will the mass of radioactive substance be halved?

Solution

It is known that $\frac{dM}{dt} = -kM$. You know that $M = Ae^{-kt}$ is a solution to this differential equation.

(a) $t = 0$, $M = 10$, so: $A = 10$

$t = 20$, $M = 9.6$: $9.6 = 10e^{-20k}$

$$\therefore -20k = \log_e 0.96$$

$$k = -\tfrac{1}{20}\log_e 0.96 \approx 0.002$$

(b) If $k = 0.002$: $M = 10e^{-0.002t}$

$M = 5$: $5 = 10e^{-0.002t}$

$$e^{-0.002t} = 0.5 \quad \textbf{or} \quad e^{0.002t} = 2$$

$$\therefore 0.002t = \log_e 2$$

$$t = 500\log_e 2 \approx 347$$

The mass of radioactive substance will be halved after about 347 years.

Note: The time taken for half of an amount of radioactive substance to decay is called the **half-life** of the substance.

EXERCISE 7.3 EXPONENTIAL GROWTH AND DECAY

1 If $\frac{dy}{dx} = 2y$ and $y = 5$ where $x = 0$, express y as a function of x.

2 If $\frac{dN}{dt} = -0.5N$ and $N = 100$ when $t = 0$, then N expressed as a function of t is:

A $N = 100e^{0.5t}$ **B** $N = 100e^{-0.5t}$ **C** $N = 0.5e^{100t}$ **D** $N = 0.5e^{-100t}$

3 If $\frac{dQ}{dt} = 0.4Q$ and $Q = 50$ when $t = 0$, express Q as a function of t.

4 If $\frac{dy}{dt} = -3y$ and $y = 20$ when $t = 0$, express y as a function of t.

5 If $\frac{dM}{dt} = -0.01M$ and $M = 10$ when $t = 0$, express M as a function of t.

6 If $y = Ae^{-kt}$, $y = 1000$ when $t = 0$, and $y = 368$ when $t = 2$, find the values of A and k.

7 If $N = Ae^{kt}$, $N = 200$ when $t = 0$, and $N = 1478$ when $t = 5$, find the values of A and k.

8 If $P = Ae^{-kh}$, $P = 76$ where $h = 0$, and $P = 28$ when $h = 5$, indicate whether each of the following statements is correct or incorrect.

(a) $A = 76$ **(b)** $e^{5k} = \frac{7}{19}$ **(c)** $k = 0.2\log_e\left(\frac{19}{7}\right)$ **(d)** $k \approx 0.20$

9 The population of a city increases at a rate that is proportional to the current population. If the population of the city was 100 000 in the year 2000 and 120 000 in the year 2010, express the population P in terms of t years after 2000.

10 In a certain bacterial culture, the rate of increase of bacteria is proportional to the number of bacteria present.

(a) If the number of bacteria doubles every 3 hours, find the hourly growth rate.
(b) If the original bacteria population is 10^4, what is the population after 9 hours?
(c) After how many hours are there 4×10^4 bacteria?

11 The rate of decay of a radioactive isotope is proportional to the amount of the isotope present at any time. If one-half of a given quantity of the isotope decays in 1600 years, what percentage will decay in 100 years?

12 The number of bacteria N in a colony after t minutes is given by $N = 10\,000e^{0.05t}$. Find:

(a) the number of bacteria after 10 minutes
(b) the time required for the original number to double.
(c) Find the rate at which the colony increases when: **(i)** $t = 10$ **(ii)** $N = 20\,000$

13 A vessel filled with liquid is being emptied. The volume V cubic metres of liquid remaining after t minutes is given by $V = V_0e^{-kt}$.

(a) Show that $\frac{dV}{dt} = -kV$.
(b) If one-quarter of the vessel is emptied in the first 5 minutes, what fraction remains after 10 minutes?
(c) At what rate is the liquid flowing out:
(i) after 10 minutes **(ii)** when one-quarter of the vessel is empty.

14 For a period of its life, the increase in the diameter of a tree approximately follows the rule $D(t) = Ae^{kt}$, where $D(t)$ is the diameter of the tree t years after the beginning of this period.

(a) If the diameter is initially 50 cm, find the value of A.
(b) If $D'(t) = 0.1D(t)$, find the value of k.
(c) After how many years is the diameter 61 cm?

15 The charge Q (measured in coulombs) on the plate of a condenser t seconds after it starts to discharge is given by the formula $Q = Ae^{-kt}$.

(a) If the original charge is 5000 coulombs, find the value of A.

(b) If $\frac{dQ}{dt} = -2000$ when $Q = 1000$, find the value of k.

(c) Find the rate of discharge when $Q = 5000$.

16 The rate of increase in the number N of bacteria in a certain culture is given by $\frac{dN}{dt} = 0.15N$, where t is time in hours.

(a) If the original number of bacteria is 1000, express N as a function of t.

(b) After how many hours has the original number of bacteria doubled? What is the rate of increase at this time?

17 Sunlight transmitted into water loses intensity as it penetrates to greater depths according to the law $I(d) = I(0)e^{-kd}$, where $I(d)$ is the intensity at depth d metres below the surface. If $I(300) = 0.3I(0)$, find:

(a) the value of k **(b)** the depth at which the intensity would be decreased by one-half.

18 The rate of increase of the population $P(t)$ of a particular island is given by the equation $\frac{d}{dt}P(t) = kP(t)$, where t is time in years. In the year 2000 the population was 1000 and in 2010 it had decreased to 800.

(a) Find k, the annual growth rate. **(b)** In how many years will the population be half that in 2000?

19 A substance decomposes at a rate equal to k times the mass of the substance present. If initially the mass is M, find the mass m at time t. If $k = 0.1$, find the value of t for which $m = \frac{M}{2}$.

20 A heated body is cooling. The excess of its temperature above that of its surroundings is $\theta = Ae^{-kt}$, where θ is measured in °C and t is in minutes.

(a) At time $t = 0$, $\theta = 80$. Find A.

(b) If the temperature of the surroundings is 20°C and the body cools to 70°C in 10 minutes, find:

(i) the body's temperature after 20 minutes **(ii)** the time taken to reach 60°C.

21 The number N of bacteria in a colony grows according to the rule $\frac{dN}{dt} = kN$. If the original number increases from 4000 to 8000 in 4 days, find the number after another 4 days.

22 A population of size N is decreasing according to the rule $\frac{dN}{dt} = -\frac{N}{100}$, where t is the time in days. If the population is initially of size N_0, find how much time it takes for the size to be halved, to the next day.

23 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 200 years, what percentage of the original radioactive mass will remain after 1000 years?

7.4 HARDER EXPONENTIAL GROWTH AND DECAY

This is an extension of the work on exponential growth and decay that you studied in the previous section, where, you considered the equation $\frac{dN}{dt} = kN$. That solution was found to be $N = Ae^{kt}$, where A is the initial value of N. It is a useful exercise to derive this result by integration.

Example 8

Show by integration that the solution to $\frac{dN}{dt} = kN$ is $N = Ae^{kt}$, where A is the value of N when $t = 0$.

Solution

$$\frac{dN}{dt} = kN$$

Reciprocal of both sides: $\frac{dt}{dN} = \frac{1}{kN}$

Integrate with respect to N: $t = \int \frac{1}{kN} dN$

$$t = \frac{1}{k}\log_e N + C$$

When $t = 0$, $N = A$: $0 = \frac{1}{k}\log_e A + C$

$$C = -\frac{1}{k}\log_e A$$

$$\therefore \quad t = \frac{1}{k}\log_e N - \frac{1}{k}\log_e A$$

$$t = \frac{1}{k}\log_e \frac{N}{A}$$

$$kt = \log_e \frac{N}{A}$$

$$\frac{N}{A} = e^{kt}$$

$$N = Ae^{kt}$$

Solution of $\frac{dN}{dt} = k(N - P)$ where k and P are constants

$\frac{dN}{dt} = k(N - P)$ means that the rate of change of N is proportional to the excess of N over a fixed quantity P. This can be applied to several real-life physical processes.

Given $\frac{dN}{dt} = k(N - P)$, where k and P are constants, it is easy to show that a solution of this equation is $N = P$.

If $N = P$, then $\frac{dN}{dt} = k(P - P) = 0$; because P is a constant, differentiating the equation $N = P$ with respect to t also gives $\frac{dN}{dt} = 0$.

It can also be shown by substitution that $N = P + Ae^{kt}$ is a solution to $\frac{dN}{dt} = k(N - P)$:

$$\text{LHS} = \frac{dN}{dt} = Ake^{kt}$$

$$\text{RHS} = k(N - P) = k(P + Ae^{kt} - P) = Ake^{kt} = \text{LHS}$$

It is important to be able to derive this result by integration.

$$\frac{dN}{dt} = k(N - P)$$

Reciprocal of both sides: $\frac{dt}{dN} = \frac{1}{k(N - P)}, \quad N \neq P$

Integrate with respect to N: $t = \frac{1}{k}\int \frac{1}{N-P}\,dN$

$$t = \frac{1}{k}\log_e(N-P) + C, \quad N > P$$

$$k(t-C) = \log_e(N-P)$$

$$N - P = e^{k(t-C)}$$

$$N - P = e^{kt} \times e^{-kC}$$

Let $A = e^{-kC}$ (a constant): $N - P = Ae^{kt}$

$$N = P + Ae^{kt}$$

If $\frac{dN}{dt} = k(N-P)$ then $N = P + Ae^{kt}$

If $k < 0$, $Ae^{kt} \to 0$ as $t \to \infty$ and hence $N \to P$ from above.

Example 9

N is increasing according to the equation $\frac{dN}{dt} = 0.4(N-50)$. If $N = 60$ when $t = 0$:

(a) show that $N = 50 + Ae^{0.4t}$ is a solution to this equation, where A is a constant

(b) calculate the value of N when $t = 20$.

Solution

(a) Differentiate $N = 50 + Ae^{0.4t}$: $\frac{dN}{dt} = 0.4Ae^{0.4t}$

Rewrite $N = 50 + Ae^{0.4t}$: $Ae^{0.4t} = N - 50$

Substitute into $\frac{dN}{dt} = 0.4Ae^{0.4t}$: $\frac{dN}{dt} = 0.4(N-50)$

Hence $N = 50 + Ae^{0.4t}$ is a solution to the equation $\frac{dN}{dt} = 0.4(N-50)$.

(b) At $t = 0$, $N = 60$: $60 = 50 + A$

$A = 10$

$\therefore \quad N = 50 + 10e^{0.4t}$

At $t = 20$: $N = 50 + 10e^{8}$

$N \approx 29\,860$

Example 10

The mass M of a particular southern right whale is modelled as $M = 55 - 54e^{-kt}$, where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

(a) Show that the rate of growth of the whale's mass is given by the differential equation $\frac{dM}{dt} = k(55-M)$.

(b) What is the birth mass of the whale? (i.e. at $t = 0$)

(c) When the whale is one year old, its mass is 10 tonnes. Show that $k = \ln\left(\frac{6}{5}\right)$.

(d) What is the mass of the whale when it is 10 years old (to the nearest tonne)?

(e) If male southern right whales grow to about 55 tonnes and females grow to about 85 tonnes, determine the gender of this whale, giving reasons for your answer.

Solution

(a) $M = 55 - 54e^{-kt}$

Differentiate with respect to t: $\dfrac{dM}{dt} = 54ke^{-sk}$

Rewrite $M = 55 - 54e^{-kt}$: $54e^{-kt} = 55 - M$

Substitute into $\dfrac{dM}{dt} = 54ke^{-kt}$: $\dfrac{dM}{dt} = k(55 - M)$

Hence the rate of growth of the whale's mass is $\dfrac{dM}{dt} = k(55 - M)$.

(b) At $t = 0$: $M = 55 - 54e^{0}$

$M = 55 - 54 = 1$

The birth mass of the whale is 1 tonne.

(c) At $t = 1$, $M = 10$: $10 = 55 - 54e^{-k}$

$54e^{-k} = 45$

$e^{-k} = \dfrac{45}{54} = \dfrac{5}{6}$

$e^{k} = \dfrac{6}{5}$

$k = \ln\left(\dfrac{6}{5}\right)$

(d) At $t = 10$: $M = 55 - 54e^{-10\ln\left(\frac{6}{5}\right)}$

$M = 55 - 54e^{10\ln\left(\frac{5}{6}\right)}$

$M \approx 46.3$

The mass of the whale to the nearest tonne is 46 tonnes.

(e) As $t \to \infty$: $M \to 55 - 54e^{-\infty}$

$M \to 55$

The limiting mass of the whale is 55 tonnes, so it is most likely to be a male.

Newton's law of cooling

Newton's law of cooling states that the cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium:

$\dfrac{dT}{dt} = -k(T - M)$ where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

Example 11

The original temperature of a body is 100°C, the temperature of its surroundings is 20°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\dfrac{dT}{dt} = -k(T - 20)$, where T is the temperature of the body at time t, find:

(a) the temperature of the body after 20 minutes **(b)** the time taken to cool from 100°C to 60°C.

Solution

$\dfrac{dT}{dt} = -k(T - 20)$

Reciprocal of both sides: $\dfrac{dt}{dT} = \dfrac{-1}{k(T - 20)}, \quad T \neq 20$

Integrate with respect to T: $t = -\dfrac{1}{k}\displaystyle\int \frac{1}{T - 20}\,dT$

$t = -\dfrac{1}{k}\log_e(T - 20) + C, \quad T > 20$

$-k(t - C) = \log_e(T - 20)$

$T - 20 = e^{-k(t - C)}$

Let $A = e^{-kC}$ (a constant): $T = 20 + Ae^{-kt}$

When $t = 0$, $T = 100$: $100 = 20 + A$

$A = 80$

When $t = 10$, $T = 70$: $70 = 20 + 80e^{-10k}$

$$e^{-10k} = \frac{50}{80} = 0.625$$

$$-10k = \log_e 0.625$$

$$-10k \approx -0.47$$

$k = 0.047$ (using the approximate logarithm value)

$\therefore\ T = 20 + 80e^{-0.047t}$

(a) When $t = 20$: $T = 20 + 80e^{-0.94}$

$T \approx 51.25$

After 20 minutes the temperature is approximately 51.25°C.

(b) When $T = 60$: $60 = 20 + 80e^{-0.047t}$

$$e^{-0.047t} = \frac{40}{80} = 0.5$$

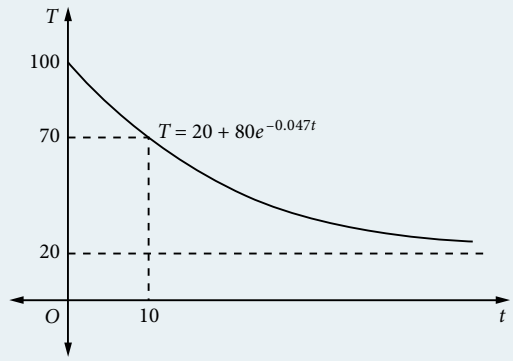
$$-0.047t = \log_e 0.5$$

$$-0.047t \approx -0.6931$$

$$t \approx 14.7$$

The temperature reaches 60°C after approximately 14.7 minutes.

The graph of $T = 20 + 80e^{-0.047t}$ shows that the temperature of the body never falls below the temperature of the surroundings. As $t \to \infty$, $T \to 20$ from above.



Wilhelmy's law

Many chemical reactions follow a law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time:

$$\frac{dx}{dt} = k(a - x),\ 0 \le x \le a$$

where a is the initial concentration and x is the amount transformed at time t.

Example 12

A chemical reaction follows the rule $\frac{dx}{dt} = k(a - x)$, where a is the initial concentration and x is the amount of the reagent transformed at time t. Thus when $t = 0$, $x = 0$. If $a = 10$ and after 2 minutes $x = 4$, find the concentration of the reagent after 5 minutes.

Solution

$$\frac{dx}{dt} = k(10-x),\ 0 \le x \le 10$$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{k(10-x)},\ 0 \le x < 10$

Integrate with respect to x: $t = \frac{1}{k}\int \frac{1}{10-x}dx$

$$kt = -\log_e(10-x) + C$$

When $t = 0, x = 0$: $0 = -\log_e 10 + C$

$$C = \log_e 10$$

$$\therefore\quad kt = \log_e 10 - \log_e(10-x)$$

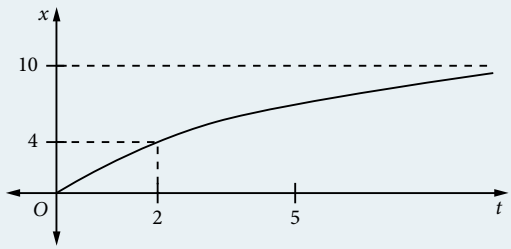
$$kt = \log_e \frac{10}{10-x}$$

Use inverse functions: $\frac{10}{10-x} = e^{kt}$

$$\frac{10-x}{10} = e^{-kt}$$

$$10 - x = 10e^{-kt}$$

$$x = 10(1 - e^{-kt}) \text{ for } t \ge 0$$



Note that the asymptote of the graph is $x = 10$, approached from below. This is consistent with the restriction $0 \le x < 10$ for $t \ge 0$.

When $t = 2, x = 4$: $4 = 10(1 - e^{-2k})$

$$0.4 = 1 - e^{-2k}$$

$$e^{-2k} = 0.6$$

Reciprocal of both sides: $e^{2k} = \frac{5}{3}$

$$2k = \log_e\left(\frac{5}{3}\right)$$

$$k = \frac{1}{2}\log_e\left(\frac{5}{3}\right)$$

When $t = 5$: $x = 10(1 - e^{-5k})$

But: $e^{-2k} = 0.6$

$$e^{-5k} = \left(e^{-2k}\right)^{\frac{5}{2}} = \left(\frac{3}{5}\right)^{\frac{5}{2}} \approx 0.279$$

$$\therefore\quad x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

or inverse functions: $-2k = \log_e 0.6$

$$-2k = -0.51$$

$$k = 0.255$$

i.e. $x = 10(1 - e^{-5k})$

$$x = 10(1 - e^{-1.275})$$

$$x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

Hence the concentration is 2.79 units after 5 minutes. (Remember that $(10 - x)$ is the concentration remaining.)

From the previous examples, it can be seen that if $k < 0$ then as $t \to \infty$, $N \to P$.

Given $\frac{dN}{dt} = k(N - P)$, where k and P are constants, if $k < 0$ then $\lim_{t\to\infty} N = P$.

EXERCISE 7.4 HARDER EXPONENTIAL GROWTH AND DECAY

1 N is decreasing according to the equation $\frac{dN}{dt} = -0.4(N-30)$. If $N = 60$ when $t = 0$:

(a) show that $N = 30 + Ae^{-0.4t}$ is a solution of this equation, where A is a constant

(b) calculate the value of N when $t = 5$.

2 N is increasing according to the equation $\frac{dN}{dt} = 0.2(N-40)$. If $N = 50$ when $t = 0$:

(a) show that $N = 40 + Ae^{0.2t}$ is a solution to this equation, where A is a constant

(b) calculate the value of N when $t = 10$.

3 The original temperature of a body is 120°C, the temperature of its surroundings is 50°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-50)$ where T is the temperature of the body at time t, find:

(a) the temperature after 20 minutes **(b)** the time taken to cool to 60°C.

4 If $N = 70$ when $t = 0$, which expression is the correct solution to $\frac{dN}{dt} = -0.5(N-20)$?

A $N = 20 + 50e^{0.5t}$ **B** $N = 20 + 50e^{-0.5t}$ **C** $N = 20 - 50e^{0.5t}$ **D** $N = 20 - 50e^{-0.5t}$

5 A metal bar has a temperature of 1230°C and cools to 1030°C in 10 minutes when the surrounding temperature is 30°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-30)$ where T is the temperature of the body at time t.

(a) Show that $T = 30 + 1200e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.

(b) Find the temperature after 20 minutes. **(c)** Find the time taken to cool from 1230°C to 80°C.

6 Water at 20°C is placed in a freezer where the air is at a constant temperature of −10°C. The temperature of the water falls to 15°C in 5 minutes. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T+10)$ where T is the temperature of the body at time t.

(a) Show that $T = -10 + 30e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.

(b) Find the temperature of the water after another 5 minutes (when $t = 10$).

7 A body whose temperature is 180°C is immersed in a liquid that is at 60°C. In 1 minute the temperature of the body has fallen to 120°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-60)$ where T is the temperature of the body at time t.

(a) Show that $T = 60 + 120e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.

(b) At what time would the temperature of the body have fallen to 90°C?

8 A current of i amperes (or 'amps') flows through a coil of inductance L henrys and resistance R ohms. The current at any time is given by $i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$, where E is the electromotive force (i.e. the voltage) in volts. Show that $L\frac{di}{dt} + Ri = E$.

9 A vessel is filled at a variable rate so that the volume of liquid in the vessel at any time t is given by $V = A(1 - e^{-kt})$.

(a) Show that $\frac{dV}{dt} = k(A - V)$.

(b) If a quarter of the vessel is filled in the first 5 minutes, what fraction is filled in the next 5 minutes?

(c) Show that $\lim_{t\to\infty} V = A$.

10 A rectangular vessel is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 20 cm, flows into the other compartment, initially empty, at a rate proportional to the difference between the levels in each compartment. The differential equation for this process is $\frac{dx}{dt} = k(20 - 2x)$, where x cm is the depth of the liquid in one of the vessels at any time t minutes.

(a) Show that $x = 10(1 - e^{-2kt})$.

(b) If the level in the second compartment rises 2 cm in the first 5 minutes, at what time will the difference in levels be 2 cm?

11 In a certain chemical process, the amount y grams of a certain substance at time t hours is given by the formula $y = 3 + e^{-kt}$.

(a) Show that $\frac{dy}{dt} = -k(y-3)$.

(b) If initially y decreases at a rate of 0.08 grams per hour, find the value of k.

(c) Find the rate of change when $y = 3.5$. **(d)** What values can y take?

7.5 RELATED RATES OF CHANGE

Rates of change were introduced earlier in this chapter. You will now look at more sophisticated applications of rates of change.

Problems with **related rates** arise when there is a function that relates two variables, e.g. x and y, where both variables are also functions of another variable, e.g. time t. For example, you may need to determine $\frac{dy}{dt}$ when $\frac{dx}{dt}$ is known. In such cases it is necessary to use the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Remember that in a context like this, 'increasing' means a positive rate of change while 'decreasing' means a negative rate of change.

Example 13

(a) If $V = \frac{4}{3}\pi r^3$ and $\frac{dr}{dt} = 5$, find the expression for $\frac{dV}{dt}$.

(b) If $S = 4\pi r^2$ and $\frac{dr}{dt} = 5$, find the expression for $\frac{dS}{dt}$.

(c) If $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, find the expression for $\frac{dV}{dS}$.

Solution

(a)
$\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2$
$\frac{dV}{dr} = 4\pi r^2$
$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
$\frac{dr}{dt} = 5$
$\frac{dV}{dt} = 4\pi r^2 \times 5$
$\frac{dV}{dt} = 20\pi r^2$

(b)
$\frac{dS}{dr} = 8\pi r$
$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$
$\frac{dr}{dt} = 5$
$\frac{dS}{dt} = 8\pi r \times 5$
$\frac{dS}{dt} = 40\pi r$

(c)
$\frac{dV}{dr} = 4\pi r^2$, $\frac{dS}{dr} = 8\pi r$
$\frac{dV}{dS} = \frac{dV}{dr} \times \frac{dr}{dS}$
$\frac{dr}{dS} = \frac{1}{8\pi r}$
$\frac{dV}{dS} = 4\pi r^2 \times \frac{1}{8\pi r}$
$\frac{dV}{dS} = \frac{r}{2}$

As $\frac{dr}{dt}$ is the same in parts **(a)** and **(b)**, these results can be used to find the answer to part **(c)** in this case.

Example 14

If $x = 5t\cos\alpha$ and $y = 5t\sin\alpha - \frac{1}{2}gt^2$, where α and g are constants, find:

(a) the expression for $\frac{dy}{dx}$ as a function of t **(b)** the expression for $\frac{dy}{dx}$ when $t = 2$

(c) if $\alpha = \frac{\pi}{4}$ and $g = 9.8$, find the value of $\frac{dy}{dx}$ when $t = 2$.

Solution

(a) $\frac{dx}{dt} = 5\cos\alpha$

$y = 5t\sin\alpha - \frac{1}{2}gt^2$

$\frac{dy}{dt} = 5\sin\alpha - gt$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

$\frac{dy}{dx} = \frac{5\sin\alpha - gt}{5\cos\alpha}$

(b) $t = 2$

$\frac{dy}{dx} = \frac{5\sin\alpha - g \times 2}{5\cos\alpha}$

$\frac{dy}{dx} = \frac{5\sin\alpha - 2g}{5\cos\alpha}$

(c) $\alpha = \frac{\pi}{4}, g = 9.8$

$\frac{dy}{dx} = \frac{5\sin\frac{\pi}{4} - 2 \times 9.8}{5\cos\frac{\pi}{4}}$

$\frac{dy}{dx} = \frac{\frac{5}{\sqrt{2}} - 19.6}{\frac{5}{\sqrt{2}}}$

$\frac{dy}{dx} = -4.544$

Example 15

Given $x = t^2 - 1$ and $y = t^3$, find as functions of t: **(a)** $\frac{dy}{dx}$ **(b)** $\frac{d^2y}{dx^2}$

Solution

(a) $\frac{dx}{dt} = 2t$

$y = t^3$

$\frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 3t^2 \times \frac{1}{2t} = \frac{3t}{2}$

(b) $\frac{dy}{dx} = \frac{3t}{2}, \frac{dx}{dt} = 2t$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2}$

$\frac{dt}{dx} = \frac{1}{2t}$

$\frac{d^2y}{dx^2} = \frac{3}{2} \times \frac{1}{2t} = \frac{3}{4t}$

Example 16

A spherical balloon is being inflated so that its radius increases at the constant rate of 3 cm/min. At what rate is its volume increasing when the radius of the balloon is 5 cm?

Solution

If r is the radius of the balloon, its volume is $V = \frac{4}{3}\pi r^3$. Given $\frac{dr}{dt} = 3$, need to find $\frac{dV}{dt}$ for $r = 5$.

By the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

But: $V = \frac{4}{3}\pi r^3$

So: $\frac{dV}{dr} = 4\pi r^2$

Thus: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 3$

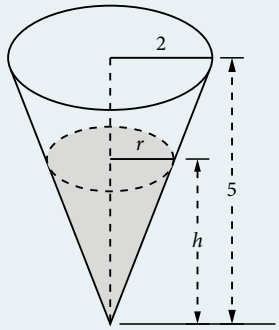
For $r = 5$: $\frac{dV}{dt} = 4\pi \times 5^2 \times 3$

$= 300\pi \text{ cm}^3/\text{min}$

Example 17

A vessel containing water has the shape of an inverted right circular cone with base radius 2 m and height 5 m. The water flows out of the apex of the cone at a constant rate of 0.2 m^3/min. Find the rate at which the water level is dropping when the depth of the water is 4 m.

Solution

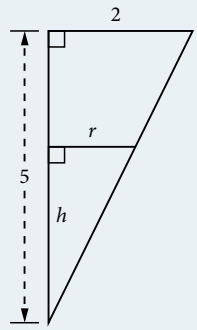


Let the depth of the water be h m, the radius of the cone at the water level be r m and the volume of the water be V m^3 at time t minutes.
The volume of the water at any time t is $V = \frac{1}{3}\pi r^2 h$.

Given $\frac{dV}{dt} = -0.2$, need to find $\frac{dh}{dt}$ when $h = 4$.

$\frac{dV}{dt}$ is negative: the volume is decreasing, because the water is flowing out of the vessel.

To find the link between r and h, use similar triangles.



From proportional sides: $\frac{r}{h} = \frac{2}{5}$

$$r = \frac{2h}{5}$$

Volume: $V = \frac{1}{3}\pi r^2 h$

Substitute $r = \frac{2h}{5}$: $V = \frac{1}{3}\pi \times \left(\frac{2h}{5}\right)^2 \times h$

$$V = \frac{4\pi h^3}{75}$$

Hence: $\frac{dV}{dh} = \frac{4\pi h^2}{25}$

Chain rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\therefore \quad -0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$

$$\frac{dh}{dt} = -\frac{5}{4\pi h^2}$$

When $h = 4$: $\frac{dh}{dt} = -\frac{5}{4\pi \times 16} = -\frac{5}{64\pi}$

$$= -0.0249 \text{ m/min}$$

Thus the water level is decreasing at a rate of 0.0249 m/min.

Alternatively: V and h are both dependent on time, so you can differentiate both sides of the volume equation with respect to time.

$$V = \frac{4\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{4\pi}{75} \times \frac{d}{dt}\left(h^3\right)$$

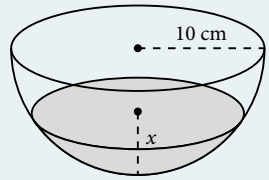
Chain rule: $= \frac{4\pi}{75} \times \frac{d}{dh}\left(h^3\right) \times \frac{dh}{dt}$

$$= \frac{4\pi}{75} \times 3h^2 \times \frac{dh}{dt}$$

$\frac{dV}{dt} = -0.2$, so: $-0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$ as before.

Example 18

The volume of water in a hemispherical bowl of radius 10 cm is $V = \frac{1}{3}\pi x^2(30-x)$, where x cm is the depth of water at time t.
The bowl is being filled at a constant rate of 2π cm^3/min.
At what rate is the depth increasing when the depth is 2 cm?



Solution

Given $\frac{dV}{dt} = 2\pi$, need to find $\frac{dx}{dt}$ when $x = 2$.
The related variables are V, x and t.

$$V = \frac{1}{3}\pi x^2(30-x)$$
$$= 10\pi x^2 - \frac{1}{3}\pi x^3$$
$$\frac{dV}{dx} = 20\pi x - \pi x^2 = \pi x(20-x)$$

Method 1

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$2\pi = \pi x(20-x) \times \frac{dx}{dt}$$
$$\frac{dx}{dt} = \frac{2}{x(20-x)}$$

Method 2

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$$
$$\frac{dx}{dt} = \frac{1}{\pi x(20-x)} \times 2\pi$$
$$\frac{dx}{dt} = \frac{2}{x(20-x)}$$

When $x = 2$: $\frac{dx}{dt} = \frac{1}{18}$ cm/s

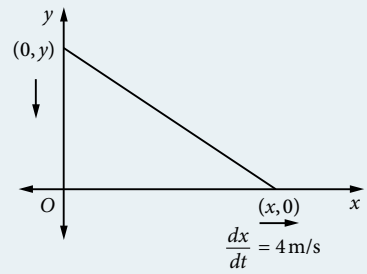
As an extension to this question, you might ask: 'At what depth is the depth increasing at a minimum rate?'
This is the same as asking: 'For what value of x is $\frac{dx}{dt}$ a minimum?'
Note that the formula for the volume of water in the hemispherical bowl at any depth a can be calculated by finding the volume generated by rotating the circle with equation $x^2 + y^2 = 100$ between $y = 10 - a$ and $y = 10$.

Example 19

A ladder 10 m long has its upper end against a vertical wall and its lower end on a horizontal floor. The lower end is slipping away from the wall at a constant speed of 4 m/s. Find the rate at which the upper end of the ladder is slipping down the wall when the lower end is 6 m from the wall. What is this rate when the upper end is very close to the ground?

Solution

At any time t, the lower end of the ladder is x m from the wall and the upper end is y m above the ground.
Given $\frac{dx}{dt} = 4$, need to find $\frac{dy}{dt}$ when $x = 6$.



By Pythagoras' theorem: $x^2 + y^2 = 100$

$$y = \sqrt{100 - x^2},\ 0 \le x \le 10$$
$$\frac{dy}{dx} = \frac{1}{2}\left(100 - x^2\right)^{-\frac{1}{2}} \times (-2x)$$
$$= \frac{-x}{\sqrt{100-x^2}},\ 0 \le x < 10$$

For $\frac{dx}{dt} = 4$:
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= \frac{-x}{\sqrt{100-x^2}} \times 4$$

At $x = 6$:
$$\frac{dy}{dt} = \frac{-4 \times 6}{\sqrt{100-36}} = -3$$

The top of the ladder is descending at the rate of 3 m/s when $x = 6$.

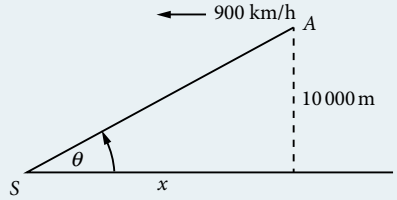
The equations show that as $y \to 0$, $x \to 10$ and $\frac{dy}{dt} \to -\infty$. This means that the speed of the upper end of the ladder is increasing as it falls (until it suddenly stops when it reaches the ground).

Example 20

A radar tracking station is located at ground level, vertically below the path of an approaching aircraft flying at 900 km/h at a constant height of 10 000 m. The tracking station transmits a radar beam that turns to follow the aircraft perfectly as it approaches. Find the rate in degrees per second at which the radar beam is turning when the aircraft is at a horizontal distance of 3 km from the station.

Solution

S represents the tracking station and x is the horizontal displacement of the aircraft at time t.



Find $\frac{d\theta}{dt}$ when $x = 3000$ m.

Converting units: $900 \text{ km/h} = 900 \times \frac{1000}{3600} \text{ m/s} = 250 \text{ m/s}$

Hence: $\frac{dx}{dt} = -250$

Right-angled triangle: $x = \frac{10000}{\tan\theta} = 10\,000(\tan\theta)^{-1}$

Chain rule differentiation: $\frac{dx}{d\theta} = 10000 \times (-1)(\tan\theta)^{-2} \times \sec^2\theta$

$$= -10000 \times \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos^2\theta}$$

$$= -10000 \operatorname{cosec}^2\theta$$

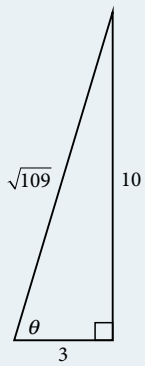
Reciprocal of both sides: $\frac{d\theta}{dx} = -\frac{\sin^2\theta}{10000}$

Chain rule: $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$$= -\frac{\sin^2\theta}{10\,000} \times \frac{dx}{dt}$$

For $\frac{dx}{dt} = -250$: $\frac{d\theta}{dt} = -\frac{\sin^2\theta}{10\,000} \times (-250) = \frac{\sin^2\theta}{40}$

From the triangle, with $x = 3000$ m: $\tan\theta = \frac{10000}{3000} = \frac{10}{3}$



From the triangle: $\sin\theta = \frac{10}{\sqrt{109}}$

Hence $\frac{d\theta}{dt} = \frac{1}{40} \times \frac{100}{109} = \frac{5}{218}$ radians per second

$$= \frac{5}{218} \times \frac{180}{\pi} \text{ degrees per second}$$

$$\approx 1.3 \text{ degrees per second}$$

Note: The initial expression for x could have been written $x = 10\,000 \cot\theta$.

EXERCISE 7.5 RELATED RATES OF CHANGE

1 If $A = \pi r^2$ and $C = 2\pi r$, the expression for $\frac{dA}{dC}$ is given by:

A 2π **B** $\frac{1}{2\pi}$ **C** r **D** $\frac{1}{r}$

2 If $x^2 + y^2 = 144$, find the value of $\frac{dy}{dt}$ when $\frac{dx}{dt} = 0.6$ and $x = 5$, given that x and y are both positive.

3 If $V = \frac{1}{3}\pi x^2(60 - x)$ and $\frac{dV}{dt} = 2\pi$, find the expression for $\frac{dx}{dt}$.

4 **(a)** If $x = 4t$ and $y = 2t^2$, find the expression for $\frac{dy}{dx}$ in terms of t.

(b) Hence find the expression for $\frac{dy}{dx}$ in terms of x.

5 Given $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{h}{4}$, answer the following questions.

(a) Find the expression for V in terms of h. **(b)** If $\frac{dV}{dt} = 0.6$, find the expression for $\frac{dh}{dt}$.

(c) Find the value of $\frac{dh}{dt}$ when $r = 1$.

6 **(a)** If $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$, find the expression for $\frac{dy}{dx}$ in terms of t.

(b) Hence find $\frac{d^2y}{dx^2}$ as a function of t.

7 **(a)** If $x = t^2 + 4t$ and $y = 3t + t^3$, find the expression for $\frac{dy}{dx}$ in terms of t.

(b) If $\frac{dy}{dx} = 1$, find the values of x. **(c)** Find $\frac{d^2y}{dx^2}$ as a function of t.

8 Water is being poured at a constant rate of $3\,\text{cm}^3/\text{s}$ into an inverted right conical vessel whose apex angle is 90°. At what rate is the water level rising when the depth is π cm?

9 A lamp is 6 m directly above a straight footpath. A person 2 m tall walks along the footpath away from the light at a constant speed of 1 m/s. At what speed is the end of the person's shadow moving along the path? At what speed is the length of the shadow increasing?

10 Given $A = x\sqrt{100 - x^2}$ and $\frac{dx}{dt} = 0.3$, the value of $\frac{dA}{dt}$ when $x = 8$ is:

A $-\frac{72}{5}$ **B** $\frac{72}{5}$ **C** $-\frac{7}{5}$ **D** $\frac{7}{5}$

11 A particle moves on the circumference of the semicircle $y = \sqrt{4 - x^2}$. If $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ at $x = 1$.

12 A melting snowball is decreasing in volume at a constant rate of $8\,\text{cm}^3/\text{min}$. If the melting snowball is always a perfect spherical shape, find the rate at which its radius is changing when the radius is 4 cm.

13 A particle is moving along a curve with equation $y = x^2$. If $\frac{dx}{dt} = 2$ at $x = 3$, find $\frac{dy}{dt}$ at that instant.

14 The perimeter of a circular sector is 20 cm. The radius is increasing at a rate of 5 cm/s.

(a) At what rate is the angle of the sector changing when the radius length is 10 cm?

(b) At what rate is the area changing when the radius is 10 cm?

15 A ladder 5 metres long is standing vertically, flat against a vertical wall, while its lower end is on the horizontal floor. The lower end moves horizontally away from the wall at a constant speed of 1 m/s while the upper end stays in contact with the wall. Find the speed at which the upper end is moving down the wall 4 seconds after the lower end has left the wall.

16 Sand is poured into a heap in the shape of a right circular cone whose semi-vertex angle is α, where $\tan\alpha = \frac{3}{4}$. When the height of the cone is 16 cm, the height is increasing at a rate of 2 cm/min. At that instant, at what rate is the volume increasing?

17 When a certain gas expands at constant temperature, its pressure P and volume V are given by the relation $PV^{1.4} = k$, a constant. At a certain instant the pressure is 25 g/cm^2 and the volume is 32 cm^3. If the volume is increasing at the rate of 5 cm^3/s, at what rate is the pressure changing at that instant?

18 A kite 50 metres above the ground is being carried horizontally by the wind at a rate of 4 m/s. If the kite's string is always perfectly straight, how fast must the length of the string be increasing when the string is 100 metres long?

19 Sand is poured into a heap at a constant rate of 4 cm^3/min, so that the heap is in the shape of a right circular cone whose height is always equal to the radius of the base. When the heap is 10 cm high:

(a) how fast is the height increasing **(b)** how fast is the area of the base increasing?

20 A boat is pulled in to a wharf by a rope at a speed of 20 m/min. If the rope is attached to a point on the boat 7 m vertically below the wharf, at what rate is the rope being drawn in when the boat is 24 m from the wharf?

21 A straight railway track and a straight road intersect at right angles. At a given instant a car travelling at 40 km/h and a train travelling at 50 km/h are moving away from the intersection and are 40 km and 30 km from the intersection respectively.

(a) The car and train continue moving in straight lines without changing their speed. One hour later, at what rate is the distance between the car and the train changing?

(b) At what rate would the distance between the car and train be changing if they were both travelling towards the intersection?

22 Two straight roads meet at an angle of 60°. Car A starts from this intersection and travels along one road at 40 km/h. One hour later, car B starts from the intersection and travels along the other road at 50 km/h. Three hours after car A starts, at what rate is the distance between cars A and B changing?

23 A conical tank with a vertical axis has a semi-vertical angle of 45°. Water, initially at a depth of 5 metres, leaks out through a hole at the bottom of the tank at a rate of $0.2\sqrt{h}$ m^3/min when the depth is h metres. Find the rate at which the depth is decreasing when the depth is 4 metres.

24 The height of a right-angled triangle is 6 cm and its base is increasing at a constant rate of 2 cm/s. At what rate is the hypotenuse increasing when its length is 10 cm?

25 A loading chute is in the shape of a square pyramid with base length 10 m and depth 8 m. Liquid is poured in at the top at a rate of 4 m^3/min. At what rate is the level rising when the depth is 4 m?

26 Grain is ejected from a chute at a rate of 0.1 m^3/min to form a heap on a flat horizontal floor. The heap is in the form of a circular cone of semi-vertical angle 45°. Find the rate (in metres per minute) at which the height of the cone is increasing at the instant 3 minutes after the opening of the chute.

27 In triangle ABC, $AB = 10$ cm, $AC = 12$ cm and angle A is increasing at the rate of 0.1 radians per second. At what rate is:

(a) the area of ΔABC increasing **(b)** the length of BC increasing, when angle A is $\frac{\pi}{3}$ radians?

28 A spherical mothball evaporates at a rate proportional to its surface area so that its volume V cm^3 and radius r after t weeks are related by the equation $\frac{dV}{dt} = -4k\pi r^2$, where k is a positive constant.

(a) Show that $\frac{dr}{dt} = -k$.

(b) If the initial radius of the mothball is 1 cm and the radius after 10 weeks is 0.5 cm, express r in terms of t.

29 The volume V and surface area S of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ respectively.

(a) Show that $\frac{dV}{dr} = S$ and $\frac{dV}{dt} = S\frac{dr}{dt}$.

(b) A spherical ball of radius 24 mm is immersed in an acid bath so that its volume decreases at a changing rate equal to three times its surface area (while remaining constantly spherical). How much time does it take to reduce to one-eighth of their original size: **(i)** the radius **(ii)** the volume

30 **(a)** Show that the formula for the volume V of a right circular cone of base radius r and height h can be expressed as $V = \frac{1}{3}\pi h^3 \tan^2 \alpha$, where α is the semi-vertex angle.

(b) Water flows out through a hole at the vertex angle of an inverted cone, whose angle is 60°, at a rate equal to π times the square root of the depth of the water at any time. At what rate (in cm/s) would the water level be dropping when the depth is 9 cm?

CHAPTER REVIEW 7

1 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \dfrac{2t}{9+t^2}$.

(a) What is the initial velocity of the particle?

(b) Find an expression for the acceleration of the particle.

(c) When is the acceleration zero?

(d) What is the maximum velocity attained by the particle and when does it occur?

2 The growth of the number of internet users in the USA was modelled as an exponential function $N = Ae^{kt}$, where N is the estimate for the number of internet users (in millions) and t is the time in years after 1 January 2001.

(a) At the start of 2001 ($t = 0$) there were 124 million internet users in the USA. At the start of 2009 there were 220 million. Find A and k.

(b) How many internet users would you expect there to be at the start of 2012?

(c) In what year would you expect the number of internet users in the USA to first exceed 300 million?

3 A full water tank holds 4000 litres. When the tap is turned on, water flows out from the tank at a rate of $\dfrac{dV}{dt} = 110 + 17t - t^2$ litres per minute, where V is the volume in litres and t is the time in minutes since the tap was turned on.

(a) At what time is the tank emptying at a rate of 50 litres per minute?

(b) At what time does the water stop flowing out of the tank?

4 Given that $N = 200 + 70e^{kt}$, which expression is equal to $\dfrac{dN}{dt}$?

A $k(200 - N)$ **B** $k(N - 200)$ **C** $k(270 - N)$ **D** $k(N - 270)$

5 The number N of bacteria in a colony grows according to the rule $\dfrac{dN}{dt} = kN$. If the original number increases from 5000 to 10 000 in 6 days, find the number of bacteria after another 6 days.

6 A population of size N is decreasing according to the rule $\dfrac{dN}{dt} = -\dfrac{N}{40}$, where t is the time in days. If the population is initially of size N_0, find how much time it takes for the size to be halved (rounded to the next day).

7 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 400 years, what percentage of the original radioactive mass will remain after 1000 years?

8 The radius of a cylinder increases at a constant rate of 0.1 cm per minute while its height remains constant at 10 cm. At what rate is the volume of the cylinder increasing when the radius is 2 cm?

9 Rain is falling and collects in an inverted cone so that the volume collected increases at a constant rate of $4\pi \text{cm}^3$ per hour. If the radius r of the cone is half its height h, find the rate (in cm per hour) at which the height is increasing when $h = 3$.

10 A cup of hot coffee at a temperature T°C loses heat in a cooler environment. It cools according to the law $\frac{dT}{dt} = -k(T - T_0)$, where t is the time elapsed in minutes, T_0 is the temperature of the environment in degrees Celsius and k is a constant.

(a) At recess, Mr Masters makes his cup of coffee with water at 100°C. The temperature in the staff room is 25°C when he places his coffee on his desk. Three minutes later his coffee is just the temperature he likes it, 75°C. Find the value of k.

(b) Before Mr Masters gets a chance to drink his coffee, he leaves the staff room to help a student with a maths problem about exponential growth and decay. What is the temperature of the coffee when he returns, 5 minutes later?

11 Cream taken out of a refrigerator has a temperature of 3°C. It is placed on the table in a room of constant temperature 24°C. After t minutes the temperature, T°C, of the cream is given by $T = A - Be^{-0.04t}$, where A and B are positive constants.

How much time does it take for the cream to reach a temperature of 8°C?

12 Given $x = 40t$ and $y = 56t - 16t^2$, then the expression for $\frac{dy}{dx}$ is given by:

A $\frac{dy}{dx} = \frac{5}{7 - 4t}$ **B** $\frac{dy}{dx} = \frac{7 - 4t}{5}$ **C** $\frac{dy}{dx} = 5(7 - 4t)$ **D** $\frac{dy}{dx} = \frac{1}{5(7 - 4t)}$

13 If $P = \frac{100}{V}$ and $\frac{dV}{dt} = 4$, find the expression for $\frac{dP}{dt}$.

14 **(a)** If $x = 2\left(t + \frac{1}{t}\right)$, $y = 2\left(t - \frac{1}{t}\right)$, find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Find $\frac{d^2y}{dx^2}$ as a function of t.

15 A spherical balloon is being filled with air at the rate of 100 cm^3/min. At what rate is the radius of the balloon increasing when the radius is 5 cm?

16 A circular oil slick floats on the surface of still water. Its area is increasing at a rate of 10 m^2/min. At what rate is the radius (r metres) increasing?

17 A point P moves on the curve $y = x^3$ so that its x-coordinate increases at a constant rate of 5 units per second. When $x = 1$: **(a)** at what rate is the y-coordinate of P increasing

(b) at what rate is the gradient of the curve increasing?

18 A vessel is shaped so that when the depth of water in it is x cm, the volume of the water is V cm^3, where $V = 108x + x^3$. Water is poured into the vessel at a constant rate of 30 cm^3/s. At what rate is the water level rising when its depth is 8 cm?

19 A trough l metres long has a cross-section in the shape of an isosceles triangle with base length $2a$ metres and height b metres. Water leaks from the trough at a constant rate of c m^3/min. Find the rate at which the water level is falling when the depth of the water is $\frac{b}{2}$ metres.

20 A metal sphere is dissolving in acid. It remains spherical and the rate at which its volume decreases is proportional to its surface area. Show that the radius of the sphere decreases at a constant rate.

21 A circular cylinder with height 6 cm and base radius 4 cm sits on a table with its axis vertical. A point source of light moves vertically up at a speed of 3 cm/s above the central axis of the cylinder, thus casting a circular shadow on the table. Find the rate at which the radius of the shadow is decreasing when the light is 4 cm above the top of the cylinder.

CHAPTER 8
Trigonometric equations

8.1 SOLVING TRIGONOMETRIC EQUATIONS USING THE AUXILIARY ANGLE METHOD

The **auxiliary angle** method of solving trigonometric equations involves changing an equation of the form $a\sin x \pm b\cos x = c$ into the form $r\sin(x \pm \alpha) = c$, which is then easier to solve. In this form, α is called the auxiliary angle.

This method can also be used to change $a\cos x \pm b\sin x = c$ into the form $r\cos(x \mp \alpha) = c$. In both cases, the constants a, b, r and α are positive real numbers.

For example, to express $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$:

Let $a\sin x + b\cos x = r\sin(x + \alpha)$

$= r(\sin x\cos\alpha + \cos x\sin\alpha)$

$= r\sin x\cos\alpha + r\cos x\sin\alpha$

This is an identity, so the coefficients of $\sin x$ and $\cos x$ on each side must be the same.

i.e. $a = r\cos\alpha$

$b = r\sin\alpha$

$\therefore\ a^2 + b^2 = r^2(\cos^2\alpha + \sin^2\alpha)$

Hence: $r^2 = a^2 + b^2$

$r = \sqrt{a^2+b^2}$ because r is a positive real number.

From the coefficients, there is also $\cos\alpha = \frac{a}{r}$ and $\sin\alpha = \frac{b}{r}$.

As a and b are positive constants, so $\cos\alpha$ and $\sin\alpha$ are also positive. This also means that α is in the first quadrant (i.e. it is an acute angle), such that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{b}{a}$.

Hence, the auxiliary angle method gives:	$a\sin x + b\cos x = r\sin(x + \alpha)$
which then allows you to obtain:	$a\sin x - b\cos x = r\sin(x - \alpha)$
	$a\cos x + b\sin x = r\cos(x - \alpha)$
	$a\cos x - b\sin x = r\cos(x + \alpha)$

In each case, $r = \sqrt{a^2+b^2}$ and α is an angle in the first quadrant such that $\tan\alpha = \frac{b}{a}$.

Example 1

Express: **(a)** $\sqrt{3}\sin x - \cos x$ in the form $r\sin(x - \alpha)$ **(b)** $3\cos x - 4\sin x$ in the form $r\cos(x + \alpha)$.

Solution

(a) $\sqrt{3}\sin x - \cos x = r\sin(x - \alpha)$

$= r(\sin x\cos\alpha - \cos x\sin\alpha)$

$= r\sin x\cos\alpha - r\cos x\sin\alpha$

Equate coefficients of $\sin x$ and $\cos x$: $r\cos\alpha = \sqrt{3}$ [1]

$r\sin\alpha = 1$ [2]

$[1]^2 + [2]^2$: $r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r^2 = 4$

$r = 2$ (as $r > 0$)

Hence from [1] and [2]: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

As $\cos\alpha$ and $\sin\alpha$ are both positive, α is in the first quadrant, such that $\tan\alpha = \frac{1}{\sqrt{3}}$, i.e. $\alpha = \frac{\pi}{6}$.

From the first equation: $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

(b) $3\cos x - 4\sin x = r\cos(x + \alpha)$

$a = 3, b = 4$: $r = \sqrt{3^2 + 4^2} = 5$

$\tan\alpha = \frac{4}{3}$: $\alpha = 53° 8'$

$\therefore\ 3\cos x - 4\sin x = 5\cos(x + 53° 8')$

Example 1 illustrates two different auxiliary angle methods that may be used. You should practise both.

Important uses of the auxiliary angle method

- Writing $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y = a\sin x + b\cos x$ much easier.
- Writing $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$ allows you to solve equations of the type $a\sin x \pm b\cos x = c$.

Example 2

Sketch the graph of $y = \sqrt{3}\sin x - \cos x$, $0 \le x \le 2\pi$.

Solution

Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$. Hence: $y = 2\sin\left(x - \frac{\pi}{6}\right)$

At the endpoints of the domain, $x = 0$ and $x = 2\pi$: $y = 2\sin\left(-\frac{\pi}{6}\right) = 2\sin\left(2\pi - \frac{\pi}{6}\right) = -1$

The greatest value of y is 2, where: $1 = \sin\left(x - \frac{\pi}{6}\right)$

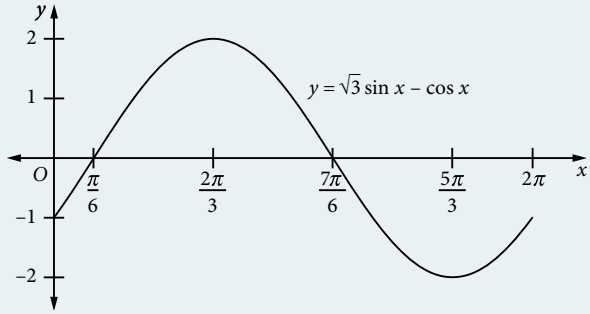
$x - \frac{\pi}{6} = \frac{\pi}{2}$

$x = \frac{2\pi}{3}$

The least value of y is -2, where: $-1 = \sin\left(x - \frac{\pi}{6}\right)$

$x - \frac{\pi}{6} = \frac{3\pi}{2}$

$x = \frac{5\pi}{3}$



The graph crosses the x-axis where: $\sin\left(x - \frac{\pi}{6}\right) = 0$

$x - \frac{\pi}{6} = 0, \pi$

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

Example 3

Solve the following equations.

(a) $\sqrt{3}\sin x - \cos x = 1,\ 0 \le x \le 2\pi$

(b) $8\cos x + 6\sin x = -3,\ 0° \le x \le 360°$

Solution

Method 1

(a) Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$.

$$\therefore \quad 2\sin\left(x - \frac{\pi}{6}\right) = 1$$
$$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$
$$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$x = \frac{\pi}{3}, \pi$$

(b) Use $a\cos x + b\sin x = r\cos(x - a)$.

$8\cos x + 6\sin x = -3$

$a = 8,\ b = 6$: $\quad r = \sqrt{8^2 + 6^2} = 10$

$\tan\alpha = \frac{6}{8} = 0.75$ so $\alpha = 36° 52'$

$\therefore\ 10\cos(x - 36° 52') = -3$

$\cos(x - 36° 52') = -0.3$

$x - 36° 52' = 107° 27', 252° 33'$

$x = 144° 19', 289° 25'$

Method 2

You can express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ for all values of x, except $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ (because $\tan\frac{\pi}{2}$ is undefined for those values).

The t formulae (see Chapter 4) give $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$.

(a) $\sqrt{3}\sin x - \cos x = 1,\ 0 \le x \le 2\pi$: $\quad \frac{2\sqrt{3}t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$, where $t = \tan\frac{x}{2}$

$$2\sqrt{3}t - (1 - t^2) = 1 + t^2$$
$$2\sqrt{3}t - 1 + t^2 = 1 + t^2$$
$$\sqrt{3}t = 1$$
$$t = \frac{1}{\sqrt{3}}$$
$$\therefore \quad \frac{x}{2} = \frac{\pi}{6} \text{ for } 0 \le \frac{x}{2} \le \pi$$
$$x = \frac{\pi}{3} \text{ for } 0 \le x \le 2\pi$$

Because $t = \tan\frac{x}{2}$ is undefined at $x = \pi$, you must now separately test whether $x = \pi$ is a solution.

$x = \pi$: $\quad \text{LHS} = \sqrt{3}\sin\pi - \cos\pi = 0 - (-1) = 1 = \text{RHS}$

Hence $x = \pi$ is also a solution. The complete solution is $x = \frac{\pi}{3}, \pi$.

(b) $8\cos x + 6\sin x = -3,\ 0° \le x \le 360°$: $\quad \frac{8(1-t^2)}{1+t^2} + \frac{12t}{1+t^2} = -3$

$$8 - 8t^2 + 12t = -3 - 3t^2$$
$$5t^2 - 12t - 11 = 0$$
$$t = \frac{12 \pm \sqrt{144 + 220}}{10}$$
$$\tan\frac{x}{2} = 3.108, -0.708 \quad \text{(to 3 d.p.)}$$
$$\frac{x}{2} = 72° 10', 144° 42' \quad \text{for } 0° \le \frac{x}{2} \le 180°$$
$$x = 144° 20', 289° 24' \quad \text{for } 0° \le x \le 360°$$

(Note the slight difference in the answers due to the rounding error when solving the quadratic equation.)
Because $t = \tan\frac{x}{2}$ is undefined at $x = \pi$, you must now separately test whether $x = 180°$ is a solution.
$x = 180°$: LHS $= 8\cos\pi + 6\sin\pi = -8 + 0 = -8 \neq$ RHS
Hence $x = 180°$ is not a solution of the equation.

Important note:
If you use the t formulae substitution to solve equations of the type $a\cos x + b\sin x = c$, you must also test to see whether $x = \pm n\pi$ is a solution of the equation. The use of the t formulae to solve a variety of equations will be covered later in this chapter.

EXERCISE 8.1 SOLVING TRIGONOMETRIC EQUATIONS USING THE AUXILIARY ANGLE METHOD

1 Express each of the following in the form $r\sin(x + \alpha)$.

(a) $\sin x + \cos x$ **(b)** $3\sin x + \sqrt{3}\cos x$
(c) $5\sin x + 12\cos x$, $0° < \alpha < 90°$ **(d)** $2\sin x + \cos x$, $0° < \alpha < 90°$

2 Express each of the following in the form $r\sin(x - \alpha)$.

(a) $\sin x - \sqrt{3}\cos x$ **(b)** $2\sin x - 3\cos x$, $0° < \alpha < 90°$
(c) $2\sin x - \cos x$, $0° < \alpha < 90°$ **(d)** $3\sin x - 3\cos x$

3 Express each of the following in the form $r\cos(x - \alpha)$.

(a) $\cos x + \sin x$ **(b)** $24\cos x + 7\sin x$, $0° < \alpha < 90°$
(c) $2\cos x + 2\sqrt{3}\sin x$ **(d)** $3\cos x + 2\sin x$, $0° < \alpha < 90°$

4 Express each of the following in the form $r\cos(x + \alpha)$.

(a) $\cos x - \sin x$ **(b)** $\sqrt{3}\cos x - \sin x$
(c) $8\cos x - \sin x$, $0° < \alpha < 90°$ **(d)** $5\cos x - 3\sin x$, $0° < \alpha < 90°$

5 Which expression is equivalent to $8\sin x - 15\cos x$?

A $17\cos(x - 61°\,56')$ **B** $17\sin(x - 61°\,56')$
C $17\cos(x + 61°\,56')$ **D** $17\sin(x + 61°\,56')$

6 Find **(i)** the maximum and **(ii)** the minimum value of the following expressions. Also find the smallest positive values of x for which the maximum and minimum occur.

(a) $\sin x - \sqrt{3}\cos x$ **(b)** $\cos x + \sin x$ **(c)** $2\sqrt{3}\cos x - 2\sin x$
(d) $5\sin x + 12\cos x$ (answer in degrees)

7 Solve:

(a) $\cos x + \sin x = 1$, $0 \leq x \leq 2\pi$ **(b)** $\cos x + \sqrt{3}\sin x = 2$, $0 \leq x \leq 2\pi$
(c) $3\cos x + 2\sin x = \sqrt{13}$, $0° \leq x \leq 360°$ **(d)** $3\sin x - \sqrt{3}\cos x = \sqrt{3}$, $0 \leq x \leq 2\pi$
(e) $6\sin x + 8\cos x = -5$, $0° \leq x \leq 360°$ **(f)** $4\cos x + 3\sin x = -1$, $0° \leq x \leq 360°$
(g) $\cos x - \sqrt{3}\sin x = 2$, $0 \leq x \leq 2\pi$ **(h)** $\cos x - \sin x = -1$, $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
(i) $3\sin x + 4\cos x = -2$, $-180° \leq x \leq 180°$ **(j)** $\sqrt{2}\sin x - \cos x = 1.5$, $0° \leq x \leq 360°$

8 Sketch the graph of $f(x) = \sqrt{3}\cos x - \sin x$, $0 \leq x \leq 2\pi$. Use your sketch to find the values of x for which:

(a) $f(x) = 1$ **(b)** $f(x) > 1$

9 Sketch the graph of $f(x) = \cos x + \sin x$, $-\pi \leq x \leq \pi$. Use your sketch to find the values of x for which:

(a) $f(x) = -1$ **(b)** $f(x) \geq -1$

8.2 SOLVING QUADRATIC TRIGONOMETRIC EQUATIONS

Example 4

(a) Solve the equation $\tan^2\theta + \tan\theta - 2 = 0,\ 0° < \theta < 360°$.

(b) Solve the equation $\cos^2 x = 2\cos x,\ -\pi \le x \le \pi$.

Solution

(a) Factorise: $(\tan\theta - 1)(\tan\theta + 2) = 0$

$\tan\theta = 1$ **or** -2

$\theta = 45°, 225°$ **or** $116° 34', 296° 34'$

Solution is $\theta = 45°, 116° 34', 225°, 296° 34'$.

(b) Rearrange: $\cos^2 x - 2\cos x = 0$

Factorise: $\cos x(\cos x - 2) = 0$

$\therefore\ \cos x = 0$ **or** 2

Because $|\cos x| \le 1$, the only solution is $\cos x = 0$.

$\therefore\ x = -\frac{\pi}{2}, \frac{\pi}{2}$

Example 5

Solve the equation $\sec^2 x - 2\tan x = 4$ for $0 \le x \le 2\pi$.
(Trigonometric values rounded to 3 d.p. where necessary.)

Solution

The trigonometric functions are different, but they can be linked by the identity $\sec^2 x = 1 + \tan^2 x$:

$$\sec^2 x - 2\tan x = 4$$
$$1 + \tan^2 x - 2\tan x = 4$$
$$\tan^2 x - 2\tan x - 3 = 0$$
$$(\tan x - 3)(\tan x + 1) = 0$$
$$\tan x = -1 \quad \textbf{or} \quad 3$$
$$x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 1.249, \pi + 1.249$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}, 1.249, 4.391$$

EXERCISE 8.2 SOLVING QUADRATIC TRIGONOMETRIC EQUATIONS

1 Solve for values between 0 and 2π inclusive:

(a) $\tan^2 x - 1 = 0$ **(b)** $\sin^2 x - \sin x = 0$ **(c)** $\cos^2\theta - 2\cos\theta + 1 = 0$

(d) $\sqrt{3}\tan^2 x + \tan x = 0$ **(e)** $4\sin^2\theta = 1$ **(f)** $\sin^2 x - \sin x\cos x = 0$

2 If $0 \le \theta \le 2\pi$, the solution to $2\cos^2\theta - 1 = 0$ is:

A $\frac{\pi}{4}, \frac{7\pi}{4}$ **B** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **C** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **D** $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

3 Solve for values between 0 and 2π inclusive:

(a) $2\cos^2\theta - 3\cos\theta - 2 = 0$ **(b)** $2\cos^2\theta + \sin\theta = 1$

(c) $2\sin^2\theta - 3\cos\theta = 2$ **(d)** $(2\cos x + 1)(\sin x - 1) = 0$

4 Solve for $0 < x < 360°$: **(a)** $2\tan^2 x + \tan x = 15$ **(b)** $5\cos^2 x + 2\sin x = 2$

5 If $0 \leq \theta \leq 2\pi$, then the solution to $3\sin^2\theta - 4\cos\theta + 1 = 0$ is (approximately or exactly):

A $\theta = 0.841, 5.442$ **B** $\theta = 1.969, 4.315$ **C** $\theta = 2.301, 3.983$ **D** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

6 Solve for $0 \leq \theta \leq 2\pi$: **(a)** $3\tan^3\theta - 3\tan^2\theta - \tan\theta + 1 = 0$ **(b)** $\cos^3\theta - 2\cos^2\theta + \cos\theta = 0$

8.3 SOLVING EQUATIONS USING ANGLE FORMULAE, INCLUDING THE *t* FORMULAE

In Chapter 4, you solved trigonometric equations using the angle formulae. More complex trigonometric equations will be considered in this section.

When you are using the *t* formulae to solve an equation, because $\tan\frac{\pi}{2}$ is undefined you must substitute $\theta = \pi$ or 180° into your equations to check that you have not missed a possible solution.

Example 6

Solve the equation $2\sin\left(x + \frac{5\pi}{6}\right) = \sin x$, for $0 \leq x \leq 2\pi$.

Solution

$$2\left(\sin x\cos\frac{5\pi}{6} + \cos x\sin\frac{5\pi}{6}\right) = \sin x$$

$$2\sin x \times \left(-\frac{\sqrt{3}}{2}\right) + 2\cos x \times \frac{1}{2} = \sin x$$

$$-\sqrt{3}\sin x + \cos x = \sin x$$

$$(1 + \sqrt{3})\sin x = \cos x$$

$$\tan x = \frac{1}{\sqrt{3}+1}$$

$$x = 0.3509, \pi + 0.3509$$

$$x = 0.351, 3.493$$

Example 7

Solve for $0 \leq x \leq 2\pi$: **(a)** $4\cos x = \operatorname{cosec} x$ **(b)** $\cos 4x - \cos 2x = 0$

Solution

(a)

$$4\cos x = \operatorname{cosec} x$$

$$4\cos x = \frac{1}{\sin x}$$

$$4\sin x\cos x = 1$$

$$2\sin 2x = 1$$

$$\sin 2x = 0.5$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

(b)

$$\cos 4x - \cos 2x = 0$$

$$2\cos^2 2x - 1 - \cos 2x = 0$$

$$2\cos^2 2x - \cos 2x - 1 = 0$$

$$(\cos 2x - 1)(2\cos 2x + 1) = 0$$

$$\cos 2x = 1, \cos 2x = -0.5$$

$$2x = 0, 2\pi, 4\pi \quad \textbf{or} \quad 2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$$

This equation could also have been solved by writing $\cos 4x = \cos 2x$.

Example 8

Solve for $0 \le x \le 2\pi$: **(a)** $\cos 3x = \cos 2x \cos x$ **(b)** $\sin 7x - \sin x = \sin 3x$

Solution

(a) $\cos 3x = \cos 2x \cos x$

$\cos 3x = \frac{1}{2}(\cos 3x + \cos x)$

$2\cos 3x = \cos 3x + \cos x$

$\cos 3x = \cos x$

$3x = x, 2\pi - x, 2\pi + x, 4\pi - x, 4\pi + x, 6\pi - x, 6\pi + x, 8\pi - x$

$2x = 0, 2\pi, 4\pi, 6\pi$ **or** $4x = 2\pi, 4\pi, 6\pi, 8\pi$

$x = 0, \pi, 2\pi$ **or** $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

The solution is $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(b) $\sin 7x - \sin x = \sin 3x$

$2\cos 4x \sin 3x = \sin 3x$

$\sin 3x(2\cos 4x - 1) = 0$

$\sin 3x = 0, \cos 4x = 0.5$

$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$ **or** $4x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3}$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ **or** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

$x = 0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{5\pi}{3}, 2\pi$

Example 9

(a) Use the expansion of $\sin(2\theta + \theta)$ to obtain an expression for $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence find the roots of $4x^3 - 3x + 0.5 = 0$.

Solution

(a) $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Let $x = \sin\theta$: $4\sin^3\theta - 3\sin\theta + 0.5 = 0$

$3\sin\theta - 4\sin^3\theta = 0.5$

$\sin 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

Hence the roots are $x = \sin\frac{\pi}{18}, \sin\frac{5\pi}{18}, \sin\frac{25\pi}{18}$.

Example 10

Solve $5\cos\theta - 2\sin\theta = 2$ for $0° \le \theta \le 360°$ using the t formulae.

Solution

$t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\tan\theta = \frac{2t}{1-t^2}$.

Substitute into the equation: $5 \times \frac{1-t^2}{1+t^2} - 2 \times \frac{2t}{1+t^2} = 2$

Simplify: $5 - 5t^2 - 4t = 2 + 2t^2$

$7t^2 + 4t - 3 = 0$

$(7t - 3)(t + 1) = 0$

$t = \frac{3}{7}, -1$

$0° \le \frac{\theta}{2} \le 180°$: $\frac{\theta}{2} = 23° 12', 180° - 45°$

$\theta = 46° 24', 270°.$

Test whether $\theta = 180°$ is a solution: LHS $= 5\cos 180° - 2\sin 180° = -5 - 0 \neq 2$

This equation could also have been solved using the auxiliary angle method.

Example 11

(a) Show that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$.

(b) By using suitable substitutions for A and B, show that $\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$.

(c) Hence solve $\sin 2x + \sin 4x = \sin 6x$ for $0 \le x \le \pi$.

Solution

(a) LHS $= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2\sin A \cos B$

(b) Let $x = A + B$ and $y = A - B$.

Adding these equations: $A = \frac{x + y}{2}$ Subtracting the equations: $B = \frac{x - y}{2}$

Substitute these results in **(a)**: $\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$

(c) Use the result in **(b)** on the LHS: $\sin 2x + \sin 4x = 2\sin 3x \cos(-x) = 2\sin 3x \cos x$

Use the double angle formula $\sin 2x = 2\sin x \cos x$ on the RHS: $\sin 6x = 2\sin 3x \cos 3x$

$2\sin 3x \cos(-x) = 2\sin 3x \cos 3x$

$2\sin 3x \cos x = 2\sin 3x \cos 3x$ ($\cos x$ is an even function, so $\cos(-x) = \cos x$)

$2\sin 3x(\cos x - \cos 3x) = 0$

$\sin 3x = 0$ **or** $\cos x - \cos 3x = 0$

$3x = 0, \pi, 2\pi, 3\pi$ **or** $\cos 3x = \cos x$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ **or** $3x = x, 2\pi - x, 2\pi + x, 4\pi - x$

$2x = 0, 4x = 2\pi, 2x = 2\pi, 4x = 4\pi$

$x = 0, \frac{\pi}{2}, \pi$

Solution is $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$.

Example 12

Solve the equation $\cos 4x + \sin 3x = 0$ for $0 \le x \le \pi$.

Solution

Rewrite equation: $\cos 4x = -\sin 3x$

Sine is an odd function, so: $-\sin 3x = \sin(-3x)$: $\cos 4x = \sin(-3x)$

Use $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to rewrite equation: $\cos 4x = \cos\left(\frac{\pi}{2} + 3x\right)$

$4x = \frac{\pi}{2} + 3x, 2\pi - \left(\frac{\pi}{2} + 3x\right), 2\pi + \left(\frac{\pi}{2} + 3x\right), 4\pi - \left(\frac{\pi}{2} + 3x\right), 6\pi - \left(\frac{\pi}{2} + 3x\right), 8\pi - \left(\frac{\pi}{2} + 3x\right)$

$x = \frac{\pi}{2}, 7x = \frac{3\pi}{2}, x = \frac{5\pi}{2}, 7x = \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}.$

$x = \frac{\pi}{2}, \frac{3\pi}{14}, \frac{\pi}{2}, \frac{11\pi}{14}, \frac{15\pi}{14}.$

As $0 \leq x \leq \pi$, the solution is $x = \frac{3\pi}{14}, \frac{\pi}{2}$ **or** $\frac{11\pi}{14}$.

EXERCISE 8.3 SOLVING EQUATIONS USING ANGLE FORMULAE, INCLUDING THE *t* FORMULAE

1 Solve for $0 \leq x \leq 2\pi$:

(a) $\sin\left(x + \frac{\pi}{3}\right) = \cos x$ **(b)** $\sin\left(x + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} - x\right)$ **(c)** $2\sin\left(x + \frac{\pi}{6}\right) = \sin x$

(d) $\sin\left(x + \frac{\pi}{4}\right) = \cos\left(x - \frac{\pi}{3}\right)$ **(e)** $4\tan\left(x - \frac{\pi}{6}\right) = \tan\left(x - \frac{\pi}{3}\right)$ **(f)** $\cot x + \cot\left(x + \frac{\pi}{4}\right) = 3$

2 Solve for $0 \leq x \leq 2\pi$:

(a) $2\cos x = \operatorname{cosec} x$ **(b)** $4\sin x = \sec x$ **(c)** $4\cos x = \sqrt{3}\operatorname{cosec} x$

(d) $3\sin x = \sec x$ **(e)** $\tan x = \operatorname{cosec} 2x$ **(f)** $\sin 2x + \cos 2x = 1$

3 Solve for $0 \leq x \leq 2\pi$, using the double angle formulae:

(a) $\cos 2x - \cos x = 0$ **(b)** $\cos 2x - \sin x = 0$ **(c)** $\sin 4x + \sin 2x = 0$ **(d)** $\tan 4x + \tan 2x = 0$

4 Solve for $0 \leq x \leq \pi$:

(a) $\cos 5x = \cos 4x \cos x$ **(b)** $\sin 5x - \sin x = \sin 3x$

(c) $\sin 7x \cos x = \sin 4x \cos 4x$ **(d)** $\sin 2x + \cos 2x = \sin x - \cos x$

5 Solve $0 \leq x \leq 2\pi$:

(a) $3\sin 2x + 4\sin^2 x - 2 = 0$ **(b)** $4(\sin^2 2x - \sin^2 x) = 1$ **(c)** $\tan 2x = \tan x$ **(d)** $\tan 2x + 3\tan x = 0$

6 Solve $0 \leq x \leq \pi$:

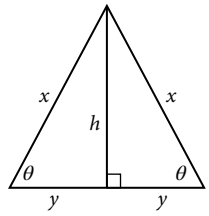
(a) $\cos^2 x - \sin^2 x = 1$ **(b)** $\tan x + \cot x = 2$ **(c)** $\tan x + 2\cot x + \sec x = 0$

7 Solve $0 \leq x \leq 2\pi$:

(a) $3\sin x + 4\cos x = 5$ **(b)** $\cos x + 3\sin x = 2$ **(c)** $10\tan x - 2\sec x = 5$ **(d)** $\sin 2x = 1 + \cos 2x$

8 Show that if $a^2 + b^2 < c^2$, the equation $a\cos\theta + b\sin\theta = c$ has no real roots.

9 The equal sides of an isosceles triangle are x cm and the third side is $2y$ cm. The equal angles are each θ and the height of the triangle is h cm, as shown.
If the perimeter of the triangle is four times the height, find the size of the angles of the triangle to the nearest minute.



10 Solve each equation:

(a) $\tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}\left(\frac{x}{3}\right) = \tan^{-1}\left(\frac{1}{5}\right)$ **(b)** $\tan^{-1}(2x) + \tan^{-1}(3x) = \tan^{-1}(1)$

11 **(a)** Use the expansion of $\cos(2\theta + \theta)$ to obtain an expression for $\cos 3\theta$ in terms of $\cos\theta$.

(b) Hence find the roots of $8x^3 - 6x - \sqrt{3} = 0$.

12 **(a)** Use the expansion of $\tan(2\theta + \theta)$ to obtain an expression for $\tan 3\theta$ in terms of $\tan\theta$.

(b) Use this result to show that $\tan 15° = 2 - \sqrt{3}$. Justify your answer.

(c) Use the expansion of $\tan(45° + 30°)$ to find the exact value of $\tan 75°$. Compare this answer to the other result obtained in **(b)**.

13 Solve each equation using the t formulae, for $0° \le \theta \le 360°$.

(a) $2\sin\theta + \cos\theta = 1$ **(b)** $5\cos\theta + 3\sin\theta = 4$ **(c)** $2\operatorname{cosec}\theta - 4\cot\theta = 3$

14 Solve each equation using the t formulae, for $0 \le x \le 2\pi$.

(a) $\sin x + \cos x = 1$ **(b)** $7\cos x - 24\sin x = 5$ **(c)** $6\sin x + 8\cos x = 5$

15 Solve for $0 \le x \le \pi$.

(a) $\cos 4x = \cos 2x$ **(b)** $\tan 2x = \cot x$ **(c)** $\sin 3x = \sin x$ **(d)** $\cos 3x = \sin 2x$

16 Solve for $0 \le \theta \le \pi$.

(a) $\cos 3\theta = \sin\left(\frac{\pi}{4} - \theta\right)$ **(b)** $\sin 2\theta = \cos\left(\theta - \frac{\pi}{4}\right)$ **(c)** $\cos 2\theta = \sin\left(\theta + \frac{\pi}{4}\right)$ **(d)** $\sin\left(\theta - \frac{\pi}{3}\right) = \cos 2\theta$

17 Solve for $0 \le x \le \pi$.

(a) $\sin 3x + \sin x = 0$ **(b)** $\sin 2x + \cos 3x = 0$ **(c)** $\tan 2x + \cot 3x = 0$

18 **(a)** Given that $2\sin\theta\cos\phi = \sin(\theta + \phi) + \sin(\theta - \phi)$, show that $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

(b) Hence solve for $0 \le \theta \le \pi$:

(i) $\sin 3\theta = \sin 5\theta + \sin\theta$ **(ii)** $\sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

(iii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$ **(iv)** $\sin 5\theta + \sin 3\theta = \sin 4\theta + \sin 2\theta$

19 **(a)** Given that $2\cos\theta\cos\phi = \cos(\theta + \phi) + \cos(\theta - \phi)$, show that $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

(b) Hence solve for $0 \le x \le 2\pi$:

(i) $\cos 5x + \cos x = \cos 3x$ **(ii)** $\cos x + \cos 3x = \cos 5x + \cos 7x$ **(iii)** $\cos 5x + \cos x + \cos 7x + \cos 3x = 0$

CHAPTER REVIEW 8

1 **(a)** Express $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) - 2\cos\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(b) Hence, or otherwise, solve $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) - 2\cos\theta = 1$ for $0 < \theta < 2\pi$.

2 **(a)** Express $3\sin x + 4\cos x$ in the form $r\sin(x + \alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$.

(b) Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \le x \le 2\pi$. Give answer(s) to two decimal places.

(c) Write the general solution for $3\sin x + 4\cos x = 5$.

3 Use $\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$ to find the exact value of $\tan\frac{\pi}{8}$.

4 Show that the cubic equation $8x^3 - 6x + 1 = 0$ can be reduced to the form $\cos 3\theta = -\frac{1}{2}$ by substituting $x = \cos\theta$. From this, deduce the following:

(a) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ **(b)** $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} = 6 + \sec\frac{\pi}{9}$

(c) $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$ **(d)** $\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} = 33$

5 It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Use this result to solve $\cos 3\theta + \cos 2\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$.

6 **(a)** Expand $\cos(2A + B)$ and hence prove that $\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$.

(b) By writing $x = k\cos\theta$ and giving k a suitable value, use the formula proved in part **(a)** to find the three roots of the equation $27x^3 - 9x = 1$. Hence write the value of the product $\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$.

7 If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation $x^3 - (a+1)x^2 + (c-a)x - c = 0$, show that $\alpha + \beta + \gamma = n\pi + \frac{\pi}{4}$.

8 Solve $\sin x = \cos 5x$ for $0 < x < \pi$.

9 **(a)** Find A and B in terms of x and y such that $\sin x + \sin y = 2\sin A\cos B$.

(b) Find the solution of $\sin\theta + \sin 2\theta + \sin 3\theta = 0$ for $0 \le \theta \le \pi$.

CHAPTER 9
Proof by mathematical induction

The nature of proof by induction

You can often see that a particular pattern seems to apply to a situation, and you may then attempt to formulate a rule that applies to that situation. For example, you might note that 1 is a factor of 420, so is 2, so is 3, so is 4 and so is 5. From these observations you might make the statement that 'every positive integer is a factor of 420', or in other words 'n is a factor of 420 for all positive integers n'. This statement is clearly wrong: $n = 9$ is a **counter-example** which proves that the statement is false (because 9 is a positive integer and not a factor of 420).

It only takes one counter-example to disprove a general statement.

For example, can you find counter-examples to disprove the following statements?

- $n^2 + n + 11$ is a prime number for all integers $n \geq 1$.
- $n^2 - n + 41$ is a prime number for all integers $n \geq 1$.

More importantly: *how do you prove that a general statement is true?* Even if you demonstrate that a statement is true for the first 1 000 000 cases, that would still not **prove** the statement to be true in all cases.

Mathematical induction is a method of proof that can be used in exactly these circumstances (for example, to prove that something is true for all positive integers n, or for all positive integers n greater than a specified starting value, or for all positive odd integers n).

A proof by induction consists of a three-step process, the final step being a conclusion.

1 Prove that the statement is true in the first case possible.

2 Prove that if the statement is true in any one case (i.e. for $n = k$), then it must also be true in the next case (i.e. for $n = k + 1$, or for $n = k + 2$ if the statement is true for even integers only, etc.).

3 Write a conclusion:

The statement is true for $n = k + 1$ if it is true for $n = k$. (As proved in step 2.)

The statement is true for $n = 1$ (i.e. for the first possible case). (As proved in step 1.)

Thus the statement is true for all integers $n \geq 1$.

In step 2 it is proven that there is the potential for a 'chain reaction' to occur: if it is true for any one case then it is also true for the next case, which means that it is also true for the case after that, and so on forever.

In step 1 it is proven that it is true in the first case: this is like flicking the switch to start a chain reaction or pushing over the first domino of an infinite line of dominoes. The infinite cascade of falling dominoes means that the statement is true for all appropriate values of n from the starting value to infinity.

You will now look at several different types of problems that require proof by induction.

9.1 MATHEMATICAL INDUCTION INVOLVING SERIES

Note the following pattern:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$

Observe that the sum of n consecutive odd numbers starting from 1 is always n^2.
Can you prove this? Yes, this can be proved by induction. (See Example **1**.)

Example 1

Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all integers $n \geq 1$.

Solution

The two methods shown below are equivalent, but Method 2 introduces a symbolic representation of the statement being proved. When you are familiar with Method 2 you will find that it requires less writing than with Method 1.

No matter which method you use, you must always write all three steps and use the appropriate words emphasised below (or their equivalent) in each step.

Method 1

Step 1 Prove that the statement is true for $n = 1$.

When $n = 1$: LHS $= 1$ RHS $= 1^2 = 1$

LHS = RHS $\therefore$ the statement is true for $n = 1$

Step 2 Assume the statement is true for $n = k$, where k is any integer greater than or equal to 1.

i.e. assume that $\boxed{1 + 3 + 5 + \ldots + (2k - 1)} = k^2$ [a]

Now prove that the statement will be true for $n = k + 1$ if it is true for $n = k$.

i.e. prove that $1 + 3 + 5 + \ldots + (2[k + 1] - 1) = (k + 1)^2$

$$\text{LHS} = 1 + 3 + 5 + \ldots + (2[k + 1] - 1)$$

$$= \boxed{1 + 3 + 5 + \ldots + (2k - 1)} + (2[k + 1] - 1)$$

using [a]: $= k^2 + (2k + 1)$

(You can only prove this is true when $n = k + 1$ if it is true when $n = k$.)

$$= (k + 1)^2$$

$$= \text{RHS}$$

Step 3 Conclusion

The statement is true for $n = k + 1$ if it is true for $n = k$ (Step 2)

The statement is true for $n = 1$ (Step 1)

$\therefore$ by induction, the statement is true for all integers $n \geq 1$.

Method 2

Let $S(n)$ be the statement that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ where n is a positive integer.

Step 1 Prove that $S(1)$ is true.

LHS $= 1$ RHS $= 1^2 = 1$

LHS = RHS $\therefore$ $S(1)$ is true

Step 2 Assume $S(k)$ is true.

i.e. assume that $\boxed{1+3+5+\ldots+(2k-1)}=k^2$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $1+3+5+\ldots+(2[k+1]-1)=(k+1)^2$

$$\begin{aligned}\text{LHS} &= 1+3+5+\ldots+(2[k+1]-1)\\ &= \boxed{1+3+5+\ldots+(2k-1)}+(2[k+1]-1)\\ &= k^2+(2k+1) \quad \text{using [a]:}\\ &= (k+1)^2\\ &= \text{RHS}\end{aligned}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 2

Prove by induction that $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Solution

Let $S(n)$ be the statement that $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$, where n is a positive integer.

Step 1 Prove that $S(1)$ is true.

$\text{LHS}=1^2=1$ $\quad \text{RHS}=\dfrac{1(1+1)(2\times1+1)}{6}=\dfrac{1(2)(3)}{6}=1$

$\text{LHS}=\text{RHS}$ $\quad \therefore S(1)$ is true

Step 2 Assume $S(k)$ is true for a positive integer k.

i.e. assume that $\boxed{1^2+2^2+3^2+\ldots+k^2}=\dfrac{k(k+1)(2k+1)}{6}$ [a]

Now, prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $1^2+2^2+3^2+\ldots+(k+1)^2=\dfrac{(k+1)(k+2)(2k+3)}{6}$

$$\begin{aligned}\text{LHS} &= \boxed{1^2+2^2+3^2+\ldots+k^2}+(k+1)^2\\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \text{using [a]:}\end{aligned}$$

Now look at the RHS: you want $\dfrac{(k+1)}{6}$ as a factor.

$$\begin{aligned}\text{LHS} &= \frac{(k+1)}{6}\big(k(2k+1)+6(k+1)\big)\\ &= \frac{(k+1)}{6}\big(2k^2+7k+6\big)\\ &= \frac{(k+1)(k+2)(2k+3)}{6}=\text{RHS}\end{aligned}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 3

Prove by induction that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for all integers $n \geq 1$.

Solution

Let $S(n)$ be the statement that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for positive integer n.

Step 1 Prove that $S(1)$ is true.

$\text{LHS} = 1 + (2 \times 1)$ (Note that the sum on the LHS starts with n and finishes with $2n$.)
$= 3$

$\text{RHS} = \frac{3 \times 1(1+1)}{2} = 3$

LHS = RHS $\therefore$ $S(1)$ is true

Step 2 Assume $S(k)$ is true for an integer $k \geq 1$.

i.e. assume that $\boxed{k + (k+1) + (k+2) + \ldots + 2k} = \frac{3k(k+1)}{2}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $(k+1) + (k+2) + \ldots + 2(k+1) = \frac{3(k+1)((k+1)+1)}{2}$

Note that when $n = k+1$, the sum on the LHS starts with $(k+1)$ and finishes with $2(k+1)$.

$$\text{LHS} = (k+1) + (k+2) + \ldots + 2k + (2k+1) + (2k+2)$$

Now you have a problem! At this point you need to use the substitution of line [a], but you are missing the first term (the k). In going from $S(k)$ to $S(k+1)$ you have lost the first term but gained two extra terms on the LHS.The solution is to break up the extra two terms $(2k+1) + (2k+2)$ as a k term and a $(3k+3)$ term.

$$\text{LHS} = \boxed{k + (k+1) + (k+2) + \ldots + 2k} + (3k+3)$$

using [a]: $= \frac{3k(k+1)}{2} + 3(k+1)$

$= \frac{3(k+1)}{2}[k+2]$ taking the common factor of $\frac{3(k+1)}{2}$

$= \text{RHS}$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

EXERCISE 9.1 MATHEMATICAL INDUCTION INVOLVING SERIES

1 If $S(n)$ is the statement that $n + 2n + 3n + \ldots + n^2 = \frac{n^2(n+1)}{2}$, then $S(5)$ represents the statement:

A $1 + 2 + 3 + \ldots + 5 = \frac{5 \times 6}{2}$ **B** $1 + 2 + 3 + \ldots + 25 = \frac{25 \times 26}{2}$

C $1 + 2 + 3 + \ldots + 25 = \frac{25 \times 6}{2}$ **D** $5 + 10 + 15 + 20 + 25 = \frac{25 \times 6}{2}$

Prove each of the following by induction for all positive integers n.

2 $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

3 $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

4 $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

5 $a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = \frac{n}{2}[2a + (n-1)d]$

6 $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$

7 $1 + r + r^2 + r^3 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}$

8 $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$

9 $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + n(n + 2) = \frac{n(n+1)(2n+7)}{6}$

10 $4 + 104 + 756 + \ldots + (n^3 + 3n^5) = \frac{n^3(n+1)^3}{2}$

11 $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

12 $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$

13 $\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \ldots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$

14 $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

15 **(a)** $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$. Hence find $\lim_{n \to \infty}\left(\frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4}\right)$

(b) Hence show that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$.

16 $(n + 1) + (n + 2) + \ldots + 2n = \frac{n(3n+1)}{2}$

17 $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$

18 $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n - 1) \times 2^n$.

19 $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(4n^2 - 1)}{3}$

20 $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = \frac{(-1)^{n-1} n(n+1)}{2}$

21 $1 \times 2 + 4 \times 2^2 + 9 \times 2^3 + \ldots + n^2 \times 2^n = (n^2 - 2n + 3) \times 2^{n+1} - 6$

22 $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \ldots + \frac{2^n - 1}{2^n} = n - 1 + \frac{1}{2^n}$

23 $2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \ldots + (n + 1) \times 2^n = n \times 2^{n+1}$

24 $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)}{30}$ (For those who like an algebraic challenge.)

9.2 PROVING DIVISIBILITY BY INDUCTION

The following points are essential for understanding a proof by induction where the assertion involves divisibility.

- If an integer N is divisible by an integer d, then $N = d \times q$, where q is an integer.
  For example, 18 is divisible by 3 because $18 = 3 \times 6$ (where 6 is an integer). But 19 is not divisible by 3 even though $19 = 3 \times 6\frac{1}{3}$, because $6\frac{1}{3}$ is not an integer.
- The integers are **closed** under addition, subtraction and multiplication. This means that whenever you add, subtract or multiply integers, the answer will always be another integer. This is not true for division: some divisions of integers give integer answers, but others do not.

Example 4

Prove by induction that $4^n - 1$ is divisible by 3 for all integers $n \geq 1$.

Solution

Let $S(n)$ be the statement that $4^n - 1$ is divisible by 3 for integer n.

Step 1 Prove that $S(1)$ is true.

$4^1 - 1 = 3$, which is divisible by 3. (*Note:* The LHS–RHS style of proof can't be used here.)

$\therefore$ $S(1)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k \geq 1$.

i.e. assume that $4^k - 1 = 3M$ where M is an integer

$4^k = 3M + 1$ [a]

Now prove that $S(k + 1)$ is true if $S(k)$ is true.

i.e. prove that $4^{k+1} - 1$ is divisible by 3

$4^{k+1} - 1 = 4 \times 4^k - 1$

using [a]: $= 4(3M + 1) - 1$

$= 12M + 3$

$= 3(4M + 1)$ which is divisible by 3, because it is a multiple of 3 and $4M + 1$ is an integer (as integers are closed under multiplication and addition).

Step 3 Conclusion

$S(k + 1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 5

Prove by induction that $3^n + 7^n$ is divisible by 10 for all positive odd integers n.

Solution

Let $S(n)$ be the statement that $3^n + 7^n$ is divisible by 10 for positive odd integer n.

Step 1 Prove that $S(1)$ is true.

$3^1 + 7^1 = 10$, which is divisible by 10.

$\therefore$ $S(1)$ is true.

Step 2 Assume $S(k)$ is true for an odd integer $k \geq 1$.

i.e. assume that $3^k + 7^k = 10M$ where M is an integer

$3^k = 10M - 7^k$ [a] (*Note*: 7^k could have been made the subject instead of 3^k.)

Now prove that $S(k + 2)$ is true if $S(k)$ is true.

(Note the $S(k+2)$ instead of $S(k+1)$, as this is working with odd integer values of n.)

i.e. prove that $3^{k+2} + 7^{k+2}$ is divisible by 10

$$3^{k+2} + 7^{k+2} = 3^2 \times 3^k + 7^2 \times 7^k$$

using [a]: $$= 9(10M - 7^k) + 49 \times 7^k$$

$$= 90M - 9 \times 7^k + 49 \times 7^k$$

$$= 90M + 40 \times 7^k$$

$$= 10(9M + 4 \times 7^k)$$

This is divisible by 10, because it is a multiple of 10 and $9M + 4 \times 7^k$ is an integer.

Step 3 Conclusion

$S(k+2)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all odd integers $n \geq 1$.

EXERCISE 9.2 PROVING DIVISIBILITY BY INDUCTION

1 If k and M are integers, which of the following expressions does *not* always generate an integer?

A $9M + 4 \times 7k$ **B** $9M - 4 \times 7k$ **C** $9M \div 4 \times 7k$ **D** $9M \times 4 \times 7k$

Prove the following by induction.

2 $5^n + 3$ is divisible by 4 for all positive integers n.

3 $3^{2n} - 1$ is divisible by 8 for all positive integers n.

4 $3^n + 2^n$ is divisible by 5 for all odd integers $n \geq 1$.

5 $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

6 **(a)** Factorise $k(k+1)(k+2) + 3(k+1)(k+2)$.

(b) Hence prove that $n(n+1)(n+2)$ is divisible by 3 for all positive integers n.

7 $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n.

8 $7^n - 2^n$ is divisible by 9 for $n \geq 2$.

9 $3^{4n} - 1$ is divisible by 80 for all positive integers n.

10 $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n.

11 $2^{3n} - 1$ is divisible by 7 for $n \geq 1$

12 $6^n + 10n - 6$ is divisible by 5 for all positive integers n.

13 $3^{2n+1} + 2^{n-1}$ is divisible by 7 for all positive integers n.

14 **(a)** Show that $(k+3)^3 = k^3 + 9k^2 + 27k + 27$.

(b) Hence prove that the sum of the cubes of three consecutive positive integers is divisible by 3.

15 Prove that the polynomial $(x-1)^{n+2} + x^{2n+1}$ is divisible by $x^2 - x + 1$ for all positive integers n.

(*Note:* In step 2, you can't say $(x-1)^{k+2} + x^{2k+1} = (x^2 - x + 1)M$ where M is an integer. You must say $(x-1)^{k+2} + x^{2k+1} = (x^2 - x + 1)M(x)$, where $M(x)$ is a polynomial, and continue this through the rest of the proof.)

16 Prove that $x^n - 1$ is divisible by $x - 1$ for all positive integers n. (Use $x^{k+1} - 1 = x^{k+1} - x^k + x^k - 1$.)

9.3 WHEN INDUCTION DOESN'T WORK

A single counter-example is enough to disprove a result.

Some proofs by mathematical induction seem to be correct, but turn out to be incorrect. The most likely reason is that you have failed to prove step 1, i.e. that the result is true at the beginning (usually for $n = 1$). This means that the assumed result ($n = k$) is false, so when you prove the result true for $n = k + 1$, you are proving it from an incorrect assumption.

Example 6

Let $S(n)$ be the statement: $n^2 - n$ is an odd integer, for all positive integers n.

(a) Show that if $S(k)$ is true, then $S(k + 1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any n?

Solution

(a) Let $S(k)$ be that $k^2 - k$ is an odd integer.
Statement: $S(k + 1)$: $(k + 1)^2 - (k + 1)$ is an odd integer.
$(k + 1)^2 - (k + 1)$
$= (k + 1)(k + 1 - 1)$
$= k(k + 1)$
$= k^2 - k + 2k$
$= S(k) + 2k$
= Odd + Even = Odd
Hence the result is true for $n = k + 1$ if it is true for $n = k$.

(b) $S(1)$ is that $1^2 - 1$ is an odd integer and since this result is 0, then $S(1)$ is false.

(c) $S(n)$ is never true because when n is even, $n^2 - n$ is even, and when n is odd, $n^2 - n$ is even.

Example 7

Use mathematical induction to prove that $2^n - 1$ is prime if n is prime.

Solution

Using a table of values to explore this for some values of n:

n	2	3	5	7
$2n - 1$	3	7	31	127

This suggests that the result works, although finding a pattern for the prime numbers for n is not possible.

The next prime values for n are 11, 13, 17, 19, 23, …

Any non-prime value for $2^n - 1$ is enough to say the original statement is false. Testing these values finds for $n = 11$: $2^{11} - 1 = 2047$. As $2047 = 23 \times 89$, this is not prime, so the initial assumption is false.

The result cannot be proved by mathematical induction.

EXERCISE 9.3 WHEN INDUCTION DOESN'T WORK

1 Let $S(n)$ be the statement: $n^2 + n$ is an odd integer.

(a) Show that if $S(k)$ is true, then $(k + 1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any n?
(d) If the statement is not true, what change do you need to make to make it true? Prove your new statement.

2 Given $1^2 + 4^2 + 7^2 + \ldots + (3n - 2)^2 = \frac{n}{2}\left(6n^2 - 3n - 1\right)$.

(a) Show that if $S(k)$ is true then $S(k + 1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any n?

3 It is stated that $n^2 - n + 41$ is prime for $n \geq 1$.

(a) Is this statement true for $n = 1$? **(b)** Is this statement true for $n = 2$?
(c) Is this statement true for $n = 5$?
(d) Is it possible to find a value of n for which this expression does not give a prime number? Justify your answer.
(e) Is the given statement true or false?

4 Let $S(n)$ be the statement: $n^2 - n$ is an odd integer.

(a) Show that if $S(k)$ is true then $S(k + 1)$ is true.
(b) Is $S(1)$ true?
(c) Is $S(n)$ true for any n?

CHAPTER REVIEW 9

1 Prove by induction that $-2 + 2 + 6 + \ldots + (4n - 6) = 2n(n - 2)$ for all integers $n \geq 1$.

2 Prove by induction that $4^{n+1} + 5^{2n-1}$ is divisible by 21 for all positive integers n.

3 Prove by induction that $7^n - 1$ is divisible by 3 for all positive integers n.

4 Prove by induction that $6 + 24 + 60 + \ldots + n(n + 1)(n + 2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n.

5 Prove by induction that $\dfrac{3}{4} + \dfrac{1}{6} + \dfrac{5}{96} + \ldots + \dfrac{n+2}{n(n+1)2^n} = 1 - \dfrac{1}{(n+1)2^n}$ for all integers $n \geq 1$.

6 Prove by induction that $5^n + 12n - 1$ is divisible by 16 for all positive integers n.

7 Prove by induction that $7^n + 6^n$ is divisible by 13 for all odd positive integers n.

8 Prove by induction that $2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n + 1)!$ for all positive integers n.

9 Prove by induction that $\dfrac{1}{4} + \dfrac{1}{28} + \dfrac{1}{70} + \ldots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$ for all integers $n \geq 1$.

CHAPTER 10
Vectors in two dimensions

10.1 INTRODUCTION TO VECTORS

Vectors give you a way to move around the Cartesian plane by moving a certain distance in a particular direction, by considering the direction to be split into horizontal and vertical parts. For example, if you walk 40 m east, then 30 m north, how far are you from your starting point, and in what direction?

This concept has many applications in Physics and is expanded to three dimensions in the Mathematics Extension 2 course. Vectors are used in kinematics and dynamics (motion), hydrodynamics (motion of a fluid), heat flow in solids, electromagnetism and many other real-life scientific applications.

Scalar quantities and vector quantities

A scalar quantity is one that is completely specified by its magnitude (size) and, if appropriate, its unit of measurement. Examples of scalar quantities include mass, length, speed, temperature and time.

Note that temperature is a scalar quantity, even though it may have a negative value (depending on the measurement scale being used). A negative temperature value indicates its relative magnitude and does not imply that temperature is oriented in any 'direction'.

A vector quantity is one that is completely described by its magnitude and direction (and, if appropriate, its unit of measurement). Examples of vector quantities include:

- displacement (change in position of an object)
- velocity (time rate of change of displacement or position)
- acceleration (time rate of change of velocity)
- force (a push or a pull that can affect the motion of an object)
- weight (the gravitational force acting on an object—this is not the same as the mass of an object, which is scalar)
- momentum (the product of an object's mass and velocity).

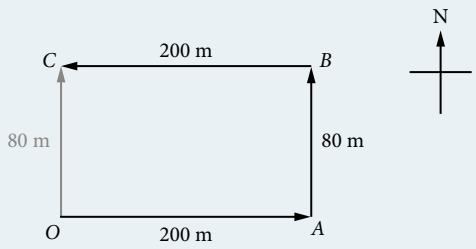
Consider the vector quantity of displacement. In order to fully specify the displacement of an object, it is necessary to know how far it is from its starting point (magnitude) and the direction in which it has moved.

Example 1

Sarah walks from point O to point C, via points A and B.

(a) What distance has she walked?

(b) What is her position relative to her starting point?



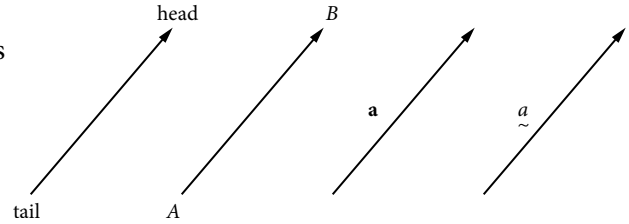
Solution

(a) Sarah walks a total distance of 480 m.

(b) Sarah is 80 m directly north of her starting point at O.
This is her displacement from O.

Vectors and vector notation

A vector quantity can be represented by a directed straight line segment with an arrowhead (on the line segment) indicating its direction and the length of the line indicating its magnitude. The beginning of the vector is called its tail and the end is called its head.



The vector with a tail at point A and head at point B is denoted $\overrightarrow{AB}$. Alternatively, it is also common to use a bold lower-case letter to denote a vector, e.g. $\mathbf{a}$, or an italic lower-case letter with a tilde underneath, e.g. $\underset{\sim}{a}$.

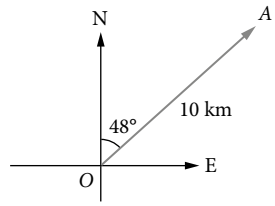
You will find it easiest to use $\overrightarrow{AB}$ or $\underset{\sim}{a}$ in handwritten text.

Magnitude of a vector

The magnitude of a vector is a measure of its size or length and is a scalar quantity.

The magnitude of vector $\overrightarrow{AB}$ is written $\left|\overrightarrow{AB}\right|$, the magnitude of vector $\underset{\sim}{a}$ is written $\left|\underset{\sim}{a}\right|$, and the magnitude of vector $\mathbf{a}$ is written $|\mathbf{a}|$.

The diagram at right shows a vector $\overrightarrow{OA}$ of magnitude 10 km in a direction N48°E.

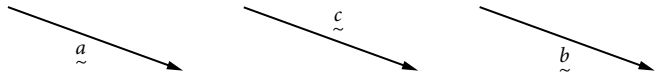


Vector algebra

Equality of vectors

Two vectors are equal if and only if they have the same magnitude and the same direction, regardless of their positions.

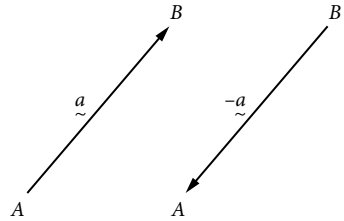
In the example below, $\underset{\sim}{a} = \underset{\sim}{b} = \underset{\sim}{c}$.



Negative of a vector

The negative of vector $\overrightarrow{AB} = \underset{\sim}{a}$ is the vector $-\overrightarrow{AB} = \overrightarrow{BA} = -\underset{\sim}{a}$.

$-\underset{\sim}{a}$ has the same magnitude as $\underset{\sim}{a}$ but with opposite direction. $\overrightarrow{BA}$ is the vector drawn from B to A, whereas $\overrightarrow{AB}$ is the vector drawn from A to B. Therefore, $\overrightarrow{BA} = -\overrightarrow{AB}$ and the direction is reversed.



The zero vector

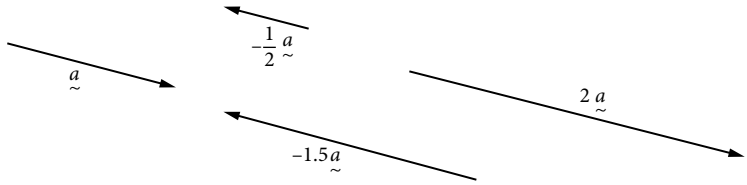
The zero vector, denoted $\underset{\sim}{0}$, is a vector of zero magnitude. Its direction cannot be defined. Adding a vector to its negative will produce the zero vector.

$\underset{\sim}{a} + (-\underset{\sim}{a}) = \underset{\sim}{0}$

Scalar multiplication of vectors

Multiplying a vector by a scalar (a number) k, where $k \in R$, the set of real numbers, $k \neq 0$, results in a vector with k times the magnitude and parallel to the original vector.

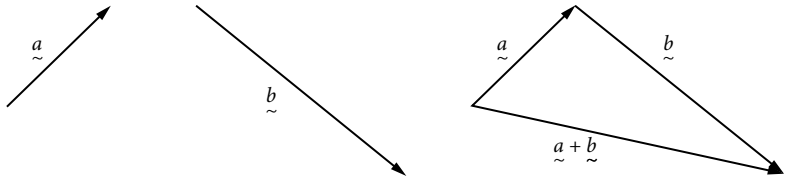
- If $k > 0$, then $k\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$, but has k times the magnitude.
- If $k = 0$, then $k\underset{\sim}{a} = \underset{\sim}{0}$.
- If $k < 0$, then $k\underset{\sim}{a}$ is in the opposite direction to $\underset{\sim}{a}$ and has k times the magnitude.



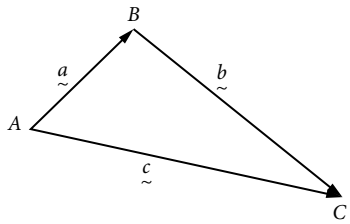
Addition of vectors

The triangle rule for the addition of vectors

To add two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the vectors are placed with the head of $\underset{\sim}{a}$ at the tail of $\underset{\sim}{b}$. The resultant vector or vector sum is a vector joining the tail of $\underset{\sim}{a}$ to the head of $\underset{\sim}{b}$. This vector addition forms a triangle, hence the **triangle rule** for vector addition, sometimes called the head-to-tail rule.



Also, for ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$.

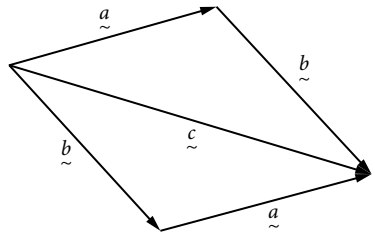


The parallelogram rule for the addition of vectors

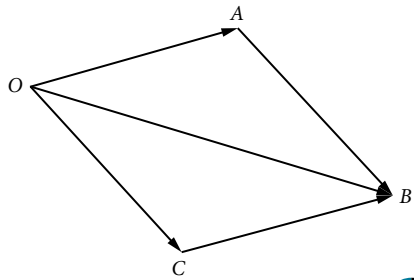
If two vectors to be added have the same initial point, they can be added by drawing a parallelogram rather than moving one vector to the head of the other vector. The diagonal of the parallelogram, drawn from the initial point to the final point, represents the vector sum. This works because adding $\underset{\sim}{b}$ to $\underset{\sim}{a}$ is the same as adding $\underset{\sim}{a}$ to $\underset{\sim}{b}$.

The diagram shows that $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a} = \underset{\sim}{c}$ and this rule is known as the **parallelogram rule** for addition of vectors.

The parallelogram rule for addition of vectors illustrates that vector addition is commutative.



For parallelogram $OABC$, you can also write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ and $\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$.



MAKING CONNECTIONS

Addition of vectors

Explore the parallelogram rule for the addition of vectors.

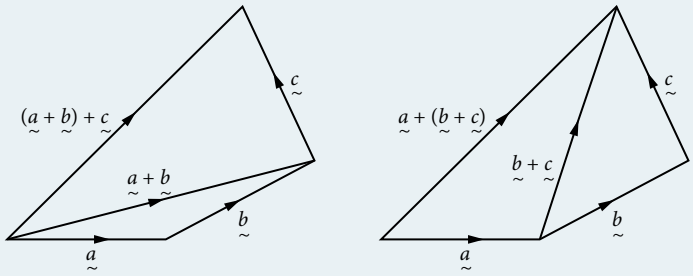
Example 2

Draw diagrams to show:

(a) $(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}$ **(b)** $\underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})$

Solution

(a) **(b)**



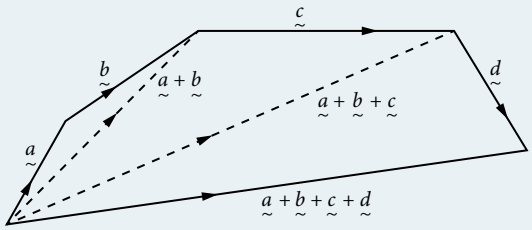
$(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}=\underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})=\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}$

Vector addition is also associative. If three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are added, then $\underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})=(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}=\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}$.
This method can be extended to any number of vectors (sometimes called the **polygon rule** for vector addition).

Example 3

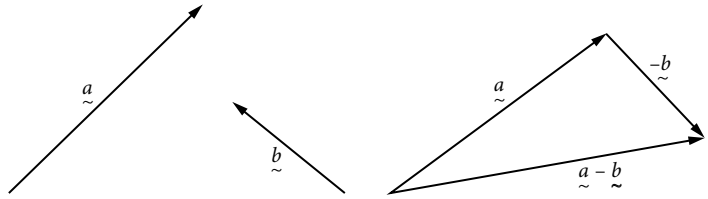
Draw a diagram to show $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}$.

Solution



Subtraction of vectors

Subtraction of a vector is defined as addition of its negative. That is, to subtract $\underset{\sim}{b}$ from $\underset{\sim}{a}$, add $-\underset{\sim}{b}$ to $\underset{\sim}{a}$, and so $\underset{\sim}{a}-\underset{\sim}{b}=\underset{\sim}{a}+(-\underset{\sim}{b})$.



MAKING CONNECTIONS

Subtraction of vectors

Use technology to explore the subtraction and addition of vectors.

If a vector is subtracted from itself, the result is the zero vector: $\underset{\sim}{a}-\underset{\sim}{a}=\underset{\sim}{0}$.

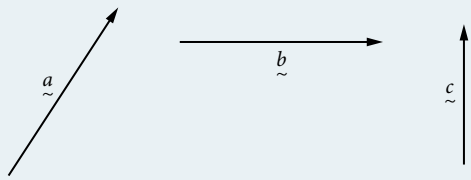
Example 4

Given the three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, as shown, construct the following:

(a) $\underset{\sim}{a}+\underset{\sim}{b}$

(b) $\underset{\sim}{c}+\underset{\sim}{a}$

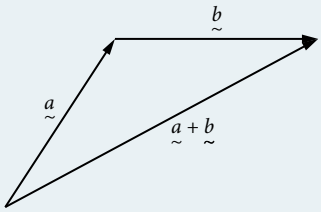
(c) $\underset{\sim}{b}-\underset{\sim}{c}$



Solution

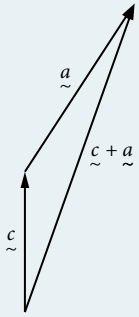
(a) To add two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the vectors are placed with the head of $\underset{\sim}{a}$ at the tail of $\underset{\sim}{b}$.

The resultant vector or vector sum is a vector joining the tail of $\underset{\sim}{a}$ to the head of $\underset{\sim}{b}$.



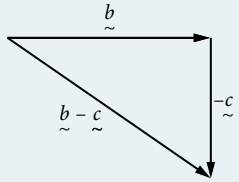
(b) To add two vectors $\underset{\sim}{c}$ and $\underset{\sim}{a}$, the vectors are placed with the head of $\underset{\sim}{c}$ at the tail of $\underset{\sim}{a}$.

The resultant vector or vector sum is a vector joining the tail of $\underset{\sim}{c}$ to the head of $\underset{\sim}{a}$.



(c) To subtract vector $\underset{\sim}{c}$ from vector $\underset{\sim}{b}$, the vector $-\underset{\sim}{c}$ is placed with the head of $\underset{\sim}{b}$ at the tail of $-\underset{\sim}{c}$.

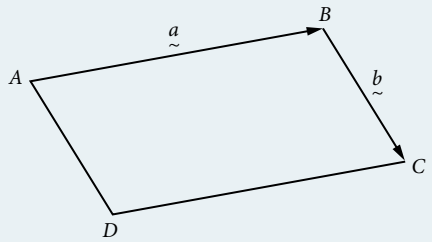
The resultant vector or vector sum is a vector joining the tail of $\underset{\sim}{b}$ to the head of $-\underset{\sim}{c}$.



Example 5

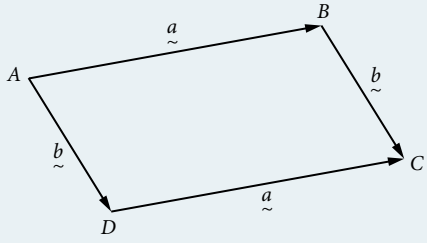
$ABCD$ is a parallelogram. If $\overrightarrow{AB}=\underset{\sim}{a}$ and $\overrightarrow{BC}=\underset{\sim}{b}$, express each of the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(a) $\overrightarrow{DA}$ **(b)** $\overrightarrow{AC}$ **(c)** $\overrightarrow{BD}$



Solution

(a) As a parallelogram has sides equal in magnitude and direction, the other sides can also be labelled.



$\overrightarrow{DA}$ has the same magnitude as $\underset{\sim}{b}$ in the opposite direction.

$\overrightarrow{DA}=-\underset{\sim}{b}$

(b) Start at A and get to C by following a path around the edges. These edges add to give $\overrightarrow{AC}$.

Either

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$

$=\underset{\sim}{a}+\underset{\sim}{b}$

or

$\overrightarrow{AC}=\overrightarrow{AD}+\overrightarrow{DC}$

$=\underset{\sim}{b}+\underset{\sim}{a}$

(c) Start at B and get to D by following a path around the edges. These edges add to give $\overrightarrow{BD}$.

$\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}$

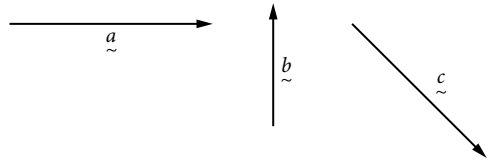
(or $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}$)

$\overrightarrow{BD}=-\underset{\sim}{a}+\underset{\sim}{b}$

$=\underset{\sim}{b}-\underset{\sim}{a}$

EXERCISE 10.1 INTRODUCTION TO VECTORS

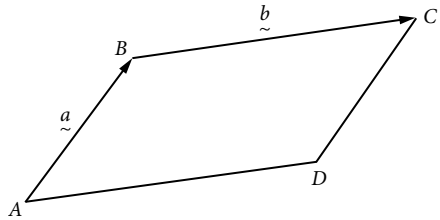
1 Given three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, as shown, construct the following:



(a) $\underset{\sim}{a}+\underset{\sim}{c}$ (b) $\underset{\sim}{c}+\underset{\sim}{b}$

(c) $\underset{\sim}{a}-\underset{\sim}{b}$ (d) $\underset{\sim}{c}-\underset{\sim}{a}$

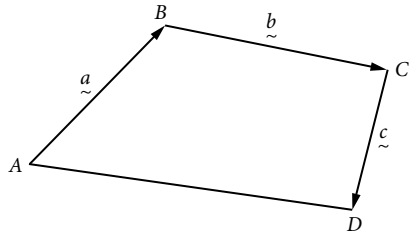
2 $ABCD$ is a parallelogram. If $\overrightarrow{AB}=\underset{\sim}{a}$ and $\overrightarrow{BC}=\underset{\sim}{b}$, express each of the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



(a) $\overrightarrow{CD}$ (b) $\overrightarrow{AD}$

(c) $\overrightarrow{CA}$ (d) $\overrightarrow{DB}$

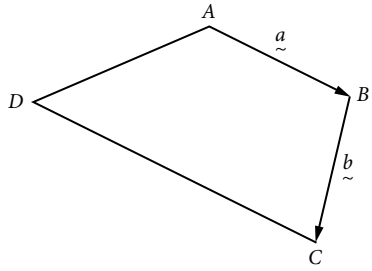
3 $ABCD$ is a quadrilateral. If $\overrightarrow{AB}=\underset{\sim}{a}$, $\overrightarrow{BC}=\underset{\sim}{b}$ and $\overrightarrow{CD}=\underset{\sim}{c}$, express each of the following vectors in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.



(a) $\overrightarrow{AC}$ (b) $\overrightarrow{AD}$

(c) $\overrightarrow{DA}$ (d) $\overrightarrow{DB}$

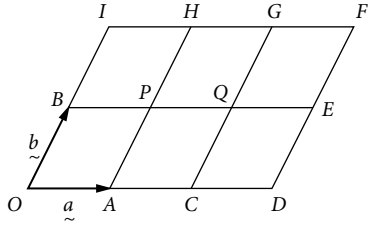
4 $ABCD$ is a trapezium with $\overline{DC}$ parallel to $\overline{AB}$ and one-and-a-half times the length of $\overline{AB}$. If $\overrightarrow{AB}=\underset{\sim}{a}$ and $\overrightarrow{BC}=\underset{\sim}{b}$, express each of the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



(a) $\overrightarrow{CD}$ (b) $\overrightarrow{CA}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{DB}$

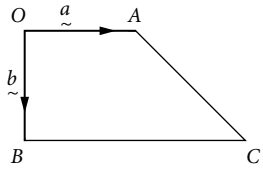
5 If all the short line segments shown are the same length, express the following in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



(a) $\overrightarrow{OP}$ (b) $\overrightarrow{OG}$ (c) $\overrightarrow{OQ}$ (d) $\overrightarrow{CE}$

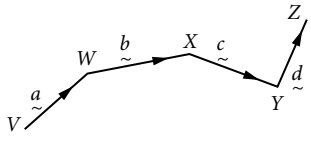
(e) $\overrightarrow{AB}$ (f) $\overrightarrow{DI}$ (g) $\overrightarrow{FQ}$ (h) $\overrightarrow{DE}+\overrightarrow{EO}$

6 $\overline{BC}$ is parallel to $\overline{OA}$ and twice its length. Express the following in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



(a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$

7 From the diagram, find the following in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$.

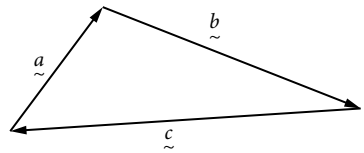


(a) $\overrightarrow{VY}$ (b) $\overrightarrow{VZ}$ (c) $\overrightarrow{WZ}$

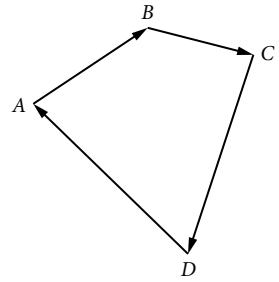
8 In ΔABC, $\overrightarrow{AB}=\underset{\sim}{a}$, $\overrightarrow{BC}=\underset{\sim}{b}$ and $\overrightarrow{CA}=\underset{\sim}{c}$. Which *one* of the following statements is true?

A $\underset{\sim}{a}+\underset{\sim}{c}=\underset{\sim}{b}$ **B** $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{0}$

C $\underset{\sim}{a}+\underset{\sim}{b}-\underset{\sim}{c}=\underset{\sim}{0}$ **D** $\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{a}$



9 In the quadrilateral $ABCD$, which *one* of the following statements is true?



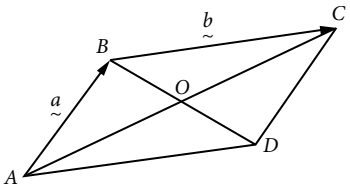
A $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CD}+\overrightarrow{DA}$

B $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CD}-\overrightarrow{DA}$

C $\overrightarrow{AB}-\overrightarrow{BC}=\overrightarrow{CD}-\overrightarrow{DA}$

D $\overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CD}-\overrightarrow{DA}$

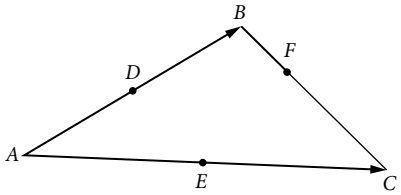
10 In the parallelogram $ABCD$ shown, the point of intersection of the diagonals is O, where O is the midpoint of both $\overline{AC}$ and $\overline{BD}$.



The vector $\overrightarrow{OC}$ is equal to:

A $\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})$ **B** $\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$ **C** $\frac{1}{2}\underset{\sim}{a}-\underset{\sim}{b}$ **D** $\frac{1}{2}\underset{\sim}{b}-\underset{\sim}{a}$

11 ΔABC is a triangle with $\overrightarrow{AB}=\underset{\sim}{a}$ and $\overrightarrow{AC}=\underset{\sim}{c}$. D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. F is a point on $\overline{BC}$ such that $\overline{FC}=2\times\overline{BF}$.



(a) Express the vectors $\overrightarrow{BC}$ and $\overrightarrow{DE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

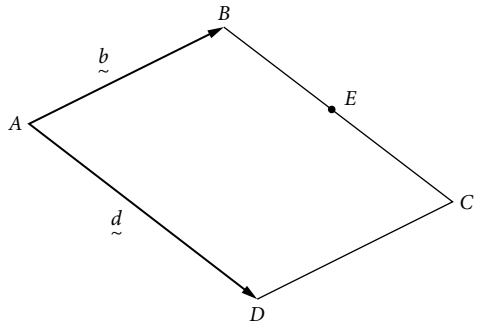
(b) Compare the vectors $\overrightarrow{BC}$ and $\overrightarrow{DE}$.

(c) What geometric property of a triangle does the answer to part **(b)** demonstrate?

(d) Express the vectors $\overrightarrow{BF}$ and $\overrightarrow{FC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(e) Show that $\overrightarrow{AF}=\frac{1}{3}(2\underset{\sim}{a}+\underset{\sim}{c})$.

12 $ABCD$ is a parallelogram in which $\overrightarrow{AB}=\underset{\sim}{b}$ and $\overrightarrow{AD}=\underset{\sim}{d}$ and E is the midpoint of $\overline{BC}$.



(a) Express $\overrightarrow{AC}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{d}$.

(b) Express $\overrightarrow{AE}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{d}$.

(c) Express $\overrightarrow{DE}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{d}$.

(d) If F is a point on $\overrightarrow{DE}$ and $\overrightarrow{DF}=\frac{2}{3}\overrightarrow{DE}$, express $\overrightarrow{DF}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{d}$.

(e) Find $\overrightarrow{AF}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{d}$, and hence show that F lies on $\overrightarrow{AC}$.

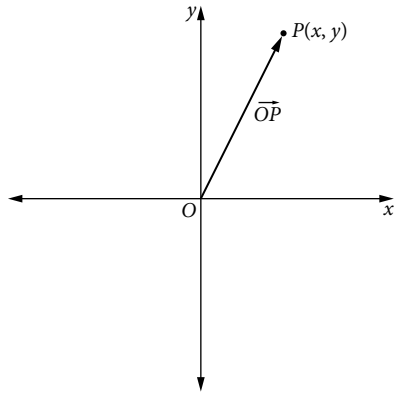
(f) Find the ratio $\overline{AF}:\overline{FC}$.

10.2 VECTORS IN TWO DIMENSIONS

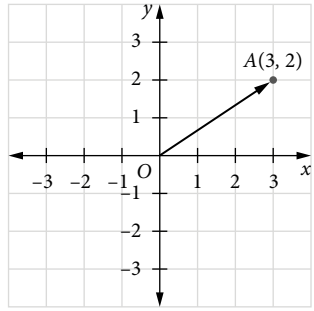
Position vectors on the Cartesian plane

A **position vector** is a vector drawn with its tail at O, the origin. Position vectors are used to represent points by a distance and a direction rather than two numbers (the coordinates). You can draw position vectors on the Cartesian plane and represent them with ordered pairs as well as a length and a direction.

The position vector of any point $P(x, y)$ relative to a fixed origin O on the Cartesian plane is uniquely specified by the vector $\overrightarrow{OP}$, as shown in the diagram. That is, the vector $\overrightarrow{OP}$ represents the position vector of point $P(x, y)$ relative to O.



For example, the diagram at right shows the position vector of point $A(3, 2)$ relative to O. The position vector is $\overrightarrow{OA}$, where A has coordinates (3, 2).



EXPLORING FURTHER

Position vectors

Use technology to explore position vectors on the Cartesian plane.

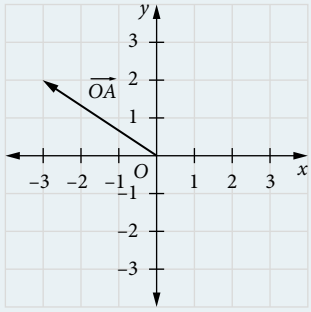
Example 6

Draw the following position vectors on the Cartesian plane.

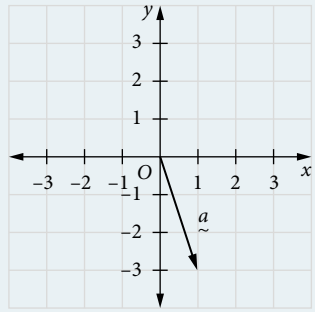
(a) $\overrightarrow{OA}$, where A is $(-3, 2)$ **(b)** $\underset{\sim}{a}$, the position vector of $(1, -3)$

Solution

(a) Vector is drawn from the initial point at the origin O to the given coordinates.



(b) Vector is drawn from the initial point at the origin O to the given coordinates.



A position vector can be represented by a coordinate pair (a, b). This represents the vector that is a units from O in the positive x direction, and b units from O in the positive y direction. This can be represented as a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ or by the coordinates (a, b). Any vector that is equivalent to a translation of a units in the positive x-direction and b units in the positive y-direction can be represented in this way.

If A has coordinates (3, 2), then the position vector $\overrightarrow{OA}$ can be represented by the column vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$: $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

The magnitude of a position vector

The magnitude of a position vector $\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix}$ can be calculated using Pythagoras' theorem.

$\left|\overrightarrow{OA}\right| = \sqrt{a^2 + b^2}$.

For example, the magnitude of the position vector $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is $\left|\overrightarrow{OA}\right| = \sqrt{3^2 + 2^2} = \sqrt{13}$ units.

Example 7

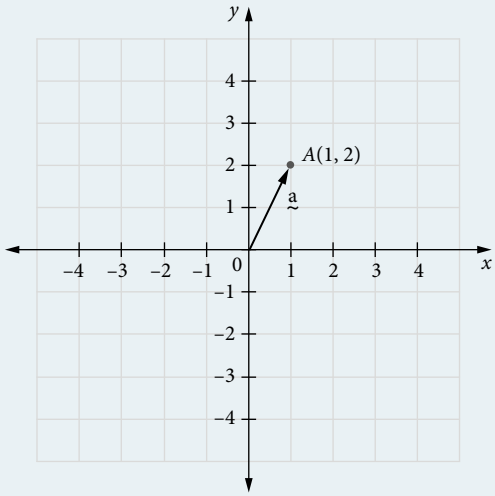
Draw the following position vectors on the Cartesian plane.

(a) $\underset{\sim}{a}$, the position vector of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. **(b)** $\overrightarrow{OB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Solution

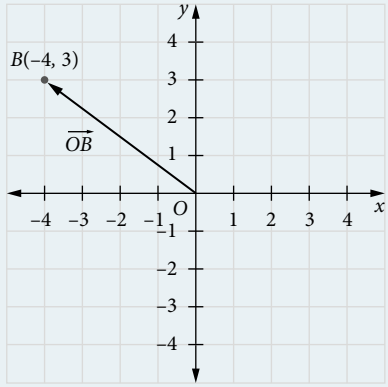
(a) $\underset{\sim}{a}$ represents a vector which is 1 unit from O in the positive x direction and 2 units from O in the positive y direction.

Vector is drawn from the origin to the point (1, 2).



(b) $\overrightarrow{OB}$ represents a vector which is −4 units from O in the positive x direction (4 units in the negative x direction) and 3 units from O in the positive y direction.

Vector is drawn from the origin to the point (−4, 3).

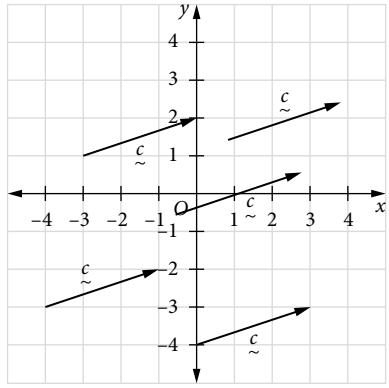


Equal vectors

Equal vectors do not have to start at the same point. To be *equal*, they only need to have the same magnitude and direction.

Although the position vector of the point $A(3, 1)$ starts at the origin, the coordinate pair (3, 1) can be used to represent any vector whose head is three units across and one unit up from its tail. However, none of the other vectors can be called position vectors.

Vectors will also be equal if and only if they are expressed using the same coordinates. For example, the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ will only be equal to the vector $\begin{bmatrix} c \\ d \end{bmatrix}$ if $a = b$ and $c = d$.



Example 8

Given $\underset{\sim}{d} = (2, 5)$, specify an ordered pair for each of the following. **(a)** $2\underset{\sim}{d}$ **(b)** $-\underset{\sim}{d}$

Solution

(a) As this vector has twice the magnitude, each coordinate will be multiplied by 2.
$2\underset{\sim}{d} = (4, 10)$.

(b) As this vector has the same magnitude but in the opposite direction to $\underset{\sim}{d}$, each coordinate must also be negated (multiplied by −1).

The vector $-\underset{\sim}{d}$ is represented by the ordered pair (−2, −5).

Example 9

Given $\underset{\sim}{e} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, specify a column vector for each of the following. **(a)** $\frac{1}{2}\underset{\sim}{e}$ **(b)** $-3\underset{\sim}{e}$

Solution

(a) As this vector has half the magnitude, each coordinate is multiplied by $\frac{1}{2}$.

The vector $\frac{1}{2}\underset{\sim}{e}$ is represented by the column vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(b) As this vector has three times the magnitude but in the opposite direction to $\underset{\sim}{e}$, each coordinate must also be negated (multiplied by -3).

The vector $-3\underset{\sim}{e}$ is represented by the column vector $\begin{pmatrix} -12 \\ 18 \end{pmatrix}$.

EXERCISE 10.2 VECTORS IN TWO DIMENSIONS

1 If vector $\underset{\sim}{a}$ is represented by the ordered pair $(2, -6)$, specify an ordered pair for each of the following vectors.

(a) $3\underset{\sim}{a}$ **(b)** $\frac{1}{2}\underset{\sim}{a}$ **(c)** $-\underset{\sim}{a}$ **(d)** $0.4\underset{\sim}{a}$

2 If vector $\underset{\sim}{b}$ is represented by the column vector $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, specify a column vector for each of the following vectors.

(a) $-2\underset{\sim}{b}$ **(b)** $5\underset{\sim}{b}$ **(c)** $\frac{1}{3}\underset{\sim}{b}$ **(d)** $-\frac{5}{4}\underset{\sim}{b}$

3 If $\underset{\sim}{c}$ is the position vector of $(6, -3)$, represent each of the following as an ordered pair.

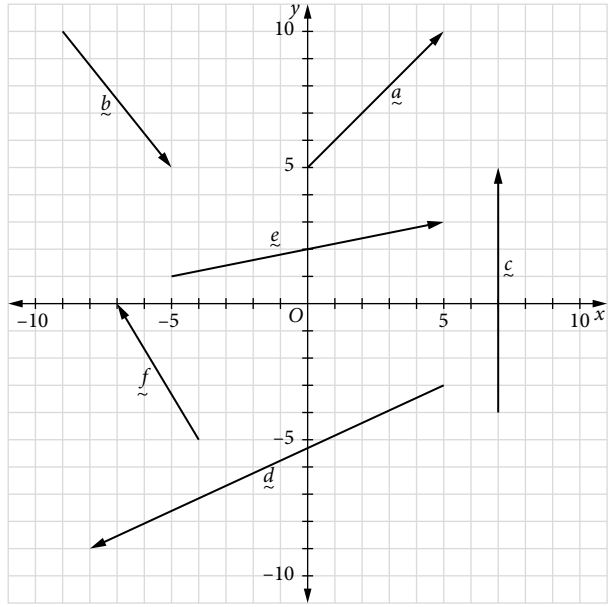
(a) $-\underset{\sim}{c}$ **(b)** $2\underset{\sim}{c}$ **(c)** $-\frac{1}{3}\underset{\sim}{c}$ **(d)** $1.5\underset{\sim}{c}$

4 If $\underset{\sim}{c}$ is the position vector of $(6, -3)$, represent each of the vectors as a column vector.

(a) $-\underset{\sim}{c}$ **(b)** $2\underset{\sim}{c}$ **(c)** $-\frac{1}{3}\underset{\sim}{c}$ **(d)** $1.5\underset{\sim}{c}$

5 Represent each of the vectors in the plane shown as an ordered pair.

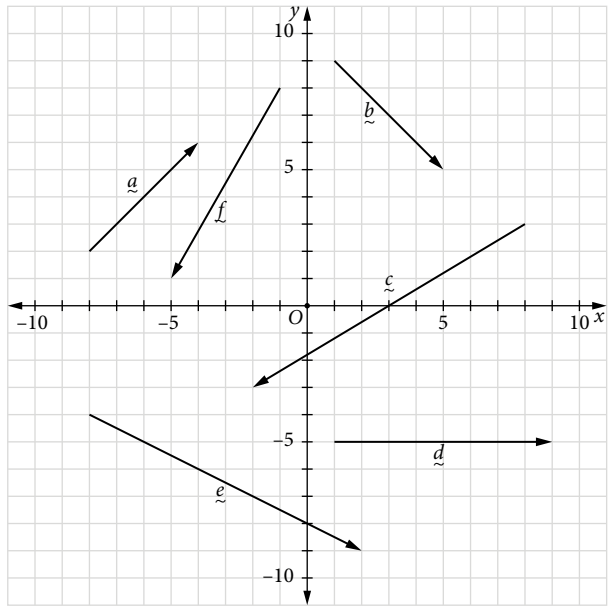
(a) $\underset{\sim}{a}$ **(b)** $\underset{\sim}{b}$ **(c)** $\underset{\sim}{c}$
(d) $\underset{\sim}{d}$ **(e)** $\underset{\sim}{e}$ **(f)** $\underset{\sim}{f}$



6 Represent each of the vectors in the plane shown as a column vector.

(a) $\underset{\sim}{a}$ **(b)** $\underset{\sim}{b}$ **(c)** $\underset{\sim}{c}$

(d) $\underset{\sim}{d}$ **(e)** $\underset{\sim}{e}$ **(f)** $\underset{\sim}{f}$



7 If vector $\underset{\sim}{a}$ is represented by the ordered pair (3, −5), then which of the following represents the vector $-2\underset{\sim}{a}$?

A (−6, 10) **B** (6, −10) **C** (10, −6) **D** (−10, 6)

8 Which of the following represents the vector from the point (2, 6) to the point (−1, 8)?

A $\begin{pmatrix}3\\2\end{pmatrix}$ **B** $\begin{pmatrix}2\\3\end{pmatrix}$ **C** $\begin{pmatrix}-3\\2\end{pmatrix}$ **D** $\begin{pmatrix}2\\-3\end{pmatrix}$

9 Draw the following vectors on the Cartesian plane.

(a) $\underset{\sim}{a}$, the position vector of (2, 4) **(b)** $\underset{\sim}{b}$, the position vector of (−4, 0)

(c) $\underset{\sim}{c}$, the position vector of (−3, −5) **(d)** $\overrightarrow{OD}$, where D is (−1, 3)

(e) $\overrightarrow{OE}$, where E is (2, −4) **(f)** $\overrightarrow{OF}$, where F is (0, −3)

10 Draw the following vectors on the Cartesian plane.

(a) $\underset{\sim}{a}$, the position vector of $\begin{pmatrix}1\\3\end{pmatrix}$ **(b)** $\underset{\sim}{b}$, the position vector of $\begin{pmatrix}-2\\2\end{pmatrix}$ **(c)** $\underset{\sim}{c}$, the position vector of $\begin{pmatrix}-3\\-4\end{pmatrix}$

(d) $\overrightarrow{OD}$, where D is $\begin{pmatrix}1\\-2\end{pmatrix}$ **(e)** $\overrightarrow{OE}$, where E is $\begin{pmatrix}0\\-5\end{pmatrix}$ **(f)** $\overrightarrow{OF}$, where F is $\begin{pmatrix}-2\\-5\end{pmatrix}$

11 **(a)** Find the column vector form of the vector $\underset{\sim}{a}$, which has an initial point (−3, 4) and finishes at the point (5, 9).

(b) Find the column vector form of the vector $\underset{\sim}{b}$, which has an initial point (5, −1) and finishes at the point (−6, −17).

(c) If the vector $\underset{\sim}{c} = \begin{pmatrix}11\\-3\end{pmatrix}$ has an initial point (2, 1), determine the coordinates of its terminal point.

(d) If the vector $\underset{\sim}{d} = \begin{pmatrix}-7\\13\end{pmatrix}$ has a terminal point (12, −2), determine the coordinates of its initial point.

(e) Sketch representations of the vector $\underset{\sim}{e} = \begin{pmatrix}3\\-2\end{pmatrix}$ with initial points at:

(i) (0, 0) **(ii)** (3, 1) **(iii)** (5, −3) **(iv)** (−5, 5) **(v)** (−2, −2).

12 The points A, B and C have coordinates $(-2, -3)$, $(2, 3)$ and $(8, -1)$ respectively.

(a) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ and express them in column vector form.

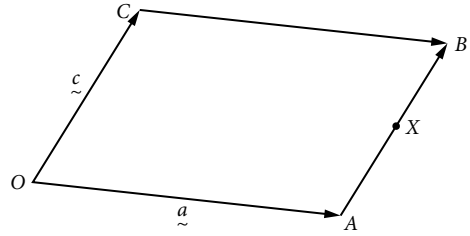
(b) Find $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{BC}\right|$ and $\left|\overrightarrow{AC}\right|$.

(c) Use Pythagoras' theorem to prove that ΔABC is a right-angled triangle.

(d) Find the coordinates of a point D such that $ABCD$ forms a square.

(e) Find the coordinates of the point of intersection of the diagonals of the square $ABCD$.

13 $OABC$ is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. X is the midpoint of $\overline{AB}$ as shown.



(a) Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

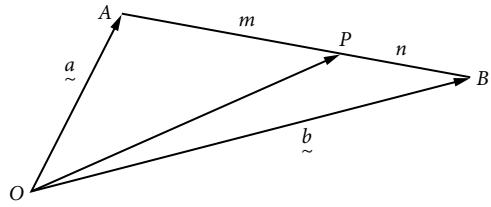
(b) Find the vector $\overrightarrow{CX}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(c) If Y is a point on $\overrightarrow{CX}$, such that $\overrightarrow{CY} = \frac{2}{3}\overrightarrow{CX}$, find $\overrightarrow{CY}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(d) Find $\overrightarrow{OY}$ and hence show that Y lies on $\overrightarrow{OB}$.

(e) Find the ratio $\overline{OY} : \overline{YB}$.

14 **(a)** The position vectors of points A and B are $(-3, 6)$ and $(2, 9)$ respectively. Find the position vector of the midpoint of $\overline{AB}$.



(b) Let the position vectors of points A and B be $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Find the position vector of the midpoint of $\overline{AB}$.

(c) The position vectors of points D and E are $(1, 7)$ and $(5, 3)$ respectively. Find the position vectors of the points of trisection of $\overline{DE}$.

(d) Let the position vectors of points D and E be $\underset{\sim}{d}$ and $\underset{\sim}{e}$ respectively. Find the position vectors of the points of trisection of $\overline{DE}$.

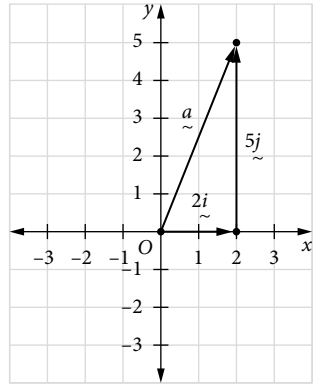
(e) Let the position vectors of points A and B be $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Let P divide $\overline{AB}$ in the ratio $m : n$ so that $\frac{\overline{AP}}{\overline{AB}} = \frac{m}{m+n}$. Show that the position vector of point P is $\frac{n}{m+n}\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}$.

10.3 VECTORS IN COMPONENT FORM

A **unit vector** is a vector with a magnitude of one unit. To obtain a unit vector from a given vector, divide that vector by its own magnitude.

If vector $\underset{\sim}{a}$ has a magnitude of $\left|\underset{\sim}{a}\right|$, then a unit vector in the direction of $\underset{\sim}{a}$, denoted by $\hat{\underset{\sim}{a}}$, can be found by dividing vector $\underset{\sim}{a}$ by its own magnitude $\left|\underset{\sim}{a}\right|$. That is, $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|}$.

The unit vector in the direction of $\underset{\sim}{a}$ is denoted $\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|}$ and $\left|\hat{\underset{\sim}{a}}\right| = 1$.

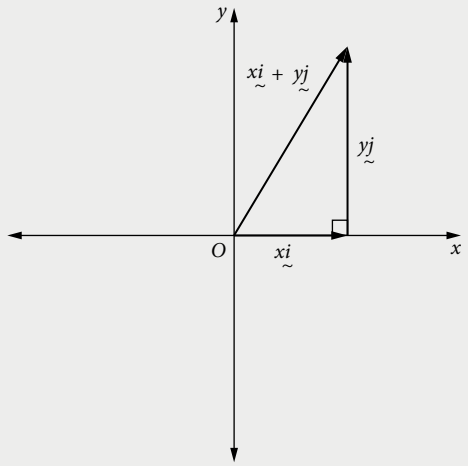


Recall that a vector can be represented as an ordered pair (x, y) or as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where the first value represents the distance parallel to the x-axis and the second value the distance parallel to the y-axis.

This information can also be defined using the vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a vector of magnitude one unit in the positive x-direction and $\underset{\sim}{j}$ is a vector of one unit magnitude in the positive y-direction. The vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors.

For example, the vector $\underset{\sim}{a}$ defined by the coordinates (2, 5) or the column vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ can be written as $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$.

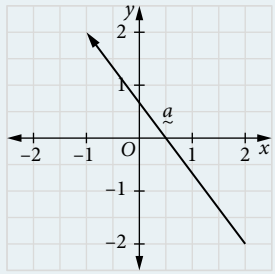
The form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is called **component form** or $\underset{\sim}{i}, \underset{\sim}{j}$ **form** of a vector. The vector $\underset{\sim}{a}$ may also be represented in column vector form as $\begin{pmatrix} x \\ y \end{pmatrix}$.



Writing vectors in component form

Example 10

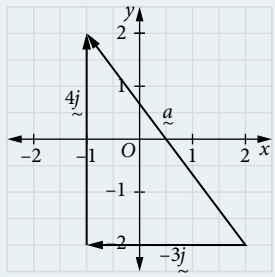
Express the vector $\underset{\sim}{a}$ in component form.



Solution

Vectors are drawn from the tail of the vector across and then up (or down) to meet the head of the original vector, labelled as $x\underset{\sim}{i}$ and $y\underset{\sim}{j}$.

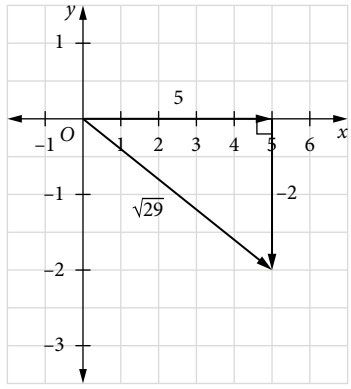
Oiginal vector in terms of the components $x\underset{\sim}{i} + y\underset{\sim}{j}$: $\underset{\sim}{a} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$



When finding the components of a vector, start at the tail end. It does not matter whether you first move parallel to the x-axis (to find the $\underset{\sim}{i}$ component) or parallel to the y-axis (to find the $\underset{\sim}{j}$ component).

Magnitude of a vector in component form

To find the magnitude in component form you can use Pythagoras' theorem, as the components form a right-angled triangle. For example, if $\underset{\sim}{a} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$:



$$|\underset{\sim}{a}| = \sqrt{5^2 + (-2)^2}$$
$$= \sqrt{29}$$

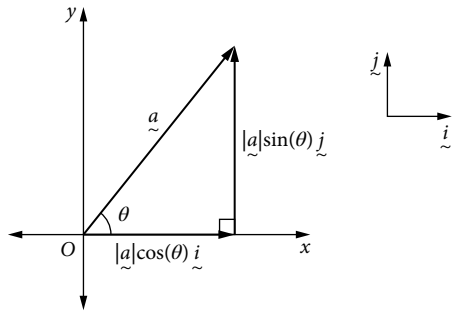
When finding the magnitude of a vector, use only the positive square root value.

Resolving vectors into component form

If a vector $\underset{\sim}{a}$ of magnitude $|\underset{\sim}{a}|$ makes an angle θ with the positive x-axis, then: $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\,\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\,\underset{\sim}{j}$.

The horizontal component of the vector $\underset{\sim}{a}$ is $|\underset{\sim}{a}|\cos\theta\,\underset{\sim}{i}$ and the vertical component is $|\underset{\sim}{a}|\sin\theta\,\underset{\sim}{j}$.

The process of specifying a vector of known magnitude and direction in component form is called **resolving the vector**.



Example 11

Resolve the vector $\underset{\sim}{a}$ into component form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, given $\underset{\sim}{a}$ has a magnitude of 6 units and has a direction of 50° to the positive x-axis. Give answers correct to two decimal places.

Solution

$|\underset{\sim}{a}| = 6$ and $\theta = 50°$

If a vector $\underset{\sim}{a}$ of magnitude $|\underset{\sim}{a}|$ makes an angle θ with the positive x-axis, then $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\,\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\,\underset{\sim}{j}$.

$$\begin{aligned}\underset{\sim}{a} &= |\underset{\sim}{a}|\cos\theta\,\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\,\underset{\sim}{j}\\ &= 6\cos(50°)\,\underset{\sim}{i} + 6\sin(50°)\,\underset{\sim}{j}\\ &= 3.86\,\underset{\sim}{i} + 4.60\,\underset{\sim}{j}\end{aligned}$$

Addition and subtraction of vectors in component form

Addition and subtraction of vectors can be done by adding or subtracting the $\underset{\sim}{i}$ components and the $\underset{\sim}{j}$ components.

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ then:

$$\begin{aligned}\underset{\sim}{a} + \underset{\sim}{b} &= \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) + \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right)\\ &= x_1\underset{\sim}{i} + x_2\underset{\sim}{i} + y_1\underset{\sim}{j} + y_2\underset{\sim}{j}\\ &= (x_1 + x_2)\,\underset{\sim}{i} + (y_1 + y_2)\,\underset{\sim}{j}\end{aligned}$$

Similarly:

$$\begin{aligned}\underset{\sim}{a} - \underset{\sim}{b} &= \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) - \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right)\\ &= (x_1 - x_2)\,\underset{\sim}{i} + (y_1 - y_2)\,\underset{\sim}{j}\end{aligned}$$

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$:

$\underset{\sim}{a} + \underset{\sim}{b} = (x_1 + x_2)\,\underset{\sim}{i} + (y_1 + y_2)\,\underset{\sim}{j}$

$\underset{\sim}{a} - \underset{\sim}{b} = (x_1 - x_2)\,\underset{\sim}{i} + (y_1 - y_2)\,\underset{\sim}{j}$

In column vector notation, this can be written as $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$ respectively.

Scalar multiplication of vectors in component form

$$\begin{aligned}k\underset{\sim}{a} &= k\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)\\ &= kx\underset{\sim}{i} + ky\underset{\sim}{j}\end{aligned}$$

In column vector notation, this can be written as $k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$.

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $k\underset{\sim}{a} = kx\underset{\sim}{i} + ky\underset{\sim}{j}$

Example 12

Given $\underset{\sim}{a} = \underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, find: **(a)** $\underset{\sim}{a} + \underset{\sim}{b}$ **(b)** $\underset{\sim}{b} - \underset{\sim}{a}$ **(c)** $-4\underset{\sim}{a} + 7\underset{\sim}{b}$

Solution

(a) Sum of the vectors in component form: $\underset{\sim}{a} + \underset{\sim}{b} = \left(\underset{\sim}{i} - 5\underset{\sim}{j}\right) + \left(-3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

Group the coefficients of the components together and simplify: $= (1-3)\underset{\sim}{i} + (-5+2)\underset{\sim}{j}$

$= -2\underset{\sim}{i} - 3\underset{\sim}{j}$

(b) Sum of the vectors in component form: $\underset{\sim}{b} - \underset{\sim}{a} = \left(-3\underset{\sim}{i} + 2\underset{\sim}{j}\right) - \left(\underset{\sim}{i} - 5\underset{\sim}{j}\right)$

Group the coefficients of the components together and simplify: $= (-3-1)\underset{\sim}{i} + (2-(-5))\underset{\sim}{j}$

$= -4\underset{\sim}{i} + 7\underset{\sim}{j}$

(c) Sum of the vectors in component form: $-4\underset{\sim}{a} + 7\underset{\sim}{b} = -4\left(\underset{\sim}{i} - 5\underset{\sim}{j}\right) + 7\left(-3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

$= -4\underset{\sim}{i} + 20\underset{\sim}{j} - 21\underset{\sim}{i} + 14\underset{\sim}{j}$

Group the coefficients of the components together and simplify: $= (-4-21)\underset{\sim}{i} + (20+14)\underset{\sim}{j}$

$= -25\underset{\sim}{i} + 34\underset{\sim}{j}$

Equality of vectors in component form

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, then $\underset{\sim}{a} = \underset{\sim}{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Example 13

Find the values of m and n if $7\underset{\sim}{i} - 5\underset{\sim}{j} = (3m+1)\underset{\sim}{i} + (4n-9)\underset{\sim}{j}$.

Solution

Equate coefficients of the vector components and solve the resulting equations:

$\underset{\sim}{i}$ components: $7 = 3m + 1$

$3m = 6$

$m = 2$

$\underset{\sim}{j}$ components: $-5 = 4n - 9$

$4n = 4$

$n = 1$

Relative position vectors

You have already looked at position vectors that represent the position of one point in relation to the origin. A **relative position vector** represents a point's position in relation to another point.

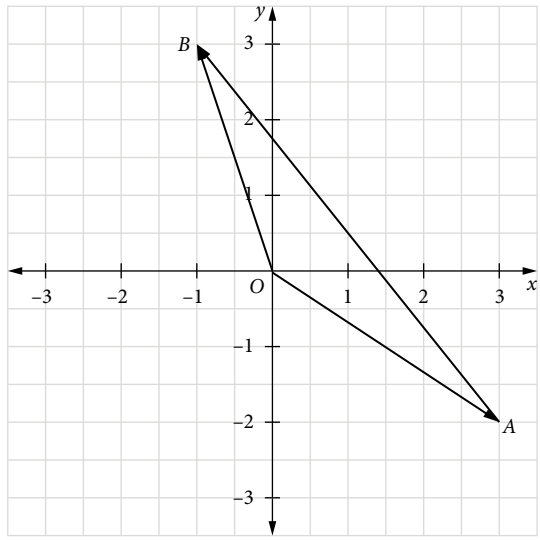
The position vector of B relative to A is given by $\overrightarrow{AB}$.

In the diagram shown $\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

The position vector of B relative to A is $\overrightarrow{AB}$, where $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now, $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$$
$$= -4\underset{\sim}{i} + 5\underset{\sim}{j}$$



Example 14

The position vector of point A on the Cartesian plane is $\overrightarrow{OA} = 12\underset{\sim}{i} - 5\underset{\sim}{j}$ and the position vector of point B is $\overrightarrow{OB} = -7\underset{\sim}{i} + 6\underset{\sim}{j}$. Find the position vector of A relative to B.

Solution

The position vector of A relative to B is $\overrightarrow{BA}$. Write the rule to find $\overrightarrow{BA}$: $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$$\therefore \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

Substitute the components and simplify: $\overrightarrow{BA} = \left(12\underset{\sim}{i} - 5\underset{\sim}{j}\right) - \left(-7\underset{\sim}{i} + 6\underset{\sim}{j}\right)$

$$= 19\underset{\sim}{i} - 11\underset{\sim}{j}$$

Parallel vectors

Two vectors are parallel if they are scalar multiples of each other:

If $\underset{\sim}{b} = k\underset{\sim}{a}$, where k is a real number, then $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

For example, if $\underset{\sim}{a} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 12\underset{\sim}{j}$, then $\underset{\sim}{b} = 4\left(\underset{\sim}{i} - 3\underset{\sim}{j}\right)$.

$\therefore \underset{\sim}{b} = 4\underset{\sim}{a}$, so $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

Example 15

Consider the three vectors $\underset{\sim}{a} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 8\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{c} = -8\underset{\sim}{i} + 12\underset{\sim}{j}$. Which two vectors are parallel?

Solution

Look at the vectors to see if a scalar multiplier exists for any of them: $\underset{\sim}{c} = -8\underset{\sim}{i} + 12\underset{\sim}{j}$

$$= 4\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$
$$= 4\underset{\sim}{a}$$

Vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are parallel.

Unit vectors in component form

Recall that a **unit vector** has a magnitude of 1 and $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

Therefore, if $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

It is usually better to express the unit vector with a rational denominator, so $\hat{\underset{\sim}{a}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

A unit vector is a 'direction finder', in that it determines a vector's direction but not its magnitude.

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$ **or** $\hat{\underset{\sim}{a}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

Example 16

Find the unit vector $\hat{\underset{\sim}{a}}$ for each of the following vectors.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -5\underset{\sim}{i} + 8\underset{\sim}{j}$

Solution

(a) Divide the original vector by its magnitude to get a unit vector: $\hat{\underset{\sim}{a}} = \frac{1}{5}\left(4\underset{\sim}{i} - 3\underset{\sim}{j}\right)$

(b) Find the magnitude of the vector: $|\underset{\sim}{a}| = \sqrt{(-5)^2 + 8^2}$

$= \sqrt{89}$

Divide the original vector by its magnitude to get a unit vector: $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{89}}\left(-5\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

$= \frac{\sqrt{89}}{89}\left(-5\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

Unit vectors can be used to find vectors in a specified direction.

Example 17

Given $\underset{\sim}{c} = 3\underset{\sim}{i} - 6\underset{\sim}{j}$:

(a) find $\hat{\underset{\sim}{c}}$ **(b)** find vector $\underset{\sim}{d}$ of magnitude 5 in the direction of $\underset{\sim}{c}$.

Solution

(a) Find the magnitude of the vector: $|\underset{\sim}{c}| = \sqrt{3^2 + (-6)^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$

Find the unit vector by dividing the vector by its magnitude: $\hat{\underset{\sim}{c}} = \frac{1}{3\sqrt{5}}\left(3\underset{\sim}{i} - 6\underset{\sim}{j}\right)$

$= \frac{\sqrt{5}}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

(b) Multiply the unit vector in the direction required by the required magnitude: $\underset{\sim}{d} = \frac{5\sqrt{5}}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

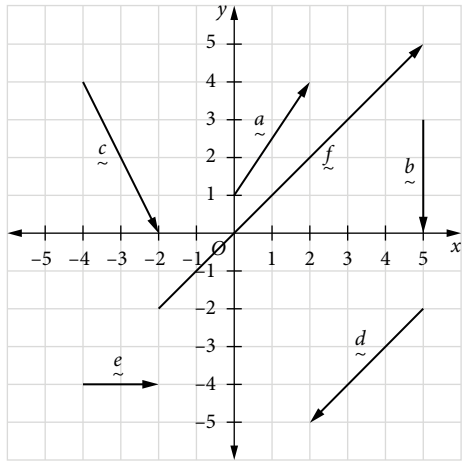
$= \sqrt{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

EXERCISE 10.3 VECTORS IN COMPONENT FORM

1 Express each vector shown in component form.

(a) $\underset{\sim}{a}$ **(b)** $\underset{\sim}{b}$ **(c)** $\underset{\sim}{c}$

(d) $\underset{\sim}{d}$ **(e)** $\underset{\sim}{e}$ **(f)** $\underset{\sim}{f}$



2 Find the magnitude of the following vectors.

(a) $\underset{\sim}{a} = 5\underset{\sim}{i} + 4\underset{\sim}{j}$ **(b)** $-4\underset{\sim}{i} + 7\underset{\sim}{j}$ **(c)** $7\underset{\sim}{i} - 24\underset{\sim}{j}$ **(d)** $-5\underset{\sim}{i}$ **(e)** $9\underset{\sim}{j}$ **(f)** $-8\underset{\sim}{i} - 5\underset{\sim}{i}$

3 Resolve the following vectors into component form $x\underset{\sim}{i} + y\underset{\sim}{j}$, correct to two decimal places.

(a) $\underset{\sim}{a}$ has a magnitude of 15 units and has a direction of 35° to the positive x-axis.

(b) $\underset{\sim}{b}$ has a magnitude of 23 units and has a direction of 121° to the positive x-axis.

(c) $\underset{\sim}{c}$ has a magnitude of 10 units and has a direction of −45° to the positive x-axis.

(d) $\underset{\sim}{d}$ has a magnitude of 36 units and has a direction of −175° to the positive x-axis.

4 Given $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, find: **(a)** $\underset{\sim}{a} + \underset{\sim}{b}$ **(b)** $\underset{\sim}{b} - \underset{\sim}{a}$ **(c)** $2\underset{\sim}{a} + 7\underset{\sim}{b}$ **(d)** $3\underset{\sim}{a} - 5\underset{\sim}{b}$

5 Find the values of m and n for $-6\underset{\sim}{i} + 7\underset{\sim}{j} = (4m - 2)\underset{\sim}{i} + (5 - 2n)\underset{\sim}{j}$.

6 Find the values of the unknown pronumerals in the following equations.

(a) $5\underset{\sim}{i} - 4\underset{\sim}{j} = 3a\underset{\sim}{i} + 2b\underset{\sim}{j}$ **(b)** $(x + 2y)\underset{\sim}{i} + y\underset{\sim}{j} = -3\underset{\sim}{i} + 7\underset{\sim}{j}$

(c) $(2g + 1)\underset{\sim}{i} - (3h - 1)\underset{\sim}{j} = 3\left(h\underset{\sim}{i} + 2g\underset{\sim}{j}\right)$ **(d)** $(3p - 18)\underset{\sim}{i} + (3q - 8)\underset{\sim}{j} = 2p\left(\underset{\sim}{i} + 2\underset{\sim}{j}\right) - q\left(2\underset{\sim}{i} - \underset{\sim}{j}\right)$

(e) $\left(x^2 + 5x\right)\underset{\sim}{i} + \left(y^3 - 1\right)\underset{\sim}{j} = -6\underset{\sim}{i} + 7\underset{\sim}{j}$ **(f)** $\left(x^2 - 4x\right)\underset{\sim}{i} + \left(y^4 + 9\right)\underset{\sim}{j} = 2\underset{\sim}{i} + 10\underset{\sim}{j}$

7 The position vectors of points A and B on the Cartesian plane are $\overrightarrow{OA} = 5\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\overrightarrow{OB} = 7\underset{\sim}{i} - 6\underset{\sim}{j}$, respectively. Find the position vector of A relative to B.

8 The position vectors of points C and D are $\overrightarrow{OC} = -8\underset{\sim}{i} + 12\underset{\sim}{j}$ and $\overrightarrow{OD} = 13\underset{\sim}{i} - 23\underset{\sim}{j}$, respectively. Find:

(a) $\overrightarrow{DC}$ **(b)** $\left|\overrightarrow{DC}\right|$

9 In parallelogram $ABCD$, $\overrightarrow{AB} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\overrightarrow{AD} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$. Find:

(a) $\overrightarrow{CD}$ **(b)** $\overrightarrow{CA}$ **(c)** $\overrightarrow{DB}$ **(d)** $\left|\overrightarrow{AB}\right|$ **(e)** $\left|\overrightarrow{CA}\right|$ **(f)** $\left|\overrightarrow{DB}\right|$

10 Given $\underset{\sim}{a} = -13\underset{\sim}{i} + 20\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 15\underset{\sim}{j}$, find:

(a) $\left|\underset{\sim}{a} - \underset{\sim}{b}\right|$ **(b)** the value of x so that the vector $x\underset{\sim}{a} + 4\underset{\sim}{b}$ is parallel to the x-axis.

11 Given vectors $\underset{\sim}{a} = 5\underset{\sim}{i} + 4\underset{\sim}{j}$, $\underset{\sim}{b} = -4\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{c} = -\underset{\sim}{i} - 9\underset{\sim}{j}$, with $3\underset{\sim}{a} + 2\underset{\sim}{b} - v\underset{\sim}{c}$ parallel to the y-axis, find the value of v.

12 Which one of the following vectors is parallel to the vector $\underset{\sim}{f} = 14\underset{\sim}{i} - 6\underset{\sim}{j}$?

A $\underset{\sim}{a} = 28\underset{\sim}{i} + 12\underset{\sim}{j}$ **B** $\underset{\sim}{b} = 14\underset{\sim}{i} + 6\underset{\sim}{j}$ **C** $\underset{\sim}{c} = -14\underset{\sim}{i} - 6\underset{\sim}{j}$ **D** $\underset{\sim}{d} = -28\underset{\sim}{i} + 12\underset{\sim}{j}$

13 Which one of the following vectors is parallel to the vector $\underset{\sim}{a} = 5\underset{\sim}{i} - 3\underset{\sim}{j}$ and has a magnitude of $2\sqrt{34}$?

A $\frac{5}{2}\underset{\sim}{i} - \frac{3}{2}\underset{\sim}{j}$ **B** $10\underset{\sim}{i} - 6\underset{\sim}{j}$ **C** $-5\underset{\sim}{i} + 3\underset{\sim}{j}$ **D** $-10\underset{\sim}{i} - 6\underset{\sim}{j}$

14 For position vectors $\overrightarrow{OF} = 2\underset{\sim}{i} - 7\underset{\sim}{j}$ and $\overrightarrow{OG} = -4\underset{\sim}{i} + 5\underset{\sim}{j}$, what is the value of $\left|\overrightarrow{FG}\right|$?

A $6\sqrt{3}$ **B** $6\sqrt{5}$ **C** $2\sqrt{10}$ **D** $2\sqrt{37}$

15 Find the unit vector parallel to each of the following vectors.

(a) $8\underset{\sim}{i} + 6\underset{\sim}{j}$ **(b)** $3\underset{\sim}{i} - 6\underset{\sim}{j}$ **(c)** $-2\underset{\sim}{i} + 9\underset{\sim}{j}$ **(d)** $-\underset{\sim}{i} - 12\underset{\sim}{j}$

16 For $\underset{\sim}{b} = 3\underset{\sim}{i} - 9\underset{\sim}{j}$:

(a) find $\hat{\underset{\sim}{b}}$ **(b)** find vector $\underset{\sim}{c}$ in the direction of $\underset{\sim}{b}$ with a magnitude of 15.

17 Given $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + \underset{\sim}{j}$, find $\hat{\underset{\sim}{c}}$ for $\underset{\sim}{c} = 2\underset{\sim}{a} - 3\underset{\sim}{b}$.

18 Given $\underset{\sim}{a} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, find the vector $\underset{\sim}{c}$ in the direction of $\underset{\sim}{b}$ where $\left|\underset{\sim}{c}\right| = \left|\underset{\sim}{a}\right|$.

19 What is the unit vector in the direction of $\underset{\sim}{a} = -2\underset{\sim}{i} + 5\underset{\sim}{j}$ is?

A $\frac{1}{7}\left(-2\underset{\sim}{i} + 5\underset{\sim}{i}\right)$ **B** $\frac{1}{29}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **C** $\frac{1}{\sqrt{29}}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **D** $\frac{1}{\sqrt{21}}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

20 Which of the following vectors is parallel to $\underset{\sim}{b} = -5\underset{\sim}{i} + 5\underset{\sim}{j}$ and has a magnitude of 5?

A $\frac{\sqrt{2}}{10}\left(-5\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **B** $5\left(-5\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **C** $\frac{\sqrt{2}}{2}\left(-5\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **D** $\frac{5\sqrt{2}}{2}\left(-5\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

21 $OABC$ is a parallelogram in which $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{CB}$. **(b)** Find the diagonals $\overrightarrow{OB}$ and $\overrightarrow{CA}$.

Let M, N and P be the midpoints of $\overline{AC}$, $\overline{OB}$ and $\overline{OA}$ respectively.

(c) Find $\overrightarrow{ON}$ and $\overrightarrow{OM}$. **(d)** Find $\overrightarrow{CP}$ and $\overrightarrow{BP}$.

22 ΔABC has vertices defined by the position vectors $\overrightarrow{OA} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\overrightarrow{OB} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\overrightarrow{OC} = 6\underset{\sim}{i} - 9\underset{\sim}{j}$.

(a) State the coordinates of the vertices of ΔABC.

(b) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$. **(c)** Find $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{BC}\right|$ and $\left|\overrightarrow{AC}\right|$.

23 ΔOAB is a triangle in which $\overrightarrow{OA} = 6\underset{\sim}{i}$ and $\overrightarrow{OB} = 4\underset{\sim}{j}$. The point M with position vector $\overrightarrow{OM} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is equidistant from O, A and B.

(a) Find the values of x and y. **(b)** Find the vectors $\overrightarrow{AM}$, $\overrightarrow{MB}$ and $\overrightarrow{OM}$.

(c) Find the values of $\left|\overrightarrow{AM}\right|$, $\left|\overrightarrow{MB}\right|$ and $\left|\overrightarrow{OM}\right|$.

24 $OABC$ is a parallelogram in which vectors $\overrightarrow{OA} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{OC} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$.

(a) Find vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$. **(b)** Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$, the diagonals of the parallelogram.

(c) Find the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where P is the midpoint of $\overline{OB}$ and Q is the midpoint of $\overline{AC}$. What can you say about the points P and Q?

(d) Find the vectors $\overrightarrow{OR}$ and $\overrightarrow{CR}$, where R is the midpoint of $\overline{AB}$.

25 $OABC$ is a square in which vectors $\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$. M is the midpoint of $\overline{AB}$ and N divides $\overline{CB}$ internally in the ratio 1 : 2.

(a) Find the vectors $\overrightarrow{OB}$, $\overrightarrow{AC}$, $\overrightarrow{OM}$, $\overrightarrow{ON}$ and $\overrightarrow{NB}$. **(b)** Find the length of the diagonals, $\left|\overrightarrow{OB}\right|$ and $\left|\overrightarrow{AC}\right|$.

26 **(a)** If $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$, find $\hat{\underset{\sim}{a}}$ and $\hat{\underset{\sim}{b}}$.

(b) Hence find the exact values of x and y, such that $2x\hat{\underset{\sim}{a}} + 3y\hat{\underset{\sim}{b}} = \underset{\sim}{i} + \underset{\sim}{j}$.

(c) If $\underset{\sim}{c} = 4\underset{\sim}{i} - 8\underset{\sim}{j}$ and $\underset{\sim}{d} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$, find $\hat{\underset{\sim}{c}}$ and $\hat{\underset{\sim}{d}}$.

(d) Hence find the exact values of v and w, such that $2v\hat{\underset{\sim}{c}} + 3w\hat{\underset{\sim}{d}} = \underset{\sim}{i} + \underset{\sim}{j}$.

27 **(a)** If $\underset{\sim}{a} = 3p\underset{\sim}{i} + 4p\underset{\sim}{j}$, $p > 0$ and $\left|\underset{\sim}{a}\right| = 2$, find the exact value of p. **(b)** Hence find $\hat{\underset{\sim}{a}}$.

(c) Find the vector $\underset{\sim}{b}$ which is parallel to $\hat{\underset{\sim}{a}}$, if $\left|\underset{\sim}{b}\right| = 10$.

(d) If $\underset{\sim}{c} = 7q\underset{\sim}{i} + 24q\underset{\sim}{j}$, $q > 0$, and $\left|\underset{\sim}{c}\right| = 4$, find the exact value of q. **(e)** Hence find $\hat{\underset{\sim}{c}}$.

(f) Find the vector $\underset{\sim}{d}$ in the direction of $\hat{\underset{\sim}{c}}$ where $\left|\underset{\sim}{d}\right| = 50$.

(g) Find the vector with magnitude 10 that is parallel to the vector $\underset{\sim}{b} + \underset{\sim}{d}$.

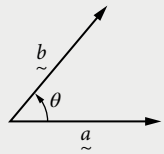
10.4 SCALAR PRODUCT OF VECTORS

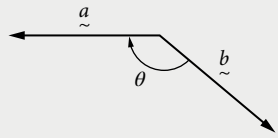
The **scalar product** (also called the **dot product** or the **inner product**) is a way of multiplying two vectors. The result of this multiplication is a scalar quantity (with magnitude but no direction). The scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is written as $\underset{\sim}{a} \bullet \underset{\sim}{b}$ (read as '$\underset{\sim}{a}$ dot $\underset{\sim}{b}$').

There is also another product operation on vectors, the **vector product** or **cross product**, which is written as $\underset{\sim}{a} \times \underset{\sim}{b}$ (read as '$\underset{\sim}{a}$ cross $\underset{\sim}{b}$'). The result of this multiplication is a vector quantity. This is beyond the scope of this course.

If θ is the angle between the positive directions of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then the scalar product is defined to be $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta$.

For a straight angle, $\theta = \pi$, so $\underset{\sim}{a} \bullet \underset{\sim}{b} = -\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$ as $\cos\pi = -1$.

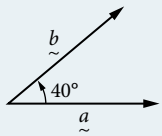




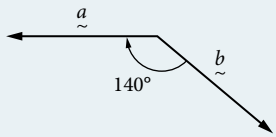
Example 18

Given $\left|\underset{\sim}{a}\right| = 5$ and $\left|\underset{\sim}{b}\right| = 6$, find the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, correct to two decimal places, in each of the following.

(a)



(b)



Solution

(a) $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta$

$= 5 \times 6 \times \cos 40°$

$= 22.98$

(b) $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta$

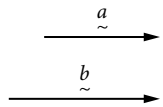
$= 5 \times 6 \times \cos 140°$

$= -22.98$

Special cases of the scalar product

Parallel vectors

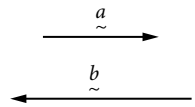
If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors in the same direction, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos 0°$.



As $\cos 0° = 1$, this means: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors but in opposite directions, then: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\pi$

$= -|\underset{\sim}{a}||\underset{\sim}{b}|$



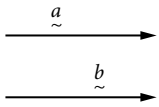
Equal vectors

If $\underset{\sim}{a} = \underset{\sim}{b}$, then $\theta = 0$ (just as for any parallel vectors), so the scalar product is the magnitude squared:

$\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}||\underset{\sim}{a}|\cos 0°$

$= |\underset{\sim}{a}||\underset{\sim}{a}|$

$= |\underset{\sim}{a}|^2$



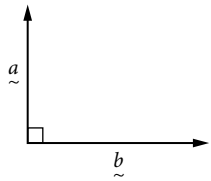
Perpendicular vectors

Two vectors are said to be perpendicular or *orthogonal* if the angle between their directions is a right angle (90°).

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular vectors, then: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\dfrac{\pi}{2}$

$= |\underset{\sim}{a}||\underset{\sim}{b}| \times 0$

$= 0$



Also, if $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = 0$, then $|\underset{\sim}{a}| = 0$ or $|\underset{\sim}{b}| = 0$ or $\theta = \dfrac{\pi}{2}$.

This leads to an important property of perpendicular vectors: if the scalar product of two non-zero vectors is zero, then the vectors are perpendicular.

If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ for non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

An important property of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ is that they are perpendicular, so that $\underset{\sim}{i} \bullet \underset{\sim}{j} = \underset{\sim}{j} \bullet \underset{\sim}{i} = 0$.

Also, using the property of parallel vectors: $\underset{\sim}{i} \bullet \underset{\sim}{i} = \underset{\sim}{j} \bullet \underset{\sim}{j} = 1$.

Scalar product for vectors in component form

In component form, for $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, the scalar product is:

$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right)$

$= (x_1x_2)\left(\underset{\sim}{i} \bullet \underset{\sim}{i}\right) + (x_1y_2)\left(\underset{\sim}{i} \bullet \underset{\sim}{j}\right) + (y_1x_2)\left(\underset{\sim}{j} \bullet \underset{\sim}{i}\right) + (y_1y_2)\left(\underset{\sim}{j} \bullet \underset{\sim}{j}\right)$

$= x_1x_2 + y_1y_2$ as $\underset{\sim}{i} \bullet \underset{\sim}{i} = \underset{\sim}{j} \bullet \underset{\sim}{j} = 1$ and $\underset{\sim}{i} \bullet \underset{\sim}{j} = \underset{\sim}{j} \bullet \underset{\sim}{i} = 0$

Consider two vectors, $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ making an angle θ_1 with the x-axis and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_1\underset{\sim}{j}$ making an angle θ_2 with the x-axis. The angle between the vectors is $\theta = \theta_1 - \theta_2$.

$\underset{\sim}{a} \bullet \underset{\sim}{b} = |a||b|\cos\theta$

$= |a||b|\cos(\theta_1 - \theta_2)$

$= |a||b|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= |a|\cos\theta_1|b|\cos\theta_2 + |a|\sin\theta_1|b|\sin\theta_2$

$= x_1x_2 + y_1y_2$

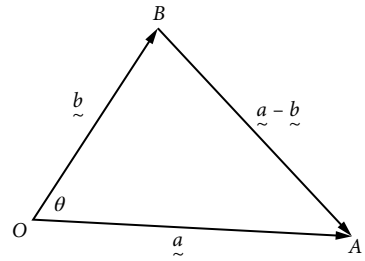
For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ the scalar product of the vectors in component form is $\underset{\sim}{a} \bullet \underset{\sim}{b} = x_1x_2 + y_1y_2$.

This means there are two basic expressions for the scalar product: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = x_1x_2 + y_1y_2$.

Geometric interpretation of the scalar product

Three vectors form the triangle ΔAOB and the length of each side is the magnitude of the vector forming that side.

The cosine rule states that $|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$.



Using the properties of the dot product, the left-hand side can be written as:

$$\begin{aligned}|\underset{\sim}{a} - \underset{\sim}{b}|^2 &= (\underset{\sim}{a} - \underset{\sim}{b}) \bullet (\underset{\sim}{a} - \underset{\sim}{b}) \\ &= \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{b} - \underset{\sim}{b} \bullet \underset{\sim}{a} + \underset{\sim}{b} \bullet \underset{\sim}{b} \\ &= |\underset{\sim}{a}|^2 - 2\underset{\sim}{a} \bullet \underset{\sim}{b} + |\underset{\sim}{b}|^2\end{aligned}$$

Rewriting the cosine rule:

$$\begin{aligned}|\underset{\sim}{a} - \underset{\sim}{b}|^2 &= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta \\ |\underset{\sim}{a}|^2 - 2\underset{\sim}{a} \bullet \underset{\sim}{b} + |\underset{\sim}{b}|^2 &= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta \\ -2\underset{\sim}{a} \bullet \underset{\sim}{b} &= -2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta \\ \underset{\sim}{a} \bullet \underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta\end{aligned}$$

Example 19

Find the scalar product $\underset{\sim}{a} \bullet \underset{\sim}{b}$, given $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$.

Solution

$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right) \bullet \left(-3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

Multiply the coefficients of the like components and sum together: $\underset{\sim}{a} \bullet \underset{\sim}{b} = 2 \times (-3) + 5 \times 4$

$= 14$

Algebraic properties of the scalar product

The scalar product has the following algebraic properties.

1. The commutative law: $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$

   This property follows immediately from the definition $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, since $|\underset{\sim}{a}||\underset{\sim}{b}| = |\underset{\sim}{b}||\underset{\sim}{a}|$.

2. The associative law, including multiplication by a scalar:

   $(m\underset{\sim}{a}) \bullet \underset{\sim}{b} = m(\underset{\sim}{a} \bullet \underset{\sim}{b})$, where m is a real number.

   This property also follows directly from the definition, although it is necessary to distinguish the cases $m < 0$ and $m > 0$.

3. The distributive law:

   $\underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$

   The distributive law can be verified using the geometric interpretation of the scalar product and assuming that the projections of $\underset{\sim}{b}$ and $\underset{\sim}{c}$ on $\underset{\sim}{a}$ are both positive.

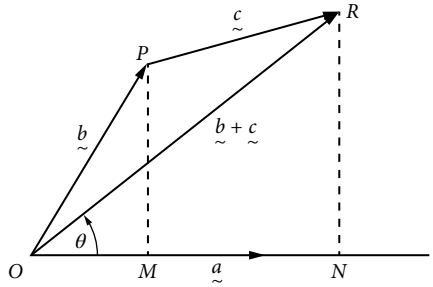
From the diagram: $\underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c}) = |\underset{\sim}{a}||\underset{\sim}{b} + \underset{\sim}{c}| \cos\theta$

$= |\underset{\sim}{a}| \times |\overrightarrow{ON}|$ as $|\underset{\sim}{b} + \underset{\sim}{c}| \cos\theta = |\overrightarrow{ON}|$

Also: $\underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c} = |\underset{\sim}{a}| \times |\overrightarrow{OM}| + |\underset{\sim}{a}| \times |\overrightarrow{MN}|$

$= |\underset{\sim}{a}| \times |\overrightarrow{ON}|$

$= \underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c})$



4 An important result involving the distributive law is $(\underset{\sim}{a} + \underset{\sim}{b})(\underset{\sim}{a} - \underset{\sim}{b}) = |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$:

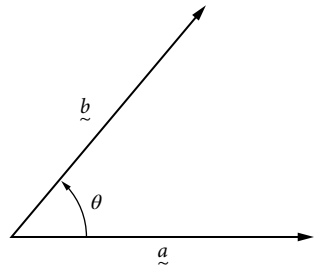
$(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b}$

$= \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b}$ since $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

Finding the angle between two vectors

The scalar product can be used to find the angle between two vectors. Let θ be the angle between the directions of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as shown.



$\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

So: $\cos\theta = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$\therefore \theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

Note: It is usual to consider θ to be the smaller of the two possible angles between the two vectors. Either way, the smaller angle and the larger (reflex) angle will be equivalent for any scalar product, because $\cos(\theta) = \cos(2\pi - \theta)$.

Example 20

Find the angle in degrees, correct to two decimal places, between vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}$.

Solution

Scalar product of the vectors: $\underset{\sim}{a} \bullet \underset{\sim}{b}$: $\underset{\sim}{a} \bullet \underset{\sim}{b} = (3\underset{\sim}{i} - 2\underset{\sim}{j})(4\underset{\sim}{i} + \underset{\sim}{j})$

$= 3 \times 4 + (-2) \times 1$

$= 10$

Magnitudes of the vectors: $|\underset{\sim}{a}| = \sqrt{3^2 + (-2)^2}$ $\quad$ $|\underset{\sim}{b}| = \sqrt{4^2 + 1^2}$

$= \sqrt{13}$ $\quad$ $= \sqrt{17}$

Substitute into the rule $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$ and simplify: $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$= \cos^{-1}\left(\dfrac{10}{\sqrt{13} \times \sqrt{17}}\right)$

$= 47.73°$ (correct to two decimal places)

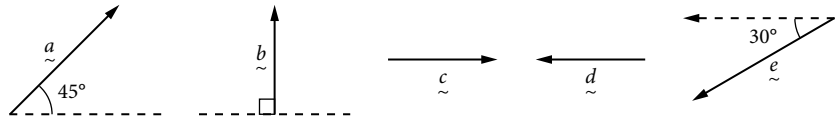
EXPLORING FURTHER

Angle between two vectors

Use technology to explore the angles between vectors.

EXERCISE 10.4 SCALAR PRODUCT OF VECTORS

1 Consider the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ and $\underset{\sim}{e}$ as shown.



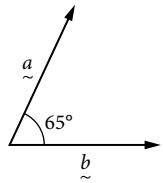
Find the angle between the following pairs of vectors.

(a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ **(b)** $\underset{\sim}{a}$ and $\underset{\sim}{c}$ **(c)** $\underset{\sim}{a}$ and $\underset{\sim}{d}$ **(d)** $\underset{\sim}{a}$ and $\underset{\sim}{e}$ **(e)** $\underset{\sim}{b}$ and $\underset{\sim}{c}$

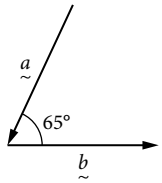
(f) $\underset{\sim}{b}$ and $\underset{\sim}{d}$ **(g)** $\underset{\sim}{b}$ and $\underset{\sim}{e}$ **(h)** $\underset{\sim}{c}$ and $\underset{\sim}{d}$ **(i)** $\underset{\sim}{c}$ and $\underset{\sim}{e}$ **(j)** $\underset{\sim}{d}$ and $\underset{\sim}{e}$

2 Given $|\underset{\sim}{a}| = 8$ and $|\underset{\sim}{b}| = 7$, find the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ for each of the following, correct to two decimal places where necessary.

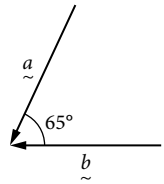
(a)



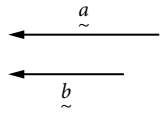
(b)



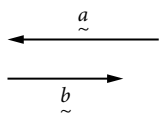
(c)



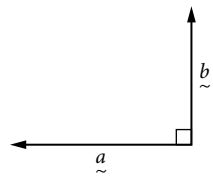
(d)



(e)



(f)



3 Find the scalar product $\underset{\sim}{a} \bullet \underset{\sim}{b}$ for the following:

(a) $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 8\underset{\sim}{j}$

(d) $\underset{\sim}{a} = -4\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} - 6\underset{\sim}{j}$ **(e)** $\underset{\sim}{a} = -2\underset{\sim}{i} - 9\underset{\sim}{j}$ and $\underset{\sim}{b} = -7\underset{\sim}{i} - 4\underset{\sim}{j}$ **(f)** $\underset{\sim}{a} = 2\underset{\sim}{i} - 12\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + \underset{\sim}{j}$

4 Show that the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} - 3\underset{\sim}{j}$ are perpendicular.

5 Find the vector $\underset{\sim}{d}$ that is perpendicular to $\underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ and has a magnitude of 10.

6 If the vectors $\underset{\sim}{e} = 7\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{f} = x\underset{\sim}{i} - 3\underset{\sim}{j}$ are perpendicular, find the value of x.

7 If $\underset{\sim}{a} = -6\underset{\sim}{i} + 2\underset{\sim}{j}$, find: **(a)** $\underset{\sim}{a} \bullet \underset{\sim}{a}$ **(b)** $|\underset{\sim}{a}|$ **(c)** $\underset{\sim}{a} \bullet \underset{\sim}{a}$ in terms of $|\underset{\sim}{a}|$

8 For any vector $\underset{\sim}{a}$, find the value of each of the following, in terms of $|a|$ where necessary.

(a) $\underset{\sim}{a} \bullet \underset{\sim}{a}$ **(b)** $\hat{\underset{\sim}{a}} \bullet \hat{\underset{\sim}{a}}$ **(c)** $\underset{\sim}{a} \bullet (-\underset{\sim}{a})$

9 If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$, express $\underset{\sim}{a} \bullet \underset{\sim}{b}$ in terms of $|a|$ and $|b|$.

10 Find the angle, correct to the nearest degree, between each of the following pairs of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

(a) $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

(d) $\underset{\sim}{a} = -2\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ **(e)** $\underset{\sim}{a} = -5\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{b} = -7\underset{\sim}{i} - 2\underset{\sim}{j}$ **(f)** $\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

11 Which vector is perpendicular to $\underset{\sim}{f} = -5\underset{\sim}{i} + 2\underset{\sim}{j}$ with magnitude 12?

A $\underset{\sim}{a} = \frac{12}{\sqrt{29}}\left(5\underset{\sim}{i} - 2\underset{\sim}{j}\right)$ **B** $\underset{\sim}{b} = \frac{12}{\sqrt{29}}\left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **C** $\underset{\sim}{c} = \frac{12}{\sqrt{29}}\left(2\underset{\sim}{i} - 5\underset{\sim}{j}\right)$ **D** $\underset{\sim}{d} = \frac{12}{\sqrt{29}}\left(-2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

12 Vectors $\underset{\sim}{a} = x\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + y\underset{\sim}{j}$ are perpendicular. What are possible values of x and y?

A $x = 1$ and $y = 3$ **B** $x = 1$ and $y = -3$ **C** $x = -2$ and $y = -6$ **D** $x = 2$ and $y = 6$

13 The angle between the vectors $\underset{\sim}{a} = -5\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$ is closest to:

A 3° **B** 40° **C** 140° **D** 177°

14 The points A, B and C have position vectors $\overrightarrow{OA} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OC} = 8\underset{\sim}{i} - \underset{\sim}{j}$.

(a) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in component form.

(b) Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ and $|\overrightarrow{AC}|$.

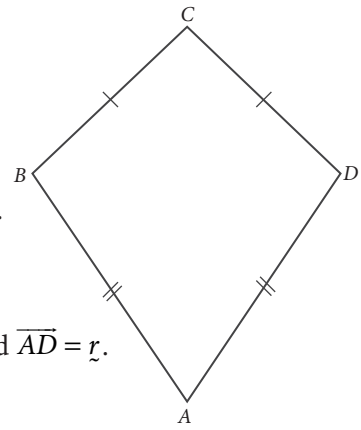
(c) Show that ΔABC is a right-angled triangle.

(d) Find the position vector of a point D such that $ABCD$ forms a square.

(e) Find the vector $\overrightarrow{BD}$, the other diagonal of the square $ABCD$.

(f) Show that the diagonals of the square $ABCD$ bisect at right angles.

15 $ABCD$ is a kite with $\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{CB} = \overrightarrow{CD}$, as shown.



(a) Show that the quadrilateral $ABCD$ whose vertices have position vectors $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$, $\overrightarrow{OC} = 8\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\overrightarrow{OD} = 5\underset{\sim}{i} + \underset{\sim}{j}$ respectively, is a kite.

(b) Find $\overrightarrow{AC}$ and $\overrightarrow{BD}$ and hence show that the diagonals of this kite cross at right angles.

(c) Express $\overrightarrow{BD}$, $\overrightarrow{BC}$ and $\overrightarrow{DC}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$, where $\overrightarrow{AB} = \underset{\sim}{p}$, $\overrightarrow{AC} = \underset{\sim}{q}$ and $\overrightarrow{AD} = \underset{\sim}{r}$.

(d) Explain why $\overrightarrow{BC} \bullet \overrightarrow{BC} = \overrightarrow{DC} \bullet \overrightarrow{DC}$ and use this equation to show that $\underset{\sim}{p} \bullet \underset{\sim}{q} = \underset{\sim}{q} \bullet \underset{\sim}{r}$.

(e) Hence show that the diagonals of a kite cross at right angles.

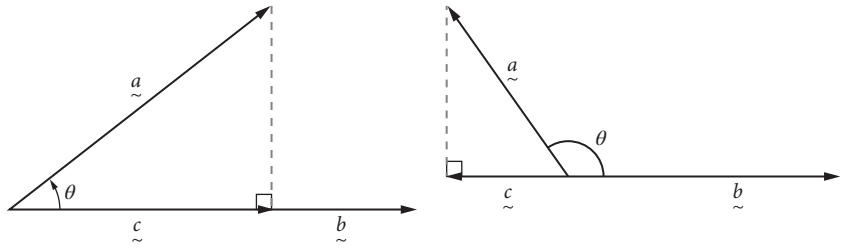
10.5 PROJECTIONS OF VECTORS

Any vector can be resolved into a sum of two vectors that are perpendicular to each other. It is usual to resolve the vector into two perpendicular components with one in a specified direction (e.g. parallel to a given vector) and the other perpendicular to that specified direction.

Scalar projection

A scalar projection of a vector can be thought of as like the shadow of a wire when the sun is overhead. (The original vector is the wire.)

In the diagrams, vector $\underset{\sim}{c}$ is the vector projection of vector $\underset{\sim}{a}$ in the direction of vector $\underset{\sim}{b}$.



The magnitude of $\underset{\sim}{c}$, $|\underset{\sim}{c}|$, is the **scalar projection** of vector $\underset{\sim}{a}$ onto vector $\underset{\sim}{b}$.

Using trigonometry, $\cos\theta = \dfrac{|\underset{\sim}{c}|}{|\underset{\sim}{a}|}$.

By definition, $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, so $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\dfrac{|\underset{\sim}{c}|}{|\underset{\sim}{a}|}$

$$= |\underset{\sim}{b}||\underset{\sim}{c}|$$

$$\therefore |\underset{\sim}{c}| = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$\text{so } |\underset{\sim}{c}| = \underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} \text{ as } \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \hat{\underset{\sim}{b}}$$

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|}$.

Example 21

Consider the vectors $\underset{\sim}{a} = 5\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) Find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

Solution

(a) Find $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$ by first finding $|\underset{\sim}{b}|$.

$$\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$$

$$|\underset{\sim}{b}| = \sqrt{3^2 + 4^2}$$
$$= 5$$

$$\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{5}\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$$

Scalar projection $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$:

$$\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \left(5\underset{\sim}{i} - \underset{\sim}{j}\right) \bullet \frac{1}{5}\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$$
$$= \frac{15 - 4}{5}$$
$$= \frac{11}{5}$$

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\dfrac{11}{5}$.

(b) Find $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ by first finding $|\underset{\sim}{a}|$.

$$\underset{\sim}{a} = 5\underset{\sim}{i} - \underset{\sim}{j} : |\underset{\sim}{a}| = \sqrt{5^2 + (-1)^2}$$
$$= \sqrt{26}$$

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{1}{\sqrt{26}}\left(5\underset{\sim}{i} - \underset{\sim}{j}\right)$$

Scalar projection $\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}}$:

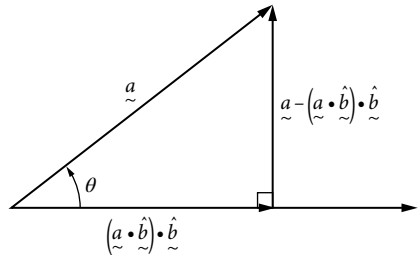
$$\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}} = \left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right) \bullet \frac{1}{\sqrt{26}}\left(5\underset{\sim}{i} - \underset{\sim}{j}\right)$$
$$= \frac{15}{\sqrt{26}} - \frac{4}{\sqrt{26}}$$
$$= \frac{11}{\sqrt{26}}$$
$$= \frac{11\sqrt{26}}{26}$$

The scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is $\dfrac{11\sqrt{26}}{26}$.

In general, the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ does *not* equal the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

Vector projection

In each of the diagrams above, vector $\underset{\sim}{c}$ is the **vector projection** of vector $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is a fraction of $\underset{\sim}{b}$, for example, $\dfrac{m}{n}\underset{\sim}{b}$ where m and n are real numbers.



Now, $\underset{\sim}{c} = |\underset{\sim}{c}|\hat{\underset{\sim}{b}}$

$$= \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

The **vector projection** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is: $\underset{\sim}{a} - \underset{\sim}{c}$

$$= \underset{\sim}{a} - \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ can also be expressed as $\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\underset{\sim}{b} \bullet \underset{\sim}{b}}\underset{\sim}{b}$.

The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$

The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ can also be expressed as $\underset{\sim}{a} - \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\underset{\sim}{b} \bullet \underset{\sim}{b}}\underset{\sim}{b}$.

Example 22

Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$.

(a) Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) Find the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

Solution

(a) Find $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$ by first finding $|\underset{\sim}{b}|$.

$\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$: $|\underset{\sim}{b}| = \sqrt{(-2)^2 + 3^2}$

$= \sqrt{13}$

$\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \dfrac{1}{\sqrt{13}}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

Scalar projection: $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$:

$\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \left(2\underset{\sim}{i} - 5\underset{\sim}{j}\right) \bullet \dfrac{1}{\sqrt{13}}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= \dfrac{-4}{\sqrt{13}} + \dfrac{-15}{\sqrt{13}}$

$= \dfrac{-19}{\sqrt{13}}$

Vector projection: $\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$:

$\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \dfrac{-19}{\sqrt{13}} \times \dfrac{1}{\sqrt{13}}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= \dfrac{-19}{13}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

The vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ onto $\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$ is $\dfrac{-19}{13}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$.

(b) Find $\underset{\sim}{a} - \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$:

$\underset{\sim}{a} - \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = 2\underset{\sim}{i} - 5\underset{\sim}{j} - \dfrac{-19}{13}\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= 2\underset{\sim}{i} - 5\underset{\sim}{j} - \dfrac{38}{13}\underset{\sim}{i} + \dfrac{57}{13}\underset{\sim}{j}$

$= -\dfrac{12}{13}\underset{\sim}{i} - \dfrac{8}{13}\underset{\sim}{j}$

$= -\dfrac{4}{13}\left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

The vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ perpendicular to $\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$ is $-\dfrac{4}{13}\left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$.

Note: The scalar product of the parallel and perpendicular vector projections should be equal to 0. This can be used to check the projections obtained.

EXERCISE 10.5 PROJECTION OF VECTORS

1 For each of the following pairs of vectors, find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 8\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 8\underset{\sim}{j}$

(d) $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ **(e)** $\underset{\sim}{a} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 5\underset{\sim}{j}$ **(f)** $\underset{\sim}{a} = -5\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{b} = -7\underset{\sim}{i} - 2\underset{\sim}{j}$

2 For each of the following pairs of vectors, find the vector projections of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

(d) $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ **(e)** $\underset{\sim}{a} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 7\underset{\sim}{j}$ **(f)** $\underset{\sim}{a} = -5\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{b} = -7\underset{\sim}{i} - 2\underset{\sim}{j}$

3 For each of the following pairs of vectors, find the vector projections of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

(d) $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ **(e)** $\underset{\sim}{a} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 7\underset{\sim}{j}$ **(f)** $\underset{\sim}{a} = -5\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{b} = -7\underset{\sim}{i} - 2\underset{\sim}{j}$

4 For the following vectors, find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

5 For the following vectors, find the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

6 For $\underset{\sim}{a} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$, the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is:

A $\frac{\sqrt{2}}{2}$ **B** $\frac{-\sqrt{2}}{2}$ **C** $\frac{-2\sqrt{13}}{13}$ **D** $\frac{2\sqrt{13}}{13}$

7 The vector projection of $3\underset{\sim}{i} + 2\underset{\sim}{j}$ onto $-\underset{\sim}{i} + 2\underset{\sim}{j}$ is $\frac{1}{5}\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)$. What is the vector projection of $3\underset{\sim}{i} + 2\underset{\sim}{j}$ perpendicular to $-\underset{\sim}{i} + 2\underset{\sim}{j}$?

A $\frac{8}{5}\left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$ **B** $\frac{8}{5}\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)$ **C** $\frac{8}{5}\left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$ **D** $2\underset{\sim}{i} + \underset{\sim}{j}$

8 Consider two vectors $\underset{\sim}{a} = -\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 4\underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. **(b)** Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(c) Find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$. **(d)** Find the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

9 Consider two vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. **(b)** Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(c) Find the vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b}$.

(d) Hence, express the vector $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ in terms of projections onto and perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

10 For any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, state the meaning of each of the following.

(a) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$ **(b)** $\hat{\underset{\sim}{a}} \bullet \underset{\sim}{b}$ **(c)** $\hat{\underset{\sim}{a}} \bullet \hat{\underset{\sim}{b}}$

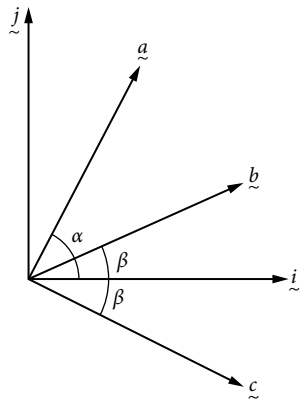
11 $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are unit vectors in the Cartesian plane.

(a) Show that $\underset{\sim}{a} = \cos\alpha\,\underset{\sim}{i} + \sin\alpha\,\underset{\sim}{j}$.

(b) Derive similar expressions for $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

(c) Find $\underset{\sim}{a} \bullet \underset{\sim}{b}$ and $\underset{\sim}{a} \bullet \underset{\sim}{c}$.

(d) Hence deduce the compound angle formulas $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.



10.6 VECTORS IN GEOMETRIC PROOFS

Many geometry theorems can be solved using vector methods.

Background knowledge

It is useful to recall the properties of triangles and special quadrilaterals, in particular:

- a **trapezium** is a quadrilateral with one pair of opposite sides parallel
- a **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. It has the properties that both pairs of opposite sides are equal, both pairs of opposite angles are equal and the diagonals bisect each other
- a **rectangle** is a parallelogram where one angle is a right angle. It has all the properties of the parallelogram, as well as that each angle is 90°, the diagonals are equal, and each diagonal divides the rectangle into a pair

of congruent triangles

- a **square** is a rectangle with a pair of equal adjacent sides. It has all the properties of a rectangle, as well as that all sides are of equal length, the diagonals bisect the angles of the square, the diagonals bisect each other at right angles, and the diagonals divide the square into four congruent right-angled triangles
- a **rhombus** is a parallelogram with a pair of adjacent sides equal. It has all the properties of the parallelogram, as well as that all sides are of equal length, the diagonals bisect the angles of the rhombus, the diagonals bisect each other at right angles, and the diagonals divide the rhombus into four congruent right-angled triangles
- a **square** can also be defined as a rhombus with one angle a right angle
- a **kite** is a quadrilateral with two pairs of adjacent sides equal. The opposite angles between the pairs of non-equal sides are equal.

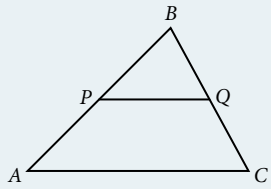
Vector proof involving sides of a triangle

Example 23

Use vector methods to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution

Consider ΔABC, where P and Q are the midpoints of sides $\overline{AB}$ and $\overline{BC}$ respectively.



Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$.

P is the midpoint of $\overline{AB}$ and Q is the midpoint of $\overline{BC}$,

so $\overrightarrow{PB} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{BQ} = \frac{1}{2}\underset{\sim}{b}$.

Express vectors required in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

$\therefore \overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$

Thus, $\overline{PQ}$ is parallel to $\overline{AC}$ and half the length of $\overline{AC}$.

Vector proof involving sides of a right-angled triangle

Pythagoras' theorem is used extensively throughout Mathematics, particularly in measurement and trigonometry. You have used it throughout this topic on vectors to calculate the magnitude of a vector. Pythagoras' theorem can be proved using vector methods.

Example 24

Use vector methods to prove Pythagoras' theorem: in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

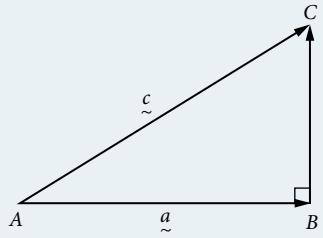
Solution

Consider ΔABC. Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{c}$.

$\therefore \overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\underset{\sim}{a} + \underset{\sim}{c}$

$= \underset{\sim}{c} - a$



By Pythagoras' theorem: $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ **or** $\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2$

Use scalar product of vectors to find $\left|\overrightarrow{AC}\right|^2$: $\left|\overrightarrow{AC}\right|^2 = \overrightarrow{AC} \bullet \overrightarrow{AC}$

$$= \underset{\sim}{c} \bullet \underset{\sim}{c}$$
$$= \left|\underset{\sim}{c}\right|^2$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$. As $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular, $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$.

That is, $\underset{\sim}{a} \bullet (\underset{\sim}{c} - \underset{\sim}{a}) = (\underset{\sim}{c} - \underset{\sim}{a}) \bullet \underset{\sim}{a} = 0$.

Now: $\underset{\sim}{a} \bullet (\underset{\sim}{c} - \underset{\sim}{a}) = \underset{\sim}{a} \bullet \underset{\sim}{c} - \underset{\sim}{a} \bullet \underset{\sim}{a} = 0$

so, $\underset{\sim}{a} \bullet \underset{\sim}{c} = \underset{\sim}{a} \bullet \underset{\sim}{a}$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$.

Now: $\underset{\sim}{a} \bullet \underset{\sim}{a} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{a}\right|$

$$= \left|\underset{\sim}{a}\right|^2$$

so $\left|\overrightarrow{AB}\right|^2 = \left|\underset{\sim}{a}\right|^2$ and $(\underset{\sim}{c} - \underset{\sim}{a}) \bullet (\underset{\sim}{c} - \underset{\sim}{a})$

$$= \underset{\sim}{c} \bullet \underset{\sim}{c} - \underset{\sim}{c} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{a}$$
$$= \underset{\sim}{c} \bullet \underset{\sim}{c} - \underset{\sim}{c} \bullet \underset{\sim}{a} \quad \text{as} \quad \underset{\sim}{a} \bullet \underset{\sim}{c} = \underset{\sim}{a} \bullet \underset{\sim}{a} \text{ from above}$$
$$= \underset{\sim}{c} \bullet \underset{\sim}{c} - \underset{\sim}{a} \bullet \underset{\sim}{a} \quad \text{as} \quad \underset{\sim}{c} \bullet \underset{\sim}{a} = \underset{\sim}{a} \bullet \underset{\sim}{a}$$
$$= \left|\underset{\sim}{c}\right|^2 - \left|\underset{\sim}{a}\right|^2$$

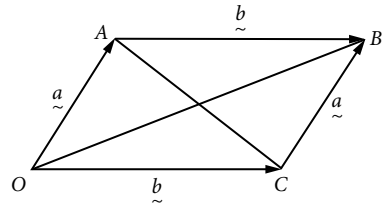
Thus $\left|\overrightarrow{BC}\right|^2 = \left|\underset{\sim}{c}\right|^2 - \left|\underset{\sim}{a}\right|^2$

So: $\left|\overrightarrow{AC}\right|^2 = \left|\underset{\sim}{c}\right|^2$

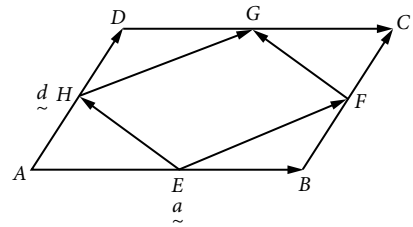
$$= \left|\underset{\sim}{a}\right|^2 + \left(\left|\underset{\sim}{c}\right|^2 - \left|\underset{\sim}{a}\right|^2\right)$$
$$= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2$$

EXERCISE 10.6 VECTORS IN GEOMETRIC PROOFS

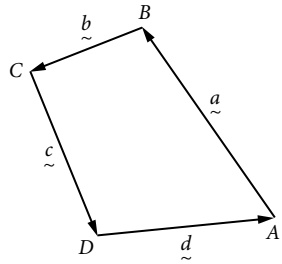
1 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{b}$. Express $\overrightarrow{OB}$ and $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Hence show that the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.



2 Consider the parallelogram $ABCD$, where $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AD} = \underset{\sim}{d}$. Prove that the midpoints of the sides of a parallelogram join to form a parallelogram.



3 Consider the quadrilateral $ABCD$, as shown.



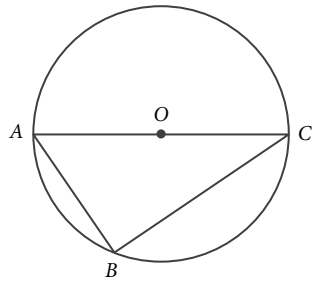
Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$, $\overrightarrow{CD} = \underset{\sim}{c}$ and $\overrightarrow{DA} = \underset{\sim}{d}$.

Which one of the following statements is correct?

A $\underset{\sim}{a} - \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{d}$ **B** $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{d}$ **D** $\underset{\sim}{a} + \underset{\sim}{c} = -\underset{\sim}{b} - \underset{\sim}{d}$

4 Consider the circle with centre O and radius $\overrightarrow{OA} = \underset{\sim}{a}$. B and C are points on the circle and $\overrightarrow{BC} = \underset{\sim}{b}$.

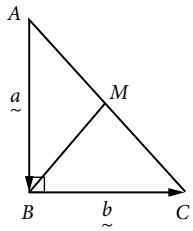


Which one of the following statements must be true?

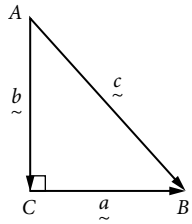
A $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$ **B** $\underset{\sim}{a} = -\frac{1}{2}\underset{\sim}{b}$

C $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{b}$ **D** $2\underset{\sim}{a} \bullet \underset{\sim}{b} = -\underset{\sim}{b} \bullet \underset{\sim}{b}$

5 Use vector methods to prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from all vertices.



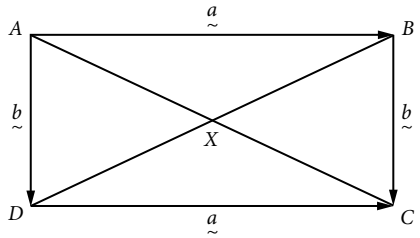
6 Use vector methods to prove that, if $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ for ΔABC, then $\angle ACB$ is a right angle.



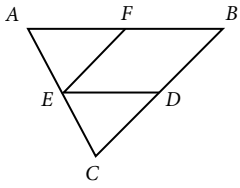
7 $ABCD$ is a rectangle.

(a) Prove that the diagonals of a rectangle bisect each other.

(b) Prove that the diagonals of a rectangle are equal in length.



8 $BDEF$ is a parallelogram contained within a triangle ABC, as shown.

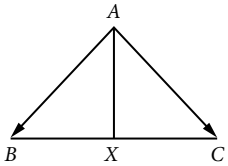


Let $\overrightarrow{AF} = \underset{\sim}{a}$, $\overrightarrow{BD} = \underset{\sim}{b}$ and F be the midpoint of $\overrightarrow{AB}$.

(a) Find the vector $\overrightarrow{AE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(b) Use vector methods to prove that $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$.

9 ABC is a triangle with $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{b}$. X is the midpoint of BC as shown.



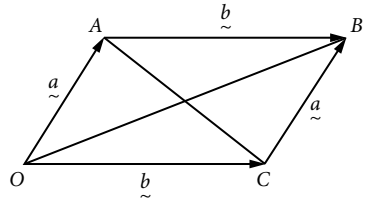
(a) Find $\overrightarrow{BX}$ and $\overrightarrow{AX}$.

(b) Find $2\left(\overrightarrow{BX} \bullet \overrightarrow{BX} + \overrightarrow{AX} \bullet \overrightarrow{AX}\right)$.

Apollonius' theorem relates to the length of a median of a triangle to the lengths of its sides. In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

(c) Prove this, i.e. prove that $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left(\left|\overrightarrow{AX}\right|^2 + \left|\overrightarrow{BX}\right|^2\right)$.

10 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{b}$.



(a) Find the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

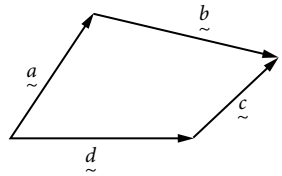
(b) Find the sum of the squares of the lengths of the sides of the parallelogram in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(c) Find the sum of the squares of the lengths of the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(d) Hence prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

CHAPTER REVIEW 10

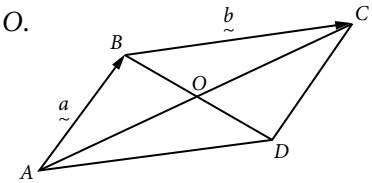
1 Four vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$, are shown in the diagram. Which one of the following statements is true?



A $\underset{\sim}{a}+\underset{\sim}{c}=\underset{\sim}{b}+\underset{\sim}{d}$ **B** $\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{c}+\underset{\sim}{d}$

C $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}=\underset{\sim}{0}$ **D** $\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{a}+\underset{\sim}{d}$

2 In the parallelogram $ABCD$ shown, the point of intersection of the diagonals is O. The vector $\overrightarrow{OD}$ is equal to:



A $\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})$ **B** $\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$

C $\frac{1}{2}\underset{\sim}{b}-\underset{\sim}{a}$ **D** $\frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})$

3 If vector $\underset{\sim}{a}$ is represented by the ordered pair $(-2, 3)$, then the vector $-3\underset{\sim}{a}$ is represented by the ordered pair:

A $(-6, 9)$ **B** $(-6, -9)$ **C** $(6, -9)$ **D** $(6, 9)$

4 The vector that runs from the point $(-3, 1)$ to the point $(3, -2)$ can be represented by the column vector:

A $\begin{pmatrix}6\\3\end{pmatrix}$ **B** $\begin{pmatrix}6\\-3\end{pmatrix}$ **C** $\begin{pmatrix}-6\\3\end{pmatrix}$ **D** $\begin{pmatrix}0\\-3\end{pmatrix}$

5 Which one of the following vectors is parallel to the vector $\underset{\sim}{f} = -6\underset{\sim}{i}+4\underset{\sim}{j}$?

A $\underset{\sim}{a}=24\underset{\sim}{i}-16\underset{\sim}{j}$ **B** $\underset{\sim}{b}=3\underset{\sim}{i}+2\underset{\sim}{j}$ **C** $\underset{\sim}{c}=-24\underset{\sim}{i}-16\underset{\sim}{j}$ **D** $\underset{\sim}{d}=-3\underset{\sim}{i}-2\underset{\sim}{j}$

6 Which one of the following vectors is parallel to the vector $\underset{\sim}{a} = -3\underset{\sim}{i}+7\underset{\sim}{j}$ and has a magnitude of $2\sqrt{58}$?

A $-24\underset{\sim}{i}+28\underset{\sim}{j}$ **B** $-\frac{3}{2}\underset{\sim}{i}+\frac{7}{2}\underset{\sim}{j}$ **C** $3\underset{\sim}{i}-7\underset{\sim}{j}$ **D** $-6\underset{\sim}{i}+14\underset{\sim}{j}$

7 Given position vectors $\overrightarrow{OA} = -3\underset{\sim}{i}+4\underset{\sim}{j}$ and $\overrightarrow{OB} = 4\underset{\sim}{i}+3\underset{\sim}{j}$, what is the value of $\left|\overrightarrow{AB}\right|$?

A $\sqrt{2}$ **B** $5\sqrt{2}$ **C** $2\sqrt{5}$ **D** $7\sqrt{2}$

8 If $\underset{\sim}{a} = -4\underset{\sim}{i}+2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i}-4\underset{\sim}{j}$, then $2\underset{\sim}{a}-\underset{\sim}{b}$ is:

A $-5\underset{\sim}{i}-2\underset{\sim}{j}$ **B** $-6\underset{\sim}{i}+10\underset{\sim}{j}$ **C** $-9\underset{\sim}{i}+8\underset{\sim}{j}$ **D** $-10\underset{\sim}{i}-4\underset{\sim}{j}$

9 What is the magnitude of the vector $\underset{\sim}{a} = 4\underset{\sim}{i}-2\underset{\sim}{j}$?

A 2 **B** $2\sqrt{3}$ **C** $2\sqrt{5}$ **D** 20

10 Which of the following vectors is parallel to the vector $2\underset{\sim}{i}+3\underset{\sim}{j}$ and has a magnitude of $2\sqrt{13}$?

A $-4\underset{\sim}{i}+6\underset{\sim}{j}$ **B** $4\underset{\sim}{i}-6\underset{\sim}{j}$ **C** $6\underset{\sim}{i}+9\underset{\sim}{j}$ **D** $-4\underset{\sim}{i}-6\underset{\sim}{j}$

11 What is the unit vector in the direction of $\underset{\sim}{a} = -3\underset{\sim}{i} - \underset{\sim}{j}$?

A $\frac{1}{\sqrt{10}}\left(-3\underset{\sim}{i} - \underset{\sim}{j}\right)$ **B** $\frac{1}{10}\left(-3\underset{\sim}{i} - \underset{\sim}{j}\right)$ **C** $\frac{1}{\sqrt{10}}\left(-3\underset{\sim}{i} + \underset{\sim}{j}\right)$ **D** $\frac{1}{\sqrt{10}}\left(3\underset{\sim}{i} - \underset{\sim}{j}\right)$

12 What vector is perpendicular to $\underset{\sim}{f} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and has magnitude 5?

A $\underset{\sim}{a} = \frac{1}{\sqrt{17}}\left(\underset{\sim}{i} + 4\underset{\sim}{j}\right)$ **B** $\underset{\sim}{b} = \frac{1}{\sqrt{17}}\left(4\underset{\sim}{i} - \underset{\sim}{j}\right)$ **C** $\underset{\sim}{c} = \frac{5}{\sqrt{17}}\left(\underset{\sim}{i} + 4\underset{\sim}{j}\right)$ **D** $\underset{\sim}{d} = \frac{5}{\sqrt{17}}\left(4\underset{\sim}{i} - \underset{\sim}{j}\right)$

13 The vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - x\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ are perpendicular. What is the value of x?

A $-\frac{15}{2}$ **B** $-\frac{6}{5}$ **C** $\frac{6}{5}$ **D** $\frac{15}{2}$

14 The angle between the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$ is closest to:

A 75° **B** 89° **C** 91° **D** 105°

15 If $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j}$, the scalar projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$ is:

A $\frac{3}{\sqrt{17}}$ **B** $\frac{3}{\sqrt{15}}$ **C** $\frac{3}{\sqrt{13}}$ **D** $\frac{3}{17}$

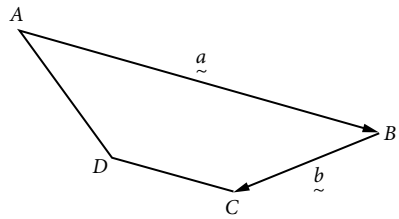
16 The vector projection of $\underset{\sim}{i} - 3\underset{\sim}{j}$ onto $\underset{\sim}{i} + 5\underset{\sim}{j}$ is $-\frac{7}{13}\left(\underset{\sim}{i} + 5\underset{\sim}{j}\right)$. What is the vector projection of $\underset{\sim}{i} - 3\underset{\sim}{j}$ perpendicular to $\underset{\sim}{i} + 5\underset{\sim}{j}$?

A $-\frac{7}{5}\left(\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ **B** $\frac{4}{13}\left(5\underset{\sim}{i} - \underset{\sim}{j}\right)$ **C** $\frac{20}{13}\left(\underset{\sim}{i} + 5\underset{\sim}{j}\right)$ **D** $\frac{12}{5}\left(\underset{\sim}{i} - 3\underset{\sim}{j}\right)$

17 $ABCD$ is a trapezium where $\overline{DC}$ is parallel to and twice the length of $\overline{AB}$. Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$. Therefore, $\overrightarrow{DA}$ is equal to:

A $\underset{\sim}{a} + \underset{\sim}{b}$ **B** $-\underset{\sim}{a} - \underset{\sim}{b}$ **C** $\underset{\sim}{b} - \underset{\sim}{a}$ **D** $\underset{\sim}{a} - \underset{\sim}{b}$

18 $ABCD$ is a trapezium with $\overline{AB}$ parallel to and three times the length of $\overline{DC}$. If $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$, express each of the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

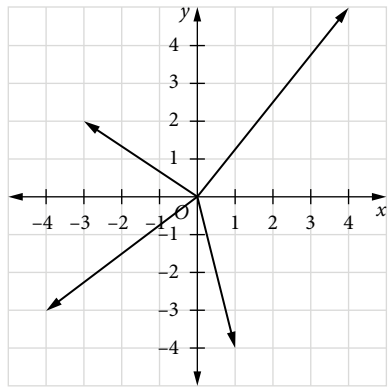


(a) $\overrightarrow{CD}$ **(b)** $\overrightarrow{AC}$

(c) $\overrightarrow{AD}$ **(d)** $\overrightarrow{DB}$

19 Label the following vectors that have been drawn on the Cartesian plane:

- $\underset{\sim}{a}$ the position vector of $(-3, 2)$
- $\overrightarrow{OB}$ where B is $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$
- $\underset{\sim}{c}$ the position vector of $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$
- $\overrightarrow{OD}$ where D is $(1, -4)$



20 Given $\underset{\sim}{a} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$, find the following vectors in component form.

(a) $\underset{\sim}{a} + \underset{\sim}{b}$ **(b)** $3\underset{\sim}{b} - 2\underset{\sim}{a}$ **(c)** $-2\underset{\sim}{a} - 7\underset{\sim}{b}$

21 Given $\underset{\sim}{a} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$, calculate the magnitude of the following vectors.

(a) $\left|\underset{\sim}{a} + \underset{\sim}{b}\right|$ **(b)** $\left|3\underset{\sim}{b} - 2\underset{\sim}{a}\right|$ **(c)** $\left|-2\underset{\sim}{a} - 7\underset{\sim}{b}\right|$

22 Given vectors $\underset{\sim}{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$, find the following vectors in column vector form.

(a) $4\underset{\sim}{c} - 3\underset{\sim}{d}$ **(b)** $7\underset{\sim}{d} - 8\underset{\sim}{c}$

23 The position vectors of points A and B are $\overrightarrow{OA} = 15\underset{\sim}{i} - 7\underset{\sim}{j}$ and $\overrightarrow{OB} = -6\underset{\sim}{i} - 19\underset{\sim}{j}$, respectively.

(a) Find the vector $\overrightarrow{AB}$ in component form. **(b)** Calculate $\left|\overrightarrow{AB}\right|$.

24 Find the magnitude of each of the following vectors.

(a) $\underset{\sim}{a} = 10\underset{\sim}{i} + 24\underset{\sim}{j}$ **(b)** $\underset{\sim}{b} = -6\underset{\sim}{i} - 3\underset{\sim}{j}$

25 Resolve the following vectors into component form $x\underset{\sim}{i} + y\underset{\sim}{j}$. Give answers correct to two decimal places.

(a) $\underset{\sim}{a}$ has a magnitude of 16 units and makes an angle of 48° to the positive x-axis.

(b) $\underset{\sim}{b}$ has a magnitude of 24 units and makes an angle of 148° to the positive x-axis.

26 Find the exact values of the unknown pronumerals in the following vector equations.

(a) $(2a - 3b)\underset{\sim}{i} - 2b\underset{\sim}{j} = 5\underset{\sim}{i} - 12\underset{\sim}{j}$ **(b)** $(2f + 5)\underset{\sim}{i} + (8 - 7g)\underset{\sim}{j} = f\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right) + 2g\left(\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

(c) $\left(a^2 - 9a\right)\underset{\sim}{i} + \left(2b^3 + 1\right)\underset{\sim}{j} = 10\underset{\sim}{i} - 5\underset{\sim}{j}$ (list multiple solutions)

27 Consider the vector $\underset{\sim}{a} = -9\underset{\sim}{i} - 3\underset{\sim}{j}$.

(a) Find $\hat{\underset{\sim}{a}}$. **(b)** Find the vector $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ with a magnitude of 5.

28 Find the scalar product $\underset{\sim}{a} \bullet \underset{\sim}{b}$, given the following pairs of vectors.

(a) $\underset{\sim}{a} = -4\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 7\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 3\underset{\sim}{i} - 7\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} - \underset{\sim}{j}$

29 Calculate the scalar product and hence show that the vectors $\underset{\sim}{a} = -3\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 10\underset{\sim}{i} + 6\underset{\sim}{j}$ are perpendicular.

30 For each of the following pairs of vectors, find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + 3\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -5\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 7\underset{\sim}{j}$

31 For $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}$, find:

(a) the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ **(b)** the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

32 The points A, B and C have coordinates (2, −5), (5, 9) and (−9, 12) respectively.

(a) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in column vector form. **(b)** Find $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{BC}\right|$ and $\left|\overrightarrow{AC}\right|$.

(c) Show that ΔABC is an isosceles triangle.

(d) Find the coordinates of a point D such that $ABCD$ forms a rhombus.

(e) Find the coordinates of the point of intersection of the diagonals of the rhombus $ABCD$.

33 **(a)** If $\underset{\sim}{a} = -4e\underset{\sim}{i} + 2e\underset{\sim}{j}$, $e > 0$ and $\left|\underset{\sim}{a}\right|^2 = 40$, find the exact value of e. **(b)** Hence, find $\hat{\underset{\sim}{a}}$.

(c) Find the vector $\underset{\sim}{b}$ that is parallel to $\hat{\underset{\sim}{a}}$ with $\left|\underset{\sim}{b}\right| = 10$.

(d) If $\underset{\sim}{c} = 4f\underset{\sim}{i} - 3f\underset{\sim}{j}$, $f > 0$ and $\left|\underset{\sim}{c}\right|^2 = 250$, find the exact value of f. **(e)** Hence find $\hat{\underset{\sim}{c}}$.

(f) Find the vector $\underset{\sim}{d}$ in the direction of $\hat{\underset{\sim}{c}}$ where $\left|\underset{\sim}{d}\right|^2 = 20$. **(g)** Find $\underset{\sim}{b} - \underset{\sim}{d}$.

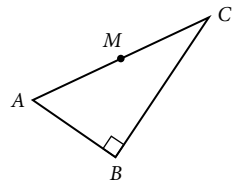
34 Consider two vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} - \underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. **(b)** Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(c) Find the vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b}$.

(d) Hence, express the vector $\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$ in terms of projections parallel to and perpendicular to $\underset{\sim}{b} = -3\underset{\sim}{i} - \underset{\sim}{j}$.

35 ΔABC is right-angled with M being the midpoint of the hypotenuse AC, as shown.
Let $\overrightarrow{AM} = \underset{\sim}{a}$ and $\overrightarrow{BM} = \underset{\sim}{b}$.



(a) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(b) Prove that M is equidistant from the three vertices of ΔABC.

36 $OABC$ is a parallelogram where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. M and N are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively.

(a) Draw a diagram of parallelogram $OABC$, showing the given vectors and midpoints.

(b) Find the vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ and show them on your diagram.

(c) Hence find the vector $\overrightarrow{MN}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(d) Find vector $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ and show this on your diagram.

(e) P is a point on $\overrightarrow{OM}$ such that $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OM}$. Find the vector $\overrightarrow{OP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(f) Q is a point on $\overrightarrow{ON}$ such that $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{ON}$. Find the vector $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

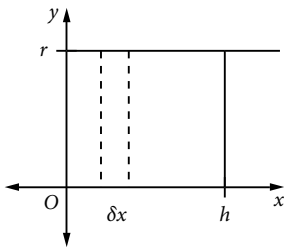
(g) Show that vector $\overrightarrow{MN}$ is parallel to and half the magnitude of $\overrightarrow{AC}$.

(h) Find vectors $\overrightarrow{AP}$, $\overrightarrow{PQ}$ and $\overrightarrow{QC}$, and hence prove that the diagonal $\overrightarrow{AC}$ is trisected at P and Q.

CHAPTER 11
Applications of calculus

11.1 VOLUMES OF SOLIDS OF REVOLUTION

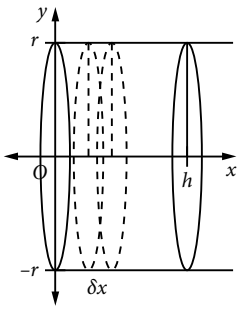
You have seen that the area of a region bounded by a line $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ can be found by adding up the areas of all the rectangles of width δx and height r between $x = 0$ and $x = h$, as δx becomes vanishingly small: $A = \lim\limits_{\delta x \to 0} \sum\limits_{0}^{h} f(x)\delta x$.



This area is given by the definite integral $A = \int_0^h r\,dx$, which is $A = \int_0^h r\,dx = [rx]_0^h = rh$.

You should recognise this as the area of a rectangle of sides r and h.

Consider what happens when the area bounded by $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ is rotated about the x-axis to form a solid of revolution, as shown in the diagram below to the left. The solid of revolution formed is a cylinder of radius r and height h.

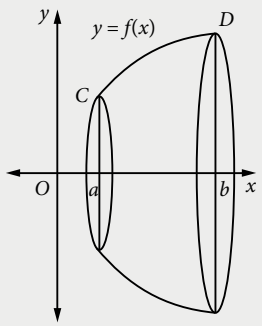


The rectangles of side r and width δx have become circular disks of radius r and thickness δx. The volume of this disk is given by $\Delta V = \pi\big(f(x)\big)^2\,\delta x$. Adding all the disks as δx gets smaller gives $V = \lim\limits_{\delta x \to 0} \sum\limits_{0}^{h} \pi\big(f(x)\big)^2\,\delta x$, which is given by the definite integral $V = \pi\int_0^h r^2\,dx$.

Thus the volume is $V = \pi\int_0^h r^2\,dx = \pi\big[r^2x\big]_0^h = \pi r^2 h$, which you should recognise as the volume of a cylinder of radius r and height h.

When the arc CD of the curve $y = f(x)$ on the interval $a \le x \le b$ is rotated about the x-axis, the volume of the solid of revolution formed is given by:

$$V = \pi\int_a^b \big(f(x)\big)^2\,dx \quad \textbf{or} \quad V = \pi\int_a^b y^2\,dx$$

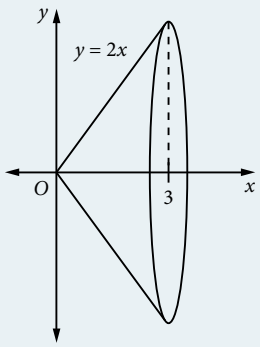


Example 1

Calculate the volume of the solid formed when the portion of the line $y = 2x$ between $x = 0$ and $x = 3$ is rotated about the x-axis. What is the name of the kind of solid formed?

Solution

Draw a diagram:



$$\text{Volume} = \pi \int_a^b y^2\,dx$$

$$= \pi \int_0^3 (2x)^2\,dx$$

$$= 4\pi \int_0^3 x^2\,dx$$

$$= 4\pi \left[\frac{x^3}{3}\right]_0^3$$

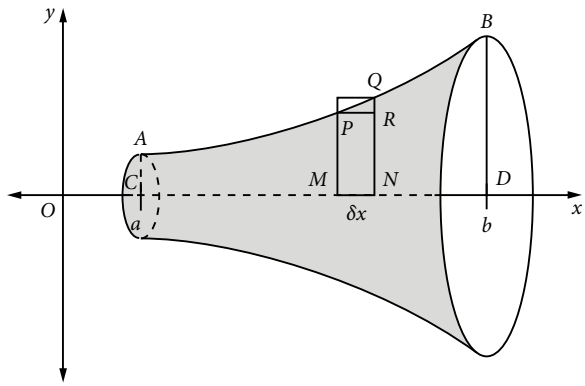
$$= 4\pi(9-0)$$

$$= 36\pi \text{ units}^3$$

The solid is a right circular cone of base radius 6 and height 3.

Volumes of solids of revolution—formal development

Consider a continuous function f in the interval $a \le x \le b$. If the plane section $ABDC$ is rotated about the x-axis then a solid is generated with circular vertical cross-sections, as shown in the diagram on the right. This solid is called a **solid of revolution**.



$P(x, y)$ is a point on the curve $y = f(x)$ and $Q(x + \delta x, y + \delta y)$ is a point close to P. The ordinate PM describes a circle of area πy^2 and QN describes a circle of area $\pi(y + \delta y)^2$.

The typical lower rectangle $PRNM$ describes a cylinder of volume $\pi y^2\,\delta x$ and the typical upper rectangle describes a cylinder of volume $\pi(y + \delta y)^2\,\delta x$. If a typical layer $PQNM$ describes a solid of volume δV, then:

$$\pi y^2\,\delta x < \delta V < \pi(y + \delta y)^2 \delta x$$

Thus: $$\sum_a^b \pi y^2\,\delta x < V < \sum_a^b \pi\left(y + \delta y\right)^2 \delta x$$

As $\delta x \to 0$: $$V = \lim_{\delta x \to 0} \sum_a^b \pi y^2\,\delta x$$

$$= \int_a^b \pi y^2\,dx$$

$$= \pi \int_a^b y^2\,dx$$

Hence, volume of a solid of revolution:

$V = \pi \int_a^b y^2\,dx$ where $y = f(x)$

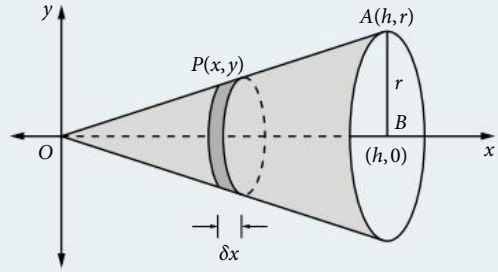
Example 2

Find the volume of a right circular cone of height h and base radius r.

Solution

The cone can be considered as a solid of revolution generated by rotating the right-angled triangle OAB about the x-axis.

The equation of OA is $y = \frac{rx}{h}$.



$$V = \pi \int_a^b y^2\,dx$$

$$V = \pi \int_0^h \frac{r^2x^2}{h^2}\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \times \frac{h^3}{3}$$

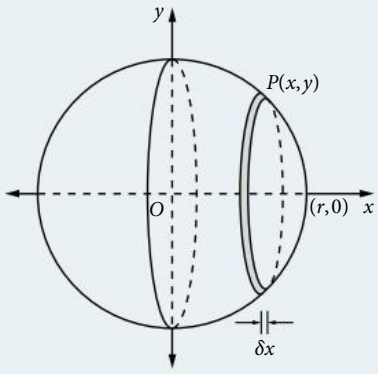
$$= \frac{1}{3}\pi r^2 h$$

Example 3

Find the volume of a sphere of radius r.

Solution

The volume of a sphere can be considered as the volume generated by rotating the semicircle defined by $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis.



Hence:

$$V = \pi \int_{-r}^{r} y^2\,dx \qquad \text{where } y = \sqrt{r^2 - x^2}$$

$$= \pi \int_{-r}^{r} \left(r^2 - x^2\right)dx \qquad \text{because } y^2 = r^2 - x^2$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right)$$

$$= \frac{4}{3}\pi r^3$$

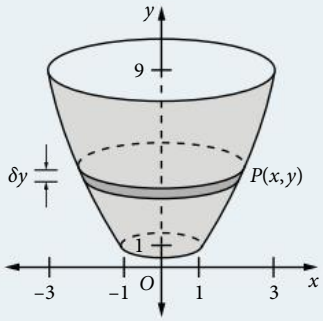
Example **3**, above, proves the formula for the volume of the sphere—a formula that you have used for many years. The formula for the area of a circle $A = \pi r^2$ can similarly be proved using calculus.

Example 4

The part of the parabola $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the y-axis. Calculate the volume generated.

Solution

$y = x^2$: $x = 1, y = 1; x = 3, y = 9$



$$V = \pi \int_1^9 x^2\,dy \qquad \text{where } x^2 = y$$

$$V = \pi \int_1^9 y\,dy$$

$$= \pi \left[\frac{y^2}{2}\right]_1^9$$

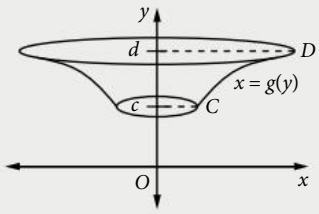
$$= \pi\left(\frac{81}{2} - \frac{1}{2}\right)$$

$$= 40\pi \text{ units}^3$$

Rotating about the *y*-axis

When the arc CD of the curve $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_c^d \big(g(y)\big)^2\,dy \qquad \textbf{or} \qquad V = \pi \int_c^d x^2\,dy$$

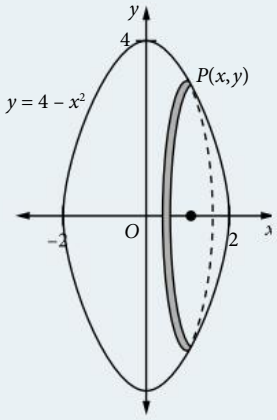


Example 5

Find the volume of the solid formed when the area bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

Solution

(a) Rotate about x-axis:



$$V = \pi \int_{-2}^{2} y^2\,dx \qquad \text{where } y = 4 - x^2$$

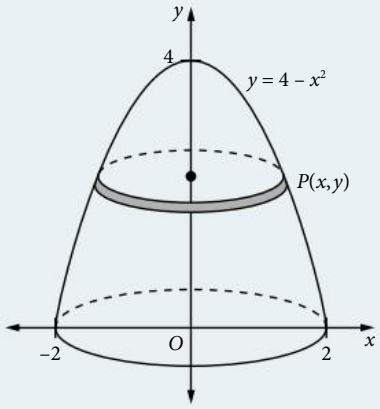
$$= \pi \int_{-2}^{2} \left(16 - 8x^2 + x^4\right)dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$$

$$= \pi\left(\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right)$$

$$= \frac{512\pi}{15} \text{ units}^3$$

(b) Rotate about y-axis:



$$V = \pi \int_0^4 x^2\,dx \qquad \text{where } x^2 = 4 - y$$

$$= \pi \int_0^4 (4 - y)\,dx$$

$$= \pi \left[4y - \frac{y^2}{2}\right]_0^4$$

$$= \pi\big((16 - 8) - 0\big)$$

$$= 8\pi \text{ units}^3$$

Example 6

Calculate: **(a)** the area bounded by the curve $y = e^{1.5x}$, the coordinate axes and the line $x = 2$

(b) the volume obtained by rotating this area about the x-axis.

Solution

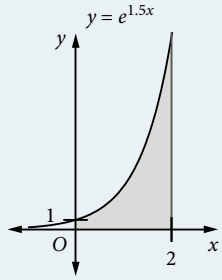
(a) $y = e^{1.5x}$, $y = 0$, $x = 2$

$$\text{Area} = \int_0^2 e^{1.5x}\,dx$$

$$= \left[\frac{2}{3}e^{1.5x}\right]_0^2$$

$$= \frac{2}{3}\left(e^3 - e^0\right)$$

$$= \frac{2\left(e^3 - 1\right)}{3} \approx 12.72 \text{ units}^2$$



(b) Volume $= \pi \int_0^2 y^2\,dx$ where $y = e^{1.5x}$.

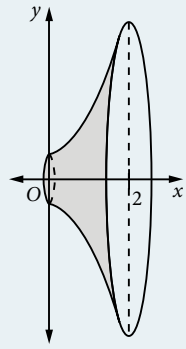
$$= \pi \int_0^2 e^{3x}\,dx$$

$$= \frac{\pi}{3}\left[e^{3x}\right]_0^2$$

$$= \frac{\pi}{3}\left(e^6 - e^0\right)$$

$$= \frac{\pi\left(e^6 - 1\right)}{3}$$

$$\approx 421.4 \text{ units}^3$$



Example 7

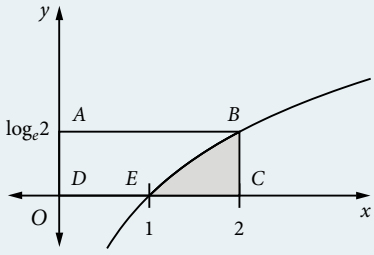
Find: **(a)** the area bounded by the curve $y = \log_e x$, the x-axis and the ordinate $x = 2$

(b) the volume of the solid of revolution formed by rotating the area bounded by the curve $y = \log_e x$, the coordinate axes and the line $y = \log_e 2$ about the y-axis.

Solution

(a) $\text{Area} = \int_1^2 \log_e x\,dx$

Instead of trying to evaluate this integral directly, draw a diagram.



This problem requires the area of the shaded region BCE. It can be obtained by finding the area of the rectangle $ABCD$ and subtracting the area $ABED$.

Because $y = \log_e x$, you can write $x = e^y$.

At $x = 2$, $y = \log_e 2$: $\quad \text{Area } ABED = \int_0^{\log_e 2} e^y\,dy$

$$= \left[e^y\right]_0^{\log_e 2}$$

$$= e^{\log_e 2} - e^0$$

$$= 2 - 1$$

$$= 1$$

$$\text{Area } ABCD = 2\log_e 2$$

$$\therefore \text{Area } BCE = 2\log_e 2 - 1$$

$$\approx 0.386 \text{ units}^2$$

(b) Volume $= \pi \int_0^{\log_e 2} x^2\, dy$ where $x = e^y$

$$= \pi \int_0^{\log_e 2} e^{2y}\, dy$$

$$= \frac{\pi}{2}\left[e^{2y}\right]_0^{\log_e 2}$$

$$= \frac{\pi}{2}\left(e^{2\log_e 2} - e^0\right)$$

$$= \frac{\pi}{2}(4-1) = \frac{3\pi}{2} \text{ units}^3$$

Example 8

Find the volume generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x}}$ between $x = 4$ and $x = 9$.

Solution

Volume $= \pi \int_4^9 y^2\, dx$ where $y = \frac{1}{\sqrt{x}}$

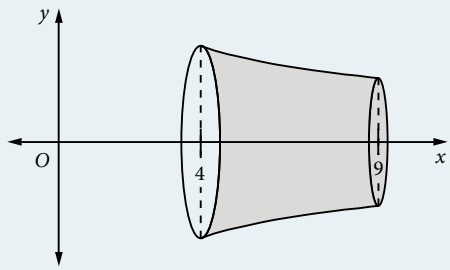
$$= \pi \int_4^9 \frac{1}{x}\, dx$$

$$= \pi \left[\log_e x\right]_4^9$$

$$= \pi\left(\log_e 9 - \log_e 4\right)$$

$$= \pi \log_e 2.25$$

$$\approx 2.548$$



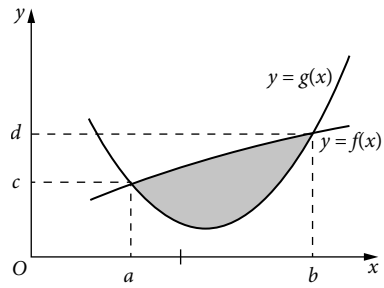
EXPLORING FURTHER

Solids of revolution

Use graphing software to graph and calculate the volume of solids of revolution.

Volume by rotating the region between two curves

When the region bounded by two curves $y = f(x)$ and $y = g(x)$ is rotated about the x-axis, the volume of the solid of revolution formed is given by $V = \pi \int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right) dx$, where a and b are the abscissae of the points of intersection of the two curves, $a < b$ and $f(x) \geq g(x)$.

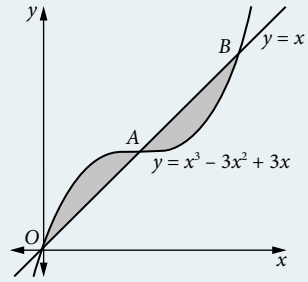


If the region is rotated about the y-axis, the equation of each curve must first be written as a function of y, i.e. $x = f{-}1(y)$ and $x = g{-}1(y)$, and the ordinates of the points of intersection used, namely c and d, as shown in the diagram.

The volume of the solid of revolution is given by $V = \pi \int_c^d \left(\left[f^{-1}(y)\right]^2 - \left[g^{-1}(y)\right]^2\right) dy$.

Example 9

The curve $y = x^3 - 3x^2 + 3x$ and the line $y = x$ intersect at $(0, 0)$, A and B.



(a) Find the coordinates of A and B.

(b) Calculate the shaded area between the curves.

(c) The shaded region between the curves from O to A is rotated about the x-axis. Calculate the exact volume of the solid formed.

(d) The shaded region between the curves from A to B is rotated about the x-axis. Calculate the exact volume of the solid formed.

(e) Hence find the volume of the solid formed when the shaded region between the curves from O to B is rotated about the x-axis.

Solution

(a)
$$\begin{aligned} x^3 - 3x^2 + 3x &= x \\ x^3 - 3x^2 + 2x &= 0 \\ x(x^2 - 3x + 2) &= 0 \\ x(x-1)(x-2) &= 0 \\ x &= 0, 1, 2 \\ y &= 0, 1, 2 \end{aligned}$$
$A(1, 1)$ and $B(2, 2)$

(b)
$$\begin{aligned} \text{Area} &= \int_0^1 \left(x^3 - 3x^2 + 3x - x\right) dx + \int_1^2 \left(x - \left(x^3 - 3x^2 + 3x\right)\right) dx \\ &= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx + \int_1^2 \left(-x^3 + 3x^2 - 2x\right) dx \\ &= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \left[-\frac{x^4}{4} + x^3 - x^2\right]_1^2 \\ &= \frac{1}{4} - 1 + 1 - 0 + \left(-4 + 8 - 4 - \left(-\frac{1}{4} + 1 - 1\right)\right) \\ &= \frac{1}{2} \text{ unit}^2 \end{aligned}$$

(c)
$$\begin{aligned} \text{Volume from } O \text{ to } A &= \pi \int_0^1 \left\{\left(x^3 - 3x^2 + 3x\right)^2 - x^2\right\} dx \\ &= \pi \int_0^1 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right) dx \\ &= \pi \left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_0^1 \\ &= \pi \left(\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3} - 0\right) \\ &= \frac{13\pi}{42} \text{ units}^3 \end{aligned}$$

(d)
$$\begin{aligned} \text{Volume from } A \text{ to } B &= \pi \int_1^2 \left\{x^2 - \left(x^3 - 3x^2 + 3x\right)^2\right\} dx \\ &= -\pi \int_1^2 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right) dx \\ &= -\pi \left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_1^2 \\ &= -\pi \left(\frac{128}{7} - 64 + 96 - 72 + \frac{64}{3} - \left[\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3}\right]\right) \\ &= \frac{29\pi}{42} \text{ units}^3 \end{aligned}$$

(e)
$$\text{Volume from } O \text{ to } B = \frac{13\pi}{42} + \frac{29\pi}{42} = \pi \text{ units}^3$$

EXERCISE 11.1 VOLUMES OF SOLIDS OF REVOLUTION

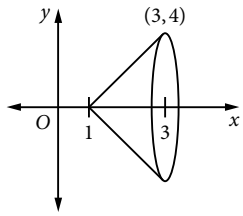
1 Find the volume of the solid of revolution formed by rotating about the x-axis the arc of the parabola $y = x^2$ between $x = 0$ and $x = 3$.

2 Find the volume of the solid of revolution formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = 4$.

3 A cone is formed by rotating about the x-axis a segment of the line $y = 3x$ between $x = 0$ and $x = 4$. The definite integral used to calculate the volume of this solid is:

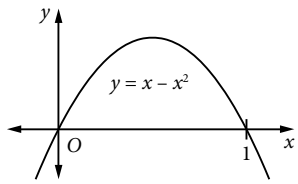
A $\int_0^4 9x^2\,dx$ **B** $\pi\int_0^4 3x^2\,dx$ **C** $\int_0^4 3x^2\,dx$ **D** $\pi\int_0^4 9x^2\,dx$

4 **(a)** Find the equation of the line passing through the points $(1, 0)$ and $(3, 4)$.
(b) A cone is formed by rotating about the x-axis the segment of the line joining the points $(1, 0)$ and $(3, 4)$. Calculate the volume of the cone.



5 The semicircle $y = \sqrt{9 - x^2}$ is rotated about the x-axis. Calculate the volume of the sphere generated.

6 The region bounded by the parabola $y = x - x^2$ and the x-axis is rotated about the x-axis. Find the volume of the solid formed.

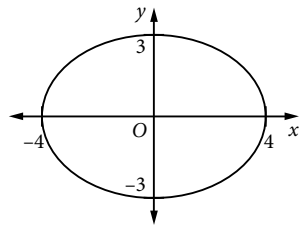


7 Find the volume of the solid formed when the region bounded by the parabola $y = 1 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

8 The region bounded by the parabola $y = (x - 2)^2$ and the coordinate axes is rotated about the x-axis. Find the volume of the solid generated.

9 Find the volume of the solid generated when the segment of the line joining the points $(0, 3)$ and $(6, 0)$ is rotated about: **(a)** the x-axis **(b)** the y-axis.

10 A rugby ball has a volume approximately the same as the volume generated by rotating the ellipse $9x^2 + 16y^2 = 144$ about the x-axis. Find its volume.

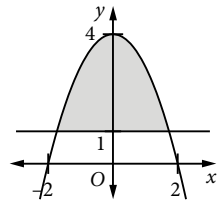


11 Find the volume of the solid formed when the region bounded by the parabola $y = 9 - x^2$ and the coordinate axes is rotated about: **(a)** the x-axis **(b)** the y-axis.

12 **(a)** Find the equation of the line through the points $(3, 0)$ and $(4, 10)$.
(b) A drinking glass has the shape of a truncated cone. The internal radii of the base and the top are 3 cm and 4 cm respectively and its depth is 10 cm. If the base of the glass sits on the x-axis, use integration to find its capacity.
(c) If the glass is filled with water to a depth of 5 cm, find the volume of water in the glass.

13 A hemispherical bowl of radius a units is filled with water to a depth of $\frac{a}{2}$ units. Use integration to find the volume of the water.

14 Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the line $y = 1$ is rotated about the y-axis.



15 A solid is formed by rotating about the y-axis the region bounded by the parabola $y = x^2 - 2$ and the x-axis. Indicate whether each statement below is a correct or incorrect step in calculating the volume of the solid formed.

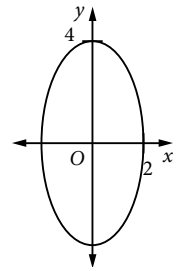
(a) $V = \pi \int_0^{\sqrt{2}} \left(x^2 - 2\right)^2 dx$ **(b)** $V = \pi \int_{-2}^{0} (y + 2)\, dy$ **(c)** $V = \pi \left[\frac{y^2}{2} + 2y \right]_{-2}^{0}$ **(d)** $V = 2\pi$

16 Use integration to find the volume of the sphere generated when the circle $x^2 + y^2 = 16$ is rotated about the x-axis.

17 The area under the curve $y = 2x\sqrt{1 - x^2}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Using the trapezoidal rule with four subintervals, find an approximation for the volume of the solid correct to two decimal places.

18 Find the volume of the solid formed when the ellipse $4x^2 + y^2 = 16$ is rotated about:

(a) the x-axis **(b)** the y-axis.

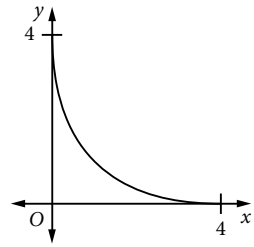


19 A region is bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes.

(a) Calculate the area of the region.

(b) Calculate the volume of the solid generated when the region is rotated about the x-axis.

(c) Calculate the volume of the solid generated when the region is rotated about the y-axis.



20 The region bounded by the curve $xy = 1$, the x-axis and the lines $x = 1$ and $x = a$, for $a > 1$, is rotated about the x-axis. Find V, the volume generated. Hence find $\lim_{a \to \infty} V$.

21 The area bounded by the parabola $y = 2x - x^2$, the y-axis and the line $y = 1$ is rotated about the x-axis. Find the volume generated.

22 Find the volume of the solid generated by rotating the region bounded by the parabola $y = 1 - x^2$ and the lines $x = 1$, $y = 1$ about: **(a)** the x-axis **(b)** the y-axis.

23 Find the volume of the cone formed by rotating the segment of the line $x + 2y = 4$ that is cut off by the axes about: **(a)** the x-axis **(b)** the y-axis.

24 Use the trapezoidal rule with five function values to estimate the volume of the solid formed by rotating the curve $y = \frac{1}{1 + x^2}$ about the x-axis between $x = -2$ and $x = 2$.

25 The area under the curve $y = e^{-x}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the volume of the solid of revolution.

26 Find the volume generated when the curve $y = e^x$, $0.5 \leq x \leq 1.5$, is rotated about the x-axis.

27 Find the volume generated when the curve $y = e^{-0.5x}$, $-2 \leq x \leq 2$, is rotated about the x-axis.

28 Find the volume generated when the curve $y = e^x + e^{-x}$ between $x = -1$ and $x = 1$ is rotated about the x-axis.

29 **(a)** Find the area of the region bounded by the curve $y = e^{-x}$, the coordinate axes and the line $x = a$, $a > 0$.
(b) Find the limit of this area as $a \to \infty$.
(c) Find the volume of the solid generated by rotating the region in **(a)** about the x-axis and find the limit of this volume as $a \to \infty$.

30 Find the volume of the solid generated by rotating about the x-axis the region enclosed by the curve $y^2 = \frac{6}{x}$, the x-axis and the ordinates $x = 1$ and $x = 3$.

31 Find the volume of the solid generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x-2}}$ between $x = 6$ and $x = 11$.

32 **(a)** Given $a > 1$, sketch the curve $y = \log_e x$ for $1 \leq x \leq a$. Find the area enclosed by the curve and the lines $y = 0$ and $x = a$.
(b) The region enclosed by the curve $y = \log_e x$ and the lines $x = 0$, $y = \log_e a$ and $y = 0$ is rotated about the y-axis to form a solid of revolution. Find the volume of this solid.

33 Sketch the curve $y = \frac{1}{x^2}$ for values of x from $x = \frac{1}{2}$ to $x = 1$. This part of the curve is rotated about the y-axis to form a solid of revolution. Find its volume.

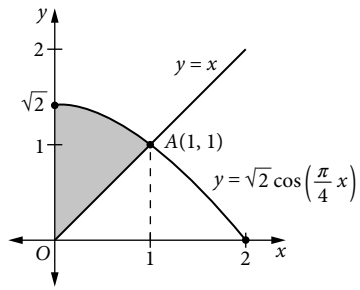
34 Sketch the curve $y = \frac{1}{\sqrt{4+x}}$ from $x = 0$ to $x = 5$. The region enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

35 The region enclosed by the curve $y = \frac{2}{\sqrt{x-7}}$ and the x-axis between $x = 8$ and $x = 10$ is rotated about the x-axis. Find the volume of the solid formed.

36 The region enclosed by the curve $y = \frac{\sqrt{x+1}}{x}$ and the x-axis between $x = 3$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

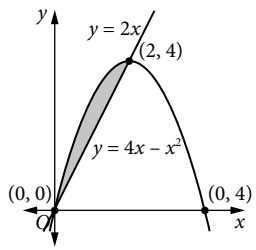
37 **(a)** Sketch the region bounded by the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$.
(b) Calculate the area of the shaded region.
(c) The region bounded by the y-axis and the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$ for $x \geq 0$, is rotated about the y-axis. Calculate the volume of the solid of revolution generated.

38 The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at the point $A(1, 1)$, as shown in the diagram.



(a) Find the exact value of the shaded area.
(b) The shaded area is rotated about the x-axis. Calculate the volume of the solid of revolution formed.
(c) The shaded area is rotated about the y-axis. Write the integral for this volume.
(d) By using a combination of exact integration and the trapezoidal rule, as appropriate, calculate the volume of the solid in **(c)**.

39 In the diagram on the right, the parabola $y = 4x - x^2$ and the line $y = 2x$ intersect at the points $(0, 0)$ and $(2, 4)$.



(a) Calculate the area of the region between the curves.
(b) The shaded region between the curves is rotated about the x-axis to form a solid of revolution. Calculate the exact volume of this solid.
(c) The shaded region between the curves is rotated about the y-axis to form a solid of revolution. Calculate the exact volume of this solid.

40 **(a)** Sketch the region bounded by the curve $y = 1 + \sqrt{x}$ and the lines $y = 1$ and $x = 4$.
(b) Calculate the area of this region.
(c) This region is rotated around the x-axis to form a solid. Calculate the volume of this solid.
(d) Calculate the volume of the solid formed if this region is rotated about the y-axis.

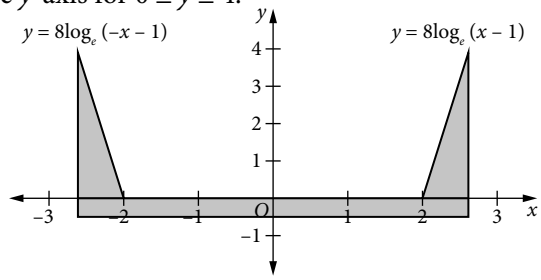
41 **(a)** On the same diagram sketch the graphs of $x^2 + y^2 = 1$ for $-1 \le x \le 0$ and $\frac{x^2}{4} + y^2 = 1$ for $0 \le x \le 2$.

(b) An egg is modelled by rotating about the x-axis the curves $x^2 + y^2 = 1$ for $-1 \le x \le 0$ and $\frac{x^2}{4} + y^2 = 1$ for $0 \le x \le 2$ to form a solid of revolution. Find the exact value of the volume of the egg.

42 A bowl is formed by rotating the curve $y = 8\log_e(x - 1)$ about the y-axis for $0 \le y \le 4$.

(a) Calculate the volume of the bowl (capacity), giving your answer correct to one decimal place.

(b) This bowl is to be moulded out of plastic with vertical sides and a solid base 0.5 units thick. The cross-section of the bowl is shown in the diagram, right. Calculate the volume of plastic used to make the bowl.



11.2 INDEFINITE INTEGRALS AND SUBSTITUTION

Some integrals can only be solved using particular substitutions for the variables. In this Mathematics Extension 1 course, any substitutions needed to find an integral are given.

Integration using a substitution can be considered as the converse of the method of differentiating a composite function—it's like using the chain rule backwards.

The aim of a substitution is to transform an integral into one that involves a standard result, e.g. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$. Variable substitution works as follows:

Let $y = \int f(u)du$ where $u = g(x)$

$\therefore \quad \frac{dy}{du} = f(u)$

But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= f(u) \times \frac{du}{dx}$

$\therefore \quad y = \int f(u) \times \frac{du}{dx}\,dx$

This 'backwards' form of the chain rule is convenient when the substitution $u = g(x)$ allows a function to be expressed as the product of $\frac{du}{dx}$ and a function of u. For example:

- If $f(x) = 2x^2(x^3 - 1)^4$ then you can substitute $u = x^3 - 1$. As $\frac{du}{dx} = 3x^2$, you can write $2x^2$ as $\frac{2}{3} \times 3x^2$, so that $f(x)$ is written: $f(x) = \frac{2}{3}(x^3 - 1)^4 \times (3x^2)$
$= \frac{2}{3}u^4 \frac{du}{dx}$ where $u = x^3 - 1$
- If $f(x) = x\sqrt{1 + x^2}$, you can see that $2x$ is the derivative of $1 + x^2$, so if you make the substitution $u = 1 + x^2$ and write x as $\frac{1}{2}(2x)$, then $f(x) = \frac{1}{2}u^{\frac{1}{2}} \frac{du}{dx}$ where $u = 1 + x^2$ and $\frac{du}{dx} = 2x$.

- If $f(x)=\dfrac{x+1}{(x^2+2x)^3}$, you can see that $2x+2$ is the derivative of x^2+2x, so if you make the substitution $u=x^2+2x$ and write $x+1$ as $\frac{1}{2}(2x+2)$, then $f(x)=\frac{1}{2}u^{-3}\dfrac{du}{dx}$.
- If $f(x)=x\sqrt{1-x}$, then you can make the substitution $u=1-x$. As $x=1-u$, $\dfrac{du}{dx}=-1$, so:

$$\begin{aligned}f(x)&=-x\sqrt{1-x}\times(-1)\\&=-(1-u)u^{\frac{1}{2}}\times\frac{du}{dx}\\&=-\left(u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)\times\frac{du}{dx}\\&=\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)\times\frac{du}{dx}\\&=f(u)\frac{du}{dx}\end{aligned}$$

Example 10

Find: **(a)** $\displaystyle\int 3x^2\left(x^3-1\right)^4\,dx$ using the substitution $u=x^3-1$

(b) $\displaystyle\int x\sqrt{1+x^2}\,dx$ using the substitution $u=1+x^2$

(c) $\displaystyle\int\frac{x+1}{\left(x^2+2x\right)^3}dx$ using the substitution $u=x^2+2x$.

Solution

(a) $u=x^3-1,\ \dfrac{du}{dx}=3x^2$

$$\begin{aligned}\int 3x^2\left(x^3-1\right)^4\,dx&=\int u^4\times\frac{du}{dx}dx\\&=\int u^4\,du\\&=\frac{1}{5}u^5+C\\&=\frac{1}{5}\left(x^3-1\right)^5+C\end{aligned}$$

(b) $u=1+x^2,\ \dfrac{du}{dx}=2x$

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx&=\frac{1}{2}\int 2x\sqrt{1+x^2}\,dx\\&=\frac{1}{2}\int u^{\frac{1}{2}}\times\frac{du}{dx}dx\\&=\frac{1}{2}\int u^{\frac{1}{2}}\,du\\&=\frac{1}{2}\times\tfrac{2}{3}u^{\frac{3}{2}}+C\\&=\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C\end{aligned}$$

(c) $u=x^2+2x,\ \dfrac{du}{dx}=2x+2$

$$\begin{aligned}\int\frac{x+1}{\left(x^2+2x\right)^3}dx&=\frac{1}{2}\int(2x+2)\left(x^2+2x\right)^{-3}\,dx\\&=\frac{1}{2}\int u^{-3}\times\frac{du}{dx}dx\\&=\frac{1}{2}\int u^{-3}\,du\\&=\frac{1}{2}\times\left(\frac{1}{-2}\right)u^{-2}+C\\&=\frac{-1}{4\left(x^2+2x\right)^2}+C\end{aligned}$$

A quick way to check your answer is to differentiate it to see that it gives the integrand.

Example 11

Find: **(a)** $\int x\sqrt{1-x}\,dx$ using the substitution $u = 1 - x$ **(b)** $\int \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(c) $\int (3x-5)^4\,dx$ using the substitution $u = 3x - 5$.

Solution

(a) $u = 1 - x, \frac{du}{dx} = -1, x = 1 - u$

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= \int (1-u)u^{\frac{1}{2}} \times \frac{du}{dx}\,dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)(-1)\,dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du \qquad (\text{note that } du = (-1)\,dx) \\ &= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C \\ &= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C\end{aligned}$$

(b) $u = 1 + t, \frac{du}{dt} = 1, t = u - 1$

$$\begin{aligned}\int \frac{t}{\sqrt{1+t}}\,dt &= \int \frac{u-1}{\sqrt{u}} \times \frac{du}{dt} \times dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)(1)\,dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \qquad (\text{note that } du = (1)\,dt) \\ &= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C\end{aligned}$$

(c) $u = 3x - 5, \frac{du}{dx} = 3$

$$\begin{aligned}\int (3x-5)^4\,dx &= \frac{1}{3}\int 3(3x-5)^4\,dx \\ &= \frac{1}{3}\int u^4 \times \frac{du}{dx}\,dx \\ &= \frac{1}{3}\int u^4\,du \\ &= \frac{1}{3} \times \frac{1}{5}u^5 + C \\ &= \frac{1}{15}(3x-5)^5 + C\end{aligned}$$

EXERCISE 11.2 INDEFINITE INTEGRALS AND SUBSTITUTION

1 Find: **(a)** $\int 2x\left(x^2-1\right)^4\,dx$ using the substitution $u = x^2 - 1$

(b) $\int 3x^2\left(x^3+4\right)^3\,dx$ using the substitution $u = x^3 + 4$

(c) $\int x^2\sqrt{x^3+1}\,dx$ using the substitution $u = x^3 + 1$.

2 Find: **(a)** $\int (2t+1)^3\,dt$ using the substitution $u = 2t + 1$

(b) $\int \frac{2x}{\sqrt{x^2-4}}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int (2x+1)\left(x^2+x+2\right)^5\,dx$ using the substitution $u = x^2 + x + 2$.

3 Using $u = 2x + 3$, $\int \frac{dx}{(2x+3)^3} = \ldots$

A $4(2x+3)^4 + C$ **B** $\frac{1}{4(2x+3)^4} + C$ **C** $-4(2x+3)^2 + C$ **D** $\frac{-1}{4(2x+3)^2} + C$

4 Find: **(a)** $\int (3-2x)^6\,dx$ using the substitution $u = 3-2x$

(b) $\int \dfrac{3x+1}{\left(3x^2+2x+5\right)^2}\,dx$ using the substitution $u = 3x^2+2x+5$

(c) $\int \left(x^2-2x\right)\left(x^3-3x^2+1\right)^4\,dx$ using the substitution $u = x^3-3x^2+1$.

5 Find: **(a)** $\int 3x^2\left(x^3+1\right)^4\,dx$ using the substitution $u = x^3+1$

(b) $\int \dfrac{t}{\sqrt{1-t^2}}\,dt$ using the substitution $u = 1-t^2$

(c) $\int (3x-5)^{\frac{2}{3}}\,dx$ using the substitution $u = 3x-5$.

6 Find: **(a)** $\int 2t\sqrt{1-t^2}\,dt$ using the substitution $u = 1-t^2$

(b) $\int x\sqrt{a^2-x^2}\,dx$ using the substitution $u = a^2-x^2$

(c) $\int z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2+1$.

7 Find: **(a)** $\int y\sqrt{y+1}\,dy$ using the substitution $u = y+1$

(b) $\int \dfrac{x}{(x-1)^3}\,dx$ using the substitution $u = x-1$

(c) $\int \dfrac{x}{\sqrt{2x-1}}\,dx$ using the substitution $u = 2x-1$.

8 **(a)** Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = 1+x^3$.

(b) Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = x^3$.

(c) Why is the substitution in **(a)** easier to use than the substitution in **(b)**?

9 $\dfrac{dy}{dx} = x\sqrt{x^2-4}$ and $y = 2$ at $x = \sqrt{5}$. Use the substitution $u = x^2-4$ to find y in terms of x.

10 If $f'(x) = \dfrac{3x}{\sqrt{x^2+1}}$ for all x and $f(0) = 2$, use the substitution $u = x^2+1$ to find $f(x)$.

11 $\dfrac{dx}{dt} = \dfrac{t-1}{\sqrt{t^2-2t+4}}$ and $x = 10$ when $t = 0$. Use the substitution $u = t^2-2t+4$ to find x in terms of t.

12 Given that $\dfrac{dr}{d\theta} = \dfrac{3}{(1-r)^4}$ and $r(0) = 0$, use the substitution $u = 1-r$ to find r in terms of θ.

13 At any point where $x > \frac{1}{2}$, the gradient of a curve is given by $\dfrac{dy}{dx} = \sqrt{2x-1}$. If the point $(2.5, 9)$ is on the curve, use the substitution $u = 2x-1$ to find the equation of the curve.

14 Given that $\dfrac{dt}{dx} = \dfrac{1}{2(4-x)^2}$ and $x = 0$ when $t = 0$, use the substitution $u = 4-x$ to find x in terms of t.

11.3 DEFINITE INTEGRALS AND SUBSTITUTION

When using a substitution to evaluate a definite integral you must take care with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Example 12

Evaluate: **(a)** $\int_1^2 2x\sqrt{x^2-1}\,dx$ using the substitution $u = x^2 - 1$

(b) $\int_{-5}^{3} x\sqrt{4-x}\,dx$ using the substitution $u = 4 - x$

(c) $\int_0^1 x^2\left(x^3+1\right)^4 dx$ using the substitution $u = x^3 + 1$.

Solution

(a) $u = x^2 - 1$, $\frac{du}{dx} = 2x$

Limits: for $x = 1$, $u = 1^2 - 1 = 0$
for $x = 2$, $u = 2^2 - 1 = 3$

$$\int_1^2 2x\sqrt{x^2-1}\,dx = \int_0^3 \sqrt{u} \times \frac{du}{dx}\,dx$$
$$= \int_0^3 u^{\frac{1}{2}}\,du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3$$
$$= \frac{2}{3}\left(3^{\frac{3}{2}} - 0\right) = 2\sqrt{3}$$

(b) $u = 4 - x$, $\frac{du}{dx} = -1$

$x = 4 - u$, so $x\sqrt{4-x} = (4-u)\sqrt{u}$
or $-x\sqrt{4-x} = (u-4)\sqrt{u}$

Limits: for $x = -5$, $u = 4 + 5 = 9$
for $x = 3$, $u = 4 - 3 = 1$

$$\int_{-5}^{3} x\sqrt{4-x}\,dx = \int_9^1 (u-4)\sqrt{u} \times \frac{du}{dx}\,dx$$
$$= \int_9^1 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} - 4 \times \frac{2}{3}u^{\frac{3}{2}}\right]_9^1$$
$$= \left(\frac{2}{5} - \frac{8}{3}\right) - \left(\frac{2}{5} \times 9^{\frac{5}{2}} - \frac{8}{3} \times 9^{\frac{3}{2}}\right)$$
$$= \frac{2}{5} - \frac{8}{3} - \frac{2}{5} \times 3^5 + \frac{8}{3} \times 3^3$$
$$= -\frac{412}{15}$$

(c) $u = x^3 + 1$, $\frac{du}{dx} = 3x^2$

Limits: for $x = 0$, $u = 1$
for $x = 1$, $u = 1^3 + 1 = 2$

$$\int_0^1 x^2\left(x^3+1\right)^4 dx = \frac{1}{3}\int_1^2 u^4 \times \frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int_1^2 u^4\,du$$
$$= \frac{1}{3}\left[\frac{1}{5}u^5\right]_1^2$$
$$= \frac{1}{15}\left[u^5\right]_1^2$$
$$= \frac{1}{15}(32-1) = \frac{31}{15}$$

Useful result

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ If you reverse the limits of integration, you change the sign of the integral.

In Example **11(b)** the integral $\int_9^1 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$ could have been written as $-\int_1^9 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$.

Check that this integral gives the same answer.

EXERCISE 11.3 DEFINITE INTEGRALS AND SUBSTITUTION

1 Evaluate: **(a)** $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $u = 1 - x^2$

(b) $\int_{-1}^{2} x\sqrt{2-x}\,dx$ using the substitution $u = 2 - x$

(c) $\int_0^2 \frac{2x}{\sqrt{x^2+1}}\,dx$ using the substitution $u = x^2 + 1$.

2 Evaluate: **(a)** $\int_{0.5}^{1.5} \frac{2-2x}{\sqrt{2x-x^2}}\,dx$ using the substitution $u = 2x - x^2$

(b) $\int_{2}^{3} x\sqrt{x-1}\,dx$ using the substitution $u = x - 1$

(c) $\int_{-0.5}^{0.5} \frac{x}{\sqrt{1-x^2}}\,dx$ using the substitution $u = 1 - x^2$.

3 Using the substitution $u = 1 + x^2$, $\int_{0}^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx = \ldots$

A 0.5 **B** 1 **C** 2 **D** $\sqrt{3}-1$

4 Evaluate: **(a)** $\int_{3}^{4} (2x-3)\left(x^2-3x+2\right)^2\,dx$ using the substitution $u = x^2 - 3x + 2$

(b) $\int_{0}^{2} \frac{x}{\left(x^2+2\right)^2}\,dx$ using the substitution $u = x^2 + 2$

(c) $\int_{1.5}^{4} \sqrt{2t+1}\,dt$ using the substitution $u = 2t + 1$.

5 Evaluate: **(a)** $\int_{0}^{1} \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(b) $\int_{0}^{1} 3x^2\left(x^3-1\right)^4\,dx$ using the substitution $u = x^3 - 1$

(c) $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$ using the substitution $u = a^2 - x^2$.

6 Evaluate: **(a)** $\int_{0}^{\sqrt{2}} z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2 + 1$

(b) $\int_{2}^{3} x\sqrt{x^2-4}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int_{1}^{5} \sqrt{2x-1}\,dx$ using the substitution $u = 2x - 1$.

7 Evaluate: **(a)** $\int_{0}^{3} y\sqrt{y+1}\,dy$ using the substitution $u = y + 1$

(b) $\int_{0}^{4} x\sqrt{16-x^2}\,dx$ using the substitution $u = 16 - x^2$

(c) $\int_{-1}^{1} x(3x-1)^4\,dx$ using the substitution $u = 3x - 1$.

8 **(a)** Evaluate $\int_{1}^{2} \frac{2x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = 4 - x^2$.

(b) Evaluate $\int_{1}^{2} \frac{2x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = x^2$.

(c) Why is the substitution in **(a)** easier to use than the substitution in **(b)**?

9 Using the substitution $u = 1 - x^2$, calculate the area bounded by the curve $y = x\sqrt{1-x^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

10 Find the area of the region bounded by the curve $y = \frac{x}{\sqrt{x^2-1}}$, the x-axis and the lines $x = \sqrt{2}$ and $x = \sqrt{5}$.

11 Sketch the curve defined by $y^2 = x^2(1-x)$ and find the area of the loop. Use the substitution $u = 1 - x$.

12 If $f(x) = x^2(x-3)$, calculate the area between the x-axis, the lines $x = 4$ and $x = 6$ and the curve $y = \sqrt{f(x)}$. Use the substitution $u = x - 3$.

13 Calculate the area of the region bounded by the curve $y = \dfrac{x}{\sqrt{x^2+4}}$, the x-axis and the lines $x = 0$ and $x = 2\sqrt{3}$. Use the substitution $u = x^2 + 4$.

Useful result

You may have noticed that for an integral of the form $\int f'(x)\big(f(x)\big)^n\,dx$, using the substitution $u = f(x)$ gives $\dfrac{du}{dx} = f'(x)$ so that the integral becomes: $\int f'(x)\big(f(x)\big)^n\,dx = \int u^n\,du$

$$= \frac{1}{n+1}u^{n+1} + C = \frac{1}{n+1}\big(f(x)\big)^{n+1} + C$$

If you can recognise that $\int f'(x)\big(f(x)\big)^n\,dx = \dfrac{1}{n+1}\big(f(x)\big)^{n+1} + C$, you can obtain your answer faster. This result will be especially useful in section 11.6 (page 250) when integrating powers of trigonometric functions.

11.4 INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

$\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ cannot be found without using a substitution. However, a useful substitution comes from the formulae for $\cos 2x$. This may not be given in a question, but you must be able to use it when necessary.

The important results are as follows: $\cos 2x = \cos^2 x - \sin^2 x$

Using $\cos^2 x + \sin^2 x = 1$: $\cos 2x = 2\cos^2 x - 1$

$\therefore$ $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Thus: $\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Similarly: $\cos 2x = 1 - 2\sin^2 x$

$\therefore$ $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Thus: $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 13

(a) Find $3\int \sin^2 x\,dx$. **(b)** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx$. **(c)** Find $\int \cos^2 3x\,dx$.

Solution

(a) $3\int \sin^2 x\,dx = 3 \times \frac{1}{2}\int (1-\cos 2x)\,dx$

$$= \frac{3}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{3x}{2} - \frac{3\sin 2x}{4} + C$$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} - 0\right)$

$= \frac{1}{2}\left(\frac{3\pi}{4} + \frac{1}{2}\right) = \frac{1}{8}(3\pi + 2)$

(c) $\int \cos^2 3x\,dx = \frac{1}{2}\int (1+\cos 6x)\,dx$

$= \frac{1}{2}\left(x + \frac{\sin 6x}{6}\right) + C$

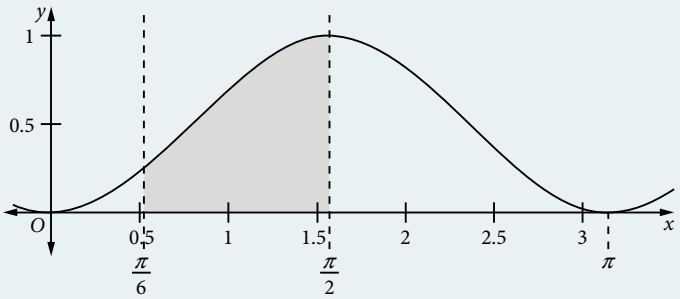
$= \frac{x}{2} + \frac{\sin 6x}{12} + C$

Example 14

Find the area bounded by the curve $y = \sin^2 x$, the x-axis and the ordinates at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Solution

Sketch the function.



$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin\pi - \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0 - \frac{\pi}{6} + \frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$

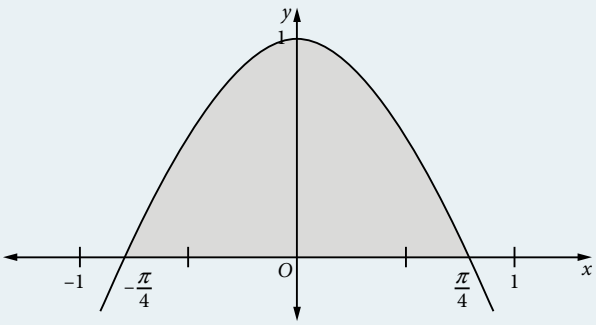
$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{4\pi + 3\sqrt{3}}{24}$ square units

Example 15

The region under the curve $y = \cos 2x$ between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the volume of the solid of revolution formed.

Solution

Sketch the function.



$\text{Volume} = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2\,dx$

$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2x\,dx$

$= \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 4x)\,dx$

$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[\frac{\pi}{4} + 0 - \left(-\frac{\pi}{4} + 0\right)\right] = \frac{\pi^2}{4}$ cubic units

EXERCISE 11.4 INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

1 Find:

(a) $\int 2\cos^2 x\,dx$ **(b)** $\int 2\sin^2 x\,dx$ **(c)** $\int \sin^2 \frac{x}{2}\,dx$

(d) $\int 2\cos^2 \frac{x}{2}\,dx$ **(e)** $\int \sin^2 3x\,dx$ **(f)** $\int \cos^2 4x\,dx$

2 Evaluate:

(a) $\int_0^{\frac{\pi}{2}} 2\sin^2 x\,dx$ **(b)** $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ **(c)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$

(d) $\int_{\frac{\pi}{2}}^{2\pi} \cos^2 \frac{x}{3}\,dx$ **(e)** $4\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 4x\,dx$ **(f)** $2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 3x\,dx$

3 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. The volume of the solid of revolution formed is given by:

A $\int_0^{\pi} \sin^2 x\,dx$ **B** $\pi\int_0^{\pi} \sin x\,dx$ **C** $\pi\int_0^{\pi} \sin^2 x\,dx$ **D** $\pi\int_0^{\pi} \sin x^2\,dx$

4 The region under the curve $y = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

5 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

6 The gradient at the point (x, y) of the function $y = f(x)$ is given by $\frac{dy}{dx} = \sin^2 x$. The curve passes through the point $(0, -2)$. Find y where $x = \frac{\pi}{4}$.

7 If $\frac{d^2y}{dx^2} = 2\cos^2 x$ and when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 0$, $y = 0$, then find y in terms of x.

8 The region bounded by the curves $y = \cos 2x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

9 The region bounded by the curves $y = \sin 3x$, the x-axis and the ordinate $x = \frac{\pi}{12}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

10 The region bounded by the curves $y = \sin x$, $y = \cos x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

(a) Find the point of intersection of the two curves.

(b) Calculate the exact value of the volume of the solid of revolution formed.

11 The region bounded by $y = \sin x$, $y = \cos x$ and the y-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

12 **(a)** Sketch the region bounded by the curve $y = \sin x + \cos x$ and the coordinate axes in the first quadrant. Show the intercepts on the axes.

(b) Calculate the area of the region in part **(a)**.

(c) The region in part **(a)** is rotated about the x-axis to form a solid of revolution. Calculate the volume of the solid formed.

11.5 INTEGRALS OF THE TYPE $\int f'(x)(f(x))^n\,dx$

You have seen this type of integral before (e.g. with integrals like $\int 2x(x^2+1)^3\,dx = \frac{1}{4}(x^2+1)^4 + C$). This section will consider trigonometric integrals such as $\int \cos x \sin^2 x\,dx$, i.e. where $f(x) = \sin x$ and $f'(x) = \cos x$. In trigonometric integrals of this type the substitution will not always be given.

Example 16

(a) Find $\int \cos x \sin^2 x \, dx$. **(b)** Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$.

Solution

(a) $\int \cos x \sin^2 x \, dx$: Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\int \cos x \sin^2 x \, dx = \int u^2 \times \frac{du}{dx} \times dx$$
$$= \int u^2 \, du$$
$$= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$$

If you can recognise that $\int \cos x \sin^2 x \, dx$ is of the form $\int f'(x)(f(x))^2 \, dx = \frac{1}{3}(f(x))^3 + C$, then you can write the answer immediately as $\int \cos x \sin^2 x \, dx = \frac{1}{3}\sin^3 x + C$.

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$: Let $u = \cos x$, $\frac{du}{dx} = -\sin x$ For $x = \frac{\pi}{3}$, $u = \frac{1}{2}$; for $x = \frac{\pi}{2}$, $u = 0$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx = \int_{\frac{1}{2}}^{0} -u^2 \times \frac{du}{dx} \times du$$
$$= \int_{\frac{1}{2}}^{0} -u^2 \, du$$
$$= \left[-\frac{1}{3}u^3\right]_{\frac{1}{2}}^{0} = 0 + \frac{1}{3} \times \left(\frac{1}{2}\right)^3 = \frac{1}{24}$$

Note: The result $\int_{\frac{1}{2}}^{0} -u^2 \, du = \int_{0}^{\frac{1}{2}} u^2 \, du$ (reversing the limits of the integral and changing the sign of the integrand) could have been used. Also: $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (-\sin x)\cos^2 x \, dx$

$$= -\left[\frac{1}{3}\cos^3 x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= -\frac{1}{3}\left(0^3 - \left(\frac{1}{2}\right)^3\right) = \frac{1}{24}$$

Substitution simplified

If you treat $\frac{du}{dx}$ like a fraction, then $\frac{du}{dx} \times dx = du$.

If $u = \sin x$, then $\frac{du}{dx} = \cos x$ can be written as $du = \cos x \, dx$.

You are really replacing $f'(x)\,dx$ by du. This makes the algebra involved much simpler.

Example 17

Find $\int \tan^2 x \sec^2 x \, dx$.

Solution

Let $u = \tan x$, $du = \sec^2 x \, dx$: $\int \tan^2 x \sec^2 x \, dx = \int u^2 \, du$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

EXERCISE 11.5 INTEGRALS OF THE TYPE $\int f'(x)(f(x))^n dx$

1 Find: **(a)** $\int \sin x \cos^2 x\, dx$ **(b)** $\int \tan x \sec^2 x\, dx$ **(c)** $\int \sin x \cos^3 x\, dx$

(d) $\int \cos x \sin^4 x\, dx$ **(e)** $\int (1+\cos 2x)\sin x\, dx$ **(f)** $\int \sin x \cos x\, dx$

2 Evaluate: **(a)** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^3 x\, dx$ **(b)** $\int_{-\pi}^{\pi} \sin^3 x \cos x\, dx$ **(c)** $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

(d) $\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x\, dx$ **(e)** $\int_0^{\pi} 2\sin\theta \cos^2\theta\, d\theta$ **(f)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(x-\frac{\pi}{4}\right) dx$

11.6 INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTION

If an expression of the form $a^2 - x^2$ occurs in the integrand, the standard trigonometric substitutions $x = a\sin\theta$ or $x = a\cos\theta$ will help to find the integral. If an expression of the form $a^2 + x^2$ occurs in the integrand, then the standard substitution is $x = a\tan\theta$.

Note: In this course, if these substitutions are needed they should be given in the question.

Example 18

Evaluate $\int_0^2 \sqrt{4-x^2}\, dx$ using the substitution $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution

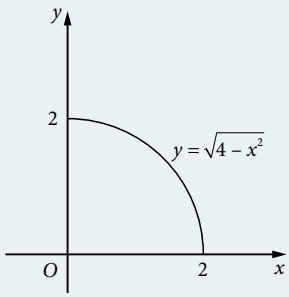
$x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $\frac{dx}{d\theta} = 2\cos\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = |2\cos\theta| = 2\cos\theta$ because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\therefore \cos\theta \ge 0$

Limit for $x = 2$: $2 = 2\sin\theta$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

Limit for $x = 0$: $0 = 2\sin\theta$, $\sin\theta = 0$, $\theta = 0$

$$\begin{aligned}\int_0^2 \sqrt{4-x^2}\, dx &= \int_0^{\frac{\pi}{2}} 2\cos\theta \times \frac{dx}{d\theta} \times d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\cos\theta \times 2\cos\theta\, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta \\ &= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= 2\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(0 + \frac{1}{2}\sin 0\right)\right] \\ &= 2\times\frac{\pi}{2} = \pi\end{aligned}$$



Geometrically, the graph of $y = \sqrt{4-x^2}$ between $x = 0$ and $x = 2$ is the quarter of a circle with equation $x^2 + y^2 = 4$ (in the first quadrant). This integration thus proves the formula for the area of a circle: $A = \pi r^2$.

Area $= \frac{1}{4}\pi r^2 = \frac{1}{4}\times\pi\times 4 = \pi$ when $r = 2$.

Note: If the restriction on θ had been different, e.g. $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, then $\cos\theta \ge 0$ would not be true and you would not be able to use $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$. Instead, you may have needed $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = -2\cos\theta$.

Remember: Always take care with trigonometric substitutions and check the sign of the function for the given domain.

Example 19

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using the substitution $x = \tan\theta$.

Solution

Method 1

$x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$: $\quad \frac{1}{1+x^2} = \frac{1}{1+\tan^2\theta} = \frac{1}{\sec^2\theta}$

Limit for $x = 1$: $\quad 1 = \tan\theta$, $\theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta$, $\theta = 0$

$$\int_0^1 \frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \frac{dx}{d\theta} \times d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} d\theta$$
$$= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

Method 2

$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

Limit for $x = 1$: $\quad 1 = \tan\theta$, $\theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta$, $\theta = 0$

$$\int_0^1 \frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta} \sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} d\theta$$
$$= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

Example 20

Find $\int \frac{1-\sin x}{x+\cos x}dx$ using the substitution $u = x + \cos x$.

Solution

$u = x + \cos x$, $du = (1 - \sin x)\, dx$: $\quad \int \frac{1-\sin x}{x+\cos x}dx = \int \frac{du}{u}$

$$= \ln|u| + C$$
$$= \ln|x + \cos x| + C$$

This example involves an indefinite integral, so you must remember to resubstitute for u at the end to make the answer a function of x.

EXERCISE 11.6 INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTION

1 Evaluate $\int_{-4}^{4} \sqrt{16-x^2}\, dx$ using the substitution $x = 4\sin\theta$. What shape have you just found the area of?

2 Evaluate $\int_0^1 x\sqrt{1-x^2}\, dx$ using the substitution $x = \sin\theta$.

3 Evaluate $\int_0^2 \frac{dx}{4+x^2}$ using the substitution $x = 2\tan\theta$.

4 Evaluate $\int_0^{0.5} \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ using the substitution $x = \sin\theta$.

5 Evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x\, dx$ using $\sin x \cos x = \frac{1}{2}\sin 2x$.

6 Evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx$ using the substitution $x=\cos\theta$.

7 Evaluate $\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}dx$ using the substitution $x=\cos\theta$.

8 Evaluate $\int_{-\frac{1}{2}}^{0} \frac{dx}{1+4x^2}$ using the substitution $x=\frac{1}{2}\tan\theta$.

9 Evaluate $\int_0^4 x\sqrt{16-x^2}\,dx$ using the substitution $x=4\sin\theta$.

10 Evaluate $\int_0^{\sqrt{3}} \frac{1}{9+x^2}dx$ using the substitution $x=3\tan\theta$.

11 Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{(\tan x+3)^4}dx$ using the substitution $u=\tan x+3$.

12 Use the substitution $u=\tan x$ to find $\int \frac{e^{\tan x}}{\cos^2 x}dx$.

13 Use the substitution $u=\cos x+\sin x$ to find $\int \frac{\sin x-\cos x}{\sin x+\cos x}dx$.

14 **(a)** By writing $\sec x=\frac{1}{\cos x}$, show that $\frac{d}{dx}(\sec x)=\sec x\tan x$.

(b) Using the substitution $u=1+\sec x$, find $\int \frac{\sec x\tan x}{1+\sec x}dx$.

Summary of integration techniques

- $\int f(u)\times\frac{du}{dx}\times dx=\int f(u)\,du$
- $\int f'(x)\big(f(x)\big)^n dx=\frac{1}{n+1}\big(f(x)\big)^{n+1}+C$
- $\int \sin^2 x\,dx=\frac{1}{2}\int(1-\cos 2x)\,dx$
- $\int \cos^2 x\,dx=\frac{1}{2}\int(1+\cos 2x)\,dx$

11.7 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

The result $\frac{dy}{dx}\times\frac{dx}{dy}=1$

Proof

Let $y=f(x)$. Differentiate both sides with respect to y:

$$\frac{d}{dy}(y)=\frac{d}{dy}\big(f(x)\big)$$

$$\therefore \quad 1=\frac{d}{dx}\big(f(x)\big)\times\frac{dx}{dy} \qquad \text{(using chain rule on RHS)}$$

$$\therefore \quad 1=\frac{dy}{dx}\times\frac{dx}{dy}$$

Hence $\frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}$

Demonstration

Let $y=x^3-1$ $\quad\therefore\quad$ $x=(y+1)^{\frac{1}{3}}$

$\frac{dy}{dx}=3x^2$ $\qquad$ $\frac{dx}{dy}=\frac{1}{3}(y+1)^{-\frac{2}{3}}$

$$\frac{dy}{dx}\times\frac{dx}{dy}=3x^2\times\frac{1}{3}(y+1)^{-\frac{2}{3}}$$

$$=\frac{x^2}{(y+1)^{\frac{2}{3}}}$$

$$=\frac{x^2}{(x^3)^{\frac{2}{3}}}$$

$$=\frac{x^2}{x^2}=1$$

Derivative of $\sin^{-1}x$

$\sin^{-1}x$ is defined for $-1 \le x \le 1$.

Let $y = \sin^{-1}x \quad \therefore \quad x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

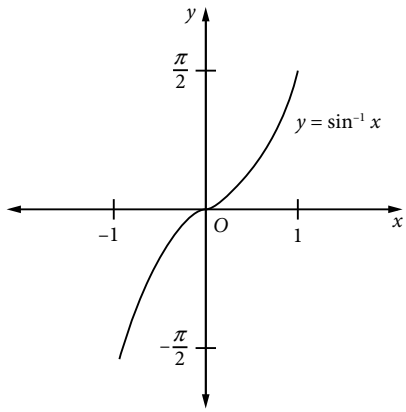
$$\frac{dx}{dy} = \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} \text{ noting that } \cos y \neq 0 \ \therefore -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Now, using $\cos^2 y + \sin^2 y = 1$: $\frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$ Which one?

As y is an angle in the first or fourth quadrants, $\cos y$ must be positive.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$



Note that the derivative is not defined at $x = \pm 1$ (the graph of $\sin^{-1}x$ has vertical tangents at its endpoints).

Also note that the derivative is positive for all x in its domain (as $\sin^{-1}x$ is an increasing function).

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

Derivative of $\sin^{-1}\frac{x}{a}$

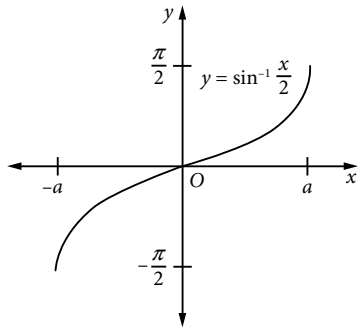
$\sin^{-1}\frac{x}{a}$ is defined for $-a \le x \le a$.

Let $y = \sin^{-1}\frac{x}{a} \quad \therefore \quad x = a\sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\frac{dx}{dy} = a\cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{a\cos y} \quad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}} \quad \text{for } -a < x < a$$



$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}, -a < x < a$$

Derivative of $\cos^{-1}x$

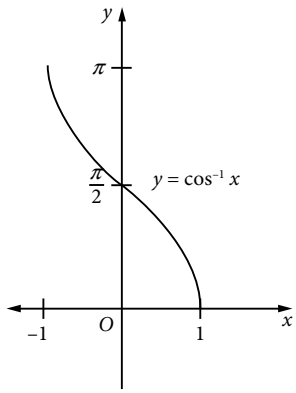
$\cos^{-1}x$ is defined for $-1 \le x \le 1$.

Let $y = \cos^{-1}x \quad \therefore \quad x = \cos y, 0 \le y \le \pi$

$$\frac{dx}{dy} = -\sin y$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sin y} \quad \text{for } 0 < y < \pi$$

$$\text{i.e. } \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$



$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

As for the derivative of $\sin^{-1}x$, the last step again uses $\cos^2 y + \sin^2 y = 1$.

Note that the gradient is negative for all x in the domain.

Note also that the domain of the derived function is different to the domain of $\cos^{-1}x$. Why?

Derivative of $\cos^{-1}\frac{x}{a}$

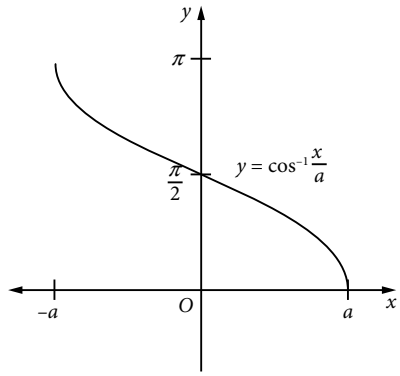
$\cos^{-1}\frac{x}{a}$ is defined for $-a \le x \le a$.

Let $y = \cos^{-1}\frac{x}{a}$ $\quad\therefore\quad x = a\cos y,\ 0 \le y \le \pi$

$$\frac{dx}{dy} = -a\sin y$$

$$\therefore \frac{dy}{dx} = -\frac{1}{a\sin y} \quad \text{for } 0 < y < \pi$$

$$\text{i.e. } \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}} \quad \text{for } -a < x < a$$



$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}},\ -a < x < a$$

Derivative of $\tan^{-1}x$

$\tan^{-1}x$ is defined for all x.

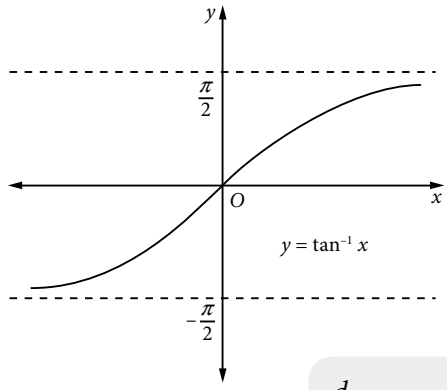
Let $y = \tan^{-1}x$ $\quad\therefore\quad x = \tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{dx}{dy} = \sec^2 y$$

$$= 1 + \tan^2 y$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + \tan^2 y} \quad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{1}{1 + x^2} \quad \text{for all } x$$



$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1 + x^2} \quad \text{for all } x$$

Derivative of $\tan^{-1}\frac{x}{a}$

$\tan^{-1}\frac{x}{a}$ is defined for all x.

Let $y = \tan^{-1}\frac{x}{a}$ $\quad\therefore\quad x = a\tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{dx}{dy} = a\sec^2 y$$

$$= a(1 + \tan^2 y)$$

$$\therefore \frac{dy}{dx} = \frac{1}{a(1 + \tan^2 y)} \quad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{a}{a^2 + x^2} \quad \text{for all } x$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2} \quad \text{for all } x$$

Example 21

Find the derivative of $\cos^{-1}(2x + 1)$, stating the values of x for which it is defined.

Solution

$\cos^{-1}(2x+1)$ is defined for $-1 \le 2x+1 \le 1$

i.e. for $-2 \le 2x \le 0$

$-1 \le x \le 0$

Hence $\cos^{-1}(2x+1)$ is defined for $-1 \le x \le 0$.

Let $y = \cos^{-1}(2x+1)$

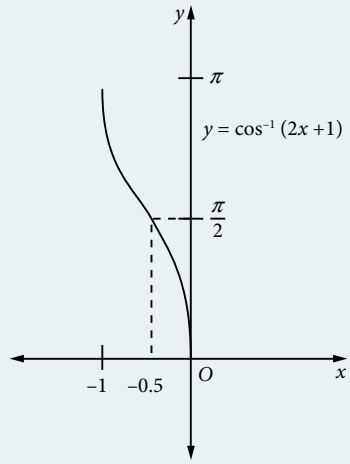
$= \cos^{-1} u$ where $u = 2x+1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}} \times 2$$

$$= -\frac{2}{\sqrt{1-(2x+1)^2}} \quad \text{provided } -1 < x < 0$$

$$= -\frac{2}{\sqrt{-4x(x+1)}}$$



Example 22

Find the derivative of $x\cos^{-1}(2x+1)$, stating the values of x for which the derivative is defined.

Solution

Following the previous example and using the product rule:

$$\frac{d}{dx}\left(x\cos^{-1}(2x+1)\right) = x \times \frac{-2}{\sqrt{-4x(x+1)}} + \cos^{-1}(2x+1)$$

$$= \frac{-2x}{\sqrt{-4x(x+1)}} + \cos^{-1}(2x+1)$$

On the RHS, the first term is defined for $-1 < x < 0$ and the second term is defined for $-1 \le x \le 0$. Thus the complete RHS is defined for $-1 < x < 0$ and these are the values for x for which the derivative is defined.

Example 23

Differentiate $\sin^{-1}(\cos x)$. Hence sketch the graph of $y = \sin^{-1}(\cos x)$ for $-\pi \le x \le \pi$.

Solution

Let $y = \sin^{-1}(\cos x)$

$= \sin^{-1} u$ where $u = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times (-\sin x)$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}}$$

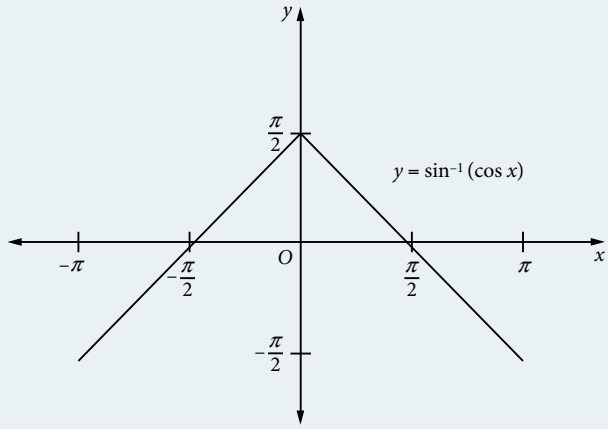
$$= \frac{-\sin x}{|\sin x|}$$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
y	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Now $|\sin x| = \sin x$ for $\sin x \geq 0$, i.e. for $0 \leq x \leq \pi$

$= -\sin x$ for $\sin x \leq 0$, i.e. for $-\pi \leq x \leq 0$

$$\therefore \frac{dy}{dx} = \begin{cases} \dfrac{-\sin x}{\sin x} = -1 & \text{for } 0 < x < \pi \\ \dfrac{-\sin x}{-\sin x} = 1 & \text{for } -\pi < x < 0 \end{cases}$$



$\frac{dy}{dx}$ is not defined when $x = -\pi, 0, \pi$. If there were no restrictions on the domain, the graph would repeat itself (i.e. it would be periodic with period 2π). The range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and the derivative is not defined for any values of $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

Note that these sharp peaks are *not* turning points, because the function here changes sharply instead of smoothly. The point where the function changes sharply is called a **cusp**.

Example 24

(a) Differentiate $\sin^{-1} x + \cos^{-1} x$. **(b)** Hence show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Solution

(a) $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$

(b) $\sin^{-1} x + \cos^{-1} x$ is a constant as its derivative is 0.

The value of the constant can be found by evaluating the function at any x in its domain.

Where $x = 0$: $\sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore$ $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (You may wish to verify this by substituting other values for x.)

EXERCISE 11.7 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

1 The derivative of $\sin^{-1} 3x$ is:

A $\dfrac{1}{\sqrt{9-x^2}}$ **B** $\dfrac{3}{\sqrt{1-9x^2}}$ **C** $\dfrac{1}{\sqrt{1-9x^2}}$ **D** $\dfrac{3}{\sqrt{9-x^2}}$

2 Differentiate the following.

(a) $y = \tan^{-1} 5x$ **(b)** $y = 3\tan^{-1}(1-x)$ **(c)** $y = \tan^{-1} x^2$ **(d)** $y = (\tan^{-1} x)^2$

(e) $y = \tan^{-1}\left(\sqrt{x}\right)$ **(f)** $y = \tan^{-1} e^x$ **(g)** $y = e^{\tan^{-1} x}$ **(h)** $y = e^x \tan^{-1} x$

(i) $y = \sin^{-1}\left(\frac{x}{4}\right)$ **(j)** $y = 2\cos^{-1}\left(\frac{3x}{2}\right)$ **(k)** $y = \log_e(\sin^{-1} x)$ **(l)** $y = \log_e(\cos^{-1} 2x)^2$

(m) $y = \sin^{-1}\left(\frac{1}{x}\right)$ **(n)** $y = \dfrac{1}{\cos^{-1} x}$ **(o)** $y = \frac{1}{x}\sin^{-1} x$ **(p)** $y = x\sin^{-1}\left(\sqrt{x}\right)$

(q) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **(r)** $y = \cos^{-1} x + \cos^{-1}(-x)$ **(s)** $y = \tan x \tan^{-1} x$ **(t)** $y = \tan^{-1}\left(\sqrt{x^2-1}\right)$

3 Find the equation of the tangent to the curve $y = 2\tan^{-1}(2x+1)$ at the point on the curve where $x = -1$.

4 **(a)** If $y = \tan^{-1}(\tan 2x)$, show that $\frac{dy}{dx} = 2$ for all x in its domain.

(b) Explain why the function is undefined at $x = \pm\frac{\pi}{4}$.

(c) Sketch the graph of $y = \tan^{-1}(\tan 2x)$ for $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

(d) Hence show that $\int_{-\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^{-1}(\tan 2x)\,dx = \frac{7\pi^2}{576}$.

5 If $y = \cos^{-1} x + \cos^{-1}(-x)$, find $\frac{dy}{dx}$ and show that $y = \pi$ for all x in the domain.

6 If $y = \sin^{-1} x + \sin^{-1}(-x)$, find $\frac{dy}{dx}$ and show that $y = 0$ for all x in the domain.

7 **(a)** Differentiate $x\tan^{-1} x$. **(b)** Hence find $\int \tan^{-1} x\,dx$.

(c) Use the substitution $u = \log_e x$ to evaluate $\int_1^e \frac{\tan^{-1}(\log_e x)}{x}\,dx$.

8 **(a)** State the domain of $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$.

(b) Find $f'(x)$. **(c)** Find $f(1)$ and $f(-1)$. **(d)** Sketch the graph of $y = f(x)$.

9 Find the coordinates of any stationary points on the curve $y = 2\sin x + 3\cos x$, $0 \le x \le \frac{\pi}{2}$, and determine their nature.

11.8 INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section the derivatives of the inverse trigonometric functions were established. From these, it follows that:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C,\ -a < x < a \qquad \int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + C,\ -a < x < a$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \text{ for all } x$$

Example 25

Find: **(a)** $\int \frac{1}{\sqrt{4 - x^2}}\,dx$ **(b)** $\int \frac{-1}{\sqrt{9 - x^2}}\,dx$ **(c)** $\int \frac{1}{4 + x^2}\,dx$ **(d)** $\int \frac{2}{\sqrt{3 - x^2}}\,dx$

Solution

(a) $\int \frac{1}{\sqrt{4 - x^2}}\,dx = \sin^{-1}\frac{x}{2} + C$ (as $a = 2$)

(b) $\int \frac{-1}{\sqrt{9 - x^2}}\,dx = \cos^{-1}\frac{x}{3} + C$ (as $a = 3$)

(c) $\int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (as $a = 2$)

(d) $\int \frac{2}{\sqrt{3 - x^2}}\,dx = 2\sin^{-1}\frac{x}{\sqrt{3}} + C$ (as $a = \sqrt{3}$)

Sometimes a slight adjustment is necessary to make the coefficient of x^2 equal to 1.

Example 26

Find: **(a)** $\int \frac{1}{\sqrt{1 - 9x^2}}\,dx$ **(b)** $\int \frac{1}{1 + 4x^2}\,dx$ **(c)** $\int \frac{2}{\sqrt{4 - 25x^2}}\,dx$

Solution

(a) $\displaystyle\int\frac{1}{\sqrt{1-9x^2}}dx=\int\frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}dx=\frac{1}{3}\int\frac{1}{\sqrt{\frac{1}{9}-x^2}}dx=\frac{1}{3}\sin^{-1}3x+C\quad\left(\text{as } a=\frac{1}{3}\right)$

(b) $\displaystyle\int\frac{1}{1+4x^2}dx=\frac{1}{4}\int\frac{1}{\frac{1}{4}+x^2}dx=\frac{1}{4}\times 2\tan^{-1}2x+C=\frac{1}{2}\tan^{-1}2x+C\quad\left(\text{as } a=\frac{1}{2}\right)$

(c) $\displaystyle\int\frac{2}{\sqrt{4-25x^2}}dx=2\int\frac{1}{\sqrt{25\left(\frac{4}{25}-x^2\right)}}dx=\frac{2}{5}\int\frac{1}{\sqrt{\frac{4}{25}-x^2}}dx=\frac{2}{5}\sin^{-1}\frac{5x}{2}+C\quad\left(\text{as } a=\frac{2}{5}\right)$

Example 27

Evaluate: **(a)** $\displaystyle\int_0^{2.5}\frac{dx}{\sqrt{25-x^2}}$ **(b)** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\frac{-1}{\sqrt{4-x^2}}dx$ **(c)** $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{1+x^2}$ **(d)** $\displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{dx}{\sqrt{1-4x^2}}$

Solution

(a) $\displaystyle\int_0^{2.5}\frac{dx}{\sqrt{25-x^2}}=\left[\sin^{-1}\frac{x}{5}\right]_0^{2.5}$

$=\sin^{-1}\frac{1}{2}-\sin^{-1}0$

$=\frac{\pi}{6}-0$

$=\frac{\pi}{6}$

(b) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\frac{-1}{\sqrt{4-x^2}}dx=\left[\cos^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}}$

$=\cos^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$=\frac{\pi}{6}-\frac{5\pi}{6}$

$=-\frac{2\pi}{3}$

(c) $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_1^{\sqrt{3}}$

$=\tan^{-1}\sqrt{3}-\tan^{-1}1$

$=\frac{\pi}{3}-\frac{\pi}{4}$

$=\frac{\pi}{12}$

(d) As $\sqrt{1-4x^2}=\sqrt{4\left(\frac{1}{4}-x^2\right)}$

$=2\sqrt{\frac{1}{4}-x^2}$

$\displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{dx}{\sqrt{1-4x^2}}=\frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{dx}{\sqrt{\frac{1}{4}-x^2}}$

$=\frac{1}{2}\left[\sin^{-1}2x\right]_{-\frac{1}{4}}^{\frac{1}{4}}$

$=\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)-\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{12}+\frac{\pi}{12}$

$=\frac{\pi}{6}$

Example 28

Evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

Solution

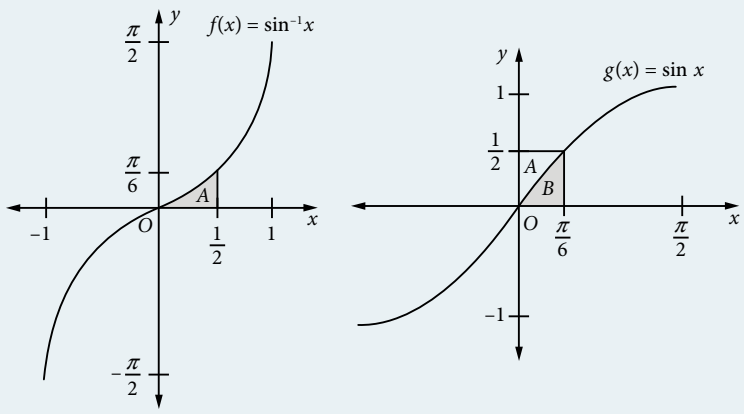
There is no standard derivative that yields $\sin^{-1} x$, so finding its primitive is beyond the scope of this Mathematics Extension 1 course. (However, the primitive can be found using the method of 'integration by parts' as studied in the Mathematics Extension 2 course.)

It is possible to evaluate this integral by an alternative method. You can regard the integral as an area and use your knowledge of inverse functions to determine this area.

The graph of $f(x) = \sin^{-1} x$ is shown in the left diagram below. The shaded area A is the area bounded by the graph, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$. This area is equal to $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

The graph of $g(x) = \sin x$ is shown at right, with the reflected area A now enclosed by this graph, $y = \frac{1}{2}$ and the y-axis. The shaded area B on the diagram can be determined by integration, and so:

$$\begin{aligned}\int_0^{\frac{1}{2}} \sin^{-1} x\,dx &= \text{Area of } A \\ &= \text{Area of rectangle} \\ &\quad \text{(i.e. } A + B) - \text{Area of } B \\ &= \frac{\pi}{6} \times \frac{1}{2} - \int_0^{\frac{\pi}{6}} \sin x\,dx \\ &= \frac{\pi}{12} - \left[-\cos x\right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\end{aligned}$$



EXERCISE 11.8 INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

1 $\int \frac{dx}{\sqrt{16-(x+1)^2}} = \ldots$ (let $u = x + 1$)

A $\sin^{-1} 4(x+1) + C$ **B** $\sin^{-1} \frac{x+1}{4} + C$ **C** $4\sin^{-1} \frac{x+1}{4} + C$ **D** $\frac{1}{4}\sin^{-1} \frac{x+1}{4} + C$

2 Find the following.

(a) $\int \frac{dx}{\sqrt{16-x^2}}$ **(b)** $\int \frac{3}{9+x^2}\,dx$ **(c)** $\int \frac{dx}{\sqrt{1-x^2}}$ **(d)** $\int \frac{-1}{\sqrt{5-x^2}}\,dx$

(e) $\int \frac{dx}{\sqrt{2-x^2}}$ **(f)** $\int \frac{5}{25+x^2}\,dx$ **(g)** $\int \frac{dx}{4+x^2}$ **(h)** $\int \frac{dx}{2+x^2}$

(i) $\int \frac{-1}{\sqrt{6-x^2}}\,dx$ **(j)** $\int \frac{dx}{\sqrt{1-4x^2}}$ **(k)** $\int \frac{dx}{1+9x^2}$ **(l)** $\int \frac{dx}{9+16x^2}$

(m) $\int \frac{dx}{x^2+16}$ **(n)** $\int \frac{dx}{\sqrt{1-3x^2}}$ **(o)** $\int \frac{-1}{\sqrt{16-9x^2}}\,dx$ **(p)** $\int \frac{dx}{\sqrt{5-4x^2}}$

(q) $\int \frac{dx}{4+(x+5)^2}$ (let $u = x + 5$) **(r)** $\int \frac{dx}{\sqrt{2-(x-3)^2}}$ (let $u = x - 3$) **(s)** $\int \frac{dx}{\sqrt{8-4(x+2)^2}}$ (let $u = x + 2$)

3 Evaluate the following.

(a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ **(b)** $\int_0^{\sqrt{3}} \frac{dx}{x^2+9}$ **(c)** $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ **(d)** $\int_{-2}^2 \frac{dx}{4+x^2}$

(e) $\int_{\frac{3}{5}}^4 \frac{dx}{1+x^2}$ **(f)** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ **(g)** $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ **(h)** $\int_0^2 \frac{-1}{\sqrt{16-x^2}}dx$

(i) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$ **(j)** $\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$ **(k)** $\int_0^1 \left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx$ **(l)** $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}}$

(m) $\int_{-4}^4 \frac{dx}{x^2+16}$ **(n)** $\int_0^{\frac{\sqrt{3}}{6}} \frac{-1}{\sqrt{1-9x^2}}dx$ **(o)** $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$ **(p)** $\int_0^{\sqrt{2}} \frac{dx}{x^2+2}$

(q) $\int_{-1}^1 \frac{dx}{\sqrt{2-x^2}}$ **(r)** $\int_{-5}^5 \frac{dx}{\sqrt{100-x^2}}$ **(s)** $\int_0^{\frac{1}{2}} \cos^{-1}x\,dx$ **(t)** $\int_0^1 \tan^{-1}x\,dx$

4 Given that $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$ and $y=1$ at $x=0$, express y as a function of x.

5 Find the area bounded by the curve $y=\frac{1}{x^2+1}$, the x-axis and the ordinates $x=1$ and $x=-1$.

6 On the same axes, sketch the graph of $y=2\sin\frac{\pi x}{4}$ for $0\le x\le 2$ and $x=2\sin\frac{\pi y}{4}$ for $0\le y\le 2$. Find the area of the region enclosed by the curves.

7 The curve $y=\frac{1}{\sqrt{1+x^2}}$ is rotated about the x-axis. Find the volume of the solid enclosed between $x=\frac{1}{\sqrt{3}}$ and $x=\sqrt{3}$.

8 Sketch the curve $y=\tan^{-1}x$ and find the area between this curve, the x-axis and the line $x=\sqrt{3}$.

9 Without evaluating the integral, explain why $\int_{-1}^1 \tan^{-1}x\,dx$ is equal to zero.

10 Without using a calculator, prove that: **(a)** $\int_{\frac{3}{5}}^4 \frac{dx}{1+x^2}=\frac{\pi}{4}$ **(b)** $\int_{-2}^3 \frac{dx}{1+x^2}=\frac{3\pi}{4}$

11 Find the volume of the solid of revolution formed by rotating the curve $y=\sin x$ between $x=0$ and $x=\frac{\pi}{2}$ about the x-axis. Hence find the volume of the solid formed by rotating the curve $y=\sin^{-1}x$ between $y=0$ and $y=\frac{\pi}{2}$ about the y-axis.

12 Differentiate $x\cos^{-1}x-\sqrt{1-x^2}$ and use the result to evaluate $\int_0^1 \cos^{-1}x\,dx$.

13 **(a)** Sketch the graph of $f(x)=1-\cos x$, $0\le x\le\pi$.

(b) Evaluate $\int_0^{\frac{\pi}{2}} f(x)dx$ and indicate the area represented by this integral on the graph of f.

(c) Find the inverse function $f^{-1}(x)$, stating its domain and range.

(d) Evaluate $\int_0^1 f^{-1}(x)dx$ and indicate the area represented by this integral on the graph of f.

14 **(a)** Prove that $\frac{d}{dx}(x\sin^{-1}x)=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$.

(b) Hence show that $\int_0^{\frac{1}{2}} \sin^{-1}x\,dx=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$. (You may use the substitution $u=1-x^2$.)

15 Use the substitution $x=\cos\theta$, where $0\le\theta\le\pi$, to show that $\int \frac{\sqrt{1-x^2}}{x^2}dx=\cos^{-1}x-\frac{\sqrt{1-x^2}}{x}+C$.

16 Find $\int x\sqrt{9-x^2}\,dx$ using the substitution $x=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Confirm your answer by using the alternative substitution $u=9-x^2$.

CHAPTER REVIEW 11

1 By integration, find the volume of the solid of revolution formed from the region bounded by:

(a) the circle $x^2 + y^2 = 1$, rotated about the x-axis

(b) the line $y = x + 3$ between $x = 0$ and $x = 2$, rotated about the x-axis

(c) the parabola $y = x^2 + 3$ between $y = 4$, $y = 12$ and the y-axis, rotated about the y-axis.

2 A speleologist is in a cave with a flat, circular floor. She walks across the diameter of the floor and measures the height of the cave ceiling every 4 metres, as recorded in the following table:

Distance from edge, x m	0	4	8	12	16	20	24
Height, y m	0	1	3	5	4	2.5	0

(a) Use the trapezoidal rule and all the values in the table to find an approximation for the area of the cave's vertical cross-section to the nearest m^2.

(b) Considering the diameter of the cave to be the x-axis (for $-12 \leq x \leq 12$), and considering the vertical at 12 m from the edge to be the y-axis, discuss how you might use the trapezoidal rule to estimate the volume of the cave.

3 Using the substitution $u = \sqrt{x}$, find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

4 **(a)** Use the substitution $u = 1 - x$ to evaluate $\int_0^1 2x\sqrt{1-x}\, dx$.

(b) Use the substitution $u = x - 3$ to evaluate $\int_3^4 x\sqrt{x-3}\, dx$.

5 Using the substitution $u = x^3 + 1$, or otherwise, evaluate $\int_0^1 x^2 e^{x^3+1} dx$.

6 Evaluate $\int_0^{\frac{\pi}{4}} \sin\theta \cos^2\theta\, d\theta$.

7 Use the substitution $u = \log_e x$ to evaluate $\int_e^{e^3} \frac{1}{x(\log_e x)^2} dx$.

8 Use the substitution $u = 9 - x^2$ to evaluate $\int_1^2 \frac{2x}{\sqrt{9-x^2}} dx$.

9 Find: **(a)** $\int_0^{\frac{\pi}{6}} \sin^2 x\, dx$ **(b)** $\int_0^{\frac{\pi}{6}} \cos^2 2x\, dx$

10 Using the substitution $u = 3x^2 + 1$, or otherwise, find $\int x(3x^2+1)^{\frac{3}{4}} dx$.

11 **(a)** Differentiate $e^{2x}(2\sin x - \cos x)$. **(b)** Hence, or otherwise, find $\int e^{2x} \sin x\, dx$.

12 Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

13 Use the substitution $u = 25 - x^2$ to evaluate $\int_2^3 \frac{2x}{\sqrt{25-x^2}} dx$.

14 **(a)** By expanding the left-hand side, show that $\sin(6x + 3x) + \sin(6x - 3x) = 2\sin 6x \cos 3x$.

(b) Hence find $\int \sin 6x \cos 3x\, dx$.

15 Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the x-axis and the line $x = \frac{\pi}{6}$ is rotated about the x-axis.

16 Evaluate $\int_0^{\frac{\pi}{9}} \frac{\sin(3x)}{\cos^2(3x)} dx$

17 **(a)** Starting from the identity $\sin(\theta+2\theta)=\sin\theta\cos 2\theta+\cos\theta\sin 2\theta$ and using the double-angle formulae, prove the identity $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

(b) Hence find $\int\left(3\sin\theta-4\sin^3\theta\right)d\theta$.

18 **(a)** Differentiate $x\cos^{-1}x$. **(b)** Hence find $\int\cos^{-1}x\,dx$. (The substitution $u=1-x^2$ will be helpful.)

19 Differentiate the following. **(a)** $\sin^{-1}x$ **(b)** $\cos^{-1}2x$ **(c)** $\tan^{-1}\frac{x}{2}$ **(d)** $e^x\sin^{-1}x$

(e) $e^{\sin^{-1}x}$ **(f)** $\tan^{-1}\frac{2}{x}$ **(g)** $x\cos^{-1}2x$ **(h)** $\sqrt{\sin^{-1}x}$ **(i)** $\log_e(\tan^{-1}x)^3$

20 Evaluate the following. **(a)** $\int_0^3\frac{dx}{x^2+9}$ **(b)** $\int_0^{\sqrt{5}}\frac{dx}{\sqrt{20-x^2}}$ **(c)** $\int_0^{\frac{\sqrt{6}}{3}}\frac{dx}{9x^2+2}$

21 Find the following integrals. **(a)** $\int\frac{dx}{\sqrt{8-x^2}}$ **(b)** $\int\frac{dx}{64+x^2}$

(c) $\int\frac{4}{4+9x^2}dx$ **(d)** $\int\frac{dx}{\sqrt{4-8x^2}}$ **(e)** $\int\frac{dx}{\sqrt{16-(x+2)^2}}$ (using $u=x+2$)

22 **(a)** Sketch $y=2\sin x$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ and show the area represented by $\int_0^1\sin^{-1}\frac{x}{2}dx$.

(b) Hence evaluate $\int_0^1\sin^{-1}\frac{x}{2}dx$.

23 **(a)** Show that $\frac{x^3+x+2}{1+x^2}=x+\frac{2}{1+x^2}$. **(b)** Hence find $\int\frac{x^3+x+2}{1+x^2}dx$.

24 **(a)** Sketch both $y=\sin^{-1}x$ and $y=\cos^{-1}x$ on the same number plane.

Hence determine the domain of the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

(b) Find the intercepts of the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ with the coordinate axes.

(c) Show that the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ is increasing for all x in its domain.

(d) Sketch the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

25 Find the following integrals. These are a mix of different integral types that you have seen in this Mathematics Extension 1 course.

(a) $\int\frac{dx}{1+x}$ **(b)** $\int\frac{dx}{1+x^2}$ **(c)** $\int\frac{x}{1+x^2}dx$ **(d)** $\int\frac{1+x^2}{x}dx$

(e) $\int\frac{x^2}{1+x^2}dx$ (Hint: $\frac{x^2}{1+x^2}=1-\frac{1}{1+x^2}$) **(f)** $\int\frac{x^3}{1+x^2}dx$ **(g)** $\int\frac{dx}{(1+x)^2}$

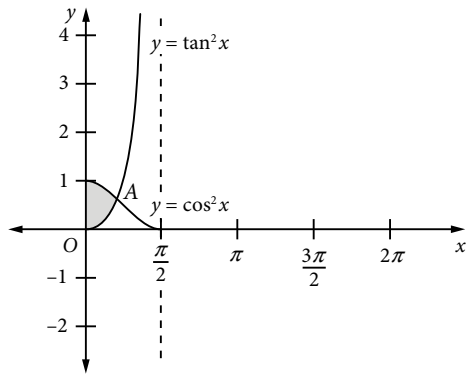
(h) $\int\frac{dx}{\sqrt{1+x}}$ **(i)** $\int\frac{dx}{\sqrt{1-x^2}}$ **(j)** $\int\frac{x}{\sqrt{1-x^2}}dx$ (using $u=1-x^2$)

(k) $\int\frac{e^x}{2+e^{2x}}dx$ (using $u=e^x$) **(l)** $\int\frac{\tan^{-1}3x}{1+9x^2}dx$ (using $u=\tan^{-1}3x$)

26 **(a)** Write the value of $\cos^{-1}x+\sin^{-1}x$ for $-1\le x\le 1$.

(b) Use the substitution $u=\sin^{-1}x$ to show that: $\int_0^{\frac{1}{\sqrt{2}}}\left(\cos^{-1}x\right)\sqrt{\frac{\sin^{-1}x}{1-x^2}}\,dx=\frac{7\pi^{\frac{5}{2}}}{240}$

27 The graph shows the curves $y = \cos^2 x$ and $y = \tan^2 x$ for $0 \leq x \leq \frac{\pi}{2}$.



(a) Show that the point of intersection A has the x-coordinate $\cos^{-1}\left(\sqrt{\frac{-1+\sqrt{5}}{2}}\right)$.

(b) Find the shaded area correct to one decimal place.

CHAPTER 12
Differential equations

Bernie Fitzpatrick, the original author of this series, once wrote: 'Mathematics is about doing, not about reading'. This chapter brings together many skills learnt across the Mathematics Advanced and Extension 1 courses and asks you to apply them in practical situations.

Differential equations involve derivatives, so to be able to solve them you must be proficient at integration. These integrations often involve the exponential and logarithmic functions, so you must know the rules needed to manipulate these functions. You must be confident with a high level of algebraic manipulation, sometimes involving several levels of algebraic fractions. When integrating the trigonometric functions you need to be able to apply the appropriate Pythagorean identities, as well as the appropriate double angle formulae. If the integration required is beyond this course, then appropriate guidance is given in the question.

We hope you enjoy the challenges involved.

12.1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

The study of differential equations began in the late 1600s with Sir Isaac Newton's investigation of the orbits of the planets about the Sun. Newton referred to these equations as 'fluxional equations'. The term **differential equation** was suggested by Newton's contemporary, Gottfried Leibniz, who did much of the early work on them.

A differential equation is an equation that relates some unknown differentiable function to one or more of its derivatives.

For example, the general form of the first-order differential equation $y'(t) = f(t, y(t))$ expresses the rate of change of a quantity $y(t)$ in terms of two variables: the time t and the value of the quantity $y(t)$ itself.

Differential equations are a powerful way to represent, understand and predict the behaviour of variable quantities, including systems that change with time.

Whereas the solution of an algebraic equation such as $x + 1 = 0$ is a number, the solution of a differential equation is a function. More specifically, the solution of a differential equation will be a differentiable function $y = g(x)$ if the differential equation is true when y and its derivatives are replaced with $g(x)$ and its derivatives.

A solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is any differentiable function $y = g(x)$ with a derivative $\frac{dy}{dx} = g'(x)$ so that $g'(x) = f(x, g(x))$ for all $x \in (a, b)$.

In other words, substituting the solution $y = g(x)$ into the differential equation will reduce $\frac{dy}{dx} = f(x, y)$ to an identity in the independent variable x, for all values of this variable in some open interval of the x-axis. Therefore:

To verify that a function $y = g(x)$ is a solution of a differential equation, you can substitute the function and its derivative(s) into both sides of the differential equation and check that both sides are identically equal.

Verification of a solution to a differential equation

Example 1

Verify by differentiation that $y = 3e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = -y$.

Solution

Calculate the LHS of the equation: $\text{LHS} = \frac{dy}{dx}$

$= 3\left(-e^{-x}\right)$

$= -3e^{-x}$

Calculate the RHS of the equation: $\text{RHS} = -y$

$= -3e^{-x}$

The LHS of the equation is identically equal to the RHS of the equation for all relevant values of the independent variable.

Graphing particular members of a general solution

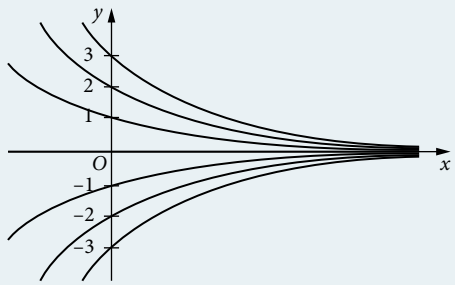
Example 2

Graph the following members of the one-parameter general solution $y = Ae^{-x}$ with $A \in \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution

The graph is drawn for $A = -3$, i.e., $y = -3e^{-x}$.

Next, separate graphs are drawn on the same axes for the other given values of A: $y = -2e^{-x}$, $y = -3e^{-x}$, etc. This should give the set of graphs shown.



A **particular solution** (or **solution curve**) of a differential equation is a unique function that is found by giving specific values to the parameters in the general solution. The parameters in the solution are chosen so that the particular solution satisfies one or more extra requirements called **initial conditions** or **initial values**. Initial conditions are also sometimes called boundary values.

Every particular solution of a first-order differential equation is the unique solution of an appropriate **initial value problem**.

An initial value problem of a first-order differential equation:

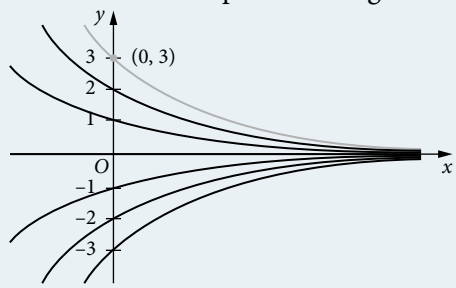
$$\begin{cases} y' = f(t, y(t)) \\ y(a) = y_a \\ t \in [a, b] \end{cases}$$

consists of the differential equation together with its initial condition, requiring you to determine a particular solution $y(t)$ over a specific interval of the independent variable $a \leq t \leq b$.

Finding the particular solution to satisfy an initial condition

Example 3

Find the particular member of the general solution $y = Ae^{-x}$, where A is a real number that passes through the point with coordinates (0, 3).



Solution

$y = Ae^{-x}$

To satisfy the relevant initial condition, substitute the given values.

$x = 0, y = 3$: $3 = Ae^0$

$\therefore A = 3$

The particular solution curve that passes through the point (0, 3) is identified.

Finding missing parameter(s) in a trial solution

A trial solution to a differential equation is a general solution with unspecified parameters that is tested to see if it satisfies an initial condition.

Example 4

Verify by differentiation that $y = ae^{\sin x}$ is a solution of the differential equation $\frac{dy}{dx} = y\cos x$, $y(0) = 2$, for a suitable choice of the parameter a.

Solution

The required derivatives of the trial solution are calculated to prove that LHS = RHS:

$$\text{LHS} = \frac{dy}{dx}$$
$$= \frac{d}{dx}\left(ae^{\sin x}\right)$$
$$= \cos x\left(ae^{\sin x}\right)$$
$$= y\cos x$$
$$= \text{RHS}$$

Check if the initial condition is satisfied, $y(0) = 2$: $y(0) = ae^{\sin 0}$

$2 = a$

$\therefore$ $y = 2e^{\sin 0}$ is a particular solution.

Example 5

Verify by differentiation that $x = Ae^{st}$ is a solution of $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$, $x(0) = 1$ and $x'(0) = 2$, for suitable values of s.

Solution

Calculate any required derivatives of the trial solution $x = Ae^{st}$:

$$\frac{dx}{dt} = sAe^{st}$$
$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d}{dt}\left(sAe^{st}\right)$$
$$= s^2Ae^{st}$$

Substitute into the LHS of the equation given, $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$:

$$\begin{aligned}\text{LHS} &= s^2Ae^{st} - 5sAe^{st} + 6Ae^{st} \\ &= Ae^{st}\left(s^2 - 5s + 6\right) \\ &= 0\end{aligned}$$

Solve for possible values of the parameter s: $Ae^{st}(s^2 - 5s + 6) = 0$

As $Ae^{st} \neq 0$, $t \in R$, so: $(s^2 - 5s + 6) = 0$

$$(s-2)(s-3) = 0$$

$$\therefore\ s = 2 \text{ or } s = 3$$

The solution could be $x = Ae^{2t}$ or $x = Ae^{3t}$.

Check initial conditions: $x(0) = 1$, $x = Ae^{st}$ $\therefore\ x(0) = Ae^0 = 1$ $\therefore\ A = 1$

$x'(0) = 2$, $x' = sAe^{st} = se^{st}$ $\therefore\ se^0 = 2$ $\therefore\ s = 2$

Hence $s = 2$ is the only solution that satisfies both the differential equation and the initial conditions.

$\therefore\ x(t) = e^{2t}$ is a particular solution.

The number of parameters in the general solution of a differential equation, which determines the number of initial conditions required to fix these parameters, depends on the **order** and the **degree** of the differential equation.

The order of a differential equation is equal to the highest order derivative of the dependent variable.
The degree of a differential equation is the highest power of that highest order derivative.

In the nth-order differential equation $\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, ..., \frac{d^{n-1}y}{dx^{n-1}}\right)$, the dependent variable (y) always appears in the numerator of any derivatives and the independent variable x appears in the denominator.

A **first-order first-degree differential equation** for the unknown dependent variable y is an equation that involves only the first derivative of y. All first-order first-degree differential equations for y can be expressed in the form $\frac{dy}{dx} = f(x, y)$ for a suitable choice of the function $f(x, y)$ of the independent variable x and dependent variable y.
The particular solution of a first-order first-degree differential equation requires a single **initial condition** $(x, y) = (x_0, y_0)$.

Classifying differential equations

Example 6

Classify the following differential equations according to their order and degree.

(a) $\frac{d\theta}{dt} = k(1 + 0.2\cos\theta)^2$ **(b)** $\left(\frac{dw}{dz}\right)^2 = 4\cos^2 z$ **(c)** $\frac{d^2x}{dt^2} - c\frac{dx}{dt} + kx = F$

Solution

(a) This equation defines a first-order first-degree differential equation because it involves only a first-order derivative to a power of one.

Variable θ is in the numerator of $\frac{d\theta}{dt}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

(b) This equation defines a first-order second-degree differential equation because it involves only a first-order derivative to a power of two.
Variable w is in the numerator of $\frac{dw}{dz}$, so it is the dependent variable, while z is in the denominator, so it is the independent variable.

(c) This equation defines a second-order first-degree differential equation because it involves a second-order derivative $\frac{d^2x}{dt^2}$ of the dependent variable to the power of one.
Variable x is in the numerator of $\frac{d^2x}{dt^2}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

Note: In each example considered so far, the solution is given in explicit form, such as $y = g(x)$.

EXERCISE 12.1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

1 Verify by differentiation that the function $y = x^n$ is a solution of the differential equation $y' - \frac{n}{x}y = 0$.

2 Use technology to graph the particular solutions that correspond to the indicated values of C.

	General solution	Differential equation	C-values
(a)	$y = Cx^3$	$x\frac{dy}{dx} - 3y = 0$	$\pm 1, \pm 2$
(b)	$x^2 - y^2 = C$	$y\frac{dy}{dx} = x$	$\pm 1, \pm 2$
(c)	$x^2 + 4y^2 = C$	$4y\frac{dy}{dx} + x = 0$	1, 2, 3, 4
(d)	$xy = C$	$x\frac{dy}{dx} + y = 0$	$\pm 1, \pm 2$

3 Verify the general solution and then specify any parameters in this solution and state the required particular solution of the initial value problem.

(a)

General solution	Differential equation	Initial condition
$y = Ae^{-2x} + 10$	$y' = 2(10 - y)$	$y(0) = 3$

(b)

General solution	Differential equation	Initial condition
$y = Ae^{-x} + 5$	$y' = 5 - y$	$y(0) = 10$

(c)

General solution	Differential equation	Initial condition
$y = \frac{e^x}{A + e^x}$	$y' = y(1 - y)$	$y(0) = 2$

(d)

General solution	Differential equation	Initial condition
$y = \frac{5e^{10x}}{A + e^{10x}}$	$y' = 2y(5 - y)$	$y(0) = 1$

(e)

General solution	Differential equation	Initial condition
$y = Ae^{2x} - \frac{x}{2} - \frac{1}{4}$	$y' = 2y + x$	$y(0) = 1$

4 Verify the general solution and then specify any parameters in this solution to find the required particular solution.

(a)

Trial solution	Differential equation	Initial condition
$y = \frac{-x}{Ax+1}$	$x^2y' = -y^2$	$y(1) = -\frac{1}{3}$

(b)

Trial solution	Differential equation	Initial condition
$y = \frac{1}{x-A}$	$y' = -y^2$	$y(1) = -1$

(c)

Trial solution	Differential equation	Initial condition
$y = Ae^{-2x} + \frac{1}{2}$	$y' = -2y + 1$	$y(0) = 1$

(d)

Trial solution	Differential equation	Initial condition
$y = e^{-x}(ax + b)$	$y'' + 2y' + y = 0$	$y(0) = 2, y'(0) = 1$

(e)

Trial solution	Differential equation	Initial condition
$y = a\sin 2x + b\cos 2x$	$\frac{d^2y}{dx^2} + 4y = 0$	$y(0) = 1, y'(0) = 2$

(f)

Trial solution	Differential equation	Initial condition
$y = ae^{-x} + be^{3x}$	$y'' - 2y' - 3y = 0$	$y(0) = 1, y'(0) = 1$

5 Classify each of the following differential equations in terms of its order, degree, dependent variable and independent variable.

(a) $(y')^2 = x^2$ **(b)** $x' = x\sin(t)$ **(c)** $\frac{d^2x}{dt^2} + kx = 0$ **(d)** $\frac{dy}{dx} = y + y^2$

6 Given that $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2y$, find the possible value(s) of k.

7 Which one of the following differential equations is satisfied by $y = \cos 2x$?

A $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 4\cos 2x$ **B** $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 4\cos 2x$

C $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 4\sin 2x$ **D** $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 4\sin 2x$

8 Verify by differentiation that the given function is a solution of the corresponding differential equation.

(a) $y = e^{x^n}$ is a solution of $y' - nx^{n-1}y = 0$ **(b)** $y = x - x^{-1}$ is a solution of $xy' + y = 2x$

(c) $y = \frac{x}{1+x}$ is a solution of $y' - (1-y)^2 = 0$ **(d)** $y = \frac{e^{rx}}{1+e^{rx}}$ is a solution of $y' = r(1-y)y$

(e) $y = \frac{1}{4-x^2}$ is a solution of $y' = 2xy^2$

9 The amount $m(t)$ of medication remaining in the bloodstream t hours after swallowing a pill can be modelled by the differential equation $\frac{dm}{dt} = -3m + 4e^{-2t}$.

The first term on the RHS represents the rate at which the medication is absorbed from the blood into the body and the second term represents the rate at which the medicine enters the bloodstream. (This is exponential because it is rapid at first, as most of the pill dissolves, then later becomes slower when only a small amount of the pill remains.)

(a) Verify by differentiation that $m(t) = 4(e^{-2t} - e^{-3t})$ is a solution of $\frac{dm}{dt} = -3m + 4e^{-2t}$.

(b) What is the initial amount of medication in the bloodstream?

(c) When is the amount of medication in the bloodstream at it greatest?

(d) What is the long-term amount of medication in the bloodstream?

10 The population of fish in a lake is initially 10 000. The population would increase at a rate of 20% per year, except that there is fishing quota of k fish per year taken from the lake. The population P after t years is modelled by the solution of the differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P - k \text{ with } P = 10\,000 \text{ when } t = 0.$$

Here, k is the constant number of fish removed from the lake each year due to fishing (the fishing quota).

(a) If the fishing quota is set at $k = 1000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000(1 + e^{\frac{t}{5}})$.

(b) Alternatively, if the fishing quota is set at $k = 3000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000(3 - e^{\frac{t}{5}})$.

More generally, you can assume an arbitrary but fixed fishing quota of k fish per year.

(c) Verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5(k + (2000 - k)e^{\frac{t}{5}})$.

(d) Hence choose a fishing quota of k fish per year to maintain the fish population at its initial value of 10 000.

12.2 DIRECTION FIELDS

Qualitative (or graphical) methods of solution

Qualitative methods are a set of graphical methods to describe the general behaviour of the solution to a differential equation without solving the equation.

Recall that $\frac{dy}{dx}$ is the slope of the curve at any point (x, y). A differential equation, such as $\frac{dy}{dx} = f(x, y)$, can be thought of as a definition of the values of the slope of the tangent to the solution curve for possible values of x and y. This enables us to sketch the graphical features of the solution. The graph showing the gradient at different points is called the **direction field** or the **slope field**.

Direction field construction on a rectangular grid

This method involves the following two steps:

1. Evaluate the derivative for a carefully selected set of points (x, y).
2. At each point (x, y), draw a short line segment of slope $\frac{dy}{dx}$.

Example 7

Construct the slope field of $\frac{dy}{dx} = xy$ on the grid: $(0,0)$ $(0,1)$ $(0,2)$ $(0,3)$
$(1,0)$ $(1,1)$ $(1,2)$ $(1,3)$
$(2,0)$ $(2,1)$ $(2,2)$ $(2,3)$
$(3,0)$ $(3,1)$ $(3,2)$ $(3,3)$

Solution

$\frac{dy}{dx} = f(x, y) = xy$ is evaluated for each point using integer values for x and y.

For example, at the point (0, 0), $\frac{dy}{dx} = xy = 0 \times 0 = 0$.

Therefore, if the curve goes through (0, 0), its gradient at that point will be 0.

At the point (2, 3), $\frac{dy}{dx} = xy = 2 \times 3 = 6$.

Therefore, if the curve goes through (2, 3), its gradient at that point will be 6.

All the gradients are calculated.

$f(0,0) = 0$ $f(0,1) = 0$ $f(0,2) = 0$ $f(0,3) = 0$

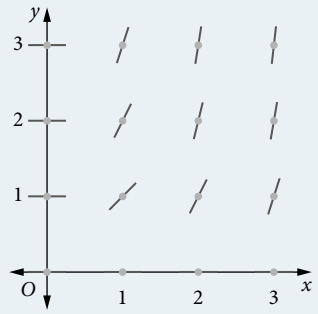
$f(1,0) = 0$ $f(1,1) = 1$ $f(1,2) = 2$ $f(1,3) = 3$

$f(2,0) = 0$ $f(2,1) = 2$ $f(2,2) = 4$ $f(2,3) = 6$

$f(3,0) = 0$ $f(3,1) = 3$ $f(3,2) = 6$ $f(3,3) = 9$

At each such point (x, y) on the grid, tangent segments of slope $\frac{dy}{dx} = f(x, y)$ are drawn using rise over run.

Having constructed a direction field, the short sloping lines can be used as a guide to draw smooth curves with the same gradients. These curves represent possible graphs generated by the differential equation. In some cases, more slopes may need to be drawn.

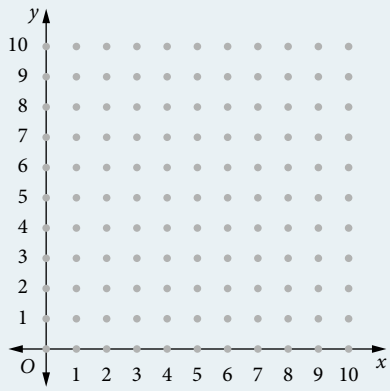


Example 8

Construct the slope field of $\frac{dy}{dx} = -2(y - 5)$ on a suitable grid for $0 \leq x \leq 10$ and $0 \leq y \leq 10$.

Solution

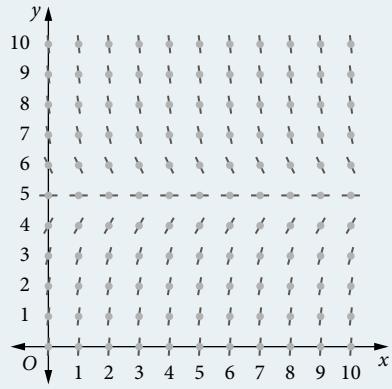
Step 1: A grid is constructed to cover the given intervals:



Step 2: At each such point (x, y) on the grid, tangent segments of the slope function are drawn. As the derivative is a function of y only, then given any specific value of y, it will be the same for all values of x. This means it only needs to be calculated for each value of y.

For $f(x, y) = -2(y - 5)$, $f(x, 0) = 10$, $f(x, 1) = 8$, $f(x, 2) = 6$, $f(x, 3) = 4$, $f(x, 4) = 2$, $f(x, 5) = 0$, $f(x, 6) = -2$, $f(x, 7) = -4$, $f(x, 8) = -6$, $f(x, 9) = -8$, $f(x, 10) = -10$.

This information is shown in the diagram at right.



Example 9

(a) Construct the slope field of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ for $-3 \le x \le 3$ and $-3 \le y \le 3$.

(b) Use the slope field to draw possible solutions to $\frac{dy}{dx} = -\frac{x}{y}$.

(c) Draw the specific solution if the curve passes through the point: **(i)** $(0, 2)$ **(ii)** $(2, -2)$.

(d) Suggest a possible equation of the general curve and test your answer by differentiation.

Solution

(a) **Step 1**: The differential equation is used to find the gradient at each point.

$(-3, -3)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-3} = -1$

$(-3, -2)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-2} = -1.5$

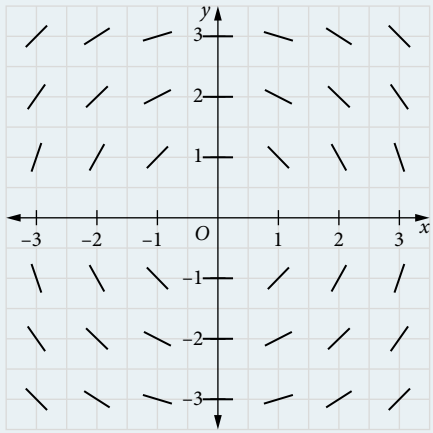
$(-3, -1)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-1} = -3$

$(-3, 0)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{0}$. This is undefined, but it may mean that if the curve goes through $(-3, 0)$, it will be vertical.

Continuing similarly for positive x values gives the following gradient values:

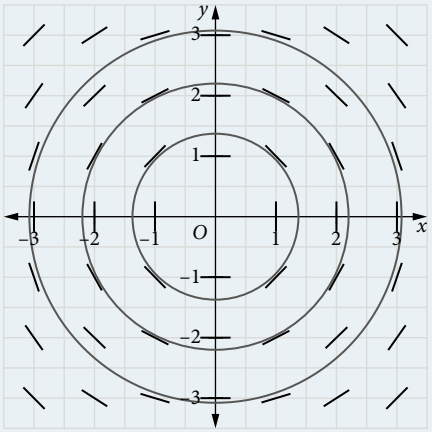
		x values						
		−3	−2	−1	0	1	2	3
y values	−3	−1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
	−2	−1.5	−1	−0.5	0	0.5	1	1.5
	−1	−3	−2	−1	0	1	2	3
	0	–	–	–	–	–	–	–
	1	3	2	1	0	−1	−2	−3
	2	1.5	1	0.5	0	−0.5	−1	−1.5
	3	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	−1

Step 2: Coordinate axes can be used to represent the slope using a short line at each point, estimating the slope using rise over run.

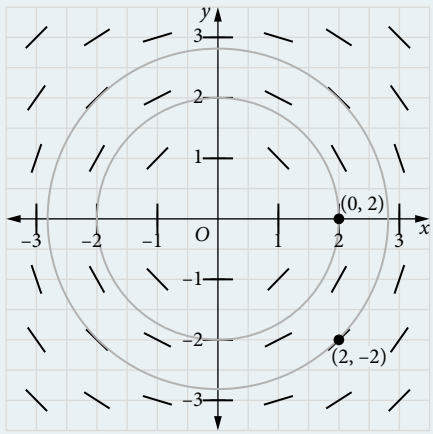


(b) The curve may be circular. Some curves can be drawn using the slopes as a guide, possibly with a compass. This diagram reinforces that the undefined slopes are at points where the curves are vertical, except for the point (0, 0). These slopes have been added to the diagram.

Circles seem to fit very well. At this point there are an infinite number of solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.



(c) A circle is drawn through each given point, as shown.



(d) The equation of a circle is $x^2 + y^2 = r^2$.

(i) In this case, $r = 2$, so the equation is $x^2 + y^2 = 4$.

If a function is required, $y = \sqrt{4 - x^2}$. The positive square root is taken because the curve goes through (0, 2).

(ii) Substitute $x = 2$, $y = -2$ in $x^2 + y^2 = r^2$.

$2^2 + (-2)^2 = r^2 \therefore r = \sqrt{8}$, so the equation is $x^2 + y^2 = 8$.

If a function is required, $y = -\sqrt{8 - x^2}$. The negative square root is taken because the curve goes through (2, −2).

The general result is tested by differentiation.

Method 1

Need to differentiate each term of $x^2 + y^2 = r^2$ with respect to x:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(r^2\right)$$

Using the chain rule,

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx} = \frac{d}{dx}\left(r^2\right)$$

Remembering that r^2 is a constant:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Method 2

Need to differentiate both sides of $y = \sqrt{r^2 - x^2}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{r^2 - x^2}\right)$$

Let $u = r^2 - x^2$, so $\frac{du}{dx} = -2x$

$$\frac{dy}{dx} = \frac{d}{du}\left(\sqrt{u}\right) \times \frac{du}{dx}$$

$$= \frac{d}{du}\left(u^{\frac{1}{2}}\right) \times \frac{du}{dx}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \times (-2x)$$

$$= -\frac{x}{\sqrt{u}}$$

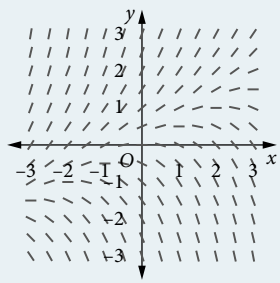
However, $y = \sqrt{r^2 - x^2} = \sqrt{u}$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

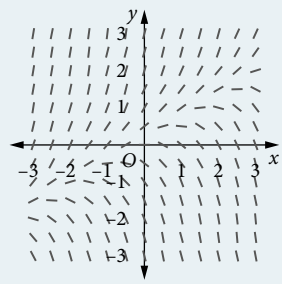
Example 10

Which of the following direction fields could have the differential equation $\frac{dy}{dx} = x + ky$ as a solution if $k > 0$?

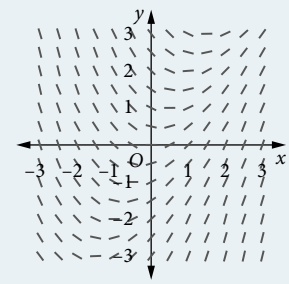
A



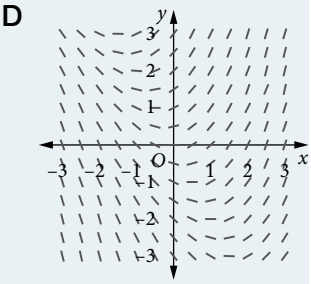
B



C



D



Solution

The behaviour of the derivative is considered for different values of x and y.

At $x = 0$ (the y-axis), $\frac{dy}{dx} = ky$, so $\frac{dy}{dx}$ will be positive where y is positive and negative where y is negative.
This eliminates C.

If $x > 0$ and $y > 0$, $\frac{dy}{dx}$ will be positive and increases as x increases. $\frac{dy}{dx}$ also increases as y increases.
This eliminates A and B.

Further analysis will show that D is consistent with $\frac{dy}{dx} = x + ky$, $k > 0$.

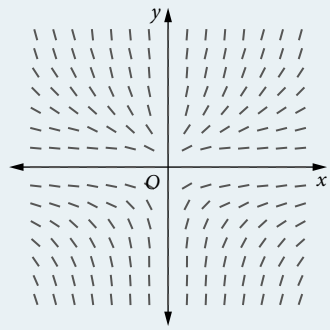
MAKING CONNECTIONS

Direction fields

Use technology to construct the direction field of a differential equation.

Example 11

The slope field for a differential equation is shown:



Which of the following could be the form of the differential equation represented by this slope field, if $k > 0$?

A $\frac{dy}{dx} = kxy^2$ **B** $\frac{dy}{dx} = kxy^3$

C $\frac{dy}{dx} = \frac{ky^2}{x}$ **D** $\frac{dy}{dx} = \frac{ky^3}{x}$

Solution

Consider how the slope lines are changing and whether this is consistent with each proposed option.

By examining option A, if $x > 0$ and $y > 0$, the slope is positive and if x is constant (any horizontal row), so the slope should increase as x increases. However, if a horizontal row is examined, the slope decreases as x increases. This also rules out option B.

Alternatively, the gradient in the slope field is undefined where $x = 0$, yet the derivatives in options A and B are defined and zero where $x = 0$.

For options C and D, if $x > 0$ and $y > 0$ the slope is expected to be positive. This is consistent with the slope field. The gradients are undefined where $x = 0$, which is also consistent with the slope field.

The difference between these two options is the power of y.

If $x > 0$ and $y > 0$, the slope should increase as y increases, which can be seen by examining any vertical column.

If $x > 0$ and $y < 0$, the slope in C will be positive and increase as y increases, but the slope in D will be negative and decrease (become more negative, i.e. steeper) as y increases. By looking at the fourth quadrant, it can be seen that option C correctly predicts the behaviour of the slope.

EXERCISE 12.2 DIRECTION FIELDS

1 Consider the differential equation $\frac{dy}{dx} = y - \frac{x}{2}$.

(a) Find the gradient of a solution curve at the point $(3, -2)$, assuming the curve goes through this point.

(b) Use integer values of x and y from -3 to 3 to construct a direction field for the differential equation $\frac{dy}{dx} = y - \frac{x}{2}$.

2 Construct direction fields for the following differential equations for $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. Use integer values of x and y.

(a) $\frac{dy}{dx} = x - y$ **(b)** $\frac{dy}{dx} = \frac{2}{x+y}$ **(c)** $\frac{dy}{dx} = \frac{x^2+y^2}{10}$ **(d)** $\frac{dy}{dx} = \frac{x+1}{y-1}$

(e) $(x+1)\frac{dy}{dx} = xy$ **(f)** $x + y + \frac{dy}{dx} = 0$ **(g)** $\frac{dy}{dx} = \sqrt{3-y}$ **(h)** $\frac{dy}{dx} = (x-1)(x+2)$

(i) $\frac{dy}{dx} = y^2 - 5$ **(j)** $\frac{dy}{dx} = \frac{x}{2} + 1$

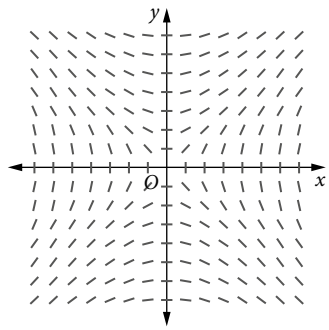
3 **(a)** Calculate the value of the derivative for each of the following values of y, and hence sketch the slope field of the differential equation $\frac{dy}{dx} = 1 + y^2$, using values of x from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, increasing by $\frac{\pi}{6}$.

(i) $y = 0$ **(ii)** $y = \pm 1$ **(iii)** $y = \pm 2$ **(iv)** $y = \pm 3$

(b) Verify by substitution that $y = \tan x$ is a solution of $\frac{dy}{dx} = 1 + y^2$ and sketch that particular solution of the curve on the slope field.

(c) State at least one point on the slope diagram which will not be on any solution of $\frac{dy}{dx} = 1 + y^2$.

4 The graph shown is the slope field of a first-order differential equation.

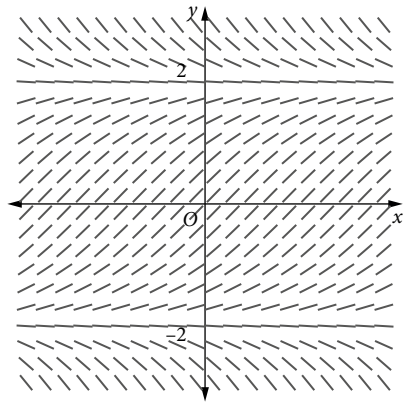


This differential equation could be:

A $y' = \frac{y}{x}$ **B** $y' = \frac{x}{y}$

C $y' = -\frac{y}{x}$ **D** $y' = -\frac{x}{y}$

5 The slope field of $\frac{dy}{dx} = f(y)$ is shown.

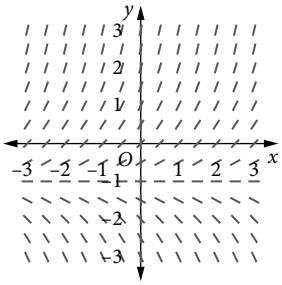


For each of the following, sketch a possible curve which is a solution to this differential equation, containing a point for which:

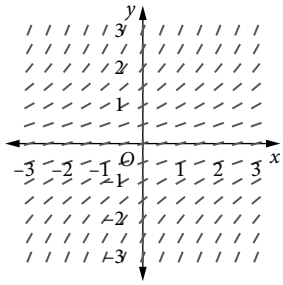
(a) $y > 2$
(b) $-2 < y < 2$
(c) $y < -2$

6 Which of the following slope fields does not represent a differential equation of the form $\frac{dy}{dx} = f(y)$?

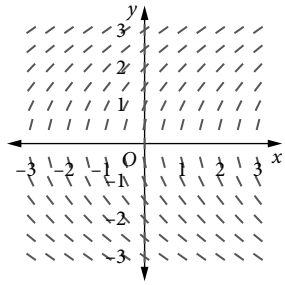
A



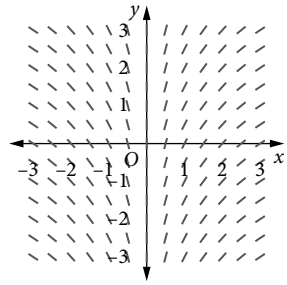
B



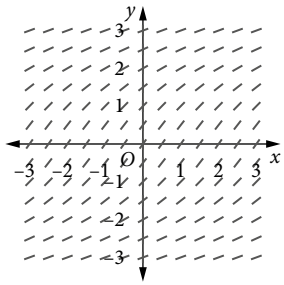
C

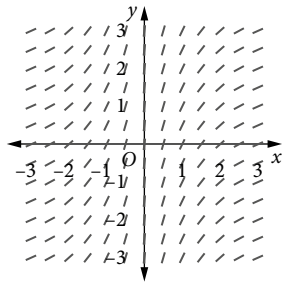


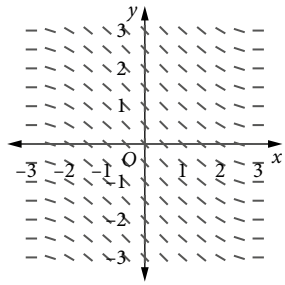
D

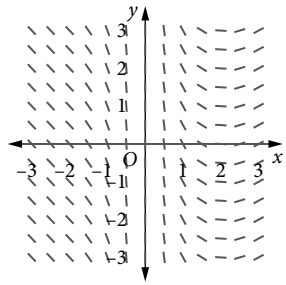


7 Which of the following slope fields represents a differential equation of the form $\frac{dy}{dx} = f(y)$?

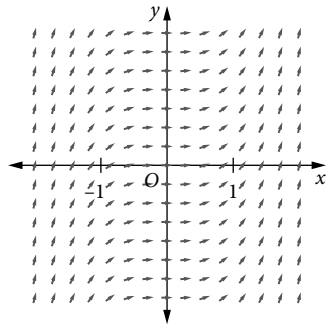
A 

B 

C 

D 

8 A first-order differential equation has a slope field as shown.



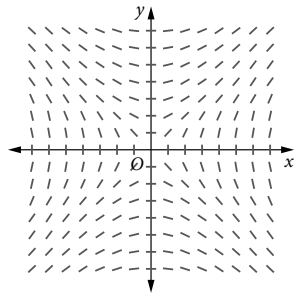
(a) Sketch three possible solutions for this differential equation.

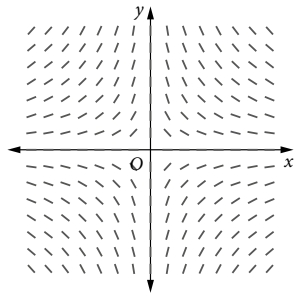
(b) Which of the following first-order differential equations is consistent with the slope field shown?

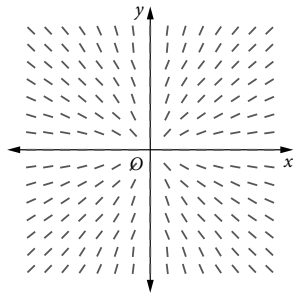
A $\frac{dy}{dx} = xy$ B $\frac{dy}{dx} = x^2$

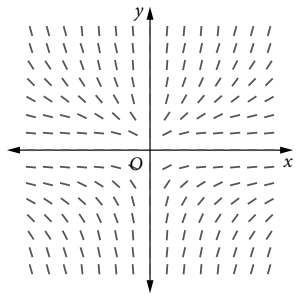
C $\frac{dy}{dx} = x^3$ D $\frac{dy}{dx} = x + y$

9 The slope field of $xy' - y = 0$ could be:

A 

B 

C 

D 

10 **(a)** Construct the direction field for the differential equation $\frac{dy}{dx} = x + y$, for $-3 \le x \le 3$ and $-3 \le y \le 3$, with x and y increasing in steps of 0.5.

(b) Draw some possible solutions to the differential equation $\frac{dy}{dx} = x + y$, including one that is a straight line, and including one that touches but does not cross the x-axis.

(c) Write the equation of the possible straight line solution.

(d) Verify whether the straight line represents a solution to the differential equation.

12.3 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

The previous section investigated techniques used to develop a graphical and/or numerical representation of the solution to a differential equation. However, the most convenient solution (when it is available) is an expression for the dependent variable as an explicit function of the independent variable or as an implicit relation between the dependent and independent variables. This section covers a number of powerful techniques capable of determining these analytical solutions.

Solving $\frac{dy}{dx} = f(x)$ given $y(a) = y_a$

Begin with the directly integrable case $\frac{dy}{dx} = f(x)$ where y is an unknown function of x and $f(x)$ is a given function of x. Wherever $f(x)$ is a continuous function over an interval, $y(x)$ can be determined by using an appropriate integration technique.

Solving a directly integrable first-order differential equation

Example 12

A one-parameter family of curves $f(x, y) = c$ has the property that the gradient of any member of the family at a point is three times the square of the x-coordinate at the point.

(a) What is the equation of the family of functions?

(b) Plot a few members of the family.

(c) What is the equation of the particular member of the family that passes through the point (1, 6)?

Solution

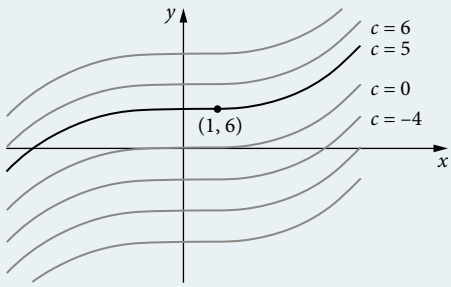
(a) Possible gradient function: $\frac{dy}{dx} = 3x^2$.

Both sides of the model are integrated with respect to the independent variable: $\int \frac{dy}{dx}dx = \int 3x^2 dx$

Equation of the family of functions: $y = x^3 + c$

Note: By the chain rule, $\int \frac{dy}{dx}dx = \int dy$. A constant of integration is required when integrating.

(b)



(c) The coordinates of the given point (1, 6) are substituted into the equation of the family of functions:

$6 = 1^3 + c$

$c = 5$

Solution: $y = x^3 + 5$

Example 13

The slope of the tangent line to an unknown curve is $\frac{1}{1+x^2}$. The y-intercept of the curve is (0, 1). Find the equation of the curve.

Solution

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\int \frac{dy}{dx}dx = \int \frac{dx}{1+x^2}$$

$$y = \tan^{-1} x + c$$

$$y(1) = 0 : 1 = \tan^{-1} 0 + c$$

$$c = 1$$

$$\therefore y = \tan^{-1} x + 1$$

Finding the particular solution of a directly integrable problem $\frac{dy}{dx} = f(x), y(a) = y_a$ involves the following two-step procedure.

1 Integrate with respect to the independent variable.

That is, $y = F(x) + c$ where $F(x) = \int f(x)dx$ and c is the constant of integration.

2 Use the initial condition to solve the constant of integration.

That is, $(x, y) = (a, y_a) \therefore y_a = F(a) + c$.

Therefore, $y = F(x) + y_a - F(a)$.

Rearranging the previous equation, the final result is $y(x) = y_a + F(x) - F(a)$.

Example 14

Given $y' = \frac{1}{\sqrt{1-x^2}}$ and $y(0) = 2$, find the solution to the differential equation.

Solution

$y' = \frac{1}{\sqrt{1-x^2}}$

$y = \int \frac{dx}{\sqrt{1-x^2}}$ $\qquad$ $y(0) = 2 : 2 = \sin^{-1} 0 + C$

$\qquad\qquad\qquad\quad C = 2$

$y = \sin^{-1} x + C$ $\qquad$ $\therefore y = \sin^{-1} x + 2$

Example 15

Find the solution to the following differential equations:

(a) $\frac{dy}{dx} = \sin x$, given $y(0) = 1$ $\qquad$ **(b)** $\frac{dy}{dx} = 2 - \cos x$, given $y(0) = 2$.

(c) Hence, use parts **(a)** and **(b)** to solve the second-order differential equation $\frac{d^2y}{dx^2} = \sin x$, given that $\frac{dy}{dx} = 1$ and $y = 2$ where $x = 0$.

Solution

(a) $\frac{dy}{dx} = \sin x$

$y = \int \sin x \, dx$

$y = -\cos x + C$

$y(0) = 1$:

$1 = -1 + C$

$C = 2$

$\therefore \quad y = 2 - \cos x$

(b) $\frac{dy}{dx} = 2 - \cos x$

$y = \int (2 - \cos x)dx$

$y = 2x - \sin x + c$

$y(0) = 2$:

$2 = 2 \times 0 - \sin 0 + c$

$c = 2$

$\therefore \quad y = 2x - \sin x + 2$

(c) Now: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

If $\frac{d^2y}{dx^2} = \sin x$ then the solution of this differential equation will give an equation of the form $\frac{dy}{dx} = f(x)$.

From part **(a)** you can obtain that $\frac{dy}{dx} = 2 - \cos x$.

From part **(b)** you can obtain the solution of this equation, it is $y = 2x - \sin x + 2$.

Hence the solution to $\frac{d^2y}{dx^2} = \sin x$ with the given initial conditions is $y = 2x - \sin x + 2$.

EXERCISE 12.3 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

1 Write the general solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x - 1$ **(b)** $f'(x) = x^2\sqrt{x}$ **(c)** $y'(x) = 2\cos 2x$ **(d)** $y'(x) = 2\cos^2 x$

(e) $\frac{dz}{dt} = \frac{1}{t^2+4}$ **(f)** $\frac{dz}{dt} = \frac{t}{t^2+4}$ **(g)** $\frac{dx}{d\theta} = \sin^2\theta + \cos^2\theta$ **(h)** $f'(x) = 1 - e^{\frac{-x}{2}}$

2 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x^3 - x + 1$, given that $y = 2$ where $x = 1$

(b) $f'(x) = \sin x(1 - \cos x)$, given that $f\left(\frac{\pi}{3}\right) = -1$

(c) $x'(t) = \sqrt{1+t}$, given that $x(0) = 3$

(d) $\frac{dV}{dt} = te^{t^2}$, given that $V = 2$ when $t = 0$

(e) $\frac{dx}{d\theta} = \frac{\sin\theta}{2+\cos\theta}$, given that $x = 1$ where $\theta = \pi$

(f) $\frac{du}{dx} = \frac{x^2}{x^2+1}$, given that $u = 5$ where $x = 0$

(g) $\frac{dy}{dt} = t\sqrt{t^2-9}$, $t \geq 3$, given that $y = 1$ when $t = 3$

3 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$, $y(0) = 1$

(b) $\frac{dx}{dt} = \frac{t}{t^2+1}$, $x = 1$ where $t = 0$

(c) $\frac{dx}{dy} = \frac{y}{2y-2}$, given that $x = 1$ where $y = 2$

(d) $\frac{dz}{dt} = \frac{t}{\sqrt{1-t}}$, $z(-2) = 0$

4 The solution to $\frac{dy}{dx} = x$ with initial condition $y(0) = -1$ is:

A always concave down **B** always concave up **C** always decreasing **D** always increasing

5 **(a)** Show that $\frac{d}{dx}\left(xe^x\right) = e^x + xe^x$. **(b)** Hence find $\int xe^x\,dx$.

(c) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x$, given $y(0) = -1$.

(d) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x - e^x$, given $y(0) = -2$.

(e) Hence find the particular solution for the second-order differential equation $\frac{d^2y}{dx^2} = xe^x$, given that $\frac{dy}{dx} = -1$ and $y = -2$ where $x = 0$.

6 Find the particular solution of the differential equation $y' = \frac{1}{x^2+1}$ if $y(0) = 1$.

7 **(a)** Show that $\frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right)\right) = \tan^{-1}x + 1$.

(b) Using **(a)**, find the particular solution of the differential equation $\frac{dy}{dx} = \tan^{-1}x + 1$ if $y(0) = 0$.

8 **(a)** If $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with initial condition $y(0) = 1$, find y.

(b) If $\frac{dz}{dx} = \frac{1}{2}\left(e^x + e^{-x}\right)$ with initial condition $z(0) = 0$, find z.

(c) Hence show that if $\frac{d^2y}{dx^2} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with $y(0) = 0$ and $y'(0) = 1$, then $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ is a particular solution of this equation.

9 A simple economic model for the rate of change $\frac{dP}{dt}$ of the price P of a product with respect to time t states that $\frac{dP}{dt}$ varies directly with the difference between the demand D and the supply S of the product. This can be written as $\frac{dP}{dt} = r(D - S)$, $P(0) = 1$ with r being a positive constant.

Let us assume the supply of widgets is given by $S(t) = 2 + \sin^2\left(\frac{\pi t}{6}\right)$ and the demand for widgets by $D(t) = 3 + \cos^2\left(\frac{\pi t}{6}\right)$, where t is the time in months since the start of the year.

(a) Express $P(t)$ in terms of r and t. The widget price after three months is \$4.

(b) Sketch the widget price over the course of the year.

10 An oil tanker hits a reef and spills oil into the sea. The oil spills from the tanker at a rate of $\frac{10^6 t}{t^4 + 16}$ litres/day, where t is the number of days since the tanker first hit the reef.

It is known that $\int \frac{t}{t^4 + 16} dt = \frac{1}{8}\arctan\left(\frac{t^2}{4}\right) + C$.

(a) If V litres is the volume of oil spilled into the sea in the first T days, find V in terms of T.

The local newspaper report stated, 'It is expected that eventually 300 000 litres of oil will spill into the sea.'

(b) Determine whether the newspaper report is in agreement with the model above.

Challenging

11 An irregularly shaped tank 10 metres high, holds 100 cubic metres of oil when full. The volume, $V(h)$ cubic metres, of oil in a partially filled tank, where the depth of oil is h metres, is given by $V(h) = Ah\sin^{-1}\left(\frac{h}{10}\right)$ where A is a real number.

(a) Find the exact value of A.

The tank is initially empty. Oil is then pumped into the tank at a constant rate of 2 cubic metres per minute.

(b) Find, in terms of h, an expression for the rate at which the volume of the oil is increasing, in metres per minute, where the depth is h metres.

(c) Find an expression for the derivative $\frac{dt}{dh}$ in terms of h.

(d) Hence find an expression for t in terms of h.

(e) Hence find the time t (in minutes) for the depth of oil depth to reach $5\sqrt{3}$ metres. Express your answer in the form $t = \frac{a\sqrt{b}}{c}$ minutes for a suitable choice of the natural numbers a, b, c.

12 A rocket has a mass 40 000 kg plus an initial mass m_F kg of fuel. At the initial time $t = 0$ seconds, the rocket is launched vertically from ground level. Ignoring the variation of gravity with height h, the altitude of the rocket is then governed by $\frac{d^2h}{dt^2} = -\frac{49}{5} + \frac{780000}{m_F - 200(t - 200)}$, $0 \le t \le 200$, starting with $\frac{dh}{dt} = 0$ and $h = 0$ when $t = 0$.

(a) Find $\frac{dh}{dt}$ (Assume $0 \le t \le 200$).

(b) Given that $\frac{d}{dt}\left(3900t - \frac{49}{10}t^2 + \frac{39}{2}\left(m_F - 200(t - 200)\right)\log_e\left|\frac{m_F - 200(t - 200)}{40000 + m_F}\right|\right)$

$$= -\frac{49t}{5} + 3900\log_e\left|\frac{m_F + 40000}{m_F - 200(t - 200)}\right|, \text{ find } h(t).$$

Burnout occurs at time $t_b = \frac{m_F}{200}$, when all the fuel has been expended.

(c) Find the altitude h_b of the rocket at the time of burnout.

12.4 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

Models that describe systems exhibiting growth or decay have the form $\frac{dy}{dx} = g(y)$. In other words, the rate of growth $\frac{dy}{dx}$ in the dependent variable y is a function $g(y)$ of the variable y alone. The function $g(y)$ on the right-hand side of such a differential equation has no explicit dependence on the independent variable x.

To solve $\frac{dy}{dx} = g(y)$ given that $y(0) = y_0$, consideration has to be given to two separate cases.

Case I: $g(y_0) = 0$

If $g(y_0) = 0$, then $y(x) = y_0$. In other words, the solution $y(x) = y_0$ is not dependent on the independent variable, so it is variously called the stationary solution, the steady state solution, or the equilibrium solution, y_s.

Case II: $g(y_0) \neq 0$

The model $\frac{dy}{dx} = g(y)$ can be transformed to a directly integrable form $\frac{1}{\left(\frac{dy}{dx}\right)} \equiv \frac{dx}{dy} = \frac{1}{g(y)}$ by taking the reciprocal of both sides, wherever $g(y_0) \neq 0$.

The solution of this new equation gives $x(y)$, so the variable x is now a function of y. In other words, the roles of the dependent and independent variables in the original equation $\frac{dy}{dx} = g(y)$ have been exchanged in the transformed equation $\frac{dx}{dy} = \frac{1}{g(y)}$.

Recall that exchanging the roles of dependent and independent variables in any relationship, such as $y = f(x)$, requires the existence of an appropriate inverse function f^{-1}. In other words, $y = f(x) \Leftrightarrow x = f^{-1}(y)$, provided f is a one-to-one function on an interval, which requires $f(x) \neq 0$ inside that interval. In this problem, $\frac{dy}{dx} = g(y)$ defines y as an invertible (i.e. one-to-one) function of x on an interval, wherever $g(y) \neq 0$ for any value of the variable y in the said solution. Therefore, the solution $x = g(y)$ of $\frac{dx}{dy} = \frac{1}{g(y)}$ can always be inverted to give the solution $y = g^{-1}(x)$ of the original problem $\frac{dy}{dx} = g(y)$ on any interval of y where $g(y) \neq 0$.

A number of these kinds of differential equations will be investigated.

To find the particular solution of $\frac{dy}{dx} = g(y)$, $y(a) = y_a$, $g(y_a) \neq 0$, requires the following five-step procedure:

- Take the reciprocal of both sides of $\frac{dy}{dx} = g(y)$ to obtain $\frac{dx}{dy} = \frac{1}{g(y)}$ with $x = a$ where $y = y_a$.
- Integrate both sides of the equation with respect to the new independent variable y:

  $\int \frac{dx}{dy}\,dy = \int \frac{1}{g(y)}\,dy$

  $x + c = G(y)$ where $G'(y) = \frac{1}{g(y)}$ and c is a constant of integration.
- Where possible, invert the equation from previous step to find the general solution for the original dependent variable y in terms of the original independent variable x:

  $x + c = G(y)$

  $\therefore\ y = G^{-1}(x + c)$
- Substitute the initial condition $(x, y) = (a, y_a)$ into the equation obtained in the previous step to evaluate the constant of integration c: $\therefore\ y_a = G^{-1}(a + c)$.
- Substitute the value of the constant into the general solution obtained in the previous step to obtain the particular solution.

Example 16

Find the solution of $\frac{dy}{dx} = 2y$, given that where $x = 0$, $y = 3$.

Solution

- Take the reciprocal of both sides of the equation:
  $\frac{dx}{dy} = \frac{1}{2y}$
- Integrate with respect to y:
  $x + c = \frac{1}{2}\log_e|y|$
  Logarithms are only defined for positive quantities, so the absolute value sign is used.
- Substitute known values of x and y, $x = 0$, $y = 3$:
  $0 + c = \frac{1}{2}\log_e 3$
- Simplify and express with y as a function of x:
  $x + \frac{1}{2}\log_e 3 = \frac{1}{2}\log_e|y|$
  $2x = \log_e|y| - \log_e 3$
  $2x = \log_e\frac{|y|}{3}$
  $\frac{|y|}{3} = e^{2x}$
  $|y| = 3e^{2x}$

This gives two solutions.

If $y > 0$, $|y| = y = 3e^{2x}$
If $y < 0$, $|y| = -y = 3e^{2x}$ $\therefore y = -3e^{2x}$
Complete solution is: $y = \pm 3e^{2x}$

Example 17

Find the particular solution of $\frac{dy}{dt} = -r(y - s)$, given that $y(0) = y_0$ with $r, s > 0$ positive constants.

Solution

Take the reciprocal of both sides and transpose $-r$:
$-r\frac{dt}{dy} = \frac{1}{y - s}$

Integrate both sides with respect to the original dependent variable y:
$-r\int\frac{dt}{dy}dy = \int\frac{1}{y - s}dy$
$-rt + c = \log_e|y - s|$

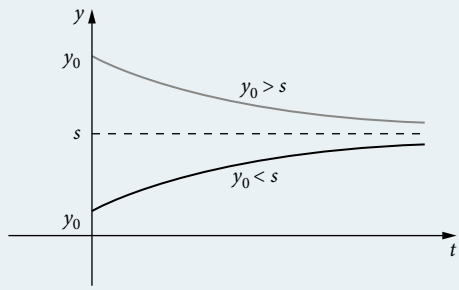
Rearrange to find the general solution:
$|y - s| = e^c e^{-rt}$
$\therefore y - s = Ae^{-rt}$
$y = s + Ae^{-rt}$ where $A = \pm e^c$.

To find the particular solution, substitute initial condition $y(0) = y_0$:
$y_0 - s = Ae^0$
$\therefore A = y_0 - s$
$\therefore y = s + (y_0 - s)e^{-rt}$

It appears that this is a single solution. However, this solution will behave differently depending on the relative size of the initial condition $y(0) = y_0$ and the steady state solution s, as shown.

In the case that $(y_0 - s) < 0$, the solution $y = s + (y_0 - s)e^{-rt}$ grows as time increases.

In the case that $(y_0 - s) > 0$, the solution $y = s + (y_0 - s)e^{-rt}$ decays as time increases.



Example **17** above is related to the model of uninhibited exponential growth/decay, $\frac{dy}{dt} = ry$, which for $y(0) = y_0$ has the general solution $y = y_0 e^{rx}$.

Assuming $r > 0$, the differential equation $\frac{dy}{dt} = -r(y - y_s)$, $y(0) = y_0$ is a model of inhibited growth for $y_0 < y_s$ and inhibited decay for $y_0 > y_s$. The rate of growth (or decay) in the dependent variable is called *inhibited* because the rate falls to zero as the dependent variable approaches its equilibrium value y_s.

The solution curve of an inhibited growth (or decay) problem is $y(t) = y_s + (y_0 - y_s)e^{-rt}$.

Example 18

(a) Show that $\frac{2}{(1-y)(1+y)} = \frac{1}{1-y} + \frac{1}{1+y}$.

(b) Find the general solution of $\frac{dy}{dx} = (1-y)(1+y)$, given that $y(0) = y_0$.

Solution

(a) $\text{RHS} = \frac{1}{1-y} + \frac{1}{1+y}$

$= \frac{1+y+1-y}{(1-y)(1+y)}$

$= \frac{2}{(1-y)(1+y)} = \text{LHS}$

(b) Take the reciprocal of both sides of the equation:

$\frac{dy}{dx} = (1-y)(1+y)$

$\frac{dx}{dy} = \frac{1}{(1-y)(1+y)}$

Both sides of the equation are integrated with respect to y:

$\int \frac{dx}{dy}\,dy = \int \frac{1}{(1-y)(1+y)}\,dy$

The result from **(a)** is used, $\frac{1}{(1-y)(1+y)} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$:

$$x = \frac{1}{2}\int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy$$

$$2x = \int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy$$

$$2x + C = \left(-\ln|1-y| + \ln|1+y|\right)$$

$$2x + C = \ln\left|\frac{1+y}{1-y}\right|$$

Note: Add the constant of integration to the side with the original independent variable.

Rearrange the equation from the previous step to find the general solution for the original dependent variable in terms of the original independent variable:

$$e^{2x+C} = \left|\frac{1+y}{1-y}\right|$$

$$\left|\frac{1+y}{1-y}\right| = e^{2x}e^{C}$$

$$\frac{1+y}{1-y} = \pm e^{2x}e^{C}$$

Let $A = \pm e^{C}$:

$$\frac{1+y}{1-y} = Ae^{2x}$$

$$1+y = Ae^{2x} - yAe^{2x}$$

$$y\left(1+Ae^{2x}\right) = Ae^{2x} - 1$$

$$y = \frac{Ae^{2x}-1}{1+Ae^{2x}}$$

The initial condition is substituted to find the particular solution, $y(0) = y_0$:

$$y_0 = \frac{A-1}{1+A}$$

$$y_0 + Ay_0 = A - 1$$

$$y_0 + 1 = A\left(1 - y_0\right)$$

$$A = \frac{1+y_0}{1-y_0}$$

Hence the solution is:

$$y = \frac{\frac{1+y_0}{1-y_0}e^{2x} - 1}{1+\frac{1+y_0}{1-y_0}e^{2x}}$$

This may also be rearranged to give:

$$y = \frac{1-\frac{1-y_0}{1+y_0}e^{-2x}}{1+\frac{1-y_0}{1+y_0}e^{-2x}}$$

The solution of the quadratic growth rate model $\frac{dy}{dx} = (1-y)(1+y)$, $y(0) = y_0$ is:

$$y = \frac{1-\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}{1+\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}.$$

EXERCISE 12.4 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

1 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -y, y(0) = 1$ **(b)** $\frac{dy}{dx} = 2y, y(0) = -1$

(c) $\frac{dy}{dx} = -2y, y(0) = -5$ **(d)** $\frac{dy}{dx} = 2y, y(0) = 3$

2 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -2(y-3), y(0) = 8$ **(b)** $\frac{dy}{dx} = -2(y(x)-8), y(0) = 3$

(c) $\frac{dy}{dx} = -4(y+3), y(0) = 5$ **(d)** $\frac{dy}{dx} = -4(y-3), y(0) = -5$

3 Given that $\frac{dy}{dx} = \cos^2 y$ and that $y = \frac{\pi}{4}$ at $x = 0$, then which of the following is true?

A $y = \frac{1}{2}y + \frac{1}{4}\sin 2y$ **B** $x = \tan\left(y + \frac{\pi}{4}\right)$ **C** $y = \tan^{-1}(x+1)$ **D** $y = \tan^{-1}\left(x - \frac{\pi}{4}\right)$

4 In Biology and Ecology, the term *desiccation* refers to the drying out (i.e. the loss of water) of the cells of a living organism. Most cells are mostly made of water. Assume that the desiccation of a cell is modelled by the solution of the following differential equation: $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, where V is the volume of the water in the cell, t is time and k is an appropriate constant of proportionality.

During the intense heat of the Australian summer, the cells of a newly fallen eucalyptus leaf still contain water, but the leaf loses this water rapidly through the process of desiccation. Suppose that each leaf cell initially contains $8\,\mu m^3$ of water, but 4 hours later each cell has only $1\,\mu m^3$ of water.

(a) Find the particular solution of $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, $V(0) = 8\,\mu m^3$.

(b) Find the time taken for the cells to lose all their water (assuming that the environmental conditions don't change over this time).

5 The pressure of the atmosphere, P kilopascals (kPa), decreases according to the height h km above sea level. The rate of change of the pressure with respect to the height above sea level is proportional to the pressure at that height.

(a) Write a differential equation to describe this situation.

(b) The pressure at sea level is 101.3 kPa and it is approximately 37.3 kPa at a height of 5 km. Solve the differential equation to find P as a function of h.

(c) Estimate the air pressure at the top of Mount Everest, which is about 9 km high.

6 In an electric circuit, a capacitor of capacitance C charged to a potential difference E is discharged through a resistance R. If q is the charge on the capacitor at time t, then $\frac{dq}{dt} = -\frac{q}{RC}$ is the differential equation describing this situation. If initially $q = EC$ then find the solution of this equation (that is, q as a function of t).

7 Newton's law of cooling states that 'the cooling rate of a body is proportional to the difference between the temperature of the body and that of the surrounding medium.' This may be written as $\frac{dT}{dt} = -k(T - M)$, where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

A pot of soup is cooked at 100°C. To cool the soup, it is placed in a room where the temperature is 20°C. After 20 minutes the temperature of the soup has dropped to 70°C.

(a) Find the general solution of the differential equation $\frac{dT}{dt} = -k(T - 20)$.

(b) Find the value of k.

(c) How much time will it take the pot of soup to cool to 25°C?

Challenging

8 Two pieces of information are needed for the safe medicinal use of a drug:

(i) dosage level required to achieve the desired effect **(ii)** time interval between doses.

The body deals with a drug by first absorbing it into body tissues, then excreting it via body organs. After being absorbed, the process of drug excretion usually begins as blood takes the drug from body tisses and flows through the kidneys. Assume that the kidney's excretion rate $\frac{dy}{dt}$ is proportional to the amount of the drug y in the body tissues, so that $\frac{dy}{dt} = -ky$, $k > 0$.

Suppose that an initial dose D of a drug is given at time $t = 0$ and that this dosage is repeated at intervals of T minutes.

(a) Show that the amount of the initial dose remaining when the second dose is given is De^{-kT}.

(b) Find an expression for the amount of drug in the body just after the nth dose has been given.

(c) Show that the maximum amount of the drug that can build up in the body from a succession of doses of quantity D is $\frac{D}{1-e^{-kt}}$. This is called the saturation level y_s and is the desired dosage level for safe maximum effectiveness.

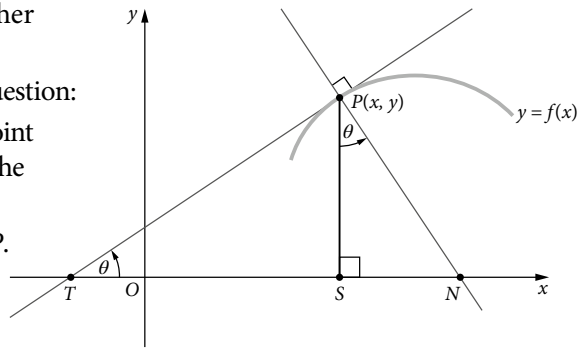
(d) The half-life for the excretion of a drug is the time for the quantity of a drug in the body to reduce from D to $\frac{D}{2}$, when no further dose is given. If the half-life is h, show that $k = \frac{\log_e 2}{h}$.

12.5 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)g(y)$ USING SEPARATION OF VARIABLES

Towards the end of his life, the great Swiss mathematician and teacher Johann Bernoulli (1667–1748) published his lectures on calculus.

In Bernoulli's discussion of integral calculus, he posed the following question:

What curve $y = f(x)$ has a subtangent ST (see diagram at right) at any point P that is always equal to a positive multiple, $a > 0$, of the abscissa OS at the point P?



The term abscissa is the formal name for the x-coordinate of point P.

In other words, Bernoulli is asking for the curves with $ST = aOS$.

Bernoulli's question can be rewritten using a more modern notation.

It follows from the right-angled ΔSPT that $ST = \frac{SP}{\tan\theta}$.

However, TP is a tangent line to the curve $y = f(x)$ at point P, given by $\tan\theta = \frac{dy}{dx}$.

SP is the ordinate (the y-value) of point P.

Therefore: $ST = \frac{PS}{\tan\theta} = \frac{y}{\left(\frac{dy}{dx}\right)}$

$ST = aOS \ \therefore \frac{y}{\left(\frac{dy}{dx}\right)} = ax$

Solving for the derivative in this equation gives $\frac{dy}{dx} = \frac{y}{ax}$.

In some ways, this problem is slightly more complicated than the equations solved so far because it has the form $\frac{dy}{dx} = f(x)g(y)$. Both the independent and the dependent variables appear as separate factors in the derivative $\frac{dy}{dx}$.

In this section you will learn how to answer Bernoulli's question. You will learn how to solve any first-order differential equation of the form $\frac{dy}{dx} = f(x)g(y)$. The key to solving these equations is the separation of the two variables onto either side of the equality.

In other words, to solve $\frac{dy}{dx} = f(x)g(y)$, transpose the equation so that all terms involving the dependent variable are on the left side of the equality and all terms involving the independent variable are on the right side.

This would give $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$.

In the table below, the original equation is in the left-hand column while the equation with variables separated is in the right-hand column.

Original differential equation	Variables	Rewritten with variables separated
$\cos x + y\frac{dy}{dx} = 0$	x and y	$y\frac{dy}{dx} = -\cos x$
$\sin v\frac{dw}{dv} - \cos v = 0$	v and w	$\frac{dw}{dv} = \cot v$
$\theta z' = z + 1$	θ and z	$\frac{1}{z+1}\frac{dz}{d\theta} = \frac{1}{\theta}$

Finding the general solution of a first-order differential equation by the method of separation of variables

Example 19

Find the general solution of the differential equation $\frac{dy}{dx} = x(y-1)$.

Solution

The dependent and independent variables are separated onto either side of the equality, with the dependent on the left and the independent variable on the right: $\frac{1}{y-1}\frac{dy}{dx} = x$.

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx = dy$ on the LHS:

$$\int \frac{1}{y-1}\frac{dy}{dx}dx = \int x\,dx$$

$$\int \frac{1}{y-1}dy = \int x\,dx$$

$$\log_e|y-1| = \frac{1}{2}x^2 + c$$

Represent the dependent variable as an explicit function of the independent variable.

Exponentiating both sides:

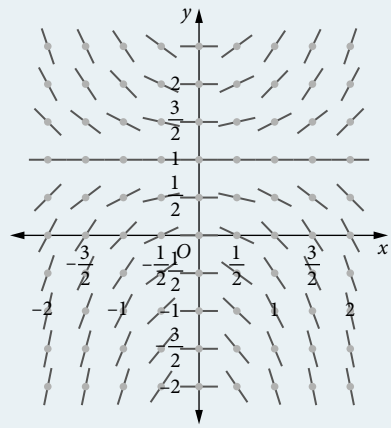
$$|y-1| = e^{\frac{1}{2}x^2+c}$$

$$= e^c e^{\frac{1}{2}x^2}$$

Removing the absolute value: $y = 1 + Ae^{\frac{1}{2}x^2}$, where $A = \pm e^c$.

It appears that the general solution of $\frac{dy}{dx} = x(y-1)$ is $y = 1 + Ae^{\frac{1}{2}x^2}$, Real A, $A \neq 0$.

The slope field of $\frac{dy}{dx} = x(y-1)$ is shown at right.



This slope field indicates that $y = 1$ should also be a solution curve for $\frac{dy}{dx} = x(y-1)$. However, the general solution $y = 1 + Ae^{\frac{1}{2}x^2}$ cannot give $y = 1$ because $A = \pm e^c$, $A \neq 0$.

In the example above, the solution $y = 1$ is an example of a so-called singular solution because it is not part of the general solution for any allowable value of the constant of integration. However, the singular solutions of $\frac{dy}{dx} = f(x)g(y)$ are often of the form $y = y^*$ for some root k of the equation $f(y^*) = 0$. The singular solutions are also usually evident from an investigation of the slope field.

Warning

To avoid missing any possible singular solutions when finding the general solution of a differential equation of the form $\frac{dy}{dx} = f(y)g(x)$, always remember to investigate the roots $\{y^* : f(y^*) = 0\}$ and/or the slope field.

Finding the particular solution of a first-order differential equation by the method of separation of variables

Example 20

Find the particular solution of Bernoulli's problem $\frac{dy}{dx} = \frac{2y}{x}$ passing through the point $(1, -4)$.

Solution

The dependent and independent variables are separated onto either side of the equality with the dependent on the left and the independent variable on the right: $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx = dy$ on the LHS of the equation: $\int \frac{1}{y}\frac{dy}{dx}dx = 2\int \frac{1}{x}dx$

$$\int \frac{1}{y}dy = 2\int \frac{1}{x}dx$$
$$\log_e|y| = 2\log_e x + c$$
$$\log_e|y| = \log_e x^2 + c$$

Solved for the dependent variable to obtain the general solution: $\log_e|y| - \log_e x^2 = c$

$$\log_e \frac{|y|}{x^2} = c$$
$$\frac{|y|}{x^2} = e^c$$

$\therefore\ y = Ax^2$, where $A = \pm e^c$

Constant of integration is found to satisfy the required initial condition $(x, y) = (1, -4)$: $-4 = A$

Constant of integration is substituted to specify the particular solution of the problem: $y = -4x^2$

To solve the first-order differential equation $\frac{dy}{dx} = f(y)g(x)$ by the method of separation of variables:

1 Solve $\{y^* : f(y^*) = 0\}$ for any steady state (or equilibrium) solutions $y = y^*$.

2 Separate the dependent and independent variables onto either side of the equality $\frac{1}{f(y)}\frac{dy}{dx} = g(x)$.

3 Integrate both sides of the equality with respect to the independent variable:

For the integral involving y terms, apply the change of variable $\int \frac{1}{f(y)}\frac{dy}{dx}dx = \int \frac{1}{f(y)}dy$.

Add the constant of integration C to the side with the independent variable. This is the general solution.

4 Wherever possible, you should represent the dependant variable as an explicit function of the independent variable. However, you may need to be satisfied with an equation that determines the dependant variable as an implicit function of the independent variable.

5 If an initial condition $y(a) = b$ is given, solve for the constant of integration C. This is the particular solution.

You have now learnt two different methods to solve the differential equation $\frac{dy}{dx} = f(y)$.

The reciprocal method

1 Take the reciprocal of both sides of $\frac{dy}{dx} = f(y)$ to obtain $\frac{dx}{dy} = \frac{1}{f(y)}$.

2 Integrate the result with respect to y to obtain $\int \frac{dx}{dy}dy = \int \frac{1}{f(y)}dy$.

The separation of variables method

1 Separate the two variables in $\frac{dy}{dx} = f(y)$ to obtain $\frac{1}{f(y)}\frac{dy}{dx} = 1$.

2 Integrate the result with respect to x to obtain $\int \frac{1}{f(y)}\frac{dy}{dx}dx = \int dx$ or $\int \frac{1}{f(y)}dy = \int dx$.

Each of the two methods above give the original independent variable x as a function of the original dependent variable y, which must then be inverted to give the required solution $y(x)$.

EXERCISE 12.5 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)g(y)$ USING SEPARATION OF VARIABLES

1 Use the method of separation of variables to find the general solution of each of the differential equations below. Where reasonable, express the family of solutions as explicit functions of x.

(a) $(x^2+4)\frac{dy}{dx} = 2xy$ **(b)** $\frac{dy}{dx} = \frac{2y}{x}$ **(c)** $\frac{dy}{dx} = (1+y^2)\sqrt{x}$ **(d)** $\frac{\sqrt{1-x^2}}{y+1}\frac{dy}{dx} = -2x$

(e) $(1+x^2)\frac{dy}{dx} = xy$ **(f)** $e^y \cos x - \frac{dy}{dx}\sin^2 x = 0$ **(g)** $(\sec x)\,y' + y^2 = 0$

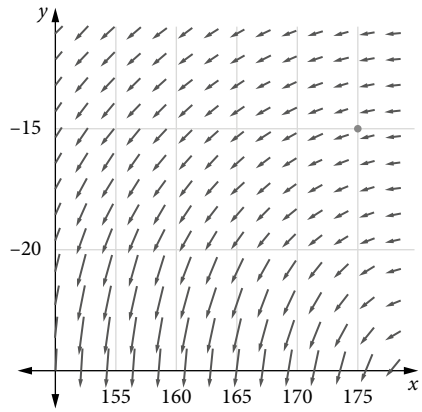
2 Find the particular solution of $e^{-x^2}yy' + xy = 0$, $y(0) = 1$.

3 Find the equation of each graph:

(a) The graph passes through (1, 2) and has a slope $\frac{3y}{x^2}$ at each point (x, y).

(b) The gradient of the tangent at point (x, y) on a graph is given by $\frac{-2y}{x}$ and the graph passes through the point (1, 2).

4 On a particular day, the synoptic scale airflow over the Coral Sea is modelled by the slope field of the following differential equation $\frac{dy}{dx} = \frac{180 - x}{4(25 + y)}$, as shown in the diagram at right.



Here, x is the longitude and y is the latitude (both are measured in degrees, with southern latitudes negative).

Tropical Cyclone Wendy is shown at the point longitude 175°E and latitude 15°S.

(a) Find the particular solution of $\frac{dy}{dx} = \frac{180 - x}{4(25 + y)}$, $y(175) = -15$.

(b) Hence, plot the path of Cyclone Wendy.

5 Consider the differential equation $\frac{dy}{dx} = 3x^2 \cos^2 y$.

(a) Find the particular solution $y = f(x)$ to the differential equation, satisfying the initial condition $f(0) = \frac{\pi}{4}$.

(b) State the domain and range of the solution found in part **(a)**.

6 An insect population P experiences a seasonal growth rate given by $\frac{dP}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right)P$, $P(0) = 1$, where P is measured in millions and t is the number of months since the beginning of spring.

Express the time variation of the insect population P and sketch this variation over the course of one year.

7 The general solution of the differential equation $\frac{dy}{dx} = y^2 \cos x$ is:

A $y = \frac{1}{\cos x - C}$ **B** $y = \frac{-1}{\sin x + C}$ **C** $y = \sqrt{2C + 2\sin x}$ **D** $y = -\sqrt{2C + 2\sin x}$

8 The particular solution of the differential equation $\frac{dy}{dx} = 2xy$, $y(0) = 2$ is:

A $y = e^{\frac{x^2}{2}}$ **B** $y = 2e^{x^2}$ **C** $y = c_1 e^{x^2}$ **D** $y = 2e^{\frac{x^2}{2}}$

Challenging

9 The altimetry equation $\frac{dp}{dz} = -\frac{g}{R}\frac{P}{(T_0 - \Gamma z)}$, relates air pressure P in pascals (Pa) to the altitude (height above sea level) z in metres (m). In this model, T_0 is the air temperature in kelvin (K) at sea level; Γ is the constant rate of temperature change as the altitude increases (K/m); P_0 is the air pressure (Pa) at sea level; R is the gas constant of air; and g is the gravitational acceleration.

(a) Find the rule for the air pressure P as a function of the altitude z.

(b) Hence, find the rule for altitude z as a function of the air pressure P.

(c) Use the table below to find the altitude corresponding to an air pressure of 61 640 Pa. Express the answer correct to the nearest metre.

g (m s^{-2})	R (J/kgK)	Γ (K/m)	P_0 (Pa)	T_0 (K)
9.80	287.053	0.0065	101325	288.15

10 Consider the differential equation $\frac{dy}{dx} = -\frac{2xy}{\log_e y}$, $y > 0$.

(a) Find the general solution $g(x, y) = c$ of this differential equation as an implicit relation between x and y, using the substitution $u = \log_e y$ to complete the integration.

(b) Find the particular solution passing through the point $(0, e)$.

(c) Explain why $x = 1$ cannot exist in the solution to part **(b)**.

11 Consider the differential equation: $\frac{dy}{dx} = \frac{5-x}{y}$

(a) Let $y = f(x)$ be the particular solution to the differential equation $0 < x < 10$, such that the line $y = 5$ is a tangent to the graph of $y = f(x)$. Find the x-coordinate of the point P of tangency.

(b) Given that $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{(5-x)^2}{y^3}$, determine the nature of the solution of the original differential equation in the vicinity of point P.

(c) Let $y = g(x)$ be the particular solution to the given differential equation for $0 < x < 10$, with the initial condition $g(5) = -5$. Find the rule $y = g(x)$.

12 The fuel–air pressure P (Pa) inside the cylinder of a combustion engine with a 10 : 1 compression ratio is related to the volume V (cm^3) of the cylinder by the differential equation $\frac{dP}{dV} = -1.400\frac{P}{V}$.

(a) Find the particular solution of this equation, if the volume of the fuel–air in the uncompressed cylinder is $600\,\text{cm}^3$ when the fuel–air pressure is 100 000 Pa.

(b) Hence, find the air pressure P when the cylinder is fully compressed to $60\,\text{cm}^3$.

13 Tropical cyclones can only form over water with surface temperatures of at least 26°C. After these cyclones have formed they tend to strengthen when passing over warmer water and weaken over colder water.

The strength of a cyclone S is related to the water surface temperature T by $\frac{dS}{dT} = k\frac{S}{T}$ where k is the constant of proportionality.

(a) What is the sign of the constant of proportionality k? Explain why this is so.

(b) Find $S(T)$ in terms of k.

14 A space probe is launched vertically upwards from the surface of a spherical planet with a radius R. If the atmospheric drag is ignored, the upwards velocity ($v\,\text{m s}^{-1}$) of the probe at height h metres above the surface of the planet is modelled by the solution of the differential equation $\frac{dv}{dh} = -\frac{gR^2}{v(R+h)^2}$, $v = u$, where $h = 0$ and g is the gravitational acceleration on the surface of the planet.

(a) Show that $v^2 = u^2 - \frac{2gR}{1+\frac{R}{h}}$.

(b) Hence find the minimum launch velocity u for the probe to escape the planet's gravity.

15 In a dry, still atmosphere, the air temperature T(K) usually decreases with the altitude h (m) above sea level according to $\frac{dT}{dh} = -0.00649$, with $T(0) = 288\text{K}$.

(a) Find a formula for the variation of temperature with height.

The barometric pressure P (Pa) also depends on the altitude above sea level h and the local air temperature T according to the differential equation $\frac{dP}{dh} = -0.0352\frac{P}{T}$, $P(0) = 101\,325$ Pa.

Whereas the temperature B at which water boils is related to air pressure P by the differential equation $\frac{dB}{dP} = 0.000189\frac{B^2}{P}$, with $B(101\,325) = 373\text{K}$.

(b) Show that $\frac{dB}{dh} = \frac{-6.65\times10^{-6}B^2}{288 - 0.00649h}$.

(c) Hence find a formula for the altitude h (m) as a function of the boiling point temperature B (K).

During a climb of Mount Everest, mountaineers observe water boiling at 84°C (that is, 357K).

(d) Find the height of the mountaineers above sea level.

16 According to Torricelli's Law, the rate of outflow from a tank filled to a depth h with a volume V of fluid draining under the influence of gravity alone is $\frac{dV}{dt} = -k\sqrt{h}$, where $k > 0$.

Consider a hemispherical tank of radius 1 m, filled to a depth h m with $V(h) = \frac{1}{3}\pi h^2(3-h)$ cubic metres of water. Assume that this tank drains through a hole at its lowest point.

(a) Use the method of related rates to show that: $\frac{dh}{dt} = -\frac{k}{\pi\sqrt{h}(2-h)}$.

(b) Initially, the tank is filled to a depth of 1 m. Find a formula for the time t taken for the water level to fall to a given depth h. (Express your answer in terms of the unknown constant of proportionality k.)

(c) Find the constant of proportionality k if it takes $\frac{187}{16}$ minutes for the depth to fall to $\frac{1}{4}$ m.

(d) Hence find the time taken (in minutes) for the tank to empty.

12.6 MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Models of uninhibited growth

The British economist Thomas R. Malthus (1766–1834) proposed a model of natural population growth that assumes that both the birth rate and the death rate of a population are proportional to the current size of the population at each instant.

Consequently, the net rate of change in population can be restated as:
'net rate of change in population = birth rate – death rate'.

In other words: Let $P(t)$ represent the size of a population at time t. Assuming that the population level is specified at the initial time $t = 0$ as $P(0) = P_0$, predict the population at later times t.

$$\frac{dP}{dt} = bP - dP$$

$$\frac{dP}{dt} = (b-d)P$$

$$\frac{dP}{dt} = rP, \text{ where } r = b - d \text{ with } P(0) = P_0.$$

In summary, Malthus' uninhibited growth model $\frac{1}{P}\frac{dP}{dt} = r$ has a constant relative growth rate r. This model results in exponential growth for P. In more general terms:

A dependent variable $y(t)$ is said to have an *exponential growth rate* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = k > 0$, or an *exponential rate of decay* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = -k < 0$ for some positive constant k.

Finding the doubling-time in a model of exponential growth

Example 21

Consider the differential equation $\frac{dy}{dt} = ky$, given that $y(0) = y_0 > 0$, with $k > 0$.

Find the time T for y to double its initial value y_0.

Solution

Rearrange to separate the two variables: $\frac{dy}{dt} = ky$

$$\frac{1}{y}\frac{dy}{dt} = k$$

Integrate with respect to the independent variable: $\int \frac{1}{y}\frac{dy}{dt}dt = \int k\,dt$

$$\log_e|y| = kt + c$$

Exponentiate: $|y| = e^{kt+c}$

The general solution is: $y = Ae^{kt}$ where $A = \pm e^c$

Apply the initial condition $y(0) = y_0$ to solve for the constant of integration: $y_0 = A$

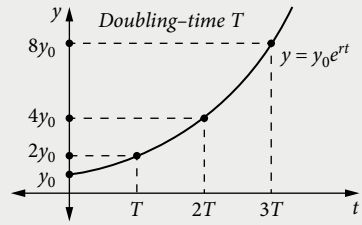
Substitute to find the model of exponential growth: $y = y_0 e^{kt}$

At time T, $y = 2y_0$: $2y_0 = y_0 e^{kT}$

$$2 = e^{kT}$$

$$T = \frac{1}{k}\log_e 2$$

For a system $y = y_0 e^{rt}$, $r > 0$ experiencing exponential growth, the doubling-time (T) is the time it takes the system to double in size. The formula for the doubling-time is $T = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2$



Radioactive decay

Example 22

The New Zealand-born physicist Ernest Rutherford is often called the father of nuclear physics. Rutherford was able to formulate a model of radioactive decay as a simple differential equation. From his laboratory experiments Rutherford discovered that the relative rate *r,* at which atoms of radioactive materials disintegrate, is proportional to the number of atoms N still present in the sample. In other words, this model states that $\frac{dN}{dt} = -rN$, $N(0) = N_0$.

(a) Show that the solution of this model is $N = N_0 e^{-rt}$.

(b) Hence find the time $t_{\frac{1}{2}}$ required for the number of atoms still present (N) to fall to $\frac{1}{2}N_0$.

Solution

(a) The two variables are separated: $\frac{1}{N}\frac{dN}{dt} = -r$

Both sides of the equation are integrated with respect to the independent variable:

$$\int \frac{1}{N}\frac{dN}{dt}dt = -\int r\,dt$$

$$\log_e|N| = -rt + c$$

$$N = \pm e^{-rt+c}$$

$$N = \pm e^c e^{-rt}$$

$\therefore \quad N = Ae^{-rt}$ where $A = \pm e^c$

The initial conditions are substituted in the last equation to find the constant of integration, $N(0) = N_0$, $N_0 > 0$:

$A = \pm e^c = N_0$

$\therefore N = N_0 e^{-rt}$ as required.

(b) The conditions required are substituted into the equation from part **(a)**: $N = N_0 e^{-rt}$ with $N = \frac{1}{2}N_0$

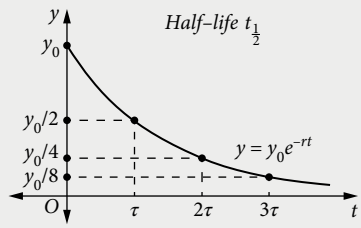
$\frac{1}{2}N_0 = N_0 e^{-rt}$

$\frac{1}{2} = e^{-rt}$

The exponential equations are changed into a logarithmic form: $t = \frac{1}{r}\log_e 2$.

Hence $t_{\frac{1}{2}} = \frac{1}{r}\log_e 2$, where $t_{\frac{1}{2}}$ is the half-life of the radioactive material.

For a system $y = y_0 e^{-rt}$, $r > 0$ experiencing exponential decay, the half-life, $t_{\frac{1}{2}}$ is the time it takes the value of y to halve. The formula for the half-life is $t_{\frac{1}{2}} = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2$.



First-order decay reactions

In a simple first-order decay reaction, the rate of decrease $-\frac{d[A]}{dt}$ of the concentration of a reactant A varies directly with the value of the concentration $[A]$ of that reactant.

That is, the concentration $[A]$ of the reactant A is modelled by the differential equation $\frac{d[A]}{dt} = -r[A]$, where the relative decay rate r is usually called the reaction rate.

Examples of some first-order decay reactions are included in the table below.

Reactant	Reaction rate r (s^{-1})	Half-life $t_{\frac{1}{2}}$
$^{238}_{92}U$	4.87×10^{-18}	4.51×10^9 years
$^{14}_{6}C$	3.83×10^{-12}	5.73×10^3 years
$^{32}_{15}P$	5.61×10^{-7}	14.3 days

Newton's law of cooling

Newton's law of cooling states that the time rate of decrease in the temperature T of an object varies directly with the difference between the object's temperature and the ambient temperature (the temperature of the surrounding medium, T_a).

Therefore: $\frac{dT}{dt} = -r(T - T_a), \quad r > 0, \quad T(0) = T_0$

If $T_0 > T_a$, the body's initial temperature is higher than the temperature of its surrounding, so the body is cooling. The model predicts that the body will cool quickly at first, reducing the excess of its temperature to that of the surrounding environment. This cooling reduces the excess temperature of the body, thus continually slowing its rate of cooling.

However, if $T_0 < T_a$, then the body's initial temperature is lower than the temperature of its surroundings, so the body is warming. Again, this change in temperature reduces the temperature difference with the surrounding medium, thus slowing the rate of warming.

Example 23

Newton's law of cooling can be used to model the temperature of a cup of coffee cooling on a kitchen bench. If the temperature of the kitchen is a constant 20°C and the initial temperature of the coffee is 95°C, after 20 minutes the coffee will have cooled to 65°C.

(a) Find the temperature of the coffee after an additional 20 minutes.

(b) Sketch a graph of the temperature of the coffee for the first 2 hours.

Solution

(a) This problem models the process of cooling, so Newton's law of cooling is: $\frac{dT}{dt} = -r(T-20)$, $r > 0$, $T(0) = 95$.

The two variables are separated: $\frac{1}{(T-20)}\frac{dT}{dt} = -r$

Both sides of the equation are integrated with respect to t and a change of variable is applied on the LHS of the equation: $\int \frac{1}{(T-20)}\frac{dT}{dt}\,dt = -\int r\,dt$

$$\int \frac{1}{(T-20)}\,dT = -\int r\,dt$$

$$\log_e|T-20| = -rt + c$$

Both sides of the equation are exponentiated and solved for the dependent variable: $T - 20 = Ae^{-rt}$, where $A = \pm e^c$.

The initial condition is substituted and the constant of integration is determined:

$T(0) = 95$: $95 - 20 = A$

$\therefore\ T = 20 + 75e^{-rt}$

$t = 20$, $T = 65$, is substituted to determine the constant of proportionality: $\log_e\left(\frac{45}{75}\right) = -20r$

$$r = \frac{1}{20}\log_e\left(\frac{5}{3}\right)$$

The constant of proportionality is substituted in the general solution: $T = 20 + 75e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)}$

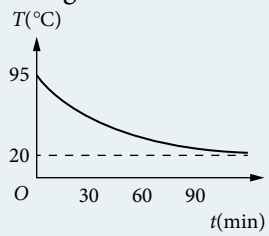
Now: $e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)} = e^{\frac{t}{20}\log_e\left(\frac{3}{5}\right)} = e^{\log_e\left(\frac{3}{5}\right)^{\frac{t}{20}}} = \left(\frac{3}{5}\right)^{\frac{t}{20}}$

Simplify: $T(t) = 20 + 75\left(\frac{3}{5}\right)^{\frac{t}{20}}$

After an additional 20 minutes, $t = 40$: $T(40) = 20 + 75\left(\frac{3}{5}\right)^2 = 20 + 27 = 47°\text{C}$

The temperature of the coffee is 47°C.

(b) Using the model from part **(a)** to plot the changing value of T:

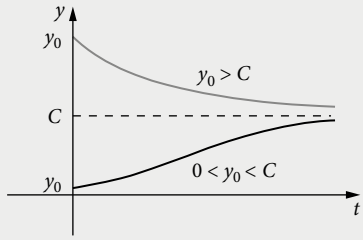


Models of modified growth and decay

The uninhibited growth model with its constant relative growth rate $r = \frac{1}{y}\frac{dy}{dt}$ does not take into account the inherent limitations on the growth of a population. In practice, most populations have a size (the carrying capacity C) beyond which their environment can no longer sustain them. To account for the finite carrying capacity inherent in most systems, the Belgian mathematician Pierre François Verhulst (1804–1849) modified the uninhibited growth model by replacing constant relative growth rate r with a damped growth rate $r\left(1-\frac{y}{C}\right)$.

Verhulst's modified growth rate model is then equivalent to $\frac{1}{y}\frac{dy}{dt} = r\left(1-\frac{y}{C}\right)$, which slows the rate of growth as the population approaches the 'carrying capacity' $y = C$ of the system.

The differential equation $\frac{dy}{dt} = ry\left(1-\frac{y}{C}\right)$, $y(0) = y_0$ where $r > 0$ is called a **logistic equation**. This models population growth in an environment with a finite carrying capacity $C > 0$.



Example 24

Rabbit Island currently has 5000 rabbits, but has sufficient space and food for 20 000 rabbits in total.

The model describing this rabbit population P is then $\frac{dP}{dt} = \frac{P}{10}\left(1-\frac{P}{20\,000}\right)$.

Find the particular solution of this differential equation, given that $\frac{20\,000}{P(20\,000-P)} = \frac{1}{P} + \frac{1}{20\,000-P}$.

Solution

Take the reciprocal of both sides of the differential equation: $\frac{dP}{dt} = \frac{P}{10}\left(1-\frac{P}{20\,000}\right)$

$$\frac{dt}{dP} = \frac{200\,000}{P(20\,000-P)}$$

Integrate both sides of the equation with respect to the original dependent variable:

Given that $\frac{200\,000}{P(20\,000-P)} = 10\left(\frac{1}{P} + \frac{1}{20\,000-P}\right)$

$$\int \frac{dt}{dP}\,dP = \int \frac{200\,000}{P(20\,000-P)}\,dP$$

$$\frac{1}{10}\int dt = \int\left(\frac{1}{P} + \frac{1}{20\,000-P}\right)dP$$

$$\frac{t}{10} + c = \log_e\left|\frac{P}{20\,000-P}\right|$$

Find the general solution: $Ae^{\frac{t}{10}} = \frac{P}{20\,000-P}$, where $A = \pm e^c$

Substitute the initial conditions $t = 0$, $P = 5000$: $A = \frac{5000}{20\,000-5000} = \frac{1}{3}$

$$P = \frac{20\,000e^{\frac{t}{10}}}{3+e^{\frac{t}{10}}}$$

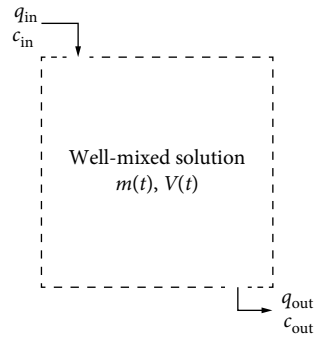
$$= \frac{20\,000}{1+3e^{\frac{-t}{10}}}$$

Modelling the time dependence of a conserved substance

Differential equations can be used to model the time behaviour of a conserved quantity within a confined region of space, if you know the rate at which the quantity flows into and out of the region. Such models are of great practical value in the fields of Science and Engineering.

If a substance m is conserved (that is, neither created nor destroyed), then the rate of change of m within a confined region (called a *control volume*) is equal to the difference between the rate at which it flows into the region (R_{in}), and the rate at which it flows out of the region (R_{out}). That is, $\frac{dm}{dt} = R_{\text{in}} - R_{out}$.

Consider the situation shown at right:



- $m(t)$ is the mass of some conserved substance (e.g. salt or sugar) in the control volume at time t
- $V(t)$ is the volume of the well-mixed solution within the control volume at time t
- q_{in}, measured in units of $\left(\frac{volume}{time}\right)$, is the rate at which the solution flows into the control volume
- q_{out}, measured in units of $\frac{volume}{time}$, is the rate at which the solution flows out of the control volume
- c_{in}, measured in units of $\left(\frac{mass}{volume}\right)$, is the concentration of the conserved substance in the inflow
- c_{out}, measured in units of $\left(\frac{mass}{volume}\right)$, is the concentration of the conserved substance in the outflow

The rate R_{in} at which the conserved substance $m(t)$ flows into the control volume is measured in units of $\frac{mass}{time}$ so that $R_{in}\left[\frac{mass}{time}\right] = q_{in}\left[\frac{volume}{time}\right] \times c_{in}\left[\frac{mass}{volume}\right]$

Similarly, the rate at which $m(t)$ flows out of the control volume is $R_{out}\left[\frac{mass}{time}\right] = q_{out}\left[\frac{volume}{time}\right] \times c_{out}\left[\frac{mass}{volume}\right]$

Assume that $m(t)$ is conserved, so it is neither created nor destroyed within the control volume. Therefore $\frac{dm}{dt} = R_{in} - R_{out}$.

This means that the net rate of change in the mass of the solute $m(t)$ within the control volume at time t, is modelled by the solution of differential equation $\frac{dm}{dt} = q_{in} \times c_{in} - q_{out} \times c_{out}$.

To solve this equation, the mass concentration of the solute in the outflow c_{out} must be expressed in terms of the dependent variable $m(t)$ and the control volume $V(t)$ as $c_{out} = \frac{m(t)}{V(t)}$.

However, the net rate of change of the volume of the solution within the control volume is modelled by the simple differential equation $\frac{dV}{dt} = q_{in} - q_{out}$, which gives the solution $V(t) = V(0) + (q_{in} - q_{out})t$.

The differential equation for the time rate of change of the conserved substance within the control volume can now be written as $\frac{dm}{dt} = q_{in} \times c_{in} - q_{out} \times \frac{m(t)}{V(0) + (q_{in} - q_{out})t}$.

Modelling the mass of a conserved quantity

Example 25

A holding tank at a desalination plant contains 200 cubic metres of brine solution (salt in water), 20 tonnes of which is salt. At time $t = 0$ minutes, seawater starts to flow into this at a rate of 10 cubic metres per minute. This seawater has a salt concentration of 35 kilograms per cubic metre. The 'well-mixed' solution (brine mixed with seawater) also flows out of the holding tank at the same rate of 10 cubic metres per minute.

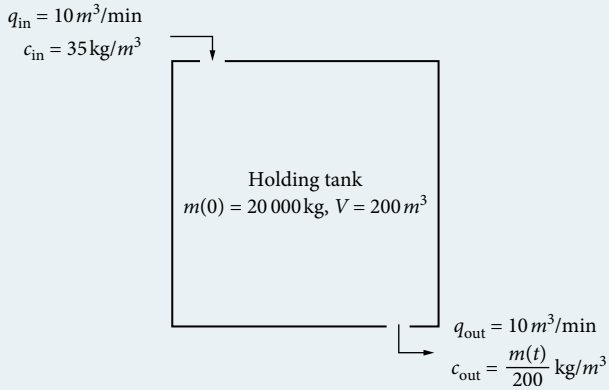
(a) Construct a differential equation to model the mass m kg of salt in the tank after t minutes.

(b) State the amount of salt within the holding tank $m(t)$ kg after t minutes.

(c) As the water continues to flow through the tank, over time the concentration of salt within the tank approaches a 'long-term' value. What is this value, in units of kg/m^3?

(d) Sketch the time dependence of the salt concentration of the 'well-mixed' outflow.

Solution



(a) Volume of the solution is fixed at 200 m^3, because $q_{in} = q_{out} = 10$ m^3/min.

Concentration of the inflow $c_{in} = 35$ kg per m^3.

Concentration of the outflow

$c_{out} = \dfrac{m(t)}{200}$ kg per m^3, where $m(t)$ is the kilograms of salt within the tank after t minutes.

Initial mass of salt is $m(0) = 20\,000$ kg.

$$\text{Net rate of change of salt}\left(\frac{\text{kg}}{\text{min}}\right) = \text{rate of inflow}\left(\frac{\text{kg}}{\text{min}}\right) - \text{rate of outflow}\left(\frac{\text{kg}}{\text{min}}\right)$$

$$\frac{dm}{dt} = 350 - \frac{m}{20}$$

$$= \frac{1}{20}(7\,000 - m) \quad \text{given that } m(0) = 20\,000 \text{ kg}$$

(b) The model is an example of the general inhibited decay model:

$$\frac{dy}{dt} = r(a - y), y(0) = y_0 \therefore y = a + (y_0 - a)e^{-rt}$$

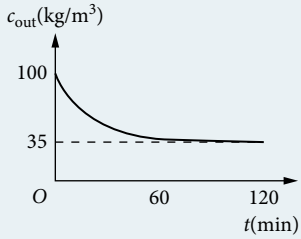
and thus, has the solution $m(t) = 7000 + 13\,000e^{\frac{-t}{20}}$

(c) Any decaying exponentials will vanish over time:

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 1000\left(7 + 13e^{\frac{-t}{20}}\right) = 7000 \text{ kg}$$

(d) The concentration c_{out} of the outflow is the mass divided by the volume of the tank: $c_{out} = \dfrac{m}{200}$ kg/m^3.

The mass exhibits an inhibited decay towards its equilibrium value of 7000 kg. This corresponds to a mass concentration of 35 kg/m^3.



EXERCISE 12.6 MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

1 Market research in a large city indicates that the maximum sales of a soon-to-be-released mobile device, is 10 truckloads per month (1 truckload = 10 000 devices).

Past experience with models iThingie1 through to iThingie6 indicates that the rate of growth in the truckloads of sales $\frac{ds}{dt}$, t months after the release of an iThingie, is directly proportional to the difference between the current sales and the maximum monthly sales.

(a) Find an equation for the rate of growth $\frac{ds}{dt}$ in the sales s as a function of the time t in months after the new product is first released onto the market. Express your answer in terms of the constant of proportionality r.

(b) Find the solution curve of your model. Express your answer in terms of the constant of proportionality r.

(c) If two truckloads are sold after one month, find the predicted number of truckloads per month after three months. (Express your answer correct to the nearest truckload.)

2 A simple model for the spread of a contagious illness assumes that the rate at which the illness spreads $\frac{dI}{dt}$ varies jointly with the product of the number of ill people I and the number of people still susceptible to the illness S. This means that $\frac{dI}{dt} = rIS$, $r > 0$.

Assume that one infected person is introduced into a fixed population of size P.
Then $P + 1 = I + S \therefore S = P + 1 - I$. Therefore, $\frac{dI}{dt} = rI(P+1-I)$, $I(0) = 1$ and $r > 0$.

(a) Show that $\frac{1}{I(P+1-I)} = \frac{1}{(1+P)}\left[\frac{1}{(1+P-I)} + \frac{1}{I}\right]$.

(b) Find I as a function of time.

3 A pond initially contains 200 000 litres of unpolluted water. A stream begins to flow into the pond at rate of 10 000 litres per day. The stream is polluted with a concentration of 2 grams of pollutant per litre. The pond also has an outlet that spills 10 000 litres of well-mixed water per day.

(a) State the initial value problem that models the mass of pollutant $m(t)$ grams in the pond, t days after the polluted stream first begins to flow into the pond.

(b) Hence find a differential equation that models the concentration of pollutant $c(t) = \frac{m(t)}{200\,000}$ grams per litre in the pond, t days after the polluted stream first begins to flow into the pond.

(c) Solve the model from part **(a)**.

(d) What is the concentration of pollutant in the pond after 10 days?

4 A tank initially contains 1000 litres of salt solution of concentration 0.01 kg/L. A solution of the same salt, but concentration 0.04 kg/L, flows into the tank at a rate of 10 litres per minute. The mixture in the tank is kept uniform by stirring and the mixture flows out at a rate of 5 litres per minute.

Let Q kg be the quantity of salt in the tank after t minutes. Set up (but do not solve) the differential equation for Q in terms of t, and specify the initial conditions.

5 Carbon monoxide (chemical symbol CO) is toxic to humans. Two hours of exposure to air with a volume concentration of CO at 0.02% will cause headaches and confusion.

During World War I, some generals commanded their soldiers from inside a bombproof bunker with an internal volume of 80 m^3. Troops resting near the air intake to the bunker would often smoke cigarettes. Unfortunately, the air intake to the bunker sucked the carbon-monoxide-filled smoke from the cigarettes back into the bunker.

Assume that smoky air is sucked into the bunker at a rate of 2 m^3/min, and that 0.03% of this air (by volume) is carbon monoxide. Ventilation fans keep the air well mixed inside the bunker, and the well-mixed air is extracted from the bunker at the same rate of 2 m^3/min. It can be shown that $\frac{dv}{dt} = 0.025(0.0003 - v)$, $v(0) = 0$.

(a) Solve this differential equation to find $v(t)$.

(b) Hence find the time for the volume fraction of carbon monoxide to reach 0.02% by volume inside the bunker. Express your answer in minutes, correct to the nearest minute.

6 A lottery winner puts \$5 000 000 in winnings into a fund that has a 5% annual rate of return, paid continuously throughout the year. Each year the winner spends \$300 000, withdrawn from the account at a continuous rate over the course of the year.

(a) Show that the differential equation to model the fund balance $x(t)$ after t years is $\frac{dx}{dt} = 0.05(x - 600\,000)$ and state the value of $x(0)$.

(b) Solve the differential equation in part **(a)**.

(c) Hence determine the balance after 20 years. Express your answer correct to the nearest 5 cents.

7 Which of the following is not a model for exponential growth y?

A $y = \frac{10}{e^{-t}}, t > 0$ **B** $y = 10(0.5)^t, t > 0$

C $\log_e \frac{1}{y} = \log_e 10 - 2t, t > 0$ **D** $\log_e(y) = \log_e(10) + t, t > 0$

8 A can of soft drink at a temperature of 24°C is placed in a freezer. The temperature inside the freezer is maintained at –4°C. When the can has been in the freezer for t minutes, the temperature of the can is T°C. The rate of decrease in the temperature of the can is proportional to the excess of its temperature over the temperature inside the freezer.

If k is a positive constant, the temperature of the can is modelled by the solution of:

A $\frac{dT}{dt} = -k(T - 24); \quad t = 0, T = -4$ **B** $\frac{dT}{dt} = -k(T - 24); \quad t = 0, T = -4$

C $\frac{dT}{dt} = -k(T + 4); \quad t = 0, T = 24$ **D** $\frac{dT}{dt} = -k(T - 4); \quad t = 0, T = 24$

9 The rate at which a rumour spreads throughout a population of 1000 students is proportional to the product of the number N of students who know the rumour and the number of students who haven't yet heard the rumour after t hours. If two students decide to start a rumour, the model that best decribes the spread of the rumour t hours later is:

A $\frac{dN}{dt} = k\frac{(1000 - N)}{1000}, \quad N(0) = 0$ **B** $\frac{dN}{dt} = k(N - 2)(1000 - N), \quad N(0) = 2$

C $\frac{dN}{dt} = kN(1000 - N), \quad N(0) = 0$ **D** $\frac{dN}{dt} = kN(1000 - N), \quad N(0) = 2$

10 A chemical dissolves in a pool at a rate equal to 10% of the amount of undissolved chemical. Initially the amount of undissolved chemical is 5 kg and after t hours x kilograms has dissolved. The differential equation that models this process is:

A $\frac{dx}{dt} = \frac{x}{10}$ **B** $\frac{dx}{dt} = \frac{5 - x}{10}$ **C** $\frac{dx}{dt} = \frac{x - 5}{10}$ **D** $\frac{dx}{dt} = 5 - \frac{x}{10}$

11 The rate of increase in the number of bacteria in a laboratory is directly proportional to the number present. If the number of bacteria triples every 2 hours, after how many hours will the number of bacteria be quadruple its initial value?

A $2\log_e \frac{4}{3}$ **B** $\frac{2\log_e 4}{\log_e 3}$ **C** $\left(\frac{\log_e 3}{2}\right)^2$ **D** $\frac{\log_e 3}{\log_e 2}$

12 In a simple model for the body mass m of an adult, the rate of change of body mass in kg/day varies directly with the difference between the total energy (food) intake C per day and the total energy expended per day. The energy expenditure per day depends upon the metabolic rate of the individual, but 165 kJ/kg of body mass is a realistic average value.

Assuming t is measured in days, the differential equation modelling the rate of change of body mass in kg/day could be:

A $\frac{dm}{dt} = k(165 - C), \quad k > 0$ **B** $\frac{dC}{dt} = k(165m - C), \quad k > 0$

C $\frac{dm}{dt} = k(C - 165m), \quad k > 0$ **D** $\frac{dC}{dt} = k(C - 165m), \quad k > 0$

13 A population $P(t)$ of an animal satisfies $\frac{dP}{dt} = \frac{1}{10}P\left(2 - \frac{P}{500}\right)$, $P(0) = 100$ with t measured in years. What is $P(t)$ as $t \to \infty$? You are given that $\frac{1}{P(1000-P)} = \frac{1}{1000}\left(\frac{1}{P} - \frac{1}{1000-P}\right)$.

A 25 **B** 250 **C** 500 **D** 1000

14 The growth rate of a tree varies jointly with the product of the current height h and the difference between the current height and the maximum height H. The differential equation modelling this growth could be:

A $\frac{dH}{dt} = r(h-H), \quad r>0$ **B** $\frac{dH}{dt} = rh(H-h), \quad r>0$

C $\frac{dh}{dt} = rh(h-H), \quad r>0$ **D** $\frac{dh}{dt} = rh(H-h), \quad r>0$

15 At any time $t \geq 0$ (in days), the rate of growth in the number of bacteria in a laboratory is directly proportional to the number N currently present. The initial population of bacteria is 1000.

(a) Assuming this growth rate continues indefinitely, write a differential equation to model the number of bacteria present in the dish after $t \geq 0$ days.

The initial population of 1000 bacteria triples during the first 2 days.

(b) Hence, show that $N(t) = a \times 3^{\frac{t}{b}}$ for a suitable choice of the positive integers a and b.

(c) By what factor will the population have increased in the first 4 days?

(d) How much time will it take for the population to grow to 10 times its initial value? Express your answer correct to the nearest hour.

Challenging

16 Almost all carbon in the world is carbon-12, which is the most common stable 'isotope' (nuclear form) of carbon. In the late 1940s the American scientist Willard Libby studied carbon-14, which is not stable: it radioactively decays according to the reaction ${}^{14}\text{C} \to {}^{14}\text{N} + \text{e}^- + \overline{\nu}_e$, in which a neutron spontaneously transforms into a proton (thus changing the atom from carbon C to nitrogen N) as it emits an electron and an antineutrino. In the upper atmosphere, carbon-12 sometimes transforms back into carbon-14 due to interactions with cosmic rays, so the proportion of both isotopes in the atmosphere stays relatively constant.

But whenever carbon is absorbed by plants to become part of living organisms in the world the carbon-12 is mostly shielded from transforming into carbon-14. This means that when an organism dies, its concentration of carbon-12 $\left(\left[{}^{12}\text{C}\right]\right)$ remains relatively constant, but its concentration of carbon-14 $\left(\left[{}^{14}\text{C}\right]\right)$ radioactively decays at the rate $\frac{d\left[{}^{14}\text{C}\right]}{dt} = -r\left[{}^{14}\text{C}\right], \quad r = 1.2097 \times 10^{-4}$ years^{-1}.

(a) Find the half-life $\left(t_{\frac{1}{2}}\right)$ of carbon-14, correct to the nearest year.

The ratio of carbon-12 to carbon-14 remains relatively constant in living organisms, roughly $R = \frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]} \approx 1.3 \times 10^{-12}$, but this ratio changes after the organism dies (because it stops absorbing new carbon-14 atoms from the atmosphere, while any carbon-14 present is still decaying). Consequently, you can determine the length of time since an organism's death by measuring how much this ratio has changed.

(b) Half the original carbon-14 has radioactively decayed. How many years ago did the tree die?

(c) Find a differential equation for the ratio $R = \frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]}$. (Hint: $[{}^{12}\text{C}]$ can be considered a constant.)

(d) Hence find a formula for the ratio R after t years in the form $R(t) = a\left(\frac{1}{2}\right)^{\frac{t}{n}}$ for a suitable choice of real number a and integer n.

(e) The skeleton of an extinct mega-marsupial is found to have a carbon ratio $\frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]} = 0.9 \times 10^{-4}$. How many years ago, to the nearest year, did the animal die?

17 Bubonic plague (known as the Black Death) ravaged Europe most severely between the years 1347 and 1351. It is estimated that the plague killed more than one in three people living in Europe at that time. The disease was survivable only in about 5% of cases.

Epidemiologists commonly use S to represent the fraction of a population that have survived an epidemic disease, t days after its arrival. The change in this fraction over time can be modelled by the differential equation: $\frac{dS}{dt} = -r(S - I)$, $S(0) = 1$, where I is the fraction of the population that ultimately recover (and hence survive).

(a) Find a formula for the survivability fraction S as a function of the time t days.

(b) Show that $\frac{d^2S}{dt^2} = r^2(S - I)$

After 1 month, the survivability fraction S approaches a steady state value of 0.05.

(c) Find I.

(d) Find the value of r if after 14 days, only 6% of the population has survived.

(e) Find the time when the death rate reaches its maximum value and state this death rate.

(f) Plot the survivability fraction over the first 2 weeks.

18 Two types of bacteria, type A and type B, coexist in a biological system. Assume that each population grows exponentially. The proportion of the total bacteria population belonging to type A is given by $p(t) = \frac{A(t)}{A(t) + B(t)}$.

(a) Express the growth rate $\frac{dp}{dt}$ in terms of $A(t)$, $B(t)$, $A'(t)$ and $B'(t)$.

(b) Given that $A'(t) = r_A A(t)$, $r_A > 0$ and $B'(t) = r_B B(t)$, $r_B > 0$, express the growth rate $\frac{dp}{dt}$ in terms r_A, r_B, A and B.

(c) Hence write the growth rate $\frac{dp}{dt}$ in terms r_A, r_B and p.

(d) If $p(0) = \frac{1}{10}$ and $r_A - r_B = \frac{1}{100}$ hour^{-1}, and given that $\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}$, find $p(10)$.

19 Toxins that enter an ecosystem are generally observed to be more concentrated per unit of biomass as you move up the food chain. This phenomenon is known as bioaccumulation.

Following one growth rate model, the length L (m) of a species of tuna at age t (years) is given by $\frac{dL}{dt} = \frac{1}{5}(3 - L)$, $L(0) = 0$, $0 \leq t \leq 20$.

(a) Find $L(t)$.

The weight W (kg) of a tuna is related to its length L by the equation $W(t) = 16L^3$. Let $H(t)$ (mg) be the accumulated mass of mercury (in milligrams) in a tuna after t years. The rate at which mercury is added to the tissue of the tuna is $\frac{dH}{dt} = \frac{1}{100}W$, $H(0) = 0$.

(b) Find $H(t)$.

(c) Find a formula for the concentration C of mercury in units of milligrams of mercury per kilogram of tuna, for a tuna of age t years.

(d) Assuming the tuna has a lifespan of 20 years, plot the mercury concentration in the tuna over its lifetime, $0 < t \leq 20$.

20 The British actuary and mathematician Benjamin Gompertz (1779–1865) proposed the following growth rate model: $\frac{dW}{dt} = rW\log_e\left(\frac{C}{W}\right)$ [1]

Numerous experimental studies have demonstrated that the growth rate $\frac{dW}{dt}$ of tumours is modelled by $\frac{dW}{dt} = \frac{1}{20}W\left(10 - \log_e W\right), \quad W(0) = e$ [2]

where W is the weight of the tumour (in milligrams) and t is the time in days.

(a) Given that W grows according to the Gompertz growth rate model [1], find the Gompertz parameters r and C from [2].

(b) Verify that $W = e^{10-9e^{\frac{-t}{20}}}$ is the solution of the differential equation [2].

(c) Find the equilibrium weight of the tumour, i.e. the weight of the tumour when it stops growing.

(d) Find the maximum growth rate for this tumour.

Treatment of many tumours is most effective when the tumour is growing at its fastest rate.

(e) After how long is the tumour growing at the fastest rate?

(f) Sketch the graph of W for the first 3 months, showing any equilibrium solutions and the point of maximum growth rate.

21 Oil is pumped from a Bass Strait oil well at a rate proportional to the volume V of oil (in units of 'barrels') remaining in the well after t years. Initially, the well had 1 000 000 barrels of oil, but 5 years later, only 600 000 barrels of oil remain. It will not be profitable to continue pumping oil when fewer than 1000 barrels of oil remain.

(a) Show that the volume of oil V remaining after t years is given by $V(t) = a\left(\frac{b}{c}\right)^{\frac{t}{d}}$ for a suitable choice of positive integers a, b, c and d.

(b) At what rate is the remaining number of barrels of oil decreasing after 5 years?

(c) For how many years will the oil well remain profitable? Express your answer to the nearest month.

22 A nature conservation group releases 21 Tasmanian devils onto a remote island off the coast of Tasmania. The group believes the island can support at most 588 Tasmanian devils. The growth rate of the Tasmanian devil population p is $\frac{dp}{dt} = rp\left(1 - \frac{p}{588}\right)$, $p(0) = 21$, with $r > 0$ and t measured in years.

(a) Show that $\frac{1}{p\left(1-\frac{p}{588}\right)} = \frac{1}{p} - \frac{1}{p-588}$.

(b) State the model for the Tasmanian devil population p in terms of r and time t.

Three years after the beginning of the breeding program, the population is 294.

(c) Find r.

(d) Use the model from part **(c)** to estimate the devil population after 6 years.

Six years after the Tasmanian devils are first taken to the island, the nature conservation group decides it is time to repopulate the mainland with the island's devils. A decision is made to take 140 devils from the island each year.

(e) Find a differential equation $\frac{dP}{dt} = f(P)$ for a suitable choice of the function $f(P)$ that models the modified growth rate in the island population $P(t)$, t years after the devils were first taken from the island back to the mainland. Assume that the value of r and the carrying capacity of devils on the island both remain unchanged.

(f) If $\frac{dP}{dt} = -K(P-a)(P-b)$, $P(0) = P_0$, find $P(t)$, given that $\frac{1}{(P-a)(P-b)} = \frac{1}{b-a}\left(\frac{1}{P-b} - \frac{1}{P-a}\right)$.

More challenging question parts:

(g) Express the function $f(P)$ found in part **(e)** in the form $f(P) = -\frac{\log_e 3}{588}(P-a)(P-b)$, $a < b$, for a suitable choice of real numbers a and b.

(h) Hence find a solution curve for the island population $P(t)$, t years after the decision to repopulate the mainland.

(i) Plot the Tasmanian devil population over the first 14 years of the breeding program. Hence, comment on the effectiveness of the breeding program.

CHAPTER REVIEW 12

1 A camera at ground level is 400 metres away from a hot air balloon just prior to the balloon lifting off. The balloon lifts off and the camera records the balloon rising into the sky at a constant rate of 10 metres per second.

(a) If θ is the angle of elevation of the balloon, express the height h of the balloon in terms of this angle.

(b) How fast is the angle of elevation θ radians changing when the balloon is 300 m above the ground?

2 Verify that $y = e^{-x}\cos x$ is a solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

3 The gradient of the tangent to a curve at any point (x, y) is $\frac{x}{x+1}$, $x > -1$. If the curve passes through the point $(1, 1)$, find the equation of the curve.

4 **(a)** If $f''(x) = \cos 2x - 3\sin x$ and $f'(0) = 3$, find $f'(x)$.

(b) Hence if $f\left(\frac{\pi}{2}\right) = 3$, find $f(x)$.

5 Solve the differential equation $\frac{dy}{dx} = 1 - 2y$, given that $y = -1$ where $x = 0$.

6 **(a)** Show that $\frac{4}{y^2 - 4} = \frac{1}{y-2} - \frac{1}{y+2}$.

(b) Find y as a function of x if y satisfies the differential equation $\frac{dy}{dx} = y^2 - 4$ and $y = 0$ where $x = 0$.

7 A species of tuna is declining so that T, the number of tuna at a time t years from now, satisfies the differential equation $\frac{dT}{dt} = -0.1T$.

(a) Write the general solution to this differential equation, where $T(0) = A > 0$ is the initial population.

(b) Find the time it will take for the numbers to fall to one-quarter of their present value.

8 Consider the initial value problem $\frac{dy}{dx} = 2x\left(1 + y^2\right)$, $y(0) = 1$. Find the exact solution to the differential equation.

9 If $\sin x = e^y$, $0 < x < \pi$, what is $\frac{dy}{dx}$ in terms of x?

A $-\cot x$ **B** $\tan x$ **C** $-\tan x$ **D** $\cot x$

10 Which one of the following differential equations is *not* satisfied by $x = e^{-3t}$?

A $\left(\frac{dx}{dt}\right)^2 - 9x^2 = 0$ **B** $\frac{d^2x}{dt^2} - 9x = 0$ **C** $\frac{d^2x}{dt^2} + 9x = 0$ **D** $\frac{dx}{dt} + 3x = 0$

11 If $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$, then the possible values of k are:

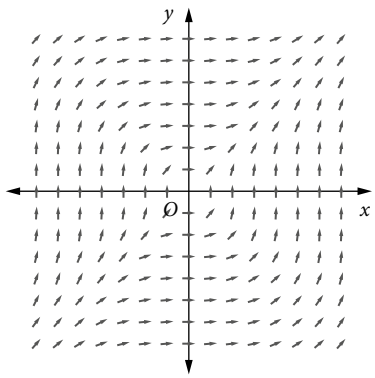
A 1 and -6 **B** -1 and 6 **C** -2 and 3 **D** 2 and -3

12 If $\frac{dy}{dx} = 2 - 3\sqrt{x}$ and $y = 3$ where $x = 1$, then y is given by the rule:

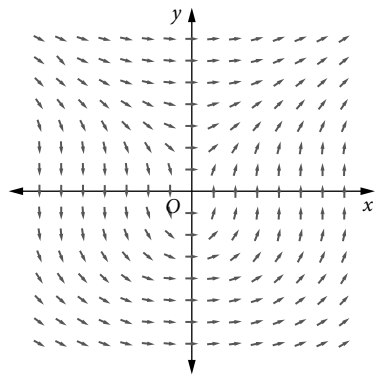
A $1 + 3x(1 - \sqrt{x})$ **B** $3 + 2x(1 - \sqrt{x})$ **C** $3 + 2x + 2x\sqrt{x}$ **D** $6\sqrt{3} - 5 + 2x - 2x\sqrt{x}$

13 What is the slope field of $y' = \frac{x^2}{y^2}$?

A



B



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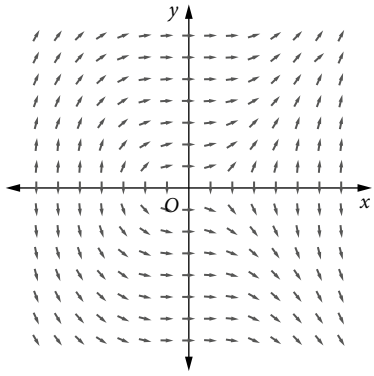

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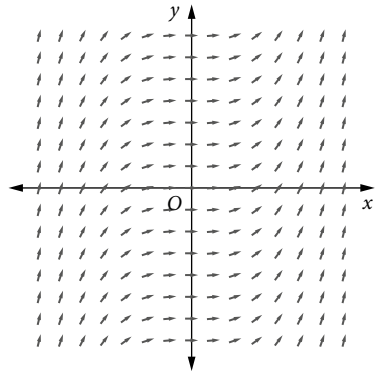


14 If $\frac{dy}{dx} = -3y$, and $y = 1$ where $x = 2$, then which of the following is true?

A $y = 2e^{3-3x}$ **B** $y = e^{6-3x}$ **C** $y = 2e^{3x-3}$ **D** $y = e^{3x-6}$

15 The general solution of $\frac{dy}{dx} = -(2 + y)$ is:

A $y = e^{-x} + A$, A is a real number **B** $y = Ae^{-x} + 2$, A is a real number

C $y = Ae^{x} + 2$, A is a real number **D** $y = Ae^{-x} - 2$, A is a real number

16 Consider the differential equation $\frac{dy}{dx} = x - 2y$, for which the solution is $g(x)$. Which of the following statements about the particular solution that contains the point $(0, -1)$ is true at $x = 0$?

A the graph is increasing and concave up **B** the graph is increasing and concave down

C the graph is decreasing and concave up **D** the graph is decreasing and concave down

Questions 17 and 18 refer to the following information.

Consider the differential equation $\frac{dy}{dx} = y \sin x$, for which the solution is $y = f(x)$. Let $f(0) = 1$.

17 Which of the following statements about the graph of $f(x)$ are true?

(i) The slope of $f(x)$ at the point $\left(\frac{\pi}{2}, 1\right)$ is 1.

(ii) $f(x)$ has a horizontal tangent where $x = 0$.

(iii) $f(x)$ has a vertical tangent where $y = 0$.

A i only **B** ii only **C** i and ii only **D** ii and iii only

18 The particular solution is:

A $y = e^{1 - \cos x}$ **B** $y = e^{\cos x - 1}$ **C** $y = e^{-\sin x}$ **D** $y = e^{\sin x}$

19 When added to water, 5 grams of a substance dissolves at a rate equal to 10% of the amount of undissolved chemical per hour. If x is the number of grams of *undissolved* chemical after t hours, then x satisfies the differential equation:

A $\frac{dx}{dt} = -\frac{1}{10}x$ **B** $\frac{dx}{dt} = -\frac{1}{5}x$ **C** $\frac{dx}{dt} = \frac{1}{5}(10-x)$ **D** $\frac{dx}{dt} = \frac{1}{10}(5-x)$

20 The population $P(t)$ of a certain species satisfies the differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{10\,000}\right)$ with an initial population $P(0) = 4000$, where t is the time in years. What is the population as t approaches infinity?

Note that $\frac{1}{P} + \frac{1}{20\,000 - P} = \frac{20\,000}{P(20\,000 - P)}$.

A 4000 **B** 5000 **C** 10 000 **D** 20 000

21 A quantity of sugar is dissolved in a tank containing 100 litres of pure water. At time $t = 0$ minutes, pure water is poured into the tank at a rate of 4 litres per minute. The tank is kept well stirred at all times. At the same time, the sugar solution is drained from a tap at the bottom of the tank at a rate of 6 litres per minute. A differential equation for the mass m grams of sugar in the tank is:

A $\frac{dm}{dt} = -6m$ **B** $\frac{dm}{dt} = 4 - \frac{3m}{50}$ **C** $\frac{dm}{dt} = -\frac{3m}{50-t}$ **D** $\frac{dm}{dt} = 4 - \frac{3m}{50-t}$

22 According to Fourier's law of heat conduction, the rate of heat transfer $\frac{dQ}{dt}$ through an ice sheet in Antarctica is given by the differential equation $\frac{dQ}{dt} = \frac{k(T_w - T_a)}{h}$, where k is the thermal conductivity of the ice, h is the thickness of the ice sheet and T_w and T_a are the temperatures at the ice/water boundary and the ice/air boundary respectively.

As the water loses Q joules of heat through the ice sheet, the rate of increase in ice thickness h is given by $\frac{dh}{dQ} = \frac{1}{L\rho}$, where L is the latent heat of sea water (in other words, the amount of heat loss required to freeze 1 kilogram of it) and ρ is the density of the ice.

(a) Find the rate of increase of the ice sheet thickness $\frac{dh}{dt}$.

(b) If $h(0) = h_0$, find $h(t)$, assuming that $\frac{k(T_w - T_a)}{L\rho}$ is a positive constant.

23 The quantity q sold of a new product is a function of the selling price p. The revenue $R(p) = p\,q(p)$ from selling q units at price p is also a function of the selling price. It can be shown that $\frac{dR}{dp} = q\left(1 + \frac{p}{q}\frac{dq}{dp}\right)$.

Economists call $E = \frac{p}{q}\frac{dq}{dp}$ the *price elasticity* of demand q with respect to price p. This price elasticity of demand measures the relative change of the quantity demanded in response to given relative change in price.

(a) Show that revenue is a maximum where $E = -1$.

The inverse demand curve of a product measures the selling price $p(q)$ in terms of the available supply q. The law of demand states that, 'when the price of a good rises, and everything else remains the same, the quantity of the good demanded will fall'. In other words, selling price varies inversely with the demanded supply. Suppose that a business has the exclusive right to import a new product into a large city. Market research indicates that the selling price \$$p$ for a new product is modelled by $p = \sqrt{25 - q^2}$, where the supply q is measured in units of 1000.

(b) State the domain of the model.

(c) Find the rate of change of the demanded quantity q with respect to the selling price p. (Express your answer in terms of q.)

(d) Hence find the selling price that will maximise revenue.

24 The relationship between air temperature T K and wind speed V m s^{-1} is modelled by the differential equation $\frac{dT}{dV} = -\frac{V}{C_P}$, where C_p is a constant known as the specific heat of the air mass.

(a) Find the general solution for the wind speed $V(T)$ as a function of temperature.

Near a particular tropical cyclone the wind speed is zero and the temperature is 308 K.

(b) Find the particular solution for the wind speed $V(T)$ as a function of temperature.

(c) If $C_p = 1004.6$, find the wind speed near the centre of a tropical cyclone if the temperature there is 307 K.

25 A mathematical model for the relationship between x the number of predators (in hundreds) and y the number of prey (in thousands), in a particular environment at time t years, gives the following pair of differential equations: $\frac{dy}{dt} = -\frac{1}{400}(x-2)$ $\quad \frac{dx}{dt} = \frac{1}{100}(y-1)$

(a) Use the chain rule to obtain a differential equation involving $\frac{dy}{dx}$, x and y only.

(b) At some point in time there are 100 predators ($x = 1$) and 3000 prey ($y = 3$). Hence show that $a^2(x-h)^2 + b^2(y-k)^2 = 1$ for real numbers a, b, h and k.

(c) The graph of the solution in part **(b)** is an ellipse with a centre at (h, k). Sketch your solution from part **(b)**.

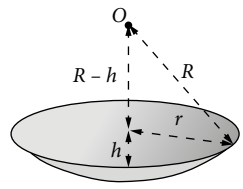
(d) From your previous answers, find the maximum and minimum number of predators the environment can support.

26 This question uses data from the table below.

$V(h) = \frac{1}{3}\pi h^2(3R-h)$	Volume of spherical cap (m^3)
$A_1(h) = \pi h(2R-h)$	Free surface area of a spherical cap (m^2)
$A_2(h) = 2\pi hR$	Curved surface area of a spherical cap (m^2)
ε	Evaporation coefficient for lake surface (m^3/m^2 day)
σ	Seepage coefficient for lake bottom (m^3/m^2 day)
Q_m	Net rate of inflow from rivers (m^3/day)

The Lake Eyre basin is the lowest point in Australia and usually dry. However, the southern lake of the Lake Eyre complex fills to a depth of about 3 metres via inflow from flooding rivers, a few times each century.

This southern lake can be modelled as the cap of a sphere of radius R (m), with a centre at O, as shown. Assume the inflow from the flooding rivers has stopped so that $Q_m \approx 0$ (m^3/day).



A possible balance for the volume of water in the southern lake of the Lake Eyre complex during a flood year is $\frac{dV}{dt} = Q_m - \varepsilon A_1(h) - \sigma A_2(h)$.

(a) Use the information from the table above to express $\frac{dV}{dt}$ as an explicit function of h.

(b) Show that $\frac{dV}{dh} = \pi h(2R-h)$ and hence find $\frac{dh}{dt}$ in simplest form.

The depth of the water in the lake can now be modelled immediately after an inflow of 1 gigalitre (1 000 000 m^3), which fills the lake to a depth of 3 m above its centre so that $h_0 = 3$ m. (Assume no further inflows or precipitation.)

(c) Find the radius R of the hemispherical cap.

27 The coroner arrives at Frogmorton Manor on a cold winter evening. Lady Frogmorton has just been found dead in her climatically controlled greenhouse by the butler. The temperature of the body is taken, and is recorded as 29°C at precisely 7:20 pm. The coroner checks the programmable thermostat and determines the greenhouse has been kept at a constant 20°C for the past 12 months. At precisely 8:20 pm, the coroner takes the temperature of the body once more, and it is recorded as 27.4°C.

Smiggins is Lady Frogmorton's chauffer. Under questioning by the police, he claims that he delivered Lady Frogmorton to the manor at around 6 pm that evening, after a day at the Chelsea Flower Show.

(a) Assuming a normal body temperature of 37°C, propose a differential equation to model the temperature $T(t)$ of the deceased at a time t hours since death.

(b) Solve the initial value problem in part **(a)**. Recall that the coroner arrived on the scene at precisely 7:20 pm, which is τ hours since death.

(c) Show that at τ hours after death $e^{r\tau} = \frac{17}{9}$.

(d) Find an expression, similar to your answer in part **(c)**, for the temperature at 8:20 pm.

(e) Hence find r.

(f) The coroner informs the police that the chauffer's story may need further investigation. Explain why this is so.

Challenging

28 Bob's credit card bill B is initially \$15 000 and he pays 18% interest on this debt per year, compounded continuously. He decides to pay it off by transferring money from his savings account continuously at the rate of \$300 per month.

(a) Find and solve a differential equation to model the credit card balance B after t years.

(b) How much time will it take to pay off the credit card bill (to the nearest day)?

(c) What is the sum total of Bob's repayments?

Assume Bob has \$40 000 in a savings account that accumulates interest at an annual rate of 6%, also compounded continuously.

(d) Find and solve a differential equation to model the balance S of Bob's savings account.

(e) How much money will Bob have in his savings account when the debt is finally paid off (assuming no other transactions)?

29 An abandoned open-cut mine just outside a large city has been purchased as a landfill for solid waste by a city council. When purchased, the open-cut mine had a volume of 1 million cubic metres. At the beginning of 2015, the landfill already had 100 000 cubic metres of solid waste. The volume of solid waste W in the landfill (measured in units of 100 000 cubic metres) t years after the beginning of 2015 is modelled by the solution of the differential equation $\frac{dW}{dt} = \frac{1}{10}(10 - W)$, $W(0) = 1$.

(a) Find the volume of solid waste in the landfill t years after 2015.

(b) Hence determine the volume of solid waste in the landfill at the beginning of 2035. (Express your answer in cubic metres, correct to the nearest cubic metre.)

30 The population $P(t)$ of penguins on an island in the Southern Ocean t years after the beginning of 2015 grows at a rate directly proportional to $1000 - P(t)$, where the constant of proportionality is k.

(a) If the population at the beginning of 2015 is 200, express the penguin population t years after the beginning of 2015 in terms of t and k.

(b) If the population after 2 years is 300, find k.

(c) Hence determine the long-term population of penguins on the island.

31 While on an unauthorised trip to the local fast food restaurant during their study period, a pair of Year 12 students are convinced that they have just seen the Prime Minister buying a hamburger. On returning to the school, their amazing discovery spreads throughout the school community at the rate $\frac{dp}{dt} = \frac{1}{10}p(1-p)$, where p is the proportion of the school community that has already heard the rumour, t minutes after their return to school.

(a) What proportion of the school community has heard the rumour when it is spreading most rapidly?
By the beginning of the afternoon period, 20% of the school community had already heard the rumour.

(b) Find $p(t)$, at time t minutes since the beginning of the afternoon period, given $\frac{1}{p(1-p)} = \frac{1}{p} - \frac{1}{p-1}$.

(c) At what time (correct to the nearest minute) is the rumour spreading most rapidly?

CHAPTER 13
Motion, forces and projectiles

13.1 PROBLEMS INVOLVING DISPLACEMENT AND VELOCITY

Displacement

When considering the motion of an object, you need to consider the position of the object, how fast it is travelling and the cause of the motion.

The position of an object must be defined relative to a reference point, which is usually its starting point. The position can be described by both its distance and its direction from the starting point. This is a vector quantity, called **displacement**. On the other hand, if you are only concerned with how far the object has travelled then the direction can be ignored, so you can consider the **distance** travelled by the object. The international system of units (SI) uses the metre (m) as the standard unit for displacement and for distance. Other common SI units include the centimetre (cm), millimetre (mm) and kilometre (km).

For example, if you walk around a square park of side length 500 m, the distance you travel is 2000 m. However, as you arrive back at your starting point, your displacement is zero.

Velocity

The rate at which the displacement of an object changes with respect to time can be described by the vector quantity **velocity**. As a vector quantity, velocity is defined by its magnitude and direction. The magnitude of velocity is the scalar quantity **speed**. The SI standard unit for velocity and speed is metres per second (m/s or m s^{-1}). Another common unit is kilometres per hour (km/h or km h^{-1}).

The average velocity of an object between two positions is defined as:

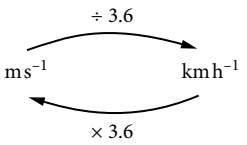
$$\text{average velocity} = \frac{\text{change in position}}{\text{time taken}} = \frac{\text{displacement}}{\text{time}}$$

On the other hand, the average speed between two points is defined as:

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Note: Speed may need to be converted from m s^{-1} to km h^{-1} or from km h^{-1} to m s^{-1}:

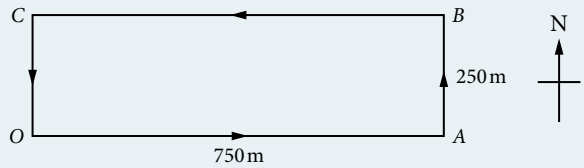
For example: $90\,\text{km h}^{-1} = 90 \times 1000 \div (60 \times 60) = 90 \div 3.6 = 25\,\text{m s}^{-1}$
and $15\,\text{m s}^{-1} = 15 \times 3.6 = 54\,\text{km h}^{-1}$.



- *Displacement* is the change in position of an object relative to its starting point. It is a vector quantity with both magnitude and direction.
- *Distance* is how far an object has travelled. It is a scalar quantity with magnitude only.
- *Velocity* is the rate at which the displacement of an object is changing with respect to time. It is a vector quantity with both magnitude and direction.
- *Speed* is the magnitude of the object's velocity. It is a scalar quantity with magnitude only.

Example 1

Marli is going for a walk around the block. She starts at O and moves to point A and then to point B, where she stops to talk to a friend.



Determine:

(a) the distance Marli travels from O to B

(b) Marli's displacement from O to B, correct to one decimal place.

If Marli takes 10 minutes to reach point B, determine:

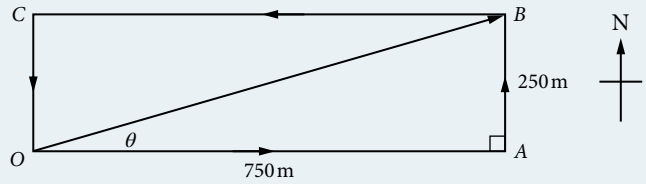
(c) her average speed in metres per second, correct to one decimal place

(d) her average velocity in metres per second, correct to one decimal place.

Solution

(a) Distance travelled from O to point A to point $B = 750 + 250 = 1000$ m

(b) Using Pythagoras' theorem to find the distance between the starting and finishing points:



$$|\overrightarrow{OB}| = \sqrt{750^2 + 250^2}$$

$$= 790.6 \text{ m}$$

Direction of motion: $\theta = \tan^{-1}\left(\frac{250}{750}\right)$

$$= 18.4°$$

Bearing is $N(90 - 18.4)°E = N71.6°E$

Marli's displacement is 790.6 m from O in a direction of N71.6°E.

(c) 10 min = 600 s

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{1000}{600}$$

$$= 1.7 \text{ m s}^{-1}$$

(d) $$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

$$= \frac{790.6}{600}$$

$$= 1.3 \text{ m s}^{-1} \text{ in the direction N71.6°E.}$$

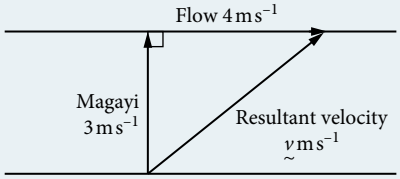
Finding resultant velocity

Example 2

Magayi can swim in still water at a rate of 3.0 m s^{-1}. If she swims in a river that is flowing at 4.0 m s^{-1} and keeps her direction (with respect to the water) perpendicular to the flow, find the magnitude of her velocity with respect to the riverbank.

Solution

Vector diagram to illustrate the situation:



Using Pythagoras' theorem to find the magnitude of the resultant velocity $\underset{\sim}{v}$:

$$|\underset{\sim}{v}| = \sqrt{3^2 + 4^2}$$

$$= 5 \text{ m s}^{-1}$$

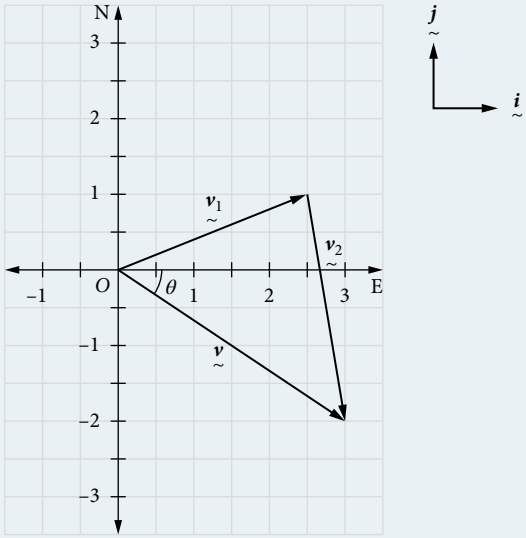
Finding resultant velocity using component form

Example 3

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the directions of east and north, respectively. Pravat is swimming in the ocean and his velocity relative to the water $\underset{\sim}{v}_1\,\text{m s}^{-1}$ is given by the vector $\underset{\sim}{v}_1 = 2.5\underset{\sim}{i} + 1.0\underset{\sim}{j}$. The ocean's current has a velocity $\underset{\sim}{v}_2\,\text{m s}^{-1}$ where $\underset{\sim}{v}_2 = 0.5\underset{\sim}{i} - 3.0\underset{\sim}{j}$. Find the magnitude and direction of Pravat's resultant velocity $\underset{\sim}{v}\,\text{m s}^{-1}$, correct to one decimal place.

Solution

Vector diagram to illustrate the situation:



Adding vectors to find the resultant velocity:

$$\begin{aligned}\underset{\sim}{v} &= \underset{\sim}{v}_1 + \underset{\sim}{v}_2\\ &= 2.5\underset{\sim}{i} + 1.0\underset{\sim}{j} + 0.5\underset{\sim}{i} - 3.0\underset{\sim}{j}\\ &= 3.0\underset{\sim}{i} - 2.0\underset{\sim}{j}\\ |\underset{\sim}{v}| &= \sqrt{(3.0)^2 + (-2.0)^2}\\ &= \sqrt{13.0}\\ &= 3.6\,\text{m s}^{-1}\end{aligned}$$

θ is an acute angle in the diagram, so use positive values for side lengths.

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{2.0}{3.0}\right)\\ &= 33.7^\circ\end{aligned}$$

The direction bearing: $90 + 33.7 = 123.7^\circ$T.

Pravat is swimming at $3.6\,\text{m s}^{-1}$ in a direction of 123.7°T.

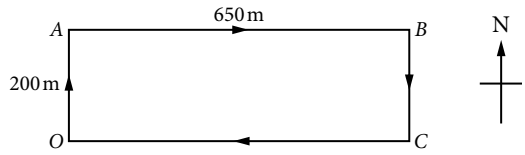
EXPLORING FURTHER

Velocity vectors

Use technology to explore combinations of velocity vectors.

EXERCISE 13.1 PROBLEMS INVOLVING DISPLACEMENT AND VELOCITY

1 Jaide is going for a walk around the block. She starts at O and moves to point A and then to point B, where she stops to have a rest.



(a) What is the distance Jaide has travelled from O to B?

(b) What is Jaide's displacement from O to B, correct to one decimal place, and what is her direction?

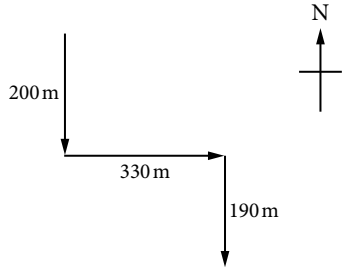
Jaide has taken 12 minutes to reach point B.

(c) What is Jaide's average speed in metres per second, correct to one decimal place?

(d) What is Jaide's average velocity in metres per second, correct to one decimal place, and what is her direction?

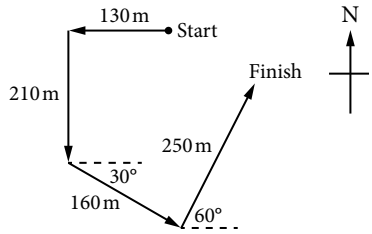
2 Courtney walks 200 m south, then 330 m east, and finally 190 m south (diagram shown is not to scale).

What is Courtney's displacement for her entire walk, correct to one decimal place, and what is the direction?



3 A mountain-climbing expedition establishes its base camp and two intermediate camps at positions A and B. Camp A is 8400 m west of and 1800 m above base camp. Camp B is 5900 m west of and 850 m higher than Camp A. What is the displacement of camp B from the base camp and what is its angle of elevation? Give answers correct to one decimal place.

4 Kenneth is taking his dog for a walk along the path in the local park, as shown in the diagram on the right (not to scale).



What is the magnitude and direction of Kenneth's resultant displacement, correct to the nearest whole number?

5 While exploring a recently discovered cave system, a spelunker (cave explorer) starts at the entrance and makes the following movements: 85 m north, 190 m east, 250 m N45°E, and 100 m south.
What is the spelunker's final displacement from the cave entrance and what is their direction?

6 Brianna can swim in still water at a rate of 6.0 km h^{-1}. If she swims in a river flowing at 4.5 km h^{-1} and keeps her direction (with respect to the water) perpendicular to the flow, then what is the magnitude of her velocity (correct to one decimal place) with respect to the riverbank?

7 **(a)** A river flows at 5 km h^{-1} and Rhani rows at 10 km h^{-1}. In what direction should Rhani row to go straight across the river?

A Angle of 30° to the riverbank downstream
B Angle of 60° to the riverbank downstream
C Angle of 30° to the riverbank upstream
D Angle of 60° to the riverbank upstream

(b) If the river flows at 10 km h^{-1} and Rhani rows at 5 km h^{-1}, then in what direction should she row to go straight across the river?

A Angle of 30° to the riverbank downstream
B Angle of 60° to the riverbank downstream
C Angle of 30° to the riverbank upstream
D It is not possible

8 Youlin and Nick are riding in a boat that has a speed relative to the water of 3.00 m s^{-1}. The boat points at an angle of 35° to the shore, moving upstream on a river that is flowing at 0.75 m s^{-1}.

How much time does it take for the boat to reach the opposite shore, correct to two decimal places, if the river is 50 m wide?

9 **(a)** Raphaela is watching an aircraft that is flying at a velocity of 100 m s^{-1} N with respect to the flow of air. If the velocity of the wind is 10 m s^{-1} north, what is the resultant velocity of the aircraft relative to Raphaela?

(b) Later, she observes a second aircraft flying with a velocity of 125 m s^{-1} N with respect to the flow of the air. If the flow of the air has a velocity of 10 m s^{-1} south, then the resultant velocity of this aircraft is 115 m s^{-1}. What is the direction of this resultant velocity, relative to Raphaela?

(c) The next day, Raphaela observes another aircraft flying with a velocity of 100 m s^{-1} N which encounters a wind coming from the side at a rate of 25 m s^{-1} W. What is the resultant velocity of this aircraft relative to Raphaela, correct to one decimal place, and what is its direction?

10 Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the directions of east and north respectively. Mitchell is swimming in the ocean and his velocity $\underset{\sim}{v}_1$ m s^{-1} is given by the vector $\underset{\sim}{v}_1 = 1.0\underset{\sim}{i} - 2.0\underset{\sim}{j}$. The ocean's current has a velocity $\underset{\sim}{v}_2$ m s^{-1} where $\underset{\sim}{v}_2 = -0.5\underset{\sim}{i} + 2.5\underset{\sim}{j}$. What is the magnitude and direction of Mitchell's resultant velocity $\underset{\sim}{v}$ m s^{-1}, correct to one decimal place, and what is its direction?

11 Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the directions of east and north respectively. Give answers correct to two decimal places where required.

(a) Express each vector in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$.

(i) $\overrightarrow{OA} = 5.0$ m at 053°T **(ii)** $\overrightarrow{AB} = 6.0$ m at 315°T **(iii)** $\overrightarrow{BC} = 4.0$ m at 240°T **(iv)** $\overrightarrow{CD} = 3.0$ m at 150°T.

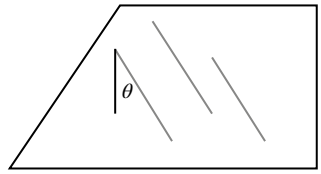
(b) Express the sum of the four displacement vectors in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$, with values correct to two decimal places.

(c) What is the magnitude and direction of the resultant vector $\overrightarrow{OD}$, correct to two decimal places, and what is its direction?

12 Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the directions of east and north respectively. Morgan and Eilish finish their last day of school and immediately decide to take a road trip across the desert in Eilish's new car. Their trip involves the following movements: 43 km at 337°T, 65 km at 270°T and 22 km at 93°T. Eilish's car breaks down after the last leg of the trip.

At this time, how far are Morgan and Eilish from the school and in what direction?

13 Johanna is driving along the freeway at 108 km h^{-1} when it begins to rain. She observes the raindrops running down the driver's side window.



Calculate the angle that the raindrops make with the vertical window, as seen by Johanna, if the raindrops have a speed of 10 m s^{-1} relative to the Earth's surface. Assume that the raindrops fall vertically down, relative to the Earth's surface.

14 Bus #1 is moving at a speed of 50 km h^{-1}, while Bus #2 is moving at 30 km h^{-1} in the opposite direction. What is the relative velocity of Bus #1 with respect to Bus #2?

A 20 km h^{-1} in the direction of Bus #1 **B** 20 km h^{-1} in the direction of Bus #2
C 80 km h^{-1} in the direction of Bus #1 **D** 80 km h^{-1} in the direction of Bus #2

15 On her way home from school, Teluila walks along three streets after exiting the school gate. She walks 240 m east, 720 m north and 75 m east. What is the magnitude of Teluila's resultant displacement?

A 555 m **B** 763 m **C** 786 m **D** 1035 m

16 Freddie rows his boat in a direction perpendicular to the riverbanks. The river is 200 m wide. The boat's speed in this direction is 2 m s^{-1}. The speed of the river's flow is 1 m s^{-1} and the riverbanks are parallel straight lines.

(a) Calculate the velocity of the boat relative to the riverbank.
(b) At what distance, in the direction downstream from his starting point, will Freddie get to the other riverbank?
(c) Calculate Freddie's final displacement from his starting point.
(d) Rohanne is going to cross the river in another rowboat. She can row at the speed of 2 m s^{-1} relative to the river's flow. At what direction should Rohanne row to get to the other riverbank at the same point as Freddie, and how much time will this take?

17 The pilot of an aircraft flying due south is notified by the flight controller that there is a second aircraft flying due north in the same general area and at the same altitude. The pilot is told that the northbound aircraft is currently located at a position that is 12.6 km, 170°T with respect to the pilot's aircraft.

(a) How many kilometres to the south is the second aircraft?
(b) How many kilometres to the east is the second aircraft?
(c) If the two aircrafts both have an airspeed of 300 km h^{-1}, how much time (in seconds) will elapse before they are *side by side*?

18 An aircraft drops a package of emergency rations to a family stranded in the floods. The aircraft is travelling horizontally at 45.0 m s^{-1} and is 100 m above the ground. A parachute allows the package to fall with constant speed and hit the ground 10 s after release.

(a) Find where the package hits the ground, relative to the point from where it was dropped, to the nearest metre.
(b) Find the velocity of the package just before it hits the ground, correct to one decimal place.

13.2 PROBLEMS INVOLVING FORCES

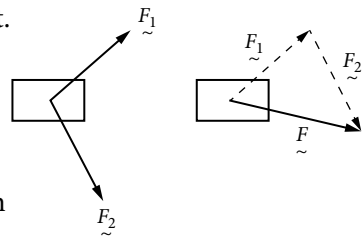
Forces exist everywhere and are fundamental to the structure of the universe and to the nature of matter.

A **force** can be thought of as a push or a pull acting on an object. For most simple objects, it is reasonable to consider the forces to be acting through a single point at the object's centre of mass. (This means the object is being considered as a **particle**, as defined earlier in Chapter 7, section 7.2 Velocity and acceleration as rates of change.)

The action of a force can affect an object by changing the speed or direction of its motion, or by deforming the object. The amount of force acting on an object is measured using the SI standard unit called the newton (N). A force of 1 newton will accelerate an object of mass 1 kilogram at a rate of 1 metre per second, so 1 N = 1 kg m s^{-2}.

Every object near the Earth's surface is subject to a force called gravity. This force is called the **weight** of the object and it acts vertically downwards on every object, towards the centre of mass of the Earth. The force due to gravity (the weight force) of a body of mass 1 kilogram is 9.8 N. For example, a person whose mass is 80 kg has a weight equal to their mass multiplied by g, where $g = 9.8\,\text{m s}^{-2}$, so an 80 kg person weighs $80 \times 9.8 = 784\,\text{N}$.

Force is a vector quantity, so it needs both magnitude and direction to fully describe it. If more than one force is acting on an object, the sum of all the forces is the **resultant force** or **net force**. The object will move as if the net force is the only force that is actually acting on the object.



For example, in the diagram on the right, the resultant force acting on the object is $\underset{\sim}{F}$, the sum of the two forces $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ acting on the object: $\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2$. This principle can be extended to involve the addition of many forces.

The resultant force or net force $\underset{\sim}{F}$ is the vector sum of all the forces acting on an object.

If forces $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3, \ldots \underset{\sim}{F}_n$ act on an object, then $\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 + \ldots \underset{\sim}{F}_n$

Calculating resultant forces

Example 4

Forces are acting on an object. Calculate the resultant force on the object for each of the following.

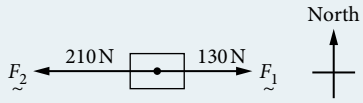
(a) $\underset{\sim}{F}_1 = 130\,\text{N}$ east, $\underset{\sim}{F}_2 = 210\,\text{N}$ west **(b)** $\underset{\sim}{F}_1 = 240\,\text{N}$ north, $\underset{\sim}{F}_2 = 165\,\text{N}$ west

Solution

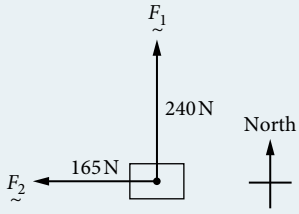
(a) Vector diagram to show the forces acting on the object:

$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2$

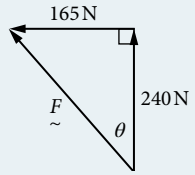
$= 130\,\text{N east} + 210\,\text{N west} = 80\,\text{N west}$



(b) Vector diagram to show the forces acting on the object:



Vector diagram to add the forces:



Using Pythagoras' theorem: $\left|\underset{\sim}{F}\right| = \sqrt{240^2 + 165^2}$

$= 291.25\ \text{N}$

Direction of resultant force: $\tan\theta = \dfrac{165}{240}$

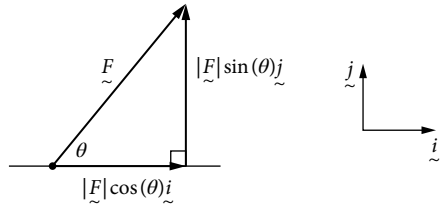
$\theta = \tan^{-1}\left(\dfrac{165}{240}\right)$

$\theta = 34.51°$

The resultant force is 291.25 N in the direction N34.51°W.

If the forces acting on an object are in the same plane, then each of the forces and the resultant force can be expressed in component form as a sum of two perpendicular vectors, using horizontal $\left(\underset{\sim}{i}\right)$ and vertical $\left(\underset{\sim}{j}\right)$ components. The force $\underset{\sim}{F}$ can be resolved into the two perpendicular components as $\underset{\sim}{F} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

For example, from the diagram: $\underset{\sim}{F} = \left|\underset{\sim}{F}\right|\cos\theta\,\underset{\sim}{i} + \left|\underset{\sim}{F}\right|\sin\theta\,\underset{\sim}{j}$.



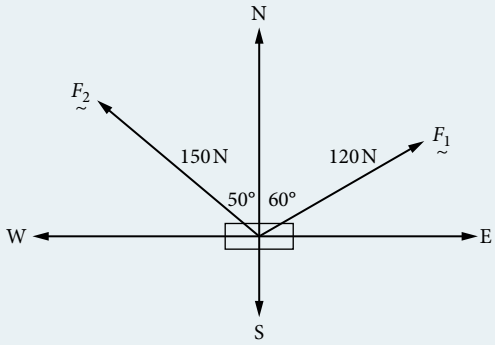
Calculating resultant forces using vector components

Example 5

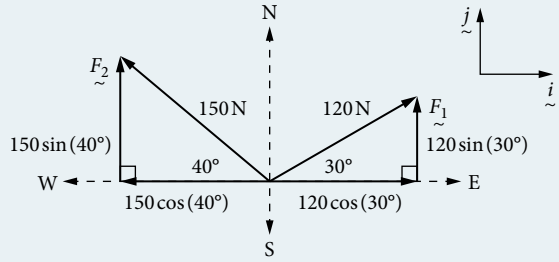
Two forces are acting on an object: $\underset{\sim}{F}_1 = 120\,\text{N}$ acting at N60°E and $\underset{\sim}{F}_2 = 150\,\text{N}$ acting at N50°W. Calculate the resultant force $\underset{\sim}{F}$ acting on the object.

Solution

Vector diagram to show the forces acting on the object:



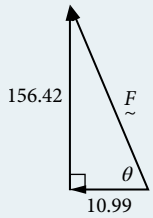
Resolving forces into horizontal $\left(\underset{\sim}{i}\right)$ and vertical $\left(\underset{\sim}{j}\right)$ components:



$$\underset{\sim}{F}_1 = 120\cos 30°\,\underset{\sim}{i} + 120\sin 30°\,\underset{\sim}{j}$$
$$= 103.92\,\underset{\sim}{i} + 60.00\,\underset{\sim}{j}$$

$$\underset{\sim}{F}_2 = -150\cos 40°\,\underset{\sim}{i} + 150\sin 40°\,\underset{\sim}{j}$$
$$= -114.91\,\underset{\sim}{i} + 96.42\,\underset{\sim}{j}$$

$$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2$$
$$= 103.92\,\underset{\sim}{i} + 60.00\,\underset{\sim}{j} - 114.91\,\underset{\sim}{i} + 96.42\,\underset{\sim}{j}$$
$$= (103.92 - 114.91)\,\underset{\sim}{i} + (60.00 + 96.42)\,\underset{\sim}{j}$$
$$= -10.98\,\underset{\sim}{i} + 156.42\,\underset{\sim}{j}$$



Using Pythagoras' theorem:

$$\left|\underset{\sim}{F}\right| = \sqrt{(-10.98)^2 + 156.42^2}$$
$$= 156.80\,\text{N}$$

$$\theta = \tan^{-1}\left(\frac{156.42}{10.98}\right)$$
$$= 86°$$

$\therefore$ Bearing is $(270 + 86) = 356°\text{T}$

$\underset{\sim}{F} = 156.80\,\text{N}$ acting in the direction 356°T.

The method shown in Example **5**, above, can be used similarly in situations where more than two forces are acting on an object.

Calculating the resultant of three forces using vector components

Example 6

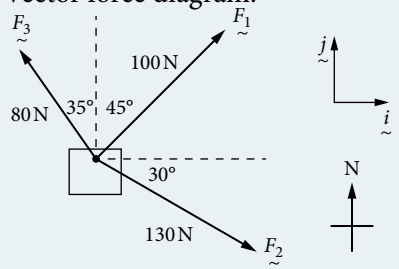
Three forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ are acting on an object: $\underset{\sim}{F}_1 = 100\,\text{N}$ at 045°T, $\underset{\sim}{F}_2 = 130\,\text{N}$ at 120°T and $\underset{\sim}{F}_3 = 80\,\text{N}$ at 325°T.

(a) Resolve each of the forces into horizontal $\underset{\sim}{i}$ components and vertical $\underset{\sim}{j}$ components.

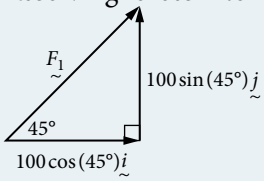
(b) Determine the resultant force $\underset{\sim}{F}$.

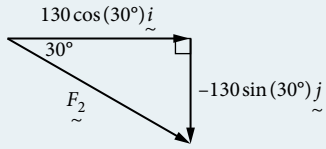
Solution

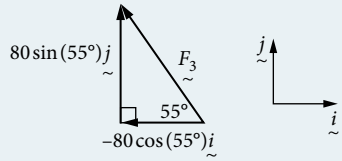
(a) Vector force diagram:



Resolving forces into horizontal $\underset{\sim}{i}$ and vertical $\underset{\sim}{j}$ components:







$$\underset{\sim}{F}_1 = 100\cos 45°\,\underset{\sim}{i} + 100\sin 45°\,\underset{\sim}{j}$$
$$= 70.71\,\underset{\sim}{i} + 70.71\,\underset{\sim}{j}$$

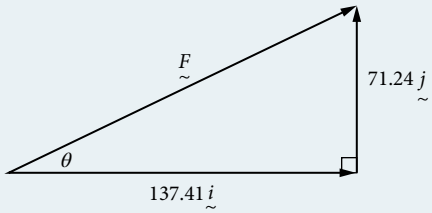
$$\underset{\sim}{F}_2 = 130\cos 30°\,\underset{\sim}{i} - 130\sin 30°\,\underset{\sim}{j}$$
$$= 112.58\,\underset{\sim}{i} - 65.00\,\underset{\sim}{j}$$

$$\underset{\sim}{F}_3 = -80\cos 55°\,\underset{\sim}{i} + 80\sin 55°\,\underset{\sim}{j}$$
$$= -45.89\,\underset{\sim}{i} + 65.53\,\underset{\sim}{j}$$

(b) Sum of horizontal $\underset{\sim}{i}$ and vertical $\underset{\sim}{j}$ components:

$$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$$
$$= (70.71 + 112.58 - 45.89)\,\underset{\sim}{i} + (70.71 - 65.00 + 65.53)\,\underset{\sim}{j}$$
$$= 137.41\,\underset{\sim}{i} + 71.24\,\underset{\sim}{j}$$

Magnitude and direction of resultant vector:



$$|\underset{\sim}{F}| = \sqrt{137.41^2 + 71.24^2}$$
$$= 154.77\ \text{N}$$
$$\tan\theta = \frac{71.24}{137.41}$$
$$\theta = 27.41°$$

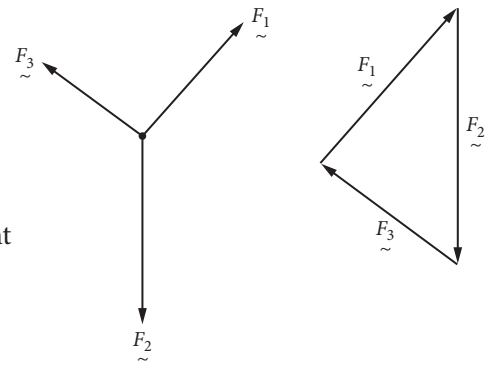
The resultant force is 154.77 N acting at an angle of 062.59°T.

Forces in equilibrium

If the resultant force acting on an object is zero, then the forces on the object are said to be in **equilibrium**. The object's motion will not change, which means that the object will remain at rest or continue with constant velocity. This is known as Newton's first law of motion.

Newton's first law of motion states that an object at rest stays at rest and an object in motion stays in motion with the same velocity, unless acted upon by an unbalanced force.

If the forces acting on an object are in equilibrium, then the resultant force or net force acting on the object is zero. For example, if the three forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ that act on a particle are in equilibrium, then $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$. The forces can be displayed using a triangle of forces, as shown at right.

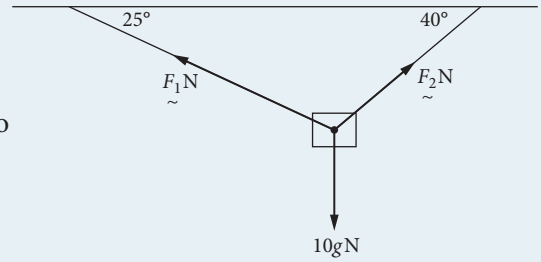


The magnitude of the forces acting and the angles between the forces can then be found using trigonometric ratios (if the triangle contains a right angle) or by using the sine or cosine rule.

In the problems in this course, strings and ropes are considered to have negligible (effectively zero) mass. A smooth, light pulley is also considered to have negligible mass and the friction between a string or rope and a pulley is considered to be negligible.

Example 7

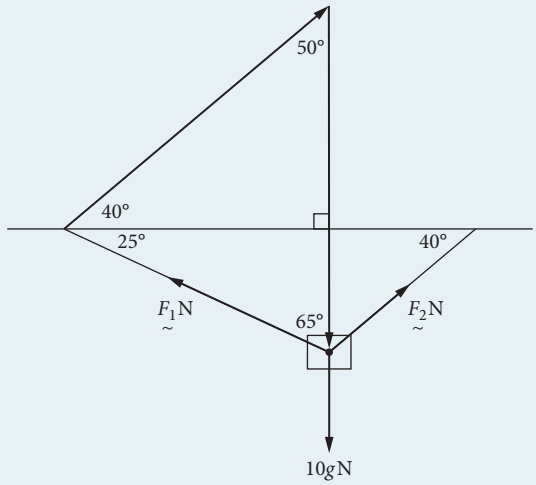
A particle of mass 10 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 25° and 40° respectively to the horizontal, find the magnitude of the tension force in each string in newtons correct to two decimal places.

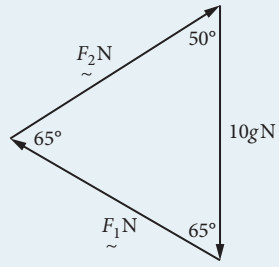


Solution

Representing the forces in a triangle:

The triangle of forces is best obtained by producing the vertical force upwards, translating $\underset{\sim}{F}_2$ to the head of $\underset{\sim}{F}_1$ and producing it until it meets the vertical line. In the original diagram the forces are not drawn to scale, so the triangle is not drawn to scale.





Using the sine rule to determine the magnitudes of the tension forces:

$$\frac{|\underset{\sim}{F}_1|}{\sin 50°} = \frac{|\underset{\sim}{F}_2|}{\sin 65°} = \frac{10g}{\sin 65°}$$

Magnitude of the first force:

$$\frac{|\underset{\sim}{F}_1|}{\sin 50°} = \frac{10g}{\sin 65°}$$

$$|\underset{\sim}{F}_1| = \frac{98}{\sin 65°} \times \sin 50°$$

$$= 82.83\,\text{N}$$

Magnitude of the second force:

$$\frac{|\underset{\sim}{F}_2|}{\sin 65°} = \frac{10g}{\sin 65°}$$

$$|\underset{\sim}{F}_2| = \frac{98}{\sin 65°} \times \sin 65°$$

$$= 98\,\text{N}$$

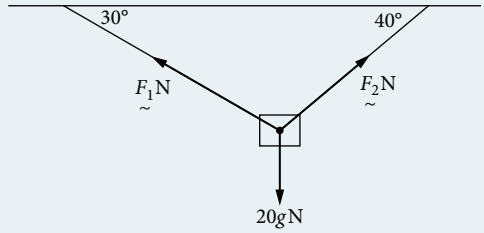
The magnitude of the forces acting and the angles between the forces can also be found by expressing vectors in component form. If there are not three forces acting on an object, then a triangle of forces obviously cannot be used.

However, if more than one force acts on an object, then the resultant force can always be determined from adding the forces. If the object is in equilibrium, then the sum of the $\underset{\sim}{i}$ components and the sum of the $\underset{\sim}{j}$ components must each be zero.

Calculating forces in equilibrium using components

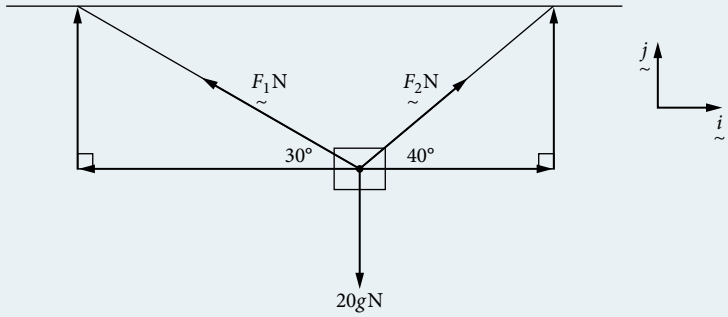
Example 8

A particle of mass 20 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° respectively to the horizontal, use component form to find the magnitude of the tension force in each string, in newtons correct to one decimal place.



Solution

Resolving the forces into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components:



Vector sum in the $\underset{\sim}{i}$ direction:

$|\underset{\sim}{F_2}|\cos 40° - |\underset{\sim}{F_1}|\cos 30° = 0$

$0.7660|\underset{\sim}{F_2}| - 0.8660|\underset{\sim}{F_1}| = 0$

$|\underset{\sim}{F_2}| = 1.1305|\underset{\sim}{F_1}|$ [1]

Vector sum in the $\underset{\sim}{j}$ direction:

$|\underset{\sim}{F_1}|\sin 30° + |\underset{\sim}{F_2}|\sin 40° - 20g = 0$

$0.5000|\underset{\sim}{F_1}| + 0.6428|\underset{\sim}{F_2}| - 196 = 0$ [2]

Substituting [1] into [2] gives: $0.5000|\underset{\sim}{F_1}| + 0.6428 \times 1.1305|\underset{\sim}{F_1}| - 196 = 0$

$1.1227|\underset{\sim}{F_1}| = 196$

$|\underset{\sim}{F_1}| = 159.8\,\text{N}$

Then, from [1]: $|\underset{\sim}{F_2}| = 1.1305 \times 159.8$

$= 180.6\,\text{N}$

The tension forces in the strings have magnitudes of $|\underset{\sim}{F_1}| = 159.8$ N and $|\underset{\sim}{F_2}| = 180.6$ N respectively.

EXPLORING FURTHER

Forces in equilibrium

Use technology to investigate forces on an object at equilibrium.

EXERCISE 13.2 PROBLEMS INVOLVING FORCES

1 Forces are acting on an object. Calculate the resultant force acting on the object for each of the following:

(a) $\underset{\sim}{F}_1 = 150\,\text{N}$ north, $\underset{\sim}{F}_2 = 120\,\text{N}$ south
(b) $\underset{\sim}{F}_1 = 67\,\text{N}$ east, $\underset{\sim}{F}_2 = 83\,\text{N}$ west
(c) $\underset{\sim}{F}_1 = 320\,\text{N}$ east, $\underset{\sim}{F}_2 = 210\,\text{N}$ west, $\underset{\sim}{F}_3 = 140\,\text{N}$ east
(d) $\underset{\sim}{F}_1 = 64\,\text{N}$ south, $\underset{\sim}{F}_2 = 56\,\text{N}$ north, $\underset{\sim}{F}_3 = 48\,\text{N}$ south

2 Forces are acting on an object. Calculate the magnitude and direction of the resultant force acting on the object for each of the following:

(a) $\underset{\sim}{F}_1 = 160\,\text{N}$ north, $\underset{\sim}{F}_2 = 120\,\text{N}$ west
(b) $\underset{\sim}{F}_1 = 80\,\text{N}$ east, $\underset{\sim}{F}_2 = 90\,\text{N}$ south
(c) $\underset{\sim}{F}_1 = 125\,\text{N}$ west, $\underset{\sim}{F}_2 = 85\,\text{N}$ south
(d) $\underset{\sim}{F}_1 = 340\,\text{N}$ north, $\underset{\sim}{F}_2 = 110\,\text{N}$ east

3 Two tugboats are applying simultaneous forces to a container ship as it attempts to come into port. One tugboat is applying a force of 3200 N in an easterly direction and the second tugboat is applying a 2300 N force directly south. What is the magnitude and direction of the resultant force being applied to the container ship?

4 Two forces are acting on an object: calculate the resultant force $\underset{\sim}{F}$ acting on the object for each of the following. Give answers correct to two decimal places.

(a) $\underset{\sim}{F}_1 = 100\,\text{N}$ acting at N40°E and $\underset{\sim}{F}_2 = 90\,\text{N}$ acting at N35°W
(b) $\underset{\sim}{F}_1 = 200\,\text{N}$ acting at N39°W and $\underset{\sim}{F}_2 = 160\,\text{N}$ acting at S40°W
(c) $\underset{\sim}{F}_1 = 15\,\text{N}$ acting at 152°T and $\underset{\sim}{F}_2 = 23\,\text{N}$ acting at 065°T
(d) $\underset{\sim}{F}_1 = 2050\,\text{N}$ acting at 037°T and $\underset{\sim}{F}_2 = 1560\,\text{N}$ acting at 226°T

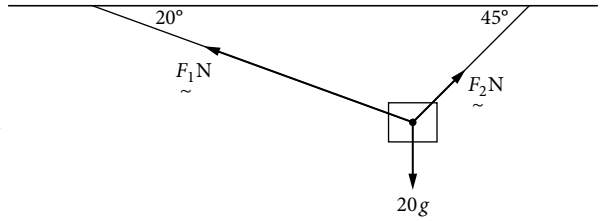
5 Three forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ are acting on an object: $\underset{\sim}{F}_1 = 200\,\text{N}$ at 057°T, $\underset{\sim}{F}_2 = 220\,\text{N}$ at 170°T and $\underset{\sim}{F}_3 = 150\,\text{N}$ at 245°T.

(a) Resolve each force into horizontal $\underset{\sim}{i}$ and vertical $\underset{\sim}{j}$ components.
(b) Determine the resultant force $\underset{\sim}{F}$. Give answers in component form.

6 Three forces are acting on an object: $\underset{\sim}{F}_1 = 250\,\text{N}$ at N45°E, $\underset{\sim}{F}_2 = 270\,\text{N}$ at S35°W and $\underset{\sim}{F}_3 = 350\,\text{N}$ at S25°E.

(a) Resolve each force into horizontal $\underset{\sim}{i}$ components and vertical $\underset{\sim}{j}$ components.
(b) Determine the resultant force $\underset{\sim}{F}$. Give answers in component form.

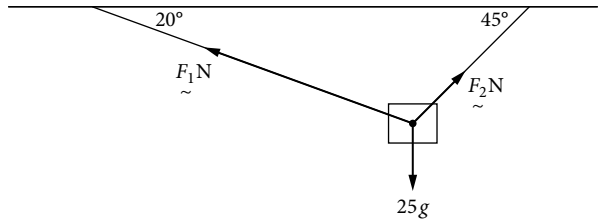
7 A particle of mass 20 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 20° and 45° respectively to the horizontal, find the magnitude of the tension force in each string, in newtons correct to two decimal places.



8 **(a)** A particle of mass 25 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 27° and 42° respectively to the horizontal, draw a force diagram and find the magnitude of the tension forces, F_1 and F_2 respectively in each string, in newtons correct to two decimal places.

(b) A particle of mass 56 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 18° and 54° respectively to the horizontal, draw a force diagram and find the magnitude of the tension forces, F_1 and F_2 respectively in each string, in newtons correct to two decimal places.

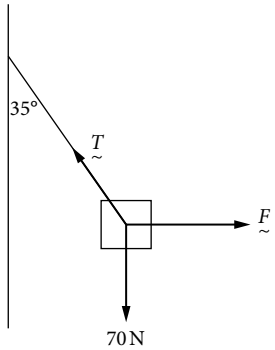
9 A particle of mass 25 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 20° and 45° respectively to the horizontal, use component form to find the magnitude of the tension forces, F_1 and F_2 respectively in each string, in newtons correct to one decimal place.



10 **(a)** A particle of mass 15 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 22° and 38° respectively to the horizontal, draw a force diagram and use component form to find the magnitude of the tension forces, F_1 and F_2 respectively in each string, in newtons correct to two decimal places.

(b) A particle of mass 45 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 36° and 34° respectively to the horizontal, draw a force diagram and use component form to find the magnitude of the tension forces, F_1 and F_2 respectively in each string, in newtons correct to two decimal places.

11 An object that is being pulled vertically downwards by a force of 70 N is attached to a wall by a string of negligible mass. The object is also being pulled to the right by a horizontal force so that the object is not moving and the string makes an angle of 35° with the vertical, as shown.



Find the magnitudes of the horizontal force $\underset{\sim}{F}$ N and the tension force in the string, $\underset{\sim}{T}$ N. Give answers correct to two decimal places.

12 An object that is being pulled vertically downwards by a force of 250 N is attached to a wall by a string of negligible mass. The object is also being pulled to the right by a horizontal force so that the object is not moving and the string makes an angle of 22° with the vertical.

(a) Complete the force diagram for this situation. Label the horizontal force $\underset{\sim}{F}$ N and the tension in the string $\underset{\sim}{T}$ N.

(b) Find the magnitudes of the horizontal force, $\underset{\sim}{F}$ N, and the tension in the string, $\underset{\sim}{T}$ N.

13 Three forces are acting on an object simultaneously in the same plane. The resultant force is $\underset{\sim}{F} = 360$ N due north. Which of the following combinations of forces will give this resultant force?

A $\underset{\sim}{F}_1 = 440$ N south, $\underset{\sim}{F}_2 = 560$ N north, $\underset{\sim}{F}_3 = 480$ N south
B $\underset{\sim}{F}_1 = 120$ N north, $\underset{\sim}{F}_2 = 250$ N south, $\underset{\sim}{F}_3 = 480$ N north
C $\underset{\sim}{F}_1 = 440$ N north, $\underset{\sim}{F}_2 = 560$ N south, $\underset{\sim}{F}_3 = 480$ N north
D $\underset{\sim}{F}_1 = 120$ N south, $\underset{\sim}{F}_2 = 240$ N north, $\underset{\sim}{F}_3 = 480$ N south

14 **(a)** Two tugboats are towing a large boat of mass 15 000 kg in an easterly direction towards a dock. The first tugboat pulls with a force of $\underset{\sim}{F}_1 = 8000$ N at an angle of 28° north of the forward motion while the second tugboat pulls with a force of $\underset{\sim}{F}_2 = 8250$ N at an angle θ° south of the forward motion. Given that there is a resistive force of 4200 N opposing the eastern motion of the large boat, calculate the total easterly force acting on the large boat, to the nearest newton.

(b) Two tugboats are towing a large boat of mass M kg back to shore. The first tugboat pulls with a force of $\underset{\sim}{F}_1$ N at an angle of 27° north of the forward motion and the second tugboat pulls with a force of $\underset{\sim}{F}_2$ N at an angle 30° south of the forward motion. If the large boat is moving with constant velocity and there is a resistive force of 5500 N opposing this motion, calculate the magnitudes of the two forces $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ to the nearest newton.

15 A section of a new bridge is being moved by four cranes that will move it horizontally into position. The chains that are connected from the cranes to the bridge section exert forces that are acting on the bridge simultaneously and in the same plane. The four forces are $\underset{\sim}{F}_1 = 2050$ N at 037°T, $\underset{\sim}{F}_2 = 1560$ N at 130°T, $\underset{\sim}{F}_3 = 1650$ N acting at 237°T and $\underset{\sim}{F}_4 = 1930$ N acting at 316°T.

(a) Resolve each force into horizontal $\underset{\sim}{i}$ and vertical $\underset{\sim}{j}$ components.

(b) Calculate the magnitude and direction of the resultant force applied to the bridge section.

13.3 PROJECTILE MOTION

Projectile motion is an example of two-dimensional motion in the vertical plane under gravity. In this course, the effect of air resistance is ignored, so projectile motion is a vector combination of horizontal constant motion with vertical motion under gravity.

The assumptions made in this model are:

- the projectile is a point
- the force due to air resistance is negligible
- the only force acting on the projectile is a constant force due to gravity, assuming that the projectile is moving close to the Earth's surface.
- the acceleration due to gravity is given by the constant $g = 9.8\,\text{m s}^{-2}$, unless given otherwise.

In this vector approach the acceleration vector will be written as $\underset{\sim}{a}(t) = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$, the velocity vector as $\underset{\sim}{v}(t) = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$, and the position vector as $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$. As the only vertical acceleration is due to gravity and there is no horizontal acceleration, the acceleration vector becomes $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$.

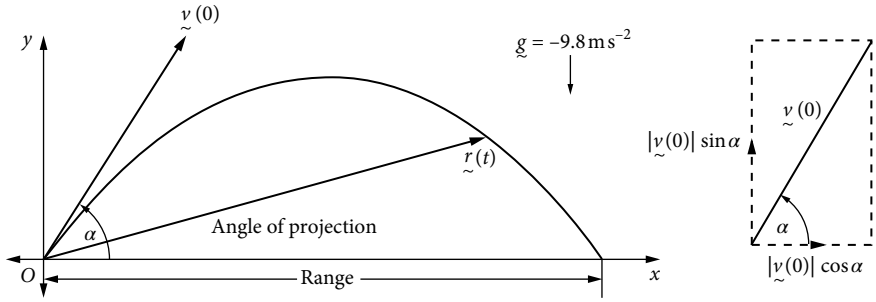
Projectile motion—a typical problem

In projectile motion, a typical problem to be considered might look like the following example:

'A particle is projected in the vertical plane from a point on horizontal ground with an initial velocity $\underset{\sim}{v}(0)\,\text{m s}^{-1}$ at an angle $\alpha°$ to the horizontal.'

In this situation, $\alpha°$ is called the angle of projection. The only force acting on the projectile is due to gravity and it acts only in the vertical plane.

It is standard to take the directions to the right and upwards as the positive directions of motion. This means that the acceleration due to gravity is negative in this case, as it is acting in the opposite direction to the motion, and so $g = -9.8\,\text{m s}^{-2}$.



A particle is projected from the point O with an initial velocity $\underset{\sim}{v}(0)$ at an angle α to the horizontal as shown. The position vector of the particle is $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$. The only force acting on the particle is the gravitational acceleration $\underset{\sim}{g}$ acting vertically downwards. When $t = 0$, $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$, while $\underset{\sim}{v}(0) = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = \underset{\sim}{0}$.

You can use integration and the initial conditions to derive the equations for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

To find velocity, integrate $\underset{\sim}{a}(t)$:

$$\begin{aligned}\underset{\sim}{v}(t) &= \int \underset{\sim}{a}(t)\,dt \\ &= \int -g\underset{\sim}{j}\,dt \\ &= -gt\underset{\sim}{j} + \underset{\sim}{c}\end{aligned}$$

$$\begin{aligned}\underset{\sim}{v}(0) &= -g\times(0)\underset{\sim}{j} + \underset{\sim}{c} \\ &= \underset{\sim}{c}\end{aligned}$$

From initial conditions:

$$\underset{\sim}{v}(0) = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$$

$$\therefore \underset{\sim}{c} = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$$

$$\begin{aligned}\therefore \underset{\sim}{v}(t) &= -gt\underset{\sim}{j} + |\underset{\sim}{v}(0)|\cos s\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j} \\ &= |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha - gt\right)\underset{\sim}{j}\end{aligned}$$

To find position, integrate $\underset{\sim}{v}(t)$:

$$\begin{aligned}\underset{\sim}{r}(t) &= \int \underset{\sim}{v}(t)\,dt \\ &= \int \left(|\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha - gt\right)\underset{\sim}{j}\right)dt \\ &= \left(|\underset{\sim}{v}(0)|\cos\alpha\, t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}\end{aligned}$$

From initial conditions: $\underset{\sim}{r}(0) = \underset{\sim}{0}$

$$\therefore \underset{\sim}{d} = \underset{\sim}{0}$$

$$\therefore \underset{\sim}{r}(t) = \left(|\underset{\sim}{v}(0)|\cos\alpha\, t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2\right)\underset{\sim}{j}$$

In component form, $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$ so $x = |\underset{\sim}{v}(0)|\cos\alpha\, t$ and $y = |\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2$.

Example 9

A particle is projected from a horizontal plane at an angle of 30° with speed of $100\,\text{m s}^{-1}$. The acceleration due to gravity is $9.8\,\text{m s}^{-2}$.

(a) Write $\underset{\sim}{v}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ in component form.

(c) Find the velocity and position after 4 seconds.

Solution

(a) $|\underset{\sim}{v}(0)| = 100$, $\alpha = 30°$: $\underset{\sim}{v}(0) = 100\cos 30°\,\underset{\sim}{i} + 100\sin 30°\,\underset{\sim}{j}$

$$= 50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j}$$

(b) Find $\underset{\sim}{a}(t)$: $\underset{\sim}{a}(t) = -9.8\,\underset{\sim}{j}$

Integrate $\underset{\sim}{a}(t)$ with respect to t:

$$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$$

$$= \int -9.8\,\underset{\sim}{j}\,dt$$

$$= -9.8t\,\underset{\sim}{j} + \underset{\sim}{c}$$

$\underset{\sim}{v}(0) = 50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j}$: $50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j} = \underset{\sim}{c}$

$\underset{\sim}{v}(t) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}$

Integrate $\underset{\sim}{v}(t)$ with respect to t:

$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$$

$$= \int \left(50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}\right)dt$$

$$= 50\sqrt{3}t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j} + \underset{\sim}{d}$$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r}(t) = 50\sqrt{3}t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$$

(c) $t = 4$: $\underset{\sim}{v}(t) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}$

$$\underset{\sim}{v}(4) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8\times 4)\,\underset{\sim}{j}$$

$$= 50\sqrt{3}\,\underset{\sim}{i} + 10.8\,\underset{\sim}{j}$$

$$\underset{\sim}{r}(t) = 50\sqrt{3}t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$$

$$\underset{\sim}{r}(4) = 50\sqrt{3}\times 4\,\underset{\sim}{i} + (50\times 4 - 4.9\times 16)\,\underset{\sim}{j}$$

$$= 200\sqrt{3}\,\underset{\sim}{i} + 121.6\,\underset{\sim}{j}$$

Equation of the path of a projectile

Previously, it was found that $\underset{\sim}{r}(t) = \left(|\underset{\sim}{v}(0)|\cos\alpha t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$, which in terms of its components gives $x = |\underset{\sim}{v}(0)|\cos\alpha t$ and $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$.

These expressions for x and y give the parametric equation of the path (or **trajectory**) of the projectile in terms of t. Combining these equations to eliminate t gives the Cartesian equation of the path of the projectile.

$x = |\underset{\sim}{v}(0)|\cos\alpha t$ gives $t = \dfrac{x}{|\underset{\sim}{v}(0)|\cos\alpha}$

Substitute this into $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$: $\quad y = |\underset{\sim}{v}(0)|\sin\alpha \times \dfrac{x}{|\underset{\sim}{v}(0)|\cos\alpha} - \frac{1}{2}g\left(\dfrac{x}{|\underset{\sim}{v}(0)|\cos\alpha}\right)^2$

$$y = x\tan\alpha - \frac{g}{2}\times\frac{x^2}{|\underset{\sim}{v}(0)|^2\cos^2\alpha}$$

The equation of the trajectory is $y = x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}$

This trajectory is a concave down parabola and the properties of the parabola may be used to answer questions about this path.

Greatest height of the projectile

A particle reaches its greatest height when it stops rising, which is when vertical component of $\underset{\sim}{v}(t)$ is zero.

Hence the greatest height occurs where $|\underset{\sim}{v}(0)|\sin\alpha - gt = 0$: $\quad t = \dfrac{|\underset{\sim}{v}(0)|\sin\alpha}{g}$

To find this greatest height, you can substitute this value of t into the vertical component of $\underset{\sim}{r}(t)$.

Vertical component: $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$

Greatest height: $y = |\underset{\sim}{v}(0)|\sin\alpha \times \dfrac{|\underset{\sim}{v}(0)|\sin\alpha}{g} - \frac{1}{2}g\left(\dfrac{|\underset{\sim}{v}(0)|\sin\alpha}{g}\right)^2$

$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g} - \frac{1}{2}\frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g}$$

$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$

Alternatively, this result could have been found using the Cartesian equation of the trajectory.

A parabola is symmetric about its turning point, so the greatest value of a concave down parabola occurs at its turning point. The abscissa (horizontal coordinate) of the turning point is the average of the abscissae of the points of intersection with the x-axis.

Substituting $y = 0$ into $y = x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}$: $\quad x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} = 0$

$$x\left(\tan\alpha - \frac{gx}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}\right) = 0$$

$$x = 0 \text{ or } x = \frac{2|\underset{\sim}{v}(0)|^2\cos^2\alpha\tan\alpha}{g}$$

$$= \frac{2|\underset{\sim}{v}(0)|^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{g}$$

$\therefore$ Greatest height occurs at $x = \dfrac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g}$

Greatest height is: $y = \dfrac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g} \times \tan\alpha - \dfrac{g}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} \times \left(\dfrac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g}\right)^2$

$$= \frac{|\underset{\sim}{v}(0)|^2 2\sin\alpha\cos\alpha}{2g} \times \frac{\sin\alpha}{\cos\alpha} - \frac{g}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} \times \left(\frac{|\underset{\sim}{v}(0)|^4 \times 4\sin^2\alpha\cos^2\alpha}{4g^2}\right)$$

$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g} - \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$

$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$

Time of flight

In the typical case, the time of flight is the time taken for the projectile to return to the ground.

This is when the vertical component of $\underset{\sim}{r}(t)$ is equal to zero.

$y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$

$$|\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2 = 0$$

$$t\left(|\underset{\sim}{v}(0)|\sin\alpha - \frac{1}{2}gt\right) = 0$$

$$t = 0 \text{ or } t = \frac{2|\underset{\sim}{v}(0)|\sin\alpha}{g}$$

$\therefore$ Projectile returns to the ground when $t = \dfrac{2|\underset{\sim}{v}(0)|\sin\alpha}{g}$.

Range of flight

The range is the horizontal distance from the point of projection to the point where the particle hits the ground.

It is the value of x, the horizontal component of $\underset{\sim}{r}(t)$ when $t = \dfrac{2|\underset{\sim}{v}(0)|\sin\alpha}{g}$.

$t = \dfrac{2|\underset{\sim}{v}(0)|\sin\alpha}{g}$: $\quad x = |\underset{\sim}{v}(0)|\cos\alpha t$

$$= |\underset{\sim}{v}(0)|\cos\alpha \times \frac{2|\underset{\sim}{v}(0)|\sin\alpha}{g}$$

$$= \frac{|\underset{\sim}{v}(0)|^2 \sin 2\alpha}{g}$$

$\therefore$ Range: $R = \dfrac{|\underset{\sim}{v}(0)|^2 \sin 2\alpha}{g}$

This result is also implied when finding the greatest height using the equation of the trajectory. It is double the horizontal distance to the greatest height, which makes sense when you consider that the trajectory is a parabola.

Maximum range

For a given speed of projection $|\underset{\sim}{v}(0)|$, the equation $R = \dfrac{|\underset{\sim}{v}(0)|^2 \sin 2\alpha}{g}$ defines the range R as a function of the angle of projection α. This function has its greatest value where $\sin 2\alpha = 1$ or $\alpha = 45°$.

Thus the greatest range of the particle occurs where $\alpha = 45°$ and is given by: $R_{\max} = \dfrac{|\underset{\sim}{v}(0)|^2}{g}$

Example 10

In Example 9, you found that $\underset{\sim}{v}(0) = 50\sqrt{3}\underset{\sim}{i} + 50\underset{\sim}{j}$, $\underset{\sim}{v}(t) = 50\sqrt{3}\underset{\sim}{i} + (50 - 9.8t)\underset{\sim}{j}$ and $\underset{\sim}{r}(t) = 50\sqrt{3}t\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$.
Use this information to find:

(a) the equation of the trajectory
(b) the greatest height reached
(c) the time of flight and the range of the flight.

Solution

(a) $\underset{\sim}{r}(t) = 50\sqrt{3}t\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$: $x = 50\sqrt{3}t$, $y = 50t - 4.9t^2$

Hence $t = \dfrac{x}{50\sqrt{3}}$ so $y = 50 \times \dfrac{x}{50\sqrt{3}} - 4.9 \times \left(\dfrac{x}{50\sqrt{3}}\right)^2$

$$= \frac{x}{\sqrt{3}} - \frac{4.9x^2}{7500}$$

$$= \frac{x\sqrt{3}}{3} - \frac{49x^2}{75000}$$

(b) $\underset{\sim}{v}(t) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}$: Greatest height occurs when $50 - 9.8t = 0$

$$t = \frac{50}{9.8} = \frac{250}{49} \text{ s}$$

$y = 50t - 4.9t^2$: Greatest height is $y_{\max} = 50 \times \frac{250}{49} - 4.9 \times \left(\frac{250}{49}\right)^2$

$$= \frac{12500}{49} - \frac{4.9 \times 250^2}{49^2}$$

$$= 127.55 \text{ m}$$

(c) $y = 50t - 4.9t^2$: Time of flight when $50t - 4.9t^2 = 0$

$$t(50 - 4.9t) = 0$$

$$t = 0 \text{ or } t = \frac{50}{4.9}$$

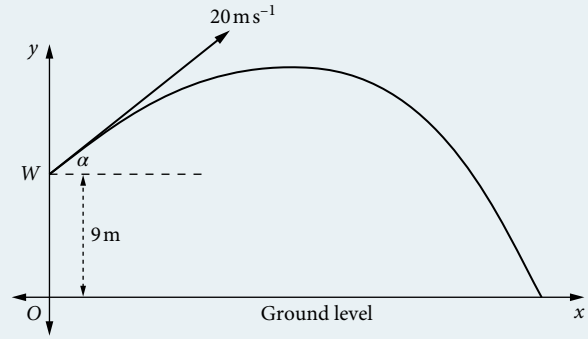
Time of flight is 10.2 s.
For the range, substitute $t = \frac{50}{4.9}$ into $x = 50\sqrt{3}\,t$: Range $= 50\sqrt{3} \times \frac{50}{4.9} = 883.7$ m

You can see from the calculations above why you are often given $g = 10 \text{ m s}^{-2}$ rather than $g = 9.8 \text{ m s}^{-2}$.

A particle is not always projected from level ground, so you will usually need to draw a diagram to show all the given information and so you can form the appropriate vector for $\underset{\sim}{a}(t)$ and the initial conditions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

Example 11

A particle is projected from a window 9 metres above the horizontal ground at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, with an initial velocity of 20 m s^{-1}. Use $g = 10 \text{ m s}^{-2}$.



(a) Write $\underset{\sim}{v}(0)$ and $\underset{\sim}{r}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) When does the particle hit the ground?

(d) What is the horizontal distance from the base of the building to where the particle hits the ground?

(e) What is the maximum height of the particle above the ground?

(f) What is the Cartesian equation of the path of the particle?

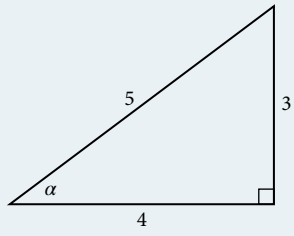
Solution

(a) $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\left|\underset{\sim}{v}(0)\right| = 20$:

$$\underset{\sim}{v}(0) = 20\cos\alpha\,\underset{\sim}{i} + 20\sin\alpha\,\underset{\sim}{j}$$

$$= 20 \times \frac{4}{5}\,\underset{\sim}{i} + 20 \times \frac{3}{5}\,\underset{\sim}{j}$$

$$= 16\underset{\sim}{i} + 12\underset{\sim}{j}$$



$t = 0$, $x = 0$, $y = 9$: $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 9\underset{\sim}{j}$

(b) $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$

$= -10\underset{\sim}{j}$

Integrate $\underset{\sim}{a}(t)$ with respect to t: $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$

$= -\int 10\underset{\sim}{j}\,dt$

$= \underset{\sim}{c} - 10t\underset{\sim}{j}$

$\underset{\sim}{v}(0) = 16\underset{\sim}{i} + 12\underset{\sim}{j}$: $16\underset{\sim}{i} + 12\underset{\sim}{j} = \underset{\sim}{c}$

$\underset{\sim}{v}(t) = 16\underset{\sim}{i} + 12\underset{\sim}{j} - 10t\underset{\sim}{j}$

$= 16\underset{\sim}{i} + (12 - 10t)\underset{\sim}{j}$

Integrate $\underset{\sim}{v}(t)$ with respect to t:

$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$

$= \int \left(16\underset{\sim}{i} + (12 - 10t)\underset{\sim}{j}\right)dt$

$= 16t\underset{\sim}{i} + \left(12t - 5t^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 9\underset{\sim}{j}$: $9\underset{\sim}{j} = \underset{\sim}{d}$

$\underset{\sim}{r}(t) = 16t\underset{\sim}{i} + \left(12t - 5t^2\right)\underset{\sim}{j} + 9\underset{\sim}{j}$

$= 16t\underset{\sim}{i} + \left(9 + 12t - 5t^2\right)\underset{\sim}{j}$

(c) The particle hits the ground where $y = 0$:

$9 + 12t - 5t^2 = 0$

$(3 + 5t)(3 - t) = 0$

$t = -\frac{3}{5}$ **or** $t = 3$

As $t \geq 0$, $t = 3$ and the particle hits the ground after 3 seconds.

(d) Find x when $t = 3$: $x = 16 \times 3 = 48$ m

The particle hits the ground 48 metres from the base of the building.

(e) The maximum height occurs when the vertical velocity is zero: $12 - 10t = 0$

$t = 1.2$ s

Find y when $t = 1.2$:

$y = 9 + 12 \times 1.2 - 5 \times 1.2^2 = 16.2$ m

(f) $\underset{\sim}{r}(t) = 16t\underset{\sim}{i} + \left(9 + 12t - 5t^2\right)\underset{\sim}{j}$:

$x = 16t$, $y = 9 + 12t - 5t^2$

$t = \frac{x}{16}$: $y = 9 + 12 \times \frac{x}{16} - 5 \times \left(\frac{x}{16}\right)^2$

$y = 9 + \frac{3x}{4} - \frac{5x^2}{256}$

MAKING CONNECTIONS

Projectile motion

Use technology to graph the motion of a projectile under the force of gravity.

Example 12

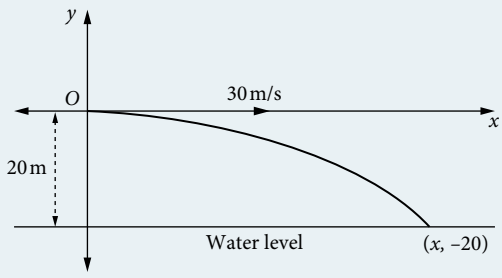
A particle is projected horizontally from the top of a cliff 20 m above the water, with a velocity of 30 m s^{-1}. Use $g = 10$ m s^{-2}.

(a) Write $\underset{\sim}{v}(0)$ and $\underset{\sim}{r}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) When does the particle hit the water?

(d) How far from the base of the cliff does the particle hit the water?



Solution

(a) Horizontal velocity, so $\underset{\sim}{v}(0) = 30\underset{\sim}{i} + 0\underset{\sim}{j}$.

The top of cliff is taken as the origin, so $\underset{\sim}{r}(0) = \underset{\sim}{0}$

The particle hits the water when $y = -20$ m.

(b) $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$

Integrate $\underset{\sim}{a}(t)$ with respect to t: $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$

$$= \int -g\underset{\sim}{j}\,dt$$

$$= -gt\underset{\sim}{j} + \underset{\sim}{c}$$

$\underset{\sim}{v}(0) = 30\underset{\sim}{i} + 0\underset{\sim}{j}$: $30\underset{\sim}{i} = \underset{\sim}{c}$

$\underset{\sim}{v}(t) = 30\underset{\sim}{i} - gt\underset{\sim}{j}$

Integrate $\underset{\sim}{v}(t)$ with respect to t: $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$

$$= \int \left(30\underset{\sim}{i} - gt\underset{\sim}{j}\right)dt$$

$$= 30t\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j} + \underset{\sim}{d}$$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = 30t\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j}$

(c) $y = -20$: $-\frac{1}{2}gt^2 = -20$

$$t^2 = \frac{40}{g}$$

$g = 10$: $t^2 = \frac{40}{10} = 4$

$t = 2$ as $t \geq 0$

The particle hits the water after 2 seconds.

(d) Find x when $t = 2$:
$x = 30t = 30 \times 2 = 60$ m
The particle hits the water 60 metres from the base of the cliff.

Note: If there is no vertical component of projection, then the velocity of projection does not affect the time taken to reach the water. If you drop an object and at the same time throw another object horizontally, you will notice that both objects reach the ground at the same time (assuming that air resistance is not significantly different for the objects). In both cases the vertical motion is governed by $y = -\frac{1}{2}gt^2$, independent of any horizontal motion, just as in the previous example.

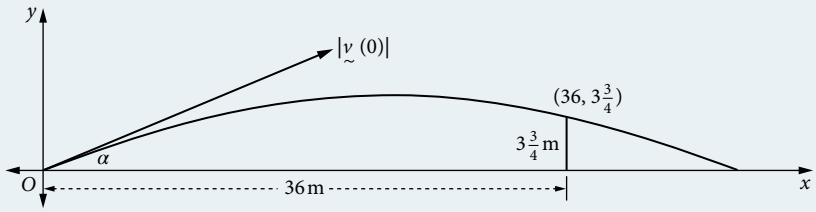
Example 13

A particle is projected from ground level and 1.5 seconds later it just clears a wall 3.75 metres high at a horizontal distance of 36 metres. If the initial velocity is $\left|\underset{\sim}{v}(0)\right| = \left|\cos\alpha\underset{\sim}{i} + \sin\alpha\underset{\sim}{j}\right|$ m s^{-1} and $\underset{\sim}{r}(t) = \left|\underset{\sim}{v}(0)\right|t\cos\alpha\,\underset{\sim}{i} + \left(\left|\underset{\sim}{v}(0)\right|t\sin\alpha - 5t^2\right)\underset{\sim}{j}$:

(a) Calculate the initial velocity and the angle of projection.

(b) Find expressions for $\underset{\sim}{r}(t)$ and $\underset{\sim}{v}(t)$ in component form.

(c) Find the range.



Solution

(a) $t = 1.5,\ x = 36,\ y = 3.75$: $\underset{\sim}{r}(t) = \left|\underset{\sim}{v}(0)\right| t\cos\alpha\,\underset{\sim}{i} + \left(\left|\underset{\sim}{v}(0)\right| t\sin\alpha - 5t^2\right)\underset{\sim}{j}$

Hence $x = \left|\underset{\sim}{v}(0)\right| t\cos\alpha$ and $y = \left|\underset{\sim}{v}(0)\right| t\sin\alpha - 5t^2$

Solve simultaneously to find the angle of projection α.

$t = 1.5,\ x = 36$: $36 = 1.5\left|\underset{\sim}{v}(0)\right|\cos\alpha$ [1]

$t = 1.5,\ y = 3.75$: $3.75 = 1.5\left|\underset{\sim}{v}(0)\right|\sin\alpha - 5\times 1.5^2$

$15 = 1.5\left|\underset{\sim}{v}(0)\right|\sin\alpha$ [2]

[2] ÷ [1]: $\dfrac{15}{36} = \dfrac{1.5\left|\underset{\sim}{v}(0)\right|\sin\alpha}{1.5\left|\underset{\sim}{v}(0)\right|\cos\alpha}$

$\tan\alpha = \dfrac{5}{12}$

$\alpha = 22°37'$

To calculate the velocity of projection $\left|v(0)\right|$ substitute into [1]: $\cos(\alpha) = \dfrac{12}{13}$, as $\tan\alpha = \dfrac{5}{12}$

$\dfrac{36}{1.5} = \left|\underset{\sim}{v}(0)\right| \times \dfrac{12}{13}$

$\left|\underset{\sim}{v}(0)\right| = \dfrac{360}{15} \times \dfrac{13}{12} = 26\,\text{m s}^{-1}$

(b) $\underset{\sim}{r}(t) = 26t\cos\alpha\,\underset{\sim}{i} + \left(26t\sin\alpha - 5t^2\right)\underset{\sim}{j}$

$= 26t \times \dfrac{12}{13}\underset{\sim}{i} + \left(26t \times \dfrac{5}{13} - 5t^2\right)\underset{\sim}{j}$

$= 24t\,\underset{\sim}{i} + \left(10t - 5t^2\right)\underset{\sim}{j}$

Differentiate $\underset{\sim}{r}(t)$ with respect to t:

$\underset{\sim}{v}(t) = 24\underset{\sim}{i} + (10 - 10t)\,\underset{\sim}{j}$

(c) $y = 0$: $10t - 5t^2 = 0$

$5t(2 - t) = 0$

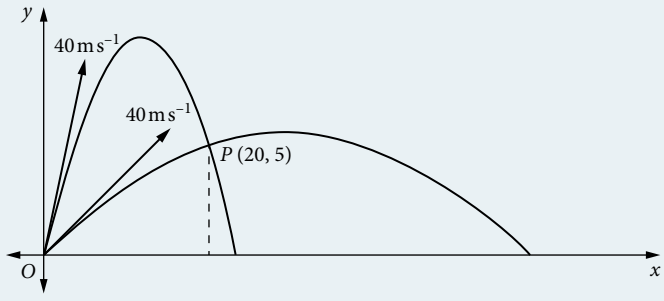
$t = 2\,\text{s}$

$x = 24 \times 2 = 48\,\text{m}$

The range is 48 metres.

Example 14

A stone is thrown to hit a small object sitting on top of a wall that is 20 metres horizontally from the point of projection and 5 metres high. If the stone is thrown from ground level with a speed of 40 m s^{-1}, show that there are two angles of projection that will allow the object to be hit.



The position vector of the particle is given by $\underset{\sim}{r}(t) = \left|\underset{\sim}{v}(0)\right| t\cos\alpha\underset{\sim}{i} + \left(\left|\underset{\sim}{v}(0)\right| t\sin\alpha - 5t^2\right)\underset{\sim}{j}$ where $\left|\underset{\sim}{v}(0)\right|$ is the initial velocity and α the angle of projection from the horizontal.

Solution

$|\underset{\sim}{v}(0)| = 40$, $\underset{\sim}{r}(t) = 40t\cos\alpha\underset{\sim}{i} + (40t\sin\alpha - 5t^2)\underset{\sim}{j}$, $x = 20$, $y = 5$.

$x = 20$: $40t\cos\alpha = 20$

$$t = \frac{1}{2\cos\alpha}$$

$y = 5$: $40t\sin\alpha - 5t^2 = 5$ [1]

Substitute $t = \frac{1}{2\cos\alpha}$ into [1]: $40\sin\alpha \times \frac{1}{2\cos\alpha} - 5\left(\frac{1}{2\cos\alpha}\right)^2 = 5$

$$20\tan\alpha - \frac{5}{4\cos^2\alpha} = 5$$

$$80\tan\alpha - 5\sec^2\alpha = 20 \quad [2]$$

Substitute the identity $\sec^2\alpha = 1 + \tan^2\alpha$ into [2] to solve for α: $80\tan\alpha - 5(1 + \tan^2\alpha) = 20$

$$5\tan^2\alpha - 80\tan\alpha + 25 = 0$$

$$\tan^2\alpha - 16\tan\alpha + 5 = 0$$

Quadratic formula: $\tan\alpha = \frac{16 \pm \sqrt{256 - 20}}{2}$

$$= \frac{16 \pm \sqrt{236}}{2}$$

$$= 8 \pm \sqrt{59}$$

$\tan\alpha = 15.68$ or 0.3189

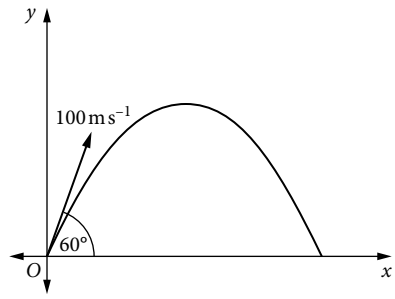
$\alpha = 86°21'$ or $17°41'$

There are two angles of projection that allow the object to be hit, 17° 41′ and 86° 21′.

EXERCISE 13.3 PROJECTILE MOTION

1 A particle is projected with a speed of 100 metres per second from a horizontal plane at an angle of 60°. Given $\underset{\sim}{v}(t) = 100\cos 60°\underset{\sim}{i} + (100\sin 60° - 10t)\underset{\sim}{j}$, find:

(a) when the particle reaches its greatest height
(b) its position vector $\underset{\sim}{r}(t)$
(c) the greatest height reached
(d) the time of flight
(e) the horizontal range
(f) the equation of the trajectory.



2 A particle is projected with a speed of 80 m s^{-1} from a horizontal plane at an angle of 30°.

(a) Write the vector for $\underset{\sim}{v}(0)$.

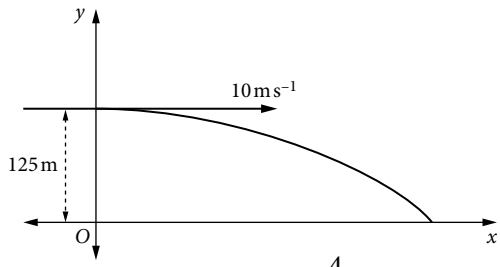
(b) Given that $\underset{\sim}{v}(t) = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + (|\underset{\sim}{v}(0)|\sin\alpha - gt)\underset{\sim}{j}$, find $\underset{\sim}{r}(t)$, using $g = 10$ m s^{-2}.

(c) Hence find the time of flight and the range of the projectile.

(d) Find the maximum height reached by the projectile.

(e) Show that the equation of the trajectory is $y = \frac{x}{\sqrt{3}} - \frac{x^2}{960}$.

3 An object is projected horizontally from the top of a building 125 m high at a speed of $10\,\text{m s}^{-1}$. Using $g = 10\,\text{m s}^{-2}$, find:



(a) $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$

(b) the time when the object hits the ground and its distance from the base of the building

(c) the maximum height reached by the object.

4 A cricket ball is hit with a velocity of $12.5\,\text{m s}^{-1}$ at an angle with the horizontal whose tangent is $\frac{4}{3}$.

(a) If $\underset{\sim}{v}(t) = |\underset{\sim}{v}(0)|\cos\alpha\,\underset{\sim}{i} + (|\underset{\sim}{v}(0)|\sin\alpha - gt)\,\underset{\sim}{j}$, find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$, given that $g = 10\,\text{m s}^{-2}$ and $\tan\alpha = \frac{4}{3}$.

(b) Find the greatest height reached.

(c) Find the time of flight and the horizontal distance travelled.

5 A particle is projected so that at time t its position is given by $\underset{\sim}{r}(t) = 36t\,\underset{\sim}{i} + \left(15t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$, where distances are in metres and time is in seconds. If α is the angle of projection and $|\underset{\sim}{v}(0)|$ is the initial velocity, which of the following statements is correct?

A $|\underset{\sim}{v}(0)| = 36\,\text{m s}^{-1}$ and $\alpha = \sin^{-1}\left(\frac{5}{13}\right)$

B $|\underset{\sim}{v}(0)| = 15\,\text{m s}^{-1}$ and $\alpha = \cos^{-1}\left(\frac{12}{13}\right)$

C $|\underset{\sim}{v}(0)| = 39\,\text{m s}^{-1}$ and $\alpha = \tan^{-1}\left(\frac{5}{13}\right)$

D $|\underset{\sim}{v}(0)| = 39\,\text{m s}^{-1}$ and $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$

6 The equation of the trajectory of a particle projected from the origin is given by $y = \frac{x}{2} - \frac{x^2}{800}$. Find:

(a) where it hits the ground

(b) the greatest height reached.

7 A particle is projected at an angle α to the horizontal with a velocity of $50\,\text{m s}^{-1}$. It passes through the point (200, 25) where distance is measured in metres and $g = 10\,\text{m s}^{-2}$.

(a) Write the expression for $\underset{\sim}{v}(0)$.

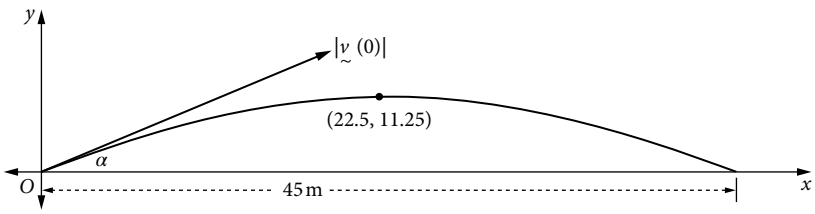
(b) Find the vectors $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) Find α given that the particle passes through the point (200, 25).

8 A ball is projected so that its horizontal range is 45 metres. It passes through a point 11.25 metres vertically above and 22.5 metres horizontally from the point of projection.

(a) If $|\underset{\sim}{v}(0)|$ is the initial velocity and α the angle of projection, find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Given $g = 10\,\text{m s}^{-2}$, find the angle of projection and the speed of projection.



9 A stone is projected from the edge of a cliff with a speed of $30\,\text{m s}^{-1}$ to hit an object 120 m horizontally from the edge and 35 m below. Given $g = 10\,\text{m s}^{-2}$, find:

(a) expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$

(b) the angle of projection.

10 A ball is thrown horizontally with speed $v\,\text{m s}^{-1}$ from a point h metres above the ground and lands at a horizontal distance d metres from the point of release. Use $g = 10\,\text{m s}^{-2}$.

(a) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Find v given $d = 3$ and $h = 1.25$.

(c) Find h given $v = 10$ and $d = 20$.

11 A particle is projected from a point whose coordinates are (0, 0) with a velocity $20\,\text{m s}^{-1}$ horizontally and $100\,\text{m s}^{-1}$ vertically. Assume $g = 10\,\text{m s}^{-2}$.

(a) What is the angle of projection? **(b)** Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) Find the time when the projectile passes through the point whose horizontal distance from O is 120 m.

(d) What is the vertical height at this time in **(c)**?

12 A particle is projected so that at any time t its position is given by $\underset{\sim}{r}(t) = 36t\underset{\sim}{i} + \left(15t - 5t^2\right)\underset{\sim}{j}$, where distances are measured in metres. Find:

(a) the velocity of projection **(b)** the angle of projection

(c) the greatest height reached **(d)** the range.

13 A particle is projected so that at any time t its position is given by $\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + \left(20t - 5t^2\right)\underset{\sim}{j}$, where distances are measured in metres. Find:

(a) the point of projection **(b)** the angle of projection

(c) the speed of projection **(d)** the coordinates of the highest point on the path.

14 A shell is fired at 200 metres per second to strike a target 2 km away. Using $g = 10\,\text{m s}^{-2}$, find:

(a) expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ **(b)** the angle of projection.

15 An aircraft travelling at $630\,\text{km h}^{-1}$ drops a package from a height of 1000 m. Using $g = 10\,\text{m s}^{-2}$, find:

(a) expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ **(b)** the time taken for the package to hit the ground

(c) the horizontal distance travelled by the package.

16 A particle is projected to just clear two walls that are each 7 m tall, and 7 m and 14 m respectively away from the point of projection. It is given that $\underset{\sim}{r}(t) = Vt\cos\alpha\underset{\sim}{i} + \left(Vt\sin\alpha - 5t^2\right)\underset{\sim}{j}$.

(a) If α is the angle of projection, prove that $\tan\alpha = 1.5$.

(b) Show that if the walls are h metres high and are respectively b metres and c metres distant from the point of projection, then $\tan\alpha = \dfrac{h(b+c)}{bc}$.

17 A particle is projected with a speed of $25\,\text{m s}^{-1}$ in the direction of a point P that is 7 m vertically above and 24 m horizontally from the point of projection. Use $g = 10\,\text{m s}^{-2}$.

(a) Draw a diagram to show this information. **(b)** What is the angle of projection?

(c) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(d) Find the coordinates of the point on the trajectory directly below P.

(e) At the instant of the particle's projection, a second particle is dropped from P. Find the position vector for this second particle.

(f) Prove that the two particles will collide (i.e. will be in the same place at the same time).

18 A particle projected at an angle of 15° has a horizontal range of 80 m. Use $g = 10\,\text{m s}^{-2}$.

(a) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$. **(b)** What is the velocity of projection?

(c) What would be the range if the angle of projection were increased to 45° with the same speed of projection?

19 A stone is projected horizontally with a velocity of $15\,\text{m s}^{-1}$ from the top of a building 30 m high. Use $g = 10\,\text{m s}^{-2}$.

(a) Find expressions for $\underset{\sim}{v}_1(t)$ and $\underset{\sim}{r}_1(t)$, the velocity and position of the stone.

At the same instant another stone is projected from the base of the same building with a velocity of $30\,\text{m s}^{-1}$ at an angle of 60° to the horizontal. The two stones collide.

(b) Find expressions for $\underset{\sim}{v}_2(t)$ and $\underset{\sim}{r}_2(t)$, the velocity and position of the second stone.

(c) If the coordinates of the base of the building are (0, 0), find the coordinates of the point of collision.

20 A golfer hits a ball from a point on a flat golf course. Two seconds later, it strikes the ground 50 m away. Using $g = 10\,\text{m s}^{-2}$, find:

(a) expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ **(b)** the velocity and angle of projection

(c) the maximum height of the ball above the golf course.

21 A particle is projected with velocity $20\,\text{m s}^{-1}$ to hit a target at a horizontal distance 20 m from the point of projection and a vertical height of 10 m. Using $g = 10\,\text{m s}^{-2}$, find:

(a) expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ **(b)** two possible angles of projection.

22 A particle is projected so that its position vector is given by $\underset{\sim}{r}(t) = 30t\,\underset{\sim}{i} + \left(40t - 5t^2\right)\underset{\sim}{j}$, where distances are measured in metres and time in seconds. Find:

(a) the expression for $\underset{\sim}{v}(t)$ **(b)** the velocity of projection **(c)** the angle of projection

(d) the Cartesian coordinates of the highest point on the path.

23 A particle projected from a point meets the horizontal plane through the point of projection after travelling a horizontal distance a. In the course of its trajectory it reaches a greatest height b above the point of projection.

(a) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Find the horizontal and vertical components of the velocity of projection in terms of a and b.

(c) Show that when the particle has travelled a horizontal distance x, it has reached a height of $\dfrac{4bx(a-x)}{a^2}$.

24 A football kicked at $16\,\text{m s}^{-1}$ just passes over a crossbar 4 m high and 16 m away.

(a) If α is the angle of projection, finds expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Using $g = 10\,\text{m s}^{-2}$, show that α satisfies $5\tan^2\alpha - 16\tan\alpha + 9 = 0$.

CHAPTER REVIEW 13

1 A car is moving at a speed of $100\,\text{km h}^{-1}$ and a truck is moving at $90\,\text{km h}^{-1}$ in the opposite direction. The relative velocity of the car with respect to the truck is:

A $10\,\text{km h}^{-1}$ in the direction of the truck **B** $10\,\text{km h}^{-1}$ in the direction of the car

C $190\,\text{km h}^{-1}$ in the direction of the truck **D** $190\,\text{km h}^{-1}$ in the direction of the car

2 On his way home from training, Mitch walks along three streets. He walks 150 m north, 300 m east and 250 m north. The magnitude of Mitch's resultant displacement is:

A 200 m **B** 400 m **C** 500 m **D** 700 m

3 Three forces are acting simultaneously in the same plane on an object. The resultant force is $\underset{\sim}{F} = 520\,\text{N}$ due east. Which of the following combinations of forces will give this resultant force?

A $\underset{\sim}{F}_1 = 350\,\text{N}$ west, $\underset{\sim}{F}_2 = 270\,\text{N}$ east, $\underset{\sim}{F}_3 = 440\,\text{N}$ west

B $\underset{\sim}{F}_1 = 350\,\text{N}$ east, $\underset{\sim}{F}_2 = 270\,\text{N}$ west, $\underset{\sim}{F}_3 = 440\,\text{N}$ east

C $\underset{\sim}{F}_1 = 350\,\text{N}$ east, $\underset{\sim}{F}_2 = 270\,\text{N}$ west, $\underset{\sim}{F}_3 = 440\,\text{N}$ west

D $\underset{\sim}{F}_1 = 350\,\text{N}$ west, $\underset{\sim}{F}_2 = 270\,\text{N}$ east, $\underset{\sim}{F}_3 = 440\,\text{N}$ east

4 A soccer ball is struck with a force of 420 N in the direction N22°E. The northerly and easterly components of this force are $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ respectively. The magnitudes of these forces are:

A $\underset{\sim}{F}_1 = 87\,\text{N}$, $\underset{\sim}{F}_2 = 411\,\text{N}$ **B** $\underset{\sim}{F}_1 = 411\,\text{N}$, $\underset{\sim}{F}_2 = 87\,\text{N}$

C $\underset{\sim}{F}_1 = 157\,\text{N}$, $\underset{\sim}{F}_2 = 389\,\text{N}$ **D** $\underset{\sim}{F}_1 = 389\,\text{N}$, $\underset{\sim}{F}_2 = 157\,\text{N}$

5 Forces are acting on an object. Calculate the resultant force acting on the object for each of the following:

(a) $\underset{\sim}{F}_1 = 250\,\text{N}$ north, $\underset{\sim}{F}_2 = 450\,\text{N}$ south, $\underset{\sim}{F}_3 = 125\,\text{N}$ north **(b)** $\underset{\sim}{F}_1 = 100\,\text{N}$ east, $\underset{\sim}{F}_2 = 100\,\text{N}$ south

6 A particle of mass 50 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings each make angles of 30° respectively to the horizontal, find the magnitude of the tension in each string, in newtons, given that $\sin 30° = \frac{1}{2}$.

7 Jaide paddles her canoe in a direction perpendicular to the riverbanks. The river is 120 m wide. The canoe's speed in this direction is $1.5\,\text{m s}^{-1}$. The water in the river is flowing at $0.8\,\text{m s}^{-1}$ and you can assume that the riverbanks are parallel straight lines.

(a) Calculate the velocity of the canoe relative to the bank. Give the angle correct to two decimal places.
(b) At what distance, in the direction downstream from the starting point, will Jaide get to the other riverbank?
(c) Calculate Jaide's final displacement from the starting point.
(d) Jake is going to cross the river in another canoe. He can paddle at the speed of $1.8\,\text{m s}^{-1}$ relative to the water's flow. At what direction should Jake row to get to the same end-point as Jaide, and how much time will this take?

8 A particle is projected with a velocity whose horizontal and vertical components are $6\,\text{m s}^{-1}$ and $4\,\text{m s}^{-1}$ respectively. Use $g = 10\,\text{m s}^{-2}$.

(a) Draw a diagram to show this information.
(b) What is the angle of projection?
(c) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.
(d) Find the greatest height reached.
(e) Find the horizontal range.

9 A cricketer hits a cricket ball off the ground towards a fielder who is 65 m away. The ball reaches a maximum height of 4.9 m and the horizontal component of the velocity is $28\,\text{m s}^{-1}$. Use $g = 9.8\,\text{m s}^{-2}$.

(a) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.
(b) How much time does it take for the ball to reach its greatest height?
(c) How far has the ball travelled horizontally when it has descended to a height of 1.3 m?
(d) Find the constant speed with which the fielder must run forward, starting at the instant the ball is hit, in order to catch the ball at a height of 1.3 m above the ground.

10 A particle is projected from a point 15 m above horizontal ground. At its highest point it just clears the top of a wall 26.25 m high and 30 m away. Use $g = 10\,\text{m s}^{-2}$.

(a) Draw a diagram to show this information.
(b) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.
(c) Find the speed and the angle of projection of the particle.

11 A tennis ball is struck with a force of 300 N in the direction S40°W. The southerly and westerly components of this force are $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ respectively. The magnitudes of these forces are:

A $\underset{\sim}{F}_1 = 230\,\text{N}, \underset{\sim}{F}_2 = 230\,\text{N}$
B $\underset{\sim}{F}_1 = 193\,\text{N}, \underset{\sim}{F}_2 = 230\,\text{N}$
C $\underset{\sim}{F}_1 = 230\,\text{N}, \underset{\sim}{F}_2 = 193\,\text{N}$
D $\underset{\sim}{F}_1 = 150\,\text{N}, \underset{\sim}{F}_2 = 260\,\text{N}$

CHAPTER 14
The binomial distribution

14.1 BERNOULLI TRIALS

When a coin is flipped, there are two outcomes possible: heads or tails. When a standard die is rolled, there are six outcomes possible: 1, 2, 3, 4, 5 or 6. If you are interested in whether or not you roll a particular number, like 6 for example, there are only two outcomes of interest in this scenario: 'getting a 6' and 'not getting a 6'.

The same argument can be applied to many practical situations: an archer hits the target or misses the target; a footballer scores a goal or misses; a medical test indicates the presence or absence of a disease. In all of these cases, you need to consider *success* and *failure*. When you attach a probability of p to success, then the probability of failure will be $1 - p$.

Bernoulli trials are a way of analysing situations like these where there are exactly two possible outcomes: success or failure. The trials are independent, so the outcome of one trial has no influence over the outcome of the next trial, and the number of trials is fixed. Bernoulli trials are named after Jacob Bernoulli (1654–1705).

Example 1

Decide whether each statement could represent a Bernoulli trial.

(a) You can either pass or fail an examination.
(b) You can either buy a particular brand of phone or not buy it.
(c) You can either walk, ride your bike or catch the bus to school.
(d) You can either get the job that you applied for or not get that job.

Solution

(a) Only two outcomes, so it is a Bernoulli trial.
(b) Only two outcomes, so it is a Bernoulli trial.
(c) Three possible outcomes, so it is not a Bernoulli trial.
(d) Only two outcomes, so it is a Bernoulli trial.

Bernoulli random variables

Associated with Bernoulli trials are Bernoulli random variables. Bernoulli random variables are often encoded using the convention that the number 1 is success, and 0 is failure. If X represents a Bernoulli random variable, and a probability of p is associated to success, where $0 < p < 1$, then:

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

The probability distribution table for this situation is:

x	0	1
$P(X = x)$	$1 - p$	p

The expected value is given by:

$$\begin{aligned} E(X) &= 0 \times (1 - p) + 1 \times p \\ &= p \end{aligned}$$

The variance is given by:

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 0^2 \times (1 - p) + 1^2 \times p - p^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

For a Bernoulli random variable X:

$E(X) = p$

$\text{Var}(X) = p(1 - p)$

EXERCISE 14.1 BERNOULLI TRIALS

1 In an examination there are 10 multiple-choice questions with four possible answers, only one of which is correct.

(a) If you just guess your answers, does this form a sequence of Bernoulli trials?

(b) Use Pascal's triangle (or the binomial expansion) to calculate the number of different ways you could get 6 out of the 10 questions correct.

2 At the recent local government elections, 20 people are selected at random after leaving the polling booth and asked if they have voted for a particular candidate. Would the list of responses represent a Bernoulli trial?

3 The babies born at a hospital in one month have their sex recorded as 'male' or 'female'.

(a) Would the list of sexes represent a Bernoulli trial?

(b) In 20 births at the hospital during the month, 11 are male. Use Pascal's triangle (or the binomial expansion) to calculate the number of different ways this result could occur.

4 A coin is tossed four times.

(a) Would the list of outcomes represent a Bernoulli trial?

(b) If the possible outcomes are recorded as H (heads) or T (tails), list all possible sets of outcomes with at least three heads.

5 Two standard dice are rolled together and the sum of the numbers rolled is noted. The result is recorded as either 7 or not 7.

(a) Does this experiment represent a Bernoulli trial?

(b) If p is the probability of a sum of 7, find the value of p.

(c) In 36 rolls of the dice, how many times would you expect the sum to be 7?

14.2 BINOMIAL DISTRIBUTION

Bernoulli trials can help you to understand the most important of the discrete probability distributions, the binomial distribution.

Suppose you are conducting an experiment, consisting of n trials, where:

- n is determined before the experiment begins
- all n trials are identical Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$
- all the trials are independent, so that the outcome from any one trial has no effect on the outcome of any other trial.

In these cases there is a **binomial random variable**, which in turn has a **binomial probability distribution**.

There is a shorthand notation to indicate a binomial distribution:

$X \sim B(n, p)$ indicates a random variable X that has a binomial distribution with n identical trials and a probability of success of p.

n and p are called the *parameters* of the distribution.

$X \sim B(n, p)$ is read as 'X is distributed as a binomial variable with parameters n and p'.

Consider the experiment of drawing three cards, one at a time with replacement, from a standard pack of 52 playing cards. If interested in the number X of hearts cards selected, then X is a binomial variable. Consider drawing a heart card to be a success. In terms of notation, $n = 3$ (the number of trials) and $p = \frac{1}{4}$ (the probability of success on any particular trial, as $P(\text{heart}) = \frac{1}{4}$.

So, $X \sim B\left(3, \frac{1}{4}\right)$.

If you were drawing three cards from a pack without replacement, then X would not be a binomial variable because the probability of success p would change with each draw.

Even if an experiment has only two possible outcomes, it is not automatically binomial.

Example 2

Find the probability of obtaining exactly two hearts in a selection of three cards if the card is replaced after each selection.

Solution

There are three ways to obtain exactly two hearts: HHN, HNH, NHH, where H represents 'getting a heart' and N represents 'not getting a heart'.

In each case, the probability is equal to $\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}$.

$$P(\text{exactly two hearts}) = 3\times\left(\frac{1}{4}\right)^2\times\frac{3}{4} = \frac{9}{64}$$

Binomial theorem

In Chapter 6 Permutations and combinations, you developed Pascal's triangle and the general expansion of $(a+b)^n$ using the binomial theorem.

This is written as:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots. + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n$$

where ${}^nC_r = \frac{n!}{(n-r)!r!}$

MAKING CONNECTIONS

Pascal's triangle and the binomial theorem

Use technology to construct Pascal's triangle and explore its relationship to the binomial theorem.

Example 3

Calculate, using technology or by hand, the coefficients in the expansion of $(a+b)^6$.

Solution

${}^6C_0 = 1$, ${}^6C_1 = 6$, ${}^6C_2 = 15$, ${}^6C_3 = 20$, ${}^6C_4 = 15$, ${}^6C_5 = 6$, ${}^6C_6 = 1$

Thus $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

It is usual to replace ${}^nC_r$ by $\binom{n}{r}$ when working with binomial probabilities. Do not mix up this notation with the column vector!

It is easy to work out simple examples like Example 2, but when finding the probability of drawing 13 hearts from 30 draws, it is not as easy to list all the possible outcomes. Instead you can use ${}^{30}C_{13}$ to find the total number of possible outcomes. With the binomial distribution, use the alternative notation $\binom{n}{r}$ or in this case, $\binom{30}{13}$.

Consider the probability of obtaining any one of these outcomes, such as 13H followed by 17N. The probability of this is $\left(\frac{1}{4}\right)^{13} \times \left(\frac{3}{4}\right)^{17}$.

When you consider all possible outcomes:

$$P(\text{exactly 13 hearts from 30 draws}) = \binom{30}{13} \times \left(\frac{1}{4}\right)^{13} \times \left(\frac{3}{4}\right)^{17}$$

Using technology to evaluate, the expression is equal to 0.013 414 448 8… which can be rounded to 0.0134.

A general expression for the probability that X takes a particular value x is as follows:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Recalling that the probability of failure is sometimes written as q, where $p + q = 1$, this can also be written as:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Example 4

A variable x follows the distribution $X \sim B(10, 0.6)$.

Find $P(X = 5)$ for this distribution, expressing your answer correct to four decimal places.

Solution

Identify n (the number of trials), x (the number of successes), p (the probability of success), and $1 - p$ (the probability of failure):

$n = 10$ $\quad X = 5$ $\quad p = 0.6$ $\quad 1 - p = 0.4$

Substitute into the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 5) = \binom{10}{5} (0.6)^5 (0.4)^5$$

$$P(X = 5) = 0.2007$$

There is a clear link between the binomial probability distribution and the binomial theorem, although the order of the terms has been reversed. You can use this link to show that the sum of the binomial probabilities equals one. Remember, it is a condition for a probability distribution that the sum of the probabilities equals one.

Using the expansion of the binomial theorem:

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} p^{n-i} (1-p)^i &= (p + 1 - p)^n \\ &= 1^n \\ &= 1\end{aligned}$$

You can also think of a binomial distribution in terms of the expansion of $(p + q)^n$, where $p + q = 1$, and p is the probability of success.

Example 5

The probability of any particular egg being cracked in a carton containing a dozen eggs is 0.05.

Find the probability that exactly three eggs are cracked, stating your answer correct to three decimal places.

Solution

Binomial probability is appropriate because there are two outcomes, a fixed probability of success and a defined number of trials.

Identify n (the number of trials), x (the number of successes, i.e. cracked eggs), p (the probability of success), and $1 - p$ (the probability of failure):

$n = 12, x = 3, p = 0.05, 1 - p = 0.95$

Substitute into the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 3) = \binom{12}{3} (0.05)^3 (0.95)^9$$

$$P(X = 3) = 0.017$$

Example 6

A particular medical test correctly identifies whether or not a person has an illness 98% of the time. If 10 people are tested, find the following probabilities, correct to three decimal places:

(a) No people are incorrectly diagnosed.

(b) At least one person is incorrectly diagnosed.

Solution

(a) Identify n, x and p: $n = 10, x = 10, p = 0.98$

Substitute into the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 10) = \binom{10}{10} (0.98)^{10} (0.02)^0$$

$$P(X = 10) = 0.817$$

(b) P(at least one incorrectly diagnosed) $= 1 - p$ (none incorrectly diagnosed)

(None incorrectly diagnosed is the same as all correctly diagnosed)

$P(X < 10) = 1 - P(X = 10)$

$= 1 - 0.817$

$= 0.183$

Note that $P(X \geq a) = 1 - P(X < a)$.

This can be a useful result in many situations.

Example 7

Accurate Andy is a darts player whose favourite shot is the triple-twenty. Andy is successful with this shot 80% of the time. In each of the following situations, Andy has five shots. In each case state your answer correct to four decimal places where necessary.

(a) Find the probability that Andy makes exactly two triple-twenties.
(b) Find the probability that Andy makes more than three triple-twenties.
(c) Find the probability that Andy makes triple-twenties on his first and fifth attempts only.

Solution

(a) $n = 5, x = 2, p = 0.8$

Substitute into the formula: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 2) = \binom{5}{2} (0.8)^2 (0.2)^3$$

$$P(X = 2) = 0.0512$$

(b) $P(\text{more than three successes}) = P(\text{four successes}) + P(\text{five successes})$

$P(X > 3) = P(X = 4) + P(X = 5)$

$$= \binom{5}{4} (0.8)^4 (0.2)^1 + \binom{5}{5} (0.8)^5 (0.2)^0$$

$$= 0.4096 + 0.32768$$

$$\approx 0.7373$$

(c) Certain conditions have been specified, therefore you cannot use a complete term from the binomial distribution. In this case, consider only the sequence SFFFS, not the more general case of two successes anywhere within the five trials.

$P(\text{triple-twenty on first and fifth only}) = 0.8 \times 0.2 \times 0.2 \times 0.2 \times 0.8$

$$= 0.00512$$

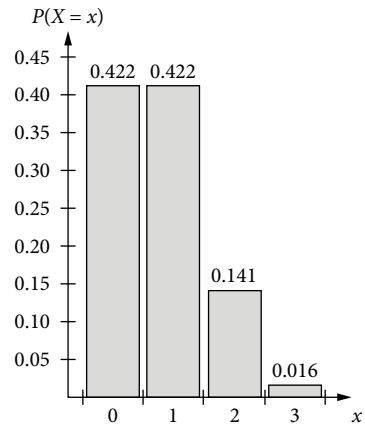
$$\approx 0.0051$$

Graph of the binomial distribution

Consider again the experiment where you draw three cards, with replacement, from a standard pack of 52 playing cards. Remember, you are interested in the number of hearts that occur in the three cards drawn, so $X \sim B(3, 0.25)$. The following table shows the distribution of the random variable:

x	0	1	2	3
$P(X = x)$	0.422	0.422	0.141	0.016

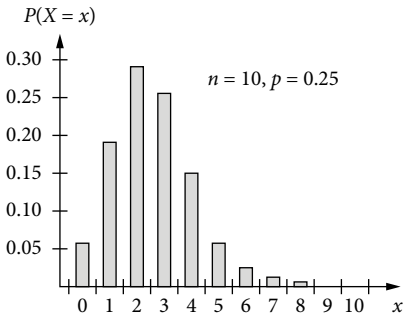
Here, the sum of the probabilities is 1.001. This is due to the rounding of the probabilities and can therefore be regarded as 1. A graph of this distribution is shown:



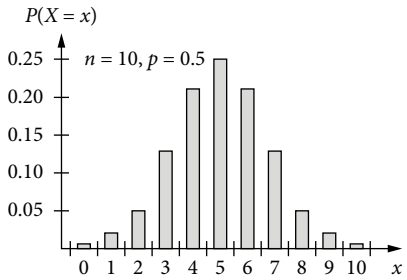
Now, consider the effect that the values n and p have on the graph of the distribution. The following diagrams will help you understand the effect each of these variables has on the graph.

Set $n = 10$.

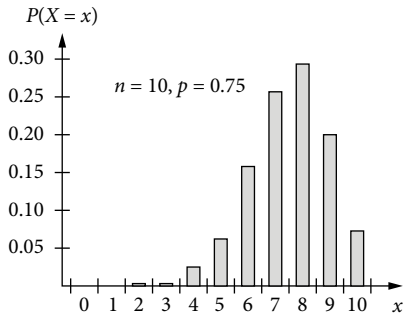
$B(10, 0.25)$



$B(10, 0.5)$



$B(10, 0.75)$



MAKING CONNECTIONS

Graphing binomial distributions

Move the sliders to explore the effects of changing n and p on the graph of a binomial distribution.

For graphs of the binomial distribution, awhere $X \sim B(n, p)$:

- If $p < 0.5$ the graph is skewed to the right (positively skewed).
- If $p = 0.5$ the graph is symmetric about the mean.
- If $p > 0.5$ the graph is skewed to the left (negatively skewed).
- As n increases, the graph clusters more tightly about the mode but retains the same shape as other distributions with the same value of p.

EXERCISE 14.2 BINOMIAL DISTRIBUTION

1 Find the stated probability for the following binomial distributions. Give your answers correct to four decimal places (d.p.).

(a) $P(X = 4)$ if $X \sim B(12, 0.7)$ **(b)** $P(X = 6)$ if $X \sim B(20, 0.45)$

(c) $P(X = 2)$ if $X \sim B(8, 0.3)$ **(d)** $P(X = 30)$ if $X \sim B(50, 0.6)$

2 A spinner is divided into eight equal sections, of which three are shaded. Find the probability, correct to four decimal places, of the spinner landing on a shaded section exactly five times out of 10 spins.

3 A standard six-sided die is rolled 50 times. Find the probability, correct to five decimal places, of the following outcomes:

(a) exactly 30 even numbers are rolled **(b)** at least one 6 is rolled.

4 Peter, a keen gardener, knows from past experience that only 60% of his tulip bulbs will flower. He plants 20 bulbs. Find the probability, correct to five decimal places, of the following outcomes:

(a) exactly 15 bulbs flower **(b)** more than 15 bulbs flower.

5 For each of the following situations, state whether or not it is suitable for binomial modelling. For those that are not, explain why the binomial model is not suitable.

(a) You are playing a game where you are required to roll a 6 before you can start. The number of rolls required for the eight players is recorded.

(b) A die is biased in such a way that $P(4) = \frac{1}{3}$. The die is rolled 40 times and the number of 4s rolled is recorded.

6 If 100 cards are drawn with replacement from a standard pack of 52 playing cards, find the probability of the following events, correct to four decimal places:

(a) exactly 25 hearts are drawn

(b) exactly 40 cards are clubs

(c) exactly 55 red cards are drawn

(d) exactly 60 cards are black.

7 The probability that an archer will shoot a bullseye with any particular arrow is $\frac{4}{5}$. The archer shoots 10 arrows. Which of the following expressions best represents the probability that the archer misses the bullseye with exactly two of the arrows?

A $\binom{10}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^8$ **B** $\binom{10}{8}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2$ **C** $\binom{8}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^8$ **D** $\binom{8}{2}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2$

8 In Amazonia, 62% of all births are girls. Find the probability, correct to four decimal places, that of eight births exactly half are girls.

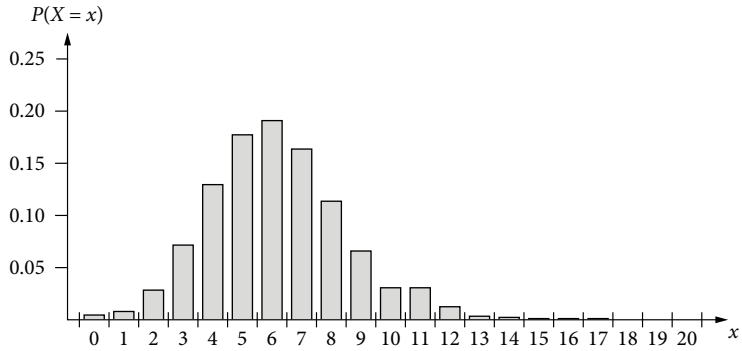
9 In the town of Sinistraville, 35% of the population is left-handed. Find the probability, correct to four decimal places, that exactly 35 people out of a random sample of 100 will be left-handed.

10 A coin is biased in such a way that $P(\text{heads}) = 3 \times P(\text{tails})$.
Find the probability, correct to three decimal places, that exactly six tosses out of 10 land on heads.

11 For the graph shown, choose the best representation for its binomial distribution.

A $X \sim B(12, 0.5)$ **B** $X \sim B(20, 0.3)$

C $X \sim B(12, 0.3)$ **D** $X \sim B(20, 0.5)$



12 Cards are dealt from a standard pack of 52 playing cards without replacement. Let X be the number of hearts dealt and assume five cards are dealt. Can this experiment be described as $X \sim B\left(5, \frac{1}{4}\right)$?

13 On the way to work, Philomena must drive through six sets of traffic lights. The lights are independent of each other and the probability that Philomena must stop at any particular set is 0.7. Find the probability, correct to four decimal places, that Philomena stops at the following sets of lights:

(a) exactly five sets of lights

(b) fewer than two sets of lights

(c) more than five sets of lights

(d) the first three sets of lights

(e) the second and sixth sets of lights only

(f) four sets of lights, including the first two.

14 Boxes of matches are supposed to contain 47 matches. Production records indicate that 80% of boxes contain 47 matches. A batch of 20 boxes is sampled. If more than four boxes do not contain exactly 47 matches, production is stopped.

(a) Find the probability, correct to four decimal places, that of the 20 boxes selected, the number of boxes that do not contain 47 matches is the following:
(i) 0 boxes **(ii)** 1 box **(iii)** 2 boxes **(iv)** 3 boxes **(v)** 4 boxes

(b) Find the probability that the number of boxes that do not have 47 matches is no more than four.

(c) Find the probability that production is stopped.

15 The local baker knows that, on average, 8% of the loaves of bread baked each day are slightly burnt and cannot be sold at full price. The likelihood of a particular loaf being slightly burnt is independent of any other loaf being slightly burnt. For each of the following, give your answers correct to four decimal places where necessary.

(a) If a random sample of 12 loaves is selected, what is the probability that exactly one loaf cannot be sold at full price?

(b) For the same random sample, calculate the probability that exactly three loaves are slightly burnt.

(c) What is the probability that all loaves in the random sample are sold at full price?

(d) To make a profit, the baker has to sell more than 40% of the loaves at full price. Calculate the probability that the baker makes a profit selling the 12 loaves.

16 A recording company receives a large number of new songs from various artists. On average, only 4% of the new songs become popular hits. One of the producers decides to select a random sample of 35 new songs.

(a) What is the probability that a maximum of three new songs from the sample will become popular hits? State your answer correct to three decimal places.

(b) What is the probability that more than four but less than six songs from the sample will become popular hits? State your answer correct to three decimal places.

(c) Determine the most likely number of songs from this sample that will become popular hits. Explain your answer using appropriate calculations.

(d) The producer decides to select 70 songs at random from the new songs received. If the probability of a new song becoming a popular hit has not changed, what is the probability that exactly one new song will become a popular hit?

(e) What would be the most likely number of new songs from this sample to become popular hits? Explain your answer using appropriate calculations.

17 Black market DVDs have a 0.15 chance of being faulty. Five DVDs are purchased independently.

(a) Draw a table to show the probability distribution of X, the number of faulty DVDs purchased.

(b) Find the probability that fewer than four faulty DVDs are purchased if you know there is at least one faulty DVD.

14.3 MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

Consider a random variable $X \sim B(5, 0.4)$. The probability distribution for X is shown in the following table:

x	0	1	2	3	4	5
$P(X = x)$	0.07776	0.2592	0.3456	0.2304	0.0768	0.01024

Using the method outlined earlier in this chapter, you can find the expected value of X, $E(X)$, and the Variance of X, $\text{Var}(X)$.

$$E(X) = \sum_{i=0}^{5} x_i p_i$$
$$= 0 \times 0.07776 + 1 \times 0.2592 + 2 \times 0.3456 + 3 \times 0.2304 + 4 \times 0.0768 + 5 \times 0.01024$$
$$= 0 + 0.2592 + 0.6912 + 0.6912 + 0.3072 + 0.0512$$
$$= 2$$

So, $[E(X)]^2 = 4$

$$E(X^2) = \sum_{i=0}^{5} x_i^2 p_i$$
$$= 0 \times 0.07776 + 1 \times 0.2592 + 4 \times 0.3456 + 9 \times 0.2304 + 16 \times 0.0768 + 25 \times 0.01024$$
$$= 0 + 0.2592 + 1.3824 + 2.0736 + 1.2288 + 0.256$$
$$= 5.2$$

So, $\text{Var}(X) = E(X^2) - [E(X)]^2$
$$= 5.2 - 4$$
$$= 1.2$$

When dealing with a binomial distribution, if X is the random variable representing the number of successes in n trials and p is the (constant) probability of success, then:

$E(X) = np = \mu$ (mean)

$\text{Var}(X) = np(1-p)$

$\sigma(X) = \sqrt{np(1-p)}$

You can check these results for the example given at the very beginning of this section, where $X \sim B(5, 0.4)$. Using the rules:

$E(X) = np$
$= 5 \times 0.4$
$= 2$

$\text{Var}(X) = np(1-p)$
$= 5 \times 0.4 \times 0.6$
$= 1.2$

The same answers are obtained.

Example 8

Find the following statistics for $X \sim B(15, 0.3)$. If necessary, give answers correct to two decimal places.

(a) $E(X)$ **(b)** $\text{Var}(X)$ **(c)** $\sigma(X)$

Solution

(a) $E(X) = np$
$= 15 \times 0.3$
$= 4.5$

(b) $\text{Var}(X) = np(1-p)$
$= 4.5 \times (1 - 0.3)$
$= 3.15$

(c) $\sigma(X) = \sqrt{np(1-p)}$
$= \sqrt{3.15}$
$= 1.77$

Example 9

Given $X \sim B(20, p)$ and $E(X) = 5$, find the value of p.

Solution

Use $E(X) = np$: $np = 5$

Substitute the known value: $20 \times p = 5$

Solve for the unknown: $p = \frac{5}{20} = \frac{1}{4} = 0.25$

Example 10

Given $X \sim B(n, p)$, $\mu = 9$ and $\sigma^2 = 6.3$, find the values of n and p.

Solution

$\mu = np$ $\qquad$ $\sigma^2 = np(1-p)$

$\mu = 9 \therefore np = 9$ $\qquad$ $\sigma^2 = 6.3 \therefore np(1-p) = 6.3$

Substitute for the known value of np in the σ^2 equation: $9(1-p) = 6.3$

$$1 - p = 0.7$$
$$p = 0.3$$

Find n: $n \times 0.3 = 9$

$$n = \frac{9}{0.3}$$
$$n = 30$$

$n = 30$, $p = 0.3$, so $X \sim B(30, 0.3)$

EXERCISE 14.3 MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

1 Find the stated values for each of the following binomial distributions.

(a) $X \sim B(20, 0.7)$. Find $E(X)$, $\text{Var}(X)$, $\sigma(X)$

(b) $X \sim B(100, 0.55)$. Find $E(X)$, $\text{Var}(X)$, $\sigma(X)$

(c) $X \sim B\left(50, \frac{2}{3}\right)$. Find $E(X)$, $\text{Var}(X)$, $\sigma(X)$

(d) $X \sim B(10, 0.5)$. Find $E(X)$, $\text{Var}(X)$, $\sigma(X)$

(e) $X \sim B(25, 0.8)$. Find $E(X)$, $\text{Var}(X)$, $\sigma(X)$

(f) $X \sim B(40, 0.4)$. Find $E(X)$, $\text{Var}(X)$

2 For each of the following, find the value of the unknown.

(a) $X \sim B(15, p)$, $E(X) = 5$, $p = ?$

(b) $X \sim B(30, p)$, $E(X) = 3$, $p = ?$

(c) $X \sim B(n, 0.2)$, $E(X) = 7$, $n = ?$

(d) $X \sim B(n, 0.4)$, $E(X) = 10$, $n = ?$

3 For each of the following, find the values of n and p for $X \sim B(n, p)$.

(a) $X \sim B(n, p)$, $\mu = 4.8$, $\sigma^2 = 2.88$

(b) $X \sim B(n, p)$, $\mu = 2$ and $\sigma^2 = 1.8$

(c) $X \sim B(n, p)$, $\mu = 4.5$ and $\sigma^2 = 3.825$

(d) $X \sim B(n, p)$, $\mu = 11$ and $\sigma^2 = 8.58$

(e) $X \sim B(n, p)$, $\mu = 65$ and $\sigma^2 = 22.75$

(f) $X \sim B(n, p)$, $\mu = 41.25$ and $\sigma^2 = 29.90625$.

4 Given $X \sim B(30, 0.4)$, which of the following pairs of values is correct?

A $E(X) = 12$, $\text{Var}(X) = \sqrt{7.2}$ $\qquad$ **B** $E(X) = 12$, $\sigma^2 = 7.2$

C $E(X) = 18$, $\sigma^2 = 7.2$ $\qquad$ **D** $E(X) = 18$, $\sigma = \sqrt{7.2}$

5 Given $X \sim B(n, p)$, $\mu = 10$ and $\sigma^2 = 8$, which of the following pairs of values is correct?

A $n = 80$, $p = 0.8$ $\qquad$ **B** $n = 50$, $p = 0.8$ $\qquad$ **C** $n = 80$, $p = 0.2$ $\qquad$ **D** $n = 50$, $p = 0.2$

6 Find $P(X = 3)$ given that $E(X) = 3$ and $\text{Var}(X) = 0.75$ and $X \sim B(n, p)$.

7 A spinner is divided into four equal sections, one of which is coloured blue. If the spinner is spun eight times, calculate the probability of obtaining less than the expected number of blue outcomes, correct to three decimal places.

8 For the variable $X \sim B(40, p)$, $\sigma^2 = 9.6$. The correct value(s) of p is:

A 0.4 $\qquad$ **B** 0.6 $\qquad$ **C** 0.4 and 0.6 $\qquad$ **D** none of these.

9 Consider $X \sim B(8, 0.25)$.

(a) Construct a table showing the probability distribution. Round probabilities to four decimal places.

(b) Use the rule $\mu = np$ to find $E(X)$.

(c) Use the rule $\sigma = \sqrt{\text{Var}(X)}$ to find $\sigma(X)$, correct to two decimal places.

10 A family has six children who are all boys or girls. Assume the probability of any child being a boy is 0.5 and that the probability is independent for each child. Find the probability of each of the following, rounding answers to four decimal places:

(a) the first two children born are male

(b) there are three boys and three girls, in any order

(c) there are more girls than boys

(d) there is at least one boy, but more girls than boys

(e) there are no consecutive births of the same sex.

11 A researcher has calculated the mean and the variance for a sample of a given random variable that has a binomial distribution.

(a) If the mean of the data set is 42 and the variance is six, determine the number of trials (n) and the probability of success (p).

(b) A new set of data is collected and the researcher notices that the mean and variance of the new set of data are double the corresponding values of the first set of data. Compare the two sets of data using appropriate calculations involving n and p.

(c) The researcher realises that the results for 10 trials have been omitted from the first set of data. When the mean is recalculated with the 10 additional results, the mean is unchanged at $\mu = 42$. What variance should the researcher expect for this set of data? Give your answer correct to two decimal places.

(d) Another researcher conducts the same experiment and records the data for 56 trials. The probability of success for this set of data is calculated to be $p = \frac{6}{7}$. What values of the mean and variance should the researcher expect to calculate for this set of data? Give your answer correct to two decimal places.

(e) Write a statement to describe the effect on the mean and variance as the number of trials increases.

14.4 NORMAL APPROXIMATION FOR THE SAMPLE PROPORTION

You can make sense of the world around you by studying particular features or characteristics that occur within populations. To study a whole population can be impractical, costly, time-consuming and sometimes impossible. In real-life situations, this means it is rare to know the probability of a certain characteristic occurring within a given population.

This issue can be overcome by obtaining a **sample** and using the information from this sample to *infer* particular characteristics about the population. In other words, you study the 'part' in order to predict information about the 'whole'. This is the basis of **statistical inference**.

In your previous studies you have analysed sets of data, summarised and then presented the information. This is known as descriptive statistics. In this chapter, you will be using inferential statistics. While descriptive statistics summarises and presents information, inferential statistics uses the statistics from a random sample to draw conclusions about the population. Recall that a sample is considered **random** if every unit of the population has an equally likely chance of being selected in the sample.

Sampling distributions and variability

The statistical results of random samples taken from a population will vary. This leads to the idea that statistics associated with samples from a population can be represented by a random variable with its own probability distribution. This applies when working with either sample means or sample proportions.

To understand the behaviour of a probability distribution of results from samples, you need to look at what happens when many samples are taken. One way of doing this is to create a **simulation** and observe the behaviour of the data.

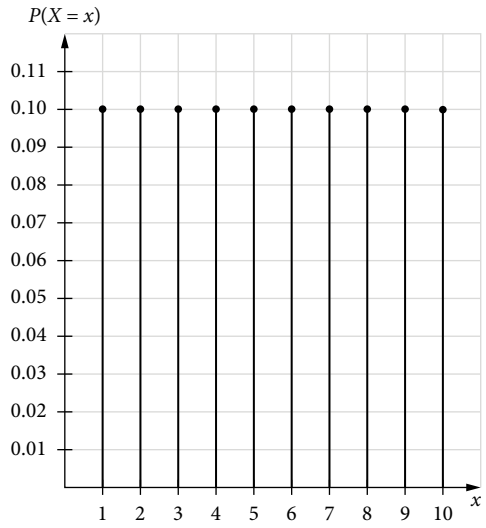
The probability distribution of a fair ten-sided die is shown on the right. This is a discrete uniform distribution.

In theory, it is expected that the results will follow a discrete uniform distribution.

Using the formulas for the expected value and standard deviation of a uniform distribution:

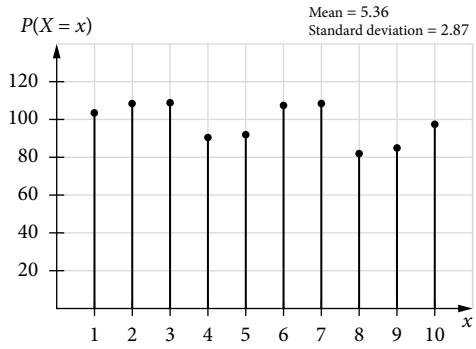
$$E(X) = \frac{u+1}{2} \quad \text{and } \sigma^2 = \frac{(u-1)^2}{12} \quad \sigma = \sqrt{\frac{81}{12}}$$
$$= \frac{10+1}{2} \qquad = \frac{(10-1)^2}{12} \qquad \approx 2.60$$
$$= \frac{11}{2} \qquad = \frac{81}{12}$$
$$= 5.5 \qquad = 6.75$$

Probability distribution of a fair ten-sided die

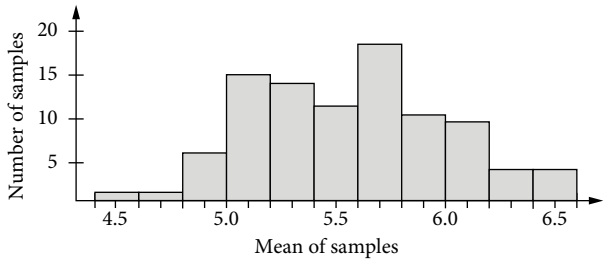


The graph on the right displays the results from 1000 observations of rolling a fair ten-sided die. This can be considered as a very large sample of a random variable from a discrete uniform distribution. The sample mean and standard deviation were calculated.

Note that this sample distribution is very close to the discrete uniform distribution, but not identical. This would be expected in most cases. Such a random sample would have a mean around 5.5 and the expected number of times of each value occurring is close to one-tenth of the sample size, in this case 100. This sample's mean is 5.36 and the standard deviation is 2.87.

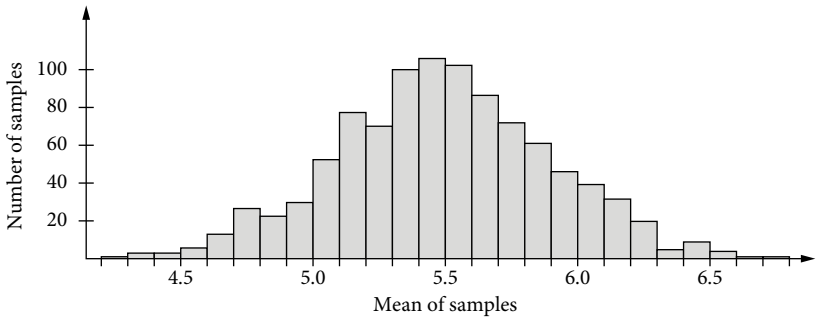


Now consider what happens if you take 100 samples of size 50 and plot the mean of each sample. Note in this case a histogram is used to plot the means of the samples, so that similar sized means are grouped within one interval, providing a better picture of the overall pattern than plotting each individual sample mean separately.



Repeating the same process 1000 times gives a better indication of the shape of the distribution of the sample means.

Here it can be seen that the distribution is becoming more bell-shaped and symmetrical, much like a normal distribution. From the histogram it can be seen that the mean is very close to 5.5, which reflects the probability of the discrete uniform distribution.



The central limit theorem

This simulation demonstrates one of the most important theorems in statistics—the **central limit theorem** (CLT). While the central limit theorem is not required knowledge for this course, it is helpful to understand the significance of this concept and the part this plays in inferential statistics. Informally, the CLT says that the sampling distribution for a *given statistic* is essentially normally distributed, regardless of the nature of the *parent* distribution. This means that you can use a normal distribution to model the sampling distribution for that statistic, and hence develop confidence intervals for estimates for a population parameter.

In most cases, the population parameters are unknown, so the CLT is used in *reverse*. Taking samples of size n from any distribution, you can calculate the mean for each sample and know that the collection of sample means will be approximately normally distributed. This implies that the average of the sample means will be approximately equal to the population mean.

Consider obtaining samples of size 50 when rolling a ten-sided fair die. When a small number of samples of size 50 are generated, the distribution of the sample means does not show a normal distribution as clearly as when larger samples are taken. The shape is more random and often can be skewed to the left or the right, or very spaced out. When a large number of samples of size 50 are generated, the distribution of the sample means starts to resemble the shape of the normal curve more closely; the larger the number of samples generated, the more evident this is.

MAKING CONNECTIONS

Central limit theorem and sample proportions

Explore how the shape of a distribution changes as more samples are taken.

If a large enough sample is taken from any population where the mean μ and the variance σ^2 are known, then the sample observations have a distribution which is approximately normal with mean $n\mu$ and variance $n\sigma^2$. In most cases, a sample size greater than or equal to 30 is usually enough to assume that the sample means will follow a normal distribution.

If $X_1, X_2, X_3, \ldots, X_n$ are independent identically distributed random variables, each with mean μ and variance σ^2, then as $n \to \infty$, $D = X_1 + X_2 + X_3 + \ldots + X_n \sim N\left(n\mu, n\sigma^2\right)$.

The central limit theorem is often expressed in another way. Instead of dealing with the sum of the samples, you can instead deal with the mean of the samples, $\overline{X}$. In this case, you have $\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$.

This means that no matter what the distribution of $X_1, X_2, X_3, \ldots, X_n$ looks like, if there are lots of distributions then the sample statistics will be approximately normal. In fact, 'lots' is often not a very large number, only $n \geq 30$ as stated above.

If the underlying population has a normal distribution, then the sample size is usually not an issue. It is only when the distribution of the underlying population is something other than normal that the sample size is more important.

Sample proportions—notation and terminology

Medical research indicates that 4% of the Australian population carry the gene for cystic fibrosis, one of the most common life-threatening genetic conditions in Australia. You could ask, 'How accurate is 4% with respect to the population?'

Realistically, the exact population parameter is unknown. Unless you are willing to perform a census, your information and subsequent calculations will come from a random sample. However, it can be shown that a proportion calculated from a simple random sample can provide a reasonable *estimate* of the population parameter.

For example, consider if a random sample of 4000 people is tested for the cystic fibrosis gene. From this sample, it is found that 140 people are carrying the gene. This gives a proportion of $\dfrac{140}{4000} = 0.035$. As this proportion is obtained from the random sample, it is called the **sample proportion**.

The table below illustrates definitions and terminology associated with sample proportions from this example.

Notation	Definition	Example
x	*Observation* For any one sample, the number of people or things that possess a certain characteristic is called the observation.	The observation is the 140 people who carry the gene.
n	*Sample size* This is the number of people or things that make up the sample.	The sample size is 4000.
$\hat{p}=\frac{x}{n}$	*Sample proportion* This is the number of people or things with a characteristic that occurs within the sample (observation), as a proportion of the number in the sample.	140 out of 4000 carry the gene. $\hat{p}=\frac{140}{4000}$ $=0.035$
p	*Population proportion* This is the true proportion of the people or things with a characteristic that occur within the whole population.	(Unknown.)

The sample proportion $\hat{p}=\frac{x}{n}$ is not the population proportion; rather, it is the best estimate of the population proportion from a single sample. Because of this, the sample proportion is sometimes called the *point estimate*.

Example 11

Consider a sample of 2000 18-year-olds in NSW. From this sample, 210 have their provisional driver's licence (P plates). Using this information, identify:

(a) the population proportion **(b)** the sample size
(c) the observation value **(d)** the sample proportion.

Solution

(a) p is the population proportion.
The population proportion, which is unknown, is the proportion of all 18-year-olds in NSW who have their provisional licence.

(b) The sample size is n, the number of people in the sample: $n = 2000$

(c) The observation value is the number of people in the sample who have their provisional driver's licence: $x = 210$

(d) The sample proportion is given by $\hat{p}=\frac{x}{n}$: $\hat{p}=\frac{x}{n}$

$$=\frac{210}{2000}$$
$$=0.105$$

The CLT and proportions

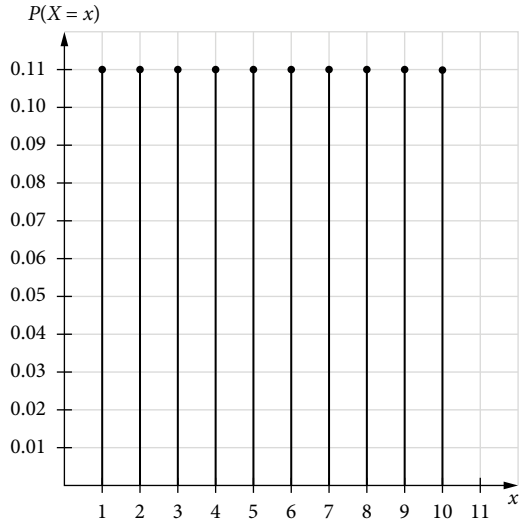
Earlier in this chapter, the variability of results between random samples taken from a single population was explored. Recall that any statistic from all of the possible samples is a random variable and each of these random variables has a probability distribution.

To understand the behaviour of a sampling distribution for proportions, a simulation has been created that will allow you to observe the behaviour of the data.

From the previous section, recall the probability distribution of a fair 10-sided die. This is a discrete uniform distribution.

Rather than considering the sampling distribution of the means, you will now consider what happens if a particular number occurs, in this case the number '3'. This simulation will observe 100 samples of size 50. In each sample, the die has been rolled 50 times and the number of threes 'recorded' for each sample.

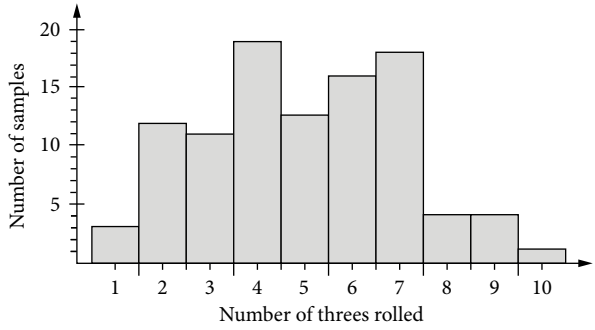
Probability distribution of a fair ten-sided die



A histogram is used to plot the occurrences of the number '3' as it provides a better picture of the whole pattern than plotting each individual result separately.

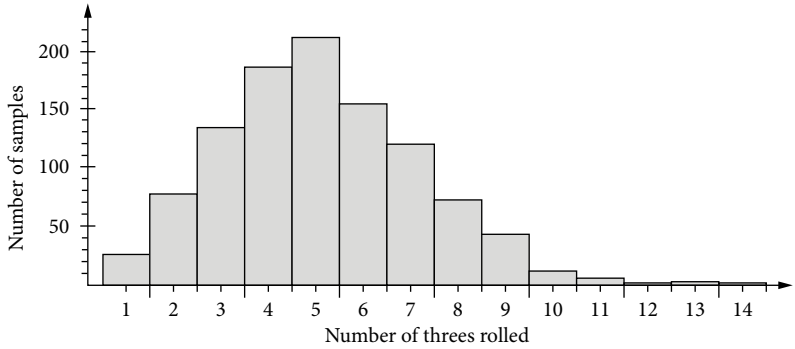
This means that in samples of 50 rolls, 3 samples had one 'three', 12 samples had 2 'threes', 11 samples had 3 'threes' (and so on); up to one sample had 10 'threes'.

Note that the distribution of the occurrence of the number 'three' does not resemble a uniform distribution.



Repeating the same process 1000 times gives a better indication of the shape of the distribution of the occurrence of the number three.

Rolling a 10-sided die 50 times, with the probability of a 'three' being 0.1 means that you would expect $50 \times 0.1 = 5$ 'threes'. In fact, you may get more or less 'threes' in some samples.



The more samples that you take, the more the distribution becomes bell-shaped and symmetrical, like a normal distribution. The tallest column on this histogram is above five, meaning that the expected result of 5 'threes' per sample more often than any other result.

The CLT applies equally to the proportion of a specific random variable in a sampling distribution, as it does to the sampling distribution of means.

Taking samples of size n, from any distribution, a specific proportion of a random variable can be calculated and it will be approximately normally distributed. This implies that the proportion of a specific random variable in a sample will be approximately equal to the proportion of the same specific random variable in the population.

MAKING CONNECTIONS

The CLT and sample proportions

The activity simulates obtaining samples of size 50 for a 10-sided fair die. Observe how the shape of the distribution of the number of threes obtained changes as more samples of size 50 are taken.

When a small number of samples of size 50 are generated, the distribution of the number of threes does not clearly show a normal distribution. The shape is more random and often skewed to the left or the right or very spaced out. When a large number of samples of size 50 are generated, the distribution of the number of threes starts to resemble the shape of a normal curve. The larger the number of samples generated, the more evident this is.

Sampling distribution for sample proportions

It is useful to explore the relationship between Bernoulli random variables and binomial distributions, and how these relate to sample proportions.

Recall from your work on discrete random variables that a Bernoulli random variable is a discrete random variable which has two outcomes: a 'success' with an associated probability of p and a 'failure' with the associated probability of $1 - p$. Within a binomial situation, if n units are chosen at random, then the variable X (the count of successes in a sample) has a binomial distribution $X \sim B(n, p)$.

The sample proportion $\hat{p} = \frac{X}{n}$ is the binomial count divided by the sample size, which gives a value between 0 and 1. This means that the set of $\hat{p}$ values are related to a random variable called $\hat{P}$ and follow a binomial distribution.

Notice the differences in notation. The values X and $\hat{p}$ are used when observing a single random sample. However, when considering the whole set of values that can be obtained from *all* samples, X and $\hat{P}$ are used.

The standard deviation of a binomial distribution can be estimated from the sample proportion $\hat{p}$ of *one sample*, rather than many. This cannot be done easily for other distributions.

As previously discussed, both the mean and the standard deviation are defined using the population parameter p. However, the value of p is unknown, so the exact mean and standard deviation cannot be calculated.

To overcome this problem, use the sample proportion $\hat{p}$ as an estimate for both the expected value and standard deviation, and replace p with $\hat{p}$. When $\hat{p}$ replaces p in the standard deviation, then it is often called the **standard error** $S(\hat{P})$ instead. This is to distinguish between the use of a parameter and a statistic.

The distribution of $\hat{P}$ for the sample proportions is defined as $\hat{P} \sim N\left(\hat{p}, \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$,

where the expected value $E(\hat{P}) \approx \hat{p}$ and the standard deviation $SD \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = S(\hat{P})$.

Example 12

A social media survey found that 134 out of 265 respondents had used some form of social media within the past year. Using this information:

(a) calculate the expected value for the proportion of people who access social media, correct to two decimal places

(b) calculate the standard error, correct to three decimal places.

Solution

(a) Use the formula for the expected value: $E\left(\hat{P}\right) \approx \hat{p}$

$$\approx \frac{x}{n}$$

Substitute the given values into the formula: $E\left(\hat{P}\right) \approx \frac{134}{265}$

$$\approx 0.51$$

The expected value $E\left(\hat{P}\right) \approx 0.51$, to two decimal places.

(b) Use the formula for the standard error: $S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$

Substitute the given values into the formula: $S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$

$$= \sqrt{\frac{0.51 \times 0.49}{265}}$$

$$\approx 0.031$$

The standard error $S\left(\hat{P}\right) = 0.031$, to three decimal places.

Example 13

A recent survey of 1500 Year 12 students showed that 990 were intending to apply for university.

(a) Calculate the expected value.

(b) Calculate the standard error, correct to four decimal places.

Solution

(a) Find the expected value using the formula: $E\left(\hat{P}\right) \approx \hat{p}$

$$\approx \frac{990}{1500}$$

$$\approx 0.66$$

(b) Use the formula for the standard error: $S\left(\hat{P}\right) = \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$

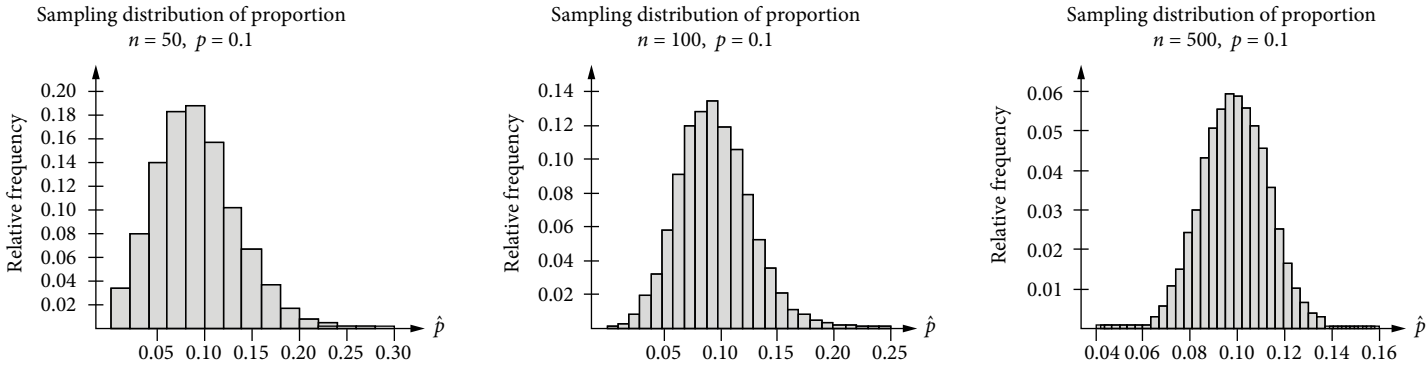
$$= \sqrt{\frac{0.66(1-0.66)}{1500}}$$

$$\approx 0.012231$$

The standard error is $S\left(\hat{P}\right) = 0.0122$.

The size of *n*

From the CLT, you know that where a random sample of size n has X 'successes', when n is sufficiently large, the sampling distribution of the sample proportion $\hat{p}$ will be approximately normal with mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.



Here you can see that for a value of $p = 0.1$ and $n = 50$, the distribution may look bell-shaped but it is noticeably skewed. However, as the value for n increases, the shape, while retaining the bell shape, becomes more symmetrical about p. This means that even if the value of p is not close to 0.5 (i.e. the population is unbalanced), the larger the sample, the more the sampling distribution of $\hat{p}$ resembles a normal distribution.

So what value of n is considered sufficiently large? The following conditions allow you to determine if the CLT can be applied.

The following two conditions must be met for the CLT to apply:

- $n\hat{p} \geq 5$
- $n(1-\hat{p}) \geq 5$

You also need to approximate the population proportion p with the sample proportion $\hat{p}$.

However, note that if the population already has a normal distribution then the conditions are not necessary.

Example 14

A survey of 25 people has obtained a sample proportion of 0.2. Determine if the distribution can be approximated by a normal distribution.

Solution

Define the parameters: $n = 25$ and $\hat{p} = 0.2$.

Test these parameters for each condition: $n\hat{p} \geq 5$

$25 \times 0.2 = 5$

$5 \geq 5$

And: $n(1-\hat{p}) \geq 5$

$25 \times 0.8 = 20$

$20 \geq 5$

Both statements are true, so the distribution can be approximated by a normal distribution.

EXERCISE 14.4 NORMAL APPROXIMATION FOR THE SAMPLE PROPORTION

1 Read through the following scenarios and identify the population proportion, sample size, observation value and sample proportion. Give answers correct to two decimal places where necessary.

(a) A report by an IT data company in the Northern Territory surveys 800 people and reports that 42% of them have received a fake email claiming they have won a large amount of money.

(b) A database search shows that approximately 215 000 Tasmanians own a Holden. A survey of 2500 Holden owners in Tasmania shows that 1893 prefer a medium- to large-sized car over a smaller model.

2 A survey has found that 64 of 104 Year 12 students are not sure what to do after they leave school. Using this information, calculate the expected value and standard error for the population proportion of Year 12 students who are not sure what to do after they leave school. Give answers correct to three decimal places.

3 The results of a survey of 334 people give a sample proportion of 0.98. Do these given values allow the distribution to be approximated by a normal distribution?

4 Calculate the expected value and standard error for the following proportion elements. Give answers to three decimal places where necessary.

(a) $x = 507$ and $n = 1300$ **(b)** $x = 415$ and $n = 500$ **(c)** $x = 50$ and $n = 1000$

5 Determine whether the following values satisfy the conditions for normality.

(a) $\hat{p} = 0.55$ and $n = 1000$ **(b)** $\hat{p} = 0.9$ and $n = 10$ **(c)** $\hat{p} = 0.1$ and $n = 50$

6 For each of the following, given the sample proportion and observation value, find the sample size. Give your answers correct to the nearest whole number.

(a) $\hat{p} = 0.7$ and $x = 91$ **(b)** $\hat{p} = 0.15$ and $x = 300$ **(c)** $\hat{p} = 0.83$ and $x = 15$

7 For each of the following, given the sample proportion and the sample size, find the observation value.

(a) $\hat{p} = 0.34$ and $n = 2500$ **(b)** $\hat{p} = 0.02$ and $n = 100$ **(c)** $\hat{p} = 0.98$ and $n = 5000$

8 The 'sampling error' is considered to be the difference between the values of the population proportion and the sample proportion. Why is the value of the sampling error difficult to calculate?

9 In the following cases, calculate the expected value, correct to two decimal places. Then calculate the standard error, correct to three decimal places. Finally, describe the shape of the sampling distribution.

(a) Jess is running for school captain. From a simple random sample of 900 students, the proportion of students who favour Jess is 0.75.
(b) A simple random sample of 50 people found that five people were prepared to quit smoking within the week.
(c) Last year 67 000 children were hospitalised. Of these admissions, 42 600 were boys.

10 Explain the difference between p and $\hat{p}$.

11 In your own words, explain the standard error.

12 What happens to the value of the standard error as the sample size increases and what does this mean?

13 According to a survey of 500 employers, it was found that 78% of employers agreed that the demand for jobs has increased. This survey was repeated many times with random samples being taken from the same population and the sample proportions recorded.

(a) Do you think that the sample proportion from each survey will always be 0.78?
(b) Would a sample proportion value of 0 be possible? What would this mean?
(c) Could some of the sample proportions be lower than 0.45? What could this be caused by?
(d) Is the sample proportion of 0.78 close to the true population proportion? Justify your answer.
(e) In what scenario could the sample proportion value not be representative of the true population proportion?

14 A random sample of 1500 patient records showed that since the introduction of heparin injections for all patients, only 7% of patients developed blood clots. This research was repeated many times with random samples being taken from the same population and the sample proportions recorded.

(a) Why do you think that the sample proportion varies for each sample?
(b) Would a sample proportion value of one be possible? What would this mean?
(c) If a sample proportion appeared abnormally high, for example 0.23, what could this indicate?
(d) A proportion of 7% is fairly small. Does this indicate a problem with the sampling process?
(e) In what scenario could the sample proportion value not be representative of the true population proportion?

15 A fitness club surveys a sample of 120 of its members on whether they are satisfied with the club's facilities or not. The survey results show that half of the members are satisfied, a third are indifferent and a sixth are dissatisfied.

(a) Calculate the observation value for each type of response.
(b) Calculate the point estimate for each type of response, correct to two decimal places where necessary.
(c) How would the point estimate of the dissatisfied members change if the sample size is doubled but the results are in the same proportion?
(d) How would the point estimate of the dissatisfied members changes if the sample size is the same but the proportion of dissatisfied customers changes to one-fifth of the sample?

16 The sales manager from a publishing company believes that 35% of the company's sales come from returning customers. A random sample of 200 sales has been chosen.

(a) For this sample, state the expected value, correct to two decimal places. Then calculate the standard error, correct to four decimal places.
(b) Assuming normality, plot the sampling distribution of $\hat{p}$.
(c) The sales manager chooses a second random sample of 200 sales and notices that only 55 of the sales are from returning customers.
(i) For the new sample, calculate the expected value, correct to three decimal places. Then, calculate the standard error, correct to four decimal places.
(ii) Assuming normality, plot the sampling distribution of this sample on the same axes as the first sample.
(d) If a third random sample is chosen and the number of sales from returning customers is 80, and assuming normality, how would the new sampling distribution compare to the other two sampling distributions? Plot the third sampling distribution on the same axes as the first and second samples.

CHAPTER REVIEW 14

1 Find the stated probability for the following binomial distributions. Express your answers as fractions in simplest form.

(a) $P(X = 3)$ if $X \sim B\left(5, \frac{3}{5}\right)$. **(b)** $P(X = 4)$ if $X \sim B\left(6, \frac{7}{10}\right)$.

2 For the variable $Y \sim B(n, p)$ it is known that $E(Y) = 32$ and $\text{Var}(Y) = 6.4$.

(a) Find the probability of success, p. **(b)** Find the number of trials, n.

3 A coin is biased in such a way that $P(\text{heads}) = 3 \times P(\text{tails})$. The coin is tossed 100 times. Let X stand for the number of tails obtained. Find the value of $E(X)$.

4 When Yehudi and Carlos play racquetball, the probability that Yehudi wins a point is 0.35.

(a) How many points would you expect Yehudi to win from the first 15 points? Give your answer to the nearest whole number of points.
(b) Choose the correct terms in the following statement.
If Yehudi won 10 out of the first 15 points, I would [not be / be slightly / be very] surprised as the number is [about the same as / just above / well above] the expected number.

5 A sample of three items is selected at random from a box containing 10 items of which three are defective. Let Y represent the number of defective items selected.

(a) Complete the table to show the probability distribution of the variable. State the probabilities in simplest fraction form.

y	0	1	2	3
$P(Y = y)$				

(b) Find the expected number of defective items, $E(Y)$.

6 A jar contains seven white marbles, three green marbles and two blue marbles. Two marbles are drawn, with replacement, from the jar. What is the probability of drawing exactly one white marble?

A $\frac{7}{12}\times\frac{7}{12}$ **B** $\left(\frac{7}{12}\times\frac{5}{11}\right)+\left(\frac{5}{12}\times\frac{7}{11}\right)$ **C** $2\times\frac{7}{12}\times\frac{5}{12}$ **D** $\frac{7}{12}\times\frac{5}{12}$

7 Find the value of t in the following probability distribution table.

x	0	1	2	3
$P(X=x)$	t	$2t$	$3t$	$4t$

A 10 **B** 0.1 **C** 6 **D** 0.6

8 A coin, which is biased so that $P(\text{heads}) = 2 \times P(\text{tails})$, is tossed eight times. The probability that the result is heads exactly three times, $P(X = 3)$, is best represented by:

A $\binom{8}{3}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3$ **B** $\binom{8}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5$ **C** $\binom{8}{5}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$ **D** $\binom{8}{3}\left(\frac{2}{5}\right)^3+\binom{8}{5}\left(\frac{1}{3}\right)^5$

9 If $Y \sim B(100, 0.2)$, what is the value of μ?

A 0.8 **B** 16 **C** 20 **D** 80

10 For a particular binomial random variable Y, it is known that $\text{Var}(Y) = 14.4$. If 60 trials are conducted, what is the probability p of success?

A 0.4 **B** 0.6 **C** 0.24 or 0.76 **D** 0.4 or 0.6

11 A die, with sides labelled 1–6, is biased so that $P(\text{odd}) = 3 \times P(\text{even})$.

(a) If rolling an odd number is considered a success, find $P(\text{success})$.

(b) The die is rolled 30 times.

(i) Write this information in the form $X \sim B(n, p)$.

(ii) What is the expected number of odd numbers that will occur in the 30 rolls?

(iii) How unusual would you consider it to roll 28 odd numbers in the 30 rolls? Explain with reference to the 95% confidence interval.

(c) **(i)** What is the probability of any pair of rolls resulting in two odd numbers?

(ii) Draw a table to show the probability distribution of the number of odd numbers in the two rolls.

12 A cereal manufacturer is running a promotion on single-serving boxes of cereal, which states that one in five cereal boxes contains a free gift. Carol likes to win free gifts and intends to buy one cereal box every day for one week. State your answers correct to three decimal places where necessary.

(a) If the variable X represents the number of cereal boxes with a free gift, calculate the probability that Carol will win exactly one free gift from the seven cereal boxes she intends to buy.

(b) What is the probability that Carol wins no free gifts from the seven cereal boxes she intends to buy?

(c) Given that Carol wins one free gift on the first day, calculate the probability that she wins exactly one more free gift in the next six days.

(d) Given that Carol does not win a free gift in the first five days, calculate the probability that she wins more than one free gift in the next two days.

(e) Carol thinks that if she buys one cereal box per day for 10 days, her chances of winning more than two free gifts will increase. Explain why Carol is correct in her thinking, using appropriate calculations.

SUMMARY

1 FURTHER WORK WITH FUNCTIONS

Properties of inequalities

- Adding or subtracting the same number from both sides does not alter an inequality.
- Multiplying both sides of an inequality by a negative number reverses the direction of the inequality.
- Multiplication by a positive number does not alter the direction of the inequality.
- Taking the reciprocal of both sides of an inequality reverses its direction when both sides have the same sign, but not if the signs are different.
- Squaring both sides of an inequality: this produces a positive number on both sides, so the direction of the inequality $a^2 \lessgtr b^2$ depends on which of $|a|$ and $|b|$ is larger.
- Square root of both sides of an inequality: this is only defined if both sides of the inequality are positive. The direction of the inequality does not change, i.e. if $a > b$ then $\sqrt{a} > \sqrt{b}$, provided a is positive and b is positive or zero.

2 POLYNOMIALS

- A real polynomial $P(x)$ is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $n, n-1, \ldots$ are all positive integers and $a_n, a_{n-1}, \ldots$ are the coefficients, which for convenience will usually be chosen as integers.
- The term $a_n x^n$ is the **leading term**. If $a_n \neq 0$, the polynomial is said to be of the n-th degree.
- $P(x)$ is defined for all real x and is a continuous and differentiable function of x.
- $P(x) = 0$ is a polynomial equation of degree n. Real numbers x that satisfy this polynomial equation are called the real **roots** of the equation or the real **zeros** of the corresponding polynomial.
- The polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ will have at most n real roots.
- Expressions are **not** polynomials if they include powers of x that are not positive integers, e.g. $x^{\frac{1}{2}} + 2 + x$, $\frac{1}{x} + 2x^{-3}$.

Division of polynomials

- If $P(x) = 3x^2 + 7x + 4$, then $P(1)$ is the value of $P(x)$ when $x = 1$, which is found by substituting $x = 1$ into the polynomial: $P(1) = 3(1)^2 + 7(1) + 4 = 14$ i.e. $P(a) = 3a^2 + 7a + 4$.
- $\frac{x^2+7x+12}{x+3}, x \neq -3$ or $(x^2 + 7x + 12) \div (x + 3)$, $x \neq -3$ are two ways of writing a long division of polynomials.
- Dividend = Divisor × Quotient + Remainder
  Degree of the remainder < degree of the divisor.
  Degree of the quotient < degree of the dividend.
- When the divisor is a linear function (first degree), the remainder will be a constant.

The remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$ until the remainder R does not contain x, then $R = P(a)$.

For any polynomial: $P(x) = (x - a)Q(x) + R$ where $Q(x)$ is another polynomial.

Note:

- If $P(x)$ is divided by $x + a$, as $x + a = x - (-a)$, then $R = P(-a)$.
- If $P(x)$ is divided by $ax - b$, as $ax - b = a\left(x - \frac{b}{a}\right)$, then $R = P\left(\frac{b}{a}\right)$.

The factor theorem

When the remainder of a division is zero, the divisor is a factor of the expression being divided:

- For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.
- Conversely: if $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

By finding the zeros of a polynomial (i.e. values of a such that $P(a) = 0$), we can **factorise** the polynomial. Zeros can be found by trial and error and then by long division.

Multiple roots of a polynomial equation

A polynomial of degree n has n zeros, but they are not necessarily all different. You say that c is a zero of multiplicity r ($r > 1$) when the factor $(x - c)$ occurs r times.

If $x = c$ is a zero of multiplicity r of the real polynomial $P(x)$, then $x = c$ is also a zero of multiplicity $(r - 1)$ of the derived polynomial $P'(x)$, a zero of multiplicity $(r - 2)$ of the second derived polynomial $P''(x)$, and so on.

If $P(x)$ is a polynomial of degree n, then $P'(x)$ must be a polynomial of degree $(n - 1)$, $P''(x)$ a polynomial of degree $(n - 2)$, and so on.

Polynomial functions

The **general polynomial function** is the function f where $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ $(a_n \neq 0)$. This function is defined for all real values of x and is continuous and differentiable.

For $a \neq 0$, we have:

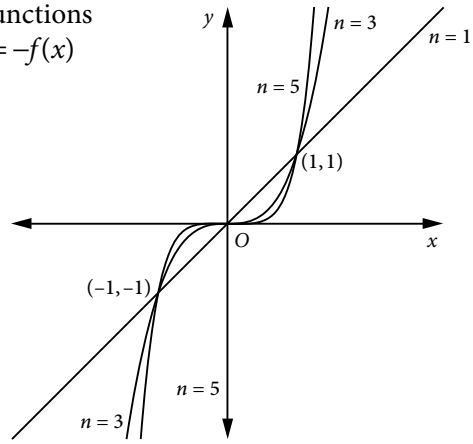
- $f(x) = ax + b$ (general linear function)
- $f(x) = ax^2 + bx + c$ (general quadratic function)
- $f(x) = ax^3 + bx^2 + cx + d$ (general cubic function)

If the coefficient of the highest power of x is unity (1), the polynomial is said to be **monic**.

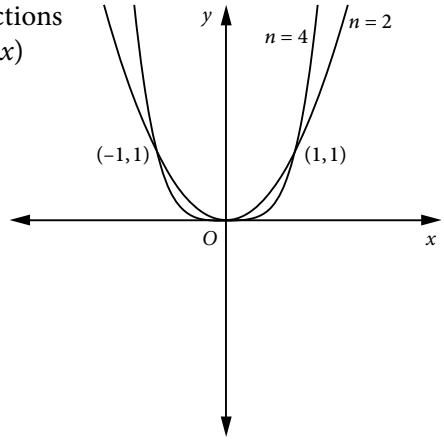
Graphs of polynomial functions

The simplest polynomial function of degree n is $f(x) = x^n$. Graphs of these basic polynomials for $n = 1, 3, 5$ follow, and for $n = 2, 4$ below:

Odd functions
$f(-x) = -f(x)$



Even functions
$f(-x) = f(x)$



Important features of $f(x) = x^n$

1 The x-axis is a tangent to each graph at the origin ($n \neq 1$).

2 $f(x) = x^n$ for even values of n defines **even functions**. For even functions, $f(-x) = f(x)$, so their graphs are symmetrical about the y-axis.

3 $f(x) = x^n$ for odd values of n defines **odd functions**. For odd functions, $f(-x) = -f(x)$. Because $f(-x)$ and $f(x)$ are opposite in sign, the graph of f for $x \leq 0$ can be obtained by rotating the graph of f for $x \geq 0$ through 180° about the origin.

4 Recognising that a function is odd or even means that you only need to draw half of the graph in detail. The other half can then be drawn using the symmetry properties.

5 Odd functions will have an inverse function, although some may require a restriction on the domain.

6 Even functions will not have a single inverse function, but can be split into two parts (by restricting the domain) so that each part has an inverse function.
(Inverse functions are covered in Chapter 5.)

7 Note that most functions are neither even nor odd, e.g. $f(x) = x^2 + x$, $f(x) = e^x$.

Cubic functions

A general cubic function is a polynomial function f of the third degree, defined by $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$.

Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$, where α is a real number.

Summary of polynomial functions

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, then:

1 For very large $|x|$, $P(x) \approx a_n x^n$.

2 A polynomial of odd degree always has at least one real zero (i.e. its graph cuts the x-axis at least once).

3 At least one maximum or minimum value of P occurs between any two distinct real zeros.

4 For a polynomial of odd degree, the ends of the graph go in opposite directions.

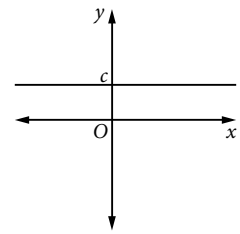
5 For a polynomial of even degree, the ends of the graph go in the same direction.

6 When the graph of a polynomial function meets the x-axis, it may cut it (single zero), touch it (double zero) or cut it at a point of inflexion (triple zero).

Polynomial graphs, standard forms

Degree 0 (constant)

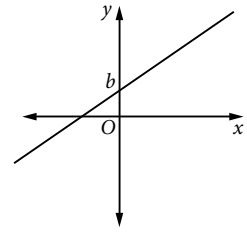
$P(x) = c$



Does not cut
No zeros

Degree 1 (linear)

$P(x) = mx + b$

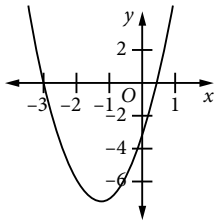


Cuts once
One zero

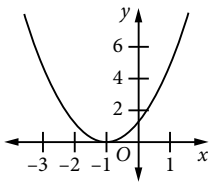
Degree 2 (quadratic)

$P(x) = ax^2 + bx + c$

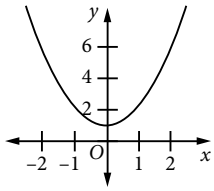
$a > 0$:



Cuts twice
$\Delta > 0$
Two zeros

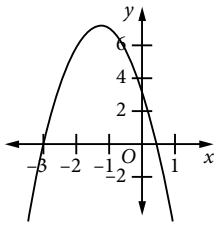


Touches
$\Delta = 0$
Double zero

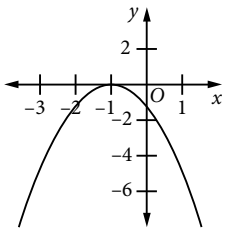


Does not cut
$\Delta < 0$
No real zeros

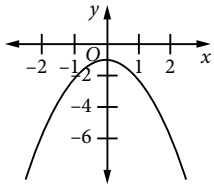
$a < 0$:



Cuts twice
$\Delta > 0$
Two zeros



Touches
$\Delta = 0$
Double zero



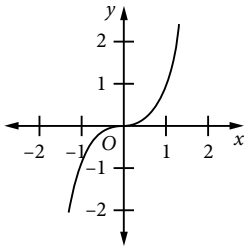
Does not cut
$\Delta < 0$
No real zeros

A quadratic polynomial may have two, one or no real zeros.

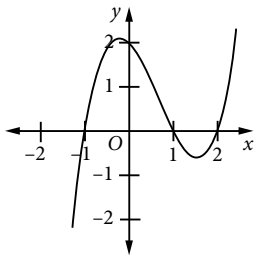
Degree 3 (cubic)

$P(x) = ax^3 + bx^2 + cx + d$

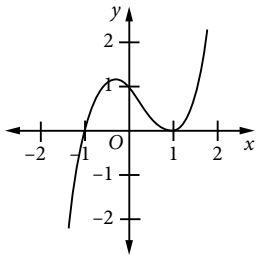
$a > 0$:



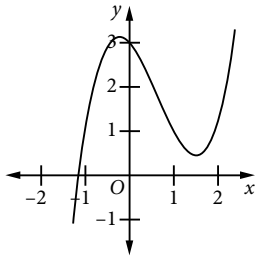
Cuts once
Triple zero



Cuts three times
Three zeros

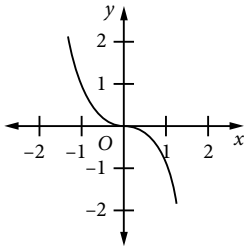


Cuts once and touches
One zero, one double zero

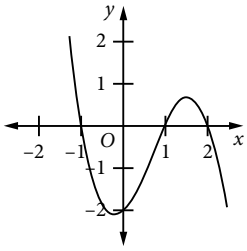


Cuts once
One zero

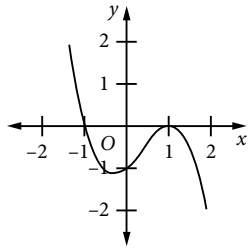
$a < 0$:



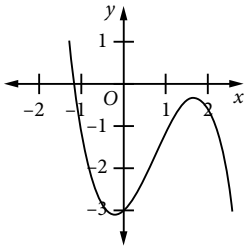
Cuts once
Triple zero



Cuts three times
Three zeros



Cuts once and touches
One zero, one double zero



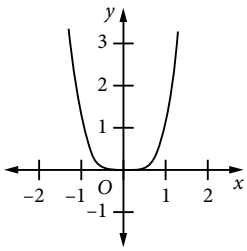
Cuts once
One zero

A cubic polynomial may have one, two or three real zeros.
A cubic polynomial always has at least one real zero.

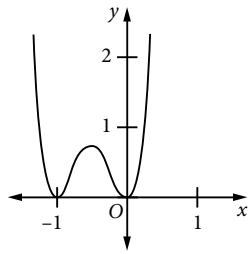
Degree 4 (quartic)

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

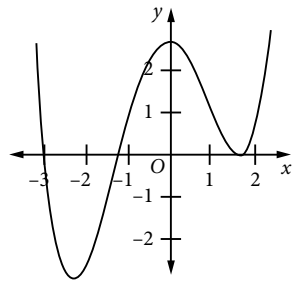
$a > 0$:



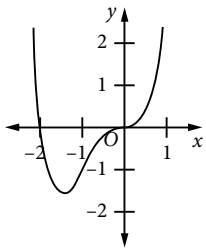
Touches
Quadruple zero



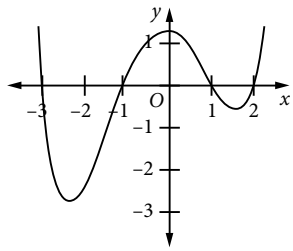
Touches twice
Two double zeros



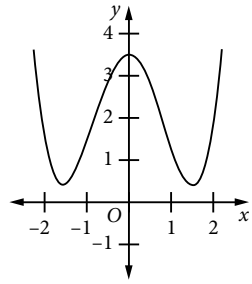
Touches once, cuts twice
Two zeros, one double zero



Cuts twice
One zero, one triple zero



Cuts four times
Four zeros



Does not cut
No real zeros

The addition of an appropriate constant to each equation can create a polynomial with no real zeros.

$a < 0$: A negative a inverts each of the six graphs above so that they open downwards, with the properties of their zeros the same.

Summary of quartic polynomials

1 A quartic polynomial may have four, three, two, one or no real zeros.

2 If a quartic polynomial has only one real zero, then it must be a quadruple zero.

3 If a quartic polynomial has only two distinct real zeros, then they are either a triple zero and a single zero or they are both double zeros.

4 If a quartic polynomial has only three distinct real zeros, then they are a double zero and two single zeros.

5 If a quartic polynomial has four distinct real zeros, then it can be factorised into four real linear factors.

6 If a quartic polynomial has no real zeros, then it cannot be factorised into any real linear factors.

Relationship between roots and coefficients

Quadratic equations

The general quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.
The roots of this equation are α and β, so that $(x - \alpha)(x - \beta) = 0$.

- Sum of roots: $\alpha + \beta = -\frac{b}{a}$
- Product of roots: $\alpha\beta = \frac{c}{a}$

Identities:

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

Cubic equations

The general cubic equation is $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$.
The roots of this equation are α, β and γ, so that $(x - \alpha)(x - \beta)(x - \gamma) = 0$.

- Sum of roots: $\alpha + \beta + \gamma = -\frac{b}{a}$
- Sum of products of pairs of roots: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
- Product of roots: $\alpha\beta\gamma = -\frac{d}{a}$

Identity:

- $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

Note: These relationships between roots and coefficients are not enough to find the roots of an equation without some additional information.

Quartic equations (4th degree)

The general quartic equation is
$ax^4 + bx^3 + cx^2 + dx + e = 0,\ a \neq 0.$

The roots of this equation are α, β, γ and δ, so that
$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0.$

- Sum of roots: $\alpha+\beta+\gamma+\delta = -\frac{b}{a}$
- Sum of products of pairs of roots:
  $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$
- Sum of products of triplets of roots:
  $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$
- Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$

3 GRAPHING FUNCTIONS

Reciprocal functions

Given the graph of $y = f(x)$, the graph of $y = \frac{1}{f(x)}$, the reciprocal function of $f(x)$.

It is important to find where $f(x) = 0$, as these x values will give the equations of the vertical asymptotes.

As $f(x) \to \pm\infty$, then $\frac{1}{f(x)} \to 0$.

Graphing polynomials by adding ordinates

Given the graphs for two polynomial functions $y = f(x)$ and $y = g(x)$, the graph of the new function, $y = f(x) + g(x)$ can be obtained by a process of adding the ordinates.

Graphing polynomials by multiplying ordinates

Given the graphs for two polynomial functions $y = f(x)$ and $y = g(x)$, the graph of the new function, $y = f(x)\,g(x)$ can be obtained by a process of multiplying the ordinates for a particular x-value.

Parametric representation

Parametric representation is when two related variables (e.g. x and y) are expressed in terms of a third variable (e.g. t or θ), so that, for example:

$x = f(t),\ y = g(t)$
$x = f(\theta),\ y = g(\theta)$

Equations like these are called **parametric equations** and the third variable (e.g. t or θ) is called the **parameter**.

- Recall that cosine and sine can be defined as x- and y-coordinates respectively of a point on the unit circle $x^2 + y^2 = 1$. The unit circle can be represented by the parametric equations $x = \cos\theta$, $y = \sin\theta$, where θ is the parameter.
- When the unit circle is described by the equation $x^2 + y^2 = 1$ (i.e. without the parameter), it is said to be in **Cartesian** form.

Parametric equation of the parabola

- The parabola $x^2 = 4ay$ can be represented by the parametric equations $x = 2at$, $y = at^2$.
- The point $(2at, at^2)$ on the parabola is frequently called 'the point t'.

4 FURTHER TRIGONOMETRIC IDENTITIES

Sum and difference of two angles

- $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
- $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
- $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
- $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
- $\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$
- $\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$

Double angle formulae

- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, $\tan\theta \neq \pm 1$

Half-angle formulae

- $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$
- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$, $\tan\frac{\theta}{2} \neq \pm 1$

t formulae $\left(\text{where } t = \tan\frac{A}{2}\right)$

- $\tan A = \frac{2t}{1-t^2}$
- $\sin A = \frac{2t}{1+t^2}$
- $\cos A = \frac{1-t^2}{1+t^2}$

Identities to simplify expressions

$\sin^2\theta + \cos^2\theta = 1 \quad \sin^2\theta = 1 - \cos^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta - \tan^2\theta = 1 \quad \sec^2\theta = 1 + \tan^2\theta$
$\tan^2\theta = \sec^2\theta - 1$

$\text{cosec}^2\theta - \cot^2\theta = 1 \quad \text{cosec}^2\theta = 1 + \cot^2\theta$
$\cot^2\theta = \text{cosec}^2\theta - 1$

Trigonometric products as sums or differences

$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$\cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\sin A + \sin B = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos B - \cos A = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

General solution of trigonometric equations

- For $\sin\theta = \sin\alpha$:
  $\theta = n\pi + (-1)^n\alpha$ (in radians)
  $\theta = n \times 180° + (-1)^n\alpha°$ (in degrees) for any integer n.
- For $\cos\theta = \cos\alpha$:
  $\theta = 2n\pi \pm \alpha$ (in radians)
  $\theta = n \times 360° \pm \theta°$ (in degrees) for any integer n.
- For $\tan\theta = \tan\alpha$:
  $\theta = n\pi + \alpha$ (in radians)
  $\theta = n \times 360° + \alpha°$ (in degrees) for any integer n.

5 INVERSE FUNCTIONS

- If any line drawn parallel to the y-axis intersects the graph of $f(x)$ no more than once, then there are no ordered pairs with the same y-coordinate. This is the **vertical line test**.
- If any line drawn parallel to the x-axis intersects the graph of $f(x)$ no more than once, then there are no ordered pairs with the same x-coordinate. This is the **horizontal line test**.
- If the graph of $f(x)$ satisfies both the vertical line test and the horizontal line test, then it is a one-to-one function. This means that for each value of x there is only one value of y, and for each value of y there is only one value of x.
- The necessary and sufficient condition for the reflection of the graph of $f(x)$ in the line $y = x$ to be the graph of a function is that $f(x)$ is one-to-one.

Increasing and decreasing functions

- A function f defined on the interval $a \le x \le b$ is said to be **strictly monotonic increasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. This means that the curve always slopes up to the right.
- A function f defined on the interval $a \le x \le b$ is said to be **strictly monotonic decreasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. This means that the curve always slopes down to the right.

Strictly monotonic increasing or decreasing functions are one-to-one.

Inverse functions

If $f(g(x)) = g(f(x)) = x$ then $g(x) = f^{-1}(x)$ is the inverse function of f.

- If d is the domain and r is the range of f, then r is the domain and d is the range of f^{-1}. Thus, if (a, b) is a point on f then (b, a) is a point on f^{-1}.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, when the inverse exists. This means that if $y = f(x)$ and $y = f^{-1}(x)$ intersect, then they do so on $y = x$.

Existence of inverse functions

A function f will have an inverse function f^{-1} if and only if f is a one-to-one function.

Exponential and logarithmic functions

$\log_a a^x = x$ for all real x (i.e. for all x in the domain of the exponential function), and $a^{\log_a x} = x$ for all $x > 0$ (i.e. for all x in the domain of the logarithmic function). Hence $y = \log_a x$ and $y = a^x$ are inverse functions.

The inverse sine function

By restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, a one-to-one function is formed whose inverse is that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
$y = \sin^{-1} x$ is written to mean 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$'.

Note that $y = \sin^{-1} x$ **does not mean** $y = (\sin x)^{-1}$, as that would be $y = \operatorname{cosec} x$.

Features of $\sin^{-1} x$:

- increasing function
- domain is $-1 \le x \le 1$ (you can only take $\sin^{-1}$ of values from -1 to 1 inclusive)
- range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
  ($\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive)
- vertical tangents at the endpoints
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function. Hence $\sin^{-1}(-x) = -\sin^{-1} x$.
- $y = \sin(\sin^{-1} x)$ exists only for values of x between -1 and 1 inclusive; $y = \sin(\sin^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \sin^{-1}(\sin x)$ exists for all real x, but it is equivalent to $y = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- its graph has a 'sawtooth' shape.

The inverse cosine function

By restricting the domain of $y = \cos x$ to $0 \le x \le \pi$, a one-to-one function is formed whose inverse is that part of $x = \cos y$ for which $0 \le y \le \pi$.

$y = \cos^{-1} x$ is written to mean 'that part of $x = \cos y$ for which $0 \le y \le \pi$'.

Features of $\cos^{-1} x$:

- decreasing function
- domain is $-1 \le x \le 1$ (you can only take $\cos^{-1}$ of values from -1 to 1 inclusive)
- range is $0 \le y \le \pi$
  ($\cos^{-1}$ gives values from 0 to π inclusive)
- vertical tangents at the endpoints
- gradient of -1 at $\left(0, \frac{\pi}{2}\right)$
- the function is neither even nor odd, but it does have rotational symmetry about its y-intercept.
- in particular: for any x in the domain, the sum of the function's heights at x and $-x$ will always be π:
  $\cos^{-1}(-x) + \cos^{-1} x = \pi$
  $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $y = \cos(\cos^{-1} x)$ exists only for values of x between -1 and 1 inclusive; $y = \cos(\cos^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \cos^{-1}(\cos x)$ exists for all real x, but it is equivalent to $y = x$ only for $0 \le x \le \pi$
  Its graph has a 'sawtooth' shape.

The inverse tangent function

By restricting the domain of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, a one-to-one function is formed whose inverse is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \tan^{-1} x$ is writtten to mean
'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

Features of $\tan^{-1} x$:

- increasing function
- domain is all real x
  (you can take $\tan^{-1}$ of any real value)
- range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$
  ($\tan^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ not inclusive)
- horizontal asymptotes at $y = \pm\frac{\pi}{2}$
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function. Hence $\tan^{-1}(-x) = -\tan^{-1} x$.
- $y = \tan(\tan^{-1} x)$ is equivalent to $y = x$ for all real x.
- $y = \tan^{-1}(\tan x)$ exists for all real x except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
- it is equivalent to $y = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- its graph is like an infinite set of parallel intervals 'with open circles on each end' (i.e. missing the values that are multiples of $\pi \pm \frac{\pi}{2}$).

6 PERMUTATIONS AND COMBINATIONS

Fundamental counting principle

If an outcome can happen in m different ways, and a second outcome can happen in n different ways, then the total number of ways in which the two outcomes can happen together is $m \times n$.

Pigeonhole principle

If $(n + 1)$ items occupy n containers, then at least one of the containers must contain at least two items.

If n items are sitting in k containers, where $n > k$, then there is at least one container with at least $\frac{n}{k}$ items in it.

Permutations

A **permutation** is an ordered selection or arrangement of all or part of a set of objects.

Number of permutations ${}^nP_r$

The symbol ${}^nP_r$ means the number of arrangements (i.e. ordered subsets, permutations) of any n different objects taken r at a time.

This denotes the number of ways of filling r places with the n different objects that are available ($r \le n$).

- ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$

Arrangements in a circle

- The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, is $\frac{n!}{n} = (n-1)!$.
- In a circular arrangement there is no start or finish, so to count the arrangements we need to fix one object as our 'starting' object.

Mutually exclusive operations—the addition principle

Mutually exclusive arrangements cannot both be true at the same time.

For example, 'A is first and B is fifth' excludes the possibility that 'A is first and B is sixth'—these two descriptions cannot both be true at the same time.

Arrangement of *n* objects when some are identical

The number of ways of arranging n objects in a row when p of the objects are identical, and q of the objects are identical (but different to the others), and so on, is $\frac{n!}{p!q!\ldots}$.

Combinations

A **combination** (or 'selection') is an unordered permutation of all or part of a set of objects. The order is not important.

Number of combinations ${}^nC_r$ or $\binom{n}{r}$

The symbol ${}^nC_r$ or $\binom{n}{r}$ denotes the number of combinations of n different objects taken r at a time.

An unordered sample can be ordered in $r!$ ways:

- $${}^nC_r = \binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$
- An ordered sample of r objects taken from n different objects can be chosen in ${}^nP_r$ ways.
- An unordered sample of r objects taken from n different objects can be chosen in ${}^nC_r$ ways.

Important result:

- $${}^nC_{n-r} = \frac{n!}{(n-r)!r!}$$
  so ${}^nC_r = {}^nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

For each set of r objects selected, there is left behind a set containing $(n-r)$ objects. Thus there must be the same number of sets containing $(n-r)$ objects as there are containing r objects.

Combinations of any number of objects

The symbol ${}^nC_r$ also represents the number of r-subsets in a given set, where $0 \le r \le n$.

If no specific value of r is stated, r can assume any value between 0 and n inclusive. Hence the number of subsets that each contain at least one element, i.e. the number of non-empty subsets in an n-set, is:

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r$$

From a set of n objects, each object can be dealt with in two ways: it can be included or it can be excluded. The number of ways of dealing with the n objects is thus $2 \times 2 \times 2 \times \ldots$ to n factors, or 2^n. But this includes the case when all n objects are excluded, so the number of combinations of at least one object is $2^n - 1$.

$$\therefore {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r = 2^n - 1$$

Expansion of $(1+x)^n$, Pascal's triangle

The coefficients of the successive powers of the $(1+x)^n$ expansion can be arranged in a triangular pattern called **Pascal's triangle**, as shown on the right.

$(1+x)^0$						1						
$(1+x)^1$					1		1					
$(1+x)^2$				1		2		1				
$(1+x)^3$			1		3		3		1			
$(1+x)^4$		1		4		6		4		1		
$(1+x)^5$	1		5		10		10		5		1	
$(1+x)^6$	1		6		15		20		15		6	1

Properties of Pascal's triangle

1. The first and last number in each row is 1.
2. Every number not at the end of a row is the sum of the two numbers in the row above it to the left and right (e.g. $10 = 4 + 6$, $15 = 5 + 10$, $20 = 10 + 10$).
3. The number of terms in each row is one more than the value of n.
4. The second and second-last terms in each row are the value of n.

Binomial expansion

(a) $$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_rx^r$$

(b) $$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

(c) ${}^nC_0 = 1$, ${}^nC_n = 1$, ${}^nC_r = {}^nC_{n-r}$

(d) ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 \le r \le n-1$

(e) The general term in the expansion of $(a+b)^n$ is $T_{r+1} = {}^nC_ra^{n-r}b^r$

(f) The ratio of successive terms of the binomial expansion of $(a+b)^n$ is $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{b}{a}$

Alternate notation:

- $${}^nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$
- $$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$$

Useful results:

(a) $x = 1$:

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} \times 1 + \binom{n}{2} \times 1^2 + \ldots + \binom{n}{r} \times 1^r + \ldots + \binom{n}{n} \times 1^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$$

$$2^n - 1 = \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n} = \sum_{r=1}^{n} \binom{n}{r} \quad \text{as } \binom{n}{0} = 1$$

(b) $x = -1$:

$$(1-1)^n = \binom{n}{0} + \binom{n}{1} \times (-1) + \binom{n}{2} \times (-1)^2 + \binom{n}{3} \times (-1)^3 + \ldots + \binom{n}{r} \times (-1)^r + \ldots + \binom{n}{n} \times (-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + \binom{n}{r} \times (-1)^r + \ldots + \binom{n}{n} \times (-1)^n$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = \frac{1}{2} \times 2^n = 2^{n-1}$$

In the expansion of $(1 + x)^n$: Sum of coefficients of even terms = Sum of coefficients of odd terms = 2^{n-1}

Probability

- $P(A) = \dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$, $0 \le P(A) \le 1$
- $P(\overline{A}) = 1 - P(A)$
- Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$
- Not mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

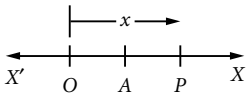
7 RATES OF CHANGE AND THEIR APPLICATION

Motion of a particle in a straight line

A **particle** is a body that behaves such that all forces acting on it can be regarded as acting through a single point. This means we can represent the body as a single point, regardless of its actual size and shape. Quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement

- For a particle represented by a point P, moving in a straight line $X'OX$:



The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O, so it does not necessarily represent the total **distance** travelled. Unlike displacement, distance is always a positive quantity.

- Note that the origin of the motion (when $t = 0$) is not necessarily at O.

Velocity

- **Velocity** is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction:

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

- Velocity can be positive or negative, depending on the direction of travel. Speed is the magnitude of the velocity and is always positive.

Acceleration

- **Acceleration** is defined as the rate of change of velocity with respect to time:

$$a(t) = v'(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

- Acceleration can be positive or negative. Positive acceleration means the velocity is increasing (i.e. in the direction of positive displacement), while negative acceleration means the velocity is decreasing, which is often called deceleration or retardation.

Important motion terms

- 'initially': $t = 0$
- 'at the origin': $x = 0$
- 'at rest': $v = 0$
- 'velocity is constant': $a = 0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

- Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
- Constant acceleration due to gravity $= 9.8\ \text{m s}^{-2}$ $(\approx 10\ \text{m s}^{-2})$

Displacement, velocity, acceleration—important links

- Given the displacement function, you can find velocity and acceleration functions by differentiating with respect to time.
- Given the velocity function, you can find the displacement function by integrating with respect to time; you can find the acceleration function by differentiating with respect to time.
- If motion information is given as a graph, remember that the definite integral is linked to the area under the graph, while the derivative is linked to the gradient of the curve.
- For example: given the graph of velocity $\frac{dx}{dt}$ against time, then the value of the definite integral gives the displacement and the slope of the curve gives the acceleration.

Related rates

Problems with **related rates** arise when there is a function that relates two variables, e.g. x and y, where both x and y are also functions of another variable, e.g. time t. For example, we may need to determine $\frac{dy}{dt}$ when $\frac{dx}{dt}$ is known. In such cases it is necessary to use the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Exponential growth and decay

$\frac{dy}{dx} = ky$ means that the rate of change of a quantity y with respect to another quantity x is proportional to y. The solution of this differential equation is the exponential rate of change:

- $y = Ae^{kx}$ for exponential growth
- $y = Ae^{-kx}$ for exponential decline (or 'exponential decay').

For $y = Ae^{kx}$, we have $\frac{dy}{dx} = kAe^{kx} = ky$:

- A is the value of y when $x = 0$ (i.e. the initial value)
- k is the **growth rate**.

The equation $\frac{dN}{dt} = k(N - P)$, where k, P are constants

- $N = Ae^{kt}$ is the solution of the equation $\frac{dN}{dt} = kN$, where A is the initial value of N.
- $N = P + Ae^{kt}$ is a solution of the equation $\frac{dN}{dt} = k(N - P)$, where A is a constant.
- If $k < 0$, $Ae^{kt} \to 0$ as $t \to \infty$ and hence $N \to P$ from above.

Newton's law of cooling

The cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium:

$$\frac{dT}{dt} = -k(T - M)$$

where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

Wilhelmy's law

Many chemical reactions follow the law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time:

$$\frac{dx}{dt} = k(a - x),\ 0 \le x \le a$$

where a is the initial concentration and x is the amount transformed at time t.

8 TRIGONOMETRIC EQUATIONS

t formulae $\left(\text{where } t = \tan\frac{A}{2}\right)$

- $\tan A = \dfrac{2t}{1-t^2}$
- $\sin A = \dfrac{2t}{1+t^2}$
- $\cos A = \dfrac{1-t^2}{1+t^2}$

Auxiliary angle method

In each case $r = \sqrt{a^2+b^2}$ and α is in the first quadrant such that $\tan\alpha = \dfrac{b}{a}$:

- $a\sin x + b\cos x \equiv r\sin(x+\alpha)$
- $a\sin x - b\cos x \equiv r\sin(x-\alpha)$
- $a\cos x + b\sin x \equiv r\cos(x-\alpha)$
- $a\cos x - b\sin x \equiv r\cos(x+\alpha)$

Note:

(a) Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y = a\sin x + b\cos x$ a lot easier.

(b) Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ allows you to solve equations of the type $a\sin x \pm b\cos x = c$.

(c) If you use t formulae substitution to solve equations of the type $a\cos x + b\sin x = c$, you must also test to see whether $x = \pm n\pi$ is a solution of the equation.

9 PROOF BY MATHEMATICAL INDUCTION

Note: It only takes one counter-example to disprove a general statement.

Mathematical induction is a method of proof that can be used, for example, to prove that something is true for all positive integers n, or for all positive integers n greater than a specified starting value, or for all positive odd integers n.

A proof by induction consists of a two-step process followed by a conclusion:

1 Prove that the statement is true in the first case possible.

2 Prove that if the statement is true in any one case (i.e. for $n = k$), then it must also be true in the next case (i.e. for $n = k+1$, or for $n = k+2$ if the statement is true for even integers only, etc.).

3 Write a conclusion:

The statement is true for $n = k+1$ if it is true for $n = k$. (As proved in Step 2.)

The statement is true for $n = 1$ (i.e. for the first possible case). (As proved in Step 1.)

Thus the statement is true for all integers $n \geq 1$.

When induction doesn't work

One counter-example is all that it takes to disprove a result.

Some proofs by mathematical induction seem to be correct, but turn out to be incorrect. The most likely reason is that you have failed to prove Step 1, that is, the result is true at the beginning (usually for $n = 1$). This means that the assumed result ($n = k$) is false, so when you prove the result true for $n = k+1$ you are proving it from an incorrect assumption.

10 VECTORS IN TWO DIMENSIONS

Vectors and vector notation

The magnitude of vector $\overrightarrow{AB}$ is written $\left|\overrightarrow{AB}\right|$, the magnitude of vector $\underset{\sim}{a}$ is written $\left|\underset{\sim}{a}\right|$, and the magnitude of vector $\mathbf{a}$ is written $|\mathbf{a}|$.

Two vectors are equal if, and only if, they have the same magnitude and the same direction, regardless of their positions.

The negative of vector $\overrightarrow{AB} = \underset{\sim}{a}$ is the vector $-\overrightarrow{AB} = \overrightarrow{BA} = -\underset{\sim}{a}$.

The zero vector, denoted $\underset{\sim}{0}$, is a vector of zero magnitude. Its direction cannot be defined. Adding a vector to its negative will produce the zero vector. $\underset{\sim}{a} + (-\underset{\sim}{a}) = \underset{\sim}{0}$

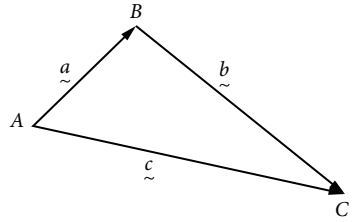
The unit vector in the direction of $\underset{\sim}{a}$ is denoted $\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|}$ and $\left|\hat{\underset{\sim}{a}}\right| = 1$.

Scalar multiplication of vectors

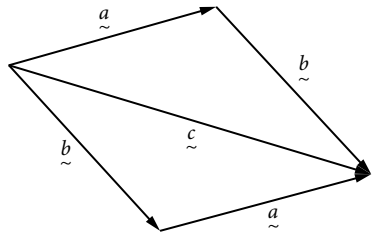
- If $k > 0$, then $k\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$ but has k times the magnitude.
- If $k = 0$, then $k\underset{\sim}{a} = \underset{\sim}{0}$.
- If $k < 0$, then $k\underset{\sim}{a}$ is in the opposite direction to $\underset{\sim}{a}$ and has k times the magnitude.

Addition of vectors

For ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$.



The parallelogram rule for the addition of vectors

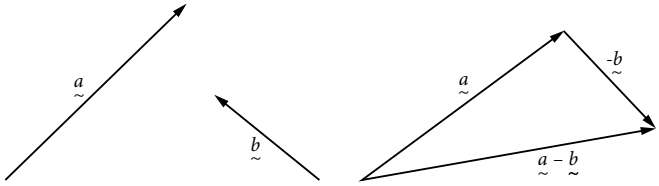


The diagram shows that $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a} = \underset{\sim}{c}$ and this rule is known as the parallelogram rule for addition of vectors.

The parallelogram rule for addition of vectors illustrates that vector addition is commutative.

Subtraction of vectors

Subtraction of a vector is defined as addition of its negative. That is, to subtract $\underset{\sim}{b}$ from $\underset{\sim}{a}$, add $-\underset{\sim}{b}$ to $\underset{\sim}{a}$, and so $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$.



Position vectors

A position vector can be represented by a coordinate pair (a, b). This represents the vector that is a units from O in the positive x direction and b units from O in the positive y direction. This can be represented as the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ or by the coordinates (a, b).

Any vector that is equivalent to a translation of a units in the positive x-direction and b units in the positive y-direction can be represented in this way.

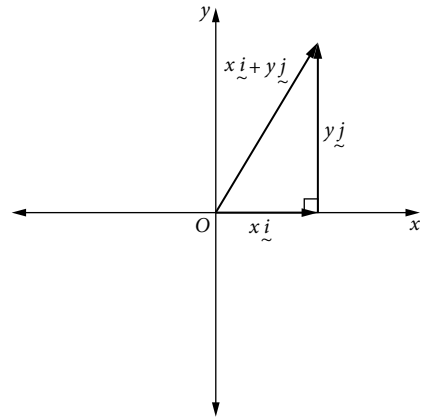
The magnitude of a vector

The magnitude of a position vector $\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix}$ can be calculated using Pythagoras' theorem.

$\left|\overrightarrow{OA}\right| = \sqrt{a^2 + b^2}$.

Vectors in component form

The form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is called **component form or $\underset{\sim}{i}, \underset{\sim}{j}$ form** of a vector. The vector $\underset{\sim}{a}$ may also be represented in column vector form as $\begin{pmatrix} x \\ y \end{pmatrix}$.



Addition and subtraction of vectors in component form

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$\underset{\sim}{a} + \underset{\sim}{b} = (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j}$

$\underset{\sim}{a} - \underset{\sim}{b} = (x_1 - x_2)\underset{\sim}{i} + (y_1 - y_2)\underset{\sim}{j}$

Scalar multiplication of vectors in component form

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $k\underset{\sim}{a} = kx\underset{\sim}{i} + ky\underset{\sim}{j}$

Equality of vectors in component form

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, then $\underset{\sim}{a} = \underset{\sim}{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Parallel vectors

Two vectors are parallel if they are scalar multiples of each other.

If $\underset{\sim}{b} = k\underset{\sim}{a}$, where $k \in R$, then $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

Unit vectors in component form

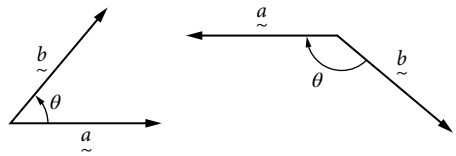
If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{x^2 + y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$ or

$\hat{\underset{\sim}{a}} = \dfrac{\sqrt{x^2 + y^2}}{x^2 + y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

Scalar product of vectors

The **scalar product** (also called the **dot product** or the **inner product**) is a way of multiplying two vectors. The result of the multiplication is a scalar quantity (magnitude but no direction). The scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is written as $\underset{\sim}{a} \bullet \underset{\sim}{b}$ (and read as '$\underset{\sim}{a}$ dot $\underset{\sim}{b}$').

If θ is the angle between the positive directions of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then the scalar product is defined to be $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta$.



Note that we always use the acute or obtuse angle, never the reflex angle.

For a straight angle, $\theta = \pi$, so $\underset{\sim}{a} \bullet \underset{\sim}{b} = -\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$, since $\cos\pi = -1$.

If the scalar product of two non-zero vectors is zero, then the vectors are perpendicular, i.e. if, for two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ the inner product formula for the scalar product of two vectors expressed in component form is $\underset{\sim}{a} \bullet \underset{\sim}{b} = x_1x_2 + y_1y_2$.

This gives use two expressions for the scalar product, $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta = x_1x_2 + y_1y_2$.

Projections of vectors

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$.

In general, the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ does not equal the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

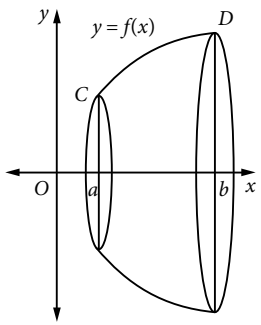
The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\underset{\sim}{b} \bullet \underset{\sim}{b}}\underset{\sim}{b}$.

11 APPLICATIONS OF CALCULUS

Volume of solids of revolution

- When the arc CD of the curve $y = f(x)$ on the interval $a \le x \le b$ is rotated about the x-axis, the volume of the solid of revolution formed is given by:

$$V = \pi\int_a^b \left(f(x)\right)^2 dx = \pi\int_a^b y^2\,dx$$



- When the arc CD of the curve $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, the volume of the solid of revolution formed is given by:

$$V = \pi\int_c^d \left(g(y)\right)^2 dy = \pi\int_c^d x^2\,dy$$

Indefinite integrals and substitution

- $\int u^n\,du = \dfrac{1}{n+1}u^{n+1} + C$
- Let $y = \int f(u)\,du$, where $u = g(x)$, so that $\dfrac{dy}{du} = f(u)$

  But $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = f(u) \times \dfrac{du}{dx}$

  so $y = \int f(u) \times \dfrac{du}{dx}\,dx = \int f(u)\,du$

  This 'backwards' form of the chain rule is convenient when the substitution of $u = g(x)$ will let you express a function being integrated as the product of $\dfrac{du}{dx}$ and a function of u.

Definite integrals and substitution

When using a substitution to evaluate a definite integral be careful with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Useful results:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$: if you reverse the limits of integration, you change the sign of the integral.
- For an integral of the form $\int f'(x)\left(f(x)\right)^n dx$,

  the substitution $u = f(x)$ gives $\dfrac{du}{dx} = f'(x)$

  so that the integral becomes:

  $\int f'(x)\left(f(x)\right)^n dx = \int u^n\,du = \dfrac{1}{n+1}u^{n+1} + C$

  $= \dfrac{1}{n+1}\left(f(x)\right)^{n+1} + C$
- If you can recognise that $\int f'(x)\left(f(x)\right)^n dx = \dfrac{1}{n+1}\left(f(x)\right)^{n+1} + C$, you can obtain your answer faster.

Integration of $\sin^2 x$ and $\cos^2 x$

$\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ can be found using substitutions from the formulae for $\cos 2x$. This will generally not be given in a question, but you must be able to use it when necessary.

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$,

  $\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

- $\sin^2 x = \dfrac{1-\cos 2x}{2}$,

  $\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Integrals of the type $\int f'(x)(f(x))^n\, dx$

In trigonometric integrals of this type such as $\int \cos x \sin^2 x\, dx$, i.e. where $f(x) = \sin x$ and $f'(x) = \cos x$, the substitution will not always be given.

- $\int \cos x \sin^2 x\, dx = \frac{1}{3}\sin^3 x + C$

Substitution simplified

If you treat $\frac{du}{dx}$ as a fraction, then $\frac{du}{dx} \times dx = du$.

If $u = \sin x$, then $\frac{du}{dx} = \cos x$ can be written as $du = \cos x\, dx$.

You are really replacing $f'(x)\, dx$ by du.

Integrals involving trigonometric substitution

If an expression of the form $a^2 - x^2$ occurs in the integrand, the standard trigonometric substitutions are $x = a\sin\theta$ or $x = a\cos\theta$.

If an expression of the form $a^2 + x^2$ occurs in the integrand, the standard substitution is $x = a\tan\theta$.

Note: If these substitutions are needed then they should be given in the question.

Differentiation of inverse trigonometric functions

Note: $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

Derivative of $\sin^{-1} x$

Let $y = \sin^{-1} x$ for $-1 \le x \le 1$

$\therefore x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\frac{dx}{dy} = \cos y \quad \therefore \frac{dy}{dx} = \frac{1}{\cos y}$ noting that $\cos y \ne 0$

$\therefore -\frac{\pi}{2} < y < \frac{\pi}{2}$

Using $\cos^2 y + \sin^2 y = 1$: $\frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$

As y is an angle in the first or fourth quadrants, $\cos y$ must be positive.

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

Note that the derivative is not defined at $x = \pm 1$ (the graph of $\sin^{-1} x$ has vertical tangents at its endpoints). Also note that the derivative is positive for all x in its domain (as $\sin^{-1} x$ is an increasing function).

Derivative of $\sin^{-1}\frac{x}{a}$

Let $y = \sin^{-1}\frac{x}{a}$ for $-a \le x \le a$

$\therefore x = a\sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\frac{dx}{dy} = a\cos y \quad \therefore \frac{dy}{dx} = \frac{1}{a\cos y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

As for the derivative of $\sin^{-1} x$:

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - a^2\sin^2 y}}$ for $-a < x < a$

$\therefore \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$

Derivative of $\cos^{-1} x$

Let $y = \cos^{-1} x$ for $-1 \le x \le 1$

$\therefore x = \cos y$ where $0 \le y \le \pi$

$\frac{dx}{dy} = -\sin y \quad \therefore \frac{dy}{dx} = -\frac{1}{\sin y}$ for $0 < y < \pi$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$ for $-1 < x < 1$

$\therefore \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

Derivative of $\cos^{-1}\frac{x}{a}$

Let $\cos^{-1}\frac{x}{a}$ for $-a \le x \le a \quad \therefore x = a\cos y, \quad 0 \le y \le \pi$

$\frac{dx}{dy} = -a\sin y \quad \therefore \frac{dy}{dx} = -\frac{1}{a\sin y}$ for $0 < y < \pi$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - a^2\cos^2 y}}$ for $-a < x < a$

$\therefore \frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$

Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x$ for all $x \quad \therefore x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y$

$\therefore \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$ for all x

$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ for all x

Derivative of $\tan^{-1}\frac{x}{a}$

Let $y = \tan^{-1}\frac{x}{a}$ for all x

$\therefore x = a\tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\frac{dx}{dy} = a\sec^2 y = a(1+\tan^2 y)$

$\therefore \frac{dy}{dx} = \frac{a}{a^2 + a^2\tan^2 y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\therefore \frac{dy}{dx} = \frac{a}{a^2 + x^2} \quad \text{for all } x$$

$$\therefore \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{a}{a^2 + x^2} \quad \text{for all } x$$

Integration involving inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + C, \quad -a < x < a$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\frac{x}{a} + C, \quad -a < x < a$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad \text{for all } x$$

12 DIFFERENTIAL EQUATIONS

A **differential equation** is an equation that relates some unknown differentiable function to one or more of its derivatives.

A differentiable function $y = g(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced with $g(x)$ and its corresponding derivatives.

To verify that a function $y = g(x)$ is a solution of a differential equation, substitute the function and its derivative(s) into both sides of the differential equation and check that both sides are identically equal.

A **particular solution** (or **solution curve**) of a differential equation is a unique function that is found by giving specific values to the parameters in the general solution. The parameters in the solution are chosen so that the particular solution satisfies one or more extra requirements called **initial conditions**. Initial conditions are sometimes called boundary values.

An initial value problem of a first-order differential equation: $\begin{cases} y' = f(t, y(t)) \\ y(a) = y_a \\ t \in [a, b] \end{cases}$

consists of the differential equation together with its initial condition, requiring you to determine a particular solution $y(t)$ over a specific interval of the independent variable $a \leq t \leq b$.

The order of a differential equation is equal to the highest order derivative of the dependent variable and the degree of the equation, is the highest power of that highest order derivative. In the n^{th} order differential equation $\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right)$, the dependent variable (y) always appears in the numerator of any derivativesand the independent variable x appears in the denominator.

Direction fields

Qualitative methods are a set of graphical methods to describe the general behaviour of the solution to a differential equation without solving the equation.

Solving differential equations of the form $\frac{dy}{dx} = f(x)$

Finding the particular solution of a directly integrable problem $\frac{dy}{dx} = f(x)$, $y(a) = y_a$ involves the following two-step procedure:

1 Integration with respect to the independent variable. That is, $y = F(x) + c$, where $F(x) = \int f(x)\,dx$ and c is the constant of integration.

2 Use the initial condition to solve for the constant of integration.

That is, $(x, y) = (a, y_a) \Rightarrow y_a = F(a) + c$.

Therefore, $y = F(x) + y_a - F(a)$.

Upon rearranging the previous equation, the final result is $y(x) = y_a + F(x) - F(a)$.

The initial condition may also be referred to as the 'boundary value'.

Solving differential equations of the form $\frac{dy}{dx} = g(y)$

To find the particular solution of $\frac{dy}{dx} = g(y)$, $y(a) = y_a$, $g(y_a) \neq 0$, requires the following five-step procedure:

1 Take the reciprocal of both sides of $\frac{dy}{dx} = g(y)$ to obtain $\frac{dx}{dy} = \frac{1}{g(y)}$ with $x = a$ where $y = y_a$.

2 Integrate both sides of the equation with respect to the new independent variable y:

$$\int \frac{dx}{dy} dy = \int \frac{1}{g(y)} dy$$

$$x + c = G(y)$$

where $G'(y) = \frac{1}{g(y)}$ and c is a constant of integration.

3 Where possible, invert the equation from previous step to find the general solution for the original dependent variable y in terms of the original independent variable x:

$$x + c = G(y)$$

$$\therefore y = G^{-1}(x + c)$$

4 Substitute the initial condition $(x, y) = (a, y_a)$ into the equation obtained in the previous step to evaluate the constant of integration c:

$$\therefore y_a = G^{-1}(a + c).$$

5 Substitute the value of the constant into the general solution obtained in the previous step to obtain the particular solution.

The solution of the uninhibited growth-decay model $\frac{dy}{dt} = ry, y(0) = y_0$ is $y = y_0 e^{rx}$.

Assuming $r > 0$, the differential equation $\frac{dy}{dt} = -r(y - y_s), y(0) = y_0$ is a model of inhibited growth if $y_0 < y_s$ and inhibited decay if $y_0 > y_s$. The rate of growth (or decay) in the dependent variable is inhibited as the dependent variable approaches its equilibrium value y_s. The solution curve of an inhibited growth (or decay) problem is $y(t) = y_s + (y_0 - y_s)e^{-rt}$.

The solution of the quadratic growth rate model $\frac{dy}{dx} = (1 - y)(1 + y), y(0) = y_0$ is

$$y = \frac{1 - \left(\frac{1 - y_0}{1 + y_0}\right)e^{-2x}}{1 + \left(\frac{1 - y_0}{1 + y_0}\right)e^{-2x}}.$$

Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables

To solve the first-order differential equation $\frac{dy}{dx} = f(y)g(x)$ by the method of separation of variables:

1 Solve $\{y^* : f(y^*) = 0\}$ for any steady state (or equilibrium) solutions $y = y^*$

2 Separate the dependent and independent variables onto either side of the equality $\frac{1}{f(y)}\frac{dy}{dx} = g(x)$

3 Integrate both sides of the equality with respect to the independent variable:

- For the integral involving y terms, apply the change of variable $\int \frac{1}{f(y)}\frac{dy}{dx}\,dx = \int \frac{1}{f(y)}\,dy$
- Add the constant of integration C to the side with the independent variable. This is the general solution.

4 Whenever possible, represent the dependent variable as an explicit function of the independent variable. However, you may need to be satisfied with an equation that determines the dependent variable as an implicit function of the independent variable.

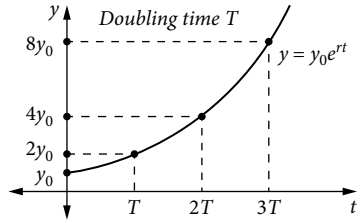
5 If an initial condition $y(a) = b$ is given, solve for the constant of integration C. This is the particular solution.

To avoid missing any possible singular solutions when finding the general solution of a differential equation of the form $\frac{dy}{dx} = f(y)g(x)$, always remember to investigate the roots $\{y^* : f(y^*) = 0\}$ and/or the slope field.

Modelling with first-order differential equations

A dependent variable $y(t)$ is said to have an *exponential growth rate* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = k > 0$ and an *exponential rate of decay* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = -k < 0$ for some positive constant k.

For a system $y = y_0 e^{rt}$, $r > 0$ experiencing exponential growth, the doubling-time (T) is the time it takes the system to double in size. The formula for the doubling-time is $T = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2$



For a system $y = y_0 e^{-rt}$, $r > 0$ experiencing exponential decay, the half-life $t_{\frac{1}{2}}$ is the time it takes the value of y to halve. The formula for the half-life is:

$t_{\frac{1}{2}} = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2.$

First-order decay reactions

In a simple first-order decay reaction, the rate of decrease $-\frac{d[A]}{dt}$ in the concentration of a reactant A, varies directly with the concentration $[A]$ of that reactant.

That is, the concentration $[A]$ of the reactant A is modelled by the differential equation $\frac{d[A]}{dt} = -r[A]$, where the relative decay rate r is usually called the reaction rate.

Newton's law of cooling

Newton's law of cooling states that the time rate of decrease in the temperature T of an object varies directly with the difference between the object's temperature and the ambient temperature (the temperature of the surrounding medium T_a).

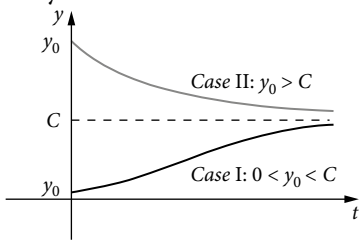
Therefore: $\frac{dT}{dt} = -r(T - T_a), \quad r > 0, \quad T(0) = T_0.$

If $T_0 > T_a$, the body's initial temperature is higher than the temperature of its surroundings, so the body is cooling. The model predicts that the body will cool quickly at first, reducing the excess of its temperature to that of the surrounding environment. This cooling reduces the excess temperature of the body, thus continually slowing its rate of cooling.

However, if $T_0 < T_a$, then the body's initial temperature is lower than the temperature of its surroundings, so the body is warming. Again, this change in temperature reduces the temperature difference with the surrounding medium, thus slowing the rate of warming.

Models of modified growth and decay

The differential equation $\frac{dy}{dt} = ry\left(1 - \frac{y}{C}\right), y(0) = y_0$ where $r > 0$ is called a **logistic equation**. This models population growth in an environment with a finite carrying-capacity $C > 0$.



Modelling the time dependence of a conserved substance

If a substance m is conserved, (that is, it is neither created, nor destroyed by chemical reaction), then the rate of change of m within a confined region (called a control volume) is equal to the difference between the rate at which it flows into the region (R_{in}), and the rate at which it flows out of the region (R_{out}). That is, $\frac{dm}{dt} = R_{in} - R_{out}$.

13 MOTION, FORCES AND PROJECTILES

Displacement is the change in position of an object relative to its starting point. It is a vector quantity, so needs to be described by both magnitude and direction.

Distance is how far an object has travelled in total. It is a scalar quantity, so has magnitude only.

Velocity is the rate at which the displacement of an object changes with respect to time. It is a vector quantity, so needs 'to be described by both magnitude and direction'.

Speed is the magnitude of the object's velocity. It is a scalar quantity, so has magnitude only.

Problems involving forces

The resultant or net force, $\underset{\sim}{F}$, is the vector sum of all the forces acting on an object.

If forces $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3, \ldots$ act on a particle, then $\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 + \ldots$

Newton's First Law of Motion states that an object at rest stays at rest and an object in motion stays in motion with the same velocity unless acted upon by an unbalanced force.

Projectile motion

Projectile motion is an example of two-dimensional motion in the vertical plane under gravity. The effect of air resistance is ignored. It is a vector combination of horizontal constant motion with vertical motion under gravity.

The assumptions made in this model are:

- the projectile is a point
- the force due to air resistance is negligible
- the only force acting on the projectile is the constant force due to gravity, assuming that the projectile is moving close to the Earth's surface
- the acceleration due to gravity is taken to be a constant, $g = 9.8\,\text{m s}^{-2}$, unless given otherwise.

In this vector approach you will use $\underset{\sim}{a}(t) = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$ as the acceleration vector, $\underset{\sim}{v}(t) = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ as the velocity vector and $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$ as the position vector. Since the only acceleration is due to gravity the acceleration vector becomes $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$ as there is no horizontal acceleration.

For a given speed of projection, $\left|\underset{\sim}{v}(0)\right|$, the equation $R = \frac{\left|\underset{\sim}{v}(0)\right|^2 \sin 2\alpha}{g}$ defines the range as a function of the angle of projection, α. This function has its greatest value when $\sin 2\alpha = 1$ or $\alpha = 45°$.

Thus, the greatest range of the particle occurs when $\alpha = 45°$ and is given by $R_{max} = \frac{\left|\underset{\sim}{v}(0)\right|^2}{g}$.

14 THE BINOMIAL DISTRIBUTION

Bernoulli trials

Bernoulli trials refer to situations where there are exactly two possible outcomes. This means there are only two possible outcomes: success and failure. The trials are independent, the outcome of one trial has no influence over the outcome of the next trial. The number of trials is fixed.

For a Bernoulli random variable X:

$E(X) = p$

$\text{Var}(X) = p(1-p)$

Binomial distribution

$X \sim B(n, p)$ indicates a random variable, X, which has a binomial distribution with n identical trials and a probability of success of p.

n and p are called the *parameters* of the distribution.

$X \sim B(n, p)$ is read: 'X is distributed as a binomial variable with parameters n and p'.

Just because an experiment has only two possible outcomes does not mean it is automatically binomial. If you were drawing the cards above without replacement, then X would not be a binomial variable because the probability of success, p, would change with each draw.

Binomial theorem

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Recalling that the probability of failure is sometimes written as q, where $p + q = 1$, this can also be written as:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Note: $P(X \geq a) = 1 - P(X < a)$.

This can be a useful result in many situations.

For graphs of the binomial distribution, where $X \sim B(n, p)$:

If $p < 0.5$ the graph is skewed to the right or positively skewed.

If $p = 0.5$ the graph is symmetric about the mean.

If $p > 0.5$ the graph is skewed to the left or negatively skewed.

As n increases, the graph clusters more tightly about the mode but retains the same shape as other distributions with the same value of p.

Mean and variance of the binomial distribution

When dealing with a binomial distribution, if X is the random variable representing the number of successes in n trials and p is the (constant) probability of success, then:

$E(X) = np = \mu = \text{mean}$

$\text{Var}(X) = np(1-p)$

$\sigma(X) = \sqrt{np(1-p)}$

Normal approximation for the sample proportion

If $X_1, X_2, X_3, \ldots, X_n$ are independent identically distributed random variables, each with mean μ and variance σ^2, then as $n \to \infty$,

$D = X_1 + X_2 + X_3 + \ldots + X_n \sim N(n\mu, n\sigma^2)$.

The central limit theorem is often expressed in another way. Instead of dealing with the sum of the samples, you can instead deal with the mean of the samples, $\bar{X}$.

In this case, you have $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$.

The sample proportion $\hat{p} = \dfrac{x}{n}$ is not the population proportion; rather it is the best estimate of the population proportion from a single sample. This leads to the sample proportion sometimes being referred to as the *point estimate*.

The CLT and proportions

The CLT applies equally to the proportion of a specific random variable in a sampling distribution, such as the number of threes obtained in samples of size of 50, as it does to the sampling distribution of means.

Sampling distribution for sample proportions

The standard deviation of a binomial distribution can be estimated from the sample proportion $\hat{p}$ of *one sample*, rather than many. This cannot be done easily for other distributions.

The distribution of $\hat{P}$ for the sample proportions is defined as $\hat{P} \sim N\left(\hat{p}, \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$,

where the expected value $E(\hat{P}) \approx \hat{p}$ and the standard deviation $SD \approx \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = S(\hat{P})$.

The following two conditions must be met for the CLT to apply:

$$n\hat{p} \geq 5$$
$$n(1-\hat{p}) \geq 5$$

You would also need to approximate the population proportion p with the sample proportion $\hat{p}$.

Mathematics Extension 1 Course Outcomes

Objective **Students:** • develop efficient strategies to solve problems using pattern recognition, generalisation, proof and modelling techniques	
Year 11 Mathematics Extension 1 outcomes A student:	**Year 12 Mathematics Extension 1 outcomes** A student:
ME11–1 uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses	**ME12–1** applies techniques involving proof or calculus to model and solve problems

Objective **Students:** • develop the ability to use concepts and skills and apply complex techniques to the solution of problems and modelling in the areas of trigonometry, functions, calculus, proof, vectors and statistical analysis	
Year 11 Mathematics Extension 1 outcomes A student:	**Year 12 Mathematics Extension 1 outcomes** A student:
ME11–2 manipulates algebraic expressions and graphical functions to solve problems	**ME12–2** applies concepts and techniques involving vectors and projectiles to solve problems
MA11–3 applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems	**ME12–3** applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations
MA11–4 applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change	**ME12–4** uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution
ME11–5 uses concepts of permutations and combinations to solve problems involving counting or ordering	**ME12–5** applies appropriate statistical processes to present, analyse and interpret data

Mathematics Extension 1 Course Outcomes

Objective **Students:** • use technology effectively and apply critical thinking to recognise appropriate times for such use	
Year 11 Mathematics Extension 1 outcomes A student:	**Year 12 Mathematics Extension 1 outcomes** A student:
ME11–6 uses appropriate technology to investigate, organise and interpret information to solve problems in a range of contexts	**ME12–6** chooses and uses appropriate technology to solve problems in a range of contexts

Objective **Students:** • develop the ability to interpret, justify and communicate mathematics in a variety of forms	
Year 11 Mathematics Extension 1 outcomes A student:	**Year 12 Mathematics Extension 1 outcomes** A student:
ME11–7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs	**ME12–7** evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms

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ANSWERS

CHAPTER 1

EXERCISE 1.1

1 $0 \le x \le 1$ **2** $-3 \le x \le 5$ **3** all real x
4 $-2 \le x \le 3$ **5** $x < -3,\ 1 < x < 2$ **6** $x \le -\frac{2}{3},\ x \ge 4$
7 $x \le 1$ **8** $x < -1,\ 2 < x < 5$ **9** $0 \le x \le 2$
10 $x = 0$ **11** $0 < x < 3$

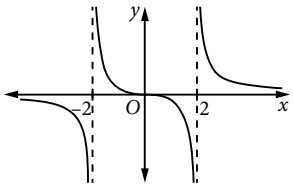
EXERCISE 1.2

1 (a) Student D
(b) Student A multiplied by $2(x-1)$, which could be positive or negative; B multiplied by $(x-1)^2$, but did not note $x \neq 1$; C took reciprocals, but can't be sure that both sides are positive.

2 $x < -1,\ x > 3$
3 $x < -3,\ x > -\frac{4}{3}$ **4** $x \le -3,\ x > -\frac{1}{2}$
5 $x < 1,\ x > 3$ **6** $\frac{5}{4} < x < \frac{13}{10}$ **7** $1 < x < 3$
8 $x < -4,\ -3 < x < 3,\ x > 4$ **9** $x < -1,\ 0 < x < 1$
10 $x < -3,\ x > \frac{10}{3}$

11 (a) $f(x) = \dfrac{x}{x^2-4}$, $f(-x) = \dfrac{-x}{(-x)^2-4} = -f(x)$

(b)

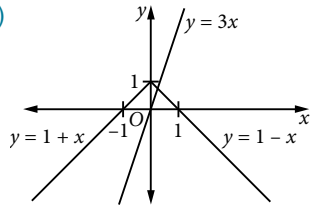


(c) $x < -2,\ -1 < x < 2,\ x > 4$

EXERCISE 1.3

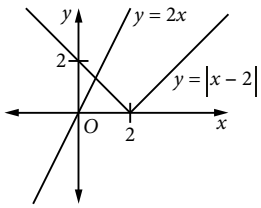
1 $x < -\frac{1}{3},\ x > \frac{1}{3}$

2 (a)



(b) $x > \frac{1}{4}$

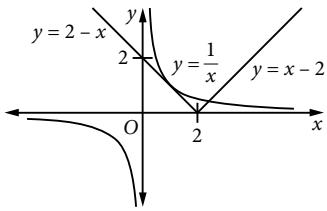
3 (a)



(b) $x > \frac{2}{3}$

4 $x < -1,\ x > 1$

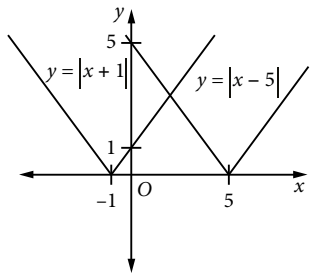
5 (a)



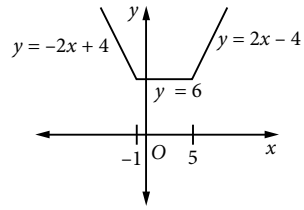
(b) $x < 0,\ x > 1 + \sqrt{2}$

6 $x > -1$ **7** $-2 \le x \le 0$ **8** $x \ge \frac{3}{2}$

9 (a)



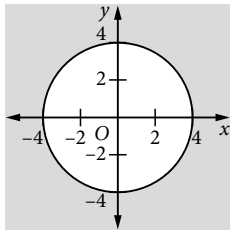
(b)



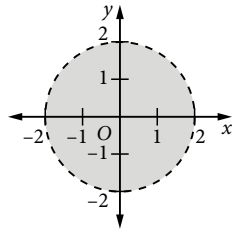
(c) $x < -\frac{3}{2},\ x > \frac{11}{2}$ (d) $-1 \le x \le 5$

EXERCISE 1.4

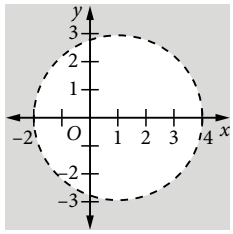
1 (a)



(b)

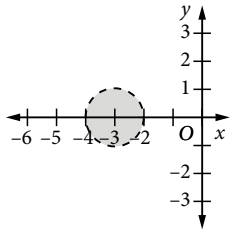


(c)



2 C

3 (a)



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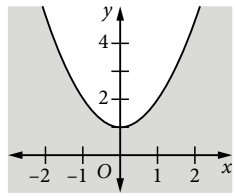

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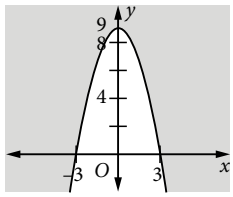

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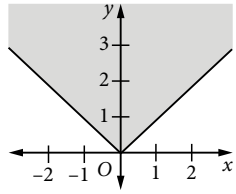

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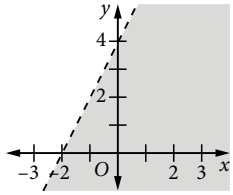

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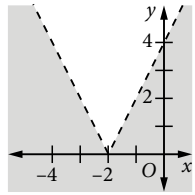
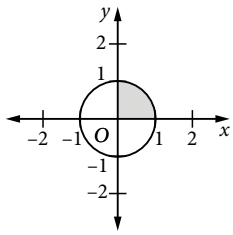
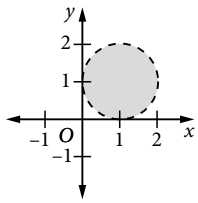


4 (a)



(b)



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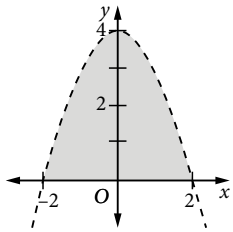

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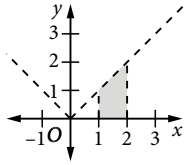
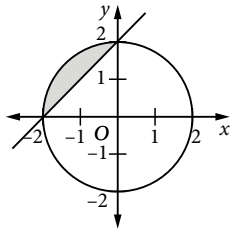


(e)



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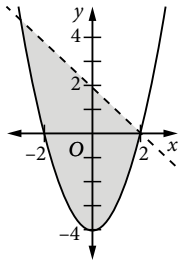
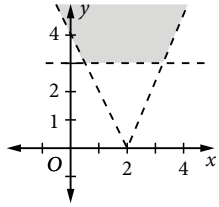


(g)



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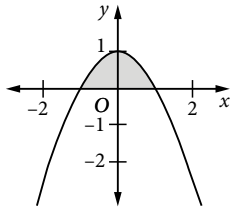
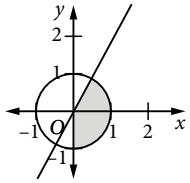
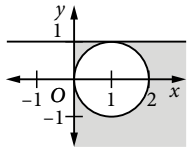


(i)

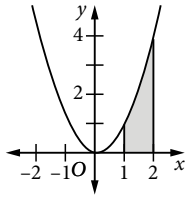


(j)

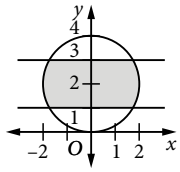


5 A

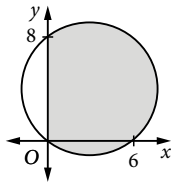
6 (a)



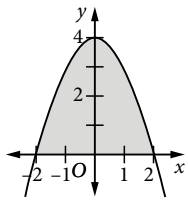
(b)



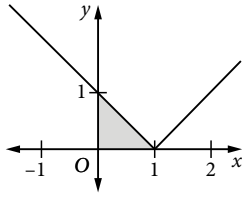
(c)



(d)



(e)



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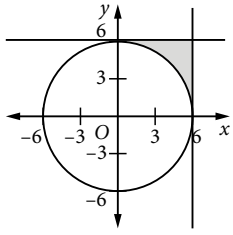

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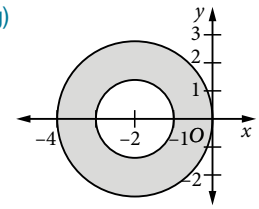
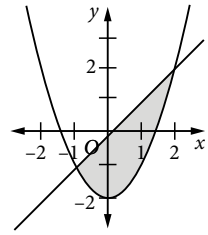
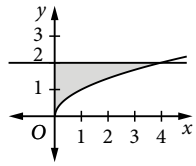


(h)



(i)



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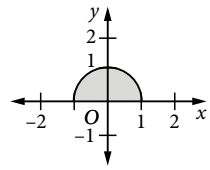


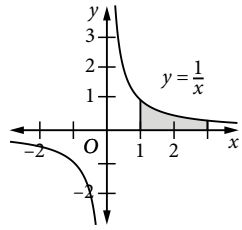
7 (a) correct (b) incorrect (c) incorrect (d) correct

CHAPTER REVIEW 1

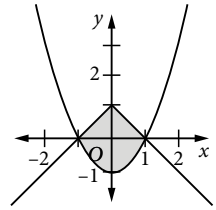
1 C

2 (a) $-1 < x < 1$ (b) $-6 \le x < -3, x > 0$ (c) $x > 0, x \ne 1$
(d) $x < -1, 0 \le x < 1$ (e) $x < -2, -1 \le x < 3, x \ge 6$ (f) $0 < x < 1$
(g) $x < -4, x > 2$ (h) $-1 - \sqrt{2} \le x \le -1 + \sqrt{2}, x = 1$

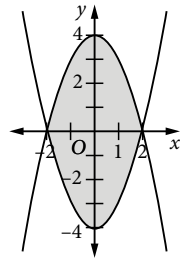
3



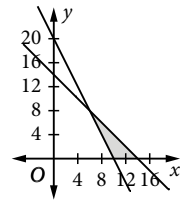
4



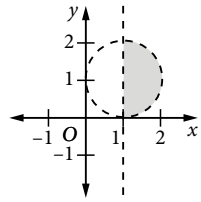
5



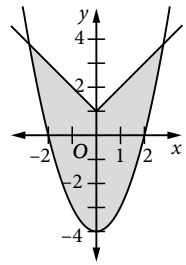
6 (6, 8) satisfies both equations:



7



8



CHAPTER 2

EXERCISE 2.1

1 C

2 $-x^3 + x^2 + 6x$
(a) 3 (b) 0 (c) 1 (d) $-x^3$ (e) 3 (f) $x = -2, 0, 3$

3 (a) $5x^3 + x^2 - 6x + 7$ (i) 3 (ii) 7 (iii) 1 (iv) not monic (v) 3
(b) $-x^3 + 27$ (i) 3 (ii) 27 (iii) 0 (iv) not monic (v) 3
(c) $ax^3 + cx^2 + bx - d$ (i) 3 (ii) $-d$ (iii) c (iv) not monic (v) 3
(d) $5x^4 - 3x^3 + 2x$ (i) 4 (ii) 0 (iii) 0 (iv) not monic (v) 4
(e) $6x^4 + 2x^3 - 15x - 5$ (i) 4 (ii) -5 (iii) 0 (iv) not monic (v) 4
(f) $9x^4 - 12x^2 + 4$ (i) 4 (ii) 4 (iii) -12 (iv) not monic (v) 4

4 (a) polynomial (b) polynomial (c) polynomial
(d) polynomial (e) not polynomial, x^{-1} term (f) polynomial
(g) not polynomial, fractional and negative powers of x
(h) not polynomial, rational function
(i) not polynomial, x in exponent

5 D

6 (a) $2x^4 - 2x^2 + x + 5$ (b) $10x^6 - 4x^5 - 20x^3 + 8x^2 - 15x + 6$
(c) $2x^5 - 2x^4 + x^2 - 3x - 7$ (d) $15x^3 - 16x^2 + 9x - 2$
(e) $-6x^4 + 18x^2 - x - 15$

7 (a) $3x^3 + 2x^2 - 9x - 6$ (b) $3x^3 + 8x^2 + 7x + 2$ (c) $-x^2 + 18x - 5$
(d) $x^3 - x^2 - 5x - 3$ (e) $9x^2 + 12x + 4$ (f) $x^4 + 5x^3 - 9x^2 - 6x + 1$

EXERCISE 2.2

1 (a) $3x + 4$, remainder 13 (b) $x^2 + 1$, reminder 0
(c) $2x^2 - 5x + 6.5$, remainder -7.5
(d) $x^3 + 3x + 2$, remainder 2 (e) $x^2 + 3x + 9$, remainder 0
(f) $x^3 + x^2 + 3x - 12$, remainder 16 (g) $-x^2 + 5x - 11$, remainder 34
(h) $x^3 + x^2 - 2x - 2$, remainder 0

(i)
$$\begin{array}{r} x+3 \\ x^2-1\overline{)\,x^3+3x^2-2x+1} \\ \underline{x^3\qquad -x} \\ 3x^2-x \\ \underline{3x^2\qquad -3} \\ -x+4 \end{array}$$
$$\frac{x^3+3x^2-2x+1}{x^2-1} = x+3-\frac{x-4}{x^2-1}$$

(j)
$$\begin{array}{r} x^2+2x-4 \\ x^2+1\overline{)\,x^4+2x^3-3x^2+x} \\ \underline{x^4\qquad +x^2} \\ 2x^3-4x^2+x \\ \underline{2x^3\qquad +2x} \\ -4x^2-x \\ \underline{-4x^2\qquad -4} \\ -x+4 \end{array}$$
$$\frac{x^4+2x^3-3x^2+x}{x^2+1} = x^2+2x-4-\frac{x-4}{x^2+1}$$

(k)
$$\begin{array}{r} x^2+x+5 \\ x^2-x+1\overline{)\,x^4\qquad +5x^2+x+1} \\ \underline{x^4-x^3+x^2} \\ x^3+4x^2+x \\ \underline{x^3-x^2\ +x} \\ 5x^2\qquad +1 \\ \underline{5x^2-5x+5} \\ 5x-4 \end{array}$$
$$\frac{x^4+5x^2+x+1}{x^2-x+1} = x^2+x+5+\frac{5x-4}{x^2-x+1}$$

(l)
$$\begin{array}{r} x^2+4x-5 \\ x^2+x+1\overline{)\,x^4+5x^3\qquad -x\ -5} \\ \underline{x^4+x^3+x^2} \\ 4x^3-x^2\ -x \\ 4x^3+4x^2+4x \\ -5x^2-5x-5 \\ \underline{-5x^2-5x-5} \\ 0 \end{array}$$
$$\frac{x^4+5x^3-x-5}{x^2+x+1} = x^2+4x-5$$

2 $-1, -97, 2a^3 - 4a^2 + 2a - 1, -37$ **3** $0, 0, a^4 - 2a^3 + a^2 - 4$
4 A **5** $p = 2, q = 3$ **6** $a = 3, b = 4, c = 12$
7 (a) -7 (b) 74 (c) 4 (d) 142 (e) $-30\frac{15}{16}$ (f) $1\frac{3}{16}$
8 14 **9** $a = 3, b = 8$ **10** $m = 2, n = 1$ **11** 11
12 $a = 6, b = -2$ **13** $a = 4, b = -9$ **14** $a = 3$, remainder $= -5$

EXERCISE 2.3

1 (a) correct (b) incorrect (c) correct (d) incorrect
2 (a) $(x+1)(x-2)(x+5)$ (b) $(x+1)(x+7)(x-6)$
(c) $(x-2)(x+4)(2x-3)$ (d) $(x-2)(3x+2)(2x+1)$
(e) $(x+1)(3x+1)(x+2)$ (f) $(x+4)(2x+3)(x-2)$
3 (a) $(x-5)(2x+1)(3x-1)$ (b) $(x+3)(2x+1)(2x-1)$
(c) $(x+4)(3x-2)(2x+5)$ (d) $(2x+1)(3x+2)(x-6)$
(e) $x(x-2)(x-3)$ (f) $(x+2)(x-2)(x+2)$
4 (a) $(x-1)(3x-1)(2x+1)$ (b) $x(3-2x)(3x+1)$
(c) $(x-1)(2x-1)(x-2)$ (d) $x(2x-3)(x+3)$
(e) $(1+x)(2-x)(3+x)$ (f) $(x+1)(x+2)(x-4)$
5 D **6** $a = 7, b = 12$ **7** $p = -1, q = 30$
8 $a = 8, b = -4$, or $a = -\frac{1568}{25}, b = 2.8$
9 $a = -1, b = 16, (2x-1)(3x-2)$ **10** $a = -1, b = -6$
11 $(x+2)\left(x-\frac{5}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{17}}{2}\right)$
12 (a) $a = 2, b = 6$ (b) $2x + 6$ **13** $a = 2, c = -2; \beta = -5$ **14** 2

EXERCISE 2.4

1 $5x^2 + 74x + 5 = 0$
2 (a) -4 (b) 1 (c) 14 (d) -52
3 $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ **4** (a) $-\frac{q}{r}$ (b) $\frac{q^2 - 2pr}{r^2}$ **5** C
6 $x = -2, 2, 3$ **7** $x = 0.5, 2, 8$ **8** $x = -1.5, -2.5, -4$
9 $x = 6, \frac{2}{3}, -1$ **10** $x = \frac{5}{2}, \frac{5\left(2 \pm \sqrt{3}\right)}{2}$
11 $m = -12, -20; 1 \pm \sqrt{5}, -3 \pm \sqrt{5}$ **12** $\frac{1}{2}, \frac{2}{3}$
13 Let the roots be α, α, β.
$2\alpha + \beta = 0$ so $\beta = -2\alpha$
$\alpha^2 + 2\alpha\beta = q$ so $\alpha^2 - 4\alpha^2 = q, -3\alpha^2 = q, \alpha^2 = -\frac{q}{3}$
$\alpha^2\beta = -r$ so $-2\alpha^3 = -r, \alpha^3 = \frac{r}{2}$
Hence $(\alpha^2)^3 = -\frac{q^3}{27}$ so $\alpha^6 = -\frac{q^3}{27}$
and $(\alpha^3)^2 = \frac{r^2}{4}$ so $\alpha^6 = \frac{r^2}{4}$
so $-\frac{q^3}{27} = \frac{r^2}{4}$ or $4q^3 + 27r^2 = 0$
14 (a) 9 (b) -2
15 $x = 0.5, 0.5, 2$ **16** $\frac{4}{3}, -\frac{3}{2}, 1 \pm \sqrt{3}$
17 $\frac{4}{3}, \frac{3}{2}, 1 \pm \sqrt{2}$ **18** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
19 $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$ **20** $\frac{2}{3}$
21 $\sqrt{2}, -\sqrt{2}, 1+\sqrt{2}, 1-\sqrt{2}$ **22** $1+\sqrt{2}, 1-\sqrt{2}$

EXERCISE 2.5

1 B $P'(x) = 3x^2 - 2x = (3x-5)(x+1)$. $P(-1) = 0$ or $P(x)$ is monic, so $x \neq \frac{5}{3}$.

2 (a) $P(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$
$P'(x) = 4x^3 - 12x^2 + 12x - 4$. $P'(1) = 4 - 12 + 12 - 4 = 0$
$P''(x) = 12x^2 - 24x + 12$. $P''(1) = 12 - 24 + 12 = 0$
$P'''(x) = 24x - 24 = 24(x-1)$. $P'''(1) = 0$
$x = 1$ is a zero of multiplicity 4.
(b) $P(x) = (x-1)^4$

3 (a) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$
$P'(x) = 4x^3 + 3x^2 - 6x - 5$
$P''(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$
$P''(x) = 0$ when $x = \frac{1}{2}$ or $x = -1$.
Since $P(x)$ is monic the only possibility is $x = -1$.
$P(-1) = 1 - 1 - 3 + 5 - 2 = 0$
$P'(-1) = -4 + 3 + 6 - 5 = 0$
$x = -1$ is a zero of multiplicity 3.
(b) $P(x) = (x + 1)^3 (x - a)$ (c) $x = -1, 2$
But $-2 = 1^3 \times (-a)$ so $a = 2$.
$P(x) = (x + 1)^3 (x - 2)$

4 (a) $P(x) = x^3 - x^2 - 8x + 12$
$P'(x) = 3x^2 - 2x - 8 = (3x + 4)(x - 2)$
$P'(x) = 0$ when $x = -\frac{4}{3}$ or $x = 2$
Since $P(x)$ is monic the only possibility is $x = 2$.
$P(2) = 8 - 4 - 16 + 12 = 0$
$x = 2$ is a zero of multiplicity 2.
(b) $P(x) = (x - 2)^2 (x - a)$ (c) $x = 2, -3$
But $12 = (-2)^2 \times (-a)$
$a = -3$
$P(x) = (x - 2)^2 (x + 3)$

5 (a) $P(x) = 3x^4 - 8x^3 + 6x^2 - 24x + 40$
$P'(x) = 12x^3 - 24x^2 + 12x - 24 = 12(x^3 - 2x^2 + x - 2)$
$= 12[x^2(x - 2) + 1(x - 2)] = 12(x - 2)(x^2 + 1)$
The only possibility is $x = 2$
$P(2) = 48 - 64 + 24 - 48 + 40 = 0$
$x = 2$ is a zero of multiplicity 2.
(b) $P(x) = (x - 2)^2 (ax^2 + bx + c)$
Coefficient of x^4 is 3, so $a = 3$
Constant term is 40, so $(-2)^2 \times c = 40$, so $c = 10$
Hence $P(x) = (x - 2)^2 (3x^2 + bx + 10)$
$P(x) = (x^2 - 4x + 4)(3x^2 + bx + 10)$
Write an expression for the term in x: $4bx - 40x = -24x$
$4b = 16$, so $b = 4$
Hence $P(x) = (x - 2)^2 (3x^2 + 4x + 10)$
(c) $x = 2$ as $3x^2 + 4x + 10 = 0$ has no real roots
$(\Delta = 16 - 4 \times 3 \times 10 = -104 < 0)$

6 (a) $P(x) = ax^4 + bx^3 + cx^2 + e$, $P(-1) = -4$, $P(1) = P'(1) = 0$
$a - b + c + e = -4$ (i)
$a + b + c + e = 0$ (ii)
(ii) − (i): $2b = 4$, so $b = 2$
$P'(x) = 4ax^3 + 3bx^2 + 2cx$
$P'(1) = 0$: $4a + 3b + 2c = 0$
$b = 2$: $4a + 6 + 2c = 0$
$2a + c = -3$
(b) $P'(-1) = -4a + 3b - 2c$
$= -2(2a + c) + 6$
$= -2 \times (-3) + 6$
$= 12$

7 $P(x) = x^3 + ax^2 + bx - 4$, $(x - 2)^2$ is a factor.
$P(2) = 8 + 4a + 2b - 4 = 0$
$4a + 2b = -4$
$2a + b = -2$ (i)
$P'(x) = 3x^2 + 2ax + b$
$P'(2) = 12 + 4a + b = 0$
$4a + b = -12$ (ii)
(i) − (ii): $-2a = 10$
$a = -5$
Substitute in (i): $-10 + b = -2$
$b = 8$
$P(x) = x^3 - 5x^2 + 8x - 4$

8 $P(x) = ax^3 + bx + c$, $P(1) = 8$, multiple zero at -1
$P(1) = a + b + c = 8$
$sa + b + c = 8$ (i)
$P(-1) = -a - b + c = 0$
$-a - b + c = 0$ (ii)
(i) + (ii): $2c = 8$
$c = 4$
$a + b = 4$ (iii)
$P'(x) = 3ax^2 + b$
$P'(-1) = 3a + b = 0$
$3a + b = 0$ (iv)
(iv) − (iii): $2a = -4$
$a = -2$
$b = 6$
$P(x) = -2x^3 + 6x + 4$

9 (a) $P(x) = 4x^3 - 8x^2 + 5x - 1$, is the root of multiplicity 2.
$P'(x) = 12x^2 - 16x + 5$
$(6x - 5)(2x - 1) = 0$
$x = \frac{5}{6}, \frac{1}{2}$
$P\left(\frac{1}{2}\right) = 4 \times \frac{1}{8} - 8 \times \frac{1}{4} + \frac{5}{2} - 1 = \frac{1}{2} - 2 + \frac{5}{2} - 1 = 0$
$x = \frac{1}{2}$ is the root of multiplicity 2.
$P(x) = (2x - 1)^2 (x - a)$
$-1 = (-1)^2 \times a$
$a = -1$
$P(x) = (2x - 1)^2 (x + 1)$
Roots are $x = \frac{1}{2}, \frac{1}{2}, -1$

(b) $P(x) = x^4 + 4x^3 - 16x - 16$, is the root of multiplicity 3.
$P'(x) = 4x^3 + 12x^2 - 16$
$P''(x) = 12x^2 + 24x = 12x(x + 2)$
$P''(x) = 0$, $x = 0, -2$. $x = 0$ is not possible because $P(x)$ has a constant term.
$P'(-2) = -32 + 48 - 16 = 0$
$P(-2) = 16 - 32 + 32 - 16 = 0$
$P(x) = (x + 2)^3 (x - a)$
$-16 = 2^3 \times (-a)$
$a = 2$
$P(x) = (x + 2)^3 (x - 2)$
Roots are $x = -2, -2, -2, 2$

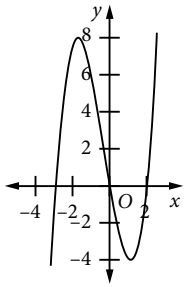
10 (a) $P(x) = 9x^3 + 12x^2 - 11x + 2$, is the root of multiplicity 2.
$P'(x) = 27x^2 + 24x - 11$
$(3x - 1)(9x + 11) = 0$
$x = \frac{1}{3}, -\frac{11}{9}$. The most likely answer is $x = \frac{1}{3}$ because $(3x - 1)^2$ generates $9x^2$ in its expansion.
$P\left(\frac{1}{3}\right) = 9 \times \frac{1}{27} + 12 \times \frac{1}{9} - \frac{11}{3} + 2 = \frac{1}{3} + \frac{4}{3} - \frac{5}{3} = 0$
$x = \frac{1}{3}$ is a root of multiplicity 2.
$P(x) = (3x - 1)^2 (x - a)$
$2 = (-1)^2 \times (-a)$
$a = -2$
$P(x) = (3x - 1)^2 (x + 2)$
Roots are $x = \frac{1}{3}, \frac{1}{3}, -2$

(b) $P(x) = 2x^4 + 5x^3 + 3x^2 - x - 1$, is the root of multiplicity 3.
$P'(x) = 8x^3 + 15x^2 + 6x - 1$
$P''(x) = 24x^2 + 30x + 6$
$6(4x^2 + 5x + 1) = 0$
$(4x + 1)(x + 1) = 0$
$x = -\frac{1}{4}, -1$. $x = -1$ is the most likely answer.
$P'(-1) = -8 + 15 - 6 - 1 = 0$
$P(-1) = 2 - 5 + 3 + 1 - 1 = 0$
$P(x) = (x + 1)^3 (2x - a)$, because the coefficient of x^4 is 2.
$-1 = 1^3 \times (-a)$
$a = 1$
$P(x) = (x + 1)^3 (2x - 1)$
Roots are $x = -1, -1, -1, \frac{1}{2}$

EXERCISE 2.6

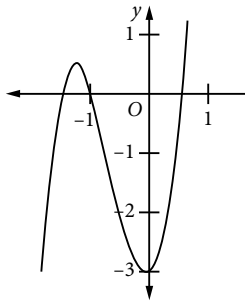
1 (a) (i) 0, 2, −3

(ii) $-3 < x < 0, x > 2$

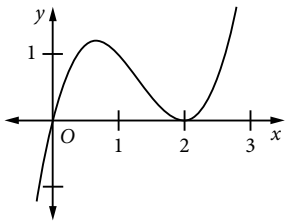


(b) (i) −1.5, −1, 0.5

(ii) $-1.5 < x < -1, x > 0.5$



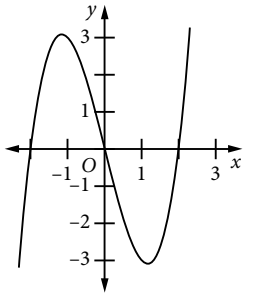
(c)

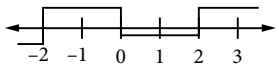


(i) 0, 2

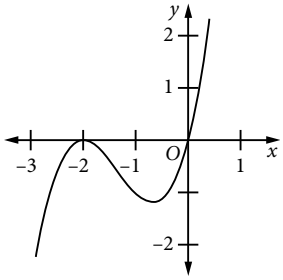
(ii) $0 < x < 2, x > 2$

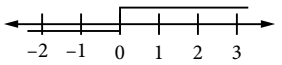
2 (a) $-2 \le x \le 0, x \ge 2$



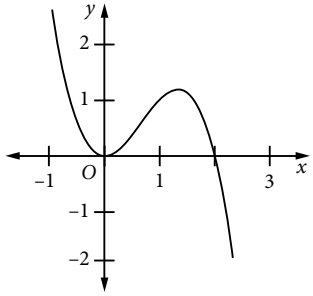


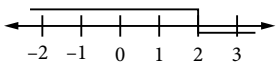
(b) $x \ge 0, x = -2$



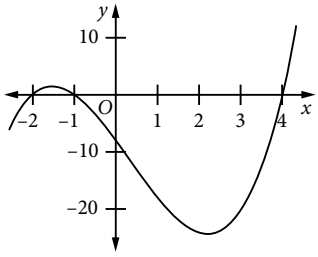


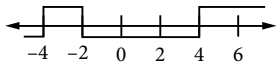
(c) $x \le 2$





(d) $-2 \le x \le -1, x \ge 4$



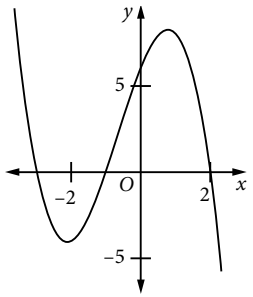


3 (a) $(x + 1)(x + 3)(2 - x)$

(b) (i) −1, −3, 2

(ii) $x < -3, -1 < x < 2$

(c)

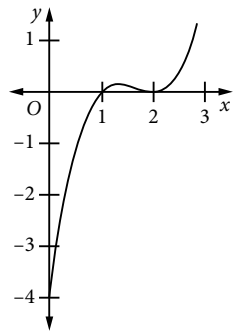


4 (a) $(x - 1)(x - 2)(x - 2)$

(b) (i) 1, 2

(ii) $1 < x < 2, x > 2$

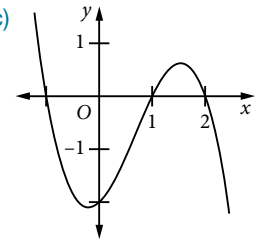
(c)



5 (a) $(1 - x)(x + 1)(x - 2)$

(b) $x < -1, 1 < x < 2$

(c)



6 D

7 (a) Shift the graph horizontally to the left 3 units (parallel to the x-axis) so that it cuts the x-axis at $x = -3$.

(b) Shift the graph horizontally to the left 3 units (parallel to the x-axis), then move it down 2 units (parallel to the y-axis). The graph now cuts the y-axis at $y = 25$.

8 $x^3 - 8 = (x - 2)(x^2 + 2x + 4) = 0$ gives $x = 2$.

For $x^2 + 2x + 4 = 0$, $\Delta = 4 - 16 < 0$, no real roots.

Hence graph of f only cuts x-axis at $x = 2$.

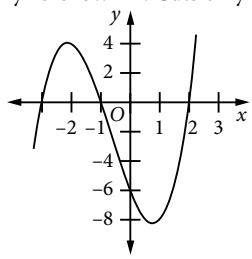
9 $f(x) = (x + 3)(x - 2)^2$

Cuts x-axis at $x = -3$, touches at $x = 2$.

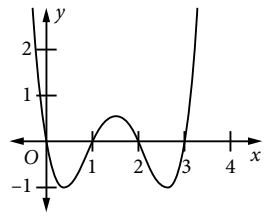
10 $f(x) = (x - 2)(x^2 - 2x + 4)$. $\Delta = -12$.

Only zero is $x = 2$. Cuts only at $x = 2$.

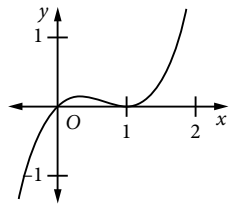
11 (a)



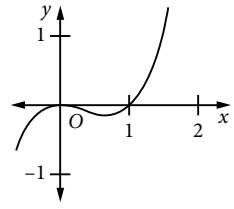
(b)



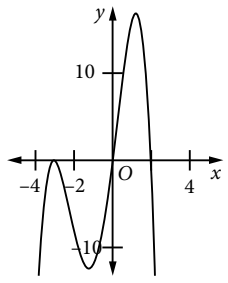
(c)



(d)



(e)



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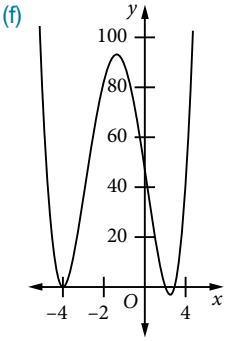
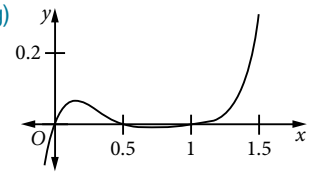
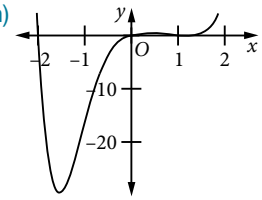


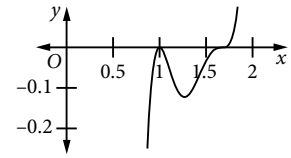
(g)



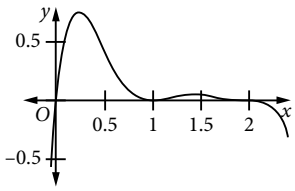
(h)



(i)

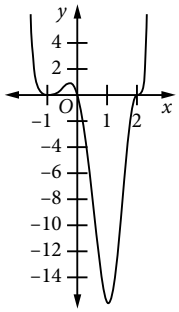


(j)

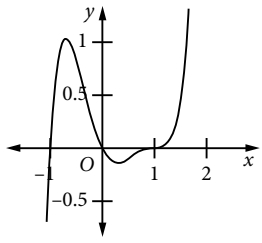


12 B

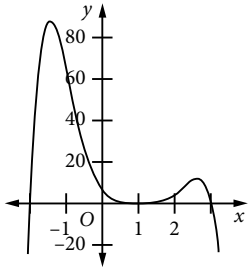
13 (a)



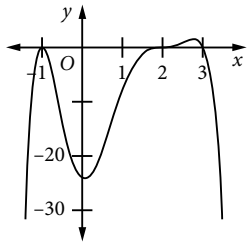
(b)



(c)



(d)



CHAPTER REVIEW 2

1 (a) $x^2 + 3x + \frac{4}{x-1}$

(b) $4x^3 - 12x^2 + 30x - 80 + \frac{200}{x+3}$

2 (a) -7 (b) -11

3 (a) $(x-1)(x+2)(x-3)$

(b) $(x+1)(x+2)(x+4)$

(c) $(x-1)(x+1)(x+5)$

(d) $(x-2)(x+2)(x+3)(x-4)$

4 (a) $a = 2, b = 4$ (b) $2x + 4$

5 $a = 2, P(-3) = -5$

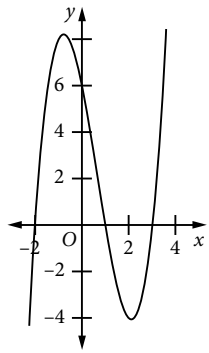
6 $a = -4, b = -2; P(3) = 17$

7 $a = 3, c = -6; \alpha = -4$

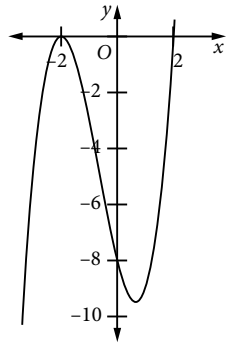
8 (a) 1 (b) 0

9 (a) 4 (b) -12 (c) -3

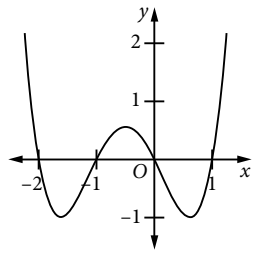
10 (a) $-2 < x < 1, x > 3$



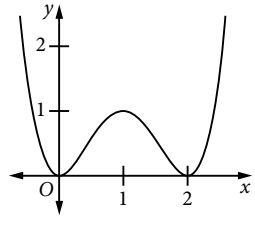
(b) $x > 2$



(c) $x < -2, -1 < x < 0, x > 1$



(d) all $x, x \neq 0, 2$

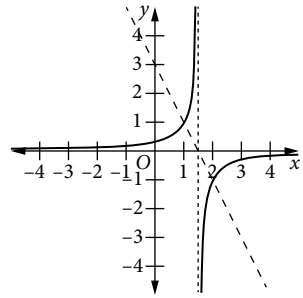


CHAPTER 3

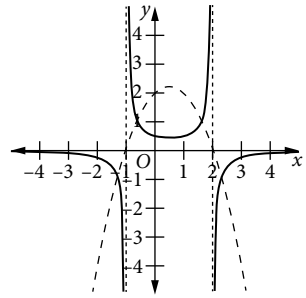
EXERCISE 3.1

1 B **2** B **3** A

4



5



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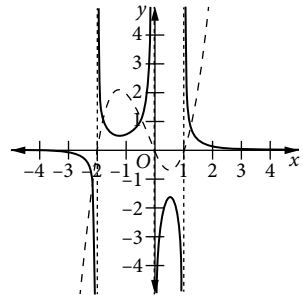

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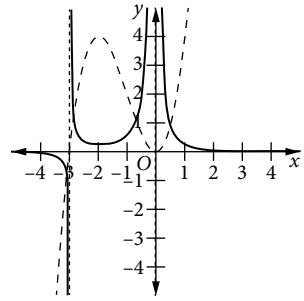

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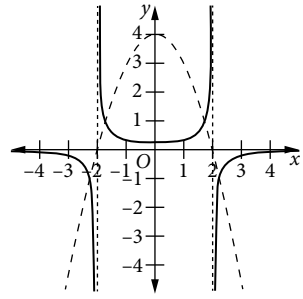

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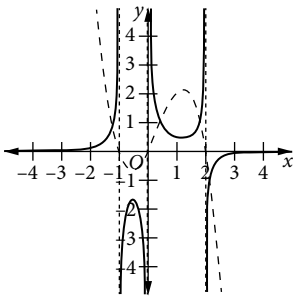

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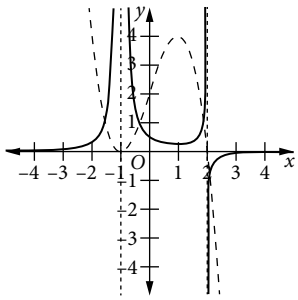
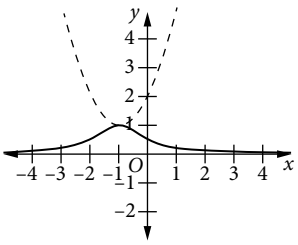
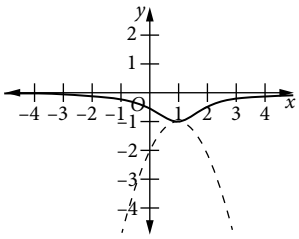


11



12



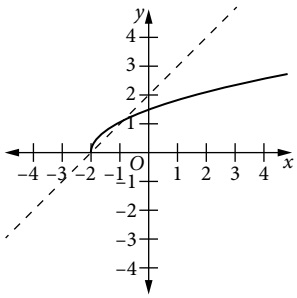
EXERCISE 3.2

1 B

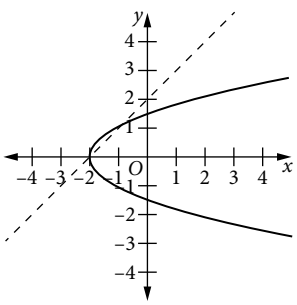
2 C

3 D

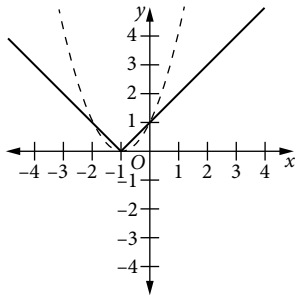
4 (a)



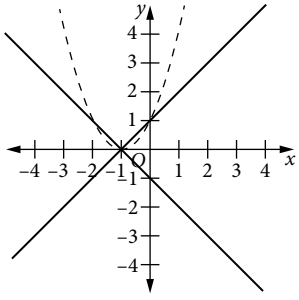
(b)



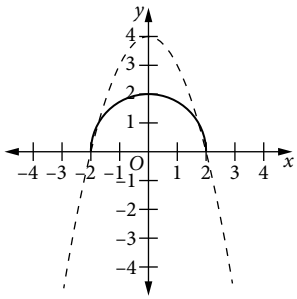
5 (a)



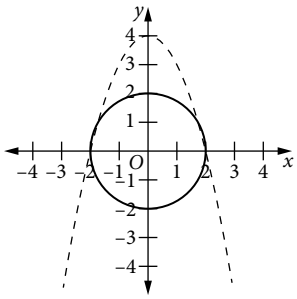
(b)



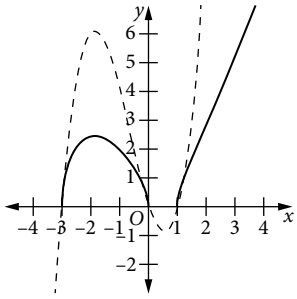
6 (a)



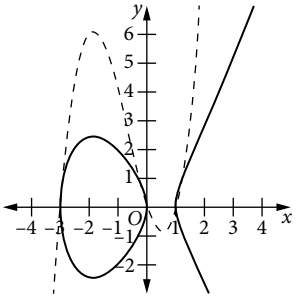
(b)



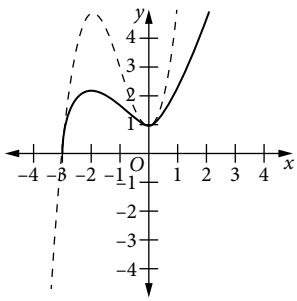
7 (a)



(b)



8 (a)



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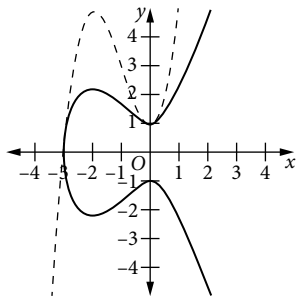
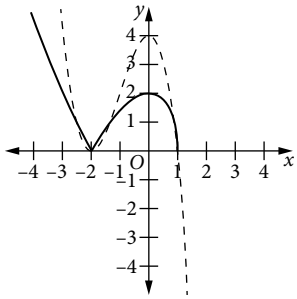
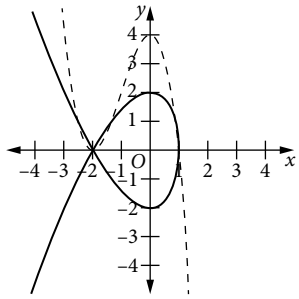


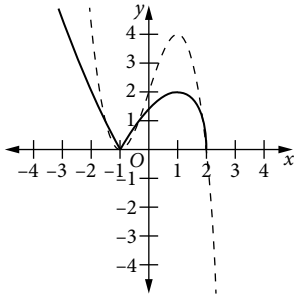
9 (a)



(b)



10 (a)



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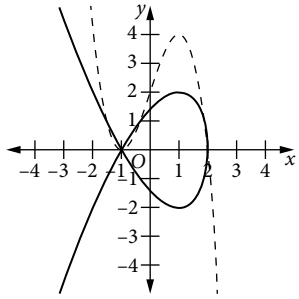
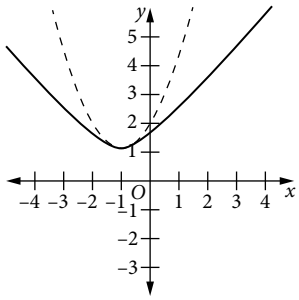
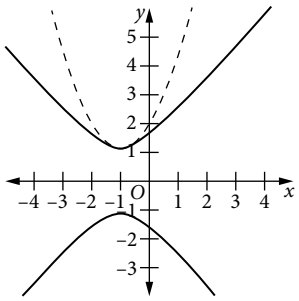


11 (a)



(b)



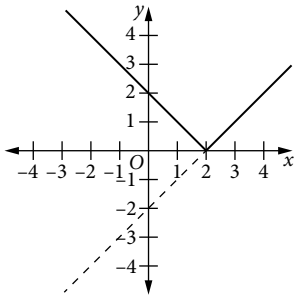
12 (a) There is no graph as the original function is always negative and hence does not have a square root.

(b) There is no graph as the original function is always negative, so cannot be equal to a positive quantity y^2.

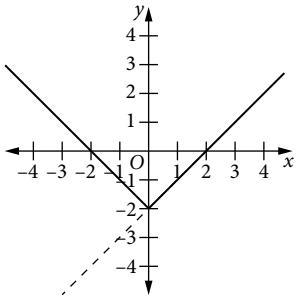
EXERCISE 3.3

1 B **2** A **3** C

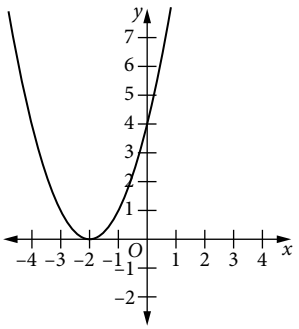
4 (a)



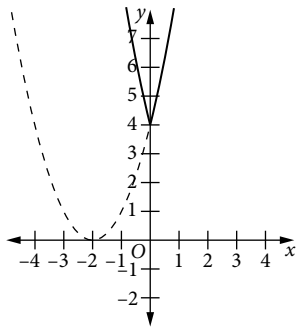
(b)



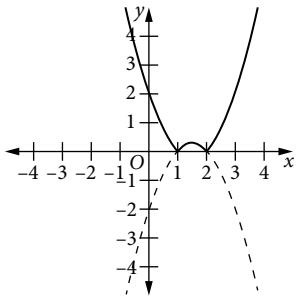
5 (a)



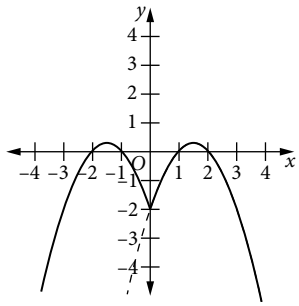
(b)



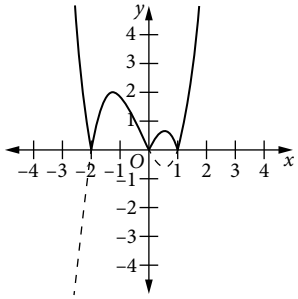
6 (a)



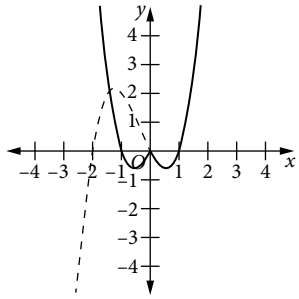
(b)



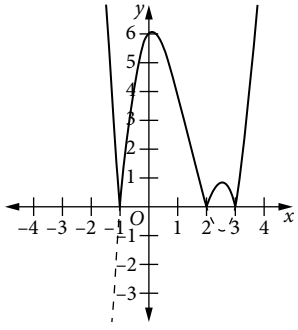
7 (a)



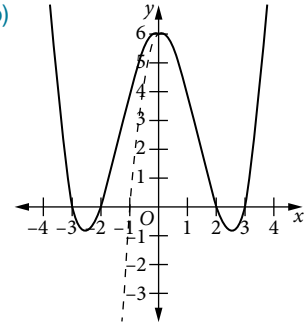
(b)



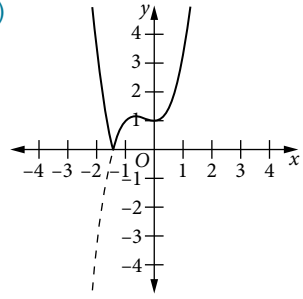
8 (a)



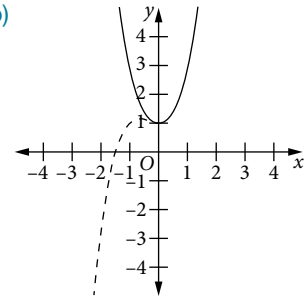
(b)



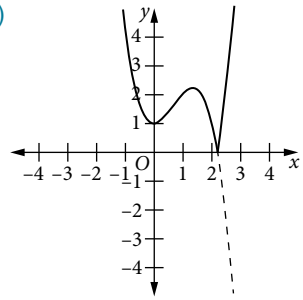
9 (a)



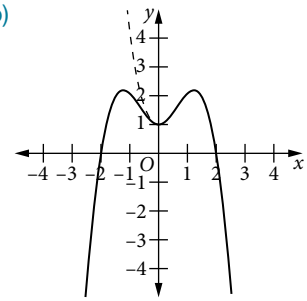
(b)



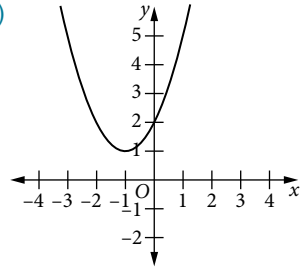
10 (a)



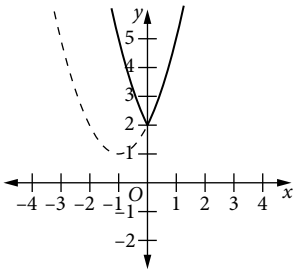
(b)



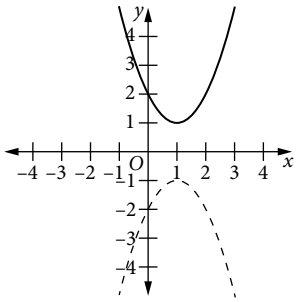
11 (a)



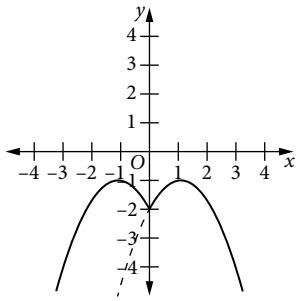
(b)



12 (a)

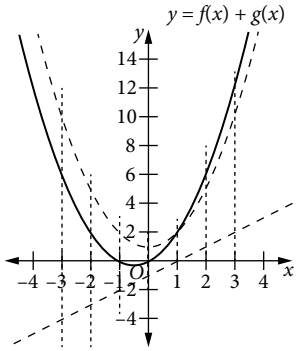


(b)



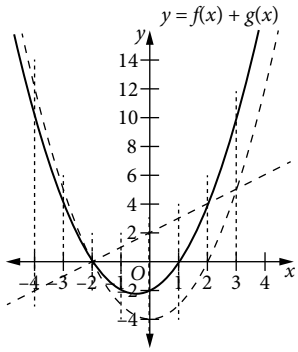
EXERCISE 3.4

1 (a)



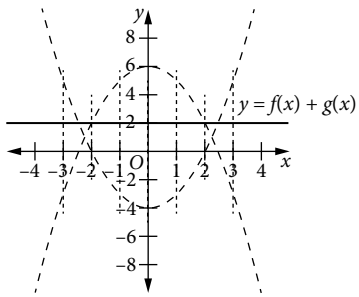
The new curve is a parabola, concave up cutting the x-axis at $(-1, 0)$ and $(0, 0)$.

(b)



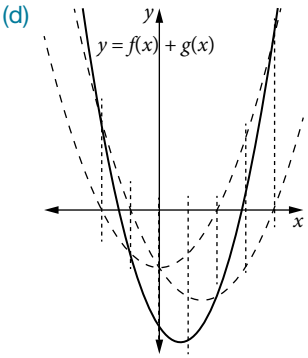
The new curve is a parabola, concave up cutting the x-axis at $(-2, 0)$ and $(1, 0)$ and the y-axis at $(0, -3)$.

(c)



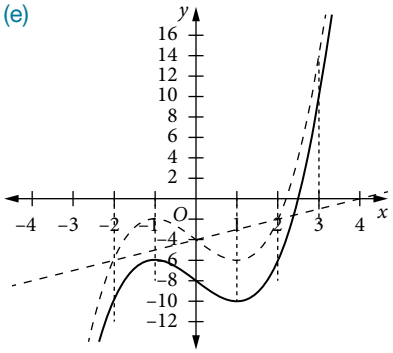
The new curve is a straight line with equation $y = 2$.

(d)



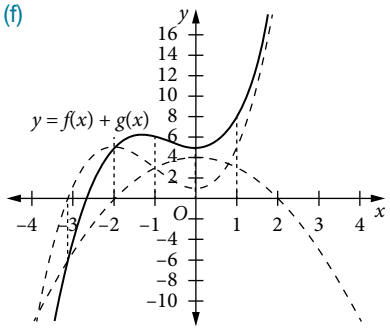
The new curve is a parabola, concave up cutting the y-axis at $(0, -4)$.

(e)



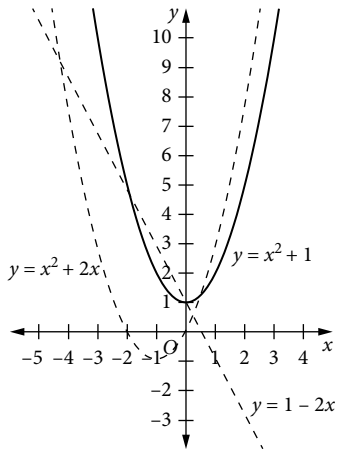
The new curve is a cubic, cutting the x-axis at about $(3.5, 0)$ and the y-axis at $(0, -8)$. It has the same shape as $y = g(x)$.

(f)

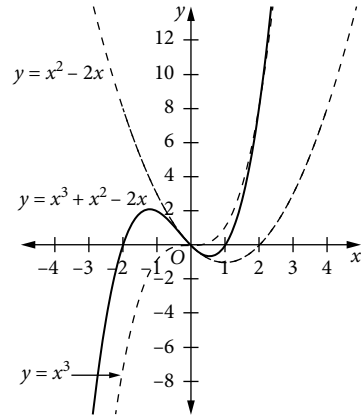


The new curve is a cubic, cutting the x-axis at about $(-2.6, 0)$ and the y-axis at $(0, 5)$. It has a similar shape to $y = g(x)$, but flattens out more between the turning points.

2



3



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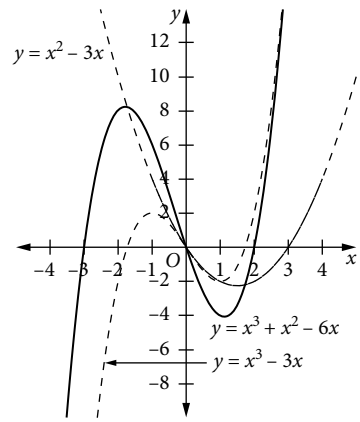
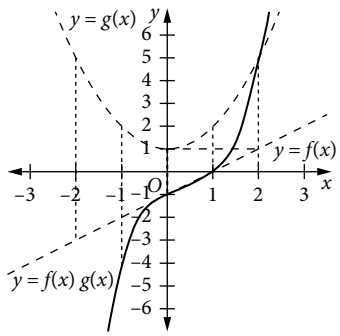


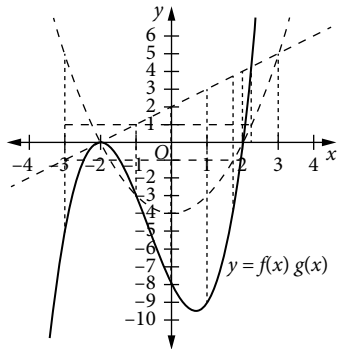
5 B

EXERCISE 3.5

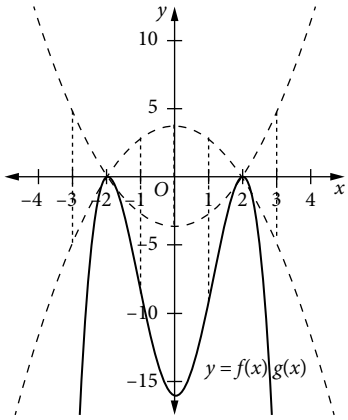
1 (a)



(b)



(c)



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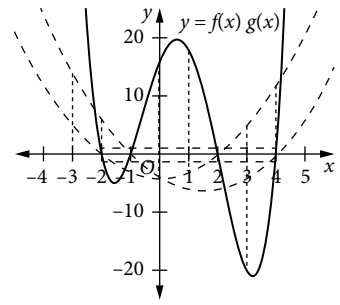

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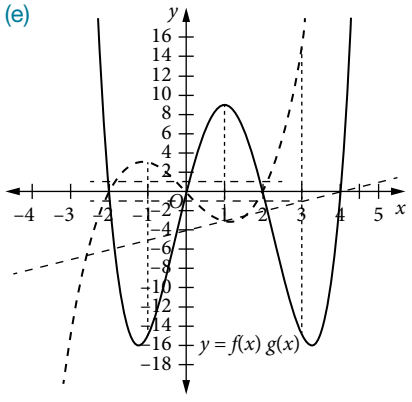

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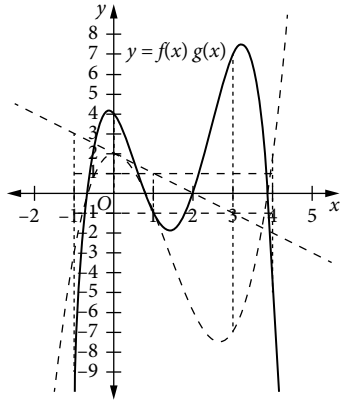

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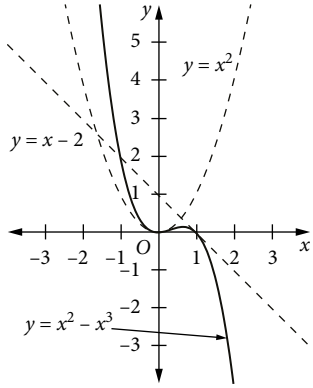
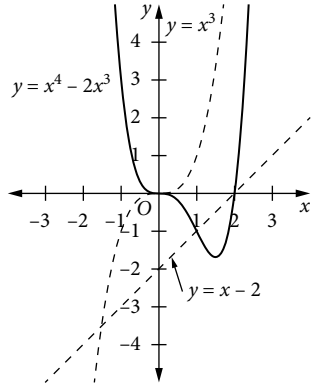


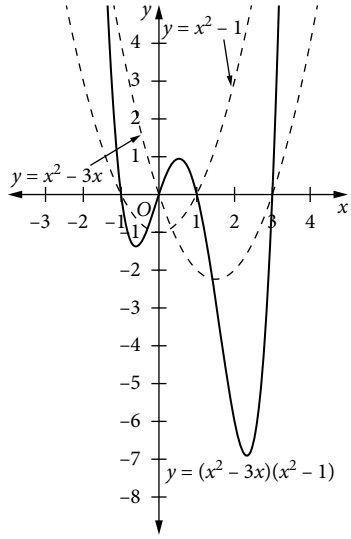
The graph is a cubic touching the x-axis at (0, 0) and cutting it at (1, 0).

3



The graph is a quartic with a horizontal point of inflection at (0, 0) and cutting the x-axis at (2, 0). It has a least value of about −1.7.

4



The graph is a quartic, cutting the x-axis at (−1, 0), (0, 0), (1, 0) and (3, 0). It has a least value of −7.

5 C

EXERCISE 3.6

1 $2y = x + 4$

2 $y = x^2$

3 $y = \frac{1}{x}, x \neq 0$

4 $x^2 + y^2 = 4, -2 \leq x \leq 2$

5 $y = \sqrt{4 - x^2}, -2 \leq x \leq 2$

6 $y = x^2 - 6x + 4, x \geq 3$

7 $2y = 3x + 8, 0 \leq x \leq 4$

8 $y = 1 - x^{\frac{2}{3}}, -1 \leq x \leq 1$

9 $y = x^2 - 4x + 3$

10 $y = x, -1 \leq x \leq 1$

11 $y^2 = 8x, x \geq 0$

12 $3x^2 + 4y^2 = 12, -2 \leq x \leq 2$

13 $y = \frac{1}{2}\sqrt{4 - x^2}, -2 \leq x \leq 2$

14 $x^2 + y^2 = 1, -1 \leq x \leq 1$

15 (a) $y = 2x + 1, y = -2x + 5$

(b) No. Boat A is at (1, 3) when $t = 2$, Boat B is at (1, 3) when $t = 3$.

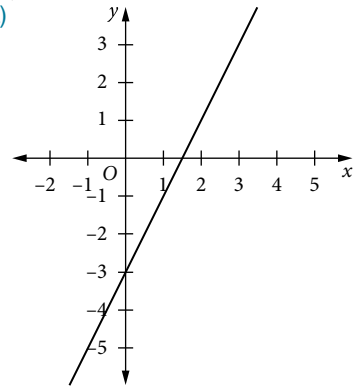
16 $x^2 - y^2 = 1, x \geq 1$

17 (a) $y = \frac{x^2}{2}$ (b) $y = \frac{x^2}{24}$

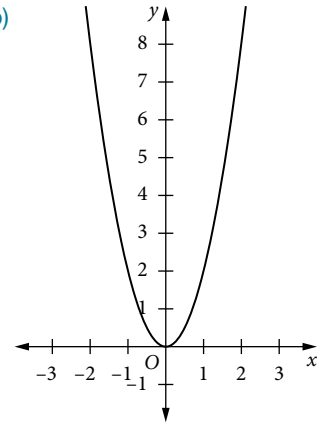
(c) $y = (x - 2)^2$ (d) $y = -\frac{x^2}{8}$

(e) $y = x^2 + 1, x \geq 0$

18 (a)



(b)



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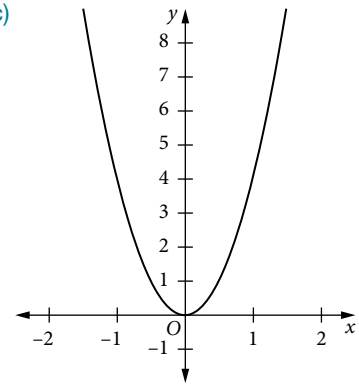

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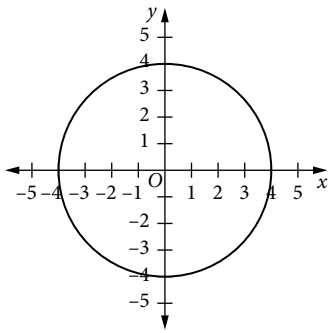

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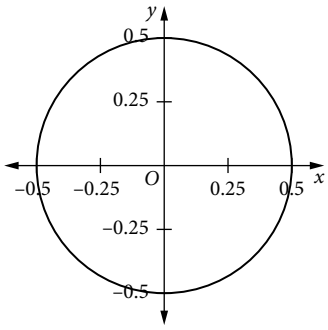
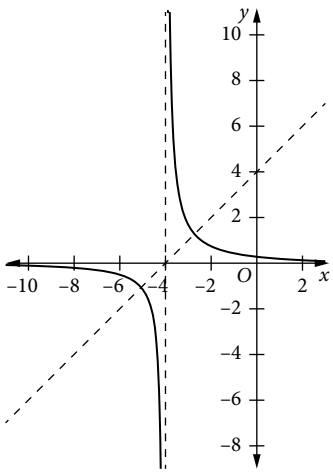


19 (a) $y = x - 1, x + y = 3$
(b) $2x + y + 24 = 0, x - 2y + 72 = 0$
(c) $y = 2x - 5, x + 2y - 5 = 0$
(d) $y = x + 2, x + y + 6 = 0$

CHAPTER REVIEW 3

1 (a)



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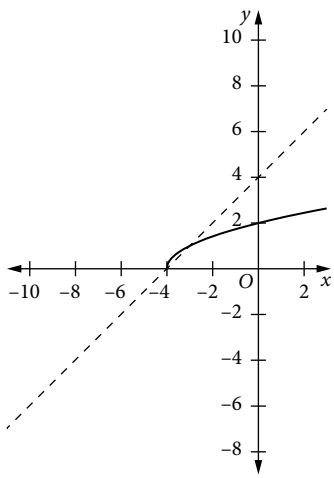

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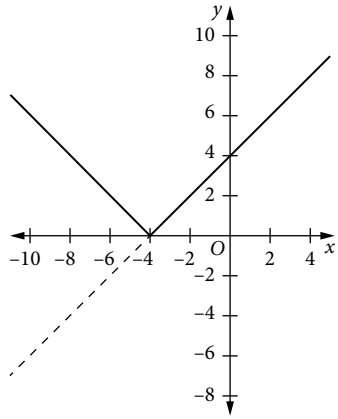

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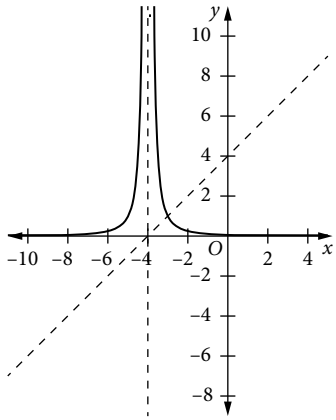

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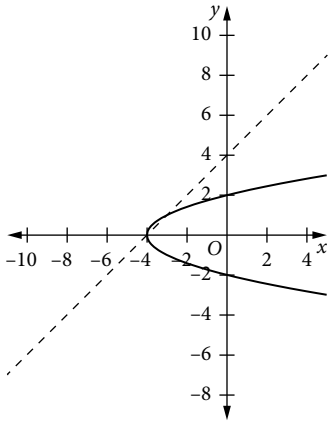

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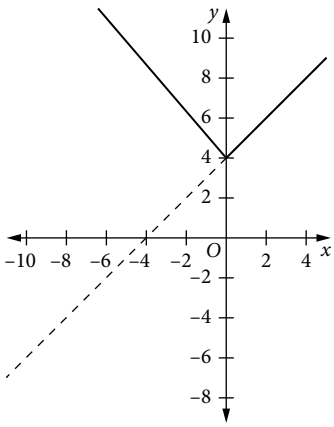

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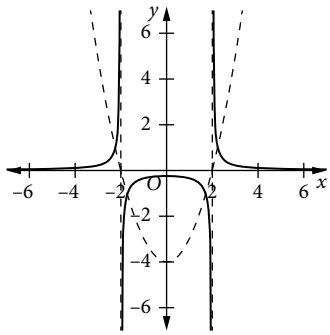

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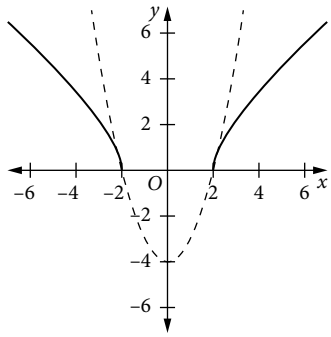

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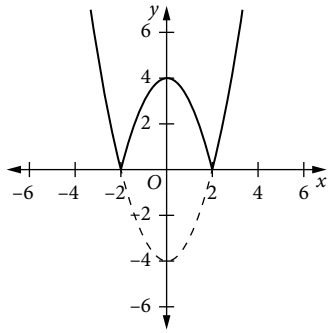

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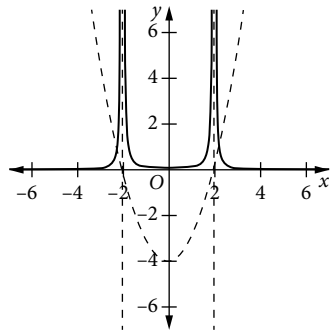

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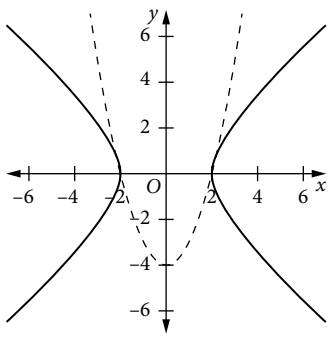

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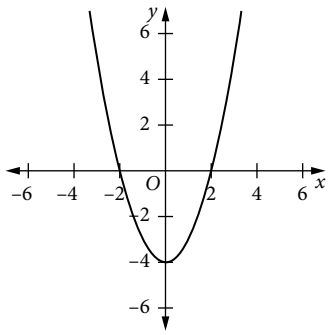

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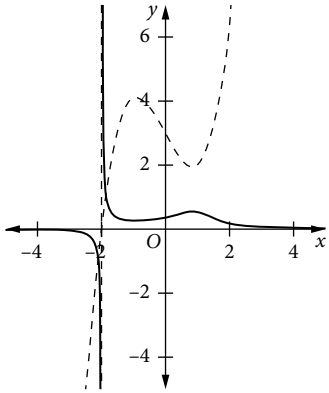

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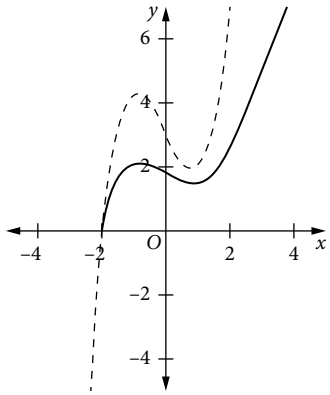

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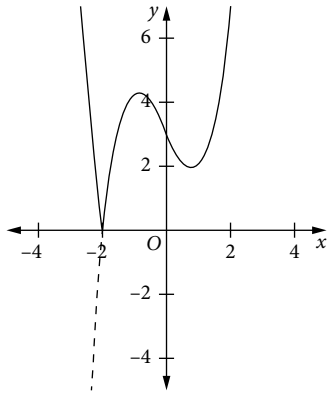

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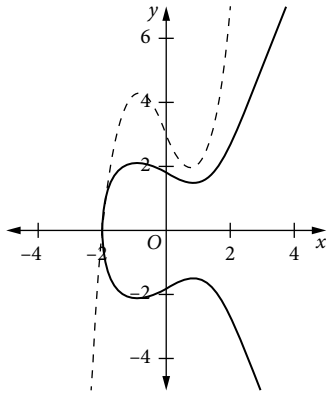

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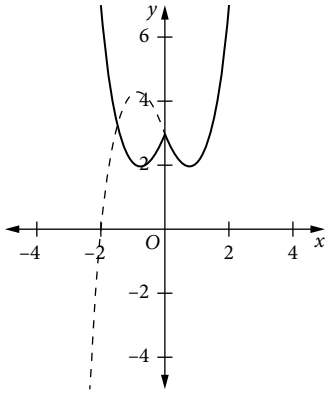

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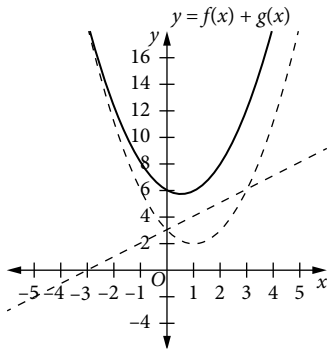

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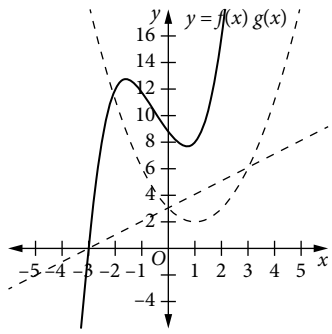
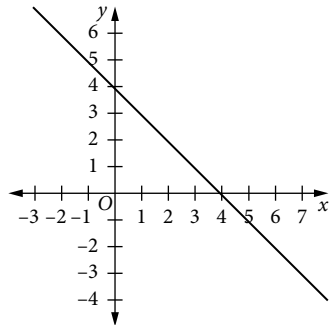
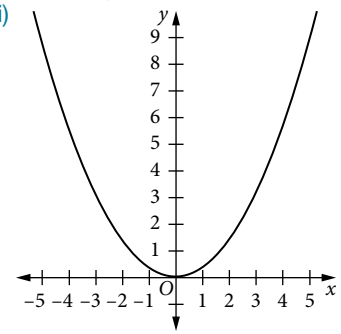


5 (a) (i)



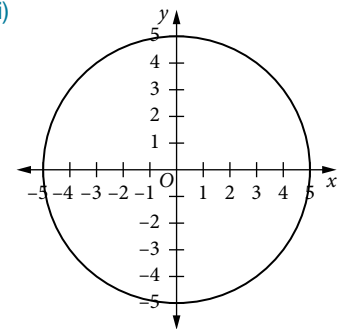
(ii) $t = 2 - x, y = 4 - x$

(b) (i)



(ii) $t = \frac{x}{3}, y = \frac{t^2}{3}$

(c) (i)



(ii) $x^2 + y^2 = 25$

6 $y = x^2 - 6x + 4$

7 (a) D (b) A (c) D (d) C

8 (a) $y = 2x^2 + 8x + 7$

CHAPTER 4

EXERCISE 4.1

1 (a) $\sin A \cos 2B + \cos A \sin 2B$
(b) $\sin 2x \cos y - \cos 2x \sin y$
(c) $\cos 2x \cos 3y + \sin 2x \sin 3y$
(d) $\frac{1}{2}\cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta$
(e) $\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$ (f) $\frac{\tan A + 1}{1 - \tan A}$

2 (a) $\sin(2A - B)$ (b) $\cos 2\alpha$
(c) $\sin A$ (d) 0 (e) $\tan(\theta - 20°)$
(f) $\tan 3\alpha$ (g) $\sin A$ (h) $\cos\theta$
(i) $\tan 2x$

3 D

4 (a) $\sin 60° = \frac{\sqrt{3}}{2}$ (b) $\tan 135° = -1$
(c) $\frac{-1 - \sqrt{3}}{2\sqrt{2}}$ (d) $2\sin x \cos 40°$

5 $\cos(A+B)=\cos A\cos B-\sin A\sin B$
$\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)$
$=\cos A\cos B+\sin A\sin B$

6 $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$
$\cos(90°-\theta-\phi)=\cos(90°-\theta)\cos\phi+\sin(90°-\theta)\sin\phi$
$\cos(90°-(\theta+\phi))=\sin\theta\cos\phi+\cos\theta\sin\phi$
$\sin(\theta+\phi))=\sin\theta\cos\phi+\cos\theta\sin\phi$

7 (a) $\frac{44}{125}$ (b) $\frac{3}{5}$ (c) $\frac{44}{117}$ **8** A **9** (a) $\frac{24}{25}$ (b) $-\frac{24}{7}$ (c) $\frac{56}{65}$

10 (a) $\sin(45°+30°)=\sin45°\cos30°+\cos45°\sin30°$
$\sin75°=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$

(b) $\tan(45°+30°)=\dfrac{\tan45°+\tan30°}{1-\tan45°\tan30°}$

$\tan75°=\dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}=\dfrac{(\sqrt{3}+1)^2}{3-1}=\dfrac{4+2\sqrt{3}}{2}=2+\sqrt{3}$

11 (a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ (b) $2-\sqrt{3}$ (c) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **12** $\dfrac{\sqrt{2}(1-\sqrt{3})}{4}$

EXERCISE 4.2

1 (a) $3\sin\theta-4\sin^3\theta$ (b) $4\cos^3\theta-3\cos\theta$ (c) $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

2 (a) $-\frac{3\sqrt{7}}{8}$ (b) $-\frac{1}{8}$ (c) $3\sqrt{7}$ (d) 3rd quadrant

3 (a) $\tan A$ (b) $\sin^2\theta$ (c) $\cos4\theta$ (d) $\frac{1}{2}$ (e) $\frac{1}{2}\sin8x$
(f) $2\sin^2\theta$ (g) $\frac{1}{4}\sin4x$ (h) $\sin4x$ (i) $1+\sin2\theta$ (j) $1-\sin2A$
(k) 1 (l) 1 (m) $\frac{1}{2}\cos2x$ (n) $\tan^2\theta$ (o) $\cos6x$

4 (a) $\frac{4}{5}$ (b) $-\frac{3}{5}$ (c) $-\frac{4}{3}$

5 (a) $\operatorname{cosec}^2\alpha$ (b) $\sin^2\theta$ (c) 1 (d) $\frac{1}{2}$ (e) $\cos^2\theta$ (f) $\sin(\theta+\phi)$

EXERCISE 4.3

1 A **2** (a) $\tan18°$ (b) $\cos30°=\frac{\sqrt{3}}{2}$ (c) $\dfrac{1}{\sin^2 45°}=\sqrt{2}$

3 (a) $\dfrac{t^2+2t-1}{1+t^2}$ (b) $\dfrac{4+6t-4t^2}{1+t^2}$ (c) $\dfrac{2-2t-2t^2}{1+t^2}$ (d) $\dfrac{1-t^2}{2t}$
(e) $\dfrac{1-6t^2+t^4}{2t(1-t^2)}$ (f) $\dfrac{1-6t^2+t^4}{1+2t^2+t^4}$ (g) $\dfrac{1}{1+t^2}$ (h) $\dfrac{1+t^2}{1-t^2}$
(i) t^2 (j) t (k) $\dfrac{1}{t}$ (l) t

4 D **5** $t=\dfrac{-1\pm\sqrt{1+d^2}}{d}$ **6** -5

7 $t=\frac{a}{b}$, LHS $=b\cos\theta+a\sin\theta$

$$=\frac{b(1-t^2)}{1+t^2}+\frac{a\times2t}{1+t^2}$$

$$=\frac{b\left(1-\frac{a^2}{b^2}\right)+\frac{2a^2}{b}}{1+\frac{a^2}{b^2}}$$

$$=\frac{b^3-a^2b+2a^2b}{b^2+a^2}$$

$$=\frac{b(b^2+a^2)}{b^2+a^2}=b=\text{RHS}$$

8 $x=\sec\theta-\tan\theta$

$$=\frac{1+t^2}{1-t^2}-\frac{2t}{1-t^2}=\frac{(1-t)^2}{1-t^2}=\frac{1-t}{1+t}$$

EXERCISE 4.4

1 LHS $=\dfrac{\sin A+\cos A\tan B}{\cos A-\sin A\tan B}=\dfrac{\frac{\sin A}{\cos A}+\tan B}{1-\frac{\sin A}{\cos A}\tan B}$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}=\tan(A+B)=\text{RHS}$$

2 LHS $=\dfrac{\sin2\theta\cos\theta-\cos2\theta\sin\theta}{\cos2\theta\cos\theta+\sin2\theta\sin\theta}$

$$=\frac{\sin(2\theta-\theta)}{\cos(2\theta-\theta)}=\frac{\sin\theta}{\cos\theta}=\tan\theta=\text{RHS}$$

3 LHS $=\dfrac{\tan A-\tan B}{\tan A+\tan B}=\dfrac{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}$

$$=\frac{\sin A\cos B-\cos A\sin B}{\sin A\cos B+\cos A\sin B}=\frac{\sin(A-B)}{\sin(A+B)}=\text{RHS}$$

4 LHS $=\sin(\theta+\alpha)\sin(\theta-\alpha)$
$=(\sin\theta\cos\alpha+\cos\theta\sin\alpha)(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$
$=\sin^2\theta\cos^2\alpha-\cos^2\theta\sin^2\alpha$
$=\sin^2\theta(1-\sin^2\alpha)-(1-\sin^2\theta)\sin^2\alpha$
$=\sin^2\theta-\sin^2\theta\sin^2\alpha-\sin^2\alpha+\sin^2\theta\sin^2\alpha$
$=\sin^2\theta-\sin^2\alpha=\text{RHS}$

5 LHS $=\dfrac{\sin3A}{\sin A}-\dfrac{\cos3A}{\cos A}$

$$=\frac{\sin3A\cos A-\cos3A\sin A}{\sin A\cos A}$$

$$=\frac{\sin(3A-A)}{\sin A\cos A}$$

$$=\frac{\sin2A}{\sin A\cos A}=\frac{2\sin A\cos A}{\sin A\cos A}=2=\text{RHS}$$

6 LHS $=\tan(45°+A)+\tan(45°-A)$

$$=\frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}$$

$$=\frac{(1+\tan A)^2+(1-\tan A)^2}{(1-\tan A)(1+\tan A)}$$

$$=\frac{1+2\tan A+\tan^2A+1-2\tan A+\tan^2A}{1-\tan^2A}$$

$$=\frac{2+2\tan^2A}{1-\tan^2A}=\frac{2\sec^2A}{1-\tan^2A}$$

$$=\frac{2}{\cos^2A-\sin^2A}=\frac{2}{\cos2A}=\text{RHS}$$

7 LHS $=\cos4x$
$=2\cos^2 2x-1$
$=2(2\cos^2x-1)^2-1$
$=8\cos^4x-8\cos^2x+1=\text{RHS}$

8 RHS $=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

$$=\frac{(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$=\frac{2\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta}{\cos^2\theta-\sin^2\theta}=\frac{\sin2\theta+1}{\cos2\theta}=\text{LHS}$$

9 LHS $=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

$$=\frac{(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$=\frac{2(\cos^2\theta+\sin^2\theta)}{\cos2\theta}=\frac{2}{\cos2\theta}=2\sec2\theta=\text{RHS}$$

10 LHS $=\dfrac{1-\cos x}{\sin x}$

$$=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}=\text{RHS}$$

11 LHS $=\dfrac{\sin A+\sin(90°-A)+1}{\sin A-\sin(90°-A)+1}$

$$=\frac{\sin A+\cos A+1}{\sin A-\cos A+1}$$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}+2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin^2\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}=\text{RHS}$$

12 $\text{LHS} = \dfrac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x}$

$= \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2} + 2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2} - 2\sin^2\frac{x}{2}}$

$= \dfrac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \dfrac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}} = \text{RHS}$

13 $\text{LHS} = \sin[(A + B) + C]$

$= \sin(A + B)\cos C + \cos(A + B)\sin C$

$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$

$= \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C = \text{RHS}$

$\sin(A + 90°) = \cos A$

14 $\text{LHS} = \cos[(A + B) + C]$

$= \cos(A + B)\cos C - \sin(A + B)\sin C$

$= (\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C$

$= \cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin C\sin A - \cos C\sin A\sin B.$

$\cos(A + 90°) = -\sin A$

15 $\text{LHS} = \tan(\theta + \alpha)\tan(\theta - \alpha)$

$= \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} \times \dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$

$= \dfrac{\tan^2\theta - \tan^2\alpha}{1 - \tan^2\theta\tan^2\alpha} = \text{RHS}$

16 $\text{LHS} = \dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

$= \dfrac{(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{\sin\theta + \cos\theta}$

$= 1 - \sin\theta\cos\theta = 1 - \frac{1}{2}\sin 2\theta = \text{RHS}$

17 $\text{LHS} = \dfrac{2\cos\frac{\theta}{2} - 1 - \cos\theta}{2\cos\frac{\theta}{2} + 1 + \cos\theta}$

$= \dfrac{2\cos\frac{\theta}{2} - 2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}}$

$= \dfrac{2\cos\frac{\theta}{2}\left(1 - \cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(1 + \cos\frac{\theta}{2}\right)} = \dfrac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}} = \text{RHS}$

18 $\text{LHS} = \cot(x + y)$

$= \dfrac{1}{\tan(x + y)} = \dfrac{1 - \tan x\tan y}{\tan x + \tan y}$

$= \dfrac{\frac{1}{\tan x\tan y} - 1}{\frac{\tan x}{\tan x\tan y} + \frac{\tan y}{\tan x\tan y}} = \dfrac{\cot x\cot y - 1}{\cot x + \cot y} = \text{RHS}$

19 $\text{LHS} = \dfrac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta}$

$= \left(\dfrac{2\tan\theta}{1 - \tan^2\theta} - \tan\theta\right) \times \dfrac{1}{\frac{2\tan\theta}{1 - \tan^2\theta} + \frac{1}{\tan\theta}}$

$= \tan\theta\left(\dfrac{2 - 1 + \tan^2\theta}{1 - \tan^2\theta}\right) \times \dfrac{\tan\theta\left(1 - \tan^2\theta\right)}{2\tan^2\theta + 1 - \tan^2\theta}$

$= \dfrac{\tan^2\theta\left(1 + \tan^2\theta\right)}{1 + \tan^2\theta} = \tan^2\theta = \text{RHS}$

20 $\text{LHS} = \tan(\theta + 45°)\tan(\theta - 45°)$

$= \dfrac{\tan\theta + 1}{1 - \tan\theta} \times \dfrac{\tan\theta - 1}{1 + \tan\theta} = -1 = \text{RHS}$

21 $\text{LHS} = \dfrac{1 - \tan\theta\tan 2\theta}{1 + \tan\theta\tan 2\theta}$

$= \dfrac{1 - \frac{\tan\theta \times 2\tan\theta}{1 - \tan^2\theta}}{1 + \frac{\tan\theta \times 2\tan\theta}{1 - \tan^2\theta}}$

$= \dfrac{1 - \tan^2\theta - 2\tan^2\theta}{1 - \tan^2\theta + 2\tan^2\theta}$

$= \dfrac{1 - 3\tan^2\theta}{1 + \tan^2\theta}$

$= \left(1 - \dfrac{3\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$

$= \cos^2\theta - 3\sin^2\theta = 4\cos^2\theta - 3 = \text{RHS}$

22 (a) p (b) q

23 $A + B + C = 180°$

Hence $\sin(A + B + C) = \sin 180° = 0$

From question **14** we have:

$\sin(A + B + C) = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C$

So: $\sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C = 0$

$\sin C\cos A\cos B - \sin A\sin B\sin C = -(\sin A\cos B\cos C + \sin B\cos C\cos A)$

$\sin C(\cos A\cos B - \sin A\sin B) = -\cos C(\sin A\cos B + \sin B\cos A)$

$\sin(180° - (A + B))(\cos A\cos B - \sin A\sin B) = -\cos C\sin(A + B)$

$\sin(A + B)(\cos A\cos B - \sin A\sin B) = -\cos C\sin(A + B)$

Hence: $\cos A\cos B - \sin A\sin B + \cos C = 0$

24 $\dfrac{1 + \sqrt{5}}{4}$ **25** $\dfrac{1 - \tan\theta}{\tan\theta(1 + \tan\theta)}$

26 $PT + TQ = PQ = x$

$PT = \dfrac{y}{\tan\alpha}, \quad TQ = \dfrac{y}{\tan\beta}$

$\dfrac{y}{\tan\alpha} + \dfrac{y}{\tan\beta} = x$

$y\left(\dfrac{\tan\beta + \tan\alpha}{\tan\alpha\tan\beta}\right) = x$

$y = \dfrac{x\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$

27 $\sqrt{2} - 1$

28 B **29** (a) $-\frac{3}{\sqrt{34}}$ (b) $-\frac{5}{\sqrt{34}}$ (c) $\frac{8}{17}$

30 (a) $\frac{\sqrt{11}}{6}$ (b) $-\frac{5\sqrt{11}}{18}$ (c) $-\frac{5\sqrt{11}}{7}$

31 (a) $\frac{15}{8}$ (b) $\frac{240}{161}$

32 (a) $\text{LHS} = 2\cos\left(\frac{\pi}{4} + x\right)\cos\left(\frac{\pi}{4} - x\right)$

$= 2\left(\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x\right)\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)$

$= 2\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$

$= \cos^2 x - \sin^2 x = \cos 2x = \text{RHS}$

(b) $\text{LHS} = \tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right)$

$= \dfrac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}} \times \dfrac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta\tan\frac{\pi}{4}}$

$= \dfrac{\tan\theta + 1}{1 - \tan\theta} \times \dfrac{\tan\theta - 1}{1 + \tan\theta} = -1 = \text{RHS}$

(c) $\text{LHS} = (\sec^2\theta - 1)\tan\left(\frac{\pi}{2} - \theta\right)$

$= \tan^2\theta\cot\theta = \tan\theta = \text{RHS}$

(d) $\text{LHS} = \tan\left(x + \frac{3\pi}{4}\right)$

$= \dfrac{\tan x + \tan\frac{3\pi}{4}}{1 - \tan x\tan\frac{3\pi}{4}} = \dfrac{\tan x - 1}{\tan x + 1} = \text{RHS}$

(e) $\text{LHS} = \dfrac{1 - \sin\left(\frac{\pi}{2} - 2x\right)}{\sin 2x}$

$= \dfrac{1 - \cos 2x}{\sin 2x}$

$= \dfrac{2\sin^2 x}{2\sin x\cos x} = \dfrac{\sin x}{\cos x} = \tan x = \text{RHS}$

(f) $\text{LHS} = \tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)$

$= \frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}$

$= \frac{(1+\tan A)^2+(1-\tan A)^2}{1-\tan^2 A}$

$= \frac{2(1+\tan^2 A)}{1-\tan^2 A}$

$= \frac{2}{\cos 2A}$ (adapting t formula)

$= 2\sec 2A = RHS$

33 $\tan(x-y) = \frac{\tan x-\tan y}{1+\tan x\tan y} = \frac{\frac{5}{4}-\frac{1}{9}}{1+\frac{5}{4}\times\frac{1}{9}} = \frac{45-4}{36+5} = 1$

$x-y = \frac{\pi}{4}$

34 (a) -1 (b) $-\frac{1}{2}$

35 $\text{LHS} = \sin\left(\theta+\frac{\pi}{6}\right)\sin\left(\theta-\frac{\pi}{6}\right)$

$= \left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)\left(\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}\right)$

$= \left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)\left(\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta\right) = \frac{3}{4}\sin^2\theta-\frac{1}{4}\cos^2\theta$

$= \sin^2\theta-\frac{1}{4}\left(\sin^2\theta+\cos^2\theta\right) = \sin^2\theta-\frac{1}{4} = \text{RHS}$

36 $\text{RHS} = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

$= \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{LHS}$

$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} = \sqrt{\frac{\left(\sqrt{2}-1\right)^2}{2-1}} = \sqrt{2}-1$

37 (a) $2\sin A\cos B$ (b) $2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

38 $\sec\theta-\tan\theta = \frac{3}{5}$

$\frac{1+t^2}{1-t^2}-\frac{2t}{1-t^2} = \frac{3}{5}$

$\frac{(1-t)^2}{1-t^2} = \frac{3}{5}$

$\frac{1-t}{1+t} = \frac{3}{5}$

$5-5t = 3+3t$

$t = \frac{1}{4}$

$\sin\theta = \frac{2t}{1+t^2} = \frac{\frac{1}{2}}{1+\frac{1}{16}} = \frac{8}{17}$

39 $4\tan(\alpha-\beta) = 3\tan\alpha$

$\frac{4(\tan\alpha-\tan\beta)}{1+\tan\alpha\tan\beta} = 3\tan\alpha$

$4\tan\alpha-4\tan\beta = 3\tan\alpha+3\tan^2\alpha\tan\beta$

$\left(3\tan^2\alpha+4\right)\tan\beta = \tan\alpha$

$\tan\beta = \frac{\tan\alpha}{3\tan^2\alpha+4} = \frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{3\sin^2\alpha}{\cos^2\alpha}+4}$

$= \frac{\sin\alpha\cos\alpha}{3\sin^2\alpha+4\cos^2\alpha} = \frac{\sin 2\alpha}{6+2\cos^2\alpha} = \frac{\sin 2\alpha}{7+\cos 2\alpha}$

40 $\text{LHS} = \cos^6\theta-\sin^6\theta$

$= \left(\cos^3\theta-\sin^3\theta\right)\left(\cos^3\theta+\sin^3\theta\right)$

$= (\cos\theta-\sin\theta)(\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta)(\cos\theta+\sin\theta)$

$\times(\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta)$

$= (\cos^2\theta-\sin^2\theta)(1+\sin\theta\cos\theta)(1-\sin\theta\cos\theta)$

$= \cos 2\theta(1-\sin^2\theta\cos^2\theta) = \left(1-\frac{1}{4}\sin^2 2\theta\right)\cos 2\theta = \text{RHS}$

41 1 or $\frac{1}{k}$

42 $B-A = C-B$ and $A = 2B-C$

$\cos(B-A) = \cos(C-B)$

$\cos(B-A) = \cos C\cos B+\sin C\sin B$

$2\sin B\cos(B-A) = 2\sin B\cos C\cos B+2\sin B\sin C\sin B$

$2\sin B\cos(B-A) = \sin 2B\cos C+2\sin^2 B\sin C$

$2\sin B\cos(B-A) = \sin 2B\cos C+(1-\cos 2B)\sin C$

$2\sin B\cos(B-A) = \sin 2B\cos C-\cos 2B\sin C+\sin C$

$2\sin B\cos(B-A) = \sin(2B-C)+\sin C$

$2\sin B\cos(B-A) = \sin A+\sin C$

43 $\frac{2lm}{l^2-m^2}, \frac{4lm(l^2-m^2)}{(l^2+m^2)^2}$

44 $\text{LHS} = (k-1)\sin(\alpha+\beta)$

$= k(\sin\alpha\cos\beta+\cos\alpha\sin\beta)-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

$= k(\sin\alpha\cos\beta-\cos\alpha\sin\beta+2\cos\alpha\sin\beta)+(\sin\alpha\cos\beta$

$-\cos\alpha\sin\beta-2\sin\alpha\cos\beta)$

$= k\sin(\alpha-\beta)+2k\cos\alpha\sin\beta+\sin(\alpha-\beta)-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta)+\frac{2\tan\alpha}{\tan\beta}\cos\alpha\sin\beta-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta)+2\sin\alpha\cos\beta-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta) = \text{RHS}$

45 $\text{LHS} = 4\sin\theta\sin\left(\theta-\frac{\pi}{3}\right)\sin\left(\theta-\frac{2\pi}{3}\right)$

$= 4\sin\theta\left(\sin\theta\cos\frac{\pi}{3}-\cos\theta\sin\frac{\pi}{3}\right)\left(\sin\theta\cos\frac{2\pi}{3}-\cos\theta\sin\frac{2\pi}{3}\right)$

$= 4\sin\theta\left(\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)\left(-\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)$

$= 4\sin\theta\left(-\frac{1}{4}\sin^2\theta+\frac{3}{4}\cos^2\theta\right) = \sin\theta\left(3\cos^2\theta-\sin^2\theta\right)$

$\text{RHS} = \sin 3\theta = \sin(2\theta+\theta)$

$= \sin 2\theta\cos\theta+\cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)\sin\theta$

$= \sin\theta(3\cos^2\theta-\sin^2\theta) = \text{LHS}$

EXERCISE 4.5

1 C

2 $\sin 6\theta+\sin 2\theta$

3 $\cos 8A+\cos 2A$

4 $\frac{1}{2}(\sin 6A-\sin 2A)$

5 $\frac{1}{2}(\sin 2\theta+\sin 2\alpha)$

6 $1-\sin 2A$

7 $\frac{1}{2}(\cos(4\theta)+\cos(2\alpha))$

8 $\sin(4\alpha+\beta)-\sin(2\alpha+3\beta)$

9 $\cos\theta-\cos 3\theta$

10 $\frac{1}{2}(\cos 6\theta-\cos 2\theta)$

11 $\frac{1}{2}(\sin 3x+\sin y)$

12 $1+\frac{\sqrt{3}}{2}$

13 $\frac{1}{2}(\cos 2\beta-\cos 2A)$

14 $\frac{\sqrt{3}-1}{4}$

15 $\sin 2\theta+\sin 2\phi$

16 $\frac{1}{2}(\sin A+\sin B)$

17 $\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\cos 50°\right)$

18 $\sin 4\theta+\sin 2\theta$

19 $\frac{1}{2}(\sin(3\theta+3\phi)+\sin(\theta-\phi))$

20 $\sin 6A-\sin 2A$

21 $\sin 10A+\sin 4A$

22 D

23 $2\cos 2x\sin x$

24 $2\cos\left(x+\frac{\alpha}{2}\right)\sin\frac{\alpha}{2}$

25 $-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

26 $2\sin\theta\cos\alpha$

27 $2\cos\frac{\theta}{2}\cos\frac{\alpha}{2}$

28 $-2\sin(A+C)\sin B$

29 $2\cos\left(\frac{3x+3y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

30 $2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

31 $2\cos 2A\cos A$

32 0

33 $-2\sin\left(\frac{3A+3B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

34 $2\cos A\cos(B+C)$

35 $2\sin\frac{5x}{2}\cos\frac{x}{2}$

36 $2\cos 2A\sin 2B$

37 $-\frac{\sqrt{2}}{2}$

38 $2\cos\left(\frac{\theta-\phi}{2}+45°\right)\sin\left(\frac{\theta+\phi}{2}-45°\right)$

39 $-\sqrt{3}\sin 15°$

40 $\sqrt{3}\cos 10°$

41 $-2\cos\left(\frac{2A-B}{2}\right)\sin\frac{B}{2}$

42 $-\sqrt{3}\sin 20°$

43 $\text{LHS} = \frac{\sin 5\theta-\sin 3\theta}{\cos 5\theta+\cos 3\theta}$
$= \frac{2\cos 4\theta\sin\theta}{2\cos 4\theta\cos\theta}$
$= \tan\theta$

44 $\text{LHS} = \frac{\sin x+\sin y}{\cos x+\cos y}$
$= \frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$
$= \tan\left(\frac{x+y}{2}\right)$

45 $\text{LHS} = \frac{\cos x-\cos 3x}{\sin x-\sin 3x}$
$= \frac{-2\sin 2x\sin(-x)}{2\cos 2x\sin(-x)}$
$= -\tan 2x$

46 $\text{LHS} = \frac{\sin 2A-\sin 2B}{\sin 2A+\sin 2B}$
$= \frac{2\cos(A+B)\sin(A-B)}{2\sin(A+B)\cos(A-B)}$
$= \frac{\tan(A-B)}{\tan(A+B)}$

47 $\text{LHS} = \frac{\cos(\theta+\alpha)-\cos(\theta-\alpha)}{\sin(\theta+\alpha)+\sin(\theta-\alpha)}$
$= \frac{-2\sin\theta\sin\alpha}{2\sin\theta\cos\alpha}$
$= -\tan\alpha$

48 $\text{LHS} = \frac{\sin x+\sin(x+y)+\sin(x+2y)}{\cos x+\cos(x+y)+\cos(x+2y)} = \tan(x+y)$
$= \frac{\sin(x+2y)+\sin x+\sin(x+y)}{\cos(x+2y)+\cos x+\cos(x+y)}$
$= \frac{2\sin(x+y)\cos y+\sin(x+y)}{2\cos(x+y)\cos y+\cos(x+y)}$
$= \frac{\sin(x+y)(2\cos y+1}{\cos(x+y)(2\cos y+1)}$
$= \tan(x+y)$ for $2\cos y+1\neq 0$

49 $\text{LHS} = \sin 5\theta+\sin 3\theta-2\sin 2\theta\cos\theta$
$= 2\sin 4\theta\cos\theta-2\sin 2\theta\cos\theta$
$= 2\cos\theta(\sin 4\theta-\sin 2\theta)$
$= 2\cos\theta\times 2\cos 3\theta\sin\theta$
$= 2\sin 2\theta\cos 3\theta$

50 $\text{LHS} = \sin 2x+\sin 4x+\sin 6x$
$= \sin 4x+\sin 2x+\sin 6x$
$= 2\sin 3x\cos x+2\sin 3x\cos 3x$
$= 2\sin 3x(\cos 3x+\cos x)$
$= 2\sin 3x\times 2\cos 2x\cos x$
$= 4\cos x\cos 2x\sin 3x$

51 $\text{LHS} = \sin^2 5\theta-\sin^2 3\theta$
$= (\sin 5\theta+\sin 3\theta)(\sin 5\theta-\sin 3\theta)$
$= 2\sin 4\theta\cos\theta\times 2\cos 4\theta\sin\theta$
$= 2\sin 4\theta\,2\cos 4\theta\times 2\sin\theta\cos\theta$
$= \sin 8\theta\sin 2\theta$

52 $\text{LHS} = \sin 35°-\sin 25°$
$= 2\cos 60°\sin 5°$
$= \sqrt{3}\sin 5°$

53 $\text{LHS} = \cos 35°+\cos 45°+\cos 75°+\cos 85°$
$= \cos 75°+\cos 45°+\cos 85°+\cos 35°$
$= 2\cos 60°\cos 15°+2\cos 60°\cos 25°$
$= \cos 25°+\cos 15°$
$= 2\cos 20°\cos 5°$

54 $\text{LHS} = 2\cos 37.5°\sin 7.5°$
$= \sin 45°-\sin 30°$
$= \frac{1}{\sqrt{2}}-\frac{1}{2}$
$= \frac{\sqrt{2}-1}{2}$

55 $\text{LHS} = \sin 25°\sin 35°-\sin 20°\sin 10°$
$= \frac{1}{2}(\cos(-10°)-\cos 60°)-\frac{1}{2}(\cos 10°-\cos 30°)$
$= \frac{1}{2}(\cos 10°-\frac{1}{2}-\cos 10°+\frac{\sqrt{3}}{2})$
$= \frac{\sqrt{3}-1}{4}$

56 $\text{LHS} = \sin 10°+\cos 40°$
$= \sin 50°+\sin 10°$
$= 2\sin 30°\cos 20°$
$= \cos 20°$
$= \sin 70°$

57 $\text{LHS} = \frac{\sin 48°+\sin 12°}{\cos 48°+\cos 12°}$
$= \frac{2\sin 30°\cos 18°}{2\cos 30°\cos 18°}$
$= \frac{1}{\sqrt{3}}$
$= \frac{\sqrt{3}}{3}$

58 $\text{LHS} = \frac{\sin 3x-\sin x}{\cos x-\cos 3x}$
$= \frac{2\cos 2x\sin x}{-2\sin 2x\sin(-x)}$
$= \frac{\cos 2x\sin x}{\sin 2x\sin x}$
$= \cot 2x$

59 $\text{LHS} = \cos(A+B)\cos(A-B)$
$= \frac{1}{2}(\cos 2A+\cos 2B)$
$= \frac{1}{2}(2\cos^2 A-1+1-2\sin^2 B)$
$= \cos^2 A-\sin^2 B$

60 $\text{LHS} = \frac{\sin\theta+\sin 7\theta}{\sin 3\theta+\sin 5\theta}$
$= \frac{2\sin 4\theta\cos 3\theta}{2\sin 4\theta\cos\theta}$
$= \frac{\cos 3\theta}{\cos\theta}$
$= \frac{\cos 2\theta\cos\theta-\sin 2\theta\sin\theta}{\cos\theta}$
$= \cos 2\theta-\frac{2\sin\theta\cos\theta\sin\theta}{\cos\theta}$
$= \cos 2\theta-2\sin^2\theta$
$= \cos 2\theta+\cos 2\theta-1$
$= 2\cos 2\theta-1$

61 $\text{LHS} = \frac{\sin(\theta+\phi)-\sin(\theta-\phi)}{\cos(\theta+\phi)-\cos(\theta-\phi)}$
$= \frac{2\cos\theta\sin\phi}{-2\sin\theta\sin\phi}$
$= -\cot\theta$

62 $\text{LHS} = \frac{\cos 75°+\cos 15°}{\sin 75°-\sin 15°}$
$= \frac{2\cos 45°\cos 30°}{2\cos 45°\sin 30°}$
$= \cot 30°$
$= \sqrt{3}$

63 $\text{LHS} = \dfrac{\sin A+\sin(A+B)+\sin(A+2B)}{\cos A+\cos(A+B)+\cos(A+2B)}$

$= \dfrac{2\sin(A+B)\cos(-B)+\sin(A+B)}{2\cos(A+B)\cos(-B)+\cos(A+B)}$

$= \dfrac{\sin(A+B)(2\cos B+1)}{\cos(A+B)(2\cos B+1)}$

$= \tan(A+B)$

64 $\text{LHS} = \dfrac{\sin(n+1)\theta+2\sin n\theta+\sin(n-1)\theta}{\cos(n-1)\theta-\cos(n+1)\theta}$

$= \dfrac{2\sin n\theta\cos\theta+2\sin n\theta}{-2\sin n\theta\sin(-\theta)}$

$= \dfrac{\cos\theta+1}{\sin\theta}$ [use $\cos\theta = 2\cos^2\frac{\theta}{2}-1$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$]

$= \dfrac{2\cos^2\frac{\theta}{2}-1+1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}$

65 $\alpha+\beta+\gamma=\pi, (\alpha+\beta)=\pi-\gamma$

$\text{LHS} = \sin 2\alpha+\sin 2\beta+\sin 2\gamma$

$= 2\sin(\alpha+\beta)\cos(\alpha-\beta)+\sin 2\gamma$

$= 2\sin(\pi-\gamma)\cos(\alpha-\beta)+\sin 2\gamma$

$= 2\sin\gamma\cos(\alpha-\beta)+2\sin\gamma\cos\gamma$

$= 2\sin\gamma(\cos(\alpha-\beta)+\cos\gamma)$

$= 2\sin\gamma\left(2\cos\left(\dfrac{\alpha-\beta+\gamma}{2}\right)\cos\left(\dfrac{\alpha-\beta-\gamma}{2}\right)\right)$

$= 4\sin\gamma\cos\left(\dfrac{\pi-2\beta}{2}\right)\cos\left(\dfrac{2\alpha-\pi}{2}\right)$

$= 4\sin\gamma\cos\left(\frac{\pi}{2}-\beta\right)\cos\left(-\frac{\pi}{2}+\alpha\right)$

$= 4\sin\gamma\sin\beta\cos\left(\frac{\pi}{2}-\alpha\right)$

$= 4\sin\alpha\sin\beta\sin\gamma.$

EXERCISE 4.6

1 (a) $x=\frac{\pi}{2}$ (b) $x=\frac{\pi}{2}, \frac{3\pi}{2}$ (c) $x=\frac{3\pi}{4}, \frac{7\pi}{4}$ (d) $x=\frac{\pi}{3}, \frac{2\pi}{3}$
(e) $x=\frac{2\pi}{3}, \frac{4\pi}{3}$ (f) $x=\frac{\pi}{6}, \frac{7\pi}{6}$
(g) $x-\frac{\pi}{6}=-\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
$x=0, \frac{4\pi}{3}, 2\pi$
(h) $x==0$ (i) $x=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (j) $x=0.4, 2.74$

2 C

3 (a) $x=0$ (b) $x=-\pi, 0, \frac{\pi}{2}, \pi$ (c) $x=-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$
(d) $x=-\frac{\pi}{2}, 0, \frac{\pi}{2}$ (e) $x=-\pi, \pi$ (f) $x=-\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$
(g) $x=-\frac{2\pi}{3}, 0, \frac{2\pi}{3}$ (h) $x=\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$
(i) $x=-\pi, -\frac{3\pi}{4}, 0, \frac{\pi}{4}, \pi$

4 D

5 (a) $\theta=\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ (b) $\theta=0, \pi, 2\pi$
(c) $x=\frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$ (d) $\theta=0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$
(e) $\theta=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (f) $\theta=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
(g) $x=0, \frac{\pi}{6}, \frac{\pi}{2}, \pi, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$

6 (a) $\theta=-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ (b) $\theta=-\frac{\pi}{3}, \frac{\pi}{3}$ (c) $\theta=0, \pm\pi$
(d) $\theta=-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ (e) $\theta=-\frac{11\pi}{12}, -\frac{3\pi}{4}, \frac{\pi}{12}, \frac{\pi}{4}$
(f) $\theta=-\frac{2\pi}{3}, 0, \frac{2\pi}{3}$

7 (a) $x=30°, 150°, 221.81°, 318.19°$
(b) $x=45°, 225°$
(c) $x=30°, 150°$

8 (a) $\theta=0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$ (b) $\theta=0, \pi, 2\pi$
(c) $\theta=\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ (d) $\theta=0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$
(e) $x=0, 2\pi, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

EXERCISE 4.7

1 (a) $\frac{\pi}{3}, \frac{2\pi}{3}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (c) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (d) $\frac{\pi}{6}, \frac{7\pi}{6}$
(e) $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$ (f) $\frac{7\pi}{6}, \frac{11\pi}{6}$
(g) $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ (h) $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

2 (a) $66°25', 293°35'$
(b) $71°34', 161°34', 251°34', 341°34'$

3 C **4** (a) $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$ (b) $-\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$

5 (a) $0, \frac{\pi}{2}, 2\pi$ (b) $0, \pi, 2\pi$ (c) $0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$
(d) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ (e) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (f) $\frac{\pi}{12}, \frac{\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}$
(g) $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ (h) $\frac{\pi}{4}, \frac{5\pi}{4}$

6 (a) $128°12', 171°48', 308°12', 351°48'$
(b) $8°26', 151°34', 188°26', 331°34'$

7 D

8 (a) $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ (b) $\frac{\pi}{3} < x < \frac{5\pi}{3}$ (c) $0 < x < \pi$
(d) $0 \le x \le \frac{\pi}{6}, \frac{11\pi}{6} < x \le 2\pi$ (e) $0 \le x \le 2\pi, x \ne \frac{\pi}{2}$
(f) $0 \le x < \frac{\pi}{2}, \frac{3\pi}{2} < x \le 2\pi$ (g) $\frac{\pi}{4} < x < \frac{\pi}{2}, \frac{5\pi}{4} < x < \frac{3\pi}{2}$
(h) $0 \le x < \frac{3\pi}{4}, \frac{5\pi}{4} < x \le 2\pi$

9 (a) $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$ (b) $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

10 (a) $-\pi < x \le -\frac{5\pi}{6}, -\frac{2\pi}{3} \le x \le \frac{\pi}{6}, \frac{\pi}{3} \le x < \pi$
(b) $-\pi < x < -\frac{2\pi}{3}, -\frac{\pi}{3} < x < \frac{\pi}{3}, \frac{2\pi}{3} < x < \pi$

EXERCISE 4.8

1 (a) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$ (c) $0, \pi, 2\pi$
(d) $45°, 63°26', 225°, 243°26'$

2 C **3** (a) $\frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$ (b) $-\pi, -\frac{5\pi}{6}, 0, \frac{\pi}{6}, \pi$

4 C **5** $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

6 $128°10', 231°50'$ **7** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **8** $0, \pi, 2\pi$

9 (a) $\theta=0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi.$
(b) $\theta=0, \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}, \pi$
(c) $\theta=0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$

10 (a) $x=0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}$ (b) $x=\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$
(c) $x=0.39, 2.23, 3.53, 5.37$ (d) $x=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

11 (a) $\theta=\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ (b) $\theta=\frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}$
(c) $\theta=0.0205, 2.074$ (d) $\theta=0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi$

12 Solve for $0 \le x \le 2\pi$, the equations:
(a) $x=0, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ (b) $x=0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{3\pi}{2}, \frac{4\pi}{3}, 2\pi$
(c) $x=\frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$ (d) $x=0, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, 2\pi$

CHAPTER REVIEW 4

1 (a) $\cos\theta$ (b) $\tan\left(\theta-\frac{\pi}{6}\right)$

(c) $\text{LHS} = \dfrac{\sin 2\theta-\sin\theta}{\cos 2\theta-\cos\theta+1}$

$= \dfrac{2\sin\theta\cos\theta-\sin\theta}{2\cos^2\theta-1-\cos\theta+1}$

$= \dfrac{\sin\theta(2\cos\theta-1)}{2\cos^2\theta-\cos\theta}$

$= \dfrac{\sin\theta(2\cos\theta-1)}{\cos\theta(2\cos\theta-1)}$

$= \tan\theta$

2 $13°17', 103°17', 193°17', 283°17'$

3 (a) $\sin\theta$ (b) $\tan\theta$

(c) $\text{Exp} = \frac{1}{2}\sin 2x\cos 2x\cos 4x$

$= \frac{1}{4}\sin 4x\cos 4x$

$= \frac{\sin 8x}{8}$

4 (a) RHS $= \cos A\cos B\left(1-\frac{\sin A}{\cos A}\times\frac{\sin B}{\cos B}\right)$

$= \cos A\cos B\left(\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}\right)$

$= \cos(A+B) = \text{LHS}$

(b) $\cos(A+B)=0$ so $A+B=\frac{\pi}{2}$

5 (a) LHS $= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$

$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$

$= \frac{1-\sin\theta}{\cos\theta} = \sec\theta-\tan\theta = \text{RHS}$

(b) RHS $= \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta = \text{LHS}$

6 $\tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°}$

$1\times(1-\tan^2 22.5°) = 2\tan 22.5°$

$1-\tan^2 22.5° = 2\tan 22.5°$

$\tan^2 22.5° + 2\tan 22.5° - 1 = 0$

$\tan 22.5° = \frac{-2\pm\sqrt{4+4}}{2}$

$\tan 22.5° = -1\pm\sqrt{2}$

Since 22.5° is an acute angle, $\tan 22.5° = \sqrt{2}-1$

Now $\tan 22.5° = \frac{2\tan 11.25°}{1-\tan^2 11.25°}$

$(\sqrt{2}-1)(1-\tan^2 11.25°) = 2\tan 11.25°$

$(\sqrt{2}-1)\tan^2 11.25° + 2\tan 11.25° + 1 - \sqrt{2} = 0$

$\tan 11.25° = \frac{-2\pm\sqrt{4-4\left(\sqrt{2}-1\right)\left(1-\sqrt{2}\right)}}{2\left(\sqrt{2}-1\right)}$

$\tan 11.25° = \frac{-1\pm\sqrt{1+\left(\sqrt{2}-1\right)^2}}{\left(\sqrt{2}-1\right)}$

$\tan 11.25° = \frac{-1\pm\sqrt{4-2\sqrt{2}}}{\left(\sqrt{2}-1\right)}$

Since 11.25° is an acute angle, $\tan 11.25° = \frac{\sqrt{4-2\sqrt{2}}-1}{\left(\sqrt{2}-1\right)}$

7 (a) $x=0, \frac{\pi}{2}, \pi$

(b) $x=\frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ or $x=\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

8 (a) $x=-\frac{\pi}{2}, 0$

(b) $x=-\frac{5\pi}{6}, -\pi, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \pi, \frac{5\pi}{6}$

(c) $x=-\frac{2\pi}{3}, 0$

CHAPTER 5

EXERCISE 5.1

1 A, B, D, E, H

2 (a) $-2\le x\le 0$ (b) $x\ge 2$ (c) $x<-2, x>-2$ (d) $x\le 1.5$ (e) $x\ge -3$

3 (a) $f^{-1}(x)=\frac{1}{2}x+2$, all real x, all real y

(b) $f^{-1}(x)=\sqrt{x+1}$, $x\ge -1$, $y\ge 0$

(c) $g^{-1}(x)=x^2+3$, $x\ge 0$, $y\ge 3$

(d) $f^{-1}(x)=-\sqrt{9-x^2}$, $0\le x\le 3$, $-3\le y\le 0$

(e) $f^{-1}(x)=x^{\frac{1}{3}}$, all real x, all real y

(f) $f^{-1}(x)=-\sqrt{x}-2$, $x\ge 0$, $y\le -2$

(g) $f^{-1}(x)=-1+\sqrt{x+1}$, $x\ge 0$, $y\ge 0$

(h) $f^{-1}(x)=e^x-1$, all real x, $y>-1$

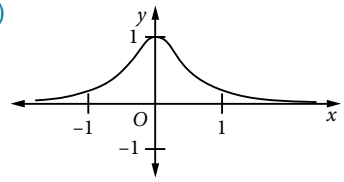
(i) $f^{-1}(x)=x^2-4x+6$, $x\le 2$, $y\ge 2$

(j) $g^{-1}(x)=-x^2-2x+4$, $x\ge -1$, $y\le 5$

(k) $f^{-1}(x)=-\log_2 x$, $0<x<1$, $y>0$

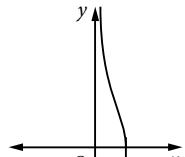
(l) $h^{-1}(x)=\frac{1}{x}-1$, $x>0$, $y>-1$

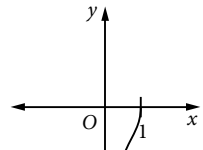
4 (a)



(b) $x\ge 0, x\le 0$

(c) $y=\sqrt{\frac{1}{x}-1}$, $0<x\le 1$, $y\ge 0$ or $y=-\sqrt{\frac{1}{x}-1}$, $0<x\le 1$, $y\le 0$





5 (a) not one-to-one, $-a\le x\le 0$, $0\le x\le a$

(b) not one-to-one, $x\le 0$, $x\ge 0$

(c) not one-to-one, $x<0$, $x>0$

6 (a) $f(g(x)) = f\left(\frac{1}{2}(x+1)\right) = 2\times\frac{1}{2}(x+1)-1 = x$,

$g(f(x)) = g(2x-1) = \frac{1}{2}((2x-1)+1) = x$

(b) $f(g(x)) = f(\log_e 2x) = \frac{1}{2}e^{\log_e 2x} = \frac{1}{2}\times 2x = x$,

$g(f(x)) = g\left(\frac{1}{2}e^x\right) = \log_e\left(2\times\frac{1}{2}e^x\right) = x$

(c) $f(g(x)) = f\left(-\sqrt{16-x^2}\right) = \sqrt{16-\left(-\sqrt{16-x^2}\right)^2}$

$= \sqrt{x^2} = x$ as $0\le x\le 4$

$g(f(x)) = g\left(\sqrt{16-x^2}\right) = -\sqrt{16-\left(\sqrt{16-x^2}\right)^2}$

$= -\sqrt{x^2} = -(-x) = x$ as $-4\le x\le 0$

(d) $f(g(x)) = f\left(1+\sqrt{1-x}\right)$

$= 2\times\left(1+\sqrt{1-x}\right)-\left(1+\sqrt{1-x}\right)^2$

$= 2+2\sqrt{1-x}-\left(1+2\sqrt{1-x}+1-x\right) = x$,

$g(f(x)) = g\left(2x-x^2\right) = 1+\sqrt{1-\left(2x-x^2\right)}$

$= 1+\sqrt{(x-1)^2} = x$ as $x\ge 1$

(e) $f(g(x)) = f\left(\frac{x+1}{2x}\right) = \frac{1}{2\times\frac{x+1}{2x}-1}$

$= \frac{1}{\frac{x+1-x}{x}} = x$,

$g(f(x)) = g\left(\frac{1}{2x-1}\right) = \frac{\frac{1}{2x-1}+1}{2\times\left(\frac{1}{2x-1}\right)} = \frac{\frac{1+2x-1}{2x-1}}{\frac{2}{2x-1}} = x$

7 D

8 (a) $\frac{dy}{dx} = 3x^2-9$ $\therefore$ stationary points at $x=\pm\sqrt{3}$

$\frac{d^2y}{dx^2} = 6x$:

Where $x=\sqrt{3}$, $\frac{d^2y}{dx^2} = 6\sqrt{3}$ which is positive

$\therefore$ max turning point at $(\sqrt{3}, 0)$

Where $x = -\sqrt{3}$, $\frac{d^2y}{dx^2} = -6\sqrt{3}$ which is negative

$\therefore$ min turning point at $(-\sqrt{3}, 12\sqrt{3})$

(b) As there is a turning point at $x = \sqrt{3}$ on the x-axis, $x = \sqrt{3}$ is a double zero of the polynomial.

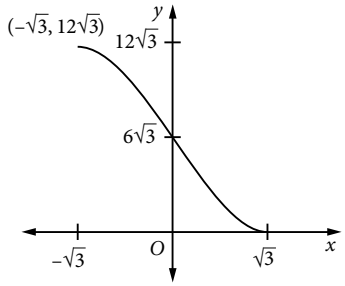
Let the zeros of the polynomial be $\sqrt{3}$, $\sqrt{3}$ and α

Using sum of roots $= -\frac{b}{a}$: $2\sqrt{3} + \alpha = 0$

$\therefore\ \alpha = -2\sqrt{3}$

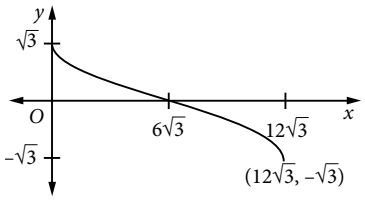
$\therefore$ other x-intercept is at $(-2\sqrt{3}, 0)$

(c)



(d) $0 \le x \le 12\sqrt{3}$, $-\sqrt{3} \le y \le \sqrt{3}$

(e)



(f) For f^{-1} the equation is $x = y^3 - 9y + 6\sqrt{3}$, but we can't make y the subject so we can't find $\frac{dy}{dx}$. However, we can find $\frac{dx}{dy} = 3y^2 - 9$, so $\frac{dy}{dx} = \frac{1}{3 \times y^2 - 9}$.

At the point $(6\sqrt{3}, 0)$: $\frac{dy}{dx} = \frac{1}{3 \times 0^2 - 9} = -\frac{1}{9}$

Thus, equation of tangent: $y - 0 = -\frac{1}{9}(x - 6\sqrt{3})$

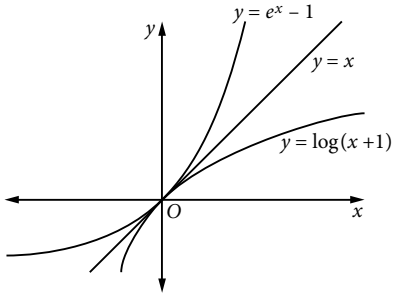
or $x + 9y = 6\sqrt{3}$

9 (a) $\frac{dy}{dx} = e^x$ which is positive for all x

$\therefore$ the function is increasing for all x

(b) Where $x = 0$, $\frac{dy}{dx} = e^0 = 1$

(c) From (b) we see that $y = x$ is a tangent to the curve at $(0, 0)$:



(d) Equation of f^{-1} is $x = e^y - 1$, which becomes $y = \log_e(x + 1)$

(e) From the graph of (c), $\log_e(x + 1) < x$ for $-1 < x < 0$, $x > 0$

EXERCISE 5.2

1 B

2 (a) $\frac{\pi}{2}$ (b) 0 (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{3}$ (e) $-\frac{\pi}{3}$ (f) 0 (g) $\frac{\pi}{2}$ (h) π
(i) $\frac{\pi}{3}$ (j) $\frac{5\pi}{6}$ (k) $\frac{\pi}{4}$ (l) $-\frac{\pi}{4}$ (m) $\frac{\pi}{3}$ (n) $-\frac{\pi}{6}$ (o) $\frac{\pi}{4}$ (p) 80°
(q) $\frac{\pi}{3}$ (r) $-\frac{\pi}{4}$ (s) $\frac{\pi}{3}$ (t) $\frac{2\pi}{3}$ (u) $\frac{\pi}{3}$ (v) $\frac{\pi}{6}$

3 (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{5}{13}$ (c) 65° (d) 180° (e) $\frac{1}{2}$ (f) $-\frac{1}{2}$
(g) $-\frac{119}{169}$ (h) $\frac{3\sqrt{2}}{4}$

4 (a) $0 < \tan^{-1}\frac{3}{5} < \tan^{-1}4 < \frac{\pi}{2}$, so $0 < \tan^{-1}4 - \tan^{-1}\frac{3}{5} < \frac{\pi}{2}$

Now $\tan(\tan^{-1}4 - \tan^{-1}\frac{3}{5})$

$$= \frac{\tan(\tan^{-1}4) - \tan(\tan^{-1}\frac{3}{5})}{1 + \tan(\tan^{-1}4)\tan(\tan^{-1}\frac{3}{5})} = \frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}} = 1$$

$\therefore\ \tan^{-1}4 - \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$

(b) $\sin^{-1}\frac{3}{5} + \sin^{-1}(-\frac{3}{5}) = \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{3}{5} = 0$

(because $\sin^{-1}x$ is an odd function)

(c) $0 < \tan^{-1}\frac{5}{12} < \frac{\pi}{2}$ and $0 < \cos^{-1}\frac{5}{13} < \frac{\pi}{2}$,

so $0 < \tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13} < \pi$

Now $\cos(\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13})$

$= \cos(\tan^{-1}\frac{5}{12})\cos(\cos^{-1}\frac{5}{13}) - \sin(\tan^{-1}\frac{5}{12})\sin(\cos^{-1}\frac{5}{13}) = 0$

$\therefore\ \tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13} = \frac{\pi}{2}$

(d) $0 < \cos^{-1}\frac{3}{5} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{3}{4} < \frac{\pi}{2}$,

so $0 < \cos^{-1}\frac{3}{5} + \tan^{-1}\frac{3}{4} < \pi$

Now $\cos(\cos^{-1}\frac{3}{5} - \tan^{-1}(-\frac{3}{4}))$

$= \cos(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{3}{4})$

$= \cos(\cos^{-1}\frac{3}{5})\cos(\tan^{-1}\frac{3}{4}) - \sin(\cos^{-1}\frac{3}{5})\sin(\tan^{-1}\frac{3}{4}) = 0$

$\therefore\ \cos^{-1}\frac{3}{5} - \tan^{-1}(-\frac{3}{4}) = \frac{\pi}{2}$

(e) $0 < \sin^{-1}\frac{3}{5} < \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$, so $0 < 2\sin^{-1}\frac{3}{5} < \frac{\pi}{2}$

Now $\sin(2\sin^{-1}\frac{3}{5}) = 2\sin(\sin^{-1}\frac{3}{5})\cos(\sin^{-1}\frac{3}{5})$

$= 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

$\therefore\ 2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25}$

(f) $0 < \sin^{-1}\frac{3}{5} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{7}{24} < \frac{\pi}{2}$,

so $0 < \sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24} < \pi$

Now $\cos(\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24}) = \cos(\sin^{-1}\frac{3}{5})$

$\cos(\tan^{-1}\frac{7}{24}) - \sin(\sin^{-1}\frac{3}{5})\sin(\tan^{-1}\frac{7}{24}) = \frac{3}{5}$

$\therefore\ \sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{3}{5}$

(g) $0 < \sin^{-1}\frac{5}{13} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{16}{63} < \frac{\pi}{2}$,

so $0 < \sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63} < \pi$

Now $\cos(\sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63})$

$= \cos(\sin^{-1}\frac{5}{13})\cos(\tan^{-1}\frac{16}{63}) - \sin(\sin^{-1}\frac{5}{13})\sin(\tan^{-1}\frac{16}{63}) = \frac{4}{5}$

$\therefore\ \sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63} = \cos^{-1}\frac{4}{5}$

5 (a) 0 (b) 0 (c) $\frac{16}{65}$ (d) $\frac{24}{25}$
(e) $\frac{63}{65}$ (f) $\frac{7}{25}$ (g) $-\frac{88}{9}$

6 (a) Let $\alpha = \sin^{-1}x$ where $0 \le x \le 1$

$\therefore\ 0 \le \alpha \le \frac{\pi}{2}$ and $\sin\alpha = x = \frac{x}{1}$

By Pythagoras' theorem: $\cos\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore\ \alpha = \cos^{-1}\sqrt{1-x^2}$ and so $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

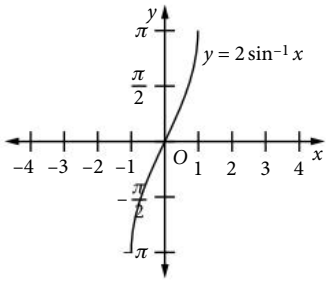
(b) Let $\alpha = \cos^{-1}x$ where $0 \le x \le 1$

$\therefore\ 0 \le \alpha \le \frac{\pi}{2}$ and $\cos\alpha = x = \frac{x}{1}$

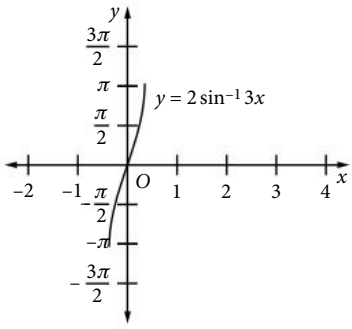
By Pythagoras' theorem: $\sin\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore\ \alpha = \sin^{-1}\sqrt{1-x^2}$ and so $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

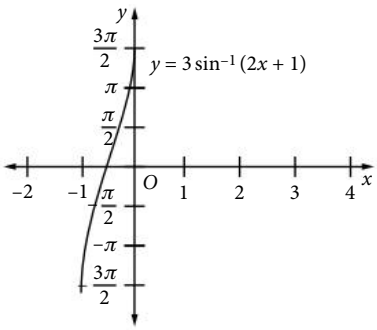
7 (a) $-1 \le x \le 1, -\pi \le y \le \pi$



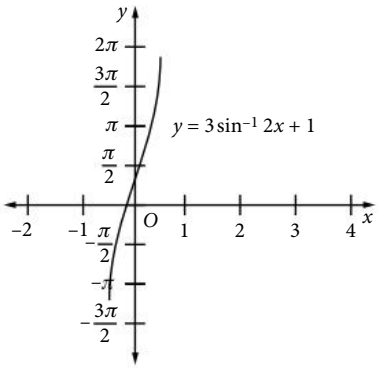
(b) $-\frac{1}{3} \le x \le \frac{1}{3}, -\pi \le y \le \pi$



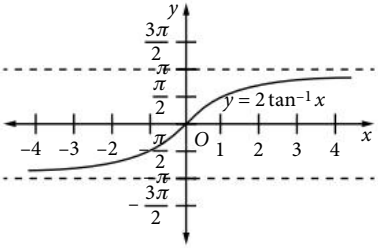
(c) $-1 \le x \le 0, -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



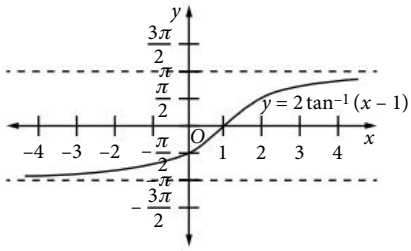
(d) $-\frac{1}{2} \le x \le \frac{1}{2}, -\frac{3\pi}{2}+1 \le y \le \frac{3\pi}{2}+1$



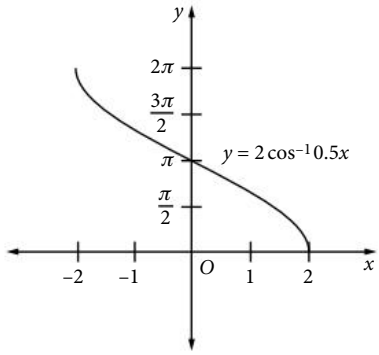
(e) all real $x, -\pi \le y \le \pi$



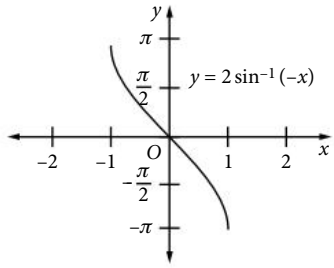
(f) all real $x, -\pi \le y \le \pi$



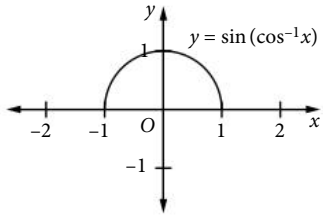
(g) $-2 \le x \le 2, 0 \le y \le 2\pi$



(h) $-1 \le x \le 1, -\pi \le y \le \pi$



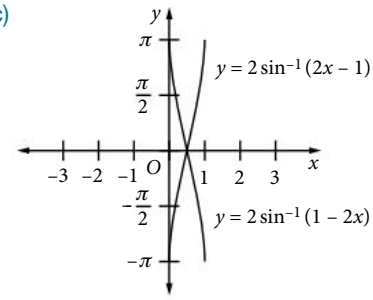
(i) Let $\alpha = \cos^{-1}x$ where $0 \le \alpha \le \pi$,
i.e. $\cos\alpha = x$
From $\sin^2\alpha + \cos^2\alpha = 1$:
$\sin\alpha = \pm\sqrt{1-x^2}$
Which one? $0 \le \alpha \le \pi$ so
$\sin\alpha = +\sqrt{1-x^2}$
Hence $y = \sin(\cos^{-1}x)$ becomes
$y = \sin\alpha$,
which becomes $y = \sqrt{1-x^2}$
i.e. a semicircle
Domain $-1 \le x \le 1$, Range $0 \le y \le 1$



8 (a) Both have domain $0 \le x \le 1$, range $-\pi \le y \le \pi$

(b) $g(x) = 2\sin^{-1}(1-2x) = 2\sin^{-1}(-[-1+2x])$
$= -2\sin^{-1}(-1+2x) = -f(x)$

(c)



9 (a) $x = 0, 1$ (b) $x = 1.5$ (c) $x = -0.47$

10 (a) $x = -\frac{1}{2}, y = \frac{1}{\sqrt{2}}$ (b) $x = -\frac{1}{\sqrt{2}}, y = \frac{\sqrt{3}}{2}$

11 (a) LHS $= \sin(\sin^{-1}x)\cos(\cos^{-1}x)$
$- \cos(\sin^{-1}x)\sin(\cos^{-1}x)$
$= x \times x - \sqrt{1-x^2} \times \sqrt{1-x^2}$
$= x^2 - (1-x^2) = 2x^2 - 1$

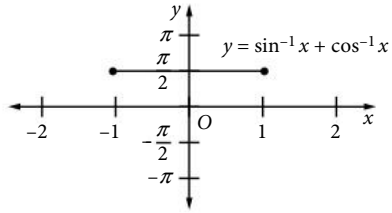
(b) $-1 \le x \le 1$
$\therefore 0 \le x^2 \le 1, 0 \le 2x^2 \le 2,$
$-1 \le 2x^2 - 1 \le 1$

(c) From (a) and (b), $\sin^{-1}x - \cos^{-1}x$
$= \sin^{-1}(2x^2 - 1)$
So the equation becomes
$\sin^{-1}(2x^2-1) = \sin^{-1}(5x-4)$
$\therefore 2x^2 - 1 = 5x - 4$
This has solutions $x = 1, \frac{3}{2}$ but the domain is $-1 \le x \le 1$
Hence the only solution is $x = 1$.

12 $f(-x) = \tan(\cos^{-1}(-x))$
$= \tan(\pi - \cos^{-1}x)$
$= -\tan(\cos^{-1}x)$
$= -f(x)$ $\therefore$ odd function

13 $y = 2\cos\frac{x}{3}, 0 \le x \le 3\pi, -2 \le y \le 2$

14 (a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \le x \le 1$, so the graph is:



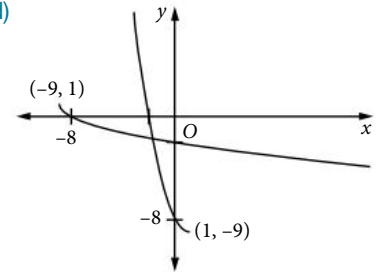
(b) $\int_{-1}^{1}(\sin^{-1}x + \cos^{-1}x)\,dx$
= Area of rectangle between interval and x-axis $= \pi$

CHAPTER REVIEW 5

1 (a) vertex $(1,-9)$, intercepts $(0,-8)$, $(4,0)$, $(-2,0)$

(b) $x \le 1$ (c) $y = 1 - \sqrt{x+9}$

(d)



(e) Intersect on line $y = x$. Solving simultaneously $y = f(x)$ and $y = x$ to find the point of intersection:

$x^2 - 3x - 8 = 0$, $x = \frac{3 \pm \sqrt{41}}{2}$

But domain of f is $x \leq 1$ $\therefore \alpha = \frac{3 - \sqrt{41}}{2}$

(f) For f^{-1}, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x+9}}$.

Evaluating this for $x = \frac{3 - \sqrt{41}}{2}$ is very difficult.

Better method: use the fact that at points of intersection of f and f^{-1}, product of their gradients is 1.

For f, $\frac{dy}{dx} = 2x - 2$. At T, $x = \frac{3 - \sqrt{41}}{2}$, gradient of $f = 2\left(\frac{3 - \sqrt{41}}{2}\right) - 2 = 1 - \sqrt{41}$

$\therefore$ gradient of f^{-1} at $T = \frac{1}{1 - \sqrt{41}} = -\frac{1 + \sqrt{41}}{40}$

2 (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) 1 (d) $\frac{\pi}{6}$ (e) $-\frac{\pi}{6}$ (f) $\frac{3}{5}$ (g) $\frac{4\sqrt{21}}{17}$

3 (a) Both $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{1}{2}$ are between 0 and $\frac{\pi}{4}$

$\therefore 0 < \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2} < \frac{\pi}{2}$

i.e. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}$ is a first quadrant angle.

Now, $\tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}\right) = 2$

using $\tan(\alpha + \beta)$ expansion

$\therefore \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2} = \tan^{-1} 2$

(b) Both $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{1}{2}$ are between 0 and $\frac{\pi}{4}$,

and $\tan^{-1}\frac{3}{4} > \tan^{-1}\frac{1}{2}$

$\therefore 0 < \tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2} < \frac{\pi}{2}$

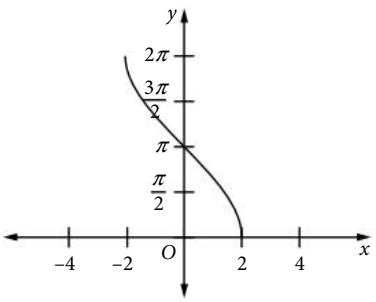
i.e. $\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2}$ is a first quadrant angle.

Now, $\sin\left(\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2}\right) = \frac{2\sqrt{5}}{25}$

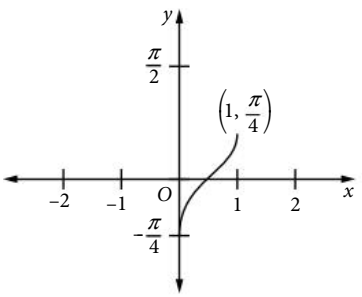
using $\sin(\alpha - \beta)$ expansion

$\therefore \tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{2\sqrt{5}}{25}$

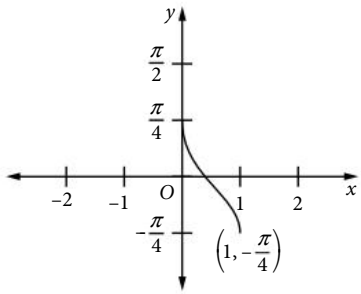
4 (a) $-2 \leq x \leq 2$, $0 \leq y \leq 2\pi$



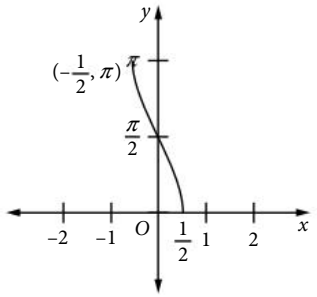
(b) $0 \leq x \leq 1$, $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$



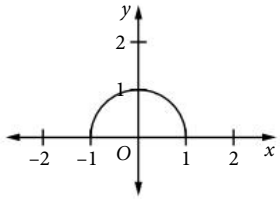
(c) $0 \leq x \leq 1$, $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$



(d) $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $0 \leq y \leq \pi$

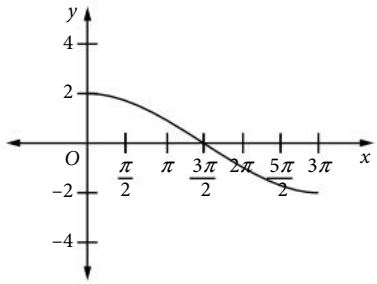


(e) $-1 \leq x \leq 1$, $0 \leq y \leq 1$

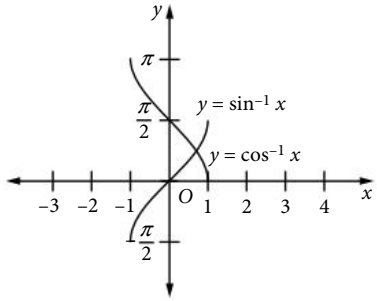


5 $y = -x - 1 - \frac{\pi}{4}$

6



7 (a) $0 < x < 1$



(b) $\left(\frac{\pi}{4}, 0\right)$

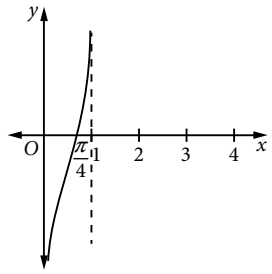
(c) $y = \log_e(\sin^{-1}x) - \log_e(\cos^{-1}x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\sin^{-1}x\sqrt{1-x^2}} + \frac{1}{\cos^{-1}x\sqrt{1-x^2}}$$

Over the domain $0 < x < 1$, every factor in the expression is positive $\therefore \frac{dy}{dx} > 0$ for all x in the domain

Hence the function is increasing for all x in the domain.

(d)



CHAPTER 6

EXERCISE 6.1

1 $5 \times 2 = 10$ **2** 72 **3** 5040 **4** 4^{20}
5 45 **6** A **7** $26^3 \times 10^2 \times 26 = 45\,697\,600$
8 $6! = 720$ **9** (a) 60 (b) 120
10 (a) 45 697 600
(b) Answers will vary. It creates about 28 million more possible number plates.
11 40 320 **12** (a) $9 \times 9! = 3\,265\,920$ (b) 9×10^9

EXERCISE 6.2

1 There are four suits, so four pigeonholes. You need to select five cards.
2 Each of the five cards can belong to one of four suits. By the pigeonhole principle, at least two of the cards must belong to the same suit: $\frac{5}{4} = 1\frac{1}{4}$.
3 There are three colours, so three pigeonholes. You need to select four socks.
4 There are six pairs of shoes, so six pigeonholes. You need to take seven shoes from the wardrobe.
5 Each student shakes hands with 24 other students, so there are 24 pigeonholes for 25 students, thus two students must have shaken hands with the same number of students.
6 In a leap year there are 366 days, so 366 pigeonholes. Since there are 400 students, then at least two of them will celebrate their birthday on the same day.
7 (a) Pairs of integers, $1 + 8 = 2 + 7 = 3 + 6 = 4 + 5 = 9$.
(b) There are four pairs of integers that add up to nine. There are four pigeonholes, so you need to pick five numbers to be sure of having two numbers that add to nine.
8 There are 12 months (pigeonholes) and 13 people (pigeons), so by the pigeonhole principle two of the people must be born in the same month.

EXERCISE 6.3

1 (a) 60 (b) 12 (c) 720 (d) 1680 (e) 9 (f) 3 628 800
(g) $n(n-1)$ (h) $n(n-1)(n-2)(n-3)$
2 (a) 120 (b) 24 (c) 40 320 (d) 5040 (e) 60 (f) 7
(g) 90 (h) $n(n-1)$
3 120 **4** C **5** 336
6 (a) $2 \times 4! = 48$ (b) ${}^3P_2 \times 3! = 36$
7 (a) $4! \times 5! = 2880$ (b) ${}^4P_2 \times 6! = 8640$ (c) $2 \times 4! \times 4! = 1152$
8 (a) 720 (b) $4! \times 3! = 144$ **9** 720, 360
10 (a) incorrect (b) correct (c) incorrect (d) correct
11 $2 \times 5! \times 5! = 28\,800$ **12** (a) 12 (b) 128 **13** 216
14 (a) 120 (b) 2880 **15** $5! \times 4! \times 2! \times 3! = 34\,560$
16 30 240 **17** (a) 1440 (b) 2520 (c) 720
18 (a) 625 (b) 1250 (c) 1875
(d) 625 (number ends in 12, 24, 32, 44, 52)
19 288 **20** $120 = 6 \times 5 \times 4$, $r = 3$ **21** 720
22 (a) 48 (b) 72 **23** 3600 **24** 144
25 3 **26** 4
27 $\text{RHS} = \frac{n!}{(n-r)!} + r \times \frac{n!}{(n-(r-1))!}$

$$= \frac{n!}{(n-r)!} + \frac{r \times n!}{(n+1-r)!}$$

$$= \frac{n!}{(n+1-r)!}[n+1-r+r]$$

$$= \frac{n!}{(n+1-r)!} \times (n+1) = \frac{(n+1)!}{(n+1-r)!} = {}^{n+1}P_r = \text{LHS}$$

28 $\text{RHS} = \frac{(n-2)!}{(n-2-r)!} + 2r \times \frac{(n-2)!}{(n-2-(r-1))!} + \frac{r(r-1)(n-2)!}{(n-2-(r-2))!}$

$$= (n-2)!\left[\frac{1}{(n-r-2)!} + \frac{2r}{(n-r-1)!} + \frac{r(r-1)}{(n-r)!}\right]$$

$$= \frac{(n-2)!}{(n-r)!}[(n-r)(n-r-1) + 2r(n-r) + r(r-1)]$$

$$= \frac{(n-2)!}{(n-r)!}\left[n^2 - 2nr - n + r^2 + r + 2nr - 2r^2 + r^2 - r\right]$$

$$= \frac{(n-2)!}{(n-r)!} \times n(n-1) = \frac{n!}{(n-r)!} = {}^nP_r = \text{LHS}$$

29 2880; 864

EXERCISE 6.4

1 60 **2** D **3** (a) $\frac{9!}{2! \times 2!} = 90\,720$ (b) 20 160
4 (a) $\frac{{}^4P_2 \times 5!}{2!} = 720$ (b) 72 **5** (a) 181 440 (b) 1440
6 (a) $\frac{{}^7P_2 \times 8!}{3! \times 2!} = 141\,120$ (b) 20 160
7 (a) incorrect (b) incorrect (c) correct (d) correct
8 37 800 **9** 35 **10** (a) 1680 (b) 140
11 840 **12** $2(n-1)!$ **13** (a) 10! (b) $10! - 2 \times 8!$
14 4^4 or $4 + 48 + 36 + 144 + 24 = 256$

EXERCISE 6.5

1 (a) 21 (b) 9 (c) 70 (d) 66 (e) 495 (f) 15 (g) 1
(h) 15 (i) $\frac{n(n-1)}{2}$ (j) $\frac{n(n-1)(n-2)(n-3)}{24}$ (k) 1
(l) $\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$
2 (a) 252 (b) 167 960 (c) 3 268 760
(d) $\frac{n(n-1)(n-2)(n-3)(n-4)}{120}$ (e) 3003 (f) 31 824
(g) 12 870 (h) 17 296
3 ${}^8C_3 = 56$ **4** 2 598 960 **5** 125 970
6 A **7** ${}^5C_2 \times {}^4C_3 = 40$ **8** 20 **9** 1470
10 (a) incorrect (b) incorrect (c) correct (d) correct
11 ${}^8C_5 \times 1 = 56$ **12** (a) ${}^8C_4 = 70$ (b) ${}^8C_6 + {}^8C_5 + {}^8C_5 = 140$
13 (a) $\binom{52}{3} - \binom{48}{3} = 4804$ (b) $\binom{48}{3} + 4\binom{48}{2} = 21\,808$
14 700 **15** $5 \times \binom{10}{3}\binom{8}{3} + 5 \times \binom{10}{4}\binom{8}{2} + 5 \times \binom{10}{5}\binom{8}{1} = 73\,080$
16 (a) 280 (b) 742 **17** 595 **18** 98
19 (a) 720 (b) 45 **20** 1946 **21** 466
22 375 **23** (a) 336 (b) $336 \times 5! = 40\,320$

24 ${}^6C_4 \times {}^5C_3 \times 7! = 756\,000$, ${}^6C_4 \times {}^5C_3 \times 4! \times 4! = 86\,400$

25 7200, ${}^5C_3 \times {}^4C_2 \times 3 \times 4! = 4320$ **26** 1440

27 $(n-4)(n-5) = 30$, $n = 10$ **28** ${}^9C_4 \times {}^5C_3 \times 1 = 1260$

29 ${}^8C_3 + {}^8C_2 + {}^8C_1 = 92$

30 ${}^8C_2 \times {}^6C_2 \times {}^4C_2 \times 1 \div 4! = 105$ **31** $2^n - 2$

32 (a) $\text{LHS} = \dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r-1)!(r+1)!}$

$= \dfrac{n!}{(n-r-1)!r!}\left[\dfrac{1}{n-r} + \dfrac{1}{r+1}\right]$

$= \dfrac{n!}{(n-r-1)!r!}\left[\dfrac{r+1+n-r}{(n-r)(r+1)}\right]$

$= \dfrac{(n+1)\times n!}{(n-r)\times(n-r-1)!\times(r+1)\times r!}$

$= \dfrac{(n+1)!}{(n-r)!\times(r+1)!} = {}^{n+1}C_{r+1} = \text{RHS}$

(b)

$\text{LHS} = \dfrac{n!}{(n-k)!k!} + 2\times\dfrac{n!}{(n-k+1)!(k-1)!} + \dfrac{n!}{(n-k+2)!(k-2)!}$

$= \dfrac{n!}{(n-k+2)!k!}\left[(n-k+2)(n-k+1) + 2k(n-k+2) + k(k-1)\right]$

$= \dfrac{n!}{(n+2-k)!k!}\left[n^2+3n+2\right]$

$= \dfrac{(n+2)(n+1)\times n!}{(n+2-k)!k!} = \dfrac{(n+2)!}{(n+2-k)!k!} = {}^{n+2}C_k = \text{RHS}$

(c) $\text{RHS} = \dfrac{n-r+1}{r}\times\dfrac{n!}{(n-r+1)!(r-1)!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r = \text{LHS}$

(d)

$\text{RHS} = \dfrac{m!}{(m-3)!3!} + \dfrac{m!}{(m-2)!2!}\times n + m\times\dfrac{n!}{(n-2)!2!} + \dfrac{n!}{(n-3)!3!}$

$= \dfrac{m(m-1)(m-2)}{6} + \dfrac{m(m-1)\times n}{2} + \dfrac{m\times n(n-1)}{2} + \dfrac{n(n-1)(n-2)}{6}$

$= \dfrac{1}{3!}\left[m^3 - 3m^2 + 2m + 3m^2n - 3mn + 3mn^2 - 3mn + n^3 - 3n^2 + 2n\right]$

$= \dfrac{1}{3!}\left[m^3 + n^3 - 3m^2 - 6mn - 3n^2 + 3m^2n + 3mn^2 + 2m + 2n\right]$

$= \dfrac{1}{3!}\left[m^3 + n^3 - 3(m+n)^2 + 3mn(m+n) + 2(m+n)\right]$

$= \dfrac{1}{3!}\left[(m+n)(m^2 - mn + n^2) - 3(m+n)^2 + 3mn(m+n) + 2(m+n)\right]$

$= \dfrac{(m+n)}{3!}\left[(m+n)^2 - 3(m+n) + 2\right]$

$= \dfrac{(m+n)(m+n-1)(m+n-2)}{3!} = {}^{m+n}C_3 = \text{LHS}$

(e)

$\text{RHS} = \dfrac{n!}{(n-r)!r!} + 3\times\dfrac{n!}{(n-r+1)!(r-1)!} + 3\times\dfrac{n!}{(n-r+2)!(r-2)!} + \dfrac{n!}{(n-r+3)!(r-3)!}$

$= \dfrac{n!}{(n-r+1)!r!}\left[n-r+1+3r\right] + \dfrac{n!}{(n-r+3)!(r-2)!}\left[\begin{matrix}3(n-r+3)\\ +r-2\end{matrix}\right]$

$= \dfrac{n!}{(n-r+3)!r!}\left[\begin{matrix}(n+2r+1)(n-r+3)(n-r+2)\\ +(3n-2r+7)(r)(r-1)\end{matrix}\right]$

$= \dfrac{n!}{(n-r+3)!r!}\left[n^3 + 6n^2 + 11n + 6\right]$

$= \dfrac{n!(n+1)(n+2)(n+3)}{(n-r+3)!r!} = \dfrac{(n+3)!}{(n+3-r)!r!} = {}^{n+3}C_r = \text{LHS}$

33 5

EXERCISE 6.6

1 (a) $\dfrac{4\times{}^{13}C_5}{{}^{52}C_5} = \dfrac{33}{16\,660}$ (b) $\dfrac{1}{54\,145}$ (c) $\dfrac{{}^4C_3\times{}^4C_2}{{}^{52}C_5} = \dfrac{1}{108\,290}$

2 D **3** $\dfrac{9}{9^3} = \dfrac{1}{81}$ **4** $\dfrac{1}{720}$ **5** (a) $\dfrac{3}{11}$ (b) $\dfrac{3}{44}$

6 $\dfrac{149}{198}$ **7** $\dfrac{{}^5C_2\times{}^6C_3}{{}^6C_3\times{}^7C_4} = \dfrac{2}{7}$ **8** $\dfrac{19}{10\,829}$ **9** $\dfrac{1}{90}$

10 $\dfrac{1}{7}$ **11** $\dfrac{1}{3}$ **12** $\dfrac{1}{35}$ **13** 17 576 000, $\dfrac{1}{67\,600}$

14 (a) correct (b) incorrect (c) correct (d) correct

15 $\dfrac{1}{220}$ **16** $\dfrac{1}{3}$

17 (a) $\dfrac{5}{9}\times\dfrac{4}{8}\times\dfrac{4}{7} = \dfrac{10}{63}$ (b) $\dfrac{10}{21}$ **18** $1 - \dfrac{12}{15}\times\dfrac{11}{14}\times\dfrac{10}{13} = \dfrac{47}{91}$

19 $\dfrac{{}^8C_4}{{}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4} = \dfrac{35}{81}$ **20** $\dfrac{5}{6}$

21 0.041 96, 0.2517, 0.4196, 0.2398, 0.0450, 0.0020

22 $\dfrac{1}{132}, \dfrac{5}{44}, \dfrac{25}{66}, \dfrac{25}{66}, \dfrac{5}{44}, \dfrac{1}{132}$ **23** $\dfrac{5}{14}, \dfrac{3}{4}$

24 $\dfrac{1}{2}$ **25** 0.3412 **26** $\dfrac{67}{165}$ **27** $\dfrac{128}{225}$

28 (a) $\dfrac{1}{2}\times\dfrac{4}{9} = \dfrac{2}{9}$ (b) $\dfrac{5}{9}$ (c) $\dfrac{8}{10}\times\dfrac{1}{9} = \dfrac{4}{45}$ (d) $\dfrac{2}{45}$

29 (a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$ **30** (a) $\dfrac{1}{35}$ (b) $\dfrac{1}{7}$ (c) $\dfrac{1}{35}$

31 $\dfrac{1}{35}$ **32** (a) $\dfrac{1}{15}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{5}$

33 $\dfrac{1}{2}$ **34** (a) $\dfrac{2717}{33\,320}$ (b) $\dfrac{1}{54\,145}$ (c) $\dfrac{2109}{9520}$

35 (a) $\dfrac{1}{{}^{44}C_6} = \dfrac{1}{7\,059\,052}$ (b) $\dfrac{4}{{}^{44}C_6} = \dfrac{1}{1\,764\,763}$

36 (a) 60 (b) 120

EXERCISE 6.7

1 (a) $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

(b) $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

(c) $1 + 7b + 21b^2 + 35b^3 + 35b^4 + 21b^5 + 7b^6 + b^7$

(d) $1 + 6x + 12x^2 + 8x^3$

(e) $1 + 20x + 160x^2 + 640x^3 + 1280x^4 + 1024x^5$

(f) $1 + 18x + 135x^2 + 540x^3 + 1215x^4 + 1458x^5 + 729x^6$

(g) $1 - 3x + 3x^2 - x^3$

(h) $1 - 8a + 28a^2 - 56a^3 + 70a^4 - 56a^5 + 28a^6 - 8a^7 + a^8$

(i) $1 - 8x + 24x^2 - 32x^3 + 16x^4$

(j) $1 + \dfrac{5x}{2} + \dfrac{5x^2}{2} + \dfrac{5x^3}{4} + \dfrac{5x^4}{16} + \dfrac{x^5}{32}$

(k) $1 - 2x + \dfrac{5x^2}{3} - \dfrac{20x^3}{27} + \dfrac{5x^4}{27} - \dfrac{2x^5}{81} + \dfrac{x^6}{729}$

(l) $1 + 2x + \dfrac{8x^2}{5} + \dfrac{16x^3}{25} + \dfrac{16x^4}{125} + \dfrac{32x^5}{3125}$

(m) $1 + 3x^2 + 3x^4 + x^6$ (n) $1 - 4x^2 + 6x^4 - 4x^6 + x^8$

(o) $x^5 + 5x^3 + 10x + \dfrac{10}{x} + \dfrac{5}{x^3} + \dfrac{1}{x^5}$

2 (a) 6 (b) 21 (c) 10 (d) 27 (e) 40 (f) 54

3 (a) 32 (b) −64 (c) $\frac{80}{27}$

4 B **5** (a) $10x^3$ (b) $-280a^3$ (c) $\dfrac{135x^3}{16}$

6 (a) $1 + 4\sqrt{2} + 6\times 2 + 4\times 2\sqrt{2} + 4 = 17 + 12\sqrt{2}$

(b) $1 - 3\sqrt{3} + 3\times 3 - 3\sqrt{3} = 10 - 6\sqrt{3}$

(c) $1 + 6\sqrt{5} + 15\times 5 + 20\times 5\sqrt{5} + 15\times 25 + 6\times 25\sqrt{5} + 125$
$= 576 + 256\sqrt{5}$

(d) $1 + 6\sqrt{3} + 36 + 24\sqrt{3} = 37 + 30\sqrt{3}$

(e) $1 - 12\sqrt{2} + 108 - 216\sqrt{2} + 324 = 433 - 228\sqrt{2}$

(f) $1 + \dfrac{5\sqrt{3}}{2} + \dfrac{15}{2} + \dfrac{15\sqrt{3}}{4} + \dfrac{45}{16} + \dfrac{9\sqrt{3}}{32} = \dfrac{362 + 209\sqrt{3}}{32}$

7 (a) $\left(1-\frac{1}{10}\right)^6$

$\approx 1-0.6+0.15-0.02+0.0015-0.00006+\ldots$

≈ 0.5314

(b) $\left(1-\frac{1}{20}\right)^5 = 1-0.25+0.025-0.00125+0.00003+\ldots \approx 0.7738$

(c) $\left(1-\frac{1}{100}\right)^4 = 1-0.04+0.0006-\ldots \approx 0.9606$

EXERCISE 6.8

1 (a) $8+12x+6x^2+x^3$

(b) $243+405x+270x^2+90x^3+15x^4+x^5$

(c) $b^4+8b^3+24b^2+32b+16$

(d) $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

(e) $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$

(f) $27-27x+9x^2-x^3$

(g) $256-1024a+1792a^2-1792a^3+1120a^4-448a^5+112a^6-16a^7+a^8$

(h) $81y^4-216y^3x+216y^2x^2-96yx^3+16x^4$

(i) $x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}$

(j) $x^{12}+6x^9+15x^6+20x^3+15+\frac{6}{x^3}+\frac{1}{x^6}$

2 (a) $16+32\sqrt{2}+24\times 2+8\times 2\sqrt{2}+4 = 68+48\sqrt{2}$

(b) $64-48\sqrt{3}+36-3\sqrt{3} = 100-51\sqrt{3}$

(c) $8+24\sqrt{10}+300+200\sqrt{10}+750+150\sqrt{10}+125$

$=1183+374\sqrt{10}$

(d) $25\sqrt{5}+125\sqrt{3}+150\sqrt{5}+150\sqrt{3}+45\sqrt{5}+9\sqrt{3}$

$=220\sqrt{5}+284\sqrt{3}$

(e) $144-288\sqrt{6}+1296-432\sqrt{6}+324 = 1764-720\sqrt{6}$

(f) $36\sqrt{6}+360\sqrt{3}+720\sqrt{6}+1440\sqrt{3}+720\sqrt{6}+288\sqrt{3}$

$=1476\sqrt{6}+2088\sqrt{3}$

3 (a) 18 (b) −84 (c) 7

4 C **5** (a) 81 (b) 20 (c) 15

EXERCISE 6.9

1 (a) 10 (b) 70 (c) 210 (d) 35 (e) 126 (f) 455

2 (a) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$

(b) $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$

(c) $x^{10}+5x^8a+10x^6a^2+10x^4a^3+5x^2a^4+a^5$

(d) $a^6+6a^4+15a^2+20+\frac{15}{a^2}+\frac{6}{a^4}+\frac{1}{a^6}$

(e) $x^8-12x^6y^2+54x^4y^4-108x^2y^6+81y^8$

(f) $\frac{32a^5}{243}-\frac{40a^4}{27b}+\frac{20a^3}{3b^2}-\frac{15a^2}{b^3}+\frac{135a}{8b^4}-\frac{243}{32b^5}$

(g) $16y^4-48y^3x^2+54y^2x^4-27yx^6+\frac{81x^8}{16}$

(h) $x^{18}-12x^{13}+60x^8-160x^3+\frac{240}{x^2}-\frac{192}{x^7}+\frac{64}{x^{12}}$

(i) $\frac{a^{10}}{32}-\frac{15a^8b}{16}+\frac{45a^6b^2}{4}-\frac{135a^4b^3}{2}+\frac{405a^2b^4}{2}-243b^5$

3 (a) 40 (b) −12 (c) −1 (d) 540 (e) 280 (f) −48 384

4 (a) $80x^2$ (b) $-108x$ (c) $-x^3$ (d) $540y^3x^3$ (e) $560x^4y^3$

(f) $-108\,864x^5z^3$ **5** A

6 (a) $\binom{8}{3}\left(\frac{m}{2}\right)^5(3n)^3 = \frac{189m^5n^3}{4}$ (b) $\binom{11}{6}\left(\frac{2x}{3}\right)^5\left(\frac{-3}{2x}\right)^6 = \frac{693}{x}$

(c) $\binom{7}{4}a^3\left(\frac{-2b}{3}\right)^4 = \frac{560}{81}a^3b^4$

(d) $\binom{n}{3}x^{n-3}\left(\frac{-1}{x}\right)^3 = -\frac{n(n-1)(n-2)x^{n-6}}{6}$

7 (a) $\binom{8}{r}x^{8-r}\left(\frac{1}{x}\right)^r$, $r=3$, coefficient $=56$

(b) $\binom{12}{r}(2x^2)^{12-r}(-3x)^r$, $r=7$,

coefficient $=-11\times 2^8\times 3^9 = -55\,427\,328$

(c) $\binom{9}{r}(3a)^{9-r}\left(-\frac{b}{3}\right)^r$, $r=4$, coefficient $=378$

(d) $\binom{13}{r}(2a)^{13-r}\left(\frac{-3}{a}\right)^r$, $r=2$,

coefficient $=13\times 2^{12}\times 3^3 = 1\,437\,696$

8 $\binom{2n}{k+1}a^{2n-(k+1)}b^{k+1} = \frac{(2n)!}{(k+1)!(2n-k-1)!}a^{2n-k-1}b^{k+1}$

CHAPTER REVIEW 6

1 tree diagram **2** (a) 24 (b) 256 **3** 120

4 (a) ${}^7P_4 = 840$ (b) $7^4 = 2401$

5 (a) $3\times {}^4P_3 + 4\times 4! = 168$

(b) $3\times 5^3 + 4\times 5^4 - 1 = 2874$ (c) 108

6 (a) 30 240 (b) 240 **7** 210 **8** 3003

9 ${}^8C_4\times {}^9C_4 = 8820$ **10** (a) 142 506 (b) 20 **11** 48

12 (a) $\binom{6}{3}\times\binom{8}{4} = 1400$ (b) $7! = 5040$

(c) ${}^4P_2\times 5! = 1440$ (d) $\frac{1440}{5040} = \frac{2}{7}$

13 0.5 **14** $\frac{1\times 6!}{8!} = \frac{1}{56}$ **15** $\frac{{}^3P_1\times {}^6P_2}{{}^7P_3} = \frac{3}{7}$

16 (a) $\frac{11!}{2!2!2!2!} = 2\,494\,800$ (b) $\frac{\frac{4!}{2!2!}\times\frac{8!}{2!2!}}{\frac{11!}{2!2!2!2!}} = \frac{4}{165}$

17 $1-\frac{{}^{48}C_3}{{}^{52}C_3} = 0.2174$ **18** $\binom{6}{3}\times\left(\frac{1}{2}\right)^6 = \frac{5}{16}$

19 (a) $\frac{7!}{2!2!} = 1260$ (b) $\frac{\frac{6!}{2!2!}+\frac{6!}{2!}}{\frac{7!}{2!2!}} = \frac{3}{7}$ (c) $\frac{\frac{2\times 6!}{2!2!}+\frac{6!}{2!}}{\frac{7!}{2!2!}} = \frac{4}{7}$

(d) $1-\frac{\frac{1\times 6!}{2!}}{\frac{7!}{2!2!}} = \frac{5}{7}$

20 (a) $\binom{6}{4}\times\left(\frac{3}{4}\right)^2\times\left(\frac{1}{4}\right)^4 = \frac{135}{4096}$

(b) $\binom{6}{4}\times\left(\frac{3}{4}\right)^2\times\left(\frac{1}{4}\right)^4 + \binom{6}{5}\times\left(\frac{3}{4}\right)\times\left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 = \frac{77}{2048}$

(c) $1-\left(\frac{1}{4}\right)^6 = \frac{4095}{4096}$

21 ${}^4C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 = \frac{5}{324}$ **22** (a) $\binom{16}{4} = 1820$ (b) $\frac{{}^9C_4}{{}^{16}C_4} = \frac{9}{130}$

23 $\frac{2\times 5!}{6!} = \frac{1}{3}$

CHAPTER 7

EXERCISE 7.1

1 C

2 (a) $\frac{dx}{dt} = 4$ (b) 4 cm s^{-1} (c) $A = (4t+1)^2$ cm^2

(d) $\frac{dA}{dt} = 32t+8$ cm^2 s^{-1}, $\frac{dA}{dt} = 32\times 5+8 = 168$ cm^2/s

3 (a) 20 litres (b) $V = 4t$ litres (c) $45 = 4t$, $t = 11.25$ minutes

4 (a) $\frac{dV}{dt} = 100000\left(1-\frac{t}{40}\right)\times\left(-\frac{1}{40}\right) = -2500\left(1-\frac{t}{40}\right)$ L per minute

(i) $\frac{dV}{dt} = -2187.5$ L min^{-1} (ii) $\frac{dV}{dt} = -1875$ L min^{-1}

(iii) $\frac{dV}{dt} = -1250$ L min^{-1}

(b) $V = 12\,500$ L

(c) $25\,000 = 50\,000\left(1 - \frac{t}{40}\right)^2$

$0.5 = \left(1 - \frac{t}{40}\right)^2$

$1 - \frac{t}{40} = \frac{1}{\sqrt{2}}$

$t = 40\left(1 - \frac{\sqrt{2}}{2}\right) = 11.7$ minutes

5 (a) $t = 0$, $V = 1000$ L (b) $\frac{dV}{dt} = \frac{t}{500} - 2$

(c) $t = 25$, $\frac{dV}{dt} = \frac{25}{500} - 2 = -1.95$ L min^{-1}

(d) $\frac{t}{500} - 2 = 0$, $t = 1000$ minutes

6 (a) $t = 0$, $\frac{dV}{dt} = \frac{1}{500} = 0.002$ mL s^{-1} (b) $t = 60$ seconds

7 (a) $\frac{dQ}{dt} = 3t^2 - 12t + 12$

$t = 0.5$, $\frac{dQ}{dt} = 6.75$ amp

(b) $t = 1$, $\frac{dQ}{dt} = 3$ amp (c) 5 C

(d) $\frac{dQ}{dt} = 0$: $3(t-2)^2 = 0$. $t = 2$ seconds

EXERCISE 7.2

1 (a) $v = 6t^2 - 10t - 4$, $a = 12t - 10$

(b) $t = 0$, $v = -4$ m s^{-1}, $a = -10$ m s^{-2}

(c) $v = 0$: $3t^2 - 5t - 2 = 0$

$(3t + 1)(t - 2) = 0$

$t = 2$ s

(d) $a = 0$: $12t - 10 = 0$

$t = \frac{5}{6}$ s

$v = 6 \times \left(\frac{5}{6}\right)^2 - 10 \times \frac{5}{6} - 4 = -8\frac{1}{6}$ m s^{-1}

$x = 2 \times \left(\frac{5}{6}\right)^3 - 5 \times \left(\frac{5}{6}\right)^2 - 4 \times \frac{5}{6} = -5\frac{35}{54}$ m

2 (a) $v = 3t^2 - 7t - 6$, $a = 6t - 7$

(b) $v = 0$: $3t^2 - 7t - 6 = 0$

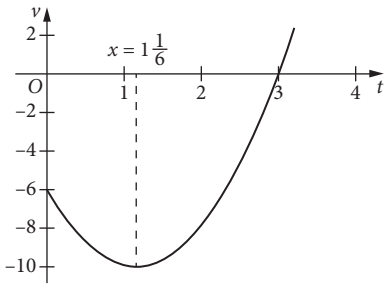
$(3t + 2)(t - 3) = 0$

$t = 3$ sec

(c) $a < 0$: $6t - 7 < 0$

$t < 1\frac{1}{6}$ sec. During the first $1\frac{1}{6}$ seconds.

(d)



When $a < 0$, the velocity is negative, it is a decreasing function reaching its least value when $a = 0$.

(e) $a > 0$ when $t > 1\frac{1}{6}$. The velocity is now an increasing function.

3 (a) $t = 0$, $v = -5$ m s^{-1}

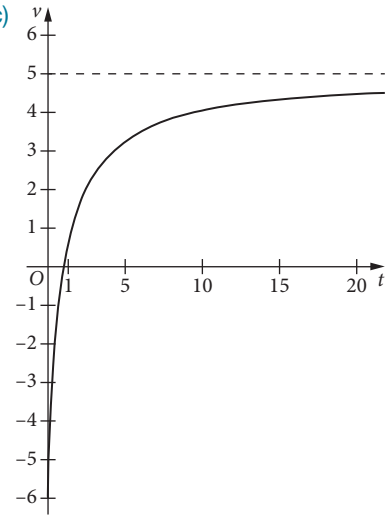
(b) $v = 0$: $5 = \frac{10}{t+1}$

$t = 1$

$a = \frac{10}{(t+1)^2}$

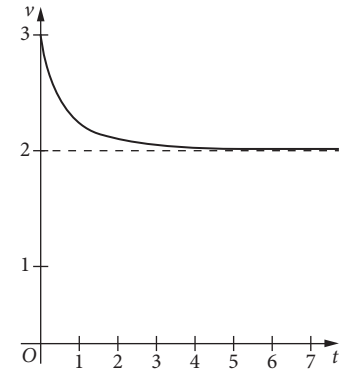
$t = 1$, $a = 2.5$ m s^{-2}

(c)



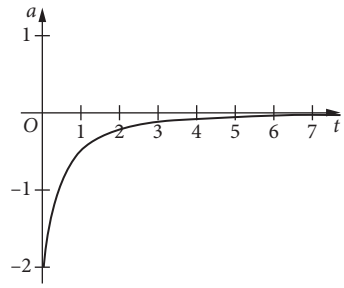
4 C

5 (a) $v = 2 + \frac{1}{(t+1)^2}$



(b) $v \to 2$ m s^{-1}

(c) $a = \frac{-2}{(t+1)^3}$



(d) Since $t \geq 0$ then $(t + 1)^3 > 0$ so $a < 0$ to all $t \geq 0$.

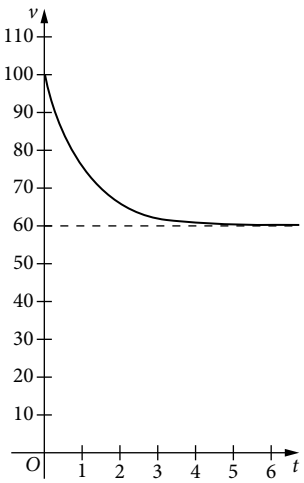
6 A

7 (a) For A, $t = 0$: $v = 100$ km h^{-1}. For B, $t = 3$,

$v = 60 + 40e^{-3} \approx 62$ km h^{-1}

(b) (i) $a = -40e^{-t}$ (ii) $a = 60 - v$

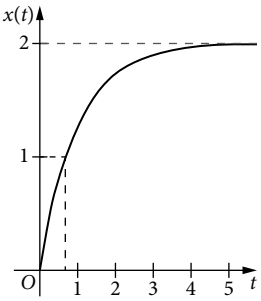
(c)



As t increases, the velocity approaches 60 km h^{-1} from above.

8 (a) $x(t) = 2(1 - e^{-t})$, $\dot{x}(t) = 2e^{-t}$, $\ddot{x}(t) = -2e^{-t}$
$x(0) = 0$, $\dot{x}(0) = 2$, $\ddot{x}(0) = -2$

(b) $x(t)$



(c) $1 = 2(1 - e^{-t})$
$e^{-t} = 0.5$
$e^{t} = 2$
$t = \ln 2 \approx 0.693$

9 (a) correct (b) incorrect (c) incorrect (d) correct

10 (a) $x = \frac{t^3}{3} - t^4$ (b) $\ddot{x} = 2t - 12t^2$

11 $a = 12t + 6$

12 (a) $v = 25 - 10t$, $x = 25t - 5t^2$ (b) 31.25 m (c) $t = 1.25$ s

13 (a) $v = 30 - 10t$ (b) $h = 30t - 5t^2$ (c) $t = 3$, $h = 45$ m (d) 6 s

14 $a = 6t - 2$

EXERCISE 7.3

Logarithm values are approximate.

1 $y = 5e^{2x}$ 2 B 3 $Q = 50e^{0.4t}$ 4 $y = 20e^{-3t}$

5 $M = 10e^{-0.01t}$ 6 $A = 1000$, $k = 0.5$ 7 $A = 200$, $k = 0.4$

8 (a) correct (b) incorrect (c) correct (d) correct

9 $P = 100\,000e^{0.018t}$ 10 (a) $\frac{dN}{dt} = 0.23N$ (b) 8×10^4 (c) 6

11 $k = \frac{\ln 2}{1600}$, 4.24%

12 (a) 16 500 (b) 13.86 min (c) (i) 824/min (ii) 1000/min

13 (a) $\frac{d}{dt}(V_0 e^{-kt}) = -kV_0 e^{-kt} = -kV$ (b) $k = \frac{1}{5}\log_e\left(\frac{4}{3}\right)$, $\frac{9}{16}$
(c) (i) $0.032V_0$ m^3 min^{-1} (ii) $0.043V_0$ m^3 min^{-1}

14 (a) 50 (b) 0.1 (c) 2 years

15 (a) 5000 (b) 2 (c) 10 000 coulombs per second

16 (a) $N = 1000e^{0.15t}$ (b) 4.62 h, 300/h

17 (a) $k = \frac{1}{300}\ln\left(\frac{10}{3}\right) \approx 0.004$ (b) 173 m

18 (a) $k = \frac{-1}{10}\ln\left(\frac{5}{4}\right) \approx -0.02$ (b) 35

19 $m = Me^{-kt}$, 6.9 20 (a) 80 (b) (i) 51.25°C (ii) 14.8 min

21 16 000 22 70 days 23 59%

EXERCISE 7.4

1 (a) $N = 30 + Ae^{-0.4t}$, $\frac{dN}{dt} = -0.4Ae^{-0.4t}$, but $Ae^{-0.4t} = N - 30$,
so $\frac{dN}{dt} = -0.4(N - 30)$
(b) 34

2 (a) $N = 40 + Ae^{0.2t}$, $\frac{dN}{dt} = 0.2Ae^{0.2t}$, but $Ae^{0.2t} = N - 40$,
so $\frac{dN}{dt} = 0.2(N - 40)$
(b) 113.9

3 $\frac{dT}{dt} = -k(T - 50)$, $\frac{dt}{dT} = \frac{-1}{k(T-50)}$, $T \neq 20$
$t = -\frac{1}{k}\int \frac{1}{T-50}dt = -\frac{1}{k}\log_e(T - 50) + C$, $T > 50$
$-k(t - C) = \log_e(T - 50)$, $T - 50 = e^{-k(t-C)}$,
$T = 50 + Ae^{-kt}$ where $A = e^{-kC}$
$t = 0$, $T = 120$: $A = 70$
$t = 10$, $T = 70$: $70 = 50 + 70e^{-10k}$, $k = \frac{1}{10}\log_e 3.5 = 0.125$,
$T = 50 + 70e^{-0.125t}$
(a) $t = 20$: $T = 50 + 70e^{-2.5} = 55.7$°C
(b) $T = 60$: $60 = 50 + 70e^{-0.125t}$, $e^{-0.125t} = \frac{1}{7}$,
$0.125t = \ln 7$, $t = 15.6$ min

4 B

5 (a) $t = 0$, $T = 1230$: RHS $= 30 + 1200 \times 1 = 1230 =$ LHS
$\frac{dT}{dt} = -1200ke^{-kt} = -k(T - 30)$
(b) $t = 10$, $T = 1030$: $1030 = 30 + 1200e^{-10k}$, $k = \frac{1}{10}\ln\left(\frac{6}{5}\right)$
$t = 20$: $T = 30 + 1200e^{-2\ln\left(\frac{6}{5}\right)} = 30 + 1200 \times \left(\frac{5}{6}\right)^2 = 863.3$°C
(c) $80 = 30 + 1200e^{-kt}$, $e^{kt} = 24$, $kt = \ln 24$, $t = \frac{10 \ln 24}{\ln 1.2} = 174$ min
$= 2$ h 54 min

6 (a) $t = 0$, $T = 20$: RHS $= -10 + 30 \times 1 = 20 =$ LHS
$\frac{dT}{dt} = -30ke^{-kt} = -k(T + 10)$
(b) $t = 5$, $T = 15$: $15 = -10 + 30e^{-5k}$, $k = \frac{1}{5}\ln\left(\frac{6}{5}\right)$
$t = 10$: $T = -10 + 30e^{-2\ln\left(\frac{6}{5}\right)} = -10 + 30 \times \left(\frac{5}{6}\right)^2$
$= 10\frac{5}{6} \approx 11$°C

7 (a) $t = 0$, $T = 180$: RHS $= 60 + 120 \times 1 = 180 =$ LHS
$\frac{dT}{dt} = -120ke^{-kt} = -k(T - 60)$
(b) $t = 1$, $T = 120$: $120 = 60 + 120e^{-k}$, $k = \ln 2$
$T = 90$: $90 = 60 + 120e^{-kt}$, $e^{-kt} = \frac{1}{4}$, $t = \frac{\ln 4}{\ln 2} = 2$ min

8 $\frac{di}{dt} = \frac{E}{R} \times \left(0 - \left(-\frac{R}{L}e^{-\frac{Rt}{L}}\right)\right) = \frac{E}{L}e^{-\frac{Rt}{L}}$
But $e^{-\frac{Rt}{L}} = 1 - \frac{Ri}{E}$ so $\frac{di}{dt} = \frac{E}{L}\left(1 - \frac{Ri}{E}\right)$
Hence $L\frac{di}{dt} = E - Ri$ or $L\frac{di}{dt} + Ri = E$

9 (a) $\frac{dV}{dt} = A(0 + ke^{-kt}) = kAe^{-kt}$
But $A - V = Ae^{-kt}$ so $\frac{dV}{dt} = k(A - V)$
(b) If total volume $= B$, then at $t = 5$: $V = \frac{B}{4}$
$\frac{B}{4} = A(1 - e^{-5k})$, $e^{-5k} = \frac{4A - B}{4A}$, $e^{5k} = \frac{4A}{4A - B}$,
$k = \frac{1}{5}\ln\left(\frac{4A}{4A - B}\right)$

At $t = 10$:

$V = A(1 - e^{-10k}) = A\left(1 - e^{-2\ln\left(\frac{4A}{4A-B}\right)}\right)$

$= A\left(1 - e^{\ln\left(\frac{4A-B}{4A}\right)^2}\right) = A\left(1 - \left(\frac{4A-B}{4A}\right)^2\right) = \frac{B}{2} - \frac{B}{16} \times \frac{B}{A}$

Now $B = \lim_{t\to\infty}\left(A(1 - e^{-kt})\right) = A$

so $\frac{B}{A} = 1$ and $V = \frac{B}{2} - \frac{B}{16} = \frac{7B}{16}$

Increase in volume $= \frac{7B}{16} - \frac{B}{4} = \frac{3B}{16}$,

fraction filled in next 5 minutes $= \frac{3}{16}$

(c) $\lim_{t\to\infty} V = \lim_{t\to\infty}\left(A(1 - e^{-kt})\right) = A\lim_{t\to\infty}\left(1 - e^{-kt}\right) = A$

10 (a) $\frac{dx}{dt} = k(20 - 2x)$, $\frac{dt}{dx} = \frac{1}{k(20-2x)}$

$t = \frac{1}{2k}\int \frac{1}{10-x}dx = -\frac{1}{2k}\log_e(10 - x) + C$

$t = 0, x = 0$: $0 = -\frac{1}{2k}\log_e 10 + C$, $C = \frac{1}{2k}\log_e 10$

$t = -\frac{1}{2k}\log_e(10 - x) + \frac{1}{2k}\log_e 10$, $-2kt = \log_e\left(\frac{10-x}{10}\right)$,

$\frac{10-x}{10} = e^{-2kt}$, $x = 10(1 - e^{-2kt})$

(b) $t = 5, x = 2$: $2 = 10(1 - e^{-10k})$, $e^{-10k} = 0.8$, $k = \frac{1}{10}\log_e 1.25$

$x = 10\left(1 - e^{-\frac{t}{5}\log_e 1.25}\right)$

Where $x = 9$: $9 = 10\left(1 - e^{-\frac{t}{5}\log_e 1.25}\right)$, $e^{-\frac{t}{5}\log_e 1.25} = \frac{1}{10}$,

$e^{\frac{t}{5}\log_e 1.25} = 10$, $\frac{t}{5}\log_e 1.25 = \log_e 10$, $t = \frac{5\log_e 10}{\log_e 1.25} = 51.6$ min

11 (a) $\frac{dy}{dt} = -ke^{-kt}$ but $e^{-kt} = y - 3$ so $\frac{dy}{dt} = -k(y - 3)$

(b) $k = 0.08$ (c) $y = 3.5$: $\frac{dy}{dt} = -0.08(3.5 - 3) = -0.04$ g/h

(d) $t = 0, y = 4$, $\lim_{t\to\infty} y = \lim_{t\to\infty}\left(3 + e^{-0.08t}\right) = 3$,

hence $3 < y \leq 4$

EXERCISE 7.5

1 C

2 $\frac{dy}{dt} = \frac{-3}{\sqrt{119}} = \frac{-3\sqrt{119}}{119}$

3 $V = \frac{1}{3}\pi x^2(60 - x)$: $V = \frac{\pi}{3}(60x^2 - x^3)$

$\frac{dV}{dx} = \frac{\pi}{3}(120x - 3x^2)$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$\frac{dV}{dt} = 2\pi$: $2\pi = \frac{\pi}{3}(120x - 3x^2) \times \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{6\pi}{\pi\left(120x - 3x^2\right)}$

$= \frac{6}{3\left(40x - x^2\right)}$

$= \frac{2}{x(40 - x)}$

4 (a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4t \times \frac{1}{4} = t$

(b) $\frac{dy}{dx} = t$, $t = \frac{x}{4}$, so $\frac{dy}{dx} = \frac{x}{4}$

5 (a) $V = \frac{1}{3}\pi r^2 h$: $V = \frac{\pi}{3} \times \left(\frac{h}{4}\right)^2 \times h = \frac{\pi h^3}{48}$

(b) $\frac{dh}{dt} = \frac{9.6}{\pi h^2}$

(c) $\frac{dh}{dt} = \frac{9.6}{\pi \times 16} = \frac{3}{5\pi}$

6 (a) $\frac{dy}{dx} = \frac{-4t}{\left(1+t^2\right)^2} \times \frac{\left(1+t^2\right)^2}{2\left(1-t^2\right)} = \frac{2t}{t^2 - 1}$

(b) $\frac{d^2y}{dx^2} = \frac{-2\left(t^2+1\right)}{\left(t^2-1\right)^2} \times \frac{\left(1+t^2\right)^2}{2\left(1-t^2\right)} = \frac{-\left(t^2+1\right)^3}{\left(t^2-1\right)^3}$

7 (a) $\frac{dy}{dx} = (3 + 3t^2) \times \frac{1}{2t+4} = \frac{3\left(1+t^2\right)}{2(t+2)}$

(b) $\frac{dy}{dx} = 1$: $\frac{3\left(1+t^2\right)}{2(t+2)} = 1$

$3 + 3t^2 = 2t + 4$

$3t^2 - 2t - 1 = 0$

$(3t + 1)(t - 1) = 0$

$t = -\frac{1}{3}, 1$

$t = -\frac{1}{3}$: $x = \frac{1}{9} - \frac{4}{3} = -\frac{11}{9}$

$t = 1$: $x = 1 + 4 = 5$

(c) $\frac{dt}{dx} = \frac{3\left(t^2 + 4t - 1\right)}{4(t+2)^3}$

8 $r = h$, $\frac{dV}{dt} = 3$; $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dh} = \pi h^2$

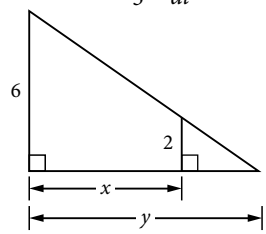
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{3}{\pi h^2}$

$h = \pi$: $\frac{dh}{dt} = \frac{3}{\pi^3}$ cm s^{-1}

9 $\frac{dx}{dt} = 1$, $\frac{y}{y-x} = \frac{6}{2}$, $y = \frac{3x}{2}$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{3}{2} \times 1 = 1.5$ m s^{-1}

Length of shadow $= y - x$

But $y - x = \frac{y}{3}$, $\frac{d}{dt}(y - x) = \frac{1}{3} \times \frac{dy}{dt} = \frac{1}{3} \times 1.5 = 0.5$ m s^{-1}



10 C

11 $\frac{dy}{dx} = \frac{-x}{\sqrt{4 - x^2}}$, $\frac{dy}{dt} = \frac{-x}{\sqrt{4 - x^2}} \times 4$

$x = 1$: $\frac{dy}{dt} = \frac{-4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$

12 $\frac{1}{8\pi}$ cm/min **13** 12

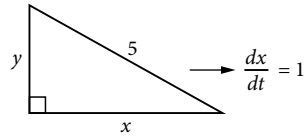
14 (a) $r = 10$: $\frac{d\theta}{dt} = -1$ radians per second

(b) $r = 10$: $\frac{dA}{dt} = -50$ cm^2 s^{-1}

15 $x^2 + y^2 = 25$, $y = \sqrt{25 - x^2}$, $\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$,

$\frac{dx}{dt} = 1$, $x = t$; at $t = 4$, $x = 4$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{-x}{\sqrt{25 - x^2}} \times 1 = \frac{-4}{\sqrt{25 - 16}} = -\frac{4}{3}$ m s^{-1}



16 $\frac{dV}{dt} = \frac{9\pi h^2}{16} \times 2 = 288\pi \text{ cm}^3/\text{min}$

17 $\frac{dP}{dt} = -\frac{4480}{32^{2.4}} \times 5 = -5\frac{15}{32} \text{ g/cm}^2$

18 At $x = 100$: $4 = \frac{100}{\sqrt{7500}} \times \frac{dx}{dt}$, $\frac{dx}{dt} = 2\sqrt{3} \text{ m s}^{-1}$

19 $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dt} = 4$, $\frac{dV}{dh} = \pi h^2$, $h = 10$

(a) $\frac{dh}{dt} = \frac{1}{25\pi}$ cm/min (b) $A = \pi h^2$, $\frac{dA}{dt} = 0.8 \text{ cm}^2/\text{min}$

20 At $x = 24$: $\frac{dy}{dt} = \frac{480}{\sqrt{625}} = 19.2$ m/min

21 (a) At $t = 1$: $\frac{dD}{dt} = 45\sqrt{2}$ km/h

(b) At $t = 1$: $\frac{dD}{dt} = 50$ km/h

22 At $t = 2$: $\frac{dD}{dt} = \frac{5 \times 96}{\sqrt{124}} \approx 43.1$ km/h

23 $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dh} = \pi h^2$, $\frac{dV}{dt} = 0.2\sqrt{h}$

At $h = 4$: $\frac{dh}{dt} = \frac{1}{40\pi}$ m/min

24 $y = 10$, $x = 8$: $\frac{dy}{dt} = 1.6 \text{ cm s}^{-1}$

25 At $h = 4$: $\frac{dh}{dt} = \frac{4}{25} = 0.16$ m/min

26 0.0732 m/min

27 (a) Area $= 60 \sin\theta$

$\frac{d(\text{Area})}{d\theta} = 60\cos\theta$, $\frac{d\theta}{dt} = 0.1$, $\frac{d(\text{Area})}{dt} = 6\cos\theta$

$\theta = \frac{\pi}{3}$: $\frac{d(\text{Area})}{dt} = 3 \text{ cm}^2 \text{ s}^{-1}$

(b) Cosine rule: $BC = \sqrt{244 - 240\cos\theta}$,

$\frac{d}{d\theta}(BC) = \frac{120\sin\theta}{\sqrt{244 - 240\cos\theta}}$

$\frac{d}{dt}(BC) = \frac{120\sin\theta}{\sqrt{244 - 240\cos\theta}} \times 0.1$

$= \frac{12\sin\theta}{\sqrt{244 - 240\cos\theta}}$

$\theta = \frac{\pi}{3}$: $\frac{d}{dt}(BC) = \frac{6\sqrt{3}}{\sqrt{244 - 120}} = \frac{3\sqrt{3}}{\sqrt{31}} \text{ cm s}^{-1}$

28 (a) $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$, $\frac{dV}{dt} = -4k\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, $-4k\pi r^2 = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = -k$

(b) $t = 0$, $r = 1$ cm; $t = 10$, $r = 0.5$ cm

$\frac{dr}{dt} = \frac{0.5 - 1}{10} = -0.05$, $r = -0.05t + C$

$1 = C$, $r = 1 - 0.05t$

29 (a) $\frac{dV}{dr} = 4\pi r^2 = S$, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, $\frac{dV}{dt} = S \times \frac{dr}{dt}$

(b) $\frac{dr}{dt} = -3$, $r = 24$

(i) $r = -3t + C$; $t = 0$, $r = 24$; $C = 24$, $r = 24 - 3t$

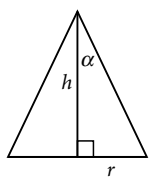
$r = 3$, $t = 7$

(ii) $V = 2304\pi$; $r = 12$, $12 = 24 - 3t$, $t = 4$

30 (a) $V = \frac{1}{3}\pi r^2 h$; $r = h\tan\alpha$:

$V = \frac{1}{3}\pi(h\tan\alpha)^2 h$

$= \frac{1}{3}\pi h^3 \tan^2\alpha$



(b) $\alpha = 30°$, $\frac{dV}{dt} = \pi\sqrt{h}$, $\frac{dV}{dh} = \pi h^2 \tan^2 30° = \frac{1}{3}\pi h^2$,

$\pi\sqrt{h} = \frac{1}{3}\pi h^2 \times \frac{dh}{dt}$

At $h = 9$: $\frac{dh}{dt} = \frac{3\sqrt{h}}{h^2} = \frac{1}{9} \text{ cm s}^{-1}$

CHAPTER REVIEW 7

1 (a) 0 (b) $a = \frac{18 - 2t^2}{(9 + t^2)^2}$ (c) $t = 3$ (d) $t = 3$, $v = \frac{1}{3} \text{ m s}^{-1}$

2 (a) $A = 124$ million, $k = \frac{1}{8}\log_e\left(\frac{55}{31}\right) \approx 0.072$

(b) 273 million (c) $t = \frac{8(\log_e 75 - \log_e 31)}{\log_e 55 - \log_e 31} \approx 12.3$, 2013

3 (a) $t = 20$ min (b) $t = 22$ min

4 B

5 $A = 5000$, $k = \frac{1}{6}\ln 2$, $N = 5000e^{2\ln 2} = 5000e^{\ln 4} = 5000 \times 4 = 20000$

6 $k = \frac{1}{40}$, $\frac{N_0}{2} = N_0 e^{-\frac{t}{40}}$, $2 = e^{\frac{t}{40}}$, $t = 40\ln 2 \approx 28$ days

7 $t = 0$, $M = A$; $k = \frac{1}{400}\ln\frac{10}{9}$

$M = A \times e^{-\frac{1000}{400}\ln\frac{10}{9}} = Ae^{2.5\ln\frac{9}{10}} = A\left(\frac{9}{10}\right)^{2.5} \approx 0.768A$; 77%

8 $\frac{dr}{dt} = 0.1$, $V = \pi r^2 h$, h constant: $\frac{dV}{dr} = 2\pi rh$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$: $\frac{dV}{dt} = 2\pi rh \times 0.1 = \frac{\pi rh}{5}$

$h = 10$, $r = 2$: $\frac{dV}{dt} = 4\pi$ cm/min

9 $r = \frac{h}{2}$, $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$, $\frac{dV}{dh} = \frac{1}{4}\pi h^2$, $\frac{dV}{dt} = 4\pi$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$: $4\pi = \frac{1}{4}\pi h^2 \times \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{16}{h^2}$

$h = 3$: $\frac{dh}{dt} = \frac{16}{9}$ cm/h

10 (a) $\frac{dT}{dt} = -k(T - 25)$, $T = 25 + Ae^{-kt}$

$t = 0$, $T = 100$: $A = 75$, $T = 25 + 75e^{-kt}$

$t = 3$, $T = 75$: $50 = 75e^{-3k}$, $k = \frac{1}{3}\ln\frac{3}{2} = 0.135$

(b) $t = 8$: $T = 25 + 75e^{-8k}$,

$T = 25 + 75e^{\ln\left(\frac{2}{3}\right)^{\frac{8}{3}}} = 25 + 75 \times \left(\frac{2}{3}\right)^{\frac{8}{3}} = 50.4°\text{C}$

11 $T = A - Be^{-0.04t}$. $t = 0$, $T = 3°\text{C}$. $A = 24°\text{C}$

$3 = 24 - B$

$B = 21$

$T = 24 - 21e^{-0.04t}$.

$T = 8°\text{C}$: $8 = 24 - 21e^{-0.04t}$

$21e^{-0.04t} = 16$

$e^{-0.04t} = \frac{16}{21}$

$-0.04t = \log_e\left(\frac{16}{21}\right)$

$t = 25\log_e\left(\frac{21}{16}\right) \approx 6.8$ minutes

12 B $x = 40t$: $\frac{dx}{dt} = 40$

$y = 56t - 16t^2$: $\frac{dy}{dt} = 56 - 32t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{8(7 - 4t)}{40} = \frac{7 - 4t}{5}$

13 $P = \frac{100}{V}$: $\frac{dP}{dV} = \frac{-100}{V^2}$

$\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$: $\frac{dP}{dt} = \frac{-100}{V^2} \times 4$

$\frac{dP}{dt} = \frac{-400}{V^2}$

14 (a) $\frac{dy}{dx} = \frac{2(t^2 + 1)}{t^2} \times \frac{t^2}{2(t^2 - 1)} = \frac{t^2 + 1}{t^2 - 1}$

(b) $\frac{d^2y}{dx^2} = \frac{-4t}{(t^2 - 1)^2} \times \frac{t^2}{2(t^2 - 1)} = \frac{-2t^3}{(t^2 - 1)^3}$

15 $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = \frac{100}{4\pi \times 25} = \frac{1}{\pi}$ cm/min

16 $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$, $\frac{dr}{dt} = \frac{5}{\pi r}$ m/min

17 (a) 15 units/s (b) 3 units/s

18 At $x = 8$: $\frac{dx}{dt} = 0.1\,\text{cm s}^{-1}$

19 At $x = \frac{b}{2}$, $\frac{dx}{dt} = -\frac{c}{al}$

Level is falling at $\frac{c}{al}$ m/min.

20 $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = -4k\pi r^2$ where k is a constant

$\frac{dV}{dr} = 4\pi r^2$, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$-4k\pi r^2 = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = -k$

Radius decreases at a constant rate.

21 Similar triangles: $\frac{y}{6} = \frac{4}{x}$, $y = \frac{24}{x}$, $\frac{dy}{dx} = -\frac{24}{x^2}$

$\frac{dy}{dt} = 3$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, $3 = -\frac{24}{x^2} \times \frac{dx}{dt}$

$y = 4$, $x = 6$: $\frac{dx}{dt} = -4.5\,\text{cm s}^{-1}$

CHAPTER 8

EXERCISE 8.1

1 (a) $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ (b) $2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$

(c) $13\sin(x + 67°\,23')$ (d) $\sqrt{5}\sin\left(x + 26°\,34'\right)$

2 (a) $2\sin\left(x - \frac{\pi}{3}\right)$ (b) $\sqrt{13}\sin\left(x - 56°19'\right)$

(c) $\sqrt{5}\sin\left(x - 26°\,34'\right)$ (d) $3\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

3 (a) $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ (b) $25\cos(x - 16°\,16')$

(c) $4\cos\left(x - \frac{\pi}{3}\right)$ (d) $\sqrt{13}\cos\left(x - 33°\,41'\right)$

4 (a) $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ (b) $2\cos\left(x + \frac{\pi}{6}\right)$

(c) $\sqrt{65}\cos\left(x + 7°\,8'\right)$ (d) $\sqrt{34}\cos\left(x + 30°\,58'\right)$

5 B

6 (a) (i) 2, $\frac{5\pi}{6}$ (ii) -2, $\frac{11\pi}{6}$ (b) (i) $\sqrt{2}$, $\frac{\pi}{4}$ (ii) $-\sqrt{2}$, $\frac{5\pi}{4}$

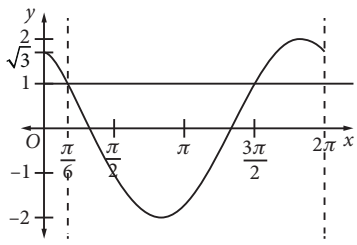
(c) (i) 4, $\frac{11\pi}{6}$ (ii) -4, $\frac{5\pi}{6}$ (d) (i) 13, 22° 37′ (ii) -13, 202° 37′

7 (a) 0, $\frac{\pi}{2}$, 2π (b) $\frac{\pi}{3}$ (c) 33° 41′ (d) $\frac{\pi}{3}$, π

(e) 156° 52′, 276° 52′ (f) 138° 24′, 295° 20′ (g) $\frac{5\pi}{3}$

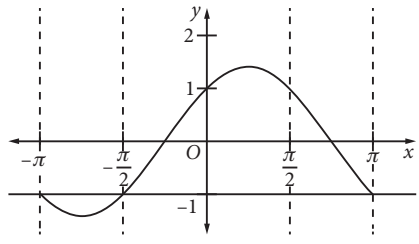
(h) $\frac{\pi}{2}$, π (i) $-76°\,43'$, 150° 27′ (j) 95° 16′, 155° 16′

8



(a) $\frac{\pi}{6}$, $\frac{3\pi}{2}$ (b) $0 \le x < \frac{\pi}{6}$, $\frac{3\pi}{2} < x \le 2\pi$

9



(a) $-\pi$, $-\frac{\pi}{2}$, π (b) $-\frac{\pi}{2} \le x \le \pi$, $x = -\pi$

EXERCISE 8.2

1 (a) $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$ (b) 0, $\frac{\pi}{2}$, π, 2π (c) 0, 2π

(d) 0, $\frac{5\pi}{6}$, π, $\frac{11\pi}{6}$, 2π (e) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$ (f) 0, $\frac{\pi}{4}$, π, $\frac{5\pi}{4}$, 2π

2 C 3 (a) $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ (b) $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$ (c) $\frac{\pi}{2}$, $\frac{3\pi}{2}$ (d) $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

4 (a) 68° 12′, 108° 26′, 248° 12′, 288° 26′

(b) 90°, 216° 52′, 323° 8′

5 A 6 (a) $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{11\pi}{6}$ (b) 0, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 2π

EXERCISE 8.3

1 (a) $x = 0.262, 3.403$ (b) $x = \frac{\pi}{4}, \frac{5\pi}{4}$ (c) $x = 2.203, 5.344$

(d) $x = 0.916, 4.058$ (e) $x = 0.2928, 2.849, 3.434, 5.990$

(f) $x = 0.372, 2.572, 3.513, 5.714$

2 (a) $x = \frac{\pi}{4}, \frac{5\pi}{4}$ (b) $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

(c) $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ (d) $x = 0.3649, 1.2059, 3.5065, 4.3475$

(e) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (f) $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

3 (a) $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ (b) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(c) $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{3\pi}{3}, 2\pi$

(d) $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$

4 (a) $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ (b) $x = 0, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \pi$

(c) $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$ (d) $x = \frac{\pi}{3}, \pi$

Only solutions in the given domain are $x = \frac{\pi}{3}, \pi$

5 (a) $x = 0.2940, 1.8648, 3.4356, 5.0064$

(b) $x = 0.9424, 2.1991, 4.0840, 5.3407$

$x = 0.3141, 2.8275, 3.4557, 5.9690$

(c) $x = 0, \pi, 2\pi$

(d) $x = 0, \pi, 2\pi$ or $x = 0.9117, 2.2299, 4.0533, 5.3714$

6 (a) $x = 0, \pi$ (b) $x = \frac{\pi}{4}, \frac{5\pi}{4}$

(c) $x = -\frac{\pi}{2}$ so no solution in the given domain.

7 (a) $x = 0.6435$ (b) $x = 0.3629, 2.1351$ (c) $x = 0.6435, 3.4254$

(d) $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$

8 $a\cos\theta + b\sin\theta = c$

$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$

$\cos(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}}$ where $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

If $a^2 + b^2 < c^2$ then $\sqrt{a^2+b^2} < c$ and $\frac{c}{\sqrt{a^2+b^2}} > 1$

Hence $\cos(\theta - \alpha) > 1$ which is not possible as by definition $-1 \le \cos x \le 1$.

Hence equation has no real roots if $a^2 + b^2 < c^2$.

9 The angles are 53° 8′, 53° 8′, 73° 44′.

10 (a) $x = 2, 3$ (b) $x = -1, \frac{1}{6}$

11 (a) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

(b) Let $x = \cos\theta$: $2(4x^3 - 3x) = \sqrt{3}$

$4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$

$\cos 3\theta = \frac{\sqrt{3}}{2}$

$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$

$x = \cos\frac{\pi}{18}, \cos\frac{11\pi}{18}, \cos\frac{13\pi}{18}$

12 (a) $\tan(3\theta) = \tan(2\theta + \theta)$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$$

$$= \frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan\theta}{1 - \left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \times \tan\theta}$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(b) Use this result to show that $\tan 15° = 2 - \sqrt{3}$. Justify your answer.

Let $\theta = 15°$ so $3\theta = 45°$ and $\tan 3\theta = 1$.

$$1 = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$1 - 3\tan^2\theta = 3\tan\theta - \tan^3\theta$

$\tan^3\theta + 1 - 3\tan^2\theta - 3\tan\theta = 0$

$(\tan\theta + 1)(\tan^2\theta - \tan\theta + 1) - 3\tan\theta(\tan\theta + 1) = 0$

$(\tan\theta + 1)(\tan^2\theta - \tan\theta + 1 - 3\tan\theta) = 0$

$(\tan\theta + 1)(\tan^2\theta - 4\tan\theta + 1) = 0$

$\tan\theta = -1$, $\tan\theta = \dfrac{4 \pm \sqrt{16-4}}{2} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$.

Since $0 < \tan 15° < 1$, then $\tan 15° = 2 - \sqrt{3}$.

(c) Use the expansion of $\tan(45° + 30°)$ to find the exact value of $\tan 75°$. Compare this answer to the other result obtained in (b).

$\tan 75° = \tan(45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\left(\sqrt{3}+1\right)^2}{3-1}$$

$$= \frac{4+2\sqrt{3}}{2}$$

$$= 2 + \sqrt{3}$$

Thus the other root from the quadratic in (b) is the exact value of $\tan 75°$.

13 (a) $\theta = 0°, 126° 52', 360°$ (b) $\theta = 77° 38', 344° 16'$
(c) $\theta = 103° 18', 330° 26'$

14 (a) $x = 0, \frac{\pi}{2}, 2\pi$ (b) $x = 0.082, 3.627$ (c) $x = 1.69, 5.88$

15 (a) $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ (b) $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$
(c) $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$ (d) $x = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$

16 (a) $\theta = \frac{\pi}{8}, \frac{7\pi}{16}, \frac{15\pi}{16}$ (b) $\theta = \frac{\pi}{4}, \frac{11\pi}{12}$
(c) $\theta = \frac{\pi}{12}, \frac{3\pi}{4}$ (d) $\theta = \frac{5\pi}{18}, \frac{17\pi}{18}$

17 (a) $x = 0, \frac{\pi}{2}, \pi$ (b) $x = \frac{\pi}{2}, \frac{3\pi}{10}, \frac{7\pi}{10}$ (c) $x = \frac{\pi}{2}$

18 (a) $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$

Let $A = \theta + \phi$ and $B = \theta - \phi$

$\dfrac{A+B}{2} = \theta$, $\dfrac{A-B}{2} = \phi$

Hence $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(b) (i) $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{\pi}{6}, \frac{5\pi}{6}$ (ii) $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$
(iii) $\theta = 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi$ (iv) $\theta = 0, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{\pi}{2}, \frac{5\pi}{7}, \pi$

19 (a) $2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi)$

Let $A = \theta + \phi$ and $B = \theta - \phi$

$\dfrac{A+B}{2} = \theta$, $\dfrac{A-B}{2} = \phi$

Hence $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(b) (i) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \frac{\pi}{2}$

(ii) $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

(iii) $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{8}, \frac{\pi}{2}, \frac{3\pi}{8}, \frac{3\pi}{2}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{4}$

CHAPTER REVIEW 8

1 (a) $\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$ (b) $2\cos\left(\theta + \frac{\pi}{3}\right) = 1$, $\theta = \frac{4\pi}{3}$

2 (a) $5\sin(x + 0.927)$ (b) 0.64 (c) $n\pi + (-1)^n \times \frac{\pi}{2} - 0.927$

3 $\tan^2\frac{\pi}{8} = \dfrac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} = \dfrac{\left(\sqrt{2}-1\right)^2}{1}$

$\tan\frac{\pi}{8} = \sqrt{2} - 1$

4 $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

Now $8\cos^3\theta - 6\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$

Hence $\cos 3\theta = -\frac{1}{2}$ $3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, $\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

$x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$, or $x = -\cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}$

(a) sum of roots: $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$,

$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = -\cos\frac{8\pi}{9} = \cos\frac{\pi}{9}$

(b) sum of products of pairs of roots:

$-\cos\frac{\pi}{9}\cos\frac{2\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\cos\frac{4\pi}{9} = -\frac{3}{4}$

$\cos\frac{\pi}{9}\cos\frac{2\pi}{9} + \cos\frac{\pi}{9}\cos\frac{4\pi}{9} = \frac{3}{4} + \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = \dfrac{3}{4\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}} + \sec\frac{\pi}{9}$

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = \dfrac{3}{4\times\frac{1}{8}} + \sec\frac{\pi}{9}$ from product of roots,

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = 6 + \sec\frac{\pi}{9}$

(c) product of roots: $\left(-\cos\frac{\pi}{9}\right)\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = -\frac{1}{8}$

So $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$, $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

(d) $\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} - \sec\frac{\pi}{9} = 6$

$\left(\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} - \sec\frac{\pi}{9}\right)^2 = 6^2$

$\sec^2\frac{4\pi}{9} + \sec^2\frac{2\pi}{9} + \sec^2\frac{\pi}{9}$

$+ 2\left(\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} - \sec\frac{\pi}{9}\sec\frac{2\pi}{9} - \sec\frac{\pi}{9}\sec\frac{4\pi}{9}\right) = 36$

$1 + \tan^2\frac{4\pi}{9} + 1 + \tan^2\frac{2\pi}{9} + 1 + \tan^2\frac{\pi}{9}$

$+ 2\left(\dfrac{\cos\frac{\pi}{9} - \cos\frac{4\pi}{9} - \cos\frac{2\pi}{9}}{\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}\right) = 36$

$\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} = 33$

5 $\theta = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{4}, \frac{7\pi}{4}$

6 (a) $\cos(2A+B) = \cos 2A\cos B - \sin 2A\sin B$
$= (2\cos^2 A - 1)\cos B - 2\sin A\cos A\sin B$
Let $A = B = \theta$:
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
$\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$

(b) $27x^3 - 9x = 1$ can be written as $\frac{27}{8}x^3 - \frac{9}{8}x = \frac{1}{8}$ or $\left(\frac{3x}{2}\right)^3 - \frac{3}{4}\times\frac{3x}{2} = \frac{1}{8}$,
so let $\frac{3x}{2} = \cos\theta$ or $x = \frac{2}{3}\cos\theta$
$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{1}{8}$, $\frac{1}{4}\cos 3\theta = \frac{1}{8}$, $\cos 3\theta = \frac{1}{2}$
$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$
Roots are $x = \frac{2}{3}\cos\frac{\pi}{9}, \frac{2}{3}\cos\frac{5\pi}{9}, \frac{2}{3}\cos\frac{7\pi}{9}$
Product of roots: $\frac{2}{3}\cos\frac{\pi}{9}\times\frac{2}{3}\cos\frac{5\pi}{9}\times\frac{2}{3}\cos\frac{7\pi}{9} = \frac{1}{27}$
$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$
Now $\cos\frac{3\pi}{9} = \cos\frac{\pi}{3} = \frac{1}{2}$ so
$\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$

7 $\tan\alpha + \tan\beta + \tan\gamma = a + 1$
$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = c - a$
$\tan\alpha\tan\beta\tan\gamma = c$
$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha)}$$
$$= \frac{a+1-c}{1-(c-a)} = 1$$
$\alpha+\beta+\gamma = n\pi + \frac{\pi}{4}$

8 $\cos 5x = \cos\left(\frac{\pi}{2} - x\right)$
$5x = \frac{\pi}{2} - x, 2\pi - \left(\frac{\pi}{2} - x\right), 2\pi + \left(\frac{\pi}{2} - x\right), 4\pi - \left(\frac{\pi}{2} - x\right),$
$4\pi + \left(\frac{\pi}{2} - x\right)$
$6x = \frac{\pi}{2}, 4x = \frac{3\pi}{2}, 6x = \frac{5\pi}{2}, 4x = \frac{7\pi}{2}, 6x = \frac{9\pi}{2}$
$x = \frac{\pi}{12}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{8}$.

9 (a) $x = A + B$, $y = A - B$ so $A = \frac{x+y}{2}$, $B = \frac{x-y}{2}$

(b) $\sin 2\theta + \sin\theta + \sin 3\theta = 0$
$\sin 2\theta + 2\sin 2\theta\cos\theta = 0$
$\sin 2\theta(1 + 2\cos\theta) = 0$
$\sin 2\theta = 0$ or $\cos\theta = -0.5$
$2\theta = 0, \pi, 2\pi$ or $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$
$\theta = 0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

CHAPTER 9

Many of the proofs below are not complete: in most cases only the outline of the proof of $S(k+1)$ is given. For more complete solutions, see the Student Worked Solutions book.

EXERCISE 9.1

1 D

2 For $S(k+1)$:
$\text{LHS} = 1 + 2 + 3 + \ldots + k + k + 1$
$= \frac{k(k+1)}{2} + k + 1$
$= \frac{(k+1)}{2}\times(k+2) = \text{RHS}$

3 For $S(k+1)$:
$\text{LHS} = 1 + 2 + 4 + \ldots + 2^{k-1} + 2^k$
$= 2^k - 1 + 2^k = 2\times 2^k - 1$
$= 2^{k+1} - 1 = \text{RHS}$

4 For $S(k+1)$:
$\text{LHS} = 2 + 5 + 8 + \ldots + (3k-1) + (3k+2)$
$= \frac{k(3k+1)}{2} + 3k + 2$
$= \frac{(k+1)(3k+4)}{2} = \text{RHS}$

5 $n = 1$: LHS $= a$. RHS $= \frac{1}{2}\times 2a = a = \text{LHS}$. Result is true for $n = 1$

$S(k) = \frac{k}{2}[2a + (k-1)d]$, $S(k+1) = \frac{(k+1)}{2}[2a + kd]$
$S(k+1) = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) + (a+kd)$
$= \frac{k}{2}[2a + (k-1)d] + (a+kd)$
$= \frac{k(2a+kd-d) + 2(a+kd)}{2}$
$= \frac{2ak + k^2d - kd + 2a + 2kd}{2}$
$= \frac{k^2d + kd + 2a + 2ak}{2}$
$= \frac{kd(k+1) + 2a(k+1)}{2}$
$= \frac{(k+1)}{2}[2a + kd] = \text{RHS}$

6 $n = 1$: LHS $= a$. RHS $= \frac{a(r-1)}{r-1} = a = \text{LHS}$. Result is true for $n = 1$

$S(k) = \frac{a(r^k - 1)}{r-1}$, $S(k+1) = \frac{a(r^{k+1} - 1)}{r-1}$
$S(k+1) = a + ar + ar^2 + \ldots + ar^{k-1} + ar^k$
$= \frac{a(r^k - 1)}{r-1} + ar^k$
$= \frac{a(r^k - 1) + ar^k(r-1)}{r-1}$
$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$
$= \frac{a(r^{k+1} - 1)}{r-1} = \text{RHS}$

7 For $S(k+1)$:
$\text{LHS} = 1 + r + r^2 + r^3 + \ldots + r^{k-1} + r^k$
$= \frac{1-r^k}{1-r} + r^k$
$= \frac{1 - r^k + r^k - r^{k+1}}{1-r}$
$= \frac{1-r^{k+1}}{1-r} = \text{RHS}$

8 $n = 1$: LHS $= 2$. RHS $= \frac{1\times 2\times 3}{3} = 2 = \text{LHS}$. Result is true for $n = 1$

$S(k) = \frac{k(k+1)(k+2)}{3}$, $S(k+1) = \frac{(k+1)(k+2)(k+3)}{3}$
$S(k+1) = 1\times 2 + 2\times 3 + 3\times 4 + \ldots + k(k+1) + (k+1)(k+2)$
$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$
$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$
$= \frac{(k+1)(k+2)(k+3)}{3} = \text{RHS}$

9 $n = 1$: LHS $= 3$. RHS $= \frac{1\times 2\times 9}{6} = 3 = \text{LHS}$. Result is true for $n = 1$

$S(k) = \frac{k(k+1)(2k+7)}{6}$, $S(k+1) = \frac{(k+1)(k+2)(2k+9)}{6}$

$S(k+1) = 1\times3 + 2\times4 + 3\times5 + \ldots + k(k+2) + (k+1)(k+3)$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$
$$= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}$$
$$= \frac{(k+1)\left[k(2k+7) + 6(k+3)\right]}{6}$$
$$= \frac{(k+1)\left[2k^2 + 7k + 6k + 18\right]}{6}$$
$$= \frac{(k+1)\left[2k^2 + 13k + 18\right]}{6}$$
$$= \frac{(k+1)(k+2)(2k+9)}{6} = \text{RHS}$$

10 $n = 1$: LHS $= 4$. RHS $= \frac{1\times2^3}{2} = 4 =$ LHS. Result is true for $n = 1$

$S(k) = \frac{k^3(k+1)^3}{2}$, $S(k+1) = \frac{(k+1)^3(k+2)^3}{2}$

$S(k+1) = 4 + 104 + 756 + \ldots + (k^3 + 3k^5) + [(k+1)^3 + 3(k+1)^5]$

$$= \frac{k^3(k+1)^3}{2} + [(k+1)^3 + 3(k+1)^5]$$
$$= \frac{k^3(k+1)^3 + 2(k+1)^3 + 6(k+1)^5}{2}$$
$$= \frac{(k+1)^3\left[k^3 + 2 + 6(k+1)^2\right]}{2}$$
$$= \frac{(k+1)^3\left[k^3 + 6k^2 + 12k + 8\right]}{2}$$
$$= \frac{(k+1)^3(k+2)^3}{2} = \text{RHS}$$

11 For $S(k+1)$:

$$\text{LHS} = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}$$

12 For $S(k+1)$:

$$\text{LHS} = \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)}$$
$$= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k(k+3)+2}{2(k+2)(k+3)} = \frac{k+1}{2(k+3)}$$
$$= \text{RHS}$$

13 For $S(k+1)$:

$$\text{LHS} = \frac{1}{1\times3} + \frac{1}{3\times5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$
$$= \text{RHS}$$

14 For $S(k+1)$:

$$\text{LHS} = \frac{1^2}{1\times3} + \frac{2^2}{3\times5} + \frac{3^2}{5\times7} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$
$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$
$$= \frac{k(k+1)(2k+3) + 2(k+1)^2}{2(2k+1)(2k+3)}$$
$$= \frac{(k+1)\left[k(2k+3) + 2(k+1)\right]}{2(2k+1)(2k+3)}$$
$$= \frac{(k+1)(k+2)(2k+1)}{2(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)} = \text{RHS}$$

15 (a) For $S(k+1)$:

$$\text{LHS} = 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k^2+4k+4)}{4}$$
$$= \frac{(k+1)^2(k+2)^2}{4} = \text{RHS, limit} = \frac{1}{4}$$

(b) In question 2 it was shown that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Hence $(1 + 2 + 3 + \ldots + n)^2 = \frac{n^2(n+1)^2}{2} = 1^3 + 2^3 + 3^3 + \ldots + n^3$ as shown in (a).

16 For $S(k)$: $(k+1) + (k+2) + \ldots + 2k = \frac{k(3k+1)}{2}$

For $S(k+1)$:

$$\text{LHS} = (k+2) + \ldots + 2k + (2k+1) + (2k+2)$$
$$= (k+1) + (k+2) + \ldots + 2k + (3k+2)$$
$$= \frac{k(3k+1)}{2} + (3k+2) = \frac{3k^2+7k+4}{2} = \frac{(k+1)(3k+4)}{2}$$
$$= \text{RHS}$$

17 For $S(k+1)$:

$$\text{LHS} = 1\times1! + 2\times2! + 3\times3! + \ldots + k\times k! + (k+1)\times(k+1)!$$
$$= (k+1)! - 1 + (k+1)\times(k+1)!$$
$$= (k+1)!\times[1 + (k+1)] - 1$$
$$= (k+1)!\times(k+2) - 1 = (k+2)! - 1 = \text{RHS}$$

18 $n = 1$: LHS $= 1$. RHS $= 1 + 0 = 1 =$ LHS. Result is true for $n = 1$

$S(k) = 1 + (k-1)\times2^k$, $S(k+1) = 1 + k\times2^{k+1}$

$$S(k+1) = 1\times2^0 + 2\times2^1 + 3\times2^2 + \ldots + k\times2^{k-1} + (k+1)\times2^k$$
$$= 1 + (k-1)\times2^k + (k+1)\times2^k$$
$$= 1 + (k - 1 + k + 1)\times2^k$$
$$= 1 + 2k\times2^k$$
$$= 1 + k\times2^{k+1} = \text{RHS}$$

19 $n = 1$: LHS $= 1$. RHS $= \frac{1\times3}{3} = 1 =$ LHS. Result is true for $n = 1$

$S(k) = \frac{k\left(4k^2-1\right)}{3} = \frac{k(2k-1)(2k+1)}{3}$,

$S(k+1) = \frac{(k+1)(2k+1)(2k+3)}{3}$

$S(k+1) = 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+1)^2$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$
$$= \frac{(2k+1)\left(k(2k-1)\right) + 3(2k+1)}{3}$$
$$= \frac{(2k+1)\left(2k^2+5k+3\right)}{3}$$
$$= \frac{(k+1)(2k+1)(2k+3)}{3} = \text{RHS}$$

20 $n = 1$: LHS $= 1$. RHS $= \frac{1\times1\times2}{2} = 1 =$ LHS.
Result is true for $n = 1$

$S(k) = \frac{(-1)^{k-1}k(k+1)}{2}$, $S(k+1) = \frac{(-1)^k(k+1)(k+2)}{2}$

$S(k+1) = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2$

$$= \frac{(-1)^{k-1}k(k+1)}{2} + (-1)^k(k+1)^2$$
$$= \frac{(-1)^{k-1}k(k+1) + 2(-1)^k(k+1)^2}{2}$$
$$= \frac{(-1)^k(k+1)\left(-k + 2(k+1)\right)}{2}$$
$$= \frac{(-1)^k(k+1)(k+2)}{2} = \text{RHS}$$

21 $n = 1$: LHS $= 2$. RHS $= 8 - 6 = 2 =$ LHS. Result is true for $n = 1$

$S(k) = (k^2 - 2k + 3)\times2^{k+1} - 6$, $S(k+1)$

$$= ((k+1)^2 - 2(k+1) + 3)\times2^{k+2} - 6$$
$$= (k^2+2)\times2^{k+2} - 6$$

$S(k+1) = 1\times2 + 4\times2^2 + 9\times2^3 + \ldots + k^2\times2^k + (k+1)^2\times2^{k+1}$

$= (k^2 - 2k + 3) \times 2^{k+1} - 6 + (k+1)^2 \times 2^{k+1}$
$= 2^{k+1}(k^2 - 2k + 3 + k^2 + 2k + 1) - 6$
$= 2^{k+1}(2k^2 + 4) - 6$
$= (k^2 + 2) \times 2^{k+2} - 6 = \text{RHS}$

22 For $S(k+1)$:

$$\text{LHS} = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \ldots + \frac{2^k - 1}{2^k} + \frac{2^{k+1} - 1}{2^{k+1}}$$
$$= k - 1 + \frac{1}{2^k} + \frac{2^{k+1} - 1}{2^{k+1}} = k - 1 + \frac{1}{2^k} + 1 - \frac{1}{2^{k+1}}$$
$$= k + \frac{2-1}{2^{k+1}} = k + \frac{1}{2^{k+1}} = \text{RHS}$$

23 $n = 1$: LHS $= 4$. RHS $= 1 \times 2^2 = 4 = $ LHS. Result is true for $n = 1$.
$S(k) = k \times 2^{k+1}$, $S(k+1) = (k+1) \times 2^{k+2}$
$S(k+1) = 2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \ldots + (k+1) \times 2^k + (k+2) \times 2^{k+1}$
$= k \times 2^{k+1} + (k+2) \times 2^{k+1}$
$= 2^{k+1}(2k+2)$
$= (k+1) \times 2^{k+2} = \text{RHS}$

24 $n = 1$: LHS $= 1$. RHS $= \frac{1 \times 2 \times 3 \times 5}{30} = 1 = $ LHS.
Result is true for $n = 1$

$$S(k) = \frac{k(k+1)(2k+1)\left(3k^2 + 3k - 1\right)}{30},$$

$$S(k+1) = \frac{(k+1)(k+2)(2k+3)\left(3k^2 + 9k + 5\right)}{30}$$

$$S(k+1) = 1^4 + 2^4 + 3^4 + \ldots + k^4 + (k+1)^4$$
$$= \frac{k(k+1)(2k+1)\left(3k^2 + 3k - 1\right)}{30} + (k+1)^4$$
$$= \frac{k(k+1)(2k+1)\left(3k^2 + 3k - 1\right) + 30(k+1)^4}{30}$$
$$= \frac{(k+1)\left[k(2k+1)\left(3k^2 + 3k - 1\right) + 30(k+1)^3\right]}{30}$$
$$= \frac{(k+1)\left[\left(2k^2 + k\right)\left(3k^2 + 3k - 1\right) + 30(k+1)^3\right]}{30}$$
$$= \frac{(k+1)\left[6k^4 + 9k^3 + k^2 - k + 30\left(k^3 + 3k^2 + 3k + 1\right)\right]}{30}$$
$$= \frac{(k+1)\left[6k^4 + 39k^3 + 91k^2 + 89k + 30\right]}{30}$$

$$\text{RHS} = \frac{(k+1)(k+2)(2k+3)\left(3k^2 + 9k + 5\right)}{30}$$
$$= \frac{(k+1)(k+2)\left(6k^3 + 27k^2 + 37k + 15\right)}{30}$$
$$= \frac{(k+1)\left(6k^4 + 39k^3 + 91k^2 + 89k + 30\right)}{30} = \text{LHS}$$

EXERCISE 9.2

1 C

2 $S(k)$ is $5^k = 4M - 3$
For $S(k+1)$: $5^{k+1} + 3 = 5 \times 5^k + 3 = 5(4M - 3) + 3$
$= 20M - 12 = 4(5M - 3)$
which is divisible by 4 as $(5M - 3)$ is an integer

3 $S(k)$ is $3^{2k} = 8M + 1$
For $S(k+1)$: $3^{2k+2} - 1 = 9 \times 3^{2k} - 1 = 9(8M + 1) - 1$
$= 72M + 8 = 8(9M + 1)$
which is divisible by 8 as $(9M + 1)$ is an integer

4 $S(k)$ is $3^k = 5M - 2^k$
For $S(k+2)$: $3^{k+2} + 2^{k+2} = 9 \times 3^k + 4 \times 2^k$
$= 9(5M - 2^k) + 4 \times 2^k = 45M - 5 \times 2^k = 5(9M - 2^k)$
which is divisible by 5 as $(9M - 2^k)$ is an integer

5 $S(k)$ is $5^k = 3M - 2 \times 11^k$
For $S(k+1)$: $5^{k+1} + 2 \times 11^{k+1}$
$= 5 \times 5^k + 22 \times 11^k = 5(3M - 2 \times 11^k) + 22 \times 11^k$
$= 15M + 12 \times 11^k = 3(5M + 4 \times 11^k)$
which is divisible by 3 as $(5M + 4 \times 11^k)$ is an integer.

6 (a) $(k+1)(k+2)(k+3)$
(b) $S(k)$ is $k(k+1)(k+2) = 3M$
For $S(k+1)$: $(k+1)(k+2)(k+3)$
$= k(k+1)(k+2) + 3(k+1)(k+2)$
$= 3M + 3(k+1)(k+2) = 3(M + (k+1)(k+2))$
which is divisible by 3 as $(M + (k+1)(k+2))$ is an integer.

7 $S(k)$ is $3^{3k} = 5M - 2^{k+2}$
For $S(k+1)$: $3^{3k+3} + 2^{k+3}$
$= 27 \times 3^{3k} + 2 \times 2^{k+2} = 27(5M - 2^{k+2}) + 2 \times 2^{k+2}$
$= 135M - 25 \times 2^{k+2} = 5(27M - 5 \times 2^{k+2})$
which is divisible by 5 as $(27M - 5 \times 2^{k+2})$ is an integer.

8 $n = 2$, Exp $= 7^2 - 2^2 = 49 - 4 = 45$, which is divisible by 9.
$S(k)$ is $7^k - 2^k = 9M$
i.e. $7^k = 9M + 2^k$
For $S(k+1)$: Exp $= 7^{k+1} - 2^{k+1}$
$= 7 \times 7^k - 2 \times 2^k$
$= 7(9M - 2^k) - 2 \times 2^k$
$= 7 \times 9M - 2^k(7 + 2)$
$= 9(7M - 2^k)$ which is divisible by 9.

9 $n = 1$, Exp $= 3^4 - 1 = 80$, which is divisible by 80.
$S(k)$ is $3^{4k} - 1 = 80M$
i.e. $3^{4k} = 80M + 1$
For $S(k+1)$: Exp $= 3^{4(k+1)} - 1$
$= 3^4 \times 3^{4k} - 1$
$= 3^4(80M + 1) - 1$
$= 81 \times 80M + 81 - 1$
$= 80(81M + 1)$, which is divisible by 80.

10 $n = 1$: Exp $= 5 + 22 = 27$, which is divisible by 3.
$S(k)$ is $5^k + 2 \times 11^k = 3M$
i.e. $5^k = 3M - 2 \times 11^k$
For $S(k+1)$: Exp $= 5^{k+1} + 2 \times 11^{k+1}$
$= 5 \times 5^k + 22 \times 11^k$
$= 5(3M - 2 \times 11^k) + 22 \times 11^k$
$= 15M + 12 \times 11^k$
$= 3(5M + 4 \times 11^k)$, which is divisible by 3.

11 $n = 1$: Exp $= 8 - 1 = 7$, which is divisible by 7.
$S(k)$ is $2^{3k} - 1 = 7M$
For $S(k+1)$: Exp $= 2^{3k+3} - 1$
$= 8 \times 2^{3k} - 1$
$= 8(2^{3k} - 1) + 7$
$= 8 \times 7M - 7$
$= 7(8M - 1)$, which is divisible by 7.

12 $n = 1$: Exp $= 6 + 10 - 6 = 10$, which is divisible by 5.
$S(k)$ is $6^k + 10k - 6 = 5M$
For $S(k+1)$: Exp $= 6^{k+1} + 10(k+1) - 6$
$= 6 \times 6^k + 10k + 10 - 6$
$= 6 \times 6^k + 6 \times 10k - 6 \times 6 \ - 50k + 40$
$= 6 \times 5M - 5(10k - 8)$
$= 5(6M - 10k + 8)$, which is divisible by 5.

13 $n = 1$: Exp $= 27 + 1 = 28$, which is divisible by 7.
$S(k)$ is $3^{2k+1} + 2^{k-1} = 7M$
is $3^{2k+1} = 7M - 2^{k-1}$
For $S(k+1)$: Exp $= 3^{2k+3} + 2^k$
$= 9 \times 3^{2k+1} + 2 \times 2^{k-1}$
$= 9(7M - 2^{k-1}) + 2 \times 2^{k-1}$
$= 63M - 9 \times 2^{k-1} + 2 \times 2^{k-1}$
$= 63M - 7 \times 2^{k-1}$
$= 7(9M - 2^{k-1})$, which is divisible by 7.

14 (a) $(k+3)^3 = (k+3)(k+3)^2 = k^3 + 9k^2 + 27k + 27$

(b) Must prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 3 for all integers $n \geq 1$.

$S(k)$ is $k^3 + (k+1)^3 + (k+2)^3 = 3M$

For $S(k+1)$: $(k+1)^3 + (k+2)^3 + (k+3)^3$

$= (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27$

$= 3M + 9k^2 + 27k + 27 = 3(M + 3k^2 + 9k + 9)$

which is divisible by 3 as $(M + 3k^2 + 9k + 9)$ is an integer

15 Prove $S(1)$ is true: $(x-1)^3 + x^3 = 2x^3 - 3x^2 + 3x - 1$

$= (x^2 - x + 1)(2x - 1)$

which is divisible by $x^2 - x + 1$

Assume $S(k)$ is true, i.e. $(x-1)^{k+2} = (x^2 - x + 1)M(x) - x^{2k+1}$

Prove $S(k+1)$ is true: $(x-1)^{k+3} + x^{2k+3}$

$= (x-1)(x-1)^{k+2} + x^2 \times x^{2k+1}$

$= (x-1)[(x^2 - x + 1)M(x) - x^{2k+1}] + x^2 \times x^{2k+1}$

$= (x-1)(x^2 - x + 1)M(x) - (x-1) \times x^{2k+1} + x^2 \times x^{2k+1}$

$= (x-1)(x^2 - x + 1)M(x) + x^{2k+1}(x^2 - x + 1)$

$= (x^2 - x + 1)[(x-1)M(x) + x^{2k+1}]$

which is divisible by $x^2 - x + 1$

16 $S(k)$ is $x^k - 1 = (x-1)M(x)$

For $S(k+1)$:

$x^{k+1} - 1 = x^{k+1} - x^k + x^k - 1$

$= x^k(x-1) + (x-1)M(x) = (x-1)[x^k + M(x)]$

which is divisible by $x - 1$

EXERCISE 9.3

1 (a) $S(k) = k^2 + k = (2N - 1)$, where N is a positive integer

$S(k+1) = (k+1)^2 + (k+1)$

$= k^2 + 3k + 2$

$= k^2 + k + 2(k+1)$

= Odd + Even = Odd, so true when $S(k)$ is true

(b) $S(1) = 1 + 1 = 2$, so result is not true.

(c) No. If n is odd, then n^2 is odd and odd + odd = even

If n is even, then n^2 is even and even + even = even

(d) The statement should be that $n^2 + n$ is an even integer for $n \geq 1$.

$S(1) = 1 + 1 = 2$, so result is true for $n = 1$

$S(k) = k^2 + k = 2N$, where N is a positive integer

$S(k+1) = (k+1)^2 + (k+1)$

$= k^2 + 3k + 2$

$= k^2 + k + 2(k+1)$

= Even + Even = Even so, result is true if $S(k)$ is true

2 (a) $S(k) = \frac{k}{2}(6k^2 - 3k - 1)$

$S(k+1) = 1^2 + 4^2 + 7^2 + \ldots + (3k-2)^2 + (3k+1)^2$

$= \frac{k}{2}(6k^2 - 3k - 1) + (3k+1)^2$

$= 3k^3 - \frac{3k^2}{2} - \frac{k}{2} + 9k^2 + 6k + 1$

$= 3k^3 + \frac{15k^2}{2} + \frac{11k}{2} + 1$

And Exp $= \frac{k+1}{2}(6(k+1)^2 - 3(k+1) - 1)$

$= \frac{k+1}{2}(6k^2 + 12k + 6 - 3k - 3 - 1)$

$= \frac{k+1}{2}(6k^2 + 9k + 2)$

$= \frac{1}{2}(6k^3 + 15k^2 + 11k + 2)$

$= 3k^3 + \frac{15k^2}{2} + \frac{11k}{2} + 1$

which was the expression obtained for $S(k+1)$

(b) $S(1)$: LHS = 1, RHS $= \frac{1}{2}(6 - 3 - 1) = 1$ = LHS so true

(c) Hence result is true for all $n \geq 1$

3 (a) $S(1) = 1 - 1 + 41 = 41$ which is prime

(b) $S(2) = 4 - 2 + 41 = 43$ which is prime

(c) $S(5) = 25 - 5 + 41 = 61$ which is prime

(d) $S(42) = 41^2 - 41 + 41 = 41^2$ which has a factor of 41 so is not prime

(e) Original statement is false

4 (a) $S(k) = k^2 - k$ is an odd integer.

Show that $S(k+1) = (k+1)^2 - (k+1)$ is an odd integer.

$S(k+1) = (k+1)^2 - (k+1)$

$= (k+1)(k+1-1)$

$= k(k+1)$

$= k^2 + k$

$= k^2 - k + 2k$

= odd integer + even integer

= odd integer

Result is true for $n = k+1$ if it is true for $n = k$.

(b) $S(1) = 1 - 1 = 0$, which is not odd

(c) It is never true. When n is off, odd^2 − odd = odd − odd = even. When n is even, even^2 − even = even.

CHAPTER REVIEW 9

1 For $S(k+1)$:

LHS $= 2k(k-2) + 4(k+1) - 6$

$= 2k^2 - 2$ = RHS

2 $S(k)$ is $4^{k+1} = 21M - 5^{2k-1}$

For $S(k+1)$: $4^{k+2} + 5^{2k+1} = 4 \times (21M - 5^{2k-1}) + 25 \times 5^{2k-1}$

$= 84M + 21 \times 5^{2k-1} = 21(4M + 5^{2k-1})$

which is divisible by 21 as $4M + 5^{2k-1}$ is an integer

3 $S(k)$ is $7^k = 3M + 1$

For $S(k+1)$: $7^{k+1} - 1 = 7(3M+1) - 1 = 3(7M+2)$

which is divisible by 3 as $7M + 2$ is an integer

4 $n = 1$: LHS = 6. RHS $= \frac{1 \times 2 \times 3 \times 4}{4} = 6$ = LHS.

Result is true for $n = 1$

$S(k) = \frac{k(k+1)(k+2)(k+3)}{4}$

$S(k+1) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$

$S(k+1) = 6 + 24 + 60 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$

$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$ = RHS

5 $n = 1$: LHS $= \frac{3}{4}$. RHS $= 1 - \frac{1}{4} = \frac{3}{4}$ = LHS.

Result is true for $n = 1$

$S(k) = 1 - \frac{1}{(k+1)2^k}$, $S(k+1) = 1 - \frac{1}{(k+2)2^{k+1}}$

$S(k+1) = \frac{3}{4} + \frac{1}{6} + \frac{5}{96} + \ldots + \frac{k+2}{k(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$

$= 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$

$= 1 - \frac{2(k+2) - (k+3)}{(k+1)(k+2)2^{k+1}}$

$= 1 - \frac{2k + 4 - k - 3}{(k+1)(k+2)2^{k+1}}$

$= 1 - \frac{k+1}{(k+1)(k+2)2^{k+1}}$

$= 1 - \frac{1}{(k+2)2^{k+1}}$ = RHS

6 $S(k)$ is $5^k = 16M - 12k + 1$
For $S(k+1)$: $5^{k+1} + 12(k+1) - 1$
$= 5(16M - 12k + 1) + 12k + 11$
$= 80M - 48k + 16 = 16(5M - 3k + 1)$
which is divisible by 16 as $5M - 3k + 1$ is an integer

7 $S(k)$ is $7^k = 13M - 6^k$
For $S(k+2)$: $7^{k+2} + 6^{k+2}$
$= 49(13M - 6^k) + 36 \times 6^k$
$= 637M - 13 \times 6^k = 13(49M - 6^k)$
which is divisible by 13 as $49M - 6^k$ is an integer

8 For $S(k+1)$:
LHS $= k(k+1)! + ((k+1)^2 + 1)(k+1)!$
$= (k+1)![k + k^2 + 2k + 2]$
$= (k+1)!(k+2)(k+1) = (k+1)(k+2)! =$ RHS

9 Prove by induction that $\frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ for all integers $n \geq 1$

$n = 1$: LHS $= \frac{1}{4}$. RHS $= \frac{1}{1 \times 4} = \frac{1}{4} =$ LHS. Result is true for $n = 1$

$S(k) = \frac{k}{3k+1}$, $S(k+1) = \frac{k+1}{3k+4}$

$S(k+1) = \frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \ldots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2 + 4k + 1}{(3k+1)(3k+4)}$

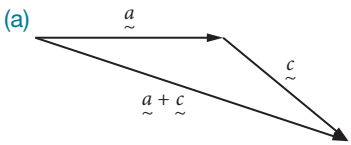
$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$= \frac{k+1}{3k+4} =$ RHS

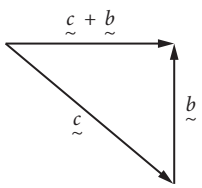
CHAPTER 10

EXERCISE 10.1

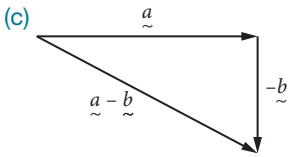
1 (a)



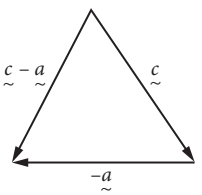
(b)



(c)



(d)



2 (a) $\overrightarrow{CD} = -\underset{\sim}{a}$ (b) $\overrightarrow{AD} = \underset{\sim}{b}$ (c) $\overrightarrow{CA} = -\underset{\sim}{a} - \underset{\sim}{b}$ (d) $\overrightarrow{DB} = \underset{\sim}{a} - \underset{\sim}{b}$

3 (a) $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$ (b) $\overrightarrow{AD} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ (c) $\overrightarrow{DA} = -\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c}$
(d) $\overrightarrow{DB} = -\underset{\sim}{b} - \underset{\sim}{c}$

4 (a) $\overrightarrow{CD} = -\frac{3}{2}\underset{\sim}{a}$ (b) $\overrightarrow{CA} = -\underset{\sim}{a} - \underset{\sim}{b}$ (c) $\overrightarrow{AD} = -\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{b}$
(d) $\overrightarrow{DB} = \frac{3}{2}\underset{\sim}{a} - \underset{\sim}{b}$

5 (a) $\overrightarrow{OP} = \underset{\sim}{a} + \underset{\sim}{b}$ (b) $\overrightarrow{OG} = 2\underset{\sim}{a} + 2\underset{\sim}{b}$ (c) $\overrightarrow{OQ} = 2\underset{\sim}{a} + \underset{\sim}{b}$
(d) $\overrightarrow{CE} = \underset{\sim}{a} + \underset{\sim}{b}$ (e) $\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$ (f) $\overrightarrow{DI} = -3\underset{\sim}{a} + 2\underset{\sim}{b}$
(g) $\overrightarrow{FQ} = -\underset{\sim}{a} - \underset{\sim}{b}$ (h) $\overrightarrow{DE} + \overrightarrow{EO} = -3\underset{\sim}{a}$

6 (a) $\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$ (b) $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

7 (a) $\overrightarrow{VY} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ (b) $\overrightarrow{VZ} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}$ (c) $\overrightarrow{WZ} = \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}$

8 B **9** D **10** B

11 (a) $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{a}$, $\overrightarrow{DE} = \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$

(b) $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

(c) The line joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.

(d) $\overrightarrow{BF} = \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{a})$. $\overrightarrow{FC} = \frac{2}{3}(\underset{\sim}{c} - \underset{\sim}{a})$

(e) $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$
$= \underset{\sim}{a} + \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{a})$
$= \underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c} - \frac{1}{3}\underset{\sim}{a}$
$= \frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c}$
$= \frac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{c})$

12 (a) $\overrightarrow{AC} = \underset{\sim}{b} + \underset{\sim}{d}$ (b) $\overrightarrow{AE} = \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{d}$ (c) $\overrightarrow{DE} = \underset{\sim}{b} - \frac{1}{2}\underset{\sim}{d}$

(d) Express the vector in terms of $\overrightarrow{AD}$ and $\overrightarrow{AB}$. $\overrightarrow{DF} = \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{d}$

(e) $\overrightarrow{AF} = \overrightarrow{AD} + \overrightarrow{DF}$
$= \underset{\sim}{d} + \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{d}$
$= \frac{2}{3}\underset{\sim}{b} + \frac{2}{3}\underset{\sim}{d}$
$= \frac{2}{3}(\underset{\sim}{b} + \underset{\sim}{d})$

Now $\overrightarrow{AC} = \underset{\sim}{b} + \underset{\sim}{d}$

$\therefore \overrightarrow{AF} = \frac{2}{3}\overrightarrow{AC}$

Hence F lies on $\overrightarrow{AC}$.

(f) $\overrightarrow{AF} : \overrightarrow{FC} = 2 : 1$

EXERCISE 10.2

1 (a) $(6, -18)$ (b) $(1, -3)$ (c) $(-2, 6)$ (d) $(0.8, -2.4)$

2 (a) $-2\underset{\sim}{b} = \begin{pmatrix} 8 \\ -10 \end{pmatrix}$ (b) $5\underset{\sim}{b} = \begin{pmatrix} -20 \\ 25 \end{pmatrix}$
(c) $\frac{1}{3}\underset{\sim}{b} = \begin{pmatrix} -\frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$ (d) $-\frac{5}{4}\underset{\sim}{b} = \begin{pmatrix} 5 \\ -\frac{25}{4} \end{pmatrix}$

3 (a) $-\underset{\sim}{c} = (-6, 3)$ (b) $2\underset{\sim}{c} = (12, -6)$
(c) $-\frac{1}{3}\underset{\sim}{c} = (-2, 1)$ (d) $1.5\underset{\sim}{c} = (9, -4.5)$

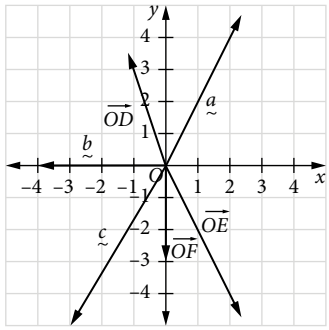
4 (a) $-\underset{\sim}{c} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$ (b) $2\underset{\sim}{c} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$
(c) $-\frac{1}{3}\underset{\sim}{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (d) $1.5\underset{\sim}{c} = \begin{pmatrix} 9 \\ -4.5 \end{pmatrix}$

5 (a) $\underset{\sim}{a} = (5, 5)$ (b) $\underset{\sim}{b} = (4, -5)$ (c) $\underset{\sim}{c} = (0, 9)$
(d) $\underset{\sim}{d} = (-13, -6)$ (e) $\underset{\sim}{e} = (10, 2)$ (f) $\underset{\sim}{f} = (-3, 5)$

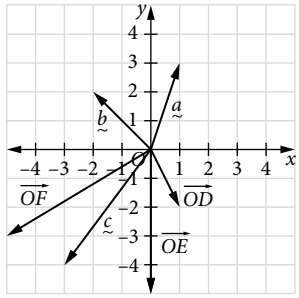
6 (a) $\underset{\sim}{a} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ (b) $\underset{\sim}{b} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ (c) $\underset{\sim}{c} = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$
(d) $\underset{\sim}{d} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ (e) $\underset{\sim}{e} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$ (f) $\underset{\sim}{f} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$

7 A **8** C

9



10



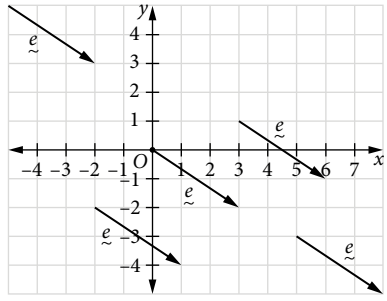
11 (a) The column vector is $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$.

(b) The column vector is $\begin{pmatrix} -11 \\ -16 \end{pmatrix}$.

(c) The coordinates of the terminal point are $(13, -2)$.

(d) The coordinates of the initial point are $(19, -15)$.

(e)



12 (a) $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$

(b) $\left|\overrightarrow{AB}\right| = 2\sqrt{13}, \left|\overrightarrow{BC}\right| = 2\sqrt{13}, \left|\overrightarrow{AC}\right| = 2\sqrt{26}$

(c) $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 = \left(\sqrt{52}\right)^2 + \left(\sqrt{52}\right)^2$

$= 104$

$= \left|\overrightarrow{AC}\right|^2$

and so ΔABC is a right–angled triangle.

(d) The coordinates of D are $(4, -7)$.

(e) The coordinates of the point of intersection of the diagonals of the square $ABCD$ are $(3, -2)$.

13 (a) $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$, $\overrightarrow{OX} = \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$ (b) $\overrightarrow{CX} = \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$

(c) $\overrightarrow{CY} = \frac{2}{3}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{c}$

(d) $\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY}$

$= \underset{\sim}{c} + \frac{2}{3}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{c}$

$= \frac{2}{3}\underset{\sim}{c} + \frac{2}{3}\underset{\sim}{a}$

$= \frac{2}{3}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)$

$= \frac{2}{3}\overrightarrow{OB}$

Hence, Y lies on $\overrightarrow{OB}$

(e) $\overrightarrow{OY} : \overrightarrow{YB} = 2 : 1$

14 (a) The position vector of the midpoint of $\overline{AB}$ is $\left(-\frac{1}{2}, \frac{15}{2}\right)$.

(b) The position vector of the midpoint of $\overline{AB}$ is $\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$.

(c) The position vectors of the points of trisection of $\overline{DE}$ are $\left(\frac{7}{3}, \frac{17}{3}\right)$ and $\left(\frac{11}{3}, \frac{13}{3}\right)$.

(d) The position vectors of the points of trisection of $\overline{DE}$ $\frac{2}{3}\underset{\sim}{d} + \frac{1}{3}\underset{\sim}{e}$ are $\frac{1}{3}\underset{\sim}{d} + \frac{2}{3}\underset{\sim}{e}$.

(e) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

$\overline{AP} = m$ and $\overline{AB} = m + n$

$\frac{\overline{AP}}{\overline{AB}} = \frac{m}{m+n}$

$\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \underset{\sim}{a} + \frac{m}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= \left(1 - \frac{m}{m+n}\right)\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}$

$= \left(\frac{m+n-m}{m+n}\right)\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}$

$= \frac{n}{m+n}\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}$

EXERCISE 10.3

1 (a) $2\underset{\sim}{i} + 3\underset{\sim}{j}$ (b) $-3\underset{\sim}{j}$ (c) $2\underset{\sim}{i} - 4\underset{\sim}{j}$ (d) $-3\underset{\sim}{i} - 3\underset{\sim}{j}$

(e) $2\underset{\sim}{i}$ (f) $7\underset{\sim}{i} + 7\underset{\sim}{j}$

2 (a) $\left|\underset{\sim}{a}\right| = \sqrt{41}$ (b) $\left|\underset{\sim}{b}\right| = \sqrt{65}$ (c) $\left|\underset{\sim}{c}\right| = 25$ (d) $\left|\underset{\sim}{d}\right| = 5$

(e) $\left|\underset{\sim}{e}\right| = 9$ (f) $\left|\underset{\sim}{f}\right| = \sqrt{89}$

3 (a) $\underset{\sim}{a} = 12.29\underset{\sim}{i} + 8.60\underset{\sim}{j}$ (b) $\underset{\sim}{b} = -11.85\underset{\sim}{i} + 19.71\underset{\sim}{j}$

(c) $\underset{\sim}{c} = 7.07\underset{\sim}{i} - 7.07\underset{\sim}{j}$ (d) $\underset{\sim}{d} = -35.86\underset{\sim}{i} - 3.14\underset{\sim}{j}$

4 (a) $7\underset{\sim}{i} - 3\underset{\sim}{j}$ (b) $-\underset{\sim}{i} + 7\underset{\sim}{j}$ (c) $29\underset{\sim}{i} + 4\underset{\sim}{j}$ (d) $-3\underset{\sim}{i} - 25\underset{\sim}{j}$

5 $m = -1$, $n = -1$

6 (a) $a = \frac{5}{3}$, $b = -2$ (b) $x = -17$, $y = 7$ (c) $g = 0$, $h = \frac{1}{3}$

(d) $p = 2$, $q = 8$ (e) $x = -3, -2$, $y = 2$

(f) $x = 2 - \sqrt{6}, 2 + \sqrt{6}$, $y = -1, 1$

7 $\overrightarrow{BA} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$

8 (a) $-21\underset{\sim}{i} + 35\underset{\sim}{j}$ (b) $7\sqrt{34}$

9 (a) $\overrightarrow{CD} = -3\underset{\sim}{i} - 4\underset{\sim}{j}$ (b) $\overrightarrow{CA} = -\underset{\sim}{i} + \underset{\sim}{j}$ (c) $\overrightarrow{DB} = 5\underset{\sim}{i} + 9\underset{\sim}{j}$

(d) $\left|\overrightarrow{AB}\right| = 5$ (e) $\left|\overrightarrow{CA}\right| = \sqrt{2}$ (f) $\left|\overrightarrow{DB}\right| = \sqrt{106}$

10 (a) $5\sqrt{10}$ (b) $x = -3$

11 $v = -7$ **12** D **13** B **14** B

15 (a) $\hat{\underset{\sim}{a}} = \frac{1}{10}\left(8\underset{\sim}{i} + 6\underset{\sim}{j}\right)$ (b) $\hat{\underset{\sim}{a}} = \frac{\sqrt{5}}{15}\left(3\underset{\sim}{i} - 6\underset{\sim}{j}\right)$

(c) $\hat{\underset{\sim}{a}} = \frac{\sqrt{85}}{85}\left(-2\underset{\sim}{i} + 9\underset{\sim}{j}\right)$ (d) $\hat{\underset{\sim}{a}} = \frac{\sqrt{145}}{145}\left(-\underset{\sim}{i} - 12\underset{\sim}{j}\right)$

16 (a) $\hat{\underset{\sim}{b}} = \frac{\sqrt{10}}{30}\left(3\underset{\sim}{i} - 9\underset{\sim}{j}\right)$ (b) $\underset{\sim}{c} = \frac{\sqrt{10}}{2}\left(3\underset{\sim}{i} - 9\underset{\sim}{j}\right)$

17 $\hat{\underset{\sim}{c}} = \frac{\sqrt{277}}{277}\left(-14\underset{\sim}{i} - 9\underset{\sim}{j}\right)$

18 $\underset{\sim}{c} = \frac{\sqrt{130}}{13}\left(-3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

19 C **20** C

21 (a) $\overrightarrow{AB} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{CB} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ (b) $\overrightarrow{OB} = 5\underset{\sim}{i} - \underset{\sim}{j}$, $\overrightarrow{CA} = \underset{\sim}{i} + 5\underset{\sim}{j}$

(c) $\overrightarrow{ON} = \frac{5}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$, $\overrightarrow{OM} = \frac{5}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$

(d) $\overrightarrow{CP} = -\frac{1}{2}\underset{\sim}{i} + 4\underset{\sim}{j}$, $\overrightarrow{BP} = -\frac{7}{2}\underset{\sim}{i} + 2\underset{\sim}{j}$

22 (a) $A(-2, 3)$, $B(4, -5)$, $C(6, -9)$

(b) $\overrightarrow{AB} = 6\underset{\sim}{i} - 8\underset{\sim}{j}$, $\overrightarrow{BC} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$, $\overrightarrow{AC} = 8\underset{\sim}{i} - 12\underset{\sim}{j}$

(c) $\left|\overrightarrow{AB}\right| = 10$, $\left|\overrightarrow{BC}\right| = 2\sqrt{5}$, $\left|\overrightarrow{AC}\right| = 4\sqrt{13}$

23 (a) $x = 3, y = 2$

(b) $\overrightarrow{OM} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{AM} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{MB} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$

(c) $\left|\overrightarrow{AM}\right| = \left|\overrightarrow{MB}\right| = \left|\overrightarrow{OM}\right| = \sqrt{13}$

24 (a) $\overrightarrow{AB} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{CB} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$

(b) Consider vector addition to obtain vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \left(2\underset{\sim}{i} - 4\underset{\sim}{j}\right) + \left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

$= 5\underset{\sim}{i} - 2\underset{\sim}{j}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \overrightarrow{AB} - \overrightarrow{CB}$

$= \left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right) - \left(2\underset{\sim}{i} - 4\underset{\sim}{j}\right)$

$= \underset{\sim}{i} + 6\underset{\sim}{j}$

$\overrightarrow{OB} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$, $\overrightarrow{AC} = \underset{\sim}{i} + 6\underset{\sim}{j}$

(c) $\overrightarrow{OP} = \frac{5}{2}\underset{\sim}{i} - \underset{\sim}{j}$, $\overrightarrow{AQ} = \frac{1}{2}\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OQ} = \frac{5}{2}\underset{\sim}{i} - \underset{\sim}{j}$

Thus, $\overrightarrow{OP} = \overrightarrow{OQ}$ and so P and Q are coincident. Therefore, the diagonals $\overline{OB}$ and $\overline{AC}$ bisect each other.

(d) $\overrightarrow{OR} = \frac{7}{2}\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{CR} = \frac{1}{2}\underset{\sim}{i} - 5\underset{\sim}{j}$

25 (a) $\overrightarrow{OB} = 5\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{AC} = -\underset{\sim}{i} + 5\underset{\sim}{j}$, $\overrightarrow{OM} = 4\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$, $\overrightarrow{ON} = 3\underset{\sim}{i} + \frac{7}{3}\underset{\sim}{j}$, $\overrightarrow{NB} = 2\underset{\sim}{i} - \frac{4}{3}\underset{\sim}{j}$

(b) $\left|\overrightarrow{OB}\right| = \sqrt{26}$ and $\left|\overrightarrow{AC}\right| = \sqrt{26}$

26 (a) $\hat{\underset{\sim}{a}} = \frac{1}{5}\left(3\underset{\sim}{i} - 4\underset{\sim}{j}\right)$ and $\hat{\underset{\sim}{b}} = \frac{1}{5}\left(-4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ (b) $x = -\frac{5}{2}, y = -\frac{5}{3}$

(c) $\hat{\underset{\sim}{c}} = \frac{\sqrt{5}}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$ and $\hat{\underset{\sim}{d}} = \frac{\sqrt{5}}{5}\left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$ (d) $v = -\frac{\sqrt{5}}{10}, w = \frac{\sqrt{5}}{5}$

27 (a) $p = \frac{2}{5}$ (b) $\frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$ (c) $6\underset{\sim}{i} + 8\underset{\sim}{j}$ (d) $q = \frac{4}{25}$

(e) $\frac{7}{25}\underset{\sim}{i} + \frac{24}{25}\underset{\sim}{j}$ (f) $14\underset{\sim}{i} + 48\underset{\sim}{j}$ (g) $\frac{10\sqrt{221}}{221}\left(5\underset{\sim}{i} + 14\underset{\sim}{j}\right)$

EXERCISE 10.4

1 (a) 45° (b) 45° (c) 135° (d) 165° (e) 90° (f) 90°
(g) 120° (h) 180° (i) 150° (j) 30°

2 (a) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 23.67$ (b) $\underset{\sim}{a} \bullet \underset{\sim}{b} = -23.67$ (c) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 23.67$
(d) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 56$ (e) $\underset{\sim}{a} \bullet \underset{\sim}{b} = -56$ (f) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

3 (a) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 21$ (b) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 14$ (c) $\underset{\sim}{a} \bullet \underset{\sim}{b} = -52$ (d) $\underset{\sim}{a} \bullet \underset{\sim}{b} = -10$
(e) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 50$ (f) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

4 Since $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} - 3\underset{\sim}{j}$ are perpendicular.

5 $\underset{\sim}{d} = \pm\left(6\underset{\sim}{i} + 8\underset{\sim}{j}\right)$ **6** $x = -\frac{15}{7}$

7 (a) 40 (b) $2\sqrt{10}$ (c) $\left|\underset{\sim}{a}\right|^2$

8 (a) $\left|\underset{\sim}{a}\right|^2$ (b) 1 (c) $-\left|\underset{\sim}{a}\right|^2$

9 If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel and in the same direction, then
$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right| \cos 0° = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$
If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel and in the opposite direction, then
$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right| \cos 180° = -\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$

10 (a) $\theta = 25°$ (b) 96° (c) $\theta = 67°$ (d) $\theta = 153°$ (e) 34°
(f) $\theta = 90°$

11 B **12** B **13** 177°

14 (a) $\overrightarrow{AB} = 4\underset{\sim}{i} + 6\underset{\sim}{j}$, $\overrightarrow{BC} = 6\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{AC} = 10\underset{\sim}{i} + 2\underset{\sim}{j}$

(b) $\left|\overrightarrow{AB}\right| = 2\sqrt{13}$, $\left|\overrightarrow{BC}\right| = 2\sqrt{13}$, $\left|\overrightarrow{AC}\right| = 2\sqrt{26}$

(c) ΔABC is a right–angled triangle if $\overrightarrow{AB} \perp \overrightarrow{BC}$ and $\angle ABC = 90°$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = 4 \times 6 + 6 \times (-4)$

$= 0$

$\therefore \angle ABC = 90°$

(d) $\overrightarrow{OD} = 4\underset{\sim}{i} - 7\underset{\sim}{j}$ (e) $\overrightarrow{BD} = 2\underset{\sim}{i} - 10\underset{\sim}{j}$

(f) Use dot product to show the diagonals are perpendicular. Let M and N be the mid points of the diagonals, then show that they are the same point.

$\overrightarrow{AC} = 10\underset{\sim}{i} + 2\underset{\sim}{j}$, and $\left|\overrightarrow{AC}\right| = \sqrt{104} = 2\sqrt{26}$

$\overrightarrow{BD} = 2\underset{\sim}{i} - 10\underset{\sim}{j}$ and $\left|\overrightarrow{BD}\right| = \sqrt{104} = 2\sqrt{26}$

$\overrightarrow{AC} \bullet \overrightarrow{BD} = 10 \times 2 + 2 \times (-10)$

$= 0$

$\therefore \overrightarrow{AC} \perp \overrightarrow{BD}$

Let M be the midpoint of $\overrightarrow{AC}$, then $\overrightarrow{AM} = \overrightarrow{MC} = 5\underset{\sim}{i} + \underset{\sim}{j}$ and so $\left|\overrightarrow{AM}\right| = \left|\overrightarrow{MC}\right| = \sqrt{26}$ and its position vector will be

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= -2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{i} + \underset{\sim}{j}$

$= 3\underset{\sim}{i} - 2\underset{\sim}{j}$

Let N be the midpoint of $\overrightarrow{BD}$, then $\overrightarrow{BN} = \overrightarrow{ND} = \underset{\sim}{i} - 5\underset{\sim}{j}$ and so $\left|\overrightarrow{BN}\right| = \left|\overrightarrow{ND}\right| = \sqrt{26}$ and its position vector will be

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$

$= 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{i} - 5\underset{\sim}{j}$

$= 3\underset{\sim}{i} - 2\underset{\sim}{j}$

Since $\overrightarrow{OM} = \overrightarrow{ON}$, the midpoints of $\overrightarrow{AC}$ and $\overrightarrow{BD}$ coincide and so the diagonals bisect each other at right angles.

15 (a) Use vector addition to calculate the vectors of the kite's sides. Then compare the magnitudes of those side vectors.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} - \left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$

$= 3\underset{\sim}{j}$

$\left|\overrightarrow{AB}\right| = 3$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 8\underset{\sim}{i} + 7\underset{\sim}{j} - \left(2\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

$= 6\underset{\sim}{i} + 3\underset{\sim}{j}$

$\left|\overrightarrow{BC}\right| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= 5\underset{\sim}{i} + \underset{\sim}{j} - \left(8\underset{\sim}{i} + 7\underset{\sim}{j}\right)$

$= -3\underset{\sim}{i} - 6\underset{\sim}{j}$

$|\overrightarrow{CD}| = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45} = 3\sqrt{5}$

$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD}$

$= 2\underset{\sim}{i} + \underset{\sim}{j} - (5\underset{\sim}{i} + \underset{\sim}{j})$

$= -3\underset{\sim}{i}$

$|\overrightarrow{DA}| = 3$

$|\overrightarrow{AB}| = 3, |\overrightarrow{BC}| = 3\sqrt{5}, |\overrightarrow{CD}| = 3\sqrt{5}, |\overrightarrow{DA}| = 3$

Adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{DA}$ are equal in length and adjacent sides $\overrightarrow{BC}$ and $\overrightarrow{CD}$ are equal in length. Thus, $ABCD$ is a kite.

(b) Use vector addition to calculate the vectors of the kite's diagonals, then calculate the dot product of those diagonals.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= 3\underset{\sim}{j} + 6\underset{\sim}{i} + 3\underset{\sim}{i}$

$= 6\underset{\sim}{i} + 6\underset{\sim}{j}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= 6\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{i} - 6\underset{\sim}{j}$

$= 3\underset{\sim}{i} - 3\underset{\sim}{j}$

$\overrightarrow{AC} \bullet \overrightarrow{BD} = 6 \times 3 + 6 \times (-3)$

$= 0$

$\therefore \overrightarrow{AC} \perp \overrightarrow{BD}$

(c) $\overrightarrow{BD} = \underset{\sim}{r} - \underset{\sim}{p}$, $\overrightarrow{BC} = \underset{\sim}{q} - \underset{\sim}{p}$ and $\overrightarrow{DC} = \underset{\sim}{q} - \underset{\sim}{r}$.

(d) Use the distributive law applied to vectors.

Now $\overrightarrow{BC} \bullet \overrightarrow{BC} = |\overrightarrow{BC}|^2$ and $\overrightarrow{DC} \bullet \overrightarrow{DC} = |\overrightarrow{DC}|^2$.

Since $|\overrightarrow{BC}| = |\overrightarrow{DC}|$, then $|\overrightarrow{BC}|^2 = |\overrightarrow{DC}|^2$ and so $\overrightarrow{BC} \bullet \overrightarrow{BC} = \overrightarrow{DC} \bullet \overrightarrow{DC}$.

Now, $\overrightarrow{BC} \bullet \overrightarrow{BC} = |\overrightarrow{BC}|^2 = |\underset{\sim}{q} - \underset{\sim}{p}|$

$= (\underset{\sim}{q} - \underset{\sim}{p}) \bullet (\underset{\sim}{q} - \underset{\sim}{p})$

$= \underset{\sim}{q} \bullet \underset{\sim}{q} - \underset{\sim}{q} \bullet \underset{\sim}{p} - \underset{\sim}{p} \bullet \underset{\sim}{q} + \underset{\sim}{p} \bullet \underset{\sim}{p}$

$= \underset{\sim}{q} \bullet \underset{\sim}{q} - 2\underset{\sim}{q} \bullet \underset{\sim}{p} + \underset{\sim}{p} \bullet \underset{\sim}{p}$

and $\overrightarrow{DC} \bullet \overrightarrow{DC} = |\overrightarrow{DC}|^2 = |\underset{\sim}{q} - \underset{\sim}{r}|^2$

$= (\underset{\sim}{q} - \underset{\sim}{r}) \bullet (\underset{\sim}{q} - \underset{\sim}{r})$

$= \underset{\sim}{q} \bullet \underset{\sim}{q} - \underset{\sim}{q} \bullet \underset{\sim}{r} - \underset{\sim}{r} \bullet \underset{\sim}{q} + \underset{\sim}{r} \bullet \underset{\sim}{r}$

$= \underset{\sim}{q} \bullet \underset{\sim}{q} - 2\underset{\sim}{q} \bullet \underset{\sim}{r} + \underset{\sim}{r} \bullet \underset{\sim}{r}$

Now, $|\overrightarrow{AB}| = |\overrightarrow{AD}|$ and so $|\underset{\sim}{p}| = |\underset{\sim}{r}|$ and $\underset{\sim}{p} \bullet \underset{\sim}{p} = \underset{\sim}{r} \bullet \underset{\sim}{r}$.

$\overrightarrow{BC} \bullet \overrightarrow{BC} = \overrightarrow{DC} \bullet \overrightarrow{DC}$

$\underset{\sim}{q} \bullet \underset{\sim}{q} - 2\underset{\sim}{q} \bullet \underset{\sim}{p} + \underset{\sim}{p} \bullet \underset{\sim}{p} = \underset{\sim}{q} \bullet \underset{\sim}{q} - 2\underset{\sim}{q} \bullet \underset{\sim}{r} + \underset{\sim}{r} \bullet \underset{\sim}{r}$

$-2\underset{\sim}{q} \bullet \underset{\sim}{p} = -2\underset{\sim}{q} \bullet \underset{\sim}{r}$ since $\underset{\sim}{p} \bullet \underset{\sim}{p} = \underset{\sim}{r} \bullet \underset{\sim}{r}$

$\therefore \underset{\sim}{q} \bullet \underset{\sim}{p} = \underset{\sim}{q} \bullet \underset{\sim}{r}$

(e) Show that the dot product of the diagonals is 0.

$\overrightarrow{AC} \bullet \overrightarrow{BD} = \underset{\sim}{q} \bullet (\underset{\sim}{r} - \underset{\sim}{p})$

$= \underset{\sim}{q} \bullet \underset{\sim}{r} - \underset{\sim}{q} \bullet \underset{\sim}{p}$

$= 0$ since $\underset{\sim}{q} \bullet \underset{\sim}{p} = \underset{\sim}{q} \bullet \underset{\sim}{r}$

$\therefore \overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular

EXERCISE 10.5

1 (a) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{8}{5}$ (b) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{18\sqrt{13}}{13}$

(c) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = 0$ (d) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = -\frac{3}{\sqrt{61}} = -\frac{3\sqrt{61}}{61}$

(e) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = -\frac{23\sqrt{41}}{41}$ (f) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{47\sqrt{53}}{53}$

2 (a) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{18}{13}(3\underset{\sim}{i} + 2\underset{\sim}{j})$ (b) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{8}{25}(3\underset{\sim}{i} + 4\underset{\sim}{j})$

(c) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \underset{\sim}{0}$ (d) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{-3}{61}(5\underset{\sim}{i} + 6\underset{\sim}{j})$

(e) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{-29}{65}(4\underset{\sim}{i} + 7\underset{\sim}{j})$ (f) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{47}{53}(-7\underset{\sim}{i} - 2\underset{\sim}{j})$

3 (a) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = -\frac{1}{13}(2\underset{\sim}{i} - 3\underset{\sim}{j})$

(b) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{1}{25}(76\underset{\sim}{i} - 57\underset{\sim}{j})$

(c) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$

(d) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{-4}{61}(42\underset{\sim}{i} - 35\underset{\sim}{j})$

(e) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{-2}{65}(7\underset{\sim}{i} - 4\underset{\sim}{j})$

(f) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{32}{53}(2\underset{\sim}{i} - 7\underset{\sim}{j})$

4 (a) $\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}} = \frac{18}{5}$ (b) $\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}} = \frac{8}{\sqrt{17}} = \frac{8\sqrt{17}}{17}$

5 (a) $(\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}} = \frac{18}{25}(4\underset{\sim}{i} + 3\underset{\sim}{j})$ (b) $(\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}} = \frac{8}{17}(4\underset{\sim}{i} - \underset{\sim}{j})$

6 B **7** C

8 (a) The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{23\sqrt{41}}{41}$.

(b) The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{23}{41}(5\underset{\sim}{i} + 4\underset{\sim}{j})$.

(c) The scalar projection of onto $\underset{\sim}{a}$ is $\frac{23\sqrt{2}}{10}$.

(d) The vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is $\frac{23}{50}(-\underset{\sim}{i} + 7\underset{\sim}{j})$.

9 (a) The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{7\sqrt{2}}{2}$.

(b) The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{7}{2}\underset{\sim}{i} - \frac{7}{2}\underset{\sim}{j}$.

(c) The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $-\frac{1}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$.

(d) $\underset{\sim}{a} = (\frac{7}{2}\underset{\sim}{i} - \frac{7}{2}\underset{\sim}{j}) + (-\frac{1}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j})$

10 (a) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$ is the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) $\hat{\underset{\sim}{a}} \bullet \underset{\sim}{b}$ is the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(c) $\hat{\underset{\sim}{a}} \bullet \hat{\underset{\sim}{b}}$ is the cosine of the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

11 (a) $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{a}|\sin\alpha\underset{\sim}{j}$

Since $\underset{\sim}{a}$ is a unit vector $|\underset{\sim}{a}| = 1$.

$\underset{\sim}{a} = 1 \times \cos\alpha\underset{\sim}{i} + 1 \times \sin\alpha\underset{\sim}{j}$

$= \cos\alpha\underset{\sim}{i} + \sin\alpha\underset{\sim}{j}$

(b) $\underset{\sim}{b} = \cos\beta\underset{\sim}{i} + \sin\beta\underset{\sim}{j}$ and $\underset{\sim}{c} = \cos\beta\underset{\sim}{i} - \sin\beta\underset{\sim}{j}$

(c) $\underset{\sim}{a} \bullet \underset{\sim}{b} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$; $\underset{\sim}{a} \bullet \underset{\sim}{c} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(d) Now $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Here $\theta = \alpha - \beta$ and $|\underset{\sim}{a}| = |\underset{\sim}{b}| = 1$.

Therefore $\underset{\sim}{a} \bullet \underset{\sim}{b} = \cos(\alpha - \beta)$

Thus, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Similarly, $\underset{\sim}{a} \bullet \underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{c}|\cos\phi$ where ϕ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

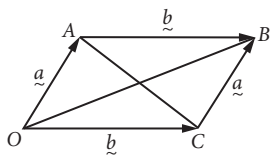
Here $\phi = \alpha + \beta$ and $|\underset{\sim}{a}| = |\underset{\sim}{c}| = 1$.

Therefore $\underset{\sim}{a} \bullet \underset{\sim}{c} = \cos(\alpha + \beta)$.

Thus, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

EXERCISE 10.6

1



Let $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{b}$.

Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

and $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{CO}$

$= \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{OB} \bullet \overrightarrow{CA} = (\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{a} - \underset{\sim}{b})$

$= \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b}$

$= \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

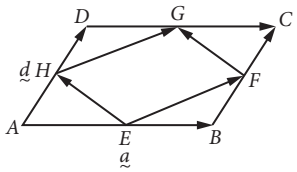
Now if $\overrightarrow{OB} \bullet \overrightarrow{CA} = 0$, then the diagonals $\overrightarrow{OB}$ and $\overrightarrow{CA}$ are perpendicular.

Thus, $\overrightarrow{OB} \bullet \overrightarrow{CA} = |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

$= 0$ if $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

Therefore, $|\overrightarrow{OA}| = |\overrightarrow{CB}| = |\overrightarrow{OC}| = |\overrightarrow{AB}|$ and $OABC$ is a rhombus.

2



Since $ABCD$ is a parallelogram, $\overrightarrow{AB} = \overrightarrow{DC} = \underset{\sim}{a}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{d}$.

Let the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ be E, F, G and H respectively. Therefore, $\overrightarrow{AE} = \overrightarrow{EB} = \overrightarrow{DG} = \overrightarrow{GC} = \frac{1}{2}\underset{\sim}{a}$ and

$\overrightarrow{AH} = \overrightarrow{HD} = \overrightarrow{BF} = \overrightarrow{FC} = \frac{1}{2}\underset{\sim}{b}$

$\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH}$

$= -\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{d}$

$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF}$

$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{d}$

$\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG}$

$= \frac{1}{2}\underset{\sim}{d} - \frac{1}{2}\underset{\sim}{a}$

$\overrightarrow{HG} = \overrightarrow{HD} + \overrightarrow{DG}$

$= \frac{1}{2}\underset{\sim}{d} + \frac{1}{2}\underset{\sim}{a}$

Now in the quadrilateral $EFGH$, opposite sides $\overrightarrow{EH} = \overrightarrow{FG}$ and opposite sides $\overrightarrow{EF} = \overrightarrow{HG}$. That is opposite sides of the parallelogram are equal in length and are parallel. Therefore, EFGH is a parallelogram.

3 D **4** D

5 $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AM} = \overrightarrow{MC}$

$= \frac{1}{2}(\overrightarrow{AC})$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{BM} = \overrightarrow{BC} - \overrightarrow{MC}$

$= \underset{\sim}{b} - \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

Since $\underset{\sim}{a} \perp \underset{\sim}{b}$ then $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{b} - \underset{\sim}{a}$ are the diagonals of a rectangle. Hence $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{b} - \underset{\sim}{a}|$ and thus $\frac{1}{2}|\underset{\sim}{a} + \underset{\sim}{b}| = \frac{1}{2}|\underset{\sim}{b} - \underset{\sim}{a}|$ and $|\overrightarrow{AM}| = |\overrightarrow{MC}| = |\overrightarrow{BM}| = \frac{1}{2}|\underset{\sim}{a} + \underset{\sim}{b}|$

6 By Pythagoras' theorem: $|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{CB}|^2$

$|\underset{\sim}{c}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2$

Now $\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$

So, $|\underset{\sim}{c}|^2 = \underset{\sim}{c} \bullet \underset{\sim}{c} = (\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{a} + \underset{\sim}{b})$

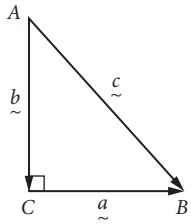
$|\underset{\sim}{c}|^2 = \underset{\sim}{a} \bullet \underset{\sim}{a} + 2\underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2\underset{\sim}{a} \bullet \underset{\sim}{b}$

$\therefore 2\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

So $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

and $\underset{\sim}{a} \perp \underset{\sim}{b}$



7 (a) $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{DB} = \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{AX} = x\overrightarrow{AC} = x(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{DX} = y\overrightarrow{DB} = y(\underset{\sim}{a} - \underset{\sim}{b})$

$\overrightarrow{AD} = \overrightarrow{AX} - \overrightarrow{DX}$

$= x(\underset{\sim}{a} + \underset{\sim}{b}) - y(\underset{\sim}{a} - \underset{\sim}{b})$

$= x\underset{\sim}{a} + x\underset{\sim}{b} - y\underset{\sim}{a} + y\underset{\sim}{b}$

$= (x - y)\underset{\sim}{a} + (x + y)\underset{\sim}{b}$

$= \underset{\sim}{b}$

$(x - y)\underset{\sim}{a} + (x + y)\underset{\sim}{b} = 0\underset{\sim}{a} + \underset{\sim}{b}$

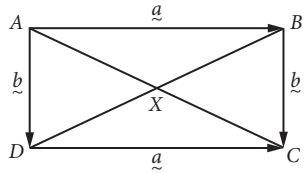
$x - y = 0$

$x + y = 1$

$2x = 1$

$x = \frac{1}{2}$

$y = \frac{1}{2}$

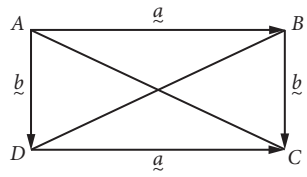


(b) $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

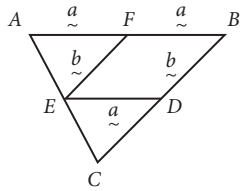
$\overrightarrow{BD} = -\underset{\sim}{a} - \underset{\sim}{b}$

$= -(\underset{\sim}{a} + \underset{\sim}{b})$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{BD}|$



8 (a) $\underset{\sim}{a} + \underset{\sim}{b}$



(b) $\overrightarrow{AC} = x\overrightarrow{AE} = x(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{BC} = y\overrightarrow{BD} = y\underset{\sim}{b}$

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = 2\underset{\sim}{a}$

$\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = 2\underset{\sim}{a}$

$2\underset{\sim}{a} = x(\underset{\sim}{a} + \underset{\sim}{b}) - y\underset{\sim}{b}$

$2\underset{\sim}{a} = x\underset{\sim}{a} + x\underset{\sim}{b} - y\underset{\sim}{b}$

$2\underset{\sim}{a} + 0\underset{\sim}{b} = x\underset{\sim}{a} + (x - y)\underset{\sim}{b}$

$x = 2, y = 2$

$\overrightarrow{BC} = 2\overrightarrow{BD}$

$\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

9 (a) $\overrightarrow{BX} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$; $\overrightarrow{AX} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

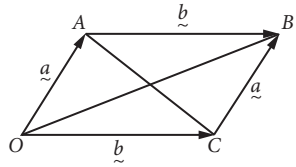
(b) $|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

(c) $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

$= 2(\overrightarrow{BX} \bullet \overrightarrow{BX} + \overrightarrow{AX} \bullet \overrightarrow{AX})$

$= 2(|\overrightarrow{AX}|^2 + |\overrightarrow{BX}|^2)$

10 (a) $\underset{\sim}{a} + \underset{\sim}{b}$; $\underset{\sim}{b} - \underset{\sim}{a}$



(b) $2(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2)$

(c) $2(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2)$

(d) The sum of the squares of the lengths of the diagonals of a parallelogram: $|\overrightarrow{OB}|^2 + |\overrightarrow{AC}|^2 = 2(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2)$

The sum of the squares of the length of the sides of a parallelogram:

$|\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 + |\overrightarrow{CB}|^2 = 2(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2)$

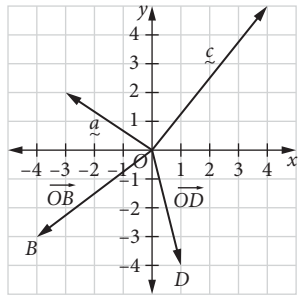
CHAPTER REVIEW 10

1	B	**2**	D	**3**	C	**4**	B
5	A	**6**	D	**7**	B	**8**	C
9	C $2\sqrt{5}$	**10**	D $-4\underset{\sim}{i} - 6\underset{\sim}{j}$	**11**	A	**12**	C
13	C	**14**	A	**15**	A	**16**	B

17 $\underset{\sim}{a} - \underset{\sim}{b}$

18 $\overrightarrow{CD} = -\frac{1}{3}\underset{\sim}{a}$, $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{AD} = \frac{2}{3}\underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{DB} = \frac{1}{3}\underset{\sim}{a} - \underset{\sim}{b}$

19



20 (a) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{i} - 6\underset{\sim}{j}$ (b) $3\underset{\sim}{b} - 2\underset{\sim}{a} = 13\underset{\sim}{i} + 7\underset{\sim}{j}$

(c) $-2\underset{\sim}{a} - 7\underset{\sim}{b} = -17\underset{\sim}{i} + 17\underset{\sim}{j}$

21 (a) $|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{37}$ (b) $|3\underset{\sim}{b} - 2\underset{\sim}{a}| = \sqrt{218}$

(c) $|-2\underset{\sim}{a} - 7\underset{\sim}{b}| = 17\sqrt{2}$

22 (a) $4\underset{\sim}{c} - 3\underset{\sim}{d} = \begin{pmatrix} 29 \\ -16 \end{pmatrix}$ (b) $7\underset{\sim}{d} - 8\underset{\sim}{c} = \begin{pmatrix} -61 \\ 40 \end{pmatrix}$

23 (a) $\overrightarrow{AB} = -21\underset{\sim}{i} - 12\underset{\sim}{j}$ (b) $|\overrightarrow{AB}| = 3\sqrt{65}$

24 (a) $|\underset{\sim}{a}| = 26$ (b) $|\underset{\sim}{b}| = 3\sqrt{5}$

25 (a) $\underset{\sim}{a} = 10.71\underset{\sim}{i} + 11.89\underset{\sim}{j}$ (b) $\underset{\sim}{b} = -20.35\underset{\sim}{i} + 12.72\underset{\sim}{j}$

26 (a) $a = \frac{23}{2}$ and $b = 6$ (b) $f = \frac{59}{19}$ and $g = \frac{18}{19}$

(c) $a = -1$, 10 and $b = \sqrt[3]{-3}$

27 (a) $\hat{\underset{\sim}{a}} = \frac{\sqrt{10}}{30}(-9\underset{\sim}{i} - 3\underset{\sim}{j})$ (b) $\underset{\sim}{b} = \frac{\sqrt{10}}{6}(-9\underset{\sim}{i} - 3\underset{\sim}{j})$

28 (a) $\underset{\sim}{a} \bullet \underset{\sim}{b} = -1$ (b) $\underset{\sim}{a} \bullet \underset{\sim}{b} = 25$

29 $\underset{\sim}{a} \bullet \underset{\sim}{b} = (-3) \times 10 + 5 \times 6 = 0$, $\therefore$ $\underset{\sim}{a} \perp \underset{\sim}{b}$

30 (a) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{2\sqrt{5}}{5}$

(b) $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{-19\sqrt{2}}{10}$

31 (a) $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{3}{17}(4\underset{\sim}{i} + \underset{\sim}{j})$

(b) $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{22}{17}(\underset{\sim}{i} - 4\underset{\sim}{j})$

32 (a) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -14 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -11 \\ 17 \end{pmatrix}$

(b) $|\overrightarrow{AB}| = \sqrt{205}$, $|\overrightarrow{BC}| = \sqrt{205}$, $|\overrightarrow{AC}| = \sqrt{410}$

(c) $\left|\overrightarrow{AB}\right| = \sqrt{3^2+14^2}$

$= \sqrt{205}$

$\left|\overrightarrow{BC}\right| = \sqrt{(-14)^2+3^2}$

$= \sqrt{205}$

$\left|\overrightarrow{AC}\right| = \sqrt{(-11)^2+17^2}$

$= \sqrt{410}$

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$

(d) $(-12, -2)$

(e) $\left(-\frac{7}{2}, \frac{7}{2}\right)$

33 (a) $e = \sqrt{2}$ (b) $\frac{\sqrt{5}}{5}\left(-2\underset{\sim}{i}+\underset{\sim}{j}\right)$ (c) $2\sqrt{5}\left(-2\underset{\sim}{i}+\underset{\sim}{j}\right)$

(d) $f = \sqrt{10}$ (e) $\frac{1}{5}\left(4\underset{\sim}{i}-3\underset{\sim}{j}\right)$ (f) $\underset{\sim}{d} = \frac{2\sqrt{5}}{5}\left(4\underset{\sim}{i}-3\underset{\sim}{j}\right)$

(g) $\frac{4\sqrt{5}}{5}\left(-7\underset{\sim}{i}+4\underset{\sim}{j}\right)$

34 (a) The scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{-\sqrt{10}}{10}$.

(b) The vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{1}{10}\left(3\underset{\sim}{i}+\underset{\sim}{j}\right)$

(c) The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\frac{17}{10}\left(\underset{\sim}{i}-3\underset{\sim}{j}\right)$.

(d) $\underset{\sim}{a} = \frac{1}{10}\left(3\underset{\sim}{i}+\underset{\sim}{j}\right)+\frac{17}{10}\left(\underset{\sim}{i}-3\underset{\sim}{j}\right)$

35 (a) $\overrightarrow{AB} = \overrightarrow{AM}+\overrightarrow{MB}$

$= \overrightarrow{AM}-\overrightarrow{BM}$

$= \underset{\sim}{a}-\underset{\sim}{b}$

$\overrightarrow{BC} = \overrightarrow{BM}+\overrightarrow{MC}$

$= \overrightarrow{BM}+\overrightarrow{AM}$

$= \underset{\sim}{a}+\underset{\sim}{b}$

(b) $\overrightarrow{AB}\cdot\overrightarrow{BC} = (\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})$

$= \left|\underset{\sim}{a}\right|^2-\left|\underset{\sim}{b}\right|^2$

$\overrightarrow{AB}\cdot\overrightarrow{BC} = 0 \because \overrightarrow{AB}\perp\overrightarrow{BC}$

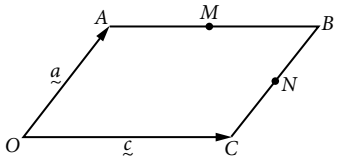
$\therefore \quad \left|\underset{\sim}{a}\right|^2-\left|\underset{\sim}{b}\right|^2 = 0$

$\left|\underset{\sim}{a}\right|^2 = \left|\underset{\sim}{b}\right|^2$

$\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

$\left|\overrightarrow{AM}\right| = \left|\overrightarrow{BM}\right| = \left|\overrightarrow{MC}\right|$

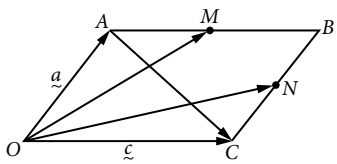
36 (a)



(b) $\overrightarrow{OM} = \underset{\sim}{a}+\frac{1}{2}\underset{\sim}{c}$ and $\overrightarrow{ON} = \frac{1}{2}\underset{\sim}{a}+\underset{\sim}{c}$

(c) $\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{c}-\underset{\sim}{a})$

(d)



(e) $\overrightarrow{OP} = \frac{2}{3}\underset{\sim}{a}+\frac{1}{3}\underset{\sim}{c}$

(f) $\overrightarrow{OQ} = \frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{c}$

(g) $\overrightarrow{AC} = \underset{\sim}{c}-\underset{\sim}{a}$

$\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{c}-\underset{\sim}{a})$

$= \frac{1}{2}\overrightarrow{AC}$

(h) $\overrightarrow{AP} = \overrightarrow{PQ} = \overrightarrow{QC} = \frac{1}{3}\overrightarrow{AC}$, hence $\overrightarrow{AC}$ is trisected at P and Q.

CHAPTER 11

EXERCISE 11.1

1 $\pi\int_0^3 x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^3 = \frac{243\pi}{5}$ **2** $\frac{256\pi}{3}$ **3** D

4 (a) $y = 2x-2$ (b) $4\pi\int_1^3 (x-1)^2\,dx = \frac{32\pi}{3}$

5 $\pi\int_{-3}^3 \left(9-x^2\right)dx = 36\pi$ **6** $\frac{\pi}{30}$

7 (a) $\pi\int_{-1}^1 \left(1-x^2\right)^2\,dx = 2\pi\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_0^1 = \frac{16\pi}{15}$

(b) $\pi\int_0^1 (1-y)\,dy = \frac{\pi}{2}$

8 $\pi\int_0^2 (x-2)^4\,dx = \frac{32\pi}{5}$

9 (a) $\pi\int_0^6 \left(3-\frac{x}{2}\right)^2 dx = \pi\left[9x-\frac{3x^2}{2}+\frac{x^3}{12}\right]_0^6 = 18\pi$

(b) $4\pi\int_0^3 (3-y)^2\,dy = 36\pi$

10 $\pi\int_{-4}^4 \left(\frac{144-9x^2}{16}\right)dx = \frac{\pi}{8}\left[144x-3x^3\right]_0^4 = 48\pi$

11 (a) $2\pi\int_0^3 \left(81-18x^2+x^4\right)dx = \frac{1296\pi}{5}$

(b) $\pi\int_0^9 (9-y)\,dy = \frac{81\pi}{2}$

12 (a) $y = 10x-30$ (b) $\pi\int_0^{10}\left(\frac{y+30}{10}\right)^2 dy = \frac{370\pi}{3}$ (c) $\frac{635\pi}{12}$

13 $\pi\int_{-a}^{\frac{-a}{2}} \left(a^2-y^2\right)dy = \frac{5a^3\pi}{24}$

14 $\pi\int_1^4 (4-y)\,dy = \frac{9\pi}{2}$

15 (a) incorrect (b) correct (c) correct (d) correct

16 $\frac{256\pi}{3}$

17 $\pi\int_0^1 \left(4x^2-4x^4\right)dx \approx 1.37$

18 (a) $\pi\int_{-2}^2 \left(16-4x^2\right)dx = \frac{128\pi}{3}$ (b) $\frac{\pi}{2}\int_0^4 \left(16-y^2\right)dy = \frac{64\pi}{3}$

19 (a) $\int_0^4 \left(2-\sqrt{x}\right)^2 dx = \int_0^4 \left(4-4\sqrt{x}+x\right)dx = 2\frac{2}{3}$

(b) $V = \pi\int_0^4 \left(4-4\sqrt{x}+x\right)^2 dx$

$= \pi\int_0^4 \left(16-32\sqrt{x}+24x-8x^{\frac{3}{2}}+x^2\right)dx = \frac{64\pi}{15}$

(c) $\frac{64\pi}{15}$

20 $V = \pi\int_1^a \frac{dx}{x^2} = \pi\left[\frac{-1}{x}\right]_1^a = \pi\left(1-\frac{1}{a}\right)$, $\lim_{a\to\infty} V = \pi$

21 $\pi-\pi\int_0^1 \left(2x-x^2\right)^2 dx = \frac{7\pi}{15}$ **22** (a) $\frac{8\pi}{12}$ (b) $\frac{\pi}{2}$

23 (a) $\pi\int_0^4 \left(\frac{4-x}{2}\right)^2 dx = \frac{16\pi}{3}$ (b) $\pi\int_0^2 (4-2y)^2\,dy = \frac{32\pi}{3}$

24 $\frac{34\pi}{25}$

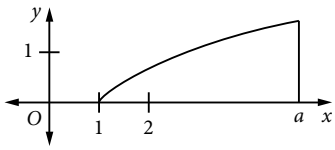
25 $V = \pi\int_0^1 e^{-2x}\,dx = \frac{\pi\left(e^2-1\right)}{2e^2}$ **26** $\frac{\pi}{2}\left(e^3-e\right)$

27 $\frac{\pi\left(e^4-1\right)}{e^2}$ **28** $V = \pi\int_{-1}^1 \left(e^{2x}+2+e^{-2x}\right)dx = \pi\left(e^2+4-e^{-2}\right)$

29 (a) $1-e^{-a}$ (b) 1 (c) $V = \frac{\pi}{2}\left(1-e^{-2a}\right)$, $\frac{\pi}{2}$

30 $\pi\int_1^3 \frac{6}{x}\,dx = 6\pi\log_e 3$ **31** $\pi\int_6^{11}\frac{dx}{x-2} = \pi\log_e 2.25$ units^3

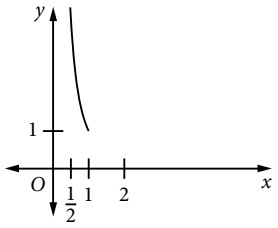
32 (a)



area $= a\ln a - a + 1$

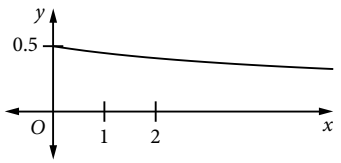
(b) $V = \pi\int_0^{\log_e a} e^{2y}\,dy = \frac{\pi}{2}\left(a^2 - 1\right)$

33



$V = \pi\ln 4$

34

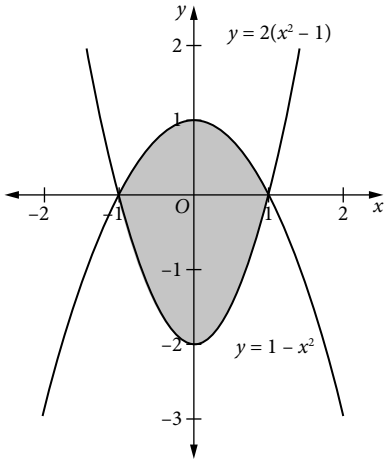


$V = \pi\ln 2.25$

35 $4\pi\ln 3$

36 $\pi\left(\frac{2}{15} + \log_e \frac{5}{3}\right)$

37 (a)



(b) Area $= \int_{-1}^{1}\left(1 - x^2 - 2\left(x^2 - 1\right)\right)dx$

$= 4$ units^2

(c) $y = 2(x^2 - 1) \Rightarrow x^2 = \frac{y}{2} + 1.\ -2 \le y \le 0$

$y = 1 - x^2 \Rightarrow x^2 = 1 - y.\ 0 \le y \le 1$

Volume $= \pi\int_a^b x^2\,dy$

Volume $= \pi\int_{-2}^{0}\left(\frac{y}{2} + 1\right)dy + \pi\int_0^1 (1 - y)\,dy$

$= \frac{3\pi}{2}$ units^3

38 (a) Area $= \int_0^1\left(\sqrt{2}\cos\left(\frac{\pi x}{4}\right) - x\right)dx$

$= \frac{8 - \pi}{2\pi}$ units^2

(b) $V = \pi\int_0^1\left(2\cos^2\left(\frac{\pi x}{4}\right) - x^2\right)dx$

$= \frac{2(\pi + 3)}{3}$ units^3

(c) $y = \sqrt{2}\cos\left(\frac{\pi x}{4}\right) \therefore \cos\left(\frac{\pi x}{4}\right) = \frac{y}{\sqrt{2}} \Rightarrow \frac{\pi x}{4} = \cos^{-1}\left(\frac{y}{\sqrt{2}}\right) \Rightarrow$

$x = \frac{4}{\pi}\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)$ and $x = y$.

Limits are 0, 1 and $\sqrt{2}$.

$V = \pi\int_0^1 y^2\,dy + \pi\int_1^{\sqrt{2}}\left(\frac{4}{\pi}\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)\right)^2 dy$

(d) $V_1 = \pi\int_0^1 y^2\,dy = \pi\left[\frac{y^3}{3}\right]_0^1 = \frac{\pi}{3}$

$V_2 = \pi\int_1^{\sqrt{2}}\left(\frac{4}{\pi}\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)\right)^2 dy$

$= \frac{\left(\sqrt{2} - 1\right)\pi^2}{4}$

$V = \frac{\pi}{3} + \frac{\left(\sqrt{2} - 1\right)\pi^2}{4}$ units^3

39 (a) $A = \int_0^2\left(4x - x^2 - 2x\right)dx$

$= \frac{4}{3}$ units^2

(b) $V = \pi\int_0^2\left\{\left(4x - x^2\right)^2 - (2x)^2\right\}dx$

$= \frac{32\pi}{5}$ units^3

(c) $y = 4x - x^2 \Rightarrow x^2 - 4x + 4 = 4 - y \Rightarrow (x - 2)^2 = 4 - y \Rightarrow$

$x - 2 = \pm\sqrt{4 - y} \Rightarrow x = 2 \pm \sqrt{4 - y}$ and since $0 \le x \le 2$ then

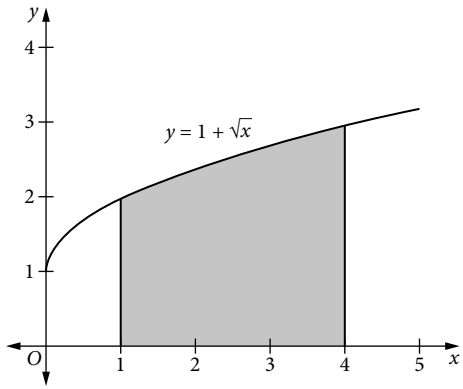
$x = 2 - \sqrt{4 - y}$.

$x = \frac{y}{2}$

$V = \pi\int_0^4\left\{\left(\frac{y}{2}\right)^2 - \left(2 - \sqrt{4 - y}\right)^2\right\}dy$

$= \frac{8\pi}{3}$ units^3

40 (a)



(b) $A = \int_1^4\left(1 + \sqrt{x}dx\right)$

$= \frac{23}{3}$ units^2

(c) $V = \pi\int_1^4\left(1 + \sqrt{x}\right)^2 dx = \frac{119\pi}{6}$

(d) $y = 1 + \sqrt{x} \Rightarrow \sqrt{x} = y - 1 \Rightarrow x = (y - 1)^2$

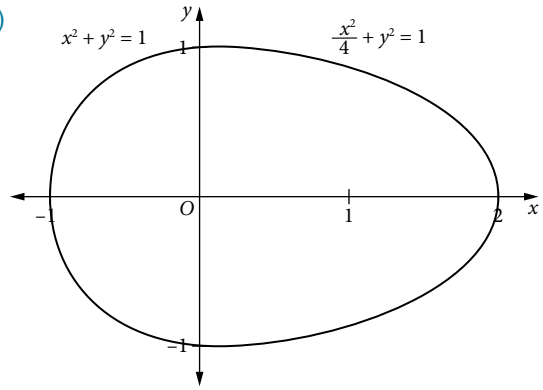
For $0 \le y \le 2$ the new solid is a piece of cylindrical pipe with volume, $V_1 = \pi(4^2 - 1^2) \times 2 = 30\pi$

For $2 \le y \le 3$ the volume of the pipe is given by

$$V_2 = \pi \int_2^3 x^2\,dy = \pi \int_2^3 (y-1)^4\,dy = \frac{31\pi}{5}$$

Total volume $= V_1 + V_2 = \frac{181\pi}{5}$ units^3

41 (a)



(b) $y^2 = 1 - x^2$ and $y^2 = 1 - \frac{x^2}{4}$

$$V = \pi \int_{-1}^{0} \left(1 - x^2\right) dx + \pi \int_0^2 \left(1 - \frac{x^2}{4}\right) dx$$

$= 2\pi$ units^3

42 (a) $y = 8\log_e(x-1) \Rightarrow \frac{y}{8} = \log_e(x-1) \Rightarrow x - 1 = e^{\frac{y}{8}} \Rightarrow$

$x = 1 + e^{\frac{y}{8}}$

$$V = \pi \int_0^4 x^2\,dy = \pi \int_0^4 \left(1 + e^{\frac{y}{8}}\right)^2 dy$$

≈ 66.8 units^3

(b) Solve $8\log_e(x-1) = 4 \Rightarrow x - 1 = e^{0.5} \Rightarrow x = 1 + e^{0.5}$ is the radius of the outer shell.

Volume of plastic = Outer Volume of cylinder − Capacity

$$= \pi\left(1 + e^{0.5}\right)^2 \times 4.5 - 66.8$$

$$= 32.4 \text{ units}^3$$

EXERCISE 11.2

1 (a) $\frac{1}{5}(x^2-1)^5 + C$ (b) $\frac{1}{4}(x^3+4)^4 + C$ (c) $\frac{2}{9}(x^3+1)^{\frac{3}{2}} + C$

2 (a) $\frac{1}{8}(2t+1)^4 + C$ (b) $2\sqrt{x^2-4} + C$ (c) $\frac{1}{6}(x^2+x+2)^6 + C$

3 D

4 (a) $-\frac{1}{14}(3-2x)^7 + C$ (b) $\frac{-1}{2(3x^2+2x+5)} + C$

(c) $\frac{1}{15}(x^3-3x^2+1)^5 + C$

5 (a) $\frac{1}{5}(x^3+1)^5 + C$ (b) $-\sqrt{1-t^2} + C$ (c) $\frac{1}{5}(3x-5)^{\frac{5}{3}} + C$

6 (a) $-\frac{2}{3}(1-t^2)^{\frac{3}{2}} + C$ (b) $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C$

(c) $\frac{3}{8}(z^2+1)^{\frac{4}{3}} + C$

7 (a) $\frac{2}{5}(y+1)^{\frac{5}{2}} - \frac{2}{3}(y+1)^{\frac{3}{2}} + C$ (b) $\frac{-1}{x-1} - \frac{1}{2(x-1)^2} + C$

(c) $\frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}} + C$

8 (a) $\frac{du}{dx} = 3x^2$, $\frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3} \times \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}(1+x^3)^{\frac{3}{2}} + C$

(b) $\frac{du}{dx} = 3x^2$, $\frac{1}{3}\int \sqrt{1+u}\,du = \frac{1}{3} \times \frac{2}{3}(1+u)^{\frac{3}{2}} + C = \frac{2}{9}(1+x^3)^{\frac{3}{2}} + C$

(c) The integration of $\sqrt{u}$ is easier than the integration of $\sqrt{1+u}$.

9 $y = \frac{1}{3}(x^2-4)^{\frac{3}{2}} + \frac{5}{3}$ **10** $f(x) = 3\sqrt{x^2+1} - 1$

11 $x = \sqrt{t^2-2t+4} + 8$ **12** $r = 1 - \sqrt[5]{1-15\theta}$

13 $y = \frac{1}{3}(2x-1)^{\frac{3}{2}} + \frac{19}{3}$ **14** $x = 4 - \frac{4}{8t+1}$

EXERCISE 11.3

1 (a) $\frac{1}{3}$ (b) $\frac{2\sqrt{3}}{5}$ (c) $2\left(\sqrt{5}-1\right)$ **2** (a) 0 (b) $\frac{44\sqrt{2}-16}{15}$ (c) 0

3 B **4** (a) $69\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $6\frac{1}{3}$ **5** (a) $\frac{4-2\sqrt{2}}{3}$ (b) $\frac{1}{5}$ (c) 0

6 (a) $\frac{9\sqrt[3]{3}-3}{8}$ (b) $\frac{5\sqrt{5}}{3}$ (c) $8\frac{2}{3}$ **7** (a) $7\frac{11}{15}$ (b) $21\frac{1}{3}$ (c) $-51\frac{1}{5}$

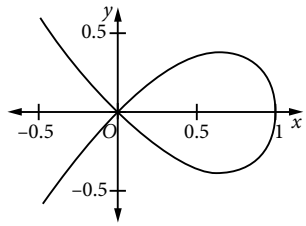
8 (a) $\int_3^0 \frac{-du}{\sqrt{u}} = \left[-2\sqrt{u}\right]_3^0 = 2\sqrt{3}$

(b) $\int_1^4 \frac{du}{\sqrt{4-u}} = \left[-2\sqrt{4-u}\right]_1^4 = -0 + 2\sqrt{3} = 2\sqrt{3}$

(c) Integrating $\sqrt{4-u}$ means that the negative sign needs to be taken into account. It is easily missed.

9 $\frac{1}{3}$ units^2 **10** 1 unit^2

11 Area $= \frac{8}{15}$ units^2



12 Area $= \int_4^6 x\sqrt{x-3}\,dx = \int_1^3 (u+3)\sqrt{u}\,du = \int_1^3 \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}}\right]_1^3 = \frac{12(4\sqrt{3}-1)}{5} \text{ units}^2$$

13 Area $= \int_0^{2\sqrt{3}} \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int_4^{16} \frac{du}{\sqrt{u}} = 2$ units^2

EXERCISE 11.4

1 (a) $x + \frac{1}{2}\sin 2x + C$ (b) $x - \frac{1}{2}\sin 2x + C$

(c) $\frac{x}{2} - \frac{1}{2}\sin x + C$ (d) $x + \sin x + C$

(e) $\frac{x}{2} - \frac{1}{12}\sin 6x + C$ (f) $\frac{x}{2} + \frac{1}{16}\sin 8x + C$

2 (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8} - \frac{1}{4}$ (c) $\frac{\pi}{12}$ (d) $\frac{3(\pi-\sqrt{3})}{4}$ (e) $\frac{\pi}{3} + \frac{\sqrt{3}}{8}$ (f) $\frac{\pi}{6}$

3 C **4** $\frac{\pi^2}{12}$ units^3 **5** $\frac{\pi^2}{2}$ units^3

6 $\frac{dy}{dx} = \frac{1}{2}(1-\cos 2x)$, $y = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$,

$y = \frac{x}{2} - \frac{1}{4}\sin 2x - 2$, $y = \frac{\pi-18}{8}$

7 $\frac{d^2y}{dx^2} = 1 + \cos 2x$, $\frac{dy}{dx} = x + \frac{1}{2}\sin 2x - \frac{\pi}{2}$,

$y = \frac{x^2}{2} - \frac{\pi x}{2} - \frac{1}{4}\cos 2x + \frac{\pi^2-2}{8}$

8 $V = \pi \int_0^{\frac{\pi}{4}} \cos^2 2x\,dx = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} (1+\cos 4x)\,dx = \frac{\pi^2}{8}$ units^3

9 $V = \pi \int_0^{\frac{\pi}{12}} \sin^2 3x\,dx = \frac{\pi}{2}\int_0^{\frac{\pi}{12}} (1-\cos 6x)\,dx = \frac{\pi}{24}(\pi-2)$ units^3

10 (a) $\sin x = \cos x$, $x = \frac{\pi}{4}$

(b) $V = \pi \int_0^{\frac{\pi}{4}} \sin^2 x\,dx + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx$

$$= \frac{\pi}{2}\left\{\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} + \left[x + \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right\}$$

$$= \frac{1}{4}(\pi^2 - 2\pi) \text{ units}^3$$

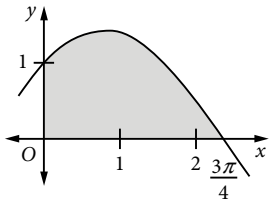
11 $V = \pi\int_0^{\frac{\pi}{4}}(\cos^2 x - \sin^2 x)\,dx$

$= \pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx$

$= \pi\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}$ units^3

12 (a)



(b) $A = \int_0^{\frac{3\pi}{4}}(\sin x + \cos x)\,dx$

$= \left[-\cos x + \sin x\right]_0^{\frac{3\pi}{4}} = 1 + \sqrt{2}$ units^2

(c) $V = \int_0^{\frac{3\pi}{4}}(\sin x + \cos x)^2\,dx = \pi\int_0^{\frac{3\pi}{4}}(1 + \sin 2x)\,dx$

$= \left[x - \frac{1}{2}\cos 2x\right]_0^{\frac{3\pi}{4}} = \frac{1}{4}\pi(3\pi + 2)$ units^3

EXERCISE 11.5

1 (a) $-\frac{1}{3}\cos^3 x + C$ (b) $\frac{1}{2}\tan^2 x + C$ (c) $-\frac{1}{4}\cos^4 x + C$
(d) $\frac{1}{5}\sin^5 x + C$ (e) $-\frac{2}{3}\cos^3 x + C$ (f) $\frac{1}{2}\sin^2 x + C$

2 (a) $\frac{1}{16}$ (b) 0 (c) 0.5 (d) 0.5 (e) $\frac{4}{3}$ (f) $\frac{\pi}{2}$

EXERCISE 11.6

1 8π, a semicircle of radius 4 **2** $\frac{1}{3}$ **3** $\frac{\pi}{8}$ **4** $\frac{\sqrt{3}}{3}$

5 $\frac{\pi+2}{64}$ **6** 0 **7** $\sqrt{3} - \frac{\pi}{3}$ **8** $\frac{\pi}{8}$ **9** $\frac{64}{3}$ **10** $\frac{\pi}{18}$

11 $du = \sec^2 x\,dx$, $\int_3^4 \frac{du}{u^4} = \left[\frac{-1}{3u^3}\right]_3^4 = \frac{37}{5184}$

12 $du = \sec^2 x\,dx$, $\int e^u\,du = e^u + C = e^{\tan x} + C$

13 $du = \cos x - \sin x$, $\int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x + \sin x| + C$

14 (a) $\frac{d}{dx}\left(\frac{1}{\cos x}\right) = -1 \times \frac{1}{\cos^2 x} \times (-\sin x)$

$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$

(b) $du = \sec x \tan x\,dx$, $\int \frac{du}{u} = \ln u + C = \ln|1 + \sec x| + C$

EXERCISE 11.7

1 B

2 (a) $\frac{5}{1+25x^2}$ (b) $\frac{-3}{x^2 - 2x + 2}$ (c) $\frac{2x}{1+x^4}$

(d) $\frac{2\tan^{-1} x}{1+x^2}$ (e) $\frac{1}{2\sqrt{x}(1+x)}$ (f) $\frac{e^x}{1+e^{2x}}$

(g) $\frac{e^{\tan^{-1} x}}{1+x^2}$ (h) $\frac{e^x}{1+x^2} + e^x \tan^{-1} x$ (i) $\frac{1}{\sqrt{16 - x^2}}$

(j) $\frac{-6}{\sqrt{4 - 9x^2}}$ (k) $\frac{1}{\sqrt{1-x^2}\sin^{-1} x}$ (l) $\frac{-4}{\sqrt{1-4x^2}\cos^{-1} 2x}$

(m) $\frac{-1}{x\sqrt{x^2 - 1}}$ (n) $\frac{1}{(\cos^{-1} x)^2\sqrt{1-x^2}}$

(o) $\frac{1}{x\sqrt{1-x^2}} - \frac{\sin^{-1} x}{x^2}$ (p) $\sin^{-1}\sqrt{x} + \frac{\sqrt{x}}{2\sqrt{1-x}}$

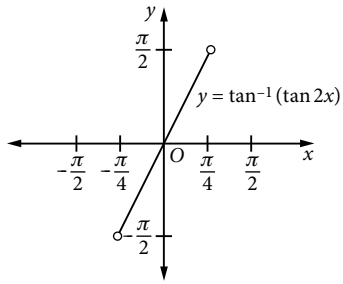
(q) $\frac{2}{1+x^2}$ if $x > 0$, $\frac{-2}{1+x^2}$ if $x < 0$, undefined at $x = 0$ (r) 0

(s) $\frac{\tan x}{1+x^2} + \tan^{-1} x \sec^2 x$ (t) $\frac{1}{x\sqrt{x^2 - 1}}$

3 $y = 2x + 2 - \frac{\pi}{2}$

4 (a) $\frac{dy}{dx} = \frac{2\sec^2 2x}{1 + \tan^2 2x} = 2$ (b) $\tan\left(\pm\frac{\pi}{2}\right)$ is undefined

(c)



(d) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^{-1}(\tan 2x)\,dx$

= area of triangle above x-axis − area of triangle below x-axis

$= \frac{1}{2} \times \frac{\pi}{6} \times \frac{\pi}{3} - \frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{4} = \frac{\pi^2}{36} - \frac{\pi^2}{64} = \frac{7\pi^2}{576}$

5 $\frac{dy}{dx} = 0$ $\therefore$ y is a constant for all x in the domain.
Where $x = 0$, $y = \cos^{-1} 0 + \cos^{-1} 0 = \pi$

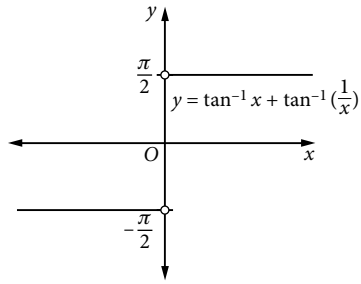
6 $\frac{dy}{dx} = 0$ $\therefore$ y is a constant for all x in the domain.
Where $x = 0$, $y = \sin^{-1} 0 + \sin^{-1} 0 = 0$

7 (a) $\frac{x}{1+x^2} + \tan^{-1} x$

(b) $x\tan^{-1} x - \frac{1}{2}\log_e(1+x^2) + C$ (c) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$

8 (a) all real x except 0 (b) 0 (c) $\frac{\pi}{2}, -\frac{\pi}{2}$

(d)



9 Max turning point at $\left(\tan^{-1}\frac{2}{3}, \sqrt{13}\right)$

EXERCISE 11.8

1 B

2 (a) $\sin^{-1}\frac{x}{4} + C$ (b) $\tan^{-1}\frac{x}{3} + C$ (c) $\sin^{-1} x + C$

(d) $\cos^{-1}\frac{x}{\sqrt{5}} + C$ (e) $\sin^{-1}\frac{x}{\sqrt{2}} + C$ (f) $\tan^{-1}\frac{x}{5} + C$

(g) $\frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (h) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$ (i) $\cos^{-1}\frac{x}{\sqrt{6}} + C$

(j) $\frac{1}{2}\sin^{-1} 2x + C$ (k) $\frac{1}{3}\tan^{-1} 3x + C$

(l) $\frac{1}{12}\tan^{-1}\frac{4x}{3} + C$ (m) $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$

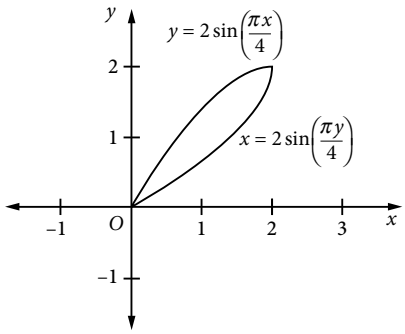
(n) $\frac{1}{\sqrt{3}}\sin^{-1}\sqrt{3}x + C$ (o) $\frac{1}{3}\cos^{-1}\frac{3x}{4} + C$ (p) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{5}} + C$

(q) $\frac{1}{2}\tan^{-1}\frac{x+5}{2} + C$ (r) $\sin^{-1}\frac{x-3}{\sqrt{2}} + C$ (s) $\frac{1}{2}\sin^{-1}\frac{x+2}{\sqrt{2}} + C$

3 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{18}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{4}$ (f) $\frac{\pi}{3}$ (g) $\frac{\pi}{4}$ (h) $-\frac{\pi}{6}$
(i) $\frac{\pi}{6}$ (j) $\frac{\pi}{12}$ (k) $\frac{\pi}{4} + \frac{1}{2}\log_e 2$ (l) $\frac{\pi}{2}$ (m) $\frac{\pi}{8}$ (n) $-\frac{\pi}{9}$
(o) $\frac{\pi}{8}$ (p) $\frac{\pi\sqrt{2}}{8}$ (q) $\frac{\pi}{2}$ (r) $\frac{\pi}{3}$ (s) $\frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$ (t) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$

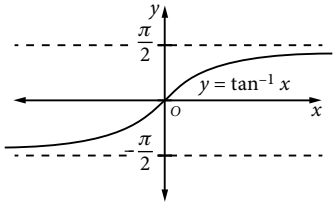
4 $y = 1 + \sin^{-1} x$ **5** $\frac{\pi}{2}$

6 $\frac{16}{\pi} - 4$



7 $\frac{\pi^2}{6}$

8 $\frac{\pi\sqrt{3}}{3} - \log_e 2$



9 When graphed, the area above the x-axis is equal to the area below the x-axis by symmetry. In general, $\int_{-a}^{a} f(x)\,dx = 0$ for every odd function $f(x)$.

10 (a) $\int_{\frac{3}{5}}^{4} \frac{dx}{x^2+1} = \tan^{-1} 4 - \tan^{-1} \frac{3}{5}$

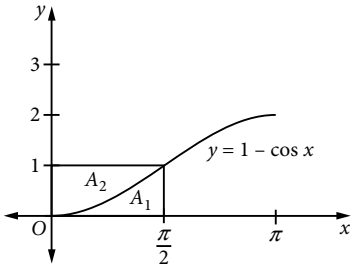
Both $\tan^{-1} 4$ and $\tan^{-1} \frac{3}{5}$ are values between 0 and $\frac{\pi}{2}$, and $\tan^{-1} 4 > \tan^{-1} \frac{3}{5}$, so $0 < \tan^{-1} 4 - \tan^{-1} \frac{3}{5} < \frac{\pi}{2}$.
Further, $\tan(\tan^{-1} 4 - \tan^{-1} \frac{3}{5}) = 1$
when evaluated using $\tan(\alpha - \beta)$ expansion.
$\therefore \tan^{-1} 4 - \tan^{-1} \frac{3}{5} = \frac{\pi}{4}$ (i.e. the first quadrant angle)

(b) $\int_{-2}^{3} \frac{dx}{x^2+1} = \tan^{-1} 3 - \tan^{-1}(-2) = \tan^{-1} 3 + \tan^{-1} 2$

Both $\tan^{-1} 3$ and $\tan^{-1} 2$ are values between $\frac{\pi}{4}$ and $\frac{\pi}{2}$
So $\frac{\pi}{2} < \tan^{-1} 3 + \tan^{-1} 2 < \pi$
Further, $\tan(\tan^{-1} 3 + \tan^{-1} 2) = -1$ when evaluated using $\tan(\alpha - \beta)$ expansion.
$\therefore \tan^{-1} 3 + \tan^{-1} 2 = \frac{3\pi}{4}$ (i.e. the scond quadrant angle)

11 $\frac{\pi^2}{4}, \frac{\pi^2}{4}$ **12** $\cos^{-1} x$, 1

13 (a)



(b) $\frac{\pi}{2} - 1$ (c) $f^{-1}(x) = \cos^{-1}(1 - x)$, $0 \le x \le 2$, $0 \le y \le \pi$
(d) 1

14 (a) Product rule: $\frac{d}{dx}(x \sin^{-1} x) = 1 \times \sin^{-1} x + x \times \frac{1}{\sqrt{1-x^2}}$

$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(b) From (a): $\int \sin^{-1} x\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx = x \sin^{-1} x$

Hence $\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int_0^{\frac{1}{2}} \sin^{-1} x\,dx = \left[x \sin^{-1} x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \sqrt{\frac{3}{4}} - (0 + \sqrt{1})$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

15 $\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\sqrt{1-\cos^2\theta}}{\cos^2\theta} \times (-\sin\theta)\,d\theta$

$= \int \frac{-\sin^2\theta}{\cos^2\theta}\,d\theta = \int -\tan^2\theta\,d\theta$

$= \int (1 - \sec^2\theta)\,d\theta$

$= \theta - \tan\theta + C = \cos^{-1} x - \tan(\cos^{-1} x) + C$

$= \cos^{-1} x - \frac{\sqrt{1-x^2}}{x} + C$

16 $-\frac{1}{3}(9 - x^2)^{\frac{3}{2}} + C$

CHAPTER REVIEW 11

1 (a) $\frac{4\pi}{3}$ (b) $\frac{98\pi}{3}$ (c) 40π

2 (a) $62\,\text{m}^2$, $64\,\text{m}^2$

(b) Rotating about the y-axis, the subintervals for the trapezia would not be equal and the curve is not the same above and below the axis. Calculating the volume for each region and taking the average would give an approximate volume:

$V = \pi \int_0^5 x^2\,dy \approx \pi\left[\frac{2.5}{2}\left(12^2 + 8^2\right) + \frac{4-2.5}{2}\left(8^2 + 4^2\right) + \frac{5-4}{2}\left(4^2 + 0^2\right)\right] \approx 288\pi$ for one side.

3 $2\,du = \frac{dx}{\sqrt{x}}$, $\int 2e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

4 (a) $2\int_1^0 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{8}{15}$ (b) $\int_0^1 (u+3)\sqrt{u}\,du = \frac{12}{5}$

5 $du = 3x^2\,dx$, $\frac{1}{3}\int_1^2 e^u\,du = \frac{1}{3}(e^2 - e)$

6 $\frac{4-\sqrt{2}}{12}$ **7** $du = \frac{dx}{x}$, $\int_1^3 \frac{du}{u^2} = \frac{2}{3}$

8 $-du = 2x\,dx$, $\int_8^5 \frac{-du}{\sqrt{u}} = \int_5^8 \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_5^8 = 4\sqrt{2} - 2\sqrt{5}$

9 (a) $\frac{2\pi - 3\sqrt{3}}{24}$ (b) $\frac{4\pi + 3\sqrt{3}}{48}$

10 $\frac{1}{6}du = x\,dx$, $\int \frac{1}{6}u^{\frac{3}{4}}\,du = \frac{2}{21}(3x^2+1)^{\frac{7}{4}} + C$

11 (a) $\frac{d}{dx}\left[e^{2x}(2\sin x - \cos x)\right] = 5e^{2x}\sin x$

(b) $\int e^{2x}\sin x\,dx = \frac{1}{5}\int \frac{d}{dx}\left\{e^{2x}(2\sin x - \cos x)\right\}dx$

$= \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$

12 $V = \pi \int_0^{\frac{\pi}{2}} \sin^2 2x\,dx = \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$ units^3

13 $\int_{16}^{21} \frac{du}{\sqrt{u}} = 2\sqrt{21} - 8$

14 (a) LHS $= \sin 6x\cos 3x + \cos 6x \sin 3x + \sin 6x \cos 3x - \cos 6x \sin 3x = 2\sin 6x \cos 3x$

(b) $\frac{1}{2}\int (\sin 9x + \sin 3x)\,dx = -\frac{1}{18}\cos 9x - \frac{1}{6}\cos 3x + C$

15 $\frac{4\pi^2 - 3\sqrt{3}\pi}{48}$ units^3

16 $\frac{1}{3}\left[\sec 3x\right]_0^{\frac{\pi}{9}} = \frac{1}{3}\left(\sec\frac{\pi}{3} - \sec 0\right) = \frac{1}{3}$

17 (a) $\sin 3\theta = \sin\theta(1 - 2\sin^2\theta) + \cos\theta \times 2\sin\theta\cos\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - 2\sin^2\theta)$
$= 3\sin\theta - 4\sin^3\theta$

(b) $\int \sin 3\theta\,d\theta = -\frac{1}{3}\cos 3\theta + C$

18 (a) $\dfrac{-x}{\sqrt{1-x^2}}+\cos^{-1}x$ (b) $x\cos^{-1}x-\sqrt{1-x^2}+C$

19 (a) $\dfrac{1}{\sqrt{1-x^2}}$ (b) $\dfrac{-2}{\sqrt{1-4x^2}}$ (c) $\dfrac{2}{4+x^2}$

(d) $e^x\sin^{-1}x+\dfrac{e^x}{\sqrt{1-x^2}}$ (e) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (f) $\dfrac{-2}{4+x^2}$

(g) $\dfrac{-2x}{\sqrt{1-4x^2}}+\cos^{-1}2x$ (h) $\dfrac{1}{2\sqrt{\sin^{-1}x}\sqrt{1-x^2}}$

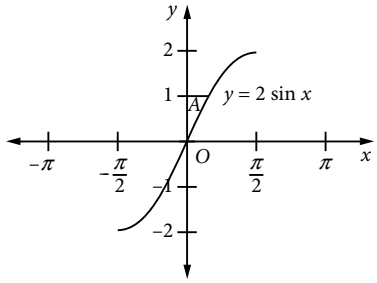
(i) $\dfrac{3}{\tan^{-1}x(1+x^2)}$

20 (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9\sqrt{2}}$

21 (a) $\sin^{-1}\frac{x}{2\sqrt{2}}+C$ (b) $\frac{1}{8}\tan^{-1}\frac{x}{8}+C$ (c) $\frac{2}{3}\tan^{-1}\frac{3x}{2}+C$

(d) $\frac{1}{2\sqrt{2}}\sin^{-1}\sqrt{2}x+C$ (e) $\sin^{-1}\frac{x+2}{4}+C$

22 (a)

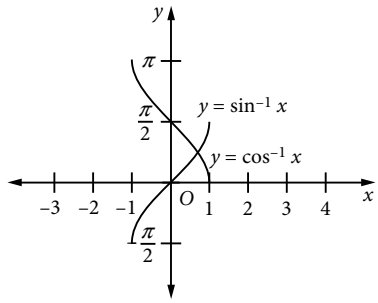


(b) $\frac{\pi}{6}+\sqrt{3}-2$

23 (a) RHS $=\dfrac{x(1+x^2)}{1+x^2}+\dfrac{2}{1+x^2}=\dfrac{x^3+x+2}{1+x^2}=$ LHS

(b) $\frac{1}{2}x^2+2\tan^{-1}x+C$

24 (a) $0<x<1$



(b) $\left(\frac{\pi}{4},0\right)$

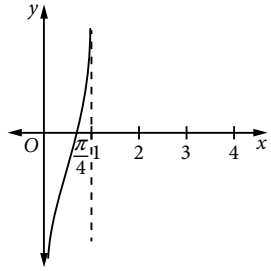
(c) $y=\log_e(\sin^{-1}x)-\log_e(\cos^{-1}x)$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{\sin^{-1}x\sqrt{1-x^2}}+\dfrac{1}{\cos^{-1}x\sqrt{1-x^2}}$

Over the domain $0<x<1$, every factor in the expression is positive $\therefore \dfrac{dy}{dx}>0$ for all x in the domain

Hence the function is increasing for all x in the domain.

(d)



25 (a) $\log_e(1+x)+C$ (b) $\tan^{-1}x+C$ (c) $\frac{1}{2}\log_e(1+x^2)+C$

(d) $\log_e x+\frac{1}{2}x^2+C$ (e) $x-\tan^{-1}x+C$

(f) $\frac{1}{2}x^2-\frac{1}{2}\log_e(x^2+1)+C$ (g) $\frac{-1}{1+x}+C$ (h) $2\sqrt{1+x}+C$

(i) $\sin^{-1}x+C$ (j) $-\sqrt{1-x^2}+C$ (k) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{e^x}{\sqrt{2}}+C$

(l) $\frac{1}{6}(\tan^{-1}3x)^2+C$

26 (a) $\frac{\pi}{2}$

(b) $u=\sin^{-1}x \quad \therefore\ du=\dfrac{1}{\sqrt{1-x^2}}dx$

Where $x=0$, $u=0$; where $x=\frac{1}{\sqrt{2}}$, $u=\frac{\pi}{4}$

$$\int_0^{\frac{1}{\sqrt{2}}}\cos^{-1}x\sqrt{\frac{\sin^{-1}x}{1-x^2}}\,dx=\int_0^{\frac{\pi}{4}}\left(\tfrac{\pi}{2}-u\right)\sqrt{u}\,du$$

$$=\left[\tfrac{\pi}{3}u^{\frac{3}{2}}-\tfrac{2}{5}u^{\frac{5}{2}}\right]_0^{\frac{\pi}{4}}=\frac{\pi^{\frac{5}{2}}}{24}-\frac{\pi^{\frac{5}{2}}}{80}=\frac{7\pi^{\frac{5}{2}}}{240}$$

27 (a) At A, $\cos^2x=\tan^2x$

$\therefore \cos^2x=\sec^2x-1$

$\cos^4x=1-\cos^2x$

$\cos^4x+\cos^2x-1=0$

$\therefore \cos^2x=\dfrac{-1\pm\sqrt{5}}{2}$

But $\dfrac{-1-\sqrt{5}}{2}$ is negative

$\therefore \cos^2x=\dfrac{-1+\sqrt{5}}{2}$

$\cos x=\pm\sqrt{\dfrac{-1+\sqrt{5}}{2}}$

But $0\le x\le\frac{\pi}{2}$

$\therefore \cos x>0$

$\therefore \cos x=\sqrt{\dfrac{-1+\sqrt{5}}{2}}$

$x=\cos^{-1}\sqrt{\dfrac{-1+\sqrt{5}}{2}}$

(b) 0.5 units^2

CHAPTER 12

EXERCISE 12.1

1 $y=x^n: \quad y'=nx^{n-1}$

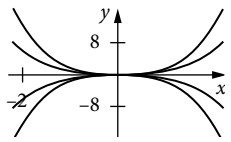
LHS $=y'-\frac{n}{x}y$

$=nx^{n-1}-\frac{n}{x}\times x^n$

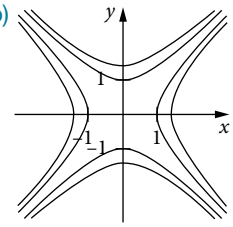
$=nx^{n-1}-nx^{n-1}$

$=0=$ RHS

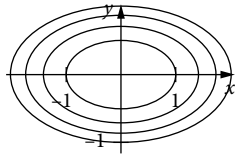
2 (a)



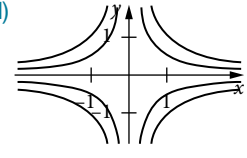
(b)



(c)



(d)



3 (a) $y=-7e^{-2x}+10$ (b) $y=5e^{-x}+5$ (c) $y=\dfrac{e^x}{e^x-\frac{1}{2}}$

(d) $y=\dfrac{5e^{10x}}{4+e^{10x}}$ (e) $y=\frac{5}{4}e^{2x}-\frac{x}{2}-\frac{1}{4}$

4 (a) $A=2 \Rightarrow y=\dfrac{-x}{1+2x}$ (b) $A=2 \Rightarrow y=\dfrac{1}{x-2}$

(c) $A=\frac{1}{2} \Rightarrow y=\frac{1}{2}e^{-2x}+\frac{1}{2}$ (d) $a=3, b=2 \Rightarrow y=e^{-x}(3x+2)$

(e) $a=1, b=1 \Rightarrow y=\sin 2x+\cos 2x$

(f) $a=\frac{1}{2}, b=\frac{1}{2} \Rightarrow y=\frac{1}{2}e^{-x}+\frac{1}{2}e^{3x}$

5 (a) The differential equation is first order, second degree, dependent variable: y, independent variable: x

(b) The differential equation is first order, first degree, dependent variable: x, independent variable: t

(c) The differential equation is second order, first degree, dependent variable: x, independent variable: t

(d) The differential equation is first order, first degree, dependent variable: y, independent variable: x

6 $\frac{dy}{dx}=ke^{kx}$, $\frac{d^2y}{dx^2}=k^2e^{kx}$. $k=-1, 2$

7 C $\frac{dy}{dx}=-2\sin 2x$, $\frac{d^2y}{dx^2}=-4\cos 2x$

8 (a) $y=e^{x^n}: y'=nx^{n-1}e^{x^n}$

$\text{LHS}=nx^{n-1}e^{x^n}-nx^{n-1}e^{x^n}=0=\text{RHS}$

(b) $y=x-x^{-1}: y'=1+x^{-2}$

$\text{LHS}=x\left(1+x^{-2}\right)+x-x^{-1}$

$=x+x^{-1}+x+x^{-1}$

$=2x=\text{RHS}$

(c) $y=\dfrac{x}{1+x}: y'=\dfrac{1}{(1+x)^2}$

$\text{LHS}=\dfrac{1}{(1+x)^2}-\left(1-\dfrac{x}{1+x}\right)^2$

$=\dfrac{1}{(1+x)^2}-\dfrac{1}{(1+x)^2}$

$=0=\text{RHS}$

(d) $y=\dfrac{e^{rx}}{1+e^{rx}}: y'=\dfrac{re^{rx}}{\left(1+e^{rx}\right)^2}$

$\text{RHS}=r\left(1-\dfrac{e^{rx}}{1+e^{rx}}\right)\left(\dfrac{e^{rx}}{1+e^{rx}}\right)$

$=\dfrac{re^{rx}}{\left(1+e^{rx}\right)^2}=\text{LHS}$

(e) $y=\dfrac{1}{4-x^2}: y'=\dfrac{2x}{\left(4-x^2\right)^2}$

$\text{RHS}=2x\times\dfrac{1}{\left(4-x^2\right)^2}$

$=\dfrac{2x}{\left(4-x^2\right)^2}=\text{LHS}$

9 (a) $m(t)=4(e^{-2t}-e^{-3t})$, $\frac{dm}{dt}=4\left(-2e^{-2t}+3e^{-3t}\right)=-8e^{-2t}+12e^{-3t}$.

$-3m+4e^{-2t}=-3\times 4\left(e^{-2t}-e^{-3t}\right)+4e^{-2t}$

$=-12e^{-2t}+12e^{-3t}+4e^{-2t}=-8e^{-2t}+12e^{-3t}=\frac{dm}{dt}$

(b) $m(0)=4(e^0-e^0)=0$

(c) $\frac{dm}{dt}=0$ when $-8e^{-2t}+12e^{-3t}=0$, $4e^{-3t}(-2e^t+3)=0$,
$t=\ln 1.5=0.4055$ h $=24.3$ minutes

(d) $m(t)=\dfrac{4\left(1-e^{-t}\right)}{e^{2t}}$, $\lim\limits_{t\to\infty} m(t)=0$.

10 (a) $P(0)=10\,000$, $P(t)=5000(1+e^{\frac{t}{5}})$: $\frac{dP}{dt}=1000e^{\frac{t}{5}}$

$\frac{1}{5}P-1000=1000\left(1+e^{\frac{t}{5}}\right)-1000=1000e^{\frac{t}{5}}=\frac{dP}{dt}$

(b) $P(t)=5000(3-e^{\frac{t}{5}})$, $\frac{dP}{dt}=-1000e^{\frac{t}{5}}$

$\frac{1}{5}P-3000=1000\left(3-e^{\frac{t}{5}}\right)-3000=-1000e^{\frac{t}{5}}=\frac{dP}{dt}$

(c) $P(t)=5(k+(2000-k)e^{\frac{t}{5}})$. $\frac{dP}{dt}=(2000-k)e^{\frac{t}{5}}$

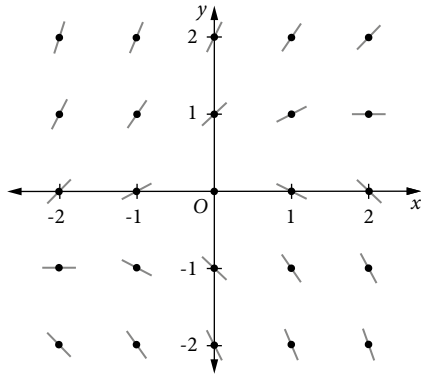
$\frac{1}{5}P-1000=\frac{1}{5}\times 5\left(k+(2000-k)e^{\frac{t}{5}}\right)-k=(2000-k)e^{\frac{t}{5}}=\frac{dP}{dt}$

(d) Require $\frac{dP}{dt}=0$ so $k=2000$

EXERCISE 12.2

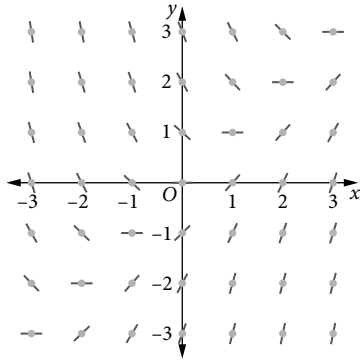
1 (a) -3.5

(b)

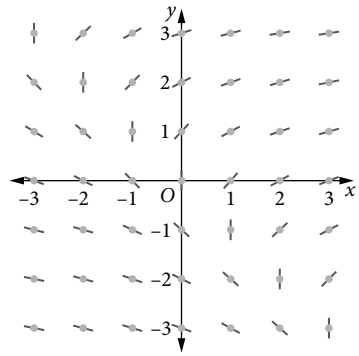


2 Undefined values of $\frac{dy}{dx}$ are represented as vertical line segments.

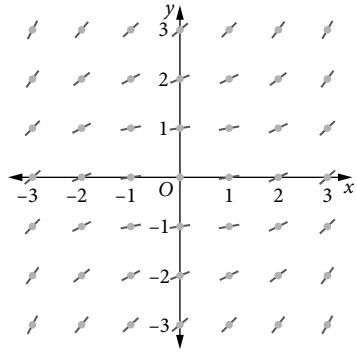
(a)



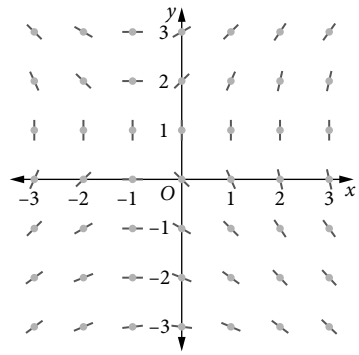
(b)



(c)



(d)



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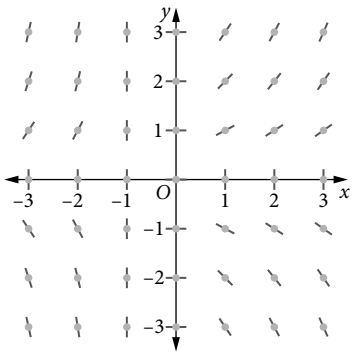
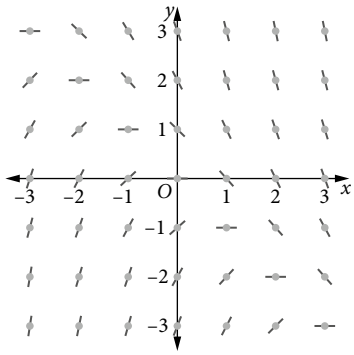
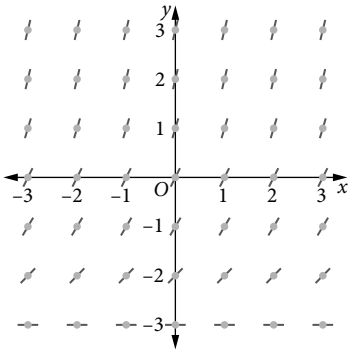


The gradient at $(-1, 0)$ cannot be determined.

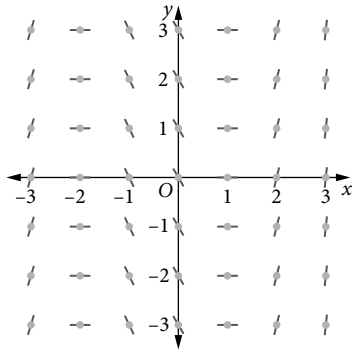
(f)



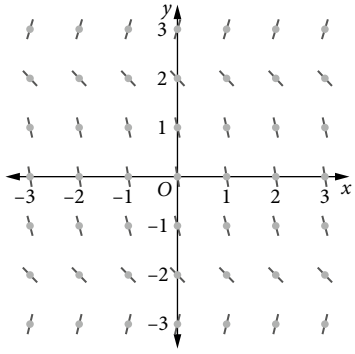
(g)



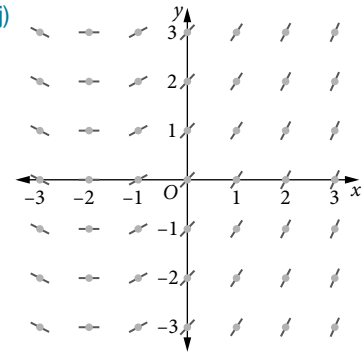
(h)



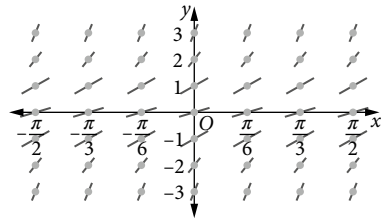
(i)



(j)

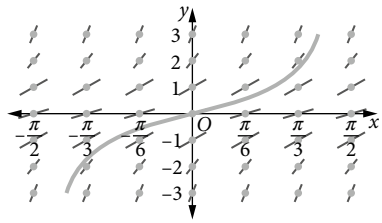


3 (a) (i) 1 (ii) 2 (iii) 5 (iv) 10



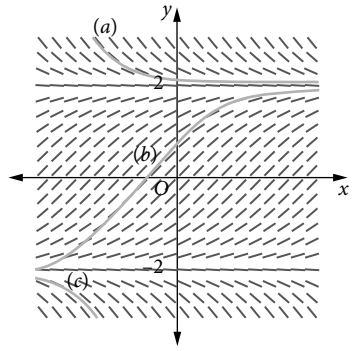
(b) $\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

(c) Any point where $x = \frac{\pi}{2}$ or $x = -\frac{\pi}{2}$



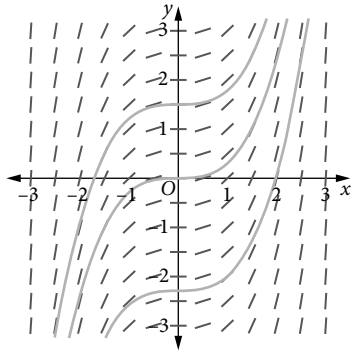
4 B

5 Refer to graph below for answers to (a), (b) and (c)



6 D **7** A

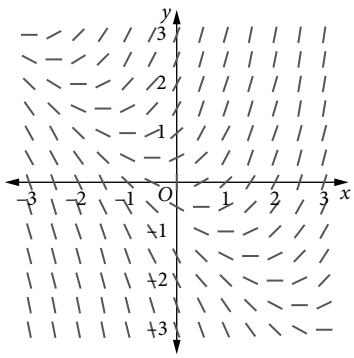
8 (a) The three lines below are an example.



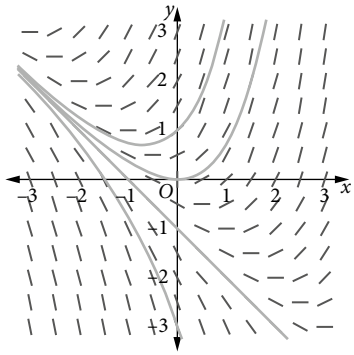
(b) B $\frac{dy}{dx} = x^2$

9 C

10 (a)



(b) Some possible solutions are shown.



(c) $y = -x - 1$ or $x + y + 1 = 0$.

(d) If $y = -x - 1$, $\frac{dy}{dx} = -1$

$y = -x - 1 \Rightarrow -1 = x + y$

$\therefore \frac{dy}{dx} = x + y$, so $y = -x - 1$ is a solution of $\frac{dy}{dx} = x + y$.

EXERCISE 12.3

1 (a) $y = x^2 - x + c$ (b) $f(x) = \frac{2}{7}x^{\frac{7}{2}} + c$

(c) $y = \sin(2x) + c$ (d) $y = x + \frac{1}{2}\sin(2x) +$

(e) $z = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$ (f) $z = \frac{1}{2}\log_e\left(x^2 + 4\right) + c$

(g) $x = \theta + c$ (h) $f(x) = x + 2e^{-\frac{x}{2}} + c$

2 (a) $y = \frac{1}{2}x^4 - \frac{1}{2}x^2 + x + 1$ (b) $f(x) = \frac{1}{2}(1 - \cos x)^2 - \frac{9}{8}$

(c) $x(t) = \frac{2}{3}(t+1)^{\frac{3}{2}} + \frac{7}{3}$ (d) $V = \frac{1}{2}e^{t^2} + \frac{3}{2}$

(e) $x = 1 - \log_e|2 + \cos\theta|$ (f) $u = x - \tan^{-1}x + 5$

(g) $y = \frac{1}{3}\left(t^2 - 9\right)^{\frac{3}{2}} + 1$

3 (a) $y = 1 + \arcsin\left(\frac{x}{3}\right)$ (b) $\frac{1}{2}\log_e\left(t^2 + 1\right) + 1$

(c) $x = \frac{1}{2}\left(y + \log_e|y - 1|\right)$ (d) $z = -2\sqrt{1-t} + \frac{2}{3}(1-t)^{\frac{3}{2}}$

4 B

5 (a) $\frac{d}{dx}\left(xe^x\right) = 1 \times e^x + x \times e^x = e^x + xe^x$

(b) $xe^x - e^x + C$ (c) $y = xe^x - e^x$

(d) $y = xe^x - 2e^x = e^x(x-2)$ (e) $y = e^x(x-2)$

6 $y = \tan^{-1}x + 1$

7 (a) $\frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right)\right)$

$= 1 + \tan^{-1}x + \frac{x}{x^2+1} - \frac{1}{2} \times \frac{2x}{x^2+1}$

$= \tan^{-1}x + 1$

(b) $\frac{dy}{dx} = \tan^{-1}x + 1$

$y = \int\left(\tan^{-1}x + 1\right)dx$

$= \int \frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right)\right)dx$

$= x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right) + C$

$y(0) = 0$: $0 = C$, $C = 0$

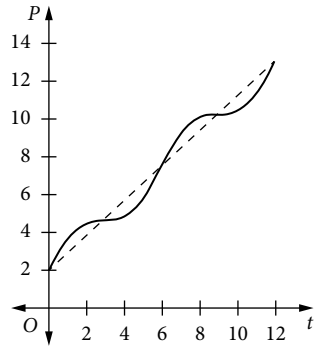
$y = x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right)$

8 (a) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ (b) $z = \frac{1}{2}\left(e^x - e^{-x}\right)$

(c) Shown in (a) and (b)

9 (a) $P(t) = 1 + rt + \frac{3r}{\pi}\sin\left(\frac{\pi t}{3}\right)$

(b) $P(t) = 1 + t + \frac{3}{\pi}\sin\left(\frac{\pi t}{3}\right)$, $0 \le t \le 12$.



The dashed line is the graph of $P(t) = 1 + t$.

10 (a) $t = T$: $V(T) = 125\,000\arctan\left(\frac{T^2}{4}\right)$

(b) Maximum amount of oil leaked $= \lim_{t\to\infty}\left(125000\tan^{-1}\left(\frac{t^2}{4}\right)\right)$

Now $\lim_{t\to\infty}\left(\tan^{-1}t\right) = \frac{\pi}{2}$

So, $\lim_{T\to\infty}V(T) = \lim_{T\to\infty}\left(125\,000\tan^{-1}\left(\frac{T^2}{4}\right)\right)$

$= 125\,000 \times \frac{\pi}{2}$

$\approx 196\,350$ litres

$< 300\,000$ litres

The report in the newspaper is incorrect.

11 (a) $A = \frac{20}{\pi}$

(b) $\frac{dV}{dh} = \frac{20}{\pi}\sin^{-1}\left(\frac{h}{10}\right) + \frac{20h}{\pi\sqrt{100-h^2}}$

(c) $\frac{dt}{dh} = \frac{10}{\pi}\sin^{-1}\left(\frac{h}{10}\right) + \frac{10h}{\pi\sqrt{100-h^2}}$

(d) $t = \frac{10h}{\pi}\sin^{-1}\left(\frac{h}{10}\right)$

(e) $h = 5\sqrt{3}$: $t = \frac{50\sqrt{3}}{3}$ minutes

12 (a) $\frac{dh}{dt} = -\frac{49t}{5} - 3900\ln\left|\frac{m_F - 200(t-200)}{m_F + 40\,000}\right|$

(b) $h(t) = 3900t - \frac{49}{10}t^2$

$+ \frac{39}{2}\left(m_F - 200(t-200)\right)\log_e\left|\frac{m_F - 200(t-200)}{40\,000 + m_F}\right|$

(c) $t_b = \frac{m_F}{200}$:

$h(t_b) = \frac{30m_F}{2} - \frac{49{m_F}^2}{400\,000} + 780\,000\log_e\left|\frac{40\,000}{40\,000 + m_F}\right|$

EXERCISE 12.4

1 (a) $y = e^{-x}$ (b) $y = -e^{2x}$ (c) $y = -5e^{-2x}$ (d) $y = 3e^{2x}$

2 (a) $y = 5e^{-2x} + 3$ (b) $y = -5e^{-2x} + 8$ (c) $y = 8e^{-4x} - 3$
(d) $y = 3 - 8e^{-4x}$

3 C

4 (a) $V(t) = \left(2 - \frac{t}{4}\right)^3$ (b) $t = 8$ hours

5 (a) $\frac{dP}{dh} = -kP$ (b) $P = 101.3e^{-0.2h}$
(c) $h = 9$: $P = 101.3e^{-1.8} = 16.7$ kPa

6 $q = ECe^{-\frac{t}{RC}}$

7 (a) $T = 20 + 80e^{-kt}$ (b) $k = \frac{1}{20}\log_e\left(\frac{8}{5}\right) \approx 0.024$
(c) $T = 25$: $25 = 20 + 80e^{-0.024t}$, $t = 118$ minutes

8 (a) $\frac{dy}{dt} = -ky$

$\frac{dy}{y} = -k\,dt$

$\int \frac{dy}{y} = -k\int dt$

$\log_e y = -kt + C$

$y = e^{-kt+C}$

$y = Ae^{-kt}$ where $A = e^C$

$t = 0, y = D$: $D = A$

$y = De^{-kt}$

$t = T$: $y = De^{-kT}$

(b) $t = 0, y = D$

$t = T$: $y = De^{-kT}$

$t = 2T$: $y = De^{-kT} + De^{-2kT}$

$t = 3T$: $y = De^{-kT} + De^{-2kT} + De^{-3kT}$

$t = (n-1)T$: $y = D + De^{-kT} + De^{-2kT} + De^{-3kT} + \ldots + De^{(-n+1)kT}$

$y = D(1 + e^{-kT} + e^{-2kT} + e^{-3kT} + \ldots + e^{(-n+1)kTT})$

Geometric series, $a = 1$, $r = e^{-kT}$.

$$y = D \times \frac{1\left(1 - e^{-nkT}\right)}{1 - e^{-kT}}$$

$$= \frac{D\left(1 - e^{-nkT}\right)}{1 - e^{-kT}}$$

(c) $$y_s = \lim_{n\to\infty}\left[\frac{D\left(1 - e^{-nkT}\right)}{1 - e^{-kT}}\right] = \frac{D}{1 - e^{-kT}}$$

(d) $t = T$, $y_s = \frac{D}{1 - e^{-kT}}$

$t = T + h$: $y = \frac{De^{-k(T+h)}}{1 - e^{-kT}}$

$\frac{De^{-k(T+h)}}{1 - e^{-kT}} = \frac{D}{2\left(1 - e^{-kT}\right)}$

$2e^{-k(T+h)} = 1$

Let $T = 0$ as you wish to measure your time from when there are no more doses:

$2e^{-kh} = 1$

$e^{kh} = 2$

$kh = \log_e 2$

$h = \frac{\log_e 2}{k}$

EXERCISE 12.5

1 (a) $y = A\left(x^2 + 4\right)$ (b) $y = Ax^2$

(c) $y = \tan\left(\frac{2}{3}x^{\frac{3}{2}} + c\right)$ (d) $y = Ae^{2\sqrt{1-x^2}} - 1$

(e) $y = A\sqrt{1 + x^2}$ (f) $y = \log_e\left(\frac{1}{\operatorname{cosec} x + c}\right)$

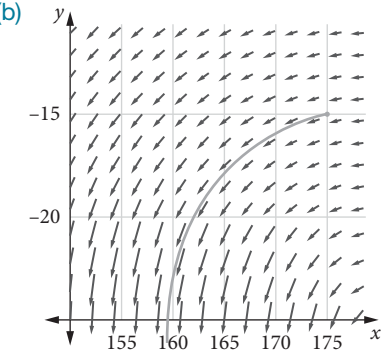
(g) $y = \frac{1}{\sin x + c}$

2 $y = \frac{3}{2} - \frac{1}{2}e^{x^2}$

3 (a) $y = 2e^{\left(3 - \frac{3}{x}\right)}$ (b) $y = \frac{2}{x^2}$

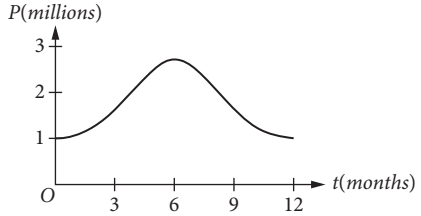
4 (a) $y = \frac{1}{2}\left(-50 + \sqrt{425 - (x - 180)^2}\right)$ as $y < 0$.

(b)



5 (a) $y = \tan^{-1}\left(x^3 + 1\right)$ (b) $R; \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6 $P = e^{\sin^2\left(\frac{\pi t}{12}\right)}$



7 B $\frac{dy}{dx} = \frac{-1 \times (-1)}{(\sin x + C)^2} \times \cos x = y^2 \cos x$

8 B $\frac{dy}{dx} = 2e^{x^2} \times 2x = 2xy$

9 (a) $P = p_0 T_0^{-\frac{g}{R\Gamma}}\left(T_0 - \Gamma z\right)^{\frac{g}{R\Gamma}}$

$$= p_0\left(1 - \frac{\Gamma z}{T_0}\right)^{\frac{g}{R\Gamma}}$$

(b) $$z = \frac{T_0}{\Gamma}\left(1 - \left(\frac{p}{p_0}\right)^{\frac{R\Gamma}{g}}\right)$$

(c) $$z = \frac{T_0}{\Gamma}\left(1 - \left(\frac{p}{p_0}\right)^{\frac{R\Gamma}{g}}\right)$$

$$= \frac{288.15}{0.0065}\left(1 - \left(\frac{61640}{101325}\right)^{\frac{287.053 \times 0.0065}{9.80}}\right)$$

≈ 4003 m

10 (a) $\frac{dy}{dx} = -\frac{2xy}{\log_e y}$

$\int \frac{\log_e y\,dy}{y} = -2\int x\,dx$

$u = \log_e y$: $du = \frac{1}{y}dy$

$\int \frac{\log_e y\,dy}{y} = \int u\,du$

$= \frac{u^2}{2}$

$= \frac{1}{2}\left(\log_e |y|\right)^2$

$\frac{1}{2}\left(\log_e |y|\right)^2 = -x^2 + c$

(b) $\left(\log_e|y|\right)^2 = 1 - 2x^2$

(c) Substitute $x = 1$ in $\left(\log_e|y|\right)^2 = 1 - 2x^2$

$\left(\log_e|y|\right)^2 = -1$ which cannot exist for y a real number.

11 (a) If $y = 5$ is a tangent to the curve at $P(x, 5)$ then $\frac{dy}{dx} = 0$ at P.

$\frac{5-x}{5} = 0$ so $x = 5$.

(b) At $P(5, 5)$: $\frac{d^2y}{dx^2} = -\frac{1}{5} < 0$ so the particular solution $y = f(x)$ passing through P has a local maximum at P.

(c) $y = -\sqrt{25 - (x-5)^2}$

12 (a) $P = \frac{7.752 \times 10^8}{V^{1.4}}$

(b) $V = 60$: $P = \frac{7.752 \times 10^8}{60^{1.4}} \approx 2.512 \times 10^6$ Pa

13 (a) As T increases then S increases, so S is an increasing function and thus $\frac{dS}{dT} > 0$. Thus $k > 0$.

(b) $S = AT^k$ where $A = e^C$ and $A > 0$ since $S > 0$ and $T > 0$.

14 (a) $\frac{dv}{dh} = -\frac{gR^2}{v(R+h)^2}$

$\int v\,dv = -gR^2 \int \frac{dh}{(R+h)^2}$

$\frac{v^2}{2} = \frac{gR^2}{R+h} + c$

$h = 0, v = u$: $\frac{u^2}{2} = gR + c$

$c = \frac{u^2}{2} - gR$

$\frac{v^2}{2} = \frac{gR^2}{R+h} + \frac{u^2}{2} - gR$

$= \frac{u^2}{2} - \frac{gR^2 + gRh - gR^2}{R+h}$

$= \frac{u^2}{2} - \frac{gRh}{R+h}$

$v^2 = u^2 - \frac{2gR}{1+\frac{R}{h}}$

(b) To escape the planet's gravity the upwards velocity v must be non-negative as $h \to \infty$.

$\lim_{h\to\infty}\left(u^2 - \frac{2gR}{1+\frac{R}{h}}\right) = u^2 - 2gR \geq 0$ when $u \geq \sqrt{2gR}$.

The minimum launch velocity is $u = \sqrt{2gR}$

15 (a) $T = 288 - 0.00649h$

(b) $\frac{dB}{dP} = 0.000189\frac{B^2}{P}$, $\frac{dP}{dh} = -0.0352\frac{P}{T}$

$\frac{dB}{dh} = \frac{dB}{dP} \times \frac{dP}{dh}$

$= 0.000189\frac{B^2}{P} \times \left(-0.0352\frac{P}{T}\right)$

$= -6.65 \times 10^{-6}\frac{B^2}{T}$

But $T = 288 - 0.006\,49h$

So $\frac{dB}{dh} = \frac{-6.65 \times 10^{-6} B^2}{288 - 0.006\,49h}$

(c) $h = \frac{288 - 3990e^{\frac{-980}{B}}}{0.006\,49}$

$h = 44\,400 - 615\,000e^{\frac{-980}{B}}$

The actual numbers obtained will depend on when you used your calculator, but should be of a similar magnitude.

(d) $B = 357°\text{K}$, $h = 44\,400 - 615\,000 \times e^{\frac{-980}{357}} \approx 4890\,\text{m} \approx 5000\,\text{m}$

16 (a) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\frac{dV}{dt} = -k\sqrt{h}$

$V(h) = \frac{1}{3}\pi h^2(3-h) = \frac{\pi}{3}\left(3h^2 - h^3\right)$

$\frac{dV}{dh} = \frac{\pi}{3}\left(6h - 3h^2\right) = \pi h(2-3h)$

$-k\sqrt{h} = \pi h(2-h) \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-k\sqrt{h}}{\pi h(2-h)}$

$= \frac{-k}{\pi\sqrt{h}(2-h)}$

(b) $t = \frac{14\pi}{15k} - \frac{2\pi}{15k}\left(10h^{\frac{3}{2}} - 3h^{\frac{5}{2}}\right)$

(c) $h = \frac{1}{4}, t = \frac{187}{16}$: $\frac{187}{16} = \frac{14\pi}{15k} - \frac{2\pi}{15k}\left(\frac{10}{8} - \frac{3}{32}\right)$

$\frac{187}{16} = \frac{14\pi}{15k} - \frac{2\pi}{15k} \times \frac{37}{32}$

$\frac{187}{16} = \frac{187\pi}{15 \times 16k}$

$k = \frac{\pi}{15}$

(d) $t = 14 - 2\left(10h^{\frac{3}{2}} - 3h^{\frac{5}{2}}\right)$

$h = 0$: $t = 14$ minutes

EXERCISE 12.6

1 (a) Maximum monthly sales = 10, Difference of current sales = $10 - s$.

$\frac{ds}{dt} = r(10-s)$, $s(0) = 0$

(b) $s = 10 - 10e^{-rt}$

(c) $s(3) = 10 - 10 \times \left(\frac{4}{5}\right)^3 = 4.88 \approx 5$ truck loads

2 (a) RHS $= \frac{1}{(1+P)}\left[\frac{1}{(1+P-I)} + \frac{1}{I}\right]$

$= \frac{1}{(1+P)}\left(\frac{I+1+P-I}{I(1+P-I)}\right)$

$= \frac{1}{(1+P)}\left(\frac{1+P}{I(1+P-I)}\right)$

$= \frac{1}{I(P+1-I)} = \text{LHS}$

(b) $I = \frac{1+P}{1+Pe^{-(1+P)rt}}$

3 (a) Net rate of change of pollutant (g/day)
= rate of inflow (g/day) − rate of outflow (g/day)

$\frac{dm}{dt} = 10\,000 \times 2 - 10\,000 \times \frac{m}{200000}$

$= 20\,000 - \frac{m}{20}$

Initially pond is unpolluted, so $m(0) = 0$

(b) $c(t) \equiv \frac{m(t)}{200\,000}$

$\frac{dc(t)}{dt} = \frac{1}{200\,000} \times \frac{dm(t)}{dt}$

$\frac{dc}{dt} = \frac{1}{200\,000} \times \left(20\,000 - \frac{m}{20}\right)$

$= \frac{1}{10} - \frac{m}{20 \times 200\,000}$

$= \frac{1}{10} - \frac{c}{20}$ where $c(0) = 0$

(c) $c = 2\left(1 - e^{-\frac{t}{20}}\right)$ (d) $c(10) = 2\left(1 - e^{-\frac{1}{2}}\right) = 0.79$ g/L

4 Net rate of change of salt (kg/min) = rate of inflow (kg/min) − rate of outflow (kg/min)

Net rate of change of volume (L/min) = rate of inflow (L/min) − rate of outflow (L/min)

$Q(0) = 1000 \times 0.01 = 10$ kg

$\frac{dV}{dt}$ = rate of inflow − rate of outflow

$= 10 - 5 = 5$ L/min

$V(0) = 1000$ L

$V(t) = 5t + 1000$

$\frac{dQ}{dt}$ = rate of inflow − rate of outflow

$= 0.04 \times 10 - \frac{Q}{V(t)} = 5$

$= 0.4 - \frac{5Q}{1000+5t}$

$= 0.4 - \frac{Q}{200+t}$

$Q(0) = 10$

5 (a) $v(t) = 0.0003\left(1 - e^{-0.025t}\right)$

(b) $v(t) = 0.0002$: $t = \frac{\log_e 3}{0.025} \approx 44$ minutes

6 (a) Net rate of change of balance ($/year) = rate of inflow ($/year) − rate of outflow ($/year)

$\frac{dx}{dt} = 0.05x - 300000$

$= 0.05(x - 600000)$ where $x(0) = 5000000$

(b) $x(t) = 1000000\left(6 - e^{0.05t}\right)$

(c) $t = 20$: $x(20) = 1000000\left(6 - e^{1}\right) = 3281718.17$

Balance is $3 281 718.15

7 B $y = 10 \times (0.5)^t$ **8** C

9 D **10** B

11 D $N = Ae^{kt}$, $3A = Ae^{2k}$, $k = \frac{1}{2}\log_e 3$, $4 = e^{\frac{t}{2}\log_e 3}$, $t = \frac{2\log_e 4}{\log_e 3}$

12 C

13 D $\frac{dP}{dt} = \frac{P}{5000}(1000 - P)$, $\int\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP = \frac{t}{5} + C$,

$\frac{P}{1000-P} = Ae^{\frac{t}{5}}$, $A = \frac{1}{9}$, $P = \frac{1000}{1 + 9e^{-\frac{t}{5}}}$

14 D

15 (a) $\frac{dN}{dt} \propto N$

$\frac{dN}{dt} = kN, \quad N(0) = 1000, \quad k > 0$

(b) $\frac{dN}{dt} = kN$

$\frac{dN}{N} = k\,dt$

$\log_e |N| = kt + C$

$N = Ae^{kt}$ where $A = \pm e^C$

$N(0) = 1000$: $A = 1000$

$N = 1000e^{kt}$

$N(2) = 3000$: $3000 = 1000e^{2k}$

$e^{2k} = 3$

$e^{k} = \sqrt{3}$

$e^{kt} = \left(\sqrt{3}\right)^t = 3^{\frac{t}{2}}$

$N = 1000 \times 3^{\frac{t}{2}}$

(c) $N(4) = 1000 \times 3^2 = 9000$

$\frac{N(4)}{N(0)} = 9$

Has increased ninefold in four days.

(d) $t = \frac{2\log_e 10}{\log_e 3} = 4.192$ days ≈ 4 days 5 hours

16 (a) $\frac{d\left[{}^{14}C\right]}{dt} = -r\left[{}^{14}C\right]$, $r = 1.2097 \times 10^{-4}$ years^{-1}

$-\frac{1}{r}\int \frac{d\left[{}^{14}C\right]}{\left[{}^{14}C\right]} = \int dt$

$t + c = -\frac{1}{r}\log_e\left(\left[{}^{14}C\right]\right)$

$t = 0$: $c = -\frac{1}{r}\log_e\left(\left[{}^{14}C\right]_0\right)$

$t = -\frac{1}{r}\log_e\left(\left[{}^{14}C\right]\right) + \frac{1}{r}\log_e\left(\left[{}^{14}C\right]_0\right)$

$= -\frac{1}{r}\log_e\left(\frac{\left[{}^{14}C\right]}{\left[{}^{14}C\right]_0}\right)$

$t_{\frac{1}{2}}$ when $\frac{\left[{}^{14}C\right]}{\left[{}^{14}C\right]_0} = \frac{1}{2}$

$t_{\frac{1}{2}} = -\frac{1}{1.2097 \times 10^{-4}}\log_e \frac{1}{2} = 5730$ years

(b) If half the original carbon-14 has radiated away then the tree died 5730 years ago.

(c) $\frac{d}{dt}\left[{}^{14}C\right] = -r\left[{}^{14}C\right]$

Since $\left[{}^{12}C\right]$ is a constant then divide the previous equation by it.

$\frac{d}{dt}\left(\frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]}\right) = -r\frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]}$ and $\frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]} = \frac{\left[{}^{14}C\right]_0}{\left[{}^{12}C\right]}$ when $t = 0$.

Let $R = \frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]}$ so $\frac{dR}{dt} = -rR$ with $R = \frac{\left[{}^{14}C\right]_0}{\left[{}^{12}C\right]}$

$= 1.3 \times 10^{-12}$ when $t = 0$.

$\frac{dR}{dt} = -1.2097 \times 10^{-4} R$ with $R = 1.3 \times 10^{-12}$ when $t = 0$.

(d) $\frac{dR}{dt} = -rR$ where $r = 1.2097 \times 10^{-4}$

$\frac{dR}{R} = -r\,dt$

$\log_e R = -rt + c$

$t = 0, R = 1.3 \times 10^{-12}$: $\log_e\left(1.3 \times 10^{-12}\right) = c$

$\log_e R = -rt + \log_e\left(1.3 \times 10^{-12}\right)$

$\log_e\left(\frac{R}{1.3 \times 10^{-12}}\right) = -rt$

$\frac{R}{1.3 \times 10^{-12}} = e^{rt}$

$R = 1.3 \times 10^{-12} e^{-rt}$

But $e^{-rt_{\frac{1}{2}}} = \frac{1}{2}$ i.e. $e^{-5370r} = \frac{1}{2}$

so $\left(e^{-5370r}\right)^{\frac{t}{5370}} = \left(\frac{1}{2}\right)^{\frac{t}{5370}}$

$R = 1.3 \times 10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5370}}$

(e) $R = \dfrac{\left[C^{14}\right]}{\left[C^{12}\right]} = 0.9 \times 10^{-4}$

$0.9 \times 10^{-4} = 1.3 \times 10^{-12} \left(\frac{1}{2}\right)^{\frac{t}{5370}}$

$\left(\frac{1}{2}\right)^{\frac{t}{5370}} = \frac{0.9}{1.3} \times 10^{8}$

$\frac{t}{5370} \log_e \left(\frac{1}{2}\right) = \log_e \left(\frac{0.9}{1.3} \times 10^{8}\right)$

$t = \dfrac{5370 \log_e \left(\frac{9}{13} \times 10^{8}\right)}{-\log_e 2} = -139\,861 \text{ years}$

17 (a) $\dfrac{dS}{dt} = -r(S - I)$

$\dfrac{dS}{S - I} = -r\,dt$

$\int \dfrac{dS}{S - I} = -r \int dt$

$\log_e |S - I| = -rt + C$

$|S - I| = e^{-rt+C}$

$S - I = Ae^{-rt}$ where $A + e^{C}$

$S(0) = 1: \quad 1 - I = A$

$S(t) = I + (1 - I)e^{-rt}$

(b) $\frac{dS}{dt} = -r(S - I)$. Use the Chain Rule.

$\frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d}{dS}\left(-r(S - I)\right) \times \frac{dS}{dt}$

$= -r \times \left(-r(S - I)\right)$

$= r^2 (S - I)$

OR $S(t) = I + (1 - I)e^{-rt}$

$\frac{dS}{dt} = -r(1 - I)e^{-rt}$

$\frac{d^2S}{dt^2} = -r(1 - I)\left(-re^{-rt}\right)$

$= r^2 (1 - I)e^{-rt}$

But $S - I = (1 - I)e^{-rt}$

So $\frac{d^2S}{dt^2} = -r^2 (S - I)$

(c) $\lim_{t \to \infty} S = 0.05$ so $\frac{dS}{dt} = 0$ as S has become a constant.

$\lim_{t \to \infty} \frac{dS}{dt} = \lim_{t \to \infty} \left[-r(0.05 - I)\right] = 0$

So $I = 0.05$

(d) $S(t) = 0.05 + 0.95e^{-rt}$

$S(14) = 0.06: \quad 0.06 = 0.05 + 0.95e^{-14r}$

$0.95e^{-14r} = 0.01$

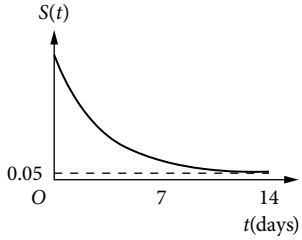
$r = -\frac{1}{14} \log_e \frac{1}{95} = \frac{1}{14} \log_e 95 = 0.3253$

(e) Death rate $= -\frac{dS}{dt}$ and is a maximum when S is a maximum

Death rate $= 0.3253(S - 0.05)$. $S(0) = 1$, which is the largest value of S so the death rate is greatest when $t = 0$.

Maximum death rate $= 0.3253(1 - 0.05) = 0.309$ per day

(f)



18 (a) $p(t) = \dfrac{A(t)}{A(t) + B(t)}$

$p'(t) = \dfrac{A'(t)\left(A(t) + B(t)\right) - A(t)\left(A'(t) + B'(t)\right)}{\left(A(t) + B(t)\right)^2}$

$= \dfrac{A'(t)A(t) + A'(t)B(t) - A'(t)A(t) - A(t)B'(t)}{\left(A(t) + B(t)\right)^2}$

$= \dfrac{A'(t)B(t) - A(t)B'(t)}{\left(A(t) + B(t)\right)^2}$

(b) $A'(t) = r_A A(t)$, $r_A > 0$ and $B'(t) = r_B B(t)$, $r_B > 0$

$p'(t) = \dfrac{r_A A(t)B(t) - r_B A(t)B(t)}{\left(A(t) + B(t)\right)^2}$

$= \dfrac{\left(r_A - r_B\right) A(t)B(t)}{\left(A(t) + B(t)\right)^2}$

(c) $p'(t) = \left(r_A - r_B\right)\left[\dfrac{A(t)}{A(t) + B(t)} \times \dfrac{B(t)}{A(t) + B(t)}\right]$

$= \left(r_A - r_B\right) \dfrac{A(t)}{A(t) + B(t)}\left[\dfrac{A(t) + B(t) - A(t)}{A(t) + B(t)}\right]$

$= \left(r_A - r_B\right) \dfrac{A(t)}{A(t) + B(t)}\left[1 - \dfrac{A(t)}{A(t) + B(t)}\right]$

$= \left(r_A - r_B\right) p(t)\left(1 - p(t)\right)$

(d) $p(0) = \frac{1}{10}$, $r_A - r_B = \frac{1}{100}$ hr^{-1}:

$\dfrac{p}{1 - p} = \dfrac{e^{\frac{t}{100}}}{9}$

$9p = e^{\frac{t}{100}} - pe^{\frac{t}{100}}$

$\left(9 + e^{\frac{t}{100}}\right) p = e^{\frac{t}{100}}$

$p(t) = \dfrac{1}{1 + 9e^{-\frac{t}{100}}}$

$p(10) = \dfrac{1}{1 + 9e^{-0.1}} \approx 0.11$

19 (a) $\dfrac{dL}{dt} = \dfrac{1}{5}(3 - L)$

$\dfrac{dL}{3 - L} = \dfrac{dt}{5}$

$\int \dfrac{dL}{3 - L} = \int \dfrac{dt}{5}$

$\log_e |3 - L| = -\dfrac{t}{5} + C$

$|3 - L| = e^{-\frac{t}{5}+C}$

$3 - L = Ae^{-\frac{t}{5}}$ where $A = \pm e^{C}$

$L(0) = 0: \quad A = 3$

$L(t) = 3 - 3e^{-\frac{t}{5}}$

$= 3\left(1 - e^{-\frac{t}{5}}\right)$

(b) $W(t) = 16L^3$, $\frac{dH}{dt} = \frac{1}{100}W$, $H(0) = 0$

$\frac{dH}{dt} = \frac{1}{100} \times 16L^3$

$= \frac{4 \times 27}{25}\left(1 - e^{-0.2t}\right)^3$

$= \frac{108}{25}\left(1 - 3e^{-0.2t} + 3e^{-0.4t} - e^{-0.6t}\right)$

$\int dH = \frac{108}{25}\int\left(1 - 3e^{-0.2t} + 3e^{-0.4t} - e^{-0.6t}\right)dt$

$H + c = \frac{108}{25}\left(t + 15e^{-0.2t} - \frac{15}{2}e^{-0.4t} + \frac{5}{3}e^{-0.6t}\right)$

$H(0) = 0$ so $c = \frac{108}{25}\left(0 + 15 - \frac{15}{2} + \frac{5}{3}\right) = \frac{108}{25} \times \frac{55}{6}$

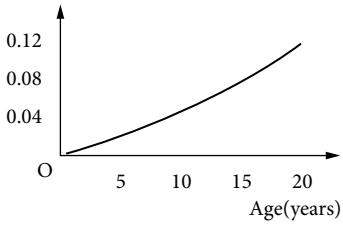
$H = \frac{108}{25}\left(t + 15e^{-0.2t} - \frac{15}{2}e^{-0.4t} + \frac{5}{3}e^{-0.6t} - \frac{55}{6}\right)$

(c) $C(t) = \dfrac{H(t)}{W(t)}$ (mg/kg)

$C = \dfrac{\frac{108}{25}\left(t + 15e^{-0.2t} - \frac{15}{2}e^{-0.4t} + \frac{5}{3}e^{-0.6t} - \frac{55}{6}\right)}{16 \times 27\left(1 - e^{-0.2t}\right)^3}$

$= \dfrac{t + 15e^{-0.2t} - \frac{15}{2}e^{-0.4t} + \frac{5}{3}e^{-0.6t} - \frac{55}{6}}{100\left(1 - e^{-0.2t}\right)^3}$

(d) Hg(mg/kg)



20 (a) $\frac{dW}{dt} = rW\log_e\left(\frac{C}{W}\right)$

$\frac{dW}{dt} = rW\left(\log_e C - \log_e W\right)$ [1]

$\frac{dW}{dt} = \frac{1}{20}W\left(10 - \log_e W\right)$ [2]

$r = \frac{1}{20}$, $\log_e C = 10$, $C = e^{10}$

(b) $W = e^{10 - 9e^{-t/20}}$

$\frac{dW}{dt} = \frac{dW}{du} \times \frac{du}{dt}$ where $u = 10 - 9e^{\frac{-t}{20}}$

$= \frac{9}{20}e^{\frac{-t}{20}} \times e^{10 - 9e^{\frac{-t}{20}}}$

$= \frac{1}{20}e^{10 - 9e^{\frac{-t}{20}}} \times 9e^{\frac{-t}{20}}$

Now $10 - \log_e\left(e^{10 - 9e^{\frac{-t}{20}}}\right) = 10 - \left(10 - 9e^{\frac{-t}{20}}\right) = 9e^{\frac{-t}{20}}$

So $\dfrac{dW}{dt} = \dfrac{1}{20}e^{10 - 9e^{\frac{-t}{20}}}\left(10 - \log_e\left(e^{10 - 9e^{\frac{-t}{20}}}\right)\right)$

And $W(0) = e^{10-9} = e$ so the initial condition is satisfied

$W = e^{10 - 9e^{\frac{-t}{20}}}$ is a solution to equation [2]

(c) $\frac{dW}{dt} = 0$

$\frac{1}{20}W\left(10 - \log_e W\right) = 0$

$10 - \log_e W = 0$

$W = e^{10}$ mg

(d) For the maximum growth rate find when $\frac{d^2W}{dt^2} = 0$

$\frac{d^2W}{dt^2} = \frac{d}{dt}\left(\frac{dW}{dt}\right)$

$= \frac{d}{dW}\left(\frac{dW}{dt}\right) \times \frac{dW}{dt}$

$= \frac{d}{dW}\left(\frac{1}{20}W\left(10 - \log_e W\right)\right) \times \frac{dW}{dt}$

$= \frac{1}{20}\left(1 \times \left(10 - \log_e W\right) + W \times \frac{-1}{W}\right) \times \left(\frac{1}{20}W\left(10 - \log_e W\right)\right)$

$= \frac{1}{20}\left(9 - \log_e W\right) \times \frac{1}{20}W\left(10 - \log_e W\right)$

$= \frac{W}{400}\left(9 - \log_e W\right)\left(10 - \log_e W\right)$

$\frac{d^2W}{dt^2} = 0$ gives $W = e^9$ or $W = e^{10}$ mg

When $W = e^{10}$ mg, this is the equilibrium position

as $t \to \infty$ and $\frac{dW}{dt} = 0$.

Hence the Greatest Growth Rate is when $W = e^9$.

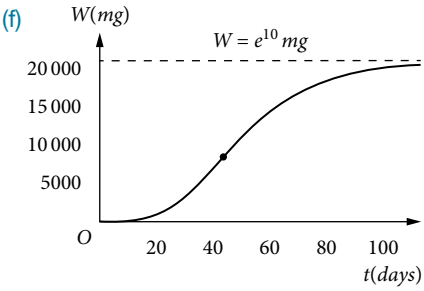
(e) $W = e^9$: $e^9 = e^{10 - 9e^{\frac{-t}{20}}}$

$9 = 10 - 9e^{\frac{-t}{20}}$

$e^{\frac{-t}{20}} = \frac{1}{9}$

$t = 20\log_e 9 \approx 44$ days

(f)



21 (a) $\frac{dV}{dt} = -kV$, $V(0) = 1000000$, $k > 0$

$\frac{dV}{V} = -kdt$

$\int\frac{dV}{V} = -k\int dt$

$\log_e|V| = -kt + C$

$|V| = e^{-kt+C}$

$V = Ae^{-kt}$ where $A = \pm e^C$

$V(0) = 1000000$: $A = 1000000$

$V = 1000000e^{-kt}$

$V(5) = 600000$: $600000 = 1000000e^{-5k}$

$e^{-5k} = \frac{3}{5}$ $\therefore e^{-k} = \left(\frac{3}{5}\right)^{\frac{1}{5}}$

$k = \frac{1}{5}\log_e\left(\frac{5}{3}\right)$

$e^{-kt} = \left(e^{-k}\right)^t = \left[\left(\frac{3}{5}\right)^{\frac{1}{5}}\right]^t = \left(\frac{3}{5}\right)^{\frac{t}{5}}$

$V = 1000000\left(\frac{3}{5}\right)^{\frac{t}{5}}$

(b) $\frac{dV}{dt} = -\frac{1}{5}\log_e\frac{5}{3} \times V$

$V = 600\,000$: $\frac{dV}{dt} = -\frac{1}{5}\log_e\frac{5}{3} \times 600000$

$= 61\,299$ barrels per year.

(c) $V = 1000$: $\quad 1000\,000\left(\frac{3}{5}\right)^{\frac{t}{5}} = 1000$

$$\left(\tfrac{3}{5}\right)^{\frac{t}{5}} = \frac{1}{1000}$$

$$\frac{t}{5} = \frac{\log_e 0.001}{\log_e 0.6}$$

$$t = 67.6$$

It will remain profitable for 67 years 7 months.

22 (a) $\text{RHS} = \dfrac{1}{p} - \dfrac{1}{p-588}$

$$= \frac{p-588-p}{p(p-588)}$$

$$= \frac{-588}{p(p-588)}$$

$$= \frac{1}{p\left(1-\frac{p}{588}\right)} = \text{LHS}$$

(b) $\dfrac{dp}{dt} = rp\left(1-\dfrac{p}{588}\right)$

$$\frac{dp}{p\left(1-\frac{p}{588}\right)} = r\,dt$$

$$\int \frac{dp}{p\left(1-\frac{p}{588}\right)} = r\int dt$$

$$\int\left(\frac{1}{p} - \frac{1}{p-588}\right)dp = r\int dt$$

$$\log_e|p| - \log_e|p-588| = rt + c$$

$$\log_e\left|\frac{p}{p-588}\right| = rt + c$$

$$\left|\frac{p}{p-588}\right| = e^{rt+c}$$

$$\frac{p}{p-588} = Ae^{rt} \quad \text{where } A = \pm e^c$$

$$p(0) = 21: \quad \frac{21}{21-588} = A \therefore A = -\frac{1}{27}$$

$$p = \frac{-p}{27}e^{rt} + \frac{588}{27}e^{rt}$$

$$p\left(27 + e^{rt}\right) = 588e^{rt}$$

$$p = \frac{588e^{rt}}{27+e^{rt}} = \frac{588}{1+27e^{-rt}}$$

(c) $p(3) = 294$: $\quad 294 = \dfrac{588}{1+27e^{-3r}}$

$$1 + 27e^{-3r} = 2$$

$$e^{-3r} = \frac{1}{27}$$

$$r = \tfrac{1}{3}\log_e 27 = \log_e 3$$

(d) $p(6) = \dfrac{588}{1+27e^{-6\log_e 3}} = \dfrac{588}{1+27\times\frac{1}{3^6}} = \dfrac{588}{1+\frac{1}{27}} = 567$

(e) Now take $P(0) = 567$ at the start of the repopulation

$\dfrac{dP}{dt}$ = rate of increase − rate of decrease

$$= P\log_e 3\left(1-\tfrac{p}{588}\right) - 140 \quad \text{where } P(0) = 567$$

(f) $\dfrac{dP}{dt} = -K(P-a)(P-b), \quad P(0) = P_0$

$$\frac{dP}{(P-a)(P-b)} = -K\,dt$$

$$\int\frac{dP}{(P-a)(P-b)} = -K\int dt$$

$$\frac{1}{b-a}\int\left(\frac{1}{P-b} - \frac{1}{P-a}\right)dP = -K\int dt$$

$$\log_e\left|\frac{P-b}{P-a}\right| = -(b-a)Kt + C$$

$$\left|\frac{P-b}{P-a}\right| = e^{-(b-a)Kt+C}$$

$$\frac{P-b}{P-a} = Ae^{-(b-a)Kt} \quad \text{where } A = \pm e^C$$

$$P(0) = P_0: \quad \frac{P_0-b}{P_0-a} = A$$

$$P - b = PAe^{-(b-a)Kt} - aAe^{-(b-a)Kt}$$

$$P\left(1 - Ae^{-(b-a)Kt}\right) = b - aAe^{-(b-a)Kt}$$

$$P = \frac{b - a\left(\frac{P_0-b}{P_0-a}\right)e^{-(b-a)Kt}}{1-\left(\frac{P_0-b}{P_0-a}\right)e^{-(b-a)Kt}}$$

(g) Require

$$P\log_e 3\left(1-\frac{P}{588}\right) - 140 \equiv -\frac{\log_e 3}{588}(P-a)(P-b), \quad a < b$$

$$P\log_e 3 - \frac{P^2\log_e 3}{588} - 140 \equiv -\frac{\log_e 3}{588}\left(P^2 - (a+b)P + ab\right)$$

$$-\frac{P^2\log_e 3}{588} + P\log_e 3 \equiv -\frac{P^2\log_e 3}{588} + \frac{(a+b)P\log_e 3}{588} - \frac{ab\log_e 3}{588}$$

Equating coefficients: $\log_e 3 = \dfrac{(a+b)\log_e 3}{588}$ and $140 = \dfrac{ab\log_e 3}{588}$

$$a + b = 588 \qquad [1]$$

$$ab = \frac{140\times 588}{\log_e 3} \qquad [2]$$

(i) $\therefore b = 588 - a$ (ii) $\therefore a(588-a) = \dfrac{140\times 588}{\log_e 3}$

$$a^2 - 588a + \frac{140\times 588}{\log_e 3} = 0$$

$$a = \frac{588 \pm \sqrt{588^2 - 4\times 1\times\frac{140\times 588}{\log_e 3}}}{2}$$

$$= \frac{588 \pm \sqrt{(4\times 49\times 3)^2 - 4\times 1\times\frac{140\times 4\times 49\times 3}{\log_e 3}}}{2}$$

$$= \frac{588 \pm 4\times 7\sqrt{49\times 3^2 - \frac{140\times 3}{\log_e 3}}}{2} = 294 \pm 14\sqrt{441 - \frac{420}{\log_e 3}}$$

$$b = 294 \mp 14\sqrt{441 - \frac{420}{\log_e 3}}$$

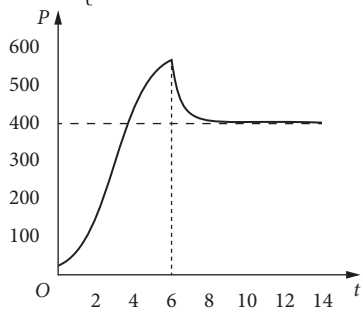
$a < b$ so $a = 294 - 14\sqrt{441 - \dfrac{420}{\log_e 3}}$ and

$$b = 294 + 14\sqrt{441 - \frac{420}{\log_e 3}}$$

(h) $K = \dfrac{\log_e 3}{588}, \quad P_0 = 567, \quad P(t) = \dfrac{b - a\left(\frac{P_0-b}{P_0-a}\right)e^{-(b-a)Kt}}{1-\left(\frac{P_0-b}{P_0-a}\right)e^{-(b-a)Kt}}$

$$P(t) = \frac{401.262 - 81.3902e^{-0.4008t}}{1 - 0.435852e^{-0.4008t}}$$

(i) $P(t) = \begin{cases} \dfrac{588}{1+27\left(\frac{1}{3}\right)^t} & 0 \le t \le 6 \\ \dfrac{401.262 - 81.3902e^{-0.4008t}}{1-0.435\,852e^{-0.4008t}} & 6 < t \le 14 \end{cases}$



The breeding program is effective as it still leaves a stable population.

CHAPTER REVIEW 12

1 (a) $\tan\theta = \frac{h}{400}$ (b) $\frac{2}{125}$ radians per second

$h = 400\tan\theta$

2 $y = e^{-x}\cos x$

$\dfrac{dy}{dx} = -e^{-x}\cos x + e^{-x}(-\sin x)$

$= -e^{-x}(\sin x + \cos x)$

$\dfrac{d^2y}{dx^2} = e^{-x}(\sin x + \cos x) - e^{-x}(\cos x - \sin x)$

$= 2e^{-x}\sin x$

$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 2e^{-x}\sin x - 2e^{-x}(\sin x + \cos x) + 2e^{-x}\cos x$

$= 0$

3 $y = x + \log_e\left(\dfrac{2}{x+1}\right), \quad x > -1$

4 (a) $f'(x) = \frac{1}{2}\sin 2x + 3\cos x$ (b) $f(x) = 3\sin x - \frac{1}{4}\cos 2x - \frac{1}{4}$

5 $y = \frac{1}{2}\left(1 - 3e^{-2x}\right)$

6 (a) $\dfrac{1}{y-2} - \dfrac{1}{y+2} = \dfrac{1(y+2)-1(y-2)}{(y-2)(y+2)}$

$= \dfrac{y+2-y+2}{(y-2)(y+2)}$

$= \dfrac{4}{y^2-4}$

(b) $\dfrac{dy}{dx} = y^2 - 4$

$\dfrac{dy}{y^2-4} = dx$

$\frac{1}{4}\int\left(\dfrac{1}{y-2} - \dfrac{1}{y+2}\right)dy = \int dx$

$\frac{1}{4}\left(\log_e|y-2| - \log_e|y+2|\right) = x + C$

$\frac{1}{4}\log_e\left|\dfrac{y-2}{y+2}\right| = x + C$

$\log_e\left|\dfrac{y-2}{y+2}\right| = 4(x+C)$

$(0,0)$: $\log_e 1 = 4C$

$C = 0$

$\left|\dfrac{y-2}{y+2}\right| = e^{4x}$

Since $(0,0)$ lies on the curve, $\left|\dfrac{y-2}{y+2}\right| = \dfrac{2-y}{2+y}$

$\dfrac{2-y}{2+y} = e^{4x}$

$2 - y = 2e^{4x} + ye^{4x}$

$y\left(e^{4x}+1\right) = 2\left(1-e^{4x}\right)$

$y = \dfrac{2\left(1-e^{4x}\right)}{1+e^{4x}}$

7 (a) $T(t) = Ae^{-0.1t}$ (b) $t = 10\log_e 4 = 13.9$ years

8 $y = \tan\left(x^2 + \frac{\pi}{4}\right)$

9 D $y = \log_e(\sin x)$, $\frac{dy}{dx} = \frac{\cos x}{\sin x}$

10 C $\frac{dx}{dt} = -3e^{-3t} = -3x$, $\frac{d^2x}{dt^2} = 9e^{-3t} = 9x$

11 D $\frac{dy}{dx} = ke^{kx} = ky$, $\frac{d^2y}{dx^2} = k^2e^{kx} = k^2y$, $k^2 + k - 6 = 0$

12 B $y = \int\left(2 - 3\sqrt{x}\right)dx = 2x - 2x^{\frac{3}{2}} + C$, $3 = C$.

$y = 3 + 2x\left(1 - \sqrt{x}\right)$

13 A

14 B $\int\dfrac{dy}{y} = -3\int dx$, $y = Ae^{-3x}$, $y = e^{6-3x}$

15 D $\int\dfrac{dy}{2+y} = -\int dx$, $\log_e|2+y| = -x + C$

16 B $(0,-1)$, $\frac{dy}{dx} = 2 > 0$.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x - 2y) = 1 - 2\frac{dy}{dx} = 1 - 2x + 4y$.

$(0,-1)$, $\frac{d^2y}{dx^2} = -3 < 0$.

17 C $\left(\frac{\pi}{2}, 1\right)$, $\frac{dy}{dx} = 1 \times \sin\frac{\pi}{2} = 1$. $x = 0$, $\frac{dy}{dx}\, y\sin 0 = 0$. $y = 0$,

$\frac{dy}{dx} = 0\sin x = 0$.

18 A $\int\dfrac{dy}{y} = \int\sin x\,dx$, $\log_e|y| = -\cos x + C$,

$y = Ae^{-\cos x}$. $A = e$. $y = e^{1-\cos x}$

19 A

20 D $\int\dfrac{dP}{P(20\,000 - P)} = \dfrac{1}{10\,000}\int dt$.

$\log_e\left|\dfrac{P}{20\,000 - P}\right| = 2t + C$,

$\dfrac{P}{20\,000 - P} = Ae^{2t}$. $A = 0.25$. $P = \dfrac{20\,000}{1 + 4e^{-2t}}$

21 C

22 (a) $\dfrac{dh}{dt} = \dfrac{k(T_w - T_a)}{L\rho h}$

(b) $h(t) = \sqrt{h_0^2 + \dfrac{2k(T_w - T_a)}{L\rho}t}$

23 (a) $\dfrac{dR}{dp} = q\left[1 + \dfrac{p}{q}\dfrac{dq}{dp}\right]$

Revenue is maximum when $\dfrac{dR}{dp} = 0$

$1 + \dfrac{p}{q}\dfrac{dq}{dp} = 0$

$\dfrac{p}{q}\dfrac{dq}{dp} = -1$ hence $E = -1$

(b) $p = \sqrt{25 - q^2}$

$q \ge 0$ and $p \ge 0$ so $25 - q^2 \ge 0$

$q^2 \le 25$

$0 \le q \le 5$ is the domain of this model.

(c) $\dfrac{dq}{dp} = \dfrac{-\sqrt{25-q^2}}{q}$

(d) Selling price is \$3.54

24 (a) $V = \sqrt{2\left(c - C_P T\right)}$

(b) $V = \sqrt{2C_P(308 - T)}$

(c) $C_P = 1004.6,\ T = 307:\ V = \sqrt{2 \times 1004.6(307 - 307)}$

$\approx 45 \text{ m s}^{-1}$

25 (a) $\dfrac{dy}{dt} = -\dfrac{1}{400}(x-2), \quad \dfrac{dx}{dt} = \dfrac{1}{100}(y-1)$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= \dfrac{-(x-2)}{400} \times \dfrac{100}{y-1}$

$= \dfrac{-(x-2)}{4(y-1)}$

(b) $\dfrac{dy}{dx} = \dfrac{-(x-2)}{4(y-1)}$

$4(y-1)dy = -(x-2)dx$

$4\int (y-1)dy = -\int (x-2)dx$

$2y^2 - 4y = -\dfrac{x^2}{2} + 2x + C$

$(1,3):\quad 18 - 12 = -\dfrac{1}{2} + 2 + C$

$C = \dfrac{9}{2}$

$2y^2 - 4y = -\dfrac{x^2}{2} + 2x + \dfrac{9}{2}$

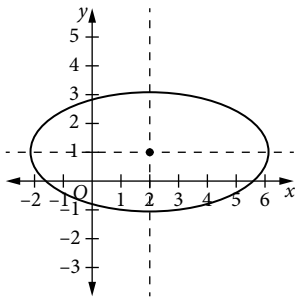
$4y^2 - 8y = -x^2 + 4x + 9$

$4\left(y^2 - 2y + 1\right) = -\left(x^2 - 4x + 4\right) + 17$

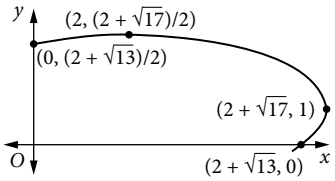
$(x-2)^2 + 4(y-1)^2 = 17$

$\left(\dfrac{1}{\sqrt{17}}\right)^2 (x-2)^2 + \left(\dfrac{2}{\sqrt{17}}\right)^2 (y-1)^2 = 1$

(c)



(d)



For maximum predators. $y = 1$, $x = 2 + \sqrt{17} \approx 6.1$, about 610.

For maximum prey, $x = 2$, $y = 1 + \frac{\sqrt{17}}{2} \approx 3.062$, about 3060.

26 (a) $\frac{dV}{dt} = Q_m - \varepsilon A_1(h) - \sigma A_2(h)$

$= 0 - \varepsilon\pi h(2R - h) - \sigma \times 2\pi hR$

$= \pi h(-2\varepsilon R + \varepsilon h - 2\sigma R)$

$= \pi h(\varepsilon h - 2R(\sigma + \varepsilon))$

(b) $V(h) = \frac{1}{3}\pi h^2(3R - h)$

$\frac{dV}{dh} = \frac{1}{3}\pi\left(6Rh - 3h^2\right)$

$\frac{dV}{dh} = \pi\left(2Rh - h^2\right)$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\pi h(\varepsilon h - 2R(\sigma + \varepsilon)) = \pi\left(2Rh - h^2\right) \times \frac{dh}{dt}$

$\dfrac{dh}{dt} = \dfrac{\pi h(\varepsilon h - 2R(\sigma + \varepsilon))}{\pi h(2R - h)}$

$\dfrac{dh}{dt} = \dfrac{\varepsilon h - 2R(\sigma + \varepsilon)}{2R - h}$

(c) $V(h) = \frac{1}{3}\pi h^2(3R - h)$

$V = 1\,000\,000,\ h = 3:\quad 1\,000\,000 = \frac{1}{3}\pi \times 9(3R - 3)$

$1\,000\,000 = 9\pi R - 9\pi$

$R = \dfrac{1\,000\,000 + 9\pi}{9\pi} \approx 35\,369 \text{ m}$

27 (a) Use Newton's law of cooling. Surrounding temperature = 20°C, $T(0) = 37$°C

$\frac{dT}{dt} = -r(T - 20)$ with $r > 0$ and $T(0) = 37$

(b) $\dfrac{dT}{T - 20} = -r\,dt$

$\int \dfrac{dT}{T - 20} = -r\int dt$

$\log_e |T - 20| = -rt + c$

$T(0) = 37:\quad \log_e 17 = c$

$\log_e \left|\dfrac{T - 20}{17}\right| = -rt$

$\left|\dfrac{T - 20}{17}\right| = e^{-rt}$

$\dfrac{T - 20}{17} = \pm e^{-rt}$

$T = 20 \pm 17e^{-rt}$

Since $t(0) = 37$, $T = 20 + 17e^{-rt}$

(c) Let 7:20 pm be τ hours after the death. Thus $T(\tau) = 29$.

$29 = 20 + 17e^{-r\tau}$

$e^{-r\tau} = \frac{9}{17}$

$e^{r\tau} = \frac{17}{9}$

(d) At 8:20 pm, $t = \tau + 1$: $27.4 = 20 + 17e^{-r(\tau+1)}$

$\frac{7.4}{17} = e^{-r(\tau+1)}$

$e^{r(\tau+1)} = \frac{17}{7.4}$

(e) $\frac{7.4}{17} = e^{-r\tau} \times e^{-r}$

$e^{-r} = \frac{7.4}{17} \times \frac{17}{9} = \frac{7.4}{9}$

$r = \log_e\left(\frac{9}{7.4}\right) = 0.196$

(f) Find the value of τ: $e^{r\tau} = \frac{17}{9}$

$r\tau = \log_e \frac{17}{9}$

$\tau = \dfrac{\log_e \frac{17}{9}}{\log_e \frac{9}{7.4}}$

$\approx 3.249 \text{ h}$

$\approx 3 \text{ h } 15 \text{ min}$

The coroner has predicted the time of death to be 3 h 15 min before 7:20 pm, that is, at 4:05 pm. The chauffer claimed that he brought Lady Frogmorton home at 6:00 pm, at which time the coroner says she was dead.

28 (a) $B = 20000 - 5000e^{\frac{9t}{50}}$

(b) $t = 7.702$ years ≈ 7 years 8.42 months

(c) Total repayments = \$3600 × time taken.

Total repayments = $\$3600 \times \frac{50}{9}\log_e 4 = \$27725.89$

(d) If S is the balance in the savings account then it increases at $\$\frac{3S}{50}$ per year, decrease by \$3600 p.a.

$$\frac{dS}{dt} = \text{ rate in } - \text{ rate out}$$

$$= \frac{3S}{50} - 3600$$

$$= \frac{3}{50}(S - 60000) \text{ with } S(0) = 40000$$

$$\frac{dS}{S - 60000} = \frac{3}{50}dt$$

$$\int \frac{dS}{S - 60000} = \frac{3}{50}\int dt$$

$$\log_e |S - 60000| = \frac{3t}{50} + C$$

$$S - 60000 = Ae^{\frac{3t}{50}} \text{ where } A = \pm e^C$$

$S(0) = 40000$: $40000 - 60000 = A$

$A = -20000$

$$S = 60000 - 20000e^{\frac{3t}{50}}$$

$$S = 20000\left(3 - e^{\frac{3t}{50}}\right)$$

(e) Find $S\left(\frac{50}{9}\log_e 4\right)$: $S = 20000\left(3 - e^{\frac{1}{3}\log_e 4}\right)$

$$= 20000\left(3 - 2^{\frac{2}{3}}\right) \approx \$28251.98$$

29 (a) $W = 10 - 9e^{\frac{-t}{10}}$

(b) $W(20) = 10 - 9e^{-2} = 8.78198$

Volume of solid waste in landfill is 878 198 cubic metres.

30 (a) $P = 1000 - 800e^{-kt}$

(b) $P(2) = 300$: $300 = 1000 - 800e^{-2k}$

$$e^{-2k} = \frac{7}{8}$$

$$k = \frac{1}{2}\log_e \frac{8}{7} \approx 0.0668$$

(c) $P(t) = 1000 - 800e^{-\frac{t}{2}\log_e \frac{8}{7}}$

$\lim_{t\to\infty} P(t) = 1000$

31 (a) $\frac{dp}{dt} = \frac{1}{10}p(1-p)$

$\frac{dp}{dt}$ is greatest when $\frac{p}{10}(1-p)$ is greatest.

This is a quadratic function so its greatest value occurs at

$p = \frac{0+1}{2}$, i.e. when $p = \frac{1}{2}$.

Half the school community.

(b) $p(t) = \dfrac{e^{\frac{t}{10}}}{4 + e^{\frac{t}{10}}} = \dfrac{1}{1 + 4e^{\frac{-t}{10}}}$

(c) From (a), most rapid spread when $p = 0.5$.

$$\frac{1}{2} = \frac{1}{1 + 4e^{\frac{-t}{10}}}$$

$$e^{\frac{-t}{10}} = \frac{1}{4}$$

$t = 10\log_e 4 = 13.9 \approx 14$ minutes

CHAPTER 13

EXERCISE 13.1

1 (a) 850 m

(b) Jaide's displacement from O to B, correct to one decimal place, is 680.1 m from O in a direction of N72.9°E.

(c) $1.2\,\text{m s}^{-1}$

(d) Jaide's average velocity, in metres per second, correct to one decimal place, is $0.9\,\text{m s}^{-1}$ in a direction of N72.9°E.

2 Courtney's displacement for her entire walk is 510.9 m in the direction S 40.2°E.

3 The displacement of camp B from the base camp is 14543.5 m at an angle of elevation of 10.5°.

4 The magnitude and direction of Kenneth's resultant displacement is 152 m in the direction S 61°E.

5 The spelunker's final displacement from the cave entrance is 401 m in the direction N66°E.

6 If she swims in a river flowing at $4.5\,\text{km h}^{-1}$ and keeps her direction (with respect to the water) perpendicular to the current, the magnitude of her velocity with respect to the riverbank is $7.5\,\text{km h}^{-1}$.

7 (a) D Head at angle of 60° to the riverbank upstream

(b) D Not possible

8 The time it takes for the boat to reach the opposite shore, if the river is 50 m wide, is 29.06 seconds.

9 (a) If the velocity of the wind is $10\,\text{m s}^{-1}$ north, the resultant velocity of the aircraft relative to Rapahela is $110\,\text{m s}^{-1}$ north.

(b) If the head wind has a velocity of $10\,\text{m s}^{-1}$ south, the resultant velocity of this aircraft relative to Rapahela is $115\,\text{m s}^{-1}$ in the direction north.

(c) The resultant velocity of this aircraft relative to Rapahela is $103.1\,\text{m s}^{-1}$ in the direction N14.0°W.

10 The magnitude and direction of Mitchell's resultant velocity, $\underset{\sim}{v}$ m s^{-1} is $\underset{\sim}{v} = 0.7\,\text{m s}^{-1}$ in the direction 045 or 45°T.

11 (a) (i) $\overrightarrow{OA} = 3.99\underset{\sim}{i} + 3.01\underset{\sim}{j}$

(ii) $\overrightarrow{AB} = -4.24\underset{\sim}{i} + 4.24\underset{\sim}{j}$

(iii) $\overrightarrow{BC} = -3.46\underset{\sim}{i} - 2\underset{\sim}{j}$

(iv) $\overrightarrow{CD} = 1.5\underset{\sim}{i} - 2.60\underset{\sim}{j}$

(b) $\overrightarrow{OD} = -2.21\underset{\sim}{i} + 2.65\underset{\sim}{j}$

(c) The magnitude and direction of the resultant vector $\overrightarrow{OD}$ is 3.45 m at 320.17°T.

12 At this time, Morgan and Eilish are 71.1 km from the school and in the direction 327.3°T.

13 72°

14 C $80\,\text{km h}^{-1}$ in the direction of bus 1

15 C 786 m

16 (a) The velocity of the boat relative to the riverbank is $2.24\,\text{m s}^{-1}$ in a direction of 63.43°.

(b) Freddie will get to the opposite riverbank 100 m downstream from point A.

(c) Freddie's final displacement from starting point D is 223.67 m at 63.42°.

(d) Rohanne will need to head upstream at 60° an angle of to the riverbank, it will take her 115.47 s to cross the river.

17 (a) The second aircraft is 12.4 km to the south.

(b) The second aircraft is 2.2 km to the east.

(c) 74.5 s will have elapsed before the aircrafts are side by side.

18 (a) The package hits the ground 450.0 m relative to the point from where it was dropped.

(b) The velocity is $46.1\,\text{m s}^{-1}$ in a direction of 77.5° to the vertical.

EXERCISE 13.2

1 (a) $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 = 30$ N, north

(b) $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 = -16$ N, west

(c) $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 250$ N, east

(d) $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 56$ N, south

2 (a) 200.00 N at N36.87°W (b) 120.42 N at S41.63°E
(c) 151.16 N at S55.78°W (d) 357.35 N at N17.93°E

3 The resultant force being applied to the container ship is 3941 N in the direction S54°E.

4 (a) $\underset{\sim}{F} = 150.86$ N, acting in the direction N4.81°E
(b) $\underset{\sim}{F} = 231.06$ N, acting in the direction N81.82°W
(c) $\underset{\sim}{F} = 28.11$ N, acting in the direction 97.2°T
(d) $\underset{\sim}{F} = 564.66$ N, acting in the direction 011.39°T

5 (a) $\underset{\sim}{F}_1 = 167.73\underset{\sim}{i} + 108.93\underset{\sim}{j}$, $\underset{\sim}{F}_2 = 38.20\underset{\sim}{i} - 216.66\underset{\sim}{j}$, $\underset{\sim}{F}_3 = -135.95\underset{\sim}{i} - 63.39\underset{\sim}{j}$

(b) $\underset{\sim}{F} = 69.99\underset{\sim}{i} - 171.12\underset{\sim}{j}$

6 (a) $\underset{\sim}{F}_1 = 176.78\underset{\sim}{i} + 176.78\underset{\sim}{j}$, $\underset{\sim}{F}_2 = -154.87\underset{\sim}{i} - 221.17\underset{\sim}{j}$, $\underset{\sim}{F}_3 = 147.92\underset{\sim}{i} - 317.21\underset{\sim}{j}$

(b) $\underset{\sim}{F} = 169.83\underset{\sim}{i} - 361.60\underset{\sim}{j}$

7 $|\underset{\sim}{F}_1| = 152.92$ N, $|\underset{\sim}{F}_2| = 203.22$ N

8 (a) $|\underset{\sim}{F}_1| = 195.02$ N; $|\underset{\sim}{F}_2| = 233.83$ N

(b) $|\underset{\sim}{F}_1| = 339.21$ N; $|\underset{\sim}{F}_2| = 548.81$ N

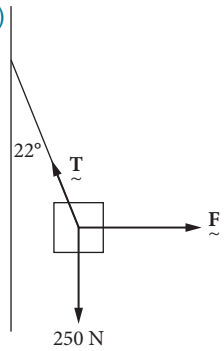
9 $|\underset{\sim}{F}_1| = 191.2$ N; $|\underset{\sim}{F}_2| = 254.0$ N

10 (a) $|\underset{\sim}{F}_1| = 133.76$ N; $|\underset{\sim}{F}_2| = 157.38$ N

(b) $|\underset{\sim}{F}_1| = 389.06$ N; $|\underset{\sim}{F}_2| = 379.68$ N

11 $|\underset{\sim}{T}| = 85.45$ N; $|\underset{\sim}{F}| = 49.01$ N

12 (a)



(b) $|\underset{\sim}{T}| = 269.63$ N; $|\underset{\sim}{F}| = 101.01$ N

13 C $\underset{\sim}{F}_1 = 440$ N north, $\underset{\sim}{F}_2 = 560$ N south, $\underset{\sim}{F}_3 = 480$ N north

14 (a) $\underset{\sim}{F} = 10\,209$ N

(b) $|\underset{\sim}{F}_1| = 3279$, $|\underset{\sim}{F}_2| = 2977$

15 (a) $\underset{\sim}{F}_1 = 1233.72\underset{\sim}{i} + 1637.20\underset{\sim}{j}$, $\underset{\sim}{F}_2 = 1195.03\underset{\sim}{i} - 1002.75\underset{\sim}{j}$, $\underset{\sim}{F}_3 = -1383.81\underset{\sim}{i} - 898.65\underset{\sim}{j}$, $\underset{\sim}{F}_4 = -1340.69\underset{\sim}{i} + 1388.33\underset{\sim}{j}$

(b) $\underset{\sim}{F} = 1162$ N at 345°T on a bearing is 345.26°T

EXERCISE 13.3

1 (a) $\underset{\sim}{v}(t) = 50\underset{\sim}{i} + \left(50\sqrt{3} - 10t\right)\underset{\sim}{j}$

Greatest height when $50\sqrt{3} - 10t = 0$, $t = 5\sqrt{3}$ s

(b) $\underset{\sim}{r}(t) = 50t\,\underset{\sim}{i} + \left(50\sqrt{3}\,t - 5t^2\right)\underset{\sim}{j}$

(c) Height $= 50\sqrt{3}\,t - 5t^2$

$t = 5\sqrt{3}$, Greatest height $= 750 - 375 = 375$ m

(d) $y = 0$: $5t\left(10\sqrt{3} - t\right) = 0$, $t = 10\sqrt{3}$ s

(e) $t = 10\sqrt{3}$: $x = 500\sqrt{3}$ m

(f) $t = \frac{x}{50}$: $y = 50\sqrt{3} \times \frac{x}{50} - 5 \times \left(\frac{x}{50}\right)^2$

$y = \sqrt{3}x - \frac{x^2}{500}$

2 (a) $|\underset{\sim}{v}(0)| = 80$, $\alpha = 30°$,
$\underset{\sim}{v}(0) = 80\cos 30°\,\underset{\sim}{i} + 80\sin 30°\,\underset{\sim}{j} = 40\sqrt{3}\underset{\sim}{i} + 40\,\underset{\sim}{j}$

(b) $\underset{\sim}{r}(t) = 40\sqrt{3}\,t\underset{\sim}{i} + \left(40t - 5t^2\right)\underset{\sim}{j}$

(c) Time of flight is 8 seconds and range is $320\sqrt{3}$ m.

(d) Greatest height when $t = 4$
Greatest height $= 40 \times 4 - 5 \times 42 = 80$ m

(e) $x = 40\sqrt{3}\,t$, $y = 40t - 5t^2$

$t = \frac{x}{40\sqrt{3}}$, $y = 40 \times \frac{x}{40\sqrt{3}} - 5 \times \left(\frac{x}{40\sqrt{3}}\right)^2$

$y = \frac{x}{\sqrt{3}} - \frac{x^2}{960}$

3 (a) $\underset{\sim}{v}(t) = 10\underset{\sim}{i} - 10t\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 10t\,\underset{\sim}{i} + \left(125 - 5t^2\right)\underset{\sim}{j}$

(b) The object hits the ground after 5 seconds, 50 m from the base of the building.

(c) Since the object is fired horizontally, its greatest height is its starting height, 125 m above the ground.

4 (a) $\underset{\sim}{v}(t) = 7.5\underset{\sim}{i} + (10 - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 7.5t\underset{\sim}{i} + \left(10t - 5t^2\right)\underset{\sim}{j}$

(b) $10 - 10t = 0$, $t = 1$ s

Greatest height $= 10 \times 1 - 5 \times 1^2 = 5$ m

(c) $y = 0$: $10t - 5t^2 = 0$

$t = 2$ s. $x = 15$ m

Time of flight is 2 seconds and the horizontal distance travelled is 15 metres.

5 D $\underset{\sim}{v}(t) = 36\,\underset{\sim}{i} + \left(15 - gt\right)\underset{\sim}{j}$. $\underset{\sim}{v}(0) = 36\,\underset{\sim}{i} + 15\,\underset{\sim}{j}$.

$|\underset{\sim}{v}(0)| = \sqrt{36^2 + 15^2} = 39$. $\tan\alpha = \frac{15}{36} = \frac{5}{12}$.

6 (a) $x = 400$ m

(b) By symmetry, greatest height is when $x = 200$.
$y = 100 - 50 = 50$ m

7 (a) $\underset{\sim}{v}(0) = 50\cos\alpha\,\underset{\sim}{i} + 50\sin\alpha\,\underset{\sim}{j}$

(b) $\underset{\sim}{v}(t) = 50\cos\alpha\,\underset{\sim}{i} + (50\sin\alpha - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 50\cos\alpha\,t\,\underset{\sim}{i} + (50\sin\alpha\,t - 5t^2)\underset{\sim}{j}$

(c) $\alpha = 36°\,52'$, $60°\,15'$

8 (0, 0) and (45, 0). (22.5, 11.25)

(a) $\underset{\sim}{v}(t) = |\underset{\sim}{v}(0)|\cos\alpha\,\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha - gt\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = |\underset{\sim}{v}(0)|\cos\alpha\,t\,\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha\,t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

(b) $\alpha = 45°$

$|\underset{\sim}{v}(0)|^2 = \dfrac{225}{\sin 45°\cos 45°}$

$|\underset{\sim}{v}(0)|^2 = 225 \times 2$

$|\underset{\sim}{v}(0)| = 15\sqrt{2}$ m s^{-1}

9 (a) Take the top of the cliff as the origin.

$\underset{\sim}{v}(t) = 30\cos\alpha\,\underset{\sim}{i} + (30\sin\alpha - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 30\cos\alpha\,t\,\underset{\sim}{i} + \left(30\sin\alpha\,t - 5t^2\right)\underset{\sim}{j}$

(b) $\alpha = 36°\,52'$

10 $|\underset{\sim}{v}(0)| = V$, $\underset{\sim}{v}(0) = V\,\underset{\sim}{i}$, $\underset{\sim}{r}(0) = H\,\underset{\sim}{j}$, $(0, D)$

(a) $\underset{\sim}{v}(0) = V\,\underset{\sim}{i}$: $\underset{\sim}{v}(t) = V\,\underset{\sim}{i} - 10t\,\underset{\sim}{j}$

$\underset{\sim}{r}(0) = H\,\underset{\sim}{j}$: $\underset{\sim}{r}(t) = Vt\,\underset{\sim}{i} + \left(H - 5t^2\right)\underset{\sim}{j}$

(b) $V = 6$ m s^{-1}

(c) H = 20 m

11 $\underset{\sim}{v}(0) = 20\underset{\sim}{i} + 100\underset{\sim}{j}$

(a) $\tan\alpha = \frac{100}{20} = 5$. $\alpha = 78°\ 41'$

(b) $\underset{\sim}{v}(0) = 20\underset{\sim}{i} + 100\underset{\sim}{j}$: $\underset{\sim}{v}(t) = 20\underset{\sim}{i} + (100 - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 20t\,\underset{\sim}{i} + (100t - 5t^2)\underset{\sim}{j}$

(c) $x = 120$ m: $20t = 120$

$t = 6$ s

(d) $t = 6$: $y = 600 - 5 \times 36 = 420$ m

12 $\underset{\sim}{r}(t) = 36t\underset{\sim}{i} + \left(15t - 5t^2\right)\underset{\sim}{j}$

(a) $\underset{\sim}{v}(t) = 36\underset{\sim}{i} + (15 - 10t)\underset{\sim}{j}$

$\left|\underset{\sim}{v}(0)\right| = 39\ \text{m s}^{-1}$

(b) $\alpha = 22°\ 37'$

(c) Greatest height = 11.25 m

(d) $y = 0$: $t = 3$

$x = 36 \times 3 = 108$ m

13 $\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + \left(20t - 5t^2\right)\underset{\sim}{j}$

(a) $t = 0$: $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$. Projected from the origin (0, 0).

(b) $\alpha = 53°\ 8'$

(c) Speed of projection is 25 m s^{-1}

(d) Greatest height is 20 m

$x = 30$

Point is (30, 20)

14 $V = 200\ \text{m s}^{-1}$, (0, 0), (2000, 0)

(a) $\underset{\sim}{v}(0) = 200\cos\alpha\,\underset{\sim}{i} + 200\sin\alpha\,\underset{\sim}{j}$:

$\underset{\sim}{v}(t) = 200\cos\alpha\,\underset{\sim}{i} + (200\sin\alpha - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 200\cos\alpha\,t\,\underset{\sim}{i} + \left(200\sin\alpha\,t - 5t^2\right)\underset{\sim}{j}$

(b) $\alpha = 15°, 75°$

15 (a) $\underset{\sim}{v}(0) = 175\underset{\sim}{i}$: $\underset{\sim}{v}(t) = 175\underset{\sim}{i} - 10t\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 1000\underset{\sim}{j}$: $\underset{\sim}{r}(t) = 175t\,\underset{\sim}{i} + \left(1000 - 5t^2\right)\underset{\sim}{j}$

(b) $y = 0$: $1000 - 5t^2 = 0$

$t = 10\sqrt{2}$ s

(c) $x = 175 \times 10\sqrt{2} = 1750\sqrt{2}$ m $= \frac{7\sqrt{2}}{4}$ km

16 $\underset{\sim}{r}(t) = Vt\cos\alpha\,\underset{\sim}{i} + \left(Vt\sin\alpha - 5t^2\right)\underset{\sim}{j}$, (7, 7), (14, 7).

(a) (7, 7): $V\,t\cos\alpha = 7$, $V\,t\sin\alpha - 5t^2 = 7$

$$t = \frac{7}{V\cos\alpha},\ V\sin\alpha \times \frac{7}{V\cos\alpha} - 5 \times \left(\frac{7}{V\cos\alpha}\right)^2 = 7$$

$$\tan\alpha - \frac{35}{V^2\cos^2\alpha} = 1$$

$$\frac{35}{V^2\cos^2\alpha} = \tan\alpha - 1$$

$$V^2\cos^2\alpha = \frac{35}{\tan\alpha - 1} \qquad [1]$$

(14, 7): $V\,t\cos\alpha = 14$, $V\,t\sin\alpha - 5t^2 = 7$

$$t = \frac{14}{V\cos\alpha}:\ V\sin\alpha \times \frac{14}{V\cos\alpha} - 5 \times \left(\frac{14}{V\cos\alpha}\right)^2 = 7$$

$$2\tan\alpha - \frac{140}{V^2\cos^2\alpha} = 1$$

$$\frac{140}{V^2\cos^2\alpha} = 2\tan\alpha - 1$$

$$V^2\cos^2\alpha = \frac{140}{2\tan\alpha - 1} \qquad [2]$$

Substitute from (1) in (2): $\frac{35}{\tan\alpha - 1} = \frac{140}{2\tan\alpha - 1}$

$2\tan\alpha - 1 = 4(\tan\alpha - 1)$

$2\tan\alpha = 3$

$\tan\alpha = 1.5$

(b) (bh), $(b + c, h)$

(b, h): $V\,t\cos\alpha = b$, $V\,t\sin\alpha - 5t^2 = h$

$$t = \frac{b}{V\cos\alpha}:\ V\sin\alpha \times \frac{b}{V\cos\alpha} - 5 \times \left(\frac{b}{V\cos\alpha}\right)^2 = h$$

$$b\tan\alpha - \frac{5b^2}{V^2\cos^2\alpha} = h$$

$$\frac{5b^2}{V^2\cos^2\alpha} = b\tan\alpha - h$$

$$V^2\cos^2\alpha = \frac{5b^2}{b\tan\alpha - h} \qquad [1]$$

(c, h): $V\,t\cos\alpha = c$, $V\,t\sin\alpha - 5t^2 = h$

$$t = \frac{c}{V\cos\alpha}:\ V\sin\alpha \times \frac{c}{V\cos\alpha} - 5 \times \left(\frac{c}{V\cos\alpha}\right)^2 = h$$

$$c\tan\alpha - \frac{5c^2}{V^2\cos^2\alpha} = h$$

$$\frac{5c^2}{V^2\cos^2\alpha} = c\tan\alpha - h$$

$$V^2\cos^2\alpha = \frac{5c^2}{c\tan\alpha - h} \qquad [2]$$

Substitute from (1) in (2): $\frac{5b^2}{b\tan\alpha - h} = \frac{5c^2}{c\tan\alpha - h}$

$b^2(\text{ctan}\,\alpha - h) = c^2(\text{btan}\,\alpha - h)$

$b^2\text{ctan}\,\alpha - b^2h = bc^2\tan\alpha - c^2h$

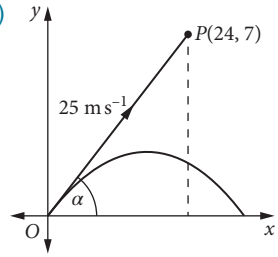
$bc((b - c)\tan\alpha = (b^2 - c^2)h$

$bc(b - c)\tan\alpha = (b + c)(b - c)h$

$\tan\alpha = \frac{(b + c)h}{bc}$

17 $\left|\underset{\sim}{v}(0)\right| = 25\ \text{m s}^{-1}$, $P(24, 7)$, $g = -10\ \text{m s}^{-2}$.

(a)



(b) $\tan\alpha = \frac{7}{24}$

$\alpha = 16°16'$

(c) $\underset{\sim}{v}(0) = 24\underset{\sim}{i} + 7\underset{\sim}{j}$: $\underset{\sim}{v}(t) = 24\underset{\sim}{i} + (7 - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 24t\,\underset{\sim}{i} + \left(7t - 5t^2\right)\underset{\sim}{j}$

(d) $x = 24$: $t = 1$

$y = 7 - 5 = 2$

The point is (24, 2)

$\underset{\sim}{r}(1) = 24\underset{\sim}{i} + 2\underset{\sim}{j}$

(e) $\underset{\sim}{r}_2(0) = 24\underset{\sim}{i} + 7\underset{\sim}{j}$, $\underset{\sim}{v}_2(0) = \underset{\sim}{0}$, $\underset{\sim}{a}_2(t) = -10\underset{\sim}{j}$

$\underset{\sim}{v}_2(t) = -10t\,\underset{\sim}{j}$

$\underset{\sim}{r}_2(t) = 24\underset{\sim}{i} + \left(7 - 5t^2\right)\underset{\sim}{j}$

(f) $t = 1$: $\underset{\sim}{r}_2(1) = 24\underset{\sim}{i} + 2\underset{\sim}{j}$

This is the same position vector as that obtained for the first particle after 1 second. Hence they are in the same place at the same time and thus collide.

18 (a) $\underset{\sim}{v}(0) = \left|\underset{\sim}{v}(0)\right|\cos 15°\,\underset{\sim}{i} + \left|\underset{\sim}{v}(0)\right|\sin 15°\,\underset{\sim}{j}$:

$\underset{\sim}{v}(t) = \left|\underset{\sim}{v}(0)\right|\cos 15°\,\underset{\sim}{i} + \left(\left|\underset{\sim}{v}(0)\right|\sin 15° - 10t\right)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = \left|\underset{\sim}{v}(0)\right|\cos 15°\,t\,\underset{\sim}{i} + \left(\left|\underset{\sim}{v}(0)\right|\sin 15°\,t - 5t^2\right)\underset{\sim}{j}$

(b) $|\underset{\sim}{v}(0)| = 40\ \text{m s}^{-1}$

(c) Range is now 160 metres.

19 $\underset{\sim}{v}_1(0) = 15\underset{\sim}{i}$, (0, 30)

(a) $\underset{\sim}{v}_1(t) = 15\underset{\sim}{i} - 10t\,\underset{\sim}{j}$

$\underset{\sim}{r}_1(0) = 30\underset{\sim}{j}$: $\underset{\sim}{r}_1(t) = 15t\,\underset{\sim}{i} + \left(30 - 5t^2\right)\underset{\sim}{j}$

(b) $\underset{\sim}{v}_2(0) = 15\underset{\sim}{i} + 15\sqrt{3}\underset{\sim}{j}$

$\underset{\sim}{v}_2(t) = 15\underset{\sim}{i} + \left(15\sqrt{3} - 10t\right)\underset{\sim}{j}$

$\underset{\sim}{r}_2(0) = \underset{\sim}{0}$: $\underset{\sim}{r}_2(t) = 15t\,\underset{\sim}{i} + \left(15\sqrt{3}\,t - 5t^2\right)\underset{\sim}{j}$

(c) The two stones collide at the point $\left(10\sqrt{3}, 23\frac{1}{3}\right)$

20 $t = 2$, (50, 0), $\underset{\sim}{r}(2) = 50\underset{\sim}{i}$. $\underset{\sim}{v}(0) = V\cos\alpha\,\underset{\sim}{i} + V\sin\alpha\,\underset{\sim}{j}$.

(a) $\underset{\sim}{v}(0) = V\cos\alpha\,\underset{\sim}{i} + V\sin\alpha\,\underset{\sim}{j}$: $\underset{\sim}{v}(t) = V\cos\alpha\,\underset{\sim}{i} + \left(V\sin\alpha - 10t\right)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = V\cos\alpha\,t\,\underset{\sim}{i} + \left(V\sin\alpha\,t - 5t^2\right)\underset{\sim}{j}$

(b) $\alpha = 21°\ 48'$

$V = 5\sqrt{29}\ \text{m s}^{-1}$

(c) 5 m

21 (a) $\underset{\sim}{v}(0) = 20\cos\alpha\,\underset{\sim}{i} + 20\sin\alpha\,\underset{\sim}{j}$:

$\underset{\sim}{v}(t) = 20\cos\alpha\,\underset{\sim}{i} + \left(20\sin\alpha - 10t\right)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 20\cos\alpha\,t\,\underset{\sim}{i} + \left(20\sin\alpha\,t - 5t^2\right)\underset{\sim}{j}$

(b) $\alpha = 45°,\ 71°\ 34'$

22 (a) $\underset{\sim}{v}(t) = 30\underset{\sim}{i} + (40 - 10t)\underset{\sim}{j}$ (b) $V = 50\ \text{m s}^{-1}$

(c) $\alpha = 53°\ 8'$ (d) Highest point is (120, 80)

23 (a) $\underset{\sim}{v}(0) = V\cos\alpha\,\underset{\sim}{i} + V\sin\alpha\,\underset{\sim}{j}$:

$\underset{\sim}{v}(t) = V\cos\alpha\,\underset{\sim}{i} + \left(V\sin\alpha - g\,t\right)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = V\cos\alpha\,t\,\underset{\sim}{i} + \left(V\sin\alpha\,t - \frac{1}{2}g\,t^2\right)\underset{\sim}{j}$

(b) Horizontal component is $V\cos\alpha = \frac{a}{2}\sqrt{\frac{g}{2b}}$,

Vertical component is $V\sin\alpha = \sqrt{2gb}$

(c) $x = Vt\cos\alpha = \frac{at}{2}\sqrt{\frac{g}{2b}} \Rightarrow t = x \times \frac{2}{a}\sqrt{\frac{2b}{g}}$

$y = V\sin\alpha\,t - \frac{1}{2}g\,t^2 = t\sqrt{2gb} - \frac{1}{2}g\,t^2$

$$t = \frac{2x}{a}\sqrt{\frac{2b}{g}}\text{: } y = \frac{2x}{a}\sqrt{\frac{2b}{g}} \times \sqrt{2gb} - \frac{1}{2}g\left(\frac{2x}{a}\sqrt{\frac{2b}{g}}\right)^2$$

$$= \frac{4bx}{a} - \frac{g}{2} \times \frac{4x^2}{a^2} \times \frac{2b}{g}$$

$$= \frac{4bx}{a} - \frac{4bx^2}{a^2}$$

$$= \frac{4bx(a-x)}{a^2}$$

Height reached is $\frac{4bx(a-x)}{a^2}$ m.

24 $V = 16\ \text{m s}^{-1}$, (16, 4)

$\underset{\sim}{v}(0) = 16\cos\alpha\,\underset{\sim}{i} + 16\sin\alpha\,\underset{\sim}{j}$

(a) $\underset{\sim}{v}(0) = 16\cos\alpha\,\underset{\sim}{i} + 16\sin\alpha\,\underset{\sim}{j}$:

$\underset{\sim}{v}(t) = 16\cos\alpha\,\underset{\sim}{i} + (16\sin\alpha - g\,t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 16\cos\alpha\,t\,\underset{\sim}{i} + \left(16\sin\alpha\,t - \frac{1}{2}g\,t^2\right)\underset{\sim}{j}$

(b) $g = 10\ \text{m s}^{-2}$, (16, 4)

$16t\cos\alpha = 16 \therefore\ t = \frac{1}{\cos\alpha}$

$t\,(16\sin\alpha - 5\text{t}) = 4$

$t = \frac{1}{\cos\alpha}$: $\frac{1}{\cos\alpha}\left(16\sin\alpha - \frac{5}{\cos\alpha}\right) = 4$

$16\tan\alpha - 5\sec^2\alpha = 4$

$16\tan\alpha - 5 - 5\tan^2\alpha = 4$

$5\tan^2\alpha - 16\tan\alpha + 9 = 0$

CHAPTER REVIEW 13

1 D 190 km h^{-1} in the direction of the car

2 C 500 m

3 B $\underset{\sim}{F}_1 = 350$ N east, $\underset{\sim}{F}_2 = 270$ N west, $\underset{\sim}{F}_3 = 440$ N east

4 D $\underset{\sim}{F}_1 = 389$ N, $\underset{\sim}{F}_2 = 157$ N

5 (a) The resultant force is 75 N south

(b) The resultant force is $100\sqrt{2}$ N, S45°E

6 $|\underset{\sim}{F}_1| = |\underset{\sim}{F}_2| = 490$ N

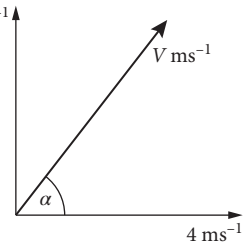
7 (a) 1.7 m s^{-1} at 61.93° to the riverbank

(b) 64 m downstream from the point O

(c) The rower's final displacement is 136 m from point O at 61.93° to the riverbank

(d) The direction is 63.61° upstream to riverbank and it will take Jake 74.42 s

8 (a) 6 ms^{-1}



(b) $\tan\alpha = 1.5$

$\alpha = 56°\ 19'$

(c) $\underset{\sim}{v}(0) = 4\underset{\sim}{i} + 6\underset{\sim}{j}$: $\underset{\sim}{v}(t) = 4\underset{\sim}{i} + (6 - 10t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 4t\,\underset{\sim}{i} + \left(6t - 5t^2\right)\underset{\sim}{j}$

(d) 1.8 m

(e) Range when $t = 1.2$ s

$x = 4.8$ m

9 $\underset{\sim}{v}(0) = 28\underset{\sim}{i} + v_y\,\underset{\sim}{j}$, $y_{\max} = 4.9$ m, $g = 9.8\ \text{m s}^{-2}$

(a) $\underset{\sim}{v}(0) = 28\underset{\sim}{i} + v_y\,\underset{\sim}{j}$: $\underset{\sim}{v}(t) = 28\underset{\sim}{i} + (v_y - 9.8t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{r}(t) = 28t\,\underset{\sim}{i} + (v_y t - 4.9t^2)\underset{\sim}{j}$

(b) $t = 1$ s

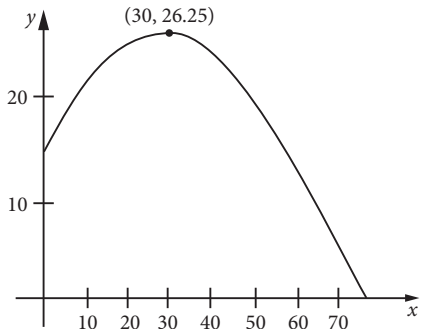
(c) $t = \frac{1}{7}$ is the time on the way up, $t = \frac{13}{7}$ is the time on the way down

$t = \frac{13}{7}$: $x = 28 \times \frac{13}{7} = 52$ m.

(d) The fielder has to run $(65 - 52) = 13$ m in $\frac{13}{7}$ s

Speed of fielder $= 13 \times \frac{7}{13} = 7\ \text{m s}^{-1}$

10 (a) $\underset{\sim}{r}(0) = 15\underset{\sim}{j}$, (30, 26.25). $y_{\max} = 26.25$ m, $g = 10\ \text{m s}^{-2}$



(b) $\underset{\sim}{v}(t) = v_x\,\underset{\sim}{i} + (v_y - 10t)\,\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 15\,\underset{\sim}{j}$: $\underset{\sim}{r}(t) = v_x\,t\,\underset{\sim}{i} + \left(15 + v_y\,t - 5t^2\right)\underset{\sim}{j}$

(c) Speed of projection = $25\,\text{m s}^{-1}$

$\alpha = 36°\ 52'$

11 C $\underset{\sim}{F}_1 = 230\,\text{N}$, $\underset{\sim}{F}_2 = 193\,\text{N}$

CHAPTER 14

EXERCISE 14.1

1 (a) Yes, only two outcomes: correct (C) or incorrect (I)

(b) ${}^{10}C_6 = {}^{10}C_4 = 210$

2 Yes, the outcomes were 'voted for' or 'didn't vote for' the candidate.

3 (a) Yes, male or female (b) ${}^{20}C_{11} = 167\,960$

4 (a) Yes, only outcomes are heads or tails

(b) HHHH, HHHT, HHTH, HTHH, THHH

5 (a) Yes, the outcomes are 'a sum of 7' or 'not a sum of 7'.

(b) $p = \frac{6}{36} = \frac{1}{6}$ (c) 6

EXERCISE 14.2

1 (a) $P(X = 4) = 0.0078$ (b) $P(X = 6) = 0.0746$

(c) $P(X = 2) = 0.2965$ (d) $P(X = 30) = 0.1146$

2 The probability, correct to four decimal places, of the spinner landing on a shaded section exactly five times out of 10 spins is $P(X = 5) = 0.1782$.

3 (a) The probability that exactly 30 even numbers are rolled is $P(X = 30) = 0.0419$.

(b) The probability that at least one 6 is rolled is $P(X \geq 1) = 0.9999$.

4 (a) The probability that exactly 15 bulbs flower is $P(X = 15) = 0.0747$

(b) The probability that more than 15 bulbs flower is $P(X > 15) = 0.0510$

5 (a) This situation is not suitable for binomial modelling because the number of trials was not determined beforehand.

(b) This situation is suitable for binomial modelling.

6 (a) The probability that exactly 25 hearts are drawn is $P(X = 25) = 0.0918$.

(b) The probability that exactly 40 cards are clubs is $P(X = 40) = 3.6263 \times 10^{-4}$.

(c) The probability that exactly 55 red cards are drawn is $P(X = 55) = 0.0485$.

(d) The probability that exactly 60 cards are black is $P(X = 60) = 0.0108$.

7 B

8 The probability that of eight births, exactly half are girls is $P(X = 4) = 0.2157$.

9 The probability that exactly 35 people out of a random sample of 100 will be left-handed is $P(X = 35) = 0.0834$.

10 The probability that exactly 6 tosses out of 10 land on heads is $P(X = 6) = 0.146$.

11 B

12 This experiment could not be described as $X \sim B\left(5, \frac{1}{4}\right)$ because the probability of success is not constant.

13 (a) The probability that Philomena stops at exactly five sets of lights is $P(X = 5) = 0.3025$.

(b) The probability that Philomena stops at fewer than two sets of lights is 0.0109.

(c) The probability that Philomena stops at more than five sets of lights is 0.1176.

(d) The probability that Philomena stops at the first three sets of lights is 0.343.

(e) The probability that Philomena stops at the second and sixth sets of lights only is 0.0040.

(f) The probability that Philomena stops at exactly four sets of lights including the first two is 0.1297.

14 (a) (i) 0.0115 (ii) 0.0576 (iii) 0.1369 (iv) 0.2054 (v) 0.2182

(b) 0.6296 (c) 0.3704.

15 (a) If a random sample of 12 loaves is selected, the probability that exactly one loaf cannot be sold at full price is 0.3837.

(b) The probability that exactly three loaves are slightly burnt is 0.0532.

(c) The probability that all loaves in the random sample are sold at full price is 0.3677.

(d) The probability that the baker would make a profit is 0.9999.

16 (a) The probability that a maximum of three new songs from the sample will become popular hits is 0.950.

(b) The probability that more than four but less than six songs from the sample will become popular hits is 0.010.

(c) It is most likely that one song will become a popular hit because it has the highest probability, $P(X = 1) = 0.349$

(d) The probability that exactly one of 70 new songs will become a popular hit is 0.167.

(e) Two songs are most likely to become popular hits because $P(X = 2) = 0.241$ has the highest probability value.

17 (a)

x	0	1	2	3	4	5
$P(X = x)$	0.4437	0.3915	0.1382	0.0244	0.0022	7.59×10^{-5}

(b) The probability that the boxed set will contain fewer than four faulty DVDs, if you know it contains at least one faulty DVD, is 0.996.

EXERCISE 14.3

1 (a) $E(X) = 14$, $\text{Var}(X) = 4.2$, $\sigma(X) = 2.05$

(b) $E(X) = 55$, $\text{Var}(X) = 24.75$, $\sigma(X) = 4.97$

(c) $E(X) = 33\frac{1}{3}$, $\text{Var}(X) = 11\frac{1}{9}$, $\sigma(X) = 3\frac{1}{3}$

(d) $E(X) = 5$, $\text{Var}(X) = 2.5$, $\sigma(X) = 1.58$

(e) $E(X) = 20$, $\text{Var}(X) = 4$, $\sigma(X) = 2$

(f) $E(X) = 16$, $\text{Var}(X) = 9.6$

2 (a) $p = \frac{1}{3}$ (b) $p = 0.1$ (c) $n = 35$ (d) $n = 25$

3 (a) $n = 12$ and $p = 0.4$ so $X \sim B(12, 0.4)$

(b) $n = 20$ and $p = 0.1$ so $X \sim B(20, 0.1)$

(c) $n = 30$ and $p = 0.15$ so $X \sim B(30, 0.15)$

(d) $n = 50$ and $p = 0.22$ so $X \sim B(50, 0.22)$

(e) $n = 100$ and $p = 0.65$ so $X \sim B(100, 0.65)$

(f) $n = 150$ and $p = 0.275$ so $X \sim B(150, 0.275)$

4 B 5 D 6 $P(X = 3) = 0.42$

7 If the spinner is spun eight times, the probability of obtaining less than the expected number of blue outcomes is 0.367.

8 C

9 (a) The probability distribution is shown in the table.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.1001	0.2670	0.3115	0.2076	0.0865	0.0231	0.0038	0.0004	0

(b) $E(X) = 2$ (c) $\sigma(X) = 1.22$

10 (a) The probability that the first two children born are male is 0.25.

(b) The probability that there are three boys and three girls, in any order, is 0.3125.

(c) The probability that there are more girls than boys is 0.3438.

(d) The probability that there is at least one boy, but more girls than boys is 0.3281.

(e) The probability that there are no consecutive births of the same gender is 0.0313.

11 (a) The number of trials is $n = 49$ with a probability of success of $p = \frac{6}{7}$.

(b) When both the mean and the variance are doubled, the probability of success does not change, $p = \frac{6}{7}$, but the number of data collected doubles to $n = 98$.

(c) The researcher should expect the variance for this set of data to be 12.10.

(d) The mean and variance the researcher would expect to calculate for this set of data is $E(X) = 48$ and $\text{Var}(X) = 6.86$.

(e) If the mean is the same but the number of trials is larger, p decreases which, in turn, produces an increase in the variance of the sample.
If the probability of success is the same but the number of trials is larger, this produces an increase in both the mean and the variance of the sample.

EXERCISE 14.4

1 (a) The population proportion is the proportion of all Northern Territory email users who have received a fake email. The sample size is 800.The observation value is 42% of $800 = 0.42 \times 800 = 336$. The sample proportion is $42\% = 0.42$.

(b) The population proportion is the proportion of Tasmanian Holden owners who prefer a medium- to large-sized car. The sample size is 2500. The observation value is 1893. The sample proportion is: $\frac{1893}{2500} = 0.76$

2 Use the formulas $E(\hat{P}) \approx \hat{p}$ and $S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

For the sample, $X = 64$ and $n = 104$.

$\hat{p} = \frac{x}{n} = \frac{64}{104} \approx 0.615$

Therefore, $E(\hat{P}) \approx \hat{p} \approx 0.615$ i.e. the expected population proportion of students not knowing what to do after they leave school, is 0.615.

$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{\frac{64}{104} \times \left(1 - \frac{64}{104}\right)}{104}} = 0.0477\ldots \approx 0.048$

i.e. the standard error of the sample proportion is 0.048.

3 The conditions are $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$.

For the sample, $n = 334$, $\hat{p} = 0.98$ and $1 - \hat{p} = 0.02$.

Hence: $n\hat{p} = 334 \times 0.98 = 327.32$. Therefore, $n\hat{p} \geq 5$.

$n(1-\hat{p}) = 334 \times 0.02 = 6.68$. Therefore, $n(1-\hat{p}) \geq 5$.

Both conditions are met. Therefore, the distribution can be approximated by a normal distribution.

4 (a) Use the formulas $E(\hat{P}) \approx \hat{p}$ and $S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. For the sample $x = 507$ and $n = 1300$.

$\hat{p} = \frac{x}{n} = \frac{507}{1300} = 0.39$

$E(\hat{P}) \approx \hat{p} \approx 0.39$

$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.39 \times (1-0.39)}{1300}} = 0.0135\ldots \approx 0.01$

(b) $x = 415$ and $n = 500$.

$\hat{p} = \frac{x}{n} = \frac{415}{500} = 0.834$

$\left(\hat{P}\right) \approx \hat{p} \approx 0.83$

$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.83 \times (1-0.83)}{500}} = 0.0168\ldots \approx 0.017$

(c) $x = 50$ and $n = 1000$.

$\hat{p} \approx \frac{x}{n} = \frac{50}{1000} = 0.05$

$E(\hat{P}) \approx \hat{p} \approx 0.05$

$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.05 \times (1-0.05)}{1000}} = 0.0069 \approx 0.007$

5 (a) The conditions are $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$

For the sample, $n = 1000$, $\hat{p} = 0.55$ and $1 - \hat{p} = 0.45$

$n\hat{p} = 1000 \times 0.55 = 550$

$n(1-\hat{p}) = 1000 \times 0.45 = 450$

Both criteria are met. Therefore the distribution can be approximated by a normal distribution.

(b) For the sample, $n = 10$, $\hat{p} = 0.9$, $1 - \hat{p} = 0.1$

$n\hat{p} = 10 \times 0.9 = 9$

$n(1-\hat{p}) = 10 \times 0.1 = 1$

Both criteria are not met. Therefore the distribution cannot be approximated by a normal distribution.

(c) For the sample $n = 50$, $\hat{p} = 0.1$, $1 - \hat{p} = 0.9$

$n\hat{p} = 50 \times 0.1 = 5$

$n(1-\hat{p}) = 50 \times 0.9 = 45$

Both criteria are met. Therefore the distribution can be approximated by a normal distribution.

6 (a) $\hat{p} = 0.7$, $x = 91$. Substitute in $\hat{p} = \frac{x}{n}$:

$0.7 = \frac{91}{n}$

$n = \frac{91}{0.7} = 130$

(b) $\hat{p} = 0.15$ and $x = 300$

$0.15 = \frac{300}{n}$

$n = \frac{300}{0.15} = 2000$

(c) $\hat{p} = 0.83$ and $x = 15$

$0.83 = \frac{15}{n}$

$n = \frac{15}{0.83} = 18.07\ldots \approx 18$

7 (a) $\hat{p} = 0.34$ and $n = 2500$. Substitute in $\hat{p} = \frac{x}{n}$

$0.34 = \frac{x}{2500}$

$x = 2500 \times 0.34 = 850$

(b) $\hat{p} = 0.02$ and $n = 100$

$0.02 = \frac{x}{100}$

$x = 100 \times 0.02 = 2$

(c) $\hat{p} = 0.98$ and $n = 5000$

$0.98 = \frac{x}{5000}$

$x = 5000 \times 0.98 = 4900$

8 Use the formula $E(\hat{P}) \approx \hat{p}$ to estimate the population proportion from the sample proportion because the population proportion is often unknown. Hence the value of the sampling error is difficult to calculate because the true value of the population proportion is usually unknown.

9 (a) For the sample, $\hat{p} = 0.75$, $1 - \hat{p} = 0.25$ and $n = 900$.

$E(\hat{P}) \approx \hat{p} = 0.75$

$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.75 \times 0.25}{900}} \approx 0.014$

The conditions for using a normal distribution are $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$.

$n\hat{p} = 900 \times 0.75 = 675$. Therefore, $n\hat{p} \geq 5$.

$n(1-\hat{p}) = 900 \times 0.25 = 225$

Both conditions are met. Therefore the distribution can be approximated by a normal distribution.

(b) A simple random sample of 50 people found that five people were prepared to quit smoking within the week.

For the sample, $x = 5$ and $n = 50$

$\hat{p} = \frac{x}{n} = \frac{5}{50} = 0.1$

Therefore, $E(\hat{P}) \approx \hat{p} = 0.1$. The expected value of people prepared to quit smoking within a week is 0.1.

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.1 \times (1-0.1)}{50}} = \sqrt{0.018} \approx 0.042$$

The conditions for using a normal distribution are $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$.

$n\hat{p} = 50 \times 0.1 = 5$. Therefore, $n\hat{p} \geq 5$

$n(1-\hat{p}) = 50 \times 0.9 = 45$. Therefore, $n(1-\hat{p}) \geq 5$.

Both conditions are met. Therefore the distribution can be approximated by a normal distribution.

(c) For the sample, $x = 42\,600$, $n = 67\,000$

$\hat{p} = \frac{x}{n} = \frac{42\,600}{67\,000} = \frac{213}{335} \approx 0.636$

$E(\hat{P}) \approx \hat{p} \approx 0.64$

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{\frac{213}{335} \times \left(1-\frac{213}{335}\right)}{67\,000}} = 0.0018\ldots \approx 0.002$$

The conditions for using a normal distribution are $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$.

$n\hat{p} = x = 42\,600$ Therefore, $n\hat{p} \geq 5$

$n(1-\hat{p}) = 67\,000 \times \left(1-\frac{213}{335}\right) = 24\,400$. Therefore,

$n(1-\hat{p}) \geq 5$.

Both conditions are met. Therefore the distribution can be approximated by a normal distribution.

10 p is the population proportion, a parameter (and numerical value) which is usually unknown and which describes a particular characteristic that occurs within the whole population. The sample proportion $\hat{p}$ is a statistic and numerical value which describes a particular characteristic that occurs within a random sample.

11 The standard error is the standard deviation or measure of variability of the statistics from sample to sample. (Your answer may differ slightly.)

12 With larger sample sizes, the graph of the distribution of sample proportions becomes more bell-shaped and symmetrical, clustering more closely around the middle value. Also, the formula for standard error of sample proportions in binomial situations is $S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, which means that the larger the value of n, the smaller the standard error. As the sample size increases, the standard error decreases.
OR The value of n appears in the denominator of the standard error formula. Therefore, as n increases, the value of the standard error decreases. This means that $\hat{P}$ is less variable in larger samples.

13 (a) No, the sample proportion can be different for each survey. However, there may be times where the sample proportion value will be the same.

(b) The standard error of the sample proportions is

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

For $\hat{p} = 0.78$, $n = 500$, $S(\hat{P}) = \sqrt{\frac{0.78 \times 0.22}{500}} \approx 0.019 \approx 0.02$

Using a normal approximation for the distribution: $P(z < -2) \approx 2.5\%$, so that $P(\hat{p} < 0.74) \approx 2.5\%$.With such a large sample size, the variability from one sample to another is tiny, hence a value of 0 from a random sample of 500 would be virtually impossible. If it did occur, you would suspect an error in the sampling process or that bias had been introduced.

(c) From part (b) $S(\hat{P}) \approx 0.02$, the standard error of the sample proportions is 0.02. Using a normal approximation for the distribution: $P(z < -2) \approx 2.5\%$, so that $P(\hat{p} < 0.74) \approx 2.5\%$. With such a large sample size, the variability from one sample to another is tiny, hence a value of 0.45 from a random sample of 500 would be virtually impossible. A large difference in sample proportion values could indicate a problem with the sampling process or bias.

(d) From part (b), $S(\hat{P}) \approx 0.02$, i.e. the standard error of the sample proportions is 0.02. Using a normal approximation for the distribution approximately 95% of the time the sample proportion will be between 0.74 and 0.82. With such a large sample size, the variability from one sample to another is very small, hence it is expected that most values will be close to 0.78.

(e) If the sample size is small or the probability is close to either 0 or 1, you will not have sample proportion values that are normally distributed so you won't know the likelihood of the statistic being representative of the parameter. There is an assumption of random selection and lack of bias whenever a statistic is to be used to estimate a parameter.

14 (a) The formula $E(\hat{P}) \approx \hat{p}$ indicates that the sample proportion gives an estimate for the expected or average value of sample proportions, which is the same as the population proportion. Hence the sample proportion is likely to be different from sample to sample. You should be suspicious if you find sample statistics from many samples are exactly the same.

(b) The standard error of the sample proportions is

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

For $\hat{p} = 0.07$, $n = 1500$: $S(\hat{P}) = \sqrt{\frac{0.07 \times 0.93}{1500}} \approx 0.007$

Using a normal approximation for the distribution, $P(z > 2) \approx 2.5\%$ so that $P(\hat{p} > 0.084) \approx 2.5\%$. Two standard deviations above the expected population proportion of 0.07 is 0.084. With such a large sample size, the variability from one sample to another is tiny, hence a value of 1 from a random sample of 1500 would be virtually impossible. A sample proportion of 1 means that 100% of a sample experienced blood clots after taking heparin.

(c) From part (b), $S(\hat{P}) \approx 0.007$, the standard error of the sample proportions is 0.007. Using a normal approximation for the distribution: $P(z > 2) \approx 2.5\%$ so that $P(\hat{p} > 0.084) \approx 2.5\%$. With such a large sample size, the variability from one sample to another is tiny, hence a value of 0.23 from a random sample of 1500 would be virtually impossible. You would suspect a problem with the sampling process or bias. It could be that those particular patients have not had a heparin shot and that there is an error within the records.

(d) For $\hat{p} = 0.07$, $n = 1500$: $n\hat{p} = 1500 \times 0.07 = 105$

$n(1-\hat{p}) = 1500 \times 0.93 = 1395$

Hence both of the criteria $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$ are satisfied. In this case, a large value of n compensates for the small sample proportion value of $\hat{p}$.

(e) If the sample proportion is close to 0 or 1 and the sample size is small, then the conditions for CLT will not be met and you would not consider the sample proportion to be representative. There is an assumption of random selection and lack of bias whenever a statistic is to be used to estimate a parameter.

15 (a) Satisfied members: $x_S = \frac{1}{2} \times 120 = 60$

Indifferent members: $x_I = \frac{1}{3} \times 120 = 40$

Dissatisfied members: $x_D = \frac{1}{6} \times 120 = 20$

(b) Satisfied members: $\hat{p}_S = \frac{x}{n} = \frac{60}{120} = 0.5$

Indifferent members: $\hat{p}_I = \frac{x}{n} = \frac{40}{120} = 0.33 (2 \text{ d.p.})$

Dissatisfied members: $\hat{p}_D = \frac{x}{n} = \frac{20}{120} = 0.17 (2 \text{ d.p.})$

(c) $x_D = \frac{1}{6} \times 240 = 40$, $\hat{p}_D = \frac{x}{n} = \frac{40}{240} = 0.17 (2 \text{ d.p.})$

The point estimate does not change because the observation value is doubled, whereas the proportion of observations doesn't change.

(d) Calculate the new observation value for the dissatisfied members.

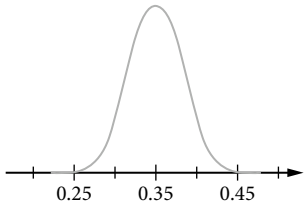
$x_D = \frac{1}{5} \times 120 = 24$, $\hat{p}_D = \frac{x}{n} = \frac{24}{120} = 0.20$

The point estimate becomes greater because the proportion of results is greater, which produces a higher observation value.

16 (a) The expected value $E(\hat{P}) \approx \hat{p}$, where $\hat{p} = 0.35$.

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.35 \times 0.65}{200}} \approx 0.0337$$

(b) Use technology to plot the sampling distribution of $\hat{p}$ with mean 0.35 and standard deviation 0.0337.



Sample distribution of $\hat{p}$

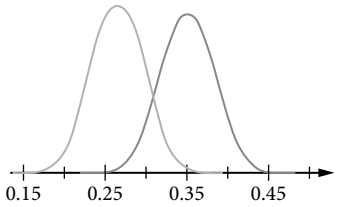
(c) (i) Let x be the observation value. The expected value $E(\hat{P}) \approx \hat{p}$ where $\hat{p} = \frac{x}{n}$. For $x = 55$, $n = 200$:

$\hat{p} = \frac{55}{200} = 0.275$

$E(\hat{P}) \approx 0.275$, Calculate the standard error using the formula:

$$S(\hat{P}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.275 \times 0.725}{200}} \approx 0.0316$$

(ii) Use technology to plot the sampling distribution of $\hat{p}$ with mean 0.275 and standard deviation 0.0316 on the same set of axes as previously.



Sample distribution of $\hat{p}$

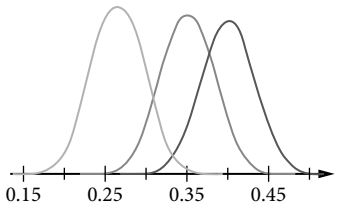
(d) Consider whether the expected value of the new sample is a higher or a lower value. A higher number of orders from returning customers will produce a higher expected value, which means that the curve will be to the right of the previous two curves.

$(\hat{P}) \approx \frac{80}{200} = 0.40$

Consider whether the standard error of the new sample is a higher or a lower value. A higher number of orders from returning customers will produce a slightly higher standard error, which means that the curve will be slightly wider.

$$S(\hat{P}) = \sqrt{\frac{0.4 \times 0.6}{200}} = 0.0346$$

Use technology to plot the sampling distribution of $\hat{p}$ with mean 0.40 and standard deviation 0.0346 on the same set of axes.



Sample distribution of $\hat{p}$

CHAPTER REVIEW 14

1 (a) $P(X = 3) = \frac{216}{625}$ (b) $P(X = 4) = \frac{64\,827}{200\,000}$

2 (a) $p = 0.8$ (b) $n = 40$

3 $E(X) = 25$

4 (a) 5 points

(b) If Yehudi won 10 of the first 15 points I would be quite surprised as the number is well above the expected number.

5 (a)

y	0	1	2	3
$P(Y = y)$	$\frac{7}{24}$	$\frac{21}{40}$	$\frac{7}{40}$	$\frac{1}{120}$

(b) $E(Y) = \frac{9}{10}$

6 C **7** B **8** B **9** C **10** D

11 (a) $P(\text{success}) = 0.75$

(b) (i) $X \sim B(30, 0.75)$

(ii) The expected number of odd numbers which will occur in the 30 rolls is $E(X) = 22.5$

(iii) To roll 28 odd numbers in the 30 rolls would be unusual as it is outside the 95% confidence limit.

(c) (i) The probability of any pair of rolls resulting in two odd numbers is 0.5625.

(ii) The probability distribution of the number of odd numbers in the two rolls

y	0	1	2
$P(Y = y)$	0.0625	0.375	0.5625

12 (a) 0.367 (b) 0.21 (c) 0.393 (d) 0.04

(e) Chance of winning more than two if buying one box per day for seven days is 0.148.

Chance of winning more than two if buying one box per day for 10 days is 0.322. Therefore, Carol is correct, her chances will increase.

GLOSSARY

A

acceleration
The rate of change of velocity with respect to time: $\ddot{x}, \dot{v}(t), \frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$, standard units m s^{-2}.

amplitude
The amplitude of a sine or cosine function is half the difference between the greatest and least values of the function. The amplitude of $y = a\sin nx$ is a.

angle of projection
The angle, measured from the horizontal, at which a particle is projected.

arc
(1) Part of the circumference of a circle.

(2) Part of a curve.

arrangements in a circle
The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, is $\frac{n!}{n} = (n-1)!$. As there is no start or finish to a circle, one object needs to be fixed as the 'starting object' for counting purposes.

asymptote
A line or curve that a function approaches but never reaches.

auxiliary angle method
Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ or $r\cos(x-\alpha)$, where $r = \sqrt{a^2+b^2}$ and α is in the first quadrant such that $\tan\alpha = \frac{b}{a}$.

B

Bernoulli distribution
The Bernoulli distribution is the probability distribution of a random variable which takes the value 1 with 'success' probability p, and the value 0 with 'failure' probability $q = 1 - p$. The Bernoulli distribution is a special case of the binomial distribution, where $n = 1$.

Bernoulli random variable
A Bernoulli random variable has two possible values, namely 0 representing failure and 1 representing success. The parameter associated with such a random variable is the probability p of obtaining a 1.

Bernoulli trial
A Bernoulli trial is an experiment with only two possible outcomes, labelled 'success' and 'failure'.

binomial coefficient
The coefficient of the term $x^{n-r}y^r$ in the expansion of $(x+y)^n$ is called a binomial coefficient. It is written as ${}^nC_r$ or $\binom{n}{r}$ where $r = 0, 1, \ldots, n$ and is given by: $\frac{n!}{r!(n-r)!}$.

binomial distribution
The binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent Bernoulli trials, each of which yields success with probability p.

binomial expansion
A binomial expansion describes the algebraic expansion of powers of a binomial expression.

binomial probability
When the same trial is repeated several times and there are only two possible outcomes in each trial (often called 'success' and 'failure'), the probability of r successes is given by $P(X = r) = \binom{n}{r}q^{n-r}p^r$, where $0 \le r \le n$, where p is the probability of a 'success' and $q = 1 - p$ is the probability of a 'failure'. This expression is the corresponding term of the binomial expansion of $(p+q)^n$.

binomial random variable
A binomial random variable X represents the number of successes in n independent Bernoulli trials. In each Bernoulli trial, the probability of success is p and the probability of failure is: $q = 1 - p$.

binomial theorem
The sum of two terms raised to the nth power can be expanded as a series of n terms according to the following formula:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

C

column vector notation

A vector, $\underset{\sim}{a}$, in two dimensions can be represented in column vector notation. For example, the ordered pair $a = (4,5)$ can be represented in column vector notation as: $a = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

combination

A combination is a selection of r distinct objects from n distinct objects, where order is not important. The number of such combinations is denoted by ${}^{n}C_{r}$ or $\begin{pmatrix} n \\ r \end{pmatrix}$, and is given by: $\frac{n!}{r!(n-r)!}$

component form of a vector

The component form of a vector, $\underset{\sim}{v}$, expresses the vector in terms of unit vectors i, a unit vector in the x-direction, and j, a unit vector in the y-direction. For example, the ordered vector pair $\underset{\sim}{v} = (4,3)$ can be represented as: $\underset{\sim}{v} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$.

cubic function *or* cubic polynomial

A polynomial function f of the third degree, defined by $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are constants, $a \neq 0$. Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$ where α is a real number.

D

decreasing function

A function f defined on the interval $a \leq x \leq b$ is said to be strictly monotonic, decreasing when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. In other words, the curve always slopes downwards to the right, so the function never has a positive gradient.

degree of a polynomial

A polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is of the nth degree if $a_n \neq 0$.

differential equation

A differential equation is any equation containing the derivative of an unknown function.

direction field

A direction field (or slope field) is a graphical representation of the tangent lines to the solutions of a first-order differential equation.

displacement

The signed (positive or negative) distance of a particle from the origin; $x(t)$.

displacement vector

A displacement vector represents the displacement from one point to another.

division algorithm

For any polynomial $P(x) = (x - a)Q(x) + R$, where $Q(x)$ is a polynomial.

E

even function

A function is even if $f(-x) = f(x)$ for all values of x in the domain. The function is also symmetrical about the y-axis.

exponential decay

Exponential decay (or 'exponential decline') occurs when a quantity decreases by a constant percentage over time.

exponential growth

Exponential growth occurs when a quantity increases by a constant percentage over time.

F

factorial (!)

The product of the first n positive integers is called the factorial of n and is denoted by $n!$.
$n! = n(n-1)(n-2)(n-3) \times \ldots \times 3 \times 2 \times 1$
By definition: $0! = 1$

factorial notation (!)

The notation $n!$ is called 'n factorial' and means $n \times (n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$.

factor theorem

The factor theorem states that a polynomial $P(x)$ has a factor $(x - k)$, if and only if, $P(k) = 0$; that is, k is a root of the equation $P(x) = 0$. The factor theorem links the factors and zeros of a polynomial.

fundamental counting principle

The fundamental counting principle states that if one event has m possible outcomes and a second independent event has n possible outcomes, then there is a total of $m \times n$ possible outcomes for the two combined events.

H

half-life

The time taken for half of the atoms in a radioactive substance to decay.

horizontal line test

To check that any horizontal line that can be drawn will only cut a curve once.

I

increasing function

A function f defined on the interval $a \leq x \leq b$ is said to be strictly monotonic, increasing when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. In other words, the curve always slopes upwards to the right, so the function never has a negative gradient.

indefinite integral

The primitive of a function, written $\int f(x)dx$.

independent events
Events are independent if the occurrence or non-occurrence of one event cannot change the probability of the occurrence of another event, i.e. the events have no effect on each other.

induction (mathematical)
A method used to prove that something is true for all positive integers n (or e.g. for all positive integers n greater than a specified starting value, or for a subset such as all positive odd integers n). First, the proposition is proved true for a starting value of n; next, it is proved that if the proposition is true for $n = k$ then it is true for $n = k + 1$; by induction, it is therefore true for all values of n greater than the starting value.

initially
When time $t = 0$ (at the beginning).

integrand
An integrand is a function that is to be integrated.

integration by substitution
Changing the variable in an integral to make it easier to determine the integral, e.g. by substituting a new variable $u = g(x)$.

inverse cosine function
$y = \cos^{-1} x$ is the inverse cosine function; it means 'that part of $x = \cos y$ for which $0 \leq y \leq \pi$'.

inverse function
If $f(g(x)) = g(f(x)) = x$ then $g(x) = f^{-1}(x)$ is the inverse function of $f(x)$. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, when the inverse function exists.

inverse sine function
$y = \sin^{-1} x$ is the inverse sine function; it means 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$'.

inverse tangent function
$y = \tan^{-1} x$ is the inverse tangent function; it means 'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

L

leading term
The term $a_n x^n$ of a polynomial, where $a_n \neq 0$.

logistic equation
The logistic equation is the differential equation $\frac{dn}{dt} = kN(P - N)$ where k, P are constants. Thus: if $N = 0$ or $N = P$, $\frac{dn}{dt} = 0$

M

mathematical induction
Mathematical induction is a method of mathematical proof used to prove statements involving the natural numbers. It is also known as proof by induction or inductive proof. The principle of induction is an axiom and so cannot itself be proven.

multiplication principle
See *fundamental counting principle*.

multiplicity of a root
Given a polynomial $P(x)$, if $P(x) = (x - a)^r Q(x)$, $Q(a) \neq 0$ and r is a positive integer, then the root $x = a$ has multiplicity r.

mutually exclusive events
Two events are mutually exclusive (or 'disjoint') if membership of one event excludes membership of the other, so that they cannot occur simultaneously.

N

Newton's law of cooling
The cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium: $\frac{dT}{dt} = -k(T - M)$, where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant). A solution of this differential equation is $T = M + Ae^{-kt}$, where A is a constant.

O

odd function
A function is odd if $f(-x) = -f(x)$ for all values of x in the domain. The function has rotational symmetry about the origin.

one-to-one function
A function for which any vertical line can only cut it once and any horizontal line can only cut it once; every one-to-one function has an inverse function.

P

parameter
(1) A parameter is a quantity that defines certain characteristics of a function or system. For example, θ is a parameter in $y = x\cos\theta$.

(2) A parameter can be a characteristic value of a situation. For example, the time taken for a machine to produce a certain product.

parametric equations
When two related variables (e.g. x and y) are expressed in terms of a third variable, the 'parameter' (e.g. t or θ), so that $x = f(t)$, $y = g(t)$ or $x = f(\theta)$, $y = g(\theta)$.

Pascal's triangle
An arrangement of numbers that gives the coefficients of the terms in the binomial expansion of $(1 + x)^n$.

permutation

A permutation is an arrangement of r distinct objects taken from n distinct objects where order is important.

The number of such permutations is denoted by ${}^nP_r$ and is equal to:

$$ {}^nP_r = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} $$

The number of permutations of n objects is $n!$

point of contact

A point where two curves touch.

polynomial

An algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n, $n-1, \ldots$ are all positive integers and a_n, $a_{n-1}, \ldots$ are the coefficients (for convenience, usually chosen as integers).

position vector

The position vector of a point P in the plane is the vector joining the origin to P.

projectile motion

The motion of a particle when it is projected in any direction and subject only to gravitational acceleration, i.e. ignoring any air resistance.

Q

quadratic function *or* quadratic polynomial

A polynomial function f of the second degree, defined by $f(x) = ax^2 + bx + c$, where a, b, c are constants, $a \neq 0$. A quadratic polynomial with no real zeros cannot be factorised into linear factors of the form $(x + \alpha)$ where α is a real number.

quartic function *or* quartic polynomial

A polynomial function f of the fourth degree, defined by $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, e are constants, $a \neq 0$. A quartic polynomial with no real zeros cannot be factorised into linear factors of the form $(x + \alpha)$ where α is a real number.

quinella

In horse racing, when the first two horses are picked correctly (in any order).

R

range on the horizontal plane

The horizontal distance between a particle's point of projection and the point where the particle meets the horizontal plane through the point of projection.

remainder theorem

The remainder theorem states if a polynomial $P(x)$ is divided by $(x - k)$, the remainder is equal to $P(k)$.

rest

When a particle is 'at rest', its velocity is zero.

roots (of an equation)

(1) The values of x for which $f(x) = 0$.

(2) The abscissae of the points at which the graph of $y = f(x)$ cuts or touches the x-axis.

S

sample proportion

The sample proportion ($\hat{p}$) is the fraction of samples out of n Bernoulli trials which were successes (x), that is:

$$ \hat{p} = \frac{x}{n} $$

For large n, $\hat{p}$ has an approximately normal distribution.

scalar

A scalar is a quantity with magnitude but no direction.

secant

The reciprocal of the cosine function. (Abbreviated 'sec'.)

solid of revolution

A three-dimensional solid formed by rotating an area under the curve about one of the coordinate axis.

statement

An assertion that can be true or false, but not both.

T

t formulae

Expressions for $\sin A$, $\cos A$ and $\tan A$ in terms of t where $t = \tan\frac{A}{2}$.

trajectory

The path followed by a particle.

trifecta

In horse racing, when the first three horses are picked in the correct order.

V

velocity

The rate of change of displacement with respect to time: $\dot{x}(t)$, $v(t)$ or $\frac{dx}{dt}$, standard units m s^{-1}.

vertical line test

To check that any vertical line that can be drawn will only cut a curve once.

If a vertical line cannot pass through a curve more than once, then the curve represents a function.

Z

zero factorial

By definition, $0! = 1$.

zeros (of a function)

The values of x for which a function $f(x) = 0$.