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Australian Curriculum

MathsWorld 10

Australian Curriculum edition

Covers Year 10 and 10A content descriptors

Jill Vincent Beth Price Natalie Caruso
Glenda Romeril David Tynan

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Jill Vincent Beth Price Natalie Caruso
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Consultants: Tracey MacBeth-Dunn and John Dowsey

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Introduction

The ***MathsWorld Australian Curriculum editions*** have been rigorously developed to cover all the content and requirements of the Australian Curriculum. The Year 10 textbook covers all content descriptors from the Australian Curriculum for both 'Year 10' and 'Year 10A'. They are carefully written to provide comprehensive and accessible textbooks, which cater for a range of ability levels. The textbooks are accompanied by a complete teacher resource package that provides extra resources, teacher notes and further support.

Access to Macmillan's innovative ebook platform, ***OneStopDigital***, is included with purchase of the student textbook or is available as standalone digital access. As part of the teacher resource package, teachers have access to the ebook versions of both the student and teacher books.

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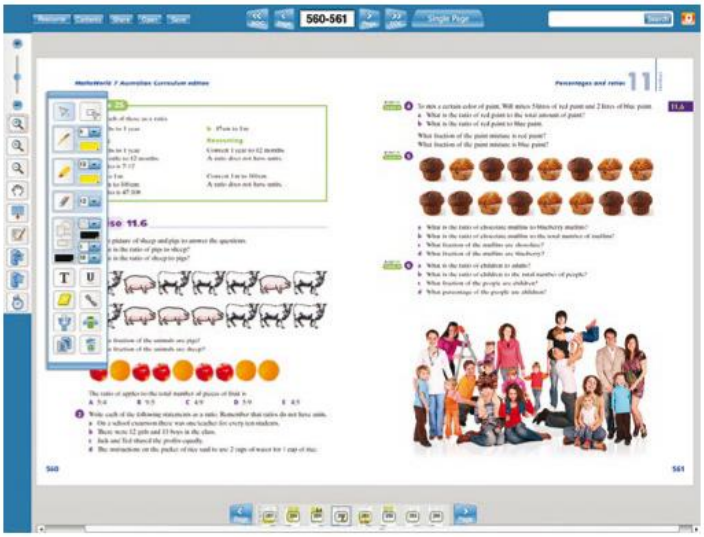
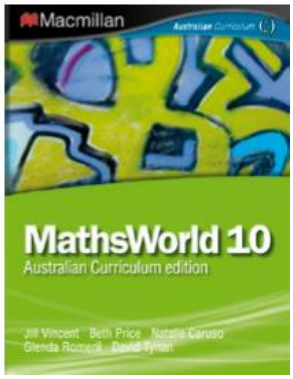


Student book

Tossing a coin

Icons throughout the book indicate links to a wide range of worksheets, technology files, tests and quizzes.

Theory and worked examples are structured into manageable sections and are designed to illustrate the main concepts and skills for each topic. Solutions to worked examples are structured into two columns, with a working column indicating what students should write and the reasoning column, showing the thought process for each major step of working.



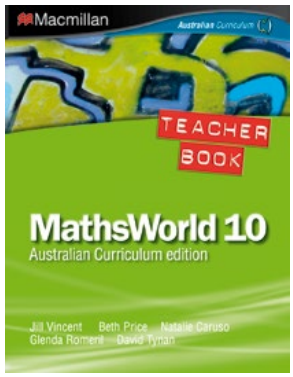
Sets of carefully graded sets of exercise questions occur at the end of each section. Exercise questions are cross-referenced to the relevant worked examples to reinforce learning and provide links where help is needed. Each exercise contains one or two clearly marked challenge questions.

At the end of each chapter, an analysis task explores and ties together concepts covered in the chapter.

Each chapter ends with a thorough review section, consisting of a concise summary, a visual map task and sets of multiple choice, short answer and extended response questions.

A practice quiz is included as a link from the ebook. The quiz can be used for revision at a later stage or could be used for individual student assignments.

Teacher book



The teacher book provides a complete package of supplementary material and support with many time-saving and customisable resources available.

Pre-tests and answers in all chapters allow teachers to assess students' prior knowledge before commencing each chapter. These are also provided as Word and PDF files linked from the student and teacher ebooks.

Chapter warm-ups assist teachers with the introduction of the 'big idea' relevant to each topic. These are also provided as Word and PDF files linked from the student and teacher ebooks.

Teaching notes are provided for each chapter.

Additional worked examples are provided in the TRB—these can be used as teaching examples during the lesson.

Blackline masters and technology files are provided as links in the ebooks, where appropriate.

The teacher book also contains answers to all student exercises and all supplementary activities.

The answers to the analysis tasks are provided in the teacher book. Also provided are additional analysis tasks and answers.

Each chapter includes two tests, in editable word format. Each chapter test includes multiple choice, short answer and extended response questions and is provided with a marking scheme.

Curriculum links and planning documents, including a suggested teaching program, are included in the teacher book.

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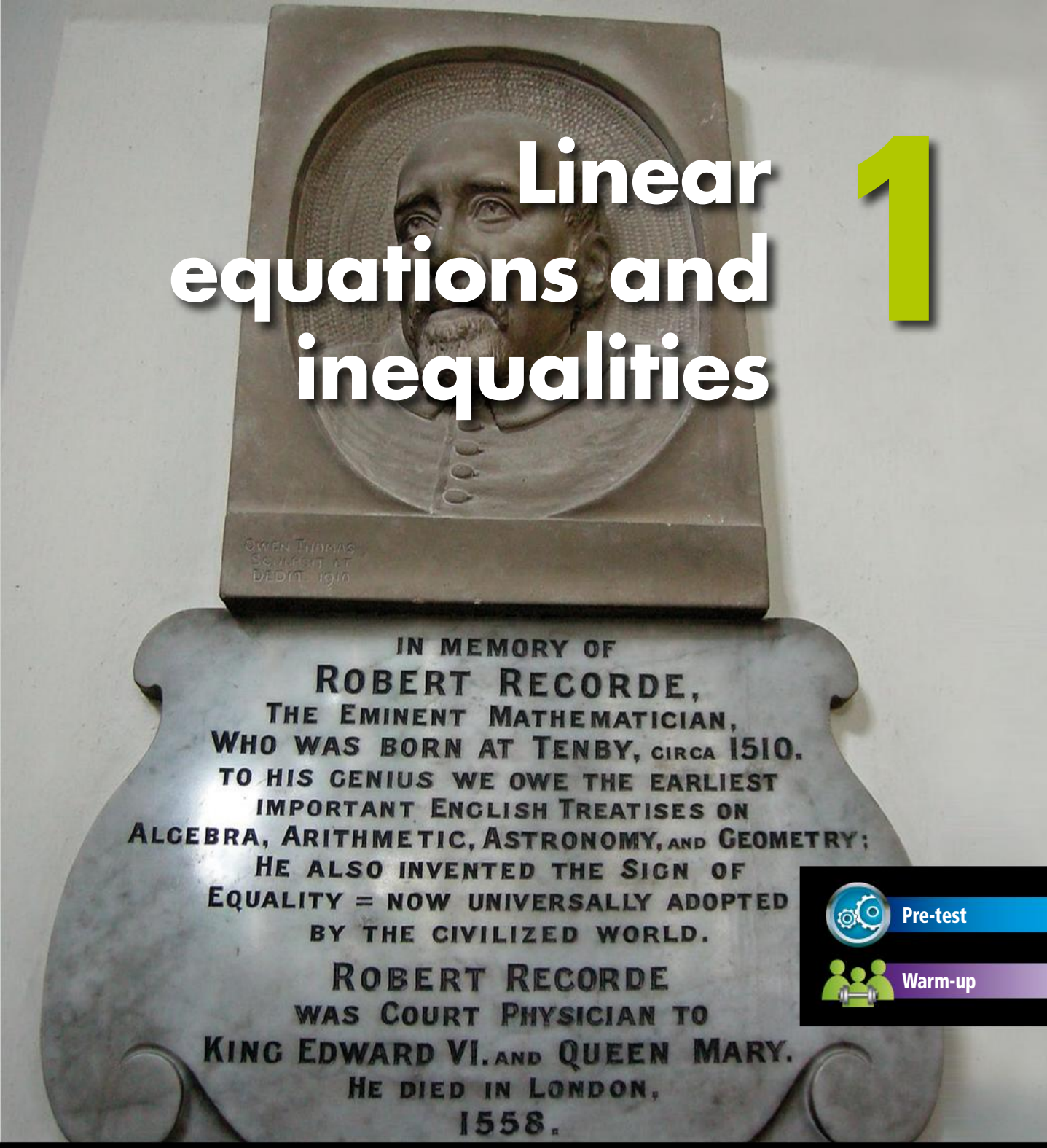
Linear equations and inequalities

1



Pre-test

Warm-up

This chapter reviews the solving of linear equations and problems based on linear equations. Both sides of an equation are equal. We can think of this as the two sides balancing each other. In this chapter, linear inequalities are also solved. In a linear inequality, the quantity on one side is greater than or less than the quantity on the other side, that is, the two sides are not balanced. Instead of a single solution, many values of the unknown will satisfy the linear inequality.

1.1 Reviewing algebraic solving strategies

Simple linear equations can be solved mentally by finding the value for the pronumeral that makes the equation a true statement.

Example 1

Solve the following equations mentally.

a $7x = 56$

b $\frac{y}{12} = 3.5$

Working

a $7x = 56$
$x = 8$

b $\frac{y}{12} = 3.5$
$y = 42$

Reasoning

$56 \div 7 = 8$ so x must be equal to 8.

y is equal to 12 times 3.5.

Algebraic methods apply a logical sequence of operations to both sides of the equation. The idea is to simplify the equation, step by step, until the pronumeral is on its own on one side of the equation. The equation is then solved.

There may be more than one sequence of operations that will successfully solve an equation. However, some sequences will be more efficient than others. As the following examples highlight, the clear recording of the sequence of operations is important.

Example 2

Solve each of the following equations algebraically. Check your solutions.

a $9x + 11 = 47$

b $7 - 2.4x = 13$

Working

a
$$9x + 11 = 47$$
$$9x + 11 - 11 = 47 - 11 \quad *$$
$$9x = 36$$
$$\frac{9x}{9} = \frac{36}{9} \quad *$$
$$x = 4$$

Reasoning

If you are confident with your solving strategies, the two lines marked * may be omitted.

Subtract 11 from both sides.

Divide both sides by 9.

continued

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Example 2 continued

Working

Check:

$$\begin{aligned} \text{LS} &= 9x + 11 \\ &= 9 \times 4 + 11 \\ &= 36 + 11 \\ &= 47 \\ &= \text{RS} \end{aligned}$$

Reasoning

Substituting 4 for x in the LS of the equation gives the same value as the RS.

b

$$\begin{aligned} 7 - 2.4x &= 13 \\ 7 - 7 - 2.4x &= 13 - 7 \\ -2.4x &= 6 \\ \frac{-2.4x}{-2.4} &= \frac{6}{-2.4} \\ x &= -2.5 \end{aligned}$$

Subtract 7 from both sides.

Divide both sides by -2.4.

Check:

$$\begin{aligned} \text{LS} &= 7 - 2.4x \\ &= 7 - 2.4 \times (-2.5) \\ &= 7 + 6 \\ &= 13 \\ &= \text{RS} \end{aligned}$$

Substituting -2.5 for x in the LS of the equation gives the same value as the RS.

Example 3

Solve each of the following equations for x. Check your solutions using substitution.

a $-\frac{2x}{3} = 14$

b $\frac{x}{6} + 13 = -8$

Working

a

$$\begin{aligned} -\frac{2x}{3} &= 14 \\ -\frac{2x}{3} \times 3 &= 14 \times 3 \\ -2x &= 42 \\ \frac{-2x}{-2} &= \frac{42}{-2} \\ x &= -21 \end{aligned}$$

Check:

$$\begin{aligned} \text{LS} &= -\frac{2x}{3} \\ &= -\frac{2 \times (-21)}{3} \\ &= -14 \\ &= \text{RS} \end{aligned}$$

Substituting -21 for x in the LS of the equation gives the same value as the RS.

Reasoning

This could also be solved in a single step by multiplying both sides by $-\frac{3}{2}$.

$$\begin{aligned} -\frac{2x}{3} &= 14 \\ -\frac{2x}{3} \times \left(-\frac{3}{2}\right) &= 14 \times \left(-\frac{3}{2}\right) \\ x &= -21 \end{aligned}$$

continued

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Example 3 continued

Working

b

$$\frac{x}{6} + 13 = -8$$
$$\frac{x}{6} + 13 - 13 = -8 - 13$$
$$\frac{x}{6} = -21$$
$$\frac{x}{6} \times 6 = -21 \times 6$$
$$x = -126$$

$$\text{LS} = \frac{x}{6} + 13$$
$$= \frac{-126}{6} + 13$$
$$= -21 + 13$$
$$= -8$$
$$= \text{RS}$$

Reasoning

The equation could also be solved by first multiplying both sides by 6, but both 13 and −8 must be multiplied by 6.

$$\frac{x}{6} + 13 = -8$$
$$x + 78 = -48$$
$$x = -126$$

Substituting −126 for x in the LS of the equation gives the same value as the RS.

Example 4

Solve the equation $\frac{3x + 3}{2} = 18$ algebraically. Check your solution.

Working

$$\frac{3x + 3}{2} = 18$$
$$\frac{3x + 3}{2} \times 2 = 18 \times 2$$
$$3x + 3 = 36$$
$$3x + 3 - 3 = 36 - 3$$
$$3x = 33$$
$$\frac{3x}{3} = \frac{33}{3}$$
$$x = 11$$

Check:

$$\text{LS} = \frac{3x + 3}{2}$$
$$= \frac{3 \times 11 + 3}{2}$$
$$= \frac{36}{2}$$
$$= 18$$
$$= \text{RS}$$

Reasoning

Doing the same thing to both sides ensures that each of the equations is equivalent to the original equation. First eliminate the denominator, 2, by multiplying both sides by 2.

Eliminate the 3 by subtracting 3 from both sides.

Divide by 3 to obtain x on its own.

Substituting $x = 11$ into the LS of the equation gives the same value as the RS.

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Equations with brackets

Brackets in an equation should be expanded and like terms collected before proceeding with solving the equation.

Example 5

Solve the equation $5(x + 2) - 2(x - 3) = 49$.

Working	Reasoning
$5(x + 2) - 2(x - 3) = 49$	Expand the brackets.
$5x + 10 - 2x + 6 = 49$	Collect like terms.
$3x + 16 = 49$	Subtract 16 from both sides.
$3x = 33$	Divide both sides by 3.
$x = 11$	

Check:

$$\begin{aligned} \text{LS} &= 5(x + 2) - 2(x - 3) \\ &= 5(11 + 2) - 2(11 - 3) \\ &= 5 \times 13 - 2 \times 8 \\ &= 65 - 16 \\ &= 49 \\ &= \text{RS} \end{aligned}$$

Substituting $x = 11$ into the LS of the equation gives the same value as the RS.

Equations with fractions

When an equation contains the addition or subtraction of fractions, multiply both sides by the lowest common denominator.

Example 6

Solve the equation $\frac{x}{4} + \frac{2x}{3} = 33$ for x.

Working	Reasoning
$\frac{x}{4} + \frac{2x}{3} = 33$	
$\frac{x}{\cancel{4}_{1}} \times \cancel{12}^{3} + \frac{2x}{\cancel{3}_{1}} \times \cancel{12}^{4} = 33 \times 12$	Multiply each term by the lowest common denominator, 12.
	Simplify the fractions.
$3x + 8x = 396$	
$11x = 396$	Collect like terms.
$x = 36$	Divide both sides by 11.

continued

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Example 6 continued

Working

Check:

$$\begin{aligned} \text{LS} &= \frac{x}{4} + \frac{2x}{3} \\ &= \frac{36}{4} + \frac{2 \times 36}{3} \\ &= 9 + 24 \\ &= 33 \\ &= \text{RS} \end{aligned}$$

Reasoning

Substituting $x = 36$ into the LS of the equation gives the same value as the RS.

Equations may include expressions rather than single terms in the numerator of fractions. Special care needs to be taken with signs, particularly if there is a negative sign before the fraction.

Example 7

Solve each of the following equations for x.

a $\dfrac{x+2}{3} - \dfrac{x-1}{5} = 1$

b $\dfrac{x}{2} + \dfrac{2x-1}{3} - \dfrac{x+2}{4} = 12$

Working

a

$$\frac{x+2}{3} - \frac{x-1}{5} = 1$$

$$\frac{(x+2)}{\cancel{3}_{1}} \times \cancel{15}^{5} - \frac{(x-1)}{\cancel{5}_{1}} \times \cancel{15}^{3} = 1 \times 15$$

$$\begin{aligned} 5(x+2) - 3(x-1) &= 15 \\ 5x + 10 - 3x + 3 &= 15 \\ 2x + 13 &= 15 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

Reasoning

Multiply each term by the lowest common denominator, 15.
Simplify fractions.
Expand brackets.
Collect like terms.
Subtract 13 from both sides.
Divide both sides by 2.

Check:

$$\begin{aligned} \text{LS} &= \frac{x+2}{3} - \frac{x-1}{5} \\ &= \frac{1+2}{3} - \frac{1-1}{5} \\ &= 1 - 0 \\ &= 1 \\ &= \text{RS} \end{aligned}$$

Substituting $x = 1$ into the LS of the equation gives the same value as the RS.

continued

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Example 7 continued

Working

b

$$\frac{x}{2} + \frac{2x-1}{3} - \frac{x+2}{4} = 12$$

$$\frac{x}{\cancel{2}_{1}} \times \cancel{12}^{6} + \frac{(2x-1)}{\cancel{3}_{1}} \times \cancel{12}^{4} - \frac{(x+2)}{\cancel{4}_{1}} \times \cancel{12}^{3} = 12 \times 12$$

$$6x + 4(2x-1) - 3(x+2) = 144$$

$$6x + 8x - 4 - 3x - 6 = 144$$

$$11x - 10 = 144$$

$$11x = 154$$

$$x = 14$$

Check:

If $x = 14$,

$$\text{LS} = \frac{14}{2} + \frac{2 \times 14 - 1}{3} - \frac{14+2}{4}$$

$$= 7 + \frac{27}{3} - \frac{16}{4}$$

$$= 7 + 9 - 4$$

$$= 12$$

$$= \text{RS}$$

Reasoning

Multiply each term by the lowest common denominator, 12. Simplify fractions.

Expand brackets.
Collect like terms.
Add 10 to both sides.
Divide both sides by 11.

Substituting $x = 14$ into the LS of the equation gives the same value as the RS.

Equations with pronumerals on both sides

To solve equations with pronumerals on both sides of the equals sign, it is first necessary to get all of the pronumerals onto one side of the equation.

Example 8

Solve the equation $2x + 3 = 2 + x$ algebraically. Then check that the solution is correct by substituting this value back into the original equation.

Working

$$2x + 3 = 2 + x$$

$$2x + 3 - 3 = 2 + x - 3$$

$$2x = x - 1$$

$$2x - x = x - 1 - x$$

$$x = -1$$

Check:

$$\text{LS} = 2(-1) + 3$$

$$= -2 + 3$$

$$= 1$$

$$\text{RS} = 2 + (-1)$$

$$= 1$$

Reasoning

Doing the same thing to both sides ensures that each of the new equations we write is equivalent to the original equation.
First the 3 is eliminated by subtracting 3 from both sides.
Subtracting x from each side presents the solution.
Alternatively, x can be subtracted from both sides first, and then 3 can be subtracted from both sides.
Substituting $x = -1$ into the LS and RS of the equation gives the same value.

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Example 9

Solve each of the following equations for x.

a $3(x - 5) = -2x - 3$

b $5(2x - 1) = 4(x + 10)$

Working

a

$$3(x - 5) = -2x - 3$$

Expand the brackets.

$$3x - 15 = -2x - 3$$

$$3x + 2x - 15 = -2x - 3 + 2x$$

Eliminate $-2x$ on the RS by adding $2x$ to both sides.

$$5x - 15 = -3$$

$$5x - 15 + 15 = -3 + 15$$

Eliminate -15 on the LS by adding 15 to both sides.

$$5x = 12$$

$$\frac{5x}{5} = \frac{12}{5}$$

Divide both sides by 5 to obtain x on its own.

$$x = 2\frac{2}{5}$$

Check:

$$\text{LS} = 3\left(2\frac{2}{5} - 5\right)$$

$$= 3 \times \left(\frac{12}{5} - \frac{25}{5}\right)$$

$$= 3\left(-\frac{13}{5}\right)$$

$$= -\frac{39}{5}$$

$$\text{RS} = -2\left(2\frac{2}{5}\right) - 3$$

$$= -\frac{24}{5} - \frac{15}{5}$$

$$= -\frac{39}{5}$$

When the solution is a fraction or decimal it is often easier to thoroughly check the working rather than substituting the solution into each side of the equation to check the solution.

b

$$5(2x - 1) = 4(x + 10)$$

Expand the brackets.

$$10x - 5 = 4x + 40$$

$$10x - 5 - 4x = 4x + 40 - 4x$$

Subtract $4x$ from both sides.
Simplify.

$$6x - 5 = 40$$

$$6x - 5 + 5 = 40 + 5$$

Add 5 to both sides.
Simplify.

$$6x = 45$$

$$\frac{6x}{6} = \frac{45}{6}$$

Divide both sides by 6.
Simplify.

$$x = \frac{15}{2}$$

continued

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Example 9 continued

Working

Check:

$$\text{LS} = 5\left(2 \times 7\frac{1}{2} - 1\right)$$
$$= 70$$
$$\text{RS} = 4\left(7\frac{1}{2} + 10\right)$$
$$= 70$$

Reasoning

Substituting $x = 7\frac{1}{2}$ into the LS and RS of the equation gives the same value.

Example 10

Solve the following equations for x.

a $5(2x + 5) = 3(4x - 1) + 13$ **b** $\dfrac{x-3}{2} = \dfrac{x+6}{3}$

Working

a

$$5(2x + 5) = 3(4x - 1) + 13$$
$$10x + 25 = 12x - 3 + 13$$
$$10x + 25 = 12x + 10$$
$$10x + 25 - 25 = 12x + 10 - 25$$
$$10x = 12x - 15$$
$$10x - 12x = 12x - 15 - 12x$$
$$-2x = -15$$
$$\frac{-2x}{-2} = \frac{-15}{-2}$$
$$x = 7\frac{1}{2}$$

Reasoning

Expand the brackets. Collect like terms.

Eliminate 25 by subtracting it from both sides.

Subtracting $12x$ from each side collects the terms involving x.

Dividing both sides by -2 presents the solution.

b

$$\frac{x-3}{2} = \frac{x+6}{3}$$
$$\frac{x-3}{\cancel{2}_{1}} \times \cancel{6}^{3} = \frac{x+6}{\cancel{3}_{1}} \times \cancel{6}^{2}$$
$$3(x - 3) = 2(x + 6)$$
$$3x - 9 = 2x + 12$$
$$3x - 9 - 2x = 2x + 12 - 2x$$
$$x - 9 = 12$$
$$x - 9 + 9 = 12 + 9$$
$$x = 21$$

Multiply each term by the lowest common denominator, 6. Simplify the fractions.

Expand brackets.

Subtract $2x$ from both sides.
Simplify.
Add 9 to both sides.

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Equations arising from substitution into formulas

When values are substituted into a formula, it may be necessary to solve an equation to calculate the required value.

Example 11

If $S = 2\pi r(r = h)$, find h if $S = 2200$ and $r = 11$. Answer correct to the nearest whole number.

Working	Reasoning
$S = 2\pi r(r + h)$	Write formula.
$2200 = 2\pi \times 11(11 + h)$	
$2200 = 22\pi(11 + h)$	Substitute in $S = 2200$ and $r = 11$.
$\frac{2200}{22\pi} = 11 + h$	
$h = \frac{2200}{22\pi} - 11$	Solve for h.
$h = 52.66$	
$h = 53$	Round to nearest whole number.

exercise 1.1

LINKS TO Example 1

1 Solve these equations mentally, and then check each answer by substitution.

a $x - 5 = 9$ **b** $\frac{x}{4} = 8$ **c** $11x = 77$

d $x + 11 = 9$ **e** $-3x = 27$ **f** $-\frac{x}{6} = 7$

g $8x = 56$ **h** $x + 15 = 7$ **i** $2x - 3 = 9$

j $4(x + 1) = 28$ **k** $-\frac{x}{5} = -60$ **l** $\frac{x}{8} = -2$

2 Which of the following is **not true** when $x = 3$?

A $5x + 2 = 17$ **B** $\frac{2x + 3}{3} = 3$ **C** $\frac{4x}{3} + 3 = 6$

D $\frac{5x - 3}{3} = 4$ **E** $\frac{9x}{3} - 3 = 6$

LINKS TO Example 2

3 Solve each of the following equations for x.

a $2x + 3 = 17$ **b** $5x - 4 = 11$ **c** $7x + 2 = 37$

d $4x + 7 = 43$ **e** $-4x - 5 = 13$ **f** $-5x - 13 = 8$

g $-2x + 4 = 10$ **h** $-3x - 5 = 22$ **i** $7 - 3x = 31$

j $8 - x = 17$ **k** $11 - 9x = 43$ **l** $3.8 - x = 5.2$

m $7x - 15 = 20$ **n** $6 - 3x = 12$ **o** $20 - 11x = 42$

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4 Thuynh solves the equation $4x + 8 = 11$ as shown at the right.

a Is the solution correct?

b Suggest an alternative way of solving the equation.

$$4x + 8 = 11$$
$$\frac{4x + 8}{4} = \frac{11}{4}$$
$$x + 2 = 2\frac{3}{4}$$
$$x = \frac{3}{4}$$

5 The solution to the equation $\frac{y}{2} + 8 = 24$ is

A $y = 12$ **B** $y = 16$ **C** $y = 20$ **D** $y = 32$ **E** $y = 40$

LINKS TO Example 3

6 Solve each of the following equations by applying the same operation to both sides.

a $\frac{4a}{7} = 32$ **b** $\frac{7x}{5} = 21$ **c** $\frac{-3x}{4} = 15$

d $\frac{-5x}{3} = 4$ **e** $\frac{9a}{11} = 18$ **f** $\frac{m}{5} - 1 = 3$

g $\frac{x}{3} - 5 = 2$ **h** $\frac{x}{6} + 6 = 17$ **i** $\frac{x}{4} - 11 = 5$

j $5 + \frac{3x}{4} = -7$ **k** $17 - \frac{6x}{7} = 14$ **l** $4 - \frac{5x}{2} = 11$

m $\frac{5a}{7} + 3 = 10$ **n** $\frac{4a}{3} + 11 = 5$ **o** $9 - \frac{x}{8} = 4$

p $7 - \frac{x}{3} = 9$ **q** $13 - \frac{x}{3} = 8$ **r** $23 - \frac{4x}{5} = 17$

LINKS TO Example 4

7 Solve each of the following equations.

a $\frac{a - 4}{2} = -5$ **b** $\frac{4x + 3}{3} = 2$ **c** $\frac{2x + 3}{5} = 3$

d $\frac{5 - 2x}{7} = 12$ **e** $\frac{2x - 3}{4} = 1$ **f** $\frac{3 + 5x}{7} = 9$

g $\frac{4 - 6x}{5} = -7$ **h** $\frac{3x - 5}{7} = 4$ **i** $\frac{7 - 2x}{3} = -13$

j $\frac{4x + 5}{3} = -1$ **k** $\frac{34 - 3x}{7} = 4$ **l** $\frac{68 - 7x}{11} = 3$

LINKS TO Example 5

8 Solve each of the following equations.

a $6(x - 3) = 12$ **b** $7(x + 4) = 42$ **c** $2(x + 3) = 7$

d $5(x - 5) + 6 = 11$ **e** $3(2 - 3x) = -4$ **f** $5(x + 3) - 9x = 11$

g $2(6x - 3) = 15$ **h** $6(a - 3) = 7$ **i** $5(4 - x) = 8$

j $9(x - 3) - 4x = 2$ **k** $4(2x - 3) - 9x = 14$ **l** $12x - 7(x - 8) = -35$

9 For which of the following equations is $x = -2$ a solution? (Try to answer this question mentally.)

a $-x - 2 = 2$ **b** $-2 = -x - 4$ **c** $3x + 2 = 4$

d $2(2 + x) = 0$ **e** $-\frac{x}{2} = 1$ **f** $\frac{2 - x}{4} = 1$

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LINKS TO Example 6

10 Solve each of the following equations.

a $\frac{x}{5} + \frac{x}{2} = 21$ **b** $\frac{x}{2} + \frac{x}{3} = 1$ **c** $\frac{3x}{4} + \frac{4x}{5} = 62$

d $\frac{4x}{3} - \frac{3x}{5} = 33$ **e** $\frac{2x}{3} - \frac{x}{6} = \frac{1}{4}$ **f** $\frac{x}{7} + \frac{2x}{5} = \frac{57}{2}$

g $\frac{7x}{12} - \frac{x}{6} = \frac{1}{12}$ **h** $\frac{2x}{3} + \frac{5x}{4} = 46$ **i** $\frac{3x}{7} + \frac{2x}{5} = 87$

j $\frac{7x}{8} - \frac{3x}{5} = 22$ **k** $\frac{8x}{3} - \frac{2x}{5} = 17$ **l** $\frac{3x}{8} - \frac{2x}{3} = -35$

LINKS TO Example 7

11 Solve each of the following equations.

a $\frac{a+2}{5} + \frac{a-1}{2} = 2$ **b** $\frac{x+3}{3} + \frac{x-1}{4} = 6$ **c** $\frac{x-4}{3} + \frac{3x+1}{2} = 12$

d $\frac{k-3}{7} + \frac{k+5}{3} = 6$ **e** $\frac{x-7}{3} + \frac{x+2}{5} = 5$ **f** $\frac{2x+1}{7} - \frac{3x+1}{5} = -1$

LINKS TO Example 8

12 Solve these linear equations.

a $3a - 2 = a + 6$ **b** $5 - 8e = 2 - 3e$

c $2d + 5 = 7d$ **d** $7m - 2m = 0$

e $5x - 2 = 3x - 3$ **f** $2m + 5 = 7m + 6$

g $16k + 8 = 8k - 12$ **h** $12 - 9x = 8 - 12x$

i $4 - 12x = 6 - 7x$ **j** $7 - 8a = 4 - 3a$

k $3y - 2 = y$ **l** $4 - 8j = j$

m $7 - 3e = 2e + 3$ **n** $7a - 21 = 75 - 5a$

o $-12 - 2b = 4b + 2$ **p** $a - 7 = -5a - 2$

13 Find m if $2 - m = 3m + 8$.

A $m = 3$ **B** $m = \frac{5}{2}$ **C** $m = \frac{2}{3}$ **D** $m = -\frac{3}{2}$ **E** $m = -3$

LINKS TO Example 9

14 Solve for x in each of the following.

a $4(x - 3) = -2(3x + 7)$ **b** $-2(2 - a) = -3(a + 4)$

c $4.3x + 7.1 = 3(0.8x + 3)$ **d** $4(2x - 3) = 9x + 14$

e $6(3 - 2x) = 5(4 - 3x)$ **f** $7(m + 2) = 5(4 + m)$

g $2(x - 10) = 4x$ **h** $21 + 3x = 3(3x - 47)$

i $2(x + 1.3) = x - 3.2$ **j** $5(3x - 4) = 4(2x - 3)$

k $-7(x - 3) = 5(x - 15)$ **l** $14(x + 2) = 4(7 - 2x)$

m $8(x + 2) = 2(2x - 5)$ **n** $5(2x - 1.3) = 3(3x - 4.2)$

LINKS TO Example 10a

15 Solve each of these equations.

a $3(2x - 1) = 4(3x + 2) - 5$ **b** $7(2 - 3m) - 5m = 2(3m - 4)$

c $9p - 4(p - 5) = 5(2p + 8)$ **d** $7 - 3(2x + 5) = -(4x - 1)$

LINKS TO Example 10b

16 Solve each of the following equations.

a $\frac{7x - 1}{5} = \frac{x + 5}{3}$ **b** $\frac{2 - x}{5} = \frac{3 + x}{2}$

c $\frac{x + 2}{3} = \frac{x + 4}{4}$ **d** $\frac{a + 5}{3} = \frac{a - 7}{2}$

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e $\dfrac{3-x}{2} = \dfrac{5+2x}{4}$

f $\dfrac{5-4m}{6} = \dfrac{3m-2}{2}$

g $\dfrac{p-3}{5} = \dfrac{p+7}{3}$

h $\dfrac{x+2}{5} = \dfrac{x-4}{2}$

i $\dfrac{2y+7}{2} = 3y-5$

j $\dfrac{x-2}{3} = \dfrac{x+2}{5}$

k $\dfrac{3m+1}{5} = \dfrac{2m+3}{4}$

l $\dfrac{8-p}{2} = p+1$

m $\dfrac{2b-7}{5} = \dfrac{b-3}{4}$

n $\dfrac{2-k}{3} = \dfrac{3k-14}{5}$

LINKS TO Example 11

17 If $P = 2(l+h)$, find h if $P = 17$ and $l = 4.5$.

18 If $M = \dfrac{a+b}{2}$, find b if $M = 31.5$ and $a = 16$.

19 If $F = \dfrac{9c}{5} + 32$, find C if $F = 70$.

20 If $I = \dfrac{PRT}{100}$, find R if $I = 320$, $P = 1000$ and $T = \dfrac{2}{3}$.

21 If $V = \dfrac{\pi r^2 h}{3}$, find r if $V = 100$ and $h = 7$. Answer correct to one decimal place.

22 Shana is asked to solve the linear equation $3(x-3) = 4x$, and her attempt is shown at the right.

a Is Shana's solution correct?

b Try to solve the problem in fewer steps.

c How would you advise Shana to solve another equation presented this way?

$$3(x-3) = 4x$$
$$x - 3 = \frac{4x}{3}$$
$$x = \frac{4x}{3} + 3$$
$$x = \frac{4x}{3} + \frac{9}{3}$$
$$x = \frac{4x+9}{3}$$
$$3x = 4x + 9$$
$$x = -9$$

23 Solve the following equations for x:

a $\dfrac{x+2}{3} + \dfrac{x-2}{4} = x+2$

b $\dfrac{2x-1}{2} - \dfrac{3-x}{3} = \dfrac{4x}{5}$

24 Explain why the equation $x - 3 = 1 + x$ has no solutions.

25 Try to solve the equation $\dfrac{6x+2}{2} = 3(x-1)$. What happens and what does this tell you about the equation?

exercise 1.1 challenge

26 Solve the equation $\dfrac{3}{x+1} = \dfrac{2}{2x-3}$ for x.

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1.2 Problem-solving with linear equations

Many problems can be solved by using algebraic techniques. The process of translating the words of a problem into algebra is called **formulation**.

As a starting point, it is important to read the question carefully and identify what you are being asked to find. Choose a pronumeral to represent the unknown, then construct an equation that can be solved.

The four-step approach to solving algebraic word problems

Step 1: Translate the words into algebra. Decide on the unknown variable and give it a letter, then formulate an equation that involves this variable.

Step 2: Solve the equation. Solve for the variable by 'doing the same to both sides'.

Step 3: Check the solution. Substitute your solution back into the original equation to check that the LS = RS.

Step 4: Translate the algebra back to words. Express your solution in terms of the original problem wording.

Example 12

There are 24 company cars in a parking lot. The cars are either black or white. There are 4 more black cars than white cars. How many white cars are in the parking lot?

Working	Reasoning
Let w be the number of white cars in the parking lot.	Select a pronumeral to represent the unknown.
There must be $w + 4$ black cars.	There are 4 more black cars than white cars.
$w + w + 4 = 24$	The total number of cars in the parking lot is 24.
$2w + 4 = 24$	Solve for w.
$2w = 20$	Subtract 4 from both sides.
$w = 10$	Divide both sides by 2.

continued

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Example 12 continued

Working	Reasoning
There are 10 white cars in the parking lot.	Express the answer in terms of the original problem.
Check: If $w = 10$ LS $= 10 + 10 + 4$ $= 24$ $=$ RS	Check the solution.

Example 13

One-third of a class studies Japanese and one-fifth of the same class studies Indonesian. This gives a total of 16 students. If no student in the class studies both languages, how many students are in the class?

Working	Reasoning
Let x be the number of students in the class.	Select a pronumeral to represent the unknown.
$\frac{x}{3} + \frac{x}{5} = 16$	One-third of the class is $\frac{x}{3}$.
$\frac{x}{\cancel{3}_1} \times \cancel{15}^5 + \frac{x}{\cancel{5}_1} \times \cancel{15}^3 = 16 \times 15$	One-fifth of the class is $\frac{x}{5}$. Multiply both sides by the lowest common denominator, 15. Simplify the fractions.
$5x + 3x = 240$	Collect like terms.
$8x = 240$	Divide both sides by 8.
$x = 30$	
There are 30 students in the class.	
Check: LS $= \frac{30}{3} + \frac{30}{5}$ $= 10 + 6$ $= 16$ $=$ RS	Check the solution.

Problems involving consecutive integers

Consecutive integers are integers that come one after the other. For example, 3 and 4 are consecutive integers. 7, 8 and 9 are also consecutive integers. Note that each integer is one more than the integer before it. If we let n stand for an integer, then the next integer will be $n + 1$, the next integer will be $n + 2$, and so on.

Consecutive even integers or consecutive odd integers are two apart. For example, 6, 8, 10 are consecutive even integers and 11, 13, 15 consecutive odd integers. If we let n stand for an

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even integer, then the next even integer will be $n + 2$, the next even integer will be $n + 4$, and so on, and similarly for odd integers.

Example 14

The sum of three consecutive even integers is 216. Find the three integers.

Working	Reasoning
Let n be the smallest of the even integers. The other two even integers are $n + 2$ and $n + 4$.	If n is the smallest of the three even integers, the next even integer will be 2 more than n and the next will be 4 more than n.
$n + (n + 2) + (n + 4) = 216$	Write an equation to show the sum of the three even integers.
$3n + 6 = 216$	
$3n = 210$	Solve for n.
$n = 70$	
The three even integers are 70, 72 and 74.	Write a sentence to answer the question.
Check: $70 + 72 + 74 = 216$.	Check the solution.

Finding unknown dimensions

Perimeters of rectangles and other shapes can be used to formulate equations and solve problems involving unknown dimensions.

Example 15

The length of a rectangle is 2.8 m longer than the width. The perimeter is 14.8 m. Find the width and length of the rectangle.

Working	Reasoning
Let x m be the width of the rectangle.	Define a variable for the width.
Length $= (x + 2.8)$ m	Express the length in terms of the width.
Perimeter $= 2(l + w)$	Write an equation using the perimeter.
$14.8 = 2(x + 2.8 + x)$	Solve the equation.
$2(2x + 2.8) = 14.8$	Divide both sides by 2.
$2x + 2.8 = 7.4$	Subtract 2.8 from both sides.
$2x = 4.6$	Divide by 2.
$x = 2.3$	Write a sentence answering the question.
If width is 2.3 m, length is $2.3 + 2.8 = 5.1$ m	In word problems we should check that we have solved the equation correctly but we should also check that the final answers to the question fit the given information.
The width of the rectangle is 2.3 m and the length is 5.1 m.	
Check:	Check the solution.
Perimeter $= 2(2.3 + 5.1)$	
$= 2 \times 7.4$	
$= 14.8$	

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Calculating ages

In problems involving people's ages, the problem may refer, for example, to the ages of two people in four years' time. It is important to remember that *both* people will be four years older in four years time.

Example 16

Lachlan is five years older than his brother Alexander and his brother Robbie is four years younger than Alexander. In three years' time Lachlan's age will be equal to the sum of the ages of Alexander and Robbie. Find the ages of the three brothers now.

Working	Reasoning
Let x years be the age of Alexander now. Lachlan is now $x + 5$ and Robbie is now $x - 4$. In three years' time, Alexander will be $x + 3$.	The ages of Lachlan and Robbie are given in terms of the age of Alexander, so define a variable for Alexander's age. Express Lachlan's and Robbie's ages in terms of x.
Lachlan will be: $(x + 5) + 3 = x + 8$ Robbie will be: $(x - 4) + 3 = x - 1$ So $x + 8 = (x + 3) + (x - 1)$ $x + 8 = 2x + 2$ $x = 6$	Write an equation to show that Lachlan's age will be equal to the sum of the ages of Alexander and Robbie in three years time.
Alexander is 6, Lachlan is 11 and Robbie is 2.	Write a sentence showing the three ages.
Check: In 3 years' time, Angus will be 9, Lachlan will be 14 and Robbie will be 5. $5 + 9 = 14$	Check the solution.

exercise 1.2

1 For each of the following, formulate an equation and then solve for the unknown.

- **a** 9 is subtracted from a to give a result of 12.
- **b** 5 is added two twice b to give a result of 19.
- **c** The difference between three times c and 7 is 20.
- **d** 6 more than the quotient of d and 2 is 10.
- **e** The sum of 4 times e and 8 is 40.
- **f** The product of 5 and three more than f is 25.
- **g** The product of 2 and the sum of g and 8 is -6.
- **h** The product of 3 and the difference between h and 4 is 12.

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LINKS TO Example 12

2 Formulate an equation that represents the information given.

a Natasha had n pot plants on the window sill. She added 3 new ones, but then half of her plants died, so she added 2 more. She now has 7 plants.

b How many pot plants did Natasha have to start with?

3 There are 51 lollies in a lolly jar. The lollies are either red or yellow. There are 5 more red lollies than yellow lollies. How many yellow lollies are in the jar?

4 A class sat a test on equations. The highest grade was 40 marks higher than the lowest grade. The sum of these two grades was 136. What was the lowest grade?

5 Marisa has a piece of string that is 40 centimetres long. She cuts it into two pieces so that one piece is 6 centimetres longer than the other. How long is each piece of string?

LINKS TO Example 13

6 One-fifth of the students in a class caught the bus to school and two-fifths of the students in the same class caught the train to school. A total of 18 students caught either the bus or the train. If no student in the class used both methods of transport, how many students are in the class?

7 One-quarter of the cars in a car park are silver and one-fifth are white. If there are 45 silver and white cars in the car park, how many cars are in the car park altogether?

8 Carla received her pay for the week and spent $\frac{1}{3}$ on rent and $\frac{1}{4}$ paying off bills. She had \$280 left. How much was she paid?

LINKS TO Example 14

9 The sum of two consecutive whole numbers is 43. Find the numbers.

10 The sum of two consecutive even numbers is 70. Find the numbers.

11 The sum of three consecutive odd numbers is 255. Find the numbers.

LINKS TO Example 15

12 The length of a rectangular field is 15 metres more than its width. If the perimeter of the field is 70 metres, find its width.

13 The length of a rectangular room is 12 metres more than twice its width. If the perimeter of the room is 144 metres, find the length and width of the room.

LINKS TO Example 16

14 Let j be Justine's age now.

a Write an expression for her age in 12 years' time.

b In 12 years, Justine will be three times as old as she was 28 years ago. Write an equation that represents this information.

c Solve this equation to find Justine's age now.

d In how many years will Justine turn 70?

15 The sum of the ages of Anna and Bella is 32. In two years, Bella will be three times as old as Anna. How old are they now?

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exercise 1.2 challenge

16 The Cheap Cars company charges \$50 per day and \$2.40 per kilometre to hire a car. The Best Hire company charges \$40 per day plus \$2.70 per kilometre. Will wishes to rent a car for 3 days. How far can he travel so that the cost would be the same from either company?

17 How many litres of solution containing 25% salt must be added to 5 litres of 35% salt solution to make a mixture containing 30% salt?

18 For thousands of years, people have enjoyed creating and solving number puzzles. Diophantus was a Greek mathematician who was born about 250 CE (just over 1750 years ago). A story similar to the one below appeared in a Greek collection of puzzles about 500 CE:

Diophantus spent $\frac{1}{6}$ of his life as a child, $\frac{1}{12}$ as a young man, then another $\frac{1}{7}$ of his life as a single man before he married. Five years after he married, he left his hometown and spent $\frac{1}{2}$ his life in another town. He returned to his hometown 4 years before he died. How old was Diophantus when he died?

a Using x to represent Diophantus' age in years when he died, write an expression in terms of x for the sum of the stages of Diophantus' life.

b Equate your expression to x (why?) and solve the equation to find x.

c How old was Diophantus when he died?

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1.3 Transposing formulas

The pronumeral on the left side of a formula is called the **subject** of the formula. In the formula $C = 2\pi r$ for calculating the circumference C of a circle of radius r, the subject of the formula is C. If we know the circumference and wish to calculate r, we can change the formula so that r is the subject by expressing it as $r = \frac{C}{2\pi}$.

We call this process **transposing** the formula to make r the subject.

The steps taken to transpose a formula are the same as those used to solve an equation.

We perform the same operations on both sides of the formula.

Example 17 shows the transposing of three well-known formulas. You will be familiar with the first of these, the formula for the perimeter of a rectangle. The second formula shows Newton's famous formula for calculating the gravitational force F between two masses m and M, where r is the distance between their centres and G is a constant. The third formula is for the total surface area S of a cylinder with radius r and height h.

Example 17

Transpose each of the following to make the pronumeral in brackets the subject.

a $P = 2(l + w)$ (l) **b** $F = \frac{GmM}{r^2}$ (r) **c** $S = 2\pi r(r + h)$ (h)

Working	Reasoning
a $P = 2(l + w)$	Making l the subject is like solving the equation to find l.
$\frac{P}{2} = l + w$	Divide both sides by 2.
$\frac{P}{2} - w = l$	Subtract w from both sides.
$l = \frac{P}{2} - w$	Make l the subject.
b $F = \frac{GmM}{r^2}$	Making r the subject is like solving the equation to find r.
$Fr^2 = GmM$	Multiply both sides by r^2, so that r is no longer in the denominator.
$r^2 = \frac{GmM}{F}$	Divide both sides by F.
$r = \sqrt{\frac{GmM}{F}}$	Take the square root of both sides. In this case, we take only the positive square root because r represents a distance and cannot be negative.

continued

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Example 17 continued

Working	Reasoning
c $S = 2\pi r(r + h)$	Making h the subject is like solving the equation to find h.
$\dfrac{S}{2\pi r} = r + h$	Divide both sides by $2\pi r$.
$\dfrac{S}{2\pi r} - r = h$	Subtract r from both sides.
$h = \dfrac{S}{2\pi r} - r$	Make h the subject.

exercise 1.3

LINKS TO Example 17

1 Transpose each of the following formulas, making the pronumeral in brackets the subject.

a $a = x + b \quad (b)$
b $F = ma \quad (a)$
c $D = ST \quad (T)$
d $I = PRT \quad (T)$
e $C = 2\pi r \quad (r)$
f $E = mc^2 \quad (m)$
g $V = \dfrac{lwh}{3} \quad (w)$
h $M = \dfrac{a + b}{2} \quad (b)$
i $P = 2(l + h) \quad (h)$
j $A = \pi r^2 \quad (r)$
k $v = u + at \quad (t)$
l $P = \dfrac{k}{V} \quad (V)$
m $A = \dfrac{(a + b)h}{2} \quad (b)$
n $S = b(2b + h) \quad (h)$
o $F = \dfrac{9C}{5} + 32 \quad (C)$
p $s = ut + \dfrac{1}{2}at^2 \quad (a)$
q $V = \dfrac{\pi r^2 h}{3} \quad (h)$
r $V = \dfrac{\pi r^2 h}{3} \quad (r)$

exercise 1.3 challenge

2 Solve the following equations for x in terms of the given pronumerals.

a $a + b + x = c$
b $\dfrac{ax}{b} = c$
c $\dfrac{ax + b}{c} = d$
d $\dfrac{a(bx + c)}{d} = e$
e $a(x + c) = bx$
f $\dfrac{ax}{b} + \dfrac{cx}{d} = e$

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1.4 Solving linear inequalities

Inequality signs

Two expressions can be linked by an inequality sign rather than an equality sign.

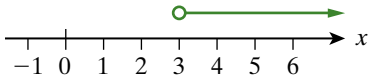
$>$ greater than
$<$ less than
$\geq$ greater than or equal to
$\leq$ less than or equal to

Such statements are called **inequalities**. For example, $2x + 3 > 5$ is an inequality.

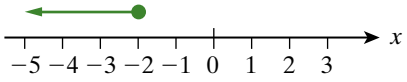
When we solve an inequality, the solution is a set of numbers rather than a single value of the variable that makes the expression true. We can represent these sets using number lines.

Small circles are used to indicate a value. A closed (filled) circle is used to indicate $\leq$ or $\geq$. For $<$ and $>$, an open circle is used.

If $x > 3$, we place a small open circle at 3 on the number line, with an arrow to the right to indicate that the required values of x are numbers greater than 3.



If $x \leq -2$, we place a closed circle at -2 on the number line, with an arrow to the left to show that the required values of x are numbers less than or equal to -2.



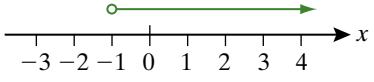
Example 18

Represent each of the following inequalities on a number line.

a $x > -1$ **b** $x \leq 4$

Working

a



Reasoning

The values of x that are greater than -1 lie to the right of -1 on the number line. Since -1 is not included we use an open circle at $x = -1$.

continued

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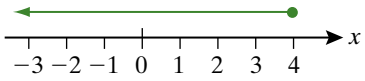
Example 18 continued

Working

b



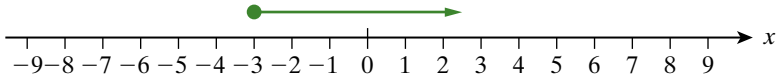
Reasoning

The values of x that are less than 4 lie to the left of 4 on the number line. Since 4 is included in the interval, we use a closed circle at $x = 4$.

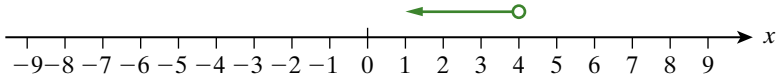
Example 19

Write an inequality to represent each of these sets of numbers.

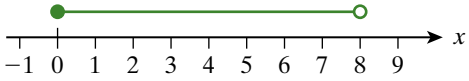
a



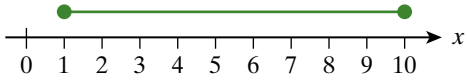
b



c



d



Working	Reasoning
a $x \geq -3$	The value $x = -3$ is marked with a closed circle so $x = -3$ is included in the set of values.
b $x < 4$	The value $x = 4$ is marked with an open circle so $x = 4$ is not included in the set of values.
c x is greater than or equal to 0 but less than 8 $0 \leq x < 8$	The value $x = 0$ is marked with a closed circle so $x = 0$ is included in the set of values. The value $x = 8$ is marked with an open circle so $x = 8$ is not included in the set of values. All numbers between 0 and 8 are also included. $0 \leq x$ is read backwards as $x \geq 0$.
d x is between 1 and 10 inclusive, that is, x is greater than or equal to 1 but less than or equal to 10. $1 \leq x \leq 10$	$x = 1$ and $x = 10$ are both marked with a closed circle, so $x = 1$ and $x = 10$ are both included in the set of values. All numbers between 1 and 10 are also included. $1 \leq x$ is read backwards as $x \geq 1$.

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Inequalities can be solved by the same algebraic methods used for equations. However, you need to take care when multiplying or dividing both sides of an inequation by a negative number. Consider the following example, which shows the consequence of dividing a 'true' inequality by a negative number.

Without sign reversal	With sign reversal
$10 > 4$	$10 > 4$
$\frac{10}{-2} > \frac{4}{-2}$ (Dividing both sides by -2) $-5 > -2$ This is a false statement.	$\frac{10}{-2} < \frac{4}{-2}$ (Dividing both sides by -2) $-5 < -2$ This is a true statement.

Multiplying or dividing by a negative number

When multiplying or dividing both sides of an inequation by a negative number, the direction of the inequality symbol must be reversed for the statement to remain true.

Example 20

Solve the following inequalities.

a $4 + 2x \geq 3$

b $-\frac{7x}{9} + 3 < 10$

In each case, represent the solution on a number line.

Working

a

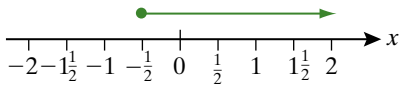
$$4 + 2x \geq 3$$

$$4 + 2x - 4 \geq 3 - 4$$

$$2x \geq -1$$

$$\frac{2x}{2} \geq \frac{-1}{2}$$

$$x \geq -\frac{1}{2}$$



Reasoning

Subtracting 4 from both sides of the equation isolates the term involving x.

Dividing both sides by 2 presents the solution.

All numbers greater than or equal to $-\frac{1}{2}$ satisfy this inequality. On a number line this set is starts at $-\frac{1}{2}$ and continues to the right of this value.

continued

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Example 20 continued

Working

b

$$-\frac{7x}{9} + 3 < 10$$

$$-\frac{7x}{9} + 3 - 3 < 10 - 3$$

$$\frac{-7x}{9} < 7$$

$$-\frac{7x}{9} \times 9 < 7 \times 9$$

$$-7x < 63$$

$$\frac{-7x}{7} > \frac{-63}{-7}$$

$$x > -9$$

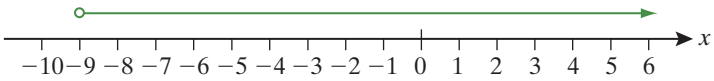
Reasoning

First eliminate the 3 by subtracting 3 from both sides.

To eliminate the fraction, multiply both sides by 9.

To isolate x, divide by -7.

As we are dividing by a negative number, the inequality symbol $<$ is reversed to give $>$.



All numbers greater than -9 satisfy this inequality. On a number line, this set starts at -9 and continues to the right of this value. The open circle at $x = -9$ indicates that this value is not included.

Example 21

Solve the following inequalities. In each case, represent your solution on a number line.

a $4x + 10 < 2x + 6$

b $\frac{k+1}{6} \leq \frac{6-k}{15}$

Working

a

$$4x + 10 < 2x + 6$$

$$4x + 10 - 2x < 2x + 6 - 2x$$

$$2x + 10 < 6$$

$$2x + 10 - 10 < 6 - 10$$

$$2x < -4$$

$$\frac{2x}{2} < \frac{-4}{2}$$

$$x < -2$$

Reasoning

Get all terms containing the unknown on one side of the inequality. Subtract $2x$ from both sides.
Simplify.
Subtract 10 from both sides.
Simplify.
Divide both sides by 2.
Simplify.

continued

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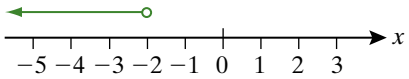
Example 21 continued

Working



Reasoning

All numbers less than -2 satisfy this inequality. On a number line this set is all values to the left of -2. The open circle at $x = -2$ indicates that this value is not included.

b

$$\frac{k+1}{6} \leq \frac{6-k}{15}$$

Multiply both sides by the lowest common denominator, which is 30.

$$\frac{k+1}{\cancel{6}_{1}} \times \cancel{30}^{5} \leq \frac{6-k}{\cancel{15}_{1}} \times \cancel{30}^{2}$$

Simplify the fractions.

$$5(k+1) \leq 2(6-k)$$

Expand brackets.

$$5k + 5 \leq 12 - 2k$$

$$5k + 5 + 2k \leq 12 - 2k + 2k$$

Add $2k$ to both sides.

$$7k + 5 \leq 12$$

Simplify.

$$7k + 5 - 5 \leq 12 - 5$$

Subtract 5 from both sides.

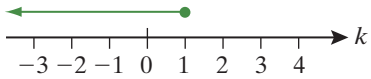
$$7k \leq 7$$

Simplify.

$$\frac{7k}{7} \leq \frac{7}{7}$$

Divide both sides by 7.

$$k \leq 1$$



All numbers less than or equal to 1 satisfy this inequality. On a number line, this set is all values to the left of 1 and 1 itself. The closed circle at $k = 1$ indicates that this value is included in the solution.

Word problems involving inequalities are solved using the same strategies as for linear equations.

Example 22

Jane's mobile phone plan has a monthly access fee of \$25 and then a flat rate of 30 cents per minute for calls. If Jane wants to keep her mobile phone bill under \$70 for the month, for how many minutes can she talk?

Working

Let m be the number of minutes of talk that Jane uses.

$$25 + 0.3m < 70$$

$$25 - 25 + 0.3m < 70 - 25$$

$$0.3m < 45$$

$$\frac{0.3m}{0.3} < \frac{45}{0.3}$$

$$m < 150$$

Jane can talk for less than 150 minutes.

Reasoning

Formulate the inequality.
Jane's phone bill is made up of the access fee of \$25 plus \$0.3 for each of the m minutes that Jane talks.
This total needs to be less than \$70.
Subtract 25 from both sides.
Divide both sides by 0.3
Write a sentence to answer the question.

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exercise 1.4

LINKS TO Example 18

1 Represent each of the following inequalities on a separate number line.

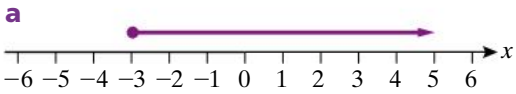
a $x > 2$ **b** $x \leq 3$ **c** $x \geq 1$ **d** $x < 1\frac{1}{2}$

e $x \geq 0$ **f** $x < -2$ **g** $x > -0.5$ **h** $x \leq -\frac{1}{2}$

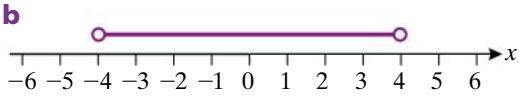
LINKS TO Example 19

2 Write an inequality to represent each of these sets of numbers.

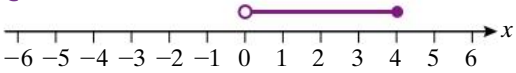
a



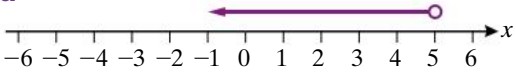
b



c



d



LINKS TO Example 20

3 Solve the following inequalities. In each case, represent the solution using a number line.

a $2x - 3 \geq 0$ **b** $3x < 2(x - 2)$ **c** $3x + 4 \geq 13$

d $\frac{5x}{4} - 3 \geq 0$ **e** $\frac{2x}{3} + 5 < 7$ **f** $-\frac{3x}{2} < 6$

g $5 - x > 6$ **h** $9 - 2x \leq 7$ **i** $2(3 - x) < 0$

j $-2(-x - 1) \leq -1$ **k** $\frac{2 - x}{3} \leq \frac{4}{3}$ **l** $\frac{5(2 - x)}{3} \geq 1$

4 The solution to $2 - 6x \leq 8$ is

A $x < -1$ **B** $x \geq -1$ **C** $x \geq 1$ **D** $x \geq -\frac{5}{3}$ **E** $x \leq -\frac{5}{3}$

5 What values of x make each of the following statements true?

a $\frac{2x + 1}{3} \leq 3$ **b** $5x + 2 > x + 8$ **c** $2x + 13 < 8 - 3x$

d $\frac{2x}{9} \leq -4$ **e** $\frac{4x + 3}{3} > 9$ **f** $5 - 3x < 17$

g $8 - 3x > 18$ **h** $\frac{-3x}{4} < 2$ **i** $\frac{1 - x}{2} > 3$

j $\frac{4 - 6x}{5} \leq -6$ **k** $-x - 5 \geq 3$ **l** $\frac{3 - 5x}{7} > 9$

LINKS TO Example 21

6 Solve the following inequalities.

a $2x - 5 < x + 2$ **b** $10m - 30 \geq 4m$

c $2p - 9 \leq 7p - 4$ **d** $4(2w + 1) < 5(3w - 2)$

e $12(x - 2) < 10(2x - 3)$ **f** $\frac{x + 1}{4} \leq \frac{x - 2}{3}$

g $\frac{2z - 1}{4} \geq \frac{z + 10}{5}$ **h** $\frac{2x + 1}{3} > \frac{x - 2}{5}$

i $\frac{y - 3}{4} \geq \frac{2y - 7}{5}$ **j** $6(y - 4) < 11(3 - y) - 6$

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7 If $2 - a < 3(2a + 3)$ then

A $a > -\frac{3}{2}$ **B** $a < \frac{3}{2}$ **C** $a > -1$ **D** $a < -1$ **E** $a > 2$

8 Raphael is a teenager. If you double his age and add 5, the total is still less than his father's age, which is 42. How old could Raphael be?

9 Solve the following inequalities.

a $2x - 3 \geq \frac{x}{2}$

b $2(3 - x) < 4(5x - 3)$

c $\frac{2 - y}{4} - \frac{1 - y}{3} \leq 5$

d $3 \geq \frac{2(x + 1)}{4} - (1 - x)$

10 What values of p make the statement $\frac{3 + p}{2} \geq 3 + p$ true?

LINKS TO Example 22

11 Nick's test grades for maths this term have been 78, 73, 86 and 90. What grade does he need to score on the fifth test if his average is to be at least 82?

12 Harry wants to hire a bike for part of a day. Cheap Bike Hire offers two bike hire plans. With Plan A you pay a fee of \$4 plus \$4.50 per hour. With Plan B you pay \$28 plus 50 cents per hour. For what numbers of hours is Plan A cheaper?

13 The cheap bike hire company offers two bike hire plans. You can pay \$25 per day plus a 15 cents per kilometre or \$10 per day plus 40 cents per kilometre. For what number of kilometres is the first plan cheaper?

14 Erryn has just started a new job in sales. She has two salary options available. With the first option she can receive a straight salary of \$800 per week and with the second option she can receive a salary of \$400 per week plus 5% of her weekly sales. What value of sales per week would make the second salary option more profitable for Erryn?

exercise 1.4 challenge

15 Write another two inequality statements that are equivalent to $\frac{x - 2}{3} > 4 - x$.

16 Consider the inequalities $2x + 5 \geq 3$ and $3 - 2x \geq -11$.

a Solve these inequalities.

b Represent the set of values that satisfy both inequalities on a number line.

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Analysis task

Mixing metal alloys

Metal alloys are mixtures of two or more different metals. Brass, for example, is a mixture of copper and zinc.

A metal alloy that contains 70% copper is to be melted and mixed with another metal alloy that contains 45% copper. A total of 30 kg of the new alloy (alloy C) is required and it needs to contain between 48% and 60% copper. What is the least and greatest amount of the 70% copper alloy that should be used?

The following steps will lead to a solution to this problem.

Let x kg be the amount of the 70% copper alloy required.

a Write an expression for the number of kilograms of copper in x kg of alloy A.

b If x kg of alloy A is used, how many kilograms of alloy B must be used in the mixture?

c Express the percentage of copper in alloy B as a decimal.

d Write an expression for the number of kilograms of copper in the amount of alloy B you have worked out in part **b**.

e If alloy C contains 48% copper, how many kilograms of copper will be in 30 kg of alloy C?

f If alloy C contains 60% copper, how many kilograms of copper will be in 30 kg of alloy C?

g Using your answers to parts **e** and **f**, copy and complete this sentence:

The number of kilograms of copper in the 30 kg of alloy C must be greater than or equal to ……. kg but less than or equal to …… kg.

h Copy the following table and use your answers to parts **a** to **f** to complete the table.

	Number of kilograms of the alloy	% copper as a decimal	Number of kilograms of copper
Alloy A (70% copper)	x	0.7	
Alloy B (42% copper)			
Alloy C (mixture of A and B)	30	between 0.48 and 0.6	

i Use your answers to parts **a** and **d** to write an expression for the total amount of copper in alloy C. Simplify the expression.

j Using your answers to parts **g** and **i**, write two inequalities.

k Solve each inequality for x.

l Hence, what is the least and greatest amount of the 70% copper alloy that should be used?

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Review *Linear equations and inequalities*

Summary

Solving equations and inequalities

- Solving an equation means finding the value of the pronumeral that makes the equation true.
- Numeric solving strategies include mental strategies, guess, check and improve and using a table of values.
- Equations can be solved algebraically by doing the same thing to both sides.
- Constructing an algebraic expression or equation from a worded problem is called formulation.
- Inequalities can be solved by doing the same thing to both sides. If you multiply or divide by a negative number in this process, the inequality sign changes direction.

Transposing formulas

- The steps taken to transpose a formula are the same as those used to solve an equation.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

equation	inequality	solution	variable
evaluate	number line	subject	
formula	pronumeral	substitution	
formulation	solving	transposing	

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Revision

Multiple-choice questions

1 The solution to the equation $\frac{2x - 11}{3} = 3x + 1$ is

A $x = -2$ **B** $x = -14$ **C** $x = -\frac{12}{7}$ **D** $x = 1$ **E** $x = -3$

2 If 7 is subtracted from the quotient of x and 2 the result is 5. The equation to solve for x is

A $\frac{x}{2} - 7 = 5$ **B** $7 - \frac{x}{2} = 5$ **C** $\frac{x - 7}{2} = 5$ **D** $\frac{7 - x}{2} = 5$ **E** $\frac{2}{x} - 7 = 5$

3 Find a if $\frac{a + 1}{2} - \frac{a - 2}{3} = 1$.

A $a = 13$ **B** $a = 7$ **C** $a = 1$ **D** $a = \frac{7}{5}$ **E** $a = -1$

4 Which one of the following does not satisfy the inequality $5 - 2x \geq -3$?

A $x = 3$ **B** $x = 4$ **C** $x = 5$ **D** $x = 0$ **E** $x = -4$

5 Find B if $A = 5B - C$.

A $B = \frac{A + C}{5}$ **B** $B = \frac{A - C}{5}$ **C** $B = \frac{5 + C}{A}$ **D** $B = \frac{C - A}{5}$ **E** $B = \frac{A}{5} + C$

Short-answer questions

6 If $y = ax + b$, find

a y when $a = 3, x = 2, b = 5$.

b b when $y = 6, a = 2, x = -1$.

7 Solve the following equations for x.

a $3x + 7 = 1$

b $\frac{x}{3} + 4 = 6$

c $3(x - 7) = 39$

d $3x - 20 = x + 4$

e $4x - 6 = 15 - 3x$

f $-12 = 2 - 7x$

g $3x + 17 = 2$

h $15 + 4x = 10 - 6x$

i $\frac{2(x - 1)}{5} - 5 = 7$

j $3\left(\frac{2x}{5} - 1\right) = 1\frac{4}{5}$

k $\frac{x}{4} + \frac{x}{3} = 21$

l $\frac{2x}{5} + \frac{3x}{4} = 46$

m $\frac{2x + 7}{3} = \frac{3}{4}$

n $\frac{x + 6}{4} = \frac{x + 7}{3}$

o $\frac{3x + 4}{5} = \frac{5 + x}{3}$

p $\frac{x + 1}{2} + \frac{x + 1}{2} = 10$

8 Transpose each formula to make the pronumeral in brackets the subject.

a $v = u + at \quad (a)$

b $S = 4\pi r^2 \quad (r)$

c $T = \sqrt{\frac{3W}{5P}} \quad (P)$

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9 Represent each of the following inequalities on separate number lines.

a $x \geq -2$ **b** $x < 2.5$

10 Solve the following linear inequalities and show your solutions on a number line.

a $2x + 3 < 45$ **b** $3x - 7 \geq 8$ **c** $\frac{x}{7} \leq 4$

d $\frac{2x}{3} \geq 5$ **e** $4 - 3x > 10$ **f** $-\frac{2x}{3} + 7 \geq 11$

g $\frac{5x - 11}{3} > -7$ **h** $9 - 13x \leq 48$ **i** $\frac{3 - x}{5} > -4$

j $5x - 25 \geq 2x + 14$ **k** $\frac{4x}{5} - \frac{x}{2} < 6$ **l** $\frac{7 - 2x}{5} \leq 7$

Extended-response questions

11 The temperature $c°$ Celsius can be converted to $f°$ Fahrenheit using the formula $f = 32 + \frac{9c}{5}$.

a Find the temperature f °F for a day with a temperature of 25 °C, that is, $c = 25$.

b Transpose the formula to make c the subject.

c What is the temperature c °C when the temperature is 100 °F? Give your answer correct to one decimal place.

d For what temperature is the number of degrees the same for both Celsius and Fahrenheit scales?

12 To book tickets over the internet there is a fixed \$7 booking fee regardless of the number of tickets bought. The cost of a ticket is \$48. Finn purchases x tickets over the internet.

a Write an expression for the total cost of the tickets.

b The total cost of Finn's purchase is \$535. Write an equation to show this.

c Solve the equation to find the number of tickets that Finn purchased.

Practice quiz
Chapter 1

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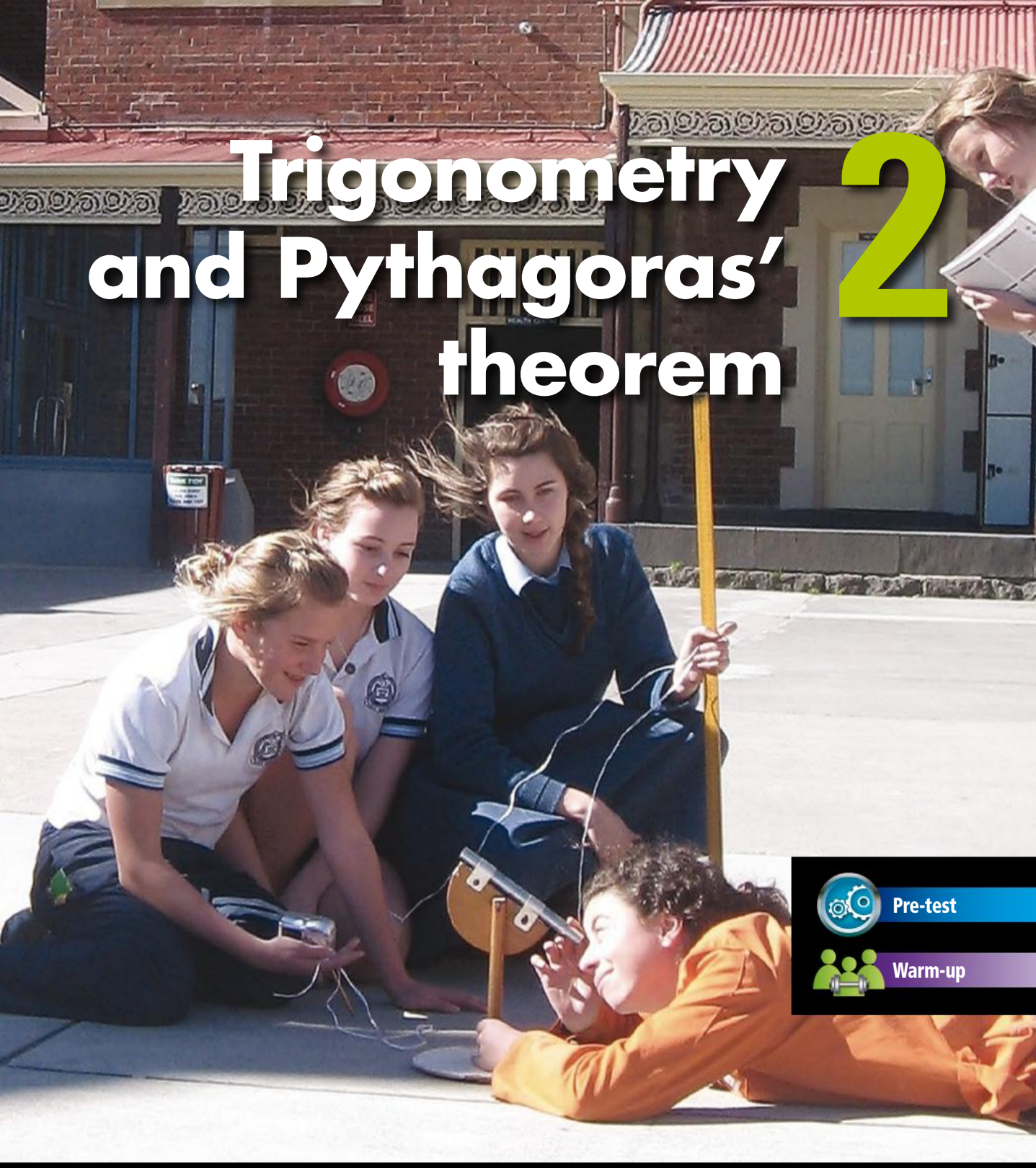
Trigonometry and Pythagoras' theorem 2



Pre-test

Warm-up

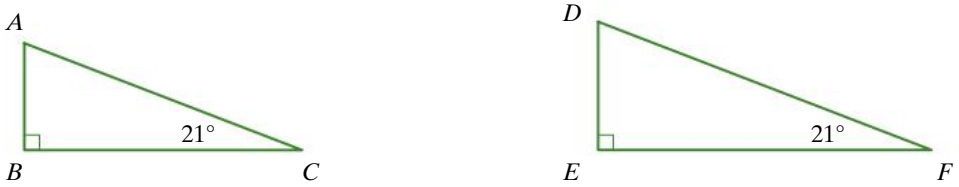
These students are measuring the angle of elevation to the top of a building. If they measure their distance from the foot of the building they can use trigonometry to calculate the height. Trigonometry is used by surveyors and engineers when marking out boundaries of land and surveying for new buildings and roads.

2.1 Review of trigonometric ratios

In *MathsWorld 9*, the trigonometric ratios—**sine**, **cosine** and **tangent**—were introduced. These ratios are abbreviated to **sin** (pronounced 'sine'), **cos** (pronounced 'coz') and **tan**, as shown on the keys of a calculator.

The trigonometric ratios can be defined in terms of the sides of right-angled triangles. All right-angled triangles that have the same angles are similar, so the ratios of their corresponding sides will be equal.

In the diagram below, $\triangle ABC$ is similar to $\triangle DEF$. So, for example, $\frac{AB}{BC} = \frac{DE}{EF}$ and $\frac{AB}{AC} = \frac{DE}{DF}$. These ratios will be the same for all right-angled triangles with an angle of 21°.



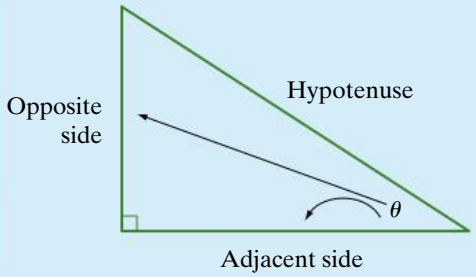
So that we can generalise the three trigonometric ratios for all right-angled triangles, we need a system for identifying or naming the sides we are referring to. The longest side of a right-angle triangle (the side that is not one of the 'arms of the right-angle) is called the **hypotenuse**. We identify the other two sides according to whether they are **adjacent** to (that is, next to) or **opposite** the angle we are referring to. This angle is often denoted by the Greek letter θ (theta).

Remember that we read 'sin' as 'sine', and 'cos' is pronounced 'coz'.

Trigonometric ratios

Trigonometric ratios

Tangent	$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$	$\tan\theta = \frac{\text{O}}{\text{A}}$
Sine	$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\sin\theta = \frac{\text{O}}{\text{H}}$
Cosine	$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$	$\cos\theta = \frac{\text{A}}{\text{H}}$



The mnemonic SOH-CAH-TOA helps us to remember the sides involved in each ratio.

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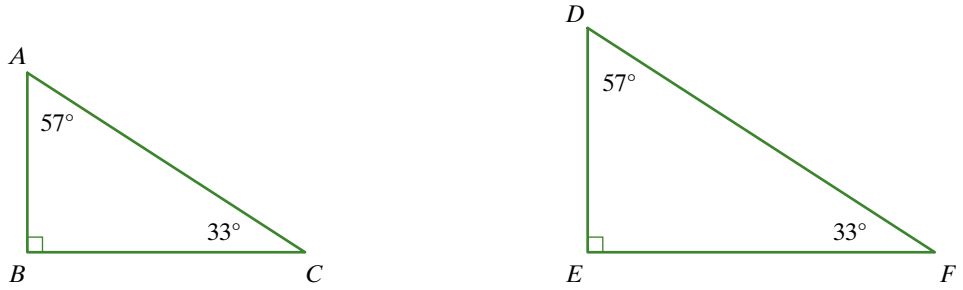
Example 1

Consider the similar triangles below.



State the ratio of sides in

a $\triangle ABC$ that is equal to $\tan 33°$

b $\triangle ABC$ that is equal to $\tan 57°$

c $\triangle DEF$ that is equal to $\dfrac{AB}{BC}$

d $\triangle ABC$ that is equal to $\sin 33°$

e $\triangle ABC$ that is equal to $\cos 57°$

f $\triangle DEF$ that is equal to $\cos 33°$

g $\triangle DEF$ that is equal to $\dfrac{AB}{AC}$

Working	Reasoning
a $\tan 33° = \dfrac{AB}{BC}$	AB is **O**pposite the angle marked 33°. BC is **A**djacent to the angle marked 33°.
b $\tan 57° = \dfrac{BC}{AB}$	BC is **O**pposite to the angle marked 57°. AB is **A**djacent the angle marked 57°.
c $\dfrac{AB}{BC} = \dfrac{DE}{EF}$	DE is **O**pposite the angle marked 33°. EF is **A**djacent to the angle marked 33°.
d $\sin 33° = \dfrac{AB}{AC}$	AB is **O**pposite the angle marked 33°. AC is the **H**ypotenuse.
e $\cos 57° = \dfrac{AB}{AC}$	AB is **A**djacent to the angle marked 57°. AC is the **H**ypotenuse.
f $\cos 33° = \dfrac{EF}{DF}$	EF is **A**djacent to the angle marked 33°. DF is the **H**ypotenuse.
g $\dfrac{AB}{AC} = \dfrac{DE}{DF}$	DE is **A**djacent to the angle marked 57°. DF is the **H**ypotenuse.

From example 1, we can see that

- the sine of an angle is equal to the cosine of its complement.
- the tangent of an angle and the tangent of its complement are reciprocals of each other.

Example 2

Without using your calculator, match the pairs that are equal in value.

$\sin 70°$ $\sin 48°$ $\cos 42°$ $\cos 20°$

continued

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Example 2 continued

Working

$\sin 70° = \cos 20°$
$\sin 48° = \cos 42°$

Reasoning

The sine of an angle is the same as the cosine of its complement.

The sine, cosine and tangent of most angles are irrational numbers so we cannot express them in exact decimal form. This means that we always need to round our calculated values to a sensible number of decimal places.

Example 3

Evaluate the following ratios to

a four decimal places.

i $\sin 15°$ **ii** $\tan 72°$

b one decimal place.

i $4.6 \cos 65°$ **ii** $\dfrac{112}{\tan 38°}$

Make sure your calculator is in degree mode!

Working

a i $\sin 15° = 0.2588$

ii $\tan 72° = 3.0777$

b i $4.6 \cos 65° = 1.9$

ii $\dfrac{112}{\tan 38°} = 143.4$

Reasoning

In calculating unknown side lengths we need to solve equations.

Example 4

Find the value of x, correct to one decimal place, in each of the following.

a $\sin 18° = \dfrac{x}{9.5}$ **b** $\cos 39° = \dfrac{6.3}{x}$

Working

a $\sin 18° = \dfrac{x}{9.5}$

$x = 9.5 \sin 18°$

$x = 2.9$ correct to one decimal place

b $\cos 39° = \dfrac{6.3}{x}$

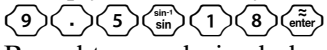
$x \cos 39° = 6.3$

$x = 6.3 \div \cos 39°$

$x = 8.1$ correct to one decimal place

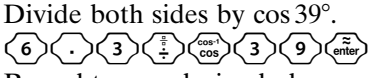
Reasoning

Multiply both sides by 9.5.



Round to one decimal place.

Multiply both sides by x.
Divide both sides by $\cos 39°$.



Round to one decimal place.

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exercise 2.1

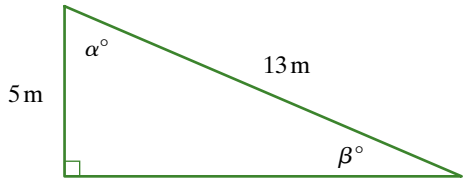
LINKS TO Example 1

1 For each of the following right-angled triangles, write the fraction for

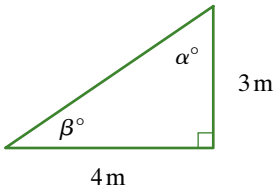
i $\sin\alpha°$ **ii** $\cos\alpha°$ **iii** $\tan\alpha°$

iv $\sin\beta°$ **v** $\cos\beta°$ **vi** $\tan\beta°$

a



b



2 Look at the fractions you wrote in question **1**.

a Which values are the same?

b Which values are reciprocals of each other?

c What do you notice? What can you generalise from this?

LINKS TO Example 2

3 Without using your calculator, match the pairs that are equal in value.

$\sin 54°$	$\cos 18°$	$\cos 83°$	$\sin 36°$	$\cos 41°$
$\sin 72°$	$\sin 49°$	$\cos 36°$	$\cos 54°$	$\sin 7°$

LINKS TO Example 3

4 Use a calculator to evaluate each of the following, correct to four decimal places.

a $\sin 81°$ **b** $\cos 14°$ **c** $\tan 75°$ **d** $\sin 30°$

e $\sin 32.48°$ **f** $\tan 19.675°$ **g** $\cos 28.56°$ **h** $\cos 0.5°$

i $\tan 89.9°$ **j** $\sin 89.9°$ **k** $\cos 89.9°$ **l** $\cos 0.8°$

LINKS TO Example 3b

5 Use your calculator to evaluate each of the following, correct to one decimal place.

a $8.6 \sin 32°$ **b** $34.9 \cos 17°$ **c** $24.5 \tan 20°$ **d** $65 \sin 22.5°$

e $76.4 \cos 18.6°$ **f** $8.63 \tan 62.5°$ **g** $18 \div \sin 24°$ **h** $260 \div \cos 75°$

i $\dfrac{54}{\tan 12°}$ **j** $\dfrac{96.5}{\sin 27°}$ **k** $\dfrac{11.8}{\cos 23°}$ **l** $\dfrac{150}{\cos 50°}$

LINKS TO Example 4a

6 Solve each of the following for x. Give each value correct to one decimal place.

a $\sin 25° = \dfrac{x}{12}$ **b** $\cos 32° = \dfrac{x}{14}$ **c** $\tan 85° = \dfrac{x}{26}$ **d** $\cos 5° = \dfrac{x}{6.5}$

e $\sin 18° = \dfrac{x}{15.6}$ **f** $\tan 45° = \dfrac{x}{18}$ **g** $\sin 56° = \dfrac{x}{45}$ **h** $\cos 72° = \dfrac{x}{96.5}$

LINKS TO Example 4b

7 Solve each of the following for x. Give each value correct to one decimal place.

a $\sin 25° = \dfrac{60}{x}$ **b** $\cos 5° = \dfrac{36}{x}$ **c** $\tan 42° = \dfrac{18}{x}$ **d** $\cos 53° = \dfrac{70}{x}$

e $\tan 12° = \dfrac{80}{x}$ **f** $\sin 65° = \dfrac{55.8}{x}$ **g** $\cos 40° = \dfrac{27.5}{x}$ **h** $\tan 30° = \dfrac{1.6}{x}$

i $\sin 48.6° = \dfrac{90}{x}$ **j** $\cos 16.5° = \dfrac{7.5}{x}$ **k** $\tan 44° = \dfrac{120}{x}$ **l** $\sin 20° = \dfrac{1.8}{x}$

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2.2 Review of calculating side lengths in right-angled triangles

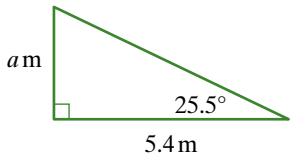
If we are given an angle and a side length in a right-angled triangle, we can find the lengths of the other sides. To calculate an unknown side length, we choose the appropriate trigonometric ratio, depending on the information that has been given.

Example 5

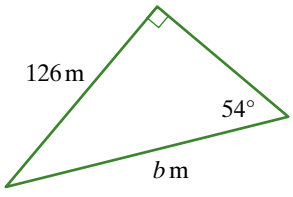
For each of these triangles

- **i** copy the diagram and mark the given and required sides as either *hypotenuse* (H) or *opposite* (O) or *adjacent* (*A*) to the given angle.
- **ii** state the appropriate trigonometric ratio, that is, sine, cosine or tangent.
- **iii** use the ratio to write an equation, and then solve the equation to find the value of the pronumeral, correct to one decimal place.

a

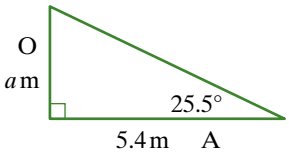


b



Working

a i



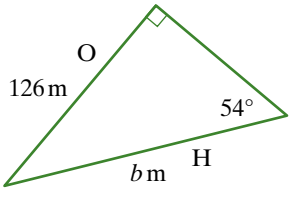
ii $\tan\theta = \dfrac{\text{A}}{\text{O}}$

iii $\tan 25.5° = \dfrac{a}{5.4}$

$a = 5.4 \tan 25.5°$

$a \approx 2.6$

b i

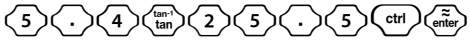


Reasoning

The given side is **A**djacent to the given angle. The required side is **O**pposite the given angle.

SOH-CAH-**TOA** reminds us to use **T**angent.

Multiply by 5.4.



The given side is **O**pposite the given angle. The required side is the **H**ypotenuse.

continued

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Example 5 continued

Working	Reasoning
ii $\sin\theta = \dfrac{O}{H}$	
iii $\sin 54° = \dfrac{126}{b}$	
$b \times \sin 54° = 126$	**SOH**-CAH-TOA reminds us to use **S**ine.
$b = \dfrac{126}{\sin 54°}$	Multiply by b.
$b \approx 155.7$	Divide by $\sin 54°$.
	1 2 6 ÷ sin 5 4 ctrl enter

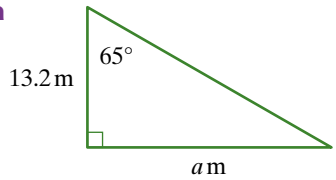
exercise 2.2

LINKS TO Example 5a

1 For each of the following triangles:

- **i** copy the diagram and label the given and required sides as hypotenuse (H), opposite (O) or adjacent (A) with respect to the given angle
- **ii** write an equation
- **iii** solve the equation to find the value of the pronumeral. Round your answers to an appropriate number of decimal places, using the number of decimal places in the given lengths as a guide.

a



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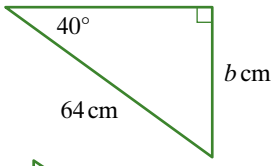

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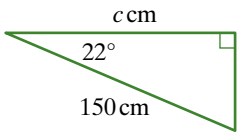

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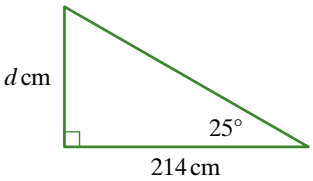

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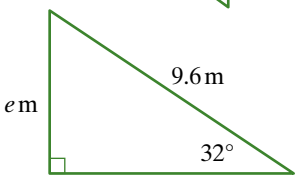

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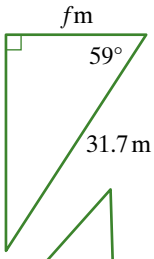

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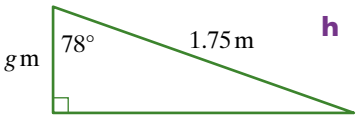

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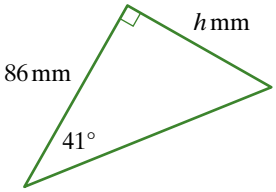

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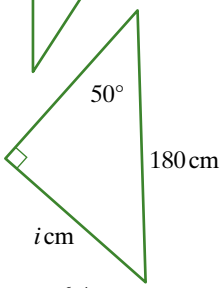

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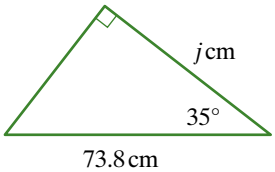

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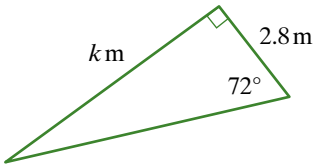

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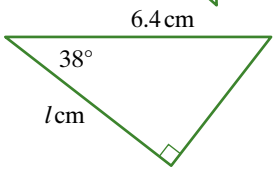

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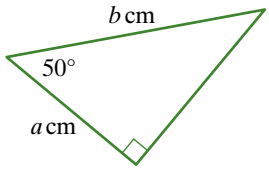
2 The value of a is

A $b\tan 50°$ **B** $b\sin 50°$ **C** $b\cos 50°$

D $\dfrac{b}{\sin 50°}$ **E** $\dfrac{b}{\cos 50°}$



LINKS TO Example 5b

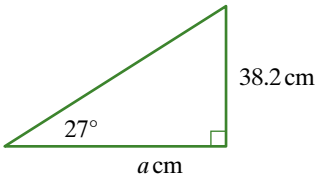
3 For each of the following triangles

i copy the diagram and label the given and required sides as either *hypotenuse* (H), *opposite* (O) or *adjacent* (A) with respect to the given angle.

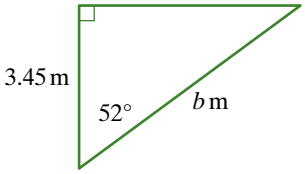
ii write an equation.

iii solve the equation to find the value of the pronumeral. Round your answers to an appropriate number of decimal places, using the number of decimal places in the given lengths as a guide.

a



b



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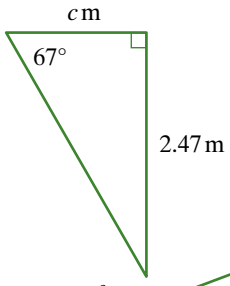

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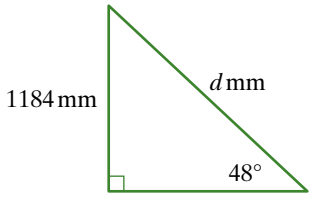

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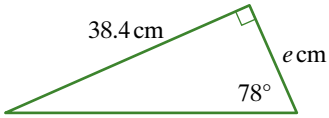

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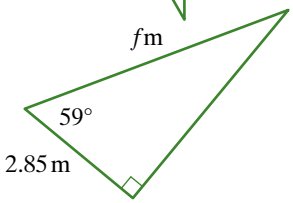

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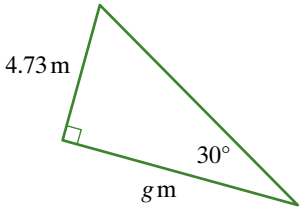

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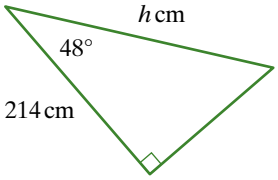

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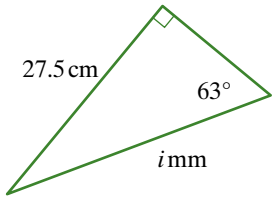

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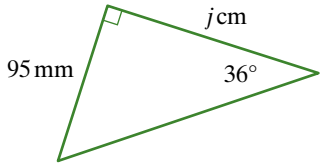

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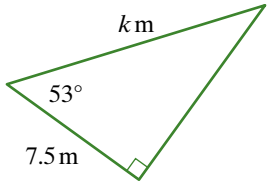

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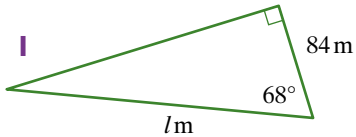
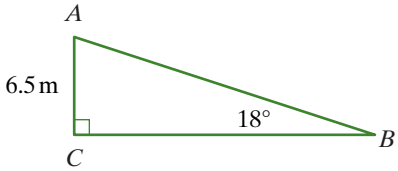


4 The length of side AB is closest to

A 6.8 m **B** 19.8 m **C** 20.0 m

D 21.0 m **E** 21.2 m



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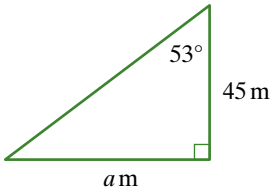
5 For each of the following triangles

i write an equation.

ii solve the equation to find the value of the pronumeral. Round your answers to an appropriate number of decimal places, using the number of decimal places in the given lengths as a guide.

a



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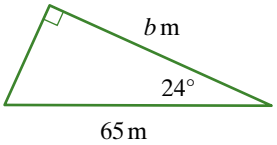

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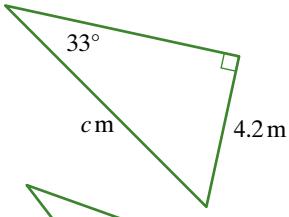

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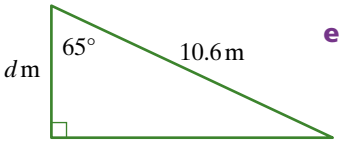

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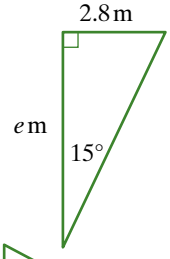

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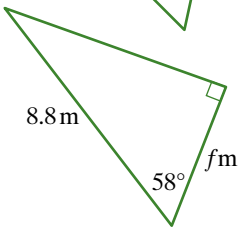

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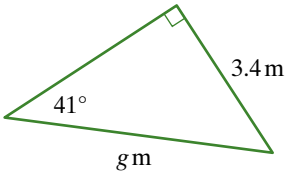

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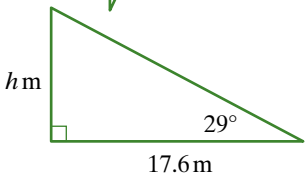

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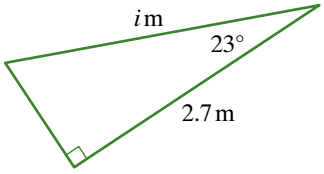

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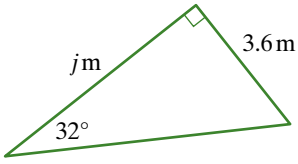

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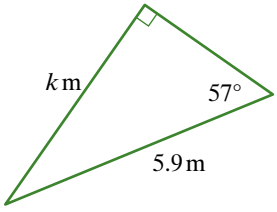

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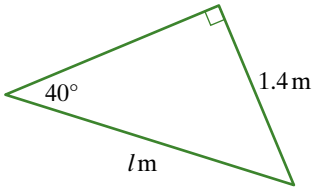
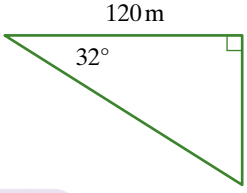


6 Find the size of the other angle and the lengths of the other sides of these triangles. Give the side lengths correct to one decimal place.

a



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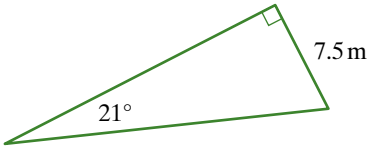
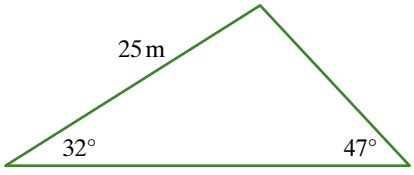


exercise 2.2 challenge

7 Find the size of the other angle and the lengths of the other sides of this triangle. Give the side lengths correct to one decimal place. Hint: It is not a right-angled triangle.



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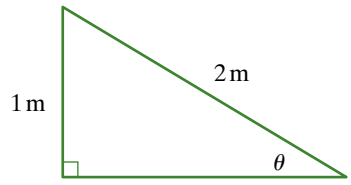
2.3 Review of calculating angles in right-angled triangles

If we know two of the sides of a right-angled triangle, we can use the trigonometric ratios to find unknown angles. For example, in the following triangle



$$\sin\theta = \frac{1}{2}$$
$$= 0.5$$

We can then write $\theta = \sin^{-1}0.5$ where $\theta = \sin^{-1}0.5$ is a shorthand way of saying "θ is the angle whose sine is 0.5."

To find the value of θ we use inverse sine on the calculator.

If we know (or can calculate) the value of the sine, cosine or tangent of an angle, we can use a calculator to find the size of the angle.

Example 6

For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees, correct to one decimal place.

a $\sin\theta = 0.7145$

b $\tan\theta = 5.6713$

Working

a $\sin\theta = 0.7145$

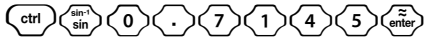
$\theta = 45.6°$ correct to one decimal place

b $\tan\theta = 5.6713$

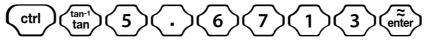
$\theta = 80.0°$ correct to one decimal place

Reasoning

$\theta = \sin^{-1}(0.7145)$



$\theta = \tan^{-1}(5.6713)$



Example 7

Use a calculator to evaluate to the nearest degree, where θ is an angle between 0° and 90°.

a $\theta = \sin^{-1}0.4798$

b $\theta = \cos^{-1}0.9153$

c $\theta = \tan^{-1}7.6145$

continued

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Example 7 continued

Working

a $\sin\theta = 0.4798$

$\theta = 28.7°$ to one decimal place

$\sin^{-1}(0.4798)$ means the angle whose sine is 0.4798

Reasoning

Press 2nd sin⁻¹/sin then type 0.4798 followed by enter.

ctrl sin⁻¹/sin 0 . 4 7 9 8 enter

Round to one decimal place.

b $\cos\theta = 0.9153$

$\theta = 23.8°$ to one decimal place

$\cos^{-1}(0.9153)$ means the angle whose cosine is 0.9153

Type 0.9153 then press 2nd cos⁻¹/cos and enter.

ctrl cos⁻¹/cos 0 . 9 1 5 3 enter

Round to one decimal place.

c $\tan\theta = 7.6145$

$\theta = \tan^{-1} 7.6145$

$\theta = 82.5°$ to one decimal place

$\tan^{-1}(7.6145)$ means the angle whose tangent is 7.6145

Type 7.6145 then press 2nd tan⁻¹/tan and enter.

ctrl tan⁻¹/tan 7 . 6 1 4 5 enter

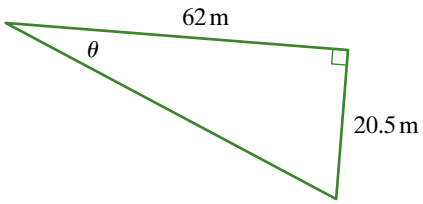
Round to one decimal place.

If we know two side lengths for a right-angled triangle, we can calculate the angles.

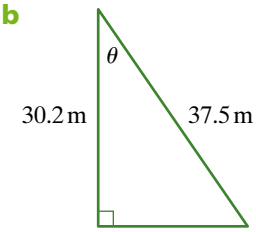
Example 8

Find the size of the angles marked θ in degrees, correct to one decimal place.

a



b



continued

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Example 8 continued

Working

a $\tan\theta = \dfrac{20.5}{62}$

$\theta = \tan^{-1}\left(\dfrac{20.5}{62}\right)$

$\theta \approx 18.3°$

Reasoning

The given sides are **O**pposite and **A**djacent to the required angle. SOH-CAH-**TOA** reminds us to use tangent.
θ is the angle whose tangent is $\dfrac{20.5}{62}$.

b $\cos\theta = \dfrac{30.2}{37.5}$

$\theta = \cos^{-1}\left(\dfrac{30.2}{37.5}\right)$

$\theta \approx 36.4°$

The given sides are the **H**ypotenuse and the side **A**djacent to the required angle. SOH-**CAH**-TOA reminds us to use cosine.
θ is the angle whose cosine is $\dfrac{30.2}{37.5}$.

exercise 2.3

LINKS TO Example 6

1 For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to the nearest degree.

a $\tan\theta = 1.3621$ **b** $\cos\theta = 0.7564$ **c** $\sin\theta = 0.2713$ **d** $\cos\theta = 0.6652$
e $\tan\theta = 1.8351$ **f** $\sin\theta = 0.9217$ **g** $\tan\theta = 13.4674$ **h** $\cos\theta = 0.5386$

LINKS TO Example 7

2 Use a calculator to evaluate to the nearest degree, where θ is an angle between 0° and 90°.

a $\theta = \sin^{-1} 0.36$ **b** $\theta = \cos^{-1} 0.45$ **c** $\theta = \sin^{-1} 0.84$ **d** $\theta = \tan^{-1} 0.6583$
e $\theta = \tan^{-1} 18.4783$ **f** $\theta = \cos^{-1} 0.06$ **g** $\theta = \sin^{-1} 0.9950$ **h** $\theta = \tan^{-1} 56.3417$
i $\theta = \cos^{-1} 0.5$ **j** $\theta = \tan^{-1} 1$ **k** $\theta = \sin^{-1} 0.7183$ **l** $\theta = \cos^{-1} 0.8582$

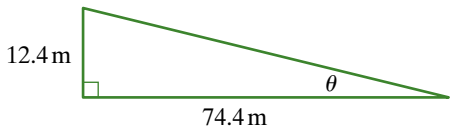
LINKS TO Example 8

3 For each of the following right-angled triangles

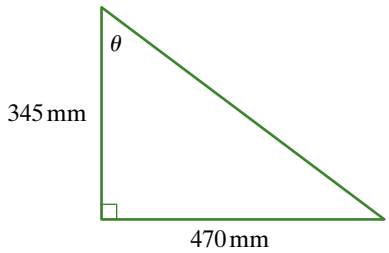
i calculate the size of angle θ, correct to the nearest degree.

ii use Pythagoras' theorem to calculate the missing side length, correct to one decimal place.

a



b



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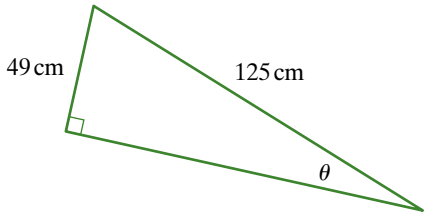
c



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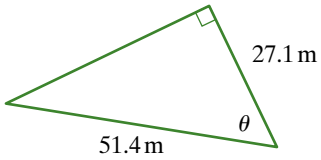

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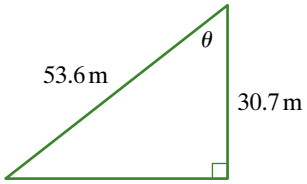

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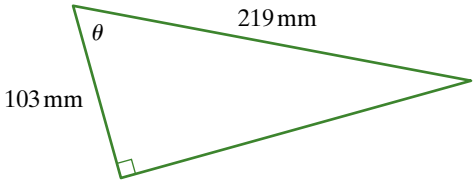

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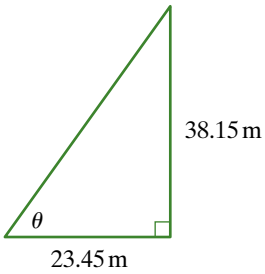

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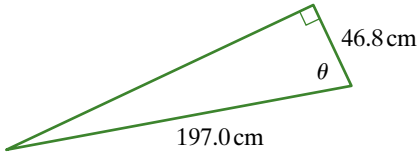
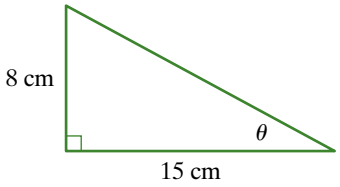


4 The side lengths of each of the following right-angled triangles form a Pythagorean triple.

- **i** Write fractions for $\sin\theta$, $\cos\theta$ and $\tan\theta$.
- **ii** Calculate the size of the angle marked θ, correct to the nearest degree.
- **iii** Give the size of the complementary angle.

a



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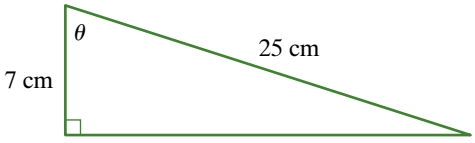
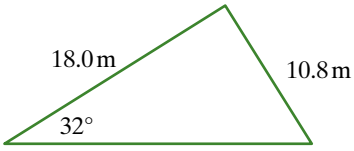


exercise 2.3 challenge

5 Find the size of the other angles and the length of the other side of this triangle. Give the angles correct to the nearest degree and the side length correct to one decimal place. Hint: It is *not* a right-angled triangle.



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2.4 Reviewing aplications of trigonometry

Trigonometry can be applied to solve problems where right-angled triangles are involved. The following steps are used.

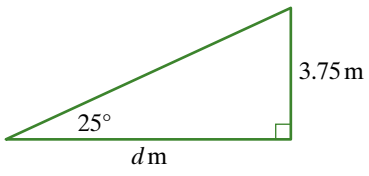
Steps for trigonometry problems

- If a diagram is not given, use the information in the question to draw a diagram.
- Label the given and required information, including whether the opposite or adjacent sides or the hypotenuse are involved.
- Decide which trigonometric ratio to use. (Use SOH-CAH-TOA to help.)
- Write an equation including the unknown side or angle and use your calculator to solve the equation.
- Round your answer to a sensible number of decimal places.
- Write a sentence, stating the answer to the question asked.

Example 9

An escalator makes an angle of 25° with the horizontal. If it rises through a vertical height of 3.75 m, what horizontal distance does the escalator cover? Give the distance correct to one decimal place.

Working



Reasoning

Include a diagram to show the given and required information.
The height 3.75 m is the side **O**pposite the given angle.
The horizontal distance d m is the side **A**djacent to the given angle.

SOH-CAH-**TOA** reminds us to use tangent.
Write ratio with given values.

$$\tan 25° = \frac{3.75}{d}$$

Solve equation.

$$d \tan 25° = 3.75$$
$$d = 3.75 \div \tan 25°$$
$$\approx 8.04$$

The escalator covers a horizontal distance of 8.0 m.

Answer the question.

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Example 10

A 6 m pole is to be supported by a cable attached to the top of the pole and to a point on the ground 1.5 m from the base of the pole.

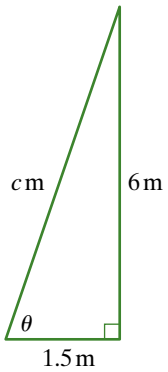
a Draw a labelled diagram.

b What angle does the cable make with the ground? Give the angle correct to the nearest degree.

c How long is the cable? Give the length in metres, correct to one decimal place.

Working

a



Reasoning

The pole is the side **O**pposite the required angle.
The distance on the ground is the side **A**djacent to the required angle.
The cable forms the **H**ypotenuse of the right-angled triangle.

b $\tan\theta = \dfrac{6}{1.5}$

$\tan\theta = 4$

$\theta = \tan^{-1}4$

$\theta = 75.96\ldots°$

The cable makes an angle of 76° with the ground.
Use SOH-CAH-**TOA** to write the ratio.

c $c^2 = 6^2 + 1.5^2$

$c^2 = 38.25$

$c = \sqrt{38.25}$

$c = 6.18\ldots$

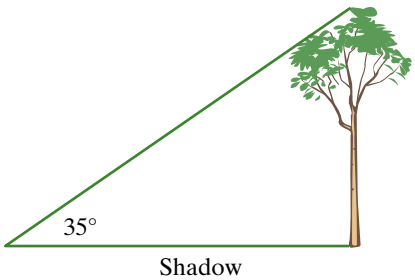
The length of the cable is 6.2 m.

Use Pythagoras' theorem as we know two sides of the right angle triangle and want the third.

exercise 2.4

LINKS TO Example 9

1 The shadow of a tree is 13 m long when the sun's rays make an angle of 35° with the ground. Find the height of the tree to the nearest metre.



2 The recommended angle for an escalator is 30°. An escalator is to be installed in a building where the height between two floors of the building is 3.4 m.

a Draw a labelled diagram.

b What horizontal distance will be needed for the escalator? Give the distance correct to two decimal places.

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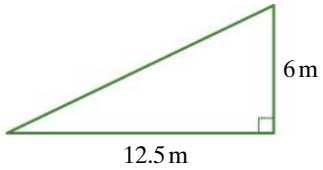
LINKS TO Example 10

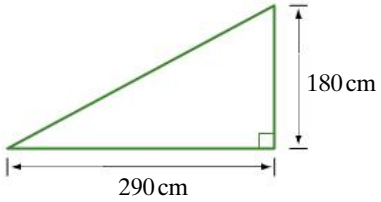
3 The diagram shows a side view of an escalator.



a What angle does the escalator make with the horizontal? Give the angle in degrees correct to one decimal place.

b What is the direct distance up the slope of the escalator? Give the distance correct to one decimal place.

4 The diagram shows a BMX ramp.



a What angle does the ramp make with the horizontal? Give the angle to the nearest degree.

b What is the distance up the ramp? Give the distance correct to the nearest centimetre.

5 A flat bank skateboard ramp is 1 m high and the length of the sloping ramp is 312 cm.

a Calculate the angle that the ramp makes with the horizontal. Give the angle to the nearest degree.

b What horizontal distance does the ramp cover? Give the distance in centimetres correct to the nearest centimetre.

6 The top of a wheelchair ramp is 60 cm above the ground level. The distance up the slope is 3.5 m.

a Draw a labelled diagram.

b What angle does the ramp make with the horizontal? Give the angle to the nearest degree.

7 A moving walkway ramp between two floors of a shopping centre covers a horizontal distance of 30 m and rises to a height of 6.4 m.

a Draw a labelled diagram.

b What angle does the walkway make with the horizontal? Give the angle correct to the nearest degree.

c What is the distance up the slope of the walkway? Give the distance in metres correct to one decimal place.

8 **a** The recommended slope for a wheelchair ramp is 1 in 20. What angle does this ramp make with the horizontal? Give the angle correct to one decimal place.

b For short ramps, slopes of 1 in 12 are considered safe. What angle does this represent?

9 Ladders are considered to be unsafe if the angle with the horizontal is greater than 70°.

a A ladder is 2.8 m long. What is the minimum distance that the foot of the ladder should be from a wall? Give the distance in centimetres to the nearest centimetre.

b For a longer ladder, would the foot of the ladder need to be closer to the wall or further away from the wall?

10 A 4-metre ladder leans against a wall with its foot 1.5 m from the wall. The angle that the ladder makes with the ground is closest to

A 3° **B** 21° **C** 22° **D** 68° **E** 69°

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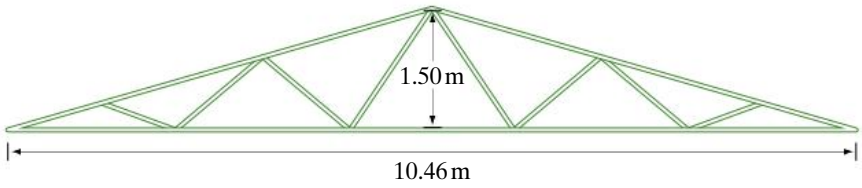
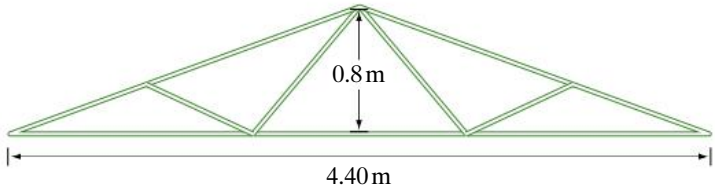
11 The pitch of a roof is the angle that the roof makes with the horizontal. Find the pitch of these roof frames to the nearest whole degree.

a



b

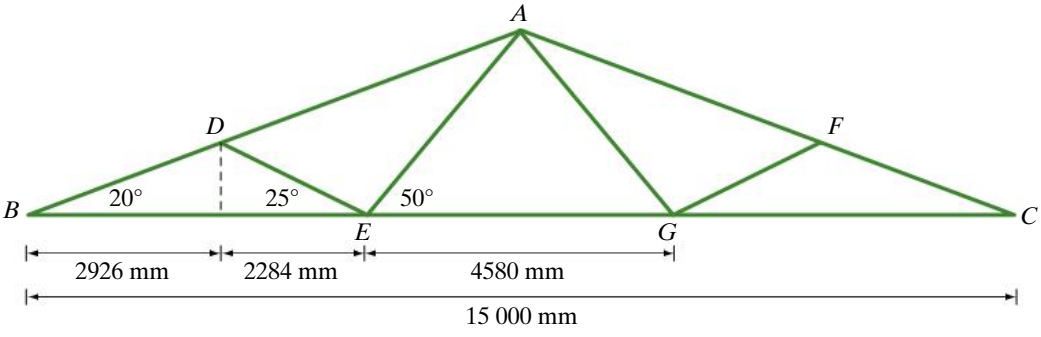


12 The shadow of a pole is 5 m long when the sun's rays make an angle of of 39° with the ground. The height of the pole is

A $5\tan 39°$ **B** $\dfrac{5}{\tan 39°}$ **C** $\dfrac{\tan 39°}{5}$ **D** $5\cos 39°$ **E** $5\sin 39°$

exercise 2.4 challenge

13



a Calculate the lengths of

i AB **ii** DE **iii** AE

b Find the total length of timber needed for the roof truss. Give the length in metres correct to one decimal place.

14 Calculate the angle that each of these roads makes with the horizontal. Give the angles correct to one decimal place.

a

b

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2.5 Pythagoras' theorem and trigonometry in surveying

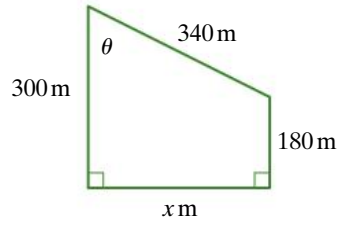
Pythagoras' theorem and trigonometry are used by surveyors when determining the dimensions of irregular pieces of land. By dividing the irregular shape into right-angled triangles, known 'base-line' measurements and angle measurements can be used to calculate unknown lengths.

Example 11

Consider the block of land shown at right.

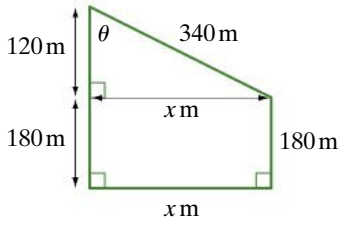
a Find the width, x m, of this block of land, correct to two decimal places.

b Find the size of the angle labelled θ, correct to one decimal place.



Working

a



Reasoning

Divide the shape into a right-angled triangle and a rectangle. The height of the right-angled triangle is 300 – 180, that is, 120 m.

$$x^2 + 120^2 = 340^2$$
$$x^2 = 340^2 - 120^2$$
$$= 101200$$
$$x = \sqrt{101200}$$
$$x \approx 318.119\ldots$$

The width of the land is 318.12 m.

Use Pythagoras' theorem to find x.

b
$$\cos\theta = \frac{120}{340}$$
$$\theta = \cos^{-1}\left(\frac{120}{340}\right)$$
$$\theta \approx 69.3°$$

The angle is 69.3° correct to one decimal place.

Use SOH-CAH-**TOA** to write the ratio.

In examples **12** and **13**, there is more than one right-angled triangle. By working first with one triangle, an unknown side length is found that allows us to work with the second triangle to find another unknown side length.

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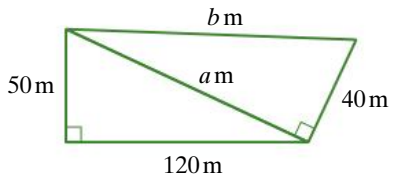
Example 12

Find the unknown side lengths in this shape, correct to the nearest centimetre.



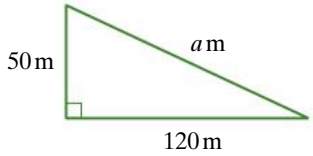
Working

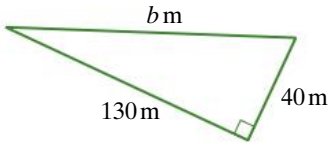
$a^2 = 50^2 + 120^2$
$a^2 = 16900$
$a = \sqrt{16900}$
$a = 130$

$b^2 = 40^2 + 130^2$
$b^2 = 18500$
$b = \sqrt{18500}$
$b \approx 136.0$

$a = 130, b = 136$
The unknown side lengths are 130 cm and 136 cm.

Reasoning



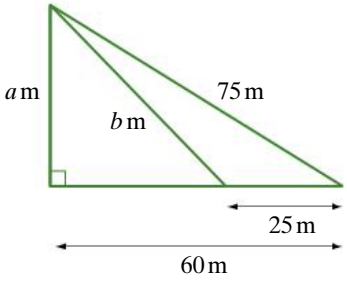


Example 13

Find the following correct nearest whole number.

a the length marked a cm

b the length marked b cm.

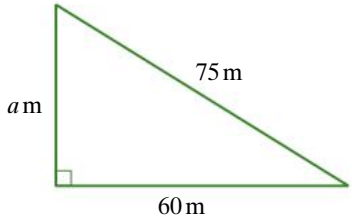


Working

a $a^2 + 60^2 = 75^2$
$a^2 = 75^2 - 60^2$
$a^2 = 2025$
$a = \sqrt{2025}$
$a = 45$

The length marked a cm is 45 cm.

Reasoning



continued

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Example 13 continued

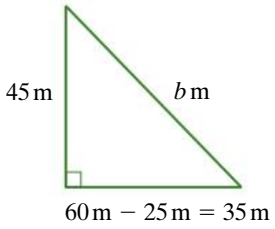
Working

b The base of the smaller triangle:
60 cm – 25 cm = 35 cm

$b^2 = 35^2 + 45^2$
$b^2 = 3250$
$b = \sqrt{3250}$
$b \approx 57.0$

The length marked b cm is 57 cm.

Reasoning

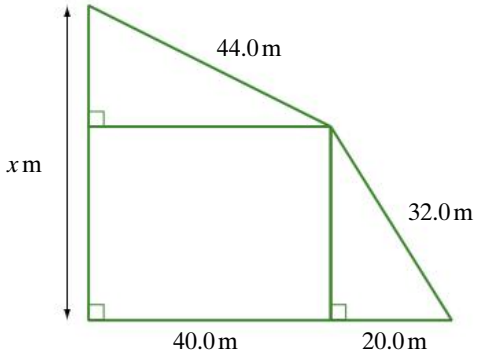


Find the value to one more decimal place than required.

In the next example, the irregular shaped piece of land is divided into a rectangle and two right-angled triangles. By applying Pythagoras' theorem to the triangles, the required side length can be found.

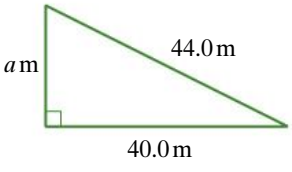
Example 14

a Find the unknown side length in this piece of land, correct to one decimal place.
b Calculate the area of the piece of land, correct to the nearest square metre.



Working

a



Reasoning

The length x m is made up of two parts. Each part is one of the shorter sides of a right-angled triangle.
Draw and label one of the right-angled triangles. Use a pronumeral for the unknown side.

continued

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Example 14 continued

Working

Let the unknown side be a m.

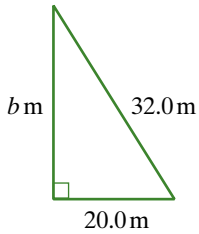
$a^2 + 40^2 = 44^2$

$a^2 = 44^2 - 40^2$

$a^2 = 336$

$a = \sqrt{336}$

$a = 18.330\ldots$



Let the unknown side be b m.

$b^2 + 20^2 = 32^2$

$b^2 = 32^2 - 20^2$

$b^2 = 624$

$b = \sqrt{624}$

$b = 24.979\ldots$

$x = a + b$

$x = 18.33 + 24.98$

$x = 43.31$

The missing side length is 43.3 m, correct to one decimal place.

b Area $= \frac{1}{2} \times a \times 40 + \frac{1}{2} \times b \times 20 + b \times 40$

$= \frac{1}{2} \times 18.33 \times 40 + \frac{1}{2} \times 24.98 \times 20 + 24.98 \times 40$

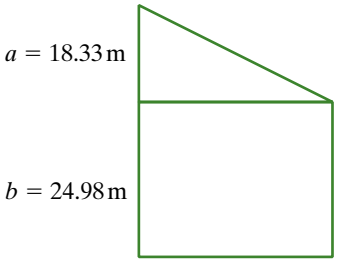
$= 1615.6$

The area is 1616 m^2.

Reasoning

Calculate to one more decimal place than is required. Leave the rounding until the two lengths have been added.

Draw and label the other right-angled triangles. Use another pronumeral for the unknown side.



Add the two lengths that have been calculated, correct to two decimal places.

Round to one decimal place.

The area is equal to the sum of the areas of the two triangles and the rectangle.

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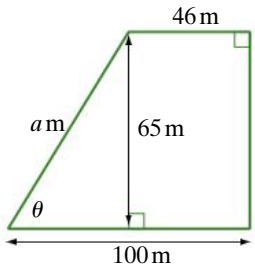
exercise 2.5

LINKS TO Example 11

1 Find the unknown lengths and angles in each of the following, giving your answers correct to one decimal place.

a



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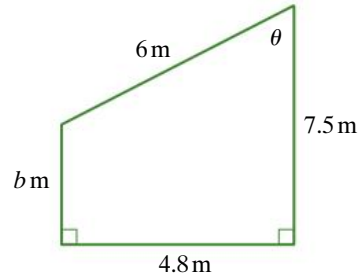

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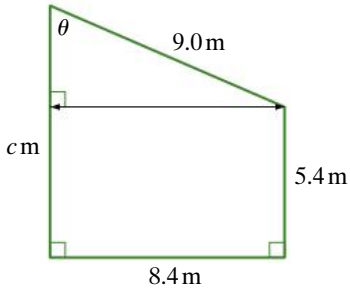

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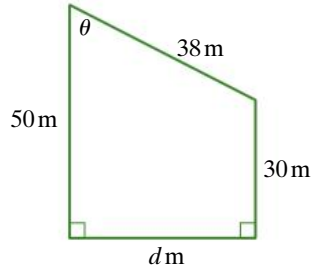
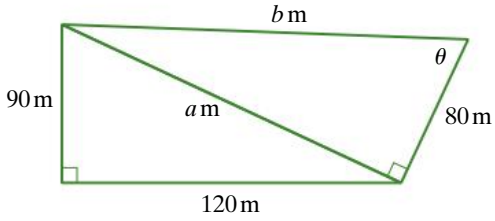


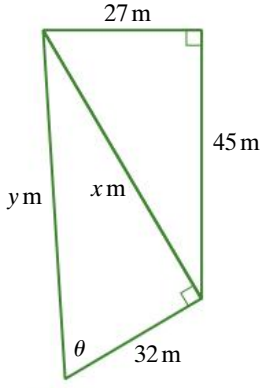
LINKS TO Examples 12, 13

2 Find the unknown lengths and angles in each of the following. Give the lengths correct to an appropriate number of decimal places and the angles correct to one decimal place.

a



b



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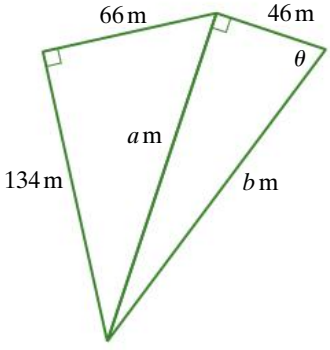

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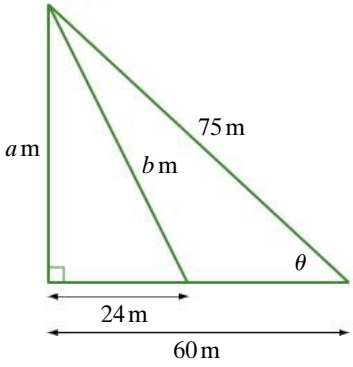

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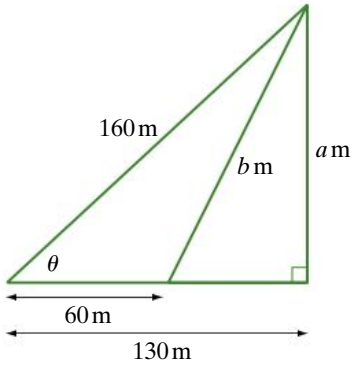

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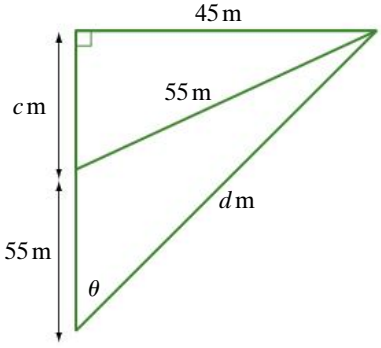

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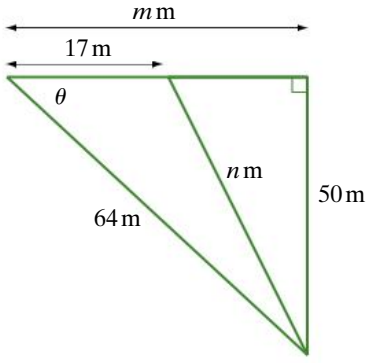

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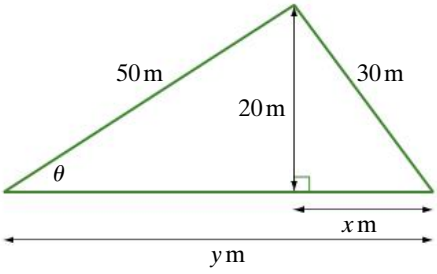
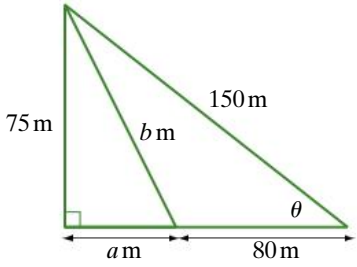


3 Consider the diagram below. The values of a, b and θ, to the nearest whole number, are

A $a = 102$, $b = 127$ and $\theta = 25°$
B $a = 88$, $b = 150$ and $\theta = 28°$
C $a = 88$, $b = 116$ and $\theta = 28°$
D $a = 50$, $b = 90$ and $\theta = 30°$
E $a = 50$, $b = 110$ and $\theta = 30°$

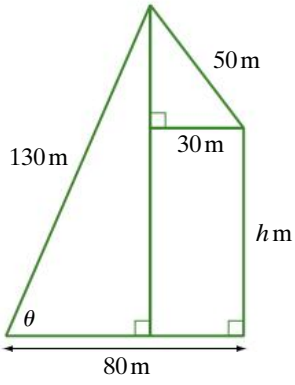


LINKS TO Example 14

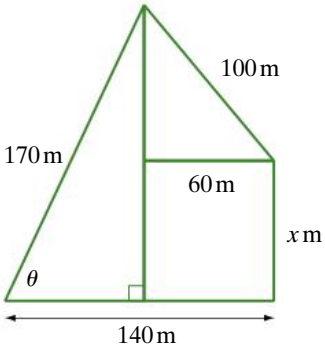
4 For each diagram below, find

i the unknown side lengths, correct to an appropriate number of decimal places.
ii the unknown angle, correct to one decimal place.
iii the area, correct to the nearest square metre.

a



b



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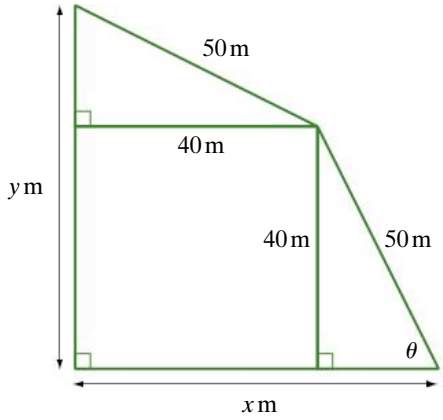
c



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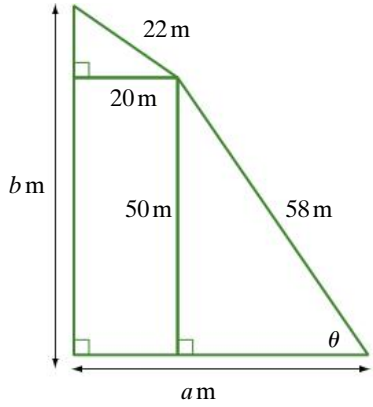
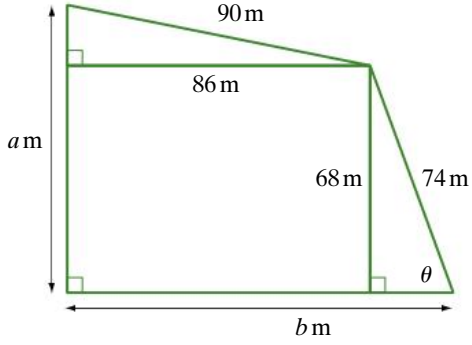


e



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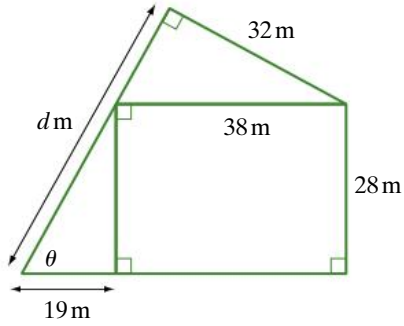
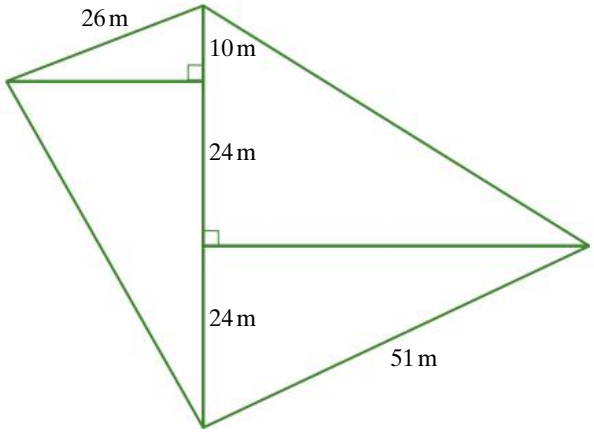


exercise 2.5 challenge

5 Find the area of this paddock.



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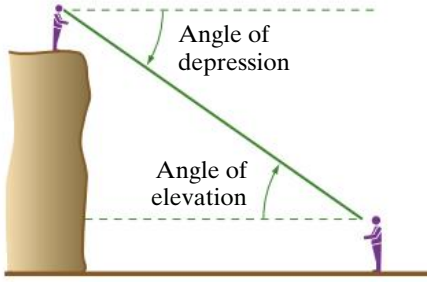
2.6 Angles of elevation and depression

If you are standing on the ground looking up at the top of a building or some other tall object, the angle through which you have to raise your line of sight is called the **angle of elevation**. Similarly if you are standing on top of a building or cliff and looking down, the angle through which you have to lower your line of sight is called the **angle of depression**.

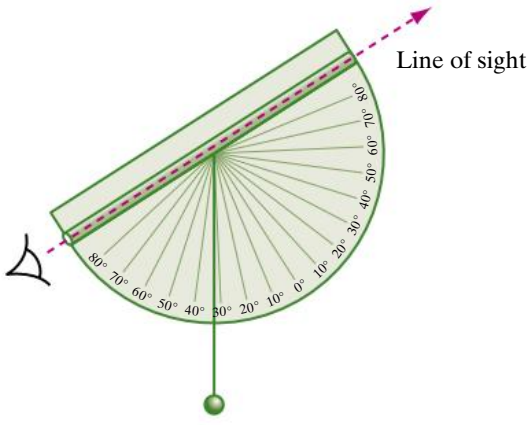


An angle of elevation and the matching angle of depression are equal because they are alternate angles between the parallel horizontal lines.

BLM
Clinometer

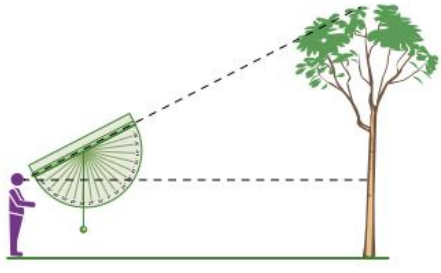
A **clinometer** (or inclinometer) is a device for measuring angles of elevation or depression. The weight on the end of the string attached to the centre of the baseline ensures that the string hangs vertically. By tilting the clinometer so that your line of sight along it is towards the top of an object such as a tree, the angle of elevation can be measured.

An incline is a slope.



Angles of elevation and depression

When you measure an angle of elevation or depression you are usually measuring from the height of your eyes above the ground. If you are using the angle to calculate the height of an object the distance of your eye level above the ground will need to be taken into account.



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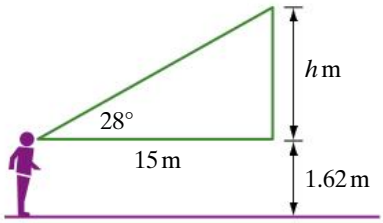
Example 15

Danny found that when he was standing 15 m from the school building, the angle of elevation to the top of the building was 28°. The height of Danny's eye above the ground was 162 cm. Find the height of the building.

Working



Reasoning

Draw a diagram and label the given information. The diagram does not need to be drawn to scale.
Convert 162 cm into metres.

$$\tan 28° = \frac{h}{15}$$
$$h = 15 \tan 28°$$
$$h \approx 7.98$$

In the right-angled triangle, h m is the side **O**pposite the known angle and 15 m is the side **A**djacent to the known angle, so use the tangent ratio.

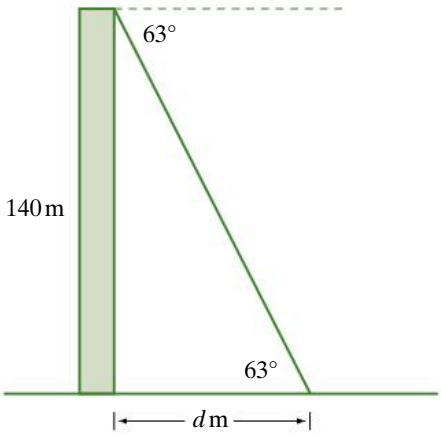
$7.98 + 1.62 = 9.60$
The height of the building is approximately 9.6 metres.

The height of Danny's eye above the ground must be added on to the calculated value for h.

Example 16

The angle of depression from the top of a 140 m building to the centre of a road intersection is 63°. How far is the centre of the intersection from the foot of the building?

Working



Reasoning

The angle of elevation from the intersection to the top of the building is 63° because the two angles are alternate angles between parallel lines.

continued

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Example 16 continued

Working

$$\tan 63° = \frac{140}{d}$$

$$d = \frac{140}{\tan 63°}$$

$$\approx 71.3$$

The centre of the intersection is approximately 71 m from the foot of the building.

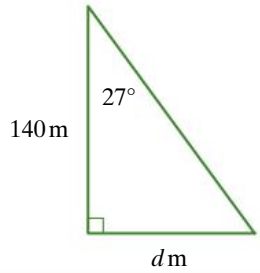
Reasoning

The complementary angle in the triangle is 27°, so an alternative calculation is:

$$\tan 27° = \frac{d}{140}$$

$$d = 140 \tan 27°$$

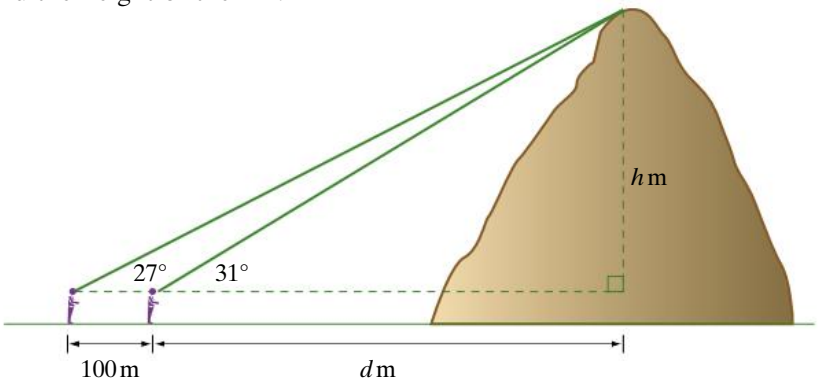
$$\approx 71.3$$



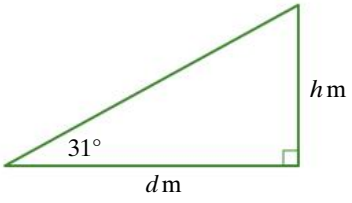
Sometimes it is not possible to measure the distance to an object, such as a hill. In these cases it may be possible to measure the angle of elevation from two different points along the same direct line to the object. The distance can then be calculated.

Example 17

A surveyor measured the angle of elevation to the top of a hill as 31°. From another point 100 m further away, the angle of elevation was 27°. The surveyor's height was 179 cm. Find the height of the hill.



Working



$$\tan 31° = \frac{h}{d}$$

$$h = d \tan 31°$$

Reasoning

Separate the diagram into two right-angled triangles.

Find an equation for h in terms of d, using the first triangle.

continued

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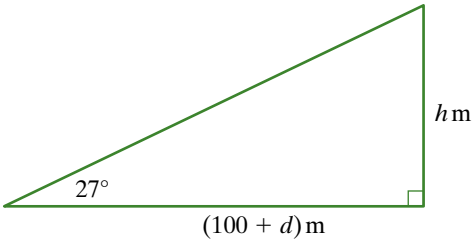
Example 17 continued

Working



Reasoning

The total distance to the base of the hill is $(100 + d)$ m.

$$\tan 27° = \frac{h}{100 + d}$$
$$h = (100 + d)\tan 27°$$

Find another equation for h in terms of d, using the second triangle.

We can use the two equations to determine the value of d before finding h.

$$d \tan 31° = (100 + d)\tan 27°$$
$$d \tan 31° = 100 \tan 27° + d \tan 27°$$

The two expressions for h are equated (put equal to each other).

$$d \tan 31° - d \tan 27° = 100 \tan 27°$$
$$d(\tan 31° - \tan 27°) = 100 \tan 27°$$

The terms containing d are arranged on the left side of the equation, and then d is taken out as a common factor.

$$d = \frac{100 \tan 27°}{(\tan 31° - \tan 27°)}$$
$$\approx 557.9$$

The coefficient of d is $(\tan 31° - \tan 27°)$, so dividing both sides by this gives the value of d.

$$h = d \tan 31°$$
$$\approx 557.9 \tan 31°$$
$$\approx 335.2$$

The value of d is then substituted into the simpler of the two original equations, $h = d \tan 31°$.

$$\text{height} = 335.2 + 1.79$$
$$= 337.0$$

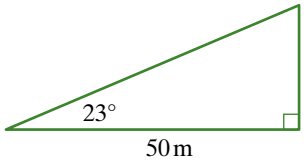
The height of the hill is approximately 337 m.

Adding on the height of the surveyor, the height of the hill is $(335.2 + 1.79)$ metres.

exercise 2.6

LINKS TO Example 15

1 The angle of elevation to the top of a tower from a point on the ground 50 m from the tower is 23°. What is the height of the tower to the nearest metre?



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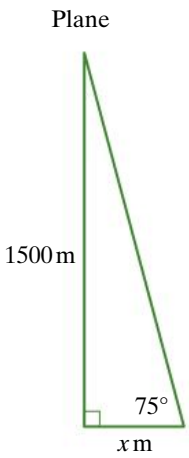
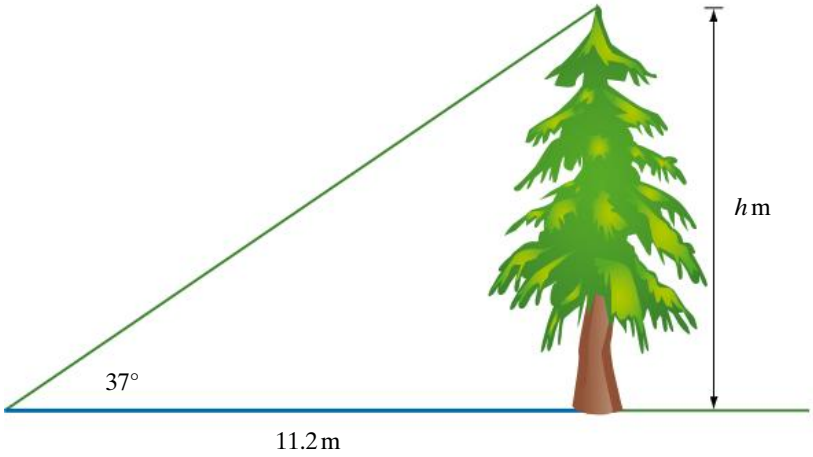
2 A plane is at a height of 1500 m. An observer on the ground measures the angle of elevation to the plane as 75°. What is the horizontal distance of the plane from the observer? Give the distance to the nearest 10 metres.



3 The angle of elevation to the top of a tree from a point on the ground 11.2 m from the tree is 37°. What is the height of the tree in metres correct to one decimal place?



4 The angle of elevation to the top of a tree from a point on the ground 120 m from the tree is 14°.

a Draw a labelled diagram.

b How high is the tree? Give the height correct to the nearest metre.

5 Weather balloons are used to collect information about wind speed and direction at different heights. The angle of elevation of the balloon is measured, and knowing the length of the tether line, the height of the balloon can be calculated.

The angle of elevation to a weather balloon is 78° and the length of the tether line is 100 m.

a Draw a labelled diagram.

b How high is the balloon? Give the height to the nearest metre.

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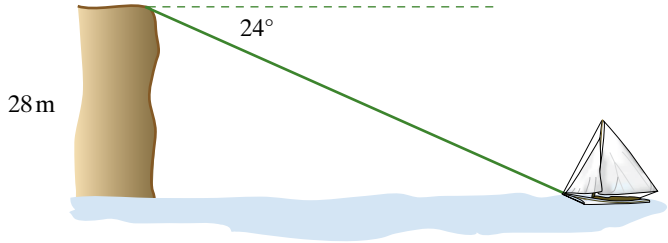
LINKS TO Example 16

6 The angle of depression from a cliff top to a yacht is 24°. If the cliff is 28 m high, how far is the yacht from the foot of the cliff?



7 The angle of depression from the top of a cliff to a sailing boat is 6°. The sailing boat is 700 m from the base of the cliff.

a Draw a labelled diagram.

b How high is the cliff?

8 The angle of depression from the top of a 12 m lookout tower to the feet of a person on the ground is 62°.

a Draw a labelled diagram.

b How far is the person from the base of the tower? Give the distance in metres correct to one decimal place.

9 The angle of depression from the top of a 24 m building to the centre of a road intersection is 8°.

a Draw a labelled diagram.

b How far is the road intersection from the base of the building? Give the distance to the nearest 10 metres.

10 From a hot air balloon the angle of depression to a person on the ground is 36°. The horizontal distance of the balloon from the person is 100 m.

a Draw a labelled diagram.

b What is the vertical height of the balloon above the ground? Give the height correct to the nearest metre.

11 Standing on top of a cliff, Luke sees a yacht out at sea on an angle of depression of 30°. Luke's eyes are 162 m above sea level.

a Draw a labelled diagram.

b How far out to sea is the yacht? Give the distance to the nearest 10 metres.

12 The angle of depression from the top of a tall building to a point on the ground 100 m from the building is 36.3°.

a Draw a labelled diagram.

b What would be the angle of depression to the same point from half way up the building? Calculate the angle correct to one decimal place.

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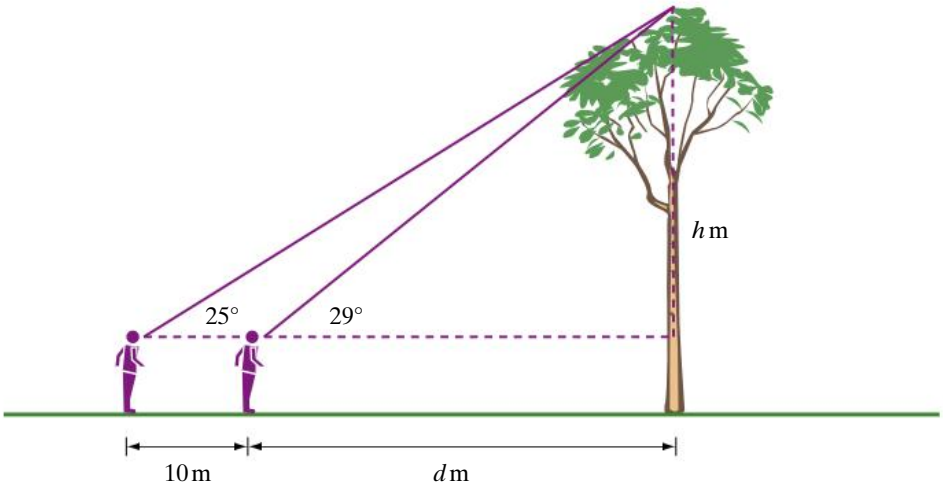
LINKS TO Example 17

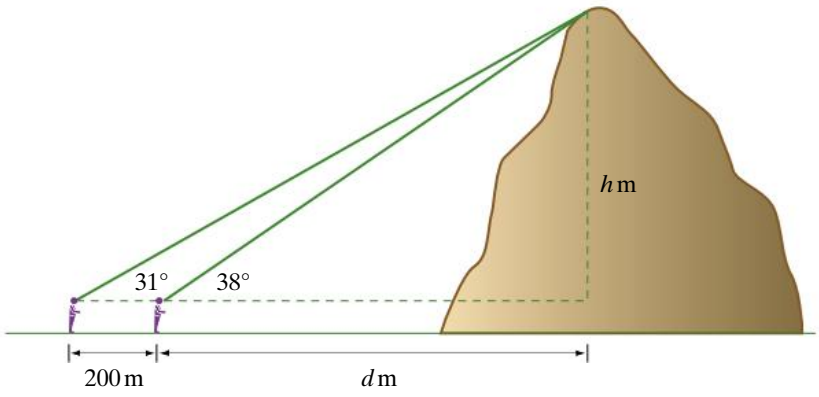
13 Alison measured the angle of elevation to the top of a tall tree as 29°. From another point 10 m further away, the angle of elevation was 25°. Alison's height is 168 cm.



a Calculate d.

b Find the height of the tree correct to the nearest metre.

14 Joel measured the angle of elevation to the top of a hill as 38°. From another point 200 m further away, the angle of elevation was 31°. Joel's height is 183 cm.



a Calculate d.

b Find the height of the hill.

c If the angles were measured again as 38.2° and 30.7°, what difference would this make to the calculated height of the hill?

d Is Joel's height significant in calculating the height of the hill?

e Allowing for small errors in measuring the angles of elevation, what is a more reasonable estimate of the height of the hill?

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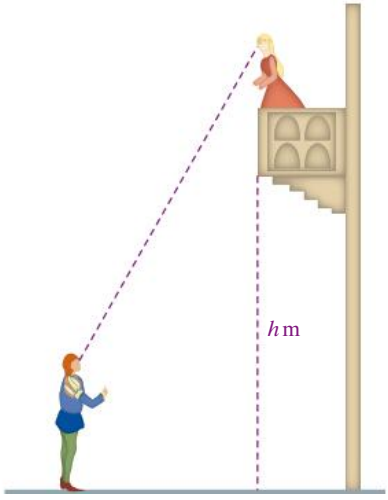
exercise 2.6 challenge

15 Juliet is standing on her balcony, which is shown in the photograph. Romeo estimates that the angle of elevation to Juliet's eyes is 67.3°. He is standing 3.5 m from the wall and the balcony overhangs the wall by 50 cm. Romeo's height is 183 cm and Juliet's height is 162 cm.

a What is the distance from Romeo's eyes to Juliet's eyes?

b What is the height, h m, of Juliet's balcony above the ground?



16 The Burj Khalifa building in Dubai is 828 m high. At a particular time of day the sun's rays make an angle of 35° with the ground.

a What is the length of the shadow cast by the Burj Khalifa building? Give the distance to the nearest 10 metres.

b What angle would the sun's rays make with the ground at a time of day when the shadow is 2 km long? Give the angle correct to one decimal place.

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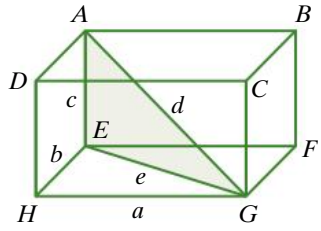
2.7 Pythagoras' theorem in three dimensions

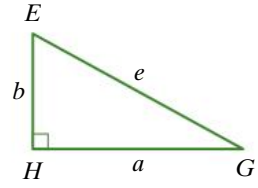
Consider the rectangular prism in the diagram below.



The length, width and height of the prism are a, b and c respectively as shown. The length of the diagonal EG is e and the length of the diagonal AG is d.

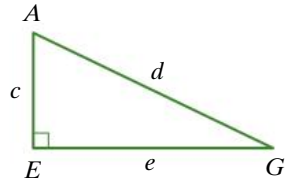
Using Pythagoras' theorem we can express e^2 in terms of a^2 and b^2.

$e^2 = a^2 + b^2$



Using Pythagoras' theorem we can express d^2 in terms of c^2 and e^2.

$d^2 = c^2 + e^2$

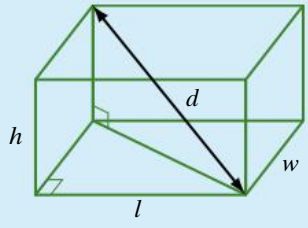


Substituting $a^2 + b^2$ for e^2, the expression for d^2 can now be written in terms of a, b and c.

$d^2 = a^2 + b^2 + c^2$

Pythagoras' theorem in three dimensions

$d^2 = l^2 + w^2 + h^2$



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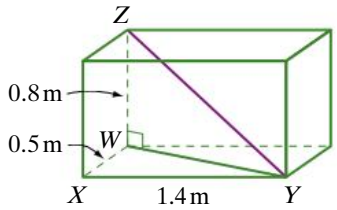
Example 18

Calculate the length of the diagonal ZY of this rectangular box.



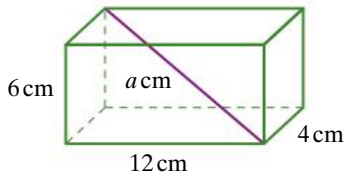
Working	Reasoning
$WY^2 = WX^2 + XY^2$ $= 0.5^2 + 1.4^2$ $= 0.25 + 1.96$ $= 2.21$	$\angle WXY = 90°$, so $\triangle WXY$ is a right-angled triangle. Use Pythagoras' theorem to find WY^2. There is no need to take the square root to find the length of WY as this length must be squared again to find the length of ZY.
$ZY^2 = ZW^2 + WY^2$ $= 0.8^2 + 2.21$ $= 0.64 + 2.21$ $= 2.85$	$\angle ZWY = 90°$ so $\triangle ZWY$ is a right-angled triangle. Use Pythagoras' theorem to find ZY^2. Remember that $WY^2 = 2.21$.
$ZY = \sqrt{2.85}$ ≈ 1.69 The length of the diagonal ZY is approximately 1.7 m.	Find the square root of 2.85.
Alternatively, $ZY^2 = ZW^2 + WX^2 + XY^2$ $= 0.8^2 + 0.5^2 + 1.4^2$ $= 0.64 + 0.25 + 1.96$ $= 2.85$ $ZY = \sqrt{2.85}$ ≈ 1.69	The two steps can be condensed into a single step because $ZY^2 = ZW^2 + WY^2$, but $WY^2 = WX^2 + XY^2$, so $ZY^2 = ZW^2 + WX^2 + XY^2$

exercise 2.7

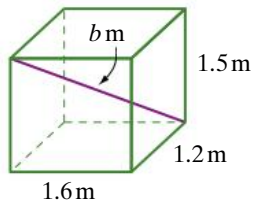
LINKS TO Example 18

1 Calculate the length of the diagonal in each of the following rectangular prisms.

a



b



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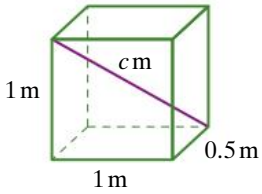
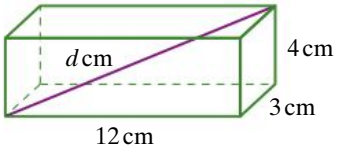
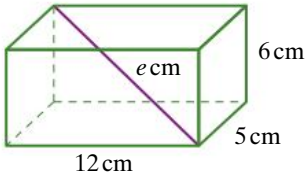
c



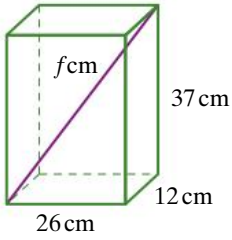
d



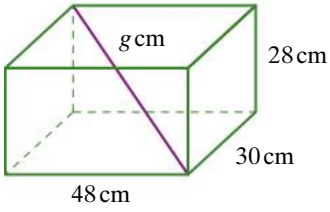
e



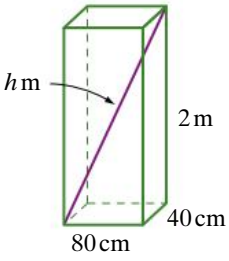
f



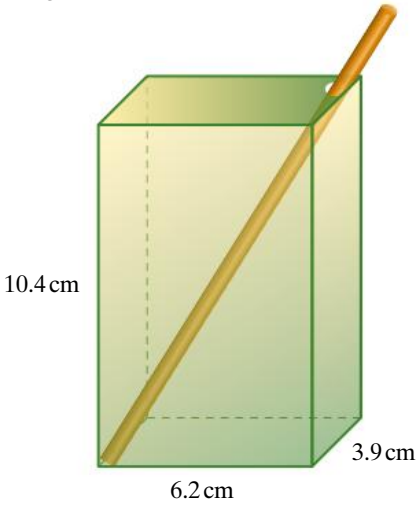
g



h



2 A 250 mL juice carton has the dimensions shown below. The straw is to protrude 1.5 cm from the carton and must be long enough to reach the diagonally opposite corner of the bottom of the carton. Calculate the length of the straw.



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2.8 Trigonometry and bearings

Many situations require that a direction is specified. For example, the paths of planes and ships, the direction along walking tracks and orienteering routes, in weather forecasting to describe the direction of storms, and definitions of property boundaries all specify a direction. You will probably be familiar with compass directions.

A compass direction can also be expressed as a **bearing**. A bearing is a direction measured in a clockwise direction from north. Bearings are expressed as angles from 0° to 360°. This illustration of the compass shows the relationship between the compass points and bearings. The bearings shown on this compass are **compass bearings** because they are measured from magnetic north, that is, the north that is indicated by a compass. Bearings are expressed using three digits. A direction that is 35 degrees in a clockwise from north is written as 035°.

(Bearings can also be expressed as **true bearings**, that is, they are measured from **true north**. True north is the position in the northern hemisphere where the Earth's axis of rotation meets the surface of the Earth. True north and magnetic north do not coincide exactly.)

Example 19

For each direction

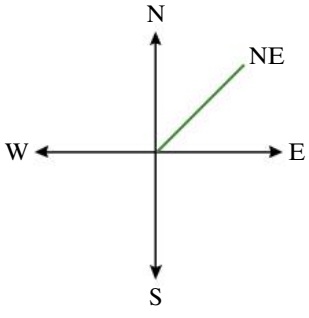
- **i** draw a diagram.
- **ii** rewrite the compass bearing as a bearing

a NE

b S40°W

Working

a i



ii 045°

Reasoning

NE is half-way between north and east.

The direction NE makes an angle of 45° with north in a clockwise direction.

continued

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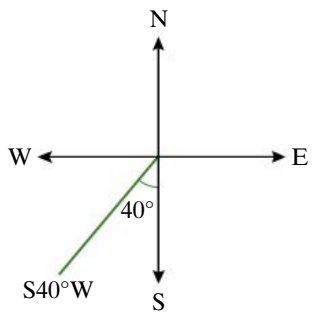
Example 19 continued

Working

b i



Reasoning

The direction S40°W is 40° towards west from south.

ii 220°

In a clockwise direction from north, the bearing is 180° + 40° = 220°.

Combined with trigonometry, bearings can be used to calculate distances in navigation and surveying.

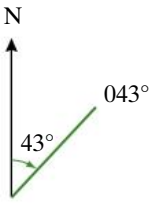
Example 20

Draw diagrams to show the following bearings.

a 043° **b** 124° **c** 345°

Working

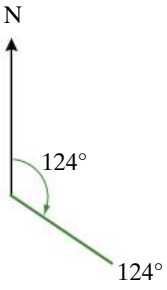
a



Reasoning

Draw a north line. Measure 43° from north in a clockwise direction.

b



Measure 124° from north in a clockwise direction.

continued

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Example 20 continued

Working

c

Reasoning

$360° - 345° = 15°$

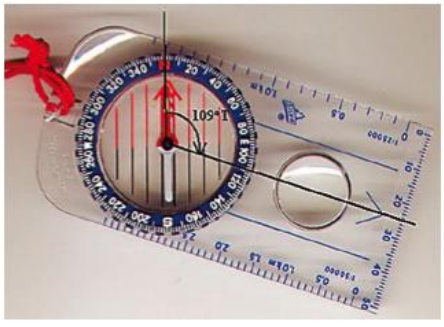
Draw a north line. Measure 15° in an anticlockwise direction from north.

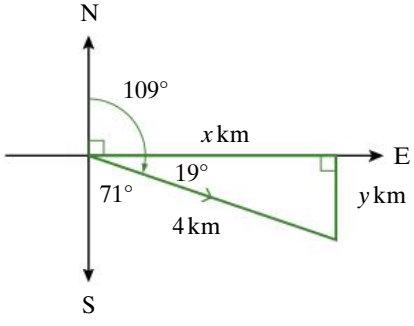
Suppose Henry walks 4 km in the direction S71°E. How far east and how far south is Henry from his starting point?

Problems of this type can be solved by constructing right-angled triangles and using trigonometric ratios.

A bearing of 109° is 19° south of east. We draw a right-angled triangle so that Henry's path is the hypotenuse of the triangle. As shown in the diagram, x km represents how far Henry is east of his starting point and y km represents how far he is south of his starting point.

As we can see on the diagram, a direction of S71°E is a bearing of 109°.





$$\cos 19° = \frac{x}{4}$$
$$x = 4 \cos 19°$$
$$\approx 3.78$$

$$\sin 19° = \frac{4}{y}$$
$$y = 4 \sin 19°$$
$$\approx 1.30$$

So Henry is approximately 3.78 km east and 1.30 km south of his starting point.

Example 21

Kacita walks for 5.6 km in a direction NNE.

a Draw a labelled diagram to show Kacita's direction.

b How far east and how far north is she of her starting point? Give the distances correct to one decimal place.

continued

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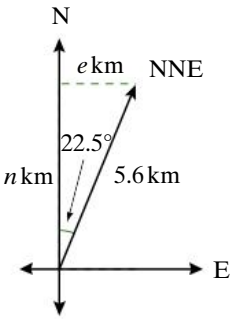
Example 21 continued

Working

a



Reasoning

NNE is between north-east and north. North-east makes an angle of 45° with north. NNE makes an angle of 22.5° with north.
Use pronumerals for the required distances north and east.

b $\sin 22.5° = \dfrac{e}{5.6}$

The distance that Kacita walks represents the **H**ypotenuse of the right-angled triangle. The distance east is the side **O**pposite the given angle.
$5.6 \times \sin(22.5°) = 2.14302$
Use **S**ine.

$e = 5.6 \sin 22.5°$
$e \approx 2.14$

$\cos 22.5° = \dfrac{n}{5.6}$
$n = 5.6 \cos 22.5°$
$n \approx 5.17$

Kacita is 2.1 km east and 5.2 km north of her starting point.

The distance north is the side **A**djacent to the given angle.
Use **C**osine.
$5.6 \times \cos(22.5°) \approx 5.17372$

Example 22

Andy sails for 24 km in a direction N55°W.

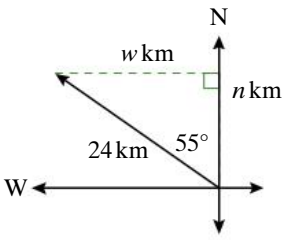
a Draw a labelled diagram of the right-angled triangle formed by Andy's route and his distances north and west of his starting point.

b How far north is Andy of his starting point?

c How far west is he of his starting point?

Working

a



Reasoning

Use pronumerals for the required distances north and west.

continued

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Example 22 continued

Working

b $\cos 55° = \dfrac{n}{24}$

$n = 24\cos 55°$

$n \approx 13.77$

Andy is 13.8 km south of his starting point.

Reasoning

The 24 km distance that Andy has sailed is the **H**ypotenuse of the right-angled triangle.
His distance north is the side **A**djacent to the angle marked 55°.
Use **C**osine.
$24 \times \cos(55) = 13.7658$

c $\sin 55° = \dfrac{w}{24}$

$w = 24\sin 55°$

$w \approx 19.66$

Andy is 19.7 km west of his starting point.

The 24 km distance that Andy has sailed is the **H**ypotenuse of the right-angled triangle.
His distance west is the side **O**pposite the angle marked 55°.
Use **S**ine.
$24 \times \sin(55) = 19.6596$

If we know the bearing on which a person has travelled, we can also use this bearing to calculate how far east (or west) and north (or south) of their starting point they have travelled.

Example 23

Maryam walks for 7 km on a bearing 124°.

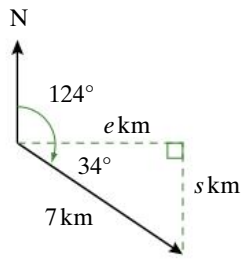
a Draw a labelled diagram of the right-angled triangle formed by Maryam's walk and her distances east and south of her starting point.

b How far south of her starting point is Maryam?

c How far east of her starting point is she?

Working

a



Reasoning

The angle in the triangle is
$124° - 90° = 34°$.
Use pronumerals for the required distances south and east.

continued

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Example 23 continued

Working

b $\sin 34° = \frac{s}{7}$

$s = 7 \sin 34°$

$s \approx 3.91$

Maryam is 3.9 km south of her starting point.

Reasoning

The 7 km distance that Maryam has walked is the **H**ypotenuse of the right-angled triangle.
Her distance south is the side **O**pposite the angle marked 34°.
Use **S**ine.
$7 \times \sin(34°) = 3.91435$

c $\cos 34° = \frac{e}{7}$

$e = 7 \cos 34°$

$e \approx 5.80$

Maryam is 5.8 km east of her starting point.

The 7 km distance that Maryam has walked is the **H**ypotenuse of the right-angled triangle.
Her distance east is the side **A**djacent to the angle marked 34°.
Use **C**osine.
$7 \times \cos(34°) = 5.80326$

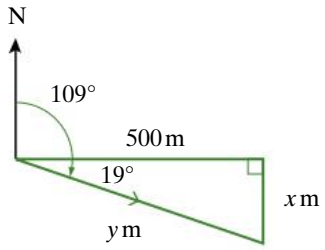
Example 24

Kim walks on a bearing 109° until he is 500 m east of his starting point.

a How far south of his starting point is Kim?

b What is his direct distance from his starting point?

Working



Reasoning

Draw a diagram to show the given and required information.
Subtract 90° from 109° to find the angle in the right-angled triangle.

a $\tan 19° = \frac{x}{500}$

$x = 500 \tan 19°$

$x \approx 172.2$

Kim is approximately 172 m south of his starting point.

The side of the triangle labelled 500 m is the side **A**djacent to the given angle.
The side of the triangle labelled x m is the side **O**pposite the given angle.
Use **T**angent.
$500 \times \tan(19°) \approx 172.1638$

continued

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Example 24 continued

b $\cos 19° = \dfrac{500}{y}$

$y = 500 \div \cos 19°$

$y \approx 528.8$

Kim's direct distance from his starting point is approximately 529 m.

The side of the triangle labelled 500 m is the side **A**djacent to the given angle.
The side of the triangle labelled y m is the **H**ypotenuse.
Use **C**osine.

Distances travelled east (or west) and north (or south) can be used to calculate bearings. Trigonometry or Pythagoras' theorem can be used to calculate distances.

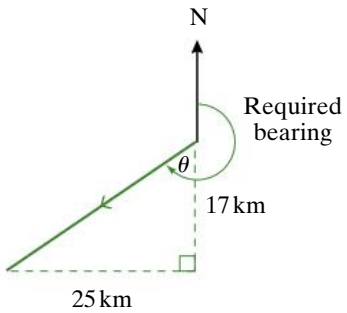
Example 25

A yacht sailed in a direction so that its final position was 25 km west and 17 km south of its starting point.

a On what bearing did it sail?

b How far was the ship from its starting point?

Working



Reasoning

Draw a diagram. Label the given and required information.

a $\tan\theta = \dfrac{25}{17}$

$\theta = \tan^{-1}\left(\dfrac{25}{17}\right)$

$\theta \approx 55.8°$

$180° + 55.8° = 235.8°$

The yacht sailed on a bearing 236°.

The bearing is the angle measured in a clockwise direction from north.
So, need to find θ and add 180°.

b $d = \sqrt{17^2 + 25^2}$

$d \approx 30.23$

The yacht was approximately 30.2 km from its starting point.

The distance of the yacht from its starting point is the hypotenuse of the right-angled triangle. Use Pythagoras' theorem to calculate the distance.

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Example 26

A yacht sails for 25 km on a bearing of 215°, and then for a further 28 km on a bearing of 250°.

a Draw a labelled diagram to show the yacht's route.

b Using two right-angled triangles drawn on your diagram, find how far south and how far west the yacht is from its starting point. Give the distances correct to one decimal place.

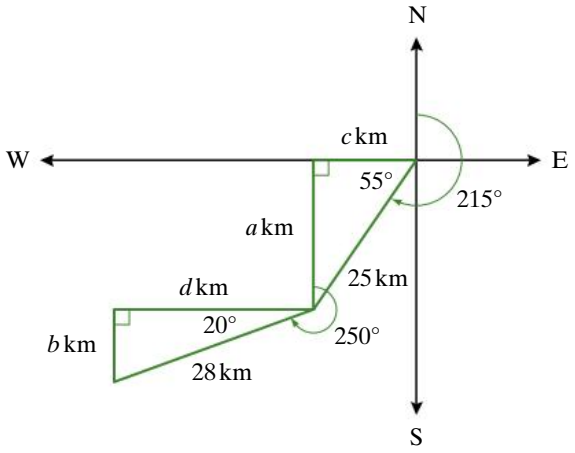
c Using your answers to part **b**, draw a new right-angled triangle to show the final distances south and west of the starting point.

d Find the distance of the ship from its starting point.

e Find the bearing of the ship from its starting point.

Working

a



Reasoning

Since west is 270° and 270° − 215° = 55°, the first leg of the path is on a bearing 55° south of west.
Similarly, the second leg of the path is 20° (270° − 250°) south of west.
To find the total distance south, we need to find a and b.
To find the total distance west, we need to find c and d.

b

$$\sin 55° = \frac{a}{25}$$
$$a = 25 \sin 55°$$
$$\approx 20.5$$

The side labelled a km is **O**pposite the 55° angle. The side labelled 25 km is the **H**ypotenuse. So use **S**ine.
$25 \times \sin(55) = 20.4788$

$$\sin 20° = \frac{b}{28}$$
$$b = 28 \sin 20°$$
$$\approx 9.6$$

The side labelled b km is **O**pposite the 20° angle. The side labelled 28 km is the **H**ypotenuse. So use **S**ine.
$28 \times \sin(20) = 9.57656$

Total distance south
$\approx 20.47\,\text{km} + 9.58\,\text{km}$
$\approx 30.05\,\text{km}$
$= 30.1\,\text{km}$ correct to one decimal place.

The total distance south is $(a + b)$ km.
Use two decimal places for a and b.
Round after adding.

continued

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Example 26 continued

Working

$$\cos 55° = \frac{c}{25}$$
$$c = 25 \cos 55°$$
$$\approx 14.3$$

Reasoning

The side labelled c km is **A**djacent to the 55° angle. The side labelled 25 km is the **H**ypotenuse. So use **C**osine.
$25 \times \cos(55) = 14.3394$

$$\cos 20° = \frac{d}{28}$$
$$d = 28 \cos 20°$$
$$\approx 26.3$$

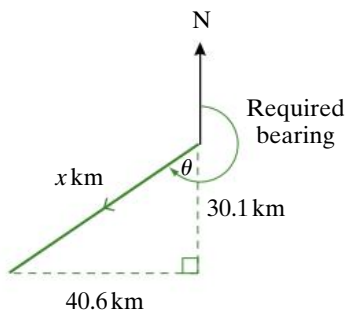
The side labelled d km is **A**djacent to the 20° angle. The side labelled 28 km is the **H**ypotenuse. So use **C**osine.
$28 \times \cos(20) = 26.3114$

Total distance west
$\approx 14.34\,\text{km} + 26.31\,\text{km}$
$\approx 40.65\,\text{km}$
$= 40.7\,\text{km}$ correct to one decimal place.

The total distance west is $(c + d)$ km.
Use two decimal places for c and d.
Round after adding.

The yacht is 30.1 km south and 40.7 km west of its starting point.

c



d $x \approx \sqrt{30.1^2 + 40.7^2}$
$x \approx 50.62$
The yacht is approximately 50.6 km from its starting point.

The distance of the yacht from its starting point is the hypotenuse of the right-angled triangle. Use Pythagoras' theorem to calculate the distance.

e $\tan\theta = \dfrac{40.7}{30.1}$
$\theta = \tan^{-1}\left(\dfrac{40.7}{30.1}\right)$
$\theta \approx 53.5°$
$53.5° + 180° = 233.5°$
The bearing of the yacht from its starting point is 233.5°.

Calculate θ.
The true bearing is the angle measured in a clockwise direction from north.

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In some two-leg journeys, the first leg of the journey may have, for example, a north component and the second leg may have a south component. The distances will then need to be subtracted to find the resulting distance north or south.

Example 27

A ship sails for 120 km on a bearing of 018°, then for a further 150 km on a bearing of 116°.

a Draw a labelled diagram to show the given and required information.

b How far is the ship east of its starting point?

c How far is the ship north of its starting point?

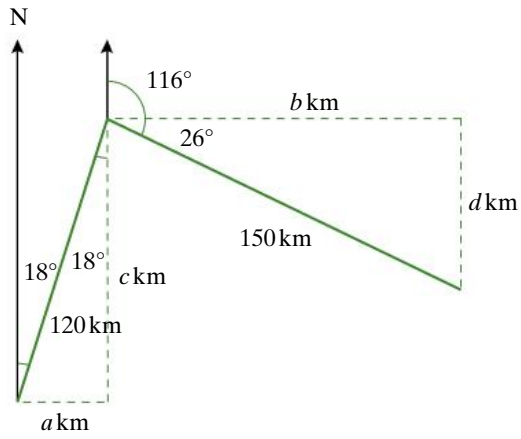
d Using your answers to parts **b** and **c**, draw a new right-angled triangle to show the final distances east and north of the starting point.

e Find the distance of the ship from its starting point.

f Find the bearing of the ship from its starting point.

Working | **Reasoning**

a



b $\sin 18° = \dfrac{a}{120}$

$a = 120 \sin 18°$

$a \approx 37.1$

$\cos 26° = \dfrac{b}{150}$

$b = 150 \cos 26°$

$b \approx 134.8$

The ship travels east on both legs of the journey.

$a + b = 37.1 + 134.8$

$a + b = 171.9$

The total distance east is approximately 172 km.

The two distances are added to find the total distance east of the starting point.

continued

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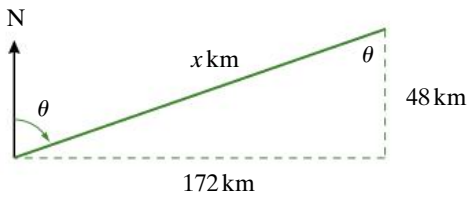
Example 27 continued

Working	Reasoning
c $\cos 18° = \dfrac{c}{120}$ $c = 120 \cos 18°$ $c \approx 114.1$ $\sin 26° = \dfrac{d}{150}$ $d = 150 \sin 26°$ $d \approx 65.8$	In the first leg of the journey, the ship travels north. In the second leg, the ship travels south.
$c - d = 114.1 - 65.8$ $c - d = 48.3$ The total distance north is approximately 48 km.	The distances are subtracted to find the distance north of the starting point.
d 	
e $x = \sqrt{172^2 + 48^2}$ $x = \sqrt{31888}$ $x \approx 178.6$ The ship is 179 km from its starting point.	Use Pythagoras' theorem to find x.
f $\tan\theta = \dfrac{171.9}{48.3}$ $\theta = \tan^{-1}\dfrac{171.9}{48.3}$ $\theta \approx 74°$ The bearing of the ship from its starting point is 074°.	The required bearing is θ. The alternate angle in the triangle is also θ. Side labelled 172 km is **O**pposite and side labelled 48 km is **A**djacent so use **T**angent.

exercise 2.8

LINKS TO Example 19

1 For each of these compass directions

i draw a diagram to show the direction in relation to the four compass points.

ii rewrite each compass direction as a bearing.

a NW	**b** SW	**c** NE	**d** SSW
e WNW	**f** SSE	**g** S15° E	**h** S75° W
i N20° W	**j** N50° E	**k** S80° E	**l** S30° W

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LINKS TO Example 20

2 Draw diagrams to match the following bearings.

a 070°	**b** 305°	**c** 270°	**d** 090°
e 240°	**f** 048°	**g** 340°	**h** 185°
i 125°	**j** 080°	**k** 180°	**l** 220°

3 A hiker walks on a bearing 090° for 3 km then on a bearing of 180° for 3 km.

a Draw a diagram to show the hiker's route.

b In which compass direction would the hiker have to walk to return directly to the starting point?

c What bearing is this?

LINKS TO Examples 21, 22

4 A boat sails for 40 km in each of these directions. For each direction

i draw a labelled diagram to show the direction and distance travelled.

ii calculate the distance the boat sails north (or south) and east (or west) of its starting point.

a N40°W	**b** S15°E	**c** N54°E	**d** N72°W
e S80°W	**f** N30°E	**g** S50°W	**h** S75°E
i N20°W	**j** S64°E	**k** N80°E	**l** S10°W

LINKS TO Example 23

5 A plane flies for 120 km in each of these directions. For each direction

i draw a labelled diagram to show the direction and distance travelled.

ii calculate the distance the plane flies north (or south) and east (or west) of its starting point

a 070°	**b** 035°	**c** 270°	**d** 090°
e 240°	**f** 048°	**g** 350°	**h** 165°
i 125°	**j** 100°	**k** 180°	**l** 220°

6 The walking track to a campsite is 8 km long, on a bearing of 142°.

a Draw a diagram showing the direction of the track.

b Using a right-angled triangle drawn on your diagram, calculate how far east and how far south the campsite is from the start of the track.

7 A yacht sails for 20 km on a bearing 320°.

a Draw a diagram showing the direction of the yacht's path.

b Using a right-angled triangle drawn on your diagram, calculate how far west and how far north the yacht is from its starting point.

8 A yacht sails for 12.6 km on a bearing of 235°.

a Draw a labelled diagram to show this information.

b Correct to one decimal place, how far

i west has the yacht sailed?

ii south has the yacht sailed?

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9 Some Year 10 students are using a GPS receiver to monitor their position on a hike. They have walked 7.3 km on a bearing of 142°.

a Draw a labelled diagram to show this information.

b Correct to one decimal place, how far

i east have they walked?

ii south have they walked?

10 A plane flies for 180 km on a bearing 235°.

a Draw a labelled diagram to show this information.

b How far south and how far west has it flown?

LINKS TO Example 24

11 A yacht is sailing on a bearing of 320° and is 65.0 km north of its starting point.

a Draw a labelled diagram to show this information.

b Find how far the yacht is west of its starting point.

c What is the direct distance of the yacht from its starting point?

LINKS TO Example 25

12 Oliver walked in a direction so that his final position was 7.5 km west and 5.8 km north of his starting point.

a Draw a labelled diagram to show this information.

b On what bearing did he walk?

13 Olivia walks 5 km north and then 3.2 km east.

a Draw a labelled diagram of the right-angled triangle formed by Olivia's route.

b What is Olivia's bearing from her starting point? Give the angle correct to the nearest degree.

14 Jack hikes 7.4 km west then 9.3 km due south.

a Draw a labelled diagram of the right-angled triangle formed by Jack's route.

b What is Jack's bearing from his starting point? Give the angle to the nearest degree.

15 A plane flies 680 km in a direct line from its starting point so that its position is now 310 km south of its starting point, and it is east of its starting point.

a Draw a labelled diagram to show this information.

b How far east is the plane of its starting point?

c On what bearing is the plane flying?

16 A plane flew in a direction so that the final position was 180 km east and 132 km south of the starting point.

a Draw a labelled diagram to show this information.

b On what bearing did the plane fly?

LINKS TO Example 26

17 A ship sails for 15 km on a bearing 070°, then changes direction and sails for a further 20 km in a direction 025°.

a Draw a diagram showing the ship's route.

b Using two right-angled triangles drawn on your diagram, calculate how far north and how far east the ship is from its starting point.

c Calculate the bearing of the ship from its starting point.

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LINKS TO Example 27

18 A plane flies for 300 km on a bearing 170° then for a further 400 km on a bearing 224°.

a Draw a labelled diagram to show the plane's route.

b Using two right-angled triangles drawn on your diagram, find the distance travelled south of the starting point.

c Using two right-angled triangles drawn on your diagram, find the distance travelled east (or west) of the starting point.

19 A plane flies 150 km on a bearing of 045°, then turns and flies for a further 200 km on a bearing of 089°.

a Draw a labelled diagram to show this information.

b How far east of its starting point is the plane?

c How far north of its starting point is the plane?

d What is the direct distance of the plane from its starting point?

e What is the bearing of the plane from its starting point?

20 A yacht sails 12.7 km on a bearing of 135°, then turns and sails for a further 14.5 km on a bearing of 205°.

a Draw a labelled diagram to show this information.

b How far south is the yacht from its starting point?

c How far east or west is the yacht from its starting point?

d Using your answers from parts **b** and **c** draw a new right-angled triangle to show the distances south and east/west.

e What is the direct distance of the yacht from its starting point?

f What is the bearing of the yacht from its starting point?

exercise 2.8 challenge

21 A yacht is taking part in a three-leg race that takes it back to its starting point. The first leg is 5 km east and the second leg is 8 km on a bearing 210°.

a Draw a labelled diagram to show this information.

b What is the length of the third leg?

c On what bearing does the yacht sail this third leg? Give the bearing correct to one decimal place.

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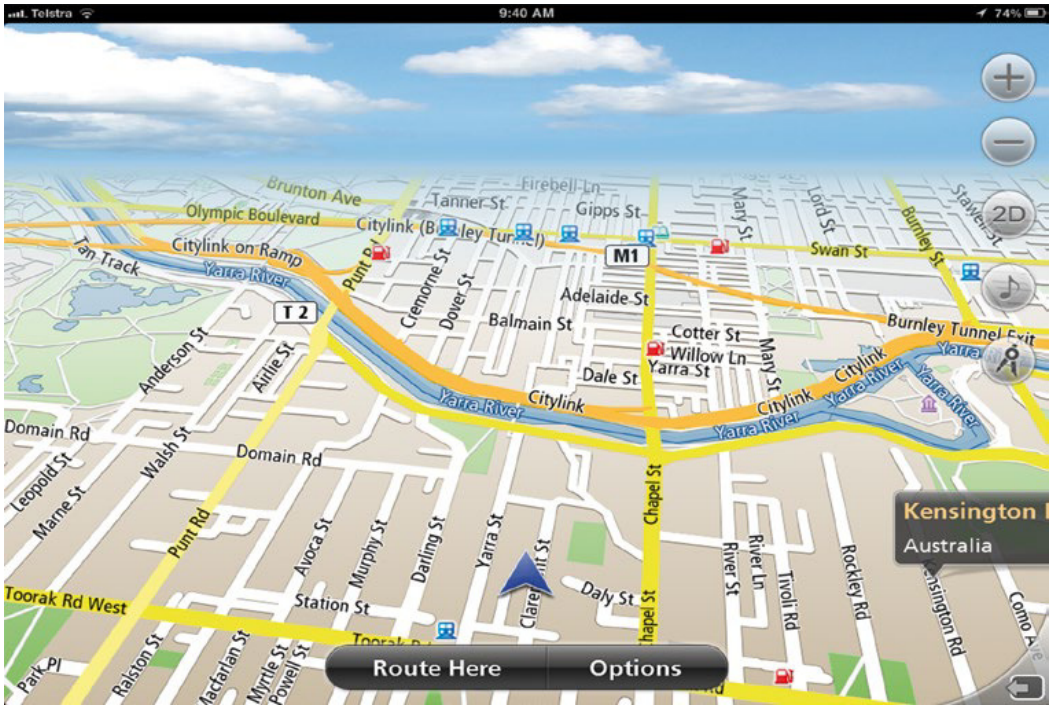
Analysis task

GPS and dead reckoning

The position of a person holding a GPS receiver may be given in terms of latitude and longitude or as coordinates. UTM (Universal Transverse Mercator) coordinates divide the Earth's surface into a grid system so that any position can be identified by an **Easting** (distance east of the reference line) and a **Northing** (distance north of the reference line). The Easting and Northing coordinates are in metres.

In-car navigation systems use a GPS receiver to track the coordinates of the position of the car. The coordinates are determined by combining information received from satellites. When a car is in a tunnel or driving through a dense urban area where the receiver cannot pick up sufficient satellite information, the car's navigation system uses **dead reckoning** sensors. These sensors calculate the position of the car by combining direction information from a compass or gyroscope and distance information from the car's odometer.



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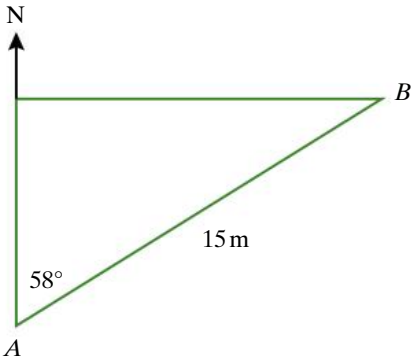
In the diagram below, a car is initially at position A, where the GPS receiver reports the coordinates as 323683.645 m, 5825587.664 m. The car enters a tunnel. At point B the compass determines that the car is travelling in a direction 058° and that it has travelled 15 m.



323683.645 is the Easting and 5825587.664 is the Northing.

a Calculate how far east the car is from A.

b Calculate how far north the car is from A.

c Determine the coordinates of point B.

A car is travelling at constant speed on a straight road. The in-car navigation system of the car gives the coordinates in metres as 592562.793, 5826950.987. After 2 minutes the coordinates are 595061.065, 5824722.601.

d Explain how you know that the car is travelling in a direction between 090° and 180°.

e Draw a labelled diagram to show the approximate direction in which the car is travelling.

f How far east has the car travelled?

g How far south has the car travelled?

h At what speed is the car travelling?

i On what bearing is the car travelling?

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Review *Trigonometry and Pythagoras' theorem*

Summary

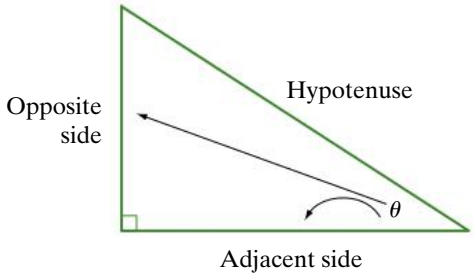
Trigonometric ratios

Right-angled triangles

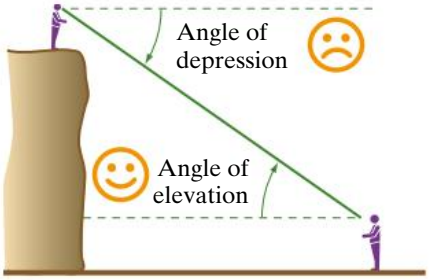
$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$



Angles of elevation and depression

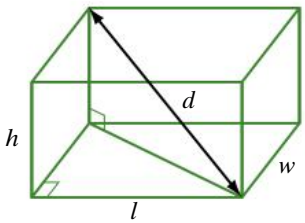


Bearings

A bearing is measured in a clockwise direction from north and is expressed as an angle form 0° to 360° using three digits, (e.g. 056°).

Pythagoras' theorem in three dimensions

$d^2 = l^2 + w^2 + h^2$



10 A
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Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

adjacent side	compass direction	inverse cosine	reciprocal
angle of depression	complementary angle	inverse sine	similar
angle of elevation	cosine	inverse tangent	sine
bearing	hypotenuse	opposite side	tangent
		ratio	

Revision

Multiple-choice questions

1 The value of a is

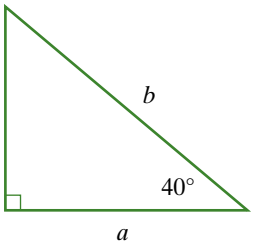
A $b\cos 40°$

B $\dfrac{b}{\cos 40°}$

C $b\sin 40°$

D $\dfrac{b}{\sin 40°}$

E $\dfrac{\sin 40°}{b}$



2 The length of the shadow of a post is 5 m when the angle of elevation of the sun is 37°. The height of the post in metres is

A $\dfrac{5}{\sin 37°}$ **B** $5 \times \sin 37°$ **C** $\dfrac{5}{\tan 37°}$ **D** $\dfrac{\tan 37°}{5}$ **E** $5 \times \tan 37°$

3 A ladder that is 2.6 m long has its foot resting on horizontal ground, with the other end leaning against a wall. The top of the ladder is 1.9 m from the ground. The angle that the ladder makes with the vertical wall is closest to

A 36° **B** 43° **C** 47° **D** 54° **E** 57°

4 To the nearest whole numbers

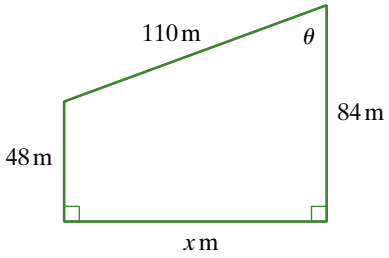
A $x = 36, \theta = 71°$

B $x = 42, \theta = 40°$

C $x = 66, \theta = 40°$

D $x = 88, \theta = 40°$

E $x = 104, \theta = 71°$



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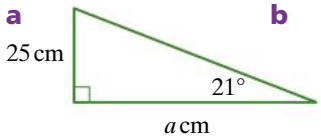
5 A plane flies on a bearing of 220° for 300 km. In relation to its starting point, the plane is now approximately

A 193 km east and 230 km south
B 193 km west and 230 km south
C 230 km east and 193 km south
D 252 km east and 358 km south
E 358 km west and 252 km south

Short-answer questions

6 Find the value of each pronumeral, correct to one decimal place.

a


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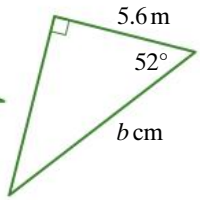

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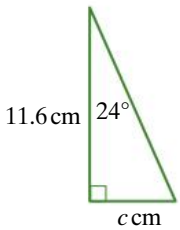

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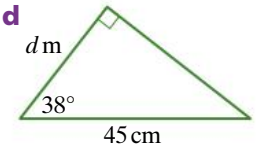
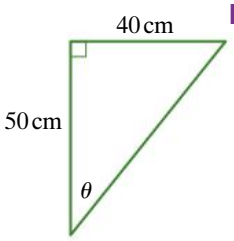


7 Find the size of the angle labelled θ, correct to one decimal place.

a


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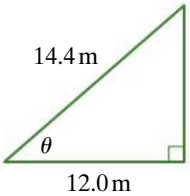

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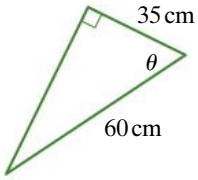

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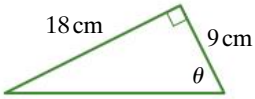
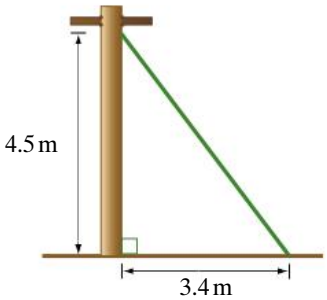


8 A kite is flying at the end of a 40 m string. The string makes an angle of 36° with the ground.

a Draw a labelled diagram.
b What is the height of the kite to the nearest metre?

9 A cable is attached to a pole 4.5 m above the ground as shown. The cable is attached to a peg in the ground 3.4 m from the base of the pole.



a What angle does the cable make with the ground? Give the angle correct to one decimal place.
b Calculate the length of the cable, correct to one decimal place
  i using Pythagoras' theorem
  ii using trigonometry and the angle you calculated in part **a**.

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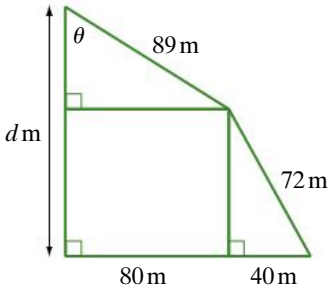
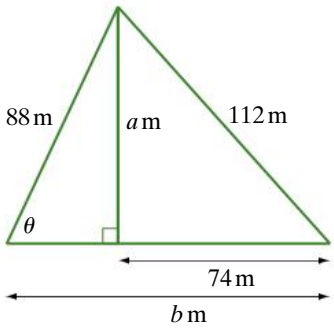
10 Find the unknown lengths and angles in each of the following. Give the lengths and angles correct to an appropriate number of decimal places.

a



b



11 The angle of depression from the top of a tower to a point on the ground 125 m from the base of the tower is 14°. What is the height of the tower, correct to the nearest metre?

12 The angle of elevation to the top of a tree from a point on the ground 50 m from the foot of the tree is 50°.

a Draw a labelled diagram.

b Calculate the height of the tree to the nearest metre.

13 For each of these compass directions

i draw a diagram to show the direction in relation to the four compass points.

ii rewrite each compass directions as a bearing.

a NE **b** WSW **c** N40°W **d** S25°E

14 Draw diagrams to match the following bearings.

a 040° **b** 260° **c** 130° **d** 320°

Extended-response questions

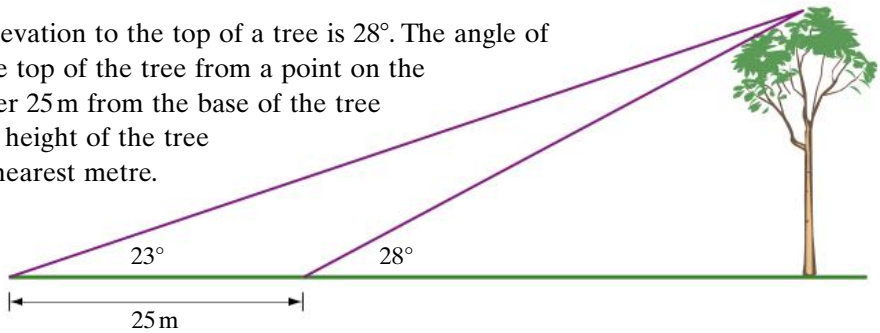
15 The angle of depression from the top of a tower to a point on the ground 30 m from the base of the tower is 23°.

a Draw a labelled diagram.

b Calculate the height of the tower, correct to one decimal place.

c Calculate the angle of depression from the top of the tower to a point on the ground 36 m from the base of the tower. Give the angle correct to the nearest degree.

16 The angle of elevation to the top of a tree is 28°. The angle of elevation to the top of the tree from a point on the ground a further 25 m from the base of the tree is 23°. Find the height of the tree correct to the nearest metre.



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17 A plane flies for 1400 km on a bearing 285°.

a Draw a labelled diagram of the right-angled triangle formed by the plane's route and the distances west and north of the starting point.

b Calculate the following, correct to one decimal place.

i How far west of the starting point is the plane?

ii How far north of the starting point is the plane?

18 Year 10 students on camp hike for 11 km on a bearing 048° then for a further 8 km on a bearing 080°.

a Draw a diagram to show the route and the two right-angled triangles to use.

b What is the total distance east that the students have hiked (correct to one decimal place)?

c What is the total distance north that they have hiked (correct to one decimal place)?

19 A ship sails for 48 km on a bearing of 124°, then turns and sails for a further 75 km on a bearing of 163°.

a Draw a careful diagram to show the ship's route, labelling all the given information.

b Calculate how far east of its starting point the ship has sailed, correct to one decimal place.

c Calculate how far south of its starting point the ship has sailed, correct to one decimal place.

d Using your answers from parts **b** and **c**, draw a new right-angled triangle to show the final distances east and south of the starting point.

e What is the direct distance of the ship from its starting point? Give the distance correct to one decimal place

f What is the bearing of the ship from its starting point? Give the bearing correct to the nearest degree.

20 A yacht sails for 8.4 km on a bearing of 310°, then turns and sails for a further 9 km on a bearing of 215°.

a Draw a careful diagram to show the yacht's route, labelling all the given information.

b Calculate how far west of its starting point the yacht has sailed, correct to one decimal place.

c Calculate how far north or south of its starting point the yacht has sailed, correct to one decimal place.

d Using your answers from parts **b** and **c**, draw a new right-angled triangle to show the final distances west and north or south of the starting point.

e What is the direct distance of the yacht from its starting point? Give the distance correct to one decimal place

f What is the bearing of the yacht from its starting point? Give the bearing correct to the nearest degree.

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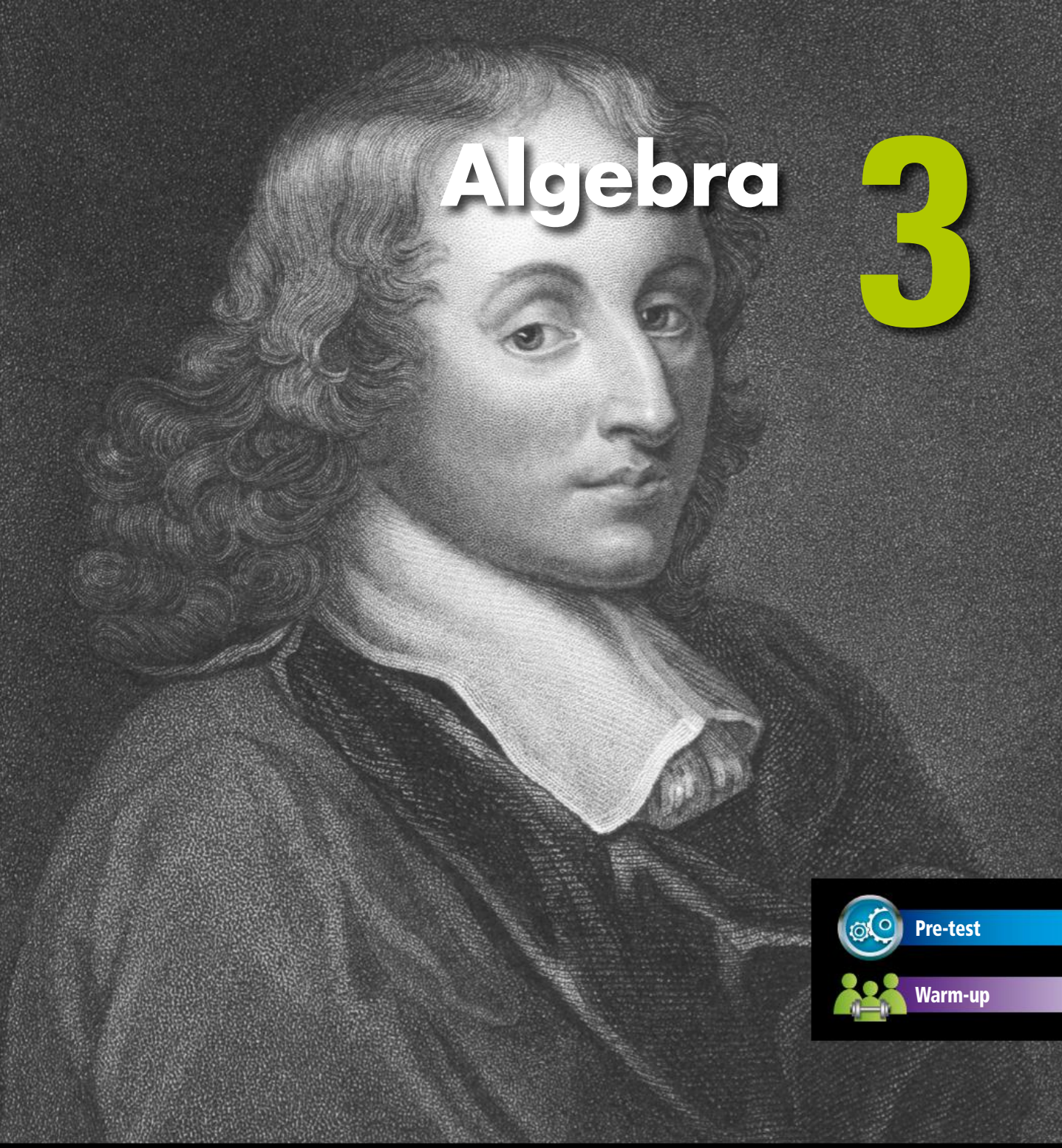
Algebra 3



Pre-test

Warm-up

Blaise Pascal was a French mathematician, physicist and philosopher. He was born in 1623 and died in 1662. During his short life he did important work in algebra, geometry, probability and physics. Pascal explored the coefficients of terms in binomial expansions of the type $(a + b)^n$. In this chapter, you will work with binomial expansions of the type $(a + b)^2$ but the analysis task at the end of the chapter will give you an opportunity to explore other binomial expansions.

3.1 Indices and algebraic fractions

This section reviews techniques for simplifying algebraic expressions, including working with positive and negative indices.

Positive and negative indices

Recall from Year 9 that indices may be positive or negative. A number or pronumeral raised to a negative power is equal to the reciprocal of that number or pronumeral raised to the positive power.

Negative indices

$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$

When simplifying expressions in index form, we normally give the simplified expression with positive indices.

Example 1

Express the following with positive indices.

a $2x^{-4}$ **b** $\frac{2m}{n^{-3}}$ **c** $\frac{a^{-2}}{b^{-3}}$

Working

a $2x^{-4} = \frac{2}{x^4}$

b $\frac{2m}{n^{-3}} = 2mn^3$

c $\frac{a^{-2}}{b^{-3}} = \frac{b^3}{a^2}$

Reasoning

Write the reciprocal of x and raise it to the positive power: $x^{-4} = \frac{1}{x^4}$

$\frac{1}{n^{-3}} = n^3$

$a^{-2} = \frac{1}{a^2}$ and $\frac{1}{b^{-3}} = b^3$

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Multiplying and dividing expressions in index form

Pronumerals with indices can be mutiplied by adding the indices provided they have the same pronumeral base. Similarly, pronumerals with indices can be divided by subtracting the indices provided they have the same pronumeral base.

Multiplication index law

When multiplying numbers or pronumerals that have the same base, add the indices.

Division index law

When dividing numbers or pronumerals that have the same base, subtract the indices.

Example 2

Use the first and second index laws to write each of the following in simpler index form.

a $-2a^3 \times 3a^5$ **b** $\dfrac{5b^{12}}{15b^4}$ **c** $\dfrac{21x^3}{9x^6}$

Working	Reasoning
a $-2a^3 \times 3a^5 = -6a^{3+5}$ $= -6a^8$	Multiply the coefficients. The bases are the same so the indices can be added.
b $\dfrac{5b^{12}}{15b^4} = \dfrac{\overset{1}{\cancel{5}}b^{12-4}}{\underset{3}{\cancel{15}}}$ $= \dfrac{b^8}{3}$	Simplify the coefficients. The bases are the same so the indices can be subtracted.
c $\dfrac{\overset{7}{\cancel{21}}x^3}{\underset{3}{\cancel{9}}x^6} = \dfrac{7x^{3-6}}{3}$ $= \dfrac{7x^{-3}}{3}$ $= \dfrac{7}{3x^3}$	Simplify the coefficients by dividing numerator and denominator by the common factor 3. The bases are the same so the indices can be subtracted. Express with positive index.

Zero index

Any pronumeral raised to the power 0 is equal to 1.

$a^0 = 1$ where $a \neq 0$

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Example 3

Simplify

a $3x^0$ **b** $(3x)^0$ **c** $-2x^{-3} \times 4x^5 \times x^{-2}$

Working	Reasoning
a $3x^0 = 3$	$x^0 = 1$
b $(3x)^0 = 1$	Only x is raised to the power 0. $(3x)$ is raised to the power 0.
c $-2x^{-3} \times 4x^5 \times x^{-2} = -8x^{-3+5-2}$ $= -8x^0$ $= -8$	Group the coefficients. Add the indices of the x terms. $x^0 = 1$

Powers with brackets

Raising a power to a power index law

When a power is raised to another power, multiply the indices.

$(a^m)^n = a^{m \times n}$

Example 4

Express the following without brackets.

a $(m^4)^3$ **b** $(-b^2)^{-2}$

Working	Reasoning
a $(m^4)^3 = m^{12}$	$(m^4)^3 = m^{4 \times 3}$
b $(-b^2)^{-2} = \dfrac{1}{(-b^2)^2}$ $= \dfrac{1}{b^4}$	$(-b^2)^2 = b^4$

Powers of products and quotients

Power of a product law index law

If a product of numbers or pronumerals in brackets is raised to a power, each number or pronumeral inside the brackets is raised to the power outside the brackets.

$(ab)^m = a^m \times b^m$

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Power of a quotient or fraction index law

When a fraction is raised to a power, both the denominator and the numerator are raised to the same power.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

When a product or quotient in brackets includes coefficients, we normally evaluate the coefficients that are powers, for example, $(-2a)^3 = (-2)^3 \times a^3 = -8a^3$.

Example 5

Express each of the following without brackets and with positive indices.

a $(-2a^{-2}b^3)^4$ **b** $(3m^{-2}n^3)^{-3}$ **c** $\left(\frac{2a}{b^{-2}}\right)^3$

Working **Reasoning**

a $(-2a^{-2}b^3)^4 = (-2)^4 \times (a^{-2})^4 \times (b^3)^4$

$= 16a^{-2\times 4} \times b^{3\times 4}$

$= 16a^{-8}b^{12}$

$= \frac{16b^{12}}{a^8}$

b $(3m^{-2}n^3)^{-3} = 3^{-3}m^{-2\times(-3)}n^{3\times(-3)}$

$= 3^{-3}m^6n^{-9}$

$= \frac{m^6}{3^{-3}n^9}$

$= \frac{m^6}{27n^9}$

c $\left(\frac{2a}{b^{-2}}\right)^3 = \frac{2^3a^3}{b^{-2\times 3}}$

$= \frac{8a^3}{b^{-6}}$

$= 8a^3b^6$

Multiplying algebraic fractions

When multiplying or dividing algebraic fractions, we use the same technique as for number fractions, dividing the numerator and the denominator by the highest common factors.

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Example 6

Simplify $\dfrac{-8xy^3}{3x} \times \dfrac{-12x}{y}$

Working

$$\frac{-8xy^3}{3x} \times \frac{-12x}{y}$$
$$= \frac{8 \times 12 \times x^2y^3}{3xy}$$
$$= 32xy^2$$

Reasoning

Simplify the numerator and simplify the denominator. The product of the two negative fractions is positive.
The highest common factor of the numerator and denominator is $3xy$.

When we simplify an algebraic expression that includes products in both the numerator and the denominator, the following order is usually appropriate:

- simplify the coefficients
- simplify the pronumeral terms in the numerator
- simplify the pronumeral terms in the denominator
- use the division index law to simplify the algebraic fraction

Example 7

Simplify each of the following, expressing with positive indices.

a $\dfrac{4a^3b^4 \times 15ab^{-2}}{6a^{-2}b^3 \times 8ab}$

b $\dfrac{(2x^2y)^{-2} \times xy^2}{(-x^{-2}y^2)^{-3} \times (-xy)^4}$

Working

a
$$\frac{4a^3b^4 \times 15ab^{-2}}{6a^{-2}b^3 \times 8ab}$$
$$= \frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{15}}a^3 \times a^1 \times b^4 \times b^{-2}}{\underset{2}{\cancel{6}} \times \underset{2}{\cancel{8}}a^{-2} \times a^1 \times b^3 \times b^1}$$
$$= \frac{5a^{3+1} \times b^{4+(-2)}}{4a^{-2+1} \times b^{3+1}}$$
$$= \frac{5a^4b^2}{4a^{-1}b^4}$$
$$= \frac{5a^{4-(-1)}b^{2-4}}{4}$$
$$= \frac{5a^5b^{-2}}{4}$$
$$= \frac{5a^5}{4b^2}$$

Reasoning

Group the coefficients and simplify by dividing numerator and denominator by common factors.
For the pronumerals, it is usually easier to simplify the numerator and simplify the denominator first before simplifying the division.
Use the multiplication index law to simplify the numerator.
Repeat for the denominator.

Use the division index law to simplify the division.

Express with positive indices.

continued

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Example 7 continued

Working	Reasoning
b $\dfrac{(2x^2y)^{-2} \times xy^2}{(-x^{-2}y^2)^{-3} \times (-xy)^4}$	Use the multiplication index law to simplify the numerator.
$= \dfrac{2^{-2}x^{-4}y^{-2} \times xy^2}{-x^6y^{-6} \times -x^4y^4}$	Repeat for the denominator.
$= \dfrac{2^{-2}x^{-4+1}y^{-2-2}}{x^{6+4}y^{-6+4}}$	Use the division index law to simplify the division.
$= \dfrac{2^{-2}x^{-3}y^0}{x^{10}y^{-2}}$	
$= 2^{-2}x^{-3-10}y^{0-(-2)}$	
$= 2^{-2}x^{-13}y^2$	
$= \dfrac{y^2}{4x^{13}}$	Express with positive indices.

Adding and subtracting algebraic fractions

Algebraic fractions can be added and subtracted in the same way as we add and subtract number fractions.

Example 8

Express as single fractions.

a $\dfrac{4x}{5} - \dfrac{x}{5}$ **b** $\dfrac{4a}{3} + \dfrac{5a}{3}$ **c** $\dfrac{4x}{9} + \dfrac{x}{4} + \dfrac{2x}{3}$

Working	Reasoning
a $\dfrac{4x}{5} - \dfrac{x}{5}$	The denominators are the same. Subtract x from $4x$ to give the numerator.
$= \dfrac{3x}{5}$	
b $\dfrac{4a}{3} + \dfrac{5a}{3}$	The denominators are the same. Add $4a$ and $5a$ to give the numerator.
$= \dfrac{9a}{3}$	Divide the numerator and denominator by the common factor 3.
$= 3a$	
c $\dfrac{4x}{9} + \dfrac{x}{4} + \dfrac{2x}{3}$	The lowest common denominator is 36. Express the fractions as equivalent fractions with the common denominator 36.
$= \dfrac{4x}{9} \times \dfrac{4}{4} + \dfrac{x}{4} \times \dfrac{9}{9} + \dfrac{2x}{3} \times \dfrac{12}{12}$	
$= \dfrac{16x}{36} + \dfrac{9x}{36} + \dfrac{24x}{36}$	Add the numerators. The denominator is 36.
$= \dfrac{49x}{36}$	

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In part **b** of the following example, notice how brackets around the numerator of the second fraction are important to avoid errors with the negative signs.

Example 9

Simplify the following:

a $\dfrac{2x+5}{4}+\dfrac{3x-1}{3}$

b $\dfrac{3x+2}{3}-\dfrac{2x-1}{6}$

Working	Reasoning
a $\dfrac{2x+5}{4}+\dfrac{3x-1}{3}$	
$\dfrac{3(2x+5)}{12}+\dfrac{4(3x-1)}{12}$	Put both fractions over the common denominator, 12. Expand the brackets.
$=\dfrac{3(2x+5)+4(3x-1)}{12}$	Write both numerators over the common denominator.
$=\dfrac{6x+15+12x-4}{12}$	Simplify the numerator.
$=\dfrac{18x+11}{12}$	
b $\dfrac{3x+2}{3}-\dfrac{2x-1}{6}$	
$=\dfrac{2(3x+2)}{6}-\dfrac{2x-1}{6}$	Put the first fraction over the common denominator, 6.
$=\dfrac{2(3x+2)-(2x-1)}{6}$	Write both numerators over the common denominator, putting brackets around the second numerator. Expand the brackets.
$=\dfrac{6x+4-2x+1}{6}$	Simplify the numerator.
$=\dfrac{4x+5}{6}$	

exercise 3.1

LINKS TO Example 1

1 Express the following with positive indices.

a $3a^{-2}$ **b** $-4y^{-5}$ **c** $\dfrac{5x^2}{y^{-3}}$ **d** $\dfrac{m^{-4}}{n^{-2}}$

LINKS TO Example 2

2 Simplify the following, expressing with positive indices.

a $-3x^4\times 2x^3$ **b** $-3m^{-6}\times 4m^4$ **c** $3x^2\times(-5x^{-3})$ **d** $-a^{-2}\times(-3a^{-1})$

e $\dfrac{-12b^7}{-3b^{-4}}$ **f** $\dfrac{12a^9}{3a^3}$ **g** $\dfrac{-15x^2}{27x^4}$ **h** $\dfrac{-15x^2}{27x^4}$

LINKS TO Example 3

3 Simplify the following.

a $-2x^0$ **b** $\dfrac{4x^0}{5}$ **c** $-3a^7\times 4a^{-2}\times a^{-5}$ **d** $\dfrac{-5b^2\times 2b^{-3}}{4b^3\times b^{-4}}$

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LINKS TO Example 4

4 Simplify the following, expressing with positive indices.

a $(a^2)^4$ **b** $(-x^{-3})^2$ **c** $(-y^{-2})^4$ **d** $(-x^{-3})^2$

LINKS TO Example 5

5 Simplify, expressing with positive indices.

a $(a^2b)^3$ **b** $(c^3d^{-2})^{-2}$ **c** $(3x^{-3}y^2)^3$ **d** $(-2a^{-4}b^2)^{-3}$

e $\left(\frac{2a}{b^2}\right)^3$ **f** $\left(\frac{-2x}{y^{-3}}\right)^2$ **g** $\left(\frac{-3m}{n^{-2}}\right)^4$ **h** $\left(\frac{5x^{-2}}{y^{-3}}\right)^{-3}$

LINKS TO Example 6

6 Simplify each of the following.

a $12a \div 3b$ **b** $8x \div 2x$ **c** $8x \div (-12y)$

d $6ab \div 8ac$ **e** $-15ab \div 10bc$ **f** $20abc \div 14ac$

g $(-100de) \div (-80efg)$ **h** $9xyz \div 12yz$ **i** $\frac{4xy}{6xz} \times \frac{10xy}{2x}$

j $\frac{4xy}{-6x} \times \frac{5xy}{2x}$ **k** $\frac{xy}{6z} \times \frac{-10xy}{-2x}$ **l** $\frac{4xy}{-6} \times \frac{-12xy}{15y}$

m $\frac{2abc}{3ad} \times \frac{-5ab}{2b}$ **n** $\frac{7fg}{-6g} \times \frac{3fx}{-2x}$ **o** $-\frac{3mn}{5mp} \times \frac{-10x}{2m}$

7 $\frac{6a^2b}{12a}$ simplifies to

A $2ab$ **B** $\frac{ab}{2}$ **C** $6ab$ **D** $\frac{ab}{6}$ **E** $\frac{b}{2a}$

LINKS TO Example 7

8 **a** $\frac{2mn^{-2} \times 14m^{-3}n}{35m^2n^{-3} \times 4mn^{-1}}$ **b** $\frac{(3xy^2)^{-3} \times 4x^2y}{(2x^2y^{-2})^2 \times (-xy)^3}$ **c** $\frac{(3ab^2)^{-2} \times (-2a^3b)^2}{(2ab^{-3})^3 \times (-ab)^{-3}}$

LINKS TO Example 8

9 Express as single fractions.

a $\frac{3x}{7} + \frac{2x}{7}$ **b** $\frac{8x}{11} - \frac{3x}{11}$ **c** $\frac{4x}{9} - \frac{x}{9} + \frac{2x}{9}$

d $-\frac{a}{6} + \frac{5a}{6} - \frac{7a}{6}$ **e** $\frac{3a}{5} + \frac{a}{3}$ **f** $\frac{5x}{8} + \frac{7x}{16}$

g $\frac{3x}{4} + \frac{x}{2} - \frac{4x}{3}$ **h** $\frac{3a}{4} + \frac{a}{3} - \frac{5a}{6}$ **i** $\frac{7x}{5} - \frac{3x}{4} + \frac{9x}{10}$

LINKS TO Example 9

10 Express as single fractions.

a $\frac{x+3}{5} + \frac{2x+1}{10}$ **b** $\frac{x-4}{6} + \frac{3x-1}{3}$ **c** $\frac{2x-3}{6} + \frac{x-5}{4}$

d $\frac{x+5}{2} + \frac{2x-7}{3}$ **e** $\frac{x+2}{3} - \frac{3x-2}{12}$ **f** $\frac{2x-5}{6} - \frac{4x-3}{12}$

g $\frac{3x-2}{4} - \frac{2x+3}{3}$ **h** $\frac{2x+1}{4} - \frac{x-3}{5}$ **i** $\frac{x+6}{3} - \frac{2x+1}{5}$

exercise 3.1 challenge

11 Simplify $\frac{3(x+3)}{2} - \frac{2(2x-1)}{3} + \frac{3x-4}{5}$

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3.2 Reviewing expanding and factorising

Expansion

Equivalent expressions give the same result for every value of the pronumerals. For example, $3(x + 5)$ is equivalent to $3x + 15$ for any value of x. We refer to $3(x + 5)$ as the **factorised form** of the expression and $3x + 15$ as the **expanded form**.

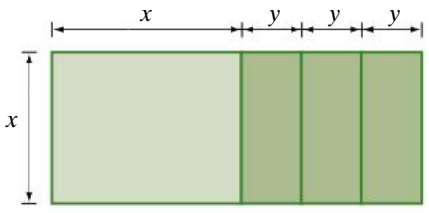
The process of going from the factorised form to the expanded form of an expression is called **expansion**.

The distributive law

In general, we can say that $a(b + c) = ab + ac$. This is called the **distributive law**.

Example 10

Write expressions for the total area of the rectangle in factorised form and in expanded form.



Working

$x(x + 3y) = x^2 + 3xy$

Reasoning

The width and length of the rectangle are x and $(x + 3y)$.
Applying the distributive law to $x(x + 3y)$:

$x(x + 3y)$

When we expand a factorised expression it is usual to simplify the expanded form, where possible.

Example 11

Expand each of the following, and simplify where possible.

a $a^2b(2a + 3b^2)$ **b** $2x(3x - 4) + 3x(x + 5)$ **c** $c(2c - 7) - (4c - 1)$

continued

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Example 11 continued

Working	Reasoning
a $a^2b(2a + 3b^2)$ $= 2a^3b + 3a^2b^3$	Multiply each term inside the brackets by a^2b. Remember that you can add indices when multiplying.
b $2x(3x - 4) + 3x(x + 5)$ $= 6x^2 - 8x + 3x^2 + 15x$ $= 9x^2 + 7x$	Multiply each term in the first set of brackets by $2x$ and each term in the second set of brackets by $3x$. Simplify by grouping like terms.
c $c(2c - 7) - (4c - 1)$ $= 2c^2 - 7c - 4c + 1$ $= 2c^2 - 11c + 1$	Expand the brackets using the distributive law. Simplify by grouping like terms.

Note that
$-(4c - 1) = -4c - (-1)$
$= -4c + 1$
Alternatively you could write
$-(4c - 1) = -1(4c - 1)$ and expand by multiplying each term in the brackets by -1

Factorisation: taking out a common factor

The expression $3(x + 5)$ is the factorised form of the expression $3x + 15$ because it is written as the product of the **factors** 3 and $x + 5$. The process of going from the expanded form to the factorised form is called **factorisation**.

To factorise $3x + 15$, the **highest common factor** (or **HCF**) of $3x$ and 15 (that is, 3) is taken out in front of brackets to give $3(x + 5)$.

Example 12

Find the highest common factor of

a $10x$ and $5x^2$ **b** $2ab$ and $6a^2b$ **c** $2x^3$ and $4x^2$ **d** $8xy$ and $-18x^2z$

Working	Reasoning
a $10x$ and $5x^2$ HCF is $5x$	HCF of 10 and 5 is 5. HCF of x and x^2 is x.
b $2ab$ and $6a^2b$ HCF is $2ab$	HCF of 2 and 6 is 2. HCF of a and a^2 is a. b is a factor of both terms.
c $2x^3$ and $4x^2$ HCF is $2x^2$	HCF of 2 and 4 is 2. HCF of x^3 and x^2 is x^2.

continued

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Example 12 continued

Working

d $8xy$ and $-18x^2z$
HCF is $2x$

Reasoning

HCF of 8 and -18 is 2.
HCF of x and x^2 is x.
y is not a factor of the second term.
z is not a factor of the first term.

Example 13

Factorise $12x^2 - 18xy$.

Working

$12x^2 - 18xy$
$= 6x(2x - 3y)$

Reasoning

The HCF of $12x^2$ and $-18xy$ is $6x$, so $6x$ goes in front of the bracket.

To find the terms inside the brackets, think: $6x \times ? = 12x^2$ and $6x \times ? = -18xy$.

When both terms are negative, and usually when the first term is negative, a negative common factor is taken out.

Example 14

Factorise.

a $-8ab - 18a^2b$

b $-6xy^2 + 21x^2y$

Working

a $-8ab - 18a^2b$
$= -2ab(4 + 9a)$

b $-6xy^2 + 21x^2y$
$= -3xy(2y - 7x)$

Reasoning

When the negative factor is taken out, the subtraction sign between the terms becomes an addition sign inside the bracket.

When the negative factor is taken out, the addition sign between the terms becomes a subtraction sign inside the bracket.

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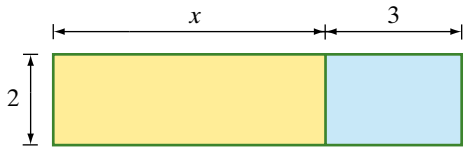
exercise 3.2

LINKS TO Example 10

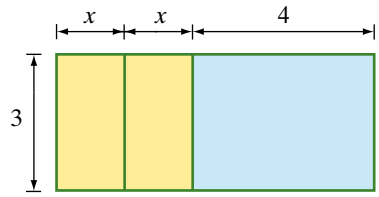
1 Find the area of each of the following rectangles, giving your answer in

i factorised form **ii** expanded form.

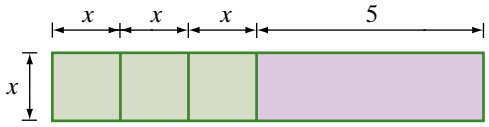
a



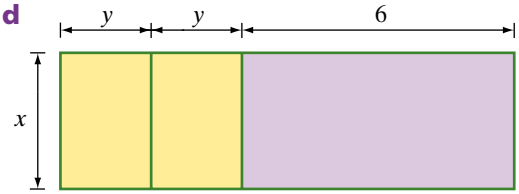
b



c



d



LINKS TO Example 11

2 Expand each of the following, and simplify where possible.

a $2(x + 4)$ **b** $3(a - 5)$ **c** $4(2b - 1)$
d $6(3m + 2)$ **e** $-3(x + 2)$ **f** $-a(a + 3)$
g $-4(b - 5)$ **h** $-2p(7 - p^2)$ **i** $2(a + 1) + 3(a + 2)$
j $7(x - 5) + 2(x + 3)$ **k** $3(x + 9) - 5(2 + x)$ **l** $5(3 - 2a) - 4(6 - 3a)$
m $k(k + 7) + 2(k + 2)$ **n** $x(x + 3) + 5(x - 4)$ **o** $m(m + 7) - 3(m + 5)$

3 Expand each of the following, and simplify where possible.

a $d(d - 2) - 3(2d + 5)$ **b** $g(3g - 8) - 2(g - 4)$ **c** $p(5 + 2p) - 3(4 - p)$
d $x(2x + 3) - 5(x - 4)$ **e** $2x(x - 7) - 3(5x - 2)$ **f** $3x(x - 4) - 5(2x + 3)$

LINKS TO Example 12

4 Find the highest common factor of these pairs of terms.

a $2a^2b$ and $4a$ **b** $10xy$ and $15x^2$ **c** $6ab$ and $2b^2$
d $-32m^2n^3$ and $-8mn^2$ **e** $-14x^2y^3$ and $-42xy$ **f** $-18ab$ and $-45a^2c$

LINKS TO Examples 13, 14

5 Factorise each of the following.

a $6b + 6$ **b** $5a - 25b$ **c** $4ab - 12b^2$
d $pq - p^2q^2$ **e** $2xy - 4x^2y^2$ **f** $6ab - 8a^2b$
g $4xy - 12x^2$ **h** $6x^2y - 10xy^2$ **i** $7m^2n^2 + 28mn$
j $12a^2b + 16a^2b^2$ **k** $5xy - 10x^2y^2$ **l** $9p^2q - 12pq$
m $10b^3 - 6a^2$ **n** $5x^2 + 15x^4$ **o** $m^2n + m^2n^2$
p $20a^2b^2 - 10ab^3$ **q** $-4a - 16b$ **r** $-7a^2b^3 - 35a^4b^2$
s $-8x^2y - 12xy^2$ **t** $-18xy^3 + 30x^3$ **u** $-48m^2n^2 + 54mn^3$

exercise 3.2 challenge

6 Find the lowest common multiple of $5x^2 - 25xy$ and $10x^2y^2 + 15xy^3$. Express your answer in both factorised form and expanded form.

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3.3 Binomial expansion

A binomial expression is an expression made up of two terms. For example, $(x + 2)$ and $(2a - 3)$ are binomial expressions.

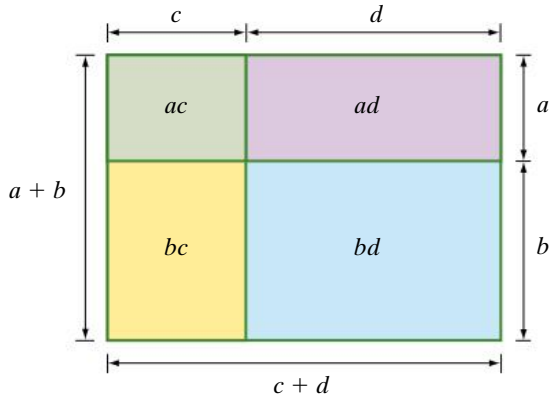
The expression $(a + b)(c + d)$ is the product of two binomial factors.

The following diagram illustrates the expansion of $(a + b)(c + d)$.

Binomial expansion

Binomial expansion

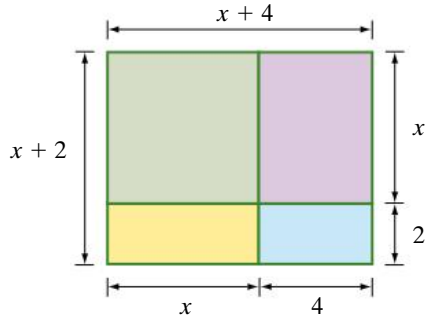
$(a + b)(c + d) = ac + ad + bc + bd$



Binomial expansions: a matching activity

Example 15

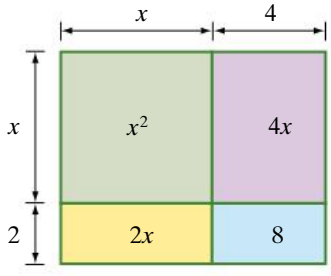
Use this diagram to find the expansion of $(x + 2)(x + 4)$.



Working

$$(x + 2)(x + 4) = x^2 + 4x + 2x + 8$$
$$= x^2 + 6x + 8$$

Reasoning



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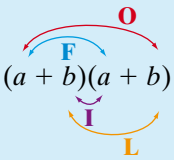
Algebraically, expressions like the previous example $(x + 2)(x + 4)$ are expanded by multiplying the whole of the second bracket by each term in the first bracket, which results in four terms. Sometimes, there are **like terms** that can be gathered.

Using FOIL to expand

To help remember the four expansion terms, you can use the word FOIL, which stands for

- **F**irst: find the product of the first term in each bracket.
- **O**utside: find the product of the first term in the first bracket and the last term in the second bracket.
- **I**nside: find the product of the last term in the first bracket and the first term in the second bracket.
- **L**ast: find the product of the last term in each bracket.



Example 16

Expand each of the following, and simplify where possible.

a $(x - 3)(x + 7)$ **b** $(2a - 5)(a - 3)$ **c** $(a - 7)(b^2 + 4)$

Working	Reasoning
a $(x - 3)(x + 7)$ $= x^2 + 7x - 3x - 21$ $= x^2 + 4x - 21$	Use FOIL and collect like terms. $(x - 3)(x + 7)$
b $(2a - 5)(a - 3)$ $= 2a^2 - 6a - 5a + 15$ $= 2a^2 - 11a + 15$	Use FOIL and collect like terms. $(2a - 5)(a - 3)$ The last term is positive because $-5 \times (-3) = 15$.
c $(a - 7)(b^2 + 4)$ $= ab^2 + 4a - 7b^2 - 28$	Use FOIL. $(a - 7)(b^2 + 4)$

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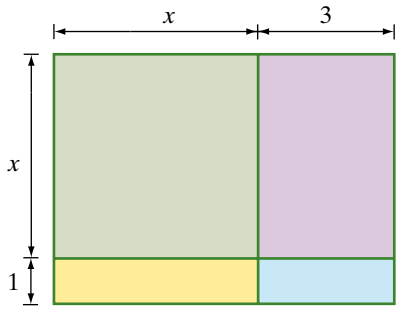
exercise 3.3

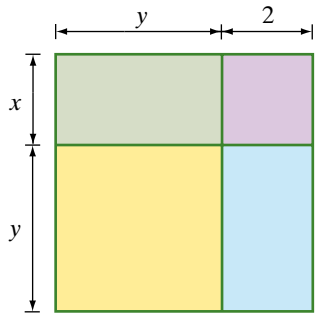
LINKS TO Example 15

1 Use each diagram to write an algebraic expression in factorised form and in expanded form.

a



b

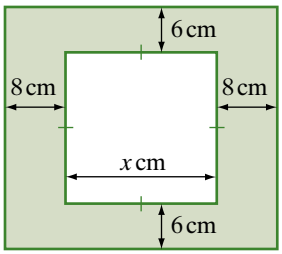


LINKS TO Example 16

2 Expand and simplify each of the following.

a $(a + 4)(a + 2)$ **b** $(b + 3)(b + 5)$ **c** $(c - 3)(c + 7)$
d $(d + 1)(d - 4)$ **e** $(e - 6)(e - 2)$ **f** $(f - 5)(f - 3)$
g $(2g + 3)(g + 5)$ **h** $(3h - 2)(2h - 1)$ **i** $(x + 3)(2x + 1)$
j $(a - 8)(3a + 4)$ **k** $(m - 7)(4m + 1)$ **l** $(2y - 3)(3y + 2)$
m $(3a - 4b)(a + 2b)$ **n** $(m - 3n)(5m - 6n)$ **o** $(2x - 1)(3x + 5)$
p $(4 - x)(5x + 2)$ **q** $(7 - a)(4 - 3a)$ **r** $(2a - 5b)(3b - 4a)$

3 A square mirror is surrounded by a frame, as shown in the diagram.



a Write down expressions for the length and width of the framed mirror in terms of x.
b Write down an expression for the area of the framed mirror in expanded form.
c Find the area of the framed mirror when $x = 40$.

exercise 3.3 challenge

4 Expand and simplify each of the following.

a $(x + 1)(x + 2)(x + 3)$ **b** $2(x - 4)(x + 1)(x - 2)$

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3.4 Common binomial factors

Some algebraic expressions have binomial expressions, rather than single terms, as factors.

In the expression $x(x - 1) + 2(x - 1)$, the common factor is $x - 1$. As shown in the following example, this term can be taken out like any other common factor.

Example 17

Factorise each of the following.

a $x(x - 1) + 2(x - 1)$

b $p(2p - 3) - 2(2p - 3)$

c $a(a - 3) + a - 3$

d $16xy(y + 3) - 24y(y + 3)$

Working	Reasoning
a $x(x - 1) + 2(x - 1)$ $= (x - 1)(x + 2)$	The common factor of $x - 1$ is taken out in front of a bracket.
b $p(2p - 3) - 2(2p - 3)$ $= (2p - 3)(p - 2)$	The common factor of $2p - 3$ is taken out in front of a bracket.
c $a(a - 3) + a - 3$ $= a(a - 3) + 1(a - 3)$ $= (a - 3)(a + 1)$	Writing $+ a - 3$ as $+ 1(a - 3)$ makes it easier to see that $a - 3$ is a common factor.
d $16xy(y + 3) - 24y(y + 3)$ $= 8y(y + 3) \times 2x + 8y(y + 3) \times -3$ $= 8y(y + 3)(2x - 3)$	We see that $y + 3$ is a common factor. Also, $8y$ is a common factor of $16xy$ and $-24y$. So $8y(y + 3)$ is taken out in front of a bracket.

Sometimes the order of two terms may need to be changed so that the terms are in the same order in both binomial factors.

Example 18

Factorise each of the following.

a $2x(x + 3) + 5(3 + x)$

b $3m(2m - 3) + 6 - 4m$

Working	Reasoning
a $2x(x + 3) + 5(3 + x)$ $= 2x(x + 3) + 5(x + 3)$ $= (x + 3)(2x + 5)$	When adding two numbers, the order doesn't matter, so $3 + x = x + 3$. The common factor $x + 3$ is taken out in front of a bracket.

continued

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Example 18 continued

Working	Reasoning
b $3m(2m - 3) + 6 - 4m$ $= 3m(2m - 3) + 2(3 - 2m)$ $= 3m(2m - 3) - 2(2m - 3)$	The expression $6 - 4m$ has a common factor of 2. When subtracting, order does matter: $(3 - 2m)$ is not the same as $(2m - 3)$. But $(3 - 2m) = -(-3 + 2m)$ $= -(2m - 3)$ Use $2(3 - 2m) = -2(2m - 3)$
$= (2m - 3)(3m - 2)$	The common factor $2m - 3$ is taken out in front of a bracket.

Factorisation by grouping 'two and two'

Consider the expression $ax + ay + 2x + 2y$. We can group the terms in pairs so that each pair has a common factor.

$ax + ay + 2x + 2y = a(x + y) + 2(x + y)$

We can then take out the binomial factor $x + y$ to show that

$ax + ay + 2x + 2y = (x + y)(a + 2)$

Sometimes we first need to rearrange the terms in order to group 'two and two'.

Is there another way of grouping the four terms in pairs so that each pair has a common factor?

Example 19

To factorise the following expressions, rearrange the terms if necessary and then group them in pairs.

a $at + aq + 4t + 4q$ **b** $p^2 - 2pq - 4p + 8q$ **c** $3ab - 2c + ac - 6b$

Working	Reasoning
a $at + aq + 4t + 4q$ $= a(t + q) + 4(t + q)$	a is a common factor of the first pair of terms and 4 is a common factor of the second pair of terms.
$= (t + q)(a + 4)$	The common factor $t + q$ is taken out in front of a bracket.
b $p^2 - 2pq - 4p + 8q$ $= p(p - 2q) - 4(p - 2q)$	p is a common factor of the first pair of terms. Instead of taking out 4 as a factor of $-4p + 8q$, we take out -4 and obtain the same binomial common factor, $p - 2q$.
$= (p - 2q)(p - 4)$	$p - 2q$ is taken out in front of a bracket.

continued

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Example 19 continued

Working	Reasoning
c $3ab - 2c + ac - 6b$	Rearrange the terms so that each pair of terms has a common factor.
$= 3ab - 6b + ac - 2c$	
$= 3b(a - 2) + c(a - 2)$	$3b$ is a common factor of the first pair of terms and c is a common factor of the second pair of terms.
$= (a - 2)(3b + c)$	The common factor $(a - 2)$ is taken out in front of a bracket.
	The four terms can be rearranged in a different way to give $(3b + c)$ as the common factor of each pair of terms. Alternatively, $3ab - 2c + ac - 6b$ $= 3ab + ac - 6b - 2c$ $= a(3b + c) - 2(3b + c)$ $= (3b + c)(a - 2)$

Example 20

Factorise each of the following.

a $(m + 5)^2 + 3(m + 5)$

b $6(x - 3) - (x - 3)^2$

Working	Reasoning
a $(m + 5)^2 + 3(m + 5)$	There is a common factor of $m + 5$.
$= (m + 5)(m + 5) + 3(m + 5)$	
$= (m + 5)(m + 5 + 3)$	$m + 5$ is taken out in front of a bracket.
$= (m + 5)(m + 8)$	Simplify the second bracket.
b $6(x - 3) - (x - 3)^2$	There is a common factor of $x - 3$.
$= 6(x - 3) - (x - 3)(x - 3)$	
$= (x - 3)[6 - (x - 3)]$	$x - 3$ is taken out in front of a bracket.
$= (x - 3)(6 - x + 3)$	Simplify the second bracket.
$= (x - 3)(9 - x)$	$-(x - 3) = -x + 3$

Remember to use brackets when subtracting a binomial expression.

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exercise 3.4

LINKS TO Example 17

1 Factorise each of the following by taking out a binomial factor.

a $a(x + 1) + 3(x + 1)$
b $4(x - 7) + x(x - 7)$
c $p(2p + 3) - 2(2p + 3)$
d $y(y - 6) - (y - 6)$
e $2w(3w - 5) + 6(3w - 5)$
f $5(3 - 4z) - 8z(3 - 4z)$
g $a^2(b - 2c) + b - 2c$
h $x(y - z) - y + z$
i $a(a + 3) + a + 3$
j $p(p - 1) - p + 1$
k $x(x + 5) + 5(x + 5)$
l $2m(2m + 3) + 3(2m + 3)$

2 Fill in the gaps to factorise by grouping the terms in pairs.

a $ax + ay + 3x + 3y$
$= __(x + y) + __(x + y)$
$= (x + y)(____)$

b $x^2 - x + yx - y$
$= __(x - 1) + __(x - 1)$
$= (x - 1)(____)$

c $3m + 18 + am + 6a$
$= __(____) + __(____)$
$= (____)(____)$

d $yz + 7z - 2y - 14$
$= z(____) - 2(____)$
$= (____)(____)$

e $3ab + 6b + 2a + 4$
$= __(____) + __(____)$
$= (____)(____)$

f $4k - 16 + km - 4m$
$= __(____) + __(____)$
$= (____)(____)$

LINKS TO Example 18

3 Factorise each of the following expressions.

a $x(x + 1) + 5(1 + x)$
b $a(a + 4) - 3(4 + a)$
c $7(x - 5) + x(x - 5)$
d $a(a - 3) + 4(3 - a)$
e $m(2m - 7) + 1(7 - 2m)$
f $2x(x - 9) - 3(9 - x)$
g $3a(a - 4) + 2(4 - a)$
h $5m(6 - 5m) - 3(5m - 6)$
i $3x(x - 3) + 6 - 2x$
j $4a(2a - 5) + 15 - 6a$
k $2p(3p - 4) + 20 - 15p$
l $3x(7 - 4x) + 8x - 14$

4 Factorise by first grouping the terms in pairs.

a $ab + ac + eb + ec$
b $tp + tq + 3p + 3q$
c $5x + 5y + kx + ky$
d $6st + 12t + 3s + 6$
e $a^2 - 3ab + 4a - 12b$
f $3x + 9 - 2xy - 6y$
g $5p + 15 - 3pq - 9q$
h $6a + 12 - 2ab - 4b$
i $a^2 - 3ab - 3a + 9b$
j $8a - 2 - 8ab + 2b$
k $a^2 - 4a + 2a - 8$
l $x^2 + 7x - 2x - 14$

LINKS TO Example 19

5 Factorise by first rearranging the terms if necessary, and then grouping them in pairs.

a $x^2 + 6y + xy + 6x$
b $4a - 3b + ab - 12$
c $2ap + 6 + 4a + 3p$
d $xy - 8 + 8y - x$
e $pq - 16r^2 + pr - 16qr$
f $p^2 - a - p + ap$
g $6m - 3n - 18 + mn$
h $1 - xy + x - y$
i $xy + 5z + 5x + yz$
j $ab + cd + ac + bd$
k $5m + 3n + mn + 15$
l $7x - 3y + xy - 21$

LINKS TO Example 20

6 Factorise each of the following.

a $5(x + 2) + (x + 2)^2$
b $(c + 3)^2 + (c + 3)$
c $(a - 4)^2 - 2(a - 4)$
d $3(r + 7) + (r + 7)^2$
e $(x - 4)^2 - 5(x - 4)$
f $5(y + 8) + (y + 8)^2$
g $9(c + 2) - (c + 2)^2$
h $3(p - 4) - (p - 4)^2$
i $2(x + 1)^2 - (x + 1)$
j $3(y - 4) - 6(y - 4)^2$
k $2(m + n)^2 - m - n$
l $2(x - 2)^2 + 3(2 - x)$

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3.5 Perfect squares

Expanding a perfect square

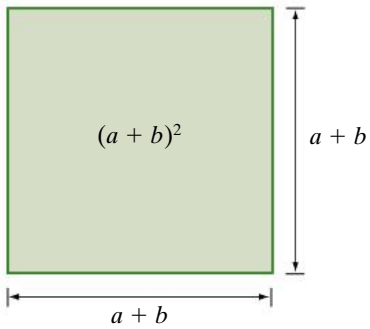
The expressions $(a + b)^2$ and $(a - b)^2$ are examples of **perfect squares**. To square a binomial expression, we multiply it by itself. For example, $(a + b)^2 = (a + b)(a + b)$. We could then use FOIL to help us expand $(a + b)(a + b)$ or similar squares.

The number patterns in these perfect squares can be used to expand a perfect square.

Factorised form	Expanded form
$(x - 1)^2$	$x^2 - 2x + 1$
$(x - 2)^2$	$x^2 - 4x + 4$
$(x + 4)^2$	$x^2 + 8x + 16$
$(x + 7)^2$	$x^2 + 14x + 49$

Factorised form	Expanded form
$(2x - 1)^2$	$4x^2 - 4x + 1$
$(3x + 2)^2$	$9x^2 + 12x + 4$
$(x + 3)^2$	$x^2 + 6x + 9$
$(x - 5)^2$	$x^2 - 10x + 25$

We can also use a diagram for $(a + b)^2$ to obtain a general rule that enables us to expand perfect squares quickly.

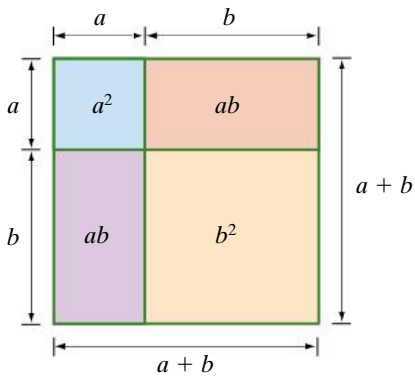


Expanding a perfect square

The square area $(a + b)^2$ is made up of

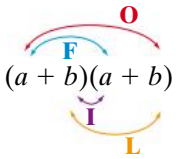
- a square with area a^2
- two rectangles, each with area ab
- another square, with area b^2.

So, $(a + b)^2 = a^2 + 2ab + b^2$.



This can also be shown algebraically using FOIL.

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$



The expression $(a - b)^2$ is also a perfect square.

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

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Example 21

Expand these perfect squares.

a $(x + 6)^2$ **b** $(3b - 2)^2$ **c** $(x + 3y)^2$

Working

a $(x + 6)^2 = x^2 + 2 \times x \times 6 + 6^2$
$= x^2 + 12x + 36$

b $(3b - 2)^2 = (3b)^2 - 2 \times 3b \times 2 + 2^2$
$= 9b^2 - 12b + 4$

c $(x + 3y)^2 = x^2 + 2 \times x \times 3y + (3y)^2$
$= x^2 + 6xy + 9y^2$

Reasoning

Use $(a + b)^2 = a^2 + 2ab + b^2$, where a is x and b is 6.

Use $(a - b)^2 = a^2 - 2ab + b^2$, where a is $3b$ and b is 2.

Remember to square both b and the coefficient of b: $(3b)^2 = 3b \times 3b = 9b^2$

Use $(a + b)^2 = a^2 + 2ab + b^2$, where a is x and b is $3y$.

Remember to square both y and the coefficient of y: $(3y)^2 = 3y \times 3y = 9y^2$

Recognising a perfect square

The following properties will help you to recognise a perfect square in expanded form.

- There must be three terms, as a perfect square is a **trinomial** expression.
- The first and last terms must both be *positive* and *perfect squares*: a^2 and b^2. (Check these terms first and identify a and b.)
- The middle term must be *double the product* of a and b (and may be either positive or negative).

Example 22

Which of the following are perfect squares? (Give your reasons.)

a $y^2 + 8y - 16$ **b** $m^2 - 6m + 9$ **c** $4bc + 4bc + 16c^2$

Working

a Not a perfect square because the last term is negative, it needs to be $+ 16$

b Perfect square because $9 = 3^2$ and $2 \times 3 = 6$

Reasoning

$y^2 + 8y + 16 = (y + 4)^2$

$m^2 - 6m + 9 = (m - 3)^2$

continued

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Example 22 continued

Working	Reasoning
c Not a perfect square, because $16c^2 = (4c)^2$ but $4bc$ does not equal $2 \times b \times 4c$.	$b^2 + 8bc + 16c^2 = (b + 4c)^2$

Factorising a perfect square

Going from factorised form to expanded form, $(a + b)^2 = a^2 + 2ab + b^2$, it follows that the reverse is also true, so $a^2 + 2ab + b^2 = (a + b)^2$. Similarly, $a^2 - 2ab + b^2 = (a - b)^2$. Therefore, if you can recognise a perfect square in expanded form, it is easy to factorise.

Expanding or factorising perfect squares

Expansion

$(a + b)^2 = a^2 + 2ab + b^2$

Factorisation

Expansion

$(a - b)^2 = a^2 - 2ab + b^2$

Factorisation

Example 23

Factorise each of the following.

a $x^2 + 14x + 49$ **b** $m^2 - 8m + 16$ **c** $18a + 81 + a^2$

Working	Reasoning
a $x^2 + 14x + 49$ $= (x + 7)^2$	$x^2 + 14x + 49$ is a perfect square of the form $a^2 + 2ab + b^2$, where a is x and b is 7.
b $m^2 - 8m + 16$ $= (m - 4)^2$	$m^2 - 8m + 16$ is a perfect square of the form $a^2 - 2ab + b^2 = (a - b)^2$, where a is m and b is 4.
c $18a + 81 + a^2$ $= a^2 + 18a + 81$ $= (a + 9)^2$	Rewrite $18a + 81 + a^2$ in the form $a^2 + 2ab + b^2$. $a^2 + 18a + 81$ is a perfect square, where a is a and b is 9.

Sometimes it is necessary to take out a common factor first. If the coefficient of the x^2 term is -1, then -1 should be taken out as a factor.

In example **24** part **a**, the expression has a common factor of 2 and in part **b**, the expression has a common factor of -1. When the common factor is taken out, the expression in the brackets can be seen to be a perfect square.

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Example 24

Factorise the following expressions.

a $2x^2 + 16x + 32$ **b** $-x^2 + 10x - 25$

Working

a $2x^2 + 16x + 32$
$= 2(x^2 + 8x + 16)$
$= 2(x^2 + 2 \times 4x + 4^2)$
$= 2(x + 4)^2$

b $-x^2 + 10x - 25$
$= -(x^2 - 10x + 25)$
$= -(x - 5)^2$

Reasoning

Take out a common factor of 2.

$x^2 + 8x + 16$ can be written in the form $a^2 + 2ab + b^2$, where $a = x$ and $b = 4$.

$a^2 + 2ab + b^2 = (a + b)^2$

When the common factor of -1 is taken outside the bracket, the signs of the terms inside the brackets change.

$x^2 - 10x + 25$ can now be factorised as a perfect square.

exercise 3.5

LINKS TO Example 21

1 Which of the following quadratic expressions are perfect squares? For each expression that is not a perfect square, show how it could be turned into a perfect square by changing one term.

a $x^2 - 12x - 36$ **b** $b^2 - 18x + 81$ **c** $h^2 + 7h + 49$
d $y^2 - 8y + 8$ **e** $m^2 + 2m + 4$ **f** $x^2 - 12x + 36$

LINKS TO Example 22

2 Copy and complete this table of perfect squares.

Factorised form	Expanded form
$(x + 1)^2$	
$(x - 2)^2$	
$(x + 3)^2$	
$(x - 4)^2$	
$(x - 5)^2$	
$(x + 6)^2$	

Factorised form	Expanded form
$(x + 7)^2$	
$(x + 8)^2$	
$(x - 9)^2$	
$(x + 10)^2$	$x^2 + 20x + 100$
$(x - 11)^2$	
$(x - 12)^2$	

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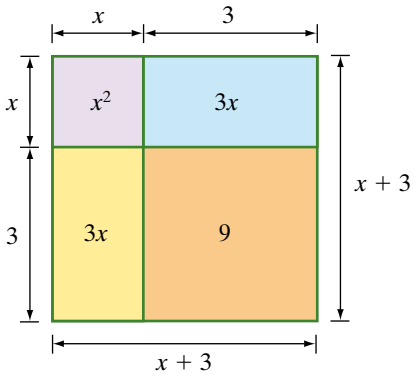
3 Use the diagram to expand $(x + 3)^2$.



4 **a** Draw a diagram that could be used to expand $(x + 4)^2$.

b Hence write down the expansion of $(x + 4)^2$.

5 Expand each of the following.

a $(a + 6)^2$ **b** $(m - 5)^2$ **c** $(x + 3)(x + 3)$
d $(c - 4)(c - 4)$ **e** $(p + 5)^2$ **f** $(m - 6)^2$
g $(a + 2)(a + 2)$ **h** $(b - 3)(b - 3)$ **i** $(3 + 2x)(3 + 2x)$
j $(4x - 3)^2$ **k** $(x + 3y)^2$ **l** $(5a - 2b)^2$
m $(7 - 2k)^2$ **n** $(8 - 3x)(8 - 3x)$ **o** $(3x + 1)^2$
p $(2 + 3x)(2 + 3x)$ **q** $(2a + 5b)^2$ **r** $(2 - 7y)^2$

LINKS TO Example 23

6 Factorise each of the following perfect squares.

a $x^2 + 8x + 16$ **b** $y^2 + 6y + 9$ **c** $p^2 - 4p + 4$
d $m^2 - 10m + 25$ **e** $k^2 + 20k + 100$ **f** $x^2 + 14x + 49$
g $a^2 - 12a + 36$ **h** $36 + 12x + x^2$ **i** $p^2 + 4pq + 4q^2$
j $16 - 8m + m^2$ **k** $x^2 - 2x + 1$ **l** $x^2 - 32x + 256$

LINKS TO Example 24

7 Factorise each of the following.

a $x^2 + 14x + 49$ **b** $y^2 + 8y + 16$ **c** $p^2 - 6p + 9$
d $k^2 - 10k + 25$ **e** $z^2 + 22z + 121$ **f** $64 + 16x + x^2$
g $m^2 + 10m + 25$ **h** $x^2 - 18x + 81$ **i** $4 + 4x + x^2$
j $24x + x^2 + 144$ **k** $y^2 - 26y + 169$ **l** $a^2 + 30a + 225$
m $400 - 40x + x^2$ **n** $36 + m^2 - 12m$ **o** $x^2 + 3x + 2.25$

8 Factorise each of the following.

a $2x^2 + 16x + 32$ **b** $3x^2 + 6x + 3$ **c** $3x^2 - 30x + 75$
d $-x^2 - 4x - 4$ **e** $-x^2 + 6x - 9$ **f** $-x^2 + 14x - 49$
g $-2x^2 + 36x - 162$ **h** $-5x^2 - 30x - 45$ **i** $3x^3 + 6x^2 + 3x$

exercise 3.5 challenge

9 A square has its length and width both decreased by 3 m. Let the side length of the square be x cm.

a What is the area of the new square? **b** By how much has the area decreased?

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3.6 Differences of two squares

Consider the expansion of $(a + b)(a - b)$, using FOIL.

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$

$(a + b)(a - b)$

The result is a **difference of two squares**, because the expression is a subtraction or difference between two terms, each of which are squares.

Note that the two middle terms have opposite signs so $+ ab - ab = 0$.

Expanding or factorising the difference of squares

Expansion

$$(a + b)(a - b) = a^2 - b^2$$

Factorisation

Example 25

Expand each of the following.

a $(x + 11)(x - 11)$

b $(2x - 3)(2x + 3)$

Working

a $(x + 11)(x - 11)$
$= x^2 - 11^2$
$= x^2 - 121$

Reasoning

Use $(a + b)(a - b) = a^2 - b^2$, where a is x and b is 11.

b $(2x - 3)(2x + 3)$
$= (2x)^2 - 3^2$
$= 4x^2 - 9$

Use $(a - b)(a + b) = a^2 - b^2$, where a is $2x$ and b is 3. Simplify each square.

In example **26** part **b**, the difference of squares includes the square of a binomial expression.

Example 26

Factorise each of the following.

a $m^2 - 16$

b $(x + 1)^2 - 9$

continued

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Example 26 continued

Working	Reasoning
a $m^2 - 16 = m^2 - 4^2$	Write the expression in the form $a^2 - b^2$.
$= (m - 4)(m + 4)$	Use $a^2 - b^2 = (a - b)(a + b)$.
b $(x + 1)^2 - 9 = (x + 1)^2 - 3^2$	$(x + 1)^2 - 9$ is in the form $a^2 - b^2$,
$= (x + 1 + 3)(x + 1 - 3)$	where a is $x + 1$ and b is 3.
$= (x + 4)(x - 2)$	Simplify each bracket.

In example **27** part **a**, each term has a coefficient that is a square number. We can write $4g^2$ as $(2g)^2$ and $25h^2$ as $(5h)^2$ so the expression is a difference of squares.

Example 27

Factorise each of the following expressions.

a $4g^2 - 25h^2$ **b** $3x^2 - 27y^2$

Working	Reasoning
a $4g^2 - 25h^2 = (2g)^2 - (5h)^2$	Write the expression in the form $a^2 - b^2$.
$= (2g - 5h)(2g + 5h)$	Use $a^2 - b^2 = (a - b)(a + b)$.
b $3x^2 - 27y^2 = 3(x^2 - 9y^2)$	Take out the common factor of 3.
$= 3[x^2 - (3y)^2]$	Write the expression inside the brackets in the form $a^2 - b^2$.
$= 3(x - 3y)(x + 3y)$	Use $a^2 - b^2 = (a - b)(a + b)$.

Factorisation over *R*

So far, all the expressions that have been factorised have involved only rational numbers. We refer to this as **factorisation over *Q***, the rational number field. Some expressions cannot be factorised using only rational numbers, but they can be factorised using irrational square roots. We refer to this as **factorisation over *R***, the real number field.

The expression $x^2 - 5$, for example, cannot be factorised over Q, since 5 is not a perfect square. However, this expression can be factorised over R using the difference of squares rule.

$$x^2 - 5 = x^2 - (\sqrt{5})^2$$
$$= (x - \sqrt{5})(x + \sqrt{5})$$

Example 28

Factorise the following expressions over R.

a $a^2 - 3$ **b** $2p^2 - 22$ **c** $(x + 1)^2 - 7$

continued

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Example 28 continued

Working	Reasoning
a $a^2 - 3 = a^2 - (\sqrt{3})^2$ $= (a - \sqrt{3})(a + \sqrt{3})$	Use the difference of squares rule $a^2 - b^2 = (a - b)(a + b)$, where $b^2 = 3$, that is, b is $\sqrt{3}$.
b $2p^2 - 22 = 2(p^2 - 11)$ $= 2[p^2 - (\sqrt{11})^2]$ $= 2(p + \sqrt{11})(p - \sqrt{11})$	First take out the common factor. Then use $a^2 - b^2 = (a - b)(a + b)$, where a is p and b is $\sqrt{11}$.
c $(x + 1)^2 - 7 = (x + 1)^2 - (\sqrt{7})^2$ $= (x + 1 - \sqrt{7})(x + 1 + \sqrt{7})$	Use $a^2 - b^2 = (a - b)(a + b)$, where a is $x + 1$ and b is $\sqrt{7}$.

exercise 3.6

LINKS TO Example 25

1 Use the 'difference of two squares' rule to expand each of the following.

a $(p - 4)(p + 4)$ **b** $(x + 5)(x - 5)$ **c** $(m - n)(m + n)$
d $(9 - m)(9 + m)$ **e** $(2x - 1)(2x + 1)$ **f** $(3 + 2x)(3 - 2x)$
g $(9 + 5y)(9 - 5y)$ **h** $(2a - 7b)(2a + 7b)$ **i** $(ab - 4)(ab + 4)$
j $(p - 7)(p + 7)$ **k** $(8 - m)(8 + m)$ **l** $(a - c)(a + c)$
m $(2x - 3)(2x + 3)$ **n** $(7 + 5k)(7 - 5k)$ **o** $(5 + 2a)(5 - 2a)$
p $(2m + 7)(2m - 7)$ **q** $(5 - 3x)(5 + 3x)$ **r** $(3 - 4a)(3 + 4a)$

2 Copy and complete this table of differences of two squares.

Factorised form	Expanded form
$(x + y)(x - y)$	
	$m^2 - n^2$
$(y + 3)(y - 3)$	
	$x^2 - 4$
$(a + 2b)(a - 2b)$	
	$9x^2 - 4y^2$

Factorised form	Expanded form
$(y + 5)(y - 5)$	
$(7 + a)(7 - a)$	
$(2y - 5x)(2y + 5x)$	
	$r^2 - t^2$
	$y^2 - 64$
	$25x^2 - 4y^2$

LINKS TO Example 26a

3 Use the 'difference of two squares' rule to factorise each of the following.

a $x^2 - 100$ **b** $p^2 - 25$ **c** $81 - y^2$ **d** $36 - x^2$
e $x^2 - 49$ **f** $1 - a^2$ **g** $y^2 - 144$ **h** $x^2 - 64$
i $4a^2 - 9$ **j** $16y^2 - 49$ **k** $k^2 - 36m^2$ **l** $4p^2 - 81q^2$
m $100 - y^2$ **n** $25 - x^2$ **o** $9a^2 - 4$ **p** $49b^2 - 16$
q $100a^2 - 49$ **r** $25x^2 - 1$ **s** $16b^2 - 25$ **t** $16 - 9x^2$
u $25 - 49m^2$ **v** $64 - 81a^2$ **w** $9x^2 - 169$ **x** $121x^2 - 49$

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4 Calculate each of the following without using a calculator.

a $17^2 - 3^2$ **b** $63^2 - 37^2$ **c** $59^2 - 41^2$ **d** $179^2 - 21^2$

LINKS TO Example 26b

5 Factorise each of the following.

a $(a + 2)^2 - 9$ **b** $(b + 3)^2 - 25$ **c** $(c + 8)^2 - 81$ **d** $(d - 5)^2 - 36$
e $(e + 7)^2 - 16$ **f** $(f - 12)^2 - 100$ **g** $(g - 4)^2 - 49$ **h** $(h + 2)^2 - 64$
i $16 - (i + 5)^2$ **j** $4 - (j + 2)^2$ **k** $25 - (k - 3)^2$ **l** $9 - (l - 2)^2$

LINKS TO Example 27

6 Factorise each of the following.

a $4x^2 - 36y^2$ **b** $3a^2 - 48b^2$ **c** $2x^2 - 98y^2$ **d** $7x^2 - 28$
e $-3x^2 + 75$ **f** $-2m^2 + 50n^2$ **g** $4x^2 - 400y^2$ **h** $6b^2 - 54$

7 Ahmed has a square vegetable garden in his backyard as illustrated below.
In a new landscape design for the backyard, the square vegetable garden is changed into a rectangular one by adding 1 m to the length and subtracting 1 m from the width.

a Write an expression in terms of x for the area of the original vegetable garden.
b Write an expression in terms of x for the area of the new vegetable garden.
c Does the new garden have a larger area, a smaller area or the same area? Justify your answer.

LINKS TO Examples 28a, b

8 Factorise each of the following over R.

a $x^2 - 5$ **b** $y^2 - 2$
c $7 - a^2$ **d** $19 - c^2$
e $p^2 - 15$ **f** $z^2 - 38$ **g** $3x^2 - 18$
h $5x^2 - 65$ **i** $25 - 5r^2$ **j** $49 - 7x^2$

R means real numbers, so the factors will include irrational numbers.

LINKS TO Example 28c

9 Factorise each of the following over R.

a $(x + 4)^2 - 13$ **b** $(x + 1)^2 - 6$ **c** $(x - 2)^2 - 5$
d $(x - 7)^2 - 10$ **e** $(x + 3)^2 - 8$ **f** $(x - 1)^2 - 18$
g $(x + 1)^2 - 3$ **h** $(y + 5)^2 - 7$ **i** $(p - 3)^2 - 11$
j $(x + 7)^2 - 8$ **k** $(x - 1)^2 - 27$ **l** $15 - (x + 2)^2$
m $18 - (k - 2)^2$ **n** $(m - 5)^2 - 5$ **o** $3(a + 2^2) - 15$

8, 18 and 27 each have a square number as a factor.
$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$
$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$
$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$

exercise 3.6 challenge

10 Factorise each of the following.

a $(4b + 3)^2 - 9b^2$ **b** $(2x - 1)^2 - 16x^2$

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3.7 Factorising quadratic trinomials

An expression of the type $x^2 + 5x + 4$ is called a **quadratic trinomial**. The word 'quadratic' comes from the Latin *quad*, meaning 'square' or 'four', and refers to the 'x squared' term. A quadratic expression has a square as the highest power of the variable. The word 'trinomial' refers to the three terms.

We know from expanding two binomial factors that $(x + 3)(x + 4) = x^2 + 7x + 12$ and $(x - 4)(x - 2) = x^2 - 6x + 8$.

How do the 3 and 4 in $(x + 3)(x + 4)$ relate to the 7 and 12 in $x^2 + 7x + 12$?

How do the -4 and -2 in $(x - 4)(x - 2)$ relate to the -6 and 8 in $x^2 - 6x + 8$?

We observe that $3 + 4 = 7$ and $3 \times 4 = 12$. We also observe that $-4 - 2 = -6$ and $-4 \times (-2) = 8$.

Factorising quadratic trinomials

The distributive law gives the following expansion for $(x + a)(x + b)$.

$$\begin{aligned}(x + a)(x + b) &= x^2 + ax + bx + ab \\ &= x^2 + (a + b)x + ab\end{aligned}$$

$x^2 + (a + b)x + ab$ is a quadratic trinomial because it has an x^2 term, an x term and a constant.

- The constant term is the *product* of the numbers in the brackets.
- The coefficient of x is the *sum* of the numbers in the brackets.

Consider $x^2 - 2x - 15$. We need to find two numbers with a product of -15 and a sum of -2. The examples in the following table show the process of trial and error.

Guess: pick numbers with a product of -15	Check: Is the sum of the numbers -2?
1 and -15	$1 + -15 = -14$: no
5 and -3	$5 + -3 = 2$: no
3 and -5	$3 + -5 = -2$: yes

Choose numbers that give the correct product and check whether they also give the correct sum.

Trial and error shows that the numbers are 3 and -5. Therefore

$$x^2 - 2x - 15 = (x + 3)(x - 5)$$

This can also be written as $(x - 5)(x + 3)$.

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The cross method

A visual representation of the trial-and-error process uses a cross as follows.

- Write the factors of the x^2 term on the left of the cross.
- Write your guesses for the numbers on the right.
- Cross-multiply to see if those numbers give the correct middle term.

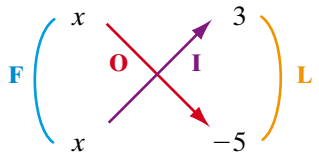
If the middle term is not correct, repeat the last two steps for another guess.

The following steps show this process for the guesses in the table above.

1st guess: 1 and −15	**2nd guess: 5 and −3**
$x^2 - 2x - 15$ multiply to give x^2 $x \quad 1$ $x \quad -15$ multiply to give -15 $-15x + 1x = -14x$ Cross-multiplying does *not* give the middle term.	$x \quad \cancel{1} \quad 5$ $x \quad \cancel{-15} \quad -3$ multiply new numbers to give -15 $-3x + 5x = 2x$ Cross-multiplying gives $2x$, not $-2x$.
3rd guess: 3 and −5	**The 'rows' in the cross diagram give the factors.**
$x \quad \cancel{1} \quad \cancel{5} \quad -5$ $x \quad \cancel{-15} \quad \cancel{-3} \quad 3$ Swap the signs from the previous guess $3x - 5x = -2x$ Cross-multiplying gives $-2x$, which is the correct middle term.	$x \quad \cancel{1} \quad \cancel{5} \quad -5$ ← One factor is $(x - 5)$ $x \quad \cancel{-15} \quad \cancel{-3} \quad 3$ ← The other factor is $(x + 3)$ $x^2 - 2x - 15 = (x - 5)(x + 3)$ or $(x + 3)(x - 5)$

Notice that the cross method is using FOIL in reverse.

$$x^2 - 2x - 15$$



$$(x + 3)(x - 5)$$

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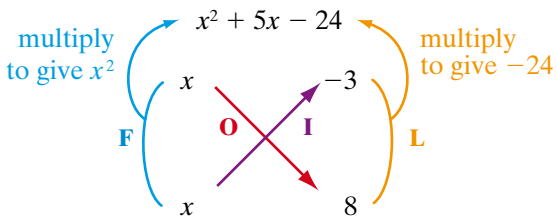
Example 29

Factorise each of the following.

a $x^2 + 11x + 24$ **b** $x^2 - 12x + 20$ **c** $x^2 + 5x - 24$

Working	Reasoning
a $x^2 + 11x + 24$	The constant term is positive, so both numbers will have the same sign. The middle term is positive, so both numbers are positive.
$= (x + 3)(x + 8)$	$3 \times 8 = 24$ and $3 + 8 = 11$
b $x^2 - 12x + 20$	The constant term is positive, so both numbers will have the same sign, but the middle term is negative, so both numbers are negative.
$= (x - 2)(x - 10)$	$-2 \times (-10) = 20$ and $-2 + (-10) = -12$
c $x^2 + 5x - 24$ $= (x - 3)(x + 8)$	The constant term is negative, so the numbers will have opposite signs.

The cross method is particularly helpful when the numbers have opposite signs.



$8x - 3x = 5x$

Cross-multiplying gives the correct middle term.

If the quadratic trinomial expression has a common factor, this should be taken out first.

If the coefficient of the x^2 term is -1, then -1 should be taken out as a factor.

Example 30

Factorise the following expressions.

a $2x^2 + 2x - 24$ **b** $5x^2 - 15x + 10$ **c** $-x^2 - 10x + 24$

Working	Reasoning
a $2x^2 + 2x - 24$ $= 2(x^2 + x - 12)$	First take out the common factor of 2. Within the brackets, the constant term is negative, so the numbers have opposite signs.
$= 2(x - 3)(x + 4)$	$-3 \times 4 = -12$ and $-3 + 4 = 1$

continued

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Example 30 continued

Working	Reasoning
b $5x^2 - 15x + 10$	First take out the common factor of 5.
$= 5(x^2 - 3x + 2)$	The constant term is positive, so both numbers will have the same sign. The middle term is negative, so both numbers are negative.
$= 5(x - 2)(x - 1)$	$-2 \times (-1) = 2$ and $-2 + (-1) = -3$
c $-x^2 - 10x + 24$	Care should be taken with the signs inside the bracket when the negative sign is taken out in front.
$= -(x^2 + 10x - 24)$	
$= -(x + 12)(x - 2)$	The factorised expression can be left as $-(x + 12)(x - 2)$ or it can be written as $(x + 12)(2 - x)$.

exercise 3.7

LINKS TO Example 29

1 Factorise each of the following.

a $x^2 + 3x + 2$ **b** $x^2 + 4x + 3$ **c** $x^2 + 12x + 36$
d $x^2 + 14x + 40$ **e** $x^2 - 9x + 20$ **f** $x^2 + 4x - 12$
g $x^2 + 18x - 63$ **h** $x^2 - 13x + 36$ **i** $x^2 - 22x + 72$
j $x^2 + 12x - 45$ **k** $x^2 - 15x + 56$ **l** $x^2 + 10x - 56$
m $a^2 - 11a + 18$ **n** $a^2 - 5a - 24$ **o** $b^2 + 14b - 72$
p $d^2 - 11d - 60$ **q** $m^2 - 22m - 48$ **r** $y^2 + 18y + 45$

LINKS TO Example 30

2 Factorise each of the following by first taking out the common factor.

a $5x^2 + 15x + 10$ **b** $3x^2 + 12x + 9$ **c** $5x^2 - 10x - 40$
d $5x^2 - 30x + 40$ **e** $4x^2 - 12x - 40$ **f** $2x^2 - 10x + 12$
g $3x^2 - 15x - 108$ **h** $5x^2 + 15x - 350$ **i** $2x^2 - 44x + 144$
j $2x^2 + 8x + 6$ **k** $3x^2 + 21x + 36$ **l** $8x^2 - 40x + 48$
m $-x^2 + 7x + 8$ **n** $-x^2 + 3x + 10$ **o** $-x^2 - 9x - 20$

exercise 3.7 challenge

3 Factorise these quadratic expression.

a $x^2 - 4x - 221$ **b** $x^2 + 26x + 133$
c $x^2 + 5x - 104$ **d** $x^2 - 30x + 189$

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3.8 Completing the square

Completing the square

Quadratic trinomials cannot always be factorised by the guess-and-check methods of inspection shown in the previous sections. In some cases, we may be able to use a method called **completing the square** to factorise a quadratic.

How to complete the square

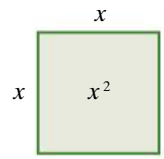
Suppose that we want to factorise $x^2 + 6x + 7$.

There are no factors of 7 that add to give 6, so we cannot factorise by inspection.

We use the first two terms of the quadratic to 'build' a perfect square, as follows.

Step 1

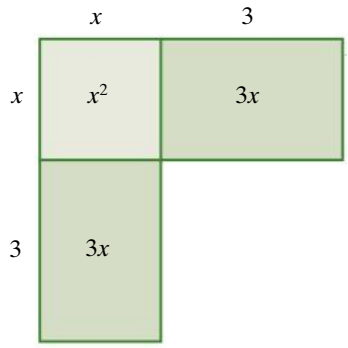
Start with a square of area x^2.



Step 2

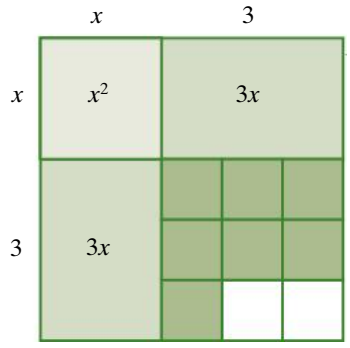
To show $6x$, add two rectangles, each of area $3x$.

This creates two sides of a square of side length $x + 3$, with a piece missing.



Step 3

The 7 units can now be added.



Step 4

The diagram shows us that we need another 2 units to complete the square with area $(x + 3)^2$.

Since the area of $x^2 + 6x + 7$ is 2 units less than the area of the square, we can say $x^2 + 6x + 7 = (x + 3)^2 - 2$

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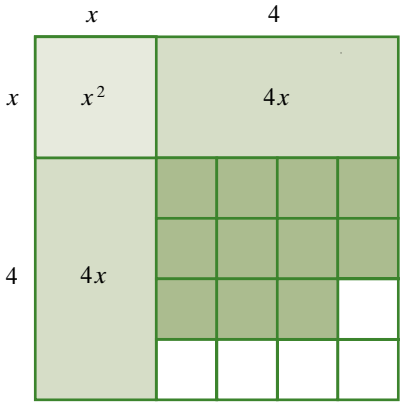
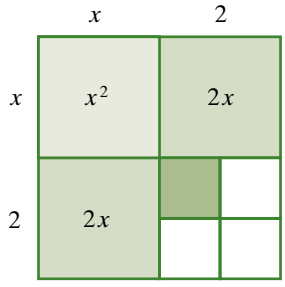
Example 31

a Use the diagram to complete the square for each of the following and express in the form $(x + h)^2 - k$.

i $x^2 + 8x + 11$



ii $x^2 + 4x + 1$



b Use a diagram to complete the square for each of the following.

i $x^2 + 6x + 2$

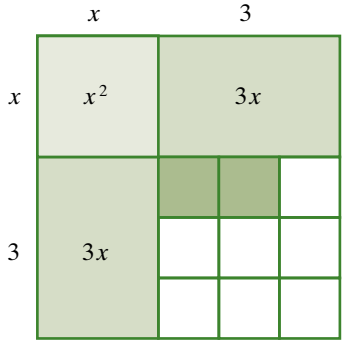
ii $x^2 + 10x + 6$

Working

a i $x^2 + 8x + 11$
$= (x + 4)^2 - 5$

ii $x^2 + 4x + 1$
$= (x + 2)^2 - 3$

b i $x^2 + 6x + 2$
$= (x + 3)^2 - 7$



ii $x^2 + 10x + 6$
$= (x + 5)^2 - 19$

Reasoning

$x^2 + 8x + 11$ is 5 less than $(x + 4)^2$.

$x^2 + 4x + 1$ is 3 less than $(x + 2)^2$.

Halve the coefficient of $6x$ and draw the diagram for $(x + 3)^2$.
$x^2 + 6x + 2$ is 7 less than $(x + 3)^2$.

Halve the coefficient of $10x$ and draw the diagram for $(x + 5)^2$.
$x^2 + 10x + 6$ is 19 less than $(x + 5)^2$.

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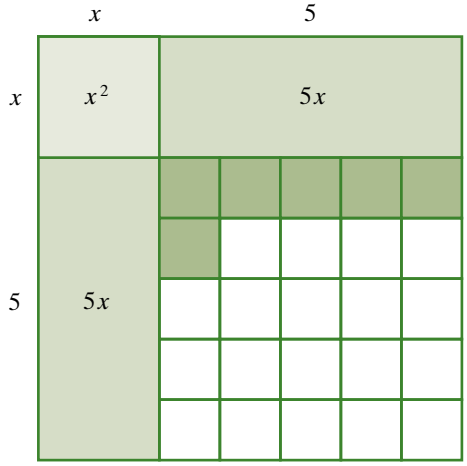
Example 31 continued

Working



Reasoning

Factorising by completing the square

We now look at the expressions we have obtained by completing the square to see how this helps us to factorise them.

We know from section **3.6** that the expression $(x + 4)^2 - 5 = (x + 4)^2 - (\sqrt{5})^2$ is a difference of squares and that we can factorise it over R as $(x + 4 - \sqrt{5})(x + 4 + \sqrt{5})$.

So, using the first expression in example **31** part **a**.

$$\begin{aligned} x^2 + 8x + 11 &= (x + 4)^2 - 5 \\ &= (x + 4 - \sqrt{5})(x + 4 + \sqrt{5}). \end{aligned}$$

In a similar way we can express the other quadratic trinomial in example **31** part **a**.

$$\begin{aligned} x^2 + 4x + 1 &= (x + 2)^2 - 3 \\ &= (x + 2)^2 - (\sqrt{3})^2 \\ &= (x + 2 - \sqrt{3})(x + 2 + \sqrt{3}) \end{aligned}$$

Example 32

Use the expressions obtained by completing the square in example **31** part **b** to express each of these quadratic trinomials in factorised form.

a $x^2 + 6x + 2$ **b** $x^2 + 10x + 6$

Working

a $x^2 + 6x + 2$
$= (x + 3)^2 - 7$
$= (x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$

Reasoning

$(x + 3)^2 - 7$ can be written as a difference of two squares:
$(x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2$

continued

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Example 32 continued

Working

b $x^2 + 10x + 6$
$= (x + 5)^2 - 19$
$= (x + 5 - \sqrt{19})(x + 5 + \sqrt{19})$

Reasoning

$(x + 5)^2 - 19$ can be written as a difference of two squares:
$(x + 5)^2 - 19 = (x + 5)^2 - (\sqrt{19})^2$

Example 33

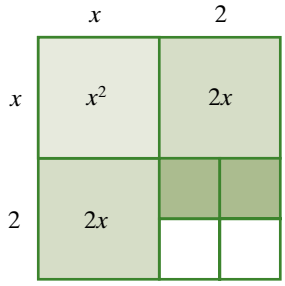
Use the method of completing the square to factorise the following expressions.

a $x^2 + 4x + 2$

b $x^2 + 10x + 12$

Working

a $x^2 + 4x + 2$
Construct a square of side length $x + 2$.



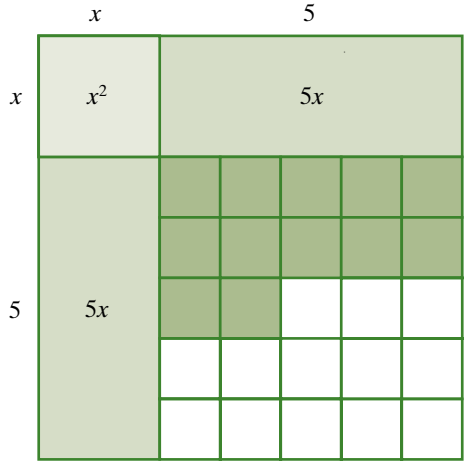
So $x^2 + 4x + 2$
$= (x + 2)^2 - 2$
$= (x + 2)^2 - (\sqrt{2})^2$
$= (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$

Reasoning

Two units are required to complete the square.
So $x^2 + 4x + 2$ is 2 units less than the square of side length $x + 2$.

This is a difference of squares.

b $x^2 + 10x + 12$
Construct a square of side length $x + 5$.



13 units are required to complete the square.
So $x^2 + 10x + 12$ is 13 units less than the square of side length $x + 5$.

continued

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Example 33 continued

Working

So $x^2 + 10x + 12$
$= (x + 5)^2 - 13$
$= (x + 5)^2 - (\sqrt{13})^2$
$= (x + 5 - \sqrt{13})(x + 5 + \sqrt{13})$

Reasoning

This is a difference of squares.

Completing the square algebraically

In order to avoid having to draw a diagram to complete the square, we can generalise from the patterns found in the numerical examples.

For a general expression of the form $x^2 + bx$, the square formed has side length $x + \frac{b}{2}$.

This results in a square of area $\left(x + \frac{b}{2}\right)^2$.

	x	$\frac{b}{2}$
x	x^2	$\frac{b}{2} \times x$
$\frac{b}{2}$	$\frac{b}{2} \times x$	$\frac{b}{2} \times \frac{b}{2}$

The region required to complete the square is $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$

Completing the square

To complete the square for $x^2 + bx$, we add the square of half the coefficient of x.

That is, $x^2 + bx + \frac{b^2}{4}$ is always a perfect square.

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2$$

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Example 34

Add the appropriate numbers to complete the square.

a $x^2 + 2x + __ = (x + __)^2$

b $x^2 + 14x + __ = (x + __)^2$

c $x^2 + 10x + __ = (x + __)^2$

Working	Reasoning
a $x^2 + 2x + 1 = (x + 1)^2$	Halve the coefficient of x and square it. $2 \div 2 = 1$ $1^2 = 1$ We must add 1 to make a perfect square.
b $x^2 + 14x + 49 = (x + 7)^2$	$14 \div 2 = 7$ $7^2 = 49$ We must add 49 to make a perfect square.
c $x^2 + 10x + 25 = (x + 5)^2$	$10 \div 2 = 5$ $5^2 = 25$ We must add 25 to make a perfect square.

Forming a difference of squares

Whenever we add a number to complete a square, we must also subtract that number to ensure that the overall expression is still equivalent to the original. As a result of that subtraction, we obtain a difference of squares. For example

$$\begin{aligned} x^2 + 6x &= \underbrace{x^2 + 6x + (3)^2} - (3)^2 \\ &= (x + 3)^2 - 9 \end{aligned}$$

To complete the square for $x^2 + 6x$, take half of 6 and square it.

Applying this method to the expression $x^2 + 6x + 1$, we obtain

$$\begin{aligned} x^2 + 6x + 1 &= \underbrace{x^2 + 6x + (3)^2} + 1 - (3)^2 \\ &= \underbrace{(x + 3)^2} + 1 - 9 \\ &= (x + 3)^2 - 8 \end{aligned}$$

We can then factorise this difference of squares over R.

$$\begin{aligned} x^2 + 6x + 1 &= (x + 3)^2 - 8 \\ &= (x + 3 + \sqrt{8})(x + 3 - \sqrt{8}) \end{aligned}$$

The method of completing the square does not always give a *difference* of two squares. Sometimes the result of completing the square is a *sum* of two squares. A sum of squares cannot be factorised.

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Example 35

For which of the following do you end up with a *sum* of squares rather than a *difference* of squares?

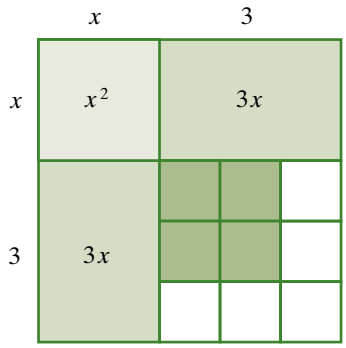
a $x^2 + 6x + 4$ **b** $x^2 + 4x + 5$ **c** $x^2 - 8x + 20$ **d** $x^2 + 2x + 3$

Working

a $x^2 + 6x + 4$
$= x^2 + 6x + 9 - 5$
$= (x + 3)^2 - 5$
This is a difference of squares.

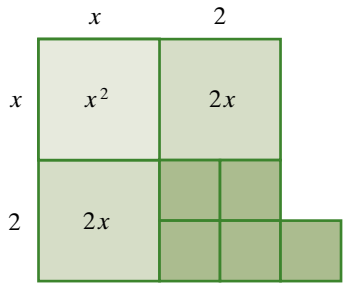
Reasoning

We need to add 5 to complete the square so we have to subtract 5 to keep the expression the same.



b $x^2 + 4x + 5$
$= x^2 + 4x + 4 + 1$
$= (x + 2)^2 + 1$
This is a sum of squares.

Instead of having to complete the square, the square is already complete with an extra 1.
$x^2 + 4x + 5$ cannot be factorised.



c $x^2 - 8x + 20$
$= x^2 - 8x + 16 + 4$
$= (x - 4)^2 + 4$
This is a sum of squares.

The square is already complete with an extra 4.
$x^2 - 8x + 20$ cannot be factorised.

d $x^2 + 2x + 3$
$= x^2 + 2x + 1 + 2$
$= (x + 1)^2 + 2$
This is a sum of squares.

The square is already complete with an extra 2.
$x^2 + 2x + 3$ cannot be factorised.

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Example 36

Use the method of completing the square to express each of the following expressions as a difference of squares. Hence factorise each expression.

a $x^2 + 8x - 5$ **b** $x^2 - 4x + 2$ **c** $x^2 + 14x + 17$

Working

a $x^2 + 8x - 5$

$= \underbrace{x^2 + 8x + (4)^2}\ \underbrace{- 5 - 16}$

$= (x + 4)^2 - 21$

$= (x + 4)^2 - (\sqrt{21})^2$

$= (x + 4 + \sqrt{21})(x + 4 - \sqrt{21})$

b $x^2 - 4x + 2$

$= \underbrace{x^2 - 4x + (2)^2}\ \underbrace{+ 2 - 4}$

$= (x - 2)^2 - 2$

$= (x - 2)^2 - (\sqrt{2})^2$

$= (x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$

c $x^2 + 14x + 17$

$= \underbrace{x^2 + 14x + (7)^2}\ \underbrace{+ 17 - 49}$

$= (x + 7)^2 - 32$

$= (x + 7)^2 - (\sqrt{32})^2$

$= (x + 7 + \sqrt{32})(x + 7 - \sqrt{32})$

$= (x + 7 + 4\sqrt{2})(x + 7 - 4\sqrt{2})$

Reasoning

To complete the square, add the square of half of 8, that is, 4^2. Subtract the same amount, that is, 16.
This gives a difference of squares.
Use $a^2 - b^2 = (a - b)(a + b)$,
where a is $x + 4$ and b is $\sqrt{21}$.

Add the square of half of 4, that is, 2^2.
Subtract the same amount, that is, 4.

This gives a difference of squares.

Add the square of half of 14, that is, 7^2.
Subtract the same amount, that is, 49.

This gives a difference of squares.

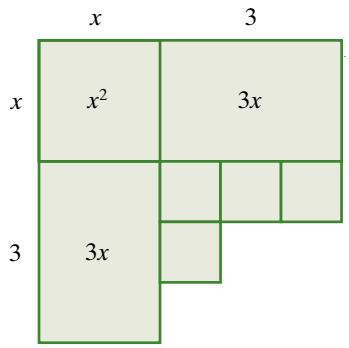
$\sqrt{32}$ can be simplified:
$\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

exercise 3.8

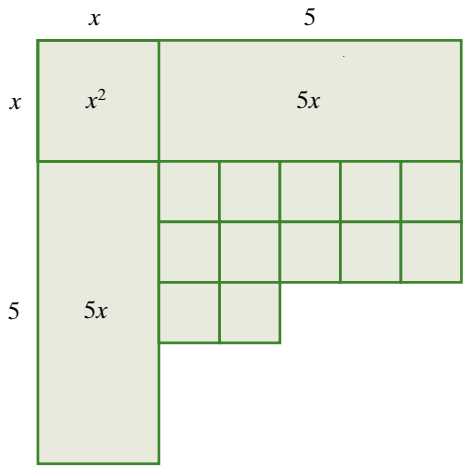
LINKS TO Example 31

1 Use the diagrams to complete the square for each of these quadratic expressions and then, write each expression in the form $(x + h)^2 - k$.

a $x^2 + 6x + 4$



b $x^2 + 10x + 12$



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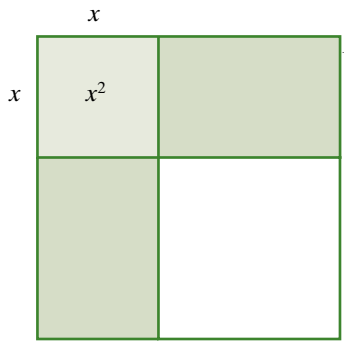
2 Use a diagram to complete the square for each of the following. Hence write each expression in the form $(x + h)^2 - k$.

a $x^2 + 12x$
b $x^2 + 16x$
c $x^2 + 20x$
d $x^2 + 8x$



3 Draw diagrams to help you complete the square for each of the following, and then write each expression in the form $(x + h)^2 + k$.

a $x^2 + 4x + 3$ **b** $x^2 + 8x + 7$ **c** $x^2 + 10x + 15$
d $x^2 + 6x + 2$ **e** $x^2 + 12x + 20$ **f** $x^2 + 2x$

LINKS TO Examples 32, 33

4 Factorise each of the following.

a $(x + 5)^2 - 6$ **b** $(x - 8)^2 - 11$ **c** $(x + 1)^2 - 20$ **d** $(x - 12)^2 - 48$
e $(x + 3)^2 - 3$ **f** $(x - 2)^2 - 5$ **g** $(x + 4)^2 - 41$ **h** $(x - 7)^2 - 32$
i $(x + 2)^2 - 29$ **j** $(x + 1)^2 - 41$ **k** $(x - 1)^2 - 51$ **l** $(x + 6)^2 - 23$

LINKS TO Example 34

5 Add the appropriate number to each of the following to complete the square.

a $x^2 + 6x + __ = (x + __)^2$ **b** $x^2 + 16x + __ = (x + __)^2$
c $x^2 + 18x + __ = (x + __)^2$ **d** $x^2 + 22x + __ = (x + __)^2$

LINKS TO Example 35

6 How would you write the following expressions in the form $(x + h)^2 + k$?

a $x^2 + 6x + 12$ **b** $x^2 + 2x + 5$

LINKS TO Example 36

7 Use the method of completing the square to factorise each of the following.

a $x^2 + 12x + 30$ **b** $x^2 - 10x + 20$ **c** $x^2 + 14x + 32$ **d** $x^2 + 4x - 1$
e $x^2 + 22x + 100$ **f** $x^2 - 6x - 2$ **g** $x^2 + 18x + 40$ **h** $x^2 + 10x - 4$
i $x^2 + 8x - 3$ **j** $x^2 - 16x + 41$ **k** $x^2 + 24x + 121$ **l** $x^2 + 8x + 5$
m $x^2 + 8x - 13$ **n** $x^2 - 2x - 5$ **o** $x^2 + 4x - 7$ **p** $x^2 + 14x - 2$

exercise 3.8 challenge

8 Use the method of completing the square to factorise each of the following.

a $x^2 + 5x + 1$ **b** $x^2 + 9x + 6$

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3.9 Further factorisation of quadratic expressions

Differences of squares: some further examples

A difference of squares may be a difference of two squared binomial expressions. It is often easier to substitute a and b for the binomial expressions, factorise $a^2 - b^2$, then substitute the binomial expressions back into $(a - b)(a + b)$ and finally simplify the expression in each bracket.

Example 37

Factorise each of the following.

a $(x + y)^2 - (2x - y)^2$ **b** $4(a + 1)^2 - 16(2a - 3)^2$ **c** $x^2 - y^2 + x + y$

Working	Reasoning
a $(x + y)^2 - (2x - y)^2$	
$= [x + y + (2x - y)][x + y - (2x - y)]$	Use $a^2 - b^2 = (a - b)(a + b)$ where a is $x + y$ and b is $2x - y$. It is useful to use two types of brackets to avoid confusion.
$= (3x)(x + y - 2x + y)$	
$= 3x(2y - x)$	
Be careful with − signs!	
b $4(a + 1)^2 - 16(2a - 3)^2$	Take out the common factor of 4.
$= 4[(a + 1)^2 - 4(2a - 3)^2]$	Use $a^2 - b^2 = (a - b)(a + b)$ to factorise the expression inside the square brackets.
$= 4[(a + 1)^2 - (2(2a - 3))^2]$	
$= 4[(a + 1 + 2(2a - 3)][a + 1 - 2(2a - 3)]$	
$= 4(a + 1 + 4a - 6)(a + 1 - 4a + 6)$	Simplify the brackets.
$= 4(5a - 5)(7 - 3a)$	Take out the common factor of 5 from the first bracket.
$= 20(a - 1)(7 - 3a)$	
c $x^2 - y^2 + x + y$	Group 'two and two', and use the difference of squares rule on the first pair of terms.
$= (x - y)(x + y) + 1(x + y)$	
$= (x + y)(x - y + 1)$	Take out the common factor of $x + y$.

Non-monic quadratic trinomials

In the quadratic trinomials factorised so far, the coefficient of the x^2 term was 1 or there was a common factor leaving a coefficient of 1 in the quadratic expression to be factorised. Quadratic trinomials where the coefficient of x^2 is 1 are called **monic quadratic trinomials**. We now look at the factorisation of **non-monic quadratic trinomials**.

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Perfect Squares

Example 38

Factorise the following perfect squares.

a $4x^2 + 28x + 49$

b $81x^2 - 198x + 121$

Working

a $4x^2 + 28x + 49$
$= (2x)^2 + 28x + 7^2$
$= (2x + 7)^2$

b $81x^2 - 198x + 121$
$= (9x)^2 - 198x + 11^2$
$= (9x - 11)^2$

Reasoning

Use $a^2 + 2ab + b^2 = (a + b)^2$ where a is $2x$ and b is 7.

Use $a^2 - 2ab + b^2 = (a - b)^2$ where a is $9x$ and b is 11.

Example 39

Factorise each of the following.

a $4x^2 + 20xy + 25y^2$

b $a^2 - \frac{2}{3}a + \frac{1}{9}$

Working

a $4x^2 + 20xy + 25y^2$
$= (2x)^2 + 2 \times 2x \times 5y + (5y)^2$
$= (2x + 5y)^2$

b $a^2 - \frac{2}{3}a + \frac{1}{9}$
$= a^2 - 2 \times a \times \frac{1}{3} + \left(\frac{1}{3}\right)^2$
$= \left(a - \frac{1}{3}\right)^2$

Reasoning

$4x^2 + 20xy + 25y^2$ is a perfect square of the form $a^2 + 2ab + b^2 = (a + b)^2$, where a is $2x$ and b is $5y$.

$a^2 - \frac{2}{3}a + \frac{1}{9}$ is a perfect square of the form $a^2 - 2ab + b^2 = (a - b)^2$, where b is $\frac{1}{3}$.

Other non-monic quadratic trinomials

It is more difficult to factorise a quadratic expression where there is no common factor and the coefficient of x^2 is not $+1$ or -1.

The cross method that was introduced in section **3.7** is useful in working out the required terms of the factors.

Example 40

Factorise the following expressions.

a $3x^2 + 2x - 8$

b $6x^2 - 13x + 5$

c $9x^2 - 21x + 10$

continued

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Example 40 continued

Working

a $3x^2 + 2x - 8$

$= (3x - 4)(x + 2)$

If the x-term is correct except for the sign, simply swap the signs of the 'lasts'.

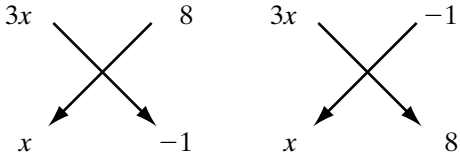
b $6x^2 - 13x + 5$

$= (3x - 5)(2x - 1)$

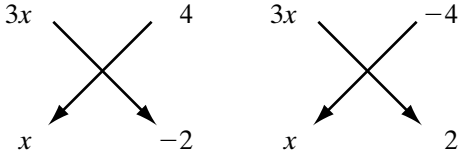
c $9x^2 - 21x + 10$
$= (3x - 2)(3x - 5)$

Reasoning

The only suitable factors of $3x^2$ are $3x$ and x, so these are the 'firsts'. The constant term is negative, so the 'lasts' will have different signs. Test pairs of numbers with a product of -8.

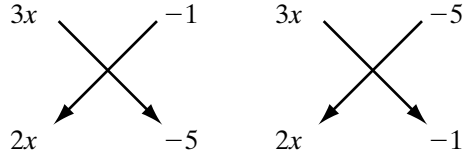


$-3x + 8x = 5x$ $\quad$ $24x - 1x = 23x$



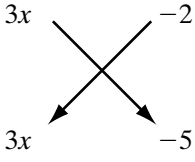
$-6x + 4x = -2x$ $\quad$ $6x - 4x = 2x$

The constant term is positive, so the 'lasts' will have the same sign.
The middle term is negative, so the 'lasts' must be -1 and -5.
The 'firsts' could be $6x$ and x or $3x$ and $2x$. Use the cross method to check.



$-15x - 2x = -17x$ $\quad$ $-3x - 10x = -13x$

Both last terms will be negative.
Using the cross method to guess and check eventually gives



$-15x - 6x = -21x$

Sometimes it is necessary to take out a common factor first and the expression may be a perfect square or a non-monic quadratic trinomial to be factorised.

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Example 41

Factorise the following expressions.

a $-27x^2 + 90xy - 75y^2$

b $24x^2 + 52x - 20$

Working

a $-27x^2 + 90xy - 75y^2$
$= -3(9x^2 - 30xy + 25y^2)$
$= -3[(3x)^2 - 2 \times 3x \times 5y + (5y)^2]$
$= -3(3x - 5y)^2$

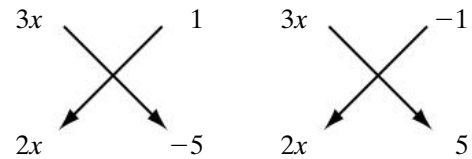
b $24x^2 + 52x - 20$
$= 4(6x^2 + 13x - 5)$
$= 4(3x - 1)(2x + 5)$

Reasoning

Take out a common factor of -3.
$9x^2 - 30xy + 25y^2$ can be written in the form $a^2 - 2ab + b^2$, where $a = 3x$ and $b = -5y$.

$a^2 - 2ab + b^2 = (a - b)^2$

Take out a common factor of 4.



$-15x + 2x = -13x$ $\quad$ $15x - 2x = 13x$

Factorising quadratic trinomials by substitution

Some trinomial expressions can be factorised by temporarily substituting a different pronumeral (such as y) for an algebraic expression (such as x^2). The following example illustrates this idea.

Example 42

Factorise each of the following expressions.

a $36x^4 - 25x^2 + 4$

b $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$

Working

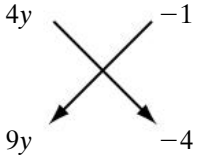
a $36x^4 - 25x^2 + 4$
$= 36(x^2)^2 - 25(x^2) + 4$
$= 36y^2 - 25y + 4$, where $y = x^2$

$= (4y - 1)(9y - 4)$

$= (4x^2 - 1)(9x^2 - 4)$
$= (2x - 1)(2x + 1)(3x - 2)(3x + 2)$

Reasoning

If x^2 is replaced by a single pronumeral, y, the expression becomes a quadratic trinomial.
Trial and error gives



$-16y - 9y = -25y$

Now substitute x^2 for y.
Each difference of squares can be factorised.

continued

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Example 42 continued

Working

b $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$
$= b^2 - 2b - 15$, where $b = x^2 + 4x$

$= (b + 3)(b - 5)$

$= (x^2 + 4x + 3)(x^2 + 4x - 5)$
$= (x + 3)(x + 1)(x + 5)(x - 1)$

Reasoning

If $x^2 + 4x$ is replaced by a single pronumeral, b, this becomes a quadratic trinomial.
Trial and error gives

$$\begin{array}{ccc} b & & 3 \\ b & & -5 \end{array}$$

$-5b + 3b = -2b$

Now substitute $x^2 + 4x$ for b.
Each of the quadratic trinomials in the brackets can now be factorised.

Completing the square with fractions

If the coefficient of x is odd, completing the square will involve fractions.

Example 43

Use the method of completing the square to express each of the following expressions as a difference of squares. Hence factorise each expression.

a $x^2 + 5x + 3$ **b** $x^2 - 7x - 3$

Working

a $x^2 + 5x + 3$

$$= \underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2} \underbrace{+ 3 - \frac{25}{4}}$$

$$= \left(x + \frac{5}{2}\right)^2 + \frac{12 - 25}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{13}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{\sqrt{13}}{2}$$

$$= \left(x + \frac{5}{2} + \frac{\sqrt{13}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{13}}{2}\right)$$

Reasoning

Add the square of half of 5, or $\left(\frac{5}{2}\right)^2$.

Subtract that amount, that is, $\frac{25}{4}$.
Write 3 as a fraction with the same denominator as $\frac{25}{4}$. Simplify the fractions.
This gives a difference of squares $a^2 - b^2$, where a is $x + \frac{5}{2}$ and b is

$$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$$

continued

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Example 43 continued

Working

b $x^2 - 7x - 3$

$$= \underbrace{x^2 - 7x + \left(\frac{7}{2}\right)^2} \underbrace{- 3 - \frac{49}{4}}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{12 + 49}{4}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{61}{4}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{\sqrt{61}}{2}$$

$$= \left(x - \frac{7}{2} + \frac{\sqrt{61}}{2}\right)\left(x - \frac{7}{2} - \frac{\sqrt{61}}{2}\right)$$

Reasoning

Add the square of half of 7, or $\left(\frac{7}{2}\right)^2$.

Subtract that amount, that is, $\frac{49}{4}$.

Write 3 as a fraction with the same denominator as $\frac{49}{4}$. Simplify the fractions.

This gives a difference of squares.

exercise 3.9

LINKS TO Example 37

1 Factorise each of the following.

a $(2x + 1)^2 - (x - 1)^2$ **b** $(a + b)^2 - (a - 2b)^2$ **c** $4(p + q)^2 - (2p + q)^2$

d $9(x - 1)^2 - 4(3 - x)^2$ **e** $x^2 - y^2 + 2x - 2y$ **f** $4x^2 - 9y^2 + 14x + 21y$

LINKS TO Example 38

2 Factorise these perfect squares.

a $4x^2 + 12xy + 9y^2$ **b** $4a^2 + 28ab + 49b^2$ **c** $9x^2 - 30xy + 25y^2$

d $16x^2 - 24xy + 9y^2$ **e** $25m^2 - 70mn + 49n^2$ **f** $49x^2 - 112xy + 64y^2$

g $x^2 - \frac{1}{4}x + \frac{1}{64}$ **h** $a^2 - \frac{1}{2}a + \frac{1}{16}$ **i** $x^2 + \frac{4}{5}x + \frac{4}{25}$

LINKS TO Example 39

3 Factorise each of the following.

a $2x^2 + 7x + 3$ **b** $3x^2 + 10x + 3$ **c** $2x^2 + 7x + 6$

d $3x^2 + 5x + 2$ **e** $2x^2 + 11x + 5$ **f** $7x^2 + 10x + 3$

g $3x^2 + 22x + 7$ **h** $2x^2 - 5x - 3$ **i** $5x^2 + 7x + 2$

j $3x^2 + x - 2$ **k** $2x^2 - 5x + 3$ **l** $3x^2 - 2x - 5$

4 Factorise each of the following.

a $12x^2 + 7x + 1$ **b** $8x^2 + 18x - 5$ **c** $8x^2 - 2x - 15$

d $15x^2 + 26x + 8$ **e** $21x^2 - 20x + 4$ **f** $35x^2 - 9x - 2$

g $15 + 2x - 8x^2$ **h** $4 - 3x - x^2$ **i** $6 - 7x - 3x^2$

j $10x^2 - x - 2$ **k** $4x^2 - 11x + 6$ **l** $3x^2 - 7x - 6$

m $5x^2 + 11x + 2$ **n** $5x^2 - 22x + 21$ **o** $3x^2 - x - 30$

p $7x^2 - 2x - 5$ **q** $4x^2 - x - 18$ **r** $10x^2 + x - 2$

s $8x^2 + 38x + 35$ **t** $6x^2 - 13x + 2$ **u** $7x^2 - 11x - 6$

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LINKS TO Example 41

5 Factorise each of the following by first taking out the common factor.

a $8x^2 + 12x - 36$ **b** $30x^2 + 55x - 35$ **c** $6x^2 + 9x - 27$
d $12x^2 + 24x + 9$ **e** $50x^2 + 15x - 5$ **f** $4 - 6x - 10x^2$
g $6x^2 - 9x - 15$ **h** $12x^2 - 68x + 40$ **i** $30x^2 - 5x - 10$

6 Factorise each of the following.

a $x^3 + 5x^2 + 4x$ **b** $3x^3 + 4x^2 - 4x$ **c** $8x^3 + 18x^2 - 5x$
d $6x^4 + 5x^3 + x^2$ **e** $10x^4 - x^3 - 2x^2$ **f** $3x^4 - 15x^3 - 72x^2$

LINKS TO Example 42

7 Use substitution to fully factorise each of the following.

a $x^4 - 5x^2 + 4$ **b** $x^4 - 29x^2 + 100$
c $16x^4 - 40x^2 + 9$ **d** $2(x + 3)^2 - (x + 3) - 6$
e $3(x - 1)^2 + 8(x - 1) + 4$ **f** $3(x + 1)^2 + 16(x + 1) + 5$
g $(x^2 + 3x)^2 - 8(x^2 + 3x) - 20$ **h** $(x^2 - x)^2 - 8(x^2 - x) + 12$

8 Fill the gaps to factorise each of the following.

a $2x^2 + 8x + 2$
$= 2[x^2 + __ + __]$
$= 2[x^2 + __ + (__)^2 + __ - (__)^2]$
$= 2[(x + __)^2 - __]$
$= 2(x + __ + __)(x + __ - __)$

b $3x^2 + 12x + 6$
$= 3[x^2 + __ + __]$
$= 3[x^2 + __ + (__)^2 + __ - (__)^2]$
$= 3[(x + __)^2 - __]$
$= 3(x + __ + __)(x + __ - __)$

c $2x^2 + 10x + 4$
$= 2[x^2 + __ + __]$
$= 2[x^2 + __ + (__)^2 + __ - (__)^2]$
$= 2[(x + __)^2 - __]$
$= 2(x + __ + __)(x + __ - __)$

d $5x^2 + 10x + 2$
$= 5[x^2 + __ + __]$
$= 5[x^2 + __ + (__)^2 + __ - (__)^2]$
$= 5[(x + __)^2 - __]$
$= 5(x + __ + __)(x + __ - __)$

9 Take out a common factor and use the method of completing the square to factorise each of the following.

a $2x^2 - 16x + 4$ **b** $6x^2 + 24x - 12$ **c** $4x^2 + 32x + 44$
d $5x^2 + 30x + 30$ **e** $3x^2 + 9x + 3$ **f** $2x^2 + 4x - 10$

LINKS TO Example 45

10 Use the method of completing the square to factorise each of the following.

a $x^2 + 11x + 20$ **b** $x^2 - 9x + 12$ **c** $x^2 + 15x + 32$
d $x^2 + 3x - 1$ **e** $x^2 + 13x - 50$ **f** $x^2 - 7x - 2$

exercise 3.9 challenge

11 A certain circle has area $A = \pi(4x^2 + 20x + 25)$. Find an expression for the radius of this circle.

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Analysis task

Pascal's triangle and binomial expansions

BLM Pascal's triangle

Pascal's triangle is named after the French mathematician Blaise Pascal (1623–1662). Pascal was the first European to make an extensive study of the patterns in this special triangle, which was discovered by the Chinese about 500 years before Pascal was born.

The following steps are used to create Pascal's triangle.

- Write the number 1 to create the apex (top) of the triangle. This is row 0.
- Write the number 1 at either end of the next row. This is row 1.
- Write the number 1 at either end of row 2. To find the number in between, add the two numbers diagonally above.

Row 0	1
Row 1	1 1
Row 2	1 2 1

- To generate each new row in Pascal's triangle, write the number 1 at either end of the row. All numbers in between are found by adding the two numbers diagonally above.

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1

a Copy Pascal's triangle above, and write the next three rows.

b Expand $(x + a)^2$. How does the expanded form relate to row 2 of Pascal's triangle? Hint: remember that the coefficient of $x^2 = 1$.

c Expand $(x + a)^3$ by using $(x + a)^3 = (x + a)(x + a)^2$, that is, multiply your previous answer by $(x + a)$. How does the expanded form relate to row 3 of Pascal's triangle?

d The expansion of $(x + a)^4$ is $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$. How does the expanded form relate to row 4 of Pascal's triangle?

e Look from one term to the next in $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$, starting with x^4.

i What pattern can you see in the powers of x?

ii What pattern can you see in the powers of a?

f Use Pascal's triangle to complete the following expansion.

$(x + a)^5 = x^5 + __x^4a + __x^3a^2 + __x^2a^3 __xa^4 + a^5$

g Use Pascal's triangle to help you expand $(x + a)^6$ and $(x + a)^7$.

Challenge

h Expand $(x - a)^2$ and $(x - a)^3$. How are the results different from those for $(x + a)^2$ and $(x + a)^3$?

i Use Pascal's triangle to help you expand $(x - a)^4$ and $(x - a)^5$.

j Use Pascal's triangle to help you expand each of the following.

i $(x + 2)^4$ **ii** $(x - 3)^5$ **iii** $(2x + 1)^3$ **iv** $(2x + 3y)^4$

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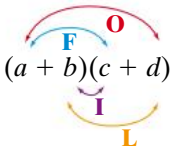
Review Algebra

Summary

Algebraic expressions: expansion and common factors

- The process of going from the factorised form of an expression to the expanded form is called expansion.
- In general, we can say that $a(b + c) = ab + ac$.
  This is the distributive law.
- When expanding two brackets,
  $(a + b)(c + d) = ac + ad + bc + bd$.
  The word FOIL reminds us to multiply
  Firsts, Outside, Inside and Lasts.



- The process of going from the expanded form to the factorised form is called factorisation.
- The highest common factor of the terms in the expression is taken out in front of brackets.

Factorisation involving binomial factors

- A binomial factor can be taken out in front of a bracket like any other common factor.
- To factorise by grouping two and two, group the terms in pairs so that each pair of terms has a common factor, and then take a binomial common factor out in front of a bracket.

Perfect squares and differences of squares

Expansion

$(a + b)^2 = a^2 + 2ab + b^2$

Factorisation

Expansion

$(a - b)^2 = a^2 - 2ab + b^2$

Factorisation

Expansion

$(a + b)(a - b) = a^2 - b^2$

Factorisation

- Not all expressions can be factorised using only rational numbers. When factorisation requires the use of irrational square roots, such as $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$, this is referred to as factorisation over R, the field of real numbers.
- Expressions that are a *sum* of two squares, for example, $x^2 + 9$, cannot be factorised.

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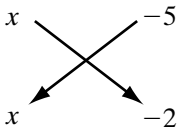
Factorising quadratic trinomials

- To factorise a quadratic trinomial of the form $x^2 + bx + c$, look for two numbers that have
  - a *product* equal to c (the constant term) and
  - a *sum* equal to b (the coefficient of x).
- The cross-method is useful in factorising many quadratic trinomials.
- For non-monic quadratic trinomials check first to see if there is a common factor.

$x^2 - 7x + 10$



Completing the square

- The method of completing the square is used to factorise quadratic trinomials that cannot be factorised using methods of inspection.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

binomial expansion	expanded form	factorised form
coefficient	expansion	monic
common factor	expression	*non-monic
complete the square	factor	perfect square
cross method	factorisation	power
difference of two squares	factorisation over *Q*	quadratic trinomial
equivalent expressions	factorisation over *R*	

Revision

Multiple-choice questions

1 When factorised, $4p^2 - 16q^2$ is equivalent to

A $4(p - 2q)^2$ **B** $16(p - 4q)^2$ **C** $2(p - 4q)(p + 4q)$
D $4(p - 4q)(p + 4q)$ **E** $4(p - 2q)(p + 2q)$

2 Which of the following expressions would be factorised by grouping 'two and two'?

A $x^2 - p^2 + 12p - 36$ **B** $x^2 - 7x - 10$ **C** $2x^2 - 6x - yx + 3y$
D $(w - 5)^2 - 25(w + 3)^2$ **E** $(b + 5) - (b + 3)(b + 5)$

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3 $(x + y)^2 - (x^2 + y^2)$ is equivalent to

A 0 **B** xy **C** $2y^2$ **D** $4xy + 2y^2$ **E** $2xy$

4 The expression $x(2x - 3)(3x + 2)$ is equal to

A $6x^2 - 5x - 1$ **B** $6x^2 - 5x - 6$ **C** $6x^3 - 5x^2 - x$
D $5x^3 - 5x^2 - 6x$ **E** $6x^3 - 5x^2 - 6x$

5 Which of the following expressions is not a perfect square?

A $4x^2 - 20x + 25$ **B** $p^2 + 6py + 9y^2$ **C** $9x^2 - 12xy + 16y^2$
D $4a^2 + 4ab + b^2$ **E** $m^2 - 10mn + 25n^2$

Short-answer questions

6 Which of the following expressions are

i perfect squares?
ii a difference of squares?

a $x^2 + 8x + 16$ **b** $a^2 - 1$ **c** $4b^2 - 12b + 9$
d $(w - 5)(w + 5)$ **e** $-z^2 + 25$ **f** $(2x + 3)(2x - 3)$

7 Expand and simplify each of the following.

a $(2a + 3y)(a - y)$
b $(2x - y)^2$
c $(2x - 5)^2 - (x + 2)(3x + 7)$

8 Factorise each of the following by taking out a common factor.

a $12a^2b^5 - 3ab^8$ **b** $4(2x + 3) - (2x + 3)^2$
c $(3y - 1)^2 + (2y + 3)(3y - 1)$ **d** $(2x + y)^2 - 6x - 3y$

9 Use your knowledge of perfect squares to factorise each of the following.

a $x^2 - 14x + 49$ **b** $x^2 + 6xy + 9y^2$

10 Factorise the following differences of squares.

a $144 - c^2$ **b** $9x^2 - 121$
c $(x + 3)^2 - 16$ **d** $25 - (a - 4)^2$
e $16(x - 3)^2 - (x - 2)^2$ **f** $4x^2 - 3$ (over R)

11 Factorise each of the following.

a $e^2 + 5e - 24$ **b** $p^2 - 5p - 6$
c $x^2 + 15x + 56$ **d** $x^2 + 2x - 63$
e $a^2 + 8a - 48$ **f** $m^2 + 18m + 45$
g $x^2 + 4x - 21$ **h** $x^2 - 10x + 21$

12 Factorise the following expressions by completing the square.

a $x^2 + 8x + 1$ **b** $m^2 - 6m + 3$

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13 Factorise each of the following expressions by first taking out the common factor.

a $5x^2 + 10x + 5$ **b** $2a^2 + 24a + 72$ **c** $6x^2 - 24x + 24$
d $3m^2 - 42m + 147$ **e** $20x^2 + 120x + 180$ **f** $-3x^2 + 30x - 75$
g $-4x^2 - 88x - 484$ **h** $-5x^2 + 90x - 405$ **i** $45x^2 + 30x + 5$
j $3a^2 - 12$ **k** $16 - 4x^2$ **l** $4y^2 - 100z^2$
m $64d^2 - 16e^2$ **n** $5x^2 - 45$ **o** $12x^2 - 75y^2$

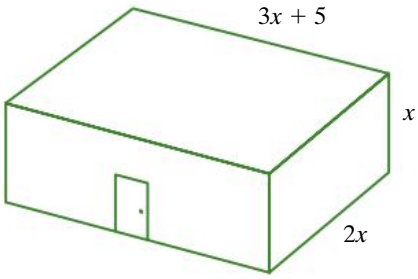
14 Use any appropriate methods to factorise the following expressions.

a $(4x + 1)^2 - (x - 3)^2$ **b** $2m^2 - 4m - 30$
c $8(5m + 3)^2 - 14(5m + 3) + 3$ **d** $25x^2 - 30xy + 9y^2$
e $y^2 - \frac{2}{5}y + \frac{1}{25}$ **f** $9a^2 - 12ab + 4b^2$
g $10w^2 - 35w - 75$ **h** $2(x - 3)^2 - 2(x - 3) - 24$

Extended-response question

15 A garden shed made of aluminium panels is designed in the shape of a rectangular prism as shown. The shed does not include a floor. Dimensions are in metres. Find

a an expanded expression for the volume of the shed.
b an expanded expression for the total surface area of the shed.
c the cost of the aluminium if $x = 3$ and the panels cost \$4.70 per square metre.



Practice quiz Chapter 3

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4 Surds

Pre-test

Warm-up

Real numbers are classified as either rational or irrational. Google has numbered its buildings at this site in California 0, 1, e and π. Which of these are irrational numbers? What is e? What do all irrational numbers have in common? What are some other irrational numbers? Are surds irrational numbers?

4.1 Real numbers: rational and irrational

Numbers such as 3, 0.3, $0.\dot{3}$, $\sqrt{3}$, $\sqrt[3]{3}$ and π are called **real numbers**. In fact all rational and irrational numbers are real numbers. As you might expect, if there are numbers that are called real numbers there are also numbers that are unreal. In fact $\sqrt{-3}$ is one of these. Mathematicians call them non-real or **imaginary numbers** rather than unreal numbers. Imaginary numbers have some important applications in mathematics and electrical engineering. However, school maths focuses on real numbers.

Real numbers are divided into two groups: **rational numbers** and **irrational numbers**. The numbers 3, 0.3 and $0.\dot{3}$ are rational numbers.

Rational numbers

A rational number is any number that can be written in the form $\frac{a}{b}$ where a and b are integers and b is not equal to 0.

Rational numbers

All rational numbers can be written in the form of a fraction $\frac{a}{b}$, where a and b are integers, and $b \neq 0$.

The set of rational numbers includes

- all the numbers we normally think of as fractions; for example, $\frac{1}{2}$, $-\frac{3}{7}$, $\frac{8}{17}$, $2\frac{3}{8} = \frac{19}{8}$.
- integers (whole numbers), because we can write whole numbers as fractions; for example, $3 = \frac{3}{1}$, $-7 = -\frac{7}{1}$.
- all terminating decimals; for example, $0.37 = \frac{37}{100}$, $2.9 = \frac{29}{10}$.
- recurring decimals, because they can be written as fractions; for example; $0.\dot{3} = 0.333\ldots = \frac{1}{3}$.

Irrational numbers

Real numbers that cannot be written in the form of a fraction $\frac{a}{b}$ (where a and b are integers) are called **irrational numbers**.

Irrational numbers

An important difference between rational and irrational numbers is that the decimal places of irrational numbers continue forever, without forming any repeating pattern.

The prefix 'ir' in front of the word 'rational' means 'not'.

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The square roots of many numbers are irrational. The square root of 2, for example, is an irrational number. If we use a calculator that displays 31 decimal places, we find that

$$\sqrt{2} \approx 1.4142135623730950488016887242097$$

No matter how many decimal places we give, we can never express $\sqrt{2}$ exactly. It is always possible to calculate more decimal places.

The square roots of 3, 5, 6, 7 and 8 are also irrational. However, the square roots of 1, 4 and 9 are rational because 1, 4 and 9 are square numbers: $\sqrt{1} = 1$, $\sqrt{4} = 2$ and $\sqrt{9} = 3$. Similarly, many cube roots, fourth roots, and so on are irrational.

In measurement calculations, we usually find **rational approximations** for irrational numbers. For example, we may use the approximate value 1.414 for $\sqrt{2}$. We say that 1.414 is a rational approximation for $\sqrt{2}$.

Example 1

Sort the following numbers into two groups: rational numbers and irrational numbers.

$\sqrt{6}$ $\quad -5 \quad 2\pi \quad 2\frac{3}{4} \quad -3.181818\ldots \quad \sqrt[3]{27} \quad 0 \quad -\frac{1}{2} \quad \sqrt[3]{2}$

Working

The rational numbers are:

$-5, 2\frac{3}{4}, -3.181818\ldots, \sqrt[3]{27}, 0, -\frac{1}{2}.$

Reasoning

A rational number can be written as $\frac{a}{b}$ where a and b are integers, and $b \neq 0$.

$-5 = \frac{-5}{1}$

$2\frac{3}{4} = \frac{11}{4}$

$-3.181818\ldots$ has a recurring pattern and so can be converted to a fraction.

$-3.\overline{18} = \frac{315}{99}$

$\sqrt[3]{27} = 3 = \frac{3}{1}$

$0 = \frac{0}{1}$

$-\frac{1}{2} = \frac{-1}{2}$

The irrational numbers are:

$\sqrt{6}, 2\pi, \sqrt[3]{2}.$

$\sqrt{6} \approx 2.449489\ldots$ has no recurring pattern. It cannot be expressed as a fraction.

π is irrational so 2π is irrational. It is approximately equal to $6.2831853\ldots$ with no recurring pattern.

$\sqrt[3]{2} \approx 1.2599210\ldots$ has no recurring pattern, so it is irrational.

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Surds

The **root** of a rational number that is irrational is called a **surd**.

Examples of surds are $\sqrt{2}$, $\sqrt{\frac{3}{7}}$, $\sqrt{5}$ and $\sqrt[3]{2}$. Examples of roots that are *not* surds are $\sqrt{1}$, $\sqrt{16}$ and $\sqrt{\frac{4}{9}}$ because these roots can all be evaluated to give rational numbers.

Most of the surds you will work with in the remainder of this chapter are **square roots**, but surds may also be cube roots, fourth roots, and so on.

Example 2

Which of the following square and cube roots are surds?

$\sqrt{2}$, $\sqrt[3]{27}$, $\sqrt{12}$, $\sqrt{25}$, $\sqrt[3]{6}$, $4\sqrt{3}$.

Working

$\sqrt{2}$, $\sqrt{12}$, $\sqrt[3]{6}$ and $4\sqrt{3}$ are surds.

Reasoning

Surds are roots that cannot be evaluated to give rational numbers.

$\sqrt[3]{27} = 3$ so it is not a surd.

$\sqrt{25} = 5$ so it is not a surd.

All of the other numbers in the list are surds.

Exact values

If we calculate the hypotenuse of a right-angled triangle to be $\sqrt{5}$, we say that $\sqrt{5}$ is the **exact value**. We can work with it, without limiting it to a number of decimal places. When we use a calculator to evaluate $\sqrt{5}$, what we have is no longer exact. It is a **rational approximation**.

In calculations, as you have seen with Pythagoras' theorem and the solution of quadratic equations, we usually need to find rational approximations of surds. However, sometimes we work with the exact surd value rather than limiting accuracy by working with an approximation. For this reason, the remainder of this chapter deals with techniques for simplifying number expressions that contain surds.

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Example 3

Without using a calculator, match each of these exact values with its rational approximation.

Exact values: $\sqrt{2}$, $\sqrt{12}$, $4\sqrt{3}$

Decimal approximations: 3.464, 6.928, 1.414

Working

$\sqrt{2} \approx 1.414$, $\sqrt{12} \approx 3.464$, and $4\sqrt{3} \approx 6.928$

Reasoning

$\sqrt{2}$ is between 1 and 2. We know this because $\sqrt{1} = 1$ and $\sqrt{4} = 2$.

$\sqrt{12}$ is between 3 and 4. We know this because $\sqrt{9} = 3$ and $\sqrt{16} = 4$.

$\sqrt{3}$ is between 1 and 2, so $4\sqrt{3}$ is between 4×1 and 4×2, that is between 4 and 8.

Equivalent surd expressions

If you use your calculator to evaluate $\sqrt{12}$ and $2\sqrt{3}$, you will find that their decimal approximations are the same. $\sqrt{12}$ and $2\sqrt{3}$ are **equivalent** surds.

$\sqrt{12}$ can be simplified by first writing it as $\sqrt{4} \times \sqrt{3}$. Then $\sqrt{4}$ can be simplified to 2.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= 2 \times \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

Notice that $\sqrt{12}$ can be simplified because 12 has the square number 4 as a factor. A surd can be simplified if the number under the square root sign has a square number as a factor.

Example 4

Show that $\sqrt{18}$ and $3\sqrt{2}$ are equivalent surds.

Working

$$\begin{aligned}\sqrt{18} &= \sqrt{9} \times \sqrt{2}\\ &= 3 \times \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

Reasoning

9 is a square number so $\sqrt{9} = 3$

We could have written $\sqrt{18}$ as $\sqrt{6} \times \sqrt{3}$ but this would have taken an extra step to get to $3\sqrt{2}$ because 6 and 3 are not square numbers.

$$\begin{aligned}\sqrt{18} &= \sqrt{6} \times \sqrt{3}\\ &= \sqrt{2} \times \underbrace{\sqrt{3} \times \sqrt{3}}_{3}\\ &= \sqrt{2} \times 3\\ &= 3\sqrt{2}\end{aligned}$$

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Simplifying surds

We found earlier that a surd can be simplified if the number under the square root sign has a square number as a factor. In example **4**, it was shown that $\sqrt{18}$ was simplified to $3\sqrt{2}$. Here, $3\sqrt{2}$ is the simpler surd. It is preferable to write surds as simply as possible.

Example 5

Simplify each of the following.

a $\sqrt{50}$ **b** $2\sqrt{45}$

Working	Reasoning
a $\sqrt{50}$	Look for a factor of 50 that is a square number. 50 is 2×25 or 5×10. 25 is a square number, so use 2×25.
50×10 is of no use because neither 5 nor 10 is a square	
	Write $\sqrt{50}$ as the product of the surds $\sqrt{2}$ and $\sqrt{25}$, with the square number surd first.
$= \sqrt{25} \times \sqrt{2}$	Calculate $\sqrt{25}$. The expression becomes a whole number multiplied by a surd.
$= 5\sqrt{2}$	$5\sqrt{2}$ is a simpler surd than $\sqrt{50}$.
b $2\sqrt{45}$	Look for a factor of 45 that is a square number. 45 is 3×15 or 5×9.
$= 2 \times \sqrt{9} \times \sqrt{5}$	9 is a square number, so use 5×9. Write $\sqrt{45}$ as the product of surds, with the square number surd first.
$= 2 \times 3 \times \sqrt{5}$	Calculate $\sqrt{9}$.
$= 6\sqrt{5}$	Multiply the two whole numbers.

Writing a surd as an entire surd

We have seen that $\sqrt{12}$ and $2\sqrt{3}$ are equal, and that $2\sqrt{3}$ is called the simpler surd because it has the smaller number under the square root sign.

A surd expressed in the form $a\sqrt{b}$, is called an **entire** surd if $a = 1$, that is the coefficient is 1. So $\sqrt{12}$ is an example of an entire surd. Example **6** shows how to write surds as entire surds.

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Example 6

Write the following as entire surds.

a $5\sqrt{2}$ **b** $2\sqrt{11}$

Working	Reasoning
a $5\sqrt{2} = 5 \times \sqrt{2}$	$5\sqrt{2}$ means $5 \times \sqrt{2}$. Write the whole number part as the square root of its square.
$= \sqrt{25} \times \sqrt{2}$	Here, write 5 as $\sqrt{25}$.
$= \sqrt{50}$	Multiply the two square roots.
b $2\sqrt{11} = 2 \times \sqrt{11}$	$2\sqrt{11}$ means $2 \times \sqrt{11}$ Write the whole number part as the square root of its square.
$= \sqrt{4} \times \sqrt{11}$	Here, write 2 as $\sqrt{4}$.
$= \sqrt{44}$	Multiply the two square roots.

Relationship between prime factors and simplified surds

In Year 7 you learnt how to use a factor ladder to express whole numbers as the product of their prime factors. For example, 45 can be expressed as $3 \times 3 \times 5 = 3^2 \times 5$, where 3 and 5 are prime numbers.

3	45
3	15
5	5
	1

We can express $\sqrt{45}$ as $\sqrt{3 \times 3 \times 5} = \sqrt{3} \times \sqrt{3} \times \sqrt{5} = 3\sqrt{5}$.

If we now compare $45 = 3^2 \times 5^1$ and $\sqrt{45} = 3\sqrt{5}$ we see that the even power, 3^2, leads to the coefficient, 3, and the odd power, 5^1, leads to $\sqrt{5}$.

When we take the square root of a number written as the product of its prime factors in index form, the factors with **even numbered powers** give the **coefficient** of the surd and the factors with **odd numbered powers** give the **number under the square root sign**.

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Example 7

Express 360 as the product of its prime factors in index form and hence express $\sqrt{360}$ in simplified surd form.

Working

2	360
2	180
2	90
3	45
3	15
5	5
	1

$$\begin{aligned} 360 &= 2^3 \times 3^2 \times 5 \\ \sqrt{360} &= \sqrt{2^3 \times 3^2 \times 5} \\ &= \sqrt{2^2 \times 2^1 \times 3^2 \times 5} \\ &= 2 \times \sqrt{2} \times 3 \times \sqrt{5} \\ &= 2 \times 3 \times \sqrt{2} \times \sqrt{5} \\ &= 6\sqrt{10} \end{aligned}$$

Reasoning

Use a factor ladder to find the prime factors of 360, starting with the smallest prime number, 2. Continue with the next prime number until 1 is obtained.

Express 360 as the product of its prime factors in index form.
Express $\sqrt{360}$ as the square root of the prime factors in index form. Express 2^3 as the product of even powered 2^2 and odd powered 2^1.
Take the square root of the even numbered powers.
Multiply the coefficients and multiply the numbers under the square root signs.

exercise 4.1

LINKS TO Example 1

1 Sort the following numbers into two groups: rational numbers and irrational numbers.
$\sqrt[3]{8}$, -6, 3π, $3\frac{1}{2}$, -5.242424…, $\sqrt{10}$, 0, $\sqrt[3]{7}$, 3.1703402…, $-\frac{5}{9}$.

LINKS TO Example 2

2 The only irrational number given below is

A $\sqrt[3]{-8}$ **B** $\sqrt{0}$ **C** $\sqrt[3]{1}$ **D** $\sqrt{9}$ **E** $\sqrt{27}$

3 Consider the square roots of the whole numbers from 1 to 20: $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{20}$.

a List the square roots and evaluate where possible, for example, $\sqrt{4} = 2$.

b Sort these square roots into two groups: surds and 'not surds'.

c Which group has more entries? Why?

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4 Draw a number line labelled 0, 1, 2, …, 5. Mark each of the surds from question **3** on this number line.

LINKS TO Example 3

5 Use a calculator to find rational approximations correct to three decimal places for the following surds

a $\sqrt{3}$ **b** $\sqrt{13}$ **c** $\sqrt{24}$ **d** $\sqrt{7}$

e $5\sqrt{2}$ **f** $5\sqrt{5}$ **g** $6\sqrt{6}$ **h** $7\sqrt{3}$

LINKS TO Example 4

6 Consider the two lists of surds below.

List 1: $\sqrt{18}$, $\sqrt{28}$, $\sqrt{300}$, $\sqrt{108}$ List 2: $2\sqrt{7}$, $6\sqrt{3}$, $3\sqrt{2}$, $10\sqrt{3}$

Match each of the surds in list 1 with a simpler surd from list 2.

7 Simplify each of the following surds to find which surd is not equal to the other four surds.

A $10\sqrt{8}$ **B** $4\sqrt{50}$ **C** $5\sqrt{32}$ **D** $\sqrt{96}$ **E** $20\sqrt{2}$

LINKS TO Example 5a

8 Simplify each of the following surds.

a $\sqrt{12}$ **b** $\sqrt{63}$ **c** $\sqrt{24}$ **d** $\sqrt{18}$

e $\sqrt{50}$ **f** $\sqrt{27}$ **g** $\sqrt{40}$ **h** $\sqrt{54}$

i $\sqrt{700}$ **j** $\sqrt{90}$ **k** $\sqrt{32}$ **l** $\sqrt{99}$

m $\sqrt{20}$ **n** $\sqrt{72}$ **o** $\sqrt{75}$ **p** $\sqrt{28}$

LINKS TO Example 5b

9 Simplify each of the following surds.

a $3\sqrt{8}$ **b** $2\sqrt{45}$ **c** $5\sqrt{27}$ **d** $3\sqrt{50}$

e $3\sqrt{44}$ **f** $-5\sqrt{28}$ **g** $10\sqrt{24}$ **h** $-2\sqrt{300}$

i $3\sqrt{20}$ **j** $-7\sqrt{18}$ **k** $-4\sqrt{90}$ **l** $3\sqrt{32}$

m $2\sqrt{12}$ **n** $4\sqrt{99}$ **o** $-2\sqrt{63}$ **p** $3\sqrt{28}$

10 When simplified, $2\sqrt{20}$ becomes

A $12\sqrt{2}$ **B** $20\sqrt{2}$ **C** $2\sqrt{5}$ **D** $4\sqrt{5}$ **E** $4\sqrt{10}$

LINKS TO Example 6

11 Express each of the following as an entire surd.

a $3\sqrt{5}$ **b** $2\sqrt{2}$ **c** $5\sqrt{3}$ **d** $3\sqrt{7}$

e $2\sqrt{11}$ **f** $-2\sqrt{6}$ **g** $10\sqrt{5}$ **h** $-3\sqrt{3}$

i $4\sqrt{3}$ **j** $-2\sqrt{5}$ **k** $-3\sqrt{11}$ **l** $6\sqrt{2}$

12 Consider the surds below.

$4\sqrt{3}$, $5\sqrt{2}$, $3\sqrt{6}$, $2\sqrt{10}$, $2\sqrt{11}$.

a Writing simplified surds as entire surds is useful when surds need to be arranged in ascending or descending order. By expressing each of the surds above as an entire surd, arrange them in descending order.

b Write a rational approximation for each of the surds, correct to two decimal places. Use this to check that your order in part **a** is correct.

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10 A

LINKS TO Example 7

13 Copy the table below. Express each of the numbers in column 1 as the product of its prime factors in index form. Write this product in column 2. In column 4 write the simplified form of the surds that are listed in column 3.

Number	Product of prime factors	Surd	Simplified surd
12		$\sqrt{12}$	
18		$\sqrt{18}$	
50		$\sqrt{50}$	
28		$\sqrt{28}$	
90		$\sqrt{90}$	

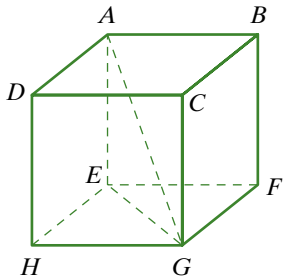
exercise 4.1 challenge

14 **a** AG is the diagonal of a cube of edge length 2 m.
Find the length of AG in simplest surd form.

b Repeat part **a** for cubes with the following edge lengths.

i 3 m **ii** 4 m

iii 8 m **iv** x m



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4.2 Square roots and cube roots as fractional powers

So far we have worked only with powers that are integers; for example, 2^5, 10^{-4}, 6^0 and x^3. However, powers can also be fractions or decimals.

If we compare the results of squaring $\sqrt{3}$ and squaring $3^{\frac{1}{2}}$ we find that we obtain 3 in both cases. To show that this is true, we square both numbers.

$$\sqrt{3} \times \sqrt{3} = 3 \text{ and } 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}}$$
$$= 3^1$$
$$= 3$$

So if $\sqrt{3} \times \sqrt{3} = 3$ and $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3$, then $\sqrt{3}$ is equivalent to $3^{\frac{1}{2}}$.

In a similar way we can show that $\sqrt[3]{2}$ is equal to $2^{\frac{1}{3}}$.

Cubing both numbers

$$2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$= 2^1$$
$$= 2 \text{ and } \sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$$

So if $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$, then $2^{\frac{1}{3}}$ is equivalent to $\sqrt[3]{2}$.

Square roots and cube roots

Raising a number to the power $\frac{1}{2}$ is the same as taking the square root of the number.

For example, $9^{\frac{1}{2}} = \sqrt{9} = 3$.

Raising a number to the power $\frac{1}{3}$ is the same as taking the cube root of the number.

For example, $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$.

This means that any integer that can be written in index form with an even index will have an integer square root. For example, $\sqrt{3^{10}} = (3^{10})^{\frac{1}{2}} = 3^5$.

Notice that we simply halve the index to obtain the square root.

However, $\sqrt{3^7}$ will not be an integer. If we write 3^7 as the product of 3^6 and 3^1, we can find the square root by halving the index of 3^6 and leaving $\sqrt{3^1}$ as a surd.

$$\sqrt{3^7} = \sqrt{3^6 \times 3^1}$$
$$= \sqrt{3^6} \times \sqrt{3^1}$$
$$= 3^3 \times \sqrt{3}$$
$$= 27\sqrt{3}$$

Fractional powers

$a^{\frac{1}{n}} = \sqrt[n]{a}$, where a is a positive number, and n is a non-zero integer.

$a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$

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Example 8

Evaluate.

a $49^{0.5}$

b $64^{\frac{1}{3}}$

Working

a $49^{0.5} = 49^{\frac{1}{2}}$
$= \sqrt{49}$
$= 7$

b $64^{\frac{1}{3}} = \sqrt[3]{64}$
$= 4$

Reasoning

A power of a half means 'square root of'.

A power of a third means 'cube root of'.

Example 9

Find the square root of 3024 in simplest surd form by first writing 3024 as the product of its prime factors.

Working

2	3024
2	1512
2	756
2	378
3	189
3	63
3	21
7	7
	1

$$3024 = 2^4 \times 3^3 \times 7$$
$$\sqrt{3024} = \sqrt{2^4 \times 3^3 \times 7}$$
$$= \sqrt{2^4 \times 3^3 \times 3^1 \times 7}$$
$$= \sqrt{2^4 \times 3^3} \times \sqrt{3 \times 7}$$
$$= 2^2 \times 3 \times \sqrt{3 \times 7}$$
$$= 12\sqrt{21}$$

Reasoning

Use a factor ladder to find the prime factors.

$$\underbrace{3^3}_{\text{odd index}} = \underbrace{3^2}_{\text{even index}} \times \underbrace{3^1}_{\text{odd index}}$$

To find the square root of numbers in index form, halve the even indices and leave the parts with odd indices as surds.

$\sqrt{2^4} = 2^2$, $\sqrt{3^2} = 3^1 = 3$

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Solving equations with indices on both sides

Equations that include surds on both sides can be solved using the following steps.

- **Step 1**: Express each side in index form with the same base.
- **Step 2**: Equate the indices.
- **Step 3**: If necessary, solve an equation formed by equating the indices.

Example 10

Solve the following equations for x.

a $5^x = \sqrt[3]{5}$

b $\sqrt{2^{x+1}} = 8$

Working

a

$5^x = \sqrt[3]{5}$

$5^x = 5^{\frac{1}{3}}$

$x = \frac{1}{3}$

b

$\sqrt{2^{x+1}} = 8$

$(2^{x+1})^{\frac{1}{2}} = 2^3$

$2^{\frac{x+1}{2}} = 2^3$

$\frac{x+1}{2} = 3$

$x + 1 = 6$

$x = 5$

Reasoning

Write the cube root of 5 as 5 to the power of a third. Now numbers on both sides of the equation have the same base.

Equate the powers.

Write each term in index form with the same base. Remember that a square root can be written as a power of a half.

Simplify the index expressions where possible.

Equate the powers.

Solve the equation.

Check: Substitute $x = 5$ on the left-hand side.

LS $= \sqrt{2^{5+1}} = \sqrt{2^6} = 2^3 = 8 =$ RS.

exercise 4.2

LINKS TO Example 8

1 Evaluate the following. Where necessary, round your answers to three decimal places.

a $4^{0.5}$ **b** $9^{\frac{1}{2}}$ **c** $16^{\frac{1}{2}}$ **d** $25^{0.5}$ **e** $\sqrt{3}$ **f** $\sqrt[3]{8}$ **g** $\sqrt[3]{50}$

2 The value of $27^{\frac{1}{3}}$ is

A 3 **B** 9 **C** $27\frac{1}{3}$ **D** $\frac{1}{19683}$ **E** $\frac{1}{81}$

LINKS TO Example 9

3 Consider the number 360.

a Use a factor ladder to find the prime factors of 360.

b Complete $360 = 2^? \times 3^? \times 5^?$.

c Complete $\sqrt{360} = \sqrt{2^? \times 3^? \times 5^?}$

$= \sqrt{\text{even powers}} \times \sqrt{\text{any factors left}}$

d Express $\sqrt{360}$ in simplest form.

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4 For each of the following numbers

i use a factor ladder to find the prime factors and then write the product of the prime factors in index form

ii use this to find the square root of the number in simplest surd form.

a 576 **b** 180 **c** 1568 **d** 864

LINKS TO Example 10

5 Solve the following equations for x.

a $10^x = \sqrt[3]{10}$ **b** $2^x = \sqrt{2}$ **c** $3^{2x} = \sqrt[3]{3}$ **d** $5^{x-2} = \sqrt{5}$

e $\sqrt{3^{x+2}} = 27$ **f** $\sqrt{7^x} = 49$ **g** $\sqrt[3]{5^{x+2}} = 125$ **h** $\sqrt{2^{x-1}} = 16$

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4.3 Fractional powers

We have seen that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Now consider $\sqrt[3]{25}$. This can be written as $\sqrt[3]{5^2}$.

In index form this is $(5^2)^{\frac{1}{3}}$, which is equal to $5^{\frac{2}{1}\times\frac{1}{3}} = 5^{\frac{2}{3}}$. So $\sqrt[3]{5^2} = 5^{\frac{2}{3}}$.

Also $5^{\frac{2}{3}} = 5^{\frac{1}{3}\times\frac{2}{1}} = (5^{\frac{1}{3}})^2 = (\sqrt[3]{5})^2$. So $5^{\frac{2}{3}} = \sqrt[3]{5^2} = (\sqrt[3]{5})^2$.

Fractional powers

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$, where a is a positive number, both m and n are integers, and $n \neq 0$.

The denominator of the fraction indicates the degree of the root. For a denominator n, we have the nth root.

The numerator tells the power to which it is raised.

It does not matter whether the root or the raising to a power is done first.

If the number that is raised to a fractional power can be expressed in index form, the index laws can also be used to simplify the fractional power.

Example 11

Evaluate the following.

a $25^{1.5}$ **b** $16^{\frac{3}{4}}$ **c** $8^{-\frac{2}{3}}$

Working	Reasoning
a $25^{1.5} = 25^{\frac{3}{2}}$	The denominator of 2 in the fractional index indicates a square root.
$= (\sqrt{25})^3$	
$= 5^3$	The numerator 3 in the fractional index indicates raising to the power of 3.
$= 125$	
b $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}}$	The denominator 4 in the fractional index indicates the fourth root of 16. $16 = 2 \times 2 \times 2 \times 2$ so $16 = 2^4$. The numerator of 3 in the fractional index indicates raising to the power 3.
$= 2^{\frac{4}{1}\times\frac{3}{4}}$	Use the power of a power index law $(a^m)^n = a^{m\times n}$
$= 2^3$	
$= 8$	

continued

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Example 11 continued

Working	Reasoning
c $8^{-\frac{2}{3}}$	Express with a positive index.
$= \dfrac{1}{8^{\frac{2}{3}}}$	The numerator 2 in the fractional index indicates raising the power of 2.
$= \dfrac{1}{\sqrt[3]{8^2}}$	The denominator 3 in the fractional index indicates the cube root.
$= \dfrac{1}{\sqrt[3]{64}}$	
$= \dfrac{1}{4}$	Alternatively $8^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}}$ $= 2^{3 \times (-\frac{2}{3})}$ $= 2^{-2}$ $= \dfrac{1}{2^2}$ $= \dfrac{1}{4}$

Example 12

Express in index form with rational powers

a $\sqrt[5]{8}$ **b** $9\sqrt{3}$ **c** Solve for x the equation $4^x = 8\sqrt{2}$

Working	Reasoning
a $\sqrt[5]{8} = \sqrt[5]{2^3}$ $= (2^3)^{\frac{1}{5}}$ $= 2^{\frac{3}{1} \times \frac{1}{5}}$ $= 2^{\frac{3}{5}}$ or $2^{0.6}$	Express the base number in index form. Represent the fifth root as a fractional power. To raise a power to a power, multiply the indices. Rational powers can be fractional or decimal.
b $9\sqrt{3} = 3^2 \times 3^{\frac{1}{2}}$ $= 3^{2+\frac{1}{2}}$ $= 3^{\frac{5}{2}}$ or $3^{2.5}$	Write both coefficient and number under the square root sign in index form with the same base. Express the square root as a power of a half. To multiply, add the indices. Write the resulting fraction in the form $\dfrac{m}{n}$ or if possible in exact decimal form.
c $4^x = 8\sqrt{2}$ $(2^2)^x = 2^3 \times 2^{\frac{1}{2}}$	Since x is a power, change each term into index form.
$2^{2x} = 2^{3\frac{1}{2}}$ $2^{2x} = 2^{\frac{7}{2}}$	Because the base numbers are the same, and the two terms are equal, it follows that the powers are also equal.

continued

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Example 12 continued

Working	Reasoning
$2x = \frac{7}{2}$ $x = \frac{7}{4}$	Equate the powers and solve the equation that is formed.

exercise 4.3

LINKS TO Example 11

1 Evaluate each of the following.

a $9^{1.5}$ **b** $8^{\frac{2}{3}}$ **c** $4^{2.5}$ **d** $1000^{\frac{5}{3}}$
e $25^{\frac{3}{2}}$ **f** $100^{\frac{5}{2}}$ **g** $81^{\frac{3}{4}}$ **h** $16^{\frac{3}{4}}$
i $27^{\frac{4}{3}}$ **j** $64^{\frac{2}{3}}$ **k** $8^{\frac{5}{3}}$ **l** $32^{0.4}$
m $25^{-\frac{3}{2}}$ **n** $32^{-\frac{2}{5}}$ **o** $64^{-\frac{5}{6}}$ **p** $100^{-1.5}$

2 The exact value of $27^{\frac{5}{3}}$ is

A 3 **B** 9 **C** 15 **D** 45 **E** 243

3 Which of the following surd terms is equivalent to $16^{\frac{2}{3}}$?

A $\sqrt[3]{4}$ **B** $4\sqrt[3]{4}$ **C** $4\sqrt{3}$ **D** $10\frac{2}{3}$ **E** 4^3

LINKS TO Example 12a

4 Express each of the following in index form with rational powers.

a $\sqrt[3]{16}$ **b** $\sqrt{125}$ **c** $(\sqrt[3]{5})^4$ **d** $\sqrt{32}$
e $(\sqrt{7})^5$ **f** $(\sqrt[5]{3})^2$ **g** $\sqrt[3]{49}$ **h** $\sqrt[5]{27}$

LINKS TO Example 12b

5 Express each of the following in index form with rational powers.

a $5\sqrt{5}$ **b** $8\sqrt{2}$ **c** $36\sqrt{6}$ **d** $16\sqrt{2}$
e $7\sqrt[4]{7}$ **f** $9\sqrt[5]{3}$ **g** $2\sqrt[3]{2}$ **h** $8\sqrt[3]{4}$

LINKS TO Example 12c

6 Solve each of the following equations for x.

a $3^x = \sqrt{27}$ **b** $5^x = 25\sqrt{5}$ **c** $2^x = \sqrt[3]{4}$ **d** $7^x = \sqrt[4]{343}$
e $4^x = 2\sqrt[3]{2}$ **f** $81^x = \sqrt{27}$ **g** $5^{x+1} = 5\sqrt[4]{5}$ **h** $7^{x-1} = 7\sqrt[3]{49}$

exercise 4.3 challenge

7 Simplify.

a $\left(\frac{1}{9}\right)^{\frac{1}{2}}$ **b** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **c** $\left(\frac{1}{9}\right)^{-\frac{3}{2}}$ **d** $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$
e $(m^6)^{\frac{2}{3}}$ **f** $(m^6)^{\frac{3}{2}}$ **g** $\left(\frac{t^4}{u^6}\right)^{\frac{3}{2}}$ **h** $\left(\frac{v^8}{w^2}\right)^{-\frac{3}{2}}$

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4.4 Adding and subtracting surds

Like surds

Like surds have the same number under the square root sign. For example, $5\sqrt{3}$ and $2\sqrt{3}$ are like surds because they both have the surd part $\sqrt{3}$.

$5\sqrt{2}$ and $2\sqrt{7}$ are **unlike surds** because they have different surd parts.

Only like surds can be added or subtracted: $5\sqrt{3} + \sqrt{3} = 6\sqrt{3}$, whereas $5\sqrt{2} + 2\sqrt{7}$ cannot be simplified.

In this way, surds behave like algebraic terms: $5x + x = 6x$, whereas $5x + 2y$ cannot be simplified. Just as $5x + x$ means five times x plus one times x, $5\sqrt{3} + \sqrt{3}$ means five times $\sqrt{3}$ plus one times $\sqrt{3}$. Therefore, $5\sqrt{3} + \sqrt{3}$ can be simplified to $6\sqrt{3}$.

Example 13

Simplify

a $3\sqrt{5} + 4\sqrt{5}$ **b** $2\sqrt{2} + \sqrt{2} - 3\sqrt{2}$ **c** $3\sqrt{7} + 2\sqrt{3} - \sqrt{7}$

Working

a $3\sqrt{5} + 4\sqrt{5}$
$= 7\sqrt{5}$

Reasoning

To add or subtract like surds, work with the whole number multiples at the front of the surd part. This tells how many of the like surds there are.

$3\sqrt{5} + 4\sqrt{5}$ means 'three root fives plus four root fives', that is, seven root fives.

$$\underbrace{\sqrt{5} + \sqrt{5} + \sqrt{5}}_{3\sqrt{5}} + \underbrace{\sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5}}_{4\sqrt{5}}$$

b $2\sqrt{2} + \sqrt{2} - 3\sqrt{2}$
$= 3\sqrt{2} - 3\sqrt{2}$
$= 0$

$$\underbrace{\sqrt{2} + \sqrt{2}}_{2\sqrt{2}} + \underbrace{\sqrt{2}}_{1\sqrt{2}} - \underbrace{\sqrt{2} - \sqrt{2} - \sqrt{2}}_{3\sqrt{2}}$$

$+\sqrt{2}$ means $+1\sqrt{2}$

c $3\sqrt{7} + 2\sqrt{3} - \sqrt{7}$

$$\underbrace{\sqrt{7} + \sqrt{7} + \sqrt{7}}_{3\sqrt{7}} + \underbrace{\sqrt{3} + \sqrt{3}}_{2\sqrt{3}} \underbrace{- \sqrt{7}}_{-\sqrt{7}}$$

$= 3\sqrt{7} - \sqrt{7} + 2\sqrt{3}$

Rearrange to put the like surds together.

$$\underbrace{\sqrt{7} + \sqrt{7} + \sqrt{7}}_{3\sqrt{7}} \underbrace{- \sqrt{7}}_{-\sqrt{7}} + \underbrace{\sqrt{3} + \sqrt{3}}_{2\sqrt{3}}$$

$= 2\sqrt{7} + 2\sqrt{3}$

Simplify by adding or subtracting the like surds.

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When an expression involves the addition of unlike surds, simplifying the surds will sometimes give rise to like surds. The surd expression can then be simplified further by collecting like terms.

Example 14

Simplify the following.

a $\sqrt{12} + \sqrt{75}$

b $2\sqrt{18} + \sqrt{45} - \sqrt{50}$

Working	Reasoning
a $\sqrt{12} + \sqrt{75}$	The surds are unlike, so check to see if they have factors that are square numbers.
$= \sqrt{4} \times \sqrt{3} + \sqrt{25} \times \sqrt{3}$	$12 = 4 \times 3, 75 = 25 \times 3$
$= 2\sqrt{3} + 5\sqrt{3}$ $= 7\sqrt{3}$	Once simplified, each term has $\sqrt{3}$ as the surd part; that is, the terms are now like surds.
b $2\sqrt{18} + \sqrt{45} - \sqrt{50}$	The surds are unlike, so check to see if they have factors that are square numbers.
$= 2\sqrt{9} \times \sqrt{2} + \sqrt{9} \times \sqrt{5} - \sqrt{25} \times \sqrt{2}$	$18 = 9 \times 2, 45 = 9 \times 5, 50 = 25 \times 2$
$= 6\sqrt{2} + 3\sqrt{5} - 5\sqrt{2}$ $= \sqrt{2} + 3\sqrt{5}$	Once simplified, the first and third terms have $\sqrt{2}$ as the surd part; that is, the terms are now like surds.

exercise 4.4

1 Copy and complete each of the following.

a $a + a + a + a = __ \times a = __ a$

b $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = __ \times __ = __\ __$

c $\sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5} = __ \times __ = __\ __$

d $\sqrt{6} + \sqrt{6} + \sqrt{6} = __ \times __ = __\ __$

2 $\underbrace{\sqrt{7} + \sqrt{7} + \sqrt{7}}_{3\sqrt{7}} \underbrace{- \sqrt{7} - \sqrt{7}}_{-2\sqrt{7}}$ shows the expression $3\sqrt{7} - 2\sqrt{7}$.

Write and then simplify the expression for each of the following.

a $\underbrace{\sqrt{5} + \sqrt{5} + \sqrt{5}}_{?} \underbrace{+ \sqrt{5} + \sqrt{5}}_{?}$

b $\underbrace{\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2}}_{?} \underbrace{- \sqrt{2} - \sqrt{2} - \sqrt{2}}_{?}$

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c $\underbrace{\sqrt{3}+\sqrt{3}}_{?}\underbrace{-\sqrt{3}-\sqrt{3}-\sqrt{3}-\sqrt{3}}_{?}\underbrace{+\sqrt{3}+\sqrt{3}}_{?}$

d $\underbrace{-\sqrt{6}-\sqrt{6}}_{?}\underbrace{+\sqrt{5}+\sqrt{5}+\sqrt{5}}_{?}\underbrace{+\sqrt{6}+\sqrt{6}+\sqrt{6}+\sqrt{6}}_{?}$

e $\underbrace{\sqrt{10}+\sqrt{10}+\sqrt{10}}_{?}\underbrace{+1+1+1}_{?}\underbrace{-\sqrt{10}-\sqrt{10}-\sqrt{10}-\sqrt{10}}_{?}\underbrace{+1+1}_{?}$

LINKS TO Example 13a

3 Simplify each of the following.

a $4\sqrt{7}+3\sqrt{7}$ **b** $6\sqrt{2}-3\sqrt{2}$ **c** $2\sqrt{6}+5\sqrt{6}$ **d** $7\sqrt{3}+\sqrt{3}$
e $5\sqrt{10}-2\sqrt{10}$ **f** $4\sqrt{11}-4\sqrt{11}$ **g** $8\sqrt{5}+3\sqrt{5}$ **h** $\sqrt{6}+7\sqrt{6}$
i $-2\sqrt{3}+8\sqrt{3}$ **j** $4\sqrt{5}+4\sqrt{5}$ **k** $-\sqrt{7}+\sqrt{7}$ **l** $2\sqrt{11}-6\sqrt{11}$
m $\sqrt{2}+\sqrt{2}$ **n** $-3\sqrt{3}-4\sqrt{3}$ **o** $3\sqrt{10}-4\sqrt{10}$ **p** $\sqrt{5}-\sqrt{5}$

LINKS TO Example 13b

4 Simplify each of the following.

a $\sqrt{7}+5\sqrt{7}-2\sqrt{7}$ **b** $-2\sqrt{5}+4\sqrt{5}-3\sqrt{5}$ **c** $8\sqrt{3}-5\sqrt{3}-4\sqrt{3}$
d $4\sqrt{2}-3\sqrt{2}+\sqrt{2}$ **e** $\sqrt{6}-2\sqrt{6}+\sqrt{6}$ **f** $-2\sqrt{11}-4\sqrt{11}+3\sqrt{11}$
g $7\sqrt{5}+3\sqrt{5}-4\sqrt{5}$ **h** $-3\sqrt{10}+6\sqrt{10}-2\sqrt{10}$ **i** $-2\sqrt{7}+\sqrt{7}+5\sqrt{7}$
j $5\sqrt{3}+2\sqrt{3}-7\sqrt{3}$ **k** $4\sqrt{2}-9\sqrt{2}+\sqrt{2}$ **l** $\sqrt{10}-3\sqrt{10}-2\sqrt{10}$
m $-2\sqrt{11}+9\sqrt{11}-6\sqrt{11}$ **n** $-6\sqrt{3}-\sqrt{3}+5\sqrt{3}$ **o** $4\sqrt{5}-4\sqrt{5}+3\sqrt{5}$
p $4\sqrt{6}+2\sqrt{6}-\sqrt{6}$ **q** $-3\sqrt{7}+4\sqrt{7}+3\sqrt{7}$ **r** $-2\sqrt{3}+7\sqrt{3}-5\sqrt{3}$

LINKS TO Example 13c

5 Simplify each of the following.

a $3\sqrt{2}+4\sqrt{5}+\sqrt{2}$ **b** $5\sqrt{3}+2\sqrt{7}+3\sqrt{7}$
c $\sqrt{6}+4\sqrt{3}+2\sqrt{6}$ **d** $7\sqrt{2}+3\sqrt{2}+4\sqrt{6}$
e $3\sqrt{10}+5\sqrt{7}-3\sqrt{10}$ **f** $6\sqrt{5}+2\sqrt{11}-4\sqrt{11}$
g $8\sqrt{5}-2\sqrt{3}-3\sqrt{5}$ **h** $\sqrt{6}+7\sqrt{6}-3\sqrt{10}$
i $4\sqrt{3}+3\sqrt{6}+2\sqrt{3}+\sqrt{6}$ **j** $2\sqrt{5}+5\sqrt{3}+\sqrt{5}-2\sqrt{3}$
k $4\sqrt{3}+6\sqrt{7}-\sqrt{3}+4\sqrt{7}$ **l** $\sqrt{7}+3\sqrt{2}-3\sqrt{2}+4\sqrt{7}$
m $\sqrt{11}+2\sqrt{11}+3\sqrt{2}-2\sqrt{11}$ **n** $2\sqrt{10}-3\sqrt{5}-3\sqrt{10}+7\sqrt{5}$
o $-2\sqrt{3}+4\sqrt{2}-3\sqrt{3}+2\sqrt{2}$ **p** $-\sqrt{3}-2\sqrt{6}+4\sqrt{3}+2\sqrt{6}$

LINKS TO Example 14a

6 Simplify each of the following.

a $\sqrt{24}+\sqrt{6}$ **b** $\sqrt{18}-\sqrt{8}$ **c** $\sqrt{27}+\sqrt{12}$ **d** $\sqrt{45}-\sqrt{20}$
e $\sqrt{28}+\sqrt{63}$ **f** $\sqrt{44}-\sqrt{99}$ **g** $\sqrt{160}-\sqrt{40}$ **h** $\sqrt{8}-\sqrt{50}$
i $2\sqrt{12}+\sqrt{27}$ **j** $3\sqrt{20}-\sqrt{125}$ **k** $\sqrt{24}+\sqrt{150}$ **l** $\sqrt{300}-\sqrt{75}$
m $\sqrt{63}-3\sqrt{7}$ **n** $3\sqrt{8}+\sqrt{50}$ **o** $\sqrt{600}-2\sqrt{24}$ **p** $3\sqrt{20}-2\sqrt{45}$

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LINKS TO Example 14b

7 Simplify each of the following.

a $\sqrt{63} + \sqrt{28} - \sqrt{7}$

b $\sqrt{200} - \sqrt{18} + \sqrt{8}$

c $\sqrt{300} + \sqrt{18} + \sqrt{27}$

d $\sqrt{99} - \sqrt{28} - \sqrt{44}$

e $\sqrt{50} - \sqrt{18} - \sqrt{8}$

f $\sqrt{54} + \sqrt{20} + \sqrt{24}$

g $\sqrt{490} - \sqrt{40} + \sqrt{90}$

h $\sqrt{12} - \sqrt{700} + \sqrt{27}$

i $\sqrt{200} - 3\sqrt{18} + \sqrt{8}$

j $-2\sqrt{24} + \sqrt{54} - \sqrt{600}$

k $\sqrt{99} - \sqrt{18} - \sqrt{44} + \sqrt{32}$

l $\sqrt{12} + \sqrt{32} - \sqrt{27} + 2\sqrt{8}$

m $\sqrt{11} + \sqrt{44} + \sqrt{18} - \sqrt{99}$

n $\sqrt{40} - \sqrt{45} - \sqrt{90} + \sqrt{20}$

o $-2\sqrt{12} + 3\sqrt{8} - \sqrt{27} + 2\sqrt{50}$

p $2\sqrt{12} + \sqrt{54} + \sqrt{75} + \sqrt{6}$

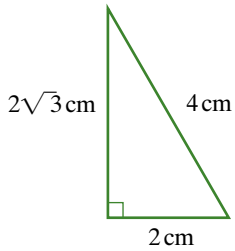
8 A square has an area of $18\,\text{cm}^2$. Giving both answers in simplest exact form, find

a the side length of the square

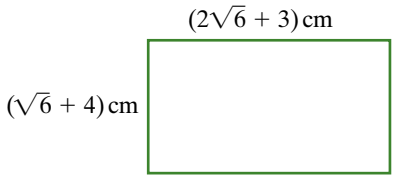
b the perimeter of the square.

9 **a** Find the perimeter of this triangle, giving your answer in exact form.

b Find the area of the triangle, giving your answer in exact form.



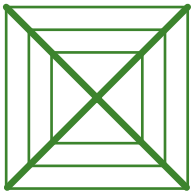
10 Find the perimeter of the rectangle shown on the right. Give the answer in exact form, and then as a decimal approximation correct to two decimal places.



exercise 4.4 challenge

11 A rotary clothes line is made with four metal arms attached to a central pole. Wire is strung between these four arms to make squares on which the clothes are hung. The four arms form the diagonals of the outer square.

The diagram shows the top view. Each of the four metal arms in this clothes line measures 2 metres.



a What length of wire is needed to make the outer square? Give your answer in exact form, as simply as possible.

b The distance from the central pole to a corner of the inner square is 1 metre and the distance from the central pole to a corner of the middle square is 1.5 metres. What length of wire is needed to make the three squares? Give your answer in exact form, as simply as possible.

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4.5 Multiplying surds

10 A

Multiplying surds involves multiplying the numbers underneath the square root signs, and writing the result underneath a single square root sign.

If the terms to be multiplied are multiples of surds, first multiply the number outside the square root signs, then multiply the numbers under the square root signs.

Multiplying surds

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

For example, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$

$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$

For example, $2\sqrt{5} \times 3\sqrt{7} = 6\sqrt{35}$

Notice that this is equivalent to multiplying the coefficients and multiplying the pronumerals when working with algebraic terms; for example, $2a \times 3b = 6ab$.

If a surd product has a square number as a factor, it can be simplified.

For example,
$$\begin{aligned}\sqrt{3} \times \sqrt{6} &= \sqrt{18} \\ &= \sqrt{9 \times 2} \\ &= 3\sqrt{2}\end{aligned}$$

Example 15

Simplify.

a $\sqrt{7} \times \sqrt{3}$

b $\sqrt{6} \times \sqrt{10}$

Working	Reasoning
a $\sqrt{7} \times \sqrt{3}$ $= \sqrt{21}$	Multiply the numbers under the square root signs.
b $\sqrt{6} \times \sqrt{10}$ $= \sqrt{60}$	Multiply the numbers under the square root signs.
$= \sqrt{4} \times \sqrt{15}$ $= 2\sqrt{15}$	If possible, simplify the resulting surd.

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Example 16

Simplify.

a $5\sqrt{2} \times 3\sqrt{7}$ **b** $2\sqrt{5} \times \sqrt{5}$ **c** $3\sqrt{2} \times 7\sqrt{6}$

Working	Reasoning
a $5\sqrt{2} \times 3\sqrt{7}$	This means $5 \times \sqrt{2} \times 3 \times \sqrt{7}$.
$= 15\sqrt{14}$	It can be regrouped as $5 \times 3 \times \sqrt{2} \times \sqrt{7}$. Multiply the numbers outside the square root sign. Multiply the numbers under the square root sign.
b $2\sqrt{5} \times \sqrt{5}$	$2\sqrt{5} \times \sqrt{5}$ means $2 \times \sqrt{5} \times \sqrt{5}$. Multiply the numbers under the square root sign.
$= 2\sqrt{25}$	
$= 2 \times 5$	$\sqrt{25}$ is not a surd. It can be evaluated.
$= 10$	
c $3\sqrt{2} \times 7\sqrt{6}$	Multiply the numbers outside the square root sign. Multiply the numbers under the square root sign.
$= 21\sqrt{12}$	12 has the square number 4 as a factor,
$= 21 \times \sqrt{4} \times \sqrt{3}$	so $\sqrt{12}$ can be simplified.
$= 21 \times 2\sqrt{3}$	
$= 42\sqrt{3}$	

Expanding surd expressions

The distributive law can be applied to expand surd expressions with brackets. Just as we can expand the algebraic expression $2(3x + 5)$ to give $6x + 10$, we can expand the surd expression $2(3\sqrt{7} + 5)$ to give $6\sqrt{7} + 10$, since $2 \times 3\sqrt{7} = 6\sqrt{7}$. Binomial surd expressions such as $(\sqrt{7} + 2)(3\sqrt{7} + 5)$ can also be expanded like algebraic expressions.

Example 17

Expand and simplify each of the following.

a $3\sqrt{2}(5\sqrt{3} + \sqrt{2} - 2)$ **b** $(\sqrt{3} + 4)(2\sqrt{3} - 5)$

c $(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})$

Working	Reasoning
a $3\sqrt{2}(5\sqrt{3} + \sqrt{2} - 2)$	Multiply each term inside the brackets by the term in front of the bracket.
$= 15\sqrt{6} + 3\sqrt{4} - 6\sqrt{2}$	
$= 15\sqrt{6} + 3 \times 2 - 6\sqrt{2}$	Where possible, simplify surds and evaluate rational square roots.
$= 15\sqrt{6} + 6 - 6\sqrt{2}$	

continued

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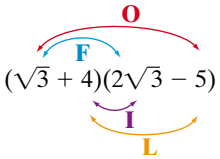
Example 17 continued

Working	Reasoning
b $(\sqrt{3} + 4)(2\sqrt{3} - 5)$	When multiplying two brackets, use FOIL.
$= 2\sqrt{9} - 5\sqrt{3} + 8\sqrt{3} - 20$	
$= 6 - 5\sqrt{3} + 8\sqrt{3} - 20$ $= -14 + 3\sqrt{3}$	Where possible, simplify surds and evaluate rational square roots. Add or subtract any like terms.
$= 3\sqrt{3} - 14$	If possible, begin the simplified expression with a positive term.
c $(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})$ $= (2\sqrt{2})^2 - (\sqrt{3})^2$ $= 8 - 3$ $= 5$	The surd expressions in the brackets are the same except for the sign in the middle of each expression. This is the factorised form of a difference of two squares. The identity $(a + b)(a - b) = a^2 - b^2$ applies here. $(2\sqrt{2})^2 = 2\sqrt{2} \times 2\sqrt{2} = 4\sqrt{4} = 4 \times 2 = 8$

exercise 4.5

LINKS TO Example 15

1 Carry out the following multiplications and write the result as a single surd. Remember to simplify or evaluate wherever possible.

a $\sqrt{11} \times \sqrt{2}$ **b** $\sqrt{5} \times \sqrt{6}$ **c** $\sqrt{3} \times \sqrt{10}$ **d** $\sqrt{2} \times \sqrt{5}$ **e** $\sqrt{7} \times \sqrt{2}$

f $\sqrt{2} \times \sqrt{3}$ **g** $\sqrt{3} \times \sqrt{7}$ **h** $\sqrt{2} \times \sqrt{2}$ **i** $\sqrt{5} \times \sqrt{5}$ **j** $\sqrt{3} \times \sqrt{3}$

k $\sqrt{2} \times \sqrt{10}$ **l** $\sqrt{6} \times \sqrt{3}$ **m** $\sqrt{6} \times \sqrt{10}$ **n** $\sqrt{15} \times \sqrt{6}$ **o** $\sqrt{7} \times \sqrt{14}$

2 The product $\sqrt{6} \times \sqrt{30}$, in simplest form, is

A 6 **B** $6\sqrt{5}$ **C** $3\sqrt{20}$ **D** $\sqrt{36}$ **E** $\sqrt{180}$

LINKS TO Example 16a

3 Carry out the following multiplications and write the result as a single surd.

a $5\sqrt{3} \times \sqrt{7}$ **b** $\sqrt{2} \times 7\sqrt{11}$ **c** $4\sqrt{3} \times 2\sqrt{5}$ **d** $5\sqrt{2} \times 4\sqrt{3}$

e $3\sqrt{2} \times 4\sqrt{5}$ **f** $2\sqrt{10} \times \sqrt{3}$ **g** $5\sqrt{3} \times 2\sqrt{11}$ **h** $\sqrt{6} \times 4\sqrt{5}$

i $2\sqrt{11} \times 5\sqrt{6}$ **j** $3\sqrt{7} \times 3\sqrt{2}$ **k** $3\sqrt{5} \times 4\sqrt{7}$ **l** $5\sqrt{7} \times 3\sqrt{10}$

m $5\sqrt{5} \times 6\sqrt{3}$ **n** $2\sqrt{6} \times 7\sqrt{5}$ **o** $\sqrt{11} \times 2\sqrt{10}$ **p** $4\sqrt{11} \times 5\sqrt{3}$

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LINKS TO Example 16b

4 Evaluate the following multiplications.

a $\sqrt{3} \times \sqrt{3}$ **b** $\sqrt{7} \times \sqrt{7}$ **c** $\sqrt{10} \times \sqrt{10}$ **d** $\sqrt{6} \times \sqrt{6}$
e $\sqrt{5} \times 6\sqrt{5}$ **f** $3\sqrt{2} \times \sqrt{2}$ **g** $3\sqrt{11} \times \sqrt{11}$ **h** $3\sqrt{10} \times \sqrt{10}$
i $5\sqrt{2} \times 3\sqrt{2}$ **j** $(2\sqrt{6})^2$ **k** $5\sqrt{3} \times 2\sqrt{3}$ **l** $(3\sqrt{7})^2$
m $(6\sqrt{10})^2$ **n** $7\sqrt{5} \times 2\sqrt{5}$ **o** $(3\sqrt{2})^2$ **p** $4\sqrt{11} \times 5\sqrt{11}$

LINKS TO Example 16c

5 Carry out the following multiplications and write the result as a single surd. Remember to evaluate or simplify wherever possible.

a $5\sqrt{3} \times \sqrt{6}$ **b** $3\sqrt{2} \times 3\sqrt{10}$ **c** $3\sqrt{6} \times 5\sqrt{2}$ **d** $4\sqrt{5} \times 3\sqrt{10}$
e $6\sqrt{2} \times \sqrt{8}$ **f** $2\sqrt{15} \times \sqrt{3}$ **g** $\sqrt{10} \times 5\sqrt{6}$ **h** $3\sqrt{14} \times 5\sqrt{2}$
i $\sqrt{12} \times 10\sqrt{2}$ **j** $2\sqrt{3} \times \sqrt{33}$ **k** $5\sqrt{2} \times 2\sqrt{10}$ **l** $2\sqrt{6} \times 6\sqrt{2}$
m $4\sqrt{10} \times 5\sqrt{2}$ **n** $3\sqrt{5} \times 2\sqrt{15}$ **o** $5\sqrt{22} \times 6\sqrt{2}$ **p** $7\sqrt{30} \times 3\sqrt{10}$

6 The product $\sqrt{8} \times 5\sqrt{2}$, in simplest form, is

A 4 **B** 20 **C** $4\sqrt{5}$ **D** $5\sqrt{16}$ **E** $\sqrt{80}$

7 Carry out the following multiplications and write the result as a single surd. Remember to evaluate or simplify wherever possible.

a $\sqrt{3} \times \sqrt{7}$ **b** $2\sqrt{3} \times 7\sqrt{3}$ **c** $3\sqrt{22} \times 3\sqrt{2}$ **d** $5\sqrt{2} \times 6\sqrt{7}$
e $2\sqrt{11} \times \sqrt{7}$ **f** $4\sqrt{5} \times 2\sqrt{10}$ **g** $\sqrt{7} \times \sqrt{10}$ **h** $2\sqrt{6} \times 5\sqrt{6}$
i $\sqrt{5} \times \sqrt{5}$ **j** $\sqrt{2} \times \sqrt{11}$ **k** $2\sqrt{10} \times 4\sqrt{7}$ **l** $2\sqrt{6} \times 3\sqrt{10}$
m $4\sqrt{10} \times 3\sqrt{2}$ **n** $2\sqrt{5} \times 3\sqrt{3}$ **o** $5\sqrt{2} \times \sqrt{8}$ **p** $\sqrt{6} \times \sqrt{5}$

LINKS TO Example 17a

8 Expand each of the following, giving answers in simplest exact form.

a $4(\sqrt{2} + 3)$ **b** $5(2 - \sqrt{7})$ **c** $3(2\sqrt{5} + 1)$
d $\sqrt{2}(\sqrt{5} + 3)$ **e** $\sqrt{3}(\sqrt{2} - \sqrt{5})$ **f** $\sqrt{5}(\sqrt{7} + \sqrt{2})$
g $2(5\sqrt{3} + \sqrt{7})$ **h** $3(\sqrt{2} - 7\sqrt{11})$ **i** $\sqrt{3}(\sqrt{3} + 5)$
j $\sqrt{5}(\sqrt{2} - \sqrt{5})$ **k** $\sqrt{3}(\sqrt{6} - \sqrt{10})$ **l** $5\sqrt{2}(3\sqrt{2} + 4)$
m $3\sqrt{6}(\sqrt{2} - \sqrt{6})$ **n** $2\sqrt{5}(3 + \sqrt{10})$ **o** $2\sqrt{3}(\sqrt{3} + \sqrt{6} + 2\sqrt{2})$

9 In simplest form, $\sqrt{3}(\sqrt{6} - \sqrt{3})$ is

A $\sqrt{12}$ **B** $3 - \sqrt{6}$ **C** $\sqrt{9} - \sqrt{6}$ **D** $3\sqrt{2} - 3$ **E** $\sqrt{18} - \sqrt{9}$

LINKS TO Examples 17b,c

10 Expand each of the following, giving answers in simplest surd form.

a $(\sqrt{2} + 3)(\sqrt{3} + 4)$ **b** $(\sqrt{5} + 6)(\sqrt{2} + 1)$
c $(\sqrt{3} + 2)(\sqrt{3} + 5)$ **d** $(\sqrt{2} + 7)(\sqrt{2} + 4)$
e $(\sqrt{3} - 4)(\sqrt{2} + 3)$ **f** $(4 - \sqrt{10})(7 - \sqrt{10})$
g $(\sqrt{2} + \sqrt{3})(\sqrt{3} + 5)$ **h** $(\sqrt{2} + \sqrt{6})(\sqrt{3} + 3)$
i $(2\sqrt{3} + 5)(\sqrt{3} + 2)$ **j** $(\sqrt{5} - 2\sqrt{2})(\sqrt{5} + 7\sqrt{2})$
k $(2\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$ **l** $(\sqrt{3} - 4)^2$

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m $(3\sqrt{7} - \sqrt{2})^2$

n $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2})$

o $(2\sqrt{10} + \sqrt{3})(2\sqrt{10} - \sqrt{3})$

p $(4\sqrt{6} + \sqrt{5})^2$

11 A rectangle has a length of $(\sqrt{10} + 1\,\text{cm})$ and a width of $(\sqrt{10} - 1\,\text{cm})$.

a Find the area of the rectangle in exact form.

b What is the side length of a square with the same area as the rectangle?

exercise 4.5 challenge

12 Consider the following multiplications.

a Find the three pairs of surds that multiply to give 6.

b Find all of the possible pairs of surds that multiply to give 12.

c Complete the brackets $(2\sqrt{3} + ___)(2\sqrt{3} - ___)$ so that the multiplication gives 3.

13 The surd expression $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$ is called a continued surd where … indicates that the pattern continues forever. Suppose that $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$. By squaring both sides we obtain $x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$. But we can replace $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$ with x, so that we obtain the quadratic equation $x^2 = 6 + x$. Solve the quadratic equation for x and hence, write down the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$ as a rational number. (Does it make sense to use the negative solutions?)

14 Repeat question **13** for the following continued surds.

a $x = \sqrt{12 + \sqrt{12 + \sqrt{12 + \ldots}}}$

b $x = \sqrt{20 + \sqrt{20 + \sqrt{20 + \ldots}}}$

c $x = \sqrt{42 + \sqrt{42 + \sqrt{42 + \ldots}}}$

d $x = \sqrt{90 + \sqrt{90 + \sqrt{90 + \ldots}}}$

e $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$

f $x = \sqrt{30 + \sqrt{30 + \sqrt{30 + \ldots}}}$

15 **a** Can you see why each of the continued surds in questions **13** and **14** can be expressed as a rational number? What is the similarity in each case?

b Suggest another continued surd that can be written as a rational number.

c Can $\sqrt{6 - \sqrt{6 - \sqrt{6 - \ldots}}}$ be written as a rational number? Explain, showing your reasoning.

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4.6 Dividing surds

To divide surds, write the numbers under the square root signs as a fraction under a single square root sign then simplify the fraction if possible.

Dividing surds

$$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

For example, $\sqrt{15} \div \sqrt{3} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}} = \sqrt{5}$

If there are numbers multiplying the surds, these can be simplified if they have common factors.

For example, $10\sqrt{12} \div 6\sqrt{8} = \frac{10\sqrt{12}}{6\sqrt{8}} = \frac{5}{3}\sqrt{\frac{12}{8}} = \frac{5}{3}\sqrt{\frac{3}{2}}$

Example 18

Simplify the following surd divisions.

a $\frac{\sqrt{10}}{\sqrt{2}}$ **b** $\sqrt{15} \div \sqrt{10}$ **c** $6\sqrt{21} \div 8\sqrt{3}$ **d** $5\sqrt{24} \div 10\sqrt{2}$

Working

a $\frac{\sqrt{10}}{\sqrt{2}}$

$= \sqrt{\frac{10}{2}}$

$= \sqrt{5}$

Reasoning

Write the numbers under the square root signs as a fraction under a single square root sign.
Simplify by dividing numerator and denominator by any common factor.
Write as a single surd fraction.

b $\sqrt{15} \div \sqrt{10}$

$= \sqrt{\frac{15}{10}}$

$= \sqrt{\frac{3}{2}}$

Simplify the fraction.

continued

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Example 18 continued

Working	Reasoning
c $6\sqrt{21} \div 8\sqrt{3}$	
$= \dfrac{6\sqrt{21}}{8\sqrt{3}}$	Write the division as a fraction and simplify if possible. First work with the whole number parts of the numerator and denominator, then work with the numbers under the square root sign.
$= \dfrac{3\sqrt{21}}{4\sqrt{3}}$	Here the whole number parts $\dfrac{6}{8}$ can be simplified to $\dfrac{3}{4}$.
$= \dfrac{3}{4}\sqrt{\dfrac{21}{3}}$	Write $\dfrac{\sqrt{21}}{\sqrt{3}}$ as a single surd, $\sqrt{\dfrac{21}{3}}$.
$= \dfrac{3\sqrt{7}}{4}$	Simplify $\sqrt{\dfrac{21}{3}}$ to $\sqrt{7}$.
d $5\sqrt{24} \div 10\sqrt{2}$	Write the division as a fraction.
$= \dfrac{5}{10}\sqrt{\dfrac{24}{2}}$	Write the surd parts as a single surd fraction.
$= \dfrac{\sqrt{12}}{2}$	Simplify the whole number parts and the surd part.
$= \dfrac{\sqrt{4 \times 3}}{2}$	Look for a square number as a factor of the surd part to see if it can be simplified.
$= \dfrac{2\sqrt{3}}{2}$	$\sqrt{12} = \sqrt{4 \times 3}$
$= \sqrt{3}$	Simplify the fraction.

exercise 4.6

LINKS TO Example 18a

1 Carry out the following divisions and write the result as a single surd. Remember to simplify wherever possible.

a $\dfrac{\sqrt{12}}{\sqrt{6}}$ **b** $\dfrac{\sqrt{15}}{\sqrt{5}}$ **c** $\dfrac{\sqrt{21}}{\sqrt{3}}$ **d** $\dfrac{\sqrt{22}}{\sqrt{2}}$

e $\dfrac{\sqrt{14}}{\sqrt{7}}$ **f** $\dfrac{\sqrt{30}}{\sqrt{3}}$ **g** $\dfrac{\sqrt{6}}{\sqrt{2}}$ **h** $\dfrac{\sqrt{55}}{\sqrt{11}}$

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i $\frac{\sqrt{80}}{\sqrt{10}}$ j $\frac{\sqrt{18}}{\sqrt{6}}$ k $\frac{\sqrt{50}}{\sqrt{2}}$ l $\frac{\sqrt{24}}{\sqrt{2}}$

m $\frac{\sqrt{36}}{\sqrt{2}}$ n $\frac{\sqrt{45}}{\sqrt{5}}$ o $\frac{\sqrt{60}}{\sqrt{3}}$ p $\frac{\sqrt{20}}{\sqrt{5}}$

LINKS TO Example 18b

2 Carry out the following divisions and write the result as a single surd. Remember to simplify wherever possible.

a $\sqrt{21} \div \sqrt{7}$ b $\sqrt{12} \div \sqrt{18}$ c $\sqrt{20} \div \sqrt{5}$ d $\sqrt{8} \div \sqrt{6}$

e $\sqrt{6} \div \sqrt{10}$ f $\sqrt{36} \div \sqrt{3}$ g $\sqrt{15} \div \sqrt{20}$ h $\sqrt{44} \div \sqrt{11}$

i $\sqrt{3} \div \sqrt{15}$ j $\sqrt{21} \div \sqrt{12}$ k $\sqrt{28} \div \sqrt{14}$ l $\sqrt{2} \div \sqrt{18}$

m $\sqrt{12} \div \sqrt{26}$ n $\sqrt{90} \div \sqrt{5}$ o $\sqrt{42} \div \sqrt{6}$ p $\sqrt{250} \div \sqrt{10}$

3 The quotient $\sqrt{7} \div \sqrt{63}$, as a single term in simplest form, is

A $\frac{1}{9}$ B $\frac{1}{3}$ C $\sqrt{7}$ D 3 E 9

LINKS TO Examples 18c, d

4 Simplify each of the following.

a $\frac{6\sqrt{14}}{3\sqrt{2}}$ b $\frac{10\sqrt{33}}{5\sqrt{3}}$ c $\frac{8\sqrt{33}}{12\sqrt{22}}$ d $\frac{14\sqrt{15}}{21\sqrt{6}}$

e $\frac{4\sqrt{20}}{8\sqrt{5}}$ f $\frac{\sqrt{45}}{6\sqrt{5}}$ g $\frac{12\sqrt{24}}{20\sqrt{10}}$ h $\frac{21\sqrt{28}}{15\sqrt{44}}$

i $\frac{14\sqrt{72}}{12\sqrt{20}}$ j $\frac{24\sqrt{54}}{18\sqrt{42}}$ k $\frac{9\sqrt{48}}{2\sqrt{54}}$ l $\frac{36\sqrt{120}}{24\sqrt{50}}$

5 In simplest form, $\frac{2\sqrt{48}}{6\sqrt{36}}$ is

A $\frac{1}{3}$ B $\frac{4}{3}$ C $\frac{2}{\sqrt{3}}$ D $\frac{2}{3\sqrt{3}}$ E $\sqrt{\frac{4}{3}}$

6 A rectangle has an area of $6\sqrt{21}$ cm^2 and a width of $2\sqrt{3}$ cm. Find the length in exact form.

exercise 4.6 challenge

7 Solve the following equation for x, giving the solution in exact form.

$$\frac{\sqrt{2}x + \sqrt{6}}{\sqrt{2}} = \sqrt{2} + 2\sqrt{3}$$

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10 A

4.7 Rationalising the denominator

Rationalising the denominator means writing an equivalent surd expression with a rational number in the denominator instead of a surd. It is a mathematical convention that surd expressions involving division are usually written with **rational denominators**.

Single-term denominators

If a surd division has a single surd term in the denominator, both the numerator and the denominator are multiplied by the surd part of the denominator.

This process works because multiplying a surd by itself is squaring it, and the square of any single surd term is a rational number. 'Squaring' and 'finding the square root' are inverse operations so they undo each other.

Example 19

Rationalise the denominator of each of the following fractions.

a $\dfrac{\sqrt{2}}{\sqrt{5}}$ **b** $\dfrac{6}{5\sqrt{3}}$

Working	Reasoning
a $\dfrac{\sqrt{2}}{\sqrt{5}}$	The surd in the denominator is $\sqrt{5}$.
$= \dfrac{\sqrt{2}}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$	Multiply both numerator and denominator by $\sqrt{5}$. (Multiplying by $\dfrac{\sqrt{5}}{\sqrt{5}}$ is the same as multiplying by 1, so the value of the original fraction is unchanged.)
$= \dfrac{\sqrt{10}}{5}$	In the denominator, $\sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$ which is a rational number.
b $\dfrac{6}{5\sqrt{3}}$	The surd part of the term in the denominator is $\sqrt{3}$.
$= \dfrac{6}{5\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$	Multiply both numerator and denominator by $\sqrt{3}$. $\sqrt{3} \times \sqrt{3} = 3$
$= \dfrac{\overset{2}{\cancel{6}} \times \sqrt{3}}{5 \times \underset{1}{\cancel{3}}}$	The whole numbers 6 in the numerator and 3 in the denominator can both be divided by 3.
$= \dfrac{2\sqrt{3}}{5}$	

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Denominators with a binomial surd expression

We saw in worked example **17** part **c** that if a binomial surd expression is multiplied by an expression that is identical except for the central sign, the result will be a whole number. This binomial surd expression that is identical except for the central sign is called the **conjugate** of the original expression. The product of a binomial surd expression and its conjugate represents the factorised form of a difference of two squares.

Sometimes the denominator of a surd division is a binomial surd, such as $\sqrt{3} + \sqrt{2}$ or $2\sqrt{5} - 4$. To rationalise the denominator, we multiply both the numerator and denominator by the **conjugate** of the surd expression. The resulting difference of two squares in the denominator is then a rational number.

Example 20

For each of these binomial surd expressions

i write the conjugate surd expression

ii multiply the surd expression by its conjugate.

a $\sqrt{7} - \sqrt{3}$ **b** $2\sqrt{3} + 1$

Working	Reasoning
a i The conjugate of $\sqrt{7} - \sqrt{3}$ is $\sqrt{7} + \sqrt{3}$.	The conjugate has the same terms as the original binomial expression but has the opposite sign in the middle.
ii $(\sqrt{7} - \sqrt{3})(\sqrt{7} + 3)$ $= 7 - 3$ $= 4$	Expand as for $(a - b)(a + b) = a^2 - b^2$ $\sqrt{7} \times \sqrt{7} = 7$ $\sqrt{3} \times \sqrt{3} = 3$
b i The conjugate of $2\sqrt{3} + 1$ is $2\sqrt{3} - 1$.	The conjugate has the same terms as the original binomial expression but has the opposite sign in the middle.
ii $(2\sqrt{3} + 1)(2\sqrt{3} - 1)$ $= 12 - 1$ $= 11$	Expand as for $(a - b)(a + b) = a^2 - b^2$ $2\sqrt{3} \times 2\sqrt{3} = 2 \times 2 \times \sqrt{3} \times \sqrt{3}$ $= 4 \times 3$ $= 12$

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Example 21

Rationalise the denominators of each of the following fractions.

a $\dfrac{2}{\sqrt{7}-\sqrt{3}}$

b $\dfrac{\sqrt{6}}{2\sqrt{3}+1}$

Working

a $\dfrac{2}{\sqrt{7}-\sqrt{3}}$

$= \dfrac{2}{\sqrt{7}-\sqrt{3}} \times \dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

$= \dfrac{2(\sqrt{7}+\sqrt{3})}{(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})}$

$= \dfrac{2(\sqrt{7}+\sqrt{3})}{7-3}$

$= \dfrac{\cancel{2}^{1}(\sqrt{7}+\sqrt{3})}{\cancel{4}_{2}}$

$= \dfrac{\sqrt{7}+\sqrt{3}}{2}$

b $\dfrac{\sqrt{6}}{2\sqrt{3}+1}$

$= \dfrac{\sqrt{6}}{2\sqrt{3}+1} \times \dfrac{2\sqrt{3}-1}{2\sqrt{3}-1}$

$= \dfrac{\sqrt{6}(2\sqrt{3}-1)}{(2\sqrt{3}+1)(2\sqrt{3}-1)}$

$= \dfrac{2\sqrt{18}-\sqrt{6}}{4\times 3-1}$

$= \dfrac{2\times 3\sqrt{2}-\sqrt{6}}{12-1}$

$= \dfrac{6\sqrt{2}-\sqrt{6}}{11}$

Reasoning

The surd expression in the denominator is $\sqrt{7}-\sqrt{3}$. To rationalise the denominator we need to multiply by the conjugate. The conjugate is $\sqrt{7}+\sqrt{3}$.

Multiply both numerator and denominator by $\sqrt{7}+\sqrt{3}$.

(Multiplying by $\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ is the same as multiplying by 1, so the value of the original fraction is unchanged.)

In the denominator, use $(a-b)(a+b) = a^2 - b^2$, that is,

$(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})$

$= (\sqrt{7})^2 - (\sqrt{3})^2$

$= 7 - 3$

$= 4$

The denominator is now a whole number, so it is rational.

The conjugate of the expression in the denominator is $2\sqrt{3}-1$.

Multiply both numerator and denominator by $2\sqrt{3}-1$, which is the same as multiplying by 1.

Using $(a-b)(a+b) = a^2 - b^2$ in the denominator gives

$(2\sqrt{3}+1)(2\sqrt{3}-1) = (2\sqrt{3})^2 - (1)^2$

$= 4\times 3 - 1$

$= 11$

In the numerator, the surd term $2\sqrt{18}$ can be simplified to $2\times 3\sqrt{2}$, that is, $6\sqrt{2}$.

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exercise 4.7

1 Carry out each of the following multiplications.

a $\sqrt{3} \times \sqrt{3}$ **b** $\sqrt{7} \times \sqrt{7}$ **c** $3\sqrt{5} \times \sqrt{5}$ **d** $9\sqrt{2} \times \sqrt{2}$

LINKS TO Example 19a

2 Express each of the following in simplest form with a rational denominator.

a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{5}}$ **c** $\frac{1}{\sqrt{7}}$ **d** $\frac{1}{\sqrt{10}}$

e $\frac{3}{\sqrt{2}}$ **f** $\frac{4}{\sqrt{3}}$ **g** $\frac{1}{\sqrt{14}}$ **h** $\frac{6}{\sqrt{7}}$

i $\frac{1}{\sqrt{6}}$ **j** $\frac{5}{\sqrt{2}}$ **k** $\frac{1}{\sqrt{15}}$ **l** $\frac{7}{\sqrt{10}}$

m $\frac{3}{\sqrt{3}}$ **n** $\frac{4}{\sqrt{2}}$ **o** $\frac{3}{\sqrt{6}}$ **p** $\frac{30}{\sqrt{15}}$

3 Express each of the following in simplest form with a rational denominator.

a $\frac{\sqrt{5}}{\sqrt{6}}$ **b** $\frac{\sqrt{3}}{\sqrt{11}}$ **c** $\frac{\sqrt{5}}{\sqrt{2}}$ **d** $\frac{\sqrt{2}}{\sqrt{3}}$

e $\frac{\sqrt{2}}{\sqrt{7}}$ **f** $\frac{\sqrt{7}}{\sqrt{10}}$ **g** $\frac{\sqrt{6}}{\sqrt{7}}$ **h** $\frac{\sqrt{3}}{\sqrt{10}}$

i $\frac{\sqrt{2}}{\sqrt{6}}$ **j** $\frac{\sqrt{10}}{\sqrt{5}}$ **k** $\frac{\sqrt{10}}{\sqrt{15}}$ **l** $\frac{\sqrt{7}}{\sqrt{11}}$

m $\frac{\sqrt{3}}{\sqrt{12}}$ **n** $\frac{\sqrt{6}}{\sqrt{20}}$ **o** $\frac{\sqrt{3}}{\sqrt{6}}$ **p** $\frac{\sqrt{15}}{\sqrt{20}}$

4 In simplest form with a rational denominator, $\frac{\sqrt{48}}{\sqrt{36}}$ is

A $\frac{\sqrt{4}}{\sqrt{3}}$ **B** $\frac{4}{3}$ **C** $\frac{2}{\sqrt{3}}$ **D** $\frac{2\sqrt{3}}{3}$ **E** $\sqrt{\frac{4}{3}}$.

LINKS TO Example 19b

5 Express each of the following in simplest form with a rational denominator.

a $\frac{1}{5\sqrt{2}}$ **b** $\frac{1}{2\sqrt{3}}$ **c** $\frac{3}{2\sqrt{5}}$ **d** $\frac{7}{2\sqrt{3}}$

e $\frac{5}{3\sqrt{2}}$ **f** $\frac{3}{2\sqrt{10}}$ **g** $\frac{4}{3\sqrt{2}}$ **h** $\frac{3}{5\sqrt{3}}$

i $\frac{2}{3\sqrt{6}}$ **j** $\frac{15}{2\sqrt{5}}$ **k** $\frac{10}{3\sqrt{15}}$ **l** $\frac{6}{7\sqrt{3}}$

m $\frac{4}{5\sqrt{2}}$ **n** $\frac{3}{2\sqrt{6}}$ **o** $\frac{3}{4\sqrt{3}}$ **p** $\frac{15}{2\sqrt{10}}$

6 Express each of the following in simplest form with a rational denominator.

a $\frac{\sqrt{3}}{5\sqrt{2}}$ **b** $\frac{\sqrt{2}}{3\sqrt{7}}$ **c** $\frac{\sqrt{7}}{2\sqrt{11}}$ **d** $\frac{\sqrt{2}}{7\sqrt{3}}$

e $\frac{\sqrt{5}}{5\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2\sqrt{5}}$ **g** $\frac{2\sqrt{6}}{3\sqrt{7}}$ **h** $\frac{2\sqrt{3}}{\sqrt{10}}$

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i $\frac{3\sqrt{2}}{\sqrt{7}}$ **j** $\frac{2\sqrt{7}}{3\sqrt{10}}$ **k** $\frac{5\sqrt{2}}{3\sqrt{5}}$ **l** $\frac{2\sqrt{3}}{3\sqrt{11}}$

m $\frac{10\sqrt{2}}{3\sqrt{5}}$ **n** $\frac{4\sqrt{3}}{5\sqrt{2}}$ **o** $\frac{3\sqrt{5}}{5\sqrt{3}}$ **p** $\frac{2\sqrt{7}}{3\sqrt{2}}$

7 In simplest form with a rational denominator, $\frac{2\sqrt{3}}{3\sqrt{7}}$ is

A 2 **B** $\frac{2}{\sqrt{7}}$ **C** $\frac{2}{\sqrt{21}}$ **D** $\frac{2\sqrt{7}}{7}$ **E** $\frac{2\sqrt{21}}{21}$.

LINKS TO Example 20

8 For each of these binomial surds expressions

i write the conjugate surd expression

ii multiply the surd expression by its conjugate

Check that each answer does not involve surds.

a $\sqrt{11} + 3$ **b** $3 - \sqrt{5}$ **c** $2\sqrt{5} + \sqrt{3}$

d $2\sqrt{3} - \sqrt{7}$ **e** $\sqrt{7} - \sqrt{3}$ **f** $2\sqrt{2} - \sqrt{5}$

LINKS TO Example 21

9 Express each of the following in simplest form with a rational denominator.

a $\frac{1}{\sqrt{7} - 2}$ **b** $\frac{1}{4 + \sqrt{10}}$ **c** $\frac{3}{\sqrt{3} + \sqrt{2}}$ **d** $\frac{4}{1 - \sqrt{3}}$

e $\frac{\sqrt{6}}{\sqrt{6} - 2}$ **f** $\frac{\sqrt{3}}{\sqrt{5} + \sqrt{3}}$ **g** $\frac{\sqrt{2}}{3 - 2\sqrt{2}}$ **h** $\frac{3}{2\sqrt{5} - 3}$

10 Express each of the following in simplest form with a rational denominator.

a $\frac{\sqrt{3} + 1}{2\sqrt{3} - 3}$ **b** $\frac{3 - \sqrt{5}}{2 + \sqrt{5}}$ **c** $\frac{\sqrt{2} + \sqrt{7}}{3\sqrt{2} + \sqrt{7}}$ **d** $\frac{\sqrt{6} - \sqrt{3}}{\sqrt{6} + \sqrt{3}}$

11 Express $\frac{7}{\sqrt{2}} - \frac{5}{\sqrt{3}}$ as a single fraction with a rational denominator. Hint: rationalise each denominator before finding a common denominator and subtracting.

12 Express $\frac{4\sqrt{3} - 3}{\sqrt{2}} - \frac{3\sqrt{2} + 1}{\sqrt{3}}$ as a single fraction with a rational denominator.

13 Simplify $\frac{3}{\sqrt{6} - 2} - \frac{1}{\sqrt{6} + 2}$.

14 Solve the equation $\frac{\sqrt{3}x}{2 - \sqrt{3}} = \frac{1 + \sqrt{3}}{3 + \sqrt{3}}$ for x. Express the solution with a rational denominator.

exercise 4.7 challenge

15 Express $\frac{1}{\sqrt{5} + 2} + \frac{1}{\sqrt{5} - 1}$ as a single fraction. Hint: rationalise the denominator of each fraction before performing the addition.

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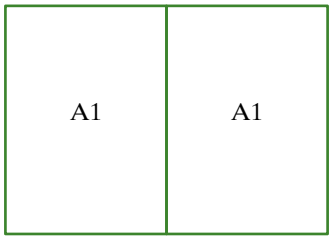
Analysis task

The A series of paper sizes

A4 paper is part of a series of related paper sizes. The largest size is A0, which has an area of exactly 1 square metre. For each paper size, the length is $\sqrt{2}$ times the width. A sheet of A1 paper is formed by cutting A0 paper in half. The width of A0 paper then becomes the length of A1 paper.



An A0 sheet of paper

a Express $\sqrt{2}$ as a rational approximation, correct to three decimal places.

b Let l m be the length and w m be the width of a sheet of A0 paper. As the length is $\sqrt{2}$ times the width, express l in terms of w.

c The area of a sheet of A0 paper is exactly 1 square metre. Write an equation for the area in terms of w and solve the equation to find a surd expression for w^2.

d Express w^2 as a power of 2 with a negative fractional index.

e Express w as a power of 2 with a negative fractional index.

f Express l as a power of 2 with a positive fractional index.

g In the sequence of A paper sizes, the width of each paper size is the length of the next smaller size, that is, the width of A0 paper is the length of A1 paper. Express the length of A1 paper as a power of 2 with a negative fractional index.

h Find an expression for the width of A1 paper.

i Continue the pattern to find the length and width of A2, A3, A4 and A5 paper, writing each as a power of 2 with negative fractional indices. Copy and complete the following table, using your calculator to find rational approximations correct to three decimal places for each length, width and area.

Paper size	Length (m)		Width (m)		Area (m^2)	
	Index form	Rational approximation	Index form	Rational approximation	Index form	Rational approximation
A0	$2^{\frac{1}{4}}$		$2^{-\frac{1}{4}}$		$2^{\frac{1}{4}} \times 2^{-\frac{1}{4}} = 2^0$	1.000
A1	$2^{-\frac{1}{4}}$					
A2						
A3						
A4						
A5						

j Measure the length and width of a piece of A4 paper and compare your measurements with the calculated values in your table.

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Review Surds

Summary

- The numbers we work with are **real numbers**. They can be separated into rational and irrational numbers.
- A **rational number** can be written as a fraction $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

  Zero, all positive and negative whole numbers, fractions, and terminating and recurring decimals are rational numbers.
- An **irrational number** cannot be written as a fraction $\frac{a}{b}$, where a and b are integers. The decimal values of all irrational numbers continue forever with no end and no pattern. Surds and π are irrational numbers.
- **Surds** are irrational numbers that in their simplest exact form involve a square root, cube root or other root.
- A **power of a half** means 'the square root of' and a **power of a third** means 'the cube root of'. In general $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- In general $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
- Surds are said to be '**entire**' if the number is entirely under the square root sign.
- Surds are said to be simplified if the number under the square root sign does not have a square number as a factor.
- Surds are called '**like**' surds if they have the same number under the square root sign.
- Only 'like' surds can be simplified by adding or subtracting.
- Any surds can be multiplied. Multiply the numbers outside the square root signs and multiply the numbers under the square root signs.
- To divide surds, write the numbers under the square root signs as a fraction under a single square root sign then simplify the fraction if possible.
- The **conjugate** of a binomial surd expression is identical except that it has the opposite sign between the terms.
- **Rationalising** the denominator of a fraction with a single term surd in the denominator involves multiplying both numerator and denominator by the surd part of the denominator, to eliminate the surd from the denominator.
- Rationalising the denominator of a fraction with a binomial surd expression in the denominator involves multiplying both numerator and denominator by the conjugate of the binomial surd expression in the denominator.

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Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

conjugate surd	integer	real number
cube root	irrational number	recurring decimal
entire surd	like surds	simplified surd
exact value	power	square number
exponent	rational approximation	square root
fraction	rational number	surd
fractional index	rationalise the denominator	terminating decimal
index		unlike surds

Revision

Multiple-choice questions

1 $8^{-\frac{2}{3}}$ is equal to

A $-8\frac{2}{3}$ **B** -4 **C** $\frac{1}{8}$ **D** $\frac{1}{4}$ **E** $\frac{1}{2}$

2 In simplest form, $2\sqrt{45}$ is

A $2\sqrt{5}$ **B** $3\sqrt{5}$ **C** $6\sqrt{5}$ **D** $5\sqrt{3}$ **E** $10\sqrt{3}$

3 In simplest form $2\sqrt{3} \times 5\sqrt{6}$ is

A $3\sqrt{2}$ **B** $30\sqrt{2}$ **C** $20\sqrt{3}$ **D** $7\sqrt{18}$ **E** $10\sqrt{18}$

4 In simplest form $2\sqrt{3} + 5\sqrt{2} - \sqrt{3} + 3\sqrt{2}$ is

A $\sqrt{3} + 8\sqrt{2}$ **B** $3\sqrt{3} + 8\sqrt{2}$ **C** $\sqrt{3} + 2\sqrt{2}$ **D** $\sqrt{6} + 8\sqrt{4}$ **E** $9\sqrt{10}$

5 In simplest form, with a rational denominator, $\frac{\sqrt{2}}{2\sqrt{3}}$ is

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{6}}$ **C** $\frac{\sqrt{3}}{3}$ **D** $\frac{\sqrt{6}}{6}$ **E** $\frac{\sqrt{6}}{2}$

Short-answer questions

6 Evaluate each of the following.

a $8^{\frac{1}{3}}$ **b** $9^{\frac{3}{2}}$ **c** $125^{-\frac{2}{3}}$

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7 Write each of the following as an entire surd.

a $5\sqrt{2}$ **b** $-3\sqrt{7}$

8 Simplify each of the following surd expressions.

a $\sqrt{5} + 4\sqrt{5} - 3\sqrt{5}$ **b** $2\sqrt{7} + 6\sqrt{7} - 9\sqrt{7}$

9 Simplify the following surds.

a $\sqrt{54}$ **b** $\sqrt{72}$ **c** $\sqrt{45}$ **d** $\sqrt{48}$

10 Simplify each of the following.

a $\sqrt{27} + \sqrt{300}$ **b** $\sqrt{18} + \sqrt{500} - \sqrt{8} + \sqrt{20}$

11 Carry out each of these surd multiplications, simplifying or evaluating where appropriate.

a $\sqrt{3} \times \sqrt{6}$ **b** $\sqrt{12} \times \sqrt{3}$ **c** $\sqrt{20} \times \sqrt{8}$ **d** $\sqrt{5} \times \sqrt{15}$

12 Expand each of the following, giving the answers in simplest surd form.

a $2(\sqrt{5} + 3\sqrt{7})$ **b** $\sqrt{3}(\sqrt{6} + 4\sqrt{2})$

c $(\sqrt{2} + \sqrt{5})(3\sqrt{2} + 2\sqrt{5})$ **d** $(6 - 5\sqrt{3})(3\sqrt{3} - 1)$

13 **a** Write the conjugate of $5 + 3\sqrt{2}$. **b** Multiply $5 + 3\sqrt{2}$ by its conjugate.

14 Express each of the following with a rational denominator.

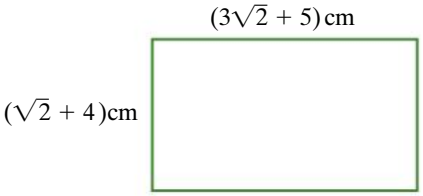
a $\dfrac{10}{\sqrt{2}}$ **b** $\dfrac{1}{\sqrt{7} + \sqrt{5}}$ **c** $\dfrac{\sqrt{3}}{2\sqrt{3} - 1}$

d $\dfrac{2}{3\sqrt{2}}$ **e** $\dfrac{\sqrt{5}}{\sqrt{5} - \sqrt{2}}$ **f** $\dfrac{3\sqrt{2} - 4}{5 - \sqrt{2}}$

Extended-response questions

15 **a** Find the perimeter of the rectangle. Give the answer in exact form, and then as a decimal approximation correct to two decimal places.

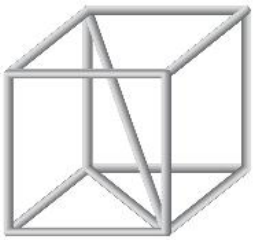
b Find the area of the rectangle. Give the answer in exact form, and then as a decimal approximation correct to two decimal places.



16 A manufacturer of wire goods is asked to make a wire-frame model of a cube, with two diagonals as shown, for use in mathematics classrooms. If the side length of the cube is 30 cm, find the total length of wire needed. Give the length

a in simplest surd form

b as a rational approximation to the nearest centimetre



Practice quiz Chapter 4

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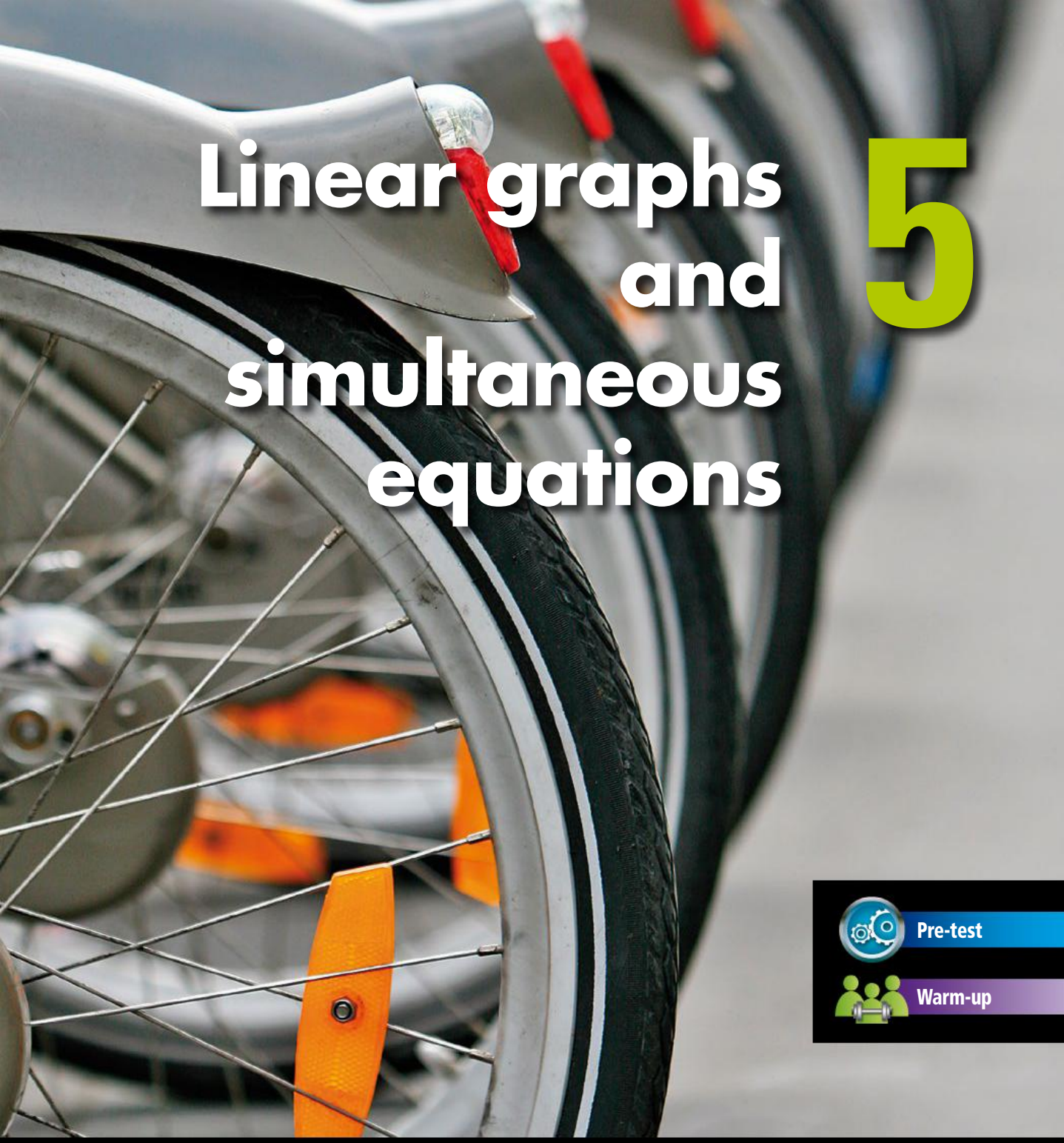


Linear graphs and simultaneous equations

5

Pre-test

Warm-up

Many rental and repair charges comprise a fixed cost and then a variable cost based on time. These bicycles can be hired at a fixed charge plus an additional charge per hour. Two hire companies may have different fixed charges and different hourly rates. Depending on the number of hours a bicycle is hired, one company may be more economical than the other. In this chapter you will learn how to solve problems of this type. The chapter starts with a review of linear graphs, including parallel and perpendicular lines.

5.1 Reviewing gradient and linear graphs

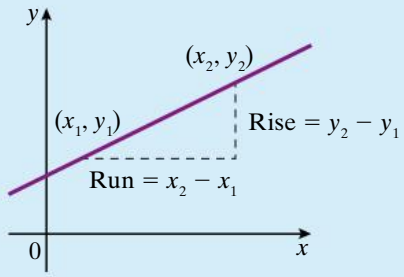
You may remember from Year 9 that the gradient, m, of a linear graph is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$$

Calculating gradients

For the two points (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Example 1

Calculate the gradient of the line passing through the points

a $(-2, 5)$ and $(3, 8)$

b $(4, -3)$ and $(-5, 6)$

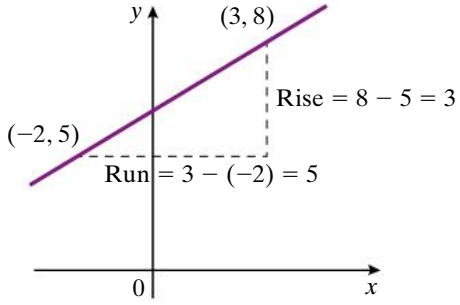
Working

a $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{8 - 5}{3 - (-2)}$

$= \dfrac{3}{5}$

Reasoning



It does not matter which of the two points we call the first point and which we call the second point.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{5 - 8}{-2 - 3}$

$= \dfrac{-3}{-5}$

$= \dfrac{3}{5}$

continued

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Example 1 continued

Working

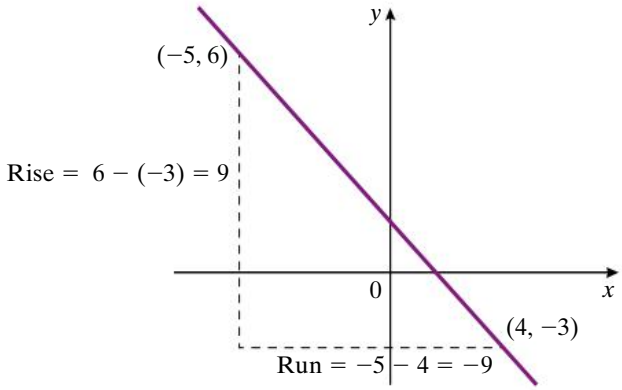
b $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{6 - (-3)}{-5 - 4}$

$= \dfrac{9}{-9}$

$= -1$

Reasoning



Gradient-intercept form of a straight line

The equation of a straight line can be written as $y = mx + b$, where

- m is the gradient
- b is the y-intercept, that is, the point where the line crosses the y-axis. We can also say that the y-intercept is the point $(0, b)$.

We refer to $y = mx + b$ as the gradient-intercept form of the straight line equation.

Example 2

Write an equation for a line that has

a a gradient of 3 and a y-intercept at $(0, 5)$

b a gradient of -4 and a y-intercept at $(0, 6)$

Working	Reasoning
a $y = mx + b$ $y = 3x + 5$	$m = 3, b = 5$
b $y = mx + b$ $y = -4x + 6$	$m = -4, b = 6$

Finding the equation given the gradient and one point on the line

The gradient formula applies to any two points on a line. If we know one point, (x_1, y_1), we can use the (x, y) to represent any other point on the line. If we also know the gradient, we can find the equation for the line by substituting for m and (x_1, y_1) in the gradient rule:

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$m = \dfrac{y - y_1}{x - x_1}$

This can be rearranged as $y - y_1 = m(x - x_1)$.

Equation of a line given the gradient and one point

$y - y_1 = m(x - x_1)$

Example 3

Find the equation of the line with gradient $\frac{1}{2}$ that passes through the point $(-1, 5)$.

Working

$y - y_1 = m(x - x_1)$

$y - 5 = \frac{1}{2}[x - (-1)]$

$y - 5 = \frac{1}{2}(x + 1)$

$2y - 10 = x + 1$

$2y = x + 11$

Reasoning

Substitute $m = \frac{1}{2}$, $x_1 = -1$ and $y_1 = 5$ in $m = \dfrac{y - y_1}{x - x_1}$.

We could put the equation into $y = mx + b$ form:

$y = \frac{1}{2}x + \frac{11}{2}$

We could also write this as $y = \frac{x}{2} + \frac{11}{2}$.

Finding the equation given two points on the line

If we know two points (x_1, y_1) and (x_2, y_2) on a line we:

- calculate the gradient using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- use the same method as for gradient and one point, choosing either one of the two points to substitute.

Example 4

Find the equation of the line passing through the points $(-3, 6)$ and $(2, -4)$.

Working

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 6}{2 - (-3)} = \frac{-10}{5} = -2$$

Reasoning

Substitute $x_1 = -3$, $y_1 = 6$, $x_2 = 2$ and $y_2 = -4$ in $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find m.

continued

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Example 4 continued

Working

Substitute $m = -2$, $x_1 = -3$, $y_1 = 6$

$$y - y_1 = m(x - x_1)$$
$$y - 6 = -2[x - (-3)]$$
$$y - 6 = -2(x + 3)$$
$$y - 6 = -2x - 6$$
$$y = -2x$$

Reasoning

Choose either of the points to substitute into $y - y_1 = m(x - x_1)$

If we had chosen the other point $(2, -4)$ to substitute:

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = -2(x - 2)$$
$$y + 4 = -2(x - 2)$$
$$y + 4 = -2x + 4$$
$$y = -2x$$

Finding the equation given the two intercepts

If the y-intercept is the point $(0, b)$ and the x-intercept is the point $(a, 0)$, we can substitute the two points into $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find m, then substitute for m and b in $y = mx + b$.

Example 5

Find the equation of the line with an x-intercept of 2 and a y-intercept at $(0, 5)$.

Working

$(0, 5)$ and $(2, 0)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 5}{2 - 0}$$
$$= \frac{-5}{2}$$
$$= -2.5$$

Substitute $m = -2.5$, $b = 5$ in $y = mx + b$

$$y = -2.5x + 5$$

Reasoning

Substitute $x_1 = 0$, $y_1 = 5$, $x_2 = 2$ and $y_2 = 0$ in $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find m.

The y-intercept is the point $(0, 5)$ so $b = 5$.

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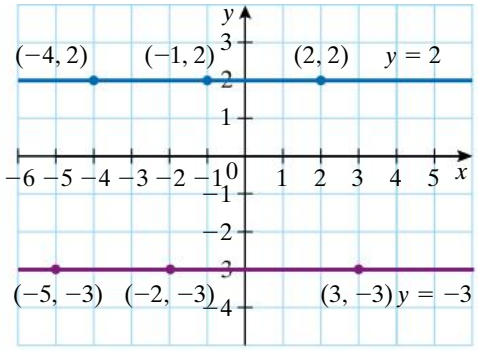
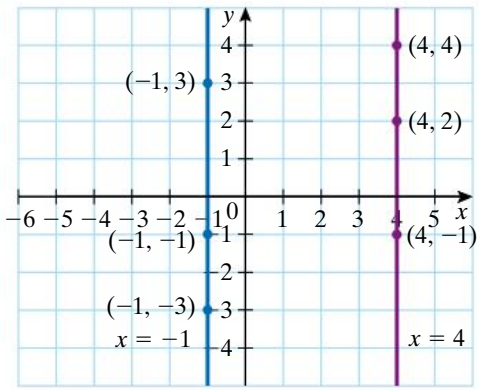
Horizontal and vertical lines

Every point on a horizontal line has the same y-coordinate, regardless of its x-coordinate. The gradient of a horizontal line is zero. Using the general equation for a straight line $y = mx + b$ we see that when $m = 0$, the equation is simply $y = b$.



The gradient of a vertical line is undefined so we cannot use $y = mx + b$. Every point on a vertical line has the same x-coordinate, regardless of its y-coordinate, so the equation is of the form $x = a$.



Example 6

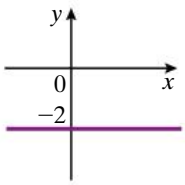
Sketch the following graphs.

a $y = -2$

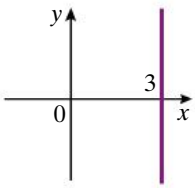
b $x = 3$

Working

a



b



Reasoning

The line $y = -2$ is a horizontal line passing through the point $(0, -2)$.

The line $x = 3$ is a vertical line passing through the point $(3, 0)$.

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General equation for a straight line

The equation $ax + by + c = 0$ is often used as a general equation for a straight line so that all possible cases can be covered by a single general equation. In this general form, a, b and c are constants. (Note that b in this general form is not the same b as in $y = mx + b$.) Either a or b can be 0, but obviously not in the same equation as there would then be no equation!

- If $a = 0$, the equation reduces to the equation for a horizontal line.
- If $b = 0$, the equation reduces to the equation for a vertical line.

Example 7

a Convert the equation $y = -\frac{3x}{4} + 3$ into the general form for a straight line.

b Convert the equation $5x - 4y + 8 = 0$ into the gradient-intercept form.

Working	Reasoning
a $y = -\frac{3x}{4} + 3$	Multiply both sides by 4 to remove the fractions.
$4y = -3x + 12$	Take all terms to the left hand-side.
$3x + 4y - 12 = 0$	
b $5x - 4y + 8 = 0$	Isolate the y term.
$4y = 5x + 8$	
$y = \frac{5x}{4} + \frac{8}{4}$	Divide both sides by 4.
$y = \frac{5}{4}x + 2$	Simplify. Alternative forms of the equation are $y = \frac{5x}{4} + 2$, or $1.25x + 2$.

Reviewing sketch graphs of linear functions

Only two points are needed to specify the position of a straight line. We can sketch the graph of any linear equation if we know two points on the line. The x-intercept and the y-intercept are useful points to choose.

Example 8

Sketch the graph of these linear functions by first finding the intercepts.

a $y = 3x + 5$

b $2x + 4y = 7$

continued

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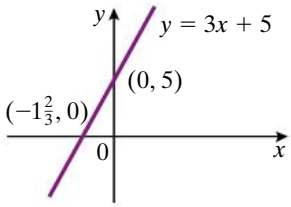
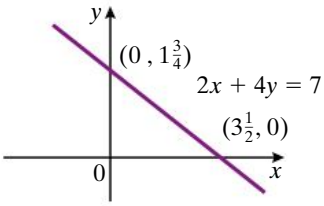
Example 8 continued

Working	Reasoning
a x-intercept: When $y = 0$, $0 = 3x + 5$ $x = -\frac{5}{3}$ $= -1\frac{2}{3}$ $\left(-1\frac{2}{3}, 0\right)$	Find the x-intercept by putting $y = 0$.
y-intercept: When $x = 0$, $y = 5$ $(0, 5)$ 	The equation is in gradient-intercept form, so y-intercept is 5. Draw and label the axes. Mark the two intercepts and draw a line through them. Label the intercepts.
b x-intercept: When $y = 0$, $2x = 7$ $x = \frac{7}{2}$ $= 3\frac{1}{2}$ $(3\frac{1}{2}, 0)$	Find the x-intercept by putting $y = 0$.
y-intercept: When $x = 0$, $4y = 7$ $y = \frac{7}{4}$ $= 1\frac{3}{4}$ $(0, 1\frac{3}{4})$	Find the y-intercept by putting $x = 0$.
	Draw and label the axes. Mark the two intercepts and draw a line through them. Label the intercepts.

For equations of the form $y = mx$, the graph passes through the origin. Using the intercepts gives only one point, so we need to choose a second point when sketching the graph.

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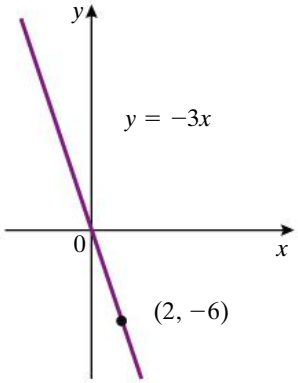
Example 9

Sketch the graph of $y = -3x$.

Working

The graph passes through (0, 0).
When $x = 2$,
$y = -3 \times 2 = -6$
The graph passes through $(2, -6)$



Reasoning

Any value may be chosen for x to find a second point. In this example, $x = 2$ was chosen.

Draw and label the axes. Draw a line passing through the origin and the point $(2, -6)$. Label the point $(2, -6)$.

Finding the equation from the graph

Linear graphs

From a linear graph we can find

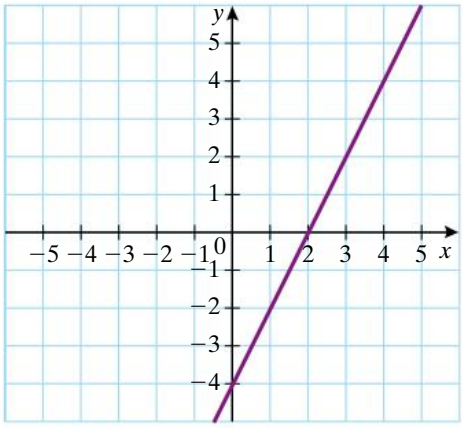
- the gradient, m
- the y-intercept, b

We can then write the equation for the line.

Example 10

Find the equation for each of these linear graphs.

a



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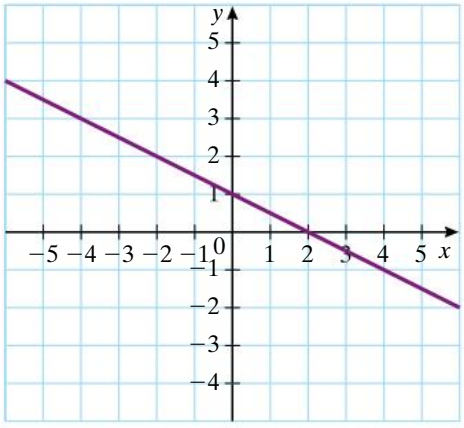

continued

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Example 10 continued

Working

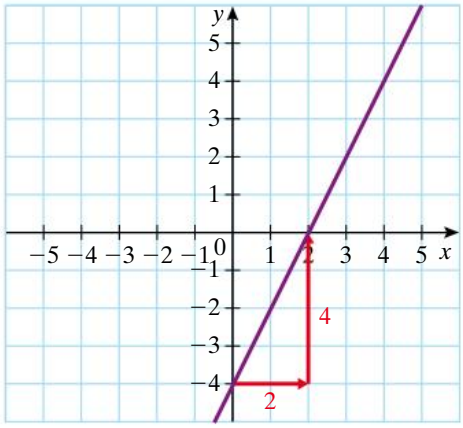
a $m = \frac{4}{2} = 2$

$b = -4$

Equation for the line is $y = 2x - 4$.

Reasoning

Draw a right-angled triangle anywhere on the graph to show the rise and run.



Gradient $= \frac{\text{rise}}{\text{run}}$

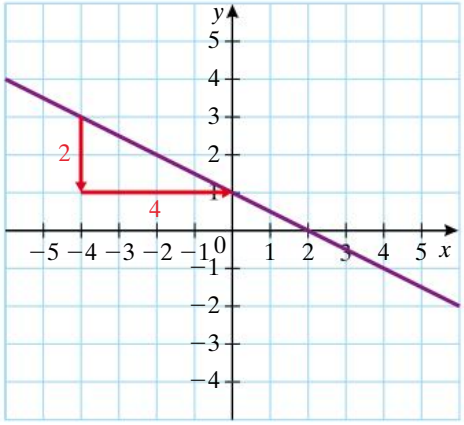
y-intercept is the point where the graph crosses the y-axis.

b $m = -\frac{2}{4} = -\frac{1}{2}$ or -0.5

$b = 1$

Equation for the line is $y = -\frac{x}{2} + 1$ or $y = -0.5x + 1$.

The graph slopes downwards to the right so the gradient is negative.



exercise 5.1

LINKS TO Example 1

1 Calculate the gradient of the lines that pass through each pair of points.

a $(-2, 1)$ and $(4, 7)$ **b** $(-6, -5)$ and $(2, 7)$ **c** $(0, -3)$ and $(2, -7)$

d $(-8, 7)$ and $(2, 2)$ **e** $(-4, -1)$ and $(2, 8)$ **f** $(-7, 3)$ and $(3, -3)$

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LINKS TO Example 2

2 Write equations for each of the following lines.

a gradient 2 and y-intercept -6
b gradient -3 and y-intercept 4
c gradient $\frac{2}{3}$ and y-intercept -1
d gradient $-\frac{4}{5}$ and y-intercept 9
e gradient $\frac{5}{7}$ and y-intercept -3
f gradient -3 and y-intercept $\frac{1}{2}$

LINKS TO Example 3

3 Find equations for each of the following lines.

a gradient 2 and passing through the point $(-1, 5)$
b gradient -5 and passing through the point $(6, 8)$
c gradient -3 and passing through the point $(-2, 4)$
d gradient 4 and passing through the point $(2, -5)$
e gradient $-\frac{5}{4}$ and passing through the point $(3, 8)$
f gradient $\frac{2}{3}$ and passing through the point $(-4, -5)$
g gradient $\frac{3}{4}$ and passing through the point $(-3, -5)$
h gradient $-\frac{1}{2}$ and passing through the point $(5, 0)$

LINKS TO Example 4

4 Find the equation of the line passing through each pair of points.

a $(1, 5)$ and $(2, -6)$
b $(-2, -1)$ and $(2, 5)$
c $(-1, -7)$ and $(4, 3)$
d $(-3, 2)$ and $(-1, -4)$
e $(-1, 5)$ and $(4, 10)$
f $(-3, 4)$ and $(3, -4)$
g $(-5, -2)$ and $(3, 4)$
h $(-2, 7)$ and $(0, 5)$
i $(1, -4)$ and $(5, 2)$
j $(-5, -4)$and $(3, 7)$
k $(-3, -5)$ and $(1, 7)$
l $(1, 2)$ and $(7, 6)$

LINKS TO Example 5

5 For each of these pairs of intercepts:

i sketch the linear graph
ii find the equation of the line.

a x-intercept -3 and y-intercept 6
b x-intercept 5 and y-intercept 3
c x-intercept 2 and y-intercept -7
d x-intercept 1 and y-intercept 5
e x-intercept -5 and y-intercept 1
f x-intercept 6 and y-intercept -4
g x-intercept 3 and y-intercept 4
h x-intercept -3 and y-intercept -1

LINKS TO Example 6

6 Sketch the graph of each of these equations.

a $y = -1$ **b** $x = 7$ **c** $y = 0$ **d** $x = -6$ **e** $x = 0$ **f** $y = 8$

7 Write an equation for a line that is

a parallel to the x-axis and passes through the point $(1, 5)$
b parallel to the x-axis and passes through the point $(-4, -3)$
c parallel to the y-axis and passes through the point $(-4, -3)$
d parallel to the x-axis and passes through the point $(-2, -6)$
e parallel to the y-axis and passes through the point $(-2, -6)$
f parallel to the y-axis and passes through the point $(6, 7)$

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LINKS TO Example 7a

8 Write each of these equations in the general form for a linear equation.

a $y = \frac{7}{3}x - 8$ **b** $y = \frac{2}{5}x - 7$ **c** $y = -\frac{2}{11}x - 1$

d $y = \frac{3x}{2} + \frac{5}{2}$ **e** $y = -\frac{4x}{5} - \frac{9}{5}$ **f** $y = -\frac{3x}{4} - \frac{5}{2}$

LINKS TO Example 7b

9 Write each of these equations in gradient-intercept form and hence state the gradient and y-intercept.

a $2x + 4y - 5 = 0$ **b** $5x - 2y - 10 = 0$ **c** $3y - 4x - 15 = 0$

d $5x - 8y - 20 = 0$ **e** $6x - 4y - 25 = 0$ **f** $7x + 4y - 10 = 0$

LINKS TO Examples 8, 9

10 Sketch graphs of the following linear functions.

a $y = 2x + 8$ **b** $y = -5x + 3$ **c** $y = -4x - 5$

d $y = -x + 3$ **e** $4x + 5y = 10$ **f** $3x - 4y = 6$

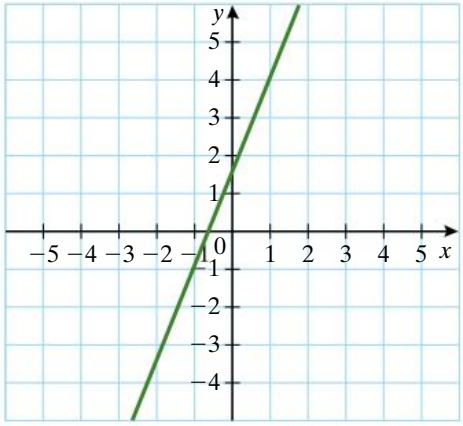
g $x - 4y = 6$ **h** $5x + 6y = 30$ **i** $2x + 5y = 8$

j $y = 4x$ **k** $y = -2x$ **l** $y = -3.5x$

LINKS TO Example 10

11 Find the equation for each of these linear graphs.

a



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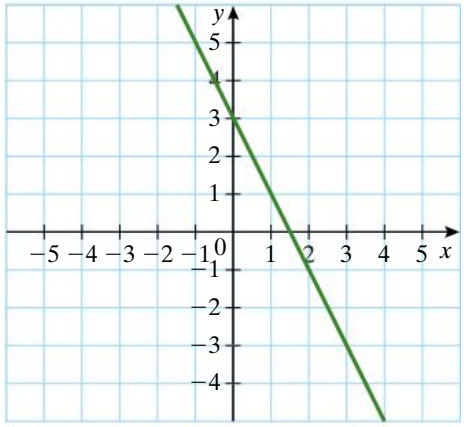

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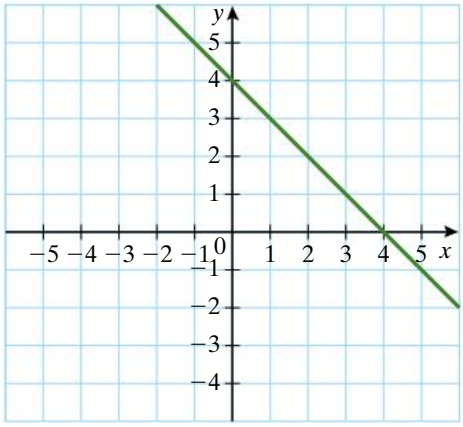

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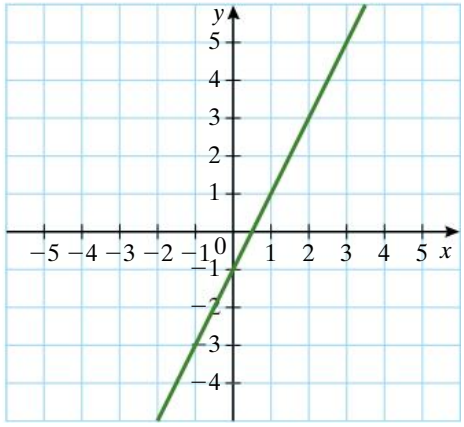

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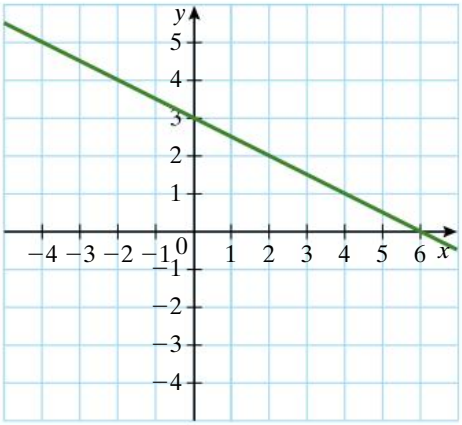
e



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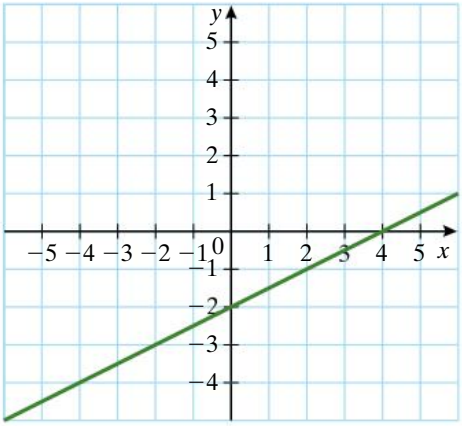

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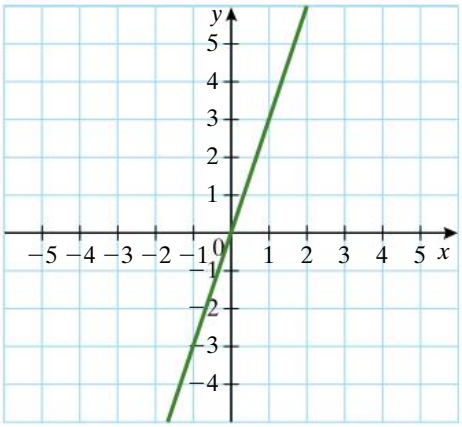

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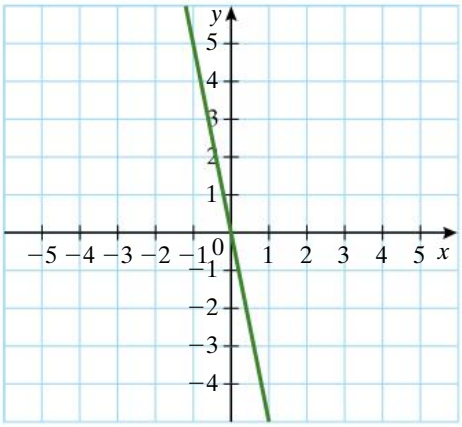

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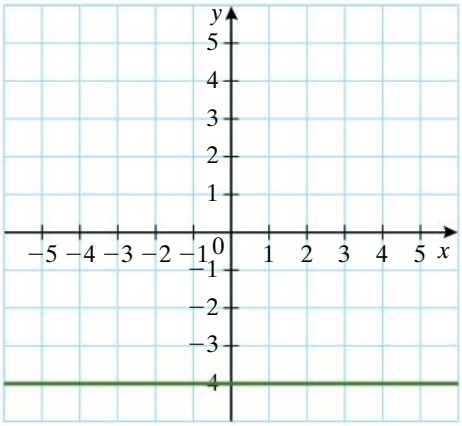

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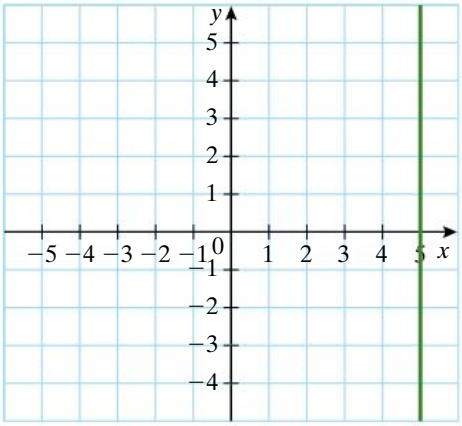

exercise 5.1 challenge

12 Find the equation for the line that passes through the point $(-1, 5)$ and is perpendicular to the line that passes through the points $(-2, -4)$ and $(3, 6)$.

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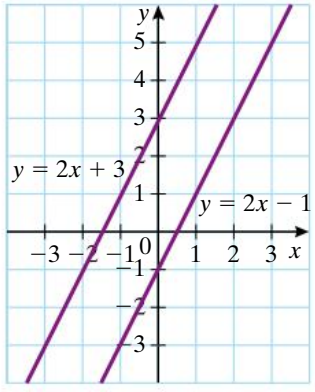
5.2 Parallel and perpendicular lines

Parallel lines

Parallel and perpendicular lines

Parallel lines

Linear graphs that have the same gradient are parallel lines. Conversely, linear graphs that are parallel have the same gradient.



Each of the lines in the graph shown on the right has a gradient of 2.

Example 11

Find the equation for the line that is parallel to the graph of $y = -4x + 5$ and passes through the point $(-1, 3)$.

Working

Gradient of $y = -4x + 5$ is -4.
Substitute $m = -4$, $x_1 = -1$ and $y_1 = 3$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -4(x - (-1))$$
$$y - 3 = -4(x + 1)$$
$$y - 3 = -4x - 4$$
$$y = -4x - 1$$

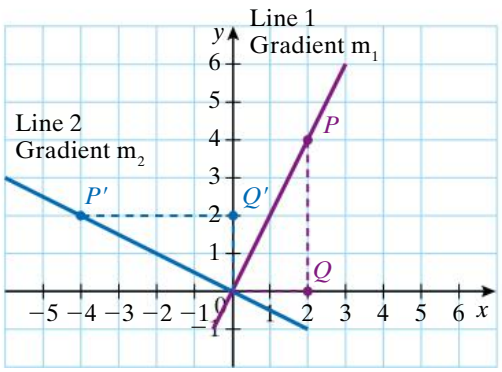
Reasoning

Parallel lines have the same gradient.
The gradient of the line $y = -4x + 5$ is -4, so the gradient of the required line is -4.
The gradient and one point are known so use $y - y_1 = m(x - x_1)$.

Perpendicular lines

Line 1 has been rotated through 90° in an anticlockwise direction about the point (0, 0) to produce Line 2. $\triangle OPQ$ is also rotated through 90° about point O in an anticlockwise direction so that P' is the image of P and Q' is the image of Q.

$OQ' = OQ$ and $P'Q' = PQ$.



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Let the gradient of Line 1 be m_1 and let the gradient of Line 2 be m_2.

$$m_1 = \frac{\text{rise}}{\text{run}} = \frac{PQ}{OQ}$$

$$m_2 = \frac{\text{rise}}{\text{run}} = \frac{-OQ'}{P'Q'} = -\frac{OQ}{PQ}$$

$$m_2 = -\frac{1}{m_1}$$

$$m_1 \times m_2 = -1$$

The product of the gradients of the two lines is -1.

This relationship between the gradients applies to all perpendicular lines except where one of the perpendicular lines is a horizontal line (gradient is zero) and the other is a vertical line (gradient is undefined).

Gradients of perpendicular lines

The product of the gradients of two perpendicular lines is -1. In other words, if two lines are perpendicular, the gradient of one line is the negative reciprocal of the gradient of the other line.

$m_1 \times m_2 = -1$ or $m_2 = -\dfrac{1}{m_1}$

Conversely, if the product of the gradients of two lines is -1, then the lines are perpendicular.

Example 12

What is the gradient of a line perpendicular to the line $y = 2x - 5$?

Working

Gradient of $y = 2x - 5$ is 2.
Gradient of a perpendicular line is $-\dfrac{1}{2}$.

Reasoning

If two lines are perpendicular, the gradient of one line is the negative reciprocal of the gradient of the other line.
The negative reciprocal of 2 is $-\dfrac{1}{2}$.

Example 13

Show that the line through the points $(-2, -1)$ and $(1, 5)$ is perpendicular to the line through the points $(-6, 2)$ and $(4, -3)$.

continued

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Example 13 continued

Working

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{1 - (-2)} = \frac{6}{3}$$

$$m_1 = 2$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 2}{4 - (-6)} = \frac{-5}{10}$$

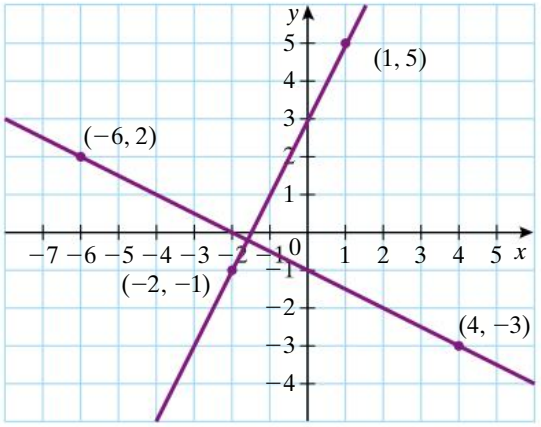
$$m_1 = -\frac{1}{2}$$

$$m_1 \times m_2 = 2 \times -\frac{1}{2} = -1$$

The two lines are perpendicular.

Reasoning

Calculate the gradient of each line.
If the product of the gradients of two lines is -1, then the lines are perpendicular.



Example 14

The line $y = 3x + 4$ is rotated though 90° anticlockwise about the y-intercept.

a What is the gradient of the rotated line?

b What is the equation of the rotated line?

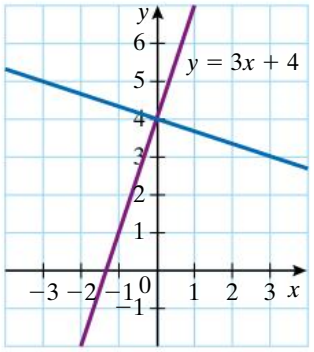
Working

a Gradient of $y = 3x + 4$ is 3.
The gradient of the rotated line is $-\frac{1}{3}$.

b The equation is $y = -\frac{1}{3}x + 4$

Reasoning

The negative reciprocal of 3 is $-\frac{1}{3}$.



The line is rotated about y-intercept so the y-intercept of the perpendicular line is 4.
It could also be written as $y = -\frac{x}{3} + 4$.

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Example 15

Find the equation of the line that passes through the point (3, 4) and is perpendicular to the line $y = -2x + 1$.

Working

Gradient of the line $y = -2x + 1$ is -2.

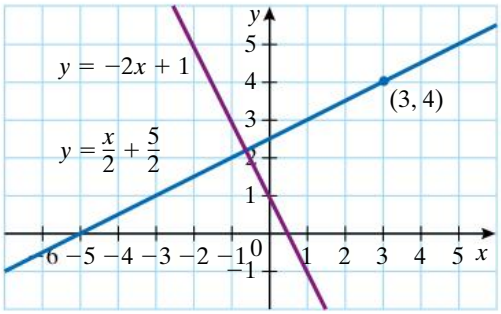
Gradient of the required line is $\frac{1}{2}$.

$$y - y_1 = m(x - x)$$
$$y - 4 = \frac{1}{2}(x - 3)$$
$$2(y - 4) = x - 3$$
$$2y - 8 = x - 3$$
$$2y = x + 5$$
$$y = \frac{1}{2}x + \frac{5}{2}$$

The equation of the line is $y = \frac{1}{2}x + \frac{5}{2}$.

Reasoning

The product of the gradients of two perpendicular lines is -1.



The equation of the perpendicular line can also be written as $2y = x + 5$.

Example 16

Show that quadrilateral $ABCD$ with vertices $A(-3, 3)$, $B(0, 6)$, $C(5, 1)$ and $D(2, -2)$ is a rectangle.

Working

Gradient of AB

$$= \frac{6 - 3}{0 - (-3)} = \frac{3}{3}$$
$$= 1$$

Gradient of BC

$$= \frac{1 - 6}{5 - 0} = \frac{-5}{5}$$
$$= -1$$

Gradient of CD

$$= \frac{-2 - 1}{2 - 5} = \frac{-3}{-3}$$
$$= 1$$

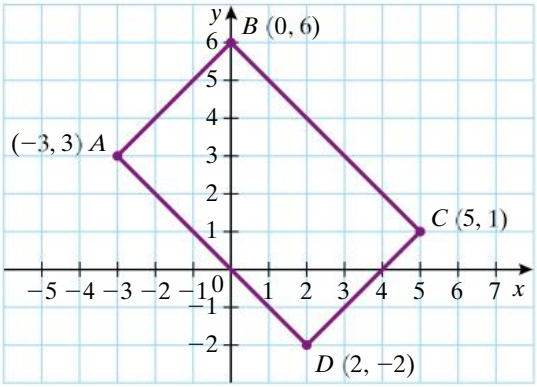
Gradient of DA

$$= \frac{3 - -2}{-3 - 2} = \frac{5}{-5}$$
$$= -1$$

$AB \parallel CD$, $BC \parallel DA$, $AB \perp BC$

Both pairs of opposite sides are parallel and adjacent sides are perpendicular, so $ABCD$ is a rectangle.

Reasoning



Gradient of AB = gradient of CD.
Gradient of BC = gradient of DA.
Gradients are negative reciprocals, so AB is perpendicular to BC.

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exercise 5.2

LINKS TO Example 11

1 Find the equation for the line that is parallel to the line $y = -4x + 5$ and passes through the point $(-1, 3)$.

2 Find the equation for the line that is parallel to the line $2y = 5x + 4$ and passes through the point $(4, -7)$.

3 Show that the line through the points $(-3, 7)$ and $(-6, -2)$ is parallel to the line $y = 3x - 5$.

LINKS TO Example 12

4 State the gradient, m, of lines that are perpendicular to the following lines.

a $y = -x + 5$ **b** $y = 4x + 2$ **c** $y = \frac{x}{2}$ **d** $y = -\frac{5x}{4} + \frac{3}{4}$

e $y = -0.1x + 5.6$ **f** $y = 3x + 7$ **g** $y = \frac{2x}{3} + 4$ **h** $y = -2x - 5$

LINKS TO Example 13

5 Show that the line $y = 4x - 1$ is perpendicular to the line $x + 4y = 7$.

LINKS TO Example 14

6 Find the equation for the lines that are perpendicular to the given line and pass through the given point.

a $y = 3x - 2$ $(-2, 0)$ **b** $y = -\frac{3x}{4} + 1$ $(5, 3)$ **c** $y = -x + 4$ $(-6, -3)$

d $y = 2x + 5$ $(4, 8)$ **e** $y = \frac{3x}{5} + 6$ $(-7, 0)$ **f** $y = \frac{x}{4} - 5$ $(3, 5)$

g $y = \frac{4x}{3} + 2$ $(4, -5)$ **h** $y = \frac{x}{3} + \frac{2}{3}$ $(0, 8)$ **i** $y = x - 6$ $(3, -6)$

j $y = -\frac{3x}{2} + 4$ $(1, 5)$ **k** $y + 3x = 7$ $(6, -4)$ **l** $2x + 7y = 3$ $(0, 0)$

LINKS TO Example 15

7 Each of the following lines are rotated through 90° about the y-intercept. For these rotated lines find

i the gradient **ii** the equation

a $y = 2x - 3$ **b** $y = -\frac{1}{2}x + 2$ **c** $y = -2x - 4$ **d** $y = x + 4$

LINKS TO Example 16

8 Quadrilateral $ABCD$ has vertices A $(-5, 2)$, B $(0, 5)$, C $(6, -5)$ and D $(1, -8)$.

a Plot the points.

b Calculate the gradient of

i AB **ii** BC **iii** CD **iv** DA

c Is $ABCD$ a rectangle? Explain.

9 A quadrilateral has the vertices (in clockwise order) $A(-1, 5)$, $B(6, 4)$, $C(4, -1)$ and $D(-3, 0)$.

a Plot the points. **b** Prove that $ABCD$ is a parallelogram.

exercise 5.2 challenge

10 The points $A(0, 5)$, $B(4, 7)$, $C(2, 3)$ and $D(-2, 1)$ are the vertices of a rhombus. Prove that the diagonals of the rhombus are perpendicular.

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5.3 Solving simultaneous linear equations: graphical method

Consider the following problem from exercise **1.4**, which was intended to be solved by setting up a linear inequality. This time we will solve the problem graphically.

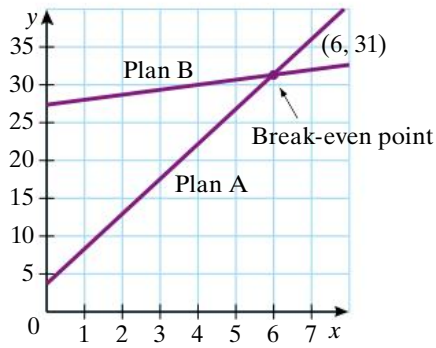
Harry wants to hire a bike for part of a day. Cheap Bike Hire offers two bike hire plans. With Plan A you pay a fee of \$4 plus \$4.50 per hour. With Plan B you pay \$28 plus 50 cents per hour. For what numbers of hours is Plan A cheaper?

If we let x be the number of hours and $\$y$ be the cost of hiring a bike, we can set up two linear equations.

Plan A $\quad y = 4.5x + 4$

Plan B $\quad y = 0.5x + 28$

Graphing these two linear equations shows that the two lines intersect at the point $(6, 31)$, that is, for 6 hours, the cost is \$31 for both plans.



In situations involving costs, this intersection point is often referred to as the **break-even point**. We can see from the graph that if Harry intends hiring the bike for less than 6 hours, he should choose Plan A. However, if he intends hiring the bike for more than 6 hours, he would be better off choosing Plan B.

Two equations that are solved together are called **simultaneous equations**. Unlike equations you have solved before, simultaneous equations have two variables. The solution represents the values of the two variables that satisfy both equations. The solution to a pair of linear simultaneous equations can sometimes be found by graphing the two equations and finding the coordinates of the point of intersection.

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Example 17

Solve this pair of simultaneous equations by making a table of values using the values $x = 0$, $x = 1$ and $x = 2$ then carefully plotting graphs of each equation on the same set of axes.

$y = x + 4$

$y = -x + 10$

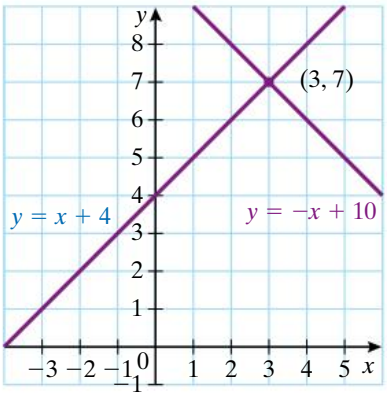
Working

$y = x + 4$

x	0	1	2
y	4	5	6

$y = -x + 10$

x	0	1	2
y	10	9	8



The solution to the pair of simultaneous equations is $x = 3$, $y = 7$.

Reasoning

The two lines intersect at the point $(3, 7)$.

Check:

$y = x + 4$

When $x = 3$,

$y = 3 + 4$

$= 7$

$y = -x + 10$

When $x = 3$,

$y = -3 + 10$

$= 7$

In each of the two situations we have looked at, the two lines intersect, giving a unique solution. This is not always the case as we see in the following examples.

Example 18

Consider each of the following pairs of simultaneous equations.

i Use a graphical approach to find any solutions.

ii What do you notice about the number of solutions in each case? What clues are there in the equations about the number of solutions?

a $x + y = 4$
$x - y = 0$

b $x + y = 4$
$x + y = 0$

c $x + y = 4$
$2x + 2y = 8$

continued

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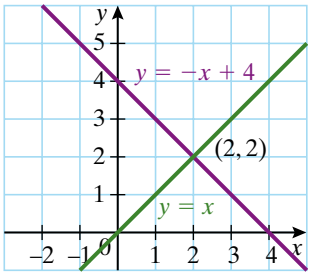
Example 18 continued

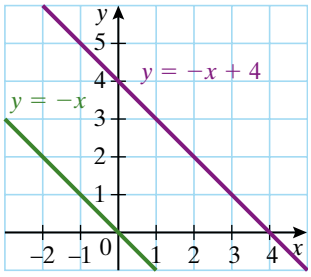
Working

a i



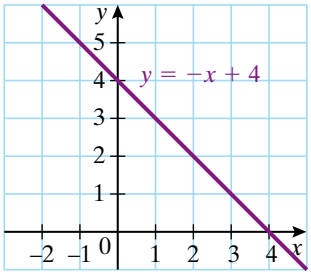
ii There is one solution:
$x = 2, y = 2$

b i



ii There are no solutions.
The lines both have gradient -1.
so they are parallel and do not intersect.

c i



ii There is an infinite number of solutions. The two equations are equivalent. If both sides of $2x + 2y = 8$ are divided by 2, $x + y = 4$, is obtained.

Reasoning

Both equations have been rearranged into $y = mx + b$ form.
$y = x + 4$
$y = x$
The lines intersect at the point $(2, 2)$.

Both equations have been rearranged into $y = mx + b$ form.
$y = -x + 4$
$y = -x$
The lines both have gradient -1.

Both equations have been rearranged into $y = mx + b$ form.
$y = -x + 4$
$2y = -2x + 8$
So $y = -x + 4$
The two lines coincide.

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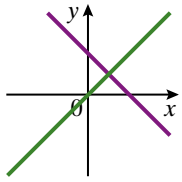
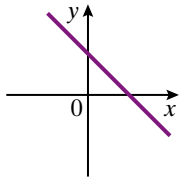
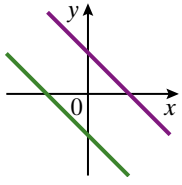
How many solutions?

If the two lines have different gradients, they intersect at one point. The pair of simultaneous linear equations has a unique solution.	1 solution	
If the two lines have the same gradient and y-intercept they coincide and intersect at every point. The pair of simultaneous linear equations has an infinite number of solutions. This occurs if the two equations are equivalent.	Infinite number of solutions	
If the two lines have the same gradient but different y-intercepts, they are parallel and never intersect. The pair of simultaneous linear equations has no solution.	No solutions	

Example 19

For each of the following pairs of equations, use your knowledge of linear graphs to determine the number of solutions. Explain your reasoning. (There is no need to find the actual solutions.)

a $y = 2x$
$y = 2x - 1$

b $2x - 7y = 3$
$2x + 7y = 3$

c $x - 2y = 2$
$2x = 4 + 4y$

d $x - y = 2$
$y - x = 2$

Working

a Since both lines have a gradient of 2, they are parallel. So there are no solutions.

Reasoning

Parallel lines never meet.

b $2x - 7y = 3$ can be written as
$y = \frac{2}{7}x - \frac{3}{7}$.
$2x + 7y = 3$ can be written as
$y = -\frac{2}{7}x + \frac{3}{7}$.
Since the lines have different gradients, there will be a unique solution, that is, one solution.

The equations can be represented as linear graphs with gradients of $\frac{2}{7}$ and $-\frac{2}{7}$ respectively.

There are two different equations.

continued

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Example 19 continued

Working

c $x - 2y = 2$ can be written as

$y = \frac{1}{2}x - 1.$

$2x = 4 + 4y$ can be written as

$y = \frac{1}{2}x - 1.$

Since both equations are identical, there will be an infinite number of solutions.

d $x - y = 2$ can be written as $y = x - 2$.
$y - x = 2$ can be written as $y = x + 2$.
Since both lines have a gradient of 1, they are parallel. So there are no solutions.

Reasoning

The two equations are equivalent, so any point on the graph of $y = \frac{1}{2}x - 1$ will provide a solution.

There is only one equation.

Parallel lines never meet.

exercise 5.3

LINKS TO Example 17

1 Solve each pair of simultaneous equations by making a table of values using $x = 1$, $x = 2$ and $x = 3$, and then plotting graphs of each equation on the same set of axes.

a $y = -x + 4$, $y = x - 2$

b $y = x - 3$, $y = -0.5x$

c $y = -x + 3$, $y = -2x + 6$

d $y = x$, $y = -x + 7$

e $y = x + 3$, $y = -2x + 9$

f $y = -6x + 13$, $y = 0.5x$

2 Use graphing software such as GeoGebra or a graphing calculator to graph the following pairs of equations and hence find the solution.

a $3x - 5y = 3$, $x = 2y$

b $5y = -x$, $5y + 15 = 2x$

c $x + 5y = 14$, $2y - x = 0$

d $y = 2x + 1$, $y = 7 - x$

e $y = x - 2$, $2y = x + 3$

f $x + y = 8$, $3x - y = 2$

LINKS TO Example 18

3 For each of the following pairs of equations, use a graphical approach to find any simultaneous solutions.

a $x - y = 1$, $x = 2y$

b $x - y = 3$, $x = y$

c $6x - 3y = 9$, $y = 2x - 3$

4 At what point do the graphs of $y = 3$ and $x = -2$ intersect?

5 Examine the following equations.

$2x + y = 5$ $\quad$ $x + 1 = 2y - 1$ $\quad$ $2y = -4x + 12$ $\quad$ $3x - 6y + 3 = 0$

a How many different pairs of equations can be chosen from these four equations?

b Which pairs of equations do you think will have no simultaneous solution? Explain.

c Rewrite the equations in the form $y = mx + b$ and graph them all on the same axes.

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d How many points of intersection are there between pairs of lines?

e Use a graphing calculator or GeoGebra to find the coordinates of each point of intersection in part **d**.

LINKS TO Example 19

6 For each of the following pairs of equations, determine the number of solutions. (Use your knowledge of linear graphs but do not actually graph the equations.)

a $y = 2 - 3x$, $y = 3x - 2$

b $2x - y = 3$, $2x = y - 5$

c $6x - 2y = 14$, $3x = 7 + y$

d $6x - 4y = 10$, $6y - 9x = 15$

7 Matt needs a plumber to repair some taps in his bathroom. He calls two plumbers for quotes.

a Pete's Plumbing charges a call-out fee of \$40 and then an hourly rate of \$40 per hour. Write the rule for the total cost T after h hours for Pete's Plumbing.

b Brad's Bathrooms charge a call-out fee of \$60 and then an hourly rate of \$30 per hour. Write the rule for the total cost T after h hours for Brad's Bathrooms.

c Matt thinks that the repair work will take one hour. Which plumber should he hire, and why?

d If Matt decides to have the taps replaced, rather than repaired, the work is likely to take 4 hours. Which plumber should he hire, and why?

e Graph both linear rules on the same set of axes. For what length of time would both plumbers charge the same total amount?

8 **a** Use a graphical approach to find a simultaneous solution to the equations $y = 8x + 2$ and $y = 9x + 3$.

b What was difficult about the graphical approach here?

9 Which pair of the following equations will have no simultaneous solutions?

$x + y = 3$ $\quad$ $y = 2$ $\quad$ $5 - y = x$ $\quad$ $2x = 3 - y$

10 Make up a pair of linear equations that you predict will have no simultaneous solution. Check your answer using a graphing calculator or CAS calculator.

exercise 5.3 challenge

11 **a** For what values of a will the simultaneous equations $x + y = 3$ and $2x + ay = 6$ have

i an infinite number of solutions? **ii** a unique solution?

b Is there a value of a for which the pair of equations would have no solutions? Explain.

12 **a** For what values of k will the simultaneous equations $x + y = 3$ and $-x - y = k$ have

i an infinite number of solutions? **ii** no solution?

b Is there a value of k for which the pair of equations would have a unique solution? Explain.

13 For what values of m and p will the simultaneous equations $x + y = 3$ and $3x + my = p$ have

a an infinite number of solutions? **b** no solution?

c a unique solution?

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5.4 Solving simultaneous linear equations: algebraic method

Using a graphical method to solve simultaneous equations is sometimes difficult. If we are using a graphing calculator to graph the equations, we may find that the intersection point is not in the viewing screen. We also find that if the equations have very similar gradients it is difficult to see the precise intersection point. Algebraic methods are a more accurate way to solve simultaneous equations. Two algebraic methods, the **substitution method** and the **elimination method**, can be used to solve two simultaneous linear equations.

The substitution method

Consider the two equations $x + 3y = 11$ and $x = 2$. These equations can be solved simultaneously in the following manner.

We can find the value of y by substituting 2 for x in the equation $x + 3y = 11$.

$$\begin{aligned} x + 3y &= 11 \\ 2 + 3y &= 11 && \text{[substituting } x = 2\text{]} \\ 3y &= 9 \\ y &= 3 \end{aligned}$$

After substituting, we have only one equation with one variable to solve.

So the solution is $x = 2$ and $y = 3$.

Now consider the equations $x + 3y = 11$ and $x = 2y + 1$.

We can find the value of y by substituting $2y + 1$ for x in the equation $x + 3y = 11$.

$$\begin{aligned} x + 3y &= 11 \\ 2y + 1 + 3y &= 11 && \text{[substituting } x = 2y + 1\text{]} \\ 5y + 1 &= 11 \\ 5y &= 10 \\ y &= 2 \end{aligned}$$

After substituting, we have only one equation with one variable to solve.

Once we know the value of y, we can find the value of x.

$$\begin{aligned} \text{Since } x &= 2y + 1 \\ x &= 2 \times 2 + 1 && \text{[substituting } y = 2\text{]} \\ &= 5 \end{aligned}$$

So the solution is $x = 5$ and $y = 2$.

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The substitution method is most commonly used when one of the unknowns is the **subject** in at least one of the equations. (That is, at least one equation begins $x = \ldots$ or $y = \ldots$) This makes it easy to substitute one equation into the other so that they can both be solved simultaneously. In the above example, the equation $x = 2y + 1$ is of this form.

Labelling simultaneous equations

When solving simultaneous equations, it is very important to set out your working in a logical manner, as shown in the following examples. It is very useful to label your equations—for example [1], [2], and so on—so that you can refer to them later in the solving process.

Example 20

Solve each pair of simultaneous equations by using the substitution method.

a $x + y = 2$
$x = -4y - 1$

b $x + 2y = 5$
$y = x + 1$

c $y = 5x + 4$
$y = 3x - 2$

Working	Reasoning
a $x + y = 2$ [1] $x = -4y - 1$ [2]	Label the equations [1] and [2].
Substitute $-4y - 1$ for x into [1].	
$-4y - 1 + y = 2$	Substituting for x gives an equation with only one unknown.
$-3y - 1 = 2$ $-3y = 3$ $y = -1$	Solve the equation for y.
Substitute $y = -1$ into [2].	Substitute the value of y into either one of the equations to find x. It makes sense to choose equation [2] because x is the subject of this equation.
$x = -4 \times (-1) - 1$ $= 4 - 1$ $= 3$	
Solution: $x = 3$ and $y = -1$	You can check your answer by substituting the values into the other equation.
Check: If $x = 3$ and $y = -1$, LS $= x + y$ $= 3 + (-1)$ $= 2$ $=$ RS	
b $x + 2y = 5$ [1] $y = x + 1$ [2]	Label the equations.
Substitute $x + 1$ for y in [1].	Substituting for y gives an equation with only one unknown.
$x + 2(x + 1) = 5$ $x + 2x + 2 = 5$	

continued

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Example 20 continued

Working	Reasoning
$3x + 2 = 5$ $3x = 3$ $x = 1$	Solve the equation for x.
Substitute $x = 1$ into [2]. $y = 1 + 1$ $= 2$	Substitute the value of x into either one of the equations to find x. It makes sense to choose equation [2] because y is the subject of this equation.
Solution: $x = 1$ and $y = 2$	
Check: If $x = 1$ and $y = 2$ LS $= x + 2y$ $= 1 + 4$ $= 5$ $=$ RS	You can check your answer by substituting the values into the other equation.
c $y = 5x + 4$ [1] $y = 3x - 2$ [2]	Label the equations.
Substitute $3x - 2$ for y in [1].	Substituting for y gives an equation with only one unknown.
$3x - 2 = 5x + 4$ $-2 - 4 = 5x - 3x$ $-6 = 2x$ $x = -3$	Solve the equation for x.
Substitute $x = -3$ into [2]. $y = 3 \times -3 - 2$ $= -9 - 2$ $= -11$	Substitute the value of x into one of the equations to find y.
Solution: $x = -3$ and $y = -11$	
Check: If $x = -3$ and $y = -11$, $y = 5x + 4$ LS $= -11$ RS $= 5 \times (-3) + 4$ $= -11$	You can check your answer by substituting the values into the other equation.

The elimination method

The **elimination method** is more commonly used to solve pairs of linear equations when neither variable is the subject of an equation. It involves combining the two equations in such a way that one of the variables can be **eliminated**.

Before we show how the elimination method is carried out, we look at the addition and subtraction of pairs of linear equations.

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We can add equations to get a new equation. For example

$$\begin{array}{lrl} & 3x + 2y = 7 & [1] \\ \text{Add} & \underline{x + 3y = 9} & [2] \\ & 4x + 5y = 16 & [1] + [2] \end{array}$$

We can also subtract equations to get a new equation. For example

$$\begin{array}{lrl} & 7x + 4y = 13 & [1] \\ \text{Subtract} & \underline{2x + 3y = 12} & [2] \\ & 5x + \ \ y = 1 & [1] - [2] \end{array}$$

We now see how this can be used to help us solve linear simultaneous equations. Notice what happens when the following pairs of equations are added. In each case the y terms are eliminated because they have the same coefficients but opposite signs. We are then left with a simple equation that can be solved for x. Once we know x we can substitute in one of the original equations to find y. In some pairs of simultaneous equations it may be the x terms that have the same coefficients and opposite signs. In this case x would be eliminated first so that we could find y.

$$\begin{array}{lrl} & 4x + y = -8 & [1] \\ \text{Add} & \underline{3x - y = -6} & [2] \\ & 7x = -14 & [1] + [2] \end{array}$$

$$\begin{array}{lrl} & 6x - 6y = 14 & [1] \\ \text{Add} & \underline{2x + 6y = 28} & [2] \\ & 8x = 42 & [1] + [2] \end{array}$$

Sometimes we may find that the coefficients of either the x terms or the y terms are the same but their signs are also the same. This means that adding the equations will not eliminate one of the unknowns. Instead, we can subtract one equation from the other to eliminate one of the unknowns.

$$\begin{array}{lrl} & 5x + 3y = 6 & [1] \\ \text{Subtract} & \underline{2x + 3y = -8} & [2] \\ & 3x = 14 & [1] - [2] \end{array}$$

$$\begin{array}{lrl} & 2x + y = 13 & [1] \\ \text{Subtract} & \underline{2x + 3y = 19} & [2] \\ & -2y = -6 & [1] - [2] \end{array}$$

The elimination method involves adding or subtracting simultaneous equations to obtain one new equation with one unknown. The equation can then be solved.

We need to be very careful with signs when we subtract equations.
$6 - (-8) = 14$

Solving simultaneous linear equations by elimination

When deciding whether to add or subtract equations to eliminate one of the variables, look at the signs of the coefficients.

- If the signs are the same, subtract the equations to eliminate a variable.
- If signs are different, add the equations.

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Algebraic method

$x + y = 4$ [1]

$x - y = 2$ [2]

$2x + 0y = 6$ [1] + [2]

$x = 3$

Substitute $x = 3$ into [1].

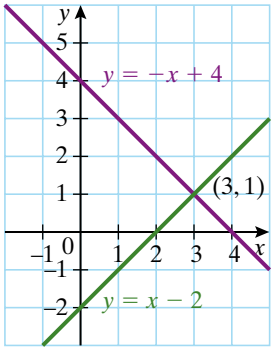
$y = 1$

So the solution is $x = 1$ and $y = 3$.

This means that both equations will be true if $x = 3$ and $y = 1$.

Graphical method

Rewrite the equations as $y = -x + 4$ and $y = x - 2$, so that you can use what you know about the gradient and y-intercept (or a calculator) to graph the lines.



The point of intersection corresponds to $x = 3$ and $y = 1$.

Example 21

Solve each pair of simultaneous equations by using the elimination method.

a $4x - 2y = -10$
$3x + 2y = 3$

b $3x + 4y = 13$
$2x + 4y = 10$

c $2x - y = 3$
$2x + 3y = -5$

Working

a $4x - 2y = -10$ [1]
$3x + 2y = 3$ [2]
$7x = -7$ [1] + [2]
$x = -1$

Substitute $x = -1$ into [2].

$3 \times (-1) + 2y = 3$
$-3 + 2y = 3$
$2y = 6$
$y = 3$

Reasoning

Label the equations.

One equation includes $-2y$ and the other includes $+2y$, so adding the equations will eliminate y. Solve for x.

Substitute this value of x into either of the original equations to find y.

continued

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Example 21 continued

Working	Reasoning
Solution: $x = -1, y = 3$ Check: Substitute $x = -1, y = 3$ in [1]. $\text{LS} = 4(-1) - 2(3)$ $= -4 - 6$ $= -10$ $= \text{RS}$	Check your solution by substituting into the other equation.
b $3x + 4y = 13$ [1] $2x + 4y = 10$ [2] $x = 3$ [1] − [2]	Label the equations. Both equations include $4y$, so subtracting will eliminate y. Here, this gives the value of x straight away.
Substitute $x = 3$ into [1]. $3(3) + 4y = 13$ $9 + 4y = 13$ $4y = 4$ $y = 1$	Substitute this value of x into either of the original equations to find y.
Solution: $x = 3, y = 1$ Check: Substitute $x = 3, y = 1$ in [2]. $\text{LS} = 2(3) + 4(1)$ $= 6 + 4$ $= 10$ $= \text{RS}$	Check your solution by substituting into the other equation.
c $2x - y = 3$ [1] $2x + 3y = -5$ [2] $-4y = 8$ [1] − [2] $y = -2$	Label the equations. Both equations include $2x$, so subtracting will eliminate x. Solve for y.
Substitute $y = -2$ into [2]. $2x + 3(-2) = -5$ $2x - 6 = -5$ $2x = 1$ $x = \frac{1}{2}$	Substitute this value of y into one of the original equations to find x.

continued

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Example 21 continued

Working

Solution: $x = \frac{1}{2}, y = -2$

Check:
Substitute $x = \frac{1}{2}, y = -2$ in [1].

$$\begin{aligned} \text{LS} &= 2\left(\frac{1}{2}\right) - (-2) \\ &= 1 + 2 \\ &= 3 \\ &= \text{RS} \end{aligned}$$

Reasoning

Note that the solutions are not always whole numbers. Check your solution by substituting into the other equation.

Extending the elimination method

It is not always possible to eliminate one of the unknowns straight away by adding or subtracting the equations. An unknown can only be eliminated if it has the same coefficient in each equation. In some cases, we need to multiply at least one of the equations by a number in order to create coefficients that are equal.

In example **22** part **a** and **22** part **b**, only one of the equations needs to be multiplied by a number to create equal coefficients. In example **22** part **c**, both equations need to be multiplied by a number to create equal coefficients.

Example 22

Solve the following pairs of simultaneous equations, using the elimination method.

a $3x + 2y = 11$
$2x - y = 5$

b $x + 3y = 5$
$3x - 4y = -11$

c $2x + 5y = 3$
$3x - 4y = -7$

Working

a

$$\begin{aligned} 3x + 2y &= 11 \quad [1] \\ 2x - y &= 5 \quad [2] \\ 4x - 2y &= 10 \quad [3] = [2] \times 2 \end{aligned}$$

$$\begin{aligned} 7x &= 21 \quad [1] + [3] \\ x &= 3 \end{aligned}$$

Substitute $x = 3$ into [1].

$$\begin{aligned} 3(3) + 2y &= 11 \\ 9 + 2y &= 11 \\ 2y &= 2 \\ y &= 1 \end{aligned}$$

Reasoning

Label the equations.

Multiply equation [2] by 2 to make the coefficient of y the same as in [1].
Number the new equation.
Adding equations [1] and [3] eliminates y, so we can solve for x.

Substitute this value of x into one of the original equations to find y.

continued

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Example 22 continued

Working	Reasoning
Solution: $x = 3, y = 1$ Check: Substitute $x = 3, y = 1$ in [2]. $\text{LS} = 2(3) - 1$ $= 6 - 1$ $= 5$ $= \text{RS}$	Check your solution by substituting in the other equation.
b $x + 3y = 5$ [1] $3x - 4y = -11$ [2] $3x + 9y = 15$ [3] = [1] × 3	Label the equations. Multiply equation [1] by 3 to make the coefficient of x the same as in [2]. Number the new equation.
$13y = 26$ [3] − [2] $y = 2$	Subtracting equation [2] from [3] eliminates x, so we can solve for y.
Substitute $y = 2$ into [1]. $x + 3(2) = 5$ $x + 6 = 5$ $x = -1$	Substitute this value of y into one of the original equations to find x.
Solution: $x = -1, y = 2$ Check: Substitute $x = -1, y = 2$ in [2]. $\text{LS} = 3(-1) - 4(2)$ $= -3 - 8$ $= -11$ $= \text{RS}$	Check your solution by substituting in the other equation.
c $2x + 5y = 3$ [1] $3x - 4y = -7$ [2] $8x + 20y = 12$ [3] = [1] × 4 $15x - 20y = -35$ [4] = [2] × 5 $23x = -23$ [3] + [4] $x = -1$	Multiplying equation [1] by 4 and equation [2] by 5 gives two new equations with $+20y$ and $-20y$. Adding the two new equations then eliminates the y terms. Solve for x.
Substitute $x = -1$ into [1]. $2(-1) + 5y = 3$ $-2 + 5y = 3$ $5y = 5$ $y = 1$	Find the value of x, then substitute into one of the original equations to find y. Here, $x = -1$ is substituted into equation [1].

continued

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Example 22 continued

Working

Solution: $x = -1, y = 1$
Check:
Substitute $x = -1, y = 1$ in [2].
$\text{LS} = 3(-1) - 4(1)$
$= -3 - 4$
$= -7$
$= \text{RS}$

Reasoning

The solution is checked by substituting the values for x and y into the other equation.

exercise 5.4

LINKS TO Example 20

1 Solve each of the following pairs of simultaneous linear equations by using the substitution method.

a $x + 2y = 8$, $y = x + 1$
b $y = 2x + 21$, $y = 3 - 7x$
c $3x + 2y = 11$, $y = -x + 4$
d $3x - 5y = 5$, $y = -x + 3$
e $x + y = 1$, $y = x - 1$
f $3x + y = 13$, $y = 4x - 8$
g $4x + y = 13$, $y = 7 - 2x$
h $3x - 2y = 5$, $x = 9 - 3y$

LINKS TO Example 21

2 Solve the following simultaneous linear equations by using the elimination method.

a $x + y = 1$, $x - y = 1$
b $4x + y = 13$, $2x + y = 7$
c $3x - 2y = 5$, $-3x - 9y = -27$
d $4x - y = 8$, $3x + y = 13$
e $2x + y = 9$, $x - y = 3$
f $3x + y = 7$, $2x - y = 8$
g $2x + 3y = 13$, $4x - 3y = -1$
h $x + y = -3$, $x - y = -1$
i $2x + y = 14$, $x - y = 1$
j $3x + y = 10$, $2x - y = 10$
k $x + y = 7$, $x - y = 1$
l $4x + 3y = 5$, $x - 3y = -10$

3 Solve the following simultaneous linear equations by using either the substitution or elimination method.

a $5m + 3n = 7$, $m - n = 3$
b $j + k = 4$, $3j + 2k = 11$
c $5q = 5 + 3r$, $5q = 10 - 3r$
d $x = 2 - y$, $y = 3 - 2x$

LINKS TO Example 22ab

4 Solve the following simultaneous equations by using the elimination method.

a $x + y = 5$, $3x + 2y = 11$
b $b - c = 9$, $2b + 2c = 2$
c $5y - 3z = 5$, $10y + 6z = 20$
d $4x + y = 17$, $5x - 2y = 31$
e $-2a + b = -3$, $5a + 2b = 12$
f $2a + 3b = 8$, $4a + 9b = 22$
g $x - 2y = 6$, $3x + y = 11$
h $5x - y = 6$, $x - 2y = 3$
i $3x + 2y = 2$, $2x - y = -8$
j $4x + 3y = -8$, $x + y = -3$
k $3x + 2y = 16$, $x - y = 2$
l $5x + 3y = 12$, $x + 4y = -1$

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LINKS TO Example 22c

5 Sometimes it is necessary to perform an operation on *both* equations before applying the elimination method. Do this to help solve the following pairs of equations simultaneously.

a $2x - 3y = 1$
$3x + 2y = 8$

b $4x + 3y = 26$
$3x - 4y = 7$

c $5x + 2y = 8$
$x - 3y = 5$

d $2x + 3y = 10$
$5x + 7y = 17$

e $-2x + 5y = 9$
$3x + 2y = 15$

f $2x - 6y = -16$
$3x + 5y = 25$

g $3x - 5y = 25$
$2x + 3y = 4$

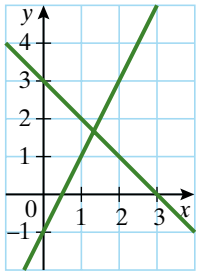
h $2x - 3y = 18$
$3x + 8y = 2$

i $4x + 3y = 11$
$3x + 4y = 10$

6 The point of intersection for the two linear graphs shown below is difficult to determine accurately from the graph.

a By finding the gradient and y-intercept, find the equation for each line.

b Solve the equations simultaneously to find the precise coordinates of the point of intersection.



exercise 5.4 challenge

7 Solve each of the following pairs of simultaneous equations.

a $-2x + 7y = 1$
$3x - 6y = 0$

b $x = 3y + 2$
$y = 2x + 3$

c $-5x + 7y = 2$
$3x - 5y = -3$

d $0.5x + 1.5y = 1$
$2.5x - 5y = 0$

8 Solve each of the following pairs of simultaneous equations.

a $\frac{x}{2} + \frac{y}{3} = -1$
$\frac{x}{5} - \frac{y}{15} = 1$

b $y = \frac{2x + 4}{3}$
$x = \frac{4y - 7}{2}$

9 To find the rule for a straight line through two given points, you can find the gradient and then substitute to find the y-intercept. The rule can also be found by using simultaneous equations as follows.

- Substitute the x- and y-coordinates from one point into the equation $y = mx + b$. For example, for the point (5, 3):

$$y = mx + b$$
$$3 = m(5) + b$$
$$5m + b = 3$$

- Repeat this step for the other point.
- Solve the simultaneous equations to find m and b.

a Use this approach to find the equation of the line that passes through the points (1, −2) and (3, 4).

b Check your answer by graphing the equation of the line that you found in part **a**.

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5.5 Applications of simultaneous linear equations

As we have seen, simultaneous equations can be solved graphically or algebraically. If the linear graphs are drawn by hand, it is not always easy to accurately determine the coordinates of the point of intersection, so the algebraic method is preferred. However, if we have access to graphing software such as GeoGebra or graphing or CAS calculators, the intersection point can be determined efficiently.

To solve a problem that is presented in words, we **formulate** a mathematical equation and then solve it. The principles that you have used previously to solve word problems that involved a single linear equation can also be applied when dealing with two variables and simultaneous equations.

Solving simultaneous equations from worded problems

Step 1 *Define the two variables*
Identify the quantities that you need to find, and choose appropriate pronumerals to represent them. Write 'let' statements that define these variables. Diagrams may be useful, especially for problems involving measurement.

Step 2 *Construct two different linear equations using the two variables*
Translate the written information into mathematical language. The objective is to develop a pair of equations, each with the two variables from step 1. There should be two pieces of information that can be used for this purpose.

Step 3 *Solve the equations simultaneously*
- Graphical solution: graph the two equations and find the point of intersection.
- Algebraic solution: use either the substitution method or the elimination method to find the point of intersection.

Step 4 *Interpret the solution within the context of the problem*
Interpret your results by expressing your answer in words, according to the original information presented in the problem.

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Example 23

The sum of two whole numbers is 18, and their difference is 6. Find the two numbers.

Working	Reasoning
Let the smaller number be x, and the larger number be y.	Step 1: Define the variables Use 'let' statements to define the two variables.
$x + y = 18$ $y - x = 6$	Step 2: Construct the equations Write equations that mean 'The sum of the numbers is 18' and 'the difference of the numbers is 6'. As the difference is (6) is positive and y is the larger number, write $y - x$ to show the difference.
$x + y = 18$ [1] $-x + y = 6$ [2] $2y = 24$ [1] + [2] $y = 12$	Step 3: Solve the equations Rearrange the equations so that the elimination method can be more easily applied. Adding the equations eliminates x, so we can solve for y.
Substituting $y = 12$ in [1] gives $x + 12 = 18$ $x = 6$	
So $x = 6$ and $y = 12$ is the solution.	
The two numbers that have a sum of 18 and a difference of 6 are 12 and 6.	Step 4: Interpret the solution The problem was to find two numbers whose sum is 18 and difference is 6. The numbers 12 and 6 fit both those requirements.

Example **24** shows question **7** in exercise **3.3** solved with simultaneous equations.

Problems that involve comparing changes based on a fixed amount and an amount based on time can be solved with simultaneous equations.

Example 24

Matt needs a plumber to repair some taps in his bathroom. He calls two plumbers for quotes.

a Pete's Plumbing charges a call-out fee of \$40 and then an hourly rate of \$40 per hour. Write the rule for the total cost T after h hours for Pete's Plumbing.

b Brad's Bathrooms charge a call-out fee of \$60 and then an hourly rate of \$30 per hour. Write the rule for the total cost T after h hours for Brad's Bathrooms.

continued

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Example 24 continued

c Matt thinks that the repair work will take one hour. Which plumber should he hire, and why?

d If Matt decides to have the taps replaced, rather than repaired, the work is likely to take 4 hours. Which plumber should he hire, and why?

e Graph both functions on the same set of axes. For what length of time would both plumbers charge the same total amount?

f Solve the two equations from parts **a** and **b** algebraically and interpret the solution.

Working	Reasoning
a $T = 40h + 40$ **b** $T = 30h + 60$	Step 1: The variables h and T are defined in the question. Step 2: Construct the two linear equations.
c Pete's Plumbing: $T = 40h + 40$ $= 40 \times 1 + 40$ $= 80$ Brad's Bathrooms: $T = 30h + 60$ $= 30 \times 1 + 60$ $= 90$ For 1 hour, Matt should choose Pete's Plumbing as it is \$10 cheaper.	Substitute $h = 1$ in each equation to find the total cost for 1 hour.
d Pete's Plumbing: $T = 40h + 40$ $= 40 \times 4 + 40$ $= 200$ Brad's Bathrooms: $T = 30h + 60$ $= 30 \times 4 + 60$ $= 180$ For 4 hours, Matt should choose Brad's Bathrooms as it is \$20 cheaper.	Substitute $h = 4$ in each equation to find the total cost for 4 hours.

continued

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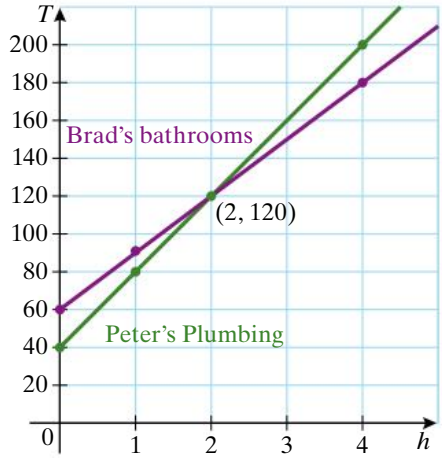
Example 24 continued

e



Step 3: Graph the equations.

Choose a suitable scale.
Label the axes.
Plot the points found in parts **c** and **d** (1, 80), (1, 90), (4, 180) and (4, 200).
Find the y-intercept for each line and plot.
Rule each line and label.
Find the coordinates of the point of intersection from the graph.

The plumbers would charge the same amount ($120) for 2 hours.

Step 4: Interpret the solution to the simultaneous equations in the context of the problem.

f $T = 40h + 40$ [1]
$T = 30h + 60$ [2]

As both equations start with $T =$, the substitution method is suitable.

Substitute $40h + 40$ for T in [2]

$40h + 40 = 30h + 60$
$10h = 20$
$h = 2$

Solve for h.

Substitute $h = 2$ in [1]

$T = 40 \times 2 + 40$
$= 120$

Substitute $h = 2$ to find T.

So both plumbers would charge the same amount of $120 for 2 hours work.

Example 25

Fran's mobile phone bill for January showed a charge of $63 for 80 minutes of calls and 60 text messages (SMS). The previous month, Fran was charged $65 for 100 minutes of call time and 20 SMS. Find the cost per minute Fran is charged for calls, and the cost for each SMS. (Assume that these rates were the same each month.)

Working

Let $m be the cost per minute of calls.
Let $s be the cost for each SMS.

Reasoning

Step 1: Define the variables
Use 'let' statements to define the variables.

continued

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Example 25 continued

Working

$80m + 60s = 63$ [1]
$100m + 20s = 65$ [2]

Reasoning

Step 2: Construct the equations
On the bill for January, 80 minutes of calls would cost \$80m, and 60 SMS would cost \$60s. As the total cost was \$63, $80m + 60s = 63$.

Using similar logic, the December bill gives the equation $100m + 20s = 65$.

$$\begin{aligned} 80m + 60s &= 63 && [1] \\ 300m + 60s &= 195 && [3] = [2] \times 3 \\ 220m &= 132 && [3] - [1] \\ m &= \frac{132}{220} = 0.6 \end{aligned}$$

Step 3: Solve the equations
Multiply equation [2] by 3 to make the coefficients of s the same in both equations. The variable s can then be eliminated by subtracting equations.

Solve for m.

Substituting $m = 0.6$ into [2] gives

$$\begin{aligned} 100(0.6) + 20s &= 65 \\ 60 + 20s &= 65 \\ s &= \frac{5}{20} = 0.25 \end{aligned}$$

So $m = 0.6$ and $s = 0.25$ is the simultaneous solution.

Check:
Substitute $m = 0.6, s = 0.25$ in [1]
$$\begin{aligned} \text{LS} &= 80(0.6) + 60(0.25) \\ &= 48 + 15 \\ &= 63 \\ &= \text{RS} \end{aligned}$$

Fran's calls cost 60c per minute, and each SMS costs 25 cents.

Step 4: Interpret the solution
The problem asked for the cost of calls and SMS. In this case, it makes sense to give the answers in cents.

Example 26

A total of 320 people attended a concert. The full price for a ticket was \$45 and the student concession price was \$30. If the total takings from ticket sales was \$11 625, how many student tickets and how many full price tickets were sold?

Working

Let x be the number of student tickets.
Let y be the number of full price tickets.

Reasoning

Define the variables.

continued

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Example 26 continued

Working	Reasoning
$x + y = 320$ [1] $30x + 45y = 11\,625$ [2]	The total number of students and full price tickets sold was 320. x tickets at \$30 = \$30x y tickets at \$45 = \$45y
$45x + 45y = 14\,400$ [3] = [1] × 45 $15x = 2775$ [3] − [2] $x = 185$	Multiply equation [1] by 45 to make the coefficients of y the same. When [2] is subtracted from [3], y is eliminated so we can solve for x.
Substitute $x = 185$ into [1] gives: $185 + y = 320$ $y = 320 - 185$ $y = 135$	We could also have multiplied [1] by 30 to make the coefficients of x the same. Then x would be eliminated and we would find y first.
There were 185 student tickets and 135 full price tickets sold.	Write a sentence to answer the question.

exercise 5.5

For each of the following word problems use these steps.

Step 1: Define two variables.

Step 2: Construct two different linear equations with those variables.

Step 3: Solve the equations simultaneously.

Step 4: Interpret the solution within the context of the problem.

LINKS TO Example 23

1. The sum of two whole numbers is 48 and their difference is 6. Find the numbers.
2. The difference of two positive numbers is 11, and one number is 3 more than twice the other. Find the numbers.
3. In a class of 24 students, there are twice as many girls as boys. How many boys and how many girls are in the class?
4. Julie's age is 2 years more than 5 times her son's age. The sum of Julie's age and her son's age is 38. How old is Julie's son?

LINKS TO Example 24

5. The Cheap Cars company charges \$50 per day and \$2.40 per kilometre to hire a car. The Best Hire company charges \$40 per day plus \$2.70 per kilometre. Will wishes to rent a car for 3 days. How far can he travel so that the cost would be the same from either company? Let \$$C$ be the cost and d km be the distance travelled and write a pair of simultaneous equations.

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LINKS TO Example 25

6 Annika uses phone cards for cheap calls to Denmark. Card A costs her 18 cents plus 2 cents per minute. Card B costs her 24 cents plus 1.5 cents per minute. For what length of call would the cost be the same for both cards?

LINKS TO Example 26

7 A total of 1200 people attended the school play. Each adult paid $5 and each student paid $2. Total ticket sales amounted to $3150. How many adults attended?

8 Luke plays cricket and has batted twice for the season so far. His batting average for these two innings is 72. If he made twice as many runs in his second innings as in his first innings, how many runs did he make in each innings?

9 Katrina and Greg have two children, Alannah and Oscar. Alannah's age is 12 years less than 4 times Oscar's age. Oscar's age is 19 years less than 3 times Alannah's age. How old are the children?

10 At one particular screening of 'Lord of the Rings', a cinema sold a total of 300 tickets and the takings amounted to exactly $2400. If adult entry was $9 and children were charged $6.50, how many adults and how many children attended the screening?

11 Rosy keeps ants and spiders as pets. She counted all their heads and legs. Altogether there were 25 heads and 166 legs. Use simultaneous equations to find how many ants and how many spiders Rosy had.

12 Kerensa counted the heads and legs of her chickens and cows. Altogether there were 18 heads and 52 legs. How many chickens and how many cows did Kerensa have?

13 Sam has been saving 10 cent coins and 20 cent coins in a jar. When he takes the jar to the bank, he finds that there are 252 coins worth a total of $43.20. How many of each coin are there?

exercise 5.5 challenge

14 Maria is now four times as old as Kate. Four years ago, Maria was six times as old as Kate. How old are Maria and Kate now?

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Analysis task

Carly's jeans

Carly is starting a new weekend job. She has been offered the choice of two different pay schemes. For both schemes, she will receive a fixed amount of money per week and then an additional amount of money for each hour that she actually works. The pay scales are shown in the table and Carly needs to decide which pay scheme she will choose.

Pay scheme	Fixed amount per week	Extra pay per hour worked
A	\$10	\$4.30
B	\$20	\$2.90

a Using $P(\$)$ for the amount that Carly earns and n (hours) for the time Carly works in a week, write an equation to show how much Carly earns for each pay scheme.

b Use the same set of axes to graph each of the equations. Use values of n from 0 to 30 and values of P from 0 to 110.

c Use the graph to find which pay scheme would be better if Carly works

- **i** 3 hours per week
- **ii** 5 hours per week
- **iii** 10 hours per week

d Use the graph to find how many hours Carly would have to work to earn the same amount of money for each pay scheme.

e Solve the equations simultaneously to check your answer to part **d**. Find the values of n and P, correct to two decimal places.

f Carly wants to buy a new pair of jeans that cost \$95. For each pay scheme calculate how many hours Carly would have to work for each pay scheme to earn this money. Round to the nearest number of whole hours.

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Review *Linear graphs and simultaneous equations*

Summary

Linear graphs

- The equation of a straight line can be written as $y = mx + b$, where m is the gradient and b is the y-intercept, that is, the point where the line crosses the y-axis. We can also say that the y-intercept is the point $(0, b)$.
- The equation $y = mx + b$ as the *gradient-intercept form* of the straight line equation.
- The equation $ax + by + c = 0$, where a, b and c are constants, is referred to as the *general equation* for a straight line.
- Parallel lines have the same gradient.
- The product of the gradients of two perpendicular lines is -1. In other words, if two lines are perpendicular, the gradient of one line is the negative reciprocal of the gradient of the other line.

  $m_1 \times m_2 = -1$ or $m_2 = -\dfrac{1}{m_1}$

  Conversely, if the product of the gradients of two lines is -1, then the lines are perpendicular.
- Only two points are needed to specify the position of a straight line. We can sketch the graph of any linear equation if we know two points on the line. The two intercepts are the easiest points to use.
- Horizontal and vertical lines

  The general equation for a vertical line is $x = a$. The gradient of a vertical line is undefined.

  The general equation for a horizontal line is $y = b$. The gradient of a horizontal line is zero.

Solving simultaneous linear equations: Graphical method

Graph each equation and locate any points of intersection. The coordinates of the point of intersection is the simultaneous solution to the pair of equations.

A pair of simultaneous equations can have

- a unique solution: lines with different gradients intersect at one point.
- an infinite number of solutions: identical lines are coincident and intersect at every point
- no solutions: lines with the same gradient are parallel and do not intersect.

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Solving simultaneous linear equations: Algebraic methods

- Substitution method: Substitute one equation into the other to create a new equation with a single variable. Solve that equation. Then substitute the solution into one of the original equations to find the other variable.
- Elimination method: Combine the equations, using addition or subtraction (and multiplication if necessary) to eliminate one of the variables. This creates a new equation with one unknown that can be solved. Use substitution to find the other unknown.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

algebraic solution	graphical solution	simultaneous equations
break-even point	intercept	sketch graph
elimination method	parallel lines	substitution method
gradient	perpendicular lines	unique solution

Revision

Multiple-choice questions

1 The gradient of the line through the points $(-4, 3)$ and $(6, 9)$ is:

A 3 **B** -3 **C** 6 **D** 0.6 **E** 1.2

2 The gradient of a line perpendicular to the line $2x + 5y = 6$ is:

A $-\frac{1}{2}$ **B** $-\frac{1}{5}$ **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** $\frac{5}{2}$

3 Which of the following is not an equivalent equation to $4y - 2x = 6$?

A $4y = 2x + 6$ **B** $2y - x = 3$ **C** $0 = x - 2y + 3$
D $y = \frac{x}{2} + 3$ **E** $y = \frac{x + 3}{2}$

4 A linear graph is parallel to the line $x - 2y = 6$ and passes through the point $(1, 1)$. The equation of the line is:

A $2y = x + 1$ **B** $x - 2y = 1$ **C** $2x - y = 1$
D $y = 5 - 2x$ **E** $2y + x = 5$

5 A linear graph is perpendicular to the line $x - 3y = 4$ and passes through the point $(1, 2)$. The equation of the line is:

A $y = \frac{x}{3} + 5$ **B** $3x + y = 5$ **C** $y = 3x + 4$
D $y = 5 - \frac{x}{3}$ **E** $x + \frac{y}{3} = 4$

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Short-answer questions

6 For each of the following linear rules, state

i the gradient

ii the y-intercept

a $y = -3x + 7$ **b** $2x - 5y = 18$ **c** $y = \frac{5x}{3}$ **d** $4x + 3y + 2 = 0$

7 Sketch the graph for each of these linear rules.

a $y = 2x + 5$ **b** $y = -4x$ **c** $2x - 3y = 6$

d $y = 3$ **e** $x = -5$ **f** $y = -2x - 4$

8 State the gradient, m, of lines that are perpendicular to the following lines.

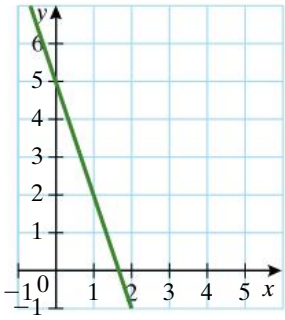
a $y = -2x + 3$ **b** $y = x + 3$ **c** $y = \frac{2x}{5} + 4$ **d** $x - 2y = 7$

9 Show that the line $y = \frac{2x}{3} - 5$ is perpendicular to the line $6x + 4y = 11$.

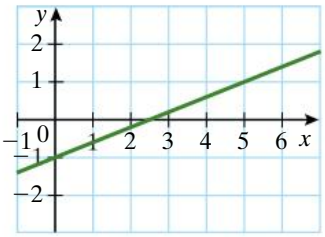
10 Find the equation for the line perpendicular to the line $y = \frac{x}{4} - 25$ and passing through the point $(-5, -2)$.

11 Find the equation for each of these linear graphs.

a



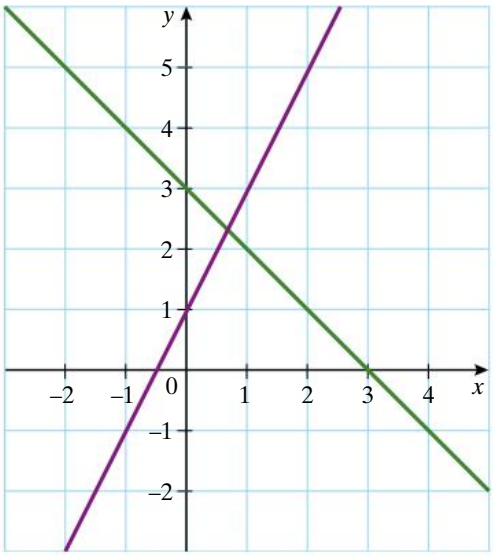
b



12 The point of intersection for the two linear graphs shown at the right is difficult to determine accurately from the graph.

a By finding the gradient and y-intercept, find the equation for each line.

b Solve the equations simultaneously to find the precise coordinates of the point of intersection.



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13 Find a simultaneous solution for each of these pairs of equations.

a $y = 2x + 3$
$y = 3 + x$

b $x = y - 6$
$x + y = 10$

c $x - y = 0$
$x + y = 0$

d $x + y = 3$
$3x - 2y = 9$

e $-2x + 4y = 5$
$22x - 8y = -19$

f $3x + 2y = 15$
$2x + 3y = 15$

g $2x - 5y = 29$
$3x + 4y = 9$

h $3x - 4y = 22$
$2x - 5y = 10$

i $3y - 7x = 3$
$7y + 3x = 65$

14 Use your knowledge of linear functions to determine the number of solutions to the following equations (without trying to solve them).

a $3x + y = -1$
$3x + y = 2$

b $4x + 8y = 40$
$x + 2y = 10$

c $x - y = 1$
$x + y = -1$

d $2x - y = 7$
$-2x + y = 7$

Extended-response question

15 Mandy is the treasurer of her tennis club. She cannot remember the amounts that were set as tournament entry fees for adults and for children. Records show that 80 adults and 50 children played on a Saturday when the total takings were \$730. On the following Saturday, 90 adults and 40 children played and the takings were \$740.

Set up and solve a pair of simultaneous equations to find the entry fees for adults and for children.

Practice quiz
Chapter 5

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Simple and compound interest

6

Pre-test

Warm-up

In this chapter, compound interest is calculated as repeated simple interest, leading into the compound interest formula.

6.1 Reviewing simple interest

Interest is charged if you borrow money. Interest is paid to you if you invest money, that is, if you lend your money. The amount of money borrowed or invested is called the **principal**. Interest that is calculated as a flat rate of the total amount borrowed or invested for the full term of the loan or investment, is called **simple interest**. The interest rate is usually advertised as a certain percentage per annum (p.a.).

Simple interest

$I = PRT$

where $\$I$ is the interest, $\$P$ the principal, R the annual interest rate and T the number of years

Note that R is a percentage, for example, if the interest rate is 6%, then $R = 6\% = \frac{6}{100} = 0.06$.

Example 1

Hannah invests \$5600 for 5 years at 6.25% p.a. interest.

a How much interest will Hannah earn?

b How much will Hannah have altogether after 5 years?

Working	Reasoning
a $I = PRT$ $= 5600 \times 6.25\% \times 5$ $= 5600 \times 0.0625 \times 5$ $= 1750$ Hannah earns \$1750 interest on the investment.	The principal is \$5600, the interest rate is 6.25% and the time is 5 years. So $P = 5600$, $R = 0.0625$, $T = 5$. Find I.
b Total $=$ \$5600 $+$ \$1750 $=$ \$7350	The principal \$5600 and the interest \$1750 is the amount altogether at the end of the investment period.

If we know any three of the four variables I, P, R or T we can calculate the fourth.

Calculating the time

Example 2 shows calculation of the time taken to earn a given amount of interest by substituting known values for I, P and R.

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Example 2

Harvey invests \$6000 at 6% per annum interest. How many years will it be before Harvey's investment has reached \$8000?

Working

$$\begin{aligned} \text{Interest} &= 8000 - \$6000 \\ &= \$2000 \\ I &= PRT \\ 2000 &= 6000 \times 0.06 \times T \\ T &= \frac{2000}{6000 \times 0.06} \\ T &= 5.55\ldots \end{aligned}$$

It would take 6 years for the investment to grow to \$8000.

Reasoning

Required interest $= \$8000 - \$6000$
$= \$2000$

Substitute $I = 2000$, $P = 6000$, $R = 0.06$. Calculate T.

Round up to 6 years. In 5 years only \$1800 would be earned.

Calculating the principal

Example 3 shows calculation of the required principal by substituting known values for I, P and T.

Example 3

What principal would earn \$450 interest if invested at 6.25% simple interest for 3 years?

Working

$$\begin{aligned} I &= PRT \\ 450 &= P \times 6.25\% \times 3 \\ 450 &= P \times 0.0625 \times 3 \\ P &= \frac{450}{0.0625 \times 3} \\ P &= 2400 \end{aligned}$$

The required principal is \$2400.

Reasoning

Find P when $I = 450$, $R = 0.0625$, $T = 3$.

Calculating the interest rate

Example 4 shows calculation of the required simple interest rate by substituting known values for I, P and T.

Example 4

What interest rate per annum would give \$300 interest if \$800 is invested for 5 years?

continued

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Example 4 continued

Working

$I = PRT$

$300 = 800 \times R \times 5$

$R = \dfrac{300}{800 \times 5}$

$R = 0.075$

The required interest rate is 7.5% p.a.

Reasoning

Find R when $I = 300$, $P = 800$, $T = 5$

Convert the interest rate to a percentage.

Loan repayment

When money is borrowed, interest is paid on the loan. Sometimes this may be simple interest, for example car loans.

Example 5

Joan borrows \$1500 at 9% per annum simple interest for 2 years.
Calculate the interest Joan pays on the loan.

Working

$I = PRT$

$= 1500 \times 9\% \times 2$

$= 1500 \times 0.09 \times 2$

$I = 270$

Joan pays \$270 interest.

Reasoning

\$1500 is the principal.
Find I when $P = 1500$, $R = 0.09$, $T = 2$

Loans are usually paid off by making equal weekly or monthly instalments. To calculate the amount of the instalments, we calculate the total amount, including interest, to be paid and divide it by the number of instalment periods.

Example 6

\$8000 is borrowed at 7.2% p.a. simple interest over 4 years. Calculate

a the interest paid

b the monthly repayments.

Working

a $P = 8000$, $R = 0.072$, $T = 4$

$I = P \times R \times T$

$= 8000 \times 0.072 \times 4$

$I = 2304$

The interest to be paid is \$2304.

Reasoning

Substitute for P, R and T.

continued

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Example 6 continued

Working

b Total amount to be repaid
= \$8000 + \$2304
= \$10 034
\$10 034 ÷ 48 = \$209.04
The estimated monthly repayment is \$209.04

Reasoning

To find the total amount to be repaid, add the interest to the amount of the loan.
To find the monthly repayment, divide by the number of months in 4 years.
4 years = 48 months
Notice that 48 payments of \$209.04 would be \$10 033.92. The first or last monthly repayment would probably be adjusted so that the total payment is exactly \$10 034.

Example 7

A home theatre system is advertised at \$4950. The Nguyen family buy the theatre system under a scheme where they pay a deposit of 20% and 24 equal monthly instalments of \$195.

a What deposit do the Nguyens pay?
b What is the total amount the Nguyens pay?
c How much more do they end up paying than if they had paid the full amount of \$4950 when they bought the system?
d The amount in part **c** represents the interest on the \$4950 paid by the Nguyens over the two years. What annual rate of simple interest on \$4950 does this represent?

Working

a 20% of \$4950 = \$990
The Nguyens pay \$990 deposit.

b 24 instalments of \$195 = \$4680
Total amount paid = \$990 + \$4680
= \$5670

c \$5670 − \$4950 = \$720
The Nguyens pay \$720 more than if they had paid the full amount at the start.

d $P = 4950, I = 720, T = 2$

$$I = P \times R \times T$$
$$720 = 4950 \times R \times 2$$
$$R = \frac{720}{4950 \times 2}$$
$$\approx 0.0727\ldots$$

The annual rate of simple interest is 7.27%.

Reasoning

The deposit is 20% of \$4950.

Multiply the amount of each instalment by the number of instalments.
Add the deposit and the amount paid in instalments.

To find the interest the Nguyens paid, subtract the cost of the theatre system from the total amount they paid.

The extra \$720 that the Nguyens paid represents the interest on \$4950 over 2 years.
Substitute for I, P, and T to find R.

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Monthly or daily interest rates

Interest rates are usually given as annual (per annum) rates. Sometimes, though, the interest may be calculated on a monthly or daily basis.

Example 8

Express the interest rate of 7.2% p.a. as

a a monthly rate.

b a daily rate correct to six decimal places.

Working

a $7.2\%\text{ p.a.} = \frac{7.2}{12}\%\text{ per month}$

$= 0.6\%\text{ per month}$

b $7.2\%\text{ p.a.} = \frac{7.2}{365}\%\text{ per day}$

$\approx 0.019726\%\text{ per day}$

Reasoning

Monthly interest rates are calculated on the basis of one-twelfth of the annual rate even though the months vary in length.

If the daily rate is quoted, then it is likely to be rounded to six decimal places.

Sometimes the exact rate of $\frac{7.2}{365}\%$ will be used.

Example 9

Calculate the interest earned on a term deposit of \$6000 invested at 5.8% for 9 months.

Working

$I = PRT$

$= 6000 \times \frac{0.058}{12} \times 9$

$I = 261$

\$261 interest is earned.

Reasoning

$5.8\% = 0.058$

Monthly interest rate $= \frac{0.058}{12}$

Alternatively, we can use the annual interest rate of 0.058. The time is then $\frac{9}{12}$ of the year. i.e, $R = 0.058$ and $T = \frac{9}{12}$.

Credit card interest

Interest on overdue credit card accounts or on credit card cash advances is charged per day.

Example 10

A credit card company has an annual interest rate of 17.74%. Calculate the interest owing on a cash advance of \$800 if the credit card account is not paid for 23 days after the cash advance was obtained.

continued

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Example 10 continued

Working

$I = PRT$

$= 800 \times \frac{0.1774}{365} \times 23$

$I = 8.9429\ldots$

\$8.94 interest is owed.

Reasoning

$17.74\% = 0.1774$

Daily interest rate $= \frac{0.1774}{365}$

exercise 6.1

LINKS TO Example 1

1 Calculate the simple interest earned on these investments.

- **a** \$580 at 4.25% interest p.a. for 4 years.
- **b** \$300 at 5.8% interest p.a. for 3 years
- **c** \$5000 at 6.4% interest p.a. for 5 years
- **d** \$12 000 at 6.25% interest p.a. for 8 years

2 Madeleine invests \$3600 for 4 years at 5.75% p.a. simple interest. How much interest will Madeleine earn?

LINKS TO Example 2

3 Alex deposits \$1500 into a fixed-interest savings account, which receives 4.75% simple interest per annum. The interest is paid at the end of each year.

- **a** How much interest will Alex earn after one year?
- **b** How many years will it take for Alex to earn over \$300 interest?

4 Carly invests \$4000 at 5.25% simple interest per annum. How long will it take for Carly's money to grow to \$6000?

5 How long would it take for an investment of \$1400 to increase to over \$1800, if the simple interest rate is 5.5% p.a.?

LINKS TO Example 3

6 Calculate the required principal for each of the following.

- **a** \$300 interest over 2 years at 5% p.a.
- **b** \$750 interest over 3 years at 4% p.a.
- **c** \$280 interest over 4 years at 7% p.a.
- **d** \$90 interest over 2 years at 4.5% p.a.

LINKS TO Example 4

7 Calculate the required simple interest rate for each of the following.

- **a** \$2000 invested for 3 years gives \$240 interest
- **b** \$3500 invested for 5 years gives \$787.50 interest
- **c** \$12 000 invested for 8 years gives \$4896 interest
- **d** \$450 invested for 2 years gives \$16.20 interest

LINKS TO Example 5

8 Jason borrows \$6000 at 9% p.a. simple interest for 2 years.

- **a** Calculate the amount of simple interest that Jason pays on the loan.
- **b** How much altogether does Jason have to pay back?

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9 Bradley borrows $2500 at 8.5% per annum simple interest for 3 years. Calculate the interest Bradley pays on the loan.

LINKS TO Example 6

10 $15 000 is borrowed at 6.8% p.a. simple interest over 5 years. Calculate

a the interest paid **b** the monthly repayments.

11 Pedro borrows $1440 at 7.2% simple interest over 3 years. Calculate

a the interest paid **b** the monthly repayments.

LINKS TO Example 7

12 Angie buys a new dishwasher for $1250. She pays a deposit of 10% then 24 monthly instalments of $55.

a What deposit does Angie pay?

b What is the total amount that Angie pays?

c How much more does she end up paying than if she had paid the full amount of $1250 when she bought the dishwasher?

d The amount in part **c** represents the interest on the $1250 that Angie paid over the two years. What annual rate of simple interest on $1250 does this represent?

13 Callum bought a new computer for $2880. He paid 15% deposit then 36 monthly instalments of $85.

a What deposit does Callum pay?

b What is the total amount that Callum pays?

c How much more does he end up paying than if he had paid the full amount of $2880 when he bought the computer?

d The amount in part **c** represents the interest on the $2880 that Callum paid over the three years. What annual rate of simple interest on $2880 does this represent?

LINKS TO Examples 8, 9

14 Calculate the simple interest earned on each of these investments.

a $8000 is invested at 5.4% p.a. for 3 months.

b $25 000 is invested at 6.1% p.a. for 90 days.

c $20 000 is invested for 30 days at 5.55% p.a.

LINKS TO Example 10

15 Lynn's credit card account of $954 was overdue by 31 days. How much interest would Lynn be charged if the daily rate is 0.045616%?

exercise 6.1 challenge

16 Chelsea was charged $1.756 interest on her credit card account of $588 that was 6 days overdue. Calculate

a the daily interest rate **b** the annual interest rate.

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6.2 Compound interest

With some bank and investment accounts, the interest earned is compounded, that is, the interest is added to the principal so that in each interest period, interest is calculated on the accumulating interest as well as on the principal. The process is an example of a **recursive process**, that is, each calculated amount becomes the value to be used in the next calculation. This form of interest is called **compound interest**. The following spreadsheet compares the growth of $2500 when it is invested for 10 years at 6% simple interest and when it is invested at 6% compound interest paid at the end of each year.

Notice how the simple interest remains constant at $150 whereas the compound interest increases each year. In the second year the interest is 6% of ($2500 + $150), that is, $159. In the third year the interest is 6% of ($2650 + $159), that is, $168.54, and so on.

	A	B	C	D	E
1	After	Simple Interest @ 6% p.a. ($)	Amount ($)	Compound Interest @ 6% p.a. ($)	Amount ($)
2	0 years		2500.00		2500.00
3	1 year	150.00	2650.00	150.00	2650.00
4	2 years	150.00	2800.00	159.00	2809.00
5	3 years	150.00	2950.00	168.54	2977.54
6	4 years	150.00	3100.00	178.65	3156.19
7	5 years	150.00	3250.00	189.37	3345.56
8	6 years	150.00	3400.00	200.73	3546.30
9	7 years	150.00	3550.00	212.78	3759.08
10	8 years	150.00	3700.00	225.54	3984.62
11	9 years	150.00	3850.00	239.08	4223.70
12	10 years	150.00	4000.00	253.42	4477.12
13		Total value of investment		Total value of investment	
14			4000.00		4477.12
15		Total interest		Total interest	
16			1500.00		1977.12

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Simple and compound interest

The graph over the page shows that for simple interest there is steady rate of increase of $150 per year in the total value of the investment. With compound interest, the total value of the investment is increasing at a faster and faster rate because the interest is itself earning interest.

Growth of this type is referred to as exponential growth because the value of the index or exponent, n, determines the new amount.

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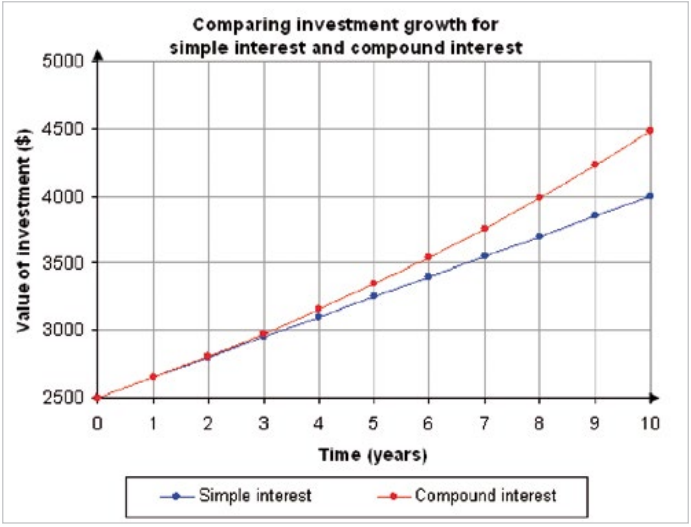

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Example 11

$5000 is invested at 6.4% compound interest for 5 years, where the interest is calculated at the end of each year on the total amount, that is the original investment plus the accumulated interest.

a Find the total value of the investment after 5 years.

b What is the total amount of interest earned?

Working

a

After	Interest ($)	Total
0 years		$5000.00
1 year	$0.064 \times 5000 = 320.00$	$5320.00
2 years	$0.064 \times 5320 = 340.48$	$5660.48
3 years	$0.064 \times 5660.48 = 362.27$	$6022.75
4 years	$0.064 \times 6022.75 = 385.46$	$6408.21
5 years	$0.064 \times 6408.21 = 410.13$	$6818.33

The total value of the investment after 5 years is $6818.34.

b Total interest = $6818.33 – $5000
= $1818.33

The total amount of interest earned is $1818.33.

Reasoning

With compound interest, the interest is added to the principal, so that in the next interest period, interest is calculated on the total amount, including interest.
Each amount has been rounded to two decimal places.

The total interest earned in 5 years is found by subtracting the principal from the total amount after 5 years.

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exercise 6.2

LINKS TO Example 11

1 $5000 is invested at 6% p.a. compound interest for 3 years. The interest is calculated and compounded at the end of each year.

a Copy and complete the following table.

b What is total value of the investment after 3 years?

c What is the total amount of interest earned?

After	Interest ($)	Total ($)
0 years		
1 year		
2 years		
3 years		

2 $12 000 is invested at 5% p.a. compound interest for 4 years. The interest is calculated and compounded at the end of each year.

a Copy and complete the following table.

b What is total value of the investment after 4 years?

c What is the total amount of interest earned?

After	Interest ($)	Total ($)
0 years		
1 year		
2 years		
3 years		
4 years		

3 For each of these investments, use tables similar to those in questions **1** and **2** to calculate

i the total value of the investment after 4 years.

ii the total amount of interest earned.

a $1000 is invested at 5% p.a. compound interest.

b $4000 is invested at 5.2% p.a. compound interest.

c $7500 is invested at 6.4% p.a. compound interest.

d $3000 is invested at 4.8% p.a. compound interest.

e $10 000 is invested at 6.25% p.a. compound interest.

f $6000 is invested at 5.8% p.a. compound interest.

4 For each of the investments in question **3**

i calculate the amount of interest that would have been earned if the interest was simple interest rather than compound interest.

ii calculate how much extra interest was earned with compound interest.

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6.3 Compound interest formula

When we calculate 5% interest on an amount then add the interest to the original amount, what we are effectively doing is multiplying the original amount by 1.05.

$\$1000 + 5\%$ of $\$1000$

$= \$1000 + 0.05 \times \$1000$

$= \$1000(1 + 0.05)$ (taking out \$1000 as a common factor)

Each time we calculate the interest and add it to the previous amount, we are multiplying the previous amount by $(1 + 0.05)$.

After n years	Total (\$)
0	1000
1	$1000 \times (1 + 0.05)$
2	$1000 \times (1 + 0.05) \times (1 + 0.05) = 1000 \times (1 + 0.05)^2$
3	$1000 \times (1 + 0.05)^2 \times (1 + 0.05) = 1000 \times (1 + 0.05)^3$
n	$1000 \times (1 + 0.05)^n$

We can now generalise for a principal $\$P$ and interest rate $R\%$, where the interest is compounded to give an amount $\$A_n$ after n years.

After n years	Interest (\$)	Total $\$(A_n)$
0		$A_0 = P$
1	$P \times R$	$A_1 = P + PR$ $= P(1 + R)$
2	$P(1 + R) \times R$	$A_2 = P(1 + R)(1 + R)$ $= P(1 + R)^2$
3	$P(1 + R)^2 \times R$	$A_3 = P(1 + R)^2(1 + R)$ $= P(1 + R)^3$
n		$A_n = P(1 + R)^n$

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The compound interest formula

When a principal $\$P$ is invested at R interest for n years, the accumulated amount $\$A_n$ is

$$A_n = P(1 + R)^n$$

The total amount of interest earned after n years is

$$I_n = A_n - P$$

Example 12

\$4000 in invested at 6.5% p.a. interest compounded annually for 5 years.

a Calculate the accumulated amount after 5 years.

b How much interest is earned?

Working	Reasoning
a $A_n = P(1 + R)^n$ $A_5 = 4000(1 + 0.065)^5$ $= 5480.346\ldots$ The amount after 5 years is \$5480.34.	$P = 4000$ $R = 6.5\% = 0.065$ $n = 5$
b $I = A - P$ $= 5480.34 - 4000$ $I = 1480.34$ The interest earned is \$1480.34.	To find the interest earned, subtract the principal from the accumulated amount.

Guess, check and improve can be used to find how long it would take for a given principal to accumulate to a particular amount.

Example 13

\$1000 is invested at 6% p.a. interest compounded annually. How many years would it be before the accumulated amount reached at least \$2000?

Working	Reasoning
$1000 \times 1.06^{10} = 1790.847\ldots$ $1000 \times 1.06^{12} = 2012.196\ldots$ $1000 \times 1.06^{11} = 1898.298\ldots$ It would take 12 years for \$1000 to accumulate to \$2000 at 6% p.a. compound interest.	Guess a value for n. Start with $n = 10$. This is too low so try $n = 12$. Check $n = 11$.

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exercise 6.3

LINKS TO Example 12

1 For each of the following

- **i** use the compound interest formula to calculate the accumulated amount if the interest is compounded annually.
- **ii** calculate how much interest is earned.

a \$1000 is invested at 5% p.a. compound interest for 4 years.
b \$3000 is invested at 4% p.a. compound interest for 3 years.
c \$7500 is invested at 6.4% p.a. compound interest for 5 years.
d \$3000 is invested at 4.8% p.a. compound interest for 4 years.
e \$10 000 is invested at 6.25% p.a. compound interest for 5 years.
f \$6000 is invested at 5.8% p.a. compound interest for 10 years.
g \$25 000 is invested at 6.8% p.a. compound interest for 8 years.
h \$4000 is invested at 4.6% p.a. compound interest for 6 years.
i \$2500 is invested at 5.2% p.a. compound interest for 5 years.
j \$3000 is invested at 4.8% p.a. compound interest for 4 years.
k \$15 000 is invested at 7.25% p.a. compound interest for 4 years.
l \$6000 is invested at 6.25% p.a. compound interest for 10 years.

LINKS TO Example 13

2 Using guess, check and improve, calculate how many years it would take for each of these investment growths if the interest is compounded at the end of each year.

a \$5000 grows to at least \$6000 at 4% p.a. interest.
b \$3000 grows to at least \$3800 at 5% p.a. interest.
c \$500 grows to at least \$700 at 6% p.a. interest.
d \$4500 grows to at least \$5000 at 4.8% p.a. interest.
e \$8000 grows to at least \$10 000 at 6.25% p.a. interest.
f \$10 000 grows to at least \$15 000 at 7.25% p.a. interest.
g \$4000 grows to at least \$8000 at 7% p.a. interest.
h \$500 grows to at least \$1000 at 5% p.a. interest.

exercise 6.3 challenge

3 How much would you need to invest now to get at least \$10 000 after 10 years at 8% p.a. compound interest?

4 What principal was invested if the total value of the investment after 5 years at 4.5% compounded annually was \$11 215.64?

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6.4 Changing the compounding period

Compound interest is not always calculated on a yearly basis. The compounding of interest may occur monthly or daily. As you will see in this section this influences the amount of interest earned or owed.

If the interest is compounded monthly, n will represent the number of months and the annual interest rate must be divided by 12 to obtain the monthly rate. For example, if the interest rate is 5% p.a. and interest is compounded monthly for 3 years, $n = 3 \times 12 = 36$ and $R = \frac{5}{12}\% = \frac{0.05}{12}$.

We can now compare the amount of interest earned when the interest is compounded at different intervals. In example **12** in the previous section we saw that \$4000 earned \$1480.34 over 5 years at 6.5% interest compounded each year. In example **14**, the same principal is again invested for 5 years, but the interest is compounded monthly.

Example 14

\$4000 in invested at 6.5% p.a. compound interest for 5 years.

a Calculate the accumulated amount if the interest is compounded monthly.

b How much interest is earned?

c Compare the interest earned with the interest earned when this amount is compounded annually.

Working	Reasoning
a $A_n = P(1 + R)^n$	$P = 4000$
$A_{60} = 4000\left(1 + \frac{0.065}{12}\right)^{60}$	The monthly interest rate is $R = \frac{0.065}{12}$.
$= 5531.269\ldots$	The number of months, n, is $n = 5 \times 12 = 60$.
The amount after 60 months is \$5531.26.	
b $I = A - P$	To find the interest earned in the 60 months, subtract the principal from the accumulated amount.
$= 5531.26 - 4000$	
$I = 1531.26$	
The interest earned is \$1531.26.	
c Compounded yearly: \$1480.34. Compounded monthly: \$1531.26. An extra \$50.92 is earned if the interest is compounded monthly.	See example **10**.

The next example shows the same principal invested for 5 years with the interest compounded daily.

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Example 15

$4000 is invested at 6.5% p.a. compound interest for 5 years.

a Calculate the accumulated amount if the interest is compounded daily.

b How much interest is earned?

Working	Reasoning
a $n = 1825$	$P = 4000$
$A_n = P(1 + R)^n$	
$A_{1825} = 4000\left(1 + \frac{0.065}{365}\right)^{1825}$	The daily interest rate is $R = \frac{0.065}{365}$.
$= 5535.962\ldots$	The number of days $= 5 \times 365 = 1825$.
The amount after 1825 days (5 years) is $5535.96.	
b $I = A - P$	To find the interest earned, subtract the principal from the accumulated amount.
$= 5535.96 - 4000$	
$I = 1535.96$	
The interest earned is $1535.96.	

Note that the interest earned increases as the compounding is done more frequently. The increase in interest earned when changing from yearly to monthly compounding is quite substantial. The increase in interest earned when changing from monthly to daily compounding is not so great. The same applies if a home loan is being paid off—it is a financial advantage to make payments fortnightly instead of monthly.

Effective and nominal rates of interest

If $4000 is invested at 6.5% interest compounded annually, each year the accumulated amount is multiplied by 1.065.

If $4000 is invested at 6.5% p.a. interest compounded monthly, each year the amount is multiplied by $\left(1 + \frac{0.065}{12}\right)^{12} = 1.06697\ldots$

This represents an annual interest rate of 0.06697, that is, approximately 6.7% p.a. This is called the **effective interest rate**. The stated rate of 6.5% p.a. is called the **nominal interest rate**.

To find the effective interest rate, calculate $\left(1 + \frac{R}{n}\right)^n$ and subtract 1.

Multiplying by 100 gives the effective rate as a percentage.

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Example 16

Calculate the effective interest rate, correct to two decimal places, if a nominal interest rate of 5.8% p.a. is compounded

a 6 monthly **b** monthly **c** daily.

Working

a $\left(1 + \frac{0.058}{2}\right)^2 = 1.05884\ldots$

The effective rate is 5.88% p.a.

b $\left(1 + \frac{0.058}{12}\right)^{12} = 1.05956\ldots$

The effective rate is 5.96% p.a.

c $\left(1 + \frac{0.058}{365}\right)^{365} = 1.05971\ldots$

The effective rate is 5.97% p.a.

Reasoning

6 monthly means that there are two compounding periods.

The 6-monthly interest rate is $\frac{0.058}{2}$.

The monthly interest rate is $\frac{0.058}{12}$.

The daily interest rate is $\frac{0.058}{365}$.

The more frequently the interest is compounded, the higher the effective rate of interest compared with the nominal rate. Banks and other financial institutions take this into account by having different rates for term deposits depending on the frequency of compounding.

Example 17

Golden Bank offers the following compound interest rates on term deposits invested for 12 months.

Calculate the effective interest rate for the monthly, quarterly and half-yearly rates.

Interest compounded	Interest rate
Monthly	5.80%
Quarterly	5.85%
Half-yearly	5.90%
On maturity (i.e., at the end of the year)	6.00%

Working

$\left(1 + \frac{0.058}{12}\right)^{12} = 1.059566\ldots$

$\left(1 + \frac{0.0585}{4}\right)^{4} = 1.059795\ldots$

$\left(1 + \frac{0.059}{2}\right)^{2} = 1.059870\ldots$

Reasoning

Monthly:
12 compounding periods.
Quarterly:
4 compounding periods.
Half-yearly:
2 compounding periods.

continued

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Example 17 continued

Working **Reasoning**

Interest compounded	Interest rate	Effective interest rate
Monthly	5.80%	5.96%
Quarterly	5.85%	5.98%
Half-yearly	5.90%	5.99%
On maturity	6.00%	6.00%

exercise 6.4

LINKS TO Examples 14, 15

1 For each of the following calculate the accumulated amount if the interest is compounded

i yearly **ii** half-yearly **iii** quarterly **iv** monthly

a \$2000 is invested at 6% p.a. compound interest for 5 years.
b \$6000 is invested at 5.2% p.a. compound interest for 4 years.
c \$2400 is invested at 6.4% p.a. compound interest for 5 years.
d \$10 000 is invested at 6.4% p.a. compound interest for 8 years.
e \$15 000 is invested at 6.25% p.a. compound interest for 4 years.
f \$7500 is invested at 5.8% p.a. compound interest for 6 years.

LINKS TO Examples 16, 17

2 Eastern Bank offers the following interest rates on term deposits. Calculate the effective interest rates for monthly, quarterly and half-yearly compounding.

Interest compounded	Interest rate	Effective interest rate
Monthly	6.43%	
Quarterly	6.48%	
Half-yearly	6.54%	
Yearly	6.65%	

3 Northern Bank offers the following interest rates on term deposits. Calculate the effective interest rates for monthly, quarterly and half-yearly compounding.

Interest compounded	Interest rate	Effective interest rate
Monthly	7.00%	
Quarterly	7.05%	
Half-yearly	7.12%	
Yearly	7.25%	

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6.5 Compound depreciation

Depreciation is the loss of value of equipment over time due to gradual wearing, newer more efficient models being sold, and so on. For taxation purposes, businesses can claim the depreciation of their equipment and vehicles against their income. Various methods are used by businesses and the Australian Tax Office to calculate depreciation.

Compound depreciation is calculated recursively in a similar way to compound interest. The depreciated value is calculated as a percentage of the value in the previous year. Unlike compound interest, the depreciation is *subtracted* each year from the original value.

Example 18

A car purchased for $24 000 is depreciated at the rate of 20% p.a.

a Calculate the depreciated value of the car after 5 years.

b By how much has the car depreciated?

Working

a

After	Depreciation	Value
0 years		$24 000
1 year	20% of $24 000 = $4800	$24 000 – $4800 = $19 200
2 years	20% of $19 200 = $3840	$19 200 – $3840 = $15 360
3 years	20% of $15 360 = $3072	$15 360 – $3072 = $12 288
4 years	20% of $12 288 = $2457.60	$12 288 – $2457.60 = $9830.40
5 years	20% of $9830.40 = $1966.08	$9830.40 – $1966.08 = $7864.32

After 5 years the value of the car is approximately $7864.

b The car has depreciated by $16 136.

Reasoning

The car depreciates 20% of its value each year. To find the depreciation during the first year, find 20% of the new value. Subtract the depreciation from the new value to find the value at the end of the first year.
To find the depreciation in the second year, calculate 20% of the value at the end of the first year.

$24 000 – $7864 = $16 136

Calculated in this way, we can see that depreciation is greatest in the first year. The older the car, the smaller the amount by which it depreciates.

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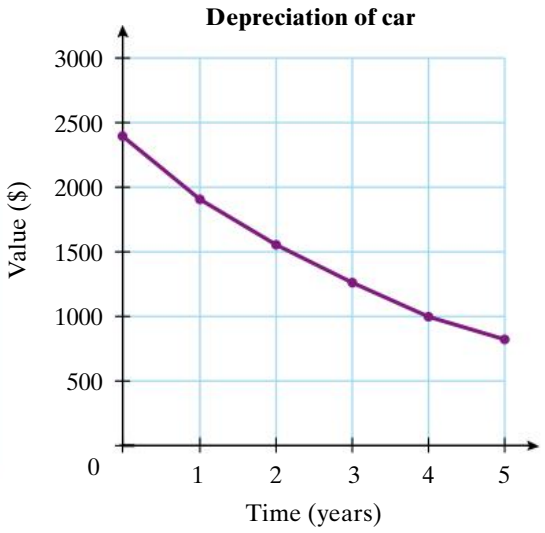
By inspecting example 18, we can see the similarity with compound interest. Unlike compound interest, where we *add* the interest, the depreciation is *subtracted* each year from the original value.

Compound depreciation

When an item of value $\$P$ depreciates at the rate of $R\%$ per year, the value $\$A_n$ after n years is given by

$$A_n = P(1 - R)^n$$

The depreciation is $P - A_n$.



Example 19

Calculate the depreciated value of a car after 8 years if the new price was \$34 000 and the rate of depreciation is 18% p.a.

Working

$A_n = P(1 - R)^n$
$A_8 = 34\,000(1 - 0.18)^8$
$= 34\,000 \times 0.82^8$
$A_8 = 6950.078\ldots$
The depreciated value after 8 years is approximately \$6950.

Reasoning

$P = 34\,000$
$n = 8$
$R = 18\% = 0.18$

exercise 6.5

LINKS TO Example 18

1 Uwimana's new car cost \$14 000. The rate of depreciation is 20% per annum.

a Copy and complete the following table.

After	Depreciation ($)	Value ($)
0 years		14 000
1 year		
2 years		
3 years		

b What is value of the car after 3 years?

c What is the amount by which the car has depreciated?

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2 An office printer cost $1200. The rate of depreciation is 20% per annum.

a Copy and complete the following table.

After	Depreciation ($)	Value ($)
0 years		1200
1 year		
2 years		
3 years		

b What is value of the printer after 3 years?

c What is the amount by which the printer has depreciated?

3 Brett paid $45 000 for his new car. The rate of depreciation is 18% per annum.

a Copy and complete the following table.

After	Depreciation ($)	Value ($)
0 years		45 000
1 year		
2 years		
3 years		

b What is value of the car after 3 years?

c What is the amount by which Brett's car has depreciated?

LINKS TO Example 19

4 Nick bought a new car that cost $36 000.

a Calculate the value of the car at the end of the first year, if it depreciates at the rate of 18% per annum.

b Use the depreciation formula to calculate the value of the car after 8 years.

5 Bakery equipment costing $240 000 depreciates at the rate of 15% per annum. Use the depreciation formula to calculate the value after 10 years.

6 Calculate the value of an office photocopier purchased for $3570 after 5 years assuming it depreciates at 15% p.a.

7 Medical equipment was purchased for a hospital at the cost of $130 000. Calculate the value of the equipment after 7 years if it depreciates at the rate of 15% p.a.

8 A farmer buys a new tractor for $43 750. If the tractor depreciates at the rate of 22.5% p.a., what is the value of the tractor after 6 years?

9 A construction company buys a new loader for $149 950. What is the value after 8 years if the loader depreciates at the rate of 18.75% p.a.?

10 A construction company buys a new excavator for $115 000. What is the value after 10 years if the rate of depreciation is 15% p.a.?

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6.6 Using online calculators

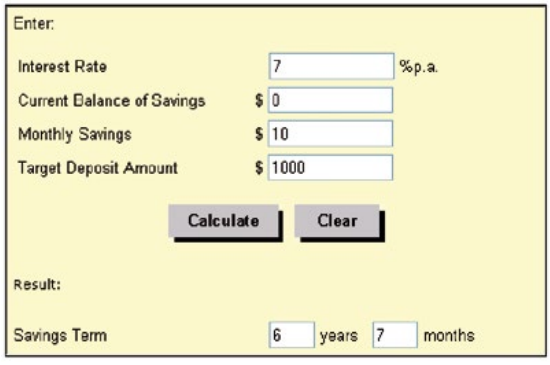
The internet is a useful source of many interactive calculators, where we simply enter the appropriate information and the calculation is done for us. Most banks have online calculators for calculating simple and compound interest for investments and loans.

exercise 6.6

1 The Commonwealth Bank has a savings calculator in which you set a target amount to reach with regular monthly savings. The calculator reports the length of time to reach this target at the nominated interest rate.

www.commbank.com.au/tools/savingstermcalc.asp

In the screen shown, monthly savings of $10 at 7% interest would reach $1000 in 6 years and 7 months.



a How much would have been deposited into the account at $10 for 6 years 7 months?

b How much interest has the bank added to this?

c Use the calculator to find how long would it take to reach a target of $3000 if you started with a current balance of zero and deposited $50 per month with an interest rate of 6.4%.

d How much interest would the bank add?

e If you have a part-time job, estimate a reasonable amount you might be able to save on a regular basis. Find out the likely interest rate you would earn. Set yourself a target and use the online calculator to see how long it would take you to reach your target.

2 Many banks have online calculators for loan repayments. To access the Bendigo Bank's Loan Repayments Calculator go to the website www.bendigobank.com.au/personal/calculators/loan_repayments_calculator.asp

The following screen shows the weekly repayments for a $10 000 loan at 8.25% interest paid back over 10 years.

The two lines on the graph represent

- the total amount still owed to the bank
- the amount of the principal still owed to the bank.

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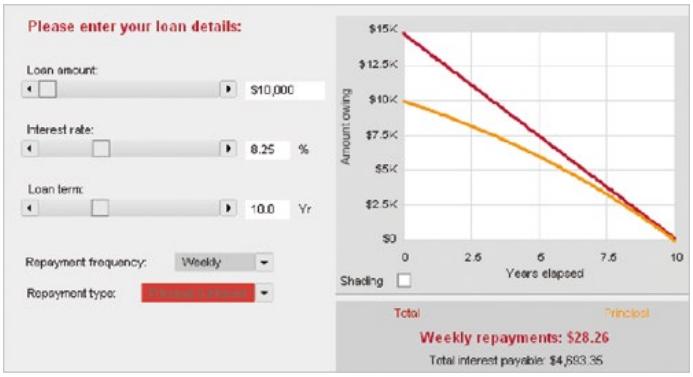


a At a weekly repayment of $28.26, how much would you pay in 10 years? (The weekly repayments are calculated on the basis of 52 weekly payments per year.)

b Subtract the principal ($10000) from this total to give the amount of interest you have paid. This should be close to the stated amount of $4693.35. There may be a small difference due to rounding of the weekly repayments.

c Use the calculator to find the monthly repayments if a loan of $25000 is paid back monthly at 8% interest over 10 years.

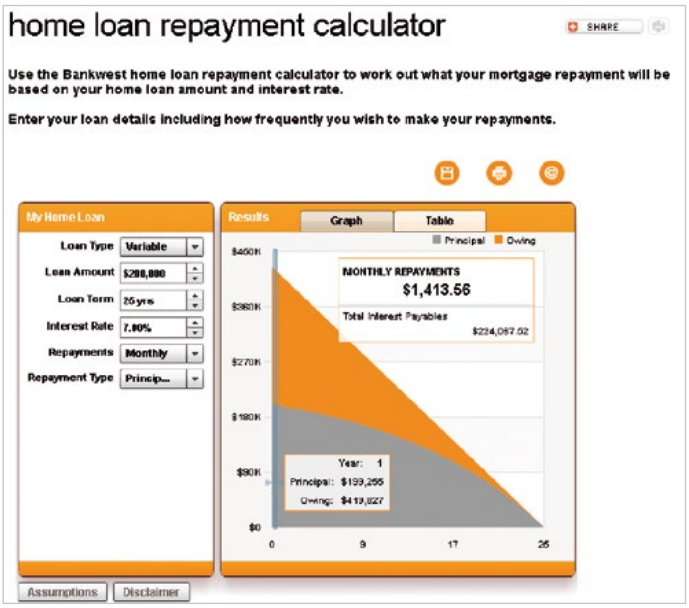
d How much interest would be paid?

e How much interest would be paid if the repayments were weekly instead of monthly? How much would be saved by weekly repayments.

3 The following Bankwest Home Loan Repayment Calculator shows the fortnightly payments for a $200000 home loan paid back fortnightly over 25 years at a fixed interest rate of 7.00%.

www.bankwest.com.au/library/html/Calculators/Home_Loan_Repayment/

As in question **2**, the two lines show the total amount and the amount of principal that is still owed.



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a Before using the online calculator, calculate the total amount paid at $652.03 per fortnight for 25 years (26 fortnightly payments per year).

b Subtract the $200 000 loan to see how much of the amount is interest. Compare your calculation with the total interest payable stated on the calculator. The small difference is probably due to rounding.

c Change the repayment frequency from fortnightly to monthly. What effect does this have on the total amount of interest paid?

d Change the repayment frequency from monthly to weekly. What effect does this have on the total amount of interest paid?

e Suppose a family takes out a $350 000 mortgage to buy a house, and plan to repay the loan over 20 years with weekly repayments. Assuming a fixed interest rate of 7.5% p.a., what would the weekly payments be?

f Using the weekly repayment amount shown on the online calculator, work out the total amount that would actually be paid over the 20 years? Hint: 52 weekly payments each year for 20 years.

g Work out how much of this would be interest.

h Use the online calculator to find the total amount paid if the family were able to pay off the loan in 10 years instead of 20 years. Work out how much of this would be interest.

4 The following savings calculator shows the amount after 10 years for an initial deposit of $10 000 with no further deposits during the 10 years (that is, deposit amount set at zero). Interest is compounded monthly.

www.bendigobank.com.au/personal/calculators/savings_plan_simulator_calculator.asp

The calculator can function as a straight compound interest calculator if you set the deposit amount to zero.

a Use the compound interest formula to show that 7% interest compounded monthly on $10 000 for 10 years (with no further deposits) would result in a total of $20 096.61.

b Compare the amounts for fortnightly and weekly compounding and check with the compound interest formula.

c How much would you have altogether if you started with $1000 and made monthly deposits of $100 for 10 years at 7% interest?

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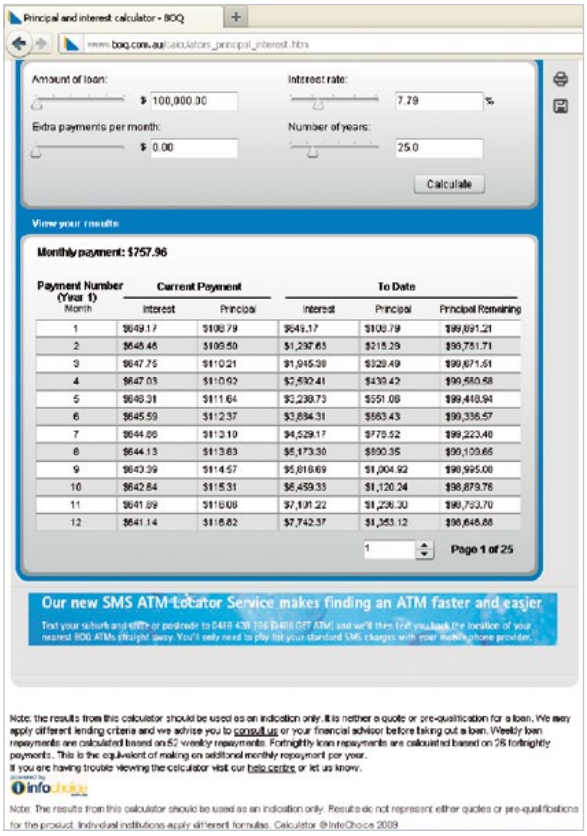
5 Consider a home loan of $100 000. Use the Bank of Queensland online calculator to answer these questions:
boq.com.au/calculators_principal_interest.htm

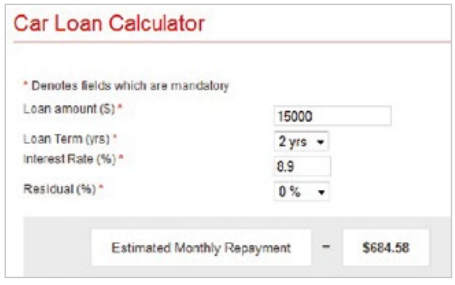
- **a** Find the monthly repayments if the loan is to be paid off over 20 years at 7% interest.
- **b** In the first month how much of this is interest and how much is principal?
- **c** In the last payment (that is after 240 months), how much of the repayment is interest and how much is principal?
- **d** Find the monthly repayments if the loan is to be paid off over 20 years at 7.1% interest.
- **e** How much extra per month is this for the 0.1% rise in interest rate?
- **f** Find the monthly repayments if the loan is to be paid off over 15 years at 7% interest.
- **g** How much more per month is this than the amount in part **a**, where the loan is paid off over 20 years?



exercise 6.6 challenge

6 Finance companies often offer special loans for buying a car. The following St George Bank Car Loan Calculator shows the monthly repayments for a $15 000 loan at 8.9% interest for 2 years.
http://www.stgeorge.com.au/help/calculators/car-loan

- **a** Using the minimum monthly repayment shown on the calculator, calculate the total payment over the 2 years.
- **b** How much interest (and fees) have been charged?
- **c** The calculator has a minimum loan amount. Check the calculator to find this minimum amount.
- **d** Use the calculator to find the minimum monthly repayment if the loan is repaid at the same interest rate but over 5 years.
- **e** What is the total repayment over the 5 years?
- **f** How much interest would be charged?



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Analysis task

Compound interest spreadsheet

In this analysis task a spreadsheet is set up to calculate compound interest on an investment of $1800 invested at 5.4% p.a. compound interest for 10 years.

	A	B	C
1		Interest rate	
2	After	Interest ($)	Amount ($)
3	0 years		
4	1 year		
5	2 years		
6	3 years		
7	4 years		
8	5 years		
9	6 years		
10	7 years		
11	8 years		
12	9 years		
13	10 years		
14		Total value of investment	
15		Total interest	

© Microsoft Corporation

a Start by setting out the columns, rows and headings as shown.

Enter the interest rate as a decimal in cell C1.

Enter the amount of the investment in cell C3.

Column B is the interest earned.

Cell B3 remains empty (why?).

b In cell B4, enter a formula that will calculate the interest on the amount in Cell C3 at the rate in cell C1. Note that when we refer to the rate in cell C1 we must use the **absolute cell reference**. We put a $ sign in front of the row number to ensure that the row number does not change as we drag the formula down. So the formula in cell B4 must refer to cell C$1.

c In cell C4, enter a formula that will add the interest in cell B4 to the previous year's amount in cell C3.

d Drag the formulas in cells B4 and C4 down as far as required for the number of years.

e Format the cells so that the amounts are correct to two decimal places.

f Use your spreadsheet to check the calculations for the investments in exercise **5.2** question **3**.

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Review Simple and compound interest

Summary

Simple interest

- $I = PRT$, where I is the interest per annum, $\$P$ the principal, R the annual interest rate and T the number of years.

Compound interest

- When a principal $\$P$ is invested at interest rate R for n years, the accumulated amount $\$A_n$ is given by
  $$A_n = P(1 + R)^n$$
- The total amount of interest earned after n years is
  $$I_n = A_n - P$$

Effective and nominal interest rates

- The stated compound interest rate is called the **nominal rate**. The **effective rate** is the simple interest rate that would result in the same amount of interest for a given investment over the same period of time.
- To find the effective interest rate, calculate $\left(1 + \frac{R}{n}\right)^n$, and subtract 1.
- Multiplying by 100 gives the effective rate as a percentage.

Compound depreciation

Compound depreciation is calculated in a similar way to compound interest. The depreciated value is calculated as a percentage of the value in the previous year. The depreciation is *subtracted* each year from the original value.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

compound depreciation	effective interest rate	nominal interest rate
compound interest	interest	principal
depreciation	interest rate	simple interest

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Revision

Multiple-choice questions

1 The simple interest on $950 borrowed at 8.5% p.a. for 3 years is

A $25.50 **B** $26.92 **C** $242.25 **D** $707.75 **E** $1192.25

2 $2000 invested at 6% p.a. simple interest would earn $360 interest after

A 2 years **B** 3 years **C** 6 years **D** 12 years **E** 60 years

3 What annual simple interest rate would give $1500 interest on an investment of $7500 for 5 years?

A 1% **B** 2% **C** 3% **D** 4% **E** 5%

4 $2400 is invested at 5% p.a. compounded annually for 10 years. The total amount to the nearest dollar after 10 years is

A $2523 **B** $3600 **C** $3909 **D** $25 200 **E** $138 396

5 $6000 is invested at 4.8% interest compounded monthly for 6 years. The total amount of the investment to the nearest dollar after 6 years is

A $6145 **B** $7099 **C** $7998 **D** $18 071 **E** $101 054

Short-answer questions

6 Alice invests $1400 for 5 years at 6.4% p.a. simple interest. How much interest will Alice earn?

7 Shawn borrows $600 to be paid back in one year with 8% simple interest.

a How much interest will Shawn pay?

b If Shawn pays back the money in 12 equal monthly instalments, how much will he pay each month?

8 $4000 is invested at 7.5% p.a. compound interest for 3 years. The interest is calculated and compounded at the end of each year.

a Copy and complete the following table.

After	Interest ($)	Amount ($)
0 years		
1 year		
2 years		
3 years		

b What is total value of the investment after 3 years?

c What is the total amount of interest earned?

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9 For each of the following investments, the interest is compounded annually. Calculate

i the final amount.

ii the amount of interest earned.

a $1500 is invested at 6.25% interest for 5 years.

b $5000 is invested at 6.4% interest for 2 years.

c $2500 is invested at 7.15% interest for 3 years.

d $12 000 is invested at 7.25% interest for 6 years.

10 For each of the investments in question **9**, calculate the final amount and the amount of interest earned, if the interest is calculated every 6 months.

11 For each of the investments in question **9**, calculate the final amount and the amount of interest earned, if the interest is calculated quarterly.

12 For each of the investments in question **9**, calculate the final amount and the amount of interest earned, if the interest is calculated monthly.

13 Calculate the effective interest for each of these nominal interest rates.

a 5% interest compounded every 6 months

b 6% interest compounded monthly

c 6.25% compounded quarterly

d 6.4% compounded monthly

14 A freight company has purchased a new truck for $165 000. The rate of depreciation is 20% per annum.

a Copy and complete the following table.

After	Depreciation ($)	Value ($)
0 years		
1 year		
2 years		
3 years		

b What is value of the truck after 3 years?

c What is the amount by which the truck has depreciated?

15 A farmer purchased a new bulldozer for $36 000. What is the value after 9 years if the rate of depreciation is 16.6% p.a.?

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Extended-response question

16 Nick bought a new car that cost $37 000, including on-road expenses. He paid a deposit of $4000 and weekly instalments of $119.20 for 5 years. At the end of that time, Nick finished paying for the car with a lump sum of $7400.

a What is the total amount Nick paid for the car?

b How much more than $37 000 did Nick pay for the car?

c The amount in part **b** represents the interest on the $33 000 that Nick owed after paying a deposit of $4000. What annual rate of simple interest on $33 000 does this represent? (Remember that Nick's loan was over 5 years.)

d Calculate the value of the car at the end of the first year, assuming depreciation at the rate of 15% per annum.

e What is the value of the car after 5 years?

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Surface area and volume

7

Pre-test

Warm-up

An historic lead shot factory stood on the site of the planned development for Melbourne Central Station and shopping complex. So that the building could be preserved, architects designed a huge glass and steel cone to enclose it. Calculation of the surface area of the cone was necessary so that estimates could be made of the quantity of materials and the cost. Similarly, calculation of the volume of the cone was necessary so that air-conditioning engineers could design a suitable system.

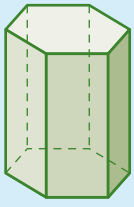
7.1 Surface area of prisms, cylinders and composite solids

The surface area of any solid such as a prism or cylinder is found by adding the areas of the separate parts of the surface. For example, a rectangular prism has six faces, so we add the areas of the six faces to find the total surface area. Prisms in general have two polygonal ends and three or more rectangular faces.

Right prisms

You might recall the formulas for calculating the surface area of prisms and cylinders from Year 9. In each case the surface area is found by adding the areas of each part of the surface.

Surface area of a prism

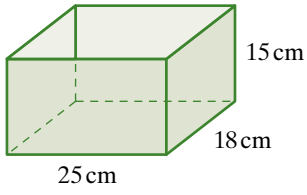


Prisms have a constant cross-section so the ends have the same area.

SA = Area of ends + Area of all rectangular faces

Example 1

Find the surface area of the outside of this box that is open at the top.



Working

Surface area
$= 25 \times 18 + 2 \times 18 \times 15 + 2 \times 25 \times 15$
$= 450 + 540 + 750$
$= 1740\,\text{cm}^2$

Reasoning

The box is a rectangular prism, but with only five surfaces: the bottom, the front and back, and two ends.

The surface area of the outside of the box is $1740\,\text{cm}^2$.

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Example 2

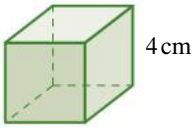
Calculate the surface area of a cube with edges of length 4 cm.

Working

Surface area $= 6 \times 4^2$
$= 96\,\text{cm}^2$

The surface area of the cube is $96\,\text{cm}^2$.

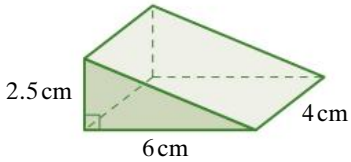
Reasoning



There are 6 faces and each face has an area of $4 \times 4\,\text{cm}^2$.

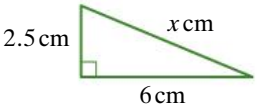
Example 3

Find the surface area of this triangular prism.



Working

$x^2 = 2.5^2 + 6^2$
$x^2 = 6.25 + 36$
$x^2 = 42.25$
$x = 6.5$



Surface area

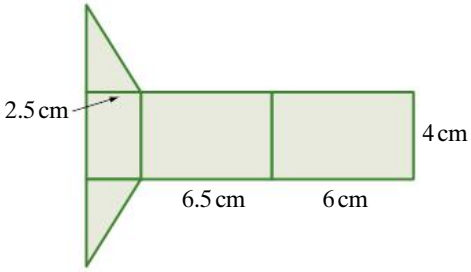
$$= 2 \times \frac{6 \times 2.5}{2} + 4 \times 2.5 + 6 \times 4 + 6.5 \times 4$$
$$= 15 + 10 + 24 + 26$$
$$= 75\,\text{cm}^2$$

The surface area of the prism is $75\,\text{cm}^2$.

Reasoning

The length of the slant edge (x) can be calculated using Pythagoras' theorem.

Notice that 2.5 and 6 are half of 5 and 12. Therefore, the sloping edge must be half of 13, since 5, 12, 13 is a Pythagorean triple.



To calculate the area of the ends of a prism with a regular polygon cross-section, we can divide the regular polygon into isosceles triangles as shown in Example 4.

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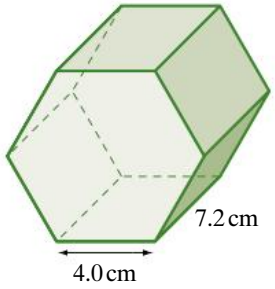
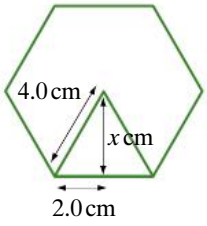
Example 4

Calculate the total surface area of this regular hexagon prism correct to the nearest square centimetre.



Working



Reasoning

Each regular hexagon end can be divided into six congruent equilateral triangles.

$x^2 = 4^2 - 2^2$
$x = \sqrt{12}$

Use Pythagoras' theorem to find the perpendicular height of each equilateral triangle.

Area of equilateral triangle
$= \frac{1}{2} \times 4 \times \sqrt{12}$
$= 2\sqrt{12}\ \text{cm}^2$

Calculate the area of each equilateral triangle, leaving as an exact value.

Area of hexagon
$= 6 \times 2\sqrt{12}$
$= 12\sqrt{12}\ \text{cm}^2$

Calculate the area of the regular hexagon.

Total surface area of prism
$= 2 \times 12\sqrt{12} + 6 \times 4 \times 7.2$
$= 24\sqrt{12} + 172.8$
$\approx 255.9\ \text{cm}^2$

The total surface area is made up of the two regular hexagon ends and six congruent rectangles.

The total surface area is approximately $256\ \text{cm}^2$.

Round to the nearest square centimetre.

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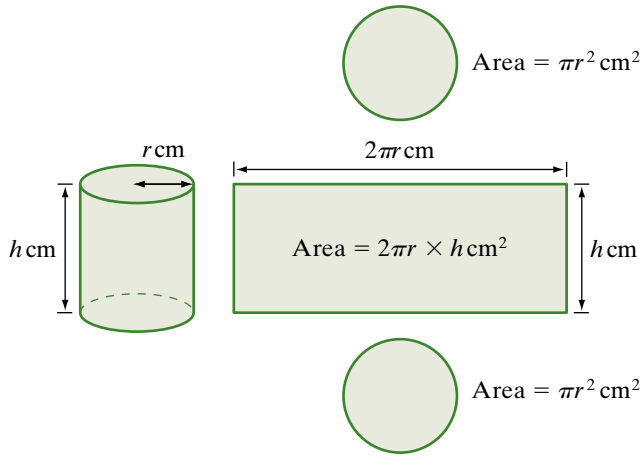
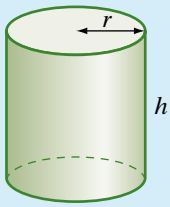
Cylinders

The surface area of a cylinder is made up of three surfaces: the flat top and bottom, and the curved surface. The top and bottom are circles. If the curved part is 'unwrapped', it forms a rectangle. The long sides of this rectangle are the same length as the circumference of the top and bottom circles.



Surface area of a cylinder



r = radius

h = height

$$SA = 2\pi r^2 + 2\pi rh$$

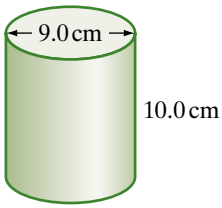
Top and bottom — Curved surface

Example 5

Calculate the following, correct to the nearest square centimetre.

a The surface area of the curved surface of this cylindrical can.

b The total surface area of the can.



Working

a Area of curved surface

$= \pi \times h$

$= 9.0 \times \pi \times 10.0$

The surface area of the curved surface of the can is approximately 283 cm^2.

b Total surface area

$= 2 \times \pi \times 4.5^2 + 282.7$

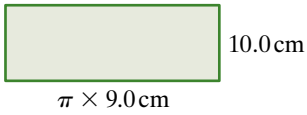
$\approx 127.2 + 282.7$

$= 409.9$ cm^2

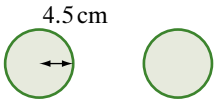
The total surface area of the can is approximately 410 cm^2.

Reasoning

The curved surface 'unwraps' to a rectangle with side lengths equal to the height and circumference of the can.



The total surface area is made up of the area of the curved surface (above) and the top and bottom circles.



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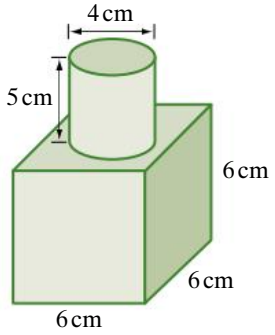
Composite solids composed of prisms and cylinders

A solid may be made up of two or more prisms or cylinders. When we calculate the total surface area, we can imagine that we are painting the entire outside of the shape. We need only the visible part of the surface. We do not include the area where the shapes are joined together.

Example 6

Calculate the total surface area of this composite shape correct to one decimal place.



Working

Visible surface area of cube
$= 6 \times 6^2 - \pi \times 2^2$
$= 216 - 4\pi$

Visible area of cylinder
$= \pi \times 4 \times 5 + \pi \times 2^2$
$= 24\pi$

Total surface area
$= 216 - 4\pi + 24\pi$
$= 216 + 20\pi$
≈ 278.83
The total surface area is $278.8\,\text{cm}^3$ correct to one decimal place.

Reasoning

The cube has 6 faces, each with area $6^2\,\text{cm}^2$. The cylinder is covering part of one face so we need to subtract the area of the circular end of the cylinder.

The curved surface and top of the cylinder are exposed.

Add the visible areas of the cube and cylinder

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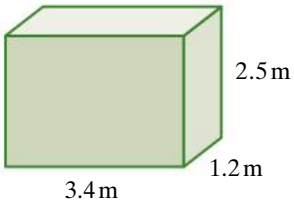
exercise 7.1

LINKS TO Examples 1, 2, 3

1 Find the surface area of each of these prisms.

a



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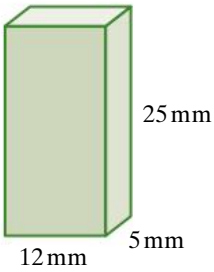

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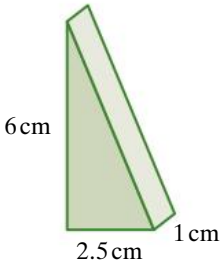

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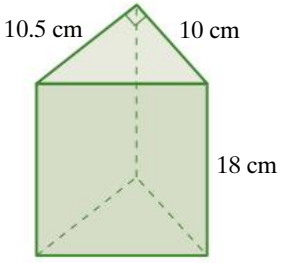

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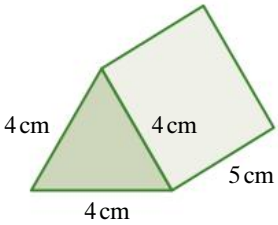

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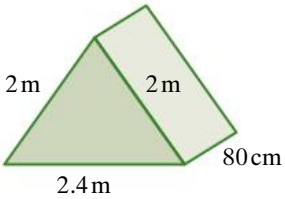
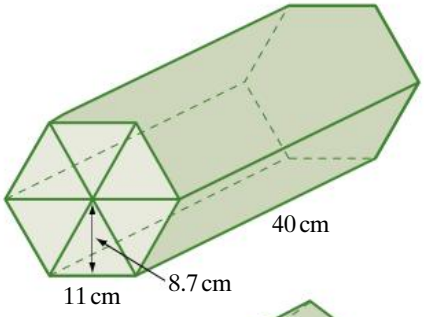


LINKS TO Example 4

2 In each of these regular polygon prisms, the polygon end has been divided into congruent isosceles triangles.
For each prism, find

- **i** the area of one of the isosceles triangles
- **ii** the area of the polygon
- **iii** the total surface area.

a



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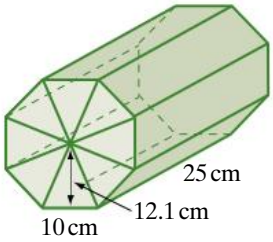

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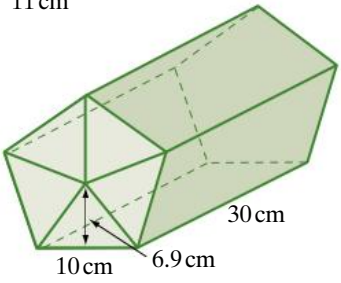

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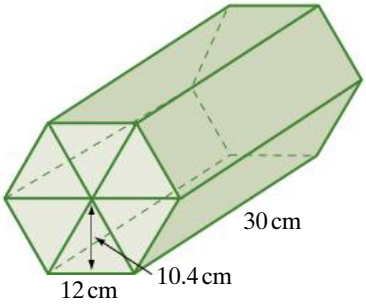

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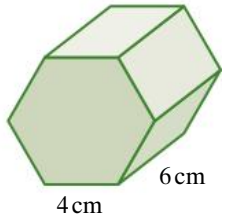
LINKS TO Example 4

3 Calculate the total surface area of this hexagonal prism.



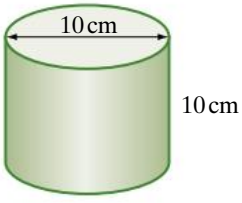
LINKS TO Example 5

4 For each of these cylinders, calculate correct to one decimal place

i the area of the curved surface.

ii the total surface area.

a



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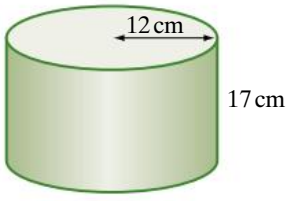

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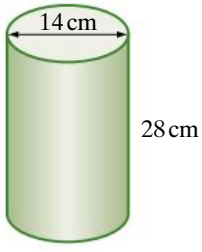

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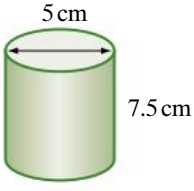

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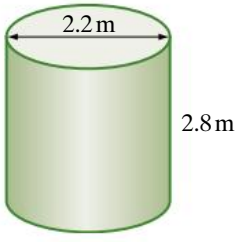

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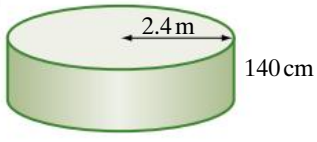
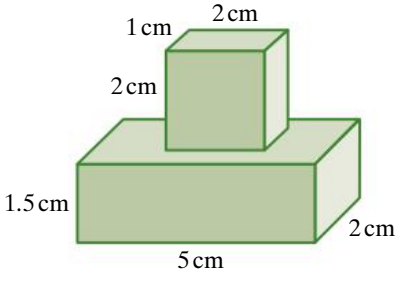


LINKS TO Example 6

5 Calculate the total surface area of each of these composite solids correct to one decimal place.

a



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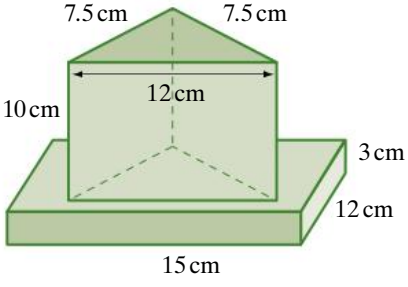

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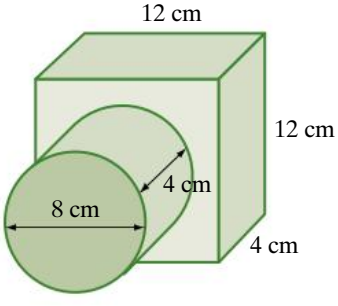

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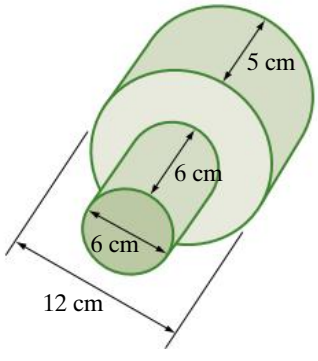

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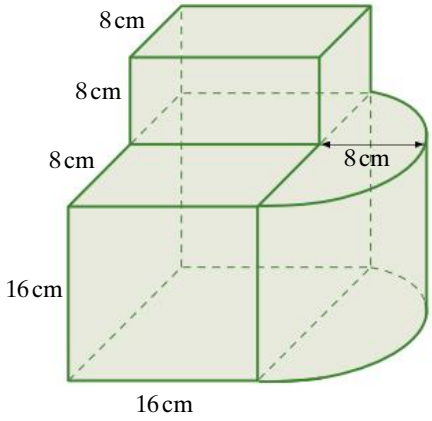
exercise 7.1 challenge

6 Calculate the total surface area of this solid correct to one decimal place.



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7.2 Volume of prisms, cylinders and composite solids

You may recall from Year 9 that the volume of any prism and of a cylinder are calculated by multiplying the area of the cross-section by the height.

Volume of prism and cylinder

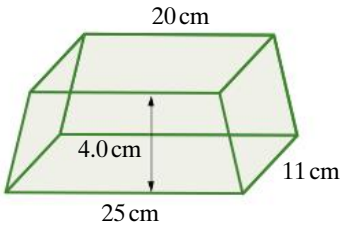
Prism: $V = Ah$ where A is the area of cross-section and h is the height

Cylinder: $Ah = \pi r^2 h$ where r is the radius and h is the height of the cylinder.

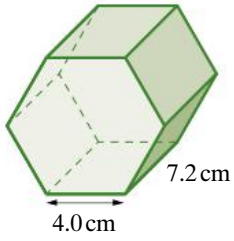
Example 7

Calculate the volume of these prisms to the nearest cubic centimetre.

a



b



Working

a Area of end $= \dfrac{(a+b)h}{2}$

$= \dfrac{20+25}{2} \times 12$

$= 270 \text{ cm}^2$

$V = Ah$

$= 270 \times 11$

$= 2970$

The volume of the prism is 2970 cm^2.

b $V = Ah$

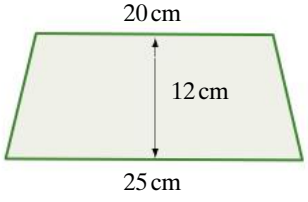
$= 12\sqrt{12} \times 7.2$

$= 299.3$

The volume is approximately 299 cm^3.

Reasoning

The end of the prism is a trapezium.



Note that h is the perpendicular height of the trapezium in this calculation.

Note that h is the height (or length) of the prism in this calculation.

See example **4** where the hexagon was divided into six congruent equilateral triangles and the area of the hexagon was calculated to be $12\sqrt{12} \text{ cm}^2$.

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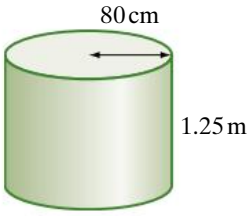
Example 8

For the cylindrical tank shown

a find its volume in cubic metres correct to one decimal place.

b find its capacity in kilolitres.

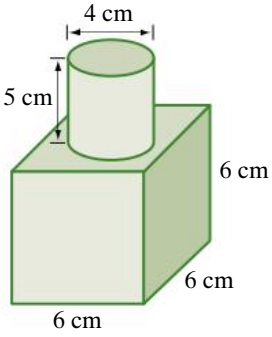


Working	Reasoning
a $V = \pi r^2 h$ $= \pi \times 0.80^2 \times 1.25$ ≈ 2.51 The volume of the tank is approximately $2.5\,\text{m}^3$.	The dimensions must be converted to the same unit, so the radius is 0.80 m.
b The capacity of the tank is 2.5 kL.	1 cubic metre is equivalent to 1 kilolitre.

When solids are made up of more than one simple solid, the volumes are added to find the total volume.

Example 9

Find the volume of this solid correct to the nearest cubic centimetre.



Working	Reasoning
Volume of cube: $V = AH$ $V = l^3$ $= 6^3$ $V = 216$	The three dimensions of the shape are each 6 cm.
Volume of cylinder: $V = AH$ $V = \pi r^2 h$ $= \pi \times 2^2 \times 5$ $= 20\pi$ $V \approx 62.8$	Diameter = 4 cm $r = 2, h = 5$
Total volume $= 216 + 62.8\ \text{cm}^3$ $= 278.8\ \text{cm}^3$	Add the two volumes to find the total volume.
The volume of the shape is $279\ \text{cm}^3$.	Round to the nearest cubic centimetre.

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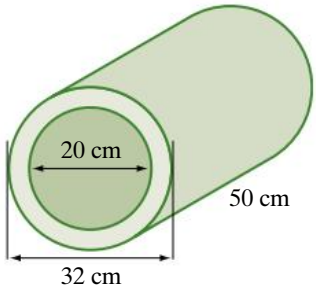
A hollow cylinder such as a concrete pipe has an annulus as its cross-section. To find the volume of material in the pipe we find the area of the annulus then multiply by the length of the pipe.

Example 10

Find the volume of concrete in this pipe correct to the nearest 100 cubic centimetres.



Working

$V = Ah$

$= (\pi \times 16^2 - \pi \times 16^2) \times 50$

$V \approx 24\,504$

The volume of concrete in the pipe is $24\,500$ cm^3.

Reasoning

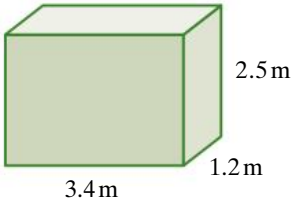
The radius of the outer circle is 16 cm.
The radius of the inner circle is 10 cm.
Area of the annulus
= area of outer circle − area of inner circle.

exercise 7.2

LINKS TO Example 7

1 Calculate the volume of each of these prisms. For part **f**, give the volume correct to one decimal place.

a



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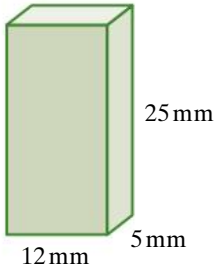

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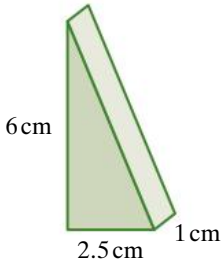

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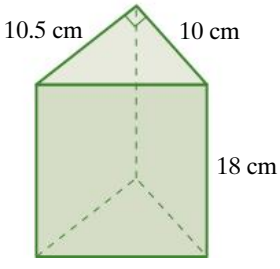

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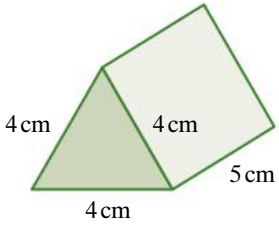

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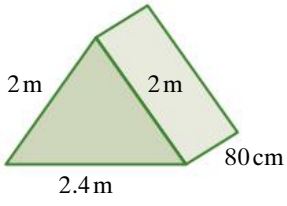

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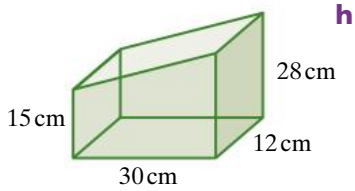

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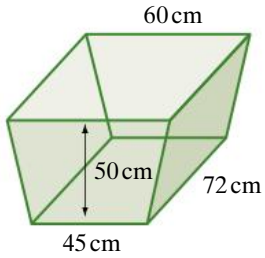

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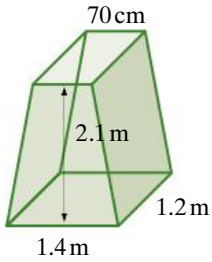
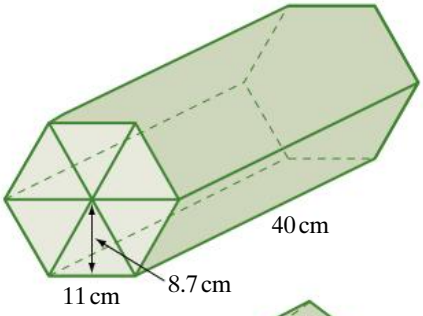


2 Calculate the volume of each of these prisms.

a



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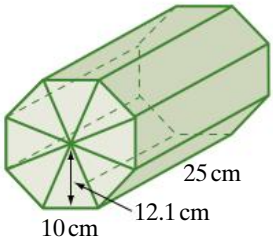

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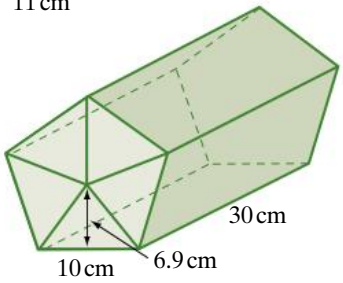

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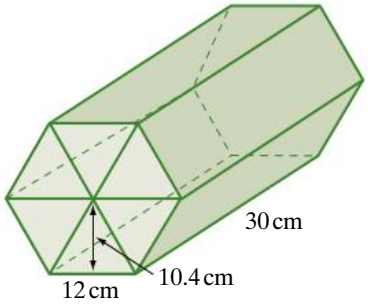
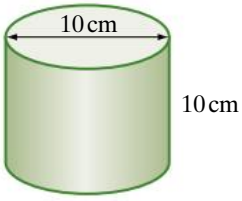


LINKS TO Example 8

3 Calculate the volume of each of these cylinders. Give each volume correct to the nearest cubic centimetre or cubic metre.

a



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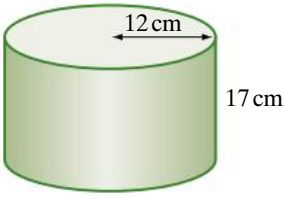

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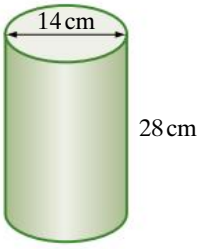

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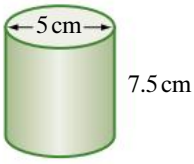

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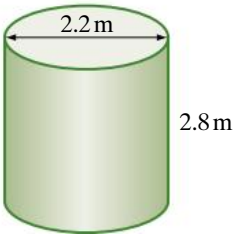

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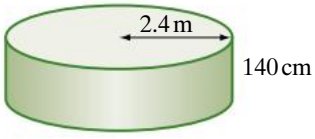

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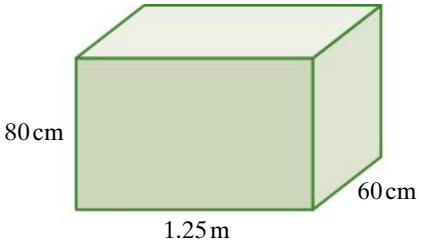
4 For each tank shown below

i find the volume in cubic metres. **ii** find the capacity in kilolitres.

Give the volume of the cylindrical tank to the nearest cubic metre.

a



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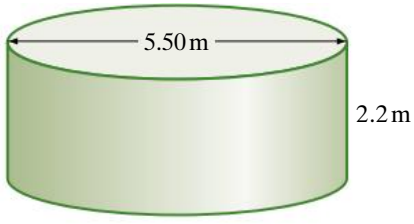
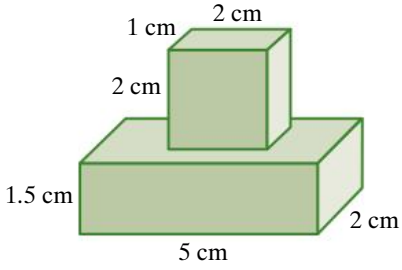


LINKS TO Example 9

5 Find the volume of each of these shapes correct to the nearest cubic centimetre.

a



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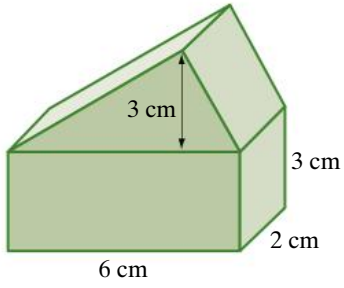

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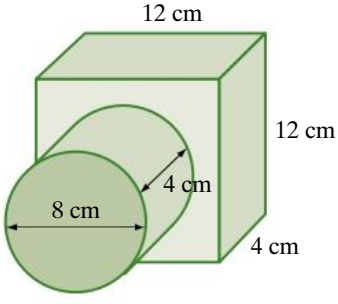

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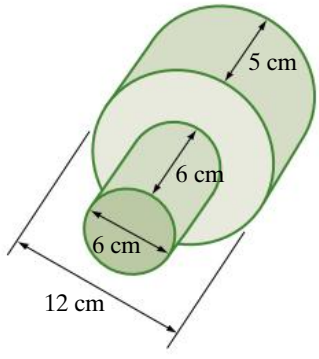
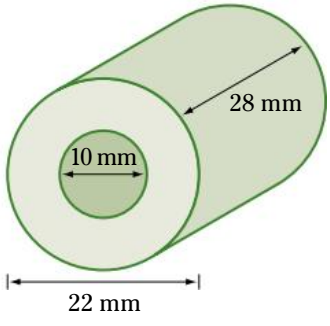


LINKS TO Example 10

6 Find the volume of metal in a hollow cylinder with inside diameter 10 mm, outside diameter 22 mm and length 28 mm. Give the volume correct to the nearest cubic centimetre.



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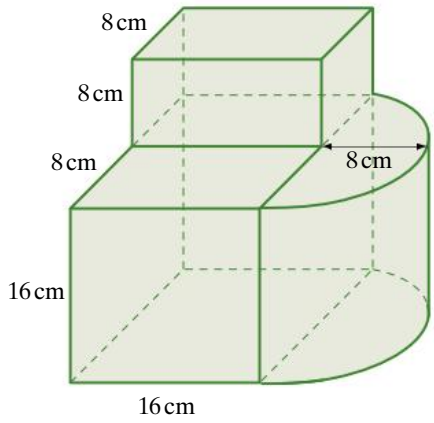
exercise 7.2 challenge

7 Calculate the volume of this solid correct to the nearest cubic centimetre.



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7.3 Surface area and volume applications for prisms, cylinders and composite solids

All the questions that follow are based on volumes of prisms and cylinders. For each question you will need to decide whether the calculation involves surface area or volume. You may also need to use the volume conversions shown below.

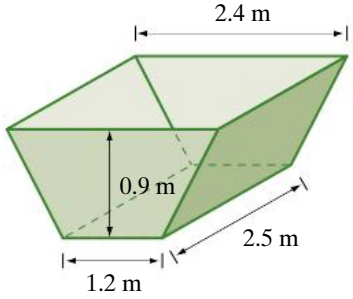
$1\ \text{cm}^3 = 1\ \text{mL}$

$1000\ \text{cm}^3 = 1\ \text{L}$

$1\ \text{m}^3 = 1\ \text{kL}$

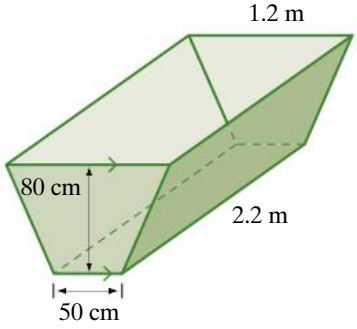
exercise 7.3

1 A waste disposal bin is in the shape of a prism with trapezium shaped ends, as shown in the diagram.



- **a** Calculate the volume of the bin
  - **i** in cubic metres.
  - **ii** in kilolitres.
- **b** Calculate the total area of sheet metal required to make the bin correct to two decimal places. (Hint: use Pythagoras' theorem to find the length of the sloping edges.)

2 A drinking trough for cattle is in the shape of a prism, as shown in the diagram.



- **a** Calculate the volume of the trough in cubic metres.
- **b** How much water would the trough hold if filled to the top? Give your answer in litres.
- **c** Calculate the area of sheet metal used to make the trough correct to two decimal places.

3 A chemical storage tank is a cylinder with height 8.6 m and radius 4.3 m.

- **a** Calculate its volume in cubic metres, to the nearest cubic metre.
- **b** Calculate its capacity in kilolitres, to the nearest kilolitre.

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4 A cylindrical storage tank has height 8.6 m and volume 1000 m^3. Calculate the radius correct to the nearest metre.

5 A cylindrical drinking glass has an internal height of 6 cm and a diameter of 7.2 cm.

a Assuming the glass is filled with orange juice to within 1.5 cm of the top, how many millilitres of juice would be required for each drink? Give the answer to the nearest millilitre.

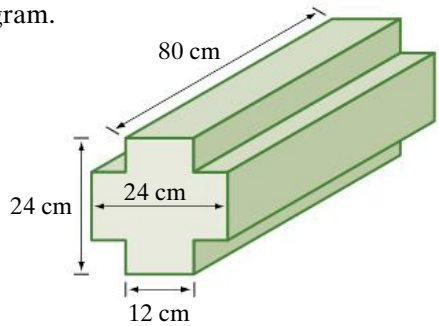
b How many similar drinks could be poured from a 2-litre container of orange juice? Give the answer to the nearest whole number.

6 Fruit juice is to be packaged in a cylindrical can with a capacity of 1 L. If the diameter of the can is 8 cm, how high would the can need to be? Give the height correct to one decimal place.

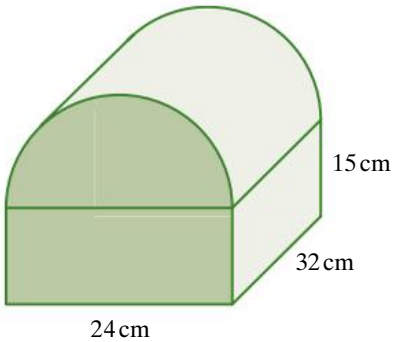
7 A steel beam has the dimensions shown on the diagram.

a Calculate the total area if the beam is to be painted on its entire surface.

b Calculate the volume of steel in the beam.

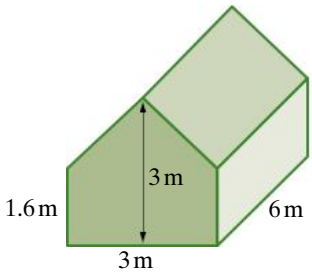


8 A letter box is in the shape of a rectangular prism and a half cylinder. Calculate the total outside surface area, including the base, correct to the nearest square centimetre.



9 Calculate the volume of each of these storage sheds correct to one decimal place.

a



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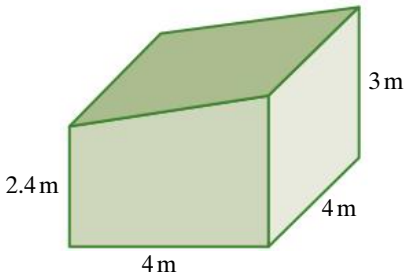

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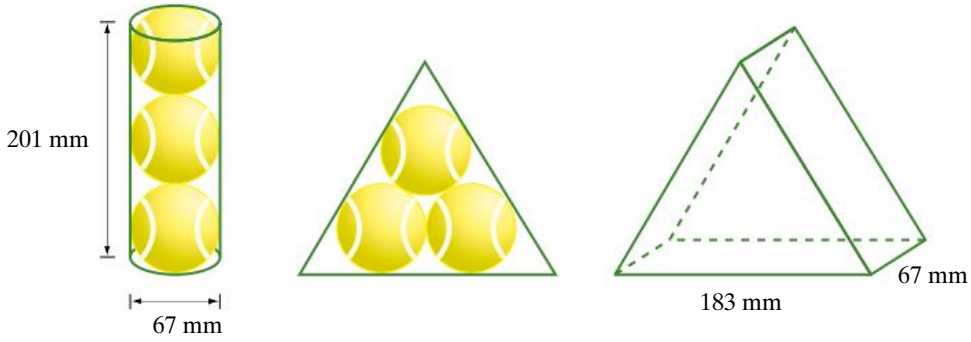
10 Tennis balls are often sold in sets of three in a cylindrical container. Another possible shape for packing three tennis balls is an equilateral triangle prism. The cost of packaging is related to the amount of material required to make the packaging.

a Calculate the total surface area of each container. Give each area correct to the nearest square millimetre.

b Which container is the most economical to make?

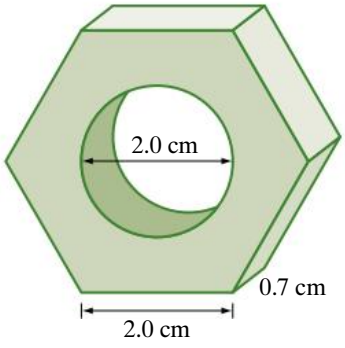


exercise 7.3 challenge

11 Steel hexagonal nuts have the dimensions shown.

a Calculate the volume of metal in each nut, correct to one decimal place.

b If the density of steel is 7.8 g/cm^3, calculate the mass of each nut, correct to one decimal place.



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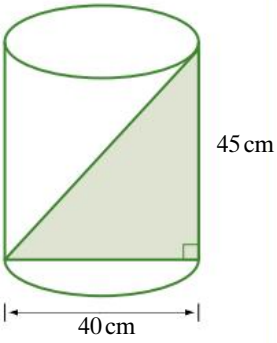
7.4 Pythagoras' theorem in cylinders, cones and pyramids

A vertical section through a cylinder is a rectangle, so a right-angled triangle is formed by the diagonal of this rectangle.

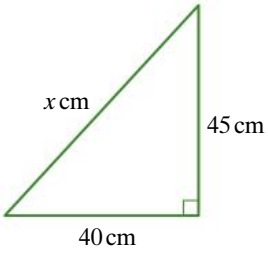
Example 11

What is the longest thin rod that will fit diagonally in this cylindrical container? Give the length correct to the nearest centimetre.



Working

Let the length of the rod be x cm.



$x^2 = 40^2 + 45^2$

$x^2 = 3625$

$x = \sqrt{3625}$

$x \approx 60.2$

The length of the longest thin rod that will fit diagonally in the cylinder is 60 cm.

Reasoning

Draw a right-angled triangle as shown.

The required length, x cm, is the hypotenuse of the right-angled triangle.

Answer the question.

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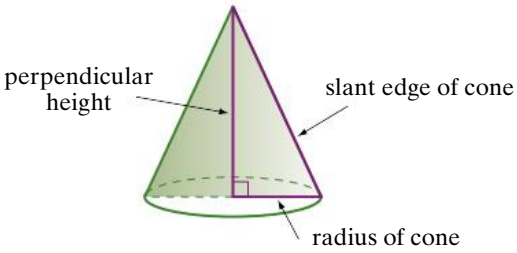
Cones

To calculate the volume of a cone we need to know the perpendicular height and the radius. To calculate the area of the curved surface we need to know the **slant edge** and the radius. Vertical sections through cones are isosceles triangles.

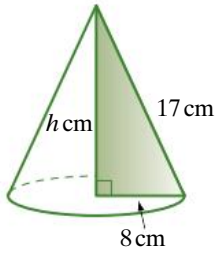
- The base of the isosceles triangle is the diameter of the circular base of the cone.
- The two equal sides of the isosceles triangle are the slant edge of the cone.
- The perpendicular height of the isosceles triangle is equal to the perpendicular height of the cone.



If we know the radius and the slant edge of the cone, we can use Pythagoras' theorem to calculate the height.

Example 12

Calculate the perpendicular height, h cm, of this cone, which has diameter 16 cm and slant edge 17 cm.



Working

$h = 15$

The perpendicular height is 15 cm.

Reasoning

Using the Pythagorean triple 8, 15, 17 we know that $h = 15$.

$h^2 = 17^2 - 8^2$

$h^2 = 225$

$h = \sqrt{225}$

$h = 15$

If we know the height and the radius, we can use Pythagoras' theorem to find the slant edge.

Example 13

a Calculate the slant edge of a cone that has diameter 20 cm and height 15 cm. Give the height correct to the nearest centimetre.

b Calculate the radius of a cone that has height 24 cm and slant edge 25 cm.

continued

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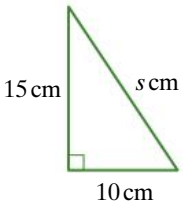
Example 13 continued

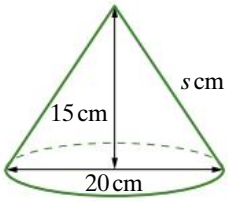
Working

a



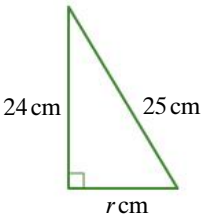
$s^2 = 15^2 + 10^2$
$s^2 = 325$
$s \approx 18.03$
The slant edge of the cone is 18 cm.

Reasoning



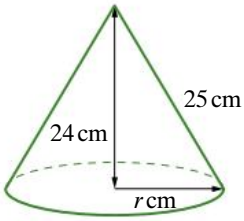
Draw the right-angled triangle.
The base of the triangle is the radius of the base of the cone.

b



$r = 7$
The radius is 7 cm.

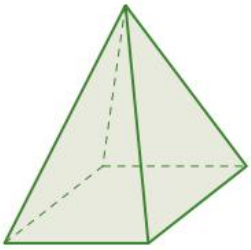
Draw the right-angled triangle.



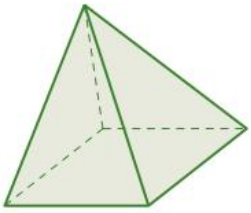
We recognise the Pythagorean triple 7, 24, 25.
$r^2 = 25^2 - 24^2$
$r^2 = 49$
$r = \sqrt{49}$
$= 7$

Pyramids

A pyramid has a polygon-shaped base and triangular faces that meet at a point called the **vertex**. A pyramid which has its vertex directly above the centre of the base is called a right pyramid. The triangular faces of a right pyramid are congruent isosceles triangles.



A right square pyramid



A square pyramid that is not a right pyramid

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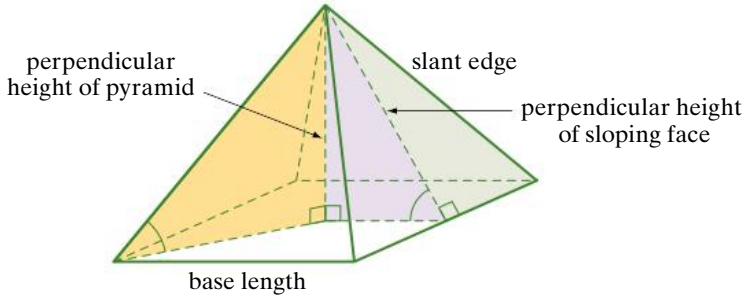
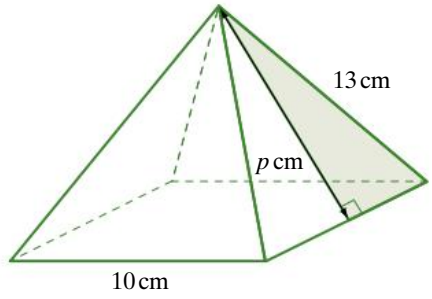
There are four different dimensions that are used in calculations with pyramids that have regular polygon bases, as shown on the square pyramid below.



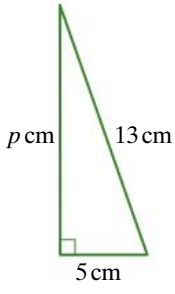
- length of the sides of the base
- height of the pyramid (from the centre of the base to the vertex)
- length of the slant edge
- perpendicular height of the isosceles triangle faces

Example 14

Calculate the perpendicular height of the faces of this square pyramid.



Working



$p = 12$

The perpendicular height is 12 cm.

Reasoning

Draw the right-angled triangle.
Base of triangle is half the side length of the square base.
Using the Pythagorean triple 5, 12, 13 we know that $p = 12$.

Alternatively

$$p^2 + 5^2 = 13^2$$
$$p^2 = 13^2 - 5^2$$
$$p^2 = 169 - 25$$
$$p^2 = 144$$
$$p = 12$$

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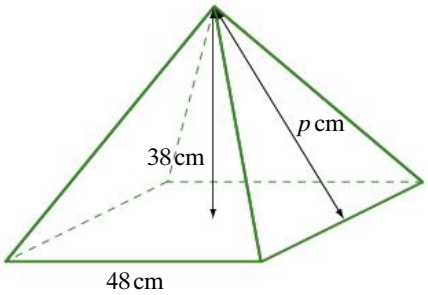
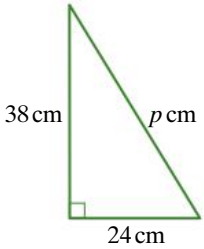
Example 15

Find the perpendicular height of the triangular faces of this square pyramid.



Working

Let p cm be the perpendicular height of the triangular face of the pyramid.



$c^2 = 24^2 + 38^2$
$c^2 = 2020$
$c = \sqrt{2020}$
$c \approx 44.94$

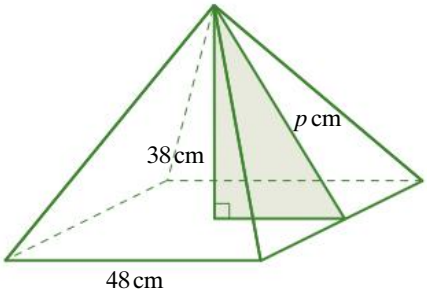
The perpendicular height of the triangular faces is 44.9 cm.

Reasoning

Draw a right-angled triangle as shown. The height of the right-angled triangle is the height of the pyramid. The base of the right-angled triangle is half the side length of the square base of the pyramid.

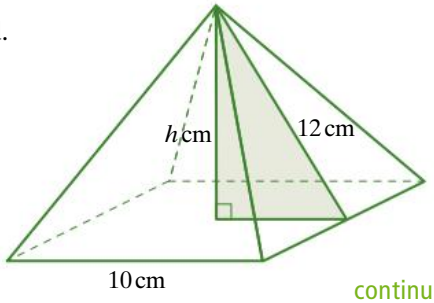
$48 \div 2 = 24$

The hypotenuse of the right-angled triangle is the perpendicular height of the triangular face of the pyramid.



Example 16

Calculate the height, h cm, of this square pyramid.



continued

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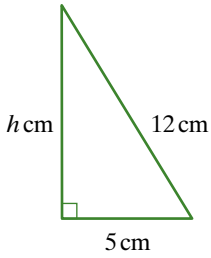
Example 16 continued

Working

$h^2 = 12^2 - 5^2$
$h^2 = 144 - 25$
$h^2 = 119$
$h = \sqrt{119}$
$h \approx 10.9087$
The height of the pyramid is approximately 10.9 cm.

Reasoning

Draw the right-angled triangle.
Base of triangle is half the side length of the square base.



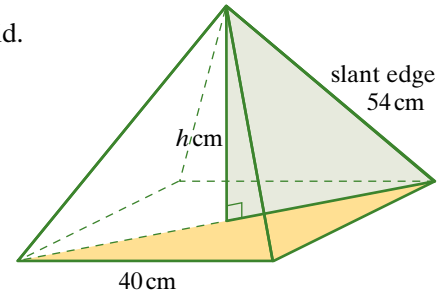
If we know the slant edge and the length of the sides of the base, to find the height of the pyramid, two steps are needed.

1 Calculate the length of the diagonal of the square base then halve it.

2 Use half of the diagonal length and the slant edge to calculate the height.

Example 17

Calculate the height, h cm, of this square pyramid.



Working

Let d cm be the length of the diagonal of the base.

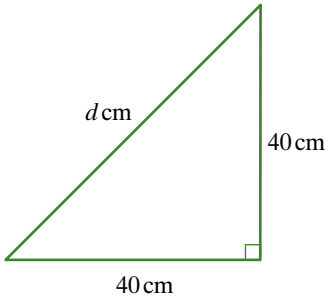
$d^2 = 40^2 + 40^2$
$d^2 = 3200$
$d = \sqrt{3200}$

Let $a = \dfrac{d}{2}$

$a = \dfrac{\sqrt{3200}}{2}$

Reasoning

Before we can calculate h, we must find the length of the diagonal of the square base, d cm.



continued

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Example 17 continued

Working

$h^2 = 54^2 - \left(\frac{\sqrt{3200}}{2}\right)^2$

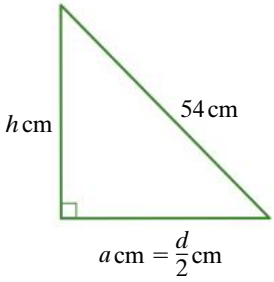
$h^2 = 2116$

$h \approx 45.99$

The height of the pyramid is approximately 46.0 cm.

Reasoning

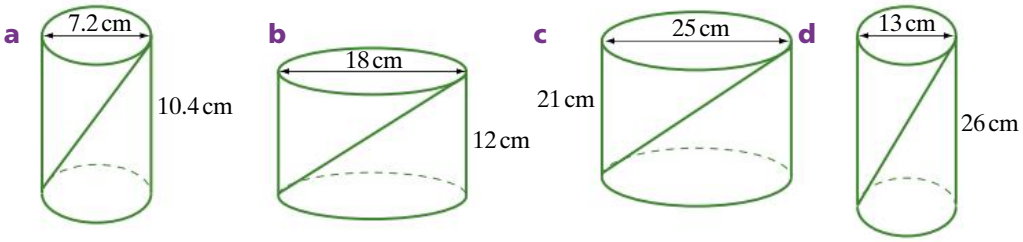
The base of the right-angled triangle is half the length of the diagonal of the square base.



exercise 7.4

LINKS TO Example 11

1 Find the diagonal length in each of these cylinders, correct to one decimal place.



2 A pencil that is 16 cm long just fits diagonally into a cylindrical container that has a diameter of 8 cm. The height of the container is closest to:

A 14 cm **B** 15 cm **C** 16 cm **D** 17 cm **E** 18 cm

3 A cylindrical pencil container is 8 cm high and has diameter 7 cm. A 16 cm pencil stands diagonally in the container.

a Draw a labelled diagram of the right-angled triangle.

b What length of the pencil is above the top of the container? Give the length correct to 1 decimal place.

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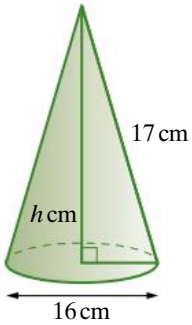
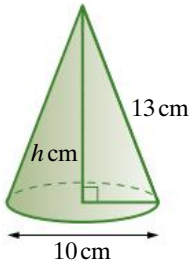
LINKS TO Example 12

4 Calculate the height of each of these cones.

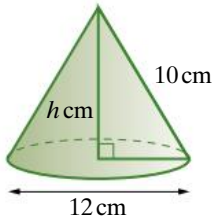
a



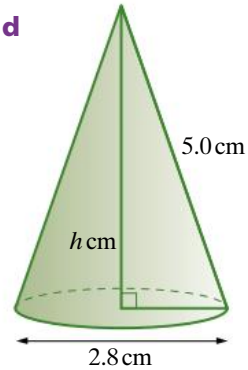
b



c



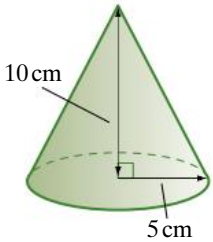
d



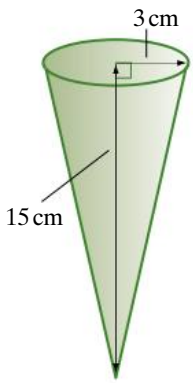
LINKS TO Example 13a

5 Calculate the slant edge of each of the following cones, correct to one decimal place.

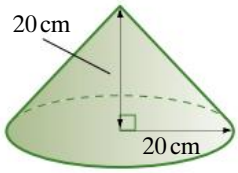
a



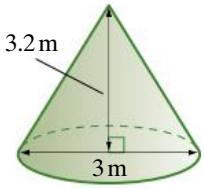
b



c



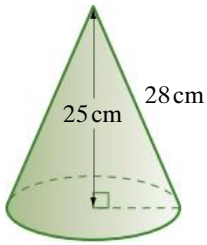
d



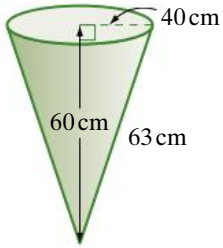
LINKS TO Example 13b

6 Calculate the radius of each of the following cones, correct to one decimal place.

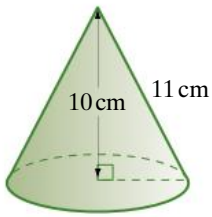
a



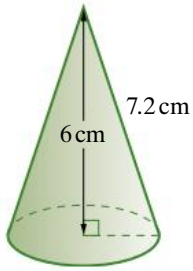
b



c



d



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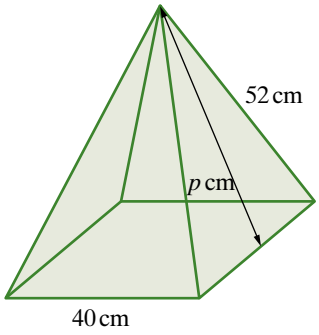
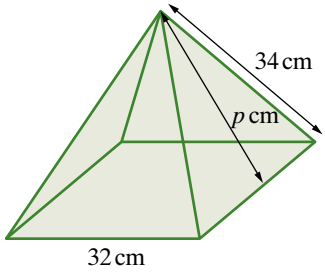
LINKS TO Examples 14, 15, 16

7 Calculate the unknown dimension in each of these square pyramids, where h cm is the height of the pyramid and p cm is the perpendicular height of the isosceles triangle faces.

a



b



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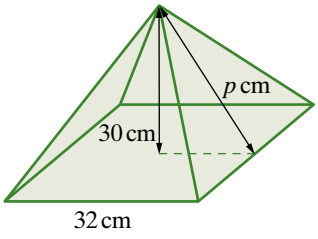

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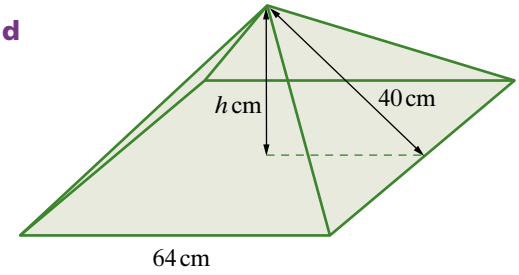
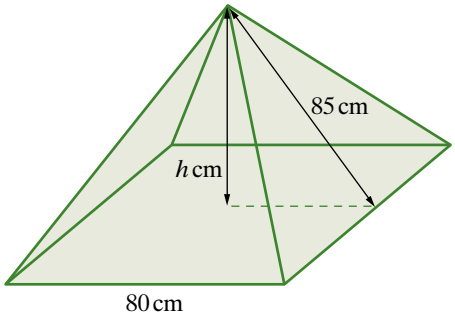
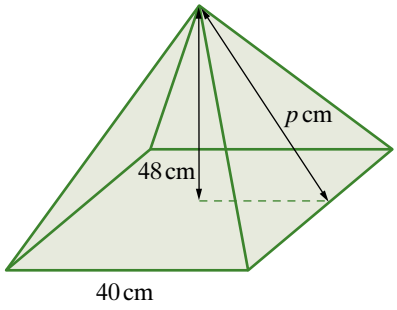


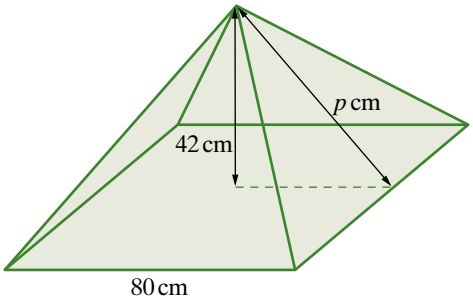
e



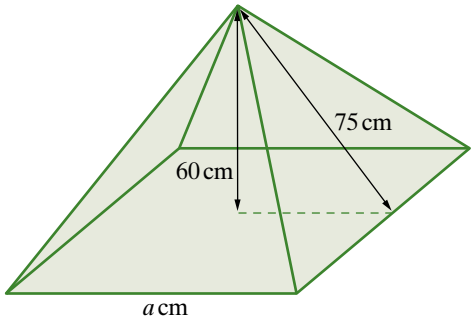
f



g



h



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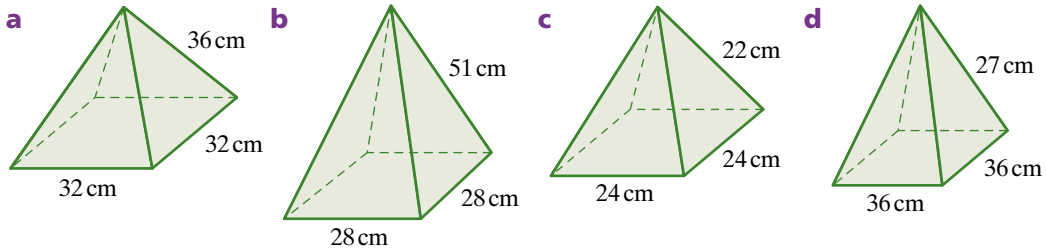
8 A pyramid is 25 cm high and the length of each side of the square base is 30 cm. The perpendicular height of the triangular faces is closest to

A 17 cm **B** 20 cm **C** 29 cm **D** 39 cm **E** 55 cm

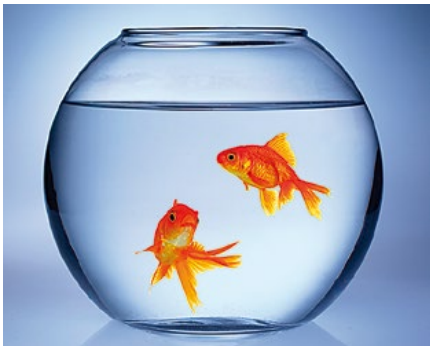
LINKS TO Example 17

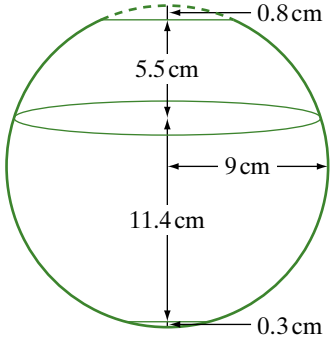
9 Calculate the height of each of these square pyramids. If you work with exact values, each height should be a whole number.



exercise 7.4 challenge

10 In a fish bowl that does not have a pump to aerate the water, oxygen is absorbed into the water from the air. Therefore, the surface area of the water and the size of the opening of the bowl are important.





a Calculate the radius of the opening at the top of the fish bowl.

b Calculate the area of the opening at the top of the bowl.

c Calculate the radius of the surface of the water.

d Calculate the area of the surface of the water.

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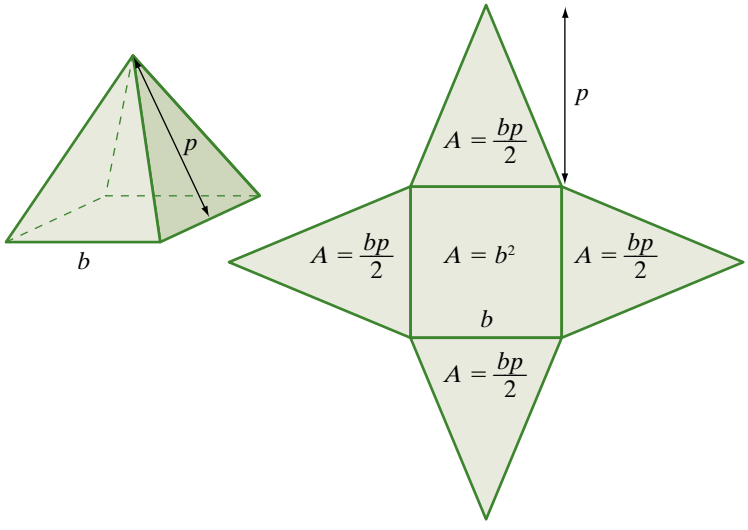
7.5 Surface area of pyramids, cones and spheres

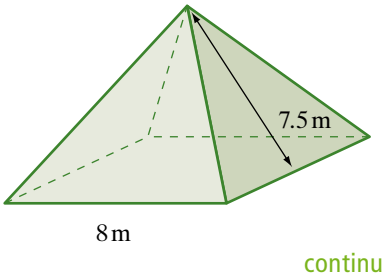
Pyramids

The surface area of a pyramid is found by adding the areas of the base and the triangular faces. The diagram below shows the net of a right square pyramid, where the side length of the square base is b units and the perpendicular height of each triangular face is p units.



Example 18

Calculate the surface area of this square pyramid.



continued

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Example 18 continued

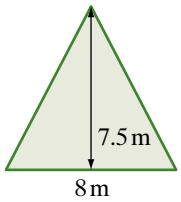
Working

Area of base $= 8^2\ \text{m}^2$
$= 64\,\text{m}^2$

Area of each triangle $= \dfrac{bh}{2}$

$= \dfrac{8 \times 7.5}{2}$

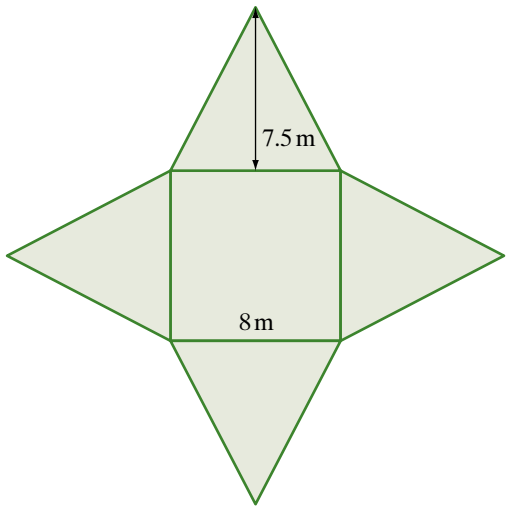
$= 30\,\text{m}^2$



Surface area $= 64 + 30 \times 4$
$= 184$

The surface area is $184\,\text{m}^2$.

Reasoning

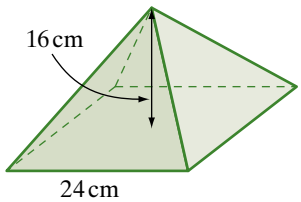


Total surface area is the area of the square base plus the area of the four triangular faces.

If we know the height of the pyramid and the length of the base, we can use Pythagoras' theorem to calculate the perpendicular height of the triangular faces.

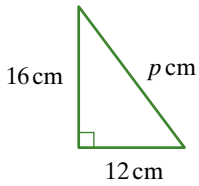
Example 19

Calculate the surface area of this square pyramid.



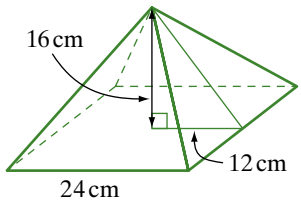
Working

Let p cm be the perpendicular height of each triangular face.



Reasoning

To calculate the area of each triangular face, the perpendicular height of each triangle is needed.



continued

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Example 19 continued

Working

$p^2 = 12^2 + 16^2$
$p^2 = 400$
$p = 20$

Surface Area $= 24^2 + 4 \times \dfrac{24 \times 20}{2}$
$= 576 + 960$
$= 1536\,\text{cm}^2$

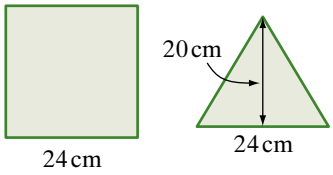
The surface area of the pyramid is $1536\,\text{cm}^2$.

Reasoning

Use Pythagoras' theorem to find p.

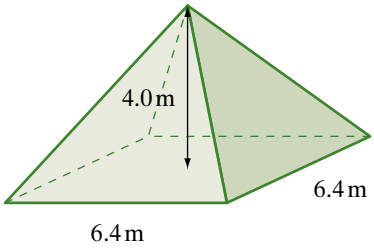
Did you recognise that 12, 16, 20 is a multiple of the Pythagorean triple 3, 4, 5?

The total surface area is equal to the area of the square base plus the area of the four triangular faces.



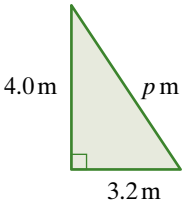
Example 20

Find the surface area of this square pyramid, correct to one decimal place.



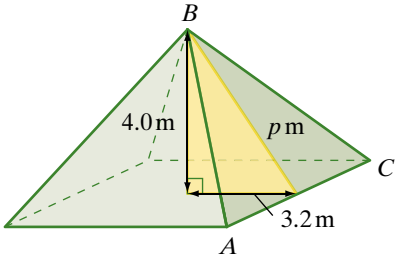
Working

$p^2 = 3.2^2 + 4.0^2$
$p^2 = 10.24 + 16.00$
$p^2 = 26.24$
$p = \sqrt{26.24}$



Reasoning

To find the area of a triangular face, we must first find p, the perpendicular height of $\triangle ABC$.



continued

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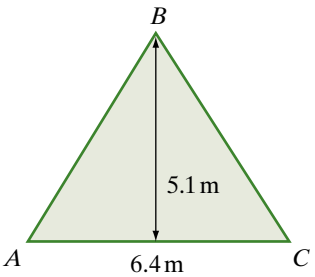
Example 20 continued

Working



Area of each triangular face

$= \frac{1}{2} \times 6.4 \times \sqrt{26.24}$

Total surface area

$= 6.4^2 + 4 \times \frac{1}{2} \times 6.4 \times \sqrt{26.24}$

$\approx 106.53 \text{ m}^2$

The surface area of the pyramid is approximately 106.5 m^2.

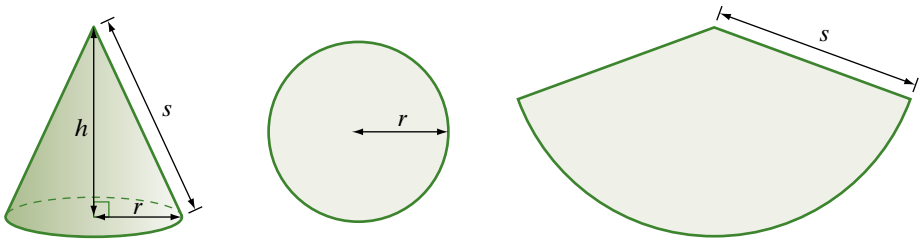
Reasoning

Total surface area is the area of thesquare base plus the area of the four triangular faces.

Use the exact value of the height, $\sqrt{26.24}$ cm, rather than using an approximate value.

Surface area of a cone

The surface of a cone is made up of two parts: the circle that forms the base, and the slanting curved surface. If this curved part is opened out flat, we see that it is a sector of a circle with radius s equal to the **slant edge** of the cone.



The circumference of the base of a cone of radius r is $2\pi r$. This is also the length around the arc of the flattened sector.

So we can see that the area of the curved surface of the cone (the green sector) is the fraction $\frac{\theta}{360}$ of the area of a circle with radius s.

So the area and the arc length are both proportional to the sector angle, θ. We can therefore write the relationship

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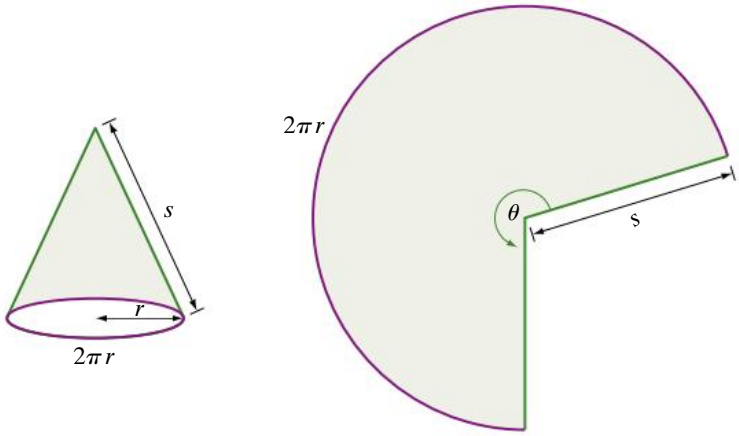
$$\frac{\text{Arc length of sector}}{\text{Circumference of circle with radius } s} = \frac{\text{Area of sector}}{\text{Area of circle with radius } s}$$

$$\frac{2\pi r}{2\pi s} = \frac{\text{Area of sector}}{\pi s^2}$$

Rearranging and simplifying, we obtain

$$\text{Area of sector} = \frac{2\pi r \times \pi s^2}{2\pi s} = \pi r s$$

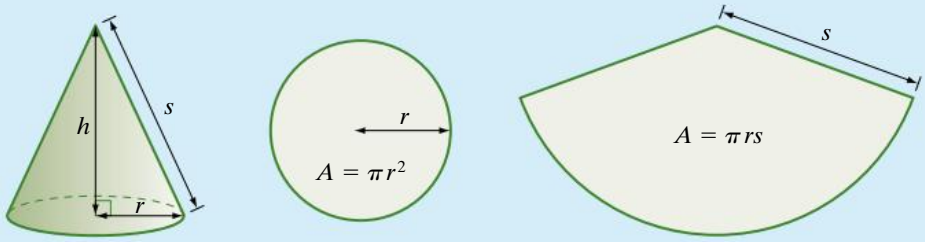


Surface area of a cone

Area of circular base of cone $= \pi r^2$

Area of curved surface of cone $= \pi r s$

Total surface area of cone $= \pi r^2 + \pi r s$



$s^2 = h^2 + r^2$
$s = \sqrt{h^2 + r^2}$

Base

Curved surface

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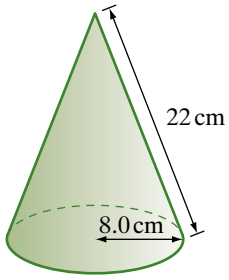
Example 21

Calculate, correct to the nearest square centimetre

a the area of the curved surface of this cone.

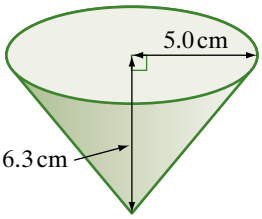
b the total surface area.



Working	Reasoning
a $r = 8.0$, $s = 22$	The radius of the base of the cone is 8.0 cm. The sloping edge is 22 cm.
Area of curved surface $= \pi rs$ $= \pi \times 8.0 \times 22$ $\approx 552.9\,\text{cm}^2$	
The area of the curved surface is approximately $553\,\text{cm}^2$.	Round to the nearest square centimetre.
b Area of base $= \pi r^2$ $= \pi \times 8.0^2$ $\approx 201.1\,\text{cm}^2$	The base is a circle with radius 8.0 cm.
Surface area = area of curved surface + area of base $= 552.9 + 201.1$ $= 754\,\text{cm}^2$	Leave the area of the base as $201.1\,\text{cm}^2$. Round after adding the two separate areas.
Surface area is approximately $754\,\text{cm}^2$.	Round to the nearest square centimetre.

Example 22

Find the area of the curved surface of this cone, correct to the nearest square centimetre.

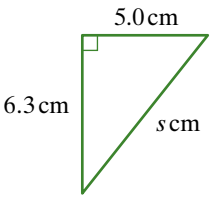


Working

$s^2 = 5.0^2 + 6.3^2$

$= 64.69$

$s = \sqrt{64.69}$



Reasoning

The slant edge s is required.

Pythagoras' theorem can be used to calculate s.

continued

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Example 22 continued

Working

Area of curved surface $= \pi rs$

$= \pi \times 5.0 \times \sqrt{64.69}$

≈ 126.339

Reasoning

Substitute $r = 5.0$ and $s = \sqrt{64.69}$.

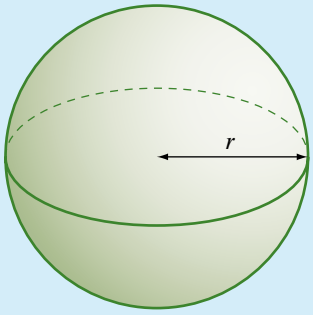
Note that the exact value is used for s to avoid rounding errors.

Surface area of a sphere

The surface area of a sphere is four times the area of a circle with the same radius.

Surface area of a sphere

$$SA = 4\pi r^2$$



Example 23

Calculate the surface area of a sphere with diameter 24 cm, to the nearest square centimetre.

Working

Surface area $= 4\pi r^2$

$= 4 \times \pi \times 12^2$

$\approx 1809.6\,\text{cm}^2$

The surface area is approximately $1810\,\text{cm}^2$.

Reasoning

The radius is 12 cm, so use $r = 12$ in the formula.

Example 24

Find the radius of a sphere that has a surface area of $1.0\,\text{m}^2$. Give the radius in centimetres to the nearest centimetre.

continued

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Example 24 continued

Working

$$\text{Surface area} = 4\pi r^2$$
$$1.0 = 4\pi r^2$$
$$r = \sqrt{\frac{1.0}{4\pi}}$$
$$\approx 0.282$$

The radius of the sphere is approximately 0.28 m or 28 cm.

Reasoning

Substitute 1.0 for the surface area and solve the equation for r.

Round to the nearest centimetre.
1 m = 100 cm

exercise 7.5

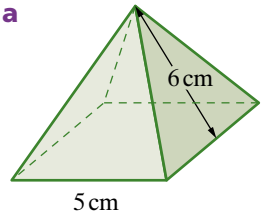
LINKS TO Example 18

1 For each of these square pyramids calculate

i the total area of the four triangular faces.

ii the total surface area of the pyramid.

a



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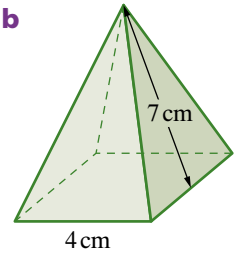

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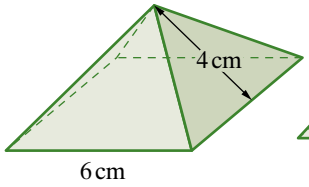

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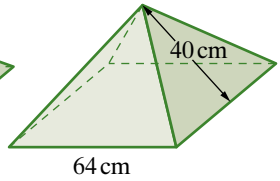

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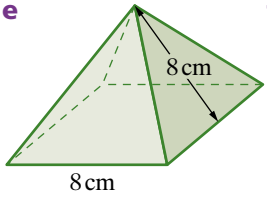

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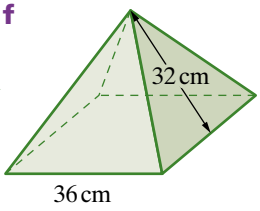

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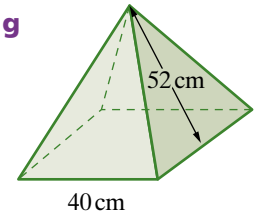

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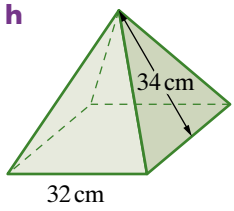
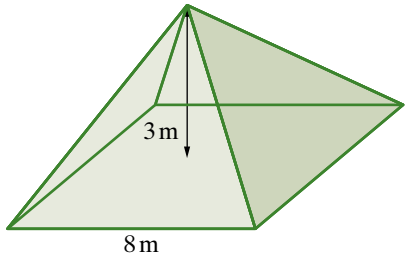


2 In square metres, the total surface area of this square pyramid is

A 80
B 112
C 144
D 160
E 224



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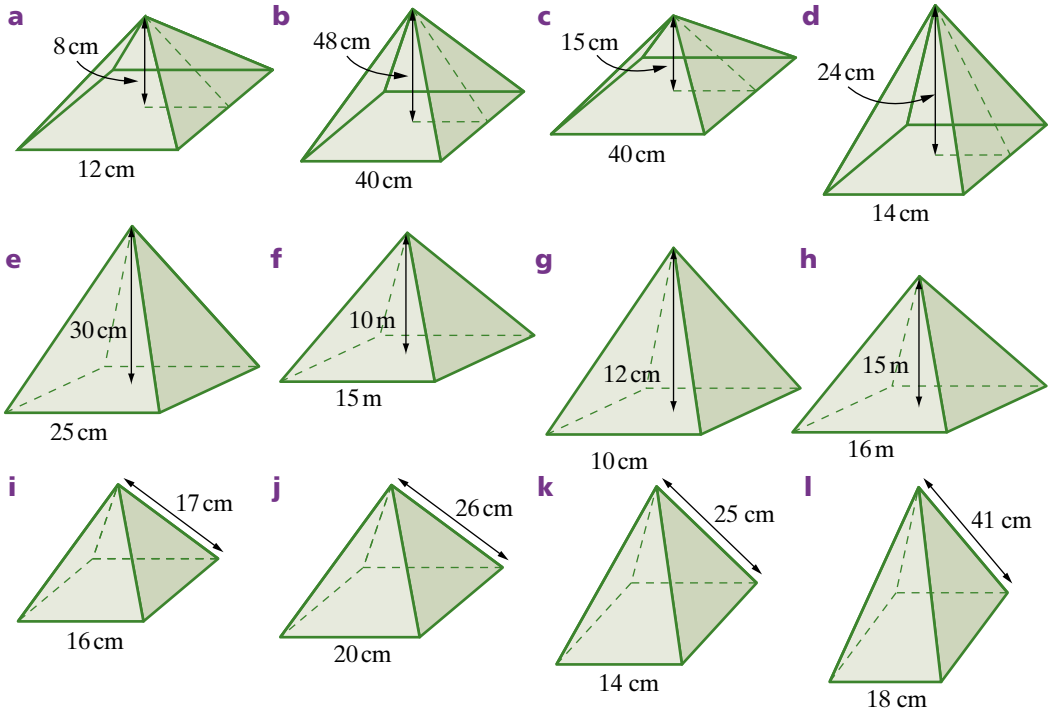
LINKS TO Examples 19, 20

3 For each of these square pyramids calculate

i the total area of the four triangular faces.

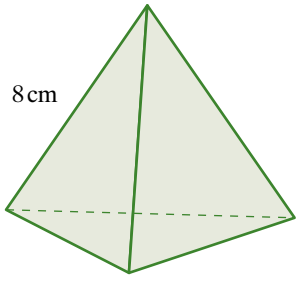
ii the total surface area of the pyramid.



4 Calculate the surface area of this regular tetrahedron (each of the faces are equilateral triangles)

a as an exact value.

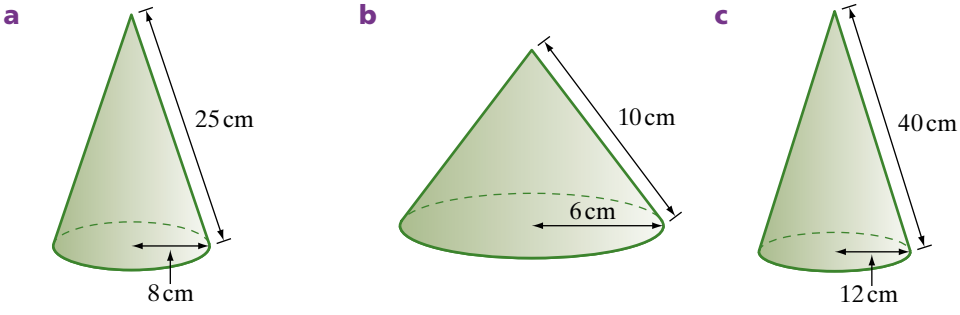
b correct to one decimal place.



LINKS TO Examples 21, 22

5 For each of these cones calculate the following, correct to the nearest square centimetre.

i the area of the curved surface. **ii** the total surface area.



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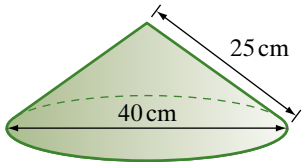

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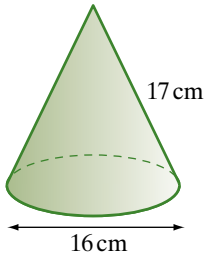

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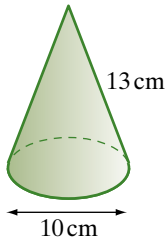

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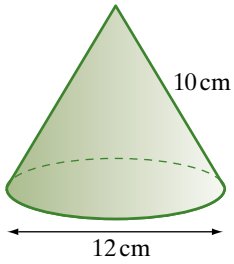

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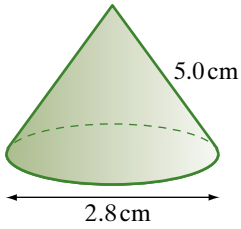

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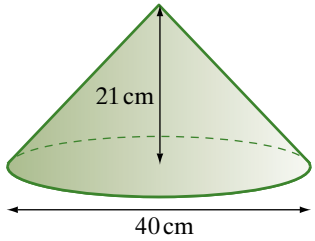

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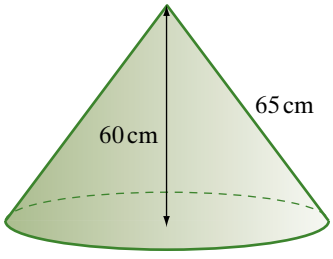

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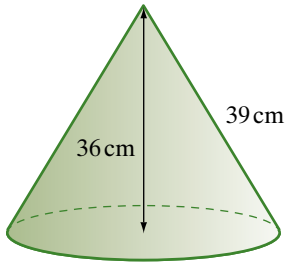

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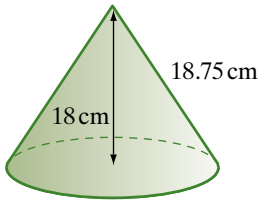
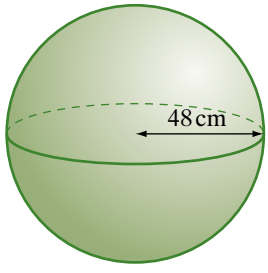


LINKS TO Example 23

6 Calculate the surface area of each of these spheres, correct to one decimal place.

a



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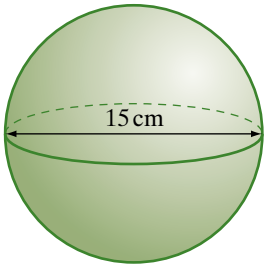

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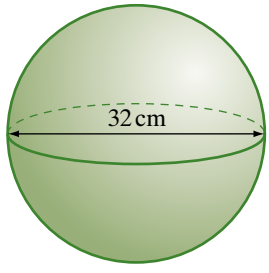

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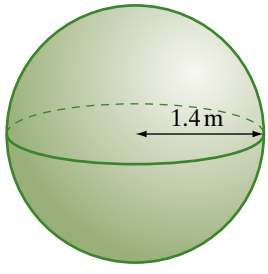

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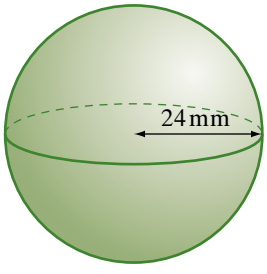

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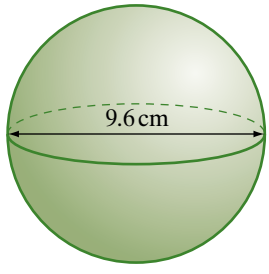

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7 Calculate the surface area of each of these spheres, correct to one decimal place.

a radius 5 cm **b** diameter 40 cm **c** diameter 8 cm
d radius 1.0 m **e** radius 25 mm **f** diameter 12 cm

LINKS TO Example 24

8 Calculate the radius of spheres with the following surface areas. Give the radius correct to one decimal place.

a $20\,\text{cm}^2$ **b** $36\,\text{cm}^2$ **c** $2827\,\text{cm}^2$
d $7854\,\text{mm}^2$ **e** $4\,\text{m}^2$ **f** $31\,416\,\text{cm}^2$

exercise 7.5 challenge

9 The triangular faces of a square pyramid are equilateral triangles. The length of each side of the square base is a units. Show that the total surface area is given by $a^2(1 + \sqrt{3})$.

10 Consider the relationship between the radius and surface area of a sphere.

a Calculate the surface areas of spheres with the following radii. Give the areas as exact values.

i 2 cm **ii** 4 cm **iii** 8 cm **iv** 16 cm

b As the radius doubles, how does the surface area increase?

11 A balloon is blown up to a radius of 5 cm. It is then blown up so that its radius increases to 10 cm. By what factor will the surface area change?

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10 A

7.6 Volume of pyramids, cones and spheres

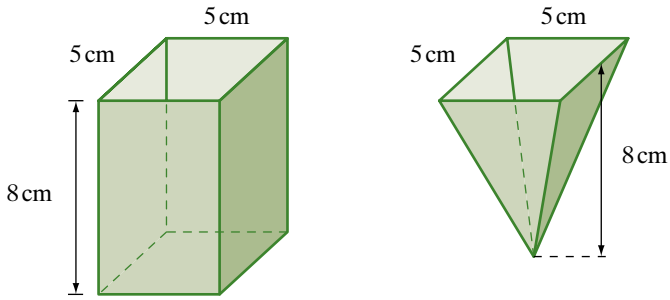
To investigate how the volume of a pyramid is related to its height and base area, we first consider a square-based pyramid. We compare the volume of a square pyramid with the volume of a square prism with the same square base and height.

Volume of a pyramid

Investigation

BLM Volume of pyramid

Using the file on the *OneStopDigital* ebook, print the net for a rectangular prism and a pyramid. To construct each shape, cut out and fold the net, and tape the edges together.



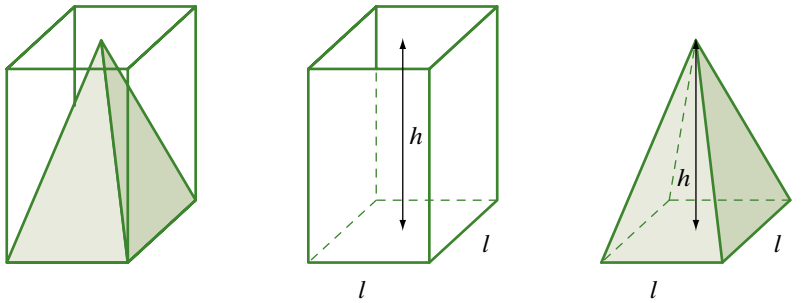
The prism and the pyramid both have the same height and the same sized square 'base'.

Carefully fill the inverted pyramid with dry material such as rice or small lentils, then tip the contents into the rectangular prism.

a How many times do you need to fill the pyramid in order to fill the rectangular prism?
b Calculate the volume of the rectangular prism.
c What do you think is the volume of the pyramid?

The investigation is not a mathematical proof but it is a useful demonstration to show the relationship between the volumes of a square prism and a square pyramid with the same base and height. With more advanced mathematics it is possible to prove the relationship.

The volume of a right square pyramid is one-third of the volume of a rectangular prism with the same square base and same height.



$V = l^2h$ $\qquad$ $V = \frac{1}{3} \times l^2h = \frac{l^2h}{3}$

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Just as the volume, V, of any prism is given by $V = Ah$, where A is the area of the base and h is the height of the prism, then the volume of a pyramid is given by $V = \frac{1}{3}Ah$.

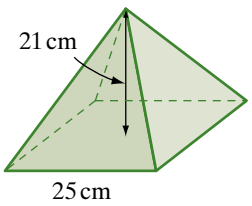
Volume of a pyramid

$V = \frac{1}{3}Ah$ where A is the area of the base and h is the height of the pyramid.

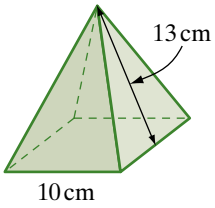
Example 25

Calculate the volume of each of these square pyramids.

a



b



Working

a $V = \frac{1}{3}Ah$

$= \frac{1}{3} \times 25^2 \times 21$

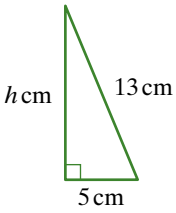
$= 4375$

The volume is $4375\,\text{cm}^3$.

Reasoning

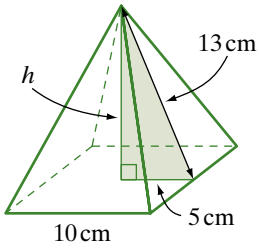
The length of the edges of the square base and the perpendicular height of the pyramid are given.

b



5, 12, 13 is a Pythagorean triple, so $h = 12$.

The perpendicular height of the pyramid is not given. This must be calculated first.



$V = \frac{1}{3}Ah$

$= \frac{1}{3} \times 10^2 \times 12$

$= 400$

The volume is $400\,\text{cm}^3$.

Substitute in the formula for the volume of a square pyramid.

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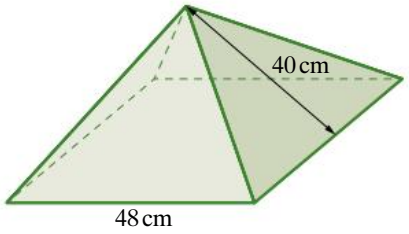
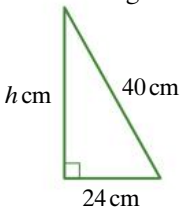
Example 26

Find the volume of this square pyramid.



Working

Let the height of pyramid be h cm.



$h^2 = 40^2 - 24^2$

$h^2 = 1600 - 576$

$h^2 = 1024$

$h = \sqrt{1024}$

$h = 32$

$V = \frac{1}{3}Ah$

$= \frac{1}{3} \times 48^2 \times 32$

$V = 24\,576$

The volume is $24\,576\,\text{cm}^3$.

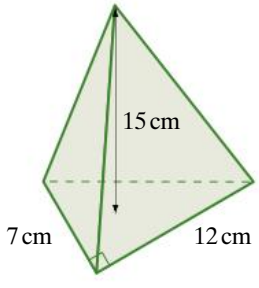
Reasoning

Draw the right-angled triangle.
The base of triangle is half the side length of the square base.
Calculate the height, h cm.

Calculate the volume, substituting $A = 48^2$ and $h = 32$.

Example 27

Calculate the volume of this triangular pyramid.



continued

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Example 27 continued

Working

$V = \frac{1}{3}Ah$

$= \frac{1}{3}\left(\frac{1}{2} \times 7 \times 12\right) \times 15$

$= 210$

The volume is 210 cm^3.

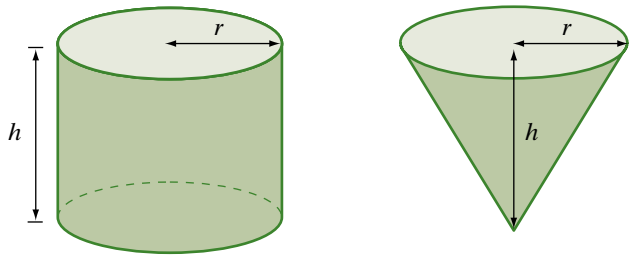
Reasoning

The volume of a pyramid.
The area of the right-angled triangle base is $\frac{1}{2} \times 7 \times 12$ cm^2.

Cones

We will now compare the volume of a cone with the volume of a cylinder with the same radius and height.

Investigation



a Based on your knowledge of the volume of a pyramid and a rectangular prism with the same base and height, make a conjecture about the volume of the cone compared with the volume of the cylinder.

BLM Volume of cone

To compare the volume of a cone with the volume of a cylinder with the same height and radius, you will need an empty 425 mL tin, such as a sliced pineapple tin, and a cone made from the net on the *OneStopDigital* ebook. Carefully fill the inverted cone with dry material such as rice or small lentils, and then tip the contents into the cylinder.

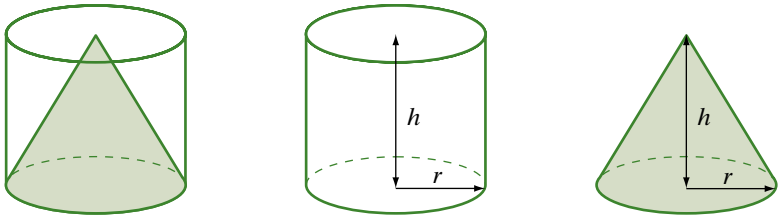
b How many times do you need to fill the cone in order to fill the cylinder?

c Calculate the volume of the cylinder.

d What do you think is the volume of the cone?

Just as for the pyramid investigation, this is not a mathematical proof of the relationship between the volume of the cone and the volume of the cylinder, but with more advanced mathematics it is possible to prove the relationship.

The volume of a cone is one-third of the volume of a cylinder with the same radius and same height.



$V = \pi r^2 h$ $\qquad$ $V = \frac{1}{3} \times \pi r^2 h = \frac{\pi r^2 h}{3}$

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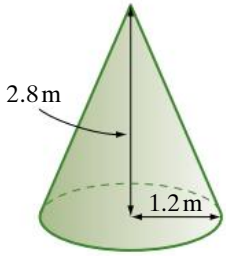
Just as the volume, V, of any cylinder is given by $V = Ah = \pi r^2 h$, where A is the area of the base and h is the height of the cylinder, then the volume of a cone is given by $V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$.

Volume of a cone

$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height of the cone.

Example 28

Calculate the volume of this cone, correct to one decimal place.



Working

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 1.2^2 \times 2.8$

≈ 4.22

The volume is approximately $4.2\,\text{m}^3$.

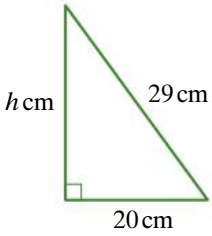
Reasoning

The required dimensions—radius and perpendicular height—are known.

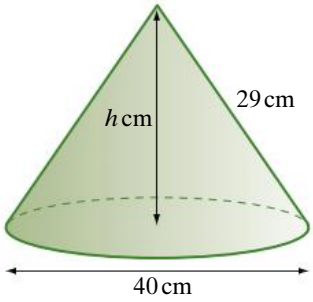
Example 29

Calculate the volume of a cone that has a slant edge of 29 cm and diameter 40 cm, correct to the nearest cubic centimetre.

Working



Reasoning



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Example 29 continued

Working

$h^2 + 20^2 = 29^2$

$h^2 = 841 - 400$

$h^2 = 441$

$h = \sqrt{441}$

$h = 21$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 20^2 \times 21$

≈ 8796.45

Volume is approximately 8796 cm^3.

Reasoning

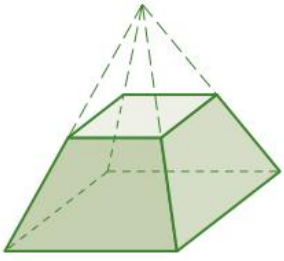
Use Pythagoras' theorem to find h.

diameter = 40 cm
radius = 20 cm

Truncated pyramids and cones

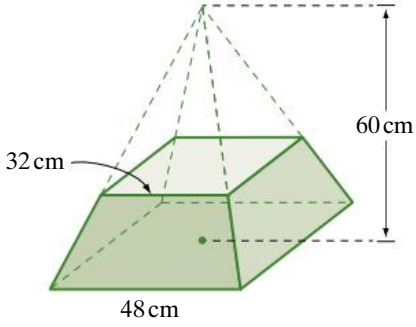
When a pyramid has the top cut off it forms a **truncated pyramid** (also called a **frustum**). The pyramid cut off is similar to the original pyramid.

When a cone has the top cut off it forms a **truncated cone** (also called a **frustum**). The cone cut off is similar to the original cone.



Example 30

Calculate the volume of this frustum, correct to the nearest cubic centimetre.
Let x cm be the height of the small cone.
Use similar triangles.



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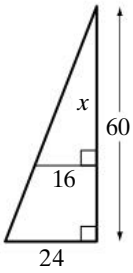
Example 30 continued

Working



$$\frac{x}{60} = \frac{16}{24}$$

$$x = \frac{16}{24} \times 60$$

$$x = 40$$

Volume of small pyramid:

$$V = \frac{1}{3}l^2h$$

$$= \frac{1}{3} \times 32^2 \times 40$$

$$V = 13\,653.3$$

Volume of large pyramid:

$$V = \frac{1}{3}l^2h$$

$$= \frac{1}{3} \times 48^2 \times 60$$

$$V = 46\,080$$

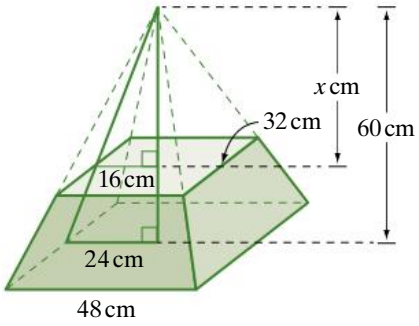
Volume of frustum:

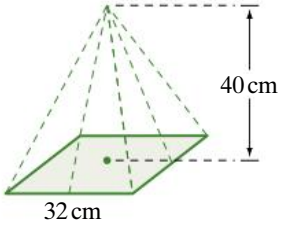
$46\,080 - 13\,653.3 = 32\,426.7$

The volume of the frustum is $32\,427\,\text{cm}^3$.

Reasoning

The two pyramids are similar.





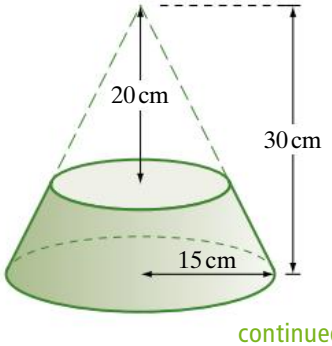
Side length of base = 48 cm
Height = 60 cm

Subtract the volume of the small pyramid from the volume of the large pyramid.

Example 31

Find the volume of this truncated cone (frustum) by first finding the volumes of the large and small cones.

Give the volume correct to the nearest cubic centimetre.



continued

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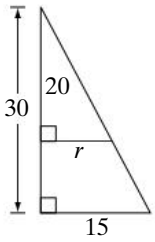
Example 31 continued

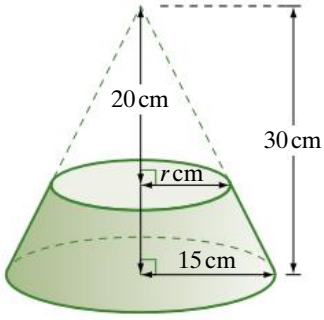
Working

Let r cm be the radius of the base of the small cone.
Using similar triangles

$$\frac{r}{15} = \frac{20}{30}$$
$$r = 10$$



Reasoning



Volume of large cone $= \frac{1}{3}\pi \times 15^2 \times 30$
≈ 7068.6

To find the volume of the large cone, substitute $r = 15$ and $h = 30$ into the formula $V = \frac{1}{3}\pi r^2 h$.

Volume of small cone $= \frac{1}{3}\pi \times 10^2 \times 20$
≈ 2094.4

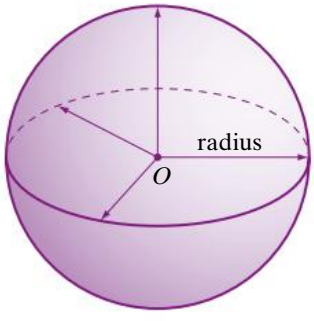
To find the volume of the small cone, substitute $r = 10$ and $h = 20$ into the formula $V = \frac{1}{3}\pi r^2 h$.

Volume of frustum $= 7068.6 - 2094.4$
$= 4974.2$

Subtract the volume of the small cone from the volume of the large cone.

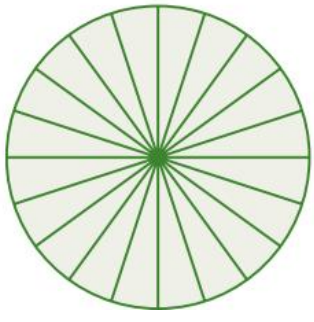
The volume of the frustum is approximately $4974\,\text{cm}^3$.

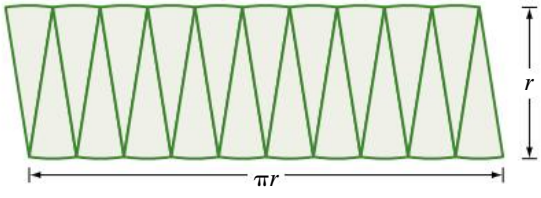
Volume of a sphere



A sphere is the three-dimensional equivalent of a circle. The distance from the centre O of the sphere to any point on the surface of the sphere is the same. This is the radius of the sphere. To indicate that a drawing represents a sphere rather than a circle, we usually draw an ellipse to represent the circumference of the sphere.

You may recall from *MathsWorld 8* how we divided the circle into a large number of sectors and arranged the sectors to form an approximate parallelogram. The area of the 'parallelogram' can be approximated by $\pi r \times r$, that is, πr^2.





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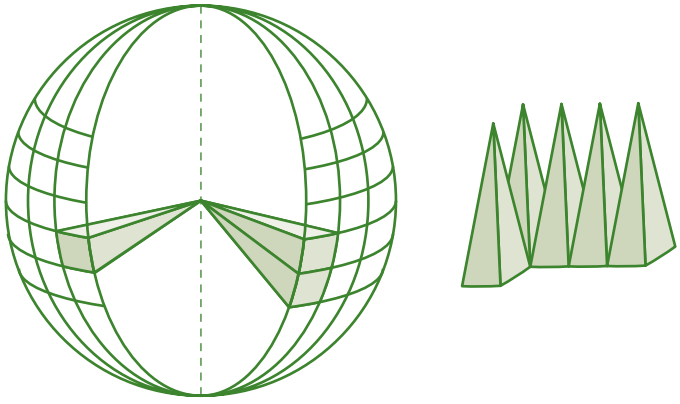
If we could divide the circle into an infinite number of infinitely thin sectors, then the area of the parallelogram would be exactly equal to the area of the circle. So the area of a circle is given by πr^2.

In a similar way we can see how the formula for the volume of a sphere comes about. Suppose that we divide the surface of a sphere into a large number of 'approximate squares'. (These shapes will actually be slightly curved rather than flat, so they will not really be squares.) Imagine that each 'square' represents the base of a pyramid, whose height is equal to the radius r of the sphere.

$$\begin{aligned}\text{Volume of each 'square pyramid'} &= \frac{1}{3} \times \text{Area of base} \times \text{Height}\\ &= \frac{1}{3} \times \text{Area of base} \times r\end{aligned}$$

If we stood all these pyramids together, the total area of their bases would be equal to the surface area of the sphere, which we know is $4\pi r^2$.



So the total volume of all the pyramids that make up the sphere is approximately equal to

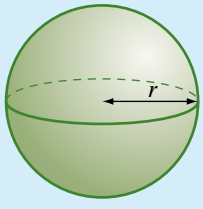
$$\begin{aligned}&\frac{1}{3} \times \text{Area of all the bases} \times r\\ &= \frac{1}{3} \times 4\pi r^2 \times r\\ &= \frac{4}{3}\pi r^3\end{aligned}$$

If we could divide the sphere into an infinite number of pyramids with infinitely small bases, then the volume of all the pyramids becomes exactly equal to the volume of the sphere. So the volume of a sphere is given by $\frac{4}{3}\pi r^3$.

Volume of a sphere

$$V = \frac{4}{3}\pi r^3$$

where r is radius of the sphere.



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Example 32

Find the volume of a sphere with radius 18.0 cm correct to the nearest cubic centimetre.

Working

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 18.0^3$$
$$\approx 24\,429.02$$

The volume of the sphere is approximately 24 429 cm^3.

Reasoning

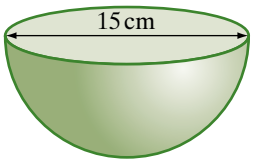
Substitute $r = 18.0$ in the formula for the volume of a sphere.

Round to the nearest cubic centimetre.

Example 33

Find the volume of a hemispherical solid with diameter 15 cm correct to the nearest cubic centimetre.

Working



$$V = \frac{1}{2} \times \frac{4}{3}\pi r^3$$
$$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 7.5^3$$
$$\approx 883.57$$

The volume of the solid is approximately 884 cm^3.

Reasoning

A hemisphere is half a sphere, so the volume of the solid will be half the volume of the sphere with the same radius.

The diameter is 15 cm so the radius is 7.5 cm.

Round to nearest cubic centimetre.

We can use the formula to find the radius of a sphere with a given volume.

Example 34

Calculate the radius of a sphere with volume 500 cm^3. Give the radius correct to one decimal place.

Working

$$V = \frac{4\pi r^3}{3}$$
$$r^3 = \frac{3V}{4\pi}$$
$$= \frac{3 \times 500}{4\pi}$$

Reasoning

Rearrange the formula to make r^3 the subject. Substitute $V = 500$.

continued

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Example 34 continued

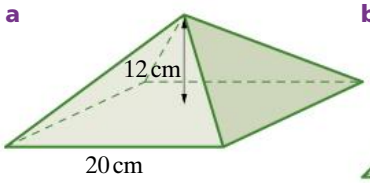
Working	Reasoning
$r^3 \approx 119.37$	Find the cube root of 119.37.
$r = \sqrt[3]{119.37}$	
$r \approx 4.92$	
The radius is approximately 4.9 cm.	Round to one decimal place.

exercise 7.6

LINKS TO Example 25

1 Calculate the volume of each of these square pyramids.

a



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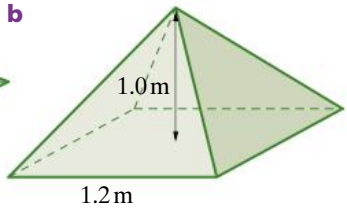

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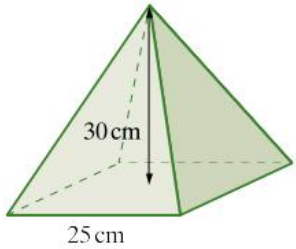

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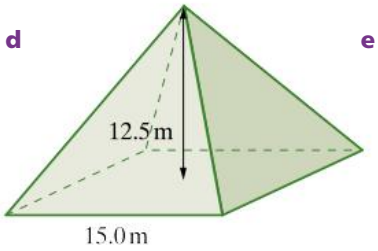

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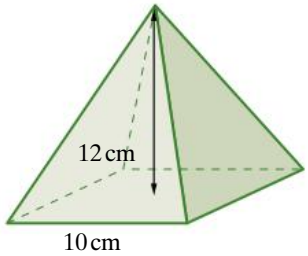

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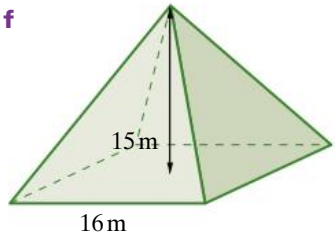

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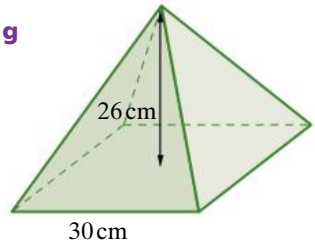

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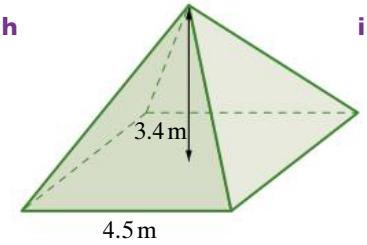

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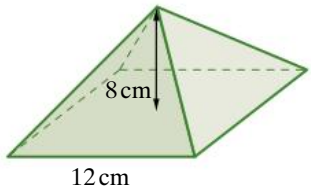

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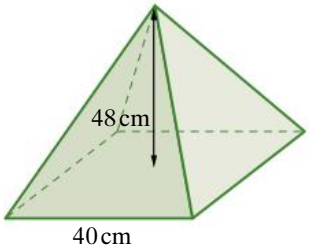

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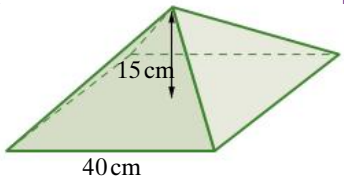

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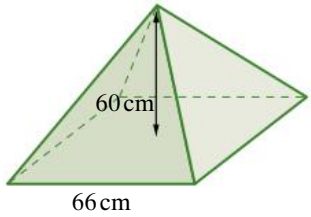

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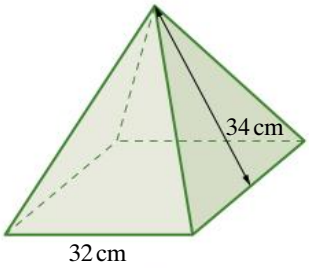
LINKS TO Example 26

2 For each of these square pyramids

i calculate the height. **ii** calculate the volume.

a



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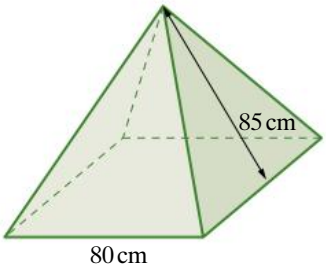

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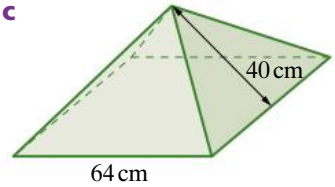

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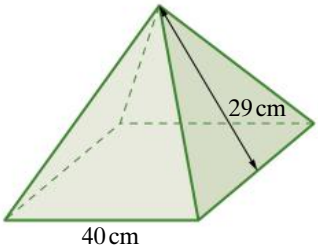

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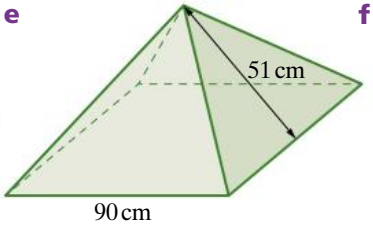

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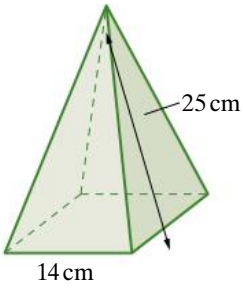
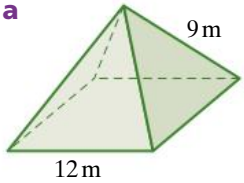


3 For each of these square pyramids

i calculate the height. **ii** calculate the volume.

a



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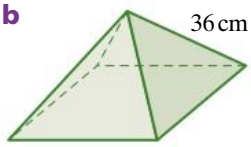

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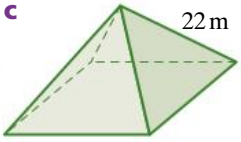

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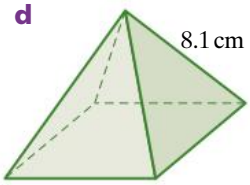
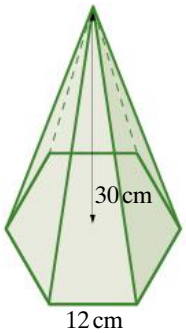


LINKS TO Example 27

4 Calculate the volume of this regular hexagonal pyramid

a as an exact value.

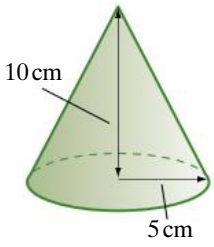
b correct to the nearest cubic centimetre.



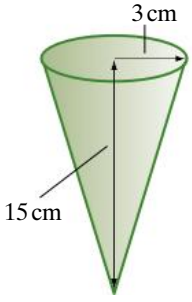
LINKS TO Example 28

5 Calculate the volume of each of these cones, correct to the nearest cubic centimetre.

a



b



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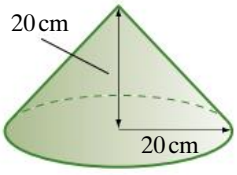

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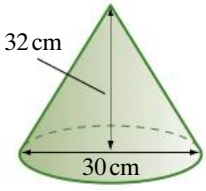

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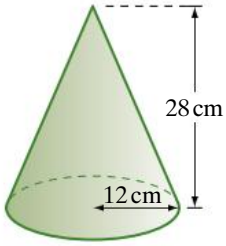
e



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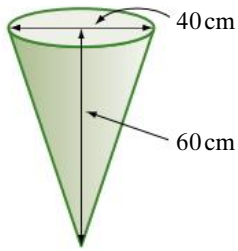

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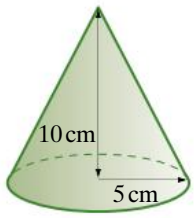

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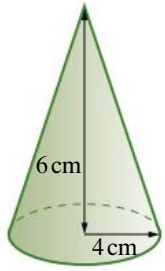

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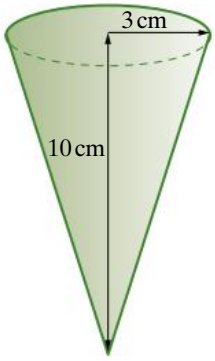

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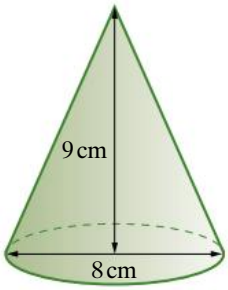

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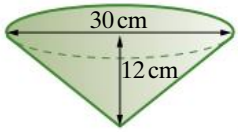

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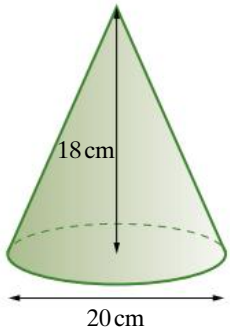


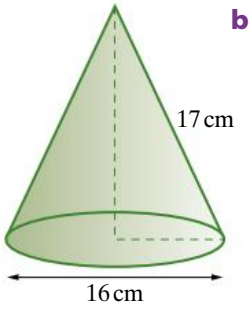
6 The diameter of the base of a cone is 6 cm and the height of the cone is 10 cm. The volume of the cone, in cubic centimetres, is

A 30π **B** 60π **C** 90π **D** 120π **E** 360π

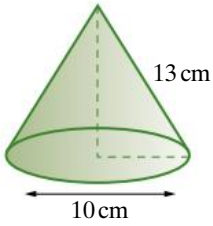
LINKS TO Example 29

7 Calculate the volume of each of these cones. Give each volume correct to the nearest cubic centimetre.

a



b



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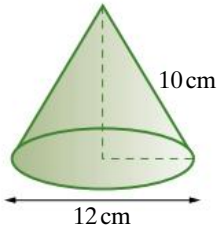

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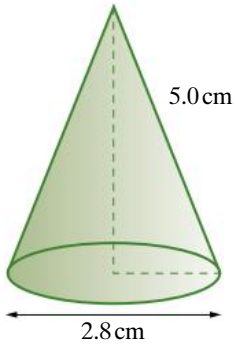
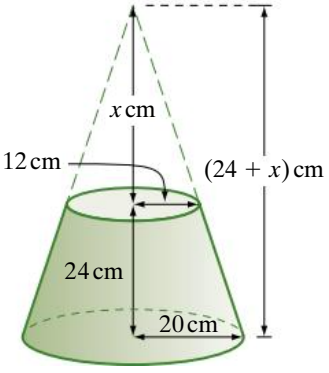
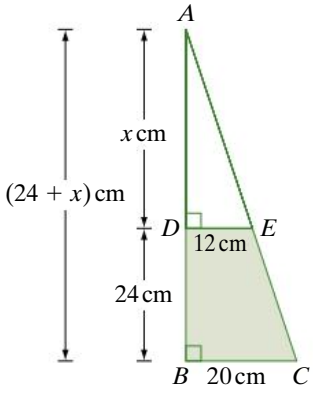


LINKS TO Example 30

8





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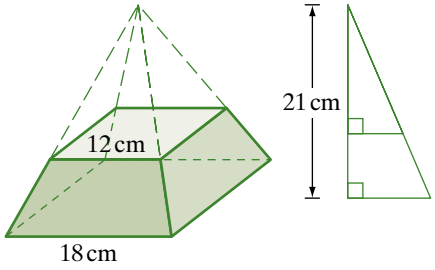
a Explain why $\triangle ADE$ and $\triangle ABC$ are similar.
b Using the fact that $\triangle ADE$ and $\triangle ABC$ are similar, write an equation that includes x.
c Solve the equation to find the value of x.
d Explain how the volume of the frustum can be calculated.
e Calculate the volume of the frustum.

LINKS TO Example 31

9 For each of these truncated pyramids, calculate:
- **i** the height of the dotted pyramid that has been removed (use similar triangles).
- **ii** the volume of the large pyramid.
- **iii** the volume of the small pyramid.
- **iv** the volume of the truncated pyramid (or frustum).

a



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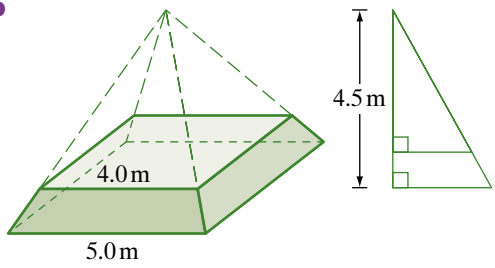
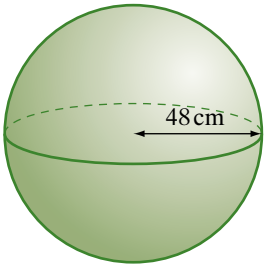


LINKS TO Example 32

10 Calculate the volume of each of these spheres, correct to the nearest cubic millimetre, centimetre or cubic metre.

a



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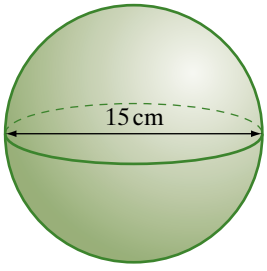

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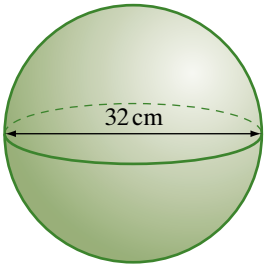

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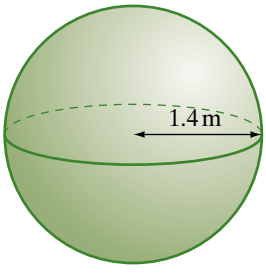

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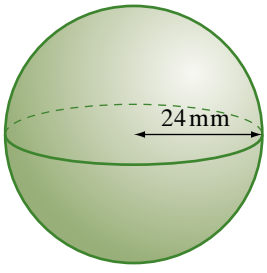

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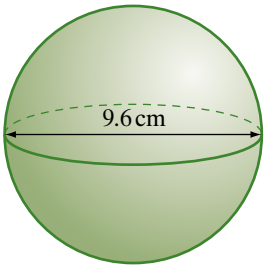

11 Calculate the volume of each of these spheres to the nearest cubic millimetre, cubic centimetre or cubic metre.

a radius 5 cm
b diameter 40 cm
c diameter 8 cm
d radius 1.0 m
e radius 25 mm
f diameter 12 cm

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LINKS TO Example 33

12 Calculate the volume of the following hemispheres, correct to one decimal place.

a radius 20 cm **b** radius 45 cm **c** radius 28 mm
d radius 1.4 m **e** diameter 80 mm **f** diameter 60 cm

LINKS TO Example 34

13 Calculate the radius of spheres with the following volumes, correct to one decimal place.

a $200\,\text{cm}^3$ **b** $2.5\,\text{m}^3$ **c** $40\,\text{cm}^3$
d $1800\,\text{mm}^3$ **e** $64\,\text{cm}^3$ **f** $325\,\text{cm}^3$

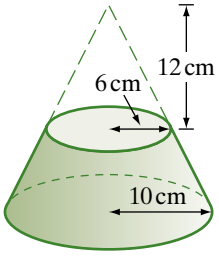
exercise 7.6 challenge

14 The triangular faces of a square pyramid are equilateral triangles. The length of each side of the square base is a units. Show that the volume is given by $\frac{\sqrt{2}a^3}{6}$.

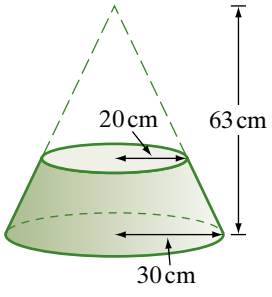
15 For each of these truncated cones, calculate

i the missing height.
ii the volume of the large cone.
iii the volume of the small cone.
iv the volume of the truncated cone (or frustum), to the nearest cubic centimetre.

a



b



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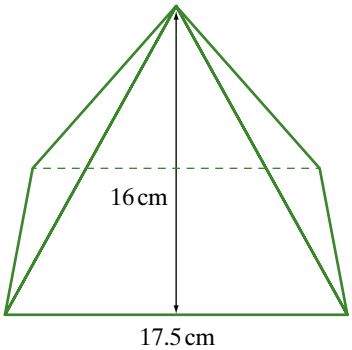
7.7 Surface area and volume applications for pyramids, cones and spheres

All the questions that follow are based on prisms, cylinders, pyramids, cones or spheres. For each question you will need to decide whether the calculation involves surface area or volume.

exercise 7.7

1 Chocolates are packaged in a transparent plastic pyramid. The length of the sides of the square base is 17.5 cm. The perpendicular height of each triangular face is 16 cm. Calculate the total surface area of the plastic pyramid. Give the area correct to the nearest square centimetre.



2 An upstairs extension has been added to a house, as shown in this photograph. The roof of the extension is a square pyramid with side length 6.5 m and slant edge 4.25 m. Calculate the surface area of the pyramid roof to the nearest square metre.

3 The Great Pyramid in Egypt was completed in 2580 BCE. It has been estimated that 4000 workers took 30 years to build the pyramid, which was constructed from approximately 2.5 million blocks of stone. The pyramid has a square base of side length 230 m and the original height is believed to be 146.5 m.

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a How many hectares does the base of the pyramid cover? (1 hectare = 10 000 m^2)

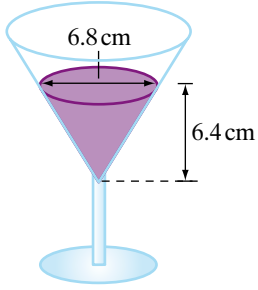
b Calculate the volume of the pyramid, correct to the nearest thousand cubic metres.

c Calculate the average volume of each block of stone.

d If the average mass of each stone block is approximately 3 tonnes, estimate the total mass of the pyramid.

e Calculate the surface area of the pyramid correct to the nearest square metre (not including the base).

4 Wine is poured into a conical glass to a depth of 6.4 cm. If the diameter of the surface of the wine is 6.8 cm, what is the volume of wine in the glass? Give your answer to the nearest millilitre.



5 The height of this party hat is 21 cm and the slant edge is 22.5 cm. It is made from a piece of card then decorated on the outside. Calculate the area to be decorated correct to the nearest square centimetre.

6 The diameter of the base of this ice-cream cone is 45 mm and the sloping edge is 112 mm.

a Calculate the outer surface area of this ice-cream cone correct to the nearest square millimetre.

b If the cone is filled with ice-cream level with the top, what volume of ice-cream would be used? Give the volume correct to the nearest millilitre.

c A hemispherical scoop of ice-cream is then placed on top. What is the total volume of ice-cream used including the ice-cream in the cone? Give the volume correct to the nearest millilitre.

7 **a** An ice-cream cone is completely filled with ice-cream and is topped with a hemispherical scoop as shown. The cone has a height of 8 cm and the diameter at the top of the cone is 5 cm. Calculate the total volume of ice-cream.

b How many litres of ice-cream would be needed for 100 ice-cream cones? Give your answer to the nearest tenth of a litre.

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LINKS TO Example 31

8 A spherical balloon has a radius of 12 cm. If the balloon is further inflated so that its radius is 24 cm, by how many times will the volume increase?

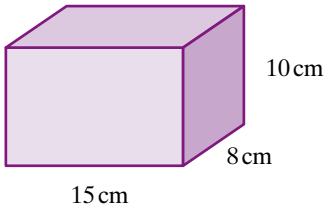
9 Chemical storage tanks for liquids or gases are usually cylinders, but spherical tanks are sometimes used.

- **a** Calculate the radius of a spherical storage tank with volume $500\,m^3$, correct to one decimal place.
- **b** Calculate the surface area of the tank, correct to one decimal place.
- **c** Calculate the height of a cylindrical tank that would have the same radius and the same capacity as the spherical tank, correct to one decimal place.
- **d** Calculate the surface area of the cylindrical tank in part **c**, correct to one decimal place.
- **e** Compare the surface areas of the two tanks.
- **f** Compare the advantages and disadvantages of these two different shapes for storage tanks.

10 The circumference of an orange measures 24.5 cm.

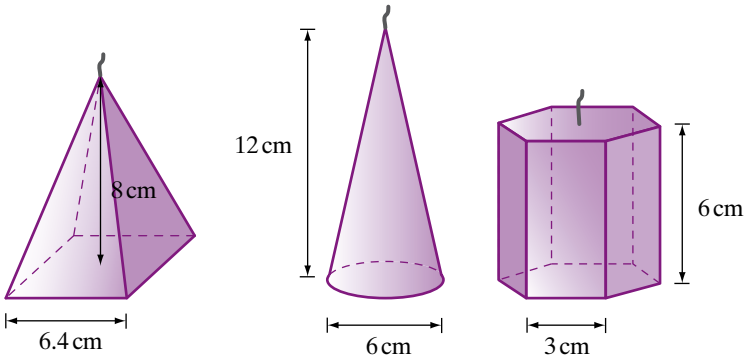
- **a** Calculate the radius of the orange, correct to one decimal place.
- **b** Calculate the volume of the orange, correct to the nearest cubic centimetre.
- **c** If the thickness of the rind is 4 mm, what is the radius of the fleshy inside of the orange?
- **d** Assuming that the fleshy inside is all juice, what is the volume of juice that could be squeezed from the orange? Give this volume correct to the nearest cubic centimetre.
- **e** When the orange was squeezed, it was found that only 160 mL of juice could be obtained. What percentage of the fleshy inside of the orange was juice?
- **f** What percentage of the total volume of the orange was juice?

11 Nelu and Ari are making candles. They have bought a rectangular slab of candle wax, which they melt and then pour into moulds.



- **a** Calculate the volume of the slab of wax.

Nelu and Ari have three different moulds: a square pyramid, a cone and a hexagonal prism. (The base of the prism is a regular hexagon.) They want to make some candles of each type.



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b Nelu and Ari start by making three pyramid candles. Calculate the total volume of wax needed for these.

c Next they make four cones. How much wax do they need for these?

d How much wax have they used altogether?

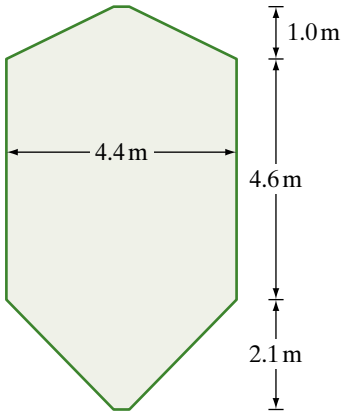
e How much wax do they have left?

f Calculate the volume of the hexagonal prism.
Hint: the hexagonal base can be divided into six equilateral triangles, and you can use Pythagoras' theorem to find the height of those triangles.

g How many hexagonal prism candles could Nelu and Ari make?

exercise 7.7 challenge

12 Many storage silos—such as those for wheat, fertilisers or animal food—are metal cylinders with a truncated cone at the top and bottom, as shown in this photograph and diagram.



Calculate the volume to the nearest cubic metre of a silo with the dimensions shown. Assume the top and the bottom of the silo are cones. The cones are actually truncated to allow for the openings at the top and bottom, so the actual volume will be slightly less than your calculated volume.

13 One of the predicted implications of global warming is an increased rate of melting of Antarctic ice, particularly the West Antarctic ice shelf. This would result in a rise in sea level of all the oceans. Although total melting of the ice might not occur for about 500 years, any melting of the ice will result in some rise in sea level. The approximate volume of ice in the West Antarctic ice shelf is 2.5 million km^3. Predicting the rise in sea level is a complex problem but we can make some rough estimates.

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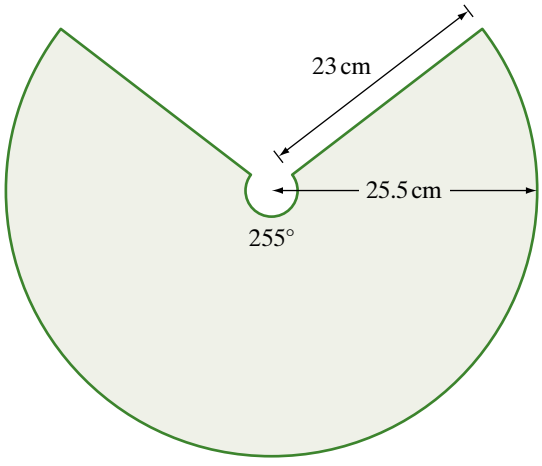
a The radius of the Earth is approximately 6370 km. Taking the Earth to be a sphere, calculate the total surface area of the Earth, including the land and the oceans, correct to the nearest million square kilometres.

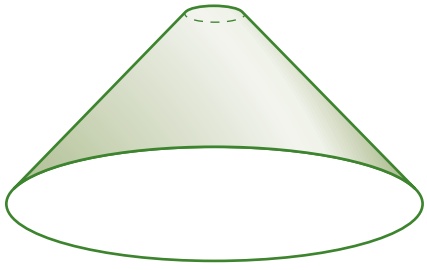
b Approximately 71% of the Earth is covered by oceans. What area of the Earth's surface does this represent, correct to the nearest million square kilometres?

c If the entire West Antarctic ice shelf melted and was distributed over all the oceans, what rise in sea level would this represent? Give the rise in metres. (This will give an overestimate as some of the current land areas would also be covered by the oceans.)

d What percentage of the West Antarctic ice shelf would need to melt to cause a rise in sea level of approximately 1 m?

14 A factory makes lightshades from sheet metal by stamping out sectors as shown in the diagram, with a smaller sector cut from the centre to allow for the electrical fittings. Each metal piece is welded into a cone shape and sprayed with coloured enamel paint on the outside and white inside.





a Calculate the area of the outside of the shade.

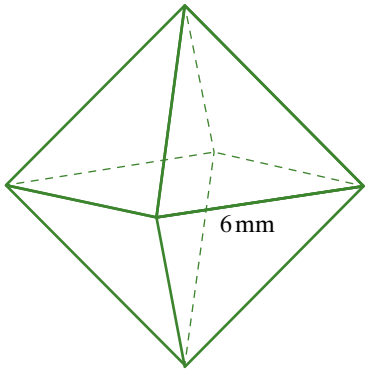
b What is the diameter of the 'base' of the finished lightshade?

15 A polished crystal of the mineral fluorite is in the shape of a regular octahedron. The regular octahedron has the structure of two square pyramids joined by their square bases. The eight faces are congruent equilateral triangles. The length of each edge is 12 mm.

Calculate as exact values

a the surface area.

b the volume.



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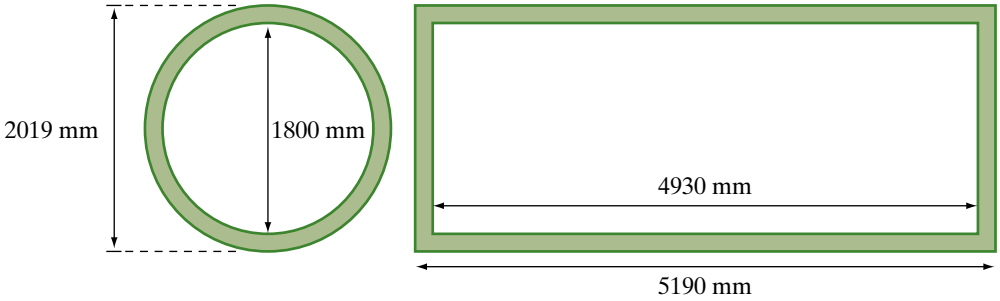
Analysis task 1

Underground rainwater tank

The photograph shows an underground concrete rainwater water storage tank being delivered to a newly constructed house. The diagram on the next page shows the external and internal dimensions of the tank.



End view Side view

a Calculate the overall volume of the tank, using the external measurements. Give the volume in cubic metres correct to one decimal place.

b What is the thickness of the concrete in the circular wall of the tank? Give the thickness to the nearest centimetre.

c Calculate the internal volume of the tank. Give the volume in cubic metres correct to one decimal place.

d What is the capacity of the tank in kilolitres? Answer to one decimal place.

e What is the volume of concrete in the tank? Give the volume in cubic metres correct to one decimal place.

f If the density of the concrete is approximately 2500 kg/m^3, calculate the approximate mass in tonnes of the empty tank. Answer to one decimal place.

g What is the approximate mass of the tank if it contains 10 000 litres of water?

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Analysis task 2

Stan's Cafe

Stan's Cafe (pronounced 'caff') is a theatre company based in Birmingham, England. One of the company's productions, called *Of all the people in all the world*, uses grains of rice to represent statistics and make them more meaningful. Each grain of rice represents one person.

Of all the people in all the world has been performed in many countries, including Australia. Over a period of days, the performers carefully weigh out quantities of rice to represent many different human statistics, such as

- the populations of particular towns and cities
- the number of doctors in the world, the number of soldiers
- the number of people born each day, the number who die
- all the people who have ever walked on the Moon
- the number of people who died in the Holocaust
- the number of children who will die this year from diseases for which a vaccine exists.

When you visit the production, you are invited to compare the one grain of rice that represents you to the millions of grains that represent other people.

Refugees (left) and millionaires (right) in the world today

People who will be born (left) and people who will die (right) in the world today

People injured in accidents involving a tea cosy in the UK last year

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Performers from Stan's Cafe are frequently asked how they create the exhibits.

> *We counted a large number of grains onto delicate scales and took an average for the number of grains per gram. We tend to count numbers up to 200 by hand and weigh out anything larger. Although this is inevitably an approximation so are most of the statistics themselves. Ultimately the show is about proportions, this beside that; using the same formula for all statistics keeps these proportions accurate.*

After Stan's Cafe has finished its production in a country, the rice is usually washed and used as food. You can read about *Of all the people in all the world* at the following website, which includes a link to a short video.

www.en.wikipedia.org/wiki/Of_All_The_People_In_All_The_World

http://www.stanscafe.co.uk/ofallthepeople/

a How many grains of rice would be needed to represent each of the following?

- **i** The students at your school
- **ii** The people living in your town or city
- **iii** The population of your state
- **iv** The population of Australia
- **v** The population of China
- **vi** All the people in the world

We will now make two assumptions.

- There are 60 grains of rice per gram
- 1 kg of rice has a volume of approximately 1000 cm^3.

b Copy and complete the following table, adding any other statistics that interest you. In each case, use the most appropriate unit, such as grams, kilograms or tonnes for mass and cubic centimetres or cubic metres for volume.

Statistic	Grains of rice	Mass	Volume
Students at our school			
People living in our town or city			
Population of your state			
Population of Australia			
Population of China			
All the people in the world			

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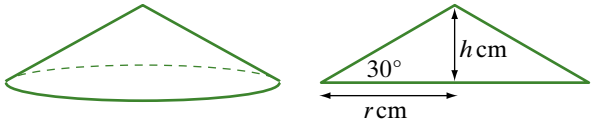
When a large quantity of small particles (such as rice grains) is poured into a heap, the particles tend to form a cone shape. In the case of rice, the angle of the sloping edge of the cone is about 30°.



c Write an expression for the height h cm in terms of the radius r cm.

d Write an expression for the volume V cm^3 of the cone in terms of the radius r cm.

e Using the appropriate volume from the table in part **b**, calculate the radius and height of the cone of rice representing the students in your school.

f Calculate the radius and height of the cones of rice for each of the other volumes in the table from part **b**. Use the most appropriate unit (centimetres or metres) for each radius and height. Record your results in a new table as shown below (or extend the previous table).

Statistic	Radius of cone	Height of cone
Students at our school		
People living in our town or city		
Population of your state		
Population of Australia		
Population of China		
All the people in the world		

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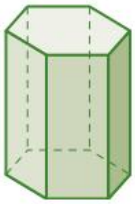
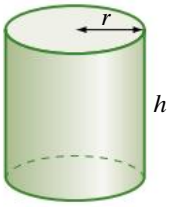
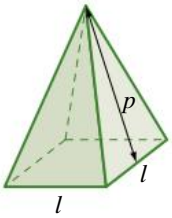
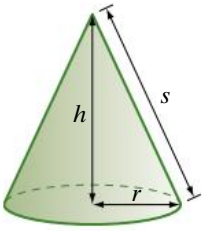
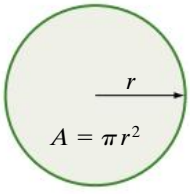
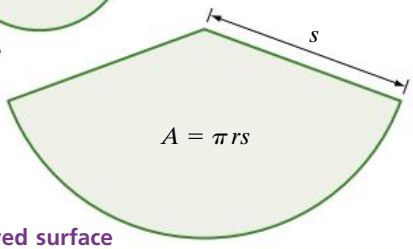
Review Surface area and volume

Summary

Three-dimensional shape	Dimensions	Surface area
Prism 	Dimensions depend on shape of cross-section	Area of ends + area of all rectangular faces
Cylinder 	r = radius h = height	$SA = 2\pi r^2 + 2\pi rh$ Top and bottom Curved surface
Square pyramid 	l = length of edge of square base p = perpendicular height of triangular faces	Area of base + area of triangular faces
Cone 	h = height r = radius s = slant edge	Area of circular base of cone = πr^2 Area of curved surface of cone = πrs Total surface area of cone = $\pi r^2 + \pi rs$  Base  Curved surface

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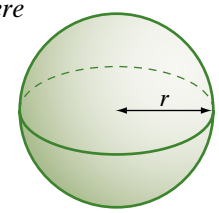
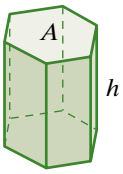
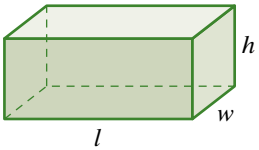
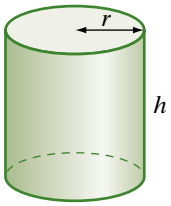
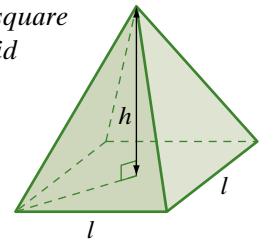
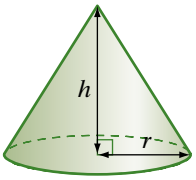
Three-dimensional shape	Dimensions	Surface area
Sphere 	r = radius	Surface area = $4\pi r^2$

Three-dimensional shape	Dimensions	Volume formula
Prism 	A = area of base h = height	$V = Ah$
Rectangular prism 	l = length w = width	$V = Ah$ $= lwh$
Cylinder 	r = radius h = height	$V = Ah$ $= \pi r^2 h$
Right square pyramid 	l = length of side of square base h = height	$V = \frac{1}{3}Ah$ $= \frac{1}{3}l^2h$
Cone 	r = radius h = height	$V = \frac{1}{3}Ah$ $= \frac{1}{3}\pi r^2 h$

10 A

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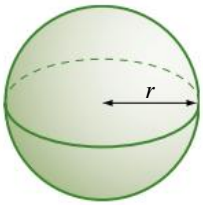
Three-dimensional shape	Dimensions	Volume formula
Sphere 	r = radius	$V = \frac{4}{3}\pi r^3$

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

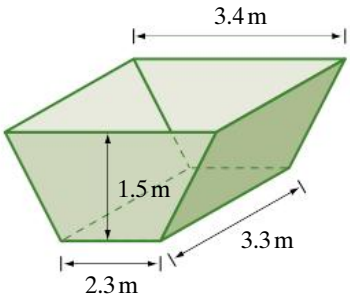
*arc length	*frustum	*slant edge
area	*hypotenuse	*slant height
capacity	perpendicular height	*sphere
circumference	prism	surface area
*cone	*pyramid	*truncated cone
curved surface	radius	*truncated pyramid
cylinder	*right pyramid	volume
*diagonal	sector	
diameter	similar triangles	

Revision

Multiple-choice questions

1 The volume of the tank shown below is closest to

A $8\,m^3$
B $10.5\,m^3$
C $11.4\,m^3$
D $14.1\,m^3$
E $16.8\,m^3$



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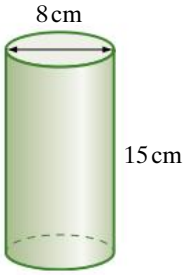
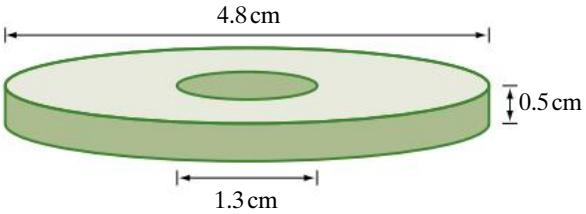
2 The total surface area (in cm^3) of this cylinder is given by



A $\pi \times 4^2 + 2 \times \pi \times 4 \times 15$
B $\pi \times 8^2 + \pi \times 4 \times 15$
C $2\pi \times 4^2 + \pi \times 4 \times 15$
D $\pi \times 8^2 + 2 \times \pi \times 8 \times 15$
E $2\pi \times 4^2 + 2 \times \pi \times 4 \times 15$

3 The volume of the circular disc shown is closest to

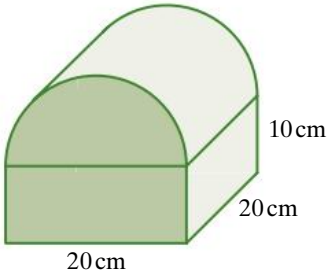


A $2.7\,cm^3$ **B** $5.5\,cm^3$
C $8.4\,cm^3$ **D** $12.6\,cm^3$
E $33.5\,cm^3$

4 The volume of a cylinder whose radius is 2 cm and height is 4 cm is equal to

A $16\pi\,cm^3$ **B** $8\pi\,cm^3$ **C** $6\pi\,cm^3$ **D** $4\pi\,cm^3$ **E** $2\pi\,cm^3$

5 A container has a rectangular base and the top is a half cylinder with a semicircular cross-section as shown. The volume of the container is

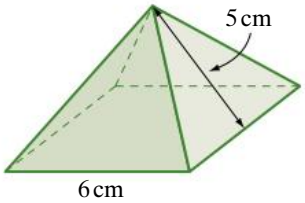


A $1200 + 400\pi\ cm^3$ **B** $1200 + 1000\pi\ cm^3$
C $4000 + 20\pi\ cm^3$ **D** $4000 + 800\pi\ cm^3$
E $4000 + 1000\pi\ cm^3$

6 The surface area of a sphere of radius 4 cm is closest to

A $50\,cm^2$ **B** $70\,cm^2$ **C** $200\,cm^2$ **D** $270\,cm^2$ **E** $800\,cm^2$

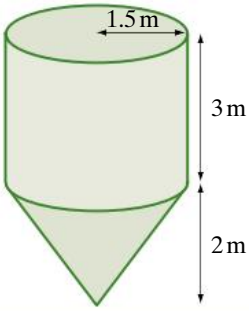
7 A right pyramid has a square base with side length 6 cm. The height of the sloping faces is 5 cm.



The volume of the pyramid is

A $48\,cm^3$ **B** $60\,cm^3$
C $80\,cm^3$ **D** $144\,cm^3$
E $180\,cm^3$

8 The volume of this solid shape is closest to



A $12\,m^3$
B $16\,m^3$
C $26\,m^3$
D $33\,m^3$
E $35\,m^3$

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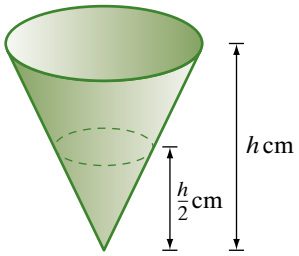
9 An inverted cone has capacity of $40\,cm^3$. It is filled with water to half its height. The volume of water in the cone is

A $20\,cm^3$ **B** $10\,cm^3$ **C** $5\,cm^3$

D $80\,cm^3$ **E** $\frac{40}{3}\,cm^3$

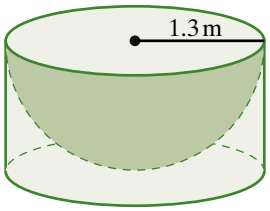


10 A concrete sculpture is designed to be a hollowed out cylinder in which a hemisphere of radius 1.3 m would fit precisely. What volume of concrete is required?

A $1.15\,cm^3$ **B** $2.30\,cm^3$

C $4.60\,cm^3$ **D** $6.90\,cm^3$

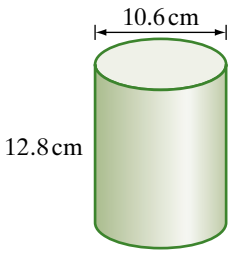
E $8.10\,cm^3$



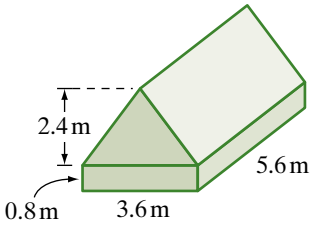
Short-answer questions

11 Calculate the volume of each of these solids, correct to the nearest cubic centimetre or cubic metre.

a



b



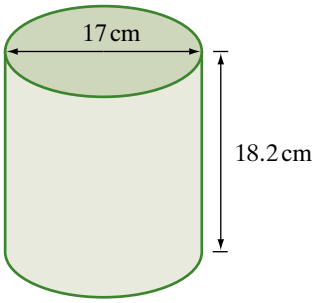
12 Calculate the total surface area of each of the solid shapes in question **11**, correct to the nearest cubic centimetre or cubic metre.

13 **a** Calculate the volume of the can of paint shown here, giving your answer correct to the nearest 10 cubic centimetres.

b What is the capacity of the can in litres? Give your answer correct to the nearest litre.

c Calculate the area of the curved surface of the can, giving your answer, correct to the nearest 10 square centimetres.

d Calculate to the nearest 10 square centimetres the total area of the sheet metal in the can, including the lid.



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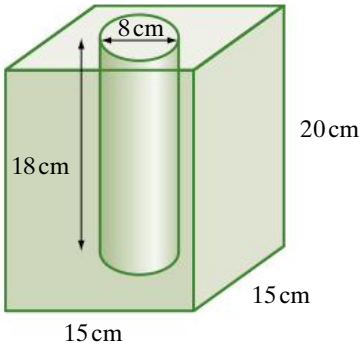
14 A glass vase is in the shape of a rectangular prism with a hollow cylinder.

a Calculate the amount of water that the vase will hold if filled to the top

i correct to the nearest cubic centimetre.

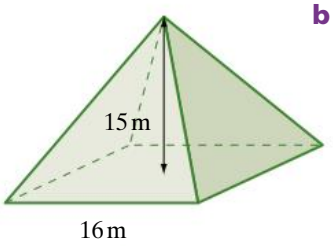
ii correct to the nearest millilitre.

b Calculate the volume of glass in the vase to the nearest cubic centimetre.

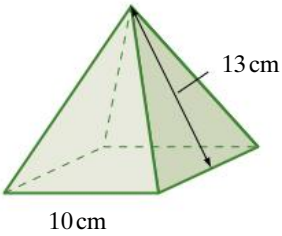


15 Calculate the volume of each of these solids, correct to the nearest cubic centimetre or cubic metre.

a



b



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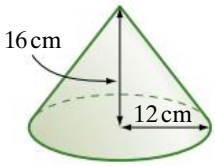
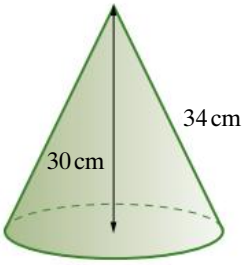
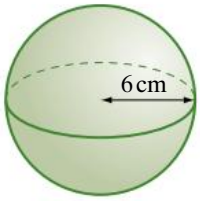


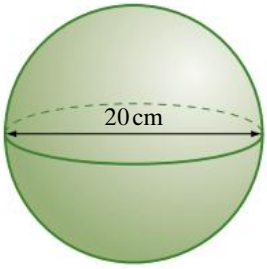
d



e



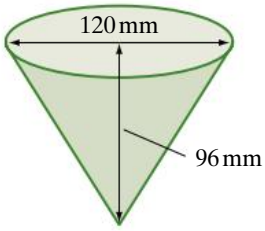
f



16 Calculate the total surface area of each of the solids in question **15**. Give each area correct to one decimal place.

17 **a** Calculate the volume of this cone. Give your answer in cubic centimetres, correct to the nearest cubic millimetre.

b Calculate the total surface area, correct to the nearest square millimetre.

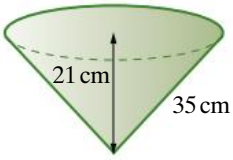


18 For each of these containers

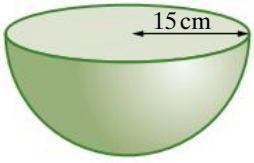
i find the volume in cubic centimetres, correct to the nearest square centimetre.

ii find the capacity in litres, correct to the nearest litre.

a



b



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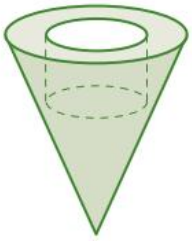
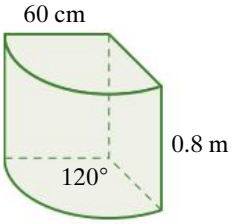
19 Find the volume of metal in a solid cone of radius 6 cm and height 10 cm from which a cylinder of radius 3 cm and height 4 cm has been removed. Give the volume to the nearest cubic centimetre.



Extended-response questions

20 A tank for tropical fish in an aquarium is in the shape of a sector of a cylinder. The sector angle is 120°, the radius is 60 cm and the height of the tank is 0.8 m.

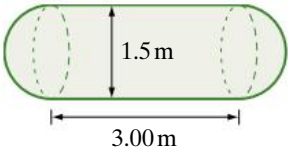


- **a** Calculate the capacity of the tank in litres, correct to the nearest litre.
- **b** If the top and curved part of the tank are glass, calculate the area of glass in the tank.
- **c** What is the mass of water in the tank? (1 litre of water has a mass of 1 kilogram)

Practice quiz Chapter 7

21 A petrol storage tank has hemispherical ends and a cylindrical centre as shown in the diagram. Calculate correct to one decimal place

- **a** the total surface area.
- **b** the volume.
- **c** the capacity in kilolitres.



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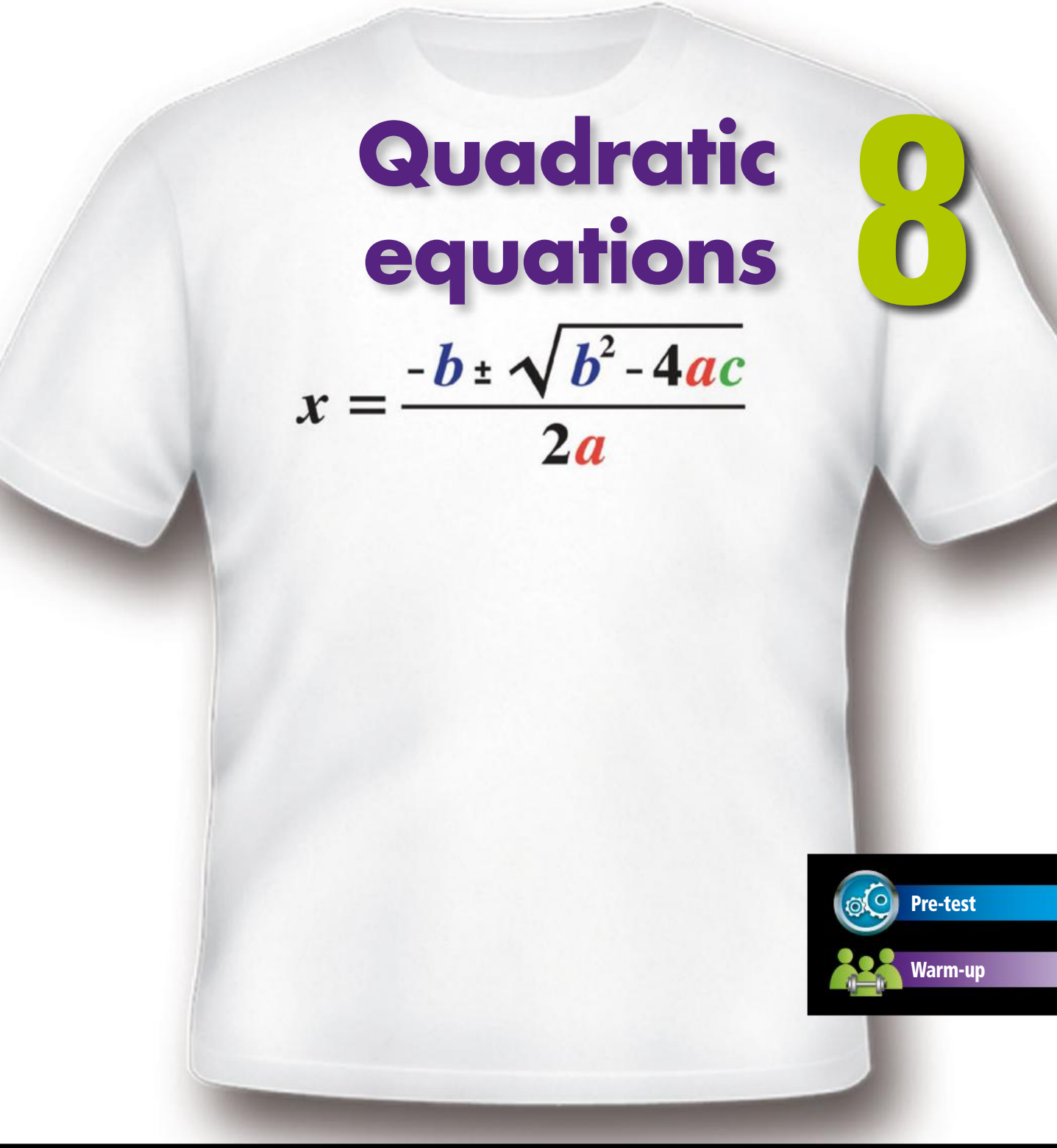
Quadratic equations 8



Pre-test

Warm-up

In this chapter you will learn by-hand methods for solving quadratic equations. For some of these equations the solutions are rational, whereas others have irrational solutions, or perhaps no real solutions. The t-shirt above displays the solution to the general quadratic equation $ax^2 + bx + c = 0$. This formula can be used to solve quadratic equations. A CAS calculator can also be used.

8.1 The null factor law

What is a quadratic equation?

In a linear equation the highest power of x is 1, for example, $3x + 4 = 16$, $\frac{4x}{3} = 2x + 5$.

In chapter 4, we used the word 'quadratic' to describe expressions such as $x^2 - 5x + 6$ or $9x^2 - 4$. 'Quad' comes from Latin and can mean 'four' or 'square'. In the word 'quadratic' the prefix 'quad' refers to the 'x squared' term. In a quadratic equation, the highest power of x is 2. In general, a quadratic equation has the form $ax^2 + bx + c = 0$. In some cases, the quadratic expression may be in factor form, and b or c can equal zero. Obviously a cannot equal zero or the equation would not be a quadratic equation. Examples of quadratic equations are $x^2 + 4x + 3 = 0$ and $x(x - 3) = 0$.

Example 1

Which of these are quadratic equations?

a $x^2 - 7x = 0$

b $k^2 = 9$

c $x^4 + 2x + 11 = 0$

d $(d - 5)(d + 3) = 0$

Working	Reasoning
a $x^2 - 7x = 0$ Yes	The equation is of the form $ax^2 + bx + c = 0$ where $c = 0$.
b $k^2 = 9$ $k^2 - 9 = 0$ Yes	Subtract 9 from both sides. The equation is of the form $ax^2 + bx + c = 0$, where $b = 0$.
c $x^4 + 2x + 11 = 0$ No	The highest power of x in a quadratic equation must be 2, that is, the x^2 term.
d $(d - 5)(d + 3) = 0$ $d^2 - 2d - 15 = 0$ Yes	Expand the left side. The equation is of the form $ax^2 + bx + c = 0$.

In chapter 1 you solved linear equations. Solving an equation means finding values of the pronumeral that make the equation true. For example, if $x + 3 = 7$, then the value $x = 4$ makes the equation true. In this chapter you will find how to solve quadratic equations. Linear equations can be solved by 'doing the same to both sides' or by inverse operations. However, we need different methods to solve quadratic equations.

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The null factor law or the principle of zero product

The word 'null' is from a Latin word and means zero or nothing (for example, the name 'Nullarbor' in Nullarbor Plain means 'no trees'). The **null factor law** states simply that if two factors multiplied together equal zero, then one or both of the factors must be equal to zero. This is also called the **principle of zero product**.

Null factor law or principle of zero product

If $ab = 0$, then $a = 0$ or $b = 0$, or both a and b equal zero.

Example 2

Find the value of x that makes these equations true. Check the solutions.

a $3x = 0$

b $2(x - 5) = 0$

Working

a $3x = 0$
$x = 0$

Check:
When $x = 0$
LS $= 3 \times 0$
$= 0$
LS = RS

b $2(x - 5) = 0$
$x - 5 = 0$
$x = 5$

Check:
When $x = 5$
LS $= 2(5 - 5)$
$= 2 \times 0$
$= 0$
LS = RS

Reasoning

If $3x = 0$ then one of the factors must be zero. Since 3 cannot equal 0, then the other factor, x, must equal 0.

If $2(x - 5) = 0$ then one of the factors must be zero. Since 2 cannot equal 0, then the other factor $(x - 5)$ must equal 0.

We can apply the same method to solve quadratic equations that are in factorised form.

Example 3

Solve these quadratic equations. Check the solutions.

a $x(x + 4) = 0$

b $(x - 2)(x + 3) = 0$

continued

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Example 3 continued

Working

a $x(x + 4) = 0$

$x = 0$ or $x + 4 = 0$

$x = 0$ or $x = -4$

Check:

When $x = 0$

$\text{LS} = 0(0 + 4)$

$= 0 \times 4$

$= 0$

$\text{LS} = \text{RS}$

When $x = -4$

$\text{LS} = -4(-4 + 4)$

$= -4 \times 0$

$= 0$

$\text{LS} = \text{RS}$

Reasoning

Either x must equal zero or the other factor $(x + 4)$ must equal zero. Solve the equation $x + 4 = 0$.
There are two possible values of x that make this equation true.

b $(x - 2)(x + 3) = 0$

$x - 2 = 0$ or $x + 3 = 0$

$x = 2$ or $x = -3$

Check:

When $x = 2$

$\text{LS} = (2 - 2)(2 + 3)$

$= 0 \times 5$

$= 0$

$\text{LS} = \text{RS}$

When $x = -3$

$\text{LS} = (-3 - 2)(-3 + 3)$

$= -5 \times 0$

$= 0$

$\text{LS} = \text{RS}$

Either $(x - 2)$ must equal zero or the other factor $(x + 3)$ must equal zero. Solve the equations $x - 2 = 0$ and $x + 3 = 0$.

Example 4

Solve these quadratic equations. Check the solutions.

a $x(2x - 7) = 0$

b $(x - 1)(3x - 4) = 0$

continued

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Example 4 continued

Working	Reasoning
a $x(2x - 7) = 0$ $x = 0$ or $2x - 7 = 0$ $x = 0$ or $2x = 7$ $x = 0$ or $x = \frac{7}{2}$ $= 3\frac{1}{2}$ Check: When $x = 0$ LS $= 0(2 \times 0 - 7)$ $= 0 \times -7$ $= 0$ LS = RS When $x = \frac{7}{2}$ LS $= \frac{7}{2}\left(\cancel{2} \times \frac{7}{\cancel{2}} - 7\right)$ $= \frac{7}{2} \times 0$ $= 0$ LS = RS	Either x must equal zero or the other factor $(2x - 7)$ must equal zero. Solve the equation $2x - 7 = 0$.
b $(x - 1)(3x - 4) = 0$ $x - 1 = 0$ or $3x - 4 = 0$ $x = 1$ or $3x = 4$ $x = 1$ or $x = \frac{4}{3}$ Check: When $x = 1$ LS $= (1 - 1)(3 \times 1 - 4)$ $= 0 \times -1$ $= 0$ LS = RS When $x = \frac{4}{3}$, LS $= \left(\frac{4}{3} - 1\right)\left(\cancel{3} \times \frac{4}{\cancel{3}} - 4\right)$ $= \frac{1}{3} \times 0$ $= 0$ LS = RS	Either $(x - 1)$ must equal zero or the other factor $(3x - 4)$ must equal zero. Solve the equations $x - 1 = 0$ and $3x - 4 = 0$.

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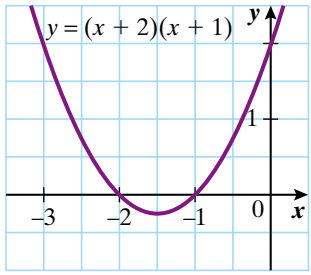
Quadratic equations and *x*-intercepts of parabolas

In Year 9, you saw that the graph of $y = x^2$ was a parabola with its turning point at (0, 0). In fact, the graphs of all quadratic functions are parabolas but the parabola is translated sideways or up and down according to the particular equation. Some parabolas are translated upwards so that they do not cross the x-axis. Many parabolas cross the x-axis at two points, that is, there are two x-intercepts. At the points where the parabola crosses the x-axis, $y = 0$. This means that we can find the x-intercepts of an equation such as $y = (x + 1)(x + 2)$ by solving the quadratic equation $(x + 1)(x + 2) = 0$. We see here that the x-intercepts are at $x = -2$ and $x = -1$ as these are the two solutions of the equation $(x + 1)(x + 2) = 0$.

The graph of $y = (x + 1)(x + 2)$ is shown at the right. Notice that the parabola crosses the x-axis at $x = -1$ and $x = -2$.



Therefore, the null factor law helps us to identify the x-intercepts—that is, the x-coordinates of the points where the graph crosses the x-axis, $(-2, 0)$ and $(-1, 0)$.

We will return to this in chapter **12** where we look at quadratic functions and their graphs.

If the quadratic expression on the left side is a perfect square and the right side is zero, the equation has only one solution. In this case, the parabola just touches the x-axis at one point.

Example 5

Solve these quadratic equations.

a $(x - 3)^2 = 0$

b $(2x - 9)^2 = 0$

Working

a

$$\begin{aligned} (x - 3)^2 &= 0 \\ (x - 3)(x - 3) &= 0 \\ x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Reasoning

$(x - 3)^2$ means $(x - 3)(x - 3)$ so
$x - 3 = 0$
There is only one solution, $x = 3$.

Check:
When $x = 3$,
$\text{LS} = (3 - 3)^2$
$= 0$
LS = RS

continued

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Example 5 continued

Working

b

$$(2x-9)^2 = 0$$
$$(2x-9)(2x-9) = 0$$
$$2x-9 = 0$$
$$2x = 9$$
$$x = \frac{9}{2}$$
$$= 4\frac{1}{2}$$

Reasoning

$(2x-9)^2$ means $(2x-9)(2x-9)$ so $2x-9=0$

$x = \frac{9}{2}$. There is only one solution,

Check:

When $x = \frac{9}{2}$

$$\text{LS} = \left(\cancel{2} \times \frac{9}{\cancel{2}} - 9\right)^2$$
$$= (9-9)^2$$
$$= 0$$
$$\text{LS} = \text{RS}$$

exercise 8.1

LINKS TO Example 1

1 State whether each of these equations is a quadratic equation or not.

a $x^2 - 64 = 0$ **b** $3y + 8 = 0$ **c** $m^2 + 5m + 4 = 0$ **d** $k^2 - 16 = 0$

e $x^2 - 5x$ **f** $\frac{x}{3} - 6 = 0$ **g** $\frac{1}{2}x^2 + 8x = 0$ **h** $2a^3 + 5a - 3 = 0$

LINKS TO Example 2

2 Solve these equations.

a $2x = 0$ **b** $5(x-3) = 0$ **c** $4(a-6) = 0$ **d** $2(y+7) = 0$

e $-4x = 0$ **f** $-3(x-11) = 0$ **g** $3(3b-5) = 0$ **h** $3(2x-15) = 0$

LINKS TO Example 3

3 Solve these quadratic equations.

a $x(x-7) = 0$ **b** $a(a+1) = 0$

c $m(m-2) = 0$ **d** $k(k+6) = 0$

e $d(d-25) = 0$ **f** $h(h-9) = 0$

g $(x-4)(x-1) = 0$ **h** $(a-7)(a-5) = 0$

i $(p+4)(p-3) = 0$ **j** $(y-6)(y-7) = 0$

k $(n+3)(n-11) = 0$ **l** $(x-2)(x+1) = 0$

m $(a+3)(a-20) = 0$ **n** $(d-5)(d-16) = 0$

o $(x-2)(x+14) = 0$ **p** $(x-15)(x-4) = 0$

LINKS TO Example 4

4 Solve these quadratic equations. Check your solutions.

a $x(2x-5) = 0$ **b** $b(2b-1) = 0$

c $y(3y-8) = 0$ **d** $a(4a-15) = 0$

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e $x(2x - 25) = 0$
f $k(3k - 7) = 0$
g $(m - 4)(2m - 1) = 0$
h $(x - 2)(2x - 9) = 0$
i $(x + 1)(5x - 8) = 0$
j $(n + 2)(4n - 9) = 0$
k $(2x + 1)(x - 5) = 0$
l $(2n - 11)(n - 1) = 0$
m $(3a - 4)(a + 7) = 0$
n $(2x + 1)(2x - 5) = 0$
o $(3x - 5)(2x - 3) = 0$
p $(4x - 7)(2x - 1) = 0$

5 The quadratic equation $(2x + 7)(5x - 8) = 0$ has solutions

A $x = -7, x = 8$
B $x = 7, x = -8$
C $x = \frac{-2}{7}, x = \frac{5}{8}$
D $x = \frac{-7}{2}, x = \frac{8}{5}$
E $x = \frac{7}{2}, x = \frac{-8}{5}$

LINKS TO Example 5

6 Solve these quadratic equations.

a $(x - 4)^2 = 0$
b $(x + 2)^2 = 0$
c $(x - 7)^2 = 0$
d $(x - 11)^2 = 0$
e $(x + 5)^2 = 0$
f $(2x - 1)^2 = 0$
g $(2x + 5)^2 = 0$
h $(3x - 4)^2 = 0$
i $(4x - 5)^2 = 0$
j $(2x + 7)^2 = 0$
k $(5x - 8)^2 = 0$
l $(4x + 1)^2 = 0$

7 Solve these quadratic equations.

a $(a - 3)(a + 7) = 0$
b $(h - 8)^2 = 0$
c $(x - 12)(x + 12) = 0$
d $(m + 2)^2 = 0$
e $(x - 5)(x + 9) = 0$
f $(2x + 3)^2 = 0$
g $b(b - 5) = 0$
h $(3x + 8)^2 = 0$
i $(3x - 5)(x + 1) = 0$
j $(2x + 3)(3x - 2) = 0$
k $3x(4x - 11) = 0$
l $(5x + 2)(2x - 7) = 0$

8 The quadratic equation $-3x(2x - 5) = 0$ has solutions

A $x = 0$ or $x = \frac{2}{5}$
B $x = 3$ or $x = \frac{5}{2}$
C $x = 0$ or $x = \frac{5}{2}$
D $x = -3$ or $x = \frac{5}{2}$
E $x = 0$ or $x = -\frac{2}{5}$

exercise 8.1 challenge

9 Write quadratic equations that would have these solutions.

a $x = 0$ or $x = -6$
b $x = 1$ or $x = \frac{3}{4}$
c $x = 0$ or $x = -\frac{6}{7}$
d $x = -\frac{2}{3}$ or $x = 2\frac{3}{4}$

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8.2 Quadratic equations and factorisation

If the quadratic expression in the equation is not already in factorised form, we can use the techniques from chapter 4 to try to factorise it. (As we will see later, there are quadratic expressions that cannot be factorised.)

Common factor

In some equations there is a common factor of x in the quadratic expression. Equations of this type always have $x = 0$ as one of the solutions.

Example 6

Solve each of these quadratic equations by first taking out a common factor on the left side. Check the solutions.

a $x^2 - 3x = 0$

b $5x^2 - 30x = 0$

Working	Reasoning
a $x^2 - 3x = 0$	
$x(x - 3) = 0$	Take out the common factor of x.
$x = 0$ or $x - 3 = 0$	Either x equals 0 or the other factor $(x - 3)$ equals zero.
$x = 0$ or $x = 3$	Solve the equation $x - 3 = 0$.
Check:	
When $x = 0$	
LS $= 0^2 - 3 \times 0$	
$= 0$	
LS = RS	
When $x = 3$	
LS $= 3^2 - 3 \times 3$	
$= 0$	
LS = RS	
b $5x^2 - 30x = 0$	
$5x(x - 6) = 0$	Take out the common factor of $5x$.
$x(x - 6) = 0$	Divide both sides of the equation by 5. The right side is still equal to zero because $0 \div 5 = 0$.
$x = 0$ or $x - 6 = 0$	Either x equals 0 or the other factor $(x - 6)$ equals zero.
$x = 0$ or $x = 6$	Solve the equation $x - 6 = 0$.

continued

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Example 6 continued

Working

Check:
When $x = 0$,
$\text{LS} = 5 \times 0^2 - 30 \times 0$
$= 0$
$\text{LS} = \text{RS}$

When $x = 6$,
$\text{LS} = 5 \times 6^2 - 30 \times 6$
$= 180 - 180$
$= 0$
$\text{LS} = \text{RS}$

Reasoning

Difference of two squares

The quadratic expression may be a difference of two squares. There may also be a common factor.

Example 7

Solve each of these quadratic equations by factorising the left side.

a $x^2 - 9 = 0$ **b** $3x^2 - 75 = 0$ **c** $4x^2 - 9 = 0$

Working

a
$x^2 - 9 = 0$
$(x - 3)(x + 3) = 0$
$x - 3 = 0$ or $x + 3 = 0$
$x = 3$ or $x = -3$

Reasoning

Factorise the difference of two squares.
Either $(x - 3)$ equals zero or $(x + 3)$ equals zero.
Solve the equations $x - 3 = 0$ and $x + 3 = 0$.
Check:
When $x = 3$,
$\text{LS} = 3^2 - 9$
$= 9 - 9$
$= 0$
$= \text{RS}$
We can see that if $x = -3$, the left side will also be zero.

continued

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Example 7 continued

Working	Reasoning
b $3x^2 - 75 = 0$ $3(x^2 - 25) = 0$ $x^2 - 25 = 0$ $(x - 5)(x + 5) = 0$ $x - 5 = 0$ or $x + 5 = 0$ $x = 5$ or $x = -5$	There is a common factor of 3. Divide both sides by 3. The right side still equals zero because $0 \div 3 = 0$ Factorise the difference of two squares. Either $(x - 5)$ equals zero or $(x + 5)$ equals zero. Solve the equations $x - 5 = 0$ and $x + 5 = 0$. Check: When $x = 5$ LS $= 3 \times 5^2 - 75$ $= 3 \times 25 - 75$ $= 75 - 75$ $= 0$ LS = RS When $x = -5$, LS $= 3 \times (-5)^2 - 75$ $= 3 \times 25 - 75$ $= 0$ LS = RS
c $4x^2 - 9 = 0$ $(2x)^2 - 3^2 = 0$ $(2x - 3)(2x + 3) = 0$ $2x - 3 = 0$ or $2x + 3 = 0$ $x = \frac{3}{2}$ or $x = -\frac{3}{2}$	The left side does not have a common factor but it is a difference of squares. Factorise then solve each of the equations. Check: When $x = \frac{3}{2}$ LS $= 4\left(\frac{3}{2}\right)^2 - 9$ $= 4 \times \frac{9}{4} - 9$ $= 9 - 9$ $= 0$ LS = RS We can see that if $x = -\frac{3}{2}$, the left side will also be zero.

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Another solving method—without factorising!

Some quadratic equations can be solved without using the null factor law. For example, part **c** of Example **7** can be solved by 'doing the same thing to both sides'. However, you must remember to take both the positive and the negative square root so that you find all possible solutions.

$$4x^2 - 9 = 0$$
$$4x^2 = 9$$

Take the square root of both sides.

$$\sqrt{4x^2} = \sqrt{9}$$
$$2x = \pm 3$$
$$x = -\frac{3}{2} \text{ or } \frac{3}{2}$$

Perfect squares

The quadratic expression may be a perfect square. This means that there is only one solution to the equation.

Example 8

Solve each of these quadratic equations by factorising the left side.

a $x^2 - 6x + 9 = 0$

b $4x^2 + 12x + 9 = 0$

Working	Reasoning
a $x^2 - 6x + 9 = 0$ $(x - 3)^2 = 0$ $x - 3 = 0$ $x = 3$	$x^2 - 6x + 9 = (x - 3)^2$ There is only one solution as both factors are the same.
b $4x^2 + 12x + 9 = 0$ $(2x + 3)^2 = 0$ $2x + 3 = 0$ $x = -\frac{3}{2}$	$4x^2 + 12x + 9 = (2x + 3)^2$ There is only one solution as both factors are the same.

Quadratic trinomials that are not perfect squares

The quadratic expression is frequently a quadratic trinomial that can be factorised into two different factors, each of which results in a solution to the equation.

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Example 9

Solve each of the following quadratic equations. Check the solutions.

a $x^2 + 3x - 10 = 0$

b $x^2 - 5x - 6 = 0$

Working

a

$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$
$$x + 5 = 0 \text{ or } x - 2 = 0$$
$$x = -5 \text{ or } \quad x = 2$$

Check:

When $x = -5$,

$$\text{LS} = (-5)^2 + 3(-5) - 10$$
$$= 25 - 15 - 10$$
$$= 0$$
$$\text{LS} = \text{RS}$$

When $x = 2$,

$$\text{LS} = (2)^2 + 3(2) - 10$$
$$= 4 + 6 - 10$$
$$= 0$$
$$\text{LS} = \text{RS}$$

b

$$x^2 - 5x = 6$$
$$x^2 - 5x - 6 = 6 - 6$$
$$x^2 - 5x - 6 = 0$$
$$(x - 6)(x + 1) = 0$$
$$x - 6 = 0 \text{ or } x + 1 = 0$$
$$x = 6 \text{ or } \quad x = -1$$

Check:

When $x = 6$,

$$\text{LS} = (6)^2 - 5(6)$$
$$= 36 - 30$$
$$= 6$$
$$\text{LS} = \text{RS}$$

When $x = -1$,

$$\text{LS} = (-1)^2 - 5(-1)$$
$$= 1 + 5$$
$$= 6$$
$$\text{LS} = \text{RS}$$

Reasoning

To factorise $x^2 + 3x - 10$, find two numbers with a product of -10 and a sum of 3. Those numbers are 5 and -2.
Solve each linear equation for x.

Substitute each value into $x^2 + 3x - 10 = 0$ to check that it makes a true statement.

We want the RS to equal zero so that we can use the null factor law. So first subtract 6 from both sides.
To factorise $x^2 - 5x - 6$, find two numbers with a product of -6 and a sum of -5. Those numbers are -6 and 1.
Solve each linear equation for x.

Substitute each x-value into $x^2 - 5x = 6$ to check that it makes a true statement.

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10 A

Example 10

Solve these quadratic equations.

a $2x^2 - 5x + 3 = 0$

b $3x^2 - 15x + 12 = 0$

Working

a

$$2x^2 - 5x + 3 = 0$$
$$(2x - 3)(x - 1) = 0$$
$$2x - 3 = 0 \text{ or } x - 1 = 0$$
$$x = \frac{3}{2} \text{ or } \quad x = 1$$

b

$$3x^2 - 15x + 12 = 0$$
$$3(x^2 - 5x + 4) = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x - 4 = 0 \text{ or } x - 1 = 0$$
$$x = 4 \text{ or } \quad x = 1$$

Reasoning

There is no common factor so factorise the quadratic trinomial.

Solve the equations $2x - 3 = 0$ and $x - 1 = 0$.

The quadratic trinomial has a common factor of 3.
Divide both sides by 3.

Factorise the quadratic trinomial.
Solve the equations $x - 4 = 0$ and $x - 1 = 0$.

exercise 8.2

LINKS TO Example 6

1 Solve each of these quadratic equations by first taking out a common factor on the left side.

a $x^2 - 6x = 0$ **b** $x^2 + 5x = 0$ **c** $a^2 - 12a = 0$

d $k^2 - 0.5k = 0$ **e** $m^2 - 8m = 0$ **f** $x^2 + \frac{1}{4}x = 0$

g $y^2 - 11y = 0$ **h** $d^2 - 7.5d = 0$ **i** $x^2 + 0.1x = 0$

j $x^2 - 14x = 0$ **k** $b^2 - 2.5b = 0$ **l** $x^2 - 2.4x = 0$

2 Solve each of these quadratic equations by first taking out the common factors on the left side.

a $2x^2 - 4x = 0$ **b** $4x^2 - 12x = 0$ **c** $3x^2 + 9x = 0$

d $2x^2 + 12x = 0$ **e** $5x^2 - 15x = 0$ **f** $11x^2 + 44x = 0$

g $7x^2 - 14x = 0$ **h** $3x^2 + 21x = 0$ **i** $4x^2 - 32x = 0$

j $9x^2 + 54x = 0$ **k** $12x^2 - 60x = 0$ **l** $3x^2 - 18x = 0$

3 The solutions to the equation $3a^2 - 48a = 0$ are

A $a = 3$ or $a = 4$ **B** $a = 0$ or $a = 4$ **C** $a = 0$ or $a = -4$

D $a = 3$ or $a = 16$ **E** $a = 0$ or $a = 16$

LINKS TO Example 7

4 For each of these equations, check if there is a common factor, then factorise the difference of squares and solve the equation.

a $x^2 - 9 = 0$ **b** $x^2 - 64 = 0$ **c** $a^2 - 49 = 0$

d $k^2 - 81 = 0$ **e** $x^2 - 36 = 0$ **f** $x^2 - 121 = 0$

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g $3x^2 - 27 = 0$ **h** $2x^2 - 50 = 0$ **i** $5x^2 - 20 = 0$
j $3x^2 - 48 = 0$ **k** $6x^2 - 6 = 0$ **l** $3x^2 - 75 = 0$

LINKS TO Example 8

5 For each of these equations, factorise the perfect square then solve the equation. How many solutions will each equation have?

a $x^2 - 2x + 1 = 0$ **b** $x^2 + 6x + 9 = 0$ **c** $x^2 + 4x + 4 = 0$
d $x^2 - 10x + 25 = 0$ **e** $x^2 - 16x + 64 = 0$ **f** $x^2 + 14x + 49 = 0$
g $x^2 - 20x + 100 = 0$ **h** $x^2 - 8x + 16 = 0$ **i** $x^2 - 12x + 36 = 0$
j $4x^2 - 4x + 1 = 0$ **k** $4x^2 - 20x + 25 = 0$ **l** $9x^2 - 12x + 4 = 0$

LINKS TO Example 9

6 For each of these equations, factorise the left side then solve the equation. How many solutions will each equation have?

a $x^2 - 7x + 6 = 0$ **b** $x^2 - 6x + 8 = 0$ **c** $x^2 - 9x + 14 = 0$
d $x^2 - 5x - 14 = 0$ **e** $x^2 - 7x + 10 = 0$ **f** $x^2 - 3x - 10 = 0$
g $x^2 - 6x - 16 = 0$ **h** $x^2 - 13x - 30 = 0$ **i** $x^2 - 7x - 18 = 0$
j $x^2 - 4x - 12 = 0$ **k** $x^2 - 5x + 6 = 0$ **l** $x^2 - 13x + 42 = 0$
m $x^2 - 13x + 36 = 0$ **n** $x^2 - 5x - 24 = 0$ **o** $x^2 + 6x - 27 = 0$

7 The solutions to the equation $x^2 - 5x - 6 = 0$ are

A $x = 3$ or $x = 2$ **B** $x = -3$ or $x = 2$ **C** $x = 3$ or $x = -2$
D $x = 6$ or $x = -1$ **E** $x = -6$ or $x = 1$

8 For each of the following quadratic equations, factorise the left side then solve the equation.

a $x^2 - x = 0$ **b** $x^2 - 4 = 0$ **c** $x^2 + 6x + 9 = 0$
d $x^2 - 8x + 12 = 0$ **e** $x^2 - 10x - 11 = 0$ **f** $x^2 + 2x - 48 = 0$
g $x^2 - 4x + 4 = 0$ **h** $x^2 - 6x + 8 = 0$ **i** $x^2 + 4x - 12 = 0$
j $x^2 + x - 12 = 0$ **k** $x^2 + 6x = 0$ **l** $x^2 - 16 = 0$
m $x^2 + 14x - 15 = 0$ **n** $x^2 - 8x = 0$ **o** $x^2 - 3x + 2 = 0$
p $4x^2 - 16 = 0$ **q** $x^2 - 8x + 16 = 0$ **r** $x^2 - 4x = 0$
s $x^2 + 5x + 6 = 0$ **t** $x^2 + 5x - 6 = 0$ **u** $2x^2 + 8x = 0$
v $x^2 + 3x - 28 = 0$ **w** $x^2 + 10x + 24 = 0$ **x** $x^2 + 10x - 24 = 0$

exercise 8.2 challenge

LINKS TO Example 10

9 For each of these equations, factorise the left side then solve the equation.

a $5x^2 - 22x + 21 = 0$ **b** $15x^2 - 26x + 8 = 0$
c $35x^2 - 9x - 2 = 0$ **d** $8x^2 - 38x + 35 = 0$

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8.3 Equations that need rearranging before factorisation

Rewriting the equation so the right side is zero

If there are terms on the right side of the equation, these terms must be brought to the left side so that the right side equals zero.

Example 11

Solve each of these equations by first moving all the terms to the left side. Check the solutions.

a $x^2 = 7x$

b $x^2 - 5x = 6$

Working

a

$$x^2 = 7x$$
$$x^2 - 7x = 0$$
$$x(x - 7) = 0$$
$$x = 0 \text{ or } x - 7 = 0$$
$$x = 0 \text{ or } \quad x = 7$$

Check:
When $x = 0$,
LS $= 0^2 = 0$
RS $= 7 \times 0 = 0$
LS = RS

When $x = 7$,
LS $= 7^2 = 49$
RS $= 7 \times 7 = 49$
LS = RS

Reasoning

Subtract $7x$ from both sides so that the right side is zero.
Take out the common factor x.

b

$$x^2 - 5x = 6$$
$$x^2 - 5x - 6 = 0$$
$$(x - 6)(x + 1) = 0$$
$$x - 6 = 0 \text{ or } x + 1 = 0$$
$$x = 6 \text{ or } \quad x = -1$$

Subtract 6 from both sides so that the right side is zero.
Factorise the left side.
Solve the equations $x - 6 = 0$ and $x + 1 = 0$.

continued

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Example 11 continued

Working

Check:
When $x = 6$,
$\text{LS} = 6^2 - 5 \times 6$
$= 36 - 30$
$= 6$
LS = RS

When $x = -1$,
$\text{LS} = (-1)^2 - 5 \times -1$
$= 1 + 5$
$= 6$
LS = RS

Reasoning

Example 12

Solve the equation $(x - 4)(x + 2) = 16$.

Working

$(x - 4)(x + 2) = 16$
$x^2 - 2x - 8 = 16$
$x^2 - 2x - 24 = 0$

$(x + 4)(x - 6) = 0$
$x + 4 = 0$ or $x - 6 = 0$
$x = -4$ or $x = 6$

Check:
When $x = -4$,
$\text{LS} = (-4 - 4)(-4 + 2)$
$= -8 \times (-2)$
$= 16$
LS = RS
When $x = 6$,
$\text{LS} = (6 - 4)(6 + 2)$
$= 2 \times 8$
$= 16$
LS = RS

Reasoning

Although the LS is factorised, the RS $\neq 0$, so we cannot use the null factor law. Expand the LS and subtract 16 from both sides to get 0 on the RS.

Now factorise the LS and use the null factor law.
Solve the equations $x + 4 = 0$ and $x - 6 = 0$.

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Expanding and simplifying the quadratic expression

The left side may need to be expanded and simplified before factorising.

Example 13

Solve the equation $(x - 3)^2 + 2x = 54$.

Working	Reasoning
$(x - 3)^2 + 2x = 54$	Expand the perfect square and subtract 54 from both sides.
$x^2 - 6x + 9 + 2x - 54 = 0$	
$x^2 - 4x - 45 = 0$	Collect like terms.
$(x - 9)(x + 5) = 0$	Factorise the quadratic trinomial.
$x - 9 = 0$ or $x + 5 = 0$	Use the null factor law.
$x = 9$ or $x = -5$	Solve for x.

10 A

Substitution

Some equations can be solved more easily by a substitution technique. The equation $(x - 3)^2 - 3(x - 3) - 10 = 0$ could be solved by first expanding the brackets and collecting like terms. However, substituting a single variable, a, for $(x - 3)$ converts the equation into a simple quadratic equation, $a^2 - 3a - 10 = 0$. Once we find the values for a, we then substitute to find x.

Example 14

Solve the equation $(x - 3)^2 - 3(x - 3) - 10 = 0$.

Working	Reasoning
$(x - 3)^2 - 3(x - 3) - 10 = 0$	The equation has the form of a quadratic equation with $(x - 3)$ instead of x.
Let $a = x - 3$.	Substitute a for $(x - 3)$.
$a^2 - 3a - 10 = 0$	
$(a + 2)(a - 5) = 0$	Solve for a.
$a = -2$ or $a = 5$	
$x - 3 = -2$ or $x - 3 = 5$	Substitute $x - 3$ for a and solve for x.
$x = 1$ or $x = 8$	

The next example shows an equation that has the form of a quadratic equation but x has been replaced by x^2. In this case we substitute a for x^2.

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Example 15

Solve the equation $x^4 - 5x^2 + 4 = 0$.

Working

$x^4 - 5x^2 + 4 = 0$
Let $a = x^2$.

$a^2 - 5a + 4 = 0$
$(a - 1)(a - 4) = 0$
$(x^2 - 1)(x^2 - 4) = 0$
$(x - 1)(x + 1)(x - 2)(x + 2) = 0$
$x = \pm 1$ or $x = \pm 2$

Reasoning

The equation has the form of a quadratic equation with x^2 instead of x.

Substitute a for x^2.
Solve for a
Substitute x^2 for a and solve for x.
Factorise each difference of squares.
There are four solutions.

Disguised quadratic equations

Sometimes a quadratic equation may not look like a quadratic equation. We may need to rearrange the equation into a form in which it can be factorised and solved.

Example 16

Convert each of the following into a quadratic equation of the form $ax^2 + bx + c = 0$. Then use factorisation techniques and the null factor law to solve the equation.

a $x + 1 = \dfrac{6}{x}$

b $x - \dfrac{4}{x - 3} = 3$

Working

a

$x + 1 = \dfrac{6}{x}$

$(x + 1) \times x = \dfrac{6}{x} \times x$

$x^2 + x = 6$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$x + 3 = 0$ or $x - 2 = 0$

$x = -3$ or $x = 2$

Reasoning

Multiply both sides by x to get rid of the fraction.

Rearrange the terms so the RS = 0.
Factorise the LS: find two numbers with a product of -6 and a sum of 1.

Solve for x.

Check:
When $x = -3$,
LS $= -3 + 1 = -2$
RS $= \dfrac{6}{-3} = -2$
LS = RS

continued

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Example 16 continued

Working

When $x = 2$,
LS $= 2 + 1 = 3$
RS $= \frac{6}{2} = 3$
LS = RS

b

$$x - \frac{4}{x-3} = 3$$

Multiply both sides by $(x - 3)$ to get rid of the fraction.

$$x \times (x-3) - \frac{4}{x-3} \times (x-3) = 3 \times (x-3)$$

$$x(x-3) - 4 = 3(x-3)$$

Expand the brackets, collect like terms, and rearrange the terms so that RS = 0.

$$x^2 - 3x - 4 = 3x - 9$$
$$x^2 - 6x + 5 = 0$$

Factorise the LS: find two numbers with a product of 5 and a sum of -6.

$$(x-1)(x-5) = 0$$

$$x - 1 = 0 \text{ or } x - 5 = 0$$
$$x = 1 \text{ or } \quad x = 5$$

Solve for x.

Check:
When $x = 1$,
LS $= 1 - \frac{4}{1-3}$
$= 1 + 2$
$= 3$
LS = RS
When $x = 5$,
LS $= 5 - \frac{4}{5-3}$
$= 5 - 2$
$= 3$
LS = RS

exercise 8.3

LINKS TO Examples 11, 12, 13

1 Solve each of these equations.

a $x^2 = 8x$
b $x^2 = -2x$
c $x^2 = -2.5x$
d $x^2 = 1.4x$
e $x^2 = \frac{1}{2}x$
f $x^2 - 3x = 4$
g $x^2 + 5x = 6$
h $x^2 - 10 = 3x$
i $x^2 - 5x = 14$
j $x^2 + 8 = 9x$
k $x^2 + 15 = 8x$
l $(x-3)^2 = 25$
m $(x+4)^2 - 2x = 43$
n $(x-1)^2 + 7x = 25$
o $(x-2)^2 + 3x = 34$

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2 The solutions to the equation $x^2 - 10x = 24$ are

A $x = 12$ or $x = -2$ **B** $x = -12$ or $x = 2$ **C** $x = 6$ or $x = -4$
D $x = 6$ or $x = 4$ **E** $x = -6$ or $x = -4$

3 The solution(s) to the equation $(x - 3)^2 - 7 = 18$ is (are)

A $x = -8$ or $x = 2$ **B** $x = -2$ or $x = 8$ **C** $x = -3$ or $x = -7$
D $x = 3$ or $x = 7$ **E** $x = 3$

4 Consider the equation $(x + 1)^2 - (x + 1) - 12 = 0$.

a Solve the equation
  i by first expanding the brackets and collecting like terms.
  ii by substituting a for $(x + 1)$.

b Which method do you prefer? Explain.

LINKS TO Example 14

5 Solve each of the following equations by substitution.

a $(x - 1)^2 - 2(x - 1) + 1 = 0$ **b** $(x - 2)^2 - 3(x - 2) + 2 = 0$
c $(3x - 1)^2 + 4(3x - 1) - 5 = 0$ **d** $(3x - 2)^2 + 4(3x - 2) - 5 = 0$
e $(2x - 3)^2 - 10(2x - 3) - 11 = 0$ **f** $(x - 3)^2 - 3(x - 3) - 10 = 0$

LINKS TO Example 15

6 Solve each of the following equations by substitution.

a $x^4 - 10x^2 + 9 = 0$ **b** $x^4 - 13x^2 + 36 = 0$
c $x^4 - 17x^2 + 16 = 0$ **d** $x^4 - 18x^2 + 81 = 0$

LINKS TO Example 16

7 Solve each of these equations by first 'removing' the fractions.

a $x + 3 = \frac{4}{x}$ **b** $x + \frac{15}{x} = 8$ **c** $x - 5 = \frac{24}{x}$
d $x - \frac{12}{x} = 11$ **e** $x - \frac{5}{x} = 4$ **f** $x + 12 = \frac{45}{x}$

exercise 8.3 challenge

8 The number of diagonals of a polygon is given by $D = \frac{n}{2}(n - 3)$, where n is the number of sides. If a polygon has 65 diagonals, how many sides does it have?

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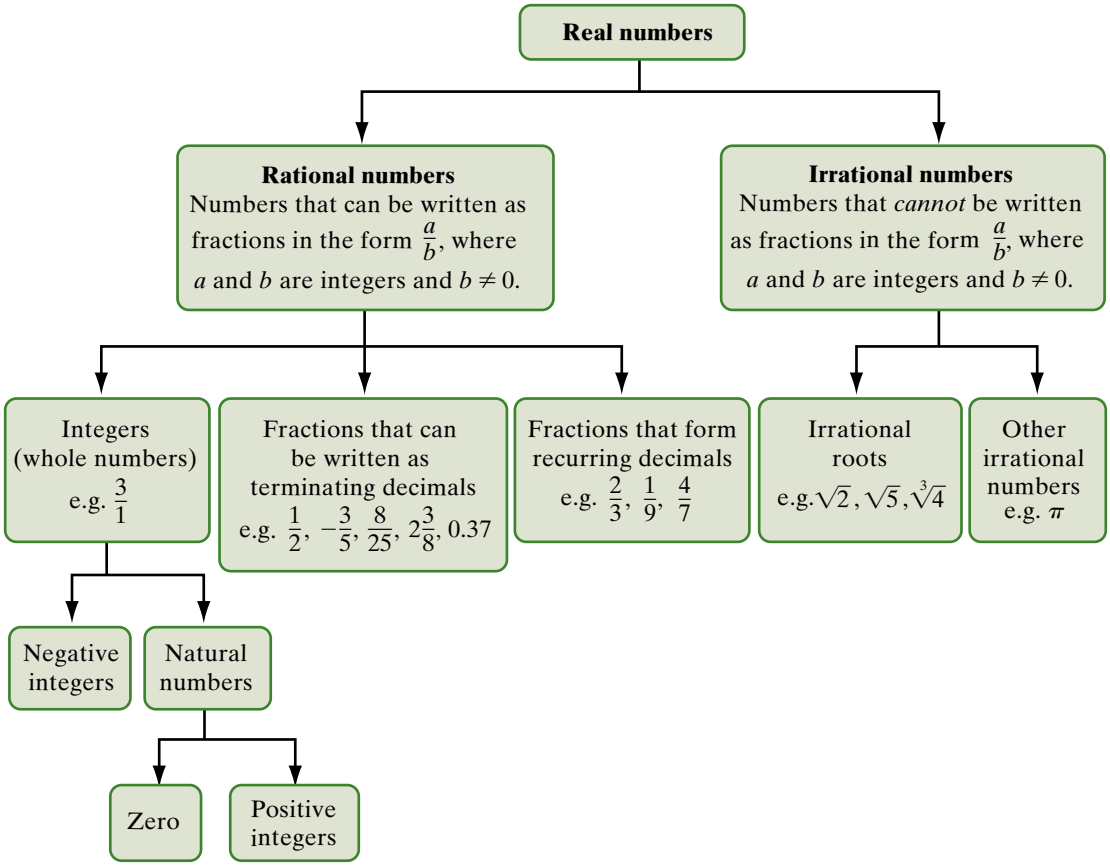
8.4 Quadratic equations with real solutions

Not all quadratic equations have real solutions. A sum of squares has no real factors, so quadratic equations such as $x^2 + 9 = 0$ or $(x - 3)^2 + 11 = 0$ have no real solutions.

In sections **8.1** to **8.3**, all of the solutions to quadratic equations were **rational** numbers. In fact, those quadratic equations were carefully selected so that the solutions were rational and relatively easy to find. However, there are many quadratics for which the solutions are **real** but **irrational**.

The diagram below reviews the relationship between real, rational and irrational numbers. Mathematicians and engineers sometimes work with 'unreal' or **imaginary numbers**, such as $\sqrt{-3}$. However, school mathematics focuses on real numbers.



In this section, we will look at general methods that can be used to find irrational solutions to quadratic equations. In section **8.8**, we will see how to identify equations that have no real solutions.

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If the solutions relate to real-world problems, the solutions may often be irrational. It is useful to find rational approximations for the solutions.

Example 17

Solve the equation $x^2 - 7 = 0$, giving the solutions:

a in exact form.

b correct to two decimal places.

Working

a

$$x^2 - 7 = 0$$
$$(x - \sqrt{7})(x + \sqrt{7}) = 0$$
$$x - \sqrt{7} = 0 \quad \text{or } x + \sqrt{7} = 0$$
$$x = \sqrt{7} \text{ or } \quad x = -\sqrt{7}$$

b $x \approx 2.65$ or $x \approx -2.65$

Reasoning

This equation could be solved by adding 7 to both sides, then taking the square root of both sides. We need to remember, though, to take the positive and negative square roots.

$x^2 = 7$

$x = \sqrt{7}$ or $x = -\sqrt{7}$.

Completing the square

In section **3.8** you used the method of 'completing the square' to factorise quadratic expressions that did not have rational factors. We can use this same technique when solving quadratic equations where the quadratic expression has no rational factors. The method of solving by completing the square can be applied to any quadratic equation, but we would not normally use this method if the quadratic expression has obvious rational factors.

The equation $x^2 - 4x - 5 = 0$ can be solved by factorising, giving the solutions $x = 5$ or $x = -1$. However, in example **18** we show how the method of solving by completing the square leads to the same result.

Example 18

Solve the equation $x^2 - 4x - 5 = 0$ by completing the square.

Working

$$x^2 - 4x - 5 = 0$$
$$x^2 - 4x + (-2)^2 - (-2)^2 - 5 = 0$$
$$x^2 - 4x + 4 - 4 - 5 = 0$$

$$(x - 2)^2 - 9 = 0$$
$$(x - 2)^2 - 3^2 = 0$$

$$x - 2 - 3 = 0 \text{ or } x - 2 + 3 = 0$$
$$x - 5 = 0 \text{ or } \quad x + 1 = 0$$
$$x = 5 \text{ or } \quad x = -1$$

Reasoning

To complete the square, halve the coefficient of x and square it. This number is added to form a perfect square but it must also be subtracted to keep the value of the left side unchanged.

Express the left side as a difference of squares.

Factorise the difference of squares.

Solve each linear equation to give the solutions to the quadratic equation

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In the next example, completing the square is used to solve an equation that cannot be easily factorised. The solutions are irrational.

Example 19

Solve the equation $x^2 - 4x - 3 = 0$ by completing the square.

Working

$$x^2 - 4x - 3 = 0$$
$$x^2 - 4x + (-2)^2 - (-2)^2 - 3 = 0$$
$$\underbrace{x^2 - 4x + 4} \quad \underbrace{-4 - 3 = 0}$$

Reasoning

To complete the square, halve the coefficient of x and square it. This number is added to form a perfect square but it must also be subtracted to keep the value of the left side unchanged.

$$(x - 2)^2 - 7 = 0$$
$$(x - 2)^2 - (\sqrt{7})^2 = 0$$
$$(x - 2 - \sqrt{7})(x - 2 + \sqrt{7}) = 0$$

Express the left side as a difference of squares.

$$x - 2 - \sqrt{7} = 0 \quad \text{or} \quad x - 2 + \sqrt{7} = 0$$
$$x = 2 + \sqrt{7} \quad \text{or} \quad x = 2 - \sqrt{7}$$

Factorise the difference of squares. Solve each linear equation to give the solutions to the quadratic equation. The solutions are irrational.

Example 20

Solve the equation $x^2 - 6x + 4 = 0$ by completing the square.

Working

$$x^2 - 6x + 4 = 0$$

Reasoning

$x^2 - 6x + 4$ does not have obvious rational factors.
Use the method of completing the square.

$$x^2 - 6x + (-3)^2 - (-3)^2 + 4 = 0$$
$$\underbrace{x^2 - 6x + 9} \quad \underbrace{-9 + 4 = 0}$$

Halve the coefficient of the x term and square it. Add $(-3)^2$ to turn $x^2 - 6x$ into a perfect square. But what we add to the left side must also be subtracted to keep the value of the left side unchanged.

$$(x - 3)^2 - 5 = 0$$
$$(x - 3)^2 - (\sqrt{5})^2 = 0$$
$$(x - 3 - \sqrt{5})(x - 3 + \sqrt{5}) = 0$$
$$x - 3 - \sqrt{5} = 0 \quad \text{or} \quad x - 3 + \sqrt{5} = 0$$
$$x = 3 + \sqrt{5} \quad \text{or} \quad x = 3 - \sqrt{5}$$

Group $-(-3)^2$ with $+ 4$ to give –5.
The left side can now be factorised as a difference of squares.
The solutions are irrational.

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If the coefficient of the x term is odd, then completing the square will involve fractions.

Example 21

Solve the equation $x^2 - 3x - 1 = 0$ by completing the square.

Working

$$x^2 - 3x - 1 = 0$$

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 - 1 = 0$$

$$\underbrace{x^2 - 3x + \frac{9}{4}} \quad \underbrace{-\frac{9}{4} - 1} = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{13}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2 = 0$$

$$x - \frac{3}{2} - \frac{\sqrt{13}}{2} = 0 \quad \text{or} \quad x - \frac{3}{2} + \frac{\sqrt{13}}{2} = 0$$

$$x = \frac{3 + \sqrt{13}}{2} \quad \text{or} \quad x = \frac{3 - \sqrt{13}}{2}$$

Reasoning

$x^2 - 3x - 1$ does not have simple factors.

Use the method of completing the square.

Halve the coefficient of the x term and square it. Add $\left(-\frac{3}{2}\right)^2$ to turn $x^2 - 3x$ into a perfect square. But what we add to the left side must also be subtracted to keep the value of the left side unchanged. Group $-\left(-\frac{3}{2}\right)^2$ with –1 to give $-\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}$.

The left side can now be factorised as a difference of squares.

The solutions are irrational.

exercise 8.4

LINKS TO Example 17

1 Solve these quadratic equations, simplifying the square root where possible.

a $x^2 - 6 = 0$ **b** $x^2 - 11 = 0$ **c** $x^2 - 5 = 0$ **d** $x^2 - 2 = 0$ **e** $x^2 - 17 = 0$
f $x^2 - 18 = 0$ **g** $x^2 - 20 = 0$ **h** $x^2 - 8 = 0$ **i** $x^2 - 12 = 0$ **j** $x^2 - 9 = 0$

LINKS TO Examples 18, 19, 20

2 Use the method of completing the square to solve each of the following quadratic equations. In parts o to r, the square root can be simplified.

a $x^2 - 2x - 9 = 0$ **b** $x^2 - 6x + 8 = 0$ **c** $x^2 - 2x - 1 = 0$
d $x^2 + 6x - 4 = 0$ **e** $x^2 - 4x + 3 = 0$ **f** $x^2 + 6x + 4 = 0$
g $x^2 - 4x + 2 = 0$ **h** $x^2 - 2x - 5 = 0$ **i** $x^2 - 16x - 5 = 0$
j $x^2 - 10x - 10 = 0$ **k** $x^2 - 8x - 20 = 0$ **l** $x^2 + 2x - 24 = 0$
m $x^2 - 4x - 9 = 0$ **n** $x^2 - 2x - 15 = 0$ **o** $x^2 - 8x - 4 = 0$
p $x^2 - 2x - 7 = 0$ **q** $x^2 - 10x - 7 = 0$ **r** $x^2 - 12x + 4 = 0$

3 Which of the equations in question **2** could have been done by simple factorising without completing the square? Explain.

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10 A

LINKS TO Example 21

4 Use the method of completing the square to solve each of the following quadratic equations.

a $x^2 - 3x - 2 = 0$ **b** $x^2 - 5x + 4 = 0$ **c** $x^2 - x - 3 = 0$
d $x^2 - x - 2 = 0$ **e** $x^2 - 5x - 2 = 0$ **f** $x^2 + x - 6 = 0$
g $x^2 - 3x - 7 = 0$ **h** $x^2 - 5x - 6 = 0$ **i** $x^2 - 3x - 5 = 0$
j $x^2 - 3x + 2 = 0$ **k** $x^2 - x - 20 = 0$ **l** $x^2 - x - 5 = 0$

5 Which of the equations in question **4** could have been done by simple factorising without completing the square? Explain.

exercise 8.4 challenge

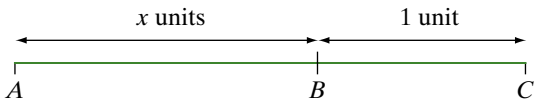
10 A

6 Harry solved $2x^2 - x - 3 = 0$ by completing the square as shown.

$$2x^2 - x - 3 = 0$$
$$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right) = 0$$
$$x^2 - \frac{x}{2} - \frac{3}{2} = 0$$
$$2\left[x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 - \frac{1}{16} - \frac{3}{2}\right] = 0$$
$$\left(x - \frac{1}{4}\right)^2 - \frac{25}{16} = 0$$
$$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$$
$$x - \frac{1}{4} = \pm\frac{5}{4}$$
$$x = \frac{5}{4} - \frac{1}{4} \text{ or } \frac{5}{4} + \frac{1}{4}$$
$$x = 1 \text{ or } 1\frac{1}{2}$$

a Are Harry's solutions correct? If not, correct the error(s) he made.
b Was this the best way of solving this equation? Explain.

7 The line segment AC is divided by point B so that $\frac{AB}{BC} = \frac{AC}{AB}$.



a Express $\frac{AB}{BC} = \frac{AC}{AB}$ in terms of x.
b Convert the equation into the form $ax^2 + bx + c = 0$.
c Solve the equation for x, giving the solutions in exact form.
d Does it make sense to use the negative solution?
e Do you recognise the solution?
f Evaluate the solution correct to three decimal places.

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8.5 Solving non-monic quadratic equations by factorisation

So far, most of the quadratic equations we have solved have been **monic quadratic equations**, that is, the coefficient of the x^2 term is ± 1 ($a = \pm 1$). The same techniques that were used in section 3.9 to factorise non-monic quadratic trinomials are now applied to solving non-monic quadratic equations.

Example 22

Solve these non-monic quadratic equations by first factorising the left side.

a $3x^2 + 2x - 8 = 0$ **b** $6x^2 - 13x + 5 = 0$ **c** $9x^2 - 21x + 10 = 0$

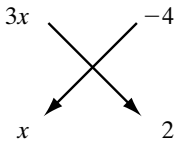
Working

a

$$3x^2 + 2x - 8 = 0$$
$$(3x - 4)(x + 2) = 0$$
$$3x - 4 = 0 \text{ or } x + 2 = 0$$
$$x = \frac{4}{3} \text{ or } \quad x = -2$$

Reasoning

Solve the equations $3x - 4 = 0$ and $x + 2 = 0$.

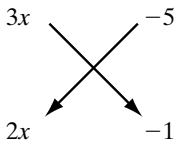


$6x - 4x = 2x$

b

$$6x^2 - 13x + 5 = 0$$
$$(3x - 5)(2x - 1) = 0$$
$$3x - 5 = 0 \text{ or } 2x - 1 = 0$$
$$x = \frac{5}{3} \text{ or } \quad x = \frac{1}{2}$$

Solve the equations $3x - 5 = 0$ and $2x - 1 = 0$.

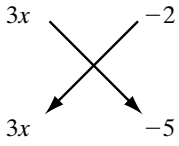


$-3x - 10x = -13x$

c

$$9x^2 - 21x + 10 = 0$$
$$(3x - 2)(3x - 5) = 0$$
$$3x - 2 = 0 \text{ or } 3x - 5 = 0$$
$$x = \frac{2}{3} \text{ or } \quad x = \frac{5}{3}$$

Solve the equations $3x - 2 = 0$ and $3x - 5 = 0$.



$-15x - 6x = -21x$

Example 23

Solve these non-monic quadratic equations by first factorising the left side.

a $4x^2 - 12x + 9 = 0$ **b** $9x^2 + 6x + 1 = 0$

continued

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Example 23 continued

Working	Reasoning
a $4x^2 - 12x + 9 = 0$	$4x^2 - 12x + 9$ is a perfect square.
$(2x - 3)^2 = 0$	
$2x - 3 = 0$	Solve the equation $2x - 3 = 0$.
$x = \frac{3}{2}$	
b $9x^2 + 6x + 1 = 0$	$9x^2 + 6x + 1$ is a perfect square.
$(3x + 1)^2 = 0$	
$3x + 1 = 0$	Solve the equation $3x + 1 = 0$.
$x = -\frac{1}{3}$	

exercise 8.5

LINKS TO Example 22

1 Solve the following non-monic quadratic equations by first factorising the left side.

a $2x^2 - x - 15 = 0$ **b** $2x^2 + x - 21 = 0$ **c** $2x^2 + 13x - 7 = 0$
d $2x^2 - 21x - 11 = 0$ **e** $4x^2 - 8x + 3 = 0$ **f** $6x^2 - 5x + 1 = 0$
g $3x^2 + x - 4 = 0$ **h** $6x^2 + 7x - 5 = 0$ **i** $12x^2 - x - 1 = 0$
j $4x^2 - 4x - 15 = 0$ **k** $6x^2 - 11x - 10 = 0$ **l** $6x^2 - 13x + 6 = 0$
m $3x^2 + 16x - 12 = 0$ **n** $3x^2 + 2x - 8 = 0$ **o** $4x^2 + 11x - 3 = 0$
p $3x^2 + x - 10 = 0$ **q** $5x^2 + 7x - 6 = 0$ **r** $4x^2 - x - 3 = 0$
s $8x^2 + 2x - 1 = 0$ **t** $12x^2 + 7x + 1 = 0$ **u** $7x^2 - 10x + 3 = 0$
v $3x^2 - 11x + 6 = 0$ **w** $2x^2 - 17x + 35 = 0$ **x** $6x^2 - 5x - 4 = 0$

LINKS TO Example 23

2 Solve the following non-monic quadratic equations by first factorising the left side.

a $9x^2 - 6x + 1 = 0$ **b** $4x^2 + 4x + 1 = 0$ **c** $16x^2 - 40x + 25 = 0$
d $9x^2 - 30x + 25 = 0$ **e** $25x^2 - 20x + 4 = 0$ **f** $4x^2 - 20x + 25 = 0$
g $25x^2 + 10x + 1 = 0$ **h** $4x^2 - 28x + 49 = 0$ **i** $9x^2 + 12x + 4 = 0$
j $64x^2 - 48x + 9 = 0$ **k** $9x^2 - 60x + 100 = 0$ **l** $16x^2 - 24x + 9 = 0$

3 Solve each of these quadratic equations.

a $6x^2 + 9x + 3 = 0$ **b** $4x^2 - 10x + 6 = 0$ **c** $10x^2 - 5x - 15 = 0$
d $6x^2 - 10x - 24 = 0$ **e** $12x^2 + 24x - 15 = 0$ **f** $9x^2 - 66x + 21 = 0$

exercise 8.5 challenge

4 For each of these equations, factorise the left side then solve the equation.

a $5x^2 - 22x + 21 = 0$ **b** $15x^2 - 26x + 8 = 0$ **c** $35x^2 - 9x - 2 = 0$
d $8x^2 - 38x + 35 = 0$ **e** $18x^2 - 21x - 72 = 0$ **f** $12x^2 - 38x - 72 = 0$

5 Solve the following equations by substitution.

a $3(2x + 1)^2 - 5(2x + 1) - 2 = 0$ **b** $2(3x + 1)^2 + 13(3x + 1) - 7 = 0$
c $6(2x - 5)^2 + 9(2x - 5) + 3 = 0$ **d** $2(4x + 3)^2 + 11(4x + 3) + 15 = 0$

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8.6 The quadratic formula

In section 8.4, the method of completing the square was used to solve quadratic equations where the quadratic expression $ax^2 + bx + c$ had no obvious rational factors. The equations that were solved in this way were all monic quadratic equations, that is, the coefficient of the x^2 term was ± 1 ($a = \pm 1$).

We will now look at the solving of non-monic quadratic equations.

Instead of having to complete the square every time we solve a non-monic quadratic equation, we can solve the general non-monic equation $ax^2 + bx + c = 0$ by completing the square and develop a formula that can be applied to any quadratic equation.

The algebraic steps for this process are quite challenging, but they result in a very useful formula that can be used to solve any quadratic equation that has real solutions. Therefore the steps are included below to show how the formula is derived.

Solving the general quadratic equation

$$ax^2 + bx + c = 0$$

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0 \qquad \text{(taking out a factor of } a\text{)}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(dividing both sides by } a\text{)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad \text{(completing the square)}$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \text{(isolating the term involving } x\text{)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that if $b^2 - 4ac$ is negative there will be no real solutions (see section 8.8).

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The quadratic formula

The solutions for any quadratic equation $ax^2 + bx + c = 0$, can be calculated (if they exist) by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 24

Use the quadratic formula to solve each of the following equations.

a $x^2 - 3x - 5 = 0$ **b** $2x^2 + 3x - 8 = 0$

Working	Reasoning
a $x^2 - 3x - 5 = 0$	
$a = 1, b = -3, c = -5$	Identify the values of a, b and c.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Substitute in the quadratic formula.
$= \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-5)}}{2 \times 1}$	Simplify the expression under the square root sign.
$x = \dfrac{3 + \sqrt{29}}{2}$ or $= \dfrac{3 - \sqrt{29}}{2}$	Separate the two solutions.
b $2x^2 + 3x - 8 = 0$	
$a = 2, b = 3, c = -8$	Identify the values of a, b and c.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Substitute in the quadratic formula.
$= \dfrac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-8)}}{2 \times 2}$	Simplify the expression under the square root sign.
$x = \dfrac{-3 + \sqrt{73}}{4}$ or $= \dfrac{-3 - \sqrt{73}}{4}$	Separate the two solutions.

Sometimes the irrational square root can be simplified, as shown in the next example.

Example 25

Use the quadratic formula to solve the equation $x^2 - 4x - 3 = 0$

Working	Reasoning
$x^2 - 4x - 3 = 0$	
$a = 1,\ b = -4$ and $c = -3$	Identify the values of a, b and c.

continued

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Example 25 continued

Working	Reasoning
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Identify the values of a, b and c.
$= \dfrac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$	Substitute the values of a, b, and c into the quadratic formula.
$= \dfrac{4 \pm \sqrt{28}}{2}$	Simplify the expression under the square root sign. $\sqrt{28} = \sqrt{4 \times 7} = \sqrt{4} \times \sqrt{7} = 2 \times \sqrt{7} = 2\sqrt{7}$
$= \dfrac{4 \pm \sqrt{4} \times \sqrt{7}}{2}$	
$= \dfrac{4 \pm 2\sqrt{7}}{2}$	
$= \dfrac{2(2 \pm \sqrt{7})}{2}$	Take out a common factor and cancel.
$x = 2 - \sqrt{7}$ or $x = 2 + \sqrt{7}$	Separate the two solutions. The solutions are the same as those found by completing the square and factorising $(x - 2)^2 - 7 = 0$.

The quadratic formula can be used with any quadratic equation, regardless of whether the quadratic expression can be solved easily by factorising.

Example 26

Solve $x^2 - 4x - 5 = 0$ by using the quadratic formula.

Working	Reasoning
For $x^2 - 4x - 5 = 0$, $a = 1$, $b = -4$ and $c = -5$	First identify the values of a, b and c.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Substitute the values of a, b, and c into the quadratic formula.
$= \dfrac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times (-5)}}{2 \times 1}$	Simplify the expression under the square root sign.
$= \dfrac{4 \pm \sqrt{36}}{2}$	Evaluate the square root.
$= \dfrac{4 \pm 6}{2}$	
$x = \dfrac{4 - 6}{2}$ or $x = \dfrac{4 + 6}{2}$	Separate the two solutions.
$x = -1$ or $x = 5$	The solutions are the same as those found by using $(x + 1)(x - 5) = 0$.

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When solving $x^2 - 4x - 5 = 0$, the square root was $\sqrt{36} = 6$, so the solutions were rational.

When solving $x^2 - 4x - 3 = 0$, the square root was $\sqrt{28} = 2\sqrt{7}$, so the solutions were irrational.

Try easier methods first!

The solutions to $x^2 - 4x - 5 = 0$ in example **30** were rational so we could have used factorisation to solve for x.

$$\begin{aligned} x^2 - 4x - 5 &= 0 \\ (x + 1)(x - 5) &= 0 \\ x &= -1 \text{ or } 5 \end{aligned}$$

Before using the quadratic formula, you could check whether a quadratic expression has simple factors.

Example 27

Solve each of these equations using the most appropriate method.

a $x^2 - 7x - 6 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $2x^2 - 11x - 16 = 0$

Working

a

$$\begin{aligned} x^2 - 7x - 6 &= 0 \\ (x - 6)(x - 1) &= 0 \\ x - 6 = 0 \text{ or } x - 1 &= 0 \\ x = 6 \text{ or } \quad x &= 1 \end{aligned}$$

Reasoning

The quadratic expression can be factorised easily.

b

$$\begin{aligned} x^2 - 8x + 1 &= 0 \\ x^2 - 8x + (-4)^2 - (-4)^2 + 1 &= 0 \\ (x - 4)^2 - 16 + 1 &= 0 \\ (x - 4)^2 - 15 &= 0 \\ (x - 4 - \sqrt{15})(x - 4 + \sqrt{15}) &= 0 \\ x - 4 - \sqrt{15} = 0 \text{ or } x - 4 + \sqrt{15} &= 0 \\ x = 4 + \sqrt{15} \text{ or } x &= 4 - \sqrt{15} \end{aligned}$$

The quadratic expression cannot be factorised easily.
Complete the square.

c

$$\begin{aligned} 2x^2 - 11x - 16 &= 0 \\ a = 2, b = -11, c &= -16 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \times 2 \times (-16)}}{2 \times 2} \\ &= \frac{11 \pm \sqrt{121 + 128}}{4} \\ &= \frac{11 \pm \sqrt{249}}{4} \\ x &= \frac{11 + \sqrt{249}}{4} \text{ or } x = \frac{11 - \sqrt{249}}{4} \end{aligned}$$

Completing the square would involve fractions. It is easier to use the quadratic formula.

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Exact and approximate solutions

In example 25 we saw that the **exact solutions** to $x^2 - 4x - 3 = 0$ were $x = 2 - \sqrt{7}$ or $2 + \sqrt{7}$. When solving practical problems, such as those involving measurements, you may be asked for an **approximate solution** (or a **rational approximation**). For example, the approximate solutions to $x^2 - 4x - 3 = 0$, correct to two decimal places, are $x = -0.65$ or 4.65.

$$\begin{aligned} x &= 2 - \sqrt{7} \\ &= 2 - 2.6457\ldots \\ &= -0.6457\ldots \\ &\approx -0.65 \end{aligned} \qquad \begin{aligned} x &= 2 + \sqrt{7} \\ &= 2 + 2.6457\ldots \\ &= 4.6457\ldots \\ &\approx 4.65 \end{aligned}$$

We say that $x = -0.65$ and $x = 4.65$ are rational approximations to the exact solution.

When solving a problem, always check whether exact or approximate solutions are required. In the case of rational approximations, you will be asked for answers correct to a certain number of decimal places.

If approximate solutions are required, evaluate the irrational solutions to the required number of decimal places.

Example 28

Use the quadratic formula to solve the equation $3x^2 - 2x - 9 = 0$, giving the approximate solutions correct to two decimal places.

Working	Reasoning
$3x^2 - 2x - 9 = 0$ $a = 3, b = -2, c = -9$	Identify a, b and c then substitute into the quadratic formula.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	
$x = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 3 \times -9}}{2 \times 3}$	
$= \dfrac{2 + \sqrt{112}}{6}$ or $x = \dfrac{2 - \sqrt{112}}{6}$	Separate the two solutions
$x \approx 2.10$ or $x \approx -1.43$ (correct to two decimal places)	Evaluate to two decimal places.

The quadratic equations arising from real-world problems often have irrational solutions, for which rational approximations are appropriate.

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Example 29

A ball is thrown vertically upwards so that its height, h metres, at time t seconds is given by the rule $h = 7t - 4.9t^2$. Find the times at which the ball is at a height of 2 metres (once on its way up and again on its way down). Give the times in seconds, correct to one decimal place.

Working	Reasoning
$7t - 4.9t^2 = 2$ $4.9t^2 - 7t + 2 = 0$	$7t - 4.9t^2 = h$ So, if $h = 2$ then $7t - 4.9t^2 = 2$
$a = 4.9, b = -7, c = 2$ $t = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 4.9 \times 2}}{2 \times 4.9}$ $= \dfrac{7 \pm \sqrt{49 - 39.2}}{9.8}$ $= \dfrac{7 \pm \sqrt{9.8}}{9.8}$ $= 0.39\ldots$ or $1.03\ldots$ The ball is at a height of 2 m after 0.4 seconds and again after 1.0 seconds.	Identify a, b and c then substitute into the quadratic formula. $t = \dfrac{7 - \sqrt{9.8}}{9.8}$ or $\dfrac{7 + \sqrt{9.8}}{9.8}$ $t \approx 0.39$ $\quad$ $t \approx 1.03$ $t = 0.4$ $\quad$ $t = 1.0$ (correct to one decimal place)

exercise 8.6

LINKS TO Examples 24, 25, 26, 27

1 Use the quadratic formula to find exact solutions to the following equations.

a $x^2 - 3x - 2 = 0$ **b** $x^2 - 5x + 3 = 0$ **c** $5x^2 = 7x - 2$
d $4x^2 - 9x + 3 = 0$ **e** $3x^2 + 2x - 6 = 0$ **f** $5x^2 - x - 3 = 0$
g $5x^2 + 8x = -2$ **h** $2x(2x - 1) = 5$ **i** $x^2 - 6x - 3 = 0$
j $x^2 + 8x - 5 = 0$ **k** $2x^2 - 6x = 7$ **l** $4x^2 - 5x - 11 = 0$
m $2x^2 + 5x = 6$ **n** $4x^2 - 2x - 3 = 0$ **o** $3x^2 + 4x - 7 = 0$
p $2x^2 = 7x - 1$ **q** $3x^2 + 5x + 2 = 0$ **r** $3x^2 + x - 3 = 0$

LINKS TO Example 28

2 Use the quadratic formula to solve these equations, giving the solutions correct to two decimal places.

a $x^2 + 4x - 9 = 0$ **b** $x^2 + 3x - 7 = 0$ **c** $a^2 + 4a + 1 = 0$
d $6y^2 - 4y - 25 = 0$ **e** $3d^2 + 7d - 18 = 0$ **f** $k^2 + 8k + 3 = 0$
g $x^2 + 5x - 15 = 0$ **h** $2h^2 - 11h - 7 = 0$ **i** $2.5a^2 + 5a - 9.5 = 0$
j $0.1x^2 + 2.4x - 6 = 0$ **k** $3y^2 - 3.2y - 12.4 = 0$ **l** $4x^2 - 3x - 20 = 0$

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3 Solve the following quadratic equations. Give the solutions in approximate form, correct to two decimal places.

a $2x^2 - 11 = 4x$ **b** $2m^2 + m = 5$ **c** $x^2 + 11x = 32$
d $x^2 - 4x = 15$ **e** $h^2 - 8 = 5h$ **f** $5x^2 + 7x = 14$
g $2x^2 + 5x = 15$ **h** $2.4m^2 + 3m = 8$ **i** $0.01a^2 - 12 = 4.5a$
j $2x^2 - 6x = 2(x + 1)$ **k** $x^2 - 4x = 5(2x + 1)$ **l** $3x(x - 3) = 4x + 7$

4 Which of the following best describes the solutions of the equation $a^2 - 4a = 2(a - 3)$?

A $a = 4.00$ or $a = 3.00$ **B** $a = -4.00$ or $a = -3.00$ **C** $a = 1.27$ or $a = 4.73$
D $a = -0.87$ or $a = 6.87$ **E** There are no real solutions.

exercise 8.6 challenge

LINKS TO Example 29

5 The speed of a train, s km/h, as it travels from station A to station B can be modelled by the equation $s = 60t - 15t^2$, where t is the time in minutes after the train leaves station A.

a Find how many minutes (correct to one decimal place) after leaving station A the train is travelling

i at 10 km/h. **ii** at 40 km/h. **iii** at 50 km/h.

b Can you suggest why there are two possible times for each speed?

c Try to find the maximum speed of train and when it is reached.

d When does the train reach station B? How do you know?

6 The height h metres of a rocket, t seconds after it is launched from sea level (0 m), is given by the function rule $h = 2t(80 - 2t)$.

a How long does the rocket take to return to sea level?

b Use the quadratic formula to determine when the rocket will be 1000 m above sea level. Answer correct to one decimal place.

7 The path of a particular golf ball could be modelled by the equation $h = -0.002d^2 + 0.4d$ where d m is the horizontal distance travelled by the ball and h m is its height above the ground.

a Find the horizontal distance (to the nearest metre) the ball has travelled when it is at a height of

i 0.5 m **ii** 5 m **iii** 15 m

b Why there are two possible distances for each height?

c Try to find the maximum height of the ball and when it is reached.

d How far has the ball travelled horizontally when it lands? How do you know?

8 A quadratic equation has solutions $x = \frac{2}{3}$ and $x = -\frac{6}{5}$. Write three possible quadratic equations.

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8.7 Solving problems with quadratic equations

When solving problems involving quadratic equations we use the following steps.

- Write a quadratic equation.
- Solve the equation.
- Interpret the solutions in terms of the problem.
- Check that the solution satisfies the information given in the problem.
- Write a sentence answering the question.

Example 30

The sum of two positive integers is 27. The product of the two integers is 126. Let n be the smaller integer.

a Write an equation in terms of n, then rearrange it so that it is in the form $ax^2 + bx + c = 0$.

b Solve the equation.

c State the values of the two integers.

Working	Reasoning
a The other integer is $27 - n$.	The sum of the two integers is 27.
$n(27 - n) = 126$	The product of the two integers is 126.
$27n - n^2 = 126$	Expand the LS.
$n^2 - 27n + 126 = 0$	Rearrange into the form $ax^2 + bx + c = 0$.
b $(n - 6)(n - 21) = 0$	Factorise.
$n = 6$ or $n = 21$	
c The integers are 6 and 21.	Check: Sum $= 6 + 21 = 27$ Product $= 6 \times 21 = 126$

Quadratic equations frequently have two solutions. When a quadratic equation is used to solve a problem, we need to consider whether both solutions make sense in the context of the problem. For example, if we are finding the length of a rectangle, it would not make sense to use a negative solution. We would select the positive solution and discard the negative solution. Although the negative solution is a solution to the quadratic equation, it is not a solution to the problem.

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Example 31

Two consecutive positive integers have a product of 210. Let n be the smaller integer. Find the two integers.

Working

$$n(n+1) = 210$$
$$n^2 + n - 210 = 0$$
$$(n+15)(n-14) = 0$$
$$n = -15 \text{ or } n = 14$$

The integers are 14 and 15.
Check:
$14 \times 15 = 210$

Reasoning

The two integers are consecutive (that is, they follow one after the other such as 4 and 5). If n is the smaller integer, the other integer is $n + 1$.

The problem states that the integers are positive so discard the negative solution.
$n = 14$ so the other integer is
$n + 1 = 14 + 1 = 15$.

Example 32

Two positive numbers are consecutive even integers. The product of the two integers is six more than three times their sum. Let n be the smaller integer.

a Write an equation in terms of n.

b Solve the equation.

c State the values of the two integers.

Working

a The other integer is $n + 2$.
$n(n+2) = 3(n+n+2) + 6$

Reasoning

The integers are even, so the larger integer must be 2 more than the smaller integer.
The product is $n(n+2)$.
The sum is $n + n + 2$.

b

$$n^2 + 2n = 3(2n+2) + 6$$
$$n^2 + 2n = 6n + 6 + 6$$
$$n^2 + 2n - 6n - 12 = 0$$
$$n^2 - 4n - 12 = 0$$
$$(n-6)(n+2) = 0$$
$$n = 6 \text{ or } n = -2$$

Expand and collect like terms.
Subtract $6n + 12$ from both sides.
Collect like terms.
Factorise.
The problem states that the integers are positive so discard the negative solution.

c The integers are 6 and 8.

If the smaller integer is 6, then the consecutive even integer is $6 + 2$.
Check:
Product $= 6 \times 8 = 48$
Six more than three times sum:
$3(6+8) + 6 = 3 \times 14 + 6 = 48$

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Example 33

The length of a rectangle is 3 m more than the width. The area is 108 m^2. Let the width of the rectangle be w m.

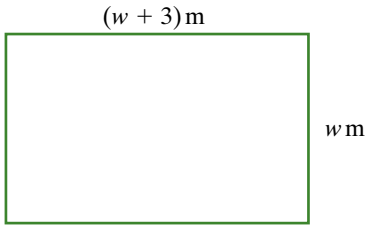
a Draw a labelled diagram.

b Write an equation for the area in terms of w.

c Solve the equation.

d Find the length and width of the rectangle.

Working	Reasoning
a Length = $(w + 3)$ m	The length is 3 m more than the width so length is 3 m more than w m.



b $w(w + 3) = 108$	Area of the rectangle = width × length.
c $w^2 + 3w = 108$	Write the equation.
$w^2 + 3w - 108 = 0$	Subtract 108 from both sides and factorise the left side.
$(w + 12)(w - 9) = 0$	
$w + 12 = 0$ or $w - 9 = 0$	
$w = -12$ or $w = 9$	The width must be a positive number so discard the negative solution.
d The width is 9 m and the length is 12 m.	Check: 12 m is 3 m more than 9 m Area = 9 m × 12 m = 108 m^2

In some problems, solving the quadratic equation to find an unknown value only partly answers the question. This value may need to be used to find a set of values. For example, solving the quadratic equation may tell us the length of one side of a triangle. Information included in the problem statement must then be used to find the other two sides.

Example 34

The hypotenuse of a right-angled triangle is 6 cm longer than the shortest side. The third side is 3 cm longer than the shortest side. Let a cm be the length of the shortest side.

a Draw a labelled diagram.

b Use Pythagoras' theorem to write an equation in terms of a.

c Solve the equation.

d Find the lengths of each of the three sides.

continued

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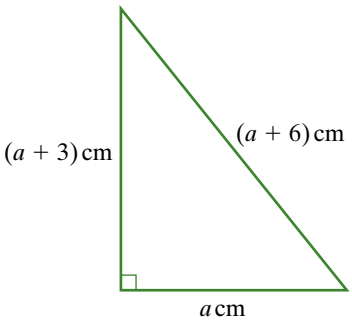
Example 34 continued

Working	Reasoning
a 	The length of the hypotenuse is $(a + 6)$ cm. The length of the third side is $(a + 3)$ cm.
b $a^2 + (a + 3)^2 = (a + 6)^2$	Use Pythagoras' theorem to express the relationships between the lengths of the three sides.
c $a^2 + a^2 + 6a + 9 = a^2 + 12a + 36$ $2a^2 + 6a + 9 = a^2 + 12a + 36$ $a^2 - 6a - 27 = 0$ $(a - 9)(a + 3) = 0$ $a - 9 = 0$ or $a + 3 = 0$ $a = 9$ or $a = -3$ Discard the solution –3.	Expand the brackets and collect like terms. Rearrange so that the right side equals zero. Factorise the quadratic expression. The length must be positive so discard the negative solution.
d The lengths of the sides are 9 cm, 12 cm and 15 cm.	The shortest side is 9 cm. The other sides are $(9 + 3)$ cm and $(9 + 6)$ cm Check: LS $= 9^2 + 12^2$ $= 81 + 144$ $= 225$ RS $= 15^2$ $= 225$ LS = RS

exercise 8.7

For each question in this exercise

- **a** write a quadratic equation.
- **b** solve the equation.
- **c** write a sentence answering the question.

LINKS TO Examples 30, 31, 32

For questions 1 to 9 let *n* be the smaller integer.

1 The sum of two integers is 23 and their product is 120. Find the integers.

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2. The sum of two integers is 25 and their product is 136. Find the integers.
3. The product of two positive consecutive integers is 156. Find the integers.
4. The product of two consecutive positive integers is five more than three times the larger integer. Find the integers.
5. The product of two consecutive even integers is 168. Find the integers.
6. The product of two consecutive odd positive integers is 143. Find the integers.
7. The product of two consecutive odd integers is one less than twice their sum. Find the integers.
8. Find two consecutive even integers, if the square of the smaller integer is 10 more than the larger integer.
9. Find three consecutive positive integers, if four times the sum of the three integers equals the product of the larger two.

LINKS TO Example 33

For questions 10 to 14, let w m be the width of the rectangle.

10. The length of a rectangle is twice its width. The area of the rectangle is $72\,m^2$. Find the dimensions of the rectangle.
11. The length of a rectangular garden is 4 m more than its width. The area of the garden is $60\,m^2$. Find the dimensions of the garden.
12. The length of a rectangular garden is 11 m more than its width. The area of the garden is $102\,m^2$. Find the dimensions of the garden.
13. The length of a rectangular garden is 20 m more than its width. The area of the garden is $69\,m^2$. Find the dimensions of the garden.
14. The perimeter of a rectangular pool is 72 m and the area is $308\,m^2$. Find the dimensions of the pool.

LINKS TO Example 34

For questions 15 to 17, let a cm be the length of the shorter side.

15. The hypotenuse of a right-angled triangle is 8 cm longer than the shortest side. The third side is 7 cm longer than the shortest side. Find the lengths of the three sides.
16. The hypotenuse of a right-angled triangle is 9 cm longer than the shortest side. The third side is 1 cm longer than the shortest side. Find the lengths of the three sides.
17. The hypotenuse of a right-angled triangle is 3 cm shorter than four times the shortest side. The third side is 3 cm longer than three times the shortest side. Find the lengths of the three sides.

exercise 8.7 challenge

18. An 18 m by 4 m rectangular garden is surrounded by a path of width w m. The area of the path is $104\,m^2$. Find the width of the path.

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8.8 The nature of solutions

You will have noticed that a quadratic equation may have 0, 1 or 2 solutions, and that solutions can be rational or irrational. The quadratic formula gives us the clue as to the number and 'nature' of solutions we can expect to find.

Using the quadratic formula, we can see that the term underneath the square root sign is the key to the number and the nature of solutions for the general equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Because $b^2 - 4ac$ is so important, this expression is given a name. It is called the **discriminant** because it discriminates between the types of solutions. The discriminant is denoted by the symbol Δ (the Greek letter *delta*), that is, $\Delta = b^2 - 4ac$. By calculating its value, we can find out the *number* of solutions and the *nature* of those solutions, that is, what type of numbers they are.

One meaning of discriminate is 'to note or observe a difference'.

Interpreting the discriminant

Discriminant Δ	Number of solutions	Solution(s)	Type of solutions
Negative $\Delta < 0$	0 solutions		No real solutions
Zero $\Delta = 0$	1 solution	$x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$	Rational
Positive $\Delta > 0$	2 solutions	$x = \frac{-b \pm \sqrt{\Delta}}{2a}$	If Δ is a perfect square: two rational solutions. If Δ is not a perfect square: two irrational solutions.

Example 35

Find the number and nature of the solutions for each of the following quadratic equations.

a $x^2 - x + 4 = 0$

b $x^2 - 8x + 16 = 0$

c $2x^2 - x - 3 = 0$

d $3x^2 - 2x - 2 = 0$

continued

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Example 35 continued

Working	Reasoning
a $x^2 - x + 4 = 0$ $a = 1, b = -1, c = 4$ $\Delta = b^2 - 4ac$ $= 1 - 16$ $= -15$ Since $\Delta < 0$, there are no real solutions.	Since the value of the discriminant is negative, there will be no real solutions. It is not possible to find a real number that is the square root of a negative number.
b $x^2 - 8x + 16 = 0$ $a = 1, b = -8, c = 16$ $\Delta = b^2 - 4ac$ $= 64 - 64$ $= 0$ Since $\Delta = 0$, there is one rational solution.	Since the value of the discriminant is zero, the square root term disappears from $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ and there will be a single (unique) solution.
c $2x^2 - x - 3 = 0$ $a = 2, b = -1, c = -3$ $\Delta = b^2 - 4ac$ $= 1 + 24$ $= 25$ Since $\Delta > 0$, there will be two solutions.	Since the value of the discriminant is greater than zero, $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ will give two different solutions.
Also, as Δ is a perfect square, the solutions will be rational.	25 is a perfect square ($\sqrt{25} = 5$), so the solutions will not involve an irrational square root.
d $3x^2 - 2x - 2 = 0$ $a = 3, b = -2, c = -2$ $\Delta = b^2 - 4ac$ $= 4 + 24$ $= 28$ Since $\Delta > 0$, there will be two solutions.	Since the value of the discriminant is greater than zero, $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ will give two different solutions.
Also, as Δ is not a perfect square, the solutions are irrational.	28 is not a perfect square, so $\sqrt{28}$ is irrational.

Example 36

Find the value of k such that $x^2 - 5x + k = 0$ has

a one solution. **b** two solutions. **c** no solutions.

continued

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Example 36 continued

Working

a The equation $x^2 - 5x + k = 0$ will have one solution if $\Delta = 0$.

$a = 1, b = -5, c = k$

$\Delta = b^2 - 4ac = 25 - 4k$

If $\Delta = 0$, then

$$25 - 4k = 0$$
$$-4k = -25$$
$$k = \frac{25}{4}$$

The equation $x^2 - 5x + k = 0$ will have one solution if $k = \frac{25}{4}$.

Reasoning

If $\Delta = b^2 - 4ac = 0$, then

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm 0}{2a} = -\frac{b}{2a}.$$

So there would be a single solution.

Write an expression for Δ in terms of k and put it equal to 0.

b The equation $x^2 - 5x + k = 0$ will have two solutions if $\Delta > 0$.

$a = 1, b = -5, c = k$

$$\Delta > 0$$
$$25 - 4k > 0$$
$$-4k > -25$$
$$k < \frac{25}{4}$$

The equation will have two solutions if $k < \frac{25}{4}$.

If $\Delta = b^2 - 4ac > 0$, then

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$
$$= \frac{-b + \sqrt{\Delta}}{2a} \text{ or } = \frac{-b - \sqrt{\Delta}}{2a}.$$

So there would be two solutions.

When dividing both sides of $-4k > -25$ by -4, don't forget to reverse the $>$ sign.

c The equation $x^2 - 5x + k = 0$ will have no solutions if $\Delta < 0$.

$$\Delta < 0$$
$$25 - 4k < 0$$
$$-4k < -25$$
$$k > \frac{25}{4}$$

The equation will have no solutions if $k > \frac{25}{4}$.

If $\Delta = b^2 - 4ac < 0$, then

$$x = \frac{-b \pm \sqrt{\text{negative number}}}{2a}$$

It is not possible to find a real number that is the square root of a negative number, so there are no real solutions.

exercise 8.8

LINKS TO Example 35

1 For each of the following equations, find the discriminant and determine the number of real solutions (0, 1 or 2).

a $x^2 - 3x - 2 = 0$ **b** $x^2 - 4x - 1 = 0$ **c** $3x^2 - 2x + 5 = 0$

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d $2x^2 - 7x - 2 = 0$ **e** $4x^2 - 12x + 9 = 0$ **f** $6x^2 - 3x + 1 = 0$

2 By finding the discriminant, determine whether the solutions of the equations are real or unreal, rational or irrational, equal or unequal.

a $x^2 - 12x + 36 = 0$ **b** $4x^2 - 9 = 0$ **c** $x^2 - 5x - 6 = 0$
d $2x^2 - 3x - 1 = 0$ **e** $4x^2 - 6x + 3 = 0$ **f** $7x^2 - 2x - 3 = 0$
g $5x^2 - 6x + 1 = 0$ **h** $4x^2 - 20x + 25 = 0$ **i** $x^2 - 7x + 15 = 0$

3 The equation $x^2 - 30x + 225 = 0$ has

A only one solution. **B** two integer solutions. **C** two real solutions.
D two rational solutions. **E** no solutions.

4 The equation $2x^2 - 7x = 5(x + 3)$ has

A only one solution. **B** two integer solutions. **C** two real solutions.
D two rational solutions. **E** no solutions.

5 Rearrange each of the following equations, and hence find the number of real solutions (but not the solutions themselves).

a $3x^2 = 5x + 1$ **b** $x(2x - 1) = 4$ **c** $2x^2 = 2 - x$

6 Find if these equations have real solutions. If there are real solutions, give them correct to two decimal places.

a $x^2 + 3x + 5 = 0$ **b** $a^2 + 12a - 7 = 0$ **c** $y^2 + 7y + 2 = 0$
d $2x^2 - 3x - 10 = 0$ **e** $2a^2 + 3a + 6 = 0$ **f** $y^2 + 6y - 2 = 0$
g $4x^2 + x - 1 = 0$ **h** $h^2 + 4h + 12 = 0$ **i** $y^2 - 6y + 30 = 0$
j $2x^2 - 3x - 15 = 0$ **k** $h^2 + 5h + 8 = 0$ **l** $2x^2 + 9x + 5 = 0$

LINKS TO Example 36

7 Find the discriminant for each of the following equations, given that k, m and n are real numbers.

a $kx^2 - 2x + 3 = 0$ **b** $2x^2 - mx - 4 = 0$
c $x^2 - 2nx + 3 = 0$ **d** $x^2 - 2x + k - 3 = 0$

8 Find the values of k if $x^2 - 3x - k = 0$ has two solutions.

9 Find the values of m for which $mx^2 - 7x + 5 = 0$ has no solutions.

10 Find the values of n for which $x^2 - (n + 1)x + 1 = 0$ has one solution.

11 Show that $3x^2 + kx - 2 = 0$ will have two real solutions for all values of k.

12 Show that $(k - 3)x^2 - 6x - (k + 3) = 0$ will have real solutions for all values of k.

exercise 8.8 challenge

13 When asked to find the value of k such that $x^2 - kx + 1 = 0$ has two solutions, Rachel said k was greater than 2, Min said k was less than -2, and Simone said k was between -2 and 2. Who was right? Explain.

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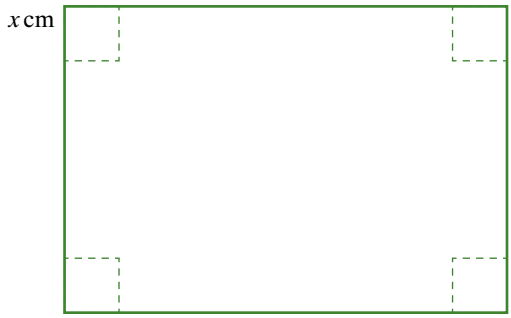
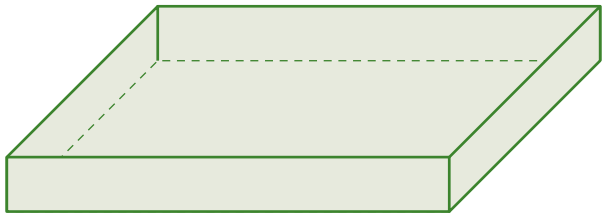
Analysis task

How big is the cut-out?

Open boxes are to be made from rectangles of cardboard by cutting out equal squares at each corner and folding up the edges. These are the sort of boxes used in cake shops.





A box is to be made from a piece of cardboard, 9 cm by 7 cm, so that the base of the box has an area of 35 cm^2.

a What are the length and width of the base of the box in terms of x when the edges are folded up?

b Write an expression for the area of the base of the box, A cm^2, using the expressions for length and width in part **a**.

c Substitute the given area for A in your rule in part **b** then rearrange the equation so that all the terms are on the left side. Divide both sides by the common factor.

d Solve the quadratic equation.

e Why must one of the solutions to the quadratic equation be discarded?

f What is the side length of the square cut out that will give the required area of the base?

g Check that this cut-out will give the required area.

h A larger box is to be made from a piece of cardboard 24 cm by 18 cm so that the area of the base is 280 cm^2. Repeat parts **a** to **g** for these values.

i Suppose a box with area 300 cm^2 is to be made from the 24 cm by 18 cm cardboard. Solve the quadratic equation, giving x correct to one decimal place. What size of cut-out would be required?

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Review Quadratic equations

Summary

- If two factors have a product of zero, then either one of the factors or both must be equal to zero.
- To solve a quadratic equation, rearrange so that all terms are on the left side and the right side is equal to zero. Factorise the left side if possible, then put each factor equal to zero and solve.
- If you cannot factorise the left side easily you can:
  - complete the square and factorise as a difference of squares.
  - use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- When solving real-world problems involving quadratic equations follow these steps.
  1. Write a quadratic equation.
  2. Solve the equation.
  3. Interpret the solutions in terms of the problem.
  4. Check that the solution satisfies the information given in the question.
  5. Write a sentence answering the question.
- Quadratic equations may have one solution, two solutions or no solutions.
  $\Delta = b^2 - 4ac$

$\Delta < 0$	no solutions
$\Delta = 0$	one solution
$\Delta > 0$	two solutions; if Δ is a perfect square, rational solutions

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

common factor	exact	quadratic formula	solution
completing the square	expand	quadratic trinomial	solve
difference of two squares	factor	quadratic equation	solving by substitution
*discriminant	factorise	rational solutions	
	irrational solutions	real solutions	
	null factor law		

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Revision

Multiple-choice questions

1 Which of the following is not a quadratic equation?

A $2x - 2 = 2x^2$ **B** $2 - x^2 = x^2 - 2$ **C** $2(x - 3) = x(8 - 3x)$
D $2 - x = 2(3 + x)$ **E** $(x - 2)(3 - x) = 0$

2 For which of the following equations is $x = -3$ a solution?

A $3 - x^2 = 0$ **B** $x^2 + 9 = 0$ **C** $3(x - 3) = 0$
D $9 = x^2$ **E** $x^2 - 2x - 3 = 0$

3 The solutions to the equation $x^2 - 6x + 5 = 0$ are

A $x = 5$ or $x = 1$ **B** $x = -5$ or $x = -1$ **C** $x = -3$ or $x = -2$
D $x = 3$ or $x = 2$ **E** $x = -3$ or $x = 2$

4 The equation $x^2 = x - 2$ has

A two rational solutions **B** exactly one solution **C** two irrational solutions
D one rational and one irrational solution **E** no solutions

5 The equation $x^2 - 4x + 2 = 0$ can also be written as

A $(x - 2)^2 - 2 = 0$ **B** $(x - 2)^2 + 6 = 0$ **C** $(x + 2)^2 - 2 = 0$
D $(x + 2)^2 + 6 = 0$ **E** $(x - 2)^2 - \sqrt{2} = 0$

Short-answer questions

6 Solve these equations.

a $-3(m - 4) = 0$ **b** $5(3x - 4) = 0$

7 Solve these quadratic equations.

a $(x + 5)(x - 8) = 0$ **b** $(b - 7)(2b + 5) = 0$
c $(b - 2.5)^2 = 0$ **d** $(2x - 7)^2 = 0$

8 Write quadratic equations in the form of $ax^2 + bx + c = 0$ that would have these solutions.

a $x = -3, x = 5$ **b** $x = 2, x = \frac{2}{3}$

9 Solve these quadratic equations.

a $4a^2 - 24a = 0$ **b** $3x^2 - 21x - 24 = 0$ **c** $6x^2 - 4x = 0$
d $k^2 - 169 = 0$ **e** $5t^2 - 45 = 0$ **f** $x^2 - 14x + 49 = 0$
g $x^2 - 8x + 15 = 0$ **h** $a^2 - a - 42 = 0$ **i** $x^2 - 16x + 63 = 0$
j $x^2 - 1.6x = 0$ **k** $b^2 - 10b - 24 = 0$ **l** $4x^2 - 20x + 25 = 0$

10 Rearrange each equation in the form $ax^2 + bx + c = 0$ and then solve.

a $x^2 + 35 = 12x$ **b** $y^2 - 24 = 5y$ **c** $(x - 3)^2 + 3x = 37$
d $(x + 1)^2 = 5(x + 11)$ **e** $x + \frac{4}{x} = 5$ **f** $x - 7 = \frac{18}{x}$

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10 A

11 Solve these equations using the method of substitution.

a $(x + 3)^2 - 2(x + 3) - 8 = 0$ **b** $(2x + 1)^2 - 3(2x + 1) - 10 = 0$

c $x^4 - 26x^2 + 25 = 0$ **d** $x^4 - 29x^2 + 100 = 0$

10A

12 Solve these quadratic equations by completing the square. Give the solutions

i in exact form.

ii as rational approximations correct to two decimal places.

a $x^2 + 4x - 13 = 0$ **b** $x^2 - 8x + 4 = 0$

c $x^2 + x - 3 = 0$ **d** $x^2 - 5x - 18 = 0$

13 Use the quadratic formula to solve these equations, giving the solutions

i in exact form.

ii correct to two decimal places.

a $x^2 + 6x - 17 = 0$ **b** $3x^2 + 2x - 7 = 0$

c $4x^2 - 7x - 1 = 0$ **d** $3x^2 - 8x + 2 = 0$

10 A

14 For each of these equations

i calculate the discriminant.

ii identify the number and type of solutions. You do not need to solve the equations.

a $x^2 + 4x - 12 = 0$ **b** $x^2 + 7x + 20 = 0$

c $2x^2 + 5x + 7 = 0$ **d** $9x^2 - 12x + 4 = 0$

15 If $x^2 - nx + 2n + 5 = 0$ has exactly one solution, find the values of n.

Extended-response question

16 For each of the following

i write a quadratic equation.

ii solve the equation.

iii write a sentence, giving the answer to the problem.

a The product of two consecutive odd integers is 12 more than the smaller of the two integers. Find the two integers if both integers are positive. (Let n be the smaller integer.)

b The length of a rectangular garden is 13 m more than its width. The area of the garden is 48 m^2. Find the dimensions of the garden. (Let the width of the garden be w m.)

c The hypotenuse of a right-angled triangle is 5 cm more than four times the length of the shortest side. The third side is 1 cm shorter than the hypotenuse. Find the length of each side of the triangle. (Let the length of the shortest side be x cm.)

Practice quiz
Chapter 8

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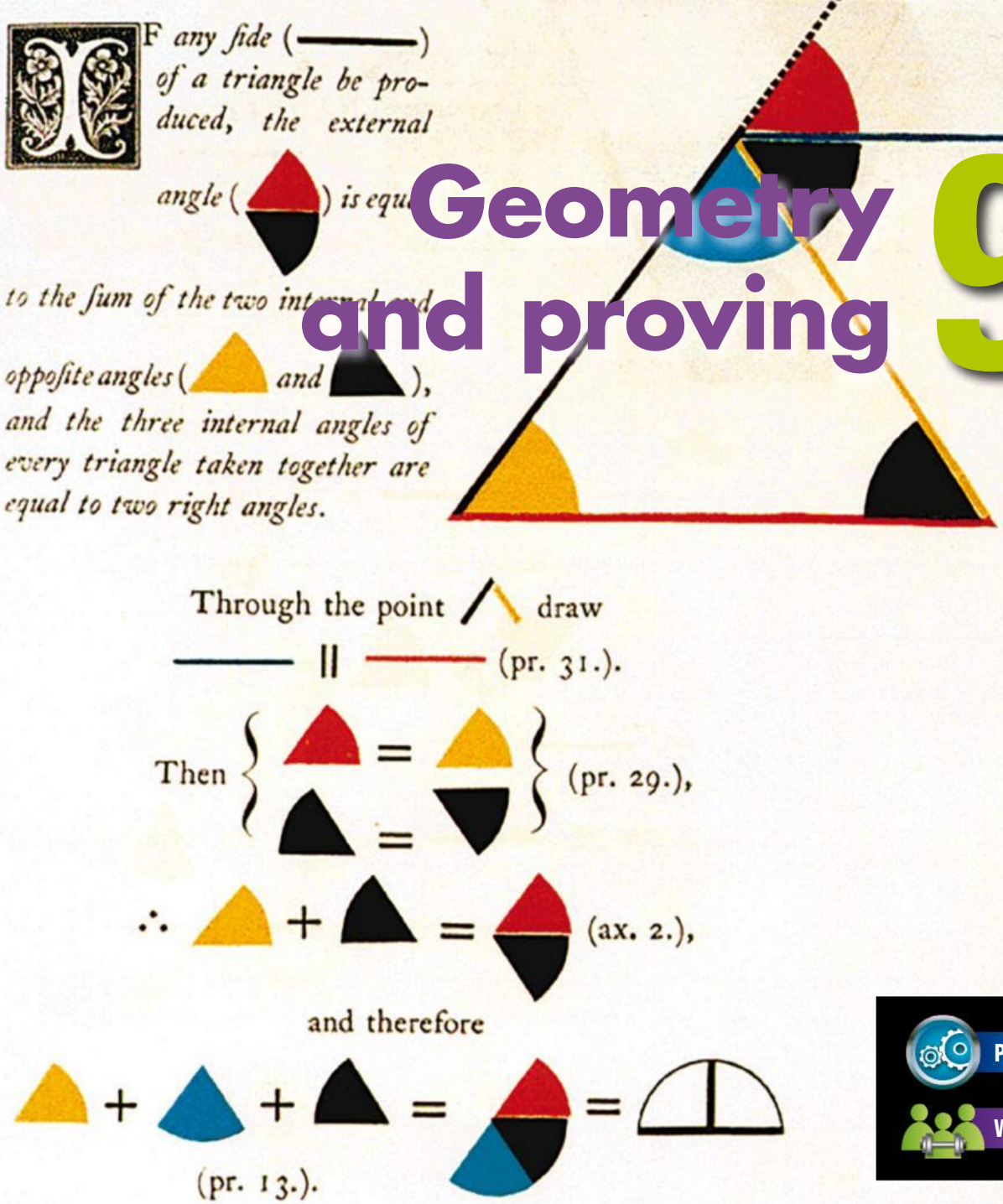
Geometry and proving 9



Pre-test

Warm-up

Mathematical proof is part of what it means to do mathematics and it is what sets mathematics apart from theories that are based on observations. About 2300 years ago, the Greek mathematician Euclid produced a famous set of 13 books titled *The Elements* in which he constructed proofs for many geometric properties. In 1847, an Irish mathematician, Oliver Byrne, produced an illustrated version of the first six books of Euclid's *Elements*. In this illustration, Byrne sets out Euclid's proof that the three angles of any triangle together make a straight angle and that an exterior angle of a triangle is equal to the sum of the interior opposite angles. In this chapter, we review the geometric reasoning with parallel lines, and congruent and similar triangles from Years 7 to 9 and introduce the geometry of circles.

9.1 Geometry and proof

A mathematical proof is true for every case – there must be no exceptions. Mathematical proofs are built up from previously known or accepted statements in a series of logical steps – we call this **deductive reasoning** because we are deducing each step from the previous step, justifying the steps as we go. The starting points for deductive reasoning are axioms, definitions, or previously proved mathematical knowledge.

Axioms

Axioms are statements that are starting points or considered self-evident – that 'go without saying', for example, if $\angle A + \angle B = \angle A + \angle C$ then $\angle B$ must equal $\angle C$.

Definitions

Some mathematical statements are true because they are definitions. For example, we know the following statements are true because that is the way we define each of these figures:

- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
- An equilateral triangle is a triangle with three equal sides.
- An isosceles triangle is a triangle with two equal sides.

The parallelogram, the equilateral triangle and the isosceles triangle obviously have other properties, but these are not part of the definitions. The opposite angles of a parallelogram are equal, for example, but this is not part of the definition of a parallelogram so it had to be proved before it was known that it was true. Once this property has been proved either by mathematicians in the past or by us, it then becomes part of our mathematical knowledge.

Theorems

A theorem is a statement that has been proved from previously accepted statements such as axioms, definitions or previously proved theorems. An example of a theorem is 'the angles of any triangle add to 180°'.

Proving that the angles of any triangle add to 180°

In Year 7, you may have torn the three angles from a triangle and placed them together to show that the three angles made a straight angle, that is, 180°.

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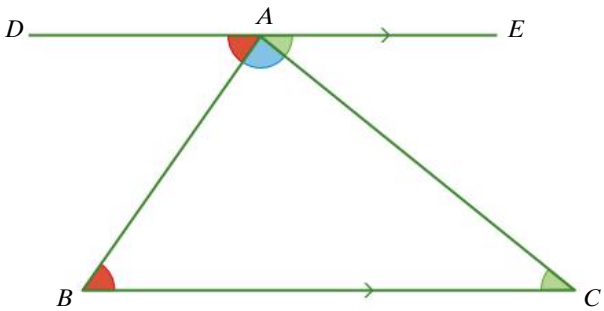
Although this is very convincing, it is not a mathematical proof. We do not know that the angles add exactly to 180° and we do not know that the same result would be obtained for every possible triangle. However, compare the arrangement of torn angles with the diagram below. Here, a line has been constructed through A so that is parallel to the base of the triangle. Although the parallel line has been drawn in a different position from the added line in Euclid's method on the opening page of this chapter, the reasoning is similar.

In the mathematical argument or proof that follows, we have used our knowledge of straight angles, parallel lines and alternate angles. Notice how the statements in each line of the proof include a reason to justify the statement.



The symbol ∥ means 'is parallel to'.

Given: ABC is a triangle.
DE is a straight line segment passing through A, $DE \parallel BC$

Prove: $\angle ABC + \angle BAC + \angle ACB = 180°$

Proof: $\angle ABC = \angle BAD$ (alternate angles equal because $DE \parallel BC$ and AB is a transversal)

$\angle ACB = \angle CAE$ (alternate angles equal because $DE \parallel BC$ and AC is a transversal)

So $\angle ABC + \angle ACB + \angle BAC = \angle BAD + \angle CAE + \angle BAC$

Sum of the angles of a triangle

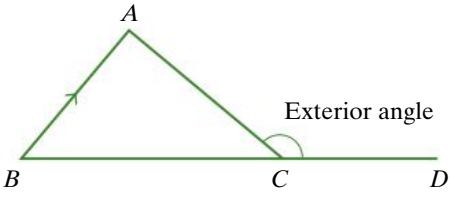
The three angles of all triangles add to 180°.

Angle sum of a triangle

This proof applies to all triangles because the mathematical argument does not depend on the particular shape of the triangle in the diagram.

Exterior angles of triangles

In this figure, ABC is a triangle, and side BC has been extended to D to make an **exterior angle**, $\angle ACD$.



Notice that the diagram used here is the same as the diagram in Euclid's method.

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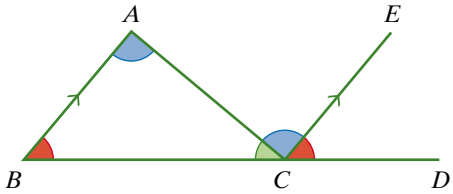
Prove: $\angle ACD = \angle BAC + \angle ABC$

Given: ABC is a triangle
$CE \parallel BA$
BC is extended to D



Proof: $\angle ABC = \angle ECD$ (Corresponding angles, because $CE \parallel BA$ and BD is a transversal)

$\angle BAC = \angle ACE$ (Alternate angles, because $CE \parallel BA$ and AC is a transversal)

So $\angle BAC + \angle ABC = \angle ACE + \angle ECD$

So $\angle BAC + \angle ABC = \angle ACD$

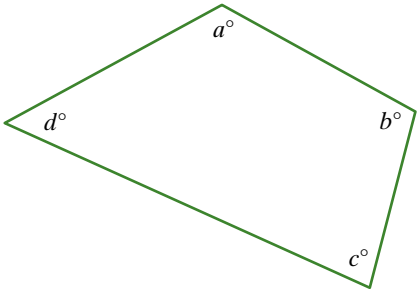
It is this type of proof, based on logical steps of reasoning, which underpins mathematical knowledge.

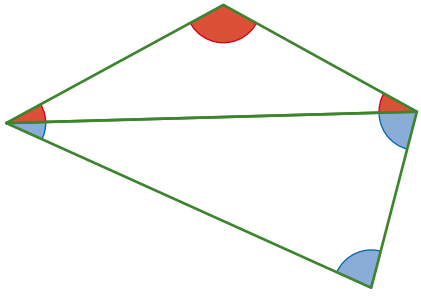
Exterior angle of a triangle

An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

The angles of any quadrilateral add to 360°

We can use the proof of the angle sum of a triangle to prove that the angle sum of any quadrilateral is 360°.





$$\boldsymbol{a + b + c + d = 360°}$$

We can easily prove this by dividing the quadrilateral into two triangles. The three angles of each triangle have a sum of 180°. (These three angles shown in red add to 180° and the three angles shown in blue add to 180°.) This makes a total of 360° for the angles of the quadrilateral.

Sum of the angles of a quadrilateral

The four angles of all quadrilaterals add to 360°.

Many geometry proofs are based on:

- angles between parallel lines
- congruent triangles
- similar triangles.

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Comparing the conditions for congruent and similar triangles

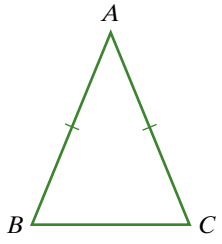
Congruent triangles	Similar triangles
Congruent triangles must satisfy both of these conditions: ■ matching angles must be equal ■ matching sides must be equal in length.	Similar triangles must satisfy both of these conditions: ■ matching angles must be equal ■ matching sides must be in the same proportion.
Conditions for congruent triangles	**Conditions for similar triangles**
■ SSS (three sides of one triangle equal to three sides of the other)	■ SSS (three sides of one triangle in proportion to three sides of the other)
■ ASA (two angles of one triangle equal to the two angles of the other triangle and a matching side equal) Note: two angles equal means that three angles are equal.	■ AAA (three angles of one triangle equal to three angles of the other)
■ SAS (two sides of one triangle equal to two sides of the other triangle and the angle between them the same)	■ SAS (two sides of one triangle in proportion to two sides of the other triangle and the angle between them the same)
■ RHS (both triangles right-angled, hypotenuse and another side of one triangle equal to the hypotenuse and another side of the other triangle)	■ RHS (both triangles right-angled, hypotenuse and another side of one triangle in proportion to the hypotenuse and another side of the other triangle)

Sometimes we need to add extra lines to a diagram, just as the parallel line was added in the angles of a triangle proof.

Example 1

Prove that the base angles of an isosceles triangle are equal.

Working



Prove: $\angle ABC = \angle ACB$.

Reasoning

Draw a diagram and state what is to be proved.

In constructing a proof, we need to have a plan of where we are going and how we are going to get there. It may be necessary to add other lines to the diagram.

continued

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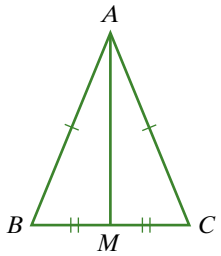
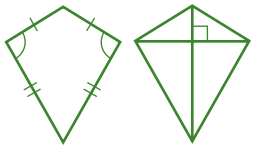
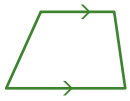
Example 1 continued

Working	Reasoning
Given: $AB = AC$ (definition of isosceles triangle) M is the midpoint of BC (construction).	By joining A to M, the midpoint of BC, we divide the isosceles triangle into two triangles. If we can prove that these triangles are congruent, then the base angles must be equal.
	
Proof: In $\triangle AMB$ and $\triangle AMC$, $AB = BC$ (definition of isosceles triangle) $BM = MC$ (M is midpoint of BC) AM is common to both triangles. $\triangle AMB \equiv \triangle AMC$ (SSS)	Two triangles are congruent if three sides of one triangle are equal to three sides of the other triangle.
So $\angle ABC = \angle ACB$ (corresponding angles in congruent triangles)	If two triangles are congruent, the angles of one triangle are equal to the corresponding (matching) angles of the other triangle.

Summary of quadrilateral properties

Properties	Name	Examples
■ Two pairs of adjacent (next to each other) sides are equal. ■ One pair of opposite angles are equal. ■ Diagonals are perpendicular. ■ One diagonal bisects the unequal angles.	Kite 'Bisect' means 'to cut in half'.	
■ One pair of opposite sides are parallel. ■ Two pairs of supplementary angles.	Trapezium	

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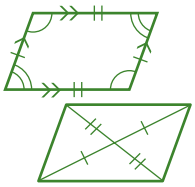
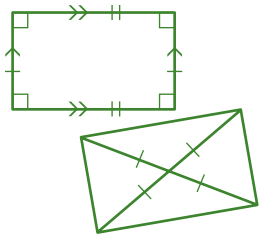
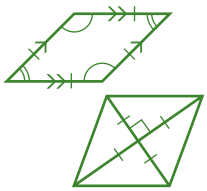
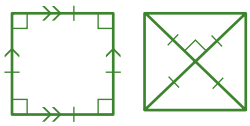
Properties	Name	Examples
■ Both pairs of opposite sides are parallel. ■ Both pairs of opposite sides are equal. ■ Opposite angles are equal. ■ Adjacent angles are supplementary. ■ Diagonals bisect each other.	Parallelogram	
■ Both pairs of opposite sides are parallel and equal. ■ All angles are right angles. ■ Diagonals are equal in length. ■ Diagonals bisect each other.	Rectangle	
■ Both pairs of opposite sides are parallel. ■ All four sides are equal. ■ Opposite angles are equal. ■ Adjacent angles are supplementary. ■ Diagonals bisect each other. ■ Diagonals are perpendicular. ■ Diagonals bisect the angles of the rhombus.	Rhombus	
■ Both pairs of opposite sides are parallel. ■ All four sides are equal. ■ All angles are right angles. ■ Diagonals are equal in length. ■ Diagonals are perpendicular. ■ Diagonals bisect each other. ■ Diagonals bisect the right angles of the square.	Square	

Parallelogram proofs

When working with triangles, we used definitions and properties that had already been proved to prove other properties. In a similar way, some properties of quadrilaterals are defined, while other properties need to be proved.

For example, a parallelogram is defined to be ‘a quadrilateral with opposite sides parallel’. We can use this definition to prove other properties of a parallelogram.

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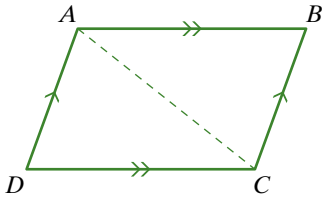
Example 2

In this figure, $ABCD$ is a parallelogram and AC is a diagonal. From the definition of a parallelogram, we know that $AB \parallel DC$ and $AD \parallel BC$. Using this information and your knowledge of angles associated with parallel lines and transversals, set out a careful mathematical argument to prove that $\triangle ABC$ is congruent to $\triangle CDA$. Justify each statement that you make.

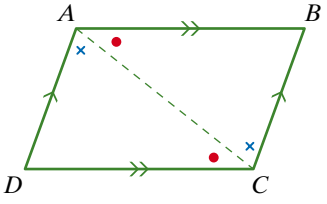


Working

Prove: $\triangle ABC \equiv \triangle CDA$
Given: $ABCD$ is a parallelogram
Proof:

In $\triangle ABC$ and $\triangle CDA$, side AC is common to both triangles.

$\angle DAC = \angle BCA$ (alternate angles, $AD \parallel BC$, AC is a transversal)
$\angle ACD = \angle CAB$ (alternate angles, $AB \parallel DC$, AC is a transversal)



So $\triangle ABC \equiv \triangle CDA$ (ASA)

Reasoning

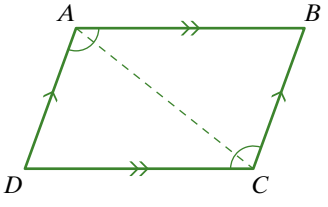
Look for clues: $\triangle ABC$ and $\triangle CDA$ share the side AC.

Diagonal AC is a transversal cutting across the parallel sides. This suggests that we could find equal angles in $\triangle ABC$ and $\triangle CDA$.

Two triangles are congruent if two angles of one triangle are equal to two angles of the other triangles and the sides joining the two equal angles are equal.

Example 3

Using the above proof that the diagonal AC divides parallelogram $ABCD$ into two congruent triangles, prove that the opposite angles of a parallelogram are equal.



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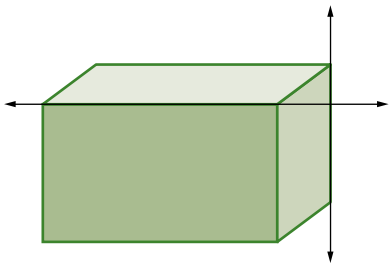
Example 3 continued

Working	Reasoning
Prove: $\angle BAD = \angle DCB$ and $\angle ABC = \angle CDA$ *Given*: $\triangle ABC \equiv \triangle CDA$ *Proof*: $\triangle ABC \equiv \triangle CDA$ (proved) So $\angle ABC = \angle CDA$ (matching angles in congruent triangles)	Something that has been proved can be used in other proofs. ASA (two angles of one triangle equal to the two angles of the other triangle and a matching side equal).
Similarly, if we draw diagonal BD we can prove that $\triangle ABD \equiv \triangle CDB$. So $\angle BAD = \angle DCB$ So the opposite angles of a parallelogram are equal.	We can see that drawing the diagonal BD would allow us to prove that $\triangle ABD \equiv \triangle CDB$ in the same way that we proved $\triangle ABC \equiv \triangle CDA$.

Counter-examples

A counter-example is an example that contradicts a statement. We need find only one example that contradicts a statement to show that the statement is false.

For example, supposing a friend tells you that lines that never meet must be parallel. We can see that the two lines along the edges of this rectangular box will never meet because they are in different planes. So the statement that '*Lines that never meet are parallel*' has been proved false by a single counter-example. The statement would need to be modified to '*Lines that are in the same plane and never meet are parallel*.'



Sometimes, though, it is not easy to find a counter-example. In 1852 it was proposed that only four colours were needed to colour in the regions of any map so that no two adjoining regions are the same colour. When the idea was first put forward, no-one could find a counter-example. That is, no-one had been able to find any examples of where more than four colours were needed. However, this was not enough to prove the idea, as it may have just meant that not enough examples were investigated. It was finally proved mathematically to be true in 1976 and is now known as the four-colour theorem.

So if we can find a counter-example, we can prove a statement to be false. But if we cannot find a counter-example, it does not necessarily mean the statement is true.

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Example 4

Is the following statement true? Can you find a counter-example?

All obtuse-angled triangles are scalene.

Working

A counter-example is an obtuse-angled triangle with angles of 120°, 30° and 30°. This is an isosceles triangle, not a scalene triangle.

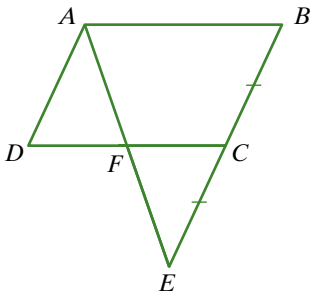
The statement is false.

Reasoning

Only one counter-example is needed to prove a statement is false.

Example 5

In the diagram below, $ABCD$ is a parallelogram and BC is extended to E so that $BC = CE$. Prove that $\triangle ADF \equiv \triangle ECF$

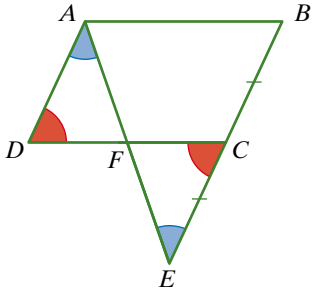


Working

In $\triangle ADF \equiv \triangle ECF$,

1. $BC = CE$ (given)
2. But $BC = AD$ (opposite sides of parallelogram)
3. So $AD = CE$
4. $\angle ADF = \angle ECF$ (alternate angles between parallel segments AD and BE cut by transversal DC)
5. $\angle DAF = \angle CEF$ (alternate angles between parallel segments AD and BE cut by transversal AE)
6. So $\triangle ADF \equiv \triangle ECF$ (ASA)

Reasoning



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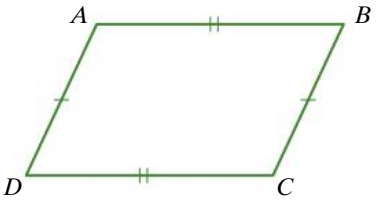
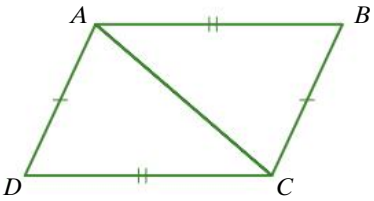
exercise 9.1

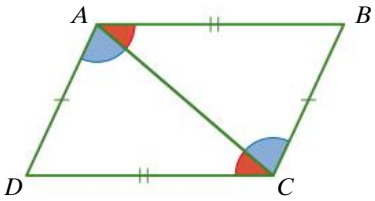
LINKS TO Example 1

1 Prove that a quadrilateral with both pairs of opposite sides equal is a parallelogram.



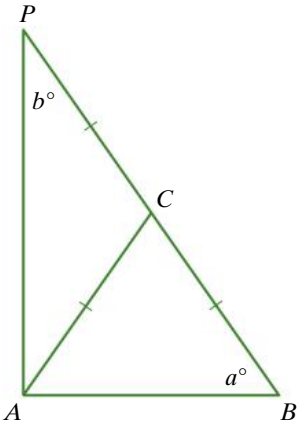
Use the following diagrams to help you with your proof. Start by proving that $\triangle ADC \equiv \triangle CBA$

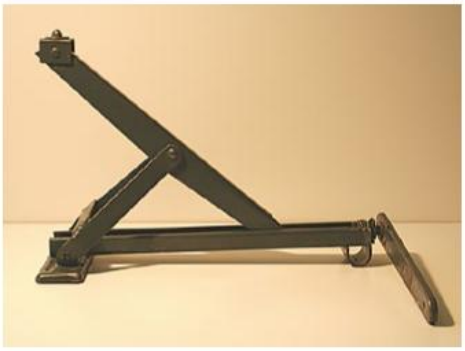




Triangle car jack

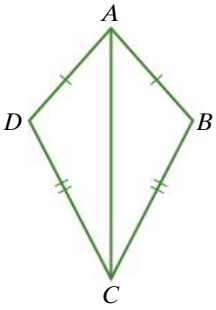
2 The construction of this car jack is based on isosceles triangles. The contact point of the jack under the wheel (point P on the diagram) moves perpendicular to the ground as the jack raises and lowers the wheel. Prove that $\angle PAB$ is a right angle.





LINKS TO Example 2

3 In the diagram on the right, $ABCD$ is a kite and AC is a diagonal. Prove that the opposite angles $\angle ADC$ and $\angle ABC$ are equal.



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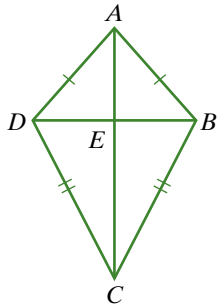
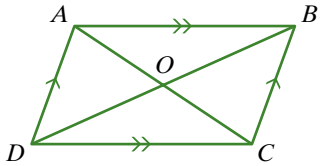
LINKS TO Example 3

4 **a** Use the diagram in question **3** to prove that the diagonal AC bisects $\angle DAB$ and $\angle DCB$.

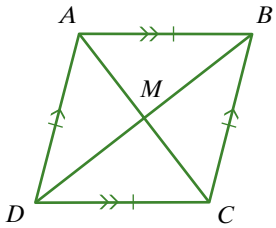
b Use your proof from part **a** to help you prove that the diagonals of a kite intersect at right angles.



5 Prove that the diagonals of a parallelogram bisect each other. Hint: which triangles will you use?

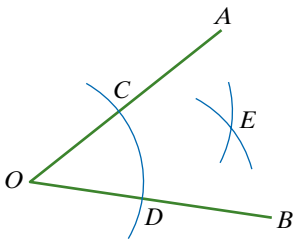


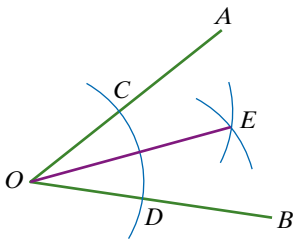
6 Using the property you proved in question **5**, prove that the diagonals of a rhombus intersect at right angles.



7 The following steps and diagrams show one method for bisecting an angle.

- The point of a compass is placed at O. An arc (part of a circle) is drawn so that it cuts the arms of $\angle AOB$ at C and D.
- The compass point is then placed at C and an arc is drawn in the space between OA and OB.
- Without changing the opening of the compass, the point is placed at D and another arc is drawn to intersect the previous arc (point E).
- The segment OE bisects $\angle AOB$, that is, $\angle AOE = \angle EOB$.





a Prove why this method works, that is, why OE bisects $\angle AOB$.

b Why is it important to use the same compass opening when drawing the arcs from C and D? That is, why must the arcs have the same radius?

c Does the first arc (drawn with the point of the compass on O) need to have the same radius as the arcs drawn from C and D?

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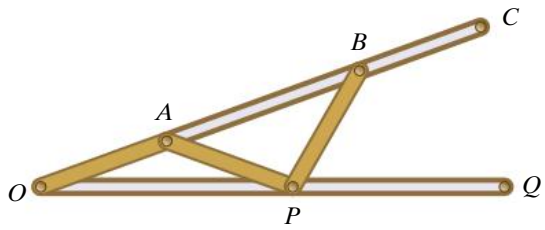
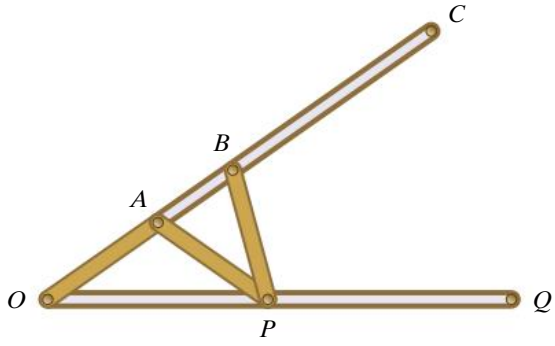
Pascal's angle trisector

8 In the 17th century, the French mathematician Blaise Pascal invented an instrument for trisecting an angle. To bisect something means to cut it into two equal parts, and to **trisect** something means to cut it into three equal parts.

Pascal's instrument consisted of hinged pieces of wood, which formed two isosceles triangles. In this figure, $OA = AP = PB$, so $\triangle OAP$ and $\triangle APB$ are isosceles triangles.

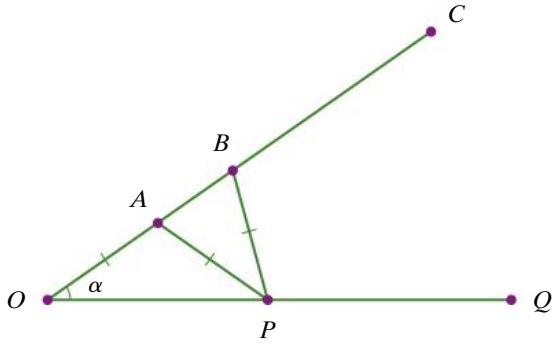


The instrument is hinged at points O, A and P. Point P can slide along a horizontal groove in the bar of wood PQ at the base of the instrument, and point B can then slide along the groove in AC.



OC is rotated until $\angle BPQ$ matches the size of the angle to be trisected. $\angle COQ = \frac{1}{3}\angle BPQ$, that is, $\angle BPQ = 3\angle COQ$.

The diagram below is a screen image from the interactive GeoGebra file Pascal's angle trisector on the *OneStopDigital* ebook. Point C can be dragged to operate the trisector.



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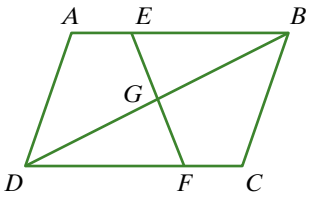
a What is the size of $\angle PAB$ in terms of α? Explain.
b What is the size of $\angle ABP$ in terms of α? Explain.
c What is the size of $\angle BPQ$ in terms of α? Explain.
d Do you think the angle trisector would have worked if Pascal's design was not based on isosceles triangles?

LINKS TO Example 4

9 Find a counter-example to show that each of the following statements is incorrect.
a An isosceles triangle cannot have a right angle.
b A scalene triangle must have three acute angles.
c If a triangle has at least one angle of 60° it must be an equilateral triangle.

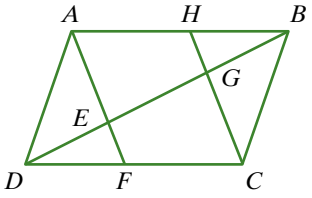
LINKS TO Example 5

10



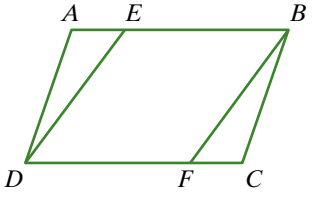
$ABCD$ is a parallelogram.
$AE = CF$
Prove that $\triangle EBG \equiv \triangle FDG$

11



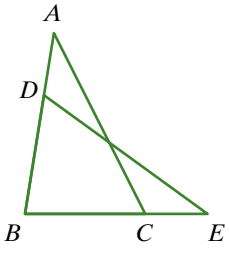
$ABCD$ is a parallelogram.
$BD \perp AF$ and $BD \perp CH$.
Prove that $\triangle ABE \equiv \triangle CDG$

12



$ABCD$ is a parallelogram.
$AE = CF$
Prove that $DEBF$ is a parallelogram.

13



$\angle BAC = \angle BED$
$AB = BE$
Prove that $AD = EC$

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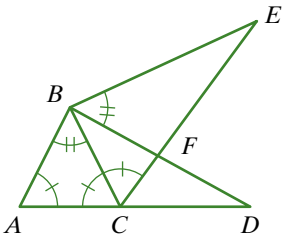
14



Prove that $\angle BEC = \angle BDA$

15 The following definition of a rectangle was given in a Year 10 mathematics book:
A rectangle is a parallelogram, one of whose angles is a right angle.

a How many right angles does a rectangle have?

b Why does the above definition need only say that *one* of the angles is a right angle?

16 The same textbook gave the following definition of a square:
A square is a rectangle in which two adjacent sides are equal.

a How many equal sides does a square have?

b Why does the above definition need only say that *two* adjacent sides are equal?

17 Carefully mark the positions of two points A and B that are 8 cm apart.

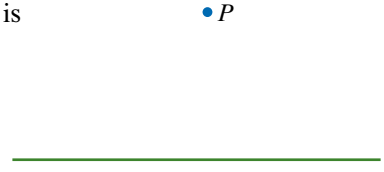
a Mark the position of a point that is exactly the same distance from A as it is from B.

b Show the positions of ten more points that are all exactly the same distance from A as they are from B.

c Describe how you found the positions of the points.

exercise 9.1 challenge

18 Prove that the shortest distance from a point P to a line is the perpendicular distance from the point to the line.



Hint: Pythagoras' theorem will help.

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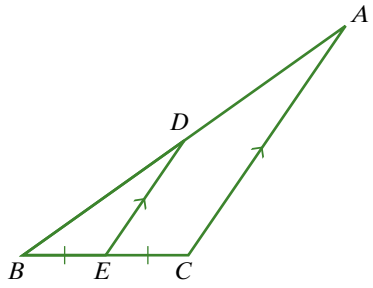
9.2 Proving with similar triangles

Using the conditions for triangles to be similar we can prove certain relationships in geometric figures by proving that two triangles are similar.

Example 6

Prove that $DE = \frac{1}{2}AC$.



Working

Prove: $DE = \frac{1}{2}AC$

Given: ABC is a triangle
$DE \parallel AC$
$BE = EC$

Proof:
In $\triangle DBE$ and $\triangle ABC$,
$\angle BDE = \angle BAC$ (corresponding angles with $AC \parallel DE$)
$\angle DEB = \angle ACB$ (corresponding angles with $AC \parallel DE$)
$\angle DBE$ is shared by both triangles.
So $\triangle DBE$ is similar to $\triangle ABC$ (AAA)
$BE = \frac{1}{2}BC$ ($BE = EC$)
So $DE = \frac{1}{2}AC$ (sides of similar triangles are in the same proportion).

Reasoning

The parallel line segments tell us to look for equal angles associated with parallel lines and a transversal.

Once we know two angles of $\triangle DBE$ are equal to two angles of $\triangle ABC$, we know all three angles of $\triangle DBE$ are equal to the three angles of $\triangle ABC$. Three angles equal is one of the conditions for triangles to be similar.

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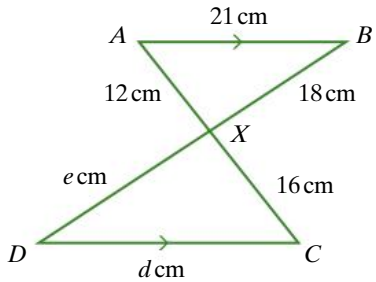
exercise 9.2

1 **a** Explain, giving reasons, why $\triangle ABX$ and $\triangle CDX$ are similar.

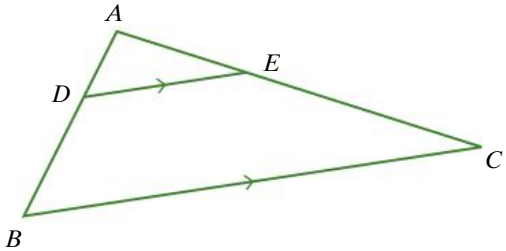
b Which side of $\triangle CXD$ corresponds to AX?

c Find the values of d and e.



LINKS TO Example 6

2 In $\triangle ABC$, segment DE is parallel to BC. Prove that $\triangle ADE$ is similar to $\triangle ABC$.



3 Using diagrams to help, explain whether each of these statements is true or false.

a All equilateral triangles are similar.

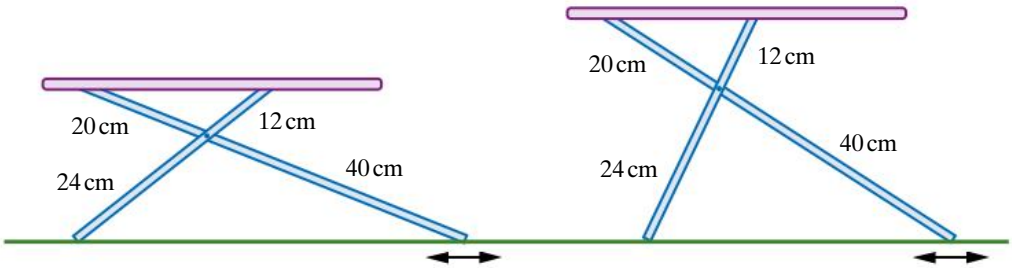
b All right-angled triangles are similar.

c All isosceles triangles are similar.

d All right-angled isosceles triangles are similar.

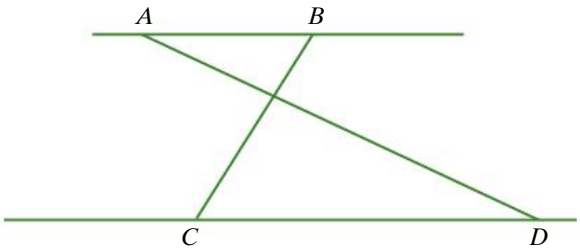
Ironing table

4 The ironing table shown below has legs that pivot. As the ironing table is raised or lowered, the top always stays parallel to the floor.



a Explain why the two triangles formed by the legs are similar.

b Explain why the top always stays parallel to the floor. Use this diagram in your explanation.



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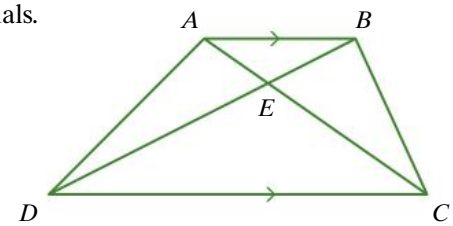
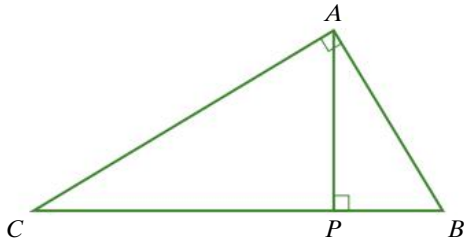
5 $ABCD$ is a trapezium. AC and BD are the diagonals. $AB = 12\,\text{cm}$, $DC = 30\,\text{cm}$, $AE = 5\,\text{cm}$ and $DE = 20\,\text{cm}$. Find the lengths of BE and CE.

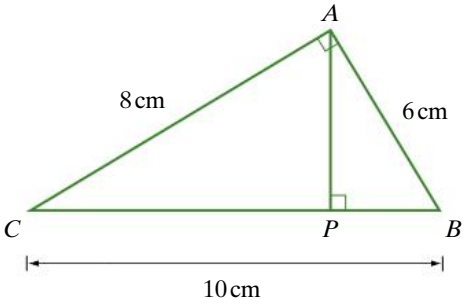


exercise 9.2 challenge

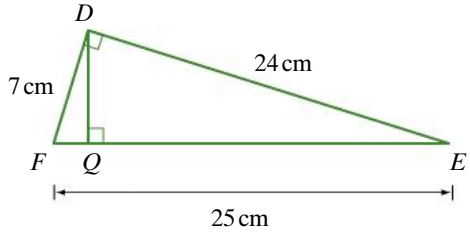
6 **a** Explain why $\triangle PBA$ and $\triangle PAC$ are similar to $\triangle ABC$.



b Find the lengths of AP and PB.



c Find the lengths of DQ and EQ.



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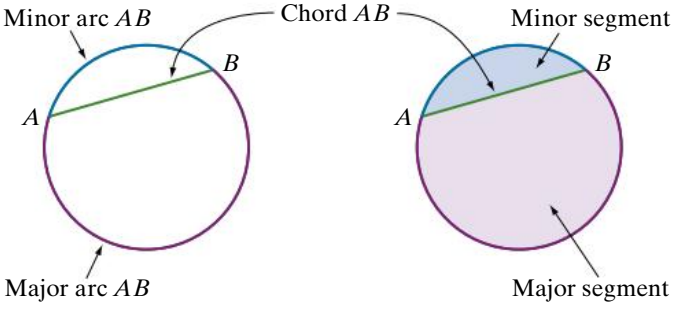
9.3 Angles in circles

Angles in circles

A **chord** is a segment that joins two points on the circumference of a circle. Part of the circumference of a circle is called an **arc**. In the diagrams below, the chord AB divides the circumference into two arcs. The smaller arc is called the **minor arc** and the larger arc is called the **major arc**.

The chord also divides the circle into two parts: a **major segment** and a **minor segment**.



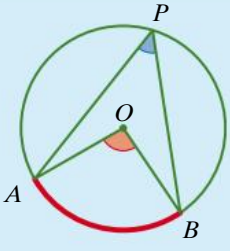
A chord divides a circle into two segments.

In the figure below, $\angle AOB$ and $\angle APB$ are both standing on the same arc (AB) of the circle. We say that the minor arc AB **subtends** $\angle AOB$ at the centre and subtends $\angle APB$ at the circumference.

Angles at the centre and circumference

When an arc subtends an angle at the centre of a circle and another angle at the circumference of a circle, the angle at the centre is always twice the angle at the circumference.

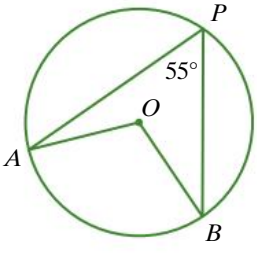
That is, $\angle AOB = 2 \times \angle APB$



Question **1** in exercise **9.3** asks you to construct a proof of this property

Example 7

Find the size of $\angle AOB$.



continued

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Example 7 continued

Working

$\angle AOB = 2 \times \angle APB$
$= 2 \times 55°$
$= 110°$

Reasoning

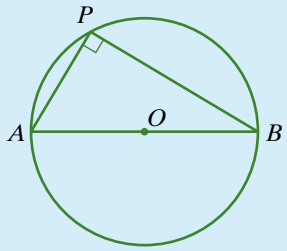
$\angle AOB$ and $\angle APB$ are standing on the same arc. The angle at the centre is twice the angle at the circumference.

In the special case where the angle at the centre is 180°, then the angle at the circumference is 90°. This is known as **Thales' Theorem**.

Angles in a semicircle

Any angle in a semicircle is a right angle.

That is, any angle subtended by the diameter of a circle is a right angle.



An alternative proof of Thales' theorem is shown here.

$AO = PO = BO$ (radii of circle)

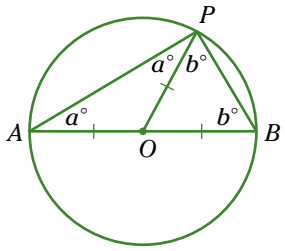
So $\triangle APO$ and $\triangle BPO$ are isosceles.

The angles marked $a°$ and $b°$ are equal.

In $\triangle APB$,
(angle sum of a triangle is 180°)

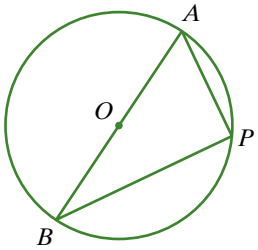
$a + a + b + b = 180$
$2a + 2b = 180$
$a + b = 90$

Hence $\angle APB$ is a right angle.



Example 8

Find the size of $\angle APB$.



Working

$\angle APB$ is subtended by a diameter of the circle.
$\angle APB = 90°$

Reasoning

AB is a diameter of the circle.

An angle subtended by a diameter is a right angle, because the angle at the centre is 180°.

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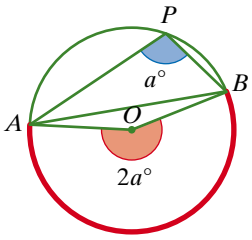
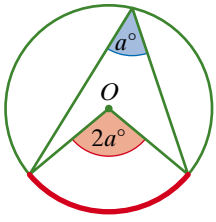
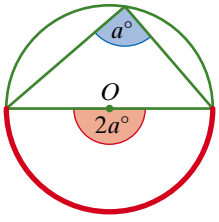
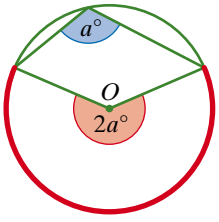
When the angle at the circumference is an obtuse angle, the angle at the centre is a reflex angle. If a line is drawn between the ends of the arc, the angle at the centre and the angle at the circumference can be seen to be in opposite segments of the circle.

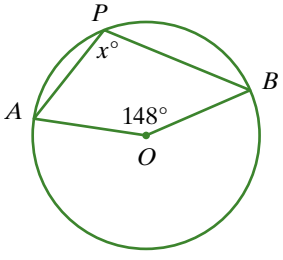


The following diagrams summarise three situations: the angle at the centre can be less than 180°, a straight angle or a reflex angle.

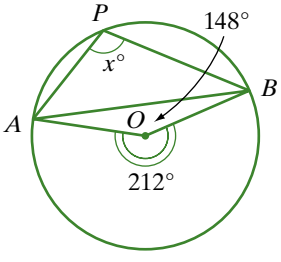
Angle at centre	Obtuse	180°	Reflex
Angle at circumference	Acute	90°	Obtuse
			

Example 9

Find the size of $\angle APB$.



Working



The reflex angle at the centre is 212°.

$$\angle APB = \frac{1}{2} \text{ reflex angle } \angle AOB$$
$$= 212° \div 2$$
$$= 106°$$

Reasoning

The angle at the circumference is standing on the major arc, so the *reflex* angle at the centre is required.
$360° - 148° = 212°$

The angle at the circumference is half the angle at the centre standing on the same arc.

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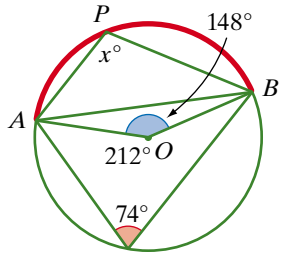
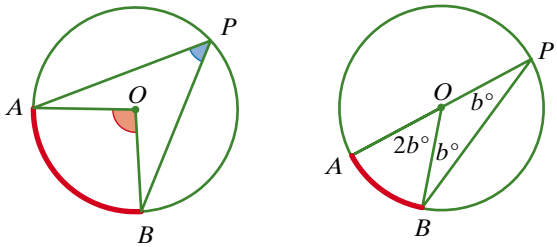
In the example above, note that the obtuse angle at the centre (148°) corresponds with the angle at the circumference (74°) standing on the *minor* arc, as shown here.



We have seen that the right angle in the semicircle is a special case of the angles at the centre and circumference. Another special case occurs when A moves around the circle towards B so that the chord AP coincides with the radius AO. We can see that the angle at the centre, $\angle AOB$, is still twice the angle at the circumference, $\angle APB$.



Proof

$\triangle OBP$ is isosceles triangle ($OB = OP =$ radius of circle)

So $\angle OBP = \angle OPB$

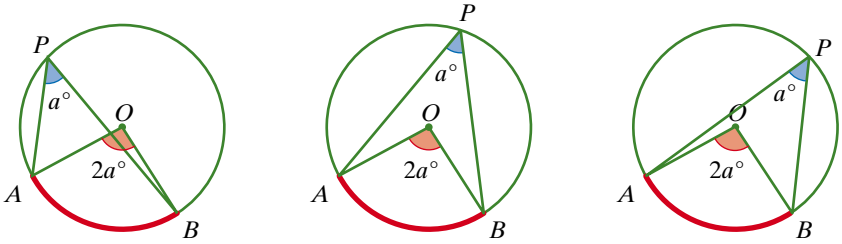
$\angle AOB = \angle OBP + \angle OPB$ ($\angle AOB$ is exterior angle to $\triangle OBP$)

$\angle AOB = 2\angle OPB$

Angles standing on the same arc

Proof

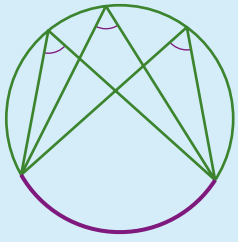
For a particular arc, the angle at the centre is always the same. And we have seen that the angle at the circumference is always half of the angle at the centre. This means that angles at the circumference that are standing on the same arc are equal. Another way of saying this is that angles subtended by the same arc are equal.



Angles on the same arc

Angles that are standing on the same arc are equal.

That is, angles subtended at the circumference by the same arc are equal.



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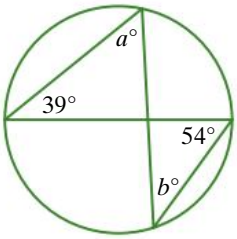
Example 10

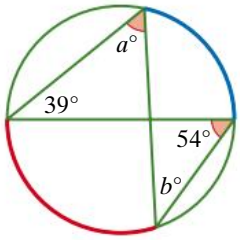
Find the values of the pronumerals.



Working

$a = 54°$
$b = 39°$

Reasoning



The angles marked 54° and $a°$ are standing on the same arc.
The angles marked 39° and $b°$ are standing on the same arc.

Cyclic quadrilaterals

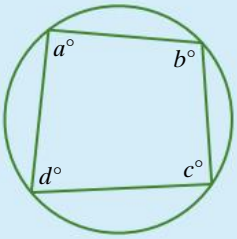
Cyclic quadrilaterals

A **cyclic quadrilateral** is a quadrilateral that has all four of its vertices on the circumference of a circle.

Cyclic quadrilaterals

The opposite angles of a cyclic quadrilateral are supplementary.

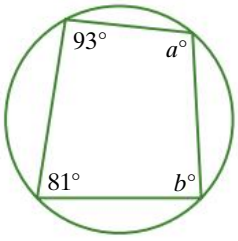
$a° + c° = 180°$ and $b° + d° = 180°$



Question **2** in exercise **9.3** asks you to construct a proof of this property.

Example 11

Find the value of a and b.



continued

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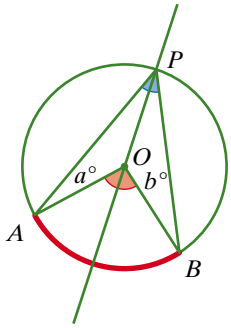
Example 11 continued

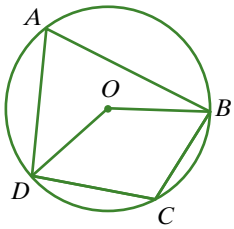
Working	Reasoning
$a + 81 = 180$ $a = 99$ $b + 93 = 180$ $b = 87$	The quadrilateral is a cyclic quadrilateral because the vertices are on the circumference of a circle. The opposite angles of cyclic quadrilaterals are supplementary

exercise 9.3

1 Use the diagram to help you construct a proof that the angle subtended at the centre is twice the angle subtended at the circumference standing on the same arc.



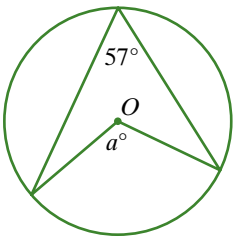
2 Use this diagram to help you construct a proof that the opposite angles of a cyclic quadrilateral are supplementary.



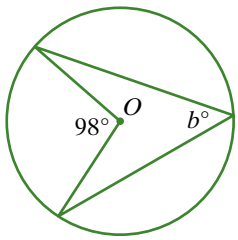
LINKS TO Examples 7, 8, 9, 10, 11

3 Find the value of the pronumerals in each of the following diagrams.

a



b



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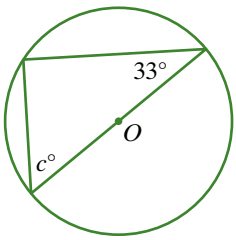

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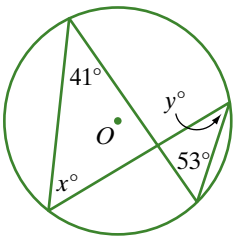

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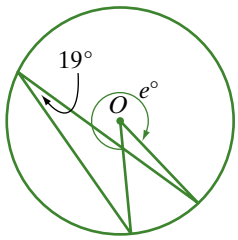

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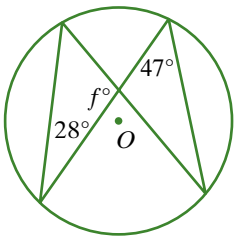

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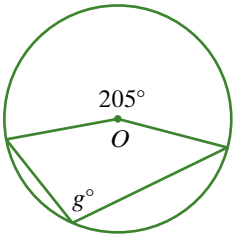

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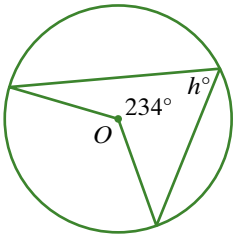

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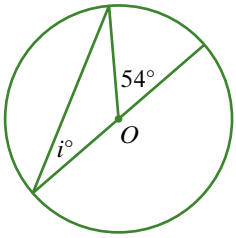

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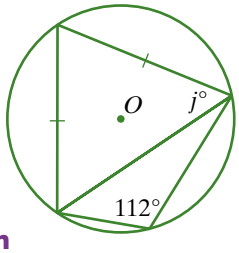

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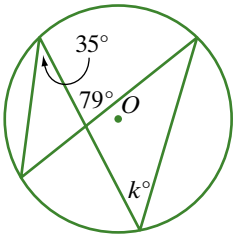

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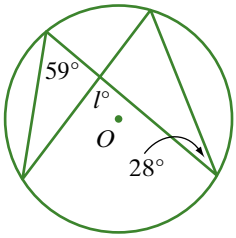

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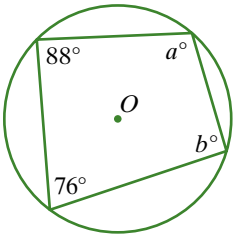

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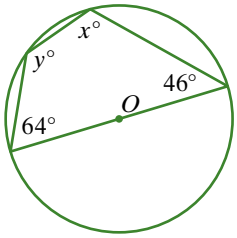

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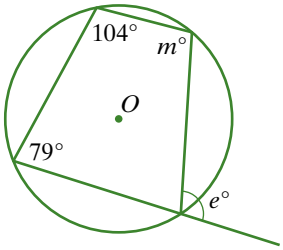
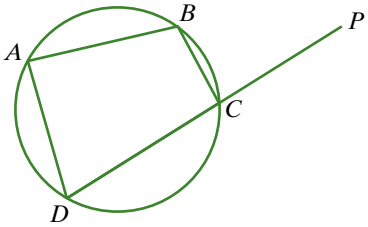


LINKS TO Example 11

4 Prove that $\angle BCP$ is equal to $\angle BAD$.



exercise 9.3 challenge

5 Prove that if both diagonals of a cyclic quadrilateral are diameters of the circle, then the cyclic quadrilateral is a rectangle.

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9.4 Circles, chords and tangents

10 A

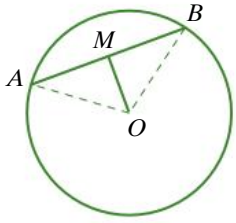
Perpendicular bisectors of chords

The line segment joining the centre of the circle to the midpoint of a chord is perpendicular to the chord.

Prove: $OM \perp AB$

Given: M is the midpoint of chord AB and O is the centre of the circle.

Proof:



In $\triangle OMA$ and $\triangle OMB$

$OA = OB$ (radii of the circle)

OM is shared by the two triangles.

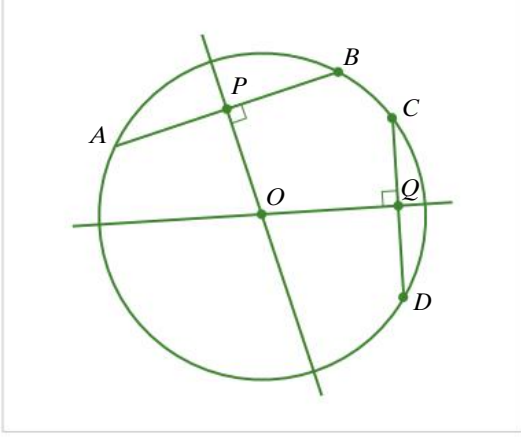
$AM = BM$ (M is the midpoint of AB)

So $\triangle OMA \equiv \triangle OMB$(SSS)

So $\angle OMA = \angle OMB = 90°$ (matching angles in congruent triangles)

Perpendicular bisectors of chords

From the above theorem, we see that the perpendicular bisector of a chord passes through the centre of the circle. This is the same for all chords and their perpendicular bisectors. Hence, we can say that the perpendicular bisectors of chords intersect at the centre of the circle. The construction to the right shows the perpendicular bisectors of chords AB and CD intersecting at O.



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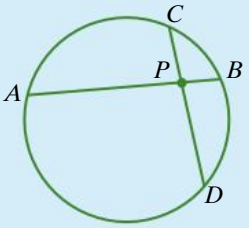
Intersecting chords

When two chords of a circle intersect, the product of the two segments of one chord is equal to the product of the segments of the other chord.

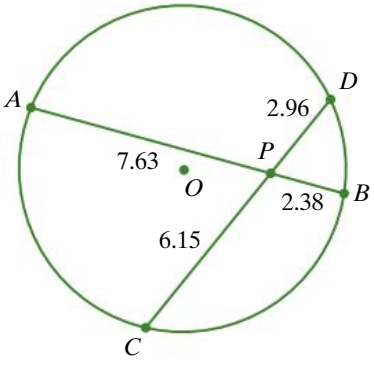
$$AP \times PB = CP \times PD$$



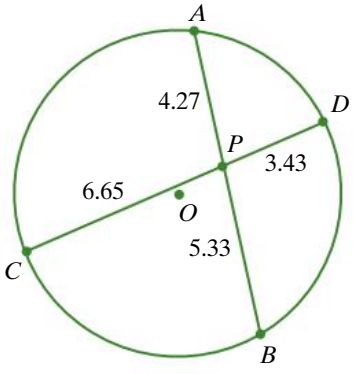
Intersecting chords

The construction below shows that the products of two intersecting chords are equal.

$AP \times BP = 6.15 \times 2.96 = 18.21$
$CP \times DP = 7.63 \times 2.38 = 18.21$



$AP \times BP = 6.65 \times 3.43 = 22.79$
$CP \times DP = 4.27 \times 5.33 = 22.79$



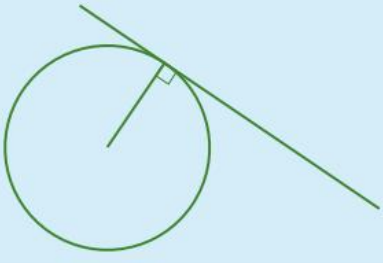
Proof

Question **1** in exercise **9.4** asks you to construct a proof of this property.

Tangents

Tangents

A tangent to a circle is a line that just touches the circle at one point, as shown in the diagram. The tangent is perpendicular to the radius at that point.



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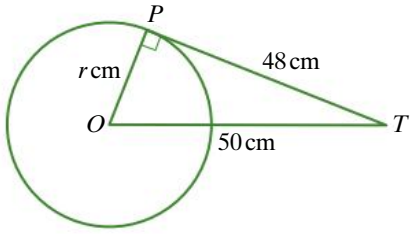
Example 12

PT is a tangent to the circle at T. Find the radius, r cm, of the circle.



Working

$\triangle OPT$ is a right-angled triangle.

$r^2 = 50^2 - 48^2$

$r^2 = 196$

$r = 14$

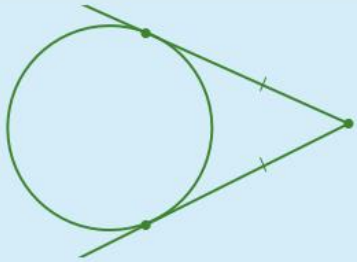
The radius is 14 cm.

Reasoning

A tangent to a circle is perpendicular to the radius at that point.

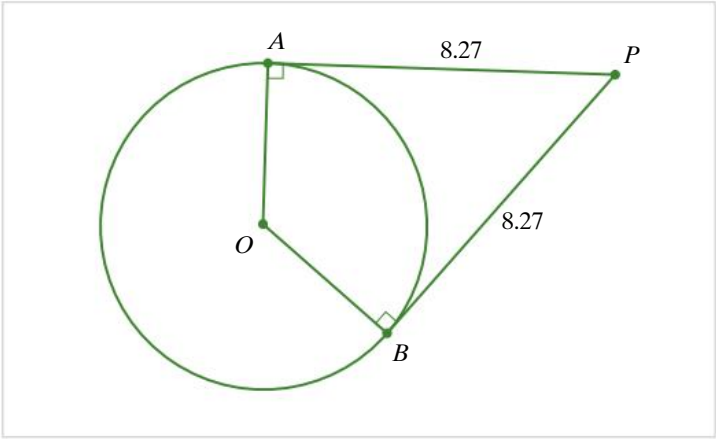
Two tangents from the same point

Two tangents to a circle from the same point are equal in length.



Tangents from a point

The construction below shows that the two tangents from the same point are equal in length.



Proof

Question **2** in exercise **9.4** asks you to construct a proof of this property.

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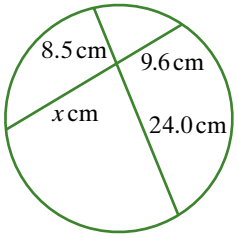
Example 13

Find the value of x.



Working

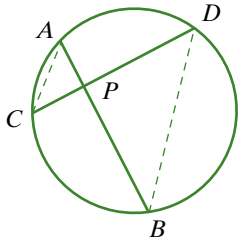
$$9.6x = 8.4 \times 24.0$$
$$x = \frac{8.4 \times 24.0}{9.6}$$
$$x = 21$$

Reasoning

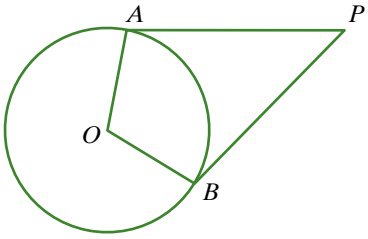
When two chords of a circle intersect, the product of the two segments of one chord is equal to the product of the segments of the other chord.

exercise 9.4

1 AB and CD are chords, intersecting at P. Prove that $AP \times PB = CP \times PD$



2 AP and BP are tangents to the circle at A and B respectively. Prove that $AP = BP$.



LINKS TO Example 12

3 Find the value of the pronumerals in the following figures.

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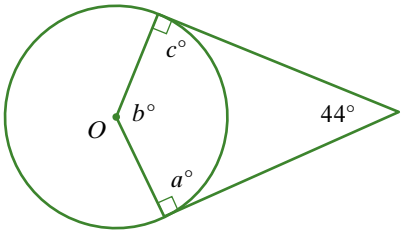

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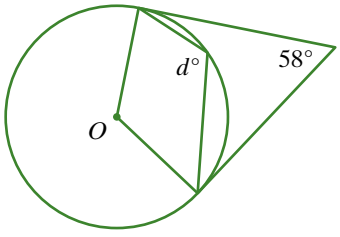

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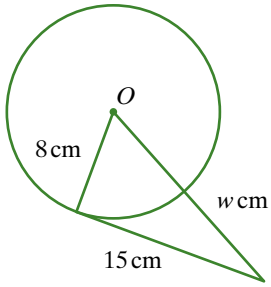
c



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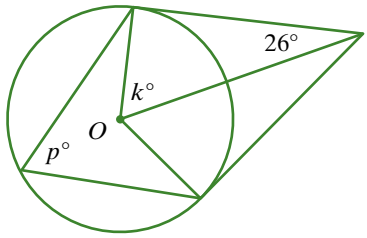
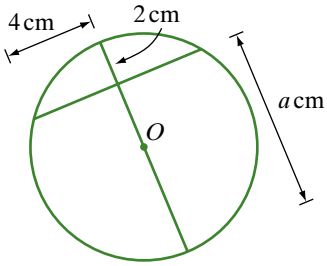
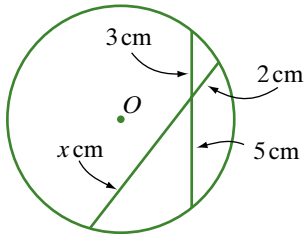


4

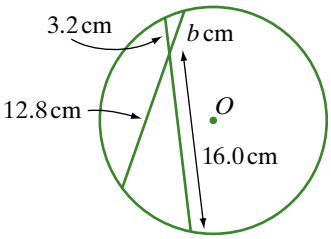
a



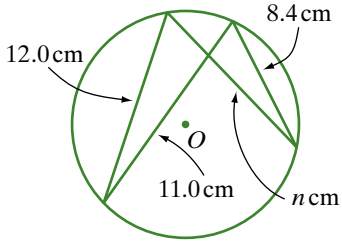
b



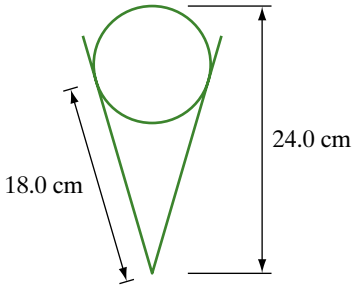
c



d



5 A ball is resting inside a cone as shown. Calculate the radius r cm of the ball.



6 Describe carefully how to find the centre of the circle by first drawing any two chords.

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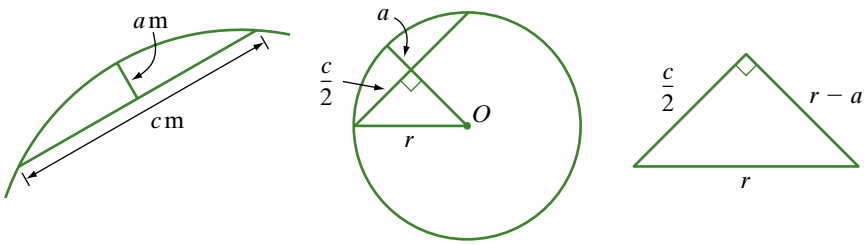
7 Archaeologists have found a fragment of a plate and wish to determine the original size of the intact plate. Copy the fragment of the plate shown below at half actual size or use a printed copy from the file in the *OneStopDigital* ebook. Using your knowledge of circles and chords, find the radius of the plate to the nearest centimetre.

BLM
Plate fragment

8 The Police Accident Division uses the following method to analyse accidents where a car has skidded out of control when taking a bend at too high a speed.

- Skid marks on a road represent the arc of a circle. Two points on the same skid mark are chosen and a line is drawn between them. This line represents a chord of the circle.
- The length of the chord c m is measured accurately.
- The perpendicular distance a m from the midpoint of the chord to the arc of the skid marks is then measured.
- The radius r m of the circle can then be calculated.



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a What property of chords is applied in this method for determining the radius of the circle?

b Write an algebraic expression to show the relationship between r, c and a. Use your algebra skills to simplify the expression and show that $r = \frac{c^2}{8a} + \frac{a}{2}$.

c In a particular accident, police measured a 30 m chord (that is, $c = 30$) and the perpendicular distance from the centre of the chord to the skid marks was 87 cm (that is, $a = 0.87$). Use the relationship from part **b** to calculate the radius r metres of the circle.

Once the radius has been determined, the speed can be calculated using the formula $s = \sqrt{fgr}$, where

s m/sec is the speed

f is the coefficient of friction

g m/sec/sec is the acceleration due to gravity ($g = 9.8$)

r m is the radius

d Using the radius calculated in part **c**, calculate the speed in m/sec if the coefficient of friction between the tyres and the road was 0.73.

e Convert the speed from metres per second to km/h. How fast was the car travelling before skidding?

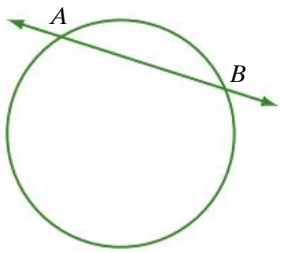
exercise 9.4 challenge

9 Using GeoGebra draw a triangle ABC. Construct a circle that passes exactly through A, B and C. Describe your method. The centre of this circle is called the **circumcentre** of the triangle.

Intersecting secants

10 In question **1**, you proved the theorem that if two chords intersect, the product of the segments of one chord is equal to the product of the segments of the other chord. The GeoGebra construction below shows that the theorem can be extended to **secants** that intersect outside the circle.

A secant is a line that cuts across a curve, in this case, across a circle. The line through AB is a secant. The segment AB of the secant is a chord.



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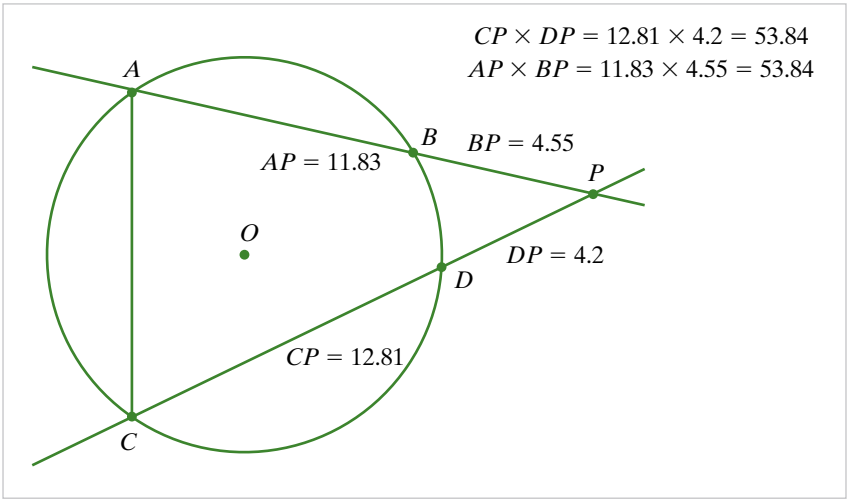
In the construction, two secants intersect at P. Even though the secants intersect outside the circle, the same relationship applies as with intersecting chords:
$AP \times BP = CP \times DP$.



In the construction above, the segments AC and BD have been included.
Prove that $\triangle PAC$ is similar to $\triangle PDB$ and hence prove that $AP \times BP = CP \times DP$.

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10 A

9.5 Conjecturing and proving

Mathematicians who are exploring new problems may make a conjecture, which they then seek to prove. Some conjectures may be proved quite simply whereas others take many years to prove. In question **1** of Exercise **9.5** you will formulate your own conjecture then prove it. Questions **2** and **3** are based on mechanical constructions that incorporate interesting geometry. The conjectures made by pairs of students working together are provided. By answering the questions, try to prove the conjectures.

In each of the three questions, GeoGebra constructions have been provided on the *OneStopDigital* ebook. Exploring the constructions by dragging and observing angle sizes or the path of points may lead you to the properties on which to base your proof. Remember to justify each statement that you make in your proof.

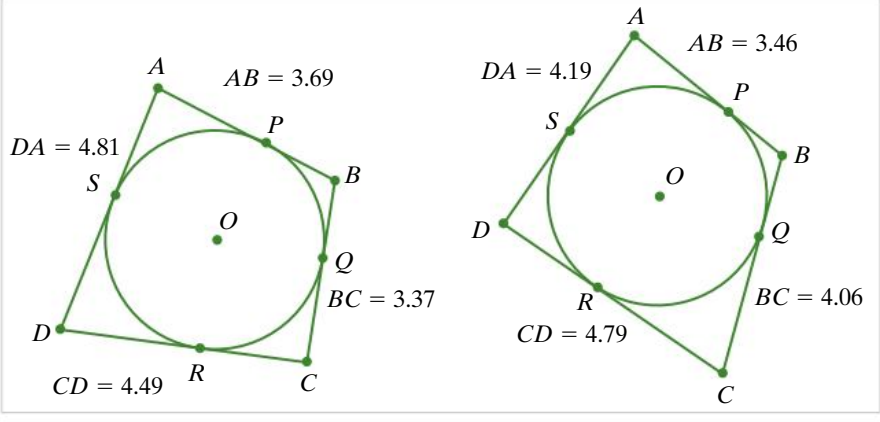
exercise 9.5

Circumscribed quadrilateral

1 In the following construction, AB, BC, CD and DA are tangents to the circle at P, Q, R and S respectively. ABCD is called a **circumscribed quadrilateral** because it has been constructed around the outside of the circle.

a Make a conjecture about the sides of quadrilateral $ABCD$. Open the GeoGebra file Quadrilateral and circle. You can drag the vertices A, B, C or D into different positions and observe the lengths of the sides of $ABCD$.

b Prove your conjecture.



Sylvester's pantograph

2 James Sylvester was a 19th-century English mathematician. Like many mathematicians at the time, he became interested in using geometry to invent mechanical linkages. One of his inventions was a pantograph that could be used to copy drawings. It was constructed from hinged bars.

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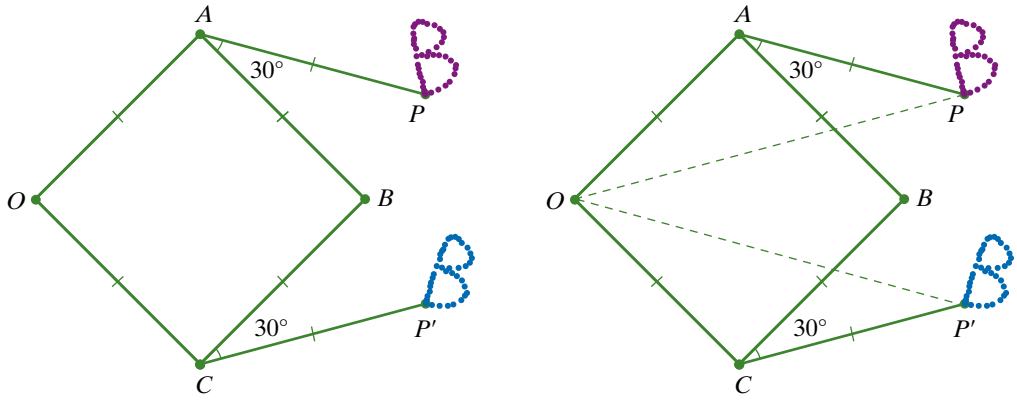
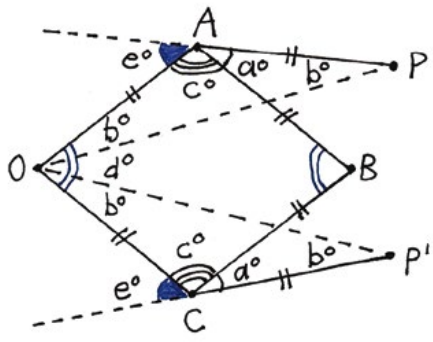
A construction of the pantograph is shown below. Point O is a fixed pivot point. $OA = OC = AB = BC = AP = CP'$. $\angle PAB = \angle P'CB$. These two angles are fixed in size (in this construction they are both 30°), although the size of the angle is not critical to the operation of the pantograph. As point P is moved, point P' follows, producing a copy of whatever has been drawn by point P.

Two students, Edward and James, noticed that the image drawn by P' was the same size and shape as the shape traced out by point P, but it was rotated. They conjectured that if the drawing and its image were the same size, P and P' must always be the same distance from the pivot point O. They decided also that the image was rotated through $\angle POP'$. Their two conjectures were:

- Conjecture 1: $OP = OP'$
- Conjecture 2: $\angle POP' = \angle PAB$ (or $\angle P'CB$)



Edward and James then used the given information to draw a diagram and label some angles, using the same letter for angles that they knew must be equal.



a What geometric shape is the quadrilateral $OABC$?

b Why did Edward and James use the same letter, a, for $\angle PAB$ and $\angle P'CB$?

c Why did they use the same letter, c, for $\angle OAB$ and $\angle OCB$?

d Why did they use the same letter, b, for the four angles $\angle AOP$, $\angle APO$, $\angle COP'$ and $\angle CP'O$?

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e Why did they use the same letter, e, for the two remaining angles at A and C?

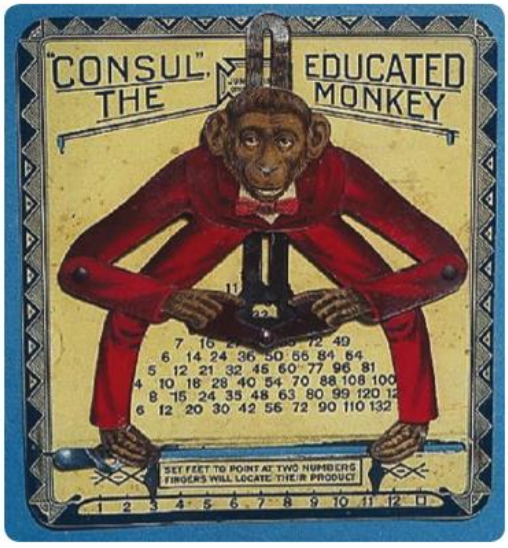
f Prove that $a = d$, that is, that $\angle POP' = \angle PAB$ (or $\angle P'CB$)

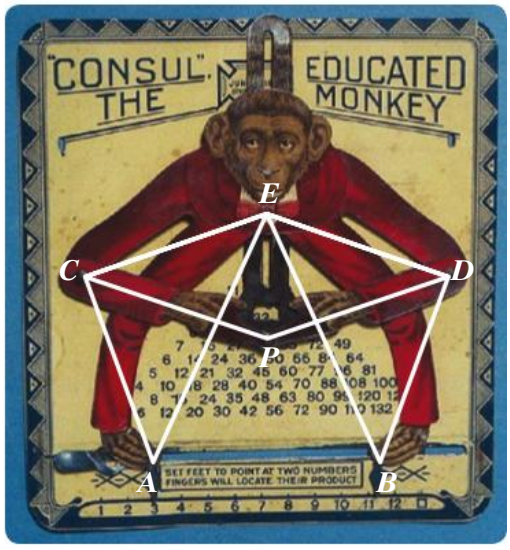
g Prove that $OP = OP'$.

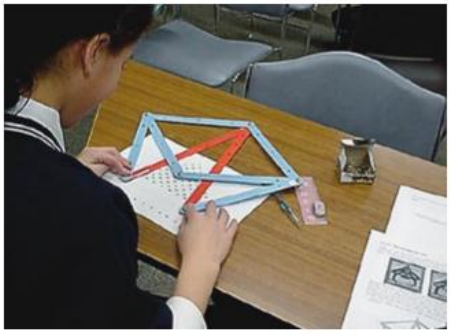
Consul

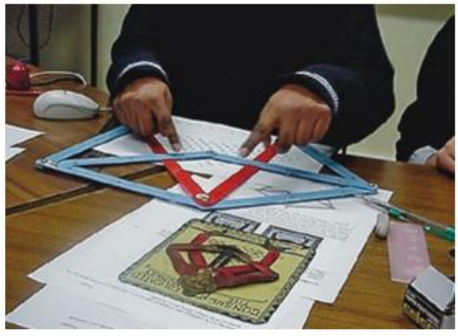
3 'Consul' the educated monkey is an American toy, invented in 1916 to teach children multiplication. When the feet are set to point to two numbers, Consul's fingers point to the product. (In the photographs below, the feet point to 3 and 11, and 33 appears in the small window between Consul's hands.)

How was the geometry of the toy designed? The geometric structure associated with the pivot points has been superimposed in white on the photograph at the right. Each upper arm and leg is constructed from a single piece of tin plate so that $\angle ACE$ and $\angle BDE$ are right angles. These two pieces of tin pivot about a point (E) beneath the bow tie. The feet, at A and B, can move only in a straight line through A and B, but can be positioned on any pair of numbers. Segments CE, DE, CP and DP (as well as AC and BD) are all equal. Quadrilateral $CEDP$ is hinged at points C and D (the monkey's elbows) and points E and P.









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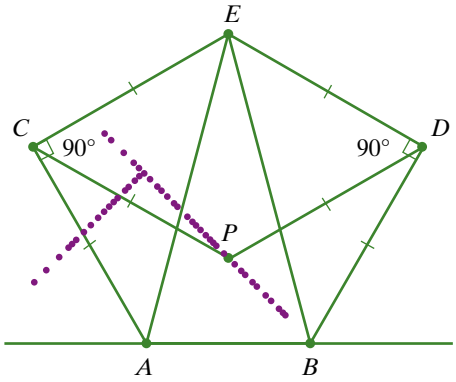
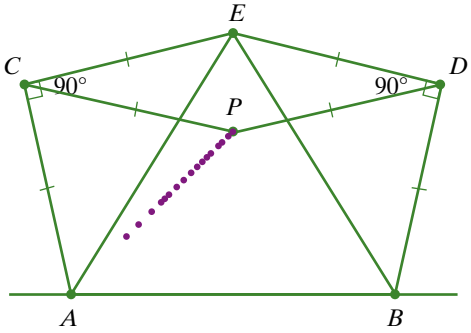




Lizzie and Ari noticed that moving A or B always moved P along a diagonal row of numbers. They made two conjectures:

- Conjecture 1: $\triangle AEB$ is similar to $\triangle ACP$ and to $\triangle BDP$
- Conjecture 2: $\angle APB = 90°$

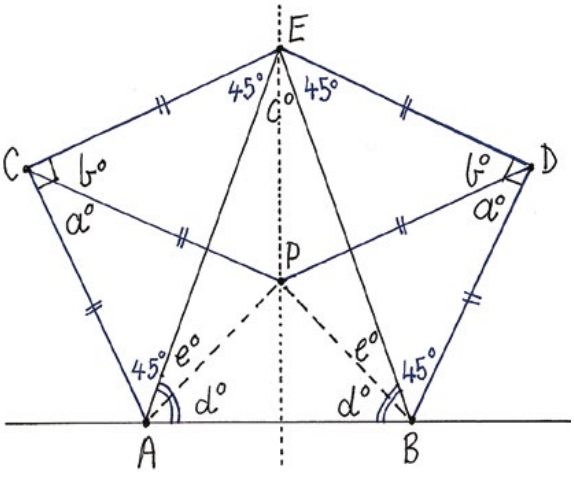
Lizzie and Ari then used a diagram of Consul to label some angles, using the same letter for angles that they knew must be equal.

a What geometric shape is quadrilateral $CEDP$? Explain.

b From the given information, what do you know about $\triangle EAB$?

c From the given information, what do you know about $\triangle ACP$ and $\triangle BDP$?

d How did Lizzie and Ari know that the four angles $\angle CEA$, $\angle EAC$, $\angle DEB$ and $\angle EBD$ are each 45°?



After unsuccessfully trying a number of different approaches to finding a proof for one of their conjectures, Lizzie and Ari decided to look at adjacent angles in quadrilateral $CEDP$ and at the two angles that make up the right angle at C. They successfully proved their second conjecture, that $\triangle AEB$ is similar to $\triangle ACP$ and to $\triangle BDP$.

e Using Lizzie and Ari's letters and the approach that they took, prove that $\angle AEB = \angle ACP$ and hence that $\triangle AEB$ is similar to $\triangle ACP$.

f Use your proof from part **e** to prove that $\angle APB = 90°$.

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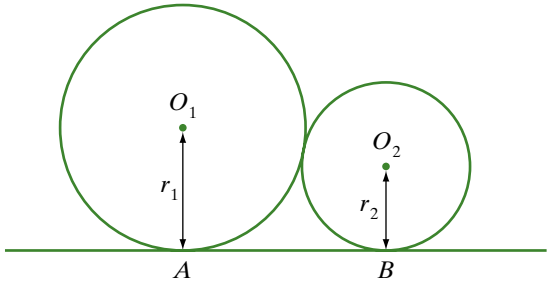
Analysis task

Sangaku

In Japan between the 17th and 19th centuries, a strange custom developed in which wooden boards inscribed with intricately decorated geometry problems were hung beneath the eaves of Shinto shrines and Buddhist temples. Some of these boards can be seen beneath the eaves of this temple. The geometry problems were known as *sangaku*. Although the exact purpose of sangaku is unclear, it seems that they were simply a joyful expression of geometry, perhaps designed to please the gods. We could compare the popularity of sangaku with the current interest in solving the number puzzles, sudoku. About 900 sangaku boards have survived, as well as several collections of sangaku problems in early 19th-century books. Three sangaku problems involving tangent circles are shown here. Tangent circles are circles that touch each other at one point. The centres of tangent circles and the point of contact are in a straight line.

a In the diagram on the right, show that $(AB)^2 = 4r_1r_2$



O_1 and O_2 are the centres of the two circles.

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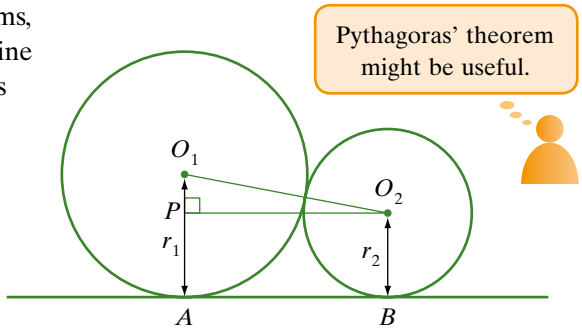
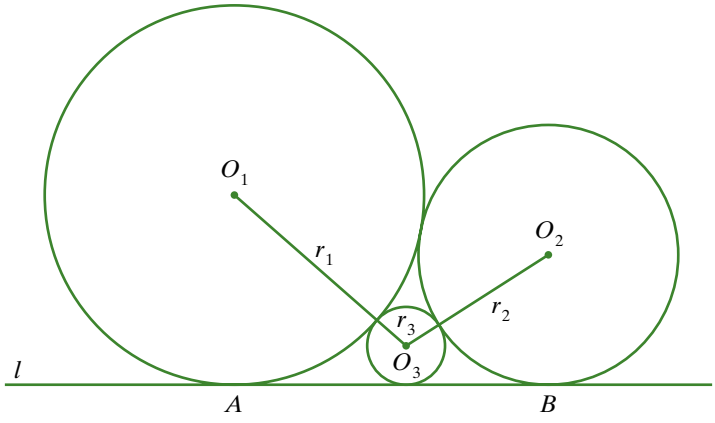
Hint: when solving geometry problems, we often need to add some further line segments. In this figure, the segments O_1O_2 and O_2P have been drawn, making a right-angled triangle O_1PO_2.

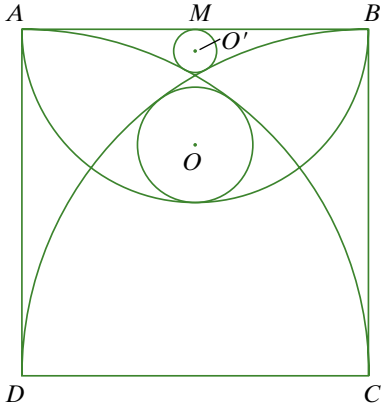


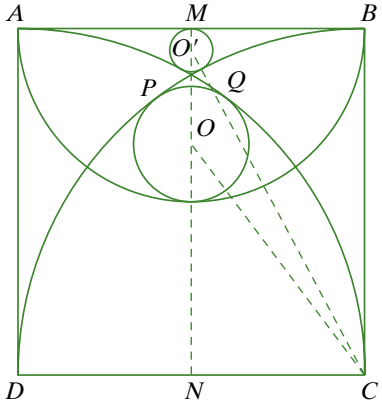
b A circle with centre O_3 touches the line l and the two tangent circles with centre O_1, radius r_1 and centre O_2, radius r_2. Show that $\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$.

Hint: Draw a line segments from O_1 to A and from O_2 to B. Draw a horizontal line segment through O_3 to meet O_1A and O_2B. Then use Pythagoras' theorem in the two right-angled triangles.



c $ABCD$ is a square of side a, and M is the centre of a circle with diameter AB; CA and DB are arcs of circles of radius a and centres D and C respectively. Find the radius, r_1, of the circle, centre O, inscribed in the area common to these circles. Find also the radius, r_2, of the circle, centre O', which touches AB and touches CA and DB externally as shown.





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Review Geometry and proving

Summary

Congruent triangles	Similar triangles
Congruent triangles must satisfy both of these conditions: ■ matching angles must be equal ■ matching sides must be equal in length.	Similar triangles must satisfy both of these conditions: ■ matching angles must be equal ■ matching sides must be in the same proportion.
Conditions for congruent triangles	**Conditions for similar triangles**
■ SSS (three sides of one triangle equal to three sides of the other)	■ SSS (three sides of one triangle in proportion to three sides of the other)
■ ASA (two angles of one triangle equal to the two angles of the other triangle and a matching side equal) Note: two angles equal means that three angles are equal.	■ AAA (three angles of one triangle equal to three angles of the other)
■ SAS (two sides of one triangle equal to two sides of the other triangle and the angle between them the same)	■ SAS (two sides of one triangle in proportion to two sides of the other triangle and the angle between them the same)
■ RHS (both triangles right-angled, hypotenuse and another side of one triangle equal to the hypotenuse and another side of the other triangle)	■ RHS (both triangles right-angled, hypotenuse and another side of one triangle in proportion to the hypotenuse and another side of the other triangle)

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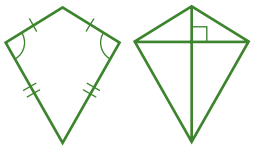
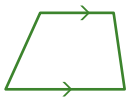
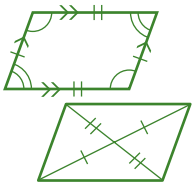
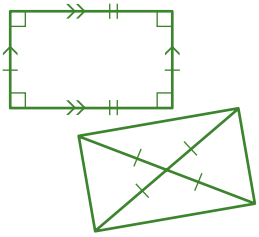
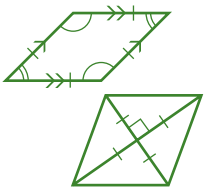
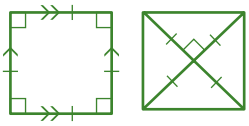
Properties	Name	Examples
■ Two pairs of adjacent (next to each other) sides are equal. ■ One pair of opposite angles are equal. ■ Diagonals are perpendicular. ■ One diagonal bisects the unequal angles.	Kite	
■ One pair of opposite sides are parallel. ■ Two pairs of supplementary angles.	Trapezium	
■ Both pairs of opposite sides are parallel. ■ Both pairs of opposite sides are equal. ■ Opposite angles are equal. ■ Adjacent angles are supplementary. ■ Diagonals bisect each other.	Parallelogram	
■ Both pairs of opposite sides are parallel and equal. ■ All angles are right angles. ■ Diagonals are equal in length. ■ Diagonals bisect each other.	Rectangle	
■ Both pairs of opposite sides are parallel. ■ All four sides are equal. ■ Opposite angles are equal. ■ Adjacent angles are supplementary. ■ Diagonals bisect each other. ■ Diagonals are perpendicular. ■ Diagonals bisect the angles of the rhombus.	Rhombus	
■ Both pairs of opposite sides are parallel. ■ All four sides are equal. ■ All angles are right angles. ■ Diagonals are equal in length. ■ Diagonals are perpendicular. ■ Diagonals bisect each other. ■ Diagonals bisect the right angles of the square.	Square	

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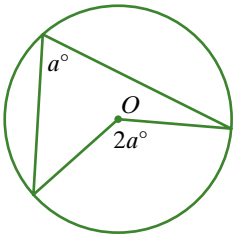
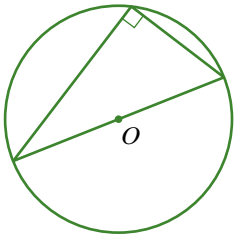
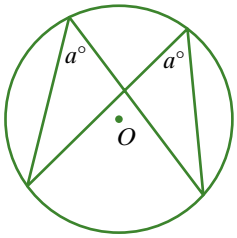
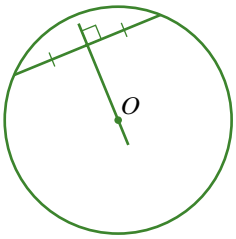
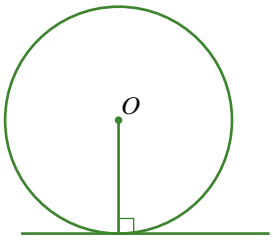
Circle properties

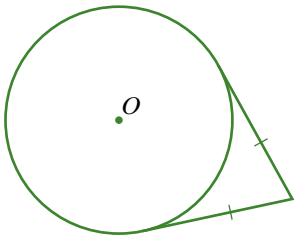


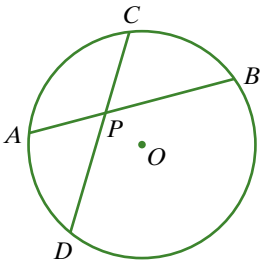




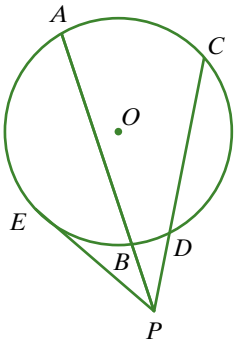






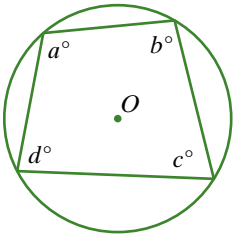


$AP \times BP = CP \times DP$



$AP \times BP = CP \times DP$

$EP^2 = AP \times BP$



$a° + c° = b° + d° = 180°$

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

*angle subtended at the centre
*angle subtended at the circumference
*arc
*chord
congruent
*cyclic quadrilateral
deductive reasoning
diagonal
*major arc
*major segment
*minor arc
*minor segment
perpendicular bisector
*secant
*segment
similar
*tangent

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Revision

Multiple-choice questions

1 What do all rectangles have that some parallelograms do not have?

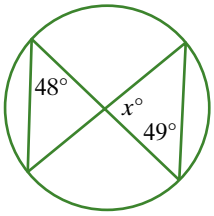
A Opposite sides equal **B** Diagonals equal **C** Opposite sides parallel
D Opposite angles equal **E** None of the above

2 Which of the following is *not* true in every rhombus?

A The two diagonals are equal in length.
B Each diagonal bisects two angles of the rhombus.
C The two diagonals are perpendicular.
D The opposite angles are equal.
E All the sides are equal.

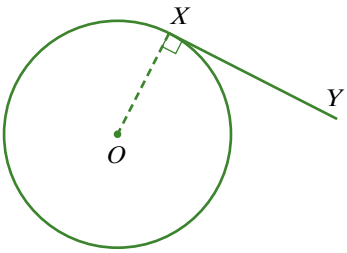
3 In the figure on the right, the value of x is

A 83 **B** 90 **C** 96
D 97 **E** 98



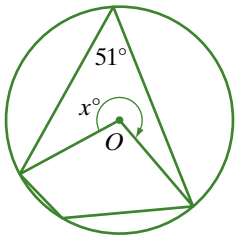
4 XY is

A an arc
B a chord
C a sector
D a segment
E a tangent



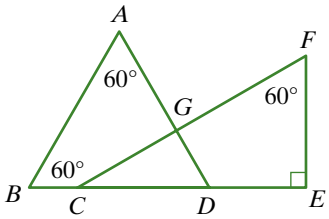
5 In this figure, the value of x is

A 102
B 129
C 231
D 258
E 309



Short-answer questions

6 Show why $\angle CGD$ is a right angle. Justify each of your statements.



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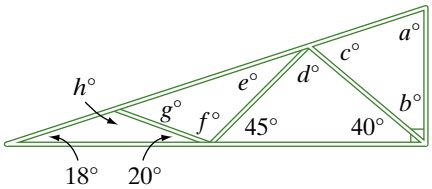
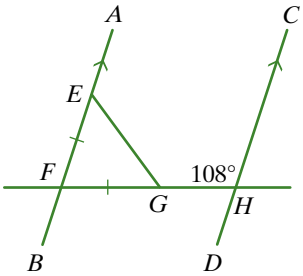
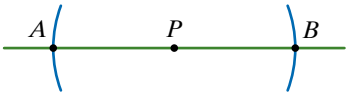
7 The diagram at right shows the design of a roof truss. Find the value of each of the pronumerals representing the angles.



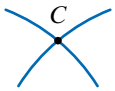
8 In the diagram, $AB \parallel CD$ and $EF = FG$. Find the size of $\angle AEG$.

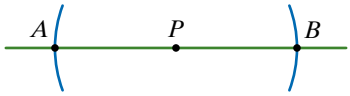


9 These three diagrams show a method for constructing a perpendicular line passing through a point P on a line. The compass point is placed at P and arcs are drawn to cut the line at A and B.

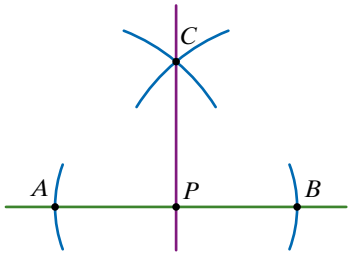


The compass point is then placed at A and an arc with radius greater than AP is drawn above the line. Keeping the compass at the same opening, another arc is drawn from B to intersect the previous arc (point C).





The line CP is perpendicular to AB.



a Why does the radius of the two arcs that intersect at C need to be greater than AP?

b Explain why this method works, that is, why is CP perpendicular to AB? Set out a logical explanation, justifying each statement you make.

c Why is it important to keep the compass open the same amount (that is, draw arcs with the same radius) when drawing the arcs from A and B?

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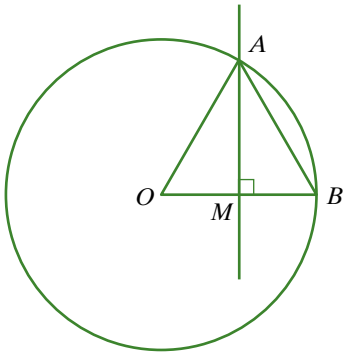
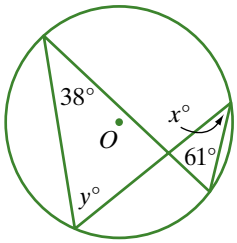
10 In the diagram, O is the centre of the circle and AM is the perpendicular bisector of OB. Explain why $\triangle AOB$ must be an equilateral triangle, justifying each statement you make.

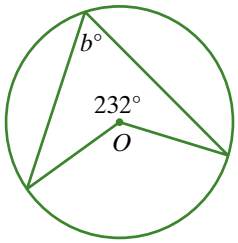


11 Find the value of the pronumeral in each of these figures.

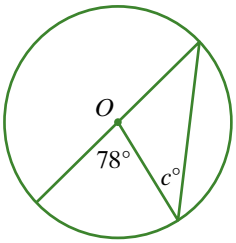
a



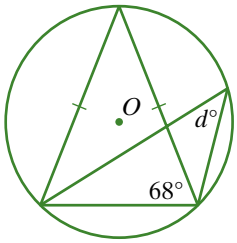
b



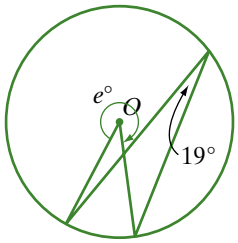
c



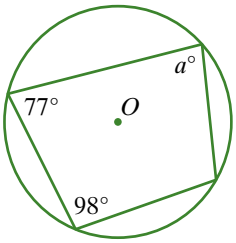
d



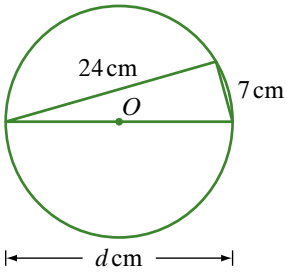
e



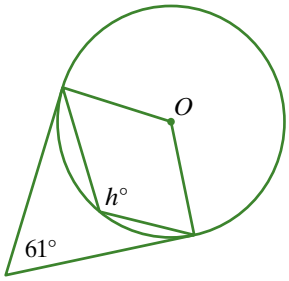
f



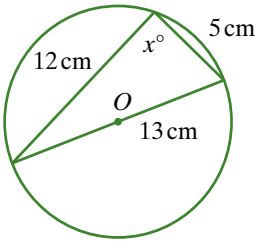
g



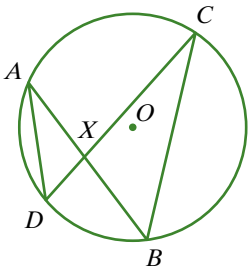
h



i



12 Show why the two triangles are similar.



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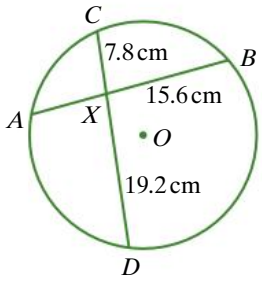
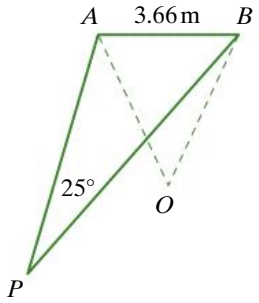
13 Calculate the length of AX.



Extended-response question

14 A hockey player at P observes that the angle subtended by the goal posts A and B is 25°. The distance AB between the goal posts is 3.66 m.



a If AB is regarded as a chord of a circle, explain why the hockey player could be at any one of many different positions around the circumference of a circle.

b What is the angle subtended by AB at the centre of the circle?

c If the centre of the circle is O, what sort of triangle is $\triangle AOB$? Explain.

d What is the size of $\angle OAB$ and $\angle OBA$?

e Using either interactive geometry software or pencil, protractor and compasses, draw a careful scale diagram to show A, B and O and the circle. Use a scale of 1 cm to represent 1 m.

f Choose a point on the major arc of the circle and label it Q. Draw the angle subtended at Q by AB. Check that $\angle AQB$ is 25°.

Practice quiz Chapter 9

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Probability 10

Pre-test

Warm-up

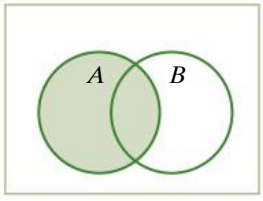
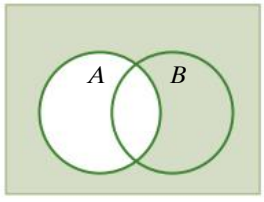
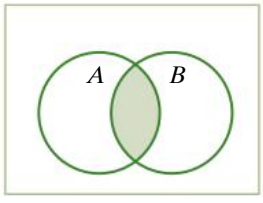
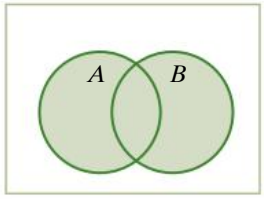
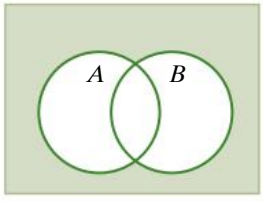
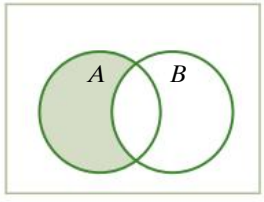
Probabilities are a way of expressing the chance of something happening. If the weather forecast for a region predicts a 20% chance of rain on a particular day, we know that for every five times when the atmospheric conditions (such as cloud types, temperature, humidity, wind direction and speed) are the same as they are now, it has rained only once. Collection of data over long periods of time allows climate scientists to make predictions about the likelihood of dry, normal or wet years.

10.1 Reviewing simple events involving 'and' and 'or'

Venn diagrams

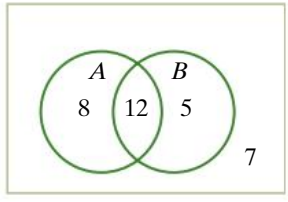
Venn diagram

For two events, A and B, the Venn diagrams below shows how shading is used to represent different outcomes.

A		A' Complement of A	
A and B		A or B	
Neither A nor B nor both		A but not B	

Example 1

The Venn diagram shows the number of favourable outcomes for two events, A and B.



$n = 32$

Use the Venn diagram to find

a $\Pr(A)$ **b** $\Pr(A \text{ or } B)$

c $\Pr(A \text{ and } B)$ **d** $\Pr(A \text{ but not } B)$

continued

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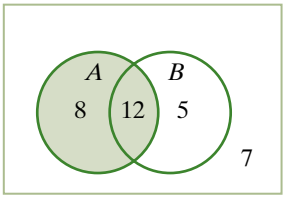
Example 1 continued

Working

a

$$\Pr(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of outcomes}}$$
$$= \frac{8 + 12}{8 + 12 + 5 + 7}$$
$$= \frac{20}{32}$$
$$= \frac{5}{8}$$

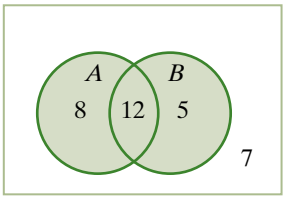
Reasoning



$n = 32$

b $\Pr(A \text{ or } B) = \dfrac{8 + 12 + 5}{32}$

$= \dfrac{25}{32}$

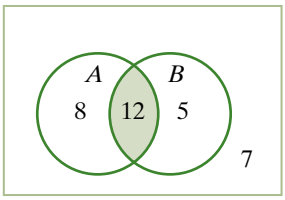


$n = 32$

$\Pr(A \text{ or } B)$ includes the outcomes favourable to A or B or both.

c $\Pr(A \text{ and } B) = \dfrac{12}{32}$

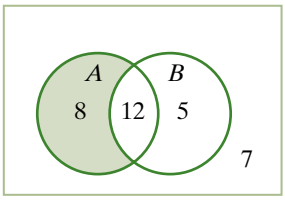
$= \dfrac{3}{8}$



$n = 32$

d $\Pr(A \text{ but not } B) = \dfrac{8}{32}$

$= \dfrac{1}{4}$



$n = 32$

Complementary events

The complement of an event is represented by the outcomes that are not included in the event.

The complement of A is written as A'. In example 1, $\Pr(A \text{ but not } B)$ could also be written as $\Pr(A \text{ and } B')$.

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Example 2

A teacher surveys her class of students about their chocolate preferences. Out of 30 students, 16 students liked dark chocolate (D) and 22 students liked milk chocolate (M). Only 4 of the students surveyed liked neither milk nor dark chocolate.

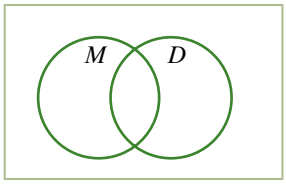
a Organise this data using a Venn diagram.

b Find the probability that a randomly selected student from the class likes

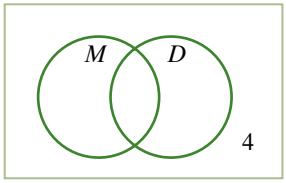
i both milk and dark chocolate **ii** milk chocolate only

Working

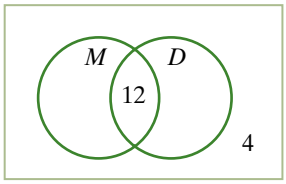
a



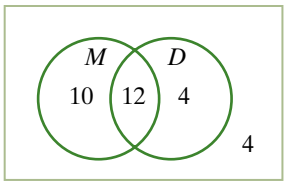
$n = 30$



$n = 30$



$n = 30$



$n = 30$

Reasoning

Draw circles to represent the students who like milk chocolate and the students who like dark chocolate. Some students like both so the circles intersect. Label the circles M and D. It is useful to write the total number in the sample set, $n = 30$.

4 students like neither type of chocolate so we write this number outside the circles representing M and D.
This leaves 26 students to add to the Venn diagram.

16 students like dark chocolate and 22 like milk chocolate, making a total of 38. However, only 26 students are unaccounted for in the Venn diagram, so 12 students must like both. We write this number in the intersection.

If 16 students like dark chocolate but 12 of these like both kinds, then 4 students must like dark chocolate only.
If 22 students like milk chocolate but 12 of these like both, then 10 students must like milk chocolate only.

continued

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Example 2 continued

Working

b i $\Pr(M \text{ and } D) = \frac{12}{30}$

$= \frac{2}{5}$

The probability that a randomly selected student will like both milk and dark chocolate is $\frac{2}{5}$.

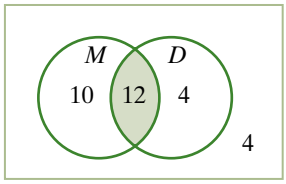
Shade the Venn diagram in part **ii** as shown:

ii $\Pr(M \text{ but not } D) = \frac{10}{30}$

$= \frac{1}{3}$

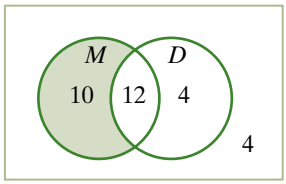
The probability that a randomly selected student will like milk chocolate but not dark chocolate is $\frac{1}{3}$.

Reasoning



$n = 30$

12 students out of 30 like both types of chocolate.

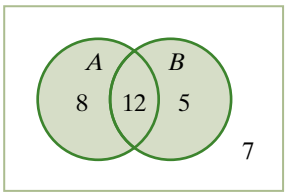


$n = 30$

10 students out of 30 like milk chocolate only.

The addition rule of probabilities

Consider again the Venn diagram in example **1**.



$n = 32$

$\Pr(A \text{ or } B)$ was calculated as $\frac{8 + 12 + 5}{32}$.

We can see from the Venn diagram that $\Pr(A) = \frac{8 + 12}{32}$ and $\Pr(B) = \frac{12 + 5}{32}$. We can also see that if we add these two together, we would be counting $\frac{12}{32}$ twice, so we would have to subtract one lot of $\frac{12}{32}$. But $\frac{12}{32}$ is equal to $\Pr(A \text{ and } B)$.

From this we can say: $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$

This is known as the **addition rule of probability**.

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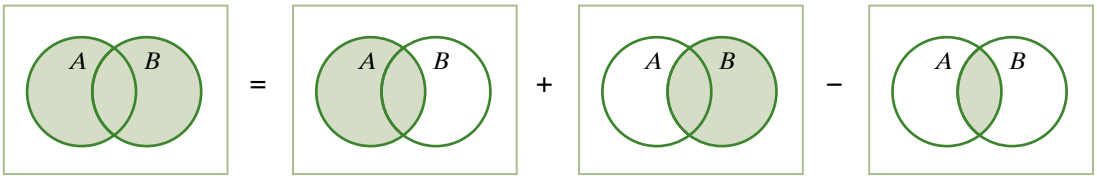
We can illustrate this with the following Venn diagrams:



Addition rule
For two events A and B, the probability of $\Pr(A \text{ or } B)$ is given by
$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$

Example 3

A number is selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let A be the event that a square number is selected and B be the event that an odd number is selected.

a List the sample space for A.

b List the sample space for B.

c Construct a Venn diagram to show the information. Shade the region that shows $(A \text{ and } B)$.

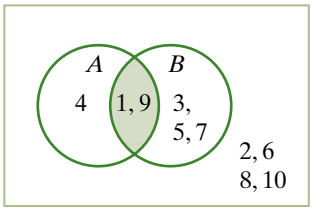
d Calculate $\Pr(A \text{ or } B)$.

Working

a $A = \{1, 4, 9\}$

b $B = \{1, 3, 5, 7, 9\}$

c



$n = 10$

d $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$

$= \frac{3}{10} + \frac{5}{10} - \frac{2}{10}$

$= \frac{6}{10}$

$= \frac{3}{5}$

Reasoning

The three square numbers are 1, 4 and 9.

The five odd numbers are 1, 3, 5, 7 and 9.

1 and 9 are square numbers and odd numbers.
The numbers 2, 6, 8 and 10 do not belong to either set.

The Venn diagram shows that the six numbers 1, 3, 4, 5, 7 and 9 satisfy 'A or B'.
When we add the separate probabilities of A and B we must subtract $\Pr(A \text{ and } B)$ so that the numbers 1 and 9 are not counted twice.

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Example 4

If $\Pr(A) = 0.6$, $\Pr(B) = 0.5$ and $\Pr(A \text{ and } B) = 0.3$, calculate $\Pr(A \text{ or } B)$.

Working

$$\begin{aligned}\Pr(A \text{ or } B) &= \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)\\ &= 0.6 + 0.5 - 0.3\\ &= 0.8\end{aligned}$$

Reasoning

Substitute the known values into the rule.

Mutually exclusive events

Consider the tossing of a coin, where the two events are 'heads' and 'tails'. These two events are said to be **mutually exclusive** because if the coin lands heads, it cannot also land tails on the same toss. Mutually exclusive events have no outcomes in common.

Similarly, suppose A is the event 'drawing a club' and B is the event 'drawing the queen of hearts' from a standard pack of cards. If the selected card satisfies A it cannot also satisfy event B, the events A and B are mutually exclusive. However, if E is the event 'drawing an even number' and R is the event 'drawing a red card', E and R are not mutually exclusive because the card could be both an even number and red.

Mutually exclusive

Clubs	Spades	Diamonds	Hearts
♣K	♠K	♦K	♥K
♣Q	♠Q	♦Q	♥Q
♣J	♠J	♦J	♥J
♣10	♠10	♦10	♥10
♣9	♠9	♦9	♥9
♣8	♠8	♦8	♥8
♣7	♠7	♦7	♥7
♣6	♠6	♦6	♥6
♣5	♠5	♦5	♥5
♣4	♠4	♦4	♥4
♣3	♠3	♦3	♥3
♣2	♠2	♦2	♥2
♣A	♠A	♦A	♥A

Not mutually exclusive

Clubs	Spades	Diamonds	Hearts
♣K	♠K	♦K	♥K
♣Q	♠Q	♦Q	♥Q
♣J	♠J	♦J	♥J
♣10	♠10	♦10	♥10
♣9	♠9	♦9	♥9
♣8	♠8	♦8	♥8
♣7	♠7	♦7	♥7
♣6	♠6	♦6	♥6
♣5	♠5	♦5	♥5
♣4	♠4	♦4	♥4
♣3	♠3	♦3	♥3
♣2	♠2	♦2	♥2
♣A	♠A	♦A	♥A

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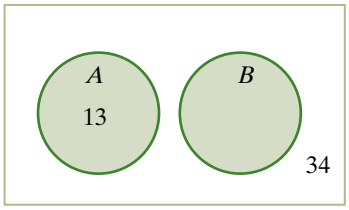
We can also show this using Venn diagrams.

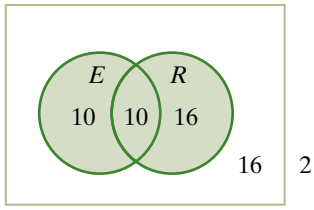
Mutually exclusive



$n = 52$

$$\begin{aligned}\Pr(A \text{ or } B) &= \Pr(A) + \Pr(B)\\ &= \frac{13}{52} + \frac{1}{52}\\ &= \frac{14}{52}\\ &= \frac{7}{26}\end{aligned}$$

Not mutually exclusive



$n = 52$

$$\begin{aligned}\Pr(E \text{ or } R) &= \Pr(E) + \Pr(R) - \Pr(E \text{ or } R)\\ &= \frac{20}{52} + \frac{26}{52} - \frac{10}{52}\\ &= \frac{36}{52}\\ &= \frac{9}{13}\end{aligned}$$

Addition rule for mutually exclusive events
For mutually exclusive events, $\Pr(A \text{ and } B) = 0$, so the addition rule simplifies to $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$.

Example 5

Determine which of the following pairs of events are mutually exclusive.

a When randomly selecting one card from a deck of 52 playing cards: 'drawing a club' and 'drawing an ace'.

b When rolling a die: 'rolling a multiple of 3' and 'rolling a number less than 3'.

Working	Reasoning
a The events 'drawing a club' and 'drawing an ace' are not mutually exclusive.	The card could be the ace of clubs.
b The events 'rolling a multiple of 3' and 'rolling a number less than 3' are mutually exclusive.	The multiples of 3 are 3 and 6. The numbers less than 3 are 1 and 2. So there is no number that satisfies both events.

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Example 6

One card is selected at random from a pack of 52 playing cards. What is the probability that the card is

a an ace or a king?

b a red card or a club?

c a black card or a red jack?

Working

a $\Pr(\text{ace or king}) = \Pr(\text{ace}) + \Pr(\text{king})$

$$= \frac{1}{13} + \frac{1}{13}$$

$$= \frac{2}{13}$$

Reasoning

The events 'drawing an ace' and 'drawing a king' are mutually exclusive because a card cannot be an ace *and* a king.
So use $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$.

b $\Pr(\text{red or club}) = \Pr(\text{red}) + \Pr(\text{club})$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= \frac{3}{4}$$

The events 'drawing a red card' and 'drawing a club' are mutually exclusive because a card cannot be red *and* a club.
So use $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$.

Alternatively,

$$\Pr(\text{black card or red jack}) = \frac{1}{2} + \frac{2}{52}$$

$$= \frac{13}{26} + \frac{1}{26}$$

$$= \frac{14}{26}$$

$$= \frac{7}{13}$$

The events 'drawing a black card' and 'drawing a *red* jack' *are* mutually exclusive, so add the probabilities of those events.

Probability tables

A **probability table** is similar to a two-way table but the cells contain probabilities rather than frequencies. The total probability, in the bottom right corner, must add to 1.

	A	A'	**Total**
B	$\Pr(A \text{ and } B)$	$\Pr(B \text{ but not } A)$	$\Pr(B)$
B'	$\Pr(A \text{ but not } B)$	$\Pr(\text{neither } A \text{ nor } B)$	$\Pr(B')$
Total	$\Pr(A)$	$\Pr(A')$	1

$\Pr(A')$ means the probability of 'not A' and $\Pr(B')$ means the probability of 'not B'.

When completing a probability table, we use the fact that the probability of an event and its complement must add to one.

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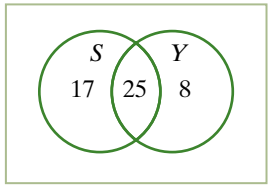
Complementary events
$\Pr(A') = 1 - \Pr(A)$
where A' is the complement of event A.

Example 7

As part of a well-being program, students can choose to do swimming or yoga—or both activities, or neither. This Venn diagram shows the choices of the 50 students.

a Complete the two-way table using the information given in the Venn diagram.

b Construct a probability table for the data.



$n = 50$

	A	A'	Total
B			
B'			
Total			

Working

a

	Y	Y'	Total
S	25	17	42
S'	8	0	8
Total	33	17	50

Reasoning

25 students chose swimming and yoga.
8 students chose yoga but not swimming.
17 students chose swimming but not yoga.
0 students chose neither swimming nor yoga.
Add the columns.
Add the rows.

b

	Y	Y'	Total
S	0.5		
S'			
Total			1

25 out of 50 students chose both yoga and swimming.

$$\Pr(Y \text{ and } S) = \frac{25}{50} = 0.5$$

	Y	Y'	Total
S	0.5		
S'	0.16		
Total			1

8 out of 50 students chose yoga but not swimming.

$$\Pr(Y \text{ but not } S) = \frac{8}{50} = 0.16$$

continued

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Example 7 continued

Working

	Y	Y'	Total
S	0.5	0.34	
S'	0.16		
Total			1

	Y	Y'	Total
S	0.5	0.34	
S'	0.16	0	
Total			1

	Y	Y'	Total
S	0.5	0.34	0.84
S'	0.16	0	0.16
Total	0.66	0.34	1

Reasoning

17 out of 50 students chose swimming but not yoga.

$\Pr(S \text{ but not } Y) = \frac{17}{50} = 0.34$

$17 + 25 + 8 = 50$ so no students did neither.
$\Pr(\text{neither } S \text{ nor } Y) = 0$

Now fill in the totals. The Total row and the Total column should each add to 1.

Example 8

If $\Pr(A) = 0.65$, $\Pr(B) = 0.38$ and $\Pr(A \text{ and } B) = 0.22$, complete the following probability table.

	A	A'	Total
B			
B'			
Total			

Working

	A	A'	Total
B	0.22		0.38
B'			
Total	0.65		1

Reasoning

Fill in the given information. Write 1 in the bottom right-hand cell.

continued

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Example 8 continued

Working

	A	A'	Total
B	0.22	0.16	0.38
B'	0.43		
Total	0.65		1

Reasoning

$\Pr(A \text{ not } B) = 0.65 - 0.22 = 0.43$
$\Pr(B \text{ not } A) = 0.38 - 0.22 = 0.16$

	A	A'	Total
B	0.22	0.16	0.38
B'	0.43		0.62
Total	0.65	0.35	1

The Total row and the Total column should each add to 1.
$\Pr(A') = 1 - 0.65 = 0.35$
$\Pr(B') = 1 - 0.38 = 0.62$

	A	A'	Total
B	0.22	0.16	0.38
B'	0.43	0.19	0.62
Total	0.65	0.35	1

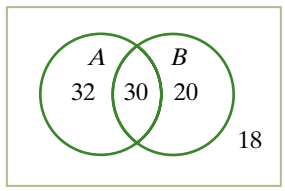
$\Pr(\text{neither } A \text{ nor } B) = 0.62 - 0.43 = 0.19$
Check:
$\Pr(\text{neither } A \text{ nor } B) = 0.35 - 0.16 = 0.19$

exercise 10.1

LINKS TO Example 1

1 Use the Venn diagram to determine each of the following probabilities.

a $\Pr(A)$ **b** $\Pr(A \text{ and } B)$
c $\Pr(\text{neither } A \text{ nor } B)$ **d** $\Pr(A \text{ or } B)$



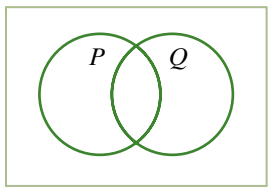
$n = 100$

LINKS TO Example 2

2 The events P and Q are such that $\Pr(P) = 0.6$, $\Pr(Q) = 0.8$ and $\Pr(P \text{ and } Q) = 0.5$.

a Copy and complete this Venn diagram for $n = 100$ trials.

b Find $\Pr(P \text{ or } Q)$.



$n = 100$

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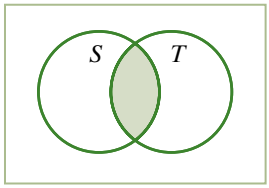
3 The letters A to Z are written on separate pieces of paper and placed in a box. Let S be the set of letters in the word 'FEBRUARY' and T be the set of letters in the word 'BIRTHDAY'.

a Copy and complete this Venn diagram, showing the number of letters in each section.



$n = 26$

b One letter is selected at random from the box. Find the probability that the letter is in the set

i T **ii** S **iii** S and T

iv S or T **v** neither S nor T **vi** S but not T

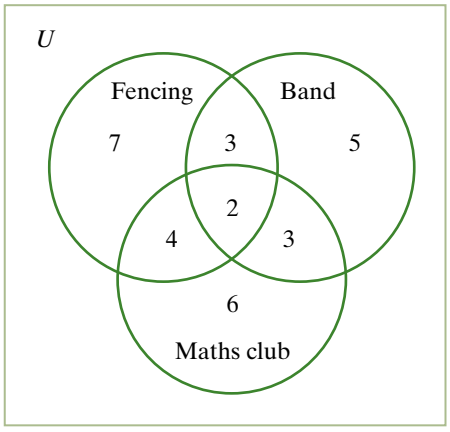
4 Some students participate in various extra-curricular activities, as shown in the Venn diagram. One of the students is chosen at random. Find the probability that the student participates in

a fencing only

b fencing and band only

c maths club but not fencing

d all three activities



$n = 30$

LINKS TO Example 3

5 The Venn diagram shows the numbers of students in a class of thirty who liked milk chocolate (M) and the number who liked dark chocolate (D).

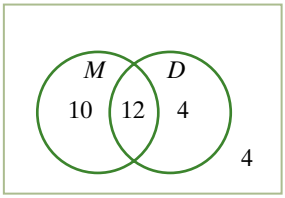
Calculate

a $\Pr(M)$ **b** $\Pr(D)$ **c** $\Pr(M \text{ and } D)$

d How many students liked either milk chocolate or dark chocolate or both?

e Calculate $\Pr(M \text{ or } D)$.

f Calculate $\Pr(M) + \Pr(D) - \Pr(M \text{ and } D)$. Is your answer the same as you obtained in part **e**? Explain.



$n = 30$

LINKS TO Example 4

6 If $\Pr(A) = 0.84$, $\Pr(B) = 0.66$ and $\Pr(A \text{ and } B) = 0.59$, calculate $\Pr(A \text{ or } B)$.

LINKS TO Example 5

7 **a** If a fair die is rolled, what is the probability of rolling

i an even number or a prime number **ii** an odd number or a factor of 4

iii a prime number or a 6 **iv** a factor of 9 or an even number

b Which of your calculations in part **a** involved mutually exclusive events?

LINKS TO Example 6

8 A card is selected from a standard pack of 52 cards. Calculate

a Pr(red) **b** Pr(5) **c** Pr(red and 5) **d** Pr(red or 5)

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9 A box contains coloured marbles. If one marble is randomly selected, the probability that it is red is 0.22 and the probability that it is blue is 0.37. If Keiko closes her eyes and selects one marble from the box, find the probability that it is

a a red marble or a blue marble **b** neither a red marble nor a blue marble

10 Two events A and B are mutually exclusive. If $\Pr(A) = 0.36$ and $\Pr(A \text{ or } B) = 0.84$, find $\Pr(B)$.

LINKS TO Example 7

11 At a particular dog show, the dogs are classified according to whether or not they are a large breed (L) and whether or not their coat is short (S). This two-way table summarises the data for all dogs at the show.

	L	L'	Total
S	50	120	170
S'	70	60	130
Total	120	180	300

One dog at the show is selected at random. Find the probability that this dog

a has a short coat **b** is a large breed with a short coat

c is a large breed without a short coat **d** is a small breed with a short coat

12 Represent the information in question **1** as

a a two-way table **b** a probability table

13 A video library surveyed 200 people regarding their movie preferences. The survey showed that 140 people liked adventure movies, 185 people enjoyed comedies, and 8 people liked neither adventure nor comedy movies.

a Represent this information as

i a Venn diagram **ii** a two-way table **iii** a probability table

b If one of the people surveyed is selected at random, find the probability that this person liked

i both comedy and adventure movies

ii adventure movies but not comedy movies

LINKS TO Example 8

14 **a** Copy and complete this probability table.

b Find

i $\Pr(X \text{ and } Y')$

ii $\Pr(X' \text{ and } Y')$

	X	X'	Total
Y	0.55		0.88
Y'			
Total		0.42	1

15 In a study of people using a local library it was found that 70% of the library users were students (S), 60% were female (F) and 42% were female students.

a Use the information to fill in this probability table.

b What is the probability that a randomly selected library user is neither female nor a student?

	S	S'	Total
F			
F'			
Total			1

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16 If $\Pr(P) = 0.75$, $\Pr(Q) = 0.42$ and $\Pr(P \text{ and } Q) = 0.35$, copy and complete the probability table on the right.

	P	P'	Total
Q			
Q'			
Total			

17 If $\Pr(A') = 0.35$, $\Pr(B) = 0.52$ and $\Pr(A \text{ and } B) = 0.21$, copy and complete this probability table.

	A	A'	Total
B			
B'			
Total			

18 If $\Pr(\text{neither } X \text{ nor } Y) = 0.10$, $\Pr(Y') = 0.2$ and $\Pr(X \text{ and } Y) = 0.35$, copy and complete the probability table on the right.

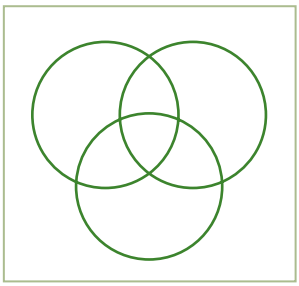
	Y	Y'	Total
X			
X'			
Total			

19 Use the information in this probability table to calculate $\Pr(A \text{ or } B)$.

	A	A'	Total
B	0.22	0.16	0.38
B'	0.43	0.19	0.62
Total	0.65	0.35	1

exercise 10.1 challenge

20 In a survey of 30 students, it was found that 21 students played tennis, 21 played cricket and 18 played hockey. While 3 students played none of these sports, 18 played both tennis and cricket, 14 played both cricket and hockey and 15 played both tennis and hockey. Find the probability that one randomly selected student from the group played all three sports.
Hint: Use a Venn diagram with three intersecting circles.



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10.2 Independent events and the multiplication rule

If we toss a coin and roll a die, we can assume that the way the coin falls does not affect how the die lands. If A is the event 'landing heads' and B is the event 'getting a 5', then we say that events A and B are **independent**.

The following **probability grid** (sometimes called an **array**) shows that there are 12 possible outcomes.

	1	**2**	**3**	**4**	**5**	**6**
H	$H, 1$	$H, 2$	$H, 3$	$H, 4$	$H, 5$	$H, 6$
T	$T, 1$	$T, 2$	$T, 3$	$T, 4$	$T, 5$	$T, 6$

The probability of any particular outcome is $\frac{1}{12}$, for example, $\Pr(H \text{ and } 2) = \frac{1}{12}$.

Notice though that $\Pr(H) = \frac{1}{2}$ and $\Pr(2) = \frac{1}{6}$. We can see then that:

$$\begin{aligned}\Pr(H \text{ and } 2) &= \Pr(H) \times \Pr(2)\\ &= \frac{1}{2} \times \frac{1}{6}\\ &= \frac{1}{12}\end{aligned}$$

Independent events

Two events are independent if the outcome of one is not influenced by the outcome of the other.

For independent events
$\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.

The converse of this is also true.
If $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$
then the events are independent.

We associate multiplication of probabilities with the word *and*.

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Example 9

For each of the following decide whether the events are independent.

a Jack tosses a coin and gets heads. Ivan tosses a coin and gets heads.

b Carmel randomly selects one ball from a bag that contains 5 green balls and 2 yellow balls. She gets a green ball and keeps it. Then Luisa randomly selects a ball and gets yellow.

Working	Reasoning
a The two events are independent.	The outcome of Ivan tossing the coin does not depend on whether Jack gets a head or a tail.
b The two events are not independent.	Carmel keeps the green ball she selects so when Luisa selects a ball there are only 6 balls left, 4 green balls and 2 yellow balls.

Example 10

Complete this probability table and hence show that events A and B are independent.

	A	A'	Total
B		0.42	0.7
B'	0.12		
Total			

Working

	A	A'	Total
B	0.28	0.42	0.7
B'	0.12	0.18	0.3
Total	0.4	0.6	1

$\Pr(A) \times \Pr(B)$
$= 0.4 \times 0.7$
$= 0.28$
$= \Pr(A \text{ and } B)$
So events A and B are independent.

Reasoning

If $\Pr(A) \times \Pr(B) = \Pr(A \text{ and } B)$ then events A and B are independent.

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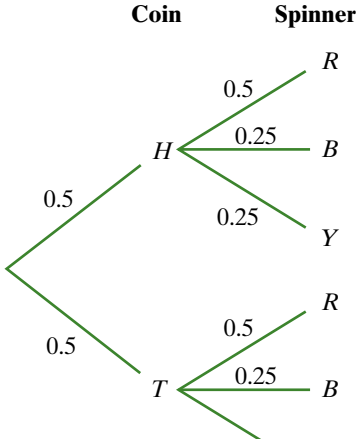
Tree diagrams and the multiplication rule

Each branch of a tree diagram can be labelled with the probability of the outcome that it represents. Probabilities can be multiplied along the branches of the tree diagram.

Consider the events tossing a coin and spinning the spinner shown.

On the tree diagram, 'heads' and 'tails' are represented by H and T respectively and red, blue and yellow are represented by R, B and Y. The probabilities are written on each branch.



To find the probability of any outcome of the two-step experiment, we multiply the probabilities along the branches for that outcome. For example,

$$\begin{aligned}\Pr(H \text{ and } R) &= 0.5 \times 0.5\\ &= 0.25\end{aligned}$$

$$\begin{aligned}\Pr(H \text{ and } B) &= 0.5 \times 0.25\\ &= 0.125\end{aligned}$$

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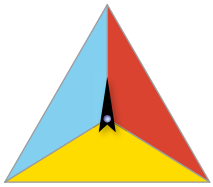
Example 11

This spinner is spun three times.

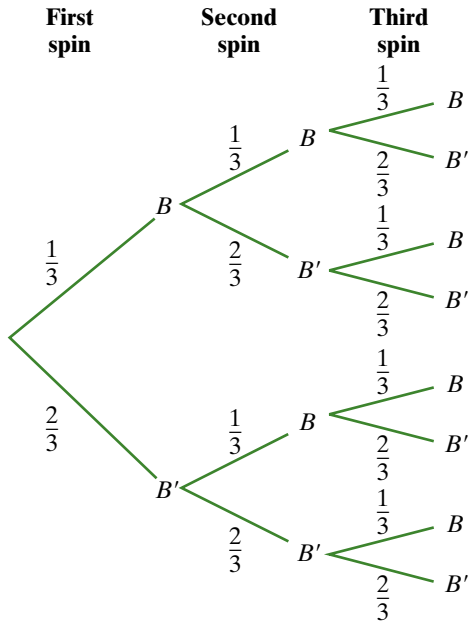


a Using B for 'blue' and B' for 'not blue', construct a tree diagram to show the outcomes.

b What is the probability that all three spins are blue?

Working

a



Reasoning

The result of any one spin is independent of the result of any other spin. So 'blue on the first spin', 'blue on the second spin' and 'blue on the third spin' are three independent events.

b $\Pr(BBB) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$

$= \frac{1}{27}$

Multiply the probabilities along the BBB branch of the tree diagram.

In the previous example we could have written $\left(\frac{1}{3}\right)^3$ rather than $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$. Using index notation is more convenient when the number of **trials** is large.

Each repeat of a probability experiment is called a trial.

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Example 12

A fair coin is tossed six times. What is the probability of tossing six tails?

Working

$$\Pr(6 \text{ tails}) = \left(\frac{1}{2}\right)^6$$
$$= \frac{1}{64}$$

Reasoning

The result of one toss is independent of the result of any other toss.

For any one toss, $\Pr(T) = \frac{1}{2}$.

A tree diagram is useful in calculating probabilities for situations involving particular outcomes, for example, 'neither' and 'at least'.

Example 13

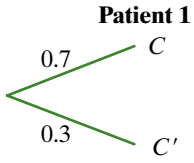
The probability that a particular drug will cure a patient of a disease is 0.7. The drug is administered to two patients.

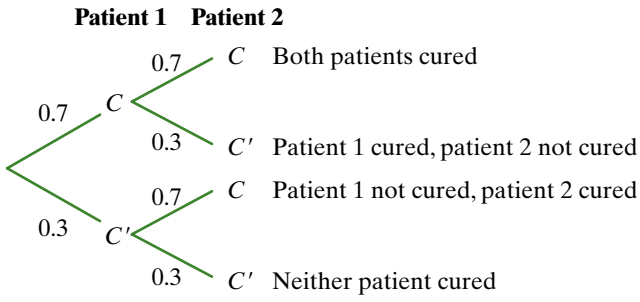
a Use a tree diagram to show the possible outcomes of treatment with the drug.

b Calculate the probability that

- **i** both patients are cured
- **ii** neither patient is cured
- **iii** only one of the patients is cured
- **iv** at least one patient is cured

Working





Reasoning

Let C represent the event 'the patient is cured'. Therefore C' is the event 'the patient is not cured'.

The first patient is either cured with a probability of 0.7 or not cured with a probability of 0.3.

The second patient has the same chance of being cured as the first patient.

For each outcome for the first patient, draw and label two similar branches for the second patient.

continued

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Example 13 continued

Working	Reasoning
b i $\Pr(C \text{ and } C) = 0.7 \times 0.7$ $= 0.49$ The probability that both patients are cured is 0.49.	(C and C) means 'patient 1 is cured' and 'patient 2 is cured'.
ii $\Pr(C' \text{ and } C') = 0.3 \times 0.3$ $= 0.09$ The probability that neither patient is cured is 0.09.	C' means 'not cured' so (C' and C') means neither patient is cured'.
iii $\Pr[(C \text{ and } C') \text{ or } (C' \text{ and } C)]$ $= 0.7 \times 0.3 + 0.3 \times 0.7$ $= 0.21 + 0.21$ $= 0.42$ The probability that only one patient is cured is 0.42.	(C and C') or (C' and C) means 'patient 1 is cured and patient 2 is not cured' or 'patient 1 is not cured and patient 2 is cured'.
iv $\Pr[(C \text{ and } C) \text{ or } (C \text{ and } C') \text{ or } (C' \text{ and } C)]$ $= 0.7 \times 0.7 + 0.7 \times 0.3 + 0.3 \times 0.7$ $= 0.49 + 0.21 + 0.21$ $= 0.91$ The probability that at least one patient is cured is 0.91.	'At least one patient is cured' means either patient 1 or patient 2 or both are cured.

In the above example, there is an easier way of calculating the probability that 'at least one patient is cured'. The complement of 'at least one patient is cured' is 'neither patient is cured'.

$$\begin{aligned}\Pr(\text{at least one cured}) &= 1 - \Pr(\text{neither cured})\\ &= 1 - 0.3 \times 0.3\\ &= 1 - 0.9\\ &= 0.91\end{aligned}$$

Example 14

A baker purchases $\frac{1}{4}$ of her flour from Wheatlovers (W) and $\frac{3}{4}$ of her flour from Bakefresh (B). The probability that a bag of flour from Wheatlovers is the correct weight (C) is $\frac{9}{10}$ and the probability that a bag from Bakefresh is the correct weight is $\frac{4}{5}$. Find the probability that a bag of flour

a is from Wheatlovers and is the correct weight

b is not the correct weight

continued

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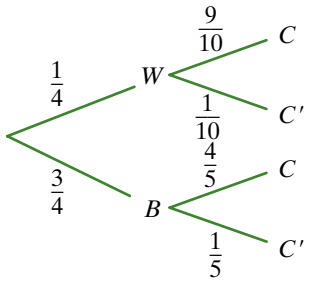
Example 14 continued

Working



Reasoning

A tree diagram is useful for organising this information.

a Pr(Wheatlovers and correct weight) $= \Pr(W \text{ and } C)$

$$= \frac{1}{4} \times \frac{9}{10}$$
$$= \frac{9}{40}$$

Multiply the probabilities along the path with branches W and then C.

b Pr(incorrect weight) $= \Pr(W \text{ and } C') + \Pr(B \text{ and } C')$

$$= \frac{1}{4} \times \frac{1}{10} + \frac{3}{4} \times \frac{1}{5}$$
$$= \frac{1}{40} + \frac{3}{20}$$
$$= \frac{7}{40}$$

Two paths through the tree give a bag of flour with incorrect weight (C'). Multiply the probabilities along each path and then add the results.

Example 15

A biased coin and an unbiased coin are tossed together. The probability that the biased coin will land heads up is 0.65. Find the probability of tossing

a two heads **b** a tail on the biased coin and a head on the unbiased coin

Working

a Let H and T represent heads and tails.
For the unbiased coin, $\Pr(H) = 0.5$
For the biased coin, $\Pr(H) = 0.65$

$\Pr(H \text{ and } H) = \Pr(H) \times \Pr(H)$
$= 0.65 \times 0.5$
$= 0.325$

Reasoning

The outcome of tossing one coin does not affect the outcome of tossing the other coin, so the events are independent.

Use $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.

b For the biased coin, $\Pr(T) = 0.35$
For the unbiased coin, $\Pr(H) = 0.5$

$\Pr(T \text{ and } H) = \Pr(T) \times \Pr(H)$
$= 0.5 \times 0.35$
$= 0.175$

The probability that the biased coin will land tails up is $1 - 0.65$.

Use $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.

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exercise 10.2

LINKS TO Example 9

1 Decide if each of these events are independent.

a Peter spins this spinner once and gets blue.
David then spins the spinner and gets yellow.

b Dean picks one card at random from a deck of 52 playing cards, and gets a heart. He replaces the card. Then Carla randomly picks a card from the same deck and gets a club.

c Jeff's alarm clock fails to ring in the morning. Jeff is late to school that day.

d A card is drawn from a pack and a die is rolled.

e The Nguyens have two daughters. They wonder if their next child will be a boy.

f To see which player starts, the players in a game must get a total score of 8 from two rolls of a die. Janie has rolled a 6 the first time.

LINKS TO Example 10

2 Complete these probability tables and hence show whether events A and B are independent.

a

	A	A'	Total
B		0.15	
B'	0.29		
Total	0.64		

b

	A	A'	Total
B		0.52	0.8
B'	0.07		
Total			

c

	A	A'	Total
B		0.14	
B'	0.25		
Total	0.80		

d

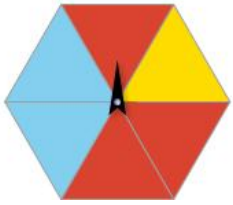
	A	A'	Total
B		0.63	0.84
B'	0.04		
Total			

3 Events P and Q are independent.
Complete the probability table.

	P	P'	Total
Q			
Q'			0.8
Total	0.25		1

LINKS TO Example 11

4 Zoe spins this spinner twice.



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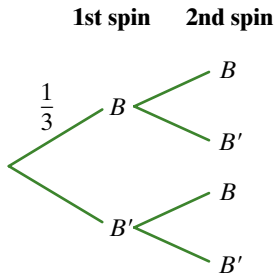
a If B is the event 'spinning blue', copy this tree diagram, writing the appropriate probability along each branch.



b Find the probability that

- **i** both spins are blue
- **ii** neither of the spins is blue
- **iii** exactly one of the spins is blue
- **iv** Zoe spins blue on the second spin only

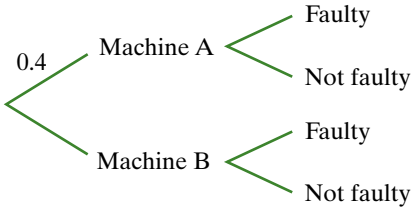
LINKS TO Example 12

5 A fair coin is tossed four times. What is the probability of tossing four heads?

LINKS TO Examples 13, 14

6 A factory that produces light globes has two machines, A and B. Machine A produces 40% of the light globes and machine B produces the rest. Of the globes produced by machine A, 10% are faulty, and for Machine B the figure is 5%. One light globe is selected at random and tested.

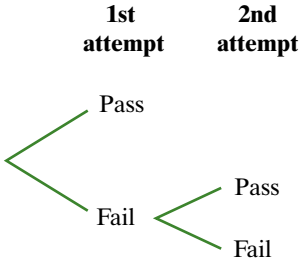
a Copy and complete the tree diagram on the right.



b Find the probability that the light globe was produced by Machine A and is not faulty.

c Find the probability that the light globe is faulty.

7 Sherene must pass an entrance test to be admitted into a certificate course. The probability that she passes the test on any attempt is 0.7. She is allowed up to two attempts.



a Copy and complete the tree diagram.

b Find the probability that Sherene

- **i** fails the test on both attempts
- **ii** is admitted into the course

8 A coin is biased so that the probability of tossing a head is 0.55. Narelle tosses this coin three times.

a Draw a tree diagram to show all possible outcomes.

b Find, correct to four decimal places, the probability that Narelle tosses

- **i** 3 tails
- **ii** exactly 2 heads
- **iii** at least 1 tail

9 A spinner marked with seven equal intervals numbered 1 to 7 is spun three times. On each spin, Marco records whether the result is odd or even.

a Draw a tree diagram to show all possible outcomes.

b Find the probability of obtaining

- **i** 3 odds
- **ii** exactly 1 even
- **iii** an odd on the second spin only

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10 In an archery contest, Amy, Justin and Matthew each fire an arrow at the same target. The probability that they hit the target is $\frac{2}{3}$ for Amy, $\frac{1}{2}$ for Justin and $\frac{1}{4}$ for Matthew. Use a tree diagram to find the probability that

a all 3 contestants hit the target
b only 1 contestant hits the target
c only Justin hits the target
d at least 1 contestant hits the target

11 A double-headed coin and a normal coin are tossed.

a Use a tree diagram to show all the possible outcomes.
b What is the probability of tossing two heads?

12 Sandra takes a plastic counter and writes 1 on one side and 2 on the other. Then she takes another counter and writes 1 on one side and 5 on the other. Sandra tosses both counters.

a Use a tree diagram to list all possible outcomes.
b Assuming that the outcomes are equally likely, find the probability that
  i both numbers are odd
  ii the sum of the two numbers is less than 6
  iii both numbers are the same

LINKS TO Example 15

13 The probability that a fair die lands on 6 is $\frac{1}{6}$. A second die is biased so that it has a greater chance of landing on 6. The probability that this biased die lands on 6 is $\frac{2}{11}$. Calculate the probability of obtaining

a two sixes
b a six on the fair die but not on the biased die.

exercise 10.2 challenge

14 In 2011, Englishman Steve Whiteley backed six winning horses at a race meeting. The odds for the six horses were 2-1, 12-1, 16-1, 16-1, 5-1 and 12-1.(Odds of 2-1 mean a chance of 1 in 3 of winning). What was the probability of Steve backing all six winners?

15 In the 17th century the French mathematicians Pascal and Fermat became involved in explaining the change in fortune of a French gambler, Antoine Gombaud (known as the Chevalier de Méré). The Chevalier had been consistently winning when he bet on at least one six in four rolls of a die. He predicted that his luck would continue if he bet on at least one double six in 24 rolls. However, he found to his surprise that he regularly lost.

a Calculate the probability of getting at least one six in four rolls of a die and hence explain why the Chevalier de Méré consistently won by repeating this strategy.
b Calculate the probability of getting at least one double six in 24 rolls of two dice and hence explain why the Chevalier de Méré consistently lost when he repeated this strategy.

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10.3 Sampling with or without replacement

Events that involve selecting an object from a collection are referred to as **sampling**. In a two-step event that involves sampling, the outcomes from the second step depend on whether the object selected at the first step is replaced. For example, if a queen is drawn from a standard pack of cards and replaced, the probability of drawing a queen the second time is still $\frac{4}{52} = \frac{1}{13}$. The two events are independent. However, if the queen is not replaced, there are now only 51 cards left, three of which are queens. The probability of drawing a queen on the second draw is now $\frac{3}{51} = \frac{1}{17}$ so the two events are not independent.

Example 16

Bianca draws a five of diamonds from a standard pack of cards and does not replace it. Tamsin then draws a card from the same pack. Find the probability that Tamsin draws

a a five

b a diamond

Working

a $\Pr(5) = \frac{3}{51}$

$= \frac{1}{17}$

b $\Pr(\text{diamond}) = \frac{12}{51}$

$= \frac{4}{17}$

Reasoning

One of the four fives has been removed. There are only 51 cards left, three of which are fives.

One of the 13 diamonds has been removed. There are only 51 cards left, 12 of which are diamonds.

Comparing sampling with and without replacement

Consider a bag containing 3 red counters and 2 blue counters. If one counter is selected at random, the probability of drawing a red counter is $\frac{3}{5}$. The selected counter is then returned to the bag and a second counter is selected. Because the first counter was replaced, the probability of drawing a red counter the second time is still $\frac{3}{5}$.

In the array below, the 3 red counters are numbered 1, 2 and 3 and the 2 blue counters are numbered 1 and 2. We can see that there is a total of 25 possible outcomes, 9 of which are 2 red counters.

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$\Pr(\text{two red counters}) = \frac{9}{25}$

Notice that $\frac{9}{25} = \frac{3}{5} \times \frac{3}{5}$ where $\frac{3}{5}$ is the probability of drawing a red counter the first time and the probability of drawing a red counter the second time.

1st selection / 2nd selection	1	2	3	1	2
1	1 1	2 1	3 1	1 1	2 1
2	1 2	2 2	3 2	1 2	2 2
3	1 3	2 3	3 3	1 3	2 3
1	1 1	2 1	3 1	1 1	2 1
2	1 2	2 2	3 2	1 2	2 2

As the first counter drawn was returned to the bag, the outcome of the first draw does not affect the outcome of the second draw. The two events, 'drawing a red counter the first time' and 'drawing a red counter the second time' are independent.

Now consider the same bag of 3 red counters and 2 blue counters but this time the counter that is drawn the first time is not replaced. The outcomes (red counter 1, red counter 1), (red counter 2, red counter 2) and so on are therefore not possible. There is now a total of only 20 outcomes, 6 of which are 2 red counters.

1st selection / 2nd selection	1	2	3	1	2
1		2 1	3 1	1 1	2 1
2	1 2		3 2	1 2	2 2
3	1 3	2 3		1 3	2 3
1	1 1	2 1	3 1		2 1
2	1 2	2 2	3 2	1 2	

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$$\Pr(2 \text{ red counters}) = \frac{6}{20} = \frac{3}{10}$$

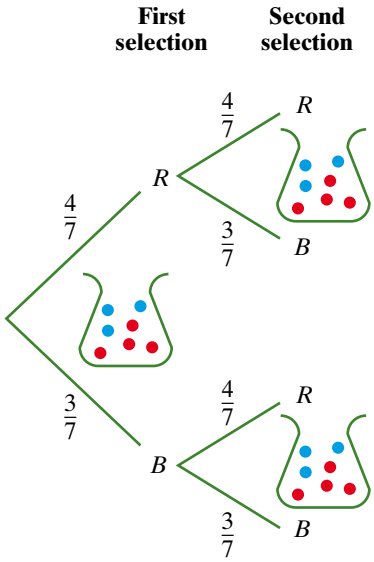
The probability of the second counter being red is actually an example of conditional probability. What we are really saying is: If the first counter is red, what is the probability of the second counter being red?

If the first counter is red, there are only 2 red counters left out of a total of 4. Hence, given that the first counter is red, then the probability of the second counter being red is $\frac{2}{4}$, that is, $\frac{1}{2}$.

$$\begin{aligned}\Pr(2 \text{ red counters}) &= \Pr(1\text{st counter red}) \times \Pr(2\text{nd counter red} | 1\text{st counter red}) \\ &= \frac{3}{5} \times \frac{1}{2} \\ &= \frac{3}{10}\end{aligned}$$

Using tree diagrams for sampling with and without replacement

Tree diagrams are also useful when comparing sampling with and without replacement.



$$\Pr(RR) = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$$

$$\Pr(RB) = \frac{4}{7} \times \frac{3}{7} = \frac{12}{49}$$

$$\Pr(BR) = \frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$$

$$\Pr(BB) = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$$

Sampling with replacement

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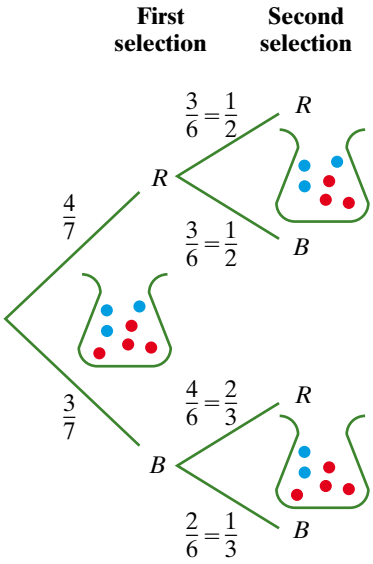


$$\Pr(RR) = \frac{4}{7} \times \frac{1}{2} = \frac{2}{7}$$

$$\Pr(RB) = \frac{4}{7} \times \frac{1}{2} = \frac{2}{7}$$

$$\Pr(BR) = \frac{3}{7} \times \frac{2}{3} = \frac{2}{7}$$

$$\Pr(BB) = \frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$$

Sampling without replacement

Example 17

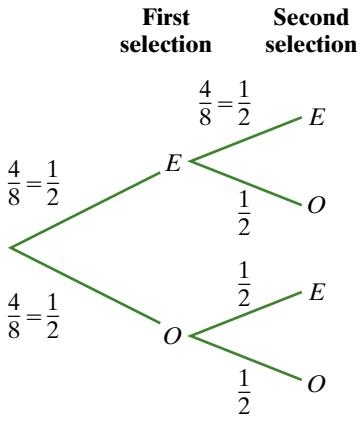
A cloth bag contains eight cards numbered from 1 to 8. Carlos draws a card from the bag, records the number then replaces the card in the bag. He then draws a second card from the bag.

a Draw a tree diagram to show the probabilities for each selection.

b Are the two events, drawing the first card and drawing the second card, independent?

c Calculate the probability of drawing two even-numbered cards.

Working



Reasoning

There are four even numbers and four odd numbers in the set of numbers 1 to 8.
If the first card is replaced there are still four even numbered cards and four odd numbered cards in the bag.

continued

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Example 17 continued

Working

b The two events 'drawing an even number the first time'and 'drawing an even number the second time' are independent.

c Pr(both numbers even)

$= \Pr(\text{first card even}) \times \Pr(\text{second card even})$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

Reasoning

The outcome of the first draw does not influence the outcome of the second draw as the card was replaced.

For independent events $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.
A is the event 'first card even'
B is the event 'second card even'

Example **18** uses the same context as example **17** but this time Carlos does not replace the card he draws.

Example 18

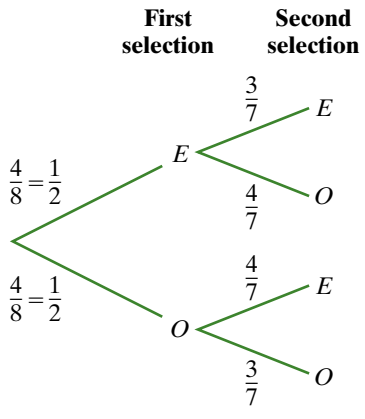
A cloth bag contains eight cards numbered from 1 to 8. Carlos draws a card from the bag without replacing it. He then draws a second card from the bag.

a Draw a tree diagram to show the probabilities for each selection.

b Are the two events, drawing the first card and drawing the second card, independent?

c Calculate the probability of drawing two even-numbered cards.

Working



Reasoning

There are four even numbers and four odd numbers in the set of numbers 1 to 8.
If the first card is not replaced there are only seven cards left in the bag.
If the first card has an even number, then there are three even numbers and four odd numbers left.
If the first card has an odd number, then there are four even numbers and three odd numbers left.

b The two events 'drawing an even number the first time' and 'drawing an even number the second time' are not independent.

The outcome of the first selection influences the outcome of the second selection as the card was not replaced.

continued

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Example 18 continued

Working

c Pr(both numbers even)

$= \Pr(\text{first card even})$

$\times \Pr(\text{second card even} | \text{first card even})$

$= \frac{1}{2} \times \frac{3}{7}$

$= \frac{3}{14}$

Reasoning

For events that are not independent,

$\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B|A)$

A is the event 'the first card is even'.

B is the event 'the second card is even given the first card is even'.

Example 19

A bag contains 6 blue balls and 4 white balls. One ball is selected from the bag and its colour noted. This ball is not replaced into the bag. A second ball is then selected from the bag and its colour noted.

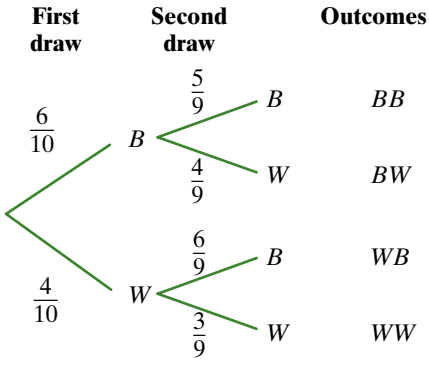
a Construct a tree diagram to show the possible outcomes of this experiment, labelling the probabilities of the events on the branches.

b Calculate the following probabilities

i both balls are blue **ii** exactly one ball is blue **iii** neither ball is blue

Working

a Let B represent drawing a blue ball. Let W represent drawing a white ball.



Reasoning

It is easier to leave the probabilities for the first draw as $\frac{6}{10}$ and $\frac{4}{10}$ rather than simplifying the fractions. This makes it easier to see the probabilities for the second draw when the ball is not replaced.

b i $\Pr(BB) = \frac{6}{10} \times \frac{5}{9}$

$= \frac{1}{3}$

The probability that both balls are blue is $\frac{1}{3}$.

Multiply the probabilities along the branch BB of the tree diagram.

continued

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Example 19 continued

Working

ii $\Pr[(BW) \text{ or } (WB)] = \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9}$

$= \frac{24}{90} + \frac{24}{90}$

$= \frac{48}{90}$

$= \frac{8}{15}$

The probability that exactly one ball is blue is $\frac{8}{15}$.

Reasoning

'Exactly one ball is blue' means 'the first ball is blue and the second white' or 'the first is white and the second is blue'.We add the two probabilities.

iii $\Pr(WW) = \frac{4}{10} \times \frac{3}{9}$

$= \frac{2}{15}$

The probability that neither ball is blue is $\frac{2}{15}$.

If neither ball is blue, both balls must be white.

exercise 10.3

LINKS TO Example 16

1 Lily selects a card at random from a standard pack of cards without replacing it. The card is the king of spades. Danny now draws a second card from the pack. Find the probability that Danny selects

a a spade

b a king

c a diamond

d the queen of spades

Clubs	Spades	Diamonds	Hearts
♣K	♠K	♦K	♥K
♣Q	♠Q	♦Q	♥Q
♣J	♠J	♦J	♥J
♣10	♠10	♦10	♥10
♣9	♠9	♦9	♥9
♣8	♠8	♦8	♥8
♣7	♠7	♦7	♥7
♣6	♠6	♦6	♥6
♣5	♠5	♦5	♥5
♣4	♠4	♦4	♥4
♣3	♠3	♦3	♥3
♣2	♠2	♦2	♥2
♣A	♠A	♦A	♥A

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2 A cloth bag contains 12 marbles. Three of these are white, four are purple and the remaining marbles are orange.

a If Angie selects a purple marble then replaces it, what is the probability that she selects a purple marble a second time?

b If Angie selects a white marble then replaces it, what is the probability that she selects an orange marble a second time?

c If Angie selects a purple marble but does not replace it, what is the probability she selects a white marble for her second selection?

LINKS TO Example 17

3 A bag contains three marbles: one red, one blue and one yellow. One marble is drawn from the bag, its colour noted then it is replaced. A second marble is then drawn from the bag and its colour is noted.

a Use a tree diagram to list all the possible outcomes of this experiment.

b What is the total number of outcomes?

c What is the probability that both marbles drawn from the bag are the same colour?

LINKS TO Example 18

4 The experiment in question **4** is repeated but this time the first marble is not replaced before drawing the second marble. Answer parts **a** and **b** for this new experiment.

5 A card is selected at random from a deck of 52 playing cards. Its suit is noted and then it is replaced in the pack before a second card is drawn. Use a suitable tree diagram to determine the probability of getting exactly one heart.

LINKS TO Example 19

6 Sofia is at the movies and has a bag containing citrus-flavoured jellies: 8 orange, 10 green and 7 yellow. Sofia selects one jelly from the bag and eats it. She then selects a second jelly and eats it.

Use a suitable tree diagram to determine the probability that

a both jellies are green.

b exactly one of the jellies is green.

c the second jelly Sofia eats is green

7 A mixed netball team consists of 4 girls and 3 boys. To select a captain and vice captain, all the players' names are put into a hat. The first name drawn at random from the hat determines the captain. The next name selected determines the vice captain. Use a tree diagram to find the probability that

a both the captain and the vice captain are girls

b the captain is a girl and the vice captain is a boy

8 A bag contains 4 blue marbles and 2 green marbles. Three marbles are selected from the bag, one at a time, without being replaced in the bag.

a Draw a tree diagram showing all possible outcomes.

b Find the probability of selecting

i 3 blue marbles

ii 2 green marbles

iii exactly 1 green marble

iv a green marble on the second selection only

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exercise 10.3 challenge

9 Amanda invites five friends to a birthday party. She assigns each of them a number from 1 to 5 and places the same numbers in a hat. Amanda then randomly selects two numbers from the hat in order to award two lucky prizes to her friends.

a If Amanda selects one number from the hat and then returns it to the hat before selecting the second number, construct a lattice diagram to show all the possible outcomes.

b What is the probability that the same person receives both prizes?

c Amanda doesn't want one of her friends to receive both prizes, so she decides not to return the first number to the hat before drawing the second number. Construct a new lattice diagram to show all the possible outcomes.

d What is the probability that person number 4 receives a prize?

e What is the probability that person number 1 does not win a prize?

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10.4 Conditional probability

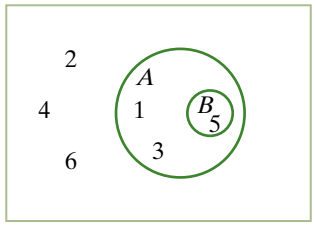
Our everyday language often includes conditional statements, for example, 'I'll come given I'm home early enough', 'We will play if it's not raining', 'I'll go on the condition you go'. Probabilities are also often based on conditional statements. For example, "What is the probability a student studies history given that they study art".

Suppose a die is thrown and the outcome is an odd number. What is the probability that it is a 5? So we are asking, if we know the number is 1, 3 or 5 what is the probability that it is 5? The Venn diagram below can be used to illustrate the situation.

In the Venn diagram, A is the event throwing 1, 3 or 5 and B is the event throwing a 5. Instead of considering the whole Venn diagram, we are restricting the sample space to considering only those outcomes that belong to events A and B. Of the three outcomes that we know are possible (1, 3 or 5), the probability that the number is 5 is $\frac{1}{3}$.

$$\Pr(5 \text{ given that odd}) = \frac{1}{3}$$

This can also be written $\Pr(5|\text{odd}) = \frac{1}{3}$.



Conditional probability
Conditional probability involves events that are not independent.
The probability that event B occurs, given that we know event A has occurred, is written as $\Pr(B|A)$.

Example 20

a Express the following in words

i $\Pr(Y|X)$ **ii** $\Pr(X|Y)$

b Express in symbols

i The probability of a card drawn from a standard pack being a queen (event Q) given that we know it is red (event R).

ii The probability of a card drawn from a standard pack being red (event R) given that we know it is a queen (event Q).

continued

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Example 20 continued

Working

a i $\Pr(Y|X)$ means the probability of event Y occurring given that we know event X has occurred.

ii $\Pr(X|Y)$ means the probability of event X occurring given that we know event Y has occurred.

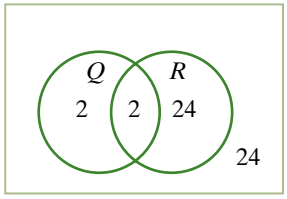
b i $\Pr(Q|R)$

ii $\Pr(R|Q)$

Reasoning

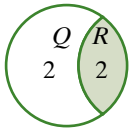
$\Pr(B|A)$ means the probability that event B occurs, given that we know event A has occurred.

It is important to note that $\Pr(B|A)$ and $\Pr(A|B)$ are not the same. Consider again the probabilities in example **20** part **b**. There are 4 queens and 26 red cards. Two cards are 'red and a queen'.



$n = 52$

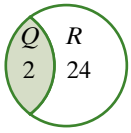
We restrict the sample space to look at the part of the Venn diagram showing all the queens.



There are 4 queens, 2 of which are red.

$$\Pr(R|Q) = \frac{2}{4} = \frac{1}{2}$$

We restrict the sample space to look at the part of the Venn diagram showing all the red cards.



There are 26 red cards, 2 of which are queens.

$$\Pr(Q|R) = \frac{2}{26} = \frac{1}{13}$$

Example 21

The two-way table shows the numbers of students who have chosen to do swimming or a yoga or both. A student is selected at random.

a If the student does yoga, what is the probability that they also do swimming?

b If the student does swimming, what is the probability that they do not do yoga?

	Y	***Y′***	**Total**
S	25	17	42
S′	8	0	8
Total	33	17	50

continued

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Example 21 continued

Working

a $\Pr(S|Y) = \frac{25}{33}$

Reasoning

Restrict the sample space to those students who do yoga:

	Y
S	25
S'	8
Total	33

33 students do yoga. Of these, 25 also do swimming.

b $\Pr(Y'|S) = \frac{17}{42}$

Restrict the sample space to those students who do swimming:

	Y	Y'	**Total**
S	25	17	42

42 students do swimming. Of these 17 do not do yoga.

A rule for calculating conditional probabilities

In the calculations shown above, we notice that $\Pr(S|Y) = \frac{25}{33}$

However, we also know that $\Pr(Y) = \frac{33}{50}$ and $\Pr(Y \text{ and } S) = \frac{25}{50}$. If we divide $\Pr(Y \text{ and } S)$ by $\Pr(Y)$, we obtain

$$\begin{aligned}\frac{\Pr(Y \text{ and } S)}{\Pr(Y)} &= \frac{25}{50} \div \frac{33}{50}\\ &= \frac{25}{50} \times \frac{50}{33}\\ &= \frac{25}{33}\end{aligned}$$

$$\text{So } \frac{\Pr(Y \text{ and } S)}{\Pr(Y)} = \Pr(S|Y)$$

In general terms, for two events A and B, the probability that B occurs, given that we know A occurs is given by:

$$\Pr(B|A) = \frac{\Pr(A \text{ and } B)}{\Pr(A)}$$

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Conditional probability calculation

For two events A and B that are not independent, the probability of event B occurring given that we know event A has occurred is given by

$$\Pr(B|A) = \frac{\Pr(A \text{ and } B)}{\Pr(A)}$$

Example 22

Use the informaton in this probability table to calculate

i $\Pr(B|A)$ **ii** $\Pr(A|B)$

	A	A'	Total
B	0.22	0.16	0.38
B'	0.43	0.19	0.62
Total	0.65	0.35	1

Working

a $\Pr(B|A) = \dfrac{\Pr(A \text{ and } B)}{\Pr(A)}$

$= \dfrac{0.22}{0.65}$

≈ 0.39

Reasoning

Restrict the sample space to the column of the table for event A.

	A
B	0.22
B'	0.43
Total	0.65

b $\Pr(A|B) = \dfrac{\Pr(A \text{ and } B)}{\Pr(B)}$

$= \dfrac{0.22}{0.38}$

≈ 0.58

Restrict the sample space to the row of the table for event B.

	A	A'	Total
B	0.22	0.16	0.38

Example 23

If $\Pr(X) = 0.75$ and $\Pr(X \text{ and } Y) = 0.15$, find $\Pr(Y|X)$.

Working

$\Pr(Y|X) = \dfrac{\Pr(X \text{ and } Y)}{\Pr(X)}$

$= \dfrac{0.15}{0.75}$

$\Pr(Y|X) = 0.2$

Reasoning

For two events A and B, $\Pr(B|A) = \dfrac{\Pr(A \text{ and } B)}{\Pr(A)}$

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Example 24

In a study of people using a local library it was found that 70% of the library users were students and 42% were female students. If a library user was selected at random what is the probability that the person is a student given that we know they are female?

Working

Let S be the event 'student'
Let F be the event 'female'
$\Pr(S) = 70\% = 0.7$
$\Pr(S \text{ and } F) = 42\% = 0.42$

$$\Pr(S|F) = \frac{\Pr(S \text{ and } F)}{\Pr(F)} = \frac{0.42}{0.7} = 0.6$$

The probability that the person is a student given that we know they are female is 0.6.

Reasoning

Use symbols for the events 'student' and 'female'.
Express the percentages as decimal probabilities.
Use the conditional probability rule:

for two events A and B, $\Pr(B|A) = \dfrac{\Pr(A \text{ and } B)}{\Pr(A)}$

exercise 10.4

LINKS TO Example 20a

1 Express the following in words

a $\Pr(M|P)$ **b** $\Pr(P|M)$

LINKS TO Example 20b

2 Express in symbols

a The probability of a card drawn from a standard pack being a heart (event H) given that we know it is red (event R).

b The probability of a card drawn from a standard pack being red (event R) given that we know it is a heart (event H).

c The probability of a number on a die being 2 (event T) given that we know it is prime (event P).

d The probability that the bus will be late (event L) given that it is Monday morning (event M).

LINKS TO Example 21

3 The following two-way table shows the number of Year 9 students in a school who study art (A) or music (M).

a If the student does music, what is the probability that they also do art?

b If the student does art, what is the probability that they do not do music?

	M	M'	Total
A	24	36	60
A'	16	4	20
Total	40	40	80

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LINKS TO Example 22

4 Two football teams, the Mudlarks and the Galahs meet in the grand final. Each team performs better according to the weather. The following probability table shows the probabilities that the Mudlarks win (M) or the Galahs win (M') if the weather is wet (W) or dry (W').

	M	M'	Total
W	0.5	0.2	0.7
W'	0.1	0.2	0.3
Total	0.6	0.4	1

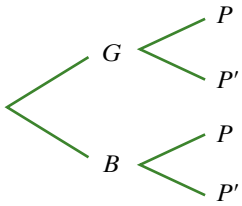
Calculate the following probabilities correct to two decimal places.

a the probability that the Mudlarks win given that the day is wet

b the probability that the Mudlarks win given that the day is dry

c the probability that the Galahs win given that the day is dry

5 Of the 16 girls in a Year 10 class, 10 play a musical instrument. Of the 14 boys in the class, 8 play a musical instrument. In the tree diagram G represents 'girl', B represents 'boy', P represents 'plays an instrument' and P' represents 'does not play an instrument'.



a Copy and complete the tree diagram, showing the probabilities on each branch.

b One student is selected at random from the class. What is the probability that the selected student

i is a girl who plays an instrument?

ii does not play an instrument?

c Given that the selected student is a girl, what is the probability that she plays an instrument?

d Given that the student plays an instrument, what is the probability that the student is a boy?

LINKS TO Example 23

6 At Green Plains High School, the probability that a Year 10 student studies both History and Art is 0.36. The probability that a student studies Art is 0.45. What is the probability that a student studies History given that the student is studying Art?

7 A bag contains red and blue marbles. Two marbles are chosen without replacement. The probability of selecting a red marble and then a blue marble is 0.42, and the probability of selecting a red marble the first time is 0.6. What is the probability of selecting a blue marble the second time, given that the first marble was red?

LINKS TO Example 24

8 At a primary school 15% of students have a skateboard and 3% have a skateboard and roller blades. What is the probability that a student selected at random has roller blades given that the student also has a skateboard?

9 If 20% of students at a school play soccer and basketball and 32% of the students play soccer, what is the probability that a randomly selected student plays basketball given that the student plays soccer?

10 Year 10 had two maths tests in September. If 80% of the class passed the first test and 75% of the class passed both tests, what percentage of students who passed the first test also passed the second test?

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exercise 10.4 challenge

11 Abdul is a contestant in a game show. One of three cups has a coin underneath. If Abdul selects the correct cup the first time, he wins a prize. If he does not select the correct cup, he has another try and if he is correct this time he wins a booby prize (the coin is not moved between attempts).

a What is the probability that Abdul selects the correct cup the first time?

b Assuming Abdul selects an incorrect cup, what is the probability that he selects the correct cup the second time.

c Using C for the event 'selecting the correct cup' and C' for the event 'selecting an incorrect cup' construct a tree diagram showing the possible outcomes after Abdul's two tries.

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Analysis task

Pascal's triangle and probabilities

BLM Pascal's triangle

Pascal's triangle is named after the French mathematician Blaise Pascal. Several rows of Pascal's triangle are shown here.

Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

To create Pascal's triangle, write the number 1 at the tip (apex) of the triangle. This is row 0. To generate each new row in Pascal's triangle, write the number 1 at either end of the row. All numbers in between are found by adding the two numbers diagonally above.

a Add the next three rows to Pascal's triangle.

b **i** Draw a tree diagram to determine all possible outcomes when a fair coin is tossed twice.

ii Use your diagram to fill in the table at the right.

Number of heads	Number of outcomes
0	
1	
2	
Total	

iii Explain how each of the numbers in the right-hand column relates to row 2 of Pascal's triangle.

c **i** Draw a tree diagram to determine all possible outcomes when a fair coin is tossed three times.

ii Use your diagram to fill in the table at the right.

iii Explain how each of the numbers in the right-hand column relates to row 3 of Pascal's triangle.

Number of heads	Number of outcomes
0	
1	
2	
3	
Total	

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d Use your results from parts **b** and **c** to complete the following tables without drawing tree diagrams.

i A fair coin is tossed 4 times.

Number of heads	Number of outcomes
0	
1	
2	
3	
4	
Total	

ii A fair coin is tossed 6 times.

Number of heads	Number of outcomes
0	
1	
2	
3	
4	
5	
6	
Total	

iii A fair coin is tossed 10 times.

Number of heads	Number of outcomes
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

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e If a fair coin is tossed twice, then each of the outcomes is equally likely, with probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

If the coin is tossed three times, then each of the outcomes is equally likely, with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Copy and complete the table at the right.

Number of tosses	Probability of each outcome
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	
5	
6	

f Use your results from parts **d** and **e** to answer the following questions.

i If a fair coin is tossed 4 times, what is the probability of getting 3 heads?

ii If a fair coin is tossed 6 times, what is the probability of getting 2 heads?

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Review Probability

Summary

- We associate addition of probabilities with the word *or*.
- For mutually exclusive events, the probability of A or B occurring is given by $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$.
- For events that are not mutually exclusive, the probability of A or B occurring is given by $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$
- Two events are said to be independent if the outcome of one event has no effect on the outcome of the other event.
- We associate multiplication of probabilities with the word *and*.
- If two events A and B are independent, then the probability of both A and B occurring is given by $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.
- In a tree diagram
  - the branches show the possible outcomes at each stage of an experiment
  - each branch represents one possible outcome
  - to calculate the probability of an outcome, multiply the probabilities from the branches along the corresponding path.
- $\Pr(B|A)$ means the probability of B occurring given that we know A has occurred. This is called conditional probability.

$$\Pr(B|A) = \frac{\Pr(A \text{ and } B)}{\Pr(A)}$$

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

array diagram	intersection	two-way table
complement	mutually exclusive events	tree diagram
conditional probability	outcome	trial
equally likely	probability	two-step experiment
experiment	probability table	Venn diagram
event	sample space	
independent events	simple event	

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Chapter revision

Multiple-choice questions

1 A four-sided die has the numbers 1, 3, 5 and 6 on its faces, and a spinner is divided into three equal sections labelled with the numbers 2, 4 and 7. For one roll of the die and one spin of the spinner, the probability of obtaining an even total is

A $\frac{3}{7}$ **B** $\frac{1}{6}$ **C** $\frac{5}{12}$ **D** $\frac{5}{7}$ **E** $\frac{1}{4}$

2 Beth and Joan play tennis each week. The probability of Beth winning any particular game is 0.6. If they play two games, the probability that Joan wins both is

A 0.64 **B** 0.6 **C** 0.4 **D** 0.36 **E** 0.16

3 If $\Pr(A) = 0.75$, $\Pr(B) = 0.36$, and $\Pr(A \text{ and } B) = 0.24$, then $\Pr(A \text{ and } B') =$

A 0.12 **B** 0.13 **C** 0.36 **D** 0.51 **E** 0.64

4 A box contains 5 red balls and 3 yellow balls. Bianca draws a ball at random from the bag and notes its colour. She then draws a second ball at random, without replacing the first. The probability that both balls are the same colour is

A $\frac{5}{7}$ **B** $\frac{5}{14}$ **C** $\frac{3}{28}$ **D** $\frac{13}{28}$ **E** $\frac{17}{32}$

5 If X and Y are mutually exclusive events, which of the following statements are always true?

i $\Pr(X \text{ or } Y) = \Pr(X) + \Pr(Y)$

ii $\Pr(X \text{ and } Y) = \Pr(X) \times \Pr(Y)$

iii $0 \leqslant \Pr(Y) \leqslant 1$

A i only **B** i and ii only **C** i and iii only

D ii and iii only **E** i, ii and iii

Short-answer questions

6 A fair coin is tossed three times. Find the probability of tossing

a 3 tails **b** exactly 1 head **c** no more than 1 tail

7 A box contains cards numbered 1 to 4. A card is randomly selected from the box and its number is noted. Then the card is returned to the box before a second card is selected.

a List all the possible outcomes.

b Find the probability that

i the first number selected is 2

ii both numbers are the same

iii the sum of the two numbers is 5

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8 June picks a raffle ticket at random from a box. The probability that she picks a red ticket is 0.24 and the probability that she picks a yellow ticket is 0.58.

a Are the events 'picking a red ticket' and 'picking a yellow ticket' mutually exclusive? Why?

b Find the probability that June picks

i a red ticket or a yellow ticket

ii neither a red ticket nor a yellow ticket

9 A and B are events such that $\Pr(A') = 0.38$, $\Pr(B) = 0.72$ and $\Pr(A \text{ or } B) = 0.95$.

a Use a probability table to find $\Pr(A \text{ and } B)$.

b Calculate $\Pr(B|A)$ correct to two decimal places.

10 X and Y are independent events. If $\Pr(X) = 0.24$ and $\Pr(Y) = 0.42$, find $\Pr(X \text{ and } Y)$.

11 Two machines A and B produce plastic bottles. A quality controller tests bottles for faults. The number of faulty bottles produced by the machines is recorded in a table.

	Machine *A*	**Machine *B***	**Total**
Faulty	28	32	60
Not faulty	172	268	440
Total	200	300	500

One bottle is selected at random from those tested. Find the probability that it is

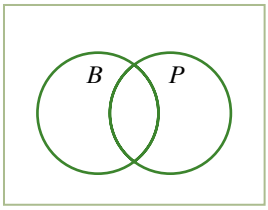
a faulty

b produced by machine B

c produced by machine A and not faulty

d produced by machine B and faulty

12 In a survey of 50 people, 28 people liked broccoli and 36 people liked peas. Six of the people surveyed liked neither vegetable.



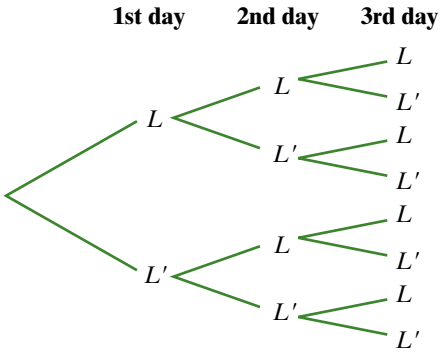
a Complete this Venn diagram to show this information.

b Use the Venn diagram to determine the probability that a person selected at random from the group

i likes broccoli **ii** likes both broccoli and peas

c What is the probability that a person likes peas given that they like broccoli?

13 The probability that Margot is late for work on any one day is 0.2. Margot works for three days in a particular week.



a Copy and complete this tree diagram.

b Use the tree diagram to determine the probability that Margot

i arrives at work on time on all three days.

ii is late to work on exactly one day.

iii is late to work on the third day only

iv is late on the second day, given that she was late on the first day.

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Extended-response questions

14 A lolly jar contains green, yellow and red lollies. The lollies are similar in size so we can assume that each lolly has an equal chance of being picked at random from the jar.

The probability that a randomly selected lolly from the jar is green is $\frac{1}{4}$ and the probability that it is yellow is $\frac{2}{3}$.

a Rosa randomly selects one lolly from the jar. Find the probability that the lolly is

i green or yellow **ii** red **iii** blue

b Now suppose that we know the jar contains 120 lollies.

i How many red lollies are in the jar?

ii $\frac{2}{5}$ of the red lollies are jelly babies and the rest are jelly beans. What is the probability of randomly selecting a red jelly bean from the jar?

15 A bag contains 3 red balls and 2 blue balls. Jim randomly selects one ball from the bag and notes its colour.

a Find the probability that the ball is

i red

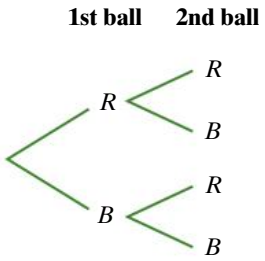
ii blue

b Are the events 'selecting a red ball' and 'selecting a blue ball' equally likely?

Jim replaces the first ball he selected. He then randomly selects a second ball from the bag and notes its colour.

c Are the events 'selecting a red ball first' and 'selecting a red ball second' independent?

d Copy and complete this tree diagram.



e What is the probability that both balls selected by Jim are blue?

f What is the probability that Jim selects one ball of each colour?

Jess performs the same experiment but she does not replace the first ball before selecting the second.

g Are the events 'selecting a red ball first' and 'selecting a red ball second' independent?

h Copy and complete another tree diagram for Jess's experiment.

i What is the probability that both balls selected by Jess are blue?

j What is the probability that Jess selects one ball of each colour?

Practice quiz
Chapter 10

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Quadratic functions 11

Pre-test

Warm-up

Objects that are thrown or projected fall under the influence of gravity and follow paths that have the shape of a parabola, that is, their paths can be represented by quadratic functions. Why do some of these water jets reach higher than others? Why do some of the jets reach further horizontally than others? How are these differences in the shapes of the parabolic paths reflected in their function rules?

11.1 Transforming the parabola

Quadratic functions

In Year 9, the graph of $y = x^2$ was introduced. The graph, called a **parabola**, is symmetrical about the y-axis and has its **turning point** (**vertex**) at (0, 0).

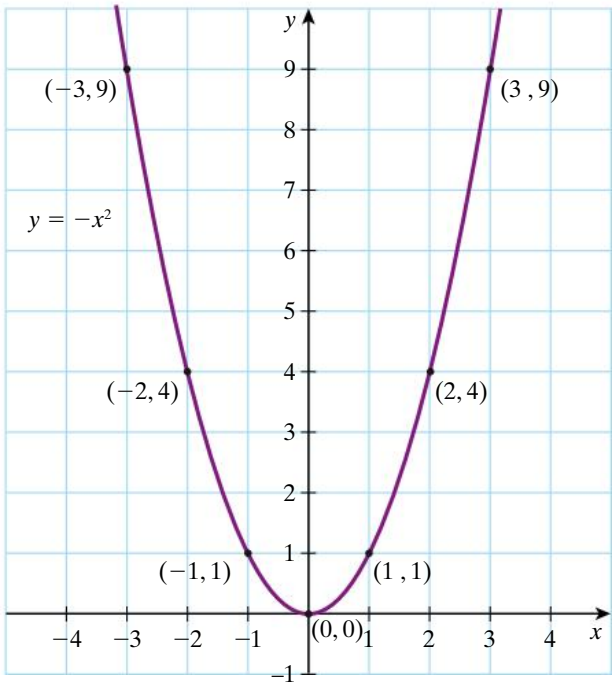
Consider the following table of values for $y = x^2$.

$y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

As the x-values increase from -3 to 0, the y-values decrease to a **minimum value** (0), then as the x-values continue to increase from 0 to 3, the y-values increase. The turning point shows the minimum value of y.

The axis of symmetry is the y-axis, that is, the line $x = 0$.



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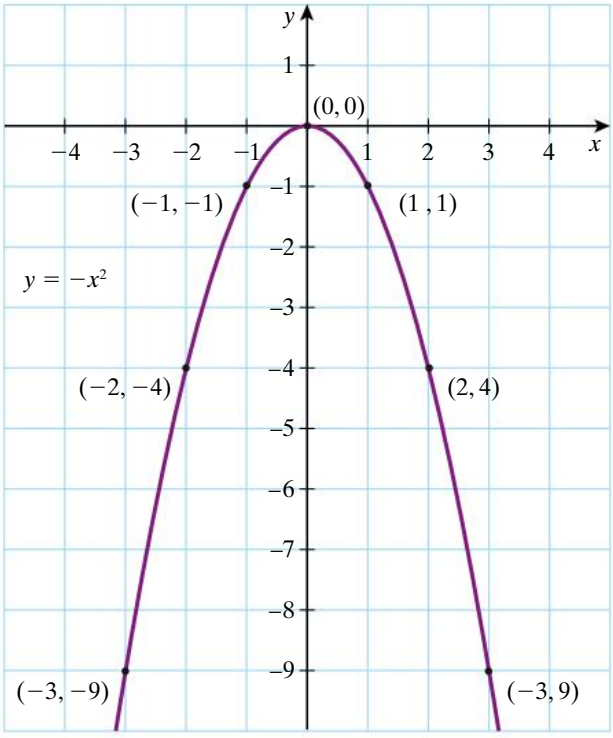
Reflecting the parabola in the *x*-axis

Parabola reflection

The graph of $y = -x^2$ is a reflection in the x-axis of the graph of $y = x^2$. For example, the point $(2, 4)$ on the graph of $y = x^2$ becomes the point $(2, -4)$ when reflected in the x-axis.

x	-3	-2	-1	0	1	2	3
y	-9	-4	-1	0	-1	-4	-9



Stretching the parabola

Parabola stretching

The effect of the coefficient of x^2 is the make the parabola 'thinner' or 'wider'. If the coefficient of x^2 is greater than 1, the parabola is stretched vertically. If the coefficient of x^2 is between 0 and 1, for example 0.5, the parabola is 'squashed' vertically, that is it becomes wider.

Example 1

For the quadratic equation $y = 2x^2$

a Complete a table of values for $-3 \le x \le 3$.

b Plot the points and join with a smooth curve.

c For comparison, show the graph of $y = x^2$ on the same set of axes.

continued

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Example 1 continued

Working

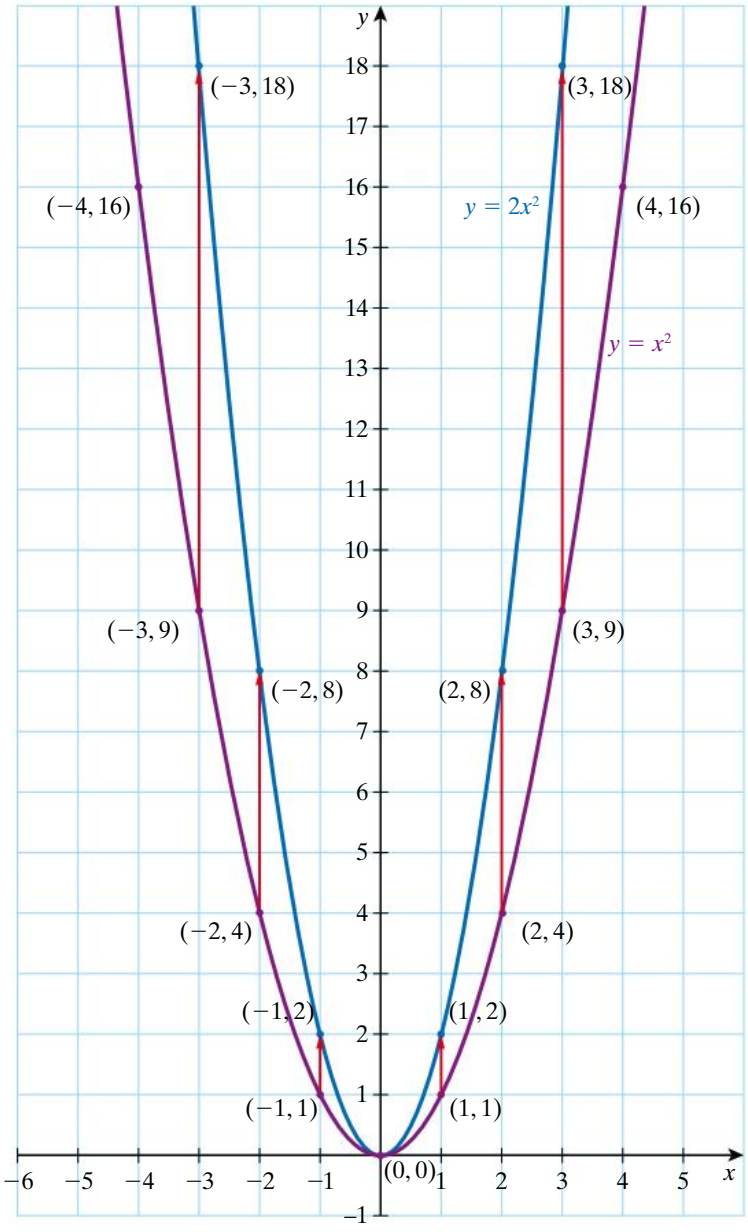
a $y = 2x^2$

x	-3	-2	-1	0	1	2	3
y	18	8	2	0	2	8	18

Reasoning

Substitute each x value into the equation $y = 2x^2$. The graph is the same shape as the graph of $y = x^2$, but the y values are increasing twice as fast so the graph appears thinner.

b–c



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Translating the parabola vertically

Parabola translation

Adding a constant term to x^2 results in a vertical translation of the parabola. Consider the graph of the quadratic function $y = x^2 + 2$.

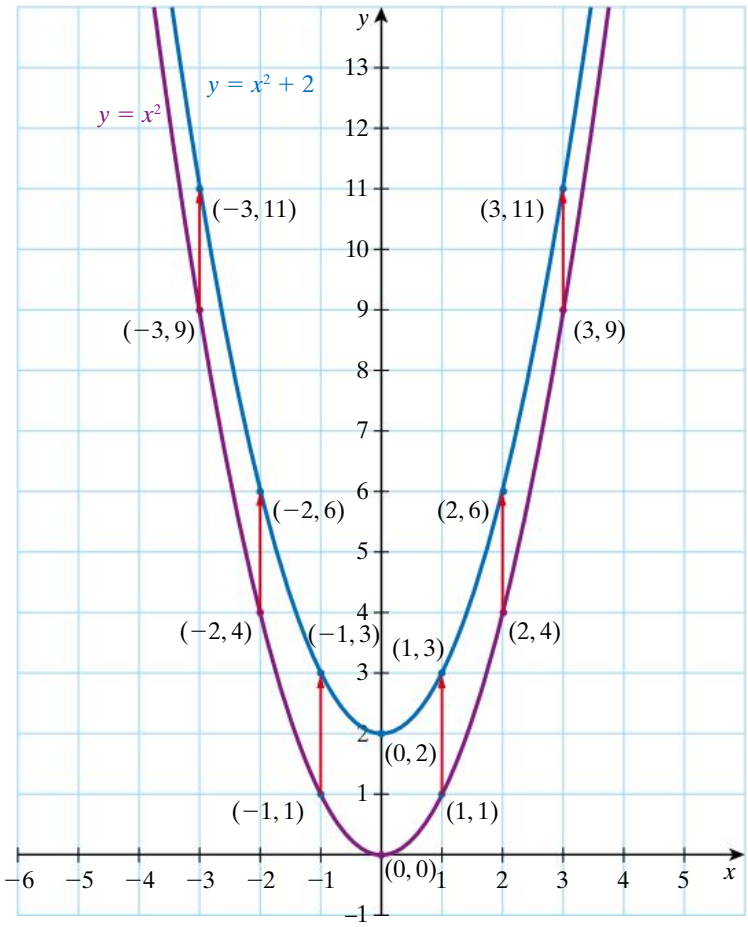
$y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

$y = x^2 + 2$

x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11

Comparing the graph of $y = x^2 + 2$ with the graph of $y = x^2$, we see that each point on the parabola $y = x^2$ is translated vertically upwards by 2 units. The graphs are the same shape but the turning point of $y = x^2 + 2$ is at (0, 2). The line $x = 0$ is still the axis of symmetry.



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Example 2

For the quadratic equation $y = x^2 - 1$

a Complete a table of values for $-3 \le x \le 3$.

b Plot the points and join with a smooth curve.

c For comparison, show the graph of $y = x^2$ on the same set of axes.

Working

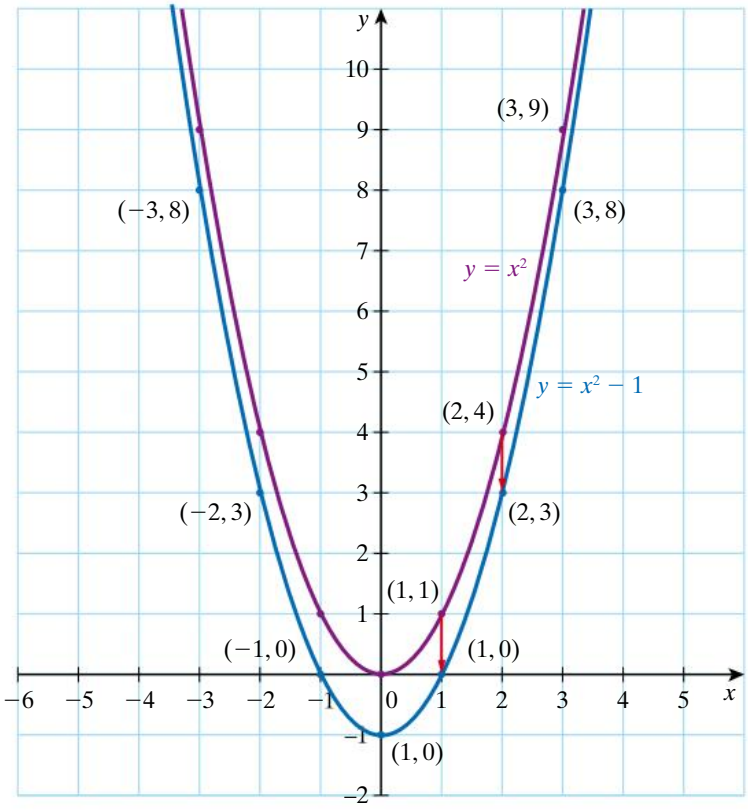
a $y = x^2 - 1$

x	-3	-2	-1	0	1	2	3
y	8	3	0	-1	0	3	8

Reasoning

Substitute each x value into the equation $y = x^2 - 1$. The graph is the same shape as the graph of $y = x^2$, but the y values are each one less so the graph is shifted down 1 unit.

b–c



When we sketch the graphs of the parabolas in example **3**, we show the position of the turning point and the points where $x = -1$ and $x = 1$. A sketch graph shows the labelled axes but does not need to be exactly to scale.

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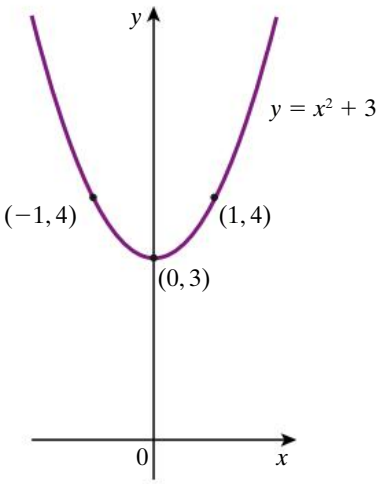
Example 3

Sketch the graphs of the following quadratic functions, showing the position of the turning point and the value of y when $x = -1$ and $x = 1$.

a $y = x^2 + 3$ **b** $y = x^2 - 2$ **c** $y = -x^2 + 1$

Working

a

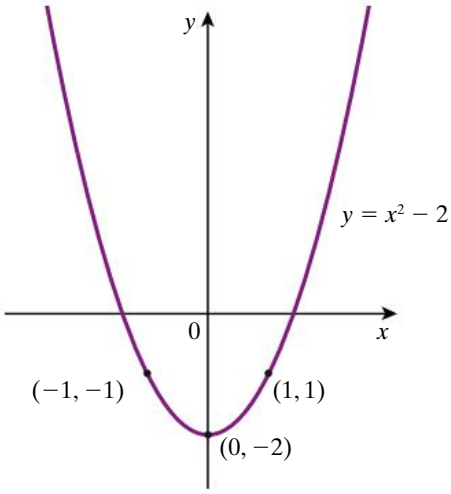


Reasoning

The graph of $y = x^2 + 3$ is the graph of $y = x^2$ translated up 3 units.
The turning point is at (0, 3).

x	−3	−2	−1	0	1	2	3
y	12	7	4	3	4	7	12

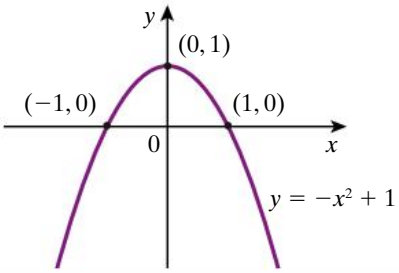
b



The graph of $y = x^2 - 2$ is the graph of $y = x^2$ translated down 2 units.
The turning point is at (0, −2).

x	−3	−2	−1	0	1	2	3
y	7	2	−1	−2	−1	2	7

c



The graph of $y = -x^2 + 1$ is the graph of $y = -x^2$ translated up 1 unit.
The turning point is at (0, 1).

x	−3	−2	−1	0	1	2	3
y	−8	−3	0	1	0	−3	−8

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Vertical translations

In general the graph of $y = x^2 + k$, where k is a constant, represents a vertical translation by k units of the graph of $y = x^2$. The turning point is the point $(0, k)$.

The graph of $y = -x^2 + k$, where k is a constant, represents a vertical translation by k units of the graph of $y = -x^2$. The turning point is the point $(0, k)$. The parabola is inverted ('upside down').

Translating the parabola horizontally

We now consider the equation $y = (x - 3)^2$ and compare the graph with the graph of $y = x^2$.

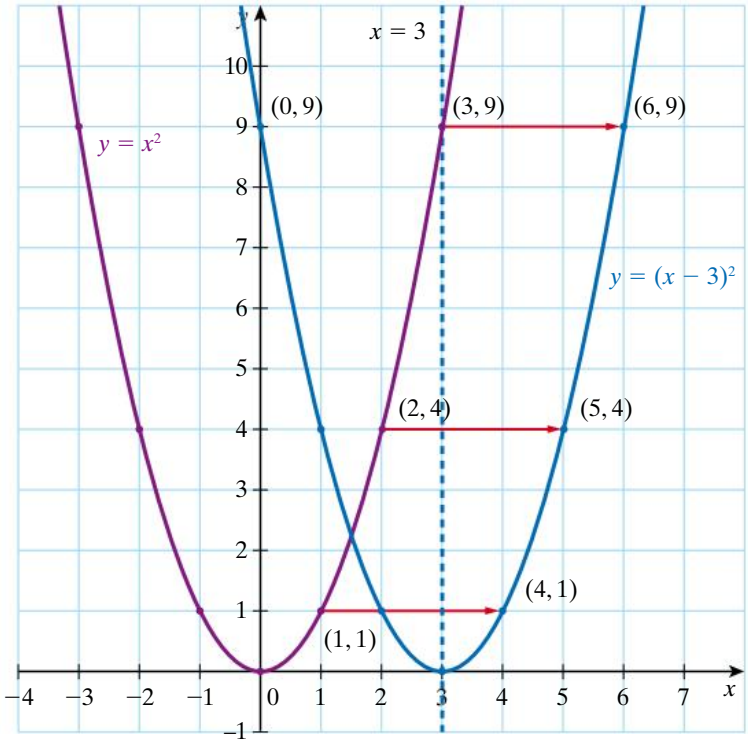
$y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

$y = (x - 3)^2$

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9

Notice how each point on the parabola $y = x^2$ has been translated 3 units to the right. The turning point is now at (3, 0) and the axis of symmetry is the line $x = 3$. The parabola crosses the y-axis at $y = 9$, that is, the y-intercept is at the point (0, 9).



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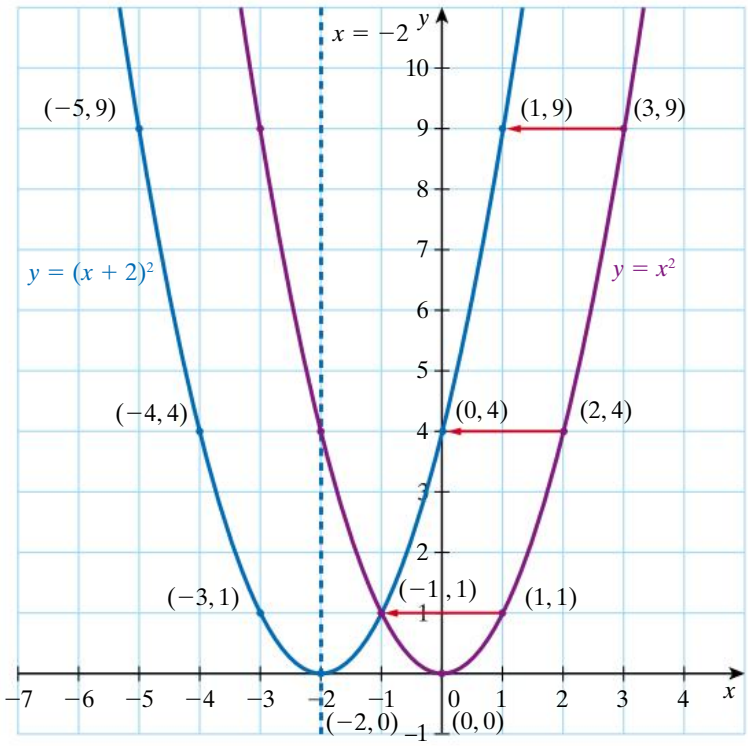
We now look at the graph of $y = (x + 2)^2$. Each point on the parabola $y = x^2$ has been translated 2 units to the left. The turning point is now at $(-2, 0)$ and the axis of symmetry is the line $x = -2$. The y-intercept is at the point $(0, 4)$.

$y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

$y = (x + 2)^2$

x	-5	-4	-3	-2	-1	0	1
y	9	4	1	0	1	4	9



Horizontal translations

In general, the graph of $y = (x - h)^2$ is the same shape as the graph of $y = x^2$ but the parabola is translated h units to the right. If h is negative, the parabola is translated to the left. The turning point is at $(h, 0)$ and the axis of symmetry is the line $x = h$.

Notice that parabolas that have been translated horizontally cross the y-axis at a point that is not the turning point. We can find the y-intercept by substituting $x = 0$ in the equation. The y-intercept should be labelled on the sketch graph of the parabola.

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Example 4

Sketch the graphs of the following quadratic functions, showing the coordinates of the y-intercept, the turning point and the equation of the axis of symmetry.

a $y = (x - 1)^2$ **b** $y = (x + 3)^2$ **c** $y = -(x + 3)^2$

Working | **Reasoning**

a $y = (x - 1)^2$

y-intercept
When $x = 0$

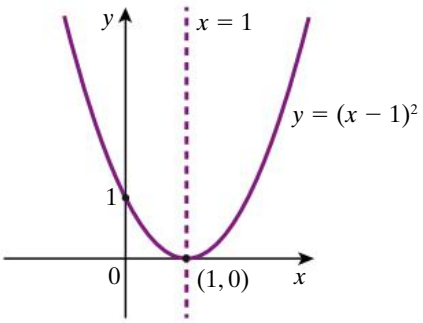
$y = (0 - 1)^2$
$y = 1$
The y-intercept is the point $(0, 1)$.

To find the y-intercept, put $x = 0$ in the equation.

The parabola $y = x^2$ is translated 1 unit to the right.

Turning point
$(1, 0)$

Axis of symmetry
$x = 1$



b $y = (x + 3)^2$

y-intercept
When $x = 0$

$y = (0 + 3)^2$
$= 9$

The y-intercept is the point $(0, 9)$.

To find the y-intercept, put $x = 0$ in the equation.

Turning point
$(-3, 0)$

Axis of symmetry
$x = -3$

The parabola $y = x^2$ is translated 3 units to the left. $y = (x - h)^2$ is the equation of a parabola that has been translated h units to the right. If the parabola is translated to the left, then $h = -3$ so $x - h$ becomes $x + 3$.

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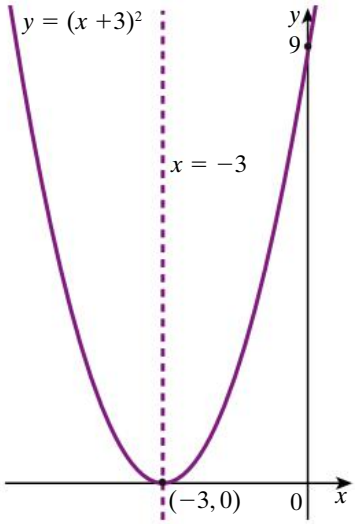
Example 4 continued

Working



c $y = -(x + 3)^2$

y-intercept

When $x = 0$

$$y = -(0 + 3)^2$$
$$= -9$$

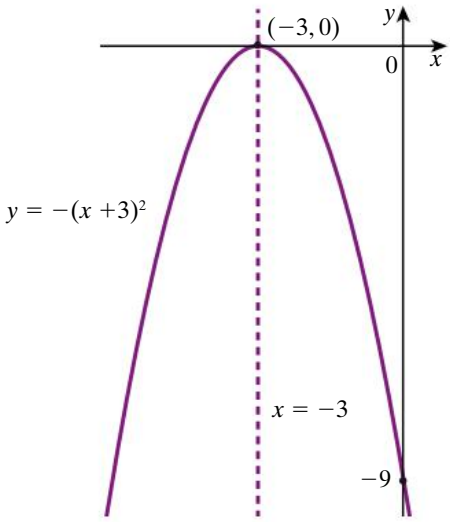
The y-intercept is the point $(0, -9)$.

Turning point

$(-3, 0)$

Axis of symmetry

$x = -3$



Reasoning

To find the y-intercept, put $x = 0$ in the equation.

The parabola $y = -x^2$ is translated 3 units to the left.

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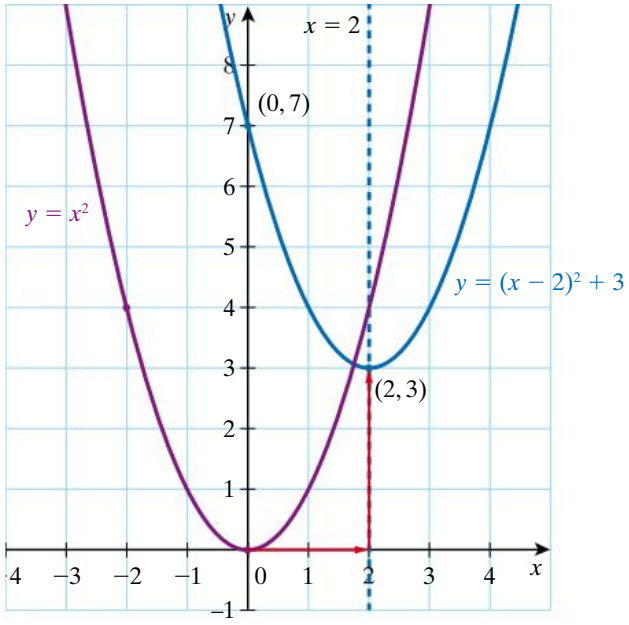
Combining vertical and horizontal translations

Consider the quadratic function $y = (x - 2)^2 + 3$. Comparing this with the functions $y = (x - 2)^2$ and $y = x^2 + 3$, we obtain the graph of $y = (x - 2)^2 + 3$ if the graph of $y = x^2$ is translated 2 units to the right and 3 units up.

The equation $y = (x - h)^2 + k$ is called the **turning point form** of a quadratic equation because we can easily tell the location of the turning point from the equation.



Turning point form

In general, the graph of $y = (x - h)^2 + k$ is the same shape as the graph of $y = x^2$ but the parabola is translated h units to the right and k units up. If h is negative, the parabola is translated to the left. If k is negative, the parabola is translated down. The turning point is at (h, k) and the axis of symmetry is the line $x = h$.

Example 5

State the coordinates of the turning point of each of the following parabolas.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 2)^2 - 7$ **c** $y = (x + 4)^2 + 1$

Working	Reasoning
a $y = (x - 3)^2 - 4$ Turning point: $(3, -4)$	$y = (x - h)^2 + k$ $h = 3, k = -4$
b $y = (x + 2)^2 - 7$ Turning point: $(-2, -7)$	$y = (x - h)^2 + k$ $h = -2, k = -7$
c $y = (x + 4)^2 + 1$ Turning point: $(-4, 1)$	$y = (x - h)^2 + k$ $h = -4, k = 1$

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Example 6

Calculate the y-intercept of each of the following parabolas.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 2)^2 - 7$ **c** $y = (x + 4)^2 + 1$

Working	Reasoning
a $y = (x - 3)^2 - 4$ When $x = 0$ $y = (0 - 3)^2 - 4$ $= 9 - 4$ $y = 5$ The y-intercept is at $(0, 5)$.	To find the y-intercept, substitute $x = 0$ in the equation.
b $y = (x + 2)^2 - 7$ When $x = 0$ $y = (0 + 2)^2 - 7$ $= 4 - 7$ $y = -3$ The y-intercept is at $(0, -3)$.	To find the y-intercept, substitute $x = 0$ in the equation.
c $y = (x + 4)^2 + 1$ When $x = 0$ $y = (0 + 4)^2 + 1$ $= 16 + 1$ $y = 17$ The y-intercept is at $(0, 17)$.	To find the y-intercept, substitute $x = 0$ in the equation.

Example 7

Sketch the graphs of the following quadratic functions, showing the coordinates of the turning point, the equation of the axis of symmetry and the y-intercept.

a $y = (x - 3)^2 + 1$ **b** $y = (x + 1)^2 - 2$ **c** $y = -(x + 3)^2 + 1$

Working	Reasoning
a $y = (x - 3)^2 + 1$	The parabola $y = x^2$ is translated 3 units to the right and 1 unit up.
Turning point $(3, 1)$	
Axis of symmetry $x = 3$	
***y*-intercept** When $x = 0$ $y = (0 - 3)^2 + 1$ $y = 10$ The y-intercept is at $(0, 10)$.	To find the y-intercept, substitute $x = 0$ in the equation.

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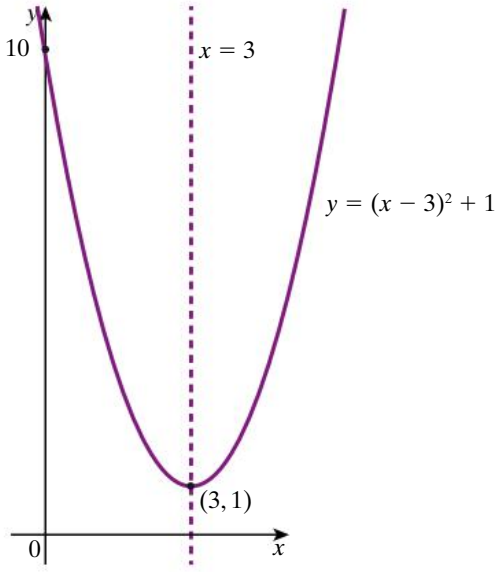
Example 7 continued

Working | **Reasoning**



b $y = (x + 1)^2 - 2$

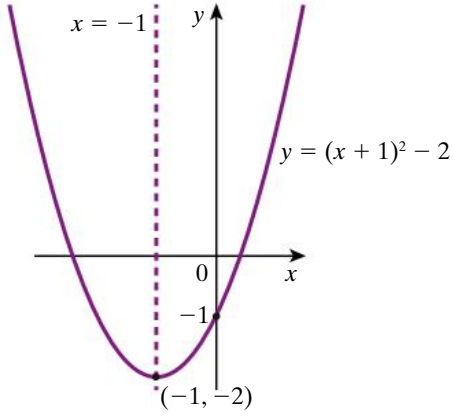
The parabola $y = x^2$ is translated 1 unit to the left and 2 units down.

Turning point
$(-1, -2)$

Axis of symmetry
$x = -1$

***y*-intercept**
When $x = 0$,
$y = (0 + 1)^2 - 2$
$y = -1$
The y-intercept is at $(0, -1)$.

To find the y-intercept, substitute $x = 0$ in the equation.



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Example 7 continued

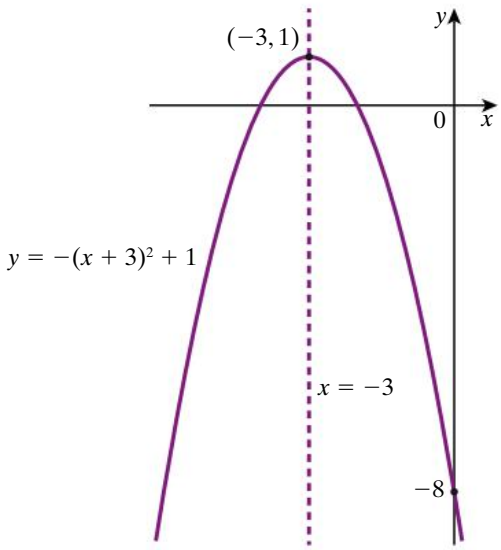
Working

c $y = -(x + 3)^2 + 1$

Turning point
$(-3, 1)$

Axis of symmetry
$x = -3$

***y*-intercept**
When $x = 0$,
$y = -(0 + 3)^2 + 1$
$y = -8$
The y-intercept is at $(0, -8)$.



Reasoning

The parabola $y = -x^2$ is translated 3 units to the left and 1 unit up.

To find the y-intercept, substitute $x = 0$ in the equation.

Example 8

The parabola $y = x^2$ is translated 3 units to the left and 4 units down. What is the equation?

Working

$y = (x + 3)^2 - 4$

Reasoning

$y = (x - h)^2 - k$ is the equation of a parabola that has been translated h units to the right and k units down. If the parabola is translated to the left, then $h = -3$ so $x - h$ becomes $x + 3$.

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exercise 11.1

LINKS TO Examples 1, 2

1 For each of these quadratic functions

i complete a table of values for $-3 \le x \le 3$

ii plot the points and join with a smooth curve

a $y = -2x^2$ **b** $y = 0.5x^2$ **c** $y = x^2 + 2$ **d** $y = (x + 1)^2$

2 For each of the following quadratic functions, state whether the parabola would be upright or inverted.

a $y = x^2 + 5$ **b** $y = -(x - 3)^2$ **c** $y = 8 - x^2$ **d** $y = (x - 4)^2 - 7$

LINKS TO Example 3

3 Sketch graphs of the following parabolas, labelling

i the turning point **ii** the axis of symmetry

a $y = x^2 + 1$ **b** $y = x^2 - 3$ **c** $y = x^2 + 6$ **d** $y = x^2 - 4$

e $y = -x^2 + 1$ **f** $y = -x^2 - 3$ **g** $y = -x^2 + 6$ **h** $y = -x^2 - 4$

LINKS TO Example 4

4 Sketch graphs of the following parabolas, labelling

i the y-intercept **ii** the turning point **iii** the axis of symmetry

a $y = (x + 1)^2$ **b** $y = (x + 4)^2$ **c** $y = (x - 2)^2$ **d** $y = (x - 4)^2$

e $y = (x + 3)^2$ **f** $y = (x - 5)^2$ **g** $y = -(x + 1)^2$ **h** $y = -(x - 2)^2$

LINKS TO Examples 5, 6, 7

5 For each of the following quadratic functions

i calculate the y-intercept

ii find the coordinates of the turning point

iii state the equation of the axis of symmetry

iv sketch the parabola, labelling the features

a $y = (x - 1)^2 + 2$ **b** $y = (x - 3)^2 - 4$ **c** $y = (x - 2)^2 + 3$

d $y = (x - 4)^2 - 2$ **e** $y = (x - 4)^2 + 7$ **f** $y = (x + 3)^2 - 5$

g $y = -(x + 1)^2 + 2$ **h** $y = -(x - 2)^2 + 3$ **i** $y = -(x + 3)^2 - 5$

LINKS TO Example 8

6 Find the equations for each of the parabolas when the parabola $y = x^2$ is translated to have the following turning points.

a $(4, -3)$ **b** $(-6, 2)$ **c** $(0, 4)$ **d** $(-3, 0)$ **e** $(2, 0)$ **f** $(0, -7)$

7 State the equations for the parabola when the parabola $y = x^2$ has the following translations.

a 4 units to the right **b** 5 units down

c 2 units to the left and 6 units up **d** 8 units to the right and 1 unit up

exercise 11.1 challenge

8 The parabola $y = (x - 2)^2 - 3$ is reflected in the x-axis.

a What is the equation of the reflected parabola?

b The reflected parabola is then reflected in the y-axis. What is the equation now?

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11.2 Quadratic functions in factorised form

When sketching a quadratic function, we get a 'complete' picture of the parabola if we know the coordinates of

- the x-intercepts
- the turning point (or vertex)
- the y-intercept

At the points where a parabola crosses the x-axis, $y = 0$. So, to find the x-intercepts, substitute $y = 0$ and use the null factor law to solve the quadratic equation for x.

Locating the x-intercepts

When the equation for a parabola is given in factorised form, it is easy to find the x-intercepts. For example, consider the parabola $y = (x + 1)(x - 3)$. The x-intercepts, found by substituting $y = 0$, are the solutions to the equation $(x + 1)(x - 3) = 0$, that is, $x = -1$ and $x = 3$.

Locating the turning point

The easiest way to find the turning point is to use the symmetry of the parabola. We know that the x-coordinate of the turning point must be half-way between the x-intercepts, that is, the average of the two x-intercepts.

$$\frac{-1 + 3}{2} = \frac{2}{2} = 1$$

So the midpoint of $x = -1$ and $x = 3$ is $x = 1$. We then substitute $x = 1$ into the equation $y = (x + 1)(x - 3)$ to find the y-coordinate of the turning point.

$$\begin{aligned} y &= (1 + 1)(1 - 3) \\ &= 2 \times (-2) \\ y &= -4 \end{aligned}$$

So the turning point is at $(1, -4)$.

Locating the y-intercept

The y-intercept is found by substituting $x = 0$ in the equation.

$$y = (x + 1)(x - 3)$$

When $x = 0$

$$\begin{aligned} y &= (0 + 1)(0 - 3) \\ &= 1 \times (-3) \\ y &= -3 \end{aligned}$$

So the y-intercept is at $(0, 3)$.

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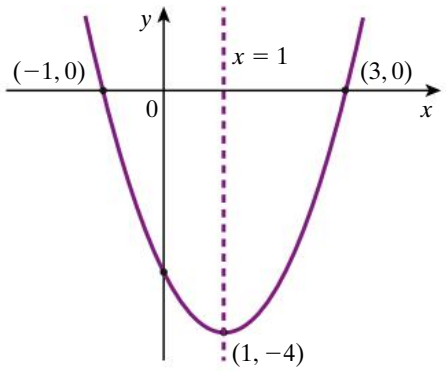
Sketching the graph



Sketching the graph of a quadratic function in factorised form

Summarising the steps for a quadratic equation in factorised form $y = (x - m)(x - n)$

- y-intercept: Substitute $x = 0$ in the equation $y = (x - m)(x - n)$. The y-intercept is at $(0, mn)$.
- x-intercepts: Substitute $y = 0$ and solve the equation $y = (x - m)(x - n)$. The x-intercepts are at $(m, 0)$ and $(n, 0)$.
- Turning point: The x-coordinate of the turning point is the midpoint (average) of the two x-intercepts $x = \frac{m + n}{2}$. Substitute the x-coordinate into the equation to find the y-coordinate.
- Axis of symmetry: $x = \frac{m + n}{2}$

Example 9

For each of these quadratic functions state the x-intercepts and sketch the graph showing the x-intercepts.

a $y = x(x - 2)$ **b** $y = (x - 1)(x - 2)$ **c** $y = -(x + 3)^2$

Working

a $y = x(x - 2)$

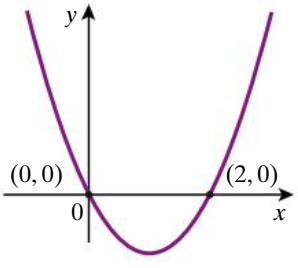
When $y = 0$,

$x(x - 2) = 0$

$x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

The x-intercepts are 0 and 2.



Reasoning

The graph crosses the x-axis when $y = 0$. Use the null factor law to solve the quadratic equation.

continued

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Example 9 continued

Working

b $y = (x - 1)(x - 2)$

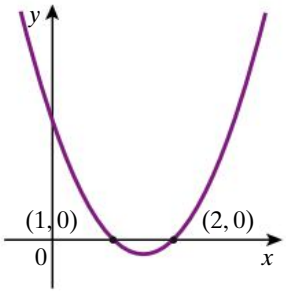
When $y = 0$,

$(x - 1)(x - 2) = 0$

$x - 1 = 0$ or $x - 2 = 0$

$x = 1$ or $x = 2$

The x-intercepts are $(1, 0)$ and $(2, 0)$.



Reasoning

The graph crosses the x-axis when $y = 0$. Use the null factor law to solve the quadratic equation.

c $y = -(x + 3)^2$

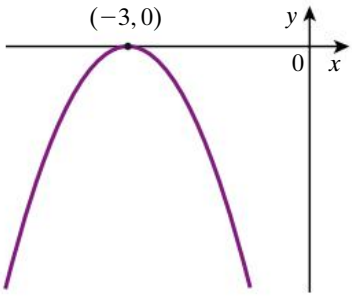
When $y = 0$,

$-(x + 3)^2 = 0$

$x + 3 = 0$

$x = -3$

The x-intercept is $(-3, 0)$.



The graph crosses the x-axis when $y = 0$. Use the null factor law to solve the quadratic equation. There is only one solution because $(x + 3)^2$ is a perfect square.

The parabola just touches the x-axis at $x = -3$.

The negative sign in $y = -(x + 3)^2$ means that the parabola is inverted.

Example 10

Sketch the following parabolas, showing the x-intercepts, the y-intercept, the turning point and the axis of symmetry.

a $y = (x - 1)(x - 9)$

b $y = (x + 2)(x - 4)$

continued

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Example 10 continued

Working

a $y = (x - 1)(x - 9)$

***x*-intercepts**

When $y = 0$,

$(x - 1)(x - 9) = 0$

$x = 1$ or $x = 9$

***y*-intercept**

When $x = 0$

$y = (0 - 1)(0 - 9)$

$y = 9$

Turning point

The midpoint of $x = 1$ and $x = 9$ is $x = 5$.

Substitute $x = 5$ into the equation

$y = (x - 1)(x - 9)$

$y = (5 - 1)(5 - 9)$

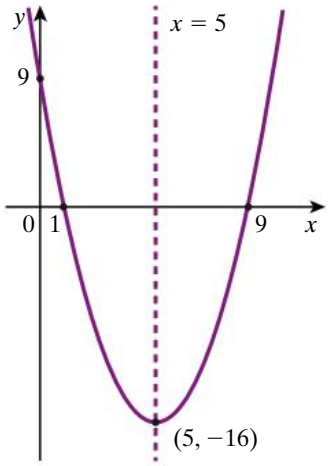
$= 4 \times (-4)$

$y = -16$

The turning point is at $(5, -16)$.

Axis of symmetry

$x = 5$



Reasoning

To sketch the parabola we need:

- the y-intercept
- the x-intercepts
- the turning point
- the axis of symmetry

Mark the positions of the turning point, the y-intercept and the x-intercepts.

Then draw a smooth parabola through the four points. Label the points.

Rule a line for the axis of symmetry and label it.

b $y = -(x + 2)(x - 4)$

***x*-intercepts**

When $y = 0$,

$-(x + 2)(x - 4) = 0$

$x = -2$ or $x = 4$

The x-intercepts are $(-2, 0)$ and $(4, 0)$

***y*-intercept**

When $x = 0$,

$y = -(0 + 2)(0 - 4)$

$y = 8$

The y-intercept is $(0, 8)$

To sketch the parabola we need

- the y-intercept
- the x-intercepts
- the turning point
- the axis of symmetry

continued

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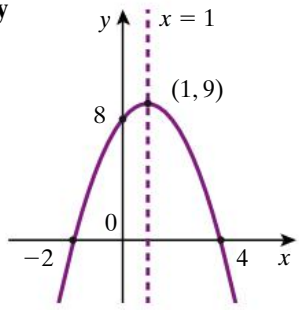
Example 10 continued

Working	Reasoning
Turning point The midpoint of $x = -2$ and $x = 4$ is $x = 1$. When $x = 1$, $y = -(x+2)(x-4)$ $y = -(1+2)(1-4)$ $= -3 \times (-3)$ $y = 9$	The x-coordinate of the turning point is midway between the x-intercepts. Substitute $x = 1$ into the equation.
The turning point is at $(1, 9)$. **Axis of symmetry** $x = 1$	Mark the positions of the turning point, the y-intercept and the x-intercepts. The negative sign in $y = -(x+2)(x-4)$ means that the parabola is inverted. Then draw a smooth parabola through the four points. Label the points. Rule a line for the axis of symmetry and label it.



exercise 11.2

LINKS TO Examples 9, 10

1 For each of the following equations

- **i** state the x-intercepts
- **ii** find the x-coordinate of the turning point
- **iii** find the y-coordinate of the turning point
- **iv** state the equation of the axis of symmetry
- **v** find the y-intercept
- **vi** sketch the graph, labelling all the features

a $y = x(x-6)$ **b** $y = x(x+3)$ **c** $y = -x(x-1)$
d $y = x(x+8)$ **e** $y = -(x-5)(x+5)$ **f** $y = (x+1)(x-3)$
g $y = (x-4)(x+2)$ **h** $y = (x-11)(x+3)$ **i** $y = -(x+2)(x-8)$
j $y = (x+3)(x-9)$ **k** $y = -(x+8)(x-8)$ **l** $y = (x-9)(x+7)$

2 Write equations in the form $y = (x-m)(x-n)$ for parabolas with the following x-intercepts.

a 5 and -1 **b** -4 and 4 **c** -3 and -1 **d** 4 and 7
e -8 and -4 **f** 3 and -3 **g** 7 and -6 **h** -2 and 0
i 0 and 8 **j** 1.5 and -1 **k** 2.5 and -3.5 **l** $\sqrt{3}$ and $-\sqrt{3}$

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11.3 Quadratic functions in turning point form

In section 11.1 we saw that the quadratic rule $y = (x - h)^2 + k$ is referred to as the turning point form of a quadratic function as the point (h, k) is the turning point of the parabola.

We find the x-intercepts by letting $y = 0$ and solving the equation for x. If k is negative, the quadratic expression is a difference of two squares, so the equation can be solved. If k is positive, the quadratic expression is a sum of two squares so the equation has no real solutions. This means there are no x-intercepts and the parabola does not cross the x-axis.

Example 11

Find the x-intercepts of each of the following parabolas.

a $y = x^2 - 9$

b $y = (x + 4)^2$

c $y = (x - 3)^2 - 1$

d $y = (x - 1)^2 - 2$

Working

a $y = x^2 - 9$

When $y = 0$

$x^2 - 9 = 0$

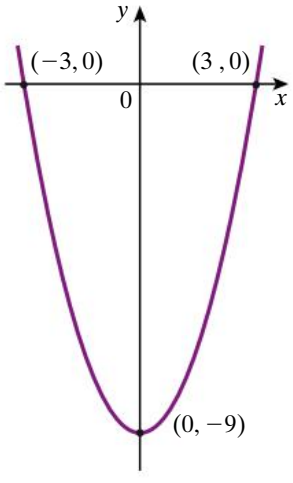
$(x + 3)(x - 3) = 0$

$x = -3$ or $x = 3$

The x-intercepts are -3 and 3.

Reasoning

Use the null factor law to solve the quadratic equation.



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Example 11 continued

Working

b $y = (x + 4)^2$

When $y = 0$

$(x + 4)^2 = 0$

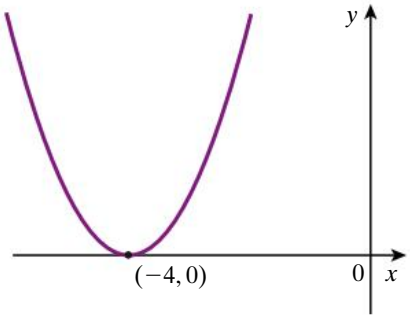
$x + 4 = 0$

$x = -4$

There is only one x-intercept, at $x = -4$.

Reasoning

Use the null factor law to solve the quadratic equation. $(x + 4)^2$ is a perfect square so there is only one solution to the equation.



c $y = (x - 3)^2 - 1$

When $y = 0$

$(x - 3)^2 - 1 = 0$

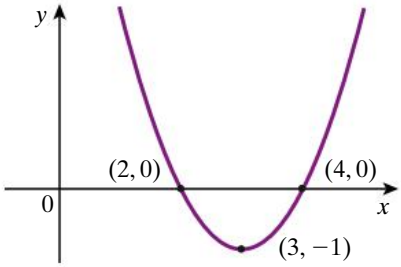
$(x - 3 + 1)(x - 3 - 1) = 0$

$(x - 2)(x - 4) = 0$

$x - 2 = 0$ or $x - 4 = 0$

$x = 2$ or $x = 4$

The x-intercepts are 2 and 4.



d $y = (x - 1)^2 - 2$

When $y = 0$

$(x - 1)^2 - 2 = 0$

$(x - 1 + \sqrt{2})(x - 1 - \sqrt{2}) = 0$

$x - 1 + \sqrt{2} = 0$ or $x - 1 - \sqrt{2} = 0$

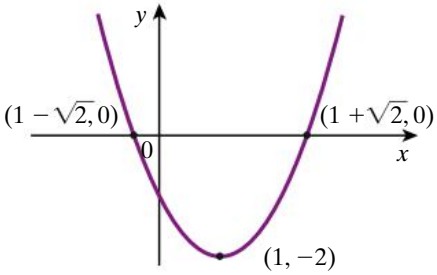
$x = 1 - \sqrt{2}$ or $x = 1 + \sqrt{2}$

The x-intercepts are $1 - \sqrt{2}$ and $1 + \sqrt{2}$.

Finding approximate values for the irrational x-intercepts allows the approximate positions of the intercepts to be located.

$1 - \sqrt{2} \approx -0.4$

$1 + \sqrt{2} \approx 2.4$



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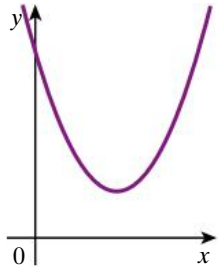
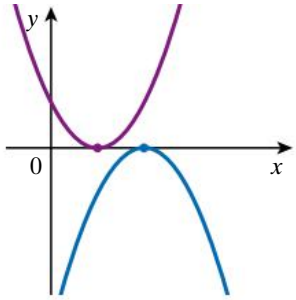
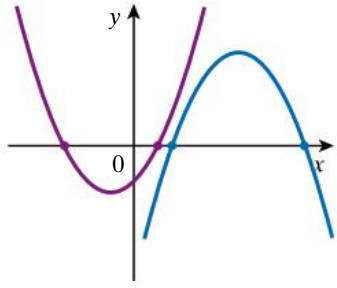
We have seen from section **11.1** that some parabolas do not have x-intercepts because they are entirely above the x-axis, or, if they are inverted parabolas, they may be entirely below the x-axis. Some parabolas have just one x-intercept because they just touch the x-axis at their turning point. Other parabolas have two x-intercepts.



No x-intercepts



One x-intercept



Two x-intercepts

Recall from section **8.4** that not all quadratic equations have real solutions. If the quadratic expression is a *sum of squares* rather than a *difference of squares*, it cannot be factorised so the equation will have no real solutions and the parabola will have no x-intercepts. We can tell, for example, that the parabola $y = (x - 3)^2 + 1$ has no x-intercepts because $(x - 3)^2 + 1$ is a sum of squares so the equation $(x - 3)^2 + 1 = 0$ has no real solutions.

Example 12

Which of these parabolas have x-intercepts?

a $y = x^2 + 4$ **b** $y = (x - 1)^2 - 16$ **c** $y = x^2 - 7$

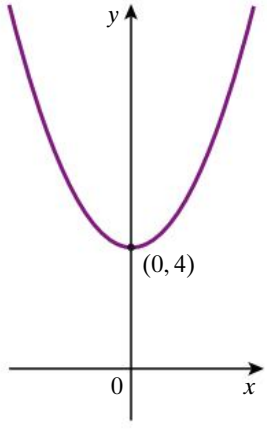
Working

a $y = x^2 + 4$

$x^2 + 4$ is a sum of squares, so the parabola does not have x-intercepts.

Reasoning

A sum of squares has no real factors so the equation $x^2 + 4 = 0$ has no real solutions.



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Example 12 continued

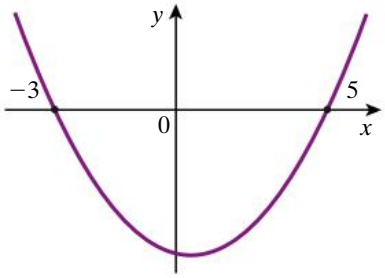
Working

b $y = (x - 1)^2 - 16$
$(x - 1)^2 - 16$ is a difference of squares so it can be factorised. The parabola $y = (x - 1)^2 - 16$ has x-intercepts.

Reasoning

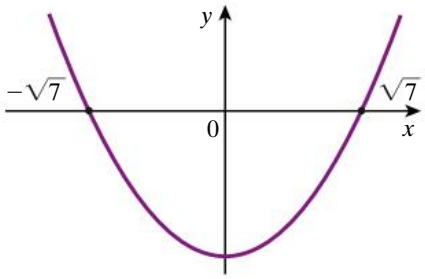
$(x - 1)^2 - 16$ is a difference of squares so the equation $(x - 1)^2 - 16 = 0$ has solutions.

$$(x - 1)^2 - 16 = 0$$
$$(x - 1 + 4)(x - 1 - 4) = 0$$
$$(x + 3)(x - 5) = 0$$
$$x = -3 \text{ and } x = 5$$



c $y = x^2 - 7$
$x^2 - 7$ is a difference of squares so it can be factorised. The parabola $y = x^2 - 7$ has x-intercepts.

$$x^2 - 7 = 0$$
$$(x + \sqrt{7})(x - \sqrt{7}) = 0$$
$$x = -\sqrt{7} \text{ and } x = \sqrt{7}$$



Sketching the graph of a quadratic function in turning point form

For a quadratic equation given in the form $y = (x - h)^2 + k$, we can now find

- the turning point, (h, k)
- the y-intercept by substituting $x = 0$ in the equation
- the x-intercepts by substituting $y = 0$ and solving the equation $(x - h)^2 + k = 0$
- the axis of symmetry: the line $x = h$

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Example 13

Sketch the following parabolas, showing the turning point, the y-intercept, the x-intercepts and the axis of symmetry.

a $y = (x - 2)^2 - 5$ **b** $y = (x + 3)^2 - 2$

Working

a $y = (x - 2)^2 - 5$

Turning point
$(2, -5)$

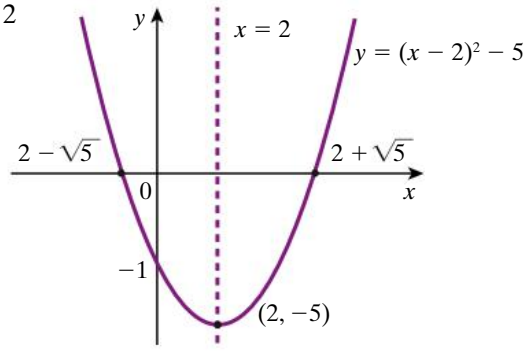
***y*-intercept**
When $x = 0$,
$y = (0 - 2)^2 - 5$
$y = -1$
The y-intercept is $(0, -1)$.

***x*-intercepts**
When $y = 0$,

$$(x - 2)^2 - 5 = 0$$
$$(x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) = 0$$
$$x - 2 + \sqrt{5} = 0 \quad \text{or } x - 2 - \sqrt{5} = 0$$
$$x = 2 - \sqrt{5} \text{ or } \quad x = 2 + \sqrt{5}$$

Approximate x-intercepts:
$(-0.24, 0)$ and $(4.24, 0)$

Axis of symmetry
$x = 2$



Reasoning

If the equation is in the form $y = (x - h)^2 + k$, the turning point is at (h, k).

Mark the positions of the turning point, the y-intercept, the x-intercepts and the axis of symmetry.
Then draw a smooth parabola through the four points. Label the points.
Rule a line for the axis of symmetry and label it.

b $y = (x + 3)^2 - 2$

Turning point
$(-3, -2)$

***y*-intercept**
When $x = 0$,
$y = (0 + 3)^2 - 2$
$= 9 - 2$
$y = 7$
The y-intercept is $(0, 7)$.

If the equation is in the form $y = (x - h)^2 + k$, the turning point is at (h, k).

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Example 13 continued

Working

x-intercepts

When $y = 0$

$$(x + 3)^2 - 2 = 0$$

$$(x + 3 + \sqrt{2})(x + 3 - \sqrt{2}) = 0$$

$$x + 3 + \sqrt{2} = 0 \quad \text{or } x + 3 - \sqrt{2} = 0$$

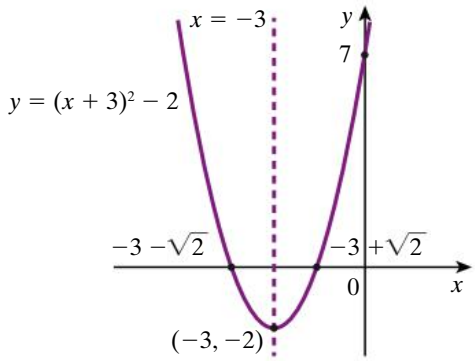
$$x = -3 - \sqrt{2} \text{ or } \quad x = -3 + \sqrt{2}$$

Approximate x-intercepts

$(-4.4, 0)$ and $(-1.6, 0)$

Axis of symmetry

$x = -3$



Reasoning

Finding approximate values of the x-intercepts helps when sketching the graph.

Mark the positions of the turning point, the y-intercept, the x-intercept and axis of symmetry. Then draw a smooth parabola through the four points. Label the points.

Rule a line for the axis of symmetry and label it.

exercise 11.3

LINKS TO Example 11

1 Find the x-intercepts of each of the following parabolas, stating whether there are one or two x-intercepts.

a $y = x^2 - 1$ **b** $y = x^2 - 25$ **c** $y = x^2 - 36$

d $y = (x + 3)^2$ **e** $y = (x - 4)^2$ **f** $y = (x + 2)^2$

g $y = (x - 2)^2$ **h** $y = (x + 1)^2 - 16$ **i** $y = (x - 3)^2 - 25$

j $y = (x + 3)^2 - 36$ **k** $y = (x - 5)^2 - 81$ **l** $y = (x + 4)^2 - 9$

LINKS TO Example 12

2 Which of these parabolas do not have x-intercepts?

a $y = x^2 + 8$ **b** $y = (x - 4)^2 + 1$ **c** $y = x^2 - 3$

d $y = x^2 + 3$ **e** $y = (x - 3)^2 + 9$ **f** $y = x^2 - 11$

LINKS TO Example 13

3 For each of the parabolas in question **1**

i state the turning point

ii find the y-intercept

iii state the equation of the axis of symmetry

iv sketch the graph, labelling the intercepts, turning point and axis of symmetry

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4 Sketch the graphs of the parabolas in question **2**, labelling the y-intercept, the turning point and the axis of symmetry.

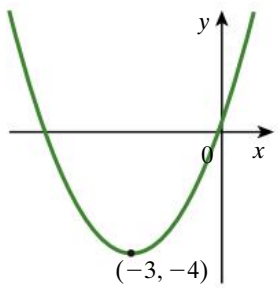
5 Sketch graphs of the following parabolas, labelling the intercepts, the turning point and the axis of symmetry.

a $y = x^2 - 5$ **b** $y = x^2 - 6$ **c** $y = x^2 - 20$
d $y = (x + 3)^2 - 5$ **e** $y = (x - 4)^2 - 3$ **f** $y = (x + 2)^2 - 2$
g $y = (x - 2)^2 - 5$ **h** $y = (x + 1)^2 - 6$ **i** $y = (x - 3)^2 - 8$
j $y = (x + 3)^2 - 3$ **k** $y = (x - 5)^2 - 10$ **l** $y = (x + 4)^2 - 5$

6 Match each of these parabolas with the correct equation.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 3)^2 - 4$
c $y = (x - 4)^2 - 3$ **d** $y = (x + 4)^2 + 3$

A



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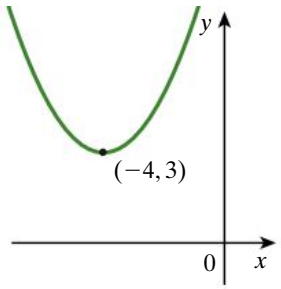

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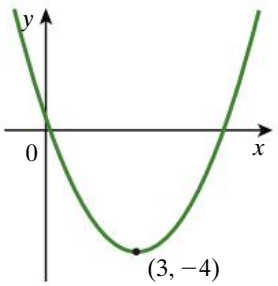

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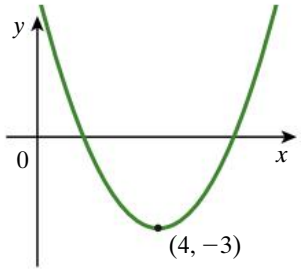

exercise 11.3 challenge

7 A parabola has an equation of the form $y = (x - h)^2 + k$. The x-intercepts are at $(-4, 0)$ and $(8, 0)$. Find the equation.

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11.4 General (expanded) form of a quadratic function

The general or expanded form of a quadratic equation is $y = ax^2 + bx + c$. In this section we will consider only those equations where $a = 1$.

Regardless of the form of the equation, we always need the same three pieces of information in order to sketch the parabola:

- y-intercept
- x-intercepts
- turning point

We start by looking at the same quadratic equation expressed in three different forms and compare the methods for finding the intercepts and turning point. Notice how each of the three forms of the equation is useful for a particular purpose.

Locating the y-intercept

Finding the y-intercept is always easy.

General form	Factorised form	Turning point form
$y = x^2 - 2x - 8$	$y = (x + 2)(x - 4)$	$y = (x - 1)^2 - 9$
When $x = 0$, $y = 0^2 - 2 \times 0 - 8$ $= -8$ The *general* or *expanded* form of the quadratic is most helpful here.	When $x = 0$, $y = (0 + 2)(0 - 4)$ $= 2 \times (-4)$ $= -8$	When $x = 0$, $y = (0 - 1)^2 - 9$ $= 1 - 9$ $= -8$

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Locating the x-intercepts

General form	Factorised form	Turning point form
$y = x^2 - 2x - 8$	$y = (x + 2)(x - 4)$	$y = (x - 1)^2 - 9$
Substitute $y = 0$ then factorise. $x^2 - 2x - 8 = 0$ $(x + 2)(x - 4) = 0$ $x + 2 = 0$ or $x - 4 = 0$ $x = -2$ or $x = 4$	Substitute $y = 0$ then solve the equation for x. $(x + 2)(x - 4) = 0$ $x + 2 = 0$ or $x - 4 = 0$ $x = -2$ or $x = 4$ The *factorised* form of the quadratic is most helpful here.	Substitute $y = 0$ then factorise the difference of squares and solve the equation for x. $(x - 1)^2 - 9 = 0$ $(x - 1 + 3)(x - 1 - 3) = 0$ $(x + 2)(x - 4) = 0$ $x + 2 = 0$ or $x - 4 = 0$ $x = -2$ or $x = 4$

Locating the turning point and axis of symmetry

General form	Factorised form	Turning point form
$y = x^2 - 2x - 8$	$y = (x + 2)(x - 4)$	$y = (x - 1)^2 - 9$
Complete the square to put the equation into turning point form. $y = x^2 - 2x - 8$ $= x^2 - 2x + 1 - 1 - 8$ $y = (x - 1)^2 - 9$ The turning point is at $(1, -9)$. The axis of symmetry is the line $x = 1$.	The x-coordinate of the turning point is the midpoint of the x-intercepts $x = \frac{-2 + 4}{2}$ $= \frac{2}{2}$ $x = 1$ Substitute $x = 1$ in the equation. $y = (1 + 2)(1 - 4)$ $= 3 \times (-3)$ $y = -9$ The turning point is at $(1, -9)$. The axis of symmetry is the line $x = 1$.	Turning point is at $(1, -9)$ The *turning-point form* of the quadratic is most helpful here. The axis of symmetry is the line $x = 1$.

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General (expanded) form

$y = x^2 - 2x - 8$

y-intercept is -8

Factorised form

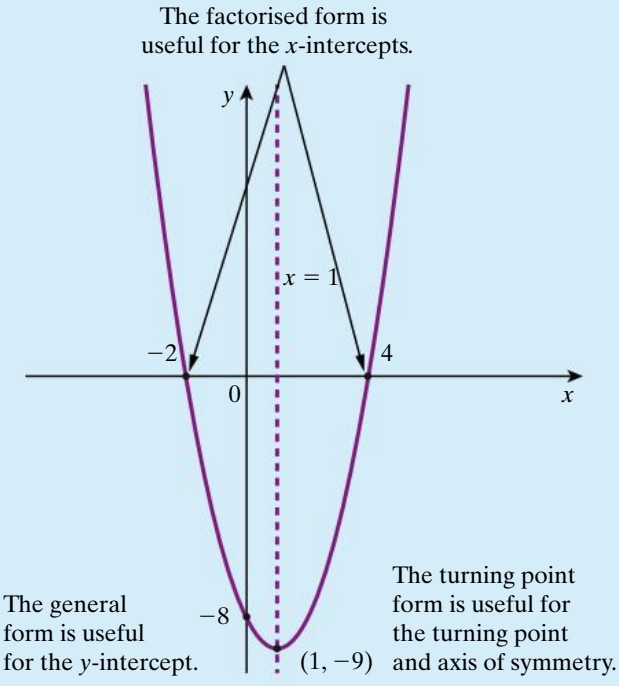
$y = (x + 2)(x - 4)$

x-intercepts are -2 and 4

Turning-point form

$y = (x - 1)^2 - 9$

Turning point (vertex) is $(1, -9)$



Example 14

Sketch the following parabolas showing the turning point, the y-intercept, the x-intercepts and the axis of symmetry.

a $y = x^2 - 8x - 20$

b $y = x^2 - 7x + 10$

Working	Reasoning
a $y = x^2 - 8x - 20$	
***y*-intercept** When $x = 0$, $y = -20$ The y-intercept is $(0, -20)$.	Substitute $x = 0$ to find the y-intercept.
***x*-intercepts** When $y = 0$, $x^2 - 8x - 20 = 0$ $(x + 2)(x - 10) = 0$ $x + 2 = 0$ or $x - 10 = 0$ $x = -2$ and $x = 10$ The x-intercepts are $(-2, 0)$ and $(10, 0)$	Substitute $y = 0$, then factorise and solve the equation to find the x-intercepts.

continued

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Example 14 continued

Working

Turning point

$x = \dfrac{-2 + 10}{2}$

$= \dfrac{8}{2}$

$x = 4$

Substitute $x = 4$,

$y = x^2 - 8x - 20$

$y = 4^2 - 8 \times 4 - 20$

$y = 16 - 32 - 20$

$y = -36$

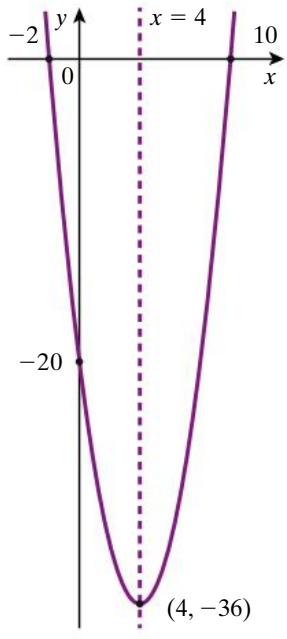
The turning point is at $(4, -36)$

Reasoning

Find the midpoint of the two x-intercepts. This gives the x-coordinate of the turning point. Substitute the value for x into the equation to find the y coordinate of the turning point.

Axis of symmetry

$x = 4$



Draw and label the axes. Mark the positions of the axis of symmetry, the intercepts and the turning point. Draw a smooth parabola through the four points.

Label the points and the axis of symmetry

b $y = x^2 - 7x + 10$

***y*-intercept**

When $x = 0$,

$y = 10$

The y-intercept is $(0, 10)$.

Substitute $x = 0$ to find the y-intercept.

***x*-intercepts**

When $y = 0$,

$x^2 - 7x + 10 = 0$

$(x - 2)(x - 5) = 0$

$x - 2 = 0$ or $x - 5 = 0$

$x = 2$ or $x = 5$

The x-intercepts are $(2, 0)$ and $(5, 0)$.

Substitute $y = 0$, then factorise and solve the equation to find the x-intercepts.

continued

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Example 14 continued

Working

Turning point

$x = \dfrac{2 + 5}{2}$

$= \dfrac{7}{2}$

$= 3.5$

Substitute $x = 3.5$

$y = x^2 - 7x + 10$

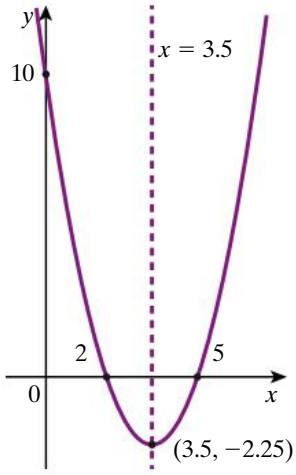
$y = 3.5^2 - 7 \times 3.5 + 10$

$y = -2.25$

The turning point is at $(3.5, -2.25)$

Axis of symmetry

$x = 3.5$



Reasoning

Find the midpoint of the two x-intercepts. This gives the x-coordinate of the turning point. Substitute the value for x into the equation to find the y coordinate of the turning point.

Draw and label the axes. Mark the positions of the axis of symmetry, the intercepts and the turning point. Draw a smooth parabola through the four points.
Label the points and the axis of symmetry

Irrational x-intercepts

The amount of work needed to find the x-intercepts and the turning point depends on whether the quadratic expression has obvious rational factors. If the quadratic expression cannot be factorised into obvious rational factors, then we use the technique of completing the square. This achieves two things: it puts the equation into turning point form and it allows the x-intercepts to be found.

Consider the equation $y = x^2 + 4x - 9$

y-intercept

Substitute $x = 0$ in the equation.

$y = -9$

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Turning point

The expression $x^2 + 4x - 9$ has no rational factors so we complete the square.

$$\begin{aligned} y &= x^2 + 4x - 9 \\ &= x^2 + 4x + 4 - 4 - 9 \\ y &= (x + 2)^2 - 13 \end{aligned}$$

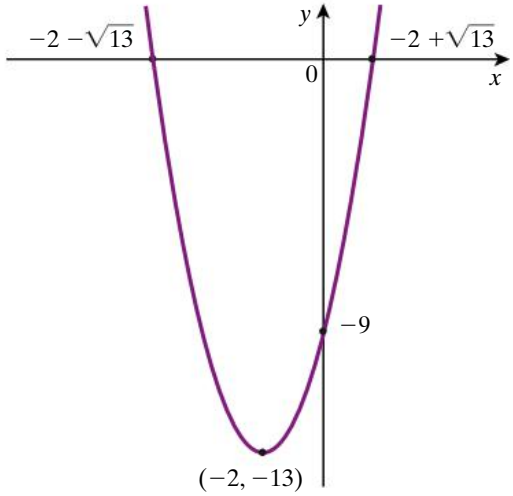
The turning point is at $(-2, -13)$.

***x*-intercepts**

We now substitute $y = 0$ to find the x-intercepts by factorising the turning point form of the equation.

$$\begin{aligned} (x + 2)^2 - 13 &= 0 \\ (x + 2 + \sqrt{13})(x + 2 - \sqrt{13}) &= 0 \\ x + 2 + \sqrt{13} = 0 \text{ or } x + 2 - \sqrt{13} &= 0 \\ x = -2 - \sqrt{13} \text{ or } x &= -2 + \sqrt{13} \end{aligned}$$

The x-intercepts are at $(-2 - \sqrt{13}, 0)$ and $(-2 + \sqrt{13}, 0)$, that is, approximately $(-5.6, 0)$ and $(1.6, 0)$.



Example 15

Sketch the following parabolas, showing the turning point, the y-intercept, the x-intercepts and the axis of symmetry.

a $y = x^2 - 2x - 7$

b $y = x^2 + 4x + 6$

Working	Reasoning
a $y = x^2 - 2x - 7$	
***y*-intercept** When $x = 0$, $y = -7$	Substitute $x = 0$ to find the y-intercept.
Turning point $y = x^2 - 2x - 7$ $= x^2 - 2x + 1 - 1 - 7$ $= (x - 1)^2 - 8$	Check first to see if the quadratic expression has rational factors. Complete the square to put the equation into turning point form.

continued

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Example 15 continued

Working

Turning point is at $(1, -8)$

x-intercepts

When $y = 0$,

$$(x - 1)^2 - 8 = 0$$

$$(x - 1 + \sqrt{8})(x - 1 - \sqrt{8}) = 0$$

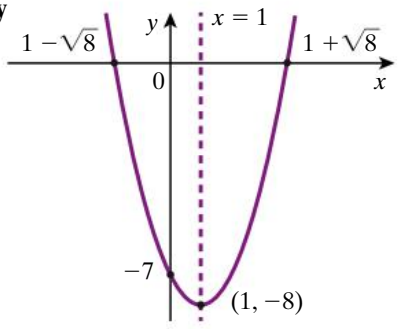
$$x - 1 + \sqrt{8} = 0 \quad \text{or } x - 1 - \sqrt{8} = 0$$

$$x = 1 - \sqrt{8} \text{ or } \quad x = 1 + \sqrt{8}$$

The x-intercepts are at $(1 - \sqrt{8}, 0)$ and $(1 + \sqrt{8}, 0)$, approximately $(-1.8, 0)$ and $(3.8, 0)$.

Axis of symmetry

$x = 1$



Reasoning

Use the turning point form to find the x-intercepts.

Factorise and solve the equation to find the x-intercepts.

b $y = x^2 + 4x + 6$

y-intercept

When $x = 0$,

$y = 6$

Substitute $x = 0$ to find the y-intercept.

Turning point

$$y = x^2 + 4x + 6$$
$$= x^2 + 4x + 4 - 4 + 6$$
$$= (x + 2)^2 + 2$$

Turning point is at $(-2, 2)$

$x^2 + 4x + 6$ has no rational factors.

Complete the square to put the equation in turning point form.

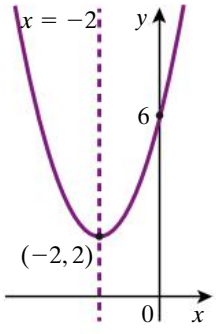
x-intercepts

$(x + 2)^2 + 2$ is a sum of squares so it cannot be factorised. There are no x-intercepts.

A sum of squares has no real factors.

Axis of symmetry

$x = -2$



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10 A

exercise 11.4

LINKS TO Example 14

1 For each of the following equations

- **i** find the y-intercept
- **ii** find the x-intercepts
- **iii** find the turning point
- **iv** state the equation of the axis of symmetry
- **v** sketch the graph, labelling all the features

a $y = x^2 + 5x + 6$ **b** $y = x^2 + 9x + 8$ **c** $y = x^2 - 8x + 7$
d $y = x^2 - 9x + 20$ **e** $y = (x - 2)(x - 4)$ **f** $y = (x - 4)(x + 5)$
g $y = (x + 2)^2 - 9$ **h** $y = (x - 5)^2 - 4$ **i** $y = (x + 7)(x - 1)$
j $y = x^2 - 5x - 36$ **k** $y = x^2 - 9x - 22$ **l** $y = x^2 + 10x + 24$
m $y = x^2 - 9x + 14$ **n** $y = (x - 4)^2 - 1$ **o** $y = x^2 - 12x + 35$

LINKS TO Example 15

2 For each of the following equations

- **i** find the y-intercept
- **ii** find the turning point by completing the square
- **iii** find the x-intercepts (if any)
- **iv** If there are x-intercepts find approximate values correct to one decimal place
- **v** state the equation of the axis of symmetry
- **vi** sketch the graph, labelling all the features

a $y = x^2 + 4x + 7$ **b** $y = x^2 + 2x - 5$ **c** $y = x^2 - 8x + 3$
d $y = x^2 - 6x + 4$ **e** $y = x^2 - 4x - 9$ **f** $y = x^2 - 2x + 5$
g $y = x^2 - 4x + 8$ **h** $y = x^2 - 8x + 13$ **i** $y = x^2 - 6x - 10$
j $y = x^2 + 2x + 6$ **k** $y = x^2 - 10x + 20$ **l** $y = x^2 + 10x + 17$

3 **a** Copy and complete the following table.

Quadratic function	Number of x-intercepts	x-intercepts	Turning point
$y = x^2 - 6x + 13$			
$y = x^2 - 6x + 9$			
$y = x^2 - 6x + 5$			
$y = x^2 - 6x + 3$			

b Sketch the graphs of the four quadratic functions on one set of axes.

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exercise 11.4 challenge

4 The parabola $y = x^2$ is translated so that it is symmetrical about the line $x = 3$ and one of the x-intercepts is 7.

a Find the equation for the parabola in factorised form.

b Find the y-intercept

c Find the turning point.

d Sketch the parabola, labelling the features.

5 The parabola $y = x^2$ is translated so that it is symmetrical about the line $x = -2$ and the y-intercept is 16.

a Find the equation for the parabola in general form.

b Find the x-intercepts

c Find the turning point.

d Sketch the parabola, labelling the features.

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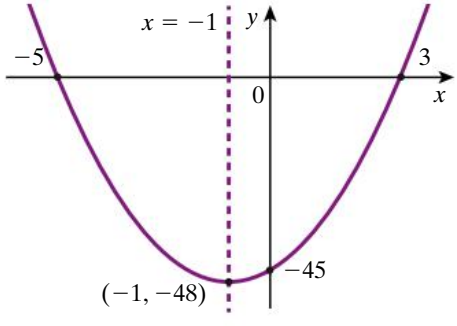
11.5 Further graphing of quadratic functions

In this section, we consider graphs of non-monic quadratic functions, that is, where the coefficient of x^2 is not 1. For quadratic expressions that can be factorised, the same techniques are used as in the previous section.

Example 16

Sketch the graph of $y = 3x^2 + 6x - 45$, showing the intercepts and turning point.

Working	Reasoning
y-intercept When $x = 0$, $y = -45$	Find the y-intercept by substituting $x = 0$.
x-intercepts When $y = 0$, $3x^2 + 6x - 45 = 0$ $3(x^2 + 2x - 15) = 0$ $3(x + 5)(x - 3) = 0$ $x + 5 = 0$ or $x - 3 = 0$ $x = -5$ or $x = 3$ The x-intercepts are $(-5, 0)$ and $(3, 0)$.	Find the x-intercepts by substituting $y = 0$ and solving the quadratic equation.
Turning point $x = \dfrac{-5 + 3}{2}$ $= \dfrac{-2}{2}$ $x = -1$	Find the x-coordinate of the turning point by finding the midpoint of $x = -5$ and $x = 3$. Substitute $x = -1$ to find the y-coordinate of the turning point.
When $x = -1$, $y = 3(-1)^2 + 6(-1) - 45$ $= 3 - 6 - 45$ $y = -48$	Axis of symmetry is found from the x-coordinate of the turning point.
The turning point is at $(-1, -48)$.	



Sketch the graph, labelling all the key points.

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Example 17

Sketch the graph of $y = 2x^2 - x - 3$, showing the intercepts and turning point.

Working

y-intercept

When $x = 0$

$y = -3$

x-intercepts

When $y = 0$,

$$2x^2 - x - 3 = 0$$
$$(2x - 3)(x + 1) = 0$$
$$2x - 3 = 0 \text{ or } x + 1 = 0$$
$$x = \frac{3}{2} \text{ or } x = -1$$

The x-intercepts are $(1\frac{1}{2}, 0)$ and $(-1, 0)$.

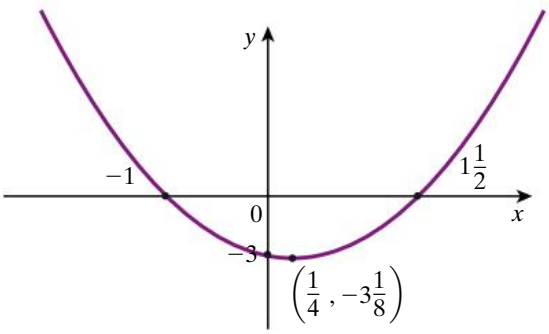
Turning point

$$x = -\frac{-1 + 1\frac{1}{2}}{2}$$

When $x = \frac{1}{4}$,

$$y = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 3$$
$$= \frac{1}{8} - \frac{2}{8} - \frac{24}{8}$$
$$= -\frac{25}{8}$$
$$y = -3\frac{1}{8}$$

The turning point is at $\left(\frac{1}{4}, -3\frac{1}{8}\right)$.



Reasoning

Find the y-intercept by substituting $x = 0$.

Find the x-intercept by substituting $y = 0$ and solving the quadratic equation.

Find the x-coordinate of the turning point by finding the midpoint of $x = -1$ and $x = 1\frac{1}{2}$.

Substitute $x = \frac{1}{4}$ to find the y-coordinate of the turning point.

Sketch the graph, labelling all the key points.

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If there are no factors and completing the square is used, the coefficient of x^2 must be taken out as a common factor first.

Example 18

For each of these quadratic functions, find the y-intercept, the x-intercepts and the coordinates of the turning point then sketch the graph.

a $y = 2x^2 + 4x - 3$ **b** $y = 2x^2 - x - 5$

Working	Reasoning
a $y = 2x^2 + 4x - 3$	
***y*-intercept** When $x = 0$, $y = -3$	Find the y-intercept by substituting $x = 0$.
Turning point $y = 2x^2 + 4x - 3$ $= 2\left[x^2 + 2x - \frac{3}{2}\right]$ $= 2\left[x^2 + 2x + 1 - 1 - \frac{3}{2}\right]$ $= 2\left[(x+1)^2 - \frac{5}{2}\right]$ $y = 2(x+1)^2 - 5$ The turning point is at $(-1, -5)$.	Put the equation into turning point form by completing the square. First, the coefficient of x^2 must be taken out as a common factor.
***x*-intercepts** When $y = 0$, $2(x+1)^2 - 5 = 0$ $(x+1)^2 - \frac{5}{2} = 0$ $\left(x + 1 + \sqrt{\frac{5}{2}}\right)\left(x + 1 - \sqrt{\frac{5}{2}}\right) = 0$ $x = -1 - \sqrt{\frac{5}{2}}$ or $x = -1 + \sqrt{\frac{5}{2}}$ The x-intercepts are at $\left(-1 - \sqrt{\frac{5}{2}}, 0\right)$ and $\left(-1 + \sqrt{\frac{5}{2}}, 0\right)$, that is, approximately $(-2.6, 0)$ and $(0.6, 0)$.	Put $y = 0$ in the turning point form of the equation. Solve the quadratic equation.

continued

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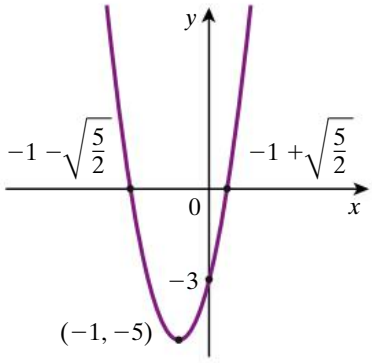
Example 18 continued

Working



Reasoning

Sketch the graph, labelling all the key points.

b $y = 2x^2 - x - 5$

***y*-intercept**

When $x = 0$,

$y = -5$

Turning point

$$y = 2x^2 - x - 5$$

$$= 2\left[x^2 - \frac{x}{2} - \frac{5}{2}\right]$$

$$= 2\left[x^2 - \frac{x}{2} + \frac{1}{16} - \frac{1}{16} - \frac{5}{2}\right]$$

$$= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5}{2}\right]$$

$$= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{41}{16}\right]$$

$$y = 2\left(x - \frac{1}{4}\right)^2 - \frac{41}{8}$$

Before completing the square, the coefficient of x^2 must be taken out as a common factor. Completing the square then puts the equation in turning point form.

Remove the square brackets by multiplying each term in the brackets by 2.

The turning point is at $\left(\frac{1}{4}, -5\frac{1}{8}\right)$.

***x*-intercepts**

When $y = 0$,

continued

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Example 18 continued

Working

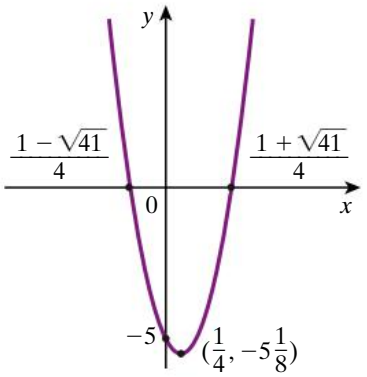
$$2\left(x - \frac{1}{4}\right)^2 - \frac{41}{8} = 0$$

$$\left(x - \frac{1}{4}\right)^2 - \frac{41}{16} = 0$$

$$\left(x - \frac{1}{4} + \frac{\sqrt{41}}{4}\right)\left(x - \frac{1}{4} - \frac{\sqrt{41}}{4}\right) = 0$$

$$x = \frac{1 - \sqrt{41}}{4} \text{ or } x = \frac{1 + \sqrt{41}}{4}$$

The x-intercepts are at $\left(\frac{1 - \sqrt{41}}{4}, 0\right)$ and $\left(\frac{1 + \sqrt{41}}{4}, 0\right)$, that is, approximately $(-1.35, 0)$ and $(1.85, 0)$.



Reasoning

x-intercepts could be found by using the quadratic formula to solve the equation $2x^2 - x - 5 = 0$ as this is probably easier than working with the turning point form.

$$x = \frac{1 \pm \sqrt{1 - 4 \times 2 \times (-5)}}{2 \times 2}$$

$$x = \frac{1 \pm \sqrt{41}}{4}$$

The general case $y = ax^2 + bx + c$

y-intercept	$y = c$
x-intercepts	$x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$
x-coordinate of turning point	$x = -\dfrac{b}{2a}$ Substitute to find y-coordinate.
Equation of axis of symmetry	$x = -\dfrac{b}{2a}$

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Using the discriminant to identify the number of x-intercepts

In section 8.8 the discriminant, $\Delta = b^2 - 4ac$, was used to find if quadratic equations had real solutions and to find whether these solutions were rational or irrational. In a similar way, the discriminant can be used to find if a parabola has x-intercepts.

Discriminant Δ	Number of solutions for $ax^2 + bx + c + 0$	Number of x-intercepts of parabola	Nature of x-intercepts
negative ($\Delta < 0$)	no real solutions	no x-intercepts	
zero ($\Delta = 0$)	one solution	one x-intercept, i.e. parabola touches the x-axis at its turning point	rational value
positive ($\Delta > 0$) (a) perfect square (b) not perfect square	Two solutions (a) two rational solutions (b) two irrational solutions	two x-intercepts	(a) rational values (b) irrational values

Example 19

For each of these functions, find the number of x-intercepts that the graph would have.

a $y = 2x^2 - 3x - 1$ **b** $y = x^2 - 6x + 9$ **c** $y = x^2 - x + 2$

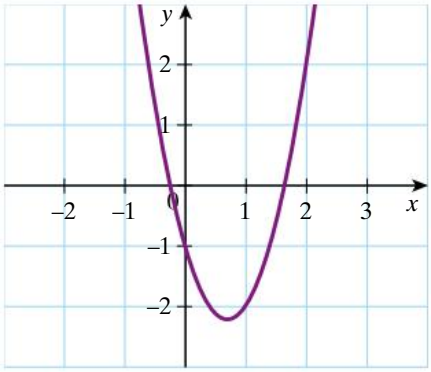
Working

a $y = 2x^2 - 3x - 1$
so $a = 2, b = -3, c = -1$

$\Delta = b^2 - 4ac$
$= 9 + 8$
$= 17$

As the discriminant is positive, the equation has two solutions and the parabola has two x-intercepts.

Reasoning



Check: Graphing the function confirms that there are two x-intercepts.

continued

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Example 19 continued

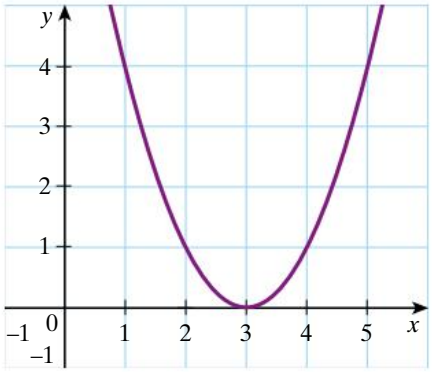
Working

b $y = x^2 - 6x + 9$
$a = 1, b = -6, c = 9$

$\Delta = b^2 - 4ac$
$= 36 - 36$
$= 0$

A zero discriminant means the equation has one solution and the parabola has one x-intercept, which will be its turning point.

Reasoning

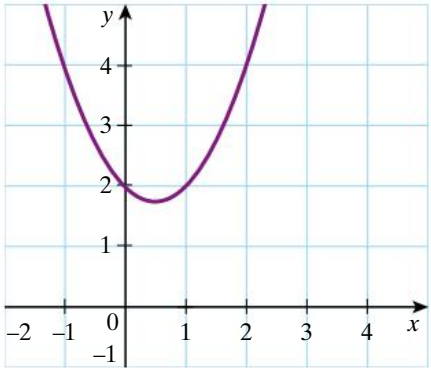


Check: Graphing the function confirms that the parabola touches the x-axis at one point, so there is one x-intercept.

c $y = x^2 - x + 2$
$a = 1, b = -1, c = 2$

$\Delta = b^2 - 4ac$
$= 1 - 8$
$= -7$

As the discriminant is negative, the equation has no solutions and the parabola does not cross the x-axis. So it has no x-intercepts.



Check: Graphing the function confirms that the parabola does not cross the x-axis.

Example 20

Sketch the graph of the parabola $y = 2x^2 - 4x - 1$, labelling the y-intercept, the x-intercepts and the coordinates of the turning point.

Working

$y = 2x^2 - 4x - 1$

y-intercept

When $x = 0$,
$y = -1$

Reasoning

Find the y-intercept by substituting $x = 0$

continued

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Example 20 continued

Working

Turning point

$x = -\dfrac{b}{2a}$

$= -\dfrac{-4}{2 \times 2}$

$x = 1$

When $x = 1$,

$y = 2 \times 1^2 - 4 \times 1 - 1$

$= 2 - 4 - 1$

$y = -3$

The turning point is at $(1, -3)$.

x-intercepts

When $y = 0$,

$2x^2 - 4x - 1 = 0$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 2 \times (-1)}}{2 \times 2}$

$= \dfrac{4 \pm \sqrt{16 + 8}}{4}$

$= \dfrac{4 \pm \sqrt{24}}{4}$

$= \dfrac{4 \pm 2\sqrt{6}}{4}$

$x = \dfrac{2 \pm \sqrt{6}}{2}$

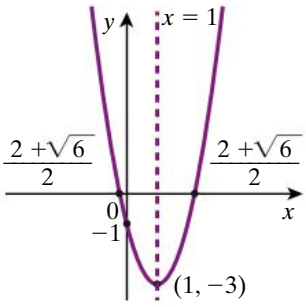
The x-intercepts are at

$\left(\dfrac{2 - \sqrt{6}}{2}, 0\right)$ and $\left(\dfrac{2 + \sqrt{6}}{2}, 0\right)$, that is,

approximately $(-0.2, 0)$ and $(2.2, 0)$.

Axis of symmetry

$x = 1$



Reasoning

Substitute $a = 2$ in $x = \dfrac{b}{2a}$ to find the x-coordinate of the turning point.

Substitute $x = 1$ in the equation to find the y-coordinate of the turning point.

Find the x-intercepts by substituting $y = 0$ and solving the quadratic equation.

The axis of symmetry is found from the x-coordinate of the turning point.

Sketch the graph, labelling all key features.

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10 A

exercise 11.5

1 Find the x-intercepts of the following quadratic functions.

a $y = x^2 - 11x + 10$ **b** $y = 2x^2 - x - 3$ **c** $y = 2x^2 - 18$
d $y = 12 - x - x^2$ **e** $y = x^2 + 4x - 5$ **f** $y = 2x^2 - 6x + 4$

LINKS TO Examples 16, 17

2 Sketch each of the quadratic functions in question **1**, labelling the y-intercept, the x-intercepts and the turning point.

LINKS TO Example 18

3 For each of the following quadratic functions sketch the graph, labelling the y-intercept, the x-intercepts and the turning point.

a $y = 2x^2 - 8x + 1$ **b** $y = 2x^2 + 12x + 7$
c $y = 2x^2 + 20x + 47$ **d** $y = 3x^2 + 6x - 2$

LINKS TO Example 19

4 Find the number of x-intercepts of each of the following quadratic functions.

a $y = x^2 - 4x - 5$ **b** $y = 2x^2 - x - 1$ **c** $y = 4x^2 - 3$
d $y = 3x^2 - 4x + 5$ **e** $y = 3x^2 + 6x + 2$ **f** $y = 3x^2 - 12x + 12$

LINKS TO Example 20

5 For each of the quadratic functions in question **4**, sketch the graph, labelling the y-intercept, the x-intercepts and the turning point.

6 On the same axes, sketch the 'family' of quadratic functions with the rule $y = a(x + 3)^2 - 2$, where

a $a = 1$ **b** $a = 3$ **c** $a = \frac{1}{2}$

Label the turning point and the y-intercept of each quadratic function (but not the x-intercepts). What is the same about each of the graphs?

7 On the same axes, sketch four members of the family of quadratic functions with the rule $y = a(x - 3)^2 - 2$.

8 Each graph in a family of quadratic functions just touches the x-axis at $(3, 0)$.

a Write down a general rule for this family of quadratics.
b If one parabola from this family passes through the point $(0, -27)$, find its rule.
c Sketch the graph of the function from part **b**.

9 Find the range of values m can take so that the parabola $y = x^2 + 3x + m - 3$ has two x-intercepts.

10 Find the values of n such that the parabola $y = nx^2 - 5x - 2$ has no x-intercepts.

11 Find the values of k such that the parabola $y = x^2 - (k + 1)x + k + 4$ touches the x-axis.

12 Show that the parabola $y = 3x^2 + (k + 2)x - 2$ has x-intercepts for all values of k.

13 Find the values of k so that the parabola $y = kx^2 - 2x + 1$ has one x-intercept.

exercise 11.5 challenge

14 What can be said of the parabola $y = (k + 3)x^2 - 2(k + 1)x + k - 1$? Explain.

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11.6 Applications of quadratic functions

Quadratic functions can be used to model many real-world situations. They can be used, for example, to model the curves of bridge arches. The parabola can be used to find maximum or minimum values. If an object is thrown, it moves in a curved path that can be modelled as a parabola. If we know certain positions of a ball after it is thrown, we can find the equation that represents the path of the ball.

For each of the different forms of the quadratic rule, we need to find three values in order to determine the rule.

Form	Rule	Find
Factorised form	$y = a(x - m)(x - n)$	a, m, n
General form	$y = ax^2 + bx + c$	a, b, c
Turning-point form	$y = a(x - h)^2 + k$	a, h, k

Using the factorised form to find the rule

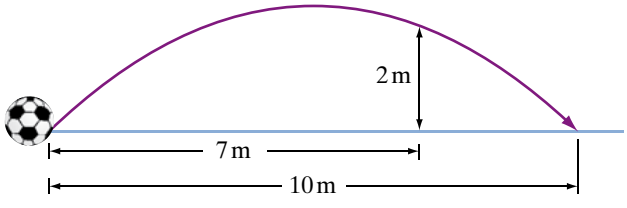
The following example shows how the factorised form can be used to find the rule when the two x-intercepts and the coordinates of one other point are known.

Example 21

A soccer ball is kicked from the ground and travels a horizontal distance of 10 m until it hits the ground again. It is just tipped by the fingers of the goalkeeper, who is 7 m horizontally from the point where the ball was kicked. The goalie's fingertips are 2 m vertically above the ground when the ball is touched.

a Find the rule for the path of the soccer ball.

b Find the maximum height reached by the ball.



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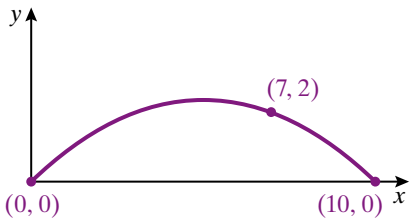
Example 21 continued

Working	Reasoning
a Set the origin at the point where the ball was kicked.	The origin and axes must be set before the rule can be found.
	Show the coordinates of the points that we know from the question and/or the diagram above.
The graph shows the x-intercepts, so use the rule $y = a(x - m)(x - n)$.	Since the x-intercepts are known, find the *factorised form* of the quadratic equation.
The x-intercepts are 0 and 10, so the rule becomes $y = a(x - 0)(x - 10)$ $= ax(x - 10)$	Substitute for the x-intercepts m and n in the factorised form. We still need to find a.
The ball passes through the point $(7, 2)$, so when $x = 7$, $y = 2$. $2 = a \times 7 \times (7 - 10)$ $2 = a \times 7 \times -3$ $2 = -21a$ $a = -\frac{2}{21}$	The information about the goalie tipping the ball gives us a third point on the path of the ball: $(7, 2)$. Substitute the x- and y-coordinates of $(7, 2)$ into the rule, and solve the equation to find the value of a.
The rule in factorised form is $y = -\frac{2}{21}x(x - 10)$.	Substitute the a-value into the rule. If desired, expand the brackets so that the rule is given in general form: $y = -\frac{2}{21}x^2 + \frac{20}{21}x$
b At maximum height of the ball $x = 5$. $y = -\frac{2}{21} \times 5(5 - 10)$ ≈ 2.4 The maximum height is 2.4 m.	By the symmetry of the parabola, the maximum height will be reached at the x-coordinate halfway between the x-intercepts. Substitute $x = 5$ into the rule.

Using the turning-point form to find the rule

The next example shows how the turning-point form can be used to find the rule when the coordinates of the turning point and one other point are known.

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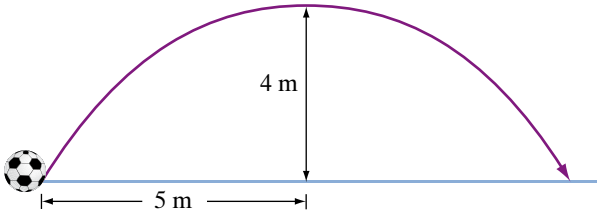
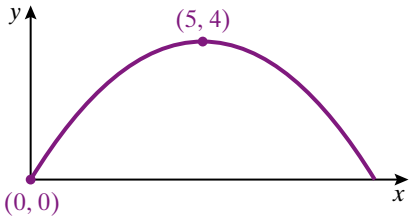
Example 22

A soccer ball is kicked from the ground and travels along a parabolic path. It reaches its greatest height of 4 m at a point that is 5 m horizontally from where it was kicked. Find the rule for the path of the soccer ball.



Working

Set the origin at the point where the ball was kicked.



The graph shows only one of the x-intercepts, but it also shows the turning point, so use the rule $y = a(x - g)^2 + h$.

Using the coordinates of the turning point (5, 4), the rule is $y = a(x - 5)^2 + 4$.

The ball starts from the point (0, 0), so when $x = 0$, $y = 0$.

$0 = a(0 - 5)^2 + 4$

$0 = 25a + 4$

$a = -\dfrac{4}{25}$

Reasoning

Before the rule can be found we must decide on the position of the origin and axes.

Show the coordinates of the points that we know from the question and diagram above.

The point where the ball reaches its maximum height is (5, 4). This is the turning point (h, k), so use the turning-point form of the rule.

Substitute the coordinates of the turning point for h and k in the general equation.

We still need to find a.

The ball was kicked from (0, 0) so this gives us another point on the path of the ball. (Alternatively, we could use the symmetry of the parabola to identify that the point (10, 0) is on the path.) Substitute the x- and y-coordinates of (0, 0) into the rule, and solve the equation to find the value of a.

continued

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Example 22 continued

Working

The rule in turning-point form is

$y = -\frac{4}{25}(x - 5)^2 + 4.$

Reasoning

Substitute the value of a into the rule. If desired, expand the brackets so that the rule is given in general form:

$$y = -\frac{4}{25}(x^2 - 10x + 25) + 4$$
$$= -\frac{4}{25}x^2 + \frac{40}{25}x - 4 + 4$$
$$= -\frac{4}{25}x^2 + \frac{8}{5}x$$

Using the general form to find the rule

The next example shows how a given point can be substituted into the general rule $y = ax^2$ to find the value of a if the origin is at the turning point.

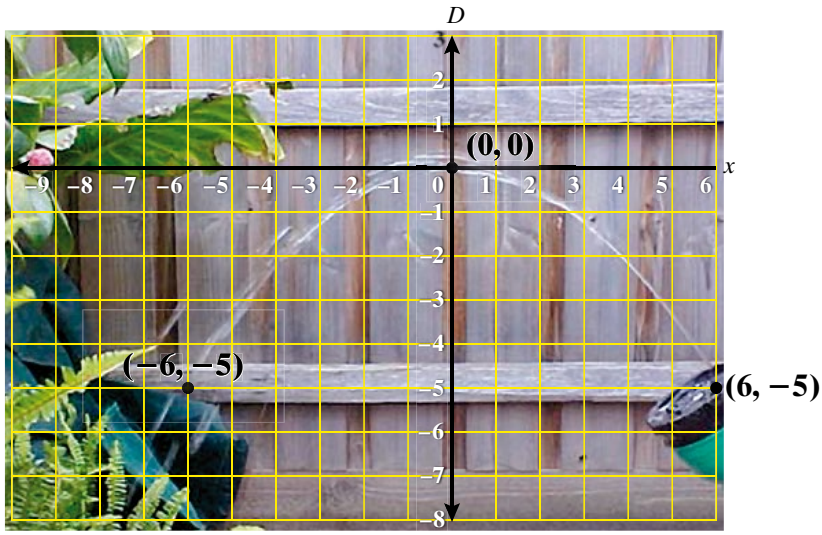
Example 23

A set of axes has been superimposed over this photograph of a jet of water, with the turning point at (0, 0).

a Find a rule for the parabola formed by the jet of water, with the turning point at (0, 0). using x for the horizontal distance from the origin and D for the vertical distance.

b Use the rule to find the value of D when $x = -3$.

c At what other point does D have the same value as your answer to part **b**?



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Example 23 continued

Working	Reasoning
a $D = ax^2$	The general rule for a parabola with its turning point at the origin is $y = ax^2$.
Substitute $x = 6$ and $D = -5$.	The coordinates of any point on the parabola can be substituted. $(6, -5)$ is one of the given points.
$-5 = a \times 6^2$	
$a = \frac{-5}{36}$	
$D = -\frac{5x^2}{36}$	
b Substitute $x = -3$	$(-3)^2 = 9$
$D = \frac{-5 \times (-3)^2}{36}$	
$= -\frac{5}{4}$	
$= -1\frac{1}{4}$	
$D = -1\frac{1}{4}$ when $x = -3$	
c $D = -1\frac{1}{4}$ when $x = 3$.	The parabola is symmetrical about the y-axis, so D has the same value for $x = -3$ and $x = 3$.

exercise 11.6

LINKS TO Example 21

1 A soccer ball is kicked from the ground and travels a horizontal distance of 12 m until it hits the ground again. It is just tipped by the fingers of the goalkeeper 7 m horizontally from the point where it was kicked. The goalie's fingertips are 2 m vertically above the ground when it is tipped. Let x m represent the horizontal distance and y m represent the vertical distance.

a Sketch the parabolic path of the ball. Choose a suitable position for the origin and draw the x- and y-axes on your diagram.

b Mark on the diagram the coordinates of the point where the ball was tipped by the goalie.

c Find the rule for the path of the soccer ball in factorised form.

LINKS TO Example 22

2 A cannon has a range of 50 m. Assume that a cannon ball is fired from ground level. The cannon ball is to hit a target 12 m high, at a horizontal distance of 30 m from the cannon.

a Draw the parabolic path of the cannon ball. Choose a suitable position for the origin and draw the x- and y-axes on your diagram.

b Mark on the diagram the coordinates corresponding to the information given. Do these points include x-intercepts or a turning point?

c Find the rule for the path of the cannon ball, using factorised or turning-point form. (Choose the form that is more useful.)

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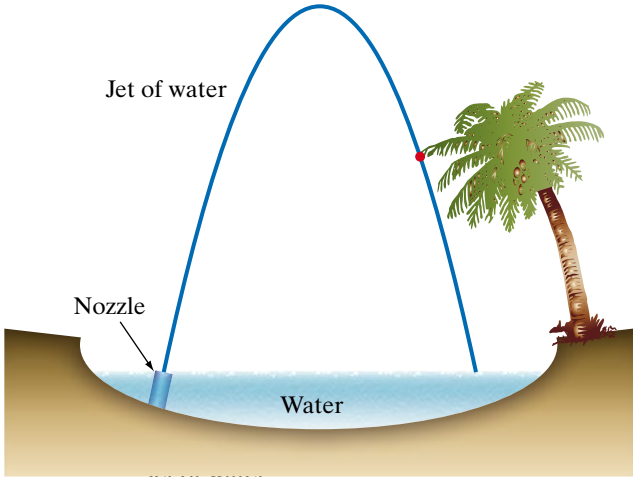
3 A fountain is designed so that its nozzle is at water level and the jet of water follows a parabolic path that lands 4 m from the nozzle. The jet of water just touches the frond of a palm tree that is 3 m above the surface of the water. This frond overhangs the water so that it is 1 m horizontally from the spot where the jet hits the water.



a Sketch the path of the jet of water. Using the nozzle as the origin, add the axes and mark the coordinates of any known points.

b Find the rule that describes the path of the water jet.

c How high above the surface of the water does the jet reach?

4 Road surfaces are often curved in a parabolic shape so that water drains off the road efficiently. A road is 8 m wide and the edges are 6 cm lower than the centre. Find the function rule for a parabola that models the shape of the road if the origin is set at the left edge of the road.

5 A football is kicked from a point 1 m above the ground. It reaches a maximum height of 3 m when it has travelled 5 m horizontally.

a Sketch the path of the football. Choose a suitable position for the origin and add the axes and the coordinates of known points.

b Find the equation of the path of the football in turning-point form.

6 Chuol was photographed making a long jump. He took off from a marker board at one end of the long jump pit and the length of his jump was recorded as 8 m. The photograph showed that when Chuol had travelled 2 m horizontally, he had reached a height of 1 m. Using the take-off board as the origin, find the rule that describes the path of Chuol's jump.

7 Lauren threw a javelin which, like any object that is thrown, followed a parabolic path. Lauren released the javelin when her hand was 2 m above the ground. When the javelin had travelled 6 m horizontally from her hand, it had risen another 5 m to reach its highest point.

a Sketch the path of the javelin.

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b The horizontal axis will be the horizontal ground. Mark this on your diagram and also show the vertical axis and the origin.

c Mark the coordinates of the point where the javelin was released and the highest point.

d Find the rule for the path of the javelin. Leave your rule in turning-point form.

e How far did the javelin travel horizontally from Lauren's hand to the ground?

8 As part of a fireworks display, a rocket is designed to be ignited from a position on the bank of a river, 2 m above the water. The rocket is to reach its maximum height 100 m above the river and 10 m out from the bank into the river. The coordinates of the maximum point on its path will be (10, 100) if the origin is at water level 2 m below the ignition point at (0, 2).

a Sketch the path of the rocket.

b Mark the coordinates of the point where the rocket is ignited and of the highest point.

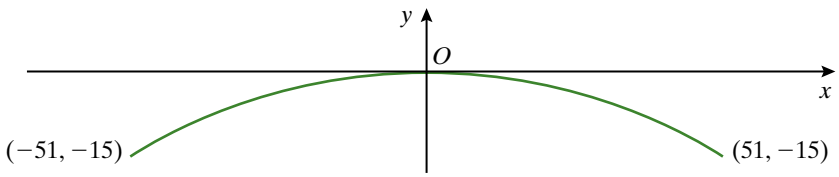
c Find the rule for the parabolic path followed by the rocket.

d How far horizontally from the ignition point will the rocket land?

LINKS TO Example 23

9 This photograph shows the Goodwill Bridge across the Brisbane River. The arch can be modelled by a parabola with a rule of the form $y = -ax^2$, where the origin is placed at the turning point of the arch.

a Find the rule for the parabola if the arch is 102 m across and 15 m high at the centre.



b Use graph paper or GeoGebra to graph your equation. Use the same scale on each axis so that the shape is not distorted.

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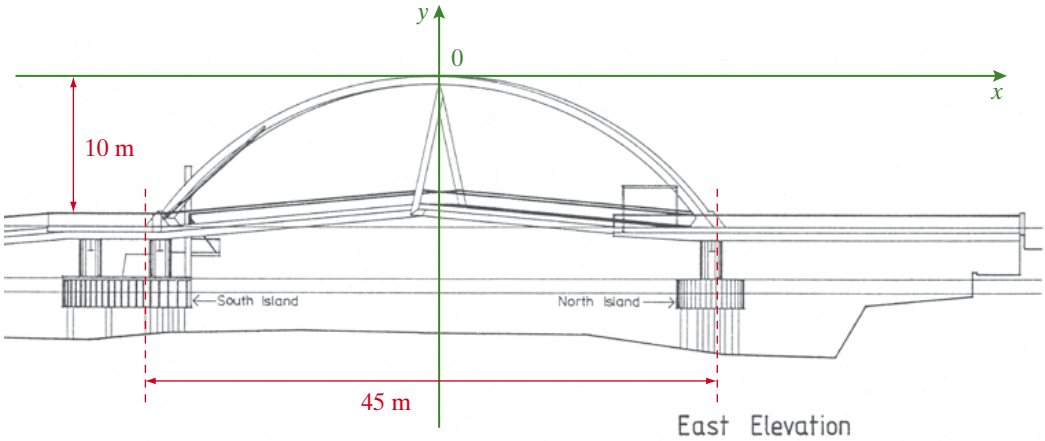
10 The photograph below shows the footbridge over the River Yarra at Southgate in Melbourne.

Below is the architects' drawing of the bridge with coordinate axes and approximate dimensions superimposed. The bridge can be modelled as an inverted parabola with a rule of the form $y = ax^2$, where a is a negative number and the turning point of the parabola is at the highest point of the arch.



a Using the coordinate axes and dimensions shown on the architects' drawing of the pedestrian bridge, find the rule for the parabolic shape of the bridge with the origin in the position shown.

b Use graph paper or GeoGebra to graph your equation. Use the same scale on each axis so that the shape is not distorted.

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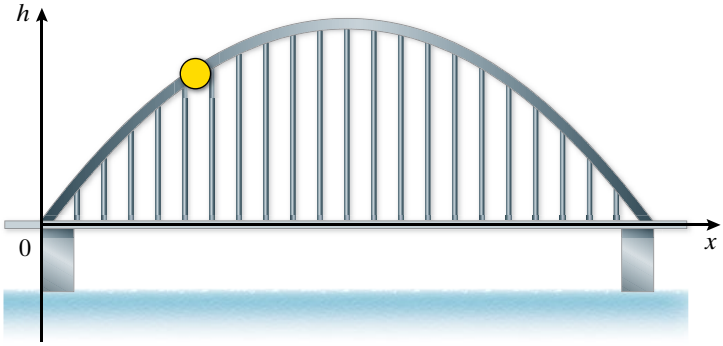
11 The arch of the bridge in this diagram can be modelled by a parabola. The origin has been positioned as shown, with the horizontal axis along a road and the vertical axis at one end of the arch. The ends of the arch are 160 m apart along the road. A circular sign is positioned 40 m horizontally from the left end of the arch and 40 m vertically above the road.



a Find the function rule for the height h of the arch.

b How far horizontally from the left end of the arch is the highest point?

c How far above the road is the highest point of the arch?

12 An archer fires an arrow towards a target 100 m away. The centre of the target is 1 m above the ground. The arrow follows a parabolic path through the air, as does any object that is shot or thrown. The archer estimates that, in order to hit the target, the arrow must be fired so that it will reach a maximum height of 15 m at a distance of 49 m.

a Sketch the path of the arrow.

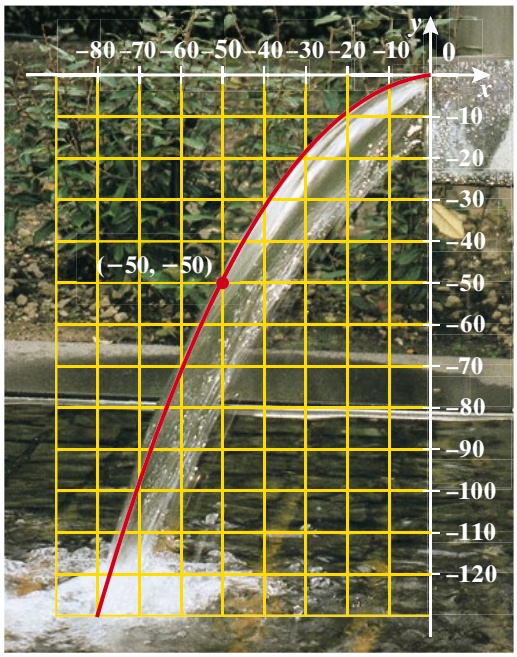
b The horizontal axis will be the horizontal ground. Mark this on your diagram and also show the vertical axis and the origin.

c Mark the coordinates of the highest point and the point where the arrow hits the centre of the target.

d Find the rule for the parabolic path of the arrow. Give the value of a as a decimal correct to four decimal places.

e How high above the ground was the arrow at the instant it was fired from the bow? Give the height correct to one decimal place.

13 This photograph shows the jet of water that occurs when water flowing in a channel spills over the open end of the channel. A Cartesian grid and a curve have been superimposed over the jet of water. The scale divisions are in centimetres.



a Find the function rule that can be used to model the path of the water.

b Use the rule to calculate the value of y when $x = -80$.

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14 To enclose the four sides of a rectangular yard, 200 m of fencing is available.

a If the length of the yard is x m, write an expression for the width in terms of x.

b Write the area, y m^2, of the rectangular yard as a function of x.

c Sketch the graph showing the x-intercepts.

d What is the x-coordinate of the turning point?

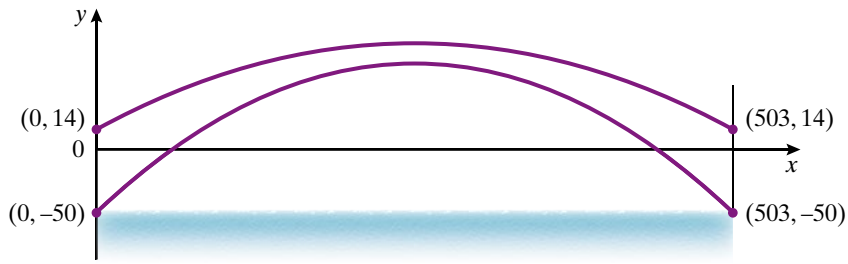
e Calculate the y-coordinate of the turning point.

f What is the maximum area that can be enclosed with the 200 m of fencing?

g Draw a labelled diagram to show the dimensions of the yard that correspond to this maximum area.

exercise 11.6 challenge

15 The two arches of the Sydney Harbour Bridge are approximately parabolic. The horizontal length of the span between the two pylons is 503 m. The road is 50 m above water level. The centre of the upper arch is 134 m above the water level and the centre of the lower arch is approximately 118 m above the water.



a Copy the diagram and label the coordinates of the turning point of each arch.

b Find the rule for a parabola that approximates the lower arch of the bridge. Give the value of a (the coefficient of x^2) correct to four decimal places.

c Repeat part **b** for the upper arch of the bridge.

d Use graph paper or GeoGebra to graph your equation. Use the same scale on each axis so that the shape is not distorted.

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e It is possible to participate in an organised climb over the top of the upper arch of the Sydney Harbour Bridge. How far above water level are the climbers when they are 100 m horizontally from one of the pylons? Give the height to the nearest metre.

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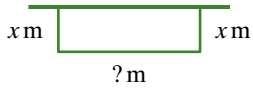
Analysis task 1

Fencing the chicken run

Imagine that you are constructing a rectangular run for two bantams (small chickens). You have 12 metres of wire netting with which to construct three sides of this rectangular run, using an existing fence as the fourth side.

Let the width of the run be x metres.



a Find an algebraic expression for the length of the chicken run, in terms of x.

b Write an equation for the area A squares metres of the chicken run in terms of x.

c Use the equation to find the area if $x = 1$. Draw a sketch to show what the chicken run would look like in this case.

d What is the largest value x could have? Explain.

e What is the value of A if $x = 0$

f Construct a table of values using integer values for x from 0 to 6.

g Explain how you know the graph will be a parabola.

h Will the parabola be upright or inverted? Explain how you know.

i Use graph paper or a graphing calculator to graph the area function.

j Identify where the graph crosses the x-axis. What do these points mean in the context of the chicken run?

k What are the coordinates of the turning point?

l What is the significance of the turning point in terms of the chicken run?

m What dimensions would you choose for the chicken run with the 12 metres of wire netting? Explain.

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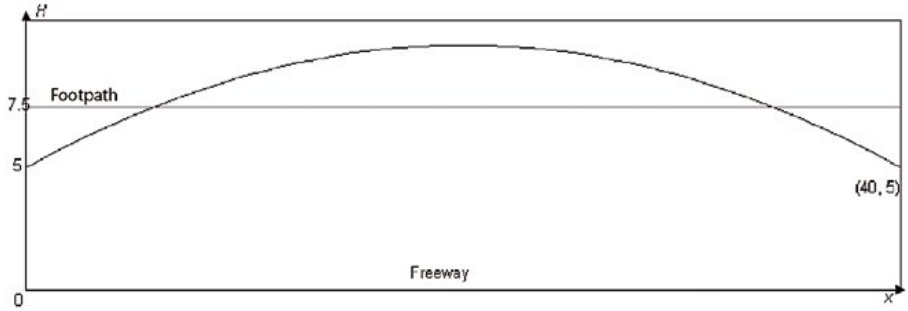
Analysis task 2

Freeway footbridge

The arch of a footbridge over a freeway is approximately parabolic in shape. The width of the bridge is 40 metres and the horizontal footpath is 7.5 metres above the freeway.



Let H metres be the height of the arch above the freeway, where x metres is the horizontal distance along the bridge from the left end. This height can be modelled with the function rule $H = -\dfrac{x(x-40)}{80} + 5$.

a Use GeoGebra to draw the graph of $H = -\dfrac{x(x-40)}{80} + 5$.

b From the graph of H, find where the arch is 9 metres above the freeway.

c Use the quadratic formula to check your answer to part **b**.

d Without using the graph, how could you determine the x-value that gives the maximum height?

e What is the maximum height of the arch above the freeway?

f Use GeoGebra or a CAS calculator to enter a function rule corresponding to the footpath, which has a height of 7.5 metres above the freeway. From the two graphs, find where the arch crosses the footpath.

g Use your answers to part **f** to work out the length of footpath between the two places where the arch crosses the footpath.

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Review Quadratic functions

Summary

Sketching graphs of quadratic equations in factorised form

- y-intercept: Substituting $x = 0$ in the equation $y = (x - m)(x - n)$. The y-intercept is at $(0, mn)$
- x-intercepts: Substituting $y = 0$ and solve the equation $y = (x - m)(x - n)$. The x-intercepts are at $(m, 0)$ and $(n, 0)$.
- turning point: The x-coordinate of the turning point is the midpoint (average) of the two x-intercepts $x = \frac{m + n}{2}$. Substitute the x-coordinate into the equation to find the y-coordinate.
- axis of symmetry: the line $x = \frac{m + n}{2}$

Sketching graphs of quadratic equations in turning point form

- In general, the graph of $y = (x - h)^2 + k$ is the same shape as the graph of $y = x^2$ but the parabola is translated h units to the right and k units up. If h is negative, the parabola is translated to the left. If k is negative, the parabola is translated down. The turning point is at (h, k) and the axis of symmetry is the line $x = h$.
- For a quadratic equation given in the form $y = (x - h)^2 + k$, we can now find
  - the turning point, (h, k)
  - the y-intercept by substituting $x = 0$ in the equation
  - the x-intercepts by substituting $y = 0$ and solving the equation $(x - h)^2 + k = 0$
  - the axis of symmetry: the line $x = h$

The general case $y = ax^2 + bx + c$

y-intercept	$y = c$
x-intercepts	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
x-coordinate of turning point	$x = -\frac{b}{2a}$ Substitute to find y-coordinate.
Equation of axis of symmetry	$x = -\frac{b}{2a}$

10 A

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The discriminant

Discriminant Δ	Number of solutions for $ax^2 + bx + c + 0$	Number of x-intercepts of parabola	Nature of x-intercepts
negative ($\Delta < 0$)	no real solutions	no x-intercepts	
zero ($\Delta = 0$)	one solution	one x-intercept, i.e. parabola touches the x-axis at its turning point	rational value
positive ($\Delta > 0$) (a) perfect square (b) not perfect square	two solutions (a) two rational solutions (b) two irrational solutions	two x-intercepts	(a) rational values (b) irrational values

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

axis of symmetry	null factor law	table of values
completing the square	parabola	turning point
*discriminant	quadratic equation	turning point form
factorised form	quadratic function	vertex
factors	roots	x-intercept
general form	rational approximation	y-intercept
irrational solution	rational solution	

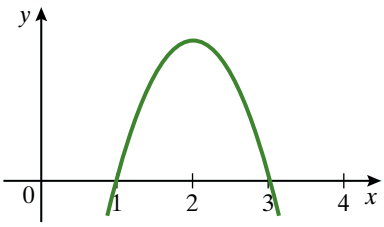
Revision

Multiple-choice questions

1 Consider the function rule $y = -(x - 2)(1 - x)$. When $x = 0$, y is equal to

A -2 **B** 1 **C** -1 **D** 2 **E** 0

2 Which of the following rules might describe this graph?



A $y = -(x + 3)(x + 1)$
B $y = (x - 3)(x - 1)$
C $y = (x + 3)(x + 1)$
D $y = (x + 3)(x - 1)$
E $y = -(x - 3)(x - 1)$

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3 The x-intercepts for the graph of the quadratic function $y = x^2 - 8x - 20$ are

A $x = -10$ and $x = -2$ **B** $x = -20$ **C** $x = -10$ and $x = 2$
D $x = -5$ and $x = -4$ **E** $x = 10$ and $x = -2$

4 The graph of the function with rule $y = (x - 3)^2 - 7$ has a turning point (or vertex) at

A $(-3, 0)$ **B** $(3, -7)$ **C** $(-3, -7)$
D $(3, 7)$ **E** $(-3, 7)$

5 The equation $y = x^2 - 4x + 2$ can also be written as

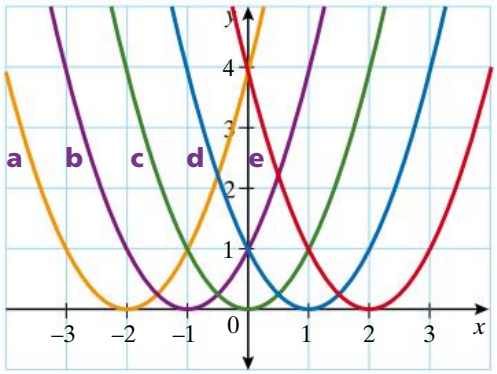
A $y = (x - 2)^2 - 2$ **B** $y = (x - 2)^2 + 6$ **C** $y = (x + 2)^2 - 2$
D $y = (x + 2)^2 + 6$ **E** $y = (x - 2)^2 - \sqrt{2}$

Short-answer questions

6 If a quadratic function has a rule $y = (x + 1)^2 - 16$, find

a the value of y when $x = 0$
b the solution to the equation $y = 0$
c the coordinates of the turning point of its graph

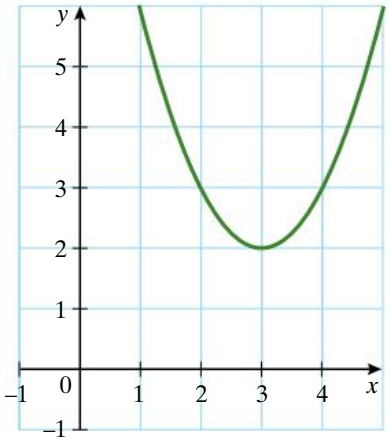
7 Write the rules for the five graphs shown here.



8 Match each of these equations with the parabola.

a $y = x^2 - 5x + 6$
b $y = (x - 5)^2 + 6$
c $y = (x - 3)^2 + 2$
d $y = (x + 3)^2 + 2$

A



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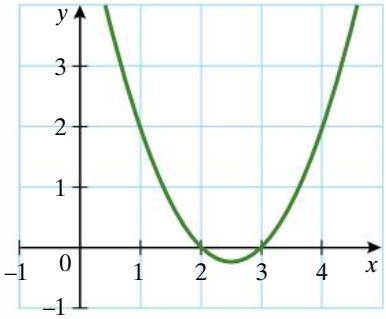

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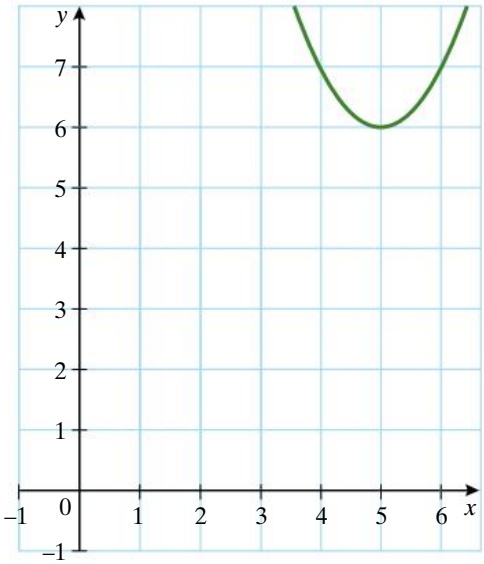
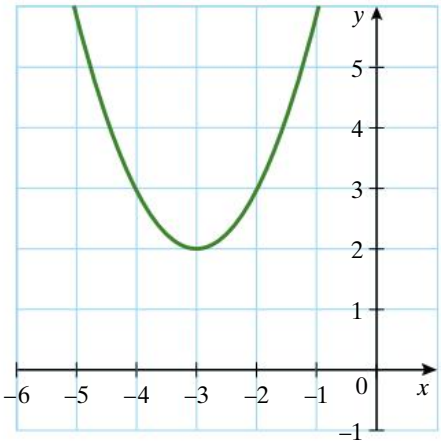
C



D



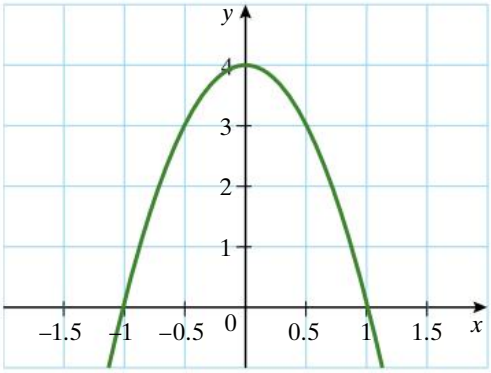
9 The parabola $y = x^2$ is translated. The axis of symmetry is the line $x = 1$. Write three possible equations for the parabola and sketch their graphs.

10 The rule for this graph is given by
$y = a(b - x^2)$

a Find the value of b.

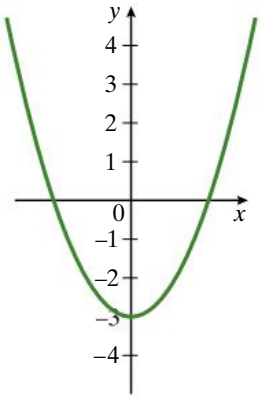
b Find the value of a.

c Use your rule to find x when $y = 3.36$.



11 **a** Give a possible rule for this parabola.

b Explain why more than one answer is possible to part **a**.



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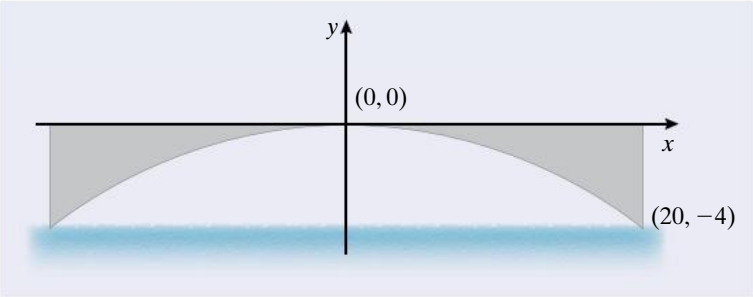
12 On the same set of axes, sketch the graphs given by the rule $y = k - x^2$, where $k = 1, 2$ or 3. Label the x and y intercepts of each of the graphs.

13 The arch of a particular bridge is a parabola. Axes have been superimposed on a drawing of the arch, with the origin at the highest point of the arch. A point 20 m horizontally from the centre of the arch is 4 m below the top of the arch. Find the rule for the parabola.



Extended-response questions

14 Sketch the following parabolas, labelling the x- and y-intercepts and the coordinates of the turning point and the axis of symmetry.

a $y = x(x - 2)$ **b** $y = x^2 - 25$ **c** $y = x^2 - 6x - 16$ **d** $y = -(x + 2)^2$
e $y = (x - 4)^2 + 1$ **f** $y = x^2 + 4$ **g** $y = -x^2 + 9$ **h** $y = x^2 - 3x + 2$

15 Sketch the following parabolas, labelling the x- and y-intercepts and the coordinates of the turning point.

a $y = -x(x - 2)$ **b** $y = 2x^2 - 2$ **c** $y = \frac{x^2}{2}$ **d** $y = 8x^2 - 8x - 16$

16 Suppose that 200 m of fencing is available to enclose a rectangular yard that borders an existing fence along one side. This means that the 200 m of fencing is used to form the other three sides of the rectangular yard.

a If the length of the yard along the existing fence is x m, write an expression for the width in terms of x.
b Write the area y m^2 of the rectangular yard as a function of x.
c Sketch the graph showing the x-intercepts.
d What is the x-coordinate of the turning point?
e Calculate the y-coordinate of the turning point.
f What is the maximum area that can be enclosed with the 200 m of fencing?
g Draw a labelled diagram to show the dimensions of the yard that correspond to this maximum area.

Practice quiz
Chapter 11

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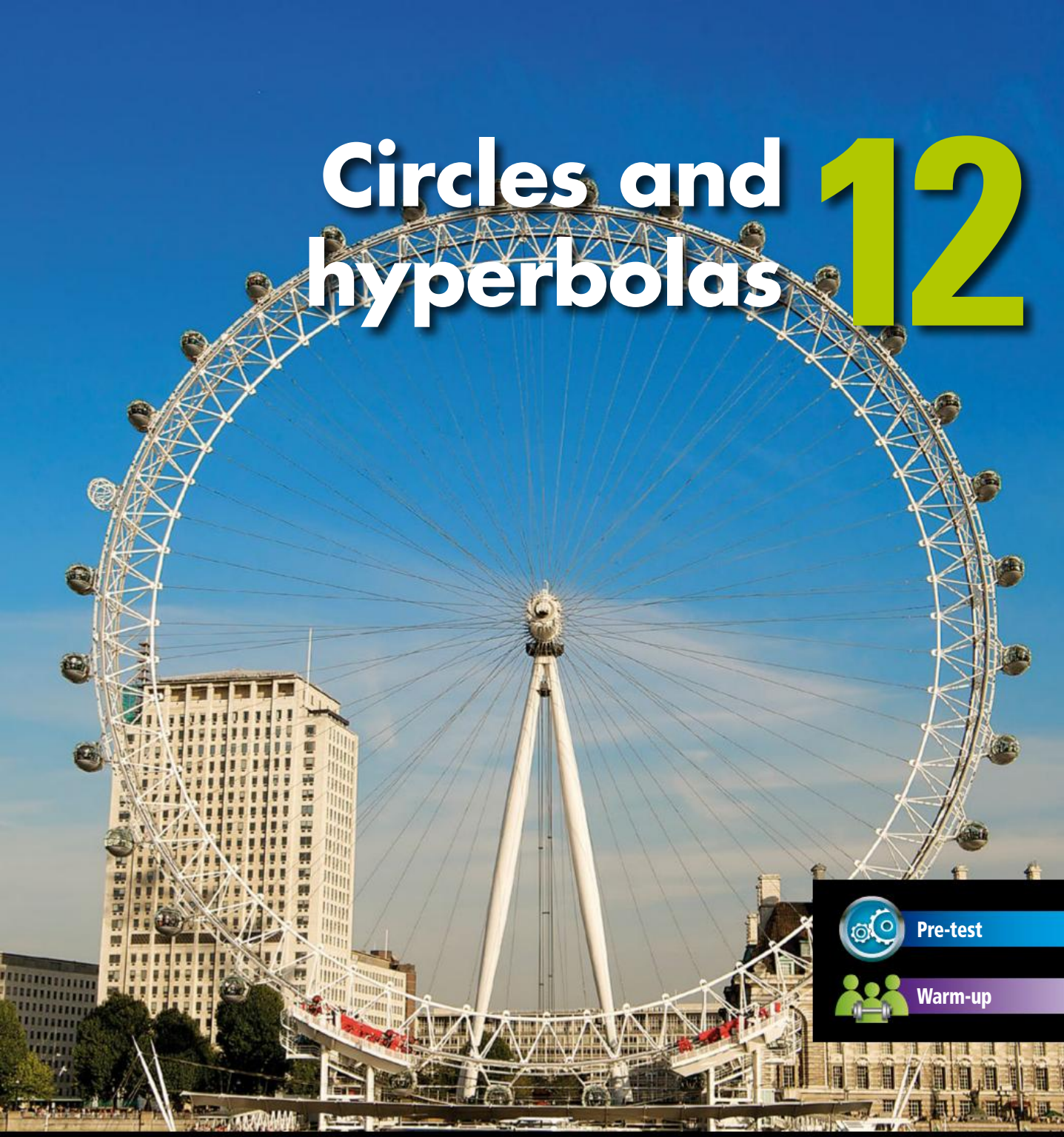
Circles and hyperbolas 12



Pre-test

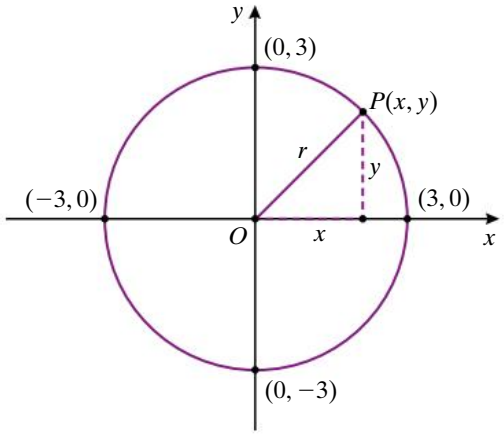
Warm-up

The rim of the London Eye has a diameter of 135 m, and 32 capsules that each hold 25 passengers. The wheel takes 30 minutes for a complete rotation and passengers have a 40 km view from the top. Many cities around the world now have giant wheels from which passengers can view the city. By drawing axes with the origin at the centre of the wheel we can express the position (x, y) of any point on the outer rim of the wheel in terms of the equation for the circle. How will the height of the boarding platform above the ground affect the equation for the circle?

12.1 Circles

We define a circle as the set of all points in a plane that are the same distance from a fixed point. This fixed point is the centre of the circle, and the distance of the points from the centre is the radius of the circle.

The circle on the right has centre at (0, 0) and radius 3 units. P is any point on the circle, with coordinates (x, y). The radius OP is the hypotenuse of the right-angled triangle. By Pythagoras' theorem we can see that $x^2 + y^2 = 3^2$. We can write this as $x^2 + y^2 = 9$. This is the equation for the circle.



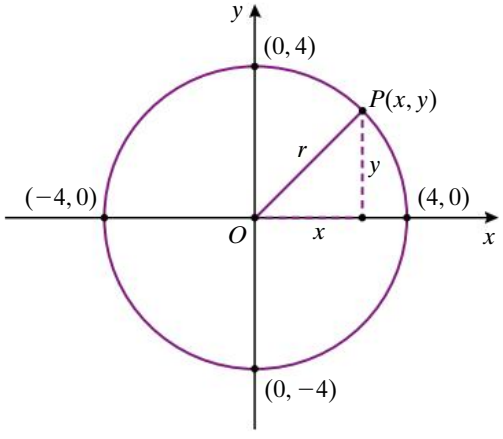
Example 1

What is the equation of a circle with centre (0, 0) and radius 4 units?

Working

$x^2 + y^2 = 4^2$
$x^2 + y^2 = 16$

Reasoning



The radius of the circle is the hypotenuse of the right-angled triangle.

Points whose coordinates satisfy the equation for the circle lie on the circle.

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Example 2

Consider a circle with centre (0, 0) and radius $\sqrt{20}$.

a What is the equation of a circle with centre (0, 0) and radius $\sqrt{20}$ units?

b Show that the points (2, 4), (4, 2) and (−2, −4) lie on the circle.

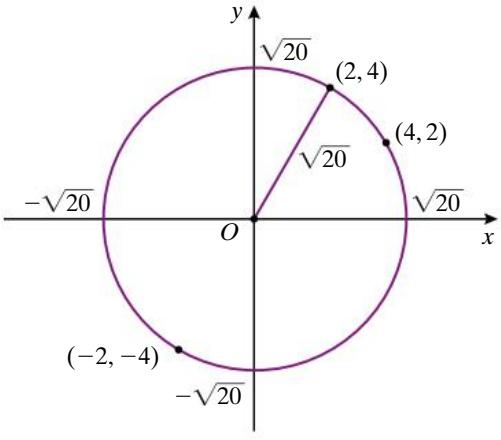
c Is the point (3, 3) on the circle?

Working

a $x^2 + y^2 = (\sqrt{20})^2$

$x^2 + y^2 = 20$

Reasoning



b (2, 4) (4, 2)

Substitute $x = 2$, $y = 4$

$$\begin{aligned} \text{LS} &= x^2 + y^2 \\ &= 2^2 + 4^2 \\ &= 4 + 16 \\ &= 20 \\ &= \text{RS} \end{aligned} \qquad \begin{aligned} \text{LS} &= x^2 + y^2 \\ &= 4^2 + 2^2 \\ &= 16 + 4 \\ &= 20 \\ &= \text{RS} \end{aligned}$$

(−2, −4)

$$\begin{aligned} \text{LS} &= x^2 + y^2 \\ &= (-2)^2 + (-4)^2 \\ &= 4 + 16 \\ &= 20 \\ &= \text{RS} \end{aligned}$$

The three points all lie on the circle because their coordinates satisfy the equation $x^2 + y^2 = 20$.

Substitute the x- and y-coordinates into the left-hand side. If the left-hand side is equal to the right-hand side (in this case 20), then the point is on the circle.

c (3, 3)

$$\begin{aligned} \text{LS} &= x^2 + y^2 \\ &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18 \\ &\neq \text{RS} \end{aligned}$$

The point (3, 3) is not on the circle.

For the point (3, 3), $x^2 + y^2 \neq 20$

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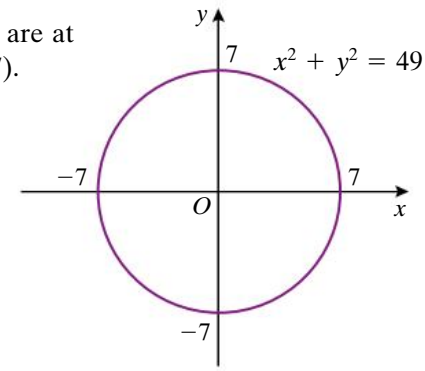
Sketching the graph of a circle

- Find the radius.
- Find the coordinates of the centre.
- Find the four intercepts.
- Draw the axes then draw a circle, marking the centre and intercepts.

Example 3

Sketch the graph of the circle with equation $x^2 + y^2 = 49$.

Working	Reasoning
Radius = 7 Centre is at (0, 0).	$x^2 + y^2 = 49$ $x^2 + y^2 = 7^2$
x-intercepts: Substitute $y = 0$ $x^2 + 0^2 = 49$ $x^2 = 49$ $x = \pm 7$ The x-intercepts are at $(-7, 0)$ and $(7, 0)$.	Where a graph crosses the x-axis, $y = 0$. The circle crosses the x-axis twice, that is, there are two x-intercepts. Include the positive and negative values for x.
y-intercepts: Substitute $x = 0$ $0^2 + y^2 = 49$ $y^2 = 49$ $y = \pm 7$ The y-intercepts are at $(0, -7)$ and $(0, 7)$.	Where a graph crosses the y-axis, $x = 0$. The circle crosses the y-axis twice, that is, there are two y-intercepts. Include the positive and negative values for x.
	Draw the axes then draw a circle, marking the centre and intercepts.

Horizontal translation of the circle

Circle translation

Circle graph reflection

The circle may be translated either horizontally or vertically (or in both directions) so that the centre is not at (0, 0).

When we consider translations of the circle $x^2 + y^2 = r^2$ it is useful to compare translation of the parabola $y = x^2$. The graph of $y = x^2$ becomes the graph of $y = (x - 3)^2$ when it is translated 3 units to the right. Similarly the graph of $x^2 + y^2 = 7^2$ becomes the graph of $(x - 3)^2 + y^2 = 7^2$ when it is translated 3 units to the right.

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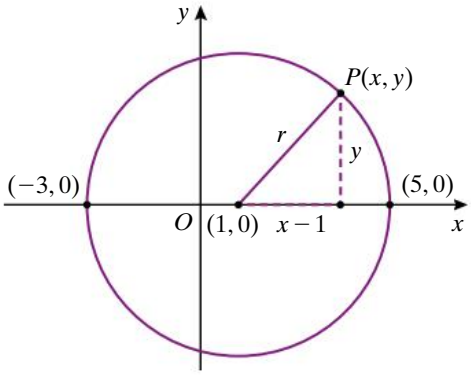
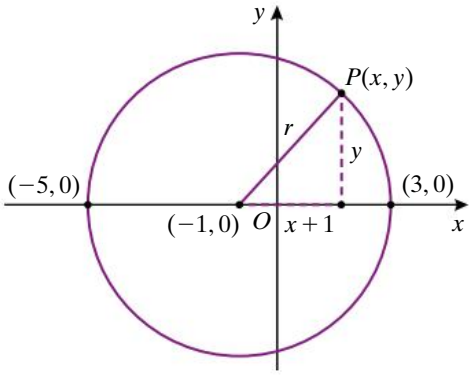
Circle A has been translated 1 unit to the right. It still has a radius of 4 but the centre is now $(1, 0)$. Note that the sides of the right-angled triangle are now $(x - 1)$ and y instead of x and y, where P is the point (x, y). Using Pythagoras' theorem, $(x - 1)^2 + y^2 = 4^2$. This could be written as $(x - 1)^2 + y^2 = 16$.

Circle B has been translated 1 unit to the left. It still has a radius of 4 but the centre is now $(1, 0)$. The sides of the right-angled triangle are now $(x + 1)$ and y instead of x and y. Using Pythagoras' theorem, $(x + 1)^2 + y^2 = 4^2$ or $(x + 1)^2 + y^2 = 16$.



Circle A
centre $(1, 0)$; radius 4
$(x - 1)^2 + y^2 = 16$



Circle B
centre $(-1, 0)$; radius 4
$(x + 1)^2 + y^2 = 16$

Example 4

a What is the equation of the circle with centre $(3, 0)$ and radius 4?
b Find the x-intercepts?
c Find the y-intercepts?
d Sketch the graph.

Working	Reasoning
a $(x - 3)^2 + y^2 = 4^2$ $(x - 3)^2 + y^2 = 16$	The general equation for a circle with radius r and centre $(0, 0)$ is $x^2 + y^2 = r^2$. The circle has been translated 3 units to the right.
b Substitute $y = 0$ $(x - 3)^2 + y^2 = 16$ $(x - 3)^2 = 16$ $(x - 3)^2 - 16 = 0$ $(x - 3 + 4)(x - 3 - 4) = 0$ $(x + 1)(x - 7) = 0$ The x-intercepts are at $x = -1$ and $x = 7$.	We find the x-intercepts for any graph by substituting $y = 0$ and solving for x. Factorise the difference of squares. Solve for x.

continued

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Example 4 continued

Working

c Substitute $x = 0$

$$(x - 3)^2 + y^2 = 16$$
$$(0 - 3)^2 + y^2 = 16$$
$$9 + y^2 = 16$$
$$y^2 - 7 = 0$$
$$(y + \sqrt{7})(y - \sqrt{7}) = 0$$

The y-intercepts are at $y = -\sqrt{7}$ and $y = \sqrt{7}$

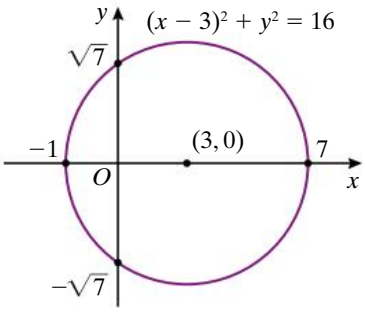
Reasoning

To find the y-intercepts substitute $x = 0$ in the equation and solve for y.

If the answer is irrational then evaluate to one decimal place to see approximately where it is, *but* label with exact value.

Factorise the difference of squares.

d



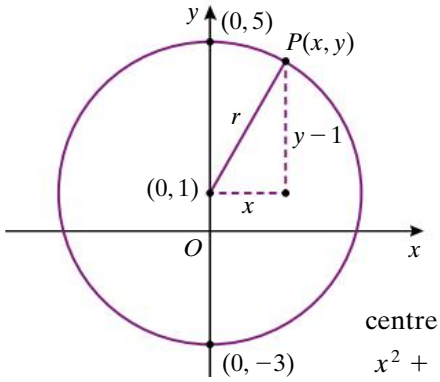
Draw and label the axes.
Mark and label the centre at $(3, 0)$.
Mark and label the two x-intercepts at $(-1, 0)$ and $(7, 0)$.
Mark the two y-intercepts at $(0, \sqrt{7})$ and $(0, -\sqrt{7})$.
Draw the circle.
$\sqrt{7} \sim 2.6$

Vertical translation of the circle

Circle translation

To translate the circle in a vertical direction we must add or subtract a number from y inside the brackets in the same way as we did for translation in the horizontal direction. For example, the graph of $x^2 + y^2 = 7^2$ becomes the graph of $x^2 + (y - 3)^2 = 7^2$ when it is translated 3 units up.

Circle C has been translated 1 unit up. The sides of the right-angled triangle are now x and $(y - 1)$ instead of x and y. Using Pythagoras' theorem, $x^2 + (y - 1)^2 = 4^2$ or $x^2 + (y - 1)^2 = 16$.



Circle C
centre (0, 1); radius 4
$x^2 + (y - 1)^2 = 16$

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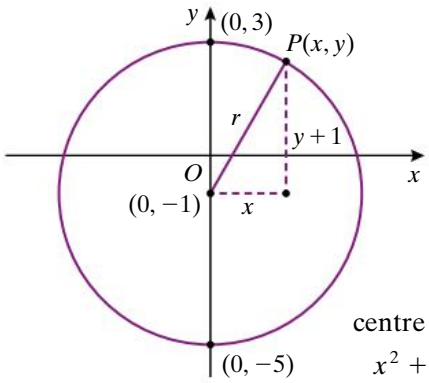
Circle D has been translated 1 unit down. The sides of the right-angled triangle are now x and $(y + 1)$ instead of x and y. Using Pythagoras' theorem, $x^2 + (y + 1)^2 = 4^2$ or $x^2 + (y + 1)^2 = 16$.



Circle D

centre $(0, -1)$; radius 4

$x^2 + (y + 1)^2 = 16$

Example 5

a What is the equation of the circle with centre $(0, -2)$ and radius 3?

b Find the x-intercepts.

c Find the y-intercepts.

d Sketch the graph.

Working

a $x^2 + (y + 2)^2 = 3^2$

$x^2 + (y + 2)^2 = 9$

b Substitute $y = 0$,

$$x^2 + (y + 2)^2 = 9$$
$$x^2 + (0 + 2)^2 = 9$$
$$x^2 + 4 = 9$$
$$x^2 - 5 = 0$$
$$(x + \sqrt{5})(x - \sqrt{5}) = 0$$

The x-intercepts are at $x = -\sqrt{5}$ and $x = \sqrt{5}$.

c Substitute $x = 0$,

$$x^2 + (y + 2)^2 = 9$$
$$0 + (y + 2)^2 = 9$$
$$(y + 2)^2 - 9 = 0$$
$$(y + 2 + 3)(y + 2 - 3) = 0$$
$$(y + 5)(y - 1) = 0$$

The y-intercepts are at $y = -5$ and $y = 1$.

Reasoning

The general equation for a circle with radius r and centre $(0, 0)$ is $x^2 + y^2 = r^2$. The circle has been translated 2 units down.

We find the x-intercepts for any graph by substituting $y = 0$ and solving for x.

Factorise the difference of squares.

To find the y-intercepts substitute $x = 0$ in the equation and solve for y.

Factorise the difference of squares.

Alternative method:
The y-intercepts of the circle $x^2 + y^2 = 3^2$ are at $y = -3$ and $y = 3$. If the circle is translated 2 units down, the y-intercepts will be at $y = -5$ and $y = 1$.

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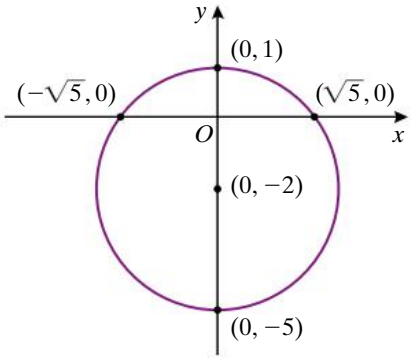
Example 5 continued

Working

d



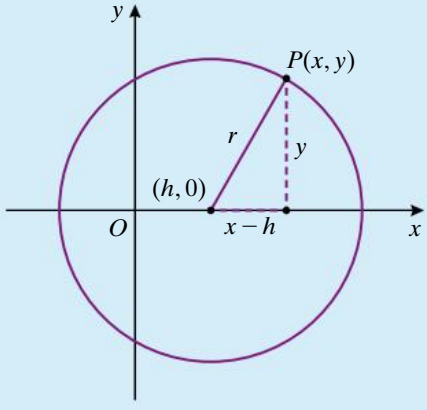
Reasoning

Draw and label the axes.
Mark and label the centre at $(0, -2)$.
Mark and label the two x-intercepts at $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$.
$\sqrt{5} \simeq 2.2$
Mark and label the two y-intercepts at $(0, 1)$ and $(0, -5)$.
Draw the circle.

Horizontal translation of a circle

If the circle $x^2 + y^2 = r^2$ is translated h units to the right, the equation is $(x - h)^2 + y^2 = r^2$.

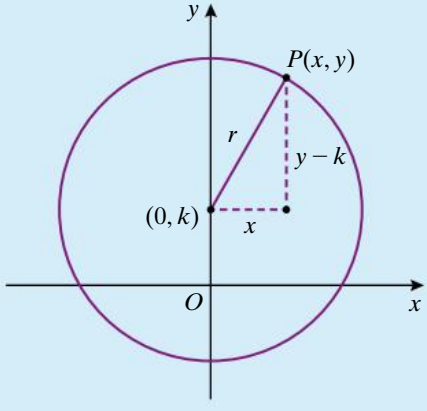
The centre is at the point $(h, 0)$.



Vertical translation of a circle

If the circle $x^2 + y^2 = r^2$ is translated k units up, the equation is $x^2 + (y - k)^2 = r^2$.

The centre is at the point $(0, k)$.



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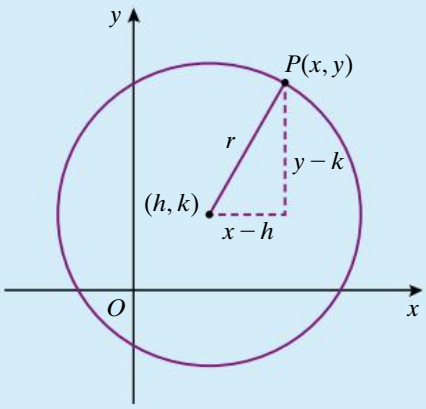
If the circle is translated both horizontally and vertically, the equation is $(x - h)^2 + (y - k)^2 = r^2$.

Horizontal and vertical translation of a circle

If the circle $x^2 + y^2 = r^2$ is translated h units to the right and k units up, the equation is $(x - h)^2 + (y - k)^2 = r^2$.

The centre is at the point (h, k).



Example 6

a What is the equation for a circle with radius 7 and centre $(-3, 4)$?

b Find any intercepts.

c Sketch the graph.

Working

a

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - (-3))^2 + (y - 4)^2 = 7^2$$
$$(x + 3)^2 + (y - 4)^2 = 49$$

b Substitute $y = 0$

$$(x + 3)^2 + (y - 4)^2 = 49$$
$$(x + 3)^2 + (0 - 4)^2 = 49$$
$$(x + 3)^2 + 16 = 49$$
$$(x + 3)^2 - 33 = 0$$
$$(x + 3 + \sqrt{33})(x + 3 - \sqrt{33}) = 0$$

$x = -3 - \sqrt{33}$ and $x = -3 + \sqrt{33}$

The x-intercepts are at $-3 - \sqrt{33}$ and $-3 + \sqrt{33}$.

Reasoning

The equation for a circle with centre at (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

Find the x-intercepts by substituting $y = 0$ and solving for x.

continued

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Example 6 continued

Working

Substitute $x = 0$

$$(x + 3)^2 + (y - 4)^2 = 49$$
$$(0 + 3)^2 + (y - 4)^2 = 49$$
$$9 + (y - 4)^2 = 49$$
$$(y - 4)^2 - 40 = 0$$

$$(y - 4 + \sqrt{40})(y - 4 - \sqrt{40}) = 0$$

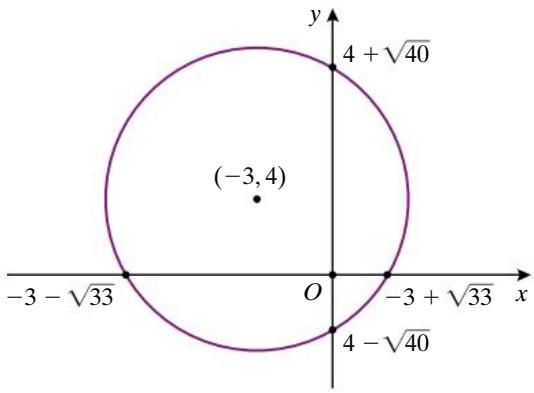
$y = 4 - \sqrt{40}$ and $y = 4 + \sqrt{40}$

The y-intercepts are at $4 - \sqrt{40}$ and $4 + \sqrt{40}$.

Reasoning

Find the y-intercepts by substituting $x = 0$ and solving for y.

c $(x + 3)^2 + (y - 4)^2 = 49$



Draw and label the axes.
Mark and label the centre at $(-3, 4)$.
Mark and label the two x-intercepts at $-3 - \sqrt{33}$ and $-3 + \sqrt{33}$.
$-3 - \sqrt{33} \simeq -5.4$ and
$-3 + \sqrt{33} \simeq 2.7$
Mark and label the two y-intercepts at $4 - \sqrt{40}$ and $4 + \sqrt{40}$.
$4 - \sqrt{40} \simeq -2.3$ and
$4 + \sqrt{40} \simeq 10.3$
Draw the circle.

Given horizontal and vertical translations of a circle, a circle may just touch an axis at one point. In some cases, the circle may just touch both axes.

Example 7

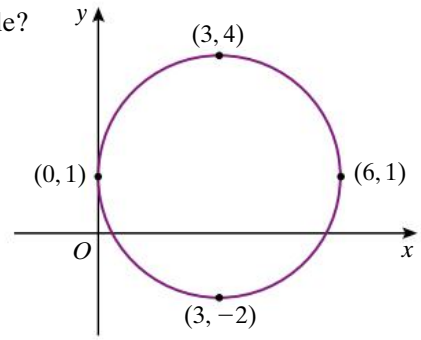
a What are the coordinates of the centre of this circle?

b What is the radius?

c What is the equation of the circle?

d Find the x-intercepts.

e Show that the circle just touches the y-axis at one point.



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Example 7 continued

Working	Reasoning
a Centre: (3, 1)	The x-coordinate of the centre is the midpoint of $x = 0$ and $x = 6$. The y-coordinate of the centre is the midpoint of $y = -2$ and $y = 4$.
b Radius: 3	The diameter of the circle is 6.
c Equation: $(x - 3)^2 + (y - 1)^2 = 9$	The equation for a circle with centre at (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.
d Substitute $y = 0$ $(x - 3)^2 + (y - 1)^2 = 9$ $(x - 3)^2 + (0 - 1)^2 = 9$ $(x - 3)^2 + 1 = 9$ $(x - 3)^2 - 8 = 0$ $(x - 3 - \sqrt{8})(x - 3 + \sqrt{8}) = 0$ The x-intercepts are at $x = 3 + \sqrt{8}$ and $x = 3 - \sqrt{8}$.	Find the x-intercepts by substituting $y = 0$ and solving for x. Factorise the difference of squares.
e Substitute $x = 0$, $(x - 3)^2 + (y - 1)^2 = 9$ $(0 - 3)^2 + (y - 1)^2 = 9$ $9 + (y - 1)^2 = 9$ $(y - 1)^2 = 0$ $y = 1$ There is only one solution, so the circle just touches the y-axis at $y = 1$.	Find any y-intercepts by substituting $x = 0$ and solving for y.

Sometimes the circle does not cross or touch either axis. In these cases, when we try to solve the equation to find the x-intercepts or y-intercepts, we obtain a sum of squares.

Example 8

Show that the circle with equation $(x - 4)^2 + (y + 5)^2 = 9$ does not cross or touch either axis.

Working	Reasoning
Substitute $y = 0$ $(x - 4)^2 + (y + 5)^2 = 9$ $(x - 4)^2 + (0 + 5)^2 = 9$ $(x - 4)^2 + 25 = 9$ $(x - 4)^2 + 16 = 0$	Find the x-intercepts by substituting $y = 0$ and solving for x.
A sum of squares has no real factors so there are no x-intercepts.	A sum of squares has no real factors. The sum of squares cannot be factorised.

continued

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Example 8 continued

Working

Substitute $x = 0$

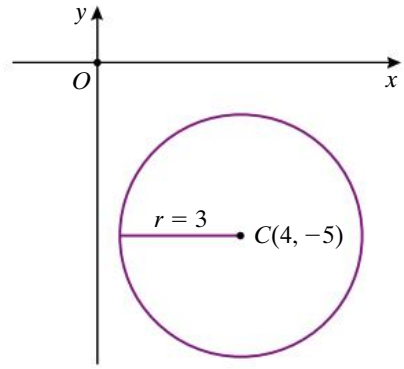
$$(x - 4)^2 + (y + 5)^2 = 9$$
$$(0 - 4)^2 + (y + 5)^2 = 9$$
$$16 + (y + 5)^2 = 9$$
$$(y + 5)^2 + 7 = 0$$

A sum of squares has no real factors so there are no y-intercepts.

Thus, the circle with equation $(x - 4)^2 + (y + 5)^2 = 9$ does not cross or touch either axis.

Reasoning

Find the y-intercepts by substituting $x = 0$ and solving for y.



If the equation for a circle is not in the form $(x - h)^2 + (y - k)^2 = r^2$ it can be converted to this form by completing both squares. The centre and radius can then be identified.

Example 9

Complete both squares of $x^2 + 6x + y^2 - 4y = 12$ to rearrange the equation into the form $(x - h)^2 + (y - k)^2 = r^2$ and hence, find the centre and radius of the circle.

Working

$$x^2 + 6x + y^2 - 4y = 12$$
$$x^2 + 6x + 3^2 - 3^2 + y^2 - 4y + (-2)^2 - (-2)^2 = 12$$
$$(x + 3)^2 - 9 + (y - 2)^2 - 4 = 12$$
$$(x + 3)^2 + (y - 2)^2 = 12 + 4 + 9$$
$$(x + 3)^2 + (y - 2)^2 = 25$$

The centre is at $(-3, 2)$ and the radius is 5.

Reasoning

Halve the coefficient of the x term and square it. Add and subtract 3^2. Halve the coefficient of the y term and square it. Add and subtract $(-2)^2$.

Finding rules for graphs reflected in the axes

Circle reflection

In *MathsWorld 7* reflections on the Cartesian plane showed that:

- reflection of a point in the y-axis results in a change in sign of the x-coordinate while the y-coordinate stays the same
- reflection of a point in the x-axis results in a change in sign of the y-coordinate while the x-coordinate stays the same.

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Reflection in the axes

When a graph is reflected in the y-axis, the equation for the reflected graph is found by replacing x by $-x$.

When a graph is reflected in the x-axis, the equation for the reflected graph is found by replacing y by $-y$.

Example 10

Consider the following graph.

a Find the coordinates of the centre.

b Find the equation for the graph.

c If the circle is reflected in the y-axis, find

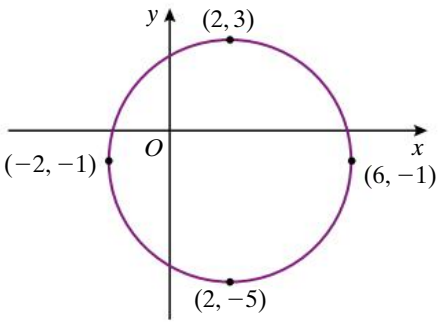
i the equation

ii the coordinates of the centre

d If the circle is reflected in the x-axis, find

i the equation

ii the coordinates of the centre



Working

a $(2, -1)$

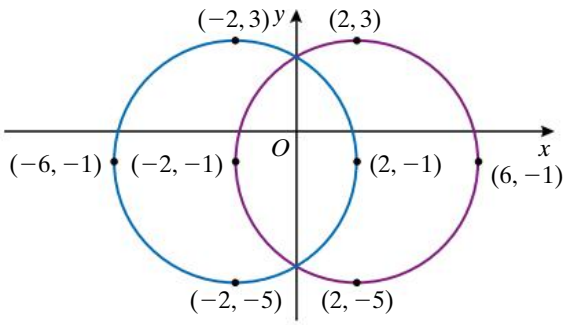
b $(x - 2)^2 + (y + 1)^2 = 16$

c i

$$(-x - 2)^2 + (y + 1)^2 = 16$$
$$(-(x + 2))^2 + (y + 1)^2 = 16$$
$$(x + 2)^2 + (y + 1)^2 = 16$$

Reasoning

Replace x by $-x$.
Take out the negative sign. Squaring $-(x + 2)$ gives $(x + 2)^2$.



ii Centre: $(-2, -1)$

d i

$$(x - 2)^2 + (-y + 1)^2 = 16$$
$$(x - 2)^2 + (-(y - 1))^2 = 16$$
$$(x - 2)^2 + (y - 1)^2 = 16$$

Replace y by $-y$.
Take out the negative sign. Squaring $-(y - 1)$ gives $(y - 1)^2$.

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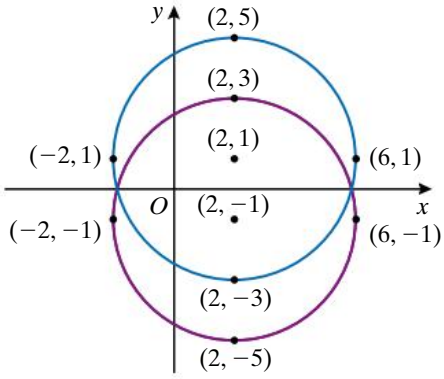
Example 10 continued

Working **Reasoning**

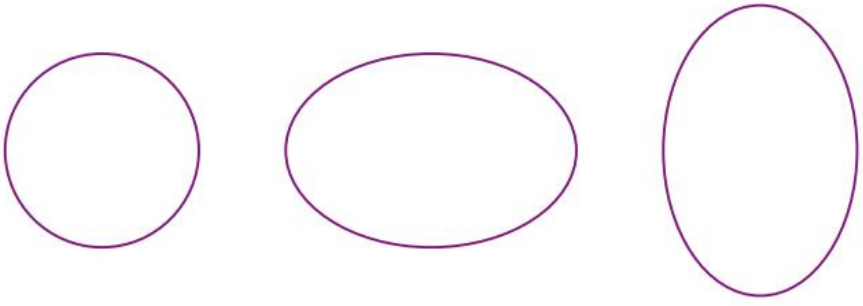


ii Centre: (2,1)

Stretching the circle: the ellipse

Circle stretching

If we stretch a circle, the shape formed is called an **ellipse**.



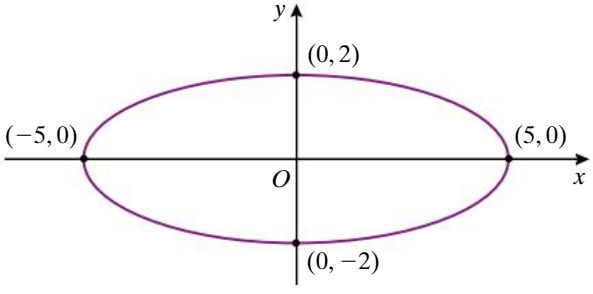
Stretched horizontally Stretched vertically

We now look at the relationship between the equation for a circle and the equation for an ellipse.

For a circle with its centre at the origin, the equation is $x^2 + y^2 = r^2$, where r is the radius.

Dividing both sides by r^2 we obtain $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$. In the case of the ellipse, the radius is not the same at every point on the ellipse.

Consider the ellipse on the coordinate plane, with centre $(0, 0)$ intercepts $(-5, 0)$, $(5, 0)$, $(0, -2)$ and $(0, 2)$.



The rule for this ellipse is $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

This can also be written as $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$.

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The general equation for an ellipse with centre at $(0, 0)$ and intercepts $(-a, 0), (a, 0), (0, -b)$ and $(0, b)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If $a = b$, the equation is the equation for a circle.

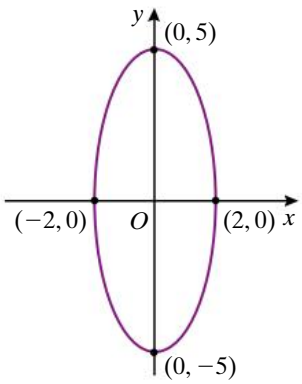
If $a > b$, the circle is stretched horizontally to become an ellipse.

If $a < b$, the circle is stretched vertically to become an ellipse.

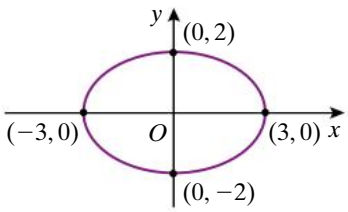
Example 11

State the rule for each of these ellipses.

a



b



Working

a $\frac{x^2}{4} + \frac{y^2}{25} = 1$

b $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Reasoning

$a = 2, b = 5$ in the rule $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$a = 3, b = 2$ in the rule $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

exercise 12.1

LINKS TO Examples 1, 2a

1 Write the equation for each of the following circles.

a centre $(0, 0)$, radius 3 **b** centre $(0, 0)$, radius 7 **c** centre $(0, 0)$, radius 6

d centre $(0, 0)$, radius 10 **e** centre $(0, 0)$, radius $\sqrt{12}$ **f** centre $(0, 0)$, radius $\sqrt{50}$

LINKS TO Examples 2b, 2c

2 **a** Show that the points $(2, 5)$ and $(5, 2)$ lie on the circle $x^2 + y^2 = 29$

b Show that the points $(-1, 4)$ and $(-4, -1)$ lie on the circle $x^2 + y^2 = 17$

c Show that the point $(4, -1)$ does not lie on the circle $x^2 + y^2 = 15$

LINKS TO Example 3

3 For each of these circles

i find the radius **ii** sketch the graph

a $x^2 + y^2 = 64$ **b** $x^2 + y^2 = 81$ **c** $x^2 + y^2 = 1$

d $x^2 + y^2 = 25$ **e** $x^2 + y^2 = 30$ **f** $x^2 + y^2 = 10$

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LINKS TO Example 4

4 For each of these circles

i state the equation **ii** find the x-intercepts

iii find the y-intercepts **iv** sketch the graph.

a centre $(1, 0)$, radius 3 **b** centre $(-3, 0)$, radius 4 **c** centre $(-4, 0)$, radius 6

d centre $(3, 0)$, radius 5 **e** centre $(3, 0)$, radius 4 **f** centre $(-5, 0)$, radius 5

LINKS TO Example 5

5 For each of these circles

i state the equation **ii** find the x-intercepts

iii find the y-intercepts **iv** sketch the graph.

a centre $(0, 4)$, radius 4 **b** centre $(0, -2)$, radius 7 **c** centre $(0, -5)$, radius 5

d centre $(0, 3)$, radius 3 **e** centre $(0, 3)$, radius 5 **f** centre $(0, -4)$, radius 4

6 The equation of this circle is

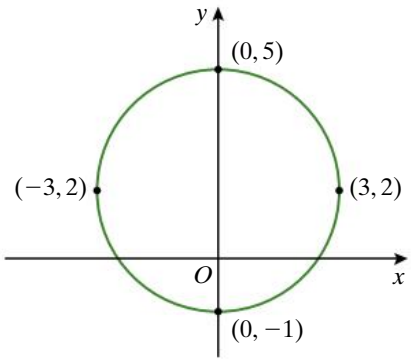
A $(x + 2)^2 + y^2 = 9$

B $(x - 3)^2 + y^2 = 9$

C $x^2 + (y - 3)^2 = 9$

D $x^2 + (y - 2)^2 = 9$

E $x^2 + (y + 2)^2 = 9$



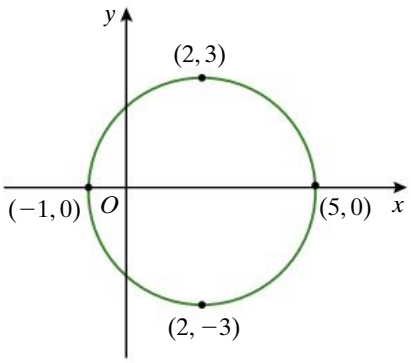
7 For each of these circles

i find the coordinates of the centre

ii find the radius

iii write the equation

a



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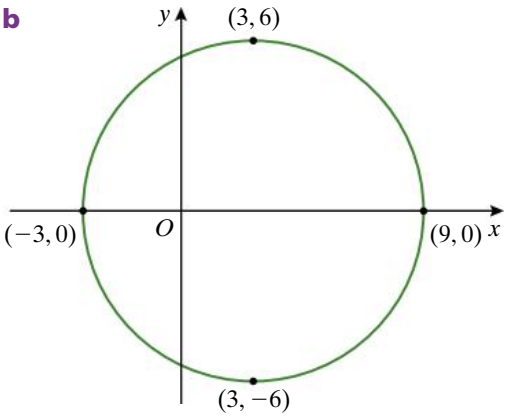

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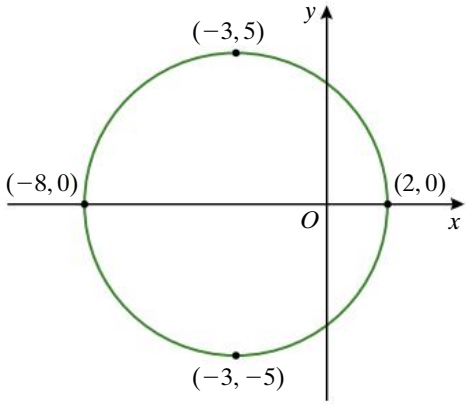
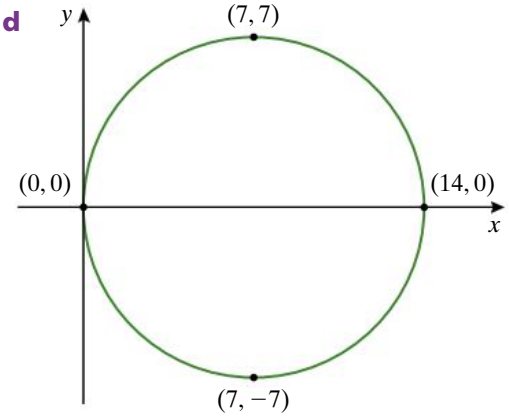
c



d



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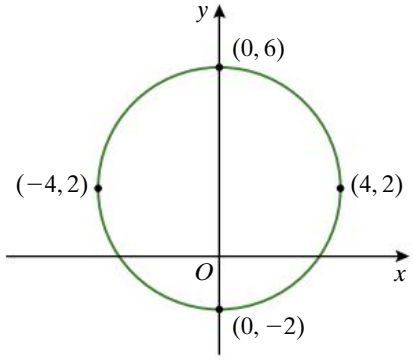

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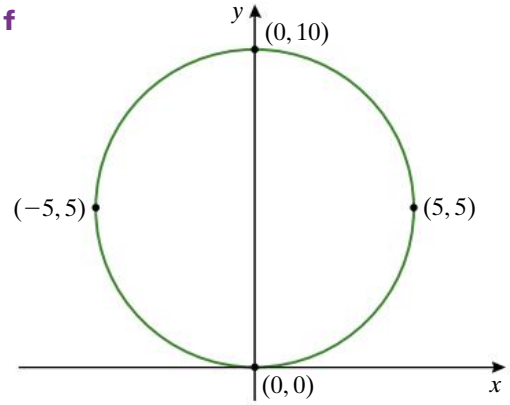

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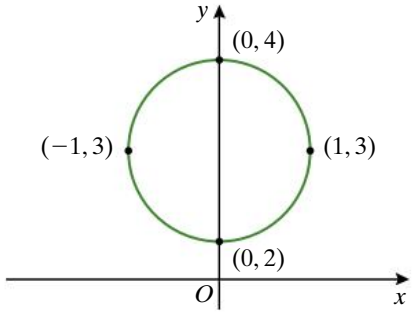

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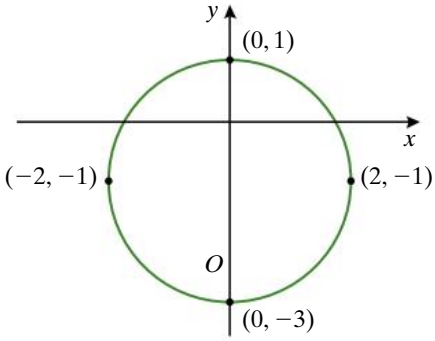

LINKS TO Example 6

8 For each of these circles

i state the equation

ii find any intercepts

iii sketch the graph.

a centre $(2, 3)$ and radius 4

b centre $(-1, 5)$ and radius 2

c centre $(3, -2)$ and radius 5

d centre $(2, 4)$ and radius 7

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9 For each of these circles

i find the centre and radius **ii** find the intercepts

iii sketch the graph, labeling the centre and intercepts

a $(x - 3)^2 + y^2 = 9$
b $x^2 + (y + 4)^2 = 36$
c $x^2 + (y + 1)^2 = 25$
d $(x + 4)^2 + y^2 = 25$
e $x^2 + (y - 2)^2 = 16$
f $(x + 1)^2 + y^2 = 16$
g $x^2 + (y - 5)^2 = 9$
h $(x - 5)^2 + y^2 = 36$
i $(x - 3)^2 + (y - 3)^2 = 9$
j $(x - 4)^2 + (y - 2)^2 = 16$
k $(x + 1)^2 + (y + 2)^2 = 16$
l $(x + 5)^2 + (y - 1)^2 = 25$
m $(x + 4)^2 + (y + 4)^2 = 36$
n $(x - 3)^2 + (y + 4)^2 = 16$
o $(x - 5)^2 + (y + 2)^2 = 9$
p $(x + 5)^2 + (y - 3)^2 = 4$

10 A Ferris wheel has radius 15 m and the base of the wheel is 5 m above the ground.

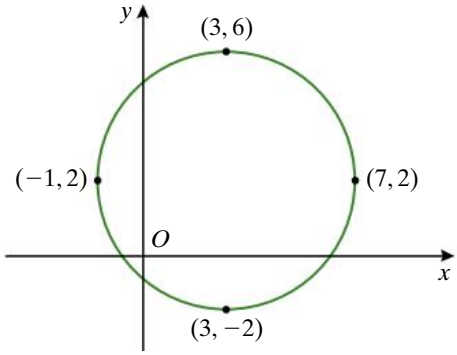
a The origin is placed at the centre of the wheel. Sketch the graph of the circle that represents the outer rim of the wheel.

b What is the equation for the outer rim of the wheel?

c If the x-coordinate of a particular gondola (passenger capsule) is 12 m horizontally from the centre of the wheel, what are the two possible heights of the gondola above the ground?

11 The equation of this circle is

A $(x + 2)^2 + (y - 3)^2 = 4$
B $(x - 2)^2 + (y + 3)^2 = 16$
C $(x - 2)^2 + (y - 3)^2 = 16$
D $(x + 3)^2 + (y - 2)^2 = 4$
E $(x - 3)^2 + (y - 2)^2 = 16$



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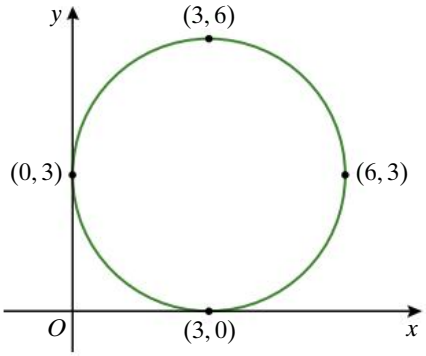
LINKS TO Example 7

12 For each of these circles

- **i** find the coordinates of the centre
- **ii** find the radius
- **iii** write the equation

a



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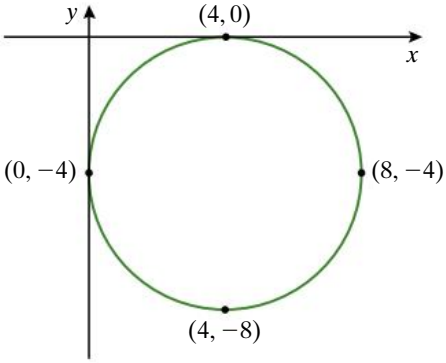

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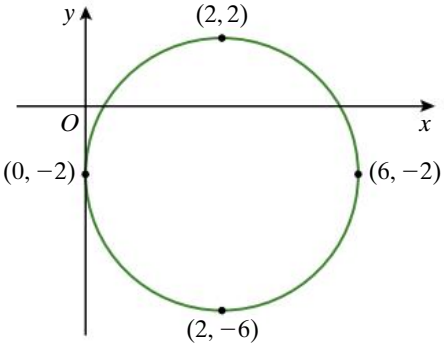

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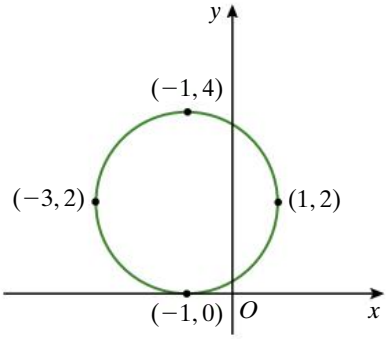

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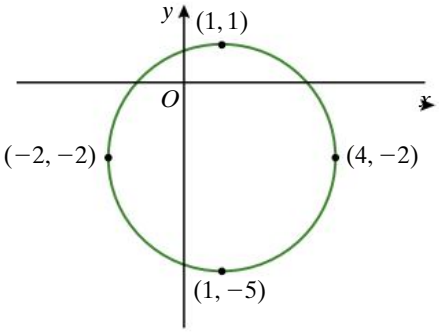

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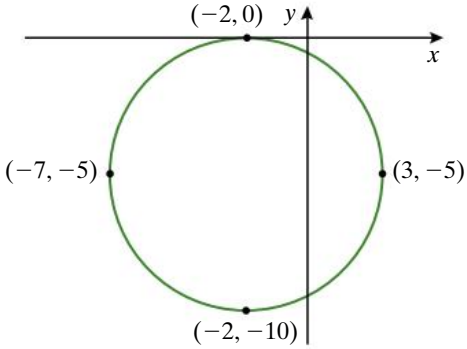

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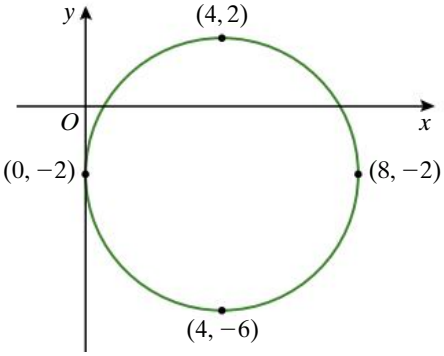

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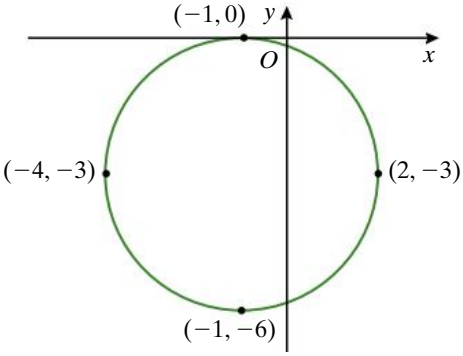

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10 A

LINKS TO Example 8

13 Show that the circle with equation $(x + 4)^2 + (y - 6)^2 = 9$ does not cross or touch either axis.

LINKS TO Example 9

14 For each of these equations

i put the equation into the form $(x - h)^2 + (y - k)^2 = r^2$.

ii state the centre and radius

a $x^2 + 6x + y^2 + 2y = 6$

b $x^2 - 8x + y^2 + 4y = 29$

c $x^2 + 2x + y^2 + 14y = 14$

d $x^2 - 4x + y^2 + 6y = 12$

e $x^2 + 10x + y^2 - 12y = 3$

f $x^2 - 8x + y^2 - 2y = 19$

LINKS TO Example 10

15 For each of the circles in question **12**, state the equation if the circle is reflected

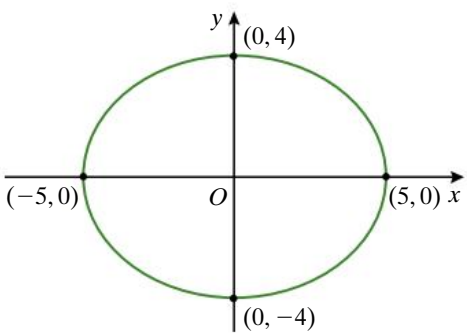
i in the y-axis

ii in the x-axis

LINKS TO Example 11

16 State the equation for each of these ellipses.

a



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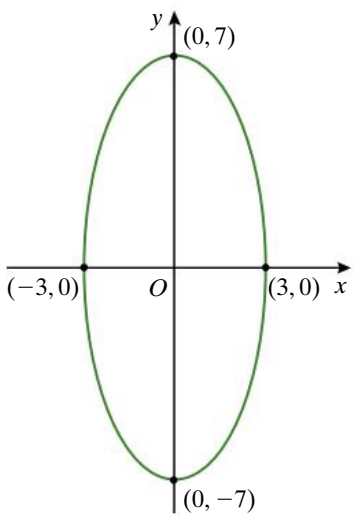

exercise 12.1 challenge

17 $P(a, b)$ where $b \neq 0$, is a point on the circle $(x - 2)^2 + y^2 = 36$. The circle cuts the x-axis at points A and B. Use gradients to show that $\angle APB$ is a right angle.

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12.2 Rectangular hyperbolas

The hyperbola $y = \frac{1}{x}$

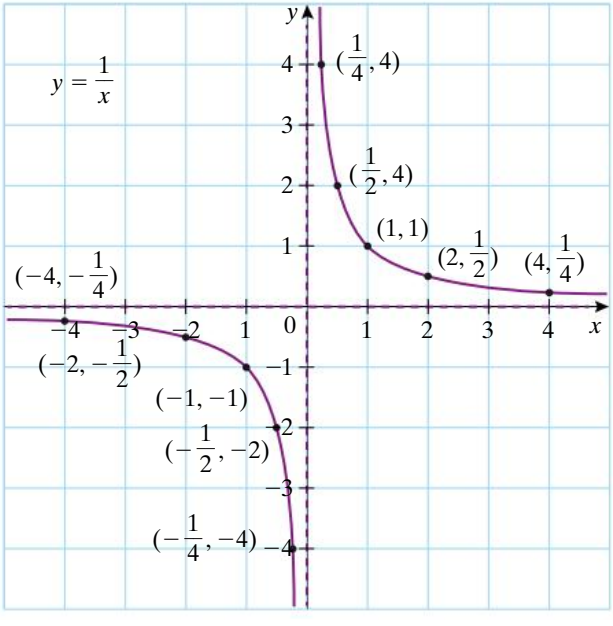
Hyperbola stretching

The rule $y = \frac{1}{x}$ shows a reciprocal relationship between x and y. As the value of x increases, the value of y decreases. The corresponding graph is called a **rectangular hyperbola**. From now on we will refer to these as hyperbolas. ('Rectangular' refers to the fact that the asymptotes are at right angles to each other. There are other hyperbolas where this is not the case). For simplicity, from now on we will use the name hyperbola rather than rectangular hyperbola.

When $x = 0$, $y = \frac{1}{0}$, that is, y is undefined. This means that the graph does not cross the y-axis.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-5	5	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

We see that the hyperbola has two separate branches. When x is positive, y is positive so one branch of the graph is in the first quadrant. When x is negative, y is negative, so there is another branch of the graph in the third quadrant. The x- and y-axes are asymptotes, that is, the graph approaches each axis but never touches it.



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10 A

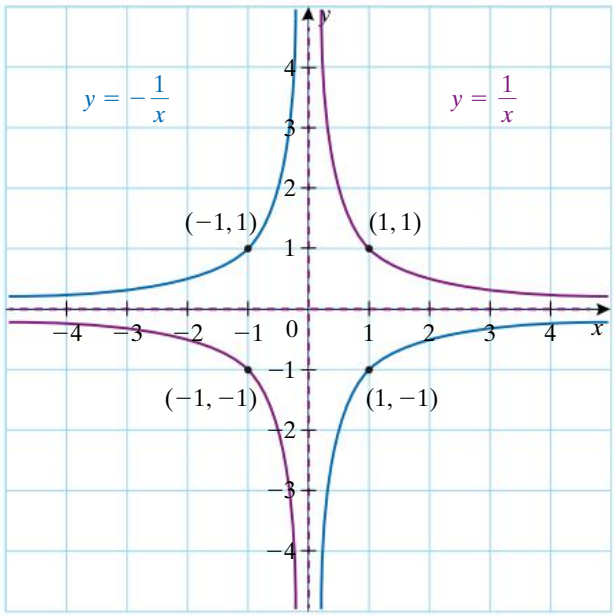
Hyperbola reflection

The hyperbola $y = \frac{a}{x}$

Reflecting the graph of $y = \frac{1}{x}$ in the axes

As we saw in section 12.1, when a graph is reflected in the y-axis, the equation for the reflected graph is found by replacing x by $-x$ and when the graph is reflected in the x-axis, the equation for the reflected graph is found by replacing y by $-y$.

In the case of $y = \frac{1}{x}$, reflection in the y-axis gives $y = \frac{1}{-x}$, that is, $y = -\frac{1}{x}$. Reflection in the x-axis gives $-y = \frac{1}{x}$, which is also $y = -\frac{1}{x}$.



Example 12

Complete the table of values for the rule $y = -\frac{1}{x}$ then plot the graph. Join the points with a smooth curve.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y										

continued

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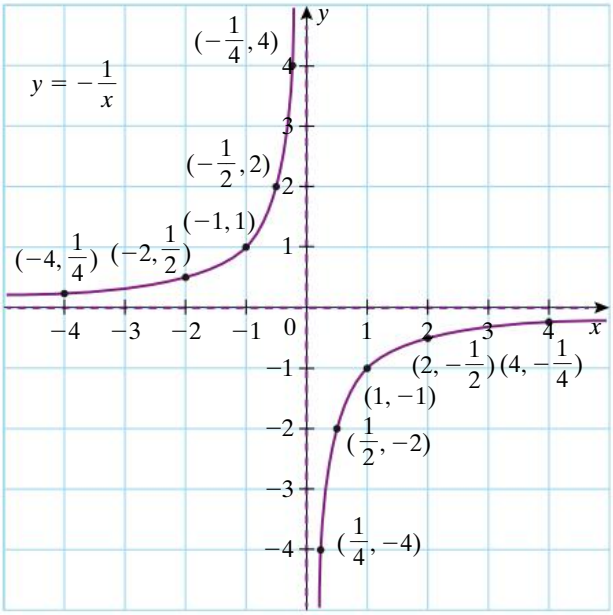
Example 12 continued

Working

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$

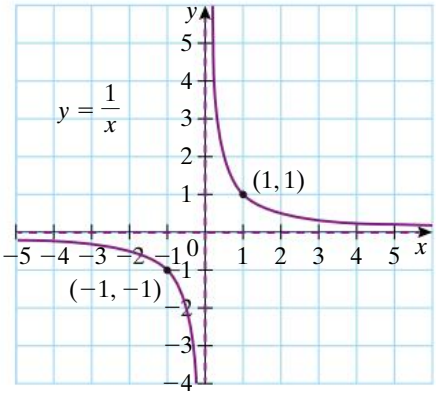


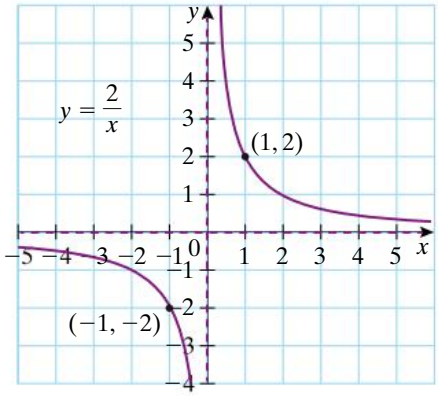
Reasoning

When x is positive, y is negative so one branch of the graph is in the fourth quadrant. When x is negative, y is positive so the other branch of graph is in the second quadrant.
The x- and y-axes are asymptotes.

Stretching the hyperbola

Comparing the graphs of $y = \frac{1}{x}$ and $y = \frac{2}{x}$, we see that each y-coordinate has been multiplied by 2 in the graph of $y = \frac{2}{x}$, that is, the graph has been stretched by a factor of 2 in the vertical direction.





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In general, the graph of $y = \frac{a}{x}$ passes through the points $(1, a)$ and $(-1, -a)$. The x-axis and the y-axis are the asymptotes.
For $a > 0$, the graph is in the first and third quadrants.
For $a < 0$, the graph is in the second and fourth quadrants.

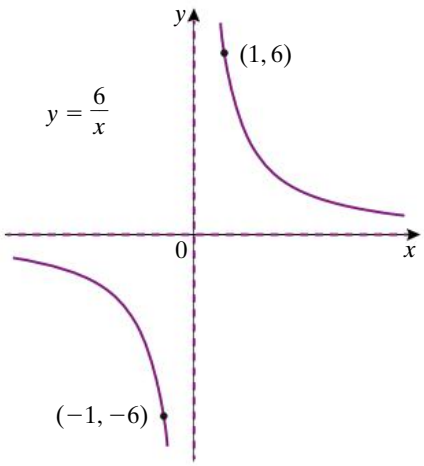
Sketching a hyperbola

When sketching a graph of a hyperbola, we sketch the shape of the hyperbola then mark two points on the graph, one on each branch. The asymptotes are shown as dotted lines.

Example 13

Sketch the graph of $y = \frac{6}{x}$, labelling one point on each branch.

Working



Reasoning

The x and y axes are asymptotes. The graph approaches the axes but never quite reaches them.
When x is positive, y is positive.
When x is negative, y is negative.

If $x = 1$, $y = \frac{6}{1} = 6$

If $x = -1$, $y = \frac{6}{-1} = -6$

Label $(1, 6)$ and $(-1, -6)$.

Horizontal translation of the hyperbola

Hyperbola translation

The hyperbola may be translated either horizontally or vertically (or in both directions).
It is useful to compare translations of the hyperbola with translations of the parabola $y = x^2$.
We know that the graph of $y = x^2$ becomes the graph of $y = (x - 3)^2$ when it is translated 3 units to the right. We saw similar translations with circles in section 12.1. Similarly the graph of $y = \frac{1}{x}$ becomes the graph of $y = \frac{1}{x - 3}$ when it is translated 3 units to the right.
The horizontal asymptote stays the same, but the vertical asymptote is now the line $x = 3$.
Each point on the graph has been translated 3 units to the right.

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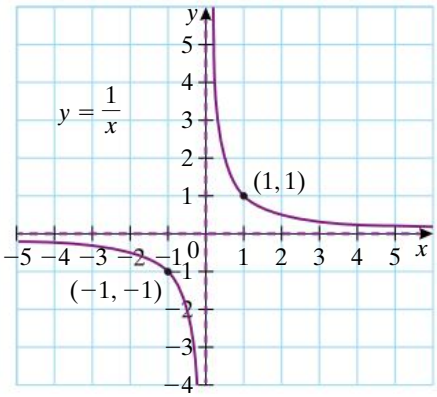

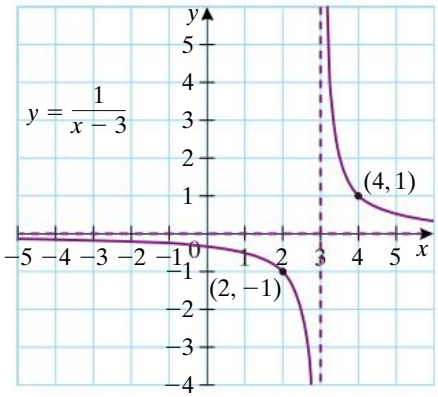

Example 14

Sketch the graph of these hyperbolas

a $y = \dfrac{5}{x-1}$

b $y = -\dfrac{2}{x+4}$

Working

a $y = \dfrac{5}{x-1}$

The asymptotes are the lines $x = 1$ and $y = 0$.

There are no x-intercepts.

y-intercept:

When $x = 0$,

$y = \dfrac{5}{0-1}$

$y = -5 \qquad \therefore (0, -5)$

Two additional points on the graph:

When $x = 6$,

$y = \dfrac{5}{6-1}$

$y = 1 \qquad \therefore (6, 1)$

When $x = -4$,

$y = \dfrac{5}{-4-1}$

$y = -1 \qquad \therefore (-4, -1)$

Reasoning

The graph of $y = \dfrac{5}{x}$ is moved 1 unit to the right.

To find the y-intercept, substitute $x = 0$.

Choose one point on each branch of the hyperbola.

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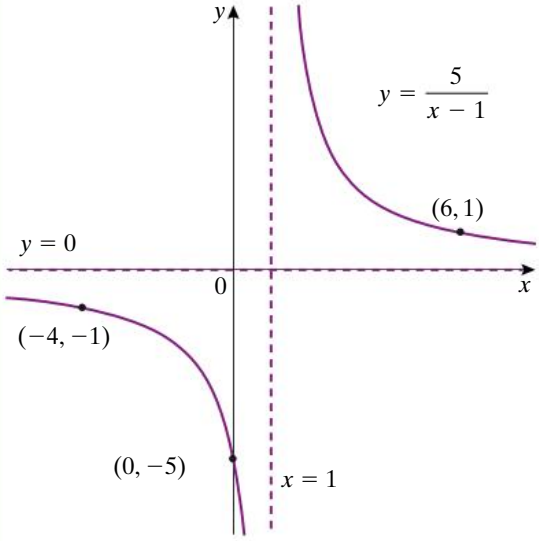
Example 14 continued

Working | **Reasoning**



b $y = -\frac{2}{x + 4}$

The asymptotes are the lines $x = -4$ and $y = 0$.

There are no x-intercepts.

The graph of $y = -\frac{2}{x}$ is moved 4 units to the left.

y-intercept:

When $x = 0$,

$y = -\frac{2}{0 + 4}$

$y = -\frac{1}{2} \quad \therefore \left(0, -\frac{1}{2}\right)$

To find the y-intercept, substitute $x = 0$.

Two additional points on the graph:

When $x = -6$,

$y = -\frac{2}{-6 + 4}$

$y = 1 \quad \therefore (-6, 1)$

When $x = -2$,

$y = -\frac{2}{-2 + 4}$

$y = -1 \quad \therefore (-2, -1)$

Choose one point on each branch of the hyperbola.

continued

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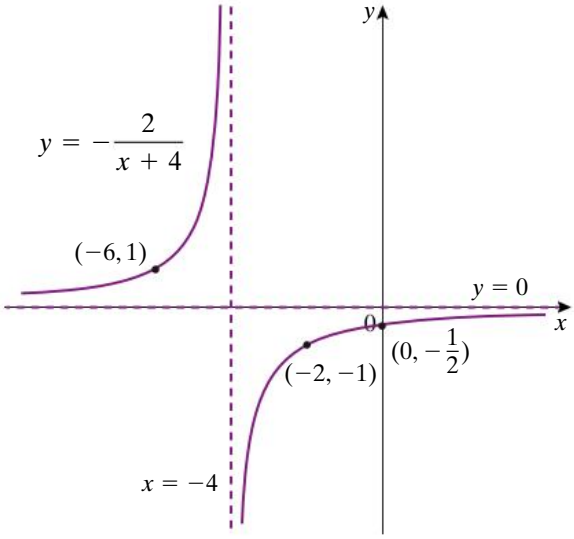
Example 14 continued

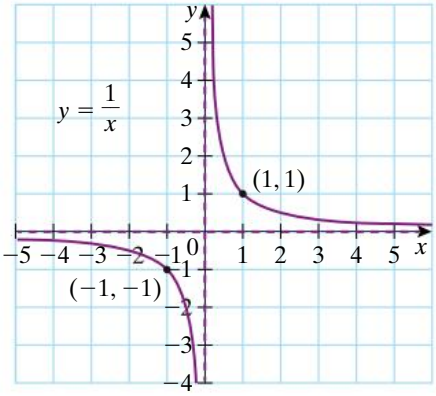
Working **Reasoning**

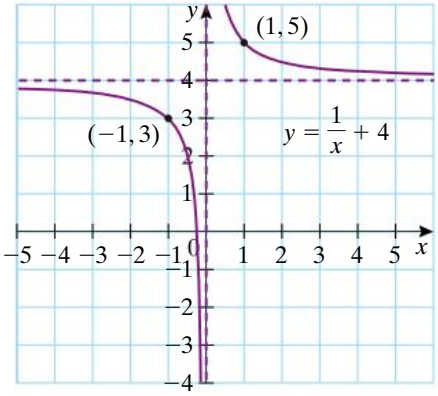


Vertical translation of the hyperbola

Hyperbola translation

Comparing again with the graph of the parabola $y = x^2$, we know that the graph of $y = x^2$ becomes the graph of $y = x^2 + 4$ if it is translated 4 units up. Similarly, adding or subtracting a constant term to $y = \frac{1}{x}$ translates the rectangular hyperbola in a vertical direction. For example, the graph of $y = \frac{1}{x}$ becomes the graph of $y = \frac{1}{x} + 4$, if it is translated 4 units up. The vertical asymptote stays the same, but the horizontal asymptote is now the line $y = 4$. Each point on the graph has been translated 4 units up.





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Example 15

Sketch the graph of these hyperbolas

a $y = \frac{5}{x} + 1$

b $y = \frac{2}{x} - 4$

Working

a $y = \frac{5}{x} + 1$

The vertical asymptote is the line $x = 0$.
The horizontal asymptote is the line $y = 1$.
There is no y-intercept.
x-intercept:

When $y = 0$,

$$0 = \frac{5}{x} + 1$$
$$-1 = \frac{5}{x}$$
$$-x = 5$$
$$x = -5$$

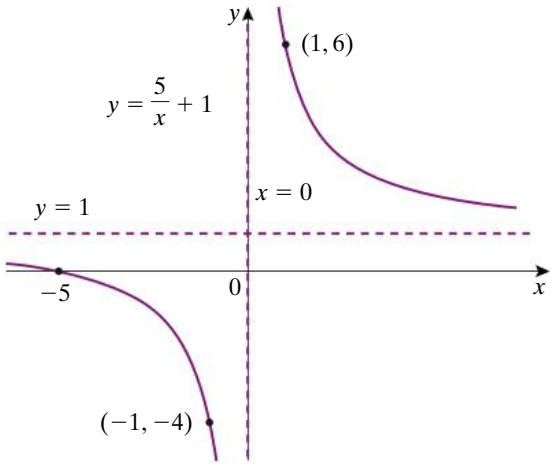
The x-intercept is at $(-5, 0)$
Two points on the graph:

When $x = 1$,

$$y = \frac{5}{1} + 1$$
$$= 6 \qquad \therefore (1,6)$$

When $x = -1$,

$$y = \frac{5}{-1} + 1$$
$$= -4 \qquad (-1,-4)$$



Reasoning

The graph of $y = \frac{5}{x}$ is translated 1 unit up.

To find two points on the graph, substitute, for example, $x = 1$ and $x = -1$.

To find the x-intercept, substitute $y = 0$.

Mark the points $(1,6)$ and $(-1,-4)$.
Mark the x-intercept at $x = -5$.

The graph of $y = \frac{5}{x}$ is translated 1 unit up.

continued

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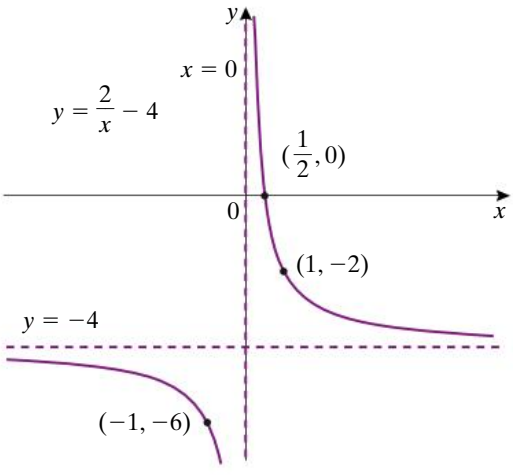
Example 15 continued

Working	Reasoning
b $y = \frac{2}{x} - 4$	The graph of $y = \frac{2}{x}$ is translated 4 units down.
The vertical asymptote is the line $x = 0$.	
The horizontal asymptote is the line $y = -4$.	
x-intercept:	To find the x-intercept, substitute $y = 0$.
When $y = 0$,	
$0 = \frac{2}{x} - 4$	
$\frac{2}{x} = 4$	
$2 = 4x$	
$x = \frac{1}{2}$	
The x-intercept is at $\left(\frac{1}{2}, 0\right)$	
There is no y-intercept.	
Two points on the graph:	To find two points on the graph, substitute, for example, $x = 1$ and $x = -1$.
When $x = 1$,	
$y = \frac{2}{1} - 4$	
$= -2 \quad \therefore (1, -2)$	
When $x = -1$,	
$y = \frac{2}{-1} - 4$	
$= -6 \quad \therefore (-1, -6)$	
	Mark the points $(1, -2)$ and $(-1, -6)$. Mark the x-intercept at $x = \frac{1}{2}$.

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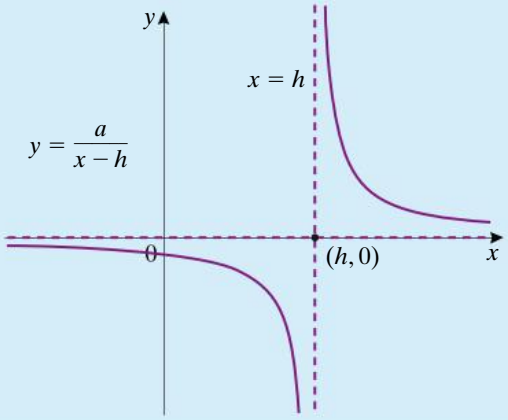
Horizontal translations of hyperbolas

If the graph of the hyperbola $y = \frac{a}{x}$ is translated h units horizontally, the equation is $y = \frac{a}{x - h}$

The vertical asymptote is the line $x = h$.

The horizontal asymptote is the line $y = 0$.

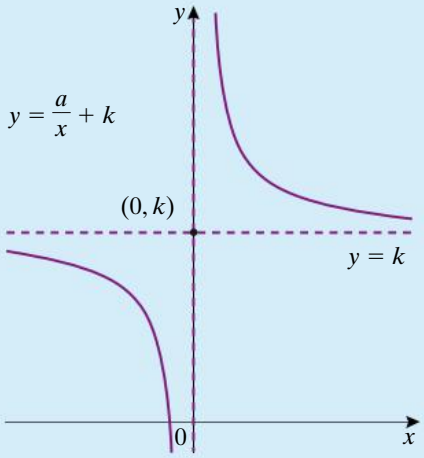


Vertical translations of hyperbolas

If the graph of the hyperbola $y = \frac{a}{x}$ is translated k units vertically, the equation is $y = \frac{a}{x} + k$

The vertical asymptote is the line $x = 0$.

The horizontal asymptote is the line $y = k$.



Translation horizontally and vertically

The graph of the hyperbola $y = \frac{a}{x - h} + k$ is translated h units horizontally and k units vertically.

Reflecting the hyperbola in the axes

Hyperbola reflection

To reflect the graph of a hyperbola in the y-axis, replace x with $-x$ in the equation.

To reflect the graph of a hyperbola in the x-axis, replace y with $-y$ in the equation.

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Example 16

Consider the hyperbola $y = \dfrac{1}{x-2} + 3$

a State the equations of the asymptotes

b If the hyperbola is reflected in the y-axis, find

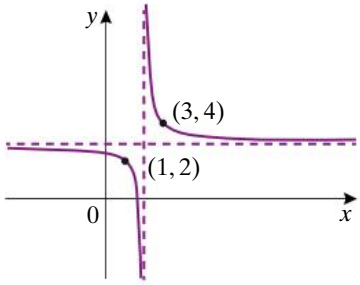
i the equation

ii the equations of the asymptotes

c If the hyperbola is reflected in the x-axis, find

i the equation

ii the equations of the asymptotes



Working

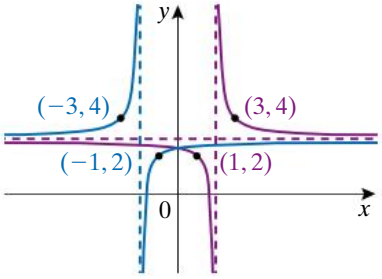
a The asymptotes are the lines $x = 2$, $y = 3$

b i $y = \dfrac{1}{-x-2} + 3$

$y = -\dfrac{1}{x+2} + 3$

Reasoning

Replace x by $-x$.
Take out the negative sign.

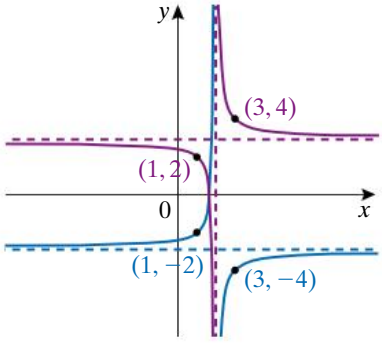


ii The asymptotes are the lines $x = -2$, $y = 3$

c i $-y = \dfrac{1}{x-2} + 3$

$y = -\dfrac{1}{x-2} - 3$

Replace y by $-y$. Divide both sides by -1.



ii The asymptotes are the lines $x = 2$, $y = -3$

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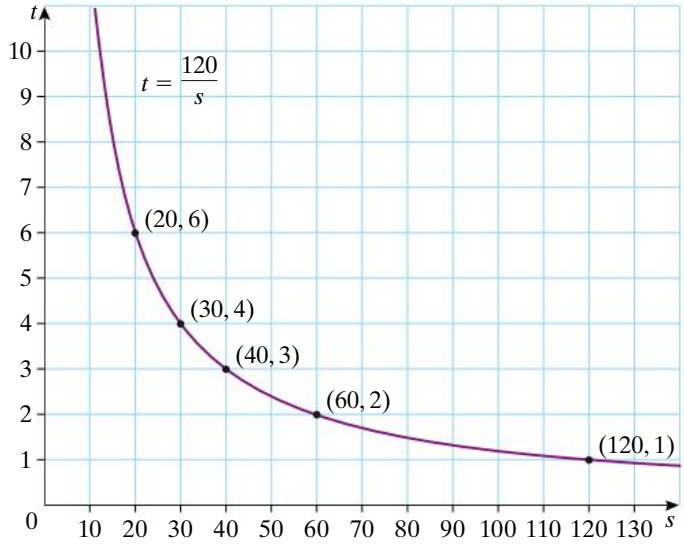
Applications of hyperbolas

If we consider the relationship between speed and time for a given distance travelled, the time taken decreases as the speed increases. If the speed doubles, the time taken is halved. Time taken is proportional to the reciprocal of the speed so the graph of time taken against speed is a hyperbola. In this case, we are interested only in positive values of speed and time, that is, the branch of the hyperbola that is in the first quadrant. Neither the time nor the speed can be zero if the given distance is to be travelled so the axes are the asymptotes of the graph and the graph of distance against speed is a hyperbola.

The hyperbola below shows the time taken, t hours to travel a distance of 120 km at speed s km/h and has the equation $t = \frac{120}{s}$.

Note that the product of the speed and time is always equal to 120.



Example 17

If 20 people work on a particular cleaning task it can be finished in one day. The task will take twice as long if the number of people is halved. Represent the relationship between the number of people who work on the task and the number of days it will take as

a a table of values

b a plotted graph

c a sketch graph

d a rule.

Working

Let p be the number of people working.
Let d be the number of days taken.

Reasoning

First define the variables. Note that p must be a whole number, but d can be a decimal as it is possible for people to work for part of a day.

continued

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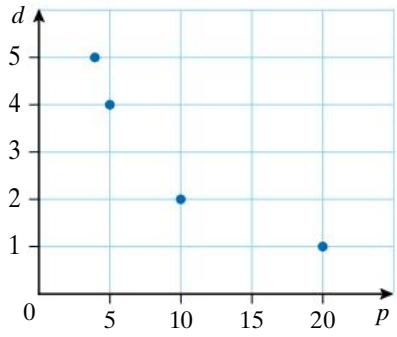
Example 17 continued

Working

a

p	20	10	5	4
d	1	2	4	5

b



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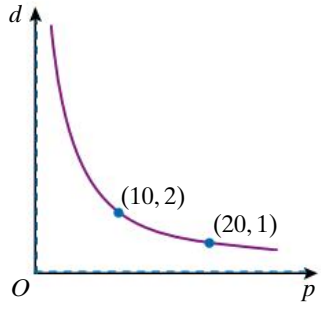

d The rule is $d = \dfrac{20}{p}$.

Reasoning

The number of people is the independent variable. Choose some appropriate values for p and write these in the first row. Work out the matching values for the dependent variable d. Write these values in the second row.

Choose a suitable scale. Label the horizontal axis with the independent variable p and the vertical axis with the dependent variable d.
Plot each point on the grid.
If doesn't make sense to join the points as the number of people must be a whole number. However, points could be added for other integer values of p between 1 and 20.

We can tell from the table that this is a reciprocal function, because each pair of values multiplies to give 20. Also, when the value of p is halved (20, 10, 5), the value of d doubles (1, 2, 4).
Draw and label axes, and add dotted lines to show that they are asymptotes.
Mark two points and draw a hyperbola through them.

The constant product of each pair of values is 20, so the rule is $pd = 20$.
The rule for a function usually begins with the dependent variable, so divide both sides by p to isolate d.

exercise 12.2

LINKS TO Example 12

1 For each of these equations.

i construct a table of values using values of x from -3 to 3

ii plot the points and join with smooth curves, labelling the asymptotes

a $y = \dfrac{6}{x}$ **b** $y = \dfrac{24}{x}$ **c** $y = \dfrac{18}{x}$ **d** $y = \dfrac{30}{x}$

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LINKS TO Example 13

2 Sketch the graph for each of the following rules. Make sure that the axes are labelled correctly. Include one point on each branch of the hyperbola.

a $y = \frac{24}{x}$ **b** $t = \frac{20}{s}$ **c** $P = \frac{16}{V}$ **d** $C = \frac{36}{p}$

LINKS TO Example 14

3 Sketch the graph of these hyperbolas, labelling one point on each branch of the hyperbola, the asymptotes and the y-intercept.

a $y = -\frac{1}{x-1}$ **b** $y = \frac{1}{x-3}$ **c** $y = \frac{1}{x-4}$ **d** $y = \frac{1}{x+4}$

e $y = -\frac{1}{x+4}$ **f** $y = -\frac{1}{x+2}$ **g** $y = -\frac{1}{x-5}$ **h** $y = -\frac{1}{x+6}$

i $y = \frac{5}{x-1}$ **j** $y = \frac{2}{x+4}$ **k** $y = -\frac{5}{x-1}$ **l** $y = -\frac{2}{x-4}$

m $y = -\frac{5}{x+1}$ **n** $y = -\frac{2}{x+4}$ **o** $y = -\frac{4}{x-2}$ **p** $y = \frac{4}{x-2}$

LINKS TO Example 15

4 Sketch the graph of these hyperbolas, labelling one point on each branch of the hyperbola, the asymptotes and the x-intercept.

a $y = \frac{1}{x} + 3$ **b** $y = -\frac{1}{x} + 3$ **c** $y = \frac{1}{x} - 3$ **d** $y = -\frac{1}{x} - 3$

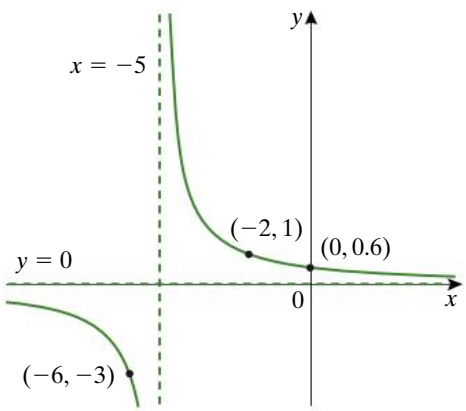
e $y = \frac{3}{x} - 2$ **f** $y = \frac{4}{x} - 1$ **g** $y = -\frac{5}{x} - 3$ **h** $y = \frac{10}{x} + 5$

5 Match each of these equations with one of the graphs.

A $y = \frac{8}{x}$ **B** $y = \frac{3}{x} + 5$ **C** $y = -\frac{2}{x} + 1$ **D** $y = \frac{8}{x-1}$

E $y = -\frac{3}{x+1}$ **F** $y = -\frac{6}{x-2}$ **G** $y = \frac{3}{x+5}$ **H** $y = -\frac{3}{x} + 1$

a



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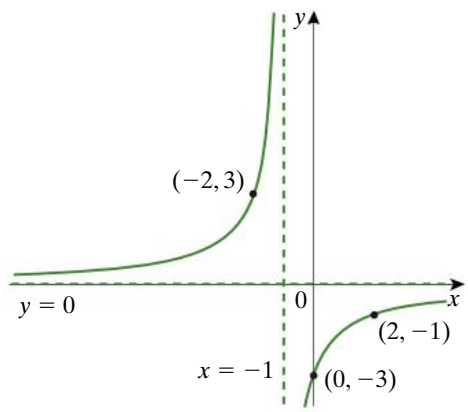

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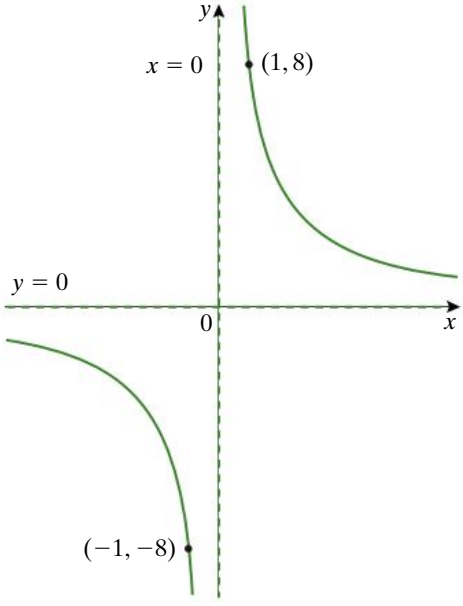
c



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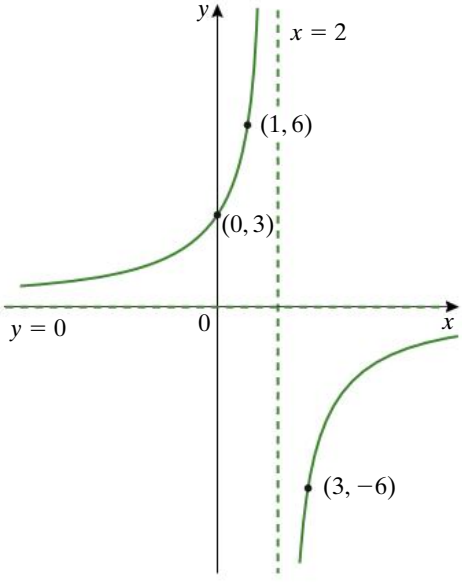

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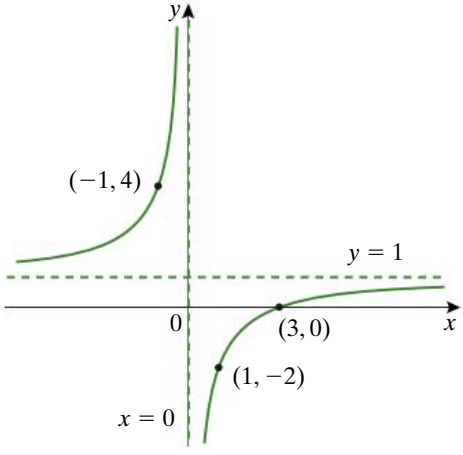

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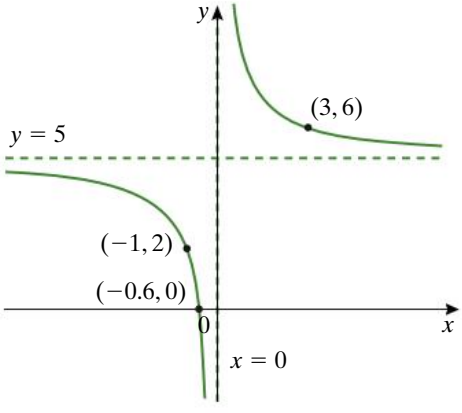

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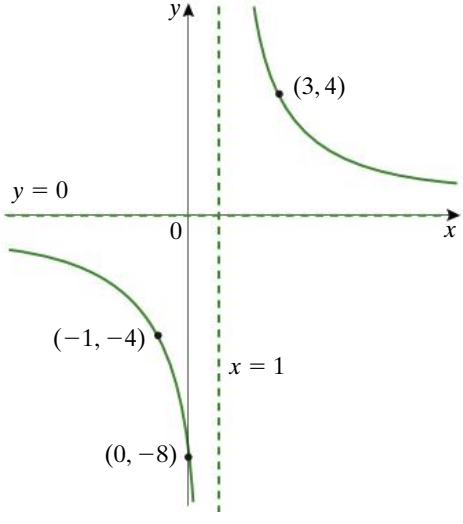

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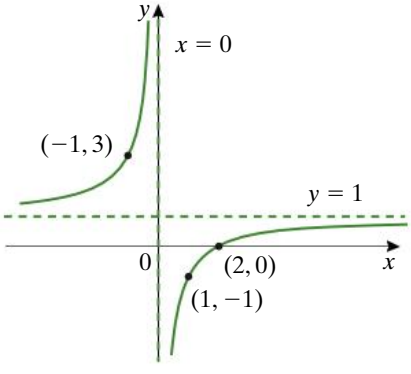

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LINKS TO Example 16

6 For each of the equations in question **5**, state the equation if the graph is reflected

i in the y-axis

ii in the x-axis

LINKS TO Example 17

7 Seong Lee is going to drive 720 km to visit relatives in South Australia.

a If Seong Lee drives at an average speed of 100 km/h, how long will the trip take?

b If the trip takes 10 hours, what is Seong Lee's average speed?

c Make a table of values to show the relationship between the average speed, s km/h, and the time taken, t hours. Use average speeds of 60, 80, 90, 100, 120 km/h.

d Write the rule for the relationship.

e Sketch a graph for this function.

8 A piece of liquorice is 48 cm long.

a If the liquorice is divided between 6 people, what length will each person get?

b If it is divided between 4 people, what length will each person get?

c State the rule connecting the length, y cm, and the number of people, x, in the form $y = \frac{k}{x}$.

d Complete the following table of values.

x	1	2	3	4	6	8	12	16	24	48
y										

e Choose a suitable scale and plot the points. Join the points of the hyperbola with a smooth curve.

9 A train travels between two towns that are 360 km apart.

a Complete this table to show how long the journey would take if the train travelled at S km/h for t hours.

S	60	72	80	90	100	120
t						

b Write a rule connecting t and S.

c Plot the points and draw a smooth curve through them.

d Use the rule to find the time when the speed is 96 km/h then check this on your graph.

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exercise 12.2 challenge

10 In the word game Scrabble™, letters of the alphabet are printed on tiles that are used to build words. The number of tiles for each letter is related to the frequency of that letter in the English language. For example, E is the most commonly occurring letter, so there are more Es than any other letter in a set of Scrabble tiles.

On the other hand, a higher score is obtained for using a less common letter, such as Q. In general, the more commonly used the letter, the lower the score for that letter.

Amanda is designing her own version of Scrabble. For each letter of the alphabet, the product of the number of tiles (T) and the score for that letter (S) will be 42. Both T and S are whole numbers.

a List all the possible values for the ordered pairs (T, S).

b Write a rule for the relationship between S and T in the form $S =$ ___.

c Plot a graph of the function. Is it appropriate to join the points?

d Would it be possible for any letter to have a zero score? Explain.

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Analysis task

Current and resistance

When electricity flows through an electrical conductor, the size of the electric current depends on the voltage of the electric supply and on the resistance to the current provided by the conductor. For domestic appliances, the voltage of the electricity supply is normally constant, so that the current depends on the resistance.

Variables associated with electricity have units that are named after 18th-century scientists who made important contributions to the understanding of electricity.

Variable	Symbol used for variable	Unit	Scientist after whom unit is named
Voltage	V	volt	Alessandro Volta
Current	I	amp	André Ampère
Resistance	R	ohm	Georg Ohm

a What is the voltage of electricity supplied to homes in Australia?

b Given a constant voltage, which variable, I or R, is the independent variable?

The table shows the relationship between current and resistance for a number of electrical appliances.

Appliance	Resistance (R ohm)	Current (I amp)
Kettle	19.2	12.5
Electric frypan	40	6.0
Heater element	57	4.2
Washing machine	60	4.0
Light globe (1)	570	0.42
Light globe (2)	960	0.25

c As the resistance increases, what happens to the size of the current?

d What do you notice about the product of I and R?

e How does this product compare with your answer to part **a**?

f Plot a graph of the current against the resistance.

g Write a rule in the form $IR =$ ___.

h Rewrite the rule in the form $I =$ ___.

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Review Circles and hyperbolas

Summary

Circles

- Circle with centre $(0, 0)$ and radius r $x^2 + y^2 = r^2$.
- Circle with centre $(h, 0)$ and radius r $(x - h)^2 + y^2 = r^2$.
- Circle with centre $(0, k)$ and radius r $x^2 + (y - k)^2 = r^2$.
- Circle with centre (h, k) and radius r $(x - h)^2 + (y - k)^2 = r^2$.
- Ellipse with intercepts $(-a, 0)$, $(a, 0)$, $(0, -b)$ and $(0, b)$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Rectangular hyperbolas

- $y = \frac{a}{x}$ is called a reciprocal function because y is the reciprocal of x.
- The graph of $y = \frac{a}{x}$, where a is a constant, is called a hyperbola.

  The x-axis and the y-axis are the asymptotes.
  The two branches of the graph are in the first and third quadrants.
- The two branches of the graph of $y = -\frac{a}{x}$ are in the second and fourth quadrants.
- If the graph of the hyperbola $y = \frac{a}{x}$ is translated h units horizontally, the equation is $y = \frac{a}{x - h}$.

  The vertical asymptote is the line $x = h$.
  The horizontal asymptote is the line $y = 0$.
- If the graph of the rectangular hyperbola $y = \frac{a}{x}$ is translated k units vertically, the equation is $y = \frac{a}{x} + k$.

  The vertical asymptote is the line $x = 0$.
  The horizontal asymptote is the line $y = k$.
- For the equation $y = \frac{a}{x - h} + k$, the graph of the hyperbola $y = \frac{a}{x}$ is translated h units horizontally and k units vertically.

  The vertical asymptote is the line $x = h$.
  The horizontal asymptote is the line $y = k$.

  The graph of $y = \frac{a}{x}$ passes through the points $(1, a)$ and $(-1, -a)$.

  The x-axis and the y-axis are the asymptotes.
  For $a > 0$, the graph is in the first and third quadrants.
  For $a < 0$, the graph is in the second and fourth quadrants.

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Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

*asymptote	horizontal translation	reflection
centre	intercept	stretching
circle	reciprocal	translation
ellipse	*rectangular hyperbola	vertical translation

Revision

Multiple-choice questions

1 The equation of a circle with centre at $(0, -4)$ and radius 4 is

A $x^2 + y^2 = 16$
B $x^2 + (y + 4)^2 = 16$
C $x^2 + (y - 4)^2 = 16$
D $(x - 4)^2 + y^2 = 16$
E $(x + 4)^2 + y^2 = 16$

2 The circle with equation $x^2 + (y - 3)^2 = 25$ has x-intercepts at

A $x = 0$ and $x = 5$
B $x = 3$ and $x = 5$
C $x = -3$ and $x = 5$
D $x = -4$ and $x = 4$
E $x = -\sqrt{34}$ and $x = \sqrt{34}$

3 Which of the following is *not* correct? The circle with equation $(x - 6)^2 + y^2 = 36$

A has radius 6
B has x-intercepts at $x = 0$ and $x = 12$
C has y-intercepts at $y = 0$ and $y = 6$
D just touches the y-axis at one point
E has its centre at the point $(6, 0)$.

4 The asymptotes to the graph of $y = \dfrac{1}{x - 3}$ are

A $x = 3$ and $y = 3$
B $x = 3$ and $y = 0$
C $x = -3$ and $y = 0$
D $x = -3$ and $y = 3$
E $x = 0$ and $y = 3$

5 Which of the following statements is *not* true?

The hyperbola with equation $y = \dfrac{6}{x + 2}$

A has the line $x = -2$ as an asymptote
B has the line $y = 0$ as an asymptote
C crosses the x-axis at $x = -2$
D crosses the y-axis at $y = 3$
E has one branch in the first quadrant and the other branch in the third quadrant.

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Short-answer questions

6 For each of these circles

- **i** find the centre and the radius
- **ii** find any intercepts
- **iii** sketch the graph, labeling the centre and intercepts

a $x^2 + y^2 = 36$ **b** $x^2 + y^2 = 14$

c $(x - 3)^2 + y^2 = 9$ **d** $x^2 + (y - 4)^2 = 36$

7 For each of the following circles,

- **i** write the equation
- **ii** find any intercepts

a centre $(0, 0)$, radius 3 **b** centre $(0, 0)$, radius $\sqrt{18}$

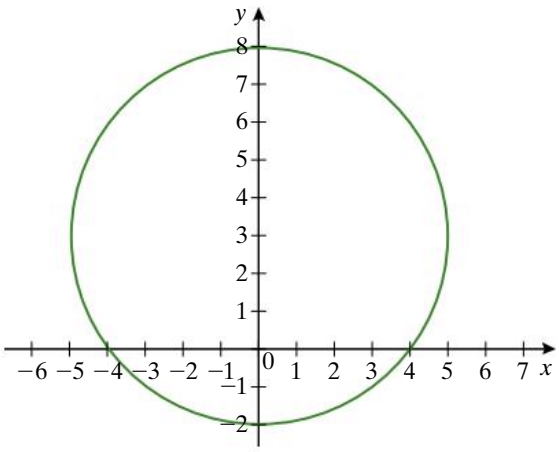
c centre $(-5, 0)$, radius 6 **d** centre $(0, 2)$, radius 5

8 For each of these circles

- **i** find the centre and radius
- **ii** find any intercepts
- **iii** sketch the graph

a $(x - 4)^2 + (y - 4)^2 = 16$ **b** $(x - 3)^2 + (y + 4)^2 = 25$

9 Find the equation for this circle graph.



10 Find the asymptotes for each of these hyperbolas.

a $y = \frac{2}{x}$ **b** $y = -\frac{3}{x + 2}$ **c** $y = \frac{4}{x} + 3$ **d** $y = -\frac{2}{x} - 5$

11 Find the x-intercept for each of these hyperbolas.

a $y = \frac{8}{x} + 2$ **b** $y = -\frac{6}{x} + 1$

12 Find the y-intercept for each of these hyperbolas.

a $y = \dfrac{4}{x + 2}$ **b** $y = -\dfrac{12}{x + 3}$

Extended-response questions

13 For each of these circles

i find the radius and centre

ii find any intercepts

iii sketch the graph, labelling the centre and intercepts

a $x^2 + y^2 = 100$ **b** $(x - 3)^2 + y^2 = 25$

c $x^2 + (y - 5)^2 = 169$ **d** $x^2 + (y - 2)^2 = 16$

14 For each of these equations

i find the asymptotes

ii find any intercepts

iii sketch the graph, labelling one point on each branch of the hyperbola, the asymptotes and any intercepts

a $y = \dfrac{10}{x}$ **b** $y = \dfrac{2}{x} + 5$ **c** $y = \dfrac{8}{x - 4}$ **d** $y = -\dfrac{4}{x} + 1$

15 The area of a rectangular yard for growing fruit trees is $600\,\text{m}^2$.

a Copy and complete this table showing possible dimensions.

Length (l m)	10	20	30	40
Width (w m)				

b What type of function is shown in this table? What feature of the values shows this?

c Write the rule for the relationship.

d Sketch the graph showing two points.

16 For each of these equations

i rearrange the equation into the form $(x - h)^2 + (y - k)^2 = r^2$

ii state the centre and radius.

a $x^2 - 6x + y^2 - 14y = 6$ **b** $x^2 - 4x + y^2 + 10y = 7$

17 Sketch the graphs of

a $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$ **b** $\dfrac{x^2}{16} + \dfrac{y^2}{81} = 1$

Practice quiz Chapter 12

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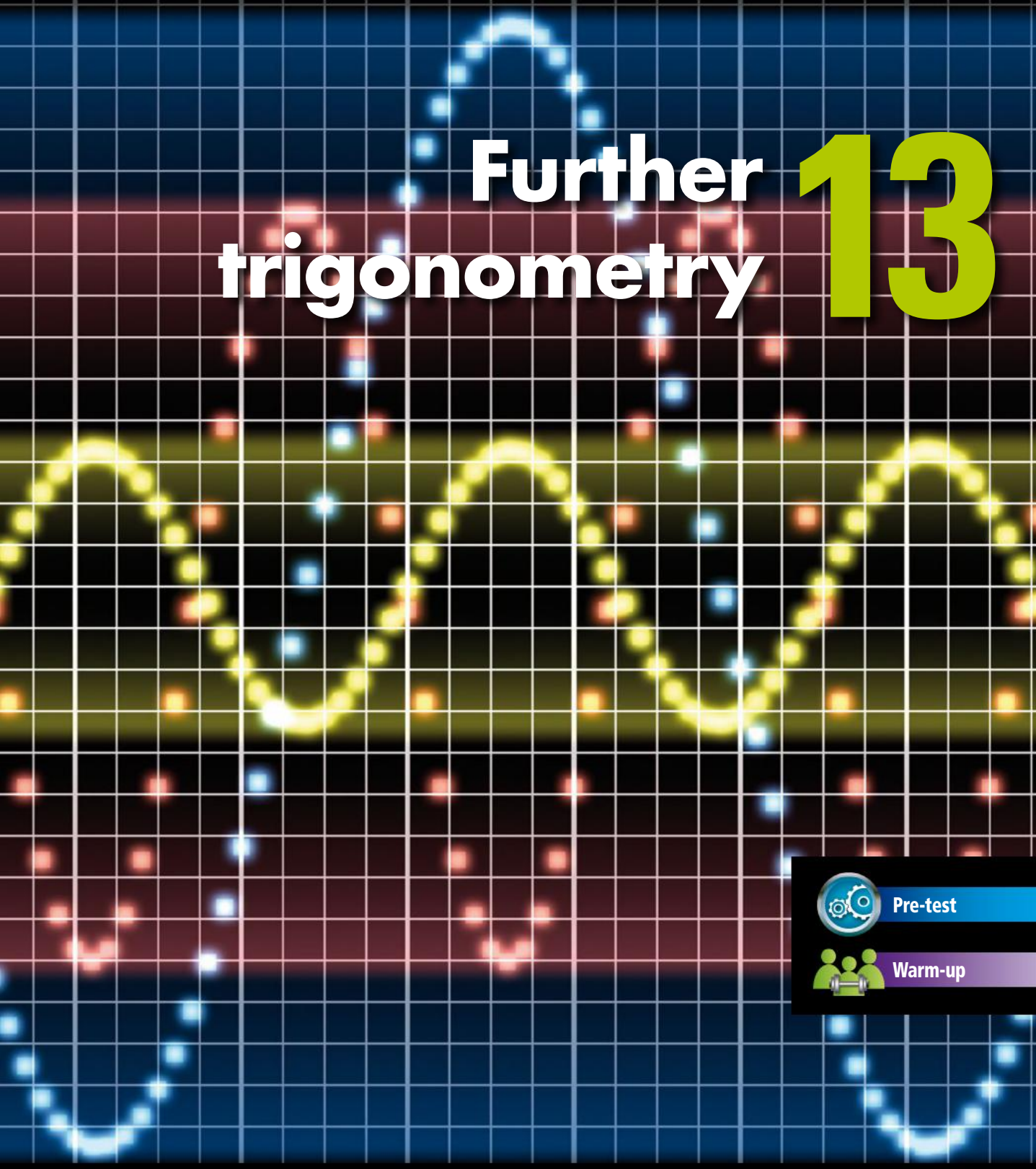
Further trigonometry 13



Pre-test

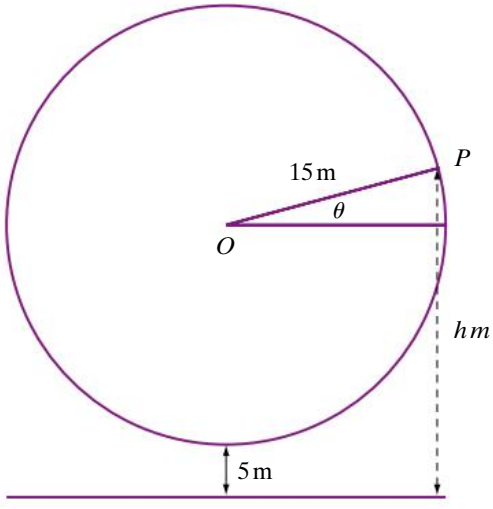
Warm-up

The curves shown on this screen have the form of sine functions. The graph of the sine function can be used to model ocean waves, sound waves, changes in average daily temperature during a year and many other natural changes that occur in a repeating pattern.

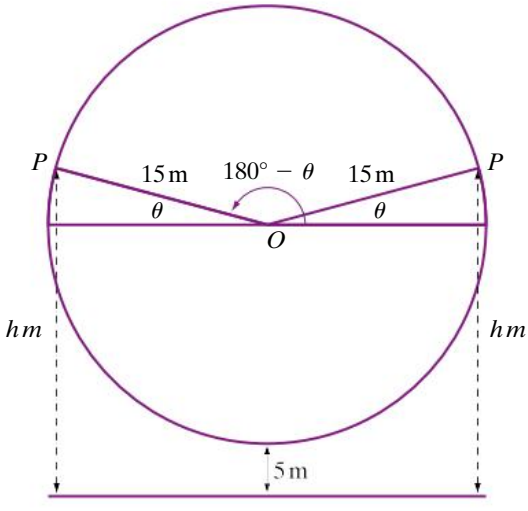
13.1 The unit circle

10 A

The photograph shows the Ferris wheel at Luna Park in Sydney. Each of the gondolas (or passenger cars) is suspended from a point at the end of a radial arm 15 m long. There are 24 gondolas, so the radial arms are 15° apart. As the wheel turns, the angle θ that a radial arm makes with the horizontal changes. The height above the ground, h m, of the suspension point P at the top of a gondola also changes. We will assume that when the Ferris wheel is turning, the radial arms clear the ground by 5 m.



As the gondola rotates, we can use symmetry to find the angle at which point P is the same height above the ground. Measuring in an anticlockwise direction, point P will be the same height above the ground when the angle is $180° - \theta$.



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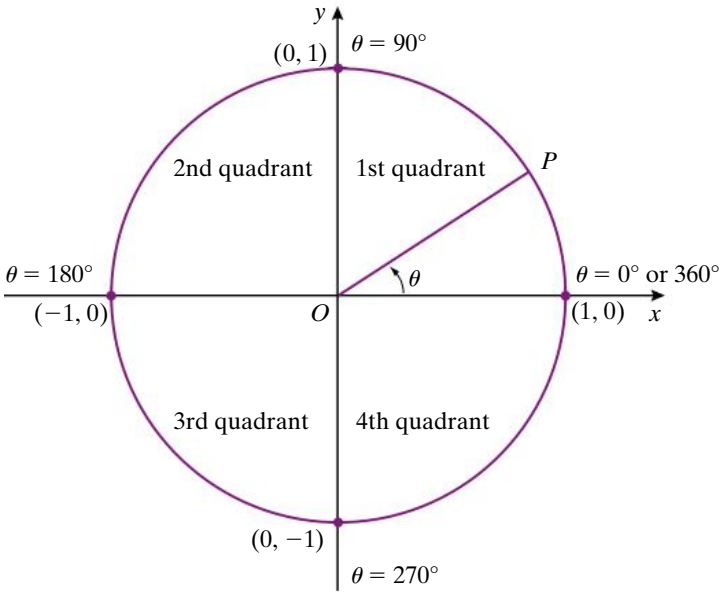
For every position of the gondola where θ is an acute angle, there will be a symmetrical position where θ is the supplementary angle. We will now build on this idea to see how we can make use of the same symmetry to define the trigonometric ratios for angles greater than 90°.

Trigonometric ratios for angles greater than 90°: the unit circle

When trigonometric ratios are defined in terms of a right-angled triangle, we are restricted to angles of less than 90°. However, we also need to be able to use trigonometric ratios for angles greater than 90°.

We can make use of the same symmetry that we saw with the Ferris wheel to define the trigonometric ratios for angles greater than 90°. The **unit circle** is a circle on the Cartesian plane with centre at the point (0, 0) and with radius 1 unit. Just as we considered the position of point P on the gondola as it moved around in a circle, we can consider a point P as it moves around the unit circle in an anticlockwise direction so that OP makes an angle θ with the positive direction of the x-axis. The x-axis and y-axis divide the unit circle into four **quadrants**, which are numbered in an anticlockwise direction as shown.



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Sine and cosine

Unit circle

In the following diagram, a right-angled triangle with hypotenuse OP has been constructed.

In $\triangle OAP$,

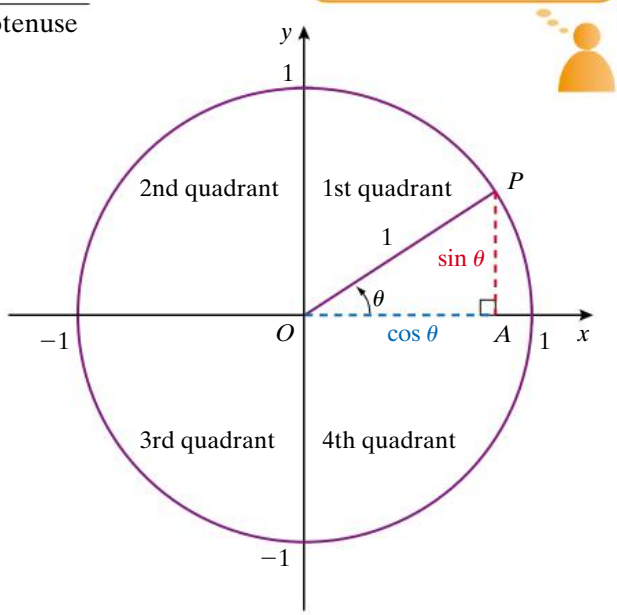
$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{OA}{OP} = \frac{OA}{1}$$

$$\cos\theta = OA$$

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{AP}{OP} = \frac{AP}{1}$$

$$\sin\theta = AP$$

$OP = 1$ as it is the radius of the unit circle.



But OA and AP are respectively the x- and y-coordinates of P. Hence in the unit circle, we can define the cosine and sine of θ in terms of the x- and y-coordinates of the point P.

$\cos\theta = x$-coordinate of P

$\sin\theta = y$-coordinate of P

Tangent

Unit circle tan

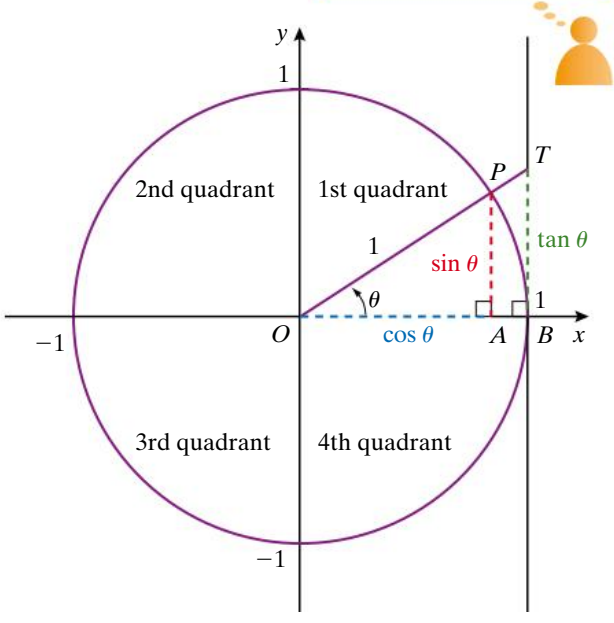
A tangent is a line that touches a circle at one point. To define $\tan\theta$, we draw a tangent to the circle through the point $(1, 0)$, labelled B. The radius OP is extended to meet the tangent at T. The segment BT is perpendicular to OB and is therefore parallel to AP.

In $\triangle OBT$,

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{BT}{OB} = \frac{BT}{1}$$

$$\tan\theta = BT$$

$OB = 1$ as radius of unit circle.



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But BT is the y-coordinate of T. Hence we can define $\tan\theta$ in terms of the y-coordinate of the point T.

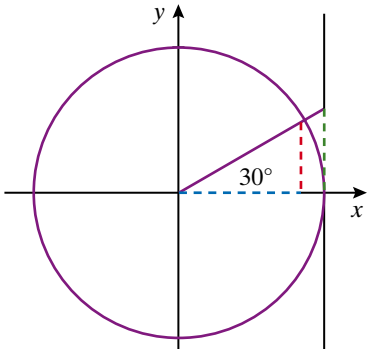
$\tan\theta = y$-coordinate of T

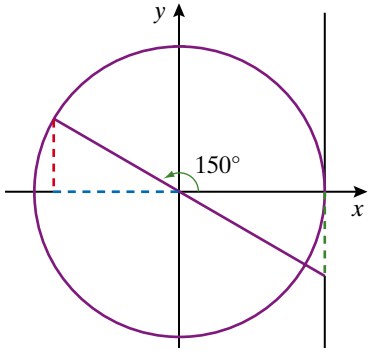
As point P moves around the circle into the second, third and fourth quadrants, we see that the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ change.

	$\cos\theta$	$\sin\theta$	$\tan\theta$
1st quadrant	+	+	+
2nd quadrant	−	+	−
3rd quadrant	−	−	+
4th quadrant	+	−	−

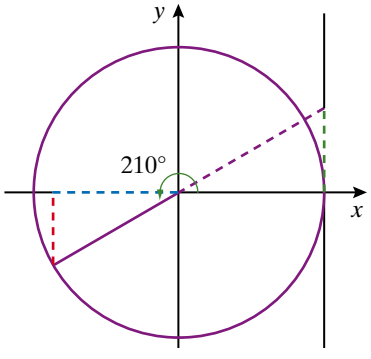
The word CAST helps us remember the quadrants where sine, cosine and tangent are positive.

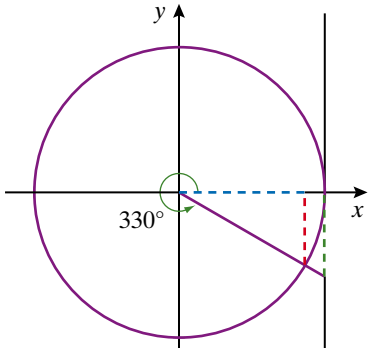
The four unit circle diagrams below show the angles 30°, 150°, 210° and 330°.





From the symmetry of the diagrams above, we can see that $\sin 150° = \sin 30°$.
We can generalise this relationship to say that $\sin(180° - \theta) = \sin\theta$.





We can also see that $\sin 210° = -\sin 30°$ and $\sin 330° = -\sin 30°$.
Generalising, we say $\sin(180° + \theta) = -\sin\theta$ and $\sin(360° - \theta) = -\sin\theta$.

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We can generalise the following relationships.

2nd quadrant	$\sin(180° - A) = \sin A$	$\cos(180° - A) = -\cos A$	$\tan(180° - A) = -\tan A$
3rd quadrant	$\sin(180° + A) = -\sin A$	$\cos(180° + A) = -\cos A$	$\tan(180° + A) = \tan A$
4th quadrant	$\sin(360° - A) = -\sin A$	$\cos(360° - A) = \cos A$	$\tan(360° - A) = -\tan A$

Example 1

For each of the following angles state which quadrant the angle is in then state the sign (positive or negative) of the

i sine **ii** cosine **iii** tangent

a 75° **b** 310° **c** 247° **d** 128°

Working

a 1st quadrant
- **i** sin 75° is positive
- **ii** cos 75° is positive
- **iii** tan 75° is positive

b 4th quadrant
- **i** sin 310° is negative
- **ii** cos 310° is positive
- **iii** tan 310° is negative

c 3rd quadrant
- **i** sin 247° is negative
- **ii** cos 247° is negative
- **iii** tan 247° is positive

d 2nd quadrant
- **i** sin 128° is positive
- **ii** cos 128° is negative
- **iii** tan 128° is negative

Reasoning

1st quadrant angles are between 0° and 90°.

4th quadrant angles are between 270° and 360°.

3rd quadrant angles are between 180° and 270°.

2nd quadrant angles are between 90° and 180°.

Example 2

Consider the sine of angle 40°.

a Use a calculator to find the value of sin 40°, correct to two decimal places.

b Draw a unit circle and mark the position of the angle 40°.

c For each of the other three quadrants, draw a unit circle, mark the related angle to 40° and state the sine of the related angle.

continued

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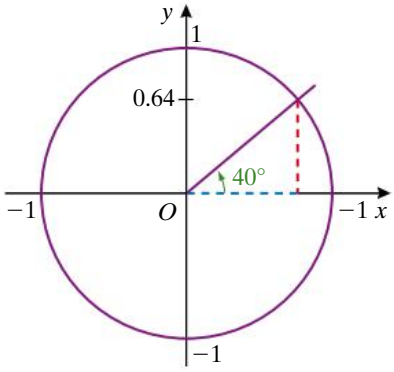
Example 2 continued

Working

a $\sin 40° \approx 0.64$

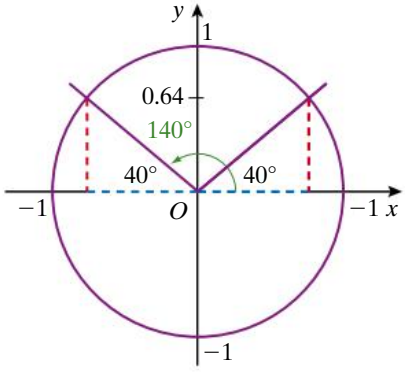
b



Reasoning

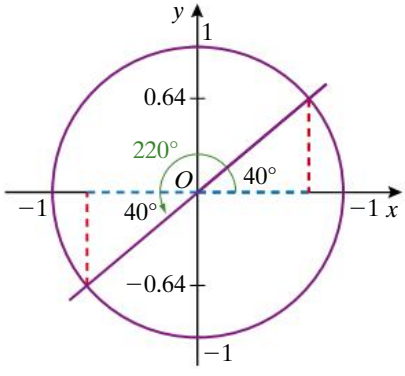
In the unit circle, the sine of the angle is the y-coordinate.

c Second quadrant
$\sin 140° \approx 0.64$



In the second quadrant, the y-coordinate of the related angle to 40° is positive.
$\sin 140°$ is positive.

Third quadrant
$\sin 220° \approx -0.64$



In the third quadrant, the y-coordinate of the related angle to 40° is negative.
$\sin 220°$ is negative.

continued

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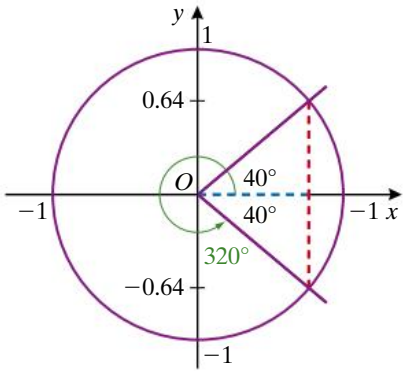
Example 2 continued

Working

Fourth quadrant
$\sin 320° \approx -0.64$



Reasoning

In the fourth quadrant, the y-coordinate of the related angle to 40° is negative.
$\sin 320°$ is negative.

Example 3

For each of the following, find all the values of θ between 0° and 360°. Give the angles correct to the nearest degree.

a $\sin\theta = 0.8290$

b $\cos\theta = -0.9563$

Working

a $\sin\theta = 0.8290$
1st quadrant angle
$\theta = 56°$

2nd quadrant
$180° - 56° = 124°$
The two possible values for θ are 56° and 124°.

b $\cos\theta = -0.9563$
The related first quadrant angle is 17°.
2nd quadrant
$180° - 17° = 163°$

3rd quadrant
$180° + 17° = 197°$

The two possible values for θ are 163° and 197°.

Reasoning

$\sin^{-1}(0.8290) = 55.9961\ldots$
Round to the nearest degree.

The sine of an angle is also positive in the second quadrant.
$\sin(180° - \theta) = \sin\theta$

Find the related first quadrant angle.
$\cos^{-1}0.9563 = 17.0009\ldots$
The cosine of an angle is negative in the second and third quadrants.
$\cos(180° - \theta) = -\cos\theta$

$\cos(180° + \theta) = -\cos\theta$

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Sine, cosine and tangent of 0° and 90°

The unit circle can be used to show the values of the sine, cosine and tangent of 0° and 90°.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0°	1	0	0
90°	0	1	undefined

Pythagoras' theorem and the unit circle

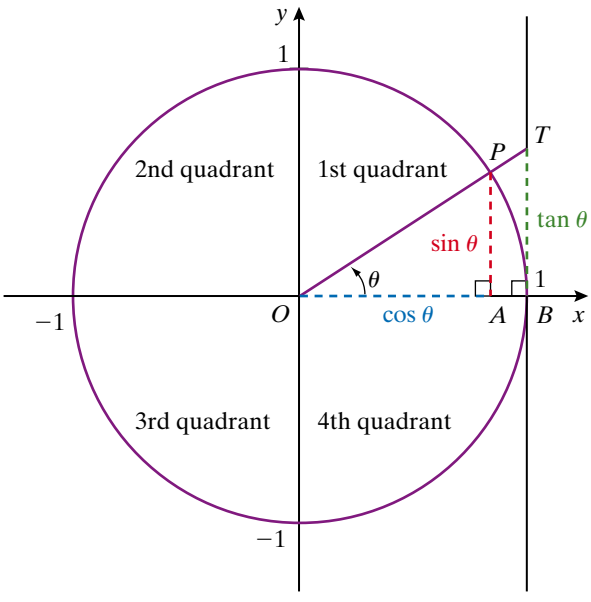
Using Pythagoras' theorem in $\triangle OAP$ gives $AP^2 + OA^2 = OP^2$. Therefore

$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$

$\sin^2\theta + \cos^2\theta \equiv 1$ for all values of θ

This is true for all values of θ so the relationship is called an **identity**.

It is the convention to write $(\sin\theta)^2$ as $\sin^2\theta$ and $(\cos\theta)^2$ as $\cos^2\theta$.
The sign $\equiv$ means 'is identical to'.



Another trigonometric identity is obtained by looking at the ratio of sides in $\triangle OAP$ and $\triangle OBT$. Because these two triangles are similar, we know that

$$\frac{BT}{OB} = \frac{AP}{OA}$$

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \text{ for all values of } \theta$$

If we know the sine of a particular angle we can use these relationships to calculate the cosine and the tangent.

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Example 4

If $\cos\theta = 0.6$, find $\sin\theta$.

Working	Reasoning
$\sin^2\theta + \cos^2\theta \equiv 1$	$\sin^2\theta + \cos^2\theta \equiv 1$ is an identity that is true for all values of θ.
$\sin^2\theta + 0.6^2 = 1$	
$\sin^2\theta = 1 - 0.6^2$	
$\sin^2\theta = 0.64$	
$\sin\theta = \sqrt{0.64}$	Both square roots must be given. A positive value for $\cos\theta$ means that θ is in the first or fourth quadrants. If θ is in the first quadrant, $\sin\theta$ is positive. But if θ is in the fourth quadrant, $\sin\theta$ is negative.
$\sin\theta = \pm 0.8$	

exercise 13.1

LINKS TO Example 1

1 State which quadrant each angle is in.

a 106° **b** 217° **c** 89° **d** 273°
e 340° **f** 173° **g** 250° **h** 184°

2 Copy and complete the table to show the sign (+ or –) for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$
First	+	+	+
Second			
Third			
Fourth			

3 State whether each of the following trigonometric ratios will be positive or negative.

a $\sin 143°$ **b** $\cos 318°$ **c** $\tan 263°$ **d** $\cos 96°$
e $\tan 345°$ **f** $\sin 246°$ **g** $\sin 91°$ **h** $\cos 189°$
i $\tan 268°$ **j** $\sin 284°$ **k** $\tan 340°$ **l** $\tan 160°$

4 Use your calculator to find the value of each of the trigonometric ratios in question **3**. Give each value correct to four decimal places

5 For each of these values of $\sin\theta$, find corresponding first and second quadrant angles. Give your answers to the nearest degree.

a $\sin\theta = 0.6018$ **b** $\sin\theta = 0.8290$ **c** $\sin\theta = 0.9703$ **d** $\sin\theta = 0.4848$

6 For each of these values of $\cos\theta$, find corresponding first and fourth quadrant angles. Give your answers to the nearest degree.

a $\cos\theta = 0.6293$ **b** $\cos\theta = 0.2079$ **c** $\cos\theta = 0.9205$ **d** $\cos\theta = 0.9925$

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7 For each of these values of $\tan\theta$, find corresponding first and third quadrant angles. Give your answers to the nearest degree.

a $\tan\theta = 1.6003$ **b** $\tan\theta = 0.7002$ **c** $\tan\theta = 0.3249$ **d** $\tan\theta = 11.4300$

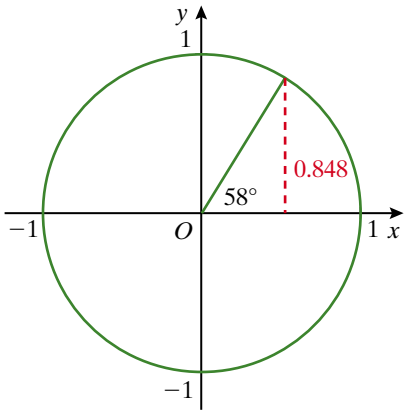
8 Give that $\sin 30° = \frac{1}{2}$ state the values of

a $\sin 150°$ **b** $\sin 210°$ **c** $\sin 330°$

LINKS TO Example 2

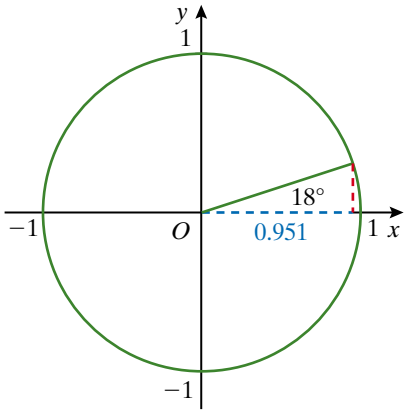
9 Correct to three decimal places, $\sin 58°$ is 0.848. Copy the unit circle diagram. Use it to help you state the value of each of the following, without using a calculator.

a $\sin 122°$
b $\sin 238°$
c $\sin 302°$
d $\cos 32°$



10 Correct to three decimal places, $\cos 18°$ is 0.951. Copy the unit circle diagram. Use it to help you state the value of each of the following, without using a calculator.

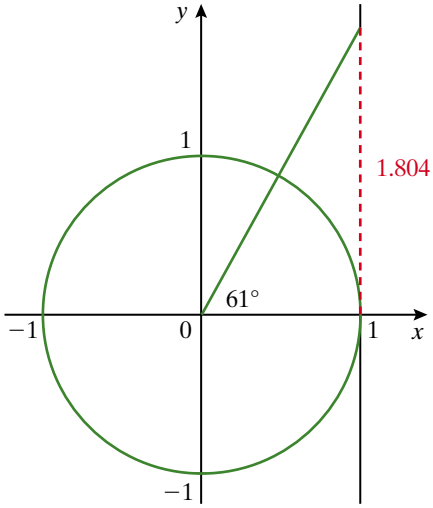
a $\cos 162°$
b $\cos 198°$
c $\cos 342°$
d $\sin 72°$



LINKS TO Example 3

11 Correct to three decimal places, $\tan 61°$ is 1.804. Copy the unit circle diagram. Use it to help you state the value of each of the following, without using a calculator.

a $\tan 119°$
b $\tan 241°$
c $\tan 299°$



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12 For each of the following, give all possible values of θ where $0° < \theta < 360°$. Give the angles to the nearest degree.

a $\cos\theta = 0.261$ **b** $\sin\theta = 0.196$ **c** $\tan\theta = 4.637$ **d** $\sin\theta = 0.794$
e $\cos\theta = 0.805$ **f** $\sin\theta = 0.943$ **g** $\tan\theta = 58.186$ **h** $\cos\theta = 0.158$

13 For each of the following, give all possible values of θ where $0° < \theta < 360°$. Give the angles to the nearest degree.

a $\cos\theta = -0.384$ **b** $\sin\theta = -0.237$ **c** $\tan\theta = -5.132$ **d** $\sin\theta = -0.649$
e $\cos\theta = -0.851$ **f** $\sin\theta = -0.898$ **g** $\tan\theta = -71.735$ **h** $\cos\theta = -0.114$

LINKS TO Example 4

14 If $\sin\theta = 0.28$, find $\cos\theta$.

15 If $\sin\theta = \frac{7}{25}$ and $\cos\theta = \frac{24}{25}$, what is the value of $\tan\theta$?

16 If $\sin\theta = \frac{20}{29}$ and $\tan\theta = \frac{20}{21}$, what is the value of $\cos\theta$?

exercise 13.1 challenge

17 If $\cos\theta = \frac{2}{\sqrt{5}}$, and θ is not in the first quadrant, find

a $\sin\theta$ **b** $\tan\theta$

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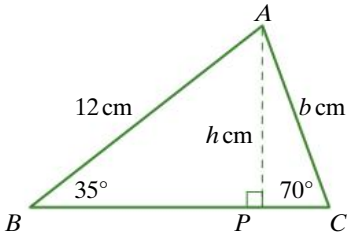
13.2 The sine rule

So far we have used trigonometry with right-angled triangles only. Now we shall see how trigonometry can be used to solve problems involving non-right-angled triangles. First we will look at an example where a non-right-angled triangle is divided into two right-angled triangles by drawing a perpendicular from one vertex to the opposite side of the triangle.

Example 5

Find the value of b in this triangle.



Working	Reasoning
In $\triangle ABP$:	Find an expression for h.
$\sin 35° = \dfrac{h}{12}$	
$h = 12 \sin 35°$	
In $\triangle ACP$:	Find b in terms of h.
$\sin 70° = \dfrac{h}{b}$	
$b = \dfrac{h}{\sin 70°}$	
$b = \dfrac{12 \sin 35°}{\sin 70°}$	Substitute $12 \sin 35°$ for h.
$b \approx 7.3$	

We can apply the same method to any triangle. In the acute-angled triangle, $\triangle ABC$, we refer to the three angles as A, B and C. The lengths of the sides opposite these angles are named a, b and c respectively.

In $\triangle APB$:

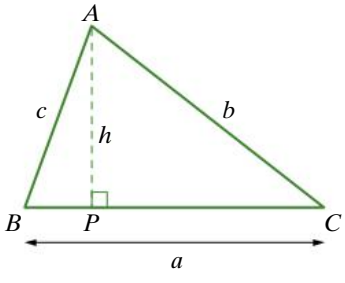
$\sin B = \dfrac{h}{c}$

$h = c \sin B$

In $\triangle APC$:

$\sin C = \dfrac{h}{b}$

$h = b \sin C$



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We now have two expressions for h. Putting these equal to each other:

$b\sin C = c\sin B$

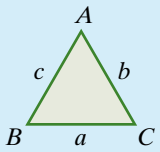
$\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

In the same way, by drawing a perpendicular from B to AC, we can show that

$\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

The sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Notice that right-angled triangles were used to derive the sine rule, but the sine rule applies to triangles that do not have right angles. The derivation still applies if $\angle BAC$ is an obtuse angle, that is, when the perpendicular height is outside the triangle (see exercise **13.2** question **2**).

We use the sine rule to find unknown lengths and angles when given

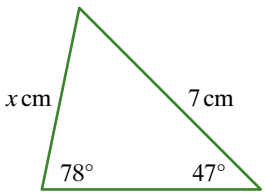
- two angles and one side length.
- two side lengths and an angle opposite one of the given sides.

Using the sine rule to find an unknown side

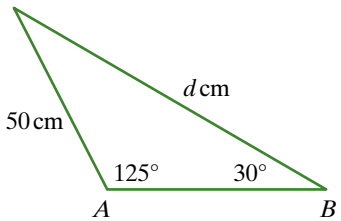
Example 6

Find the value of the pronumerals in the following triangles, correct to one decimal place.

a



b



Working

a $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{x}{\sin 47^\circ} = \dfrac{7}{\sin 78^\circ}$

$x = \dfrac{7\sin 47^\circ}{\sin 78^\circ}$

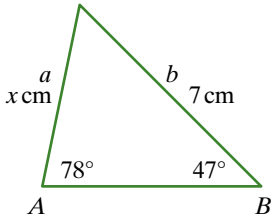
$x \approx 5.23$

$x = 5.2$ cm correct to one decimal place.

Reasoning

x divided by the sine of the opposite angle is equal to 7 divided by the sine of its opposite angle.

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$



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Example 6 continued

Working

b $\dfrac{d}{\sin 125°} = \dfrac{50}{\sin 30°}$

$d = \dfrac{50 \sin 125°}{\sin 30°}$

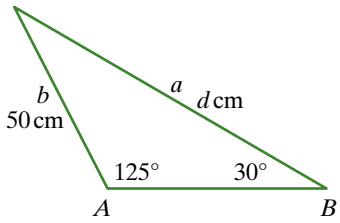
$d \approx 81.9152$

$d = 81.9\,\text{cm}$ correct to one decimal place.

Reasoning

d divided by the sine of the opposite angle is equal to 50 divided by the sine of its opposite angle.

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$



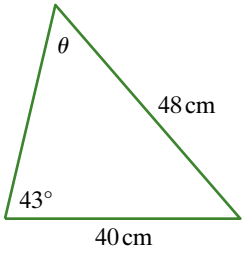
Using the sine rule to find an unknown angle

When using the sine rule to find angles, it is easier to use the reciprocal form.

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Example 7

Find the value of θ, correct to one decimal place.



Working

$\dfrac{\sin\theta}{40} = \dfrac{\sin 43°}{48}$

$\sin\theta = \dfrac{40 \sin 43°}{48}$

$\sin\theta \approx 0.56833$

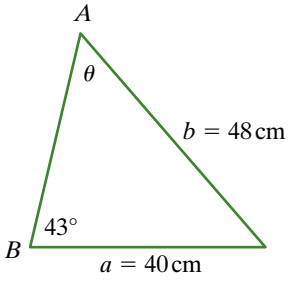
$\theta \approx 34.63°$

$\theta = 34.6°$ correct to one decimal place.

Reasoning

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\sin\theta$ divided by the opposite side is equal to $\sin 43°$ divided by its opposite side.



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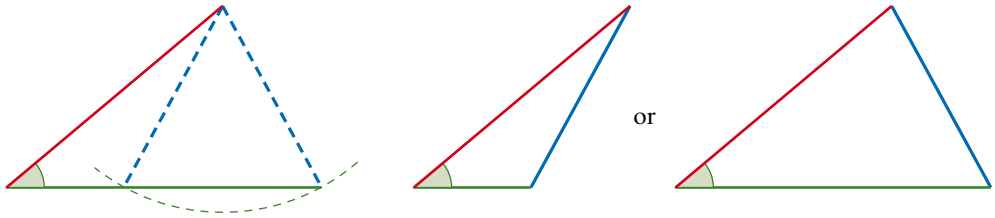
The ambiguous case

Consider the problem below in which we are given two sides and an angle, which is not between the two sides. There are two possible triangles that satisfy the given set of information.

We call this situation the **ambiguous case**, and it arises because the sine of an angle and the sine of its supplement are equal, that is, $\sin\theta = \sin(180° - \theta)$.



The ambiguous case

The ambiguous case arises

- if the given angle is less than 90°,
- if it is not the angle included between the two given sides,
- if the side opposite the given angle is smaller than the side adjacent to it.

Example 8

In $\triangle ABC$, $AB = 11$ cm, $AC = 8$ cm and $\angle ABC = 40°$. Find the size of $\angle ACB$, correct to the nearest degree.

Working

Let θ be the acute $\angle ACB$.

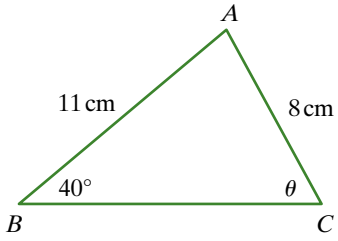
$$\frac{\sin\theta}{11} = \frac{\sin 40°}{8}$$

$$\sin\theta = \frac{11\sin 40°}{8}$$

$$\sin\theta \approx 0.883833$$

$$\theta \approx 62.1°$$

Reasoning



The given angle is less than 90°, is not the angle included between the two given sides, and the side opposite the given angle is smaller than the side adjacent to it. So this is an example of the ambiguous case. There are two possible positions for C and two possible sizes for $\angle ACB$. The two values will be supplementary angles. So there will be two possible triangles. Use the sine rule to find θ.

continued

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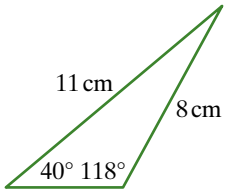
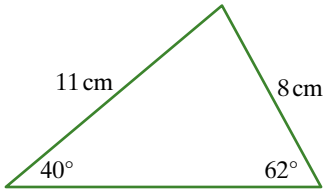
Example 8 continued

Working





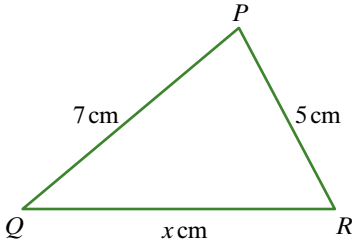
Correct to the nearest degree, the two possible values for $\angle ACB$ are 62° and 180° – 62°, that is, 118°.

Reasoning

An angle and its supplement both have the same value for their sine.

exercise 13.2

1 Using the sine rule in $\triangle PQR$



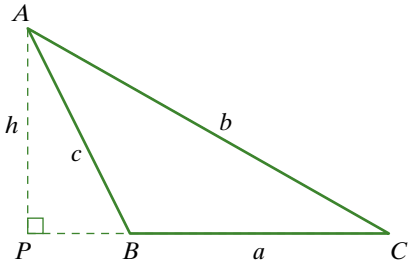
A $\dfrac{x}{\sin P} = \dfrac{5}{7}$ **B** $\dfrac{x}{\sin P} = \dfrac{5}{\sin R}$

C $\dfrac{x}{\sin P} = \dfrac{5}{\sin Q}$ **D** $\dfrac{x}{\sin P} = \dfrac{7}{\sin Q}$

E $\dfrac{x}{\sin Q} = \dfrac{5}{\sin R}$

LINKS TO Example 5

2 AP is the perpendicular height of the obtuse-angled triangle ABC.



a What is the size of $\angle ABP$ in terms of $\angle B$ where $\angle B$ is the obtuse angle in the triangle?

b What do you know about the sine of $\angle ABP$?

c Write two different expressions for h

i using side c and $\angle ABP$ in $\triangle ABP$

ii using side b and $\angle ACP$ in $\triangle ACP$

d Using your expressions from part **c**, show that the sine rule applies to the obtuse-angled triangle.

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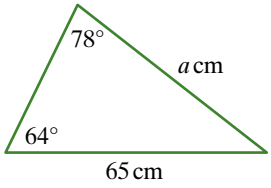
LINKS TO Example 6

3 Use the sine rule to find the value of the pronumeral in each of these triangles, correct to one decimal place.

a



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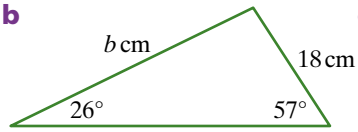

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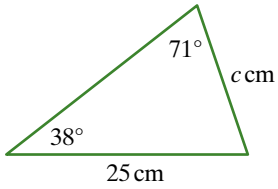

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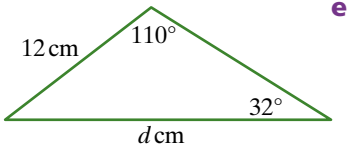

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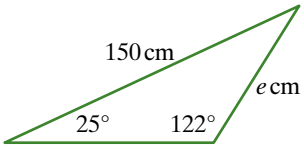

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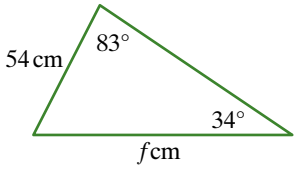

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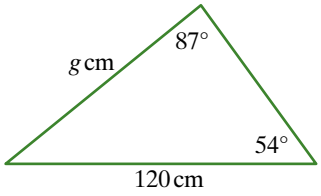

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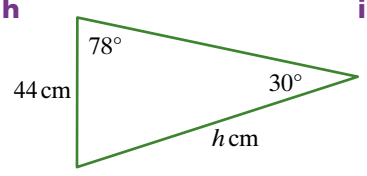

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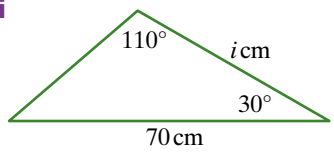

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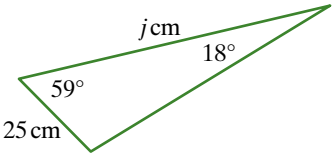

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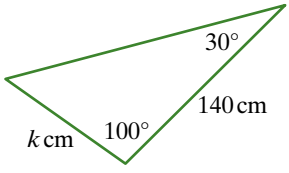

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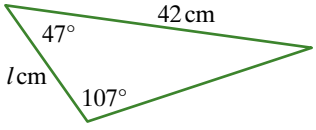


4 For each of the following find the value of the pronumeral, correct to one decimal place. (Hint: First draw a triangle and label the vertices A, B and C and label the sides opposite the angles a, b and c.)

a In $\triangle ABC$, $\angle A = 45°$, $\angle B = 80°$ and $b = 15$. Find the value of a.

b In $\triangle ABC$, $\angle A = 65°$, $\angle C = 50°$ and $a = 40$. Find the value of c.

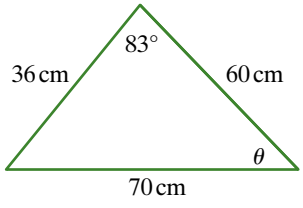
c In $\triangle ABC$, $\angle B = 72°$, $\angle C = 36°$ and $c = 24$. Find the value of b.

d In $\triangle ABC$, $\angle A = 20°$, $\angle C = 110°$ and $c = 8$. Find the value of a.

5 $\sin\theta$ is equal to

A $\dfrac{36\sin 83°}{70}$ **B** $\dfrac{70\sin 83°}{36}$ **C** $\dfrac{60\sin 83°}{36}$

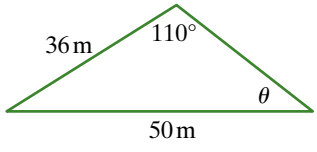
D $\dfrac{70\sin 83°}{60}$ **E** $\dfrac{70}{36\sin 83°}$



LINKS TO Examples 7, 8

6 For each triangle, first decide if the ambiguous case arises, then use the sine rule to calculate the value(s) of θ, correct to the nearest degree.

a



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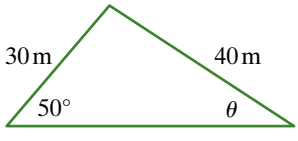

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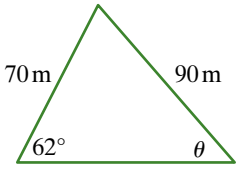

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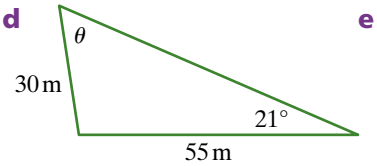

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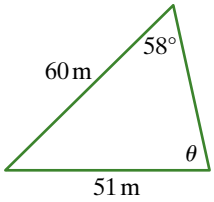

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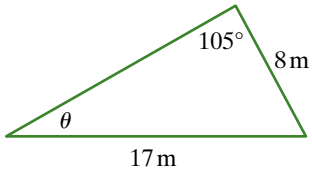

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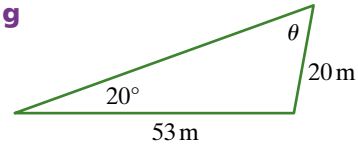

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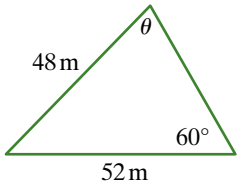

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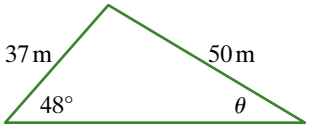

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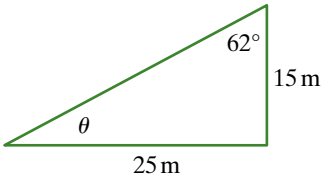

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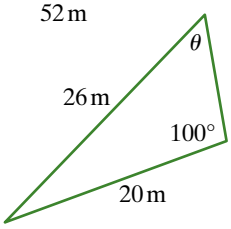

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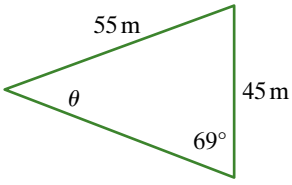

LINKS TO Examples 7, 8

7 For each triangle, first decide if the ambiguous case arises, and then use the sine rule to calculate the value(s) of θ, correct to the nearest degree.

a In $\triangle ABC$, $\angle A = 45°$, $a = 30$ and $b = 15$. Find the value of $\angle B$.

b In $\triangle ABC$, $\angle C = 25°$, $a = 50$ and $c = 35$. Find the value of $\angle A$.

c In $\triangle ABC$, $\angle B = 120°$, $b = 18$ and $c = 7$. Find the value of $\angle C$.

d In $\triangle ABC$, $\angle A = 64°$, $a = 16$ and $c = 12$. Find the value of $\angle C$.

8 The three angles of a triangle are 36°, 80° and 64°. The length of the shortest side is 80 cm.

a Draw and label the triangle.

b Calculate the lengths of the other two sides correct to one decimal place.

exercise 13.2 challenge

9 In $\triangle ABC$, $A = 30°$, $BC = 25$ cm and $AC = 35$ cm.

a Explain why two triangles are possible. Draw diagrams to illustrate your answer.

b Find the two possible values for $\angle B$ and, hence, the two possible values for $\angle C$. Give the angles correct to one decimal place. Keep the angle values in your calculator so you obtain an accurate value for the length of AB in the next calculation.

c Find the two possible values for the length of AB.

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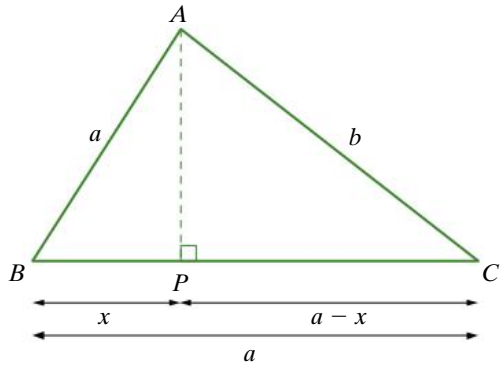
13.3 The cosine rule

10 A

We do not always have the necessary information about a triangle to use the sine rule. For example, we may know all three side lengths, but the angles are unknown. By dividing a general triangle into two right-angled triangles as we did when we derived the sine rule, a new rule can be derived that allows us to deal with different combinations of information.



Using Pythagoras' theorem in $\triangle APB$:

$h^2 = c^2 - x^2$

Using Pythagoras' theorem in $\triangle APC$:

$h^2 = b^2 - (a - x)^2$

Equating the two expressions for h^2:

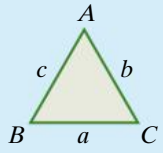
$$\begin{aligned} c^2 - x^2 &= b^2 - (a - x)^2 \\ b^2 &= c^2 + (a - x)^2 - x^2 \\ b^2 &= c^2 + a^2 - 2ax + x^2 - x^2 \\ b^2 &= c^2 + a^2 - 2ax \end{aligned}$$

But from $\triangle ABP$, $x = c \cos B$.

So $b^2 = c^2 + a^2 - 2ac \cos B$.

In a similar way we can show that $a^2 = b^2 + c^2 - 2bc \cos A$ and $c^2 = a^2 + b^2 - 2ab \cos C$

The cosine rule



The cosine rule states that:

$b^2 = c^2 + a^2 - 2ac \cos B$ and $a^2 = b^2 + c^2 - 2bc \cos A$
and $c^2 = a^2 + b^2 - 2ab \cos C$

We can also write the rules with the cosine of the angle as the subject:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ and } \cos B = \frac{a^2 + c^2 - b^2}{2ac} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The derivation still applies if $\angle ABC$ is an obtuse angle, that is, when the perpendicular height is outside the triangle (see exercise **13.3**, question **2**).

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Using the cosine rule

We use the cosine rule to find unknown lengths and angles when given

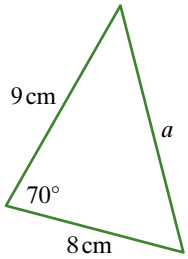
- all three side lengths
- two side lengths and the included angle.

If we know two sides and the included angle we can calculate the third side.

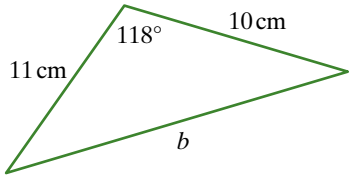
Example 9

Find the values of a and b correct to one decimal place.

a



b



Working

a $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 8^2 + 9^2 - 2 \times 8 \times 9 \times \cos 70°$

$a^2 = 64 + 81 - 144\cos 70°$

$a \approx 9.78\ldots$

$a = 9.8$ cm correct to one decimal place.

Reasoning

Two sides and the included angle are known so the cosine rule can be used.

b $b^2 = a^2 + c^2 - 2ac\cos B°$

$b^2 = 10^2 + 11^2 - 2 \times 10 \times 11 \times \cos 118°$

$b^2 = 100 + 121 - 220\cos 118°$

$b \approx 18.00\ldots$

$b = 18.0$ cm correct to one decimal place.

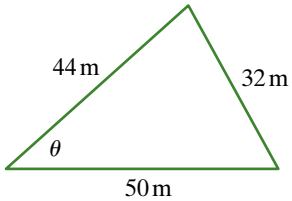
Two sides and the included angle are known so the cosine rule can be used.

If we know the lengths of the three sides of a triangle, we can use the cosine rule to find angles.

Example 10

Calculate the value of θ correct to one decimal place for each of these triangles.

a



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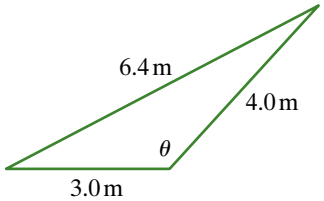

continued

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Example 10 continued

Working

a $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab\cos c}$

$\cos\theta = \dfrac{50^2 + 44^2 - 32^2}{2 \times 50 \times 44}$

$\cos\theta = \dfrac{3412}{4400}$

$\theta = \cos^{-1}\left(\dfrac{3412}{4400}\right)$

$\theta \approx 39.15°$

The angle is 39.2° correct to one decimal place.

Reasoning

We do not evaluate and approximate $\dfrac{3412}{4400}$ before we use inverse cosine. This gives a more accurate value of θ.

Alternatively, we can evaluate $\dfrac{3412}{4400}$ and keep the value in the calculator before using inverse cosine.

b $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab\cos c}$

$\cos\theta = \dfrac{3^2 + 4^2 - 6.4^2}{2 \times 3 \times 4}$

$\cos\theta = \dfrac{-15.96}{24}$

$\theta = \cos^{-1}\left(\dfrac{-15.96}{24}\right)$

$\theta \approx 131.68°$

The angle is 131.7° correct to one decimal place.

θ is an obtuse angle so the cosine is negative. We do not approximate $\dfrac{-15.96}{24}$ before we use inverse cosine.

exercise 13.3

LINKS TO Example 9

1 Applying the cosine rule in this triangle

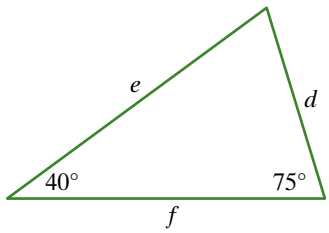
A $d^2 = e^2 + f^2 - 2ef\cos 75°$

B $e^2 = d^2 + f^2 - 2df\cos 75°$

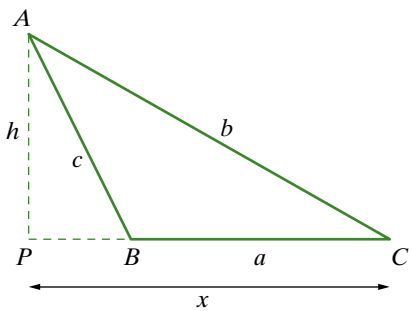
C $f^2 = d^2 + f^2 - 2de\cos 40°$

D $d^2 = e^2 + f^2 + 2ef\cos 40°$

E $f^2 = d^2 + e^2 - 2de\cos 40°$



2 Consider the diagram below, in which $\triangle APC$, $\triangle APB$ can be seen.



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a Using information from

i $\triangle APC$, write an expression for x in terms of b and $\cos C$.

ii $\triangle APC$, use Pythagoras' theorem to write an expression for h^2.

iii $\triangle APB$, what is the length of PB in terms of x and a?

iv $\triangle APB$, use Pythagoras' theorem to write an expression for h^2.

b Equate your two expressions for h^2 from parts **ii** and **iv**.

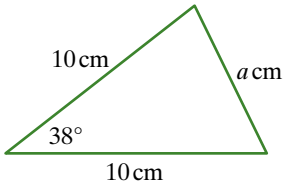
c Expand and simplify.

d Substitute your expression from part **i** for x. You should now have the cosine rule.

LINKS TO Example 9

3 Use the cosine rule to find the value of the pronumeral in each of these triangles, correct to one decimal place.

a



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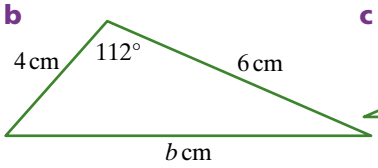

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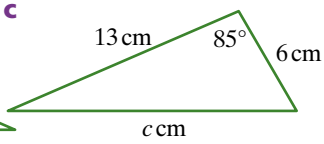

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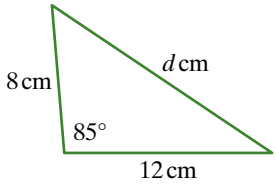

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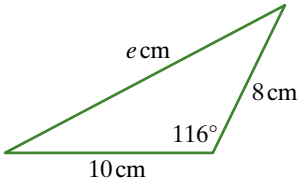

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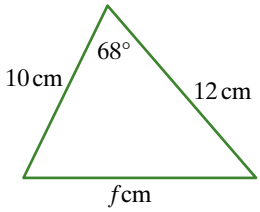

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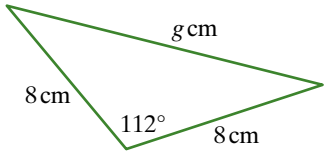

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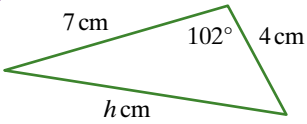

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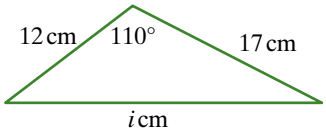

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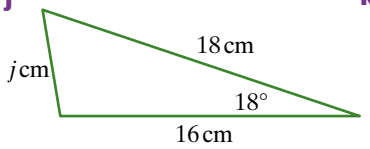

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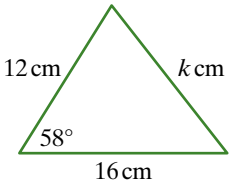

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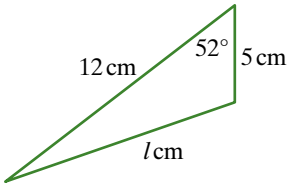

4 For each of the following draw and label a diagram of $\triangle ABC$, then find the value of the pronumeral given, correct to one decimal place.

a In $\triangle ABC$, $\angle A = 32°$, $b = 20$ and $c = 18$. Find the value of a.

b In $\triangle ABC$, $\angle C = 45°$, $a = 12$ and $b = 16$. Find the value of c.

c In $\triangle ABC$, $\angle B = 27°$, $a = 30$ and $c = 48$. Find the value of b.

d In $\triangle ABC$, $\angle C = 67°$, $a = 26$ and $b = 35$. Find the value of c.

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5 $\cos\theta$ is equal to

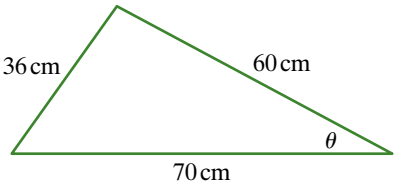
A $\dfrac{60^2 + 70^2 - 36^2}{2 \times 60 \times 70}$

B $\dfrac{36^2 + 70^2 - 60^2}{2 \times 37 \times 70}$

C $\dfrac{60^2 + 70^2 + 36^2}{2 \times 60 \times 70}$

D $\dfrac{36^2 + 60^2 + 70^2}{2 \times 36 \times 60}$

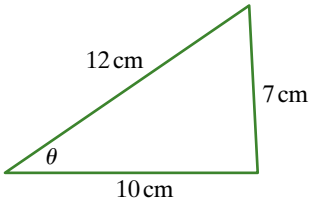
E $\dfrac{36^2 + 60^2 - 70^2}{2 \times 36 \times 60}$



LINKS TO Example 10

6 In each triangle, use the cosine rule to find the value of θ, correct to one decimal place.

a



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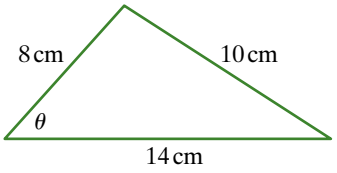

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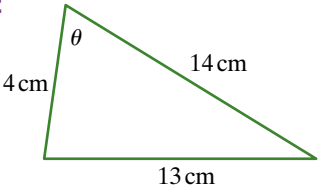

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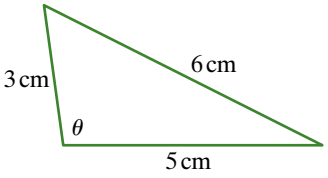

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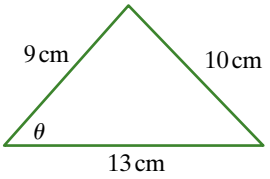

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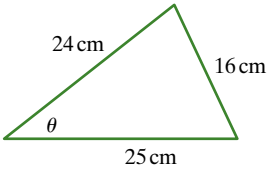

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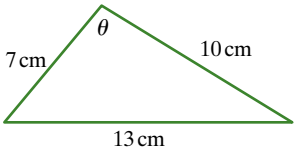

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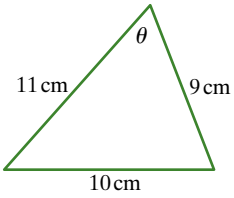

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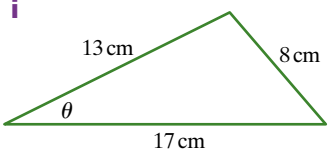

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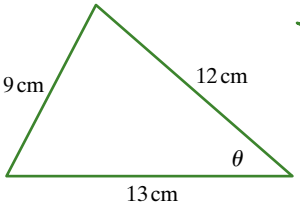

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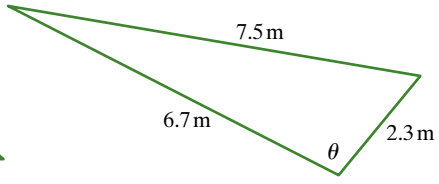

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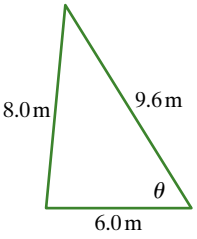

7 For each of the following, find the value of the required angle, correct to one decimal place.

a In $\triangle ABC$, $a = 30$, $b = 15$ and $c = 25$. Find the value of $\angle B$.

b In $\triangle ABC$, $a = 40$, $b = 12$ and $c = 35$. Find the value of $\angle A$.

c In $\triangle ABC$, $a = 20$, $b = 16$ and $c = 24$. Find the value of $\angle C$.

d In $\triangle ABC$, $a = 11$, $b = 7$ and $c = 12$. Find the value of $\angle C$.

exercise 13.3 challenge

8 The three sides of a triangle are 12 cm, 15 cm and 20 cm. Calculate the size of the largest angle, correct to one decimal place.

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13.4 Mixed applications of the sine and cosine rules

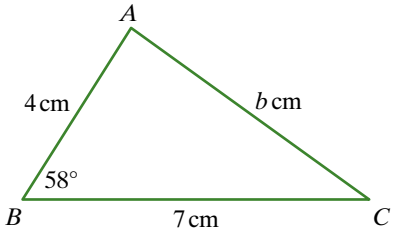
When solving triangles to find unknown sides and angles, we look at the given information to decide whether to use the sine rule or the cosine rule.

Example 11

Consider triangle ABC.

a Find the length of AC correct to one decimal place.

b Find the size of angles A and C correct to the nearest degree.



Working

a Let the length of AC be b cm.

$b^2 = a^2 + c^2 + 2ac\cos B$

$b^2 = 7^2 + 4^2 - 2 \times 7 \times 4 \times \cos 58°$

$b^2 = 49 + 16 - 56\cos 58°$

$b^2 = 35.32$

$b \approx 5.94$

AC is approximately 5.9 cm.

Reasoning

We know two sides and the included angle. Use the cosine rule to find the length of AC.

$b^2 = 65 - 56\cos 58°$

b $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$$\frac{\sin A}{7} = \frac{\sin 58°}{\sqrt{\dfrac{65 - 56\cos 58°}{65}}}$$

$$\sin A = \frac{7\sin 58°}{\sqrt{\dfrac{65 - 56\cos 58°}{65}}}$$

$$A = \sin^{-1}\left(\frac{7\sin 58°}{\sqrt{\dfrac{65 - 56\cos 58°}{65}}}\right)$$

$A \approx 87.1977\ldots$

As we now know all three sides and one angle we could use either the sine rule or the cosine rule to find $\angle A$.

$\angle A$ is approximately 87° and $\angle C$ is approximately 35°.

Find $\angle C$ by subtracting (58° + 87°) from 180°.

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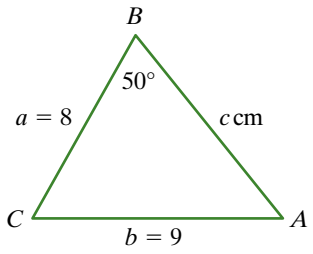
Example 12

In $\triangle ABC$ the lengths of the sides opposite $\angle A$ and $\angle B$ are 8 cm and 9 cm respectively (that is, $a = 8$, $b = 9$) and $\angle B = 50°$. Find c, $\angle A$ and $\angle C$. Give c correct to one decimal place and give the angles correct to the nearest degree.

Working



Reasoning

Draw the triangle showing the given information. Use the convention of naming the angles A, B and C and naming the lengths of the sides opposite these angles a, b and c, respectively.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{8} = \frac{\sin 50°}{9}$$

$$\sin A = \frac{8 \sin 50°}{9}$$

$$A = \sin^{-1}\left(\frac{8 \sin 50°}{9}\right)$$

We don't have three sides or two sides and the included angle, so we cannot use the cosine rule.
We have a, b and $\angle B$ so we use the sine rule to find $\angle A$.

$$A = 42.916\ldots°$$

$\angle A$ is 43° and $\angle C$ is 87°.

Use angle sum of triangle is 180°C to find $\angle C$.
Round the angles to the nearest degree.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 87°} = \frac{9}{\sin 50°}$$

$$c = \frac{9 \times \sin 87°}{\sin 50°}$$

$$c \approx 11.73$$

Now that we know b, $\angle B$ and $\angle C$, we can use the sine rule to find c.
We could also use the cosine rule because we know a, b and $\angle C$.

Round to one decimal place.

Example 13

A hiker sets out from an overnight hut and walks 3.9 km on a bearing 018° before changing direction and walking a further 8.2 km on a bearing 148°.

a Draw a labelled diagram to show this information.

b How far is the hiker from the hut? Give the distance correct to one decimal place.

c What is the bearing of the hut from the hiker? Give the bearing to the nearest degree.

continued

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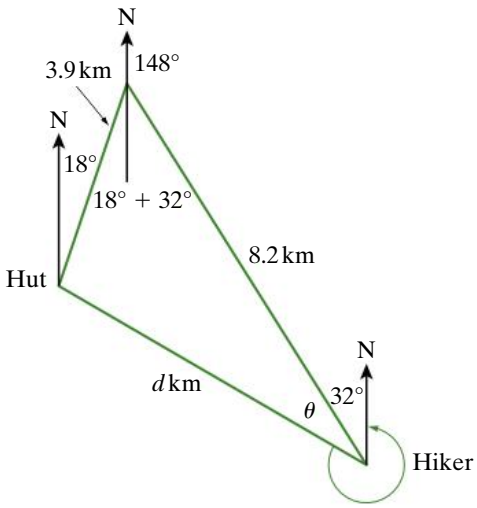
Example 13 continued

Working

a



Reasoning

The angle 50° is the sum of 18° (using alternate angles between the parallel north lines) and 32° (the supplementary angle of the bearing angle of 148°).

b Let d km be the distance of the hiker from the hut.

$d^2 = 3.9^2 + 8.2^2 - 2 \times 3.9 \times 8.2 \times \cos 50°$

$d^2 = 41.337\ldots$

$d = \sqrt{41.337\ldots}$

$d = 6.429\ldots$

The hiker is 6.4 km from the hut.

Two sides and the included angle are known, so use the cosine rule to find d.

c $\dfrac{\sin\theta}{3.9} = \dfrac{\sin 50°}{d}$

$\sin\theta = \dfrac{3.9 \times \sin 50°}{6.429\ldots}$

$\theta \approx 27.7°$

$360° - (32° + 27.7°) = 300.3$

The bearing of the hut from the hiker is 300°.

The sides opposite the angles 50° and θ are known so use the sine rule.

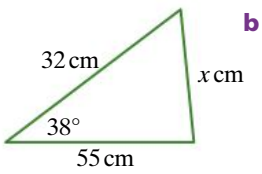
The bearing is measured in a clockwise from the north line.

exercise 13.4

LINKS TO Examples 11, 12

1 For each of these triangles, find the value of x correct to one decimal place, or the size of the angle labelled θ, correct to the nearest degree.

a



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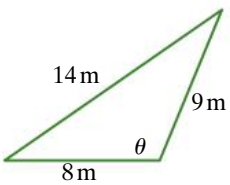

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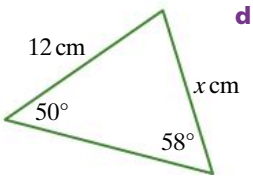

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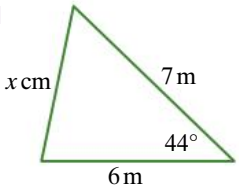

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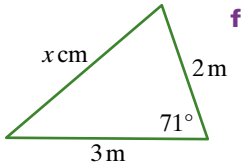

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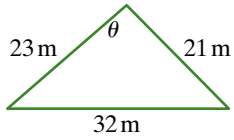

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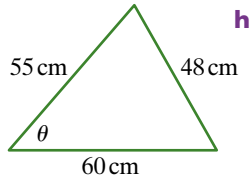

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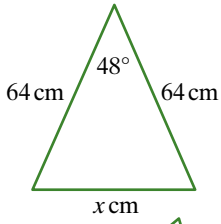

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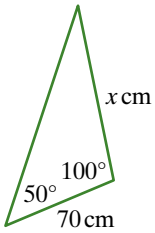

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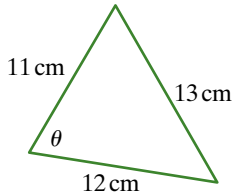

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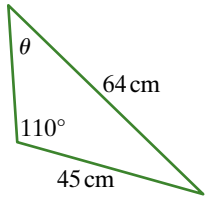

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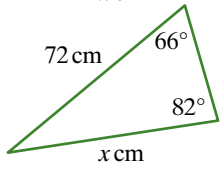
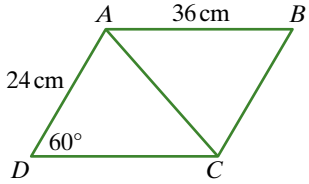


2 $ABCD$ is a parallelogram.



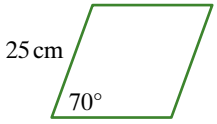
a Find the length of the diagonal AC, correct to one decimal place.

b Find the length of the diagonal BD, correct to one decimal place.

c Find the size of $\angle DAC$, correct to one decimal place.

3 Two adjacent sides of a parallelogram are 25 cm and 48 cm. The angle between them is 50°. Find the lengths of the two diagonals, correct to one decimal place.

4 Calculate the length of the longer diagonal of this rhombus correct to one decimal place.



LINKS TO Example 13

5 A yacht sails for 5 km on a bearing 048° then turns and sails for 8 km on a bearing 120°.

a Draw a labelled diagram.

b Calculate the distance of the yacht from its starting point, correct to one decimal place.

6 A ship travels 8 km from point A to point B on a bearing 060°. A second ship travels 12 km from point A to point C on a bearing of 140°.

a Draw a labelled diagram.

b Calculate the distance between the ships, correct to one decimal place.

exercise 13.4 challenge

7 In the first leg of a three-leg yacht race, a yacht sailed on a bearing 050° for 5 km. In the second leg, the yacht sailed on a bearing of 095° for 7 km. The third leg was back to the starting point.

a Draw a labelled diagram of the route.

b What was the length of the third leg?

c On what bearing did the yacht sail the third leg?

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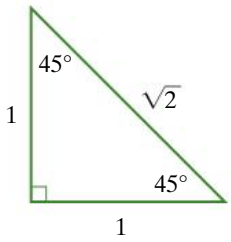
13.5 Exact values of trigonometric ratios for 30°, 45° and 60°

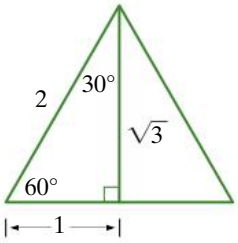
BLM
Exact values

The sine, cosine and tangent of most angles are irrational numbers, so the values given by a calculator are rational approximations. For certain angles, it is sometimes useful to know the exact values. The worksheet *Exact values* in the student ebooks leads through the derivation of these exact values. It is useful to memorise these values or to remember how they are derived as shown in the worksheet.

We can use these exact values when calculating side lengths in right-angled triangles with these angles.

	θ		
	30°	**45°**	**60°**
sin θ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan θ	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

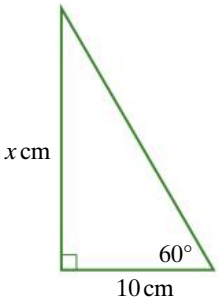




Example 14

Find the exact values of the pronumerals in these right-angled triangles.

a



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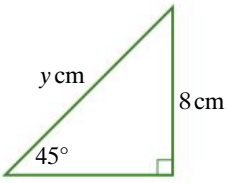

Working

a $\tan 60° = \frac{x}{10}$

$x = 10 \tan 60°$

$x = 10\sqrt{3}$

Reasoning

The exact value of $\tan 60°$ is $\sqrt{3}$.

continued

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Example 14 continued

Working

b $\sin 45° = \dfrac{8}{y}$

$\dfrac{1}{\sqrt{2}} = \dfrac{8}{y}$

$\dfrac{y}{\sqrt{2}} = 8$

$y = 8\sqrt{2}$

Reasoning

The exact value of $\sin 45°$ is $\dfrac{1}{\sqrt{2}}$.

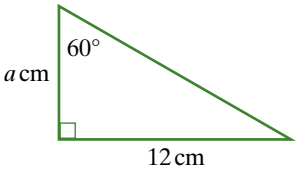
exercise 13.5

1 Complete the worksheet Exact values in the *OneStopDigital* ebook.

LINKS TO Example 14

2 Find the exact values of the pronumerals in these right-angled triangles.

a



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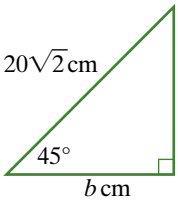

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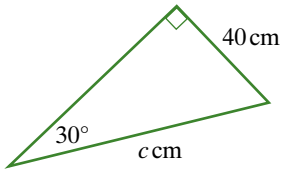

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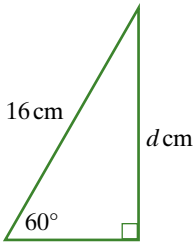

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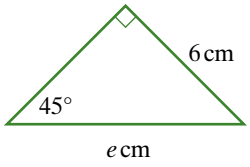

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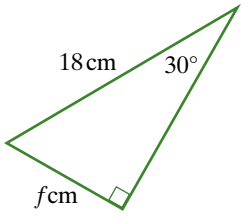

3 Use the exact values to show that $\dfrac{\sin 30°}{\cos 30°} = \tan 30°$.

4 Use the exact values to show each of the following.

a $\sin^2 30° + \cos^2 30° = 1$ **b** $\sin^2 60° + \cos^2 60° = 1$ **c** $\sin^2 45° + \cos^2 45° = 1$

exercise 13.5 challenge

5 Give exact values for each of the following.

a $\sin 135°$ **b** $\cos 225°$ **c** $\sin 150°$ **d** $\tan 135°$

e $\sin 210°$ **f** $\cos 240°$ **g** $\tan 315°$ **h** $\cos 210°$

i $\sin 300°$ **j** $\tan 240°$ **k** $\sin 120°$ **l** $\cos 150°$

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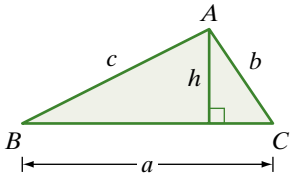
13.6 Area of a triangle

We can use trigonometry to derive a formula for the area of a triangle.

In $\triangle ABC$,

$$\sin B = \frac{h}{c}$$
$$h = c \sin B$$
$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times a \times h$$
$$= \frac{1}{2} ac \sin B$$

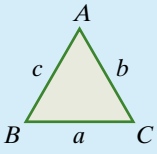


In a similar way, we can show that the area is also equal to $\frac{1}{2}bc \sin A$ or $\frac{1}{2}ab \sin C$.

Area of triangles

The area of a triangle is equal to half the product of two sides and the sine of the included angle.

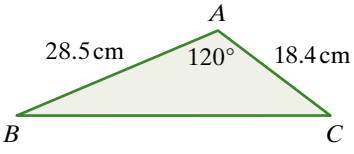
$$\text{Area} = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$



We use this method if we know the lengths of two sides of a triangle and the size of the included angle.

Example 15

Find the area of triangle ABC.



Working

$$Area = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2} \times 28.5 \times 18.4 \times \sin 120°$$
$$\approx 227.1$$

The area is approximately $227\,\text{cm}^2$.

Reasoning

Two sides and the included angle are given.

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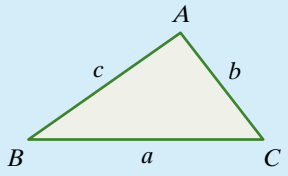
Heron's formula for the area of a triangle

This formula for the area of a triangle was developed about 2000 years ago by Heron of Alexandria, a Greek engineer and mathematician. It is a useful formula for finding the area if only the side lengths of the triangle are known.

Area of a triangle

Area $= \sqrt{s(s-a)(s-b)(s-c)}$

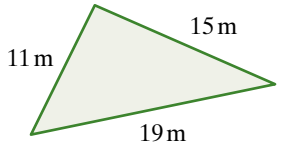
where s is the **semiperimeter** (half perimeter) of the triangle, that is $s = \dfrac{a+b+c}{2}$.



We use Heron's formula if we know the three sides of the triangle.

Example 16

Calculate the area of this triangle, correct to the nearest square metre.



Working	Reasoning
$s = \dfrac{11+15+19}{2}$ $= 22.5$	The three side lengths are known so use Heron's formula. Calculate the semiperimeter, s.
Area $= \sqrt{s(s-a)(s-b)(s-c)}$ $= \sqrt{22.5(22.5-11)(22.5-15)(22.5-19)}$ $= \sqrt{22.5 \times 11.5 \times 7.5 \times 3.5}$ $= \sqrt{6792.1875}$	Substitute values for the side lengths into Heron's formula.
≈ 82.4	Find a rational approximation.
The area is approximately $82\,\text{m}^2$.	Round your answer to the nearest whole number.

How is Heron's formula derived?

The derivation of Heron's formula is included here for those students who are interested to know how such a formula could be found, although algebra as we know it was not used by the ancient Greek mathematicians. As you read the proof, notice how the techniques of algebra are used to simplify expressions at each stage, leading eventually to a neat formula.

First, we express $s - a$, $s - b$ and $s - c$ in terms of a, b and c, by substituting $\dfrac{a+b+c}{2}$ for s.

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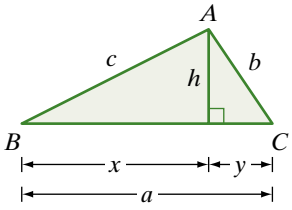
$$s - a = \frac{a + b + c}{2} - a = \frac{a + b + c - 2a}{2} = \frac{b + c - a}{2}$$

$$s - b = \frac{a + b + c}{2} - b = \frac{a + b + c - 2b}{2} = \frac{a + c - b}{2}$$

$$s - c = \frac{a + b + c}{2} - c = \frac{a + b + c - 2c}{2} = \frac{a + b - c}{2}$$

Next, the perpendicular, h, is drawn, splitting a into x and y.



Using Pythagoras' theorem, we know that $x^2 + h^2 = c^2$ and $y^2 + h^2 = b^2$.

From those two equations, we see that $c^2 - x^2 = b^2 - y^2$.

We will now eliminate y, so that we obtain x in terms of a, b and c.

From the diagram, $y = a - x$.

$$\begin{aligned} \text{So } c^2 - x^2 &= b^2 - y^2 \\ c^2 - x^2 &= b^2 - (a - x)^2 \\ c^2 - x^2 &= b^2 - a^2 + 2ax - x^2 \\ c^2 &= b^2 - a^2 + 2ax \\ x &= \frac{c^2 - b^2 + a^2}{2a} \end{aligned}$$

Using this expression for x, we find h in terms of a, b and c.

$$\begin{aligned} h^2 &= c^2 - x^2 \\ &= c^2 - \left(\frac{c^2 - b^2 + a^2}{2a}\right)^2 \\ &= \frac{4a^2c^2 - (c^2 - b^2 + a^2)^2}{4a^2} \end{aligned}$$

Factorising the numerator as a difference of two squares, then rearranging and factorising again as differences of two squares, we obtain

$$\begin{aligned} h^2 &= \frac{4a^2c^2 - (c^2 - b^2 + a^2)^2}{4a^2} \\ &= \frac{[2ac + (c^2 - b^2 + a^2)][2ac - (c^2 - b^2 + a^2)]}{4a^2} \\ &= \frac{(a^2 + 2ac + c^2 - b^2)(b^2 - a^2 + 2ac - c^2)}{4a^2} \\ &= \frac{[(a + c)^2 - b^2][b^2 - (a - c)^2]}{4a^2} \\ &= \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4a^2} \end{aligned}$$

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We can now express each of the factors in the numerator in terms of s.

$$h^2 = \frac{2s \times 2(s-b) \times 2(s-c) \times 2(s-a)}{4a^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$h = \sqrt{\frac{4s(s-a)(s-b)(s-c)}{a^2}}$$

$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

This expression for h can now be used to calculate the area of the triangle.

$$A = \frac{ah}{2}$$

$$= \frac{a}{2} \times h$$

$$= \frac{a}{2} \times \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Algebra is a powerful tool for turning a very complicated formula into a neat simple formula!

Summary of formulas for area of a triangle

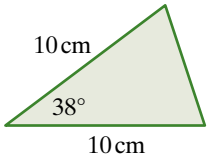
Given information	Formula
Base and perpendicular height	$Area = \frac{bh}{2}$
Three sides	$Area = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$
Two sides and included angle	$Area = \frac{1}{2}ab\sin C$

exercise 13.6

LINKS TO Example 15

1 Use the formula Area $= \frac{1}{2}ab\sin C$ to find the area of each of these triangles.

a



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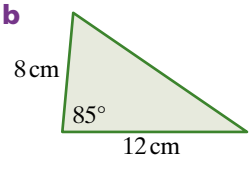

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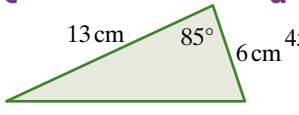

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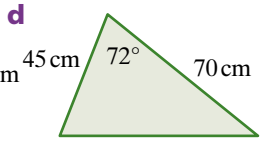

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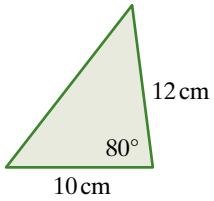

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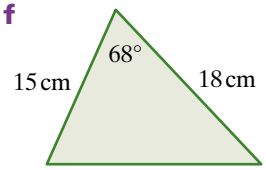

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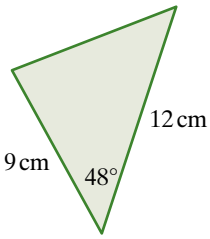

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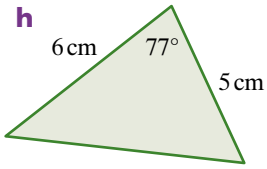

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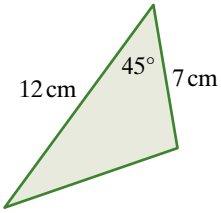

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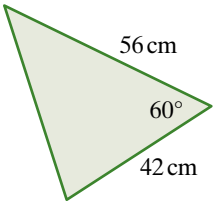

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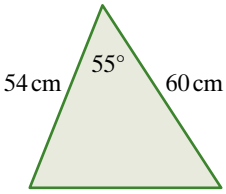

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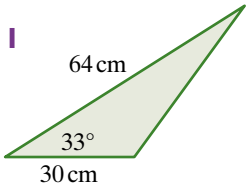

LINKS TO Example 16

2 Use Heron's formula to find the area of each of these triangles correct to one decimal place.

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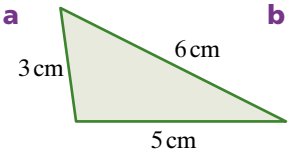

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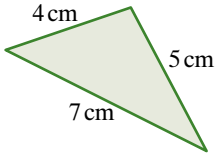

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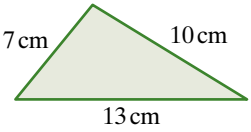

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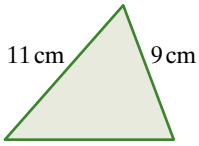

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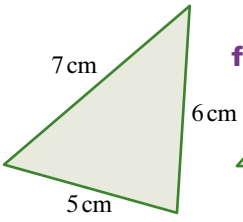

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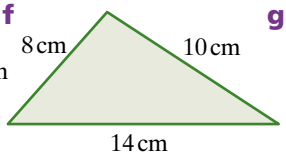

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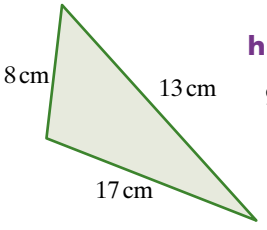

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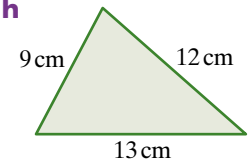

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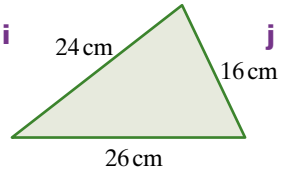

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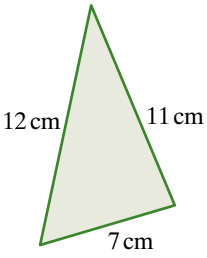

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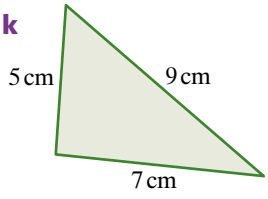

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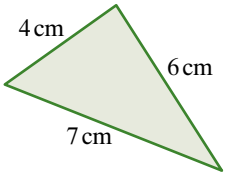
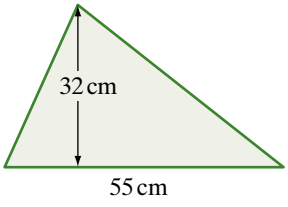


3 Calculate the area of these triangles, choosing the most appropriate formula each time.

a



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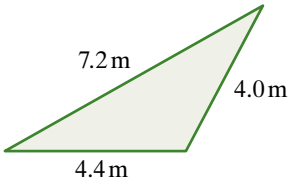

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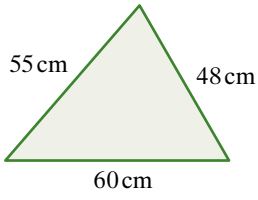

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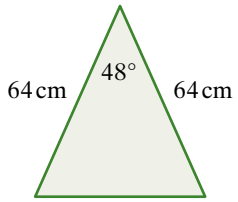

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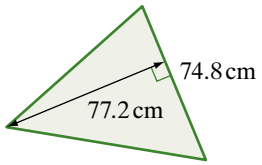

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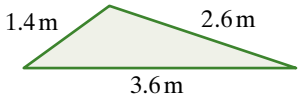

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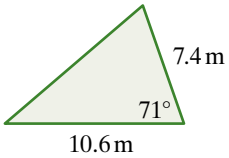

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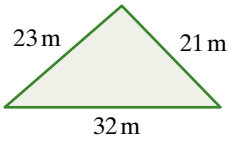

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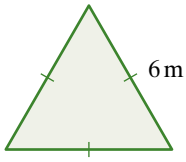

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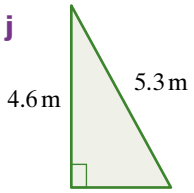
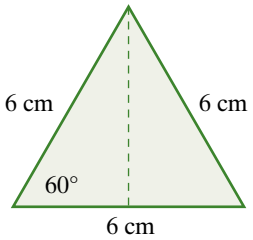


4 **a** Use each of the following methods to calculate the area of this equilateral triangle in simplest surd form.



i Use Heron's formula.

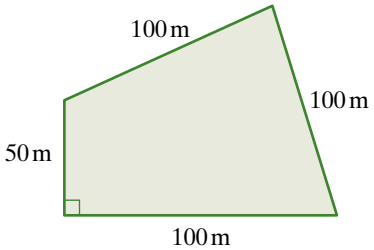
ii Calculate the perpendicular height using Pythagoras' theorem and then use the formula $A = \dfrac{b \times h}{2}$.

iii Use the lengths of two sides and the sine of the included angle.

b Give the area as a rational approximation, correct to one decimal place.

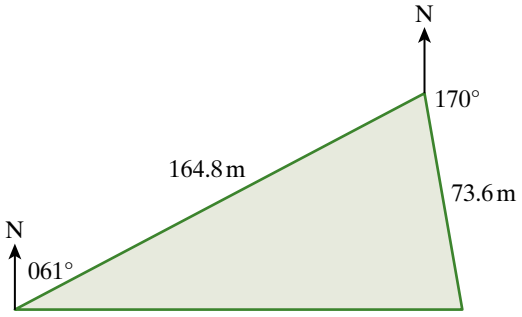
$\sin 60° = \dfrac{\sqrt{3}}{2}$

5 Find the area of this block of land to the nearest square metre.



exercise 13.6 challenge

6 The sides of a triangular block of land are defined in terms of bearings. The lengths of the sides and the bearings are shown on the plan.



a Calculate the area to the nearest square metre.

b Calculate the length of the third side of the block of land, correct to one decimal place.

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13.7 Solving trigonometric equations

A trigonometric equation may be an equation in $\sin\theta$, $\cos\theta$ or $\tan\theta$. To solve a trigonometric equation we use the following steps:
Step 1: Use the same techniques as for algebraic equations to solve for $\sin\theta$, $\cos\theta$ or $\tan\theta$.
Step 2: Find the appropriate first quadrant value for θ.
Step 3: Find any other values for θ that satisfy the equation.

Example 17

Solve the equation $2\sin\theta - 1 = 0$ for $0° \leq \theta \leq 360°$.

Working	Reasoning
$2\sin\theta - 1 = 0$ $2\sin\theta = 1$ $\sin\theta = \frac{1}{2}$	Step 1: Solve the equation for $\sin\theta$.
First quadrant: $\theta = 30°$	Step 2: Find the first quadrant angle.
As $\sin\theta$ is positive, θ is in the first or second quadrant.	Step 3: Find the related angles in the other quadrants.
Second quadrant: $\theta = 180° - 30°$ $= 150°$	Find the second quadrant angle. $\sin(180° - \theta) = \sin\theta$
The two possible values for θ are 30° and 150°.	

Example 18

Solve the equation $2\cos\theta + \sqrt{3} = 0$ for $0° \leq \theta \leq 360°$.

Working	Reasoning
$2\cos\theta + \sqrt{3} = 0$ $2\cos\theta = -\sqrt{3}$ $\cos\theta = -\frac{\sqrt{3}}{2}$	Step 1: Solve the equation for $\cos\theta$.
The related first quadrant angle is 30°.	Step 2: Find the first quadrangle. $\cos\theta$ is negative so θ is in the second or third quadrant.

continued

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Example 18 continued

Working	Reasoning
As $\cos\theta$ is negative, θ is in the second or third quadrant.	Step 3: Find the related first quadrant angle.
$\theta = 180° - 30°$ or $\theta = 180° + 30°$ $\theta = 150°$ or $\theta = 210°$	Find the second and third quadrant angles. $\cos(180° - \theta)$ and $\cos(180° + \theta)$
The two possible values for θ are 150° and 210°.	

Example 19

Solve the equation $\tan^2\theta - 1 = 2$ for $0° \le \theta \le 360°$.

Working	Reasoning
$\tan^2\theta - 1 = 2$ $\tan^2\theta - 3 = 0$ $(\tan\theta - \sqrt{3})(\tan\theta + \sqrt{3}) = 0$ $\tan\theta = \sqrt{3}$ or $\tan\theta = -\sqrt{3}$	Solve for $\tan\theta$ in the same way as for the quadratic equation $x^2 - 1 = 2$.
$\tan\theta = \sqrt{3}$ 1st quadrant: $\theta = 60°$ 3rd quadrant: $\theta = 180° + 60°$ $= 240°$	$\tan\theta$ is positive in the first and third quadrants.
$\tan\theta = -\sqrt{3}$ 2nd quadrant: $\theta = 180° - 60°$ $= 120°$ 4th quadrant: $\theta = 360° - 60°$ $= 300°$	$\tan\theta$ is negative in the second and fourth quadrants.
The possible values for θ are 60°, 120°, 240° and 300°.	List all possible values.

Simplifying trigonometric expressions

Using the trigonometric identities $\sin^2\theta + \cos^2\theta \equiv 1$ and $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ we can simplify trigonometric expressions using the algebraic techniques of factorising and expanding. For example, $\sin^2\theta - \cos^2\theta$ can be factorised as a difference of two squares:

$$\sin^2\theta - \cos^2\theta = (\sin\theta - \cos\theta)(\sin\theta + \cos\theta)$$

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Example 20

Expand and simplify the expression $(\sin\theta + \cos\theta)^2$.

Working

$(\sin\theta + \cos\theta)^2$
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$
$\equiv 1 + 2\sin\theta\cos\theta$

Reasoning

The expression can be expanded in the same way that we expand $(a + b)^2$:
Simplify using $\sin^2\theta + \cos^2\theta = 1$.

exercise 13.7

LINKS TO Example 17

1 Solve the equation $2\sin\theta + 1 = 0$ for $0° \le \theta \le 360°$.

LINKS TO Example 18

2 Solve the equation $2\cos\theta - \sqrt{3} = 0$ for $0° \le \theta \le 360°$.

LINKS TO Example 19

3 Solve the following equations for $0° \le \theta \le 360°$.

a $2\cos\theta - 1 = 0$ **b** $2\sin\theta + \sqrt{3} = 0$ **c** $2\sin^2\theta - 1 = 0$
d $\tan^2\theta - 1 = 0$ **e** $3\tan^2\theta - 1 = 0$ **f** $4\cos^2\theta - 3 = 0$

LINKS TO Example 20

4 Simplify the following expressions.

a $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$ **b** $\tan\theta \times \cos\theta$ **c** $\dfrac{\sin\theta}{\tan\theta}$

d $\dfrac{\cos^2\theta}{1 - \sin\theta}$ **e** $\cos^2\theta\tan^2\theta + \cos^2\theta$ **f** $\sin^3\theta + \sin\theta\cos^2\theta$

5 Simplify $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$.

exercise 13.7 challenge

6 Show that $\dfrac{1 + \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 - \cos\theta}$.

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Analysis task

The sine, cosine and tangent functions

Unit circle

For this analysis task you will need the GeoGebra file *Unit circle and sine, cosine graphs* on the *OneStopDigital* ebook (or a printed copy of the unit circle). You will also need the coordinate grid for plotting the sine and cosine graphs and a graphing calculator or graphing software to view the graphs.

The sine function

a Evaluate the exact values $\sin 30° = \frac{1}{2}$, $\sin 45° = \frac{1}{\sqrt{2}}$ and $\sin 60° = \frac{\sqrt{3}}{2}$, correct to two decimal places.

b Complete the following table of values, remembering that $\sin 0° = 0$ and $\sin 90° = 1$. Use the unit circle to help you with the values for angles beyond 90°. Notice that 180° has been included twice to help you with the symmetry of the values.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$									

θ	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\theta$									

BLM
Sine graph grid

c Using a sharp pencil, carefully plot the values on the graph grid provided on the *OneStopDigital* ebook. When you have finished plotting all the points draw a smooth curve through your points.

d What is the maximum value of $\sin\theta$?

Sine and cosine graphs

e What is the minimum value of $\sin\theta$?

f Describe the behaviour of $\sin\theta$ in each of these four intervals.

- **i** 0° to 90°
- **ii** 90° to 180°
- **iii** 180° to 270°
- **iv** 270° to 360°

g Label the four quadrants on the appropriate intervals of your graph, for example, label 1st quadrant on the interval from 0° to 90°.

h Just as for waves, the amount by which the graph goes above and below its mean value is called the amplitude. What is the amplitude of the sine graph?

The GeoGebra file *Sin Cos graph grid* is already set up with degrees on the x-axis. In this file the sine function can be graphed by entering $y = \sin(x°)$ in the Input bar.

i Notice that the graph forms a repeating pattern. The period of the sine graph is the number of degrees in one complete pattern of the graph. What is the period of the sine graph?

j Explain the meaning of values of x between 360° and 720° in terms of the unit circle.

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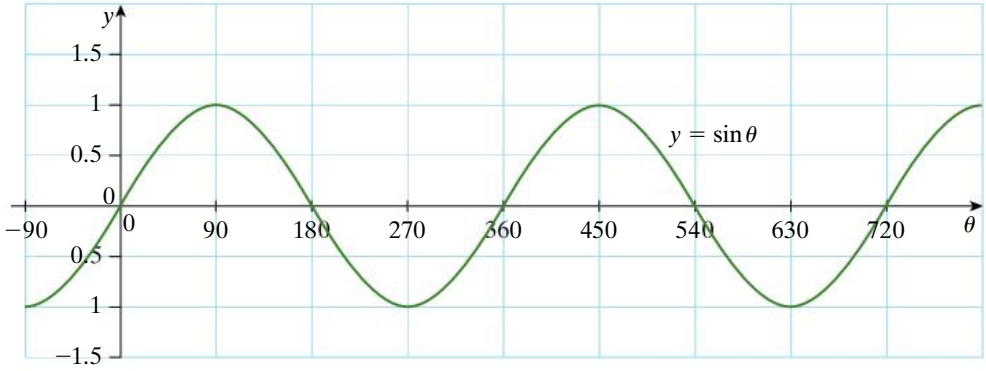
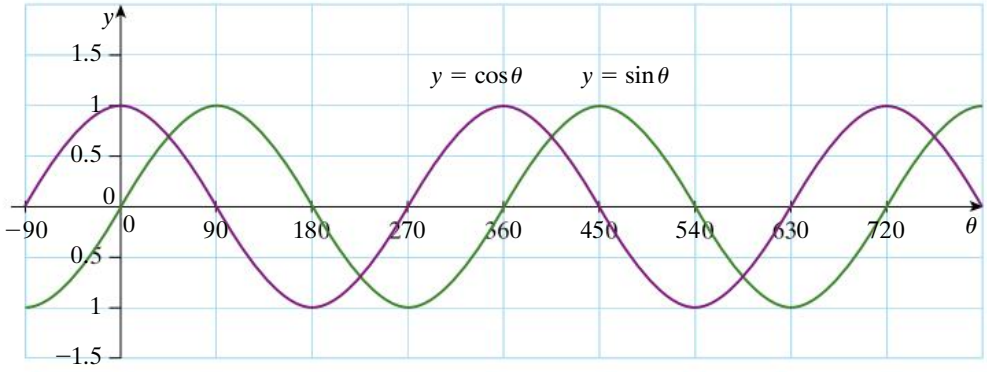
The cosine function

k Complete a table of values for $y = \cos\theta$ in the same way that you did for $y = \sin\theta$.

l Graph the function $y = \cos\theta$ on the same set of axes as your graph of $y = \sin\theta$.



m Compare the graphs of $y = \sin\theta$ and $y = \cos\theta$. Describe the similarities and differences between the graphs of the sine function and the cosine function, including a comparison of the period and amplitude.



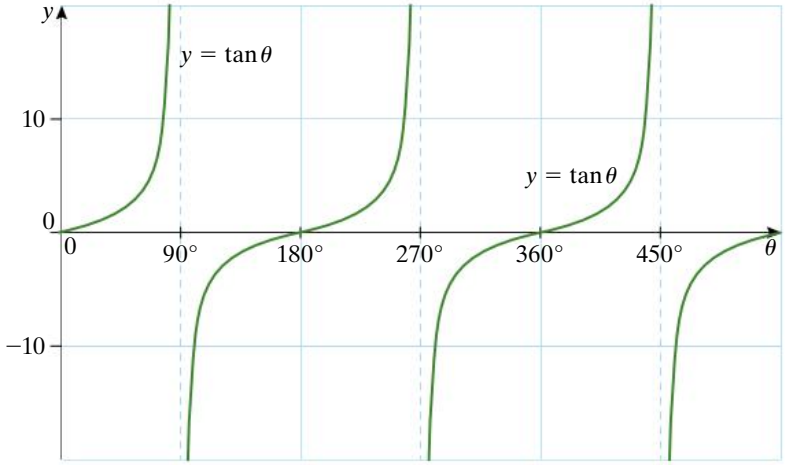
The tangent function

Tangent graph grid

n Use technology to graph $y = \tan\theta$.

o What is the period of the tangent graph?

p Is it meaningful to talk of the amplitude of the tangent graph?



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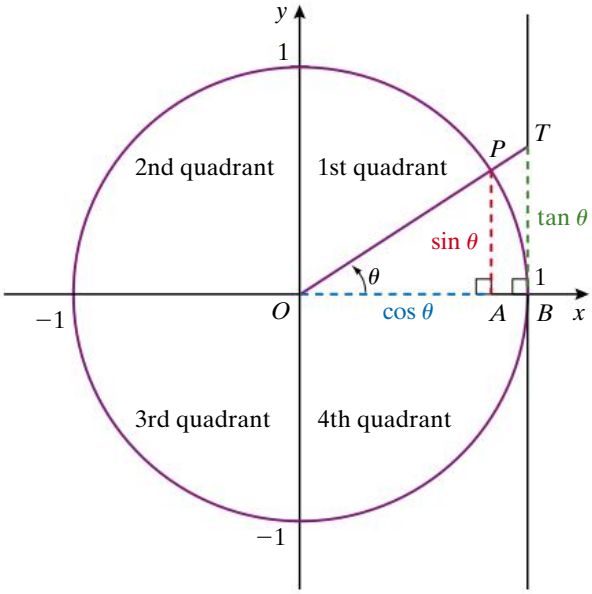
Review Further trigonometry

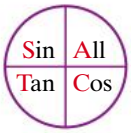
Summary

The unit circle



Summary of signs in the four quadrants

	$\cos\theta$	$\sin\theta$	$\tan\theta$
1st quadrant	+	+	+
2nd quadrant	−	+	−
3rd quadrant	−	−	+
4th quadrant	+	−	−



1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$
$\cos\theta$	$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = \cos\theta$
$\tan\theta$	$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = \tan\theta$	$\tan(360° - \theta) = -\tan\theta$

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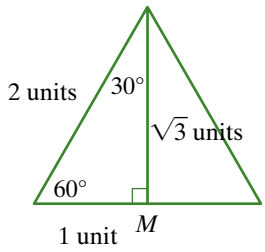
Sine rule

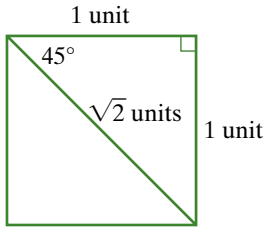
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- The sine rule is used to find unknown lengths and angles when given
  - two angles and one side length
  - two side lengths and an angle opposite one of the given sides
  - The ambiguous case arises if the given angle is less than 90°, is not the angle included between the two given sides, and the side opposite the given angle is smaller than the side adjacent to it.

Cosine rule

- $a^2 = b^2 + c^2 - 2bc\cos A$
- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- The cosine rule is used to find unknown lengths and angles when given
  - all three side lengths.
  - two side lengths and the included angle.

Exact values for 30°, 45° and 60°





	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

	sin	cos	tan
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

Trigonometric identities

- $\sin^2\theta + \cos^2\theta \equiv 1$ for all values of θ
- $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ for all values of θ

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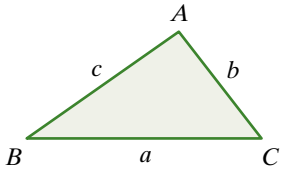
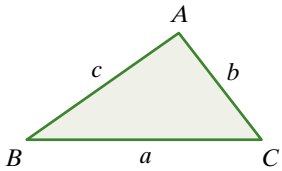
Area of a triangle

- Area $= \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C$
- Use this method if two side lengths and the size of the included angle are known.



Heron's formula

- Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where s is the **semiperimeter** (half perimeter) of the triangle, that is $s = \frac{a+b+c}{2}$
- Use this method if the three side lengths are known.



Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

ambiguous case	exact value	period
amplitude	Heron's formula	semiperimeter
asymptote	identity	sine rule
cosine rule	quadrant	unit circle

Revision

Multiple-choice questions

1 Which of the following is *not* equal to $\sin 40°$?

A $\sin 140°$ **B** $\sin 220°$ **C** $\sin 400°$ **D** $\cos 50°$ **E** $\cos 310°$

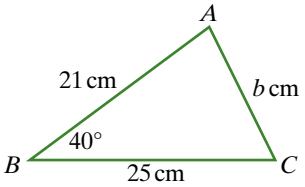
2 If $\tan\theta < 0$ and $\cos\theta > 0$ then θ is in the

A 2nd quadrant **B** 2nd or 3rd quadrants **C** 3rd quadrant
D 4th quadrant **E** 3rd or 4th quadrants

3 If $\sin\theta = \frac{7}{25}$ and θ is in the second quadrant, then $\cos\theta$ equals

A $\frac{24}{25}$ **B** $-\frac{24}{25}$ **C** $-\frac{7}{24}$ **D** $\frac{18}{25}$ **E** $-\frac{18}{25}$

4 The value of b in the triangle shown is given by



A $\frac{21\sin 40°}{25}$ **B** $\frac{25\sin 40°}{21}$

C $\frac{21}{25\sin 40°}$ **D** $\sqrt{21^2 + 25^2 - 2 \times 21 \times 25\cos 40°}$

E $\sqrt{\frac{21^2 + 25^2}{2 \times 21 \times 25\cos 40°}}$

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5 Which of the following is a solution to the equation $3\tan\theta = -\sqrt{3}$?

A $\theta = 30°$ **B** $\theta = 60°$ **C** $\theta = 120°$ **D** $\theta = 150°$ **E** $\theta = 210°$

Short-answer questions

6 State the quadrant for each of these angles.

a 260° **b** 88° **c** 320° **d** 170° **e** 196° **f** 135°

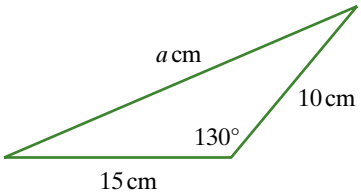
7 Find all possible values for θ, where $0° \le \theta \le 360°$. Give the angles correct to the nearest degree.

a $\sin\theta = 0.6428$ **b** $\cos\theta = -0.8829$ **c** $\tan\theta = 2.1445$

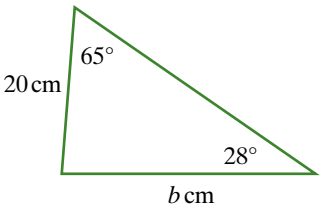
8 What is the value of $\sin^2 40° + \cos^2 40°$?

9 For the following triangles, find the value of the pronumeral, correct to one decimal place.

a

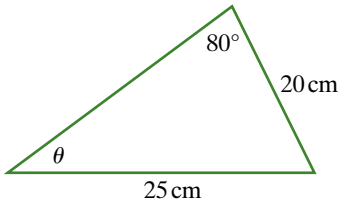


b

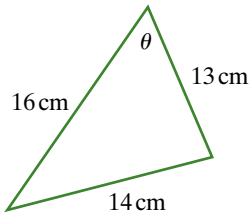


10 Find the size of the angle θ, correct to the nearest degree, in each of these triangles.

a

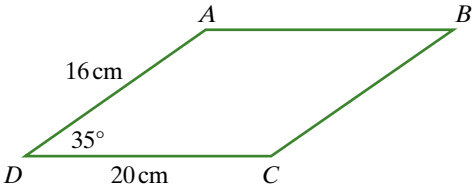


b



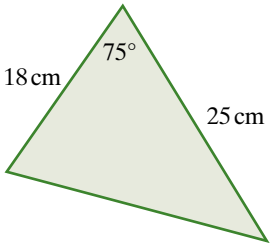
11 In a triangle ABC, $a = 8$, $A = 98°$ and $B = 41°$. Find the missing angle and side lengths. Give the lengths correct to one decimal place.

12 Calculate the length of each diagonal of this parallelogram, correct to one decimal place.

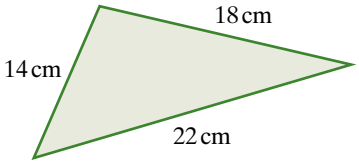


13 Find the area of each of these triangles.

a



b



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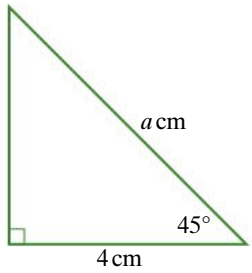
14 State the exact values for each of the following.

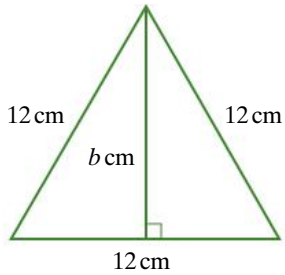
a $\sin 60°$ **b** $\cos 210°$ **c** $\tan 135°$ **d** $\sin 330°$

15 Calculate the exact values of the pronumerals in these triangles.

a



b



16 Solve the equation $2\sin\theta - \sqrt{3} = 0$ for $0° \leq \theta \leq 360°$.

17 If $\cos\theta = \dfrac{\sqrt{15}}{4}$, find $\sin\theta$.

Extended-response question

18 A yacht sails for 15 km on a bearing of 054° then turns and sails for 24 km on a bearing of 125°.

a Draw a labelled diagram to show this information.

b How far is the yacht from its starting point? Give the distance correct to one decimal place.

c On what bearing will the yacht sail to return directly to its starting point? Give the bearing to the nearest degree.

Practice quiz Chapter 13

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Statistics 14

Pre-test

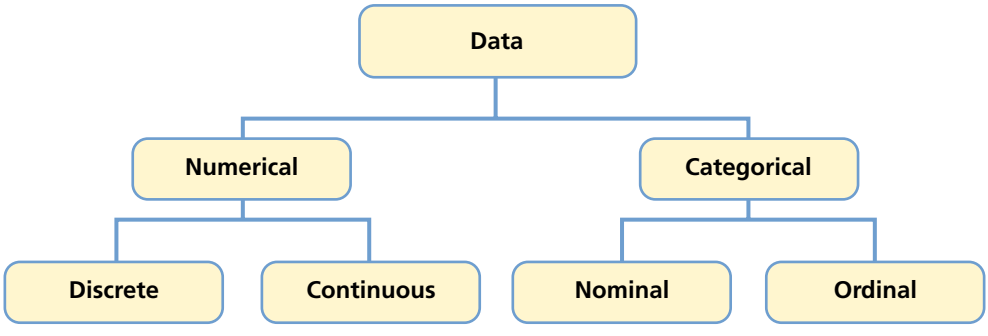
Warm-up

Data can be collected in many forms, such as numbers, words or pictures. Statistics is used to categorise, analyse and display data so that it becomes more useful. Data comes from many sources and may relate, for example, to human populations, agricultural crops, native plants and animals, manufactured goods and climate. In Years 7 to 9 you will have seen many different ways of dealing with data. This chapter introduces some further techniques for analysing, displaying and comparing data.

14.1 Reviewing visual displays of data

You have seen in previous years that statistical data may be of different types. **Numerical data** is data that can be measured or counted. If only particular, usually whole number, values are possible, the data is called **discrete numerical data**, for example the number of cars owned by a family. If all values in a particular range of values, including decimal values, are possible, the data is called **continuous numerical data**, for example, heights and weights.

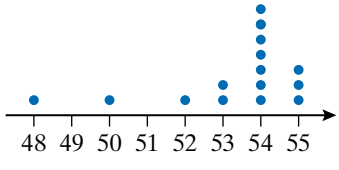
Categorical data is data that can be placed in groups or categories. If categories have an obvious order or ranking (for example, year levels such as Year 8, Year 9, Year 10), the data is described as **ordinal**. If categories are simply names for subgroups of the data (for example, New South Wales, Northern Territory, Taiwan), the data is described as **nominal**.



Different types of visual displays suit different types of data. In previous years, the visual displays that have been used are

- column graphs
- pie graphs
- dot plots
- stem-and-leaf plots (stem plots)
- histograms.

The last three of these are summarised below, showing the types of data for which they are suitable.

Dot plot	Example
■ Best for a small set of *discrete numerical* data where the values are reasonably close together in size. ■ Provides a quick view of the 'shape' of a distribution. ■ Makes it easy to identify any clusters or outliers.	 **Number of matches in a match box**

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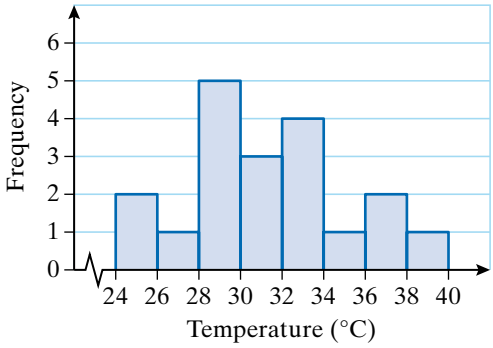
Stem plot	Example
■ Appropriate for *discrete numerical* data, where the values are reasonably close together in size. ■ The data is 'grouped' (into rows) but each of the actual values can still be read from the plot. ■ Provides a quick view of the 'shape' of a distribution; looks a little like a histogram on its side. ■ Use split stems to provide more detail. ■ To show large or small data values in a stem plot, the values may first need to be rounded. ■ Back-to-back stem plots are useful for comparing two data sets with the same numerical variable.	**Number of words per sentence** Stem | Leaf 1 | 0 1 3 4 4 1 | 5 6 6 8 9 2 | 2 3 3 4 4 2 | 5 7 7 3 | 2 2 3 3 | 7 4 | 0 1|5 means 15

Histogram	Example
■ Appropriate for *continuous numerical* data but can also be used for *discrete numerical* data. ■ The horizontal axis has a continuous numerical scale. The vertical axis shows the *frequency*. ■ The data is grouped into *intervals* (for example 24−<26, 26−<28, 28−<30, and so on). ■ The numbers along the horizontal axis label the marks *between* intervals, not the columns themselves. ■ Shows clearly the 'shape' of a distribution; for example skewed, symmetrical, bimodal. ■ To display more detail, decrease the size of the class interval.	**January temperatures in Year 10C's classroom** 

As we can see, dot plots and stem plots both allow each individual data value to be displayed as well as the overall shape of the distribution. The histogram, however, clearly shows the shape of the distribution, but we cannot observe individual data values.

Example 1

The following data show the scores of 18 students on a test.
10, 13, 10, 15, 16, 11, 11, 16, 15, 10, 10, 11, 19, 16, 15, 11, 10, 15

a Display the data as a dot plot.

b Comment on the distribution of scores.

continued

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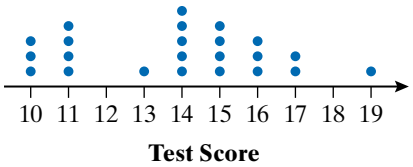
Example 1 continued

Working



b The scores ranged from 10 to 19. Half of the students scored either 14 or 15. The most common score was 14. Seven students scored less than 12.

Reasoning

It is not necessary to start the number line at 0.

The dot plot shows the distribution of scores clearly. Individual data values can be seen. The distribution is bimodal as there is a peak at 11 and another at 14.

Example 2

Students in a Year 10 class were asked to monitor the number of text messages they sent in a particular week. The results are shown below.

0, 0, 0, 2, 3, 4, 4, 5, 8, 8, 11, 11, 11, 15, 15, 16, 16, 16, 17, 17, 19, 19, 19, 23, 25, 29, 34, 58

a Display the data in a stem plot with two stems for each multiple of 10 text messages.
b What is the median number of messages?
c What is the range?
d Does the range give a good indication of the spread of the numbers of text messages?
e Construct a histogram with the same data intervals as in the stem and leaf plot.

Working

a

Stem	Leaf
0	0 0 0 2 3 4 4
0	5 8 8
1	1 1 1
1	5 5 6 6 6 7 7 9 9 9
2	3
2	5 9
3	4
3	
4	
4	
5	
5	8

5|8 means 58

Reasoning

Data values from 0–4 are placed beside the first stem 0. Data values from 5–9 are placed beside the second stem 0. This is continued for the other multiples of 10.

continued

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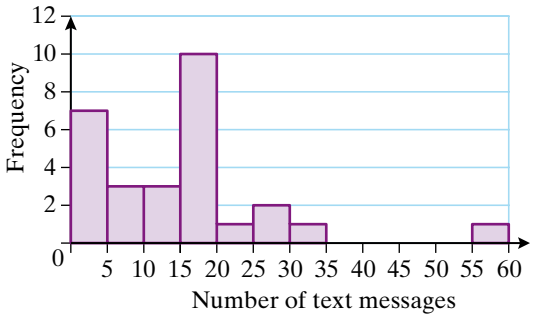
Example 2 continued

Working	Reasoning
b The median number of messages is 15.	There are 28 data values so the median is the mean of the 14th and 15th data values. The 14th and 15th data values are both 15.
c The range is 58 messages.	Range = largest data value – smallest data value = 58 – 0 = 58
d The range is misleading because most of the students sent between 0 and 34 messages.	58 is an outlier.
e **Text messages sent by 28 Year 10 students in a particular week**	Note that the data is discrete numerical data and that the stem plot and histogram display the data in a similar way.



Stem	Leaf
0	0 0 0 2 3 4 4
0	5 8 8
1	1 1 1
1	5 5 6 6 6 7 7 9 9 9
2	3
2	5 9
3	4
3	
4	
4	
5	
5	8

5|8 means 58

The stem plot shows individual data values. The histogram does not show individual data values.

Notice in the above graph that the numbers on the horizontal axis label the marks between intervals. This is different from a column graph, where the columns are labelled rather than the divisions between them.

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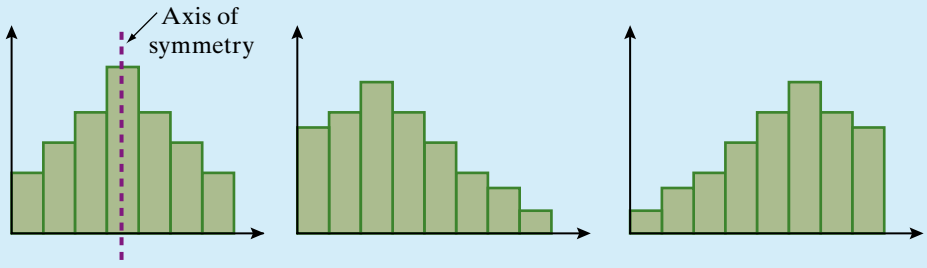
Symmetrical and skewed distributions

The overall shape of a histogram helps us analyse the 'shape' or distribution of the data.

When the data has a symmetrical distribution, the histogram has a vertical axis of symmetry. A distribution that is **asymmetrical**—that is, *not* symmetrical—is described as having either **positive skew** if it shows a 'lean' to the left so it 'tails' to the right or **negative skew** if it shows a lean to the right so it 'tails' to the left.

The shape of a distribution



The shape of distributions can also be seen in dot plots and stem plots.

Example 3

Students participating in the CensusAtSchool project were asked to complete a concentration task (unassisted, under supervision) and to record the time they took. The times (in seconds) for a sample of 30 students were: 40, 42, 129, 98, 68, 40, 43, 66, 57, 35, 51, 46, 76, 68, 37, 56, 46, 51, 23, 44, 34, 64, 52, 33, 50, 33, 42, 50, 42, 33.

a What type of statistical variable do the values represent?

b Create a histogram with an appropriate class interval to show the data.

c Comment on the distribution of this sample data.

d Suggest an alternative visual plot to view the distribution of this variable.

Working

a The time taken to complete the task could take any value within appropriate limits. Therefore, this is a continuous numerical variable.

Reasoning

Although the sample values have been rounded to the nearest second, it is possible for the task to take 'in between times' such as 65.5 seconds. So the data is not restricted to particular values.

continued

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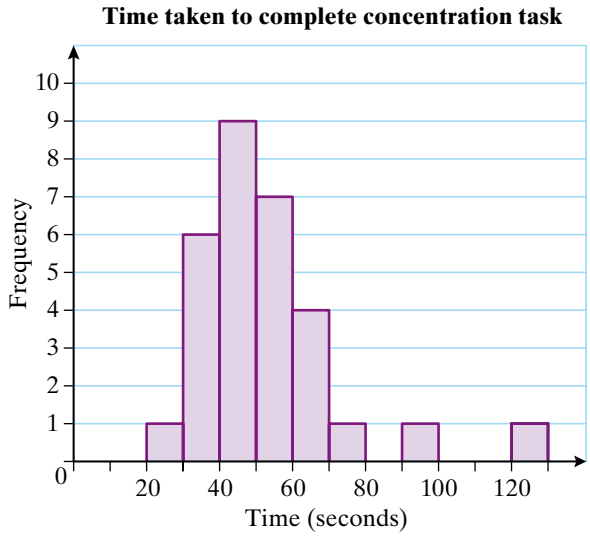
Example 3 continued

Working

b An interval of 10 or 20 seconds seems appropriate. The frequency table at the right and the histogram below show intervals of 10 seconds.

Reasoning

The time taken to complete the task can be tabulated in a frequency table.



Time taken (seconds)	Frequency
0−<10	0
10−<20	0
20−<30	1
30−<40	6
40−<50	9
50−<60	7
60−<70	4
70−<80	1
80−<90	0
90−<100	1
100−<110	0
110−<120	0
120−<130	1

c The majority of the data appears to be centred fairly symmetrically around times of 50 seconds, although there is an outlier in the 120−<130 seconds interval.

To describe the distribution, identify the centre and spread of the data, and any unusual features.

d As the values are given to the nearest second, a stem plot could be a useful display, as it would show each individual data value.

A stem plot is useful for a small set of numerical values that are reasonably close together.

Back-to-back stem and leaf plots

In Year 9 we saw how back-to-back stem and leaf plots are useful for comparing two different groups for the same numerical variable. The back-to-back stem and leaf plot allows us to find the median value for each group and to compare the shapes of the two distributions.

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Example 4

The following data show the heights in centimetres of a random sample of Year 7 boys and girls.

Boys

146, 148, 149, 150, 150, 151, 152, 153, 154, 154, 155, 156, 157, 163, 163, 164, 165, 166, 167, 170, 173, 173, 174, 175, 180

Girls

141, 146, 147, 151, 152, 153, 154, 155, 155, 155, 155, 155, 156, 156, 156, 158, 159, 160, 160, 161, 161, 163, 164, 169, 174

a Use the data to construct a back-to-back stem and leaf plot
b Compare the median height for boys and for girls
c Compare the range for the two groups
d Comment on the shape of the distributions.

Working

a **Heights of 25 Year 7 boys and 25 Year 7 girls (cm)**

Leaves	Stem	Leaves
Boys		**Girls**
	14	1
9 8 6	14	6 7
4 4 3 2 1 0 0	15	1 2 3 4
7 6 5	15	5 5 5 5 5 6 6 6 8 9
4 3 3	16	0 0 1 1 3 4
7 6 5	16	9
4 3 3 0	17	4
5	17	
0	18	

Key: 1|14| means 141 and |14|6 means 146

Reasoning

Using two stems for each 10 cm shows the shape of the distribution more clearly. The stem and leaf plot should have a title and a key.

b The median height for the boys is 157 cm.
The median height for the girls is 156 cm.

There are 25 students in each sample, so the median is the 13th data value.

c Range = largest data value – smallest data value
Boys: 180 cm – 146 cm = 34 cm
Girls: 174 cm – 141 cm = 33 cm
The range of heights for boys and for girls is 33 cm.

Boys: smallest data value is 146 cm; largest data value is 180 cm.
Girls: smallest data value is 141 cm; largest data value is 174 cm.

continued

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Example 4 continued

Working

d The boys' heights are positively skewed, with the larger heights more spread out. There is a peak in the 150 cm to 154 cm and another small peak at 170 cm to 174 cm. The distribution of the girls' heights is fairly symmetrical, with the greatest number of heights between 155 cm and 159 cm.

Reasoning

If each distribution is turned sideways, the distribution resembles a histogram.

continued

exercise 14.1

LINKS TO Example 1

1 The following data show the ages of students in a language class.
19, 15, 18, 15, 18, 19, 20, 20, 16, 20, 17, 15, 17, 18, 19, 17, 16, 19, 18, 20

a Represent the data as a dot plot.

b Comment on the shape of the distribution.

LINKS TO Example 2

2 The following data show the ages of people in a fast food outlet at a 8 am one Saturday.
27, 41, 28, 32, 21, 35, 5, 19, 27, 8, 38, 38, 49, 43, 5,
11, 21, 42, 37, 48, 62, 15, 16, 32, 3, 5, 34, 13, 13, 14

a Represent the data as a stem plot using

i a stem for each multiple of 10

ii split stems (two stems for each multiple of 10).

Use a key to indicate what the stem and leaf values represent.

b Which stem plot do you think the fast food outlet would find most useful as an indication of the age group of the customers at that time? Explain.

LINKS TO Example 3

3 Use the data from question **2** to answer the following.

a Construct a histogram using a suitable class interval.

b Describe the shape of the distribution.

LINKS TO Example 4

4 The following data show the reaction times in seconds to an online task for 25 Year 7 students using their dominant hand.

Boys

0.31, 0.32, 0.32, 0.34, 0.34, 0.34, 0.34, 0.34, 0.35, 0.37, 0.38, 0.40, 0.40, 0.40, 0.42, 0.42, 0.43, 0.43, 0.48, 0.50, 0.53, 0.56, 0.64, 0.89

Girls

0.26, 0.31, 0.32, 0.32, 0.33, 0.34, 0.34, 0.35, 0.35, 0.35, 0.36, 0.37, 0.37, 0.39, 0.39, 0.39, 0.40, 0.40, 0.40, 0.43, 0.43, 0.49, 0.54, 0.59, 0.62

a Use the data to construct a back-to-back stem and leaf plot (for the stems use 2 to represent 0.2, and so on)

b Compare the median time for boys and for girls

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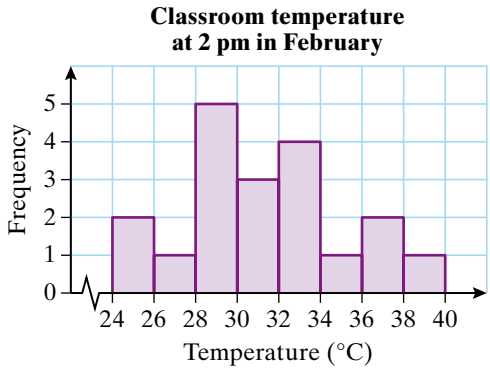
c Compare the range for the two groups

d Comment on the shape of the distributions.

5 Students at a certain school are trying to convince the principal that one of the classrooms needs an air conditioner. To support their case, they monitor the classroom temperature at 2 pm on each school day during February. This histogram shows the recorded temperatures (to the nearest degree Celsius).



a What class interval is used here?

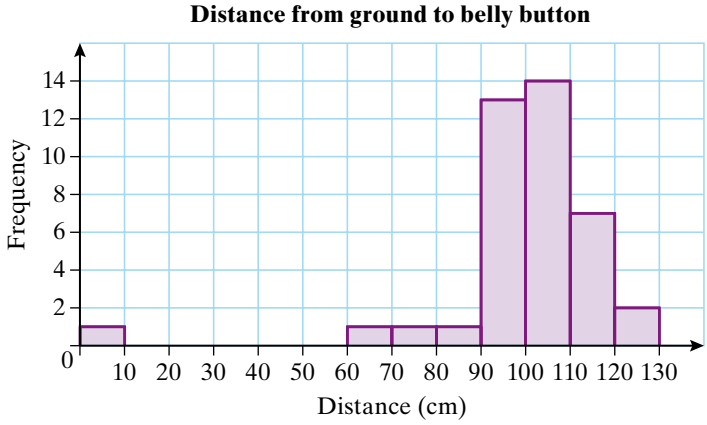
b On how many days were data recorded?

c On what proportion of the days was the temperature measured at 32°C or above?

d Redraw the histogram using a different class interval, so that the days in part **c** are shown in a single column.

exercise 14.1 challenge

6 In the CensusAtSchool project, one question asked students to measure the distance (to the nearest centimetre) from the ground to their belly button. The data from a sample of 40 students is displayed in this histogram.



a Explain why this is an example of data from a continuous numerical variable.

b What proportion of students had a measurement greater than 1 m?

c How do you think the measurement of 0–10 cm might be explained?

d Redraw the histogram using

i intervals of 20 cm **ii** intervals of 40 cm

e Explain what happens to the display of the distribution as the size of the interval is increased.

f Explain why it would not be possible to construct a histogram with an interval of 5 cm from the data displayed above.

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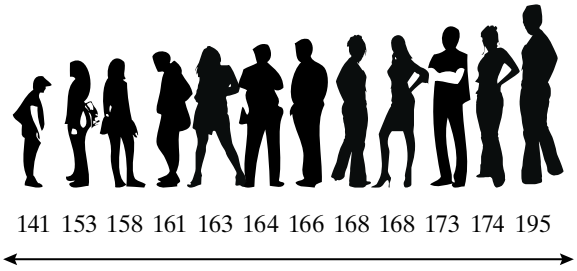
14.2 Measuring spread: interquartile range

When describing a set of data, we are often interested in the variation in the data, that is, the spread of data from the mean or median. In some data sets the data values may be bunched around the centre, whereas in other data sets the values may be distributed more widely. In *MathsWorld 9* the **range** was used as a measure of the spread of data.

Range
range = the largest data value – the smallest data value

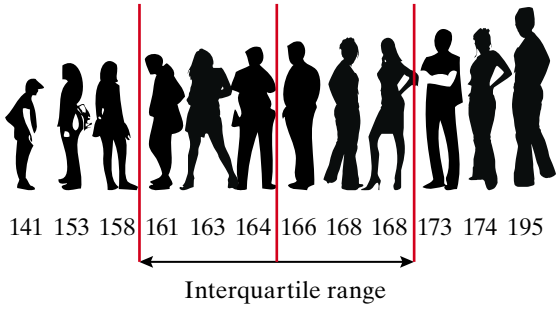
The diagram shows the heights in centimetres of 12 students. The range is 54 cm.



One problem with using the range to describe the spread of a set of data is that it can be affected by one extreme large value or small value and therefore may not be a useful representation of the variation within the data group. In the above example, the range of 54 cm is somewhat misleading since, apart from the one very short and one very tall student, the spread of heights was quite small.

Quartiles and the interquartile range

One way of overcoming this problem is to place the data in sorted order, and then divide the sorted data into **quartiles** (each holding 25% of the data values), and look at the spread within the middle two quartiles (or middle 50% of the data values). In this example, there are three data values in each quartile.



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The **median** divides a set of data into two halves because it is the middle value. The middle value of the lower half is called the **lower quartile**, Q_1. The middle value of the upper half is called the **upper quartile**, Q_3. The median is denoted by Q_2. The **interquartile range** (IQR) is the difference between Q_3 and Q_1.

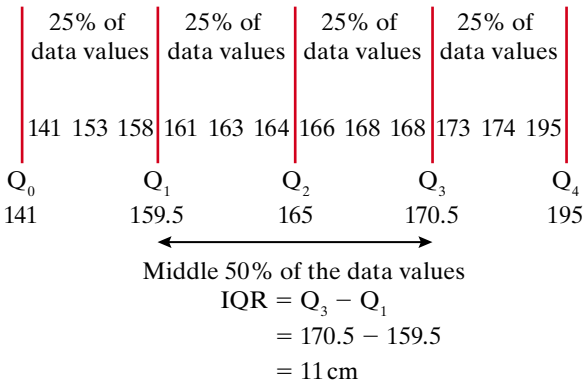
Unlike the range, the interquartile range is not affected by extreme data values because it measures the spread of the middle half of the data.

Interquartile range

Interquartile range (IQR) $= Q_3 - Q_1$

Minimum data value	Q_0	
Lower quartile	Q_1	middle value of the lower half of the data
Median	Q_2	middle value of the whole data set
Upper quartile	Q_3	middle value of the upper half of the data
Maximum data value	Q_4	

The quartiles and interquartile range for the height data are shown below.



$$\text{IQR} = Q_3 - Q_1$$
$$= 170.5 - 159.5$$
$$= 11\text{ cm}$$

Q_2 is the same as the median.

Calculating Q_1, Q_3 and the IQR

The values of Q_1 and Q_3 are found in a manner similar to the way the median is calculated.

The procedure can be summarised as follows.

- Arrange the data in ascending order.
- Divide the data into two halves (if n is odd, ignore the middle value).
  - Q_1 is the median of the lower half of data values.
  - Q3 is the median of the upper half of data values.
- Calculate Interquartile Range or IQR $= Q_3 - Q_1$.

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Example 5

Some Year 10 students were asked to record how much money they received in a particular week, from sources such as pocket money or part-time work. The results (in dollars) for 20 students were 0, 0, 25, 0, 260, 15, 30, 10, 0, 13, 0, 0, 0, 20, 30, 80, 20, 79, 5, 40.

Calculate Q_1, Q_3 and the IQR.

Working	Reasoning
0, 0, 0, 0, 0, 0, 0, 5, 10, 13, | 15, 20, 20, 25, 30, 30, 40, 79, 80, 260	Arrange the data in order from smallest to largest. Identify the median. The median is between 13 and 15.
$Q_1 = 0$ $Q_3 = 30$ $\text{IQR} = Q_3 - Q_1$ $= 30 - 0$ $= 30$	The lower quartile is the median of the lower half. The two middle values are 0 and 0. The upper quartile is the median of the upper half. The two middle values are 30 and 30.
So Q_1 is 0, Q_3 is 30 and the interquartile range is 30.	The interquartile range is the difference between the upper and lower quartiles.

Five-number summary

The five values Q_0, Q_1, Q_2, Q_3 and Q_4 are known as the **five-number summary**. Example 6 shows the calculation of the five-number summary for a data set with an odd number of data values. Note that the median in this case is an actual data value. It is not counted in the lower and upper halves of the data when finding Q_1 and Q_3.

Example 6

Calculate the five-number summary for the following data set.
20, 2, 10, 14, 20, 4, 15, 19, 11, 7, 12, 4, 16, 5, 18

Working	Reasoning
2, 4, 4, 5, 7, 10, 11, 12, 14, 15, 16, 18, 19, 20, 20	The minimum value is 2. The maximum value is 20.
The five number summary is: $Q_0 = 2$ $Q_1 = 5$ $Q_2 = 12$ $Q_3 = 18$ $Q_4 = 20$	There are 15 data values. The median is the 8th value, that is, 12. To find the lower quartile, find the middle value of the lower half of the data: 2, 4, 4, 5, 7, 10, 11

continued

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Example 6 continued

Working

Reasoning

Note that 12 is not counted in the lower or upper halves of the data.
To find the upper quartile, find the middle value of the upper half of the data:
14, 15, 16, 18, 19, 20, 20

Outliers

A data set may have one or two values that can be considered as extreme values that do not fit the general pattern of the other data values. These extreme values are called outliers.

Identifying outliers

Calculate $Q_1 - 1.5 \times IQR$ and $Q_3 + 1.5 \times IQR$. Any data value that is below $Q_1 - 1.5 \times IQR$ or above $Q_3 + 1.5 \times IQR$ is described as an outlier.

Example 7

Find if there any outliers in the data sets in

a example **5** **b** example **6**

Working

a 0, 0, 0, 0, 0, 0, 0, 5, 10, 13, 15, 20, 20, 25, 30, 30, 40, 79, 80, 260

$Q_1 = 0$

$Q_3 = 30$

$IQR = 30 - 0 = 30$

$1.5 \times IQR \text{ above } Q_3 = Q_3 + 1.5 \times IQR$
$= 30 + 1.5 \times 30$
$= 75$

$1.5 \times IQR \text{ below } Q_1 = Q_1 - 1.5 \times IQR$
$= 0 - 1.5 \times 30$
$= -45$

So 79, 80 and 260 would be considered outliers.

b 2, 4, 4, 5, 7, 10, 11, 12, 14, 15, 16, 18, 19, 20, 20

$Q_1 = 5$

$Q_3 = 18$

$IQR = 18 - 5 = 13$

Reasoning

$1.5 \times IQR$ above Q_3 and below Q_1 mark 'bounds' for the majority of the data. Any values outside these bounds are described as outliers.
The data values 79, 80 and 260 are all outside the bound of 75.

The lowest data value is 0 so there are no outliers at the lower end.

continued

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Example 7 continued

Working	Reasoning
$1.5 \times \text{IQR above } Q_3 = Q_3 + 1.5 \times \text{IQR}$ $= 18 + 1.5 \times 13$ $= 18 + 19.5$ $= 37.5$ $1.5 \times \text{IQR below } Q_1 = Q_1 - 1.5 \times \text{IQR}$ $= 5 - 1.5 \times 13$ $= 5 - 19.5$ $= -14.5$	
There are no outliers.	There are no data values lying outside -14.5 and 37.5.

exercise 14.2

LINKS TO Example 5

1 For each of the following data sets, calculate

i Q_1 and Q_3 **ii** the interquartile range

a 4, 4, 5, 5, 5, 6, 7, 7, 8, 9, 9, 10, 12, 13, 13

b 15, 15, 16, 16, 16, 16, 18, 18, 19, 21, 21, 23, 24, 24, 26

c 40, 43, 46, 48, 52, 57, 58, 60, 61, 63, 67, 71, 75, 76, 80

d 1.3, 1.3, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8, 1.8, 1.9, 1.9, 2.0, 2.0, 2.1, 2.2

e 1, 3, 3, 3, 5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 10, 16

f 12, 13, 22, 23, 30, 41, 48, 49, 50, 51, 63, 64, 70, 75, 77, 80

g 19, 15, 18, 15, 18, 19, 20, 20, 16, 20, 17, 15, 17, 18, 19, 17, 16, 19, 18, 20

h 102, 97, 95, 99, 96, 102, 102, 100, 97, 102, 99, 99, 95, 97, 102, 100, 95, 99, 101, 100

LINKS TO Example 6

2 Create a five-number summary for each of the following sets of data.

a 45, 2, 49, 92, 83, 55, 67, 40, 24, 22, 51, 88, 58, 18, 81, 67, 14, 12, 38, 15

b 26.75, 26.82, 26.98, 23.25, 24.69, 25.82, 24.34, 26.44, 24.79, 26.58, 23.21, 24.87, 24.89, 26.71, 24.38, 26.09, 26.69, 23.47, 26.40, 25.26

3 Two companies produce balls of wool that are labelled as having a mass of 50 g. A random sample of 10 balls is taken from each company, and each ball is weighed. The results (in grams) are as follows.

Company A: 50.2, 50.8, 50.2, 50.6, 49.4, 49.2, 51.6, 50.3, 47.6, 52.1

Company B: 51.2, 51.1, 51.3, 51.7, 51.5, 50.4, 50.8, 50.2, 50.7, 51.1

a For each company's data, calculate the median and the IQR.

b Do the results tell you anything about the manufacturing processes of the two companies?

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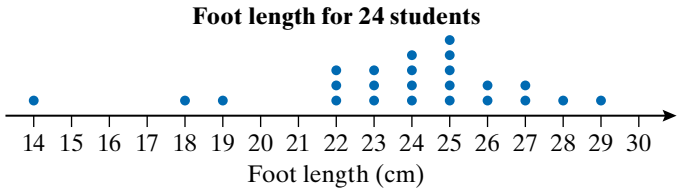
LINKS TO Example 7

4 The CensusAtSchool project asked students to record their foot length. This dot plot shows foot lengths for a sample of 24 students.



a Calculate the five-number summary for the data.

b Between which two lengths do the middle 50% of the data values lie?

c Circle each 25% of the data. Which 'tail' is longer, and what does this mean?

d Determine which lengths are outliers in this sample of students.

5 A newspaper article was analysed for sentence length. The stem plot shows the results.

Sentence length

Stem	Leaf
1	0 1 3 4 4
1	5 6 6 8 9
2	2 3 3 4 4
2	5 7 7
3	2 2 3
3	7
4	0

(Key: 1|5 means 15)

a Calculate the five-number summary for the data.

b Calculate the interquartile range.

c What are the values between which the middle 50% of sentence lengths would lie?

d Comment on the sentence length in this newspaper. Would the paper suit people with a low level of reading ability?

6 The following set of data shows the masses (in grams) of a dozen eggs sampled from a particular poultry farm:
56, 57, 62, 67, 61, 56, 63, 69, 51, 53, 69, 54.

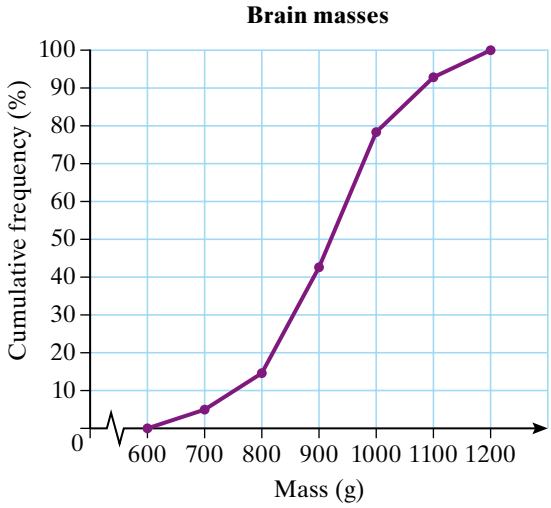
a Calculate the median mass.

b Assuming this sample is representative of the eggs produced at this farm, calculate the values between which the middle 50% of egg masses will lie.

c Would an egg that weighs 72 g be considered an outlier according to the '1.5 × IQR' rule?

exercise 14.2 challenge

7 A cumulative percentage frequency graph shows how the cumulative percentage frequency changes as the values of the data increase.



a Use the cumulative percentage frequency graph of brain masses to find each of the following.

i Q_0 (minimum value)

ii Q_1 (25th percentile)

iii Q_2 (median)

iv Q_3 (75th percentile)

v Q_4 (maximum value)

b Find the interquartile range.

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14.3 Measuring spread: standard deviation

Range and interquartile range are both based on the overall variation in the data set. It is also useful to have a measure of spread that takes into account the variation of individual data values from the mean. The **standard deviation** measures how clustered the data values are around the mean.

Consider the following data set.

13, 15, 16, 12, 19, 17, 13, 14, 20, 15

First we calculate the mean, $\bar{x}$.

$$\bar{x} = \frac{13 + 15 + 16 + 12 + 19 + 17 + 13 + 14 + 20 + 15}{10} = \frac{154}{10} = 15.4$$

To find how the individual data values vary from the mean, we subtract the mean from each data value, that is, we calculate $x - \bar{x}$ for each data value. By definition of the mean, the sum of these differences is always zero. The positive differences cancel the negative differences, as shown below. So finding the sum of the differences does not help us find a measure of spread. One way of overcoming this problem of the differences summing to zero is to square the differences before adding them. We now have a sum of positive values.

x	$x - \bar{x}$	$(x - \bar{x})^2$
13	−2.4	5.76
15	−0.4	0.16
16	0.6	0.36
12	−3.4	11.56
19	3.6	12.96
17	1.6	2.56
13	−2.4	5.76
14	−1.4	1.96
20	4.6	21.16
15	−0.4	0.16
Total	$\sum(x - \bar{x}) = 0$	$\sum(x - \bar{x})^2 = 62.4$

The sum of the squared differences is then divided by $n - 1$, where n is the number of data values in the sample. This is called the **variance** of the data. Recall that we originally squared all the differences, so it is now appropriate to take the square root of the variance. The square root of the variance is called the **standard deviation**.

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Measures of spread

Variance

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

Standard deviation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

Standard deviation is a widely used measure of the spread of data.

In the above data set:

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

$$= \sqrt{\frac{62.4}{10 - 1}}$$

$$= \sqrt{6.93}$$

$= 2.63$ to two decimal places

Note:

If data are available for a whole population then we divide by n rather than $n - 1$. In this chapter, we are dealing with samples of populations so we divide by $n - 1$.

Example 8

Calculate the variance and standard deviation for the following data set.
20, 2, 10, 14, 20, 4, 15, 19, 11, 7, 12, 4, 16, 5, 18

Working

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{20 + 2 + 10 + 14 + 20 + 4 + 15 + 19 + 11 + 7 + 12 + 4 + 16 + 5 + 18}{15}$$

$$= \frac{177}{15}$$

$$= 11.8$$

Reasoning

The mean is calculated by adding all the data values and dividing by the number of data values.

continued

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Example 8 continued

Working

x	$x - \bar{x}$	$(x - \bar{x})^2$
20	8.2	70.56
2	−9.8	96.04
10	−1.8	3.24
14	2.2	4.84
20	8.2	70.56
4	−7.8	60.84
15	3.2	10.24
19	7.2	51.84
11	−0.8	0.64
7	−4.8	23.04
12	0.2	0.04
4	−7.8	60.64
16	4.2	17.64
5	−6.8	46.24
18	6.2	38.44
Total	$\sum(x - \bar{x}) = 0$	$\sum(x - \bar{x})^2 = 548.4$

Reasoning

The mean is subtracted from each data value to find the difference. Each of the differences is squared.

The sum of the squared differences is calculated.

$$s^2 = \frac{\sum(x - \bar{x})^2}{n - 1}$$
$$= \frac{548.4}{15 - 1}$$
$$= 39.17$$

The variance is 39.17.

The sum of the squared differences is divided by $n - 1$. The result is called the variance.

Standard deviation
$= \sqrt{39.17}$
$= 6.26$ to two decimal places

The standard deviation is the square root of the variance.

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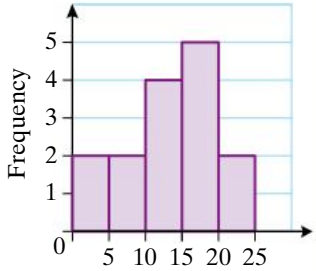
The standard deviation gives a fairly good indication of where most of the data lies. Provided the data is reasonably symmetrical (that is, it is not greatly skewed), we often find that:

- two-thirds (approximately 67%) of the data lies in the interval one standard deviation either side of the mean, that is, $\bar{x} \pm s$.
- 95% of the data lies in the interval two standard deviations either side of the mean, that is, $\bar{x} \pm 2s$.
- nearly all of the data lies in the interval three standard deviations either side of the mean, that is, $\bar{x} \pm 3s$.

Example 9

Consider the data set from example 8.
What percentage of the data values 20, 2, 10, 14, 20, 4, 15, 19, 11, 7, 12, 4, 16, 5, 18

a might you expect to lie within the interval $\bar{x} \pm s$?
b actually lie within the interval $\bar{x} \pm s$?
c might you expect to lie within the interval $\bar{x} \pm 2s$?
d actually lie within the interval $\bar{x} \pm 2s$?

Working	Reasoning
a Two-thirds (approximately 67%) of the data values might be expected to lie in the interval $\bar{x} \pm s$.	This percentage may apply if the distribution is symmetrical. A histogram of the data shows that the distribution is not symmetrical. 
b $\bar{x} \pm s = 11.8 \pm 6.3$. $11.8 + 6.3 = 18.1$ $11.8 - 6.3 = 5.5$ Eight of the data values lie between 5.5 and 18.1. This is 53% of the data values.	The data is not clustered quite as densely around the mean compared with a distribution where two-thirds (67%) of data values lie in the interval $\bar{x} \pm s$.
c 95% of the data values would be expected to lie in the interval $\bar{x} \pm 25$.	This percentage may apply if the distribution is symmetrical. The above histogram shows that the distribution is not symmetrical.
d $\bar{x} \pm 2s = 11.8 \pm 12.6$ $11.8 + 12.6 = 24.4$ $11.8 - 12.6 = -0.8$ 100% of the data values lie between -0.8 and 24.4.	The data is not spread out quite as far from the mean compared with a distribution where only 95% of data values lie in the interval $\bar{x} \pm 2s$.

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Example 10

The mean for a set of approximately symmetrical data is 25.4 and the standard deviation is 3.2.

a Between which limits would you expect approximately two-thirds of the data to lie?

b Between which limits would you expect approximately 95% of the data to lie?

Working

a Approximately two-thirds of the data would be expected to lie between 25.4 ± 3.2, that is, between 22.2 and 28.6.

b Approximately 95% of the data would be expected to lie between $25.4 \pm 2 \times 3.2$, that is, between 19.0 and 31.8.

Reasoning

Two-thirds of the data will lie in the interval one standard deviation either side of the mean, that is, $\bar{x} \pm s$.

95% of the data will lie in the interval two standard deviations either side of the mean, that is, $\bar{x} \pm 2s$.

Tech tip

Microsoft Excel can be used to calculate standard deviation.

	A	B	C
1	2	=STDEV(A1:A15)	6.258708
2	4		
3	4		
4	5		
5	7		
6	10		
7	11		
8	12		
9	14		
10	15		
11	16		
12	18		
13	19		
14	20		
15	20		

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exercise 14.3

LINKS TO Example 8

1 For each of these data sets calculate to two decimal places where necessary

i the mean **ii** the variance **iii** the standard deviation

a 4, 4, 5, 5, 5, 6, 7, 7, 8, 9, 9, 10, 12, 13, 13

b 15, 15, 16, 16, 16, 16, 18, 18, 19, 21, 21, 23, 24, 24, 26

c 40, 43, 46, 48, 52, 57, 58, 60, 61, 63, 67, 71, 75, 76, 80

d 1.3, 1.3, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8, 1.8, 1.9, 1.9, 2.0, 2.0, 2.1, 2.2

e 1, 3, 3, 3, 5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 10, 16

f 12, 13, 22, 23, 30, 41, 48, 49, 50, 51, 63, 64, 70, 75, 77, 80

g 19, 15, 18, 15, 18, 19, 20, 20, 16, 20, 17, 15, 17, 18, 19, 17, 16, 19, 18, 20

h 102, 97, 95, 99, 96, 102, 102, 100, 97, 102, 99, 99, 95, 97, 102, 100, 95, 99, 101, 100

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2 The stem-and-leaf plot shows the number of words in each of 24 sentences on a page. Calculate

a the mean

b the variance

c the standard deviation

Number of words per sentence

Stem	Leaf
1	0 1 3 3 4
1	5 6 6 6 8 8
2	2 3 3 4 4
2	5 7 7
3	2 3 3
3	7
4	0

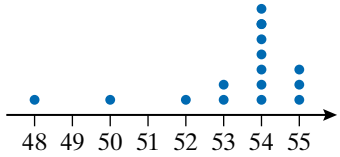
(Key: 1|0 means 10)

3 The dot plot shows the number of matches in each of 15 match boxes. Calculate

a the mean

b the variance

c the standard deviation



Number of matches in a match box

4 The following data show the heights of 25 Year 10 girls and 25 Year 10 boys.

a Calculate the mean height and standard deviation for each group of students.

Year 10 girls' heights

157, 157, 158, 159, 160, 160, 160, 161, 162, 163, 163, 163, 165, 165, 165, 165, 165, 165, 166, 166, 167, 168, 168, 174, 176

Year 10 boys' heights

151, 160, 165, 166, 167, 167, 168, 172, 173, 173, 175, 175, 175, 175, 175, 176, 177, 177, 182, 183, 184, 184, 184, 188, 190

b What conclusion can you draw about the height of Year 10 girls compared with the heights of Year 10 boys?

LINKS TO Example 9

5 In the CensusAtSchool project, students measured their reaction times (in seconds). The data values for a sample of 30 students using their right hands were as follows.

0.38, 0.37, 0.55, 0.37, 0.65, 0.39, 0.3, 0.37, 0.26, 0.66,
0.39, 0.34, 0.37, 0.35, 2.96, 0.4, 0.3, 0.42, 0.25, 0.81,
0.28, 0.29, 0.33, 0.34, 0.33, 0.32, 0.33, 0.36, 0.39, 0.32

a Use a calculator to calculate the mean and standard deviation.

b Which data value appears to be an outlier? How might this outlier have occurred?

c Delete this outlier from the data set and calculate the mean and standard deviation again.

d Using the values for the mean and standard deviation from part **c**, between which values would you expect two thirds of reaction times to lie? Compare this with the actual number who had reaction times between these values.

e Between which values would you expect 95% of reaction times to lie? Compare this with the actual number who had reaction times between these values.

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Height data

6 The following data show the heights of 50 Year 10 students and 50 Year 7 students.

Year 10 heights

151, 157, 157, 158, 159, 159, 160, 160, 160, 160, 160, 161, 162, 163, 163, 163, 165, 165, 165, 165, 165, 165, 165, 166, 166, 166, 167, 167, 167, 168, 168, 168, 170, 170, 172, 173, 173, 174, 175, 175, 175, 176, 176, 177, 177, 183, 184, 184, 188, 190

Year 7 heights

142, 145, 146, 150, 150, 150, 150, 151, 151, 152, 152, 152, 152, 152, 153, 153, 154, 154, 155, 157, 158, 158, 158, 159, 160, 160, 160, 161, 161, 161, 162, 162, 163, 163, 163, 163, 164, 164, 164, 165, 166, 166, 167, 167, 168, 172, 174, 175, 175, 186

a Use a spreadsheet to calculate the mean height and standard deviation for each group of students.

b Compare the percentage of the Year 10 heights that lie within one standard deviation either side of the mean with the expected percentage.

c Compare the percentage of the Year 7 heights that lie within one standard deviation either side of the mean with the expected percentage.

d What conclusion can you draw about the heights of the Year 10 students compared with the heights of the Year 7 students?

LINKS TO Example 10

7 In a study of an experimental herd of cows, the daily milk production of each cow was recorded. The mean was 27.0 kg/day and the standard deviation was 4.6 kg/day.

State the limits between which you would expect the following proportions of the daily milk production to lie

a approximately two-thirds

b approximately 95%

8 For each of the following data sets, state the limits between which you would expect these proportions of the data to lie

i approximately two-thirds of the data

ii approximately 95% of the data

a mean 7.5; standard deviation 1.2

b mean 0.84; standard deviation 0.15

c mean 36.8; standard deviation 2.1

d mean 34.3; standard deviation 0.2

exercise 14.3 challenge

9 A sample of 200 pumpkins had mean mass 3240 g and standard deviation 520 g.

a Between what weights would you expect two-thirds of the weights to lie?

b How many pumpkins would you expect to weight more than 4280 g?

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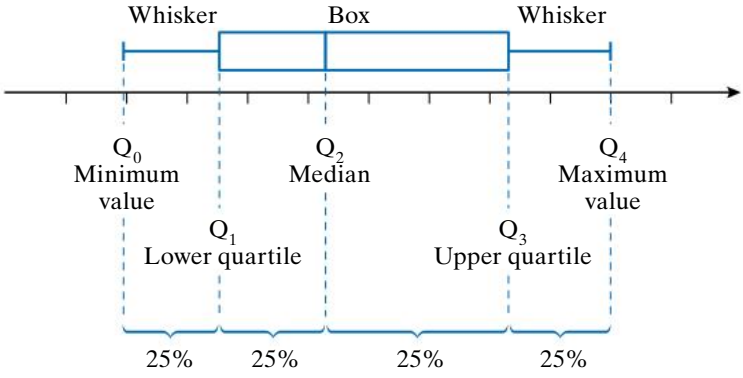
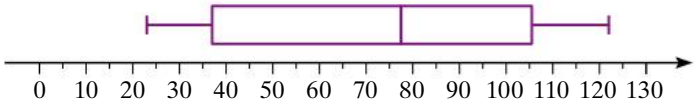
14.4 Box plots

A **box plot** (or **box and whisker plot**) uses the five-number summary to provide a visual summary of both the centre and the spread of a set of data. Each of the four sections of the box plot represents 25% of the data values.



If the five-number summary was 23, 37, 77.5, 105.5, 122, the box plot locates these five numbers along a scale, as shown here.

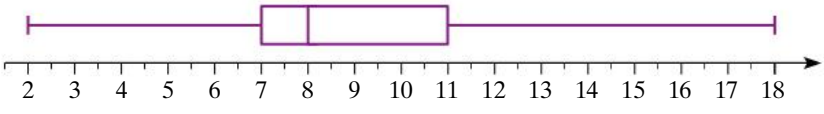


A box plot gives an overall picture of the distribution: where the centre lies, where the bulk of the data lie, and how varied the values are. For example, the box plot above shows that 25% of the data values are below 37 and 50% of the values are between 37 and 105.

Example 11

For the data displayed in this box plot state the

a median **b** range **c** upper quartile
d lower quartile **e** interquartile range
f the values between which the upper 50% of the data lie



Working

a Median = 8

Reasoning

The median is marked by the line in the box.

continued

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Example 11 continued

Working	Reasoning
b Range = 18 – 2 = 16	Range = largest data value – smallest data value.
c Upper quartile = 11	The upper quartile is indicated by the right-hand end of the box.
d Lower quartile = 7	The lower quartile is indicated by the left-hand end of the box.
e Interquartile range = 11 – 7 = 4	IQR = upper quartile – lower quartile

The box is a visual representation of where half the data values lie. The lengths of the whiskers indicate whether the data is symmetrical or skewed.

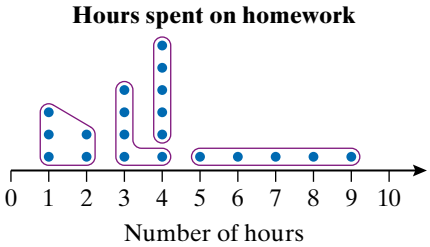
Example 12

The following dot plot shows the number of hours spent on homework by 20 Year 10 students on a particular weekend.

a Consider the four groupings on the dot plot. What do these groups show?

b Calculate the five-number summary for this data set.

c Construct a box plot for the data and compare the two plots.



Working

a Each group contains 5 out of 20 points (25%), so the groupings divide the data into quartiles. The location of the groups also shows that
- there is a clear 'centre' near 4 hours
- 50% of the students spent either 3 or 4 hours on homework
- the upper quartile is spread out further than the other quartiles, so there is a long upper 'tail'

Reasoning

The height and width of the groupings provide a visual picture of each quarter of the data.

Grouping the data into quarters highlights its spread.

continued

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Example 12 continued

Working

b Since $n = 20$, each quartile contains 5 points and the five-number summary values can be calculated from the dot plot.

Q_0 (minimum value) $= 1$

$Q_1 = \dfrac{2 + 3}{2} = 2.5$

Q_2 (median) $= 4$

$Q_3 = \dfrac{4 + 5}{2} = 4.5$

Q_4 (maximum value) $= 9$

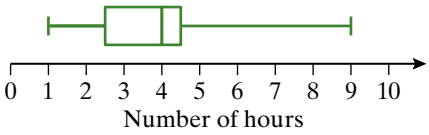
So the five-number summary is 1, 2.5, 4, 4.5, 9.

Reasoning

As $n = 20$, the upper and lower 50% of the data each contain 10 values. So Q_1 is the mean of the 5th and 6th values, and Q_3 is the mean of the 15th and 16th values.

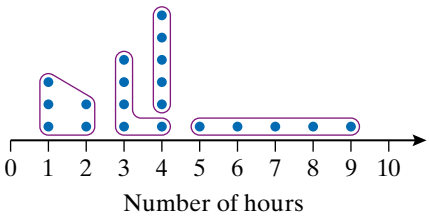
The Q values mark the start and finish of each quarter (25%) of the data.

c



Draw a number line with an appropriate scale, long enough to display the minimum and maximum values. Use the five-number summary to draw the box and whiskers.

Hours spent on homework



The box plot highlights or confirms that the centre of the data is near 4 hours (median = 4). This value is harder to 'read' on the dot plot.

The long upper tail (or whisker) is easily visible on both plots, although the dot plot shows the actual values.

There is a concentration of points between the median and Q_3, represented as a narrow box on the box plot, and as a high frequency for the score 4 on the dot plot.

When a group is 'flat', the data varies a lot in that group. When a group is 'tall', the data doesn't vary much in that group.

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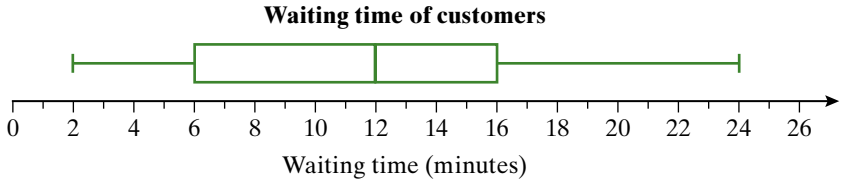
Example 13

The box plot below shows the number of minutes customers at a Chinese takeaway restaurant had to wait before receiving their orders.



a State the median waiting time and explain its significance.

b What is the range of this set of data?

c **i** What is the interquartile range of this set of data?

ii What does the interquartile range tell you about the data?

Working	Reasoning
a The median waiting time is 12 minutes. Half of the customers waited for less than 12 minutes to receive their orders and the other half of the customers waited for longer than 12 minutes.	The median value corresponds to the vertical line within the box. Half of the values in the data set lie below the median and half lie above the median.
b Range = Highest value − Lowest value = 24 − 2 = 22	The highest and lowest values correspond to the ends of the whiskers.
c **i** $Q_1 = 6$ and $Q_3 = 16$ IQR = $Q_3 - Q_1$ = 16 − 6 = 10	The lower quartile (Q_1) corresponds to the lower end of the box. The upper quartile (Q_3) corresponds to the upper end of the box.
ii Fifty per cent of the waiting times lie between 6 and 16 minutes.	The IQR represents the middle half of the values.

Handling outliers

The length of the whiskers on a box plot can sometimes give a misleading impression, because there may be a single extreme value (that is, an **outlier**) that causes a particularly long whisker.

For this reason, some box plots are constructed so that any extreme or outlying values are separated from the whiskers. We have already seen that an outlier can be defined as any value that is below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$

Any such values can be shown as separate dots on the box plot and not included in the whiskers.

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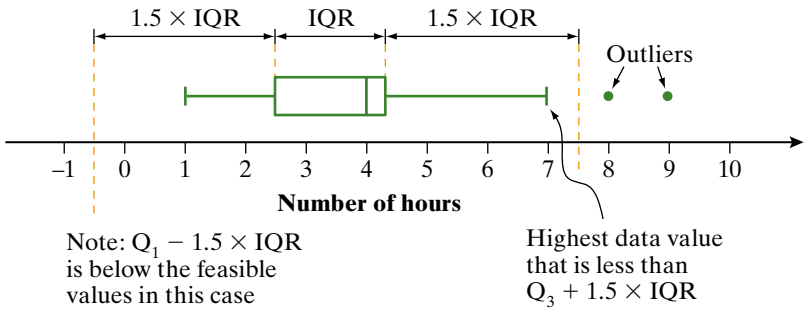
On the left side of the box plot, the whisker then extends only as far as the first data value that is greater than the outlier. On the right side, the right whisker extends only as far as the last data value that is less than the outlier.

Consider the data about the number of hours spent on homework from example **12**. In that example,
$Q_1 = 2.5$ and $Q_3 = 4.5$, so IQR $= 2$.

$Q_1 - 1.5 \times \text{IQR}$
$= 2.5 - 1.5 \times 2$
$= -0.5$

$Q_3 + 1.5 \times \text{IQR}$
$= 4.5 + 1.5 \times 2$
$= 7.5$



Example 14

The following data show, in ascending order, the ages of all the members of Amy's family who attended a birthday party.

a Construct a box plot representing this data without regard to outliers.
1, 5, 7, 15, 22, 28, 29, 32, 35, 35, 38, 40, 41, 42, 42, 43, 55, 65, 73, 88

b Calculate any outliers in the data and redraw the box plot to show the outliers.

Working

a Lowest value = 1
Highest value = 88

Reasoning

The five-number summary values required for a box plot are the lowest value, Q_1, Q_2 and Q_3 and the highest value.

The lowest and highest values of the data can be read directly from the ordered set of data.

Median

1, 5, 7, 15, 22, 28, 29, 32, 35, 35, | 38, 40, 41, 42, 42, 43, 55, 65, 73, 88

$Q_2 = \dfrac{(35 + 38)}{2}$
$= 36.5$

The median is the middle value. There are 20 values, so $n = 20$.

$\dfrac{n+1}{2} = \dfrac{20+1}{2} = 10.5$

The median (Q_2) is halfway between the 10th and 11th data values.

continued

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Example 14 continued

Working

Lower quartile

1, 5, 7, 15, 22 | 28, 29, 32, 35, 35

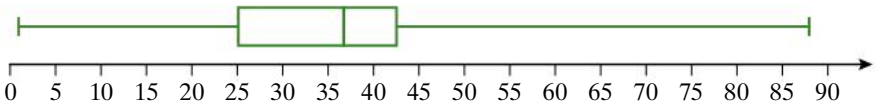
$$Q_1 = \frac{(22 + 28)}{2}$$
$$= 25$$

Upper quartile

38, 40, 41, 42, 42, | 43, 55, 65, 73, 88

$$Q_3 = \frac{(42 + 43)}{2}$$
$$= 42.5$$

The five-number summary is
1, 25, 36.5, 42.5 and 88.



Ages of family members (years)

Reasoning

Q_1 is between the two middle values of the bottom half of the data.

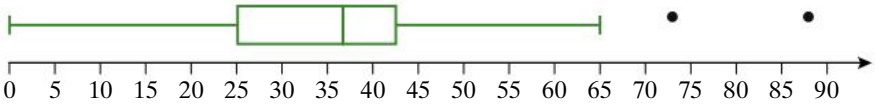
Q_3 is between the two middle values of the upper half of the data.

b $Q_1 = 25$
$Q_3 = 42.5$
$IQR = 42.5 - 25$
$= 17.5$
$1.5 \times IQR$ below $Q_1 = 25 - 1.5 \times 17.5$
$= -1.25$
$1.5 \times IQR$ above $Q_3 = 42.5 + 1.5 \times 17.5$
$= 68.75$

73 and 88 are outliers.

Calculate IQR.
Then find $Q_1 - 1.5 \times IQR$ and $Q_3 + 1.5 \times IQR$.

73 and 88 are greater than $Q_3 + 1.5 \times IQR$.
The right-hand whisker extends to the highest value that is less than or equal to $1.5 \times IQR$, which in this case is 65.



Ages of family members (years)

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Parallel box plots

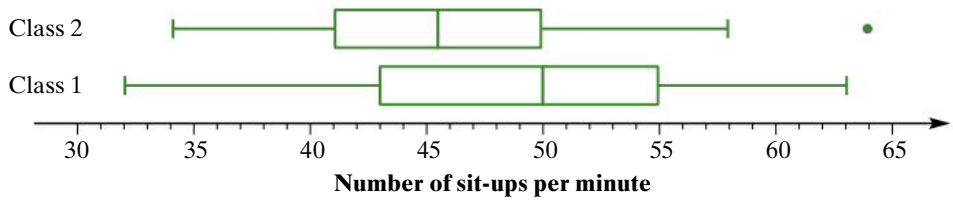
Parallel box plots are similar to back-to-back stem plots in that they display the distribution of a numerical variable across some categorical variable. However, only two categorical variables can be displayed on a back-to-back stem plot. By contrast, parallel box plots can be used to display two or more categorical variables. Unlike back-to-back stem plots, parallel box plots don't need to share a 'stem', so they can be used when the categorical variable takes more than two values.

The back-to-back stem and leaf plot on the right shows the number of sit-ups performed by two classes. The number of sit-ups is the numerical variable and the class is the categorical variable.

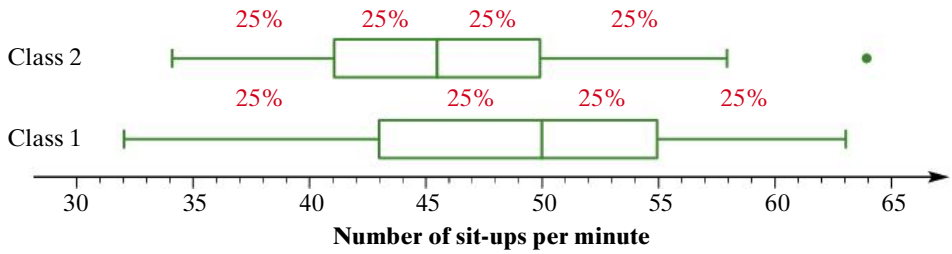
Comparison of sit-ups in 2 minutes

Class 1		Class 2
Leaf	**Stem**	**Leaf**
3 2	3	4
5 5	3	5 6 8
3 2 0	4	0 1 1 2 2 3 3
8 8 7 6 6 5	4	5 5 6 6 7 7 8 9
4 3 3 2 0 0	5	0 0 1
9 7 6 5 5	5	6 7 8
3 1 0	6	4
(LS: 2|3 means 32	Key	RS: 3|4 means 34)

The same data are now presented as parallel box plots. Notice that this display provides a more visual representation of the centre and spread of the performance of each class, although individual performances are no longer shown.



The parallel box plots show clearly that both the range and interquartile range are greater for Class 1, that is, Class 1 scores are more spread out. Class 1 also has a higher median score. In Class 1, 50% of students were able to complete more than 50 sit-ups per minute compared with only 25% in Class 2. The number of sit-ups per minute for the middle 50% of Class 1 students were spread from 43 to 55, whereas for Class 2 the number of sit-ups per minute were more bunched up, ranging from 41 to 50.



Parallel box plots are thus very useful for comparing data.

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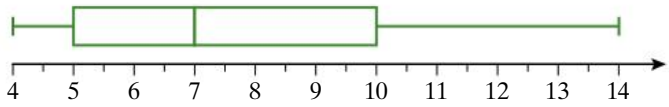
exercise 14.4

LINKS TO Example 11

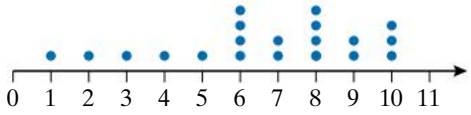
1 For the data displayed in this box plot state the



a median
b range
c interquartile range
d upper quartile
e the values between which the upper 50% of the values lie
f the values between which the middle 50% of the values lie
g the values between which the lower 25% of the values lie.

LINKS TO Example 12

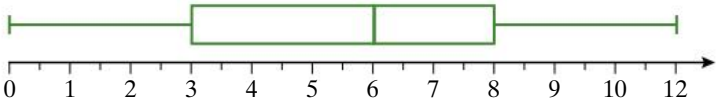
2 Copy this dot plot and group the data into four quartiles. That is, circle each quartile, as shown in example 12.



a Where (approximately) is the centre of the distribution?
b Between which two values do the middle 50% of the values lie?
c Which 25% of the data is the most spread out?
d Construct a box plot to show the data from the dot plot.

LINKS TO Example 13

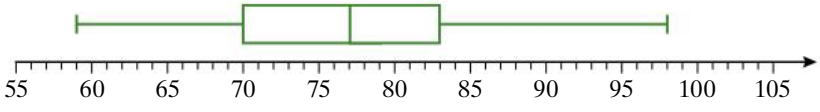
3 Some Year 10 students were asked to record how many hours they spent on homework in a certain week. This box plot shows the data for the group.



Time spent on homework (hours)

a State the median number of hours.
b What was the smallest number of hours a student spent doing homework?
c What was the greatest number of hours a student spent doing homework?
d Find the range of the values.
e State the values of Q_1 and Q_3.
f Find the interquartile range.
g Use your answers in parts **a** to **f** to describe the information presented in this box plot.

4 This box plot shows the data from a weigh-in at a health club.



Mass (kg) of health club members

a What is the median of this data?
b What is the range of this data?
c Between what two values does the middle half of the data lie?
d What percentage of people in the group weigh less than 70 kg?

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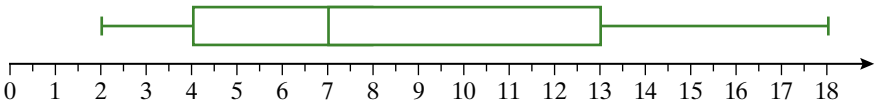
5 The ages of people queuing for a certain ride at a funfair were: 7, 8, 8, 9, 12, 12, 13, 15, 16, 18, 19, 20, 22 and 23 years.

a Find the five-number summary values. **b** Construct a box plot displaying the data.

6 Create a set of 14 pieces of data for which this could be the box plot.



LINKS TO Example 14

7 Thirty students were asked to keep a record of how many movies they watched in a week. Their results are as follows.

3, 5, 5, 7, 2, 5, 6, 4, 3, 3, 0, 1, 3, 5, 10, 0, 3, 7, 1, 4, 3, 4, 4, 1, 2, 5, 8, 3, 9, 2

a Find the five-number summary. **b** What is the range?
c What is the interquartile range? **d** Construct a box plot displaying this data.
e Determine which values are outliers. **f** Redraw the box plot showing the outliers.

8 Ten students were asked to measure their handspan in centimetres. The results were: 15, 14, 17, 19, 14, 14, 13, 16, 21, 16.

a Find the five-number summary. **b** What is the range?
c What is the interquartile range? **d** Construct a box plot.

9 The following data shows the time (in months) that patients in a ward of a public hospital had to wait for elective surgery.

2, 3, 5, 6, 5, 3, 9, 3, 4, 1, 12, 15, 5, 3, 6, 7, 9

a Find the five-number summary. **b** Calculate the interquartile range.
c Construct a box plot.
d Determine whether there are any outliers.

10 Chiara's favourite running event is the 200 metres. Last season, she competed in 20 races. Her times (in seconds) are as follows.

31.6, 28.4, 31.4, 30.2, 31.1, 29.2, 35.1, 33.7, 31.7, 33.1, 32.6, 34.6, 31.9, 39.9, 31.7, 31.6, 29.7, 33.5, 29.4, 30.2

a Find the five-number summary.
b Calculate the range and the interquartile range.
c What does the interquartile range tell you about Chiara's times for running 200 metres?
d Construct a box plot that represents the data.
e Which of Chiara's times can be considered outliers?
f Construct a new box plot showing any outliers.

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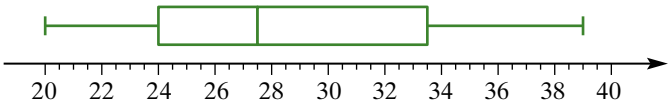
11 A Year 10 Mathematics analysis task was marked out of 40, and the results for a class of 20 students are shown in this box plot.



Analysis task score (out of 40)

a List the five-number summary on which the box plot is based.

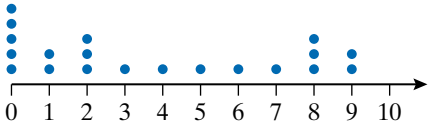
b How many students scored

i less than 24? **ii** either less than Q_1 or more than Q_3?

c What score was exceeded by the top 50% of the students?

d Draw one of the possible dot plots that could match the data in the box plot.

12 A teacher was analysing his students' performance in a technology-free Mathematics test. He counted the number of marks lost by each student due to arithmetic errors.



Number of arithmetic errors

a Find the median number of errors and the interquartile range.

b Generate a box plot from this information.

c Comment on the distribution of arithmetic errors.

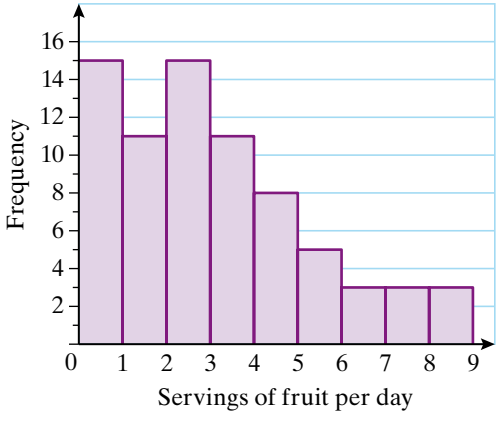
13 This histogram displays the responses of 74 teenagers who were asked to estimate how many pieces of fruit they eat on an average day.

a Convert this information into

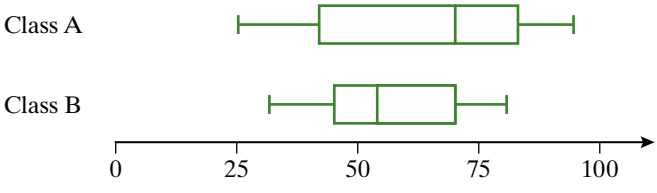
i a five-number summary

ii a box plot

b What information is more easily seen from the box plot than the bar chart?



14 The box plots below show the results of two classes on the same test. Which class performed better? Why?



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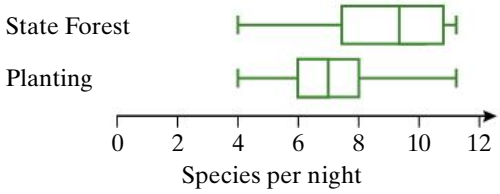
15 Scientists studying bat populations were interested to know if tree-planting in heavily cleared areas was effective for restoring habitats for bats. In a study of bats in an area of southern New South Wales, the scientists used electronic equipment to record and identify the calls of different species of bats. The numbers of bat species present in a state forest and in areas that had been re-planted were recorded each night over a season. The parallel box plots show the numbers of species recorded per night.



- **a** Compare the median number of bat species per night in the two locations.
- **b** Compare the interquartile range in the two locations.
- **c** Compare the range in the two locations.
- **d** Compare the 25% of nights when the greatest numbers of species were recorded in the two locations.
- **e** What conclusion can you draw from these data about whether tree plantings are effective in restoring bat habitats?

16 A certain airport has 25 flight departures per day. The data shows the number of minutes that the flights were delayed for two consecutive days.

Day 1: 11, 4, 2, 2, 15, 8, 0, 11, 0, 0, 0, 12, 3, 5, 2, 3, 5, 0, 6, 25, 7, 4, 9, 5, 10

Day 2: 8, 6, 5, 2, 9, 7, 4, 2, 5, 1, 2, 2, 4, 2, 5, 1, 3, 9, 7, 8, 1, 0, 4, 26, 29

- **a** Calculate the five-number summary for each day.
- **b** Determine if there are any outliers.
- **c** Draw parallel box plots (showing any outliers).
- **d** Comment on any differences between the delays for the two days.

17 The following data show the heights of 25 Year 10 girls and 25 Year 10 boys in a random sample from the 2011 CensusAtSchool.

Year 10 girls' heights

157, 157, 158, 159, 160, 160, 160, 161, 162, 163, 163, 163, 165, 165, 165, 165, 165, 165, 166, 166, 167, 168, 168, 174, 176

Year 10 boys' heights

151, 160, 165, 166, 167, 167, 168, 172, 173, 173, 175, 175, 175, 175, 175, 176, 177, 177, 182, 183, 184, 184, 184, 188, 190

- **a** For each data set
  - **i** find the five-number summary
  - **ii** calculate if there are any outliers
- **b** Draw parallel box plots for the data.
- **c** Comment on the similarities and differences between the two height distributions.

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18 The following data from the 2011 CensusAtSchool show the responses of a random sample of 50 Year 10 students to the question 'How many hours of sleep do you usually get on a school night?'

2, 3, 5, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8,
8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 10, 10, 11, 11, 16

a Find the five-number summary.
b Calculate any outliers.

The following data shows the responses of a random sample of 50 Year 7 students to the same question.

6, 6, 6, 6, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 10, 10, 10, 10, 10, 10,
10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 11, 12, 12, 12, 13, 16

c Repeat parts **a** and **b** for this data.
d Draw parallel box plots for the two sets of data.
e Comment on any differences between the two sets of data.

19 The following data from the 2011 CensusAtSchool show the responses of a random samples of 25 Year 10 girls and 25 Year 10 boys to the question 'How many hours of sleep do you usually get on a school night?'

Year 10 girls
7, 7, 7, 6, 7, 10, 9, 8, 8, 9, 8, 8, 8, 5, 8, 9, 9, 9, 10, 9, 8, 7, 9, 8, 8
Year 10 boys
7, 5, 8, 10, 7, 9, 8, 8, 16, 4, 8, 9, 8, 10, 8, 10, 6, 10, 9, 8, 8, 9, 8, 11, 8

a For each set of data
  i find the five-number summary **ii** calculate if there are any outliers
b Draw parallel box plots for the two sets of data.
c Comment on any differences between the two sets of data.

Height data

20 The following data from the 2011 CensusAtSchool show the heights of random samples of 50 Year 10 students and 50 Year 7 students.

Year 10 heights
151, 157, 157, 158, 159, 159, 160, 160, 160, 160, 160, 161, 162, 163, 163, 163, 165, 165, 165, 165, 165, 165, 165, 166, 166, 166, 167, 167, 167, 168, 168, 168, 170, 170, 172, 173, 173, 174, 175, 175, 175, 176, 176, 177, 177, 183, 184, 184, 188, 190

Year 7 heights
142, 145, 146, 150, 150, 150, 150, 151, 151, 152, 152, 152, 152, 152, 153, 153, 154, 154, 155, 157, 158, 158, 158, 159, 160, 160, 160, 161, 161, 161, 162, 162, 163, 163, 163, 163, 164, 164, 164, 165, 166, 166, 167, 167, 168, 172, 174, 175, 175, 186

a For each data set
  i find the five-number summary **ii** calculate if there are any outliers
b Draw parallel box plots for the data.
c Comment on the similarities and differences between the two height distributions.

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21 The following table shows the five-number summaries for the ages of all Australians and for Aboriginal and Torres Strait Island Australians in 2006. The quartiles were estimated from the grouped age data reported by the Australian Bureau of Statistics based on the 2006 census. Australians over the age of 85 were grouped in the same category so it was not possible to obtain the maximum age from this data. However, we know from other sources that the oldest Australian was over 110, and we know from Australian Bureau of Statistics data that there was at least one death at the age of about 100 of an indigenous person between 2007 and 2009.

	Aboriginal and Torres Strait Islander Australians	All Australians (2006)
Lowest age	0	0
Lower quartile	9	19
Median age	21	37
Upper quartile	39	54
Maximum age	~ 100	~ 110

Based on Australian Bureau of Statistics data

a Construct parallel box plots to show the age distributions for all Australians and for Aboriginal and Torres Strait Islander Australians.

b Compare the median ages for all Australians and for Aboriginal and Torres Strait Islanders.

c Compare the interquartile range.

d Complete the following sentences:

Approximately ___% of all Australians are under the age of 20.

Approximately ___% of Aboriginals and Torres Strait Islanders are under the age of 20.

Approximately ___% of all Australians are over the age of 40.

Approximately ___% of Aboriginals and Torres Strait Islanders are over the age of 40.

e What conclusion can you draw from these data?

f The Australian Human Rights Commission Social Justice Report of 2008 contains an appendix with a statistical overview of Aboriginal and Torres Strait Islander peoples in Australia. The appendix can be downloaded from the website www.hreoc.gov.au/social_justice/sj_report/sjreport08/app2.html. Read the sections on Growth of the Indigenous population and Health, and list some of the reasons for the difference in life expectancy of all Australians and Aboriginal and Torres Strait Islander people.

exercise 14.4 challenge

22 Construct a set of 20 data values for which there will be no right-hand whisker.

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14.5 Bivariate numerical data

So far in this chapter we have focused on **univariate numerical data**, that is, a single numerical variable was analysed. In some of those situations we have considered the combination of a numerical variable and a categorical variable. For example, we compared the number of sit-ups for students in two different classes. Here the numerical variable was the number of sit-ups and the categorical variable was the class that the students were in. Parallel box plots and back-to-back stem and leaf plots are suitable for displaying the combination of a numerical variable and a categorical variable.

There are many situations, however, where we are interested in comparing two numerical variables, that is, **bivariate numerical data**. We may wish, for example, to investigate the heights and weights of members of a football team.

Scatter plots

An important tool for exploring bivariate data is a scatter plot. This gives a visual display of the relationship between two numerical variables. To construct a scatter plot, values of one variable are shown on the horizontal axis and values of the other variable are shown on the vertical axis. Each observation is represented by a point on the scatter plot.

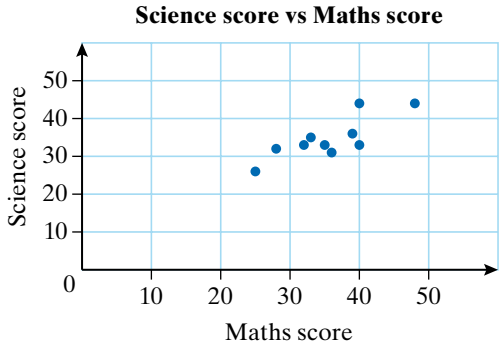
Example 15

The results of a Maths test and a Science test for a group of Year 10 students are shown in the table below.

Maths	33	40	39	40	25	28	35	48	36	32
Science	35	33	36	44	26	32	33	44	31	33

Draw a scatter plot for this set of data, with Maths on the horizontal axis and Science on the vertical axis.

Working



Reasoning

Each pair of test scores is represented by a point on the scatter plot.

In this scatter plot, each of the axes begins from 0. This means that there is a lot of 'wasted' space.

continued

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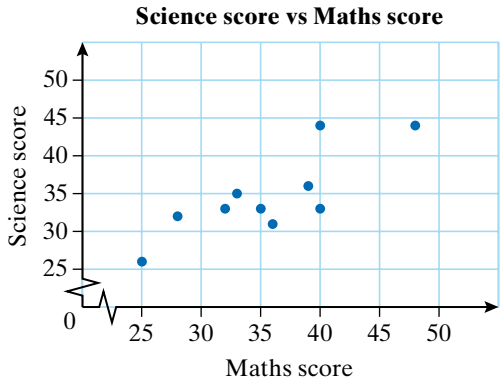
Example 15 continued

Working



Reasoning

The axes need not start at 0. Using 'broken' axes allows us to 'zoom in' and see the association between the two variables more clearly.

'Broken axes' allow a better view of this data.

Dependent and independent variables

When analysing bivariate data, the key question is whether the two variables tend to vary together in some way or whether they behave independently of each other. In some sets of bivariate data, one variable clearly depends—or is likely to depend—on the other. This variable is referred to as the **dependent variable**. The other variable, on which the dependent variable seems to depend, is called the **independent variable**.

For example, suppose that we are analysing experimental data that shows how high a ball bounces when it is dropped from different heights. The height the ball bounces obviously depends on the height from which it is dropped. The height from which the ball is dropped is clearly the independent variable, and this goes on the x-axis.

In other cases, we may be justified in assuming that one variable depends on the other, even though we may not be sure. Sometimes, though, there is no obvious dependency of one variable on the other, and in fact both variables may depend on some other variable. In this case it does not matter which variable we put on the x-axis.

When constructing a scatter plot, the independent variable is assigned to the horizontal or x-axis, and the dependent variable is assigned to the vertical or y-axis.

Association

Association is a word used to describe the nature of the relationship between two sets of data. The pattern of points in a scatter plot indicates the strength of the association. The following scatter plots show how we describe the association between two variables in terms of a linear relationship.

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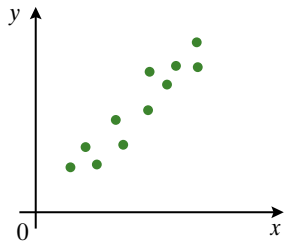
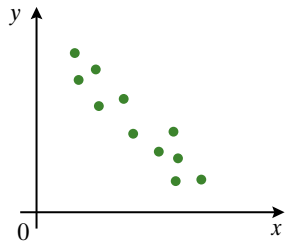
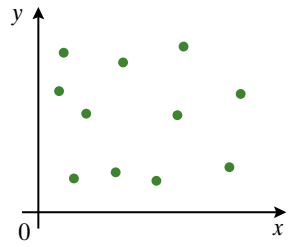
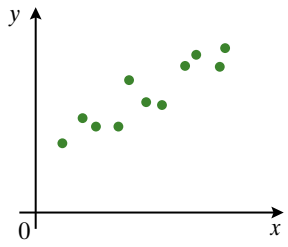
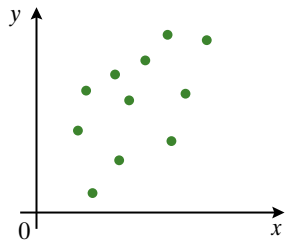
		
Positive association As x increases, so does y.	**Negative association** As x increases, y decreases.	**No association** There is no relationship between the two variables.

The strength of the association between the variables can be described as strong or weak.

	
Strong positive association	**Weak positive association**

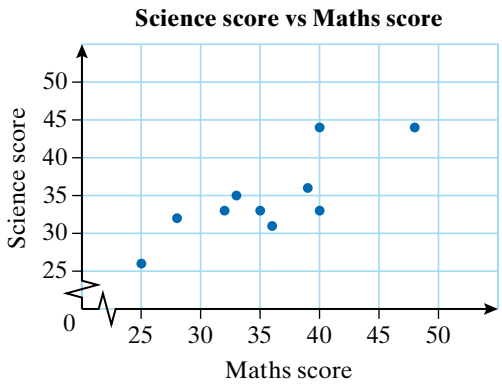
Example 16

Describe the association between the Science scores and the Maths scores in example **15**.

Working

There is a fairly strong positive relationship between the Science scores and the Maths scores.

Reasoning



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Causal effect

If one variable is likely to directly influence the other variable, we say that there is a **causal effect**, that is, one variable is caused by the other. In many scatter plots, though, we need to be very cautious in assuming that there is a causal effect. More likely, both variables are related to a third variable, or there may be several interacting variables. The apparent relationship may also be due to chance.

The following table shows data for 16 countries for fertility rate (the mean number of children per woman), and life expectancy at birth (the age in years) to which a newborn baby is likely to live.

When a cause-and-effect relationship is not clear, it does not matter which variable we put on the horizontal axis. In the scatter plot below, the mean number of children has been put on the horizontal axis and life expectancy on the vertical axis. The scatter plot shows that there is a moderately strong negative association between the two variables.

	Fertility rate (Mean number of children per woman)	Life expectancy at birth (years)
Afghanistan	5.6	50
Australia	1.8	82
Bangladesh	2.6	70
Chad	4.9	49
China	1.6	75
Ethiopia	6.0	57
Guatemala	3.2	71
India	2.8	67
Indonesia	2.2	72
Malawi	5.4	52
Mozambique	5.4	52
Nigeria	5.4	52
Somalia	6.3	51
Sudan	4.2	63
United States	2.1	79
Vietnam	1.9	72

https://www.cia.gov/library/publications/the-world-factbook/index.html

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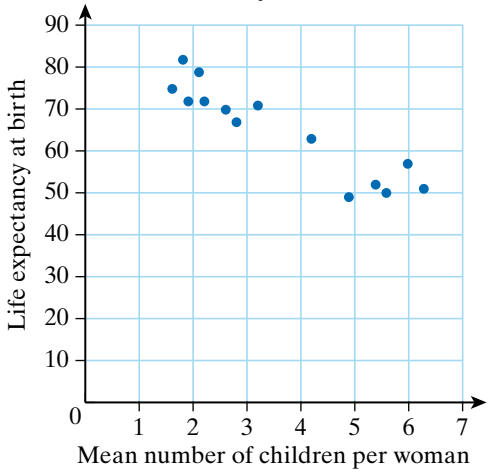
There is tendency for those countries where the fertility rate is lowest, such as Australia and China, to have higher life expectancies than countries where the fertility rate is higher, for example, Ethiopia and Somalia. However, the scatter plot does not show that the fertility rate is directly responsible for the life expectancy.

Further studies in each country would be needed to investigate the reason for the relationship. For example, women may choose to have more children if the life expectancy is lower, but both variables may be due to some third variable such levels of poverty or wealth. This would directly affect life expectancy, but would also affect fertility rate as families in poor countries rely on children to work and support the family. Religious beliefs and availability of contraceptives may also be factors contributing to higher fertility in poorer countries.

Relationship between life expectancy and female fertility rate for 16 countries



Example 17

This data for 12 countries shows the life expectancy (in years, as calculated at birth) and the pupil-to-teacher ratio in primary school.

a Construct a scatter plot of the data with pupil-to-teacher ratio in primary school on the horizontal axis.

b Describe the association between the two variables.

c Discuss whether you think there is a causal relationship between the variables, that is, whether one variable depends on the other.

Country	Primary pupil-teacher ratio	Life expectancy at birth
Burundi	51	59
Cambodia	50	63
China	17	75
Eritrea	38	63
France	19	81
Guatemala	28	71
Indonesia	16	71
Madagascar	40	64
Mozambique	58	52
Sudan	38	63
United States	14	79
Vietnam	20	72

http://stats.uis.unesco.org/unesco/TableViewer/tableView.aspx?ReportID=165
http://data.worldbank.org/indicator/SE.PRM.ENRL.TC.ZS

continued

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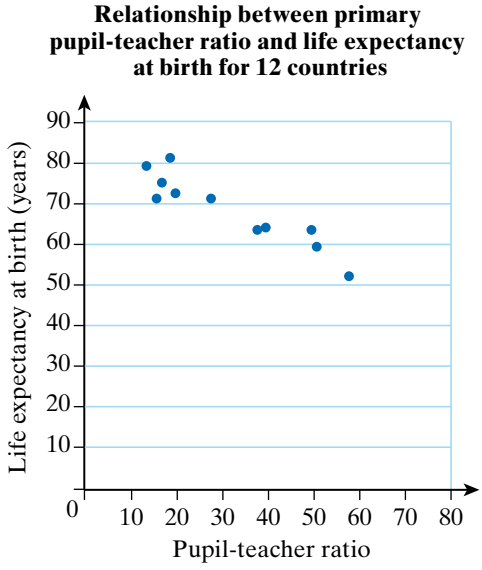
Example 17 continued

Working

a



Reasoning

The data can be plotted on graph paper or a spreadsheet can be used. The axes do not need to start at zero. Changing the scale on the axes may show the association more clearly.
(Note that two of the 12 countries had a pupil–teacher ratio of 38 and a life expectancy of 63.)

b There is moderately strong negative association between the two variables. As the pupil-teacher ratio increases, there is a tendency for life expectancy at birth to decrease.

The relationship is negative, because the pupil-to-teacher ratio decreases as life expectancy increases.

c Both variables are likely to depend on the economic wealth of the country. Poor countries are likely to have poor health services so the life expectancy is low. These countries are also unlikely to be able to afford to train or employ adequate numbers of teachers, so there will be a higher pupil-to-teacher ratio. If classes are large, many children may not learn to read and write. As adults they may not have the knowledge to care for their health or the health of their children.

Social issues may involve many different variables.

Line of good fit

Not all relationships are linear. If there does appear to be a linear relationship between two variables we can draw a line of good fit through the points. This line is sometimes called a **trend line**, as it shows the general trend of the path along which the points lie. It is likely that very few—or even none—of the points actually lie on the line, but they should be scattered fairly evenly on each side of the line.

10 A
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We can find the rule for the line of good fit by finding

- the gradient and y-intercept
- two points on the line.

Example 18

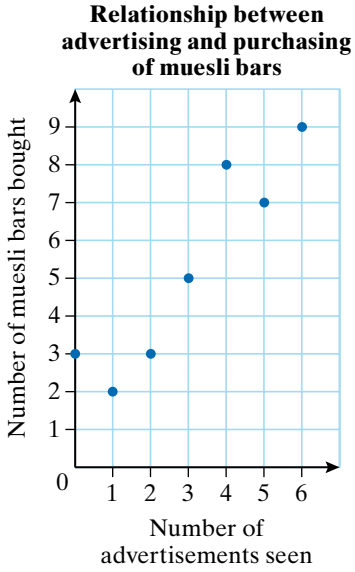
A food manufacturer studied the effect of an advertising campaign for its new muesli bars. People were selected at random and contacted by phone. They were asked how many packets of muesli bars they had bought in the last week and how many advertisements for the muesli bars they had seen in the last week. The survey results were as follows.

Number of advertisements	0	1	2	3	4	5	6
Number of packets of muesli bars bought	3	2	3	5	8	7	9

a Construct a scatter plot of the data with the number of advertisements seen on the horizontal axis.
b Describe the association.
c Draw a line of good fit and find its equation.
d Interpret the equation in terms of the context.

Working

a



Reasoning

Use the number of advertisements as the independent variable.

b There is strong positive association between the variables.

As the number of advertisements seen increases, the number of muesli bars bought increases.

continued

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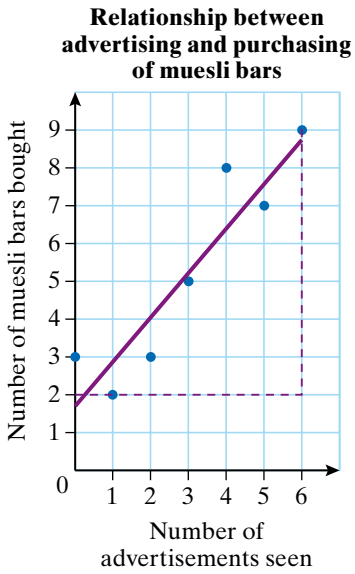
Example 18 continued

Working

c



The rule for a straight line is $y = mx + b$, where m is the gradient and b is the y-intercept.

The y-intercept is approximately 2, so $b = 2$.

Gradient $= \frac{7}{6} \approx 1.2$

So $m = 1.2$

The rule for the line of good fit is $y = 1.2x + 2$.

d People were likely to have bought two packets of muesli bars without having seen any advertisements. For every advertisement that people saw, they bought on average an extra 1.2 packets.

Reasoning

Draw a line of good fit so there is a balance of points on either side of the line.

Since we are drawing the line of good fit 'by eye', different people may draw the line in a slightly different position. The y-intercept and gradient, and hence the equation, will vary.

Let x be the number of advertisements seen and y be the number of packets of muesli bars bought.

The line of best fit meets the y-axis at approximately 2.

Gradient $= \frac{\text{rise}}{\text{run}}$

The y-intercept is 2 and the gradient is 1.2.

Interpolation and extrapolation

The equation of a line of good fit can be used to make predictions. We can look at a particular value of one variable on the line of good fit and read a value for the other variable. If we know the equation for the line of good fit, we can substitute a value for one variable and predict the value of the other variable. Predicting from a scatter plot in this way is called

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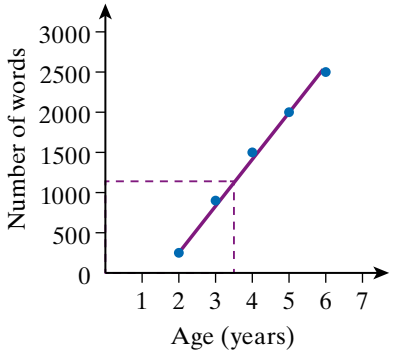
interpolation and **extrapolation**. Interpolation means predicting from the line of good fit within the range of the given data. Extrapolation means predicting beyond the data values. In general, we can be more confident of predictions when interpolating than when extrapolating. This is because we cannot be sure that the data will follow a similar pattern beyond the range of the data we have.

Children's vocabulary increases as they increase in age. The following graph shows the average number of words in the vocabulary of a sample of children from age 2 to 6 years. We could use this graph to predict that a child aged $3\frac{1}{2}$ might have a vocabulary of about 1200 words. This is an example of interpolation. However, although there seems to be linear relationship between the ages of 2 and 6, we cannot be sure that this linear pattern will continue beyond 6 years. There may, for example, be a rapid increase in vocabulary beyond 6 years, then a slowing down as children reach teenage and adult years. So we cannot be confident of our predictions if we extrapolate beyond the given data.

Average number of words at different ages



Example 19

When anthropologists analyse human skeletal remains, one of the pieces of information they try to work out is the height of the person. The lengths of small bones such as the metacarpals (finger bones linking the knuckles with the hand) can be used as predictors of height. To test the strength of this association, data was collected from a sample of ten adults.

Metacarpal length (mm)	52	45	51	39	41	48	49	45	43	47
Height (cm)	180	171	178	157	163	172	183	174	168	174

a Construct a scatter plot of the data with length of metacarpal bone on the horizontal axis.

b Describe the association.

c Draw a line of good fit and find its rule.

d Use the line of good fit to predict the height of a person if the length of their metacarpal bone is

i 40 mm **ii** 59 mm

e How confident are you of each of the predictions in part **d**?

continued

10 A

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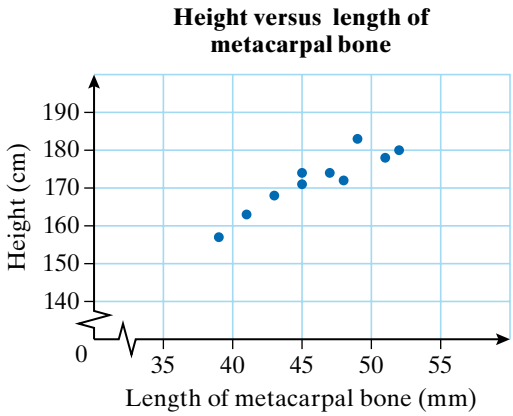
Example 19 continued

Working

a



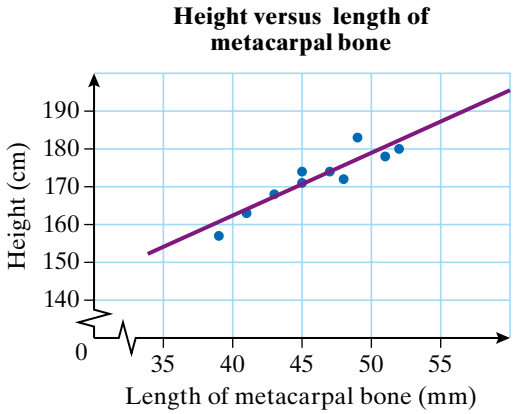
Reasoning

It is appropriate to use the horizontal axis to represent the length of the metacarpal bone because the intention is to predict height from metacarpal bone length.

b There is a strong positive association between the two variables.

As the length of the metacarpal bone increases, the height increases, so the association is positive. The points lie reasonably close to a straight line.

c



Draw a line of good fit so that there is a good balance of points on either side of it.

Two points on the line of good fit shown above are (35, 154) and (45, 170). Therefore

Choose any two convenient points on or close to the line of good fit.

$154 = 35m + b$ [1]
$170 = 45m + b$ [2]

Substitute the coordinates of the points into the general rule for a straight line, $y = mx + b$.

Subtract equation [1] from [2].
$16 = 10m$
$m = 1.6$

Solve the pair of simultaneous equations for m by subtracting to eliminate b.

continued

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Example 19 continued

Working	Reasoning
Substitute $m = 1.6$ in equation [1]. $154 = 35 \times 1.6 + b$ $154 = 56 + b$ $b = 98$	Substitute the value for m into either one of the equations to find the value of b. Note that the broken axes mean that we cannot find the y-intercept by extending the graph to meet the y-axis.
The rule for line of good fit is $y = 1.6x + 98$.	Substitute the values found for m and b in the general equation of a line.
d i When $x = 40$, $y = 1.6 \times 40 + 98 = 162$ The predicted height is 162 cm. **ii** When $x = 59$, $y = 1.6 \times 59 + 98 = 192.4$ The predicted height is 192 cm.	Substitute each of the values for x into the rule for the line of good fit, $y = 1.6x + 98$.
e A metacarpal bone of length 40 mm is within the range of the data, so the prediction is likely to be reasonably accurate. However, 59 mm is beyond the range of the data, so we cannot be sure that the relationship between the two variables continues in the same approximately linear way.	**Interpolating** (predicting within the range of the data) is likely to be more accurate than **extrapolating** (predicting beyond the range of the data).

Bivariate data where the independent variable is time

There are many situations where we wish to observe changes over time in a particular numerical variable, such as changes in retail prices, populations, or number of species of plants. We can construct a scatter plot of the data, with time in years as the independent variable. We refer to this type of scatter plot as a time series and we normally join the points. The time series will indicate if there are any long-term changes, for example, upward or downward trends.

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Example 20

The following data show the number of international passengers carried by an airline each January from 2000 to 2012.

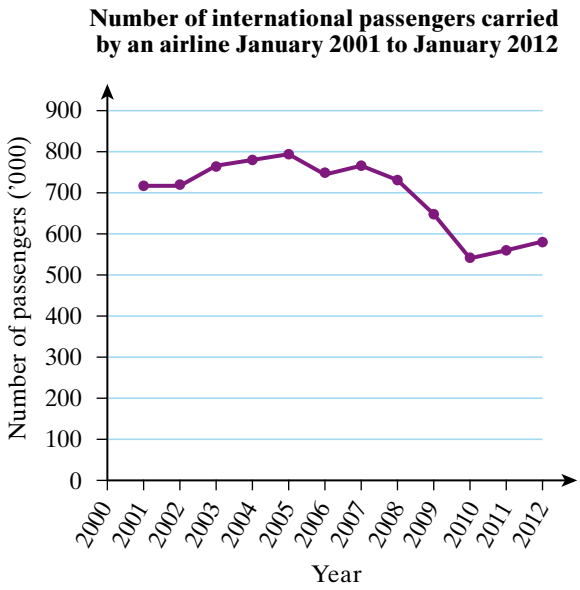
a Construct a time series graph

b Comment on any trend in the number of passengers.

Year	Number of international passengers in January ('000)
2001	717
2002	720
2003	764
2004	780
2005	794
2006	749
2007	766
2008	731
2009	648
2010	542
2011	560
2012	580

Working

a



Reasoning

Plot the points, with time on the horizontal axis and the number of passengers on the vertical axis.

continued

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Example 20 continued

Working | **Reasoning**

b Overall, there has been a general downward trend in the number of passengers. From 2002 until 2005, the number of passengers increased each year, but then declined until 2010, when the numbers have increased slightly. Data for the next few years would show whether this upward trend will continue.

exercise 14.5

LINKS TO Example 15

1 During summer, the manager of a milk bar noticed a relationship between the maximum daily temperature and the number of drink cans sold. He recorded this information on seven randomly selected days.

a Which variable should go on the horizontal axis? Why?

b Construct a scatter graph of the data on graph paper, using broken axes.

c Describe what the graph is showing.

Temperature (°C)	Number of cans sold
25	259
28	314
29	325
31	403
34	480
36	498
37	526

LINKS TO Example 16

2 This table shows the number of hours of sunshine and the amount of solar energy (in megajoules per square metre) for one week in Melbourne during June.

a Construct a scatter plot of the data with hours of sunshine on the horizontal axis..

b Describe the association.

Hours of sunshine	Solar energy (MJ/m^2)
7.3	8.3
0.2	3.8
4.8	6.9
9.0	9.7
4.2	7.9
6.6	7.6
0.1	4.0

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Student data

3 The file Student data in the *OneStopDigital* ebook contains data from the 2011 CensusAtSchool for a random sample of 25 Year 10 students for the following four questions.

Question 9	How tall are you without your shoes on? (nearest cm)
Question 12	What is the length of your right foot without a shoe? (nearest cm)
Question 13	Use your dominant hand to test your reaction time (seconds)
Question 14	Use your non-dominant hand to test your reaction time (seconds)

a Use a spreadsheet to construct a scatter plot for the data from questions 9 and 12. Describe the association between height and length of right foot.

b Use a spreadsheet to construct a scatter plot for the data from questions 13 and 14. Describe the association between reaction time for dominant and non-dominant hands.

c Use a spreadsheet to construct a scatter plot for the data from questions 9 and 13. Describe the association between height and reaction time with the dominant hand.

LINKS TO Example 17

4 A mathematics test was given to 12 students. The foot length of each student was also measured. It was suggested that if a strong association between the foot lengths and test results was found, then measuring foot length would be an easier way of assessing mathematics students than giving tests. This table shows the resulting data.

Foot length (x cm)	21	25	21	24	16	16	17	18	20	17	21	21
Maths score (y%)	65	74	69	77	46	52	53	48	65	53	67	55

a On graph paper, make a scatter plot of the data, starting each axis at zero with foot length on the horizontal axis.

b Describe the association between foot length and Maths score.

c Do you think there is a causal relationship between the two variables?

d Suggest a possible reason for the association in the collected data. What does the data *not* tell us?

Female literacy

5 The female literacy rate is the percentage of women who can read and write. The infant death rate is the number of children out of every 1000 live births who die before the age of one year. The table shows these rates for 15 countries.

a Use a spreadsheet to produce a scatter plot of the data with female literacy rate on the horizontal axis.

b Describe the association between the variables.

c Do you think there could be a causal relationship between the two variables? Explain.

d Suggest other likely reasons for the association.

e Mozambique has a female literacy rate of 33%. Estimate the infant death rate in Mozambique.
How confident are you of your estimate?

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	Female literacy rate (%)	Infant death rate (per 1000 live births)
Afghanistan	13	149
Australia	99	5
Bangladesh	41	51
Chad	13	95
China	89	16
Ethiopia	35	77
Guatemala	63	26
India	48	48
Indonesia	87	28
Malawi	50	81
Nigeria	61	92
Somalia	26	106
Sudan	51	68
United States	99	6
Vietnam	92	21

6 The following table shows the female literacy rate (that is, the percentage of women who can read and write) and the pupil–teacher ratio in primary schools.

a Use a spreadsheet to produce a scatter plot of the data with female literacy rate on the horizontal axis..

b Describe the association between the variables.

c Do you think there could be a causal relationship between the two variables? Explain

d Suggest other likely reasons for the association.

	Female literacy rate (% who can read and write)	Primary pupil–teacher ratio
Afghanistan	13	64
Bangladesh	41	57
Chad	13	69
China	89	18
Guatemala	63	31
India	48	40
Indonesia	87	20
Mozambique	33	67
Nigeria	61	43
Sudan	51	34
United States	99	14
Vietnam	92	21

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LINKS TO Example 18

7 Look back at your scatter plot for question **1**. Using a ruler and pencil, draw a line of good fit through the points.

a Use the line of good fit to predict the sales on days when the maximum temperature is

i 20 °C **ii** 40 °C

b How confident are you of these predictions?

LINKS TO Example 19

Hand span data

8 The following table shows the hand span and height for 27 Year 8 students.

Hand span (cm)	Height (cm)	Hand span (cm)	Height (cm)	Hand span (cm)	Height (cm)
19.5	160	18.9	162	19.0	154
18.5	162	20.1	165	20.6	168
20.1	163	16.5	157	22.5	166
21.0	170	20.5	156	15.5	152
19.0	160	20.5	170	19.8	163
20.0	150	18.5	160	18.0	165
20.1	164	19.2	149	18.5	162
19.5	162	19.5	161	16.7	158
18.5	156	22.0	155	16.0	156

a Use a spreadsheet to construct a scatter plot of the data, starting each axis at an appropriate value with hand span on the horizontal axis.

b Describe the association.

c Print your scatter plot and use a ruler to draw a line of good fit.

d Use the line of good fit to predict the height of a student with a hand span of 21.5 cm. How confident would you be of your prediction?

e Do you think hand span would be a good predictor of height?

Salination data

9 Accumulation of salt in the soil (salination) is a serious problem for farmers. The following table shows the salt concentration in the soil (as a percentage) and the yield of maize in tonnes per hectare.

a Use a spreadsheet to construct a scatter plot of the data with concentration of salt in soil on the horizontal axis.

b Describe the association.

c Print your scatter plot and use a ruler to draw a line of good fit.

d Find the rule for your line of good fit.

e What does the y-intercept represent?

f What does the gradient of the line of good fit represent?

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Concentration of salt in soil (%)	Yield of maize (tonnes per hectare)
0.76	1.316
1.15	0.570
0.62	1.930
0.83	1.205
0.34	2.941
0.81	0.970
0.54	1.852
0.35	2.150
0.51	1.961
0.72	2.389
0.02	4.170
0.22	3.945

Student data

10 The following CensusAtSchool data shows the heights and arm spans for a random sample of 40 Year 10 students.

a Use the file in the *OneStopDigital* ebook to construct a scatter plot of the data with height on the horizontal axis.

b Print your graph and draw a line of good fit.

c Use the line of good fit to predict the arm span of a student with height

i 188 cm **ii** 195 cm

iii 140 cm

d How confident are you of your predictions?

e Using your Excel graph insert a linear trend line. The minimum height in the data set is 151 cm, so extend the trend line back 151 units and select the option to display the equation on the graph. What is the equation of the linear trend line?

f Use the equation to predict the arm span of a student with height

i 188 cm **ii** 195 cm

ii 140 cm

g Compare the predictions with those you made from your hand-drawn line of good fit in part **c**.

Height (cm)	Arm span (cm)	Height (cm)	Arm span (cm)
175	175	176	176
157	157	157	161
163	155	177	180
183	183	167	163
175	160	167	171
151	157	172	167
160	164	159	164
160	159	168	167
163	163	166	164
165	162	176	183
173	173	190	175
165	170	188	188
158	158	165	167
161	164	165	162
168	155	166	152
165	160	160	158
174	171	184	177
167	163	168	175
166	166	160	160
176	176	170	175

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LINKS TO Example 20

Elephant seals

11 The following table shows the estimated population of southern elephant seals at a Southern Ocean island between 1988 and 2002.

Year	Population
1988	23 100
1989	23 500
1990	22 300
1991	21 200
1992	21 100
1993	20 600
1994	19 900
1995	19 600
1996	19 800
1997	19 800
1998	18 900
1999	18 700
2000	19 200
2001	19 300
2002	20 300

a Use a spreadsheet to construct a joined scatter plot of the data.

b Describe what the graph shows.

12 The following scatter plot shows the fastest 100 m Olympic sprint times for men and for women from 1928 until 2008. Linear trend lines have been added to both sets of data.

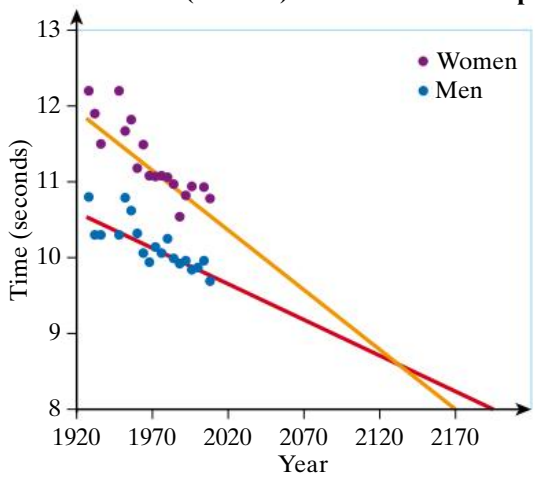
a What is the general trend in times for the 100 m sprint times?

b What do you think the graph is trying to show?

c Comment on the appropriateness of linear trend lines.

d Sketch graphs to show a more realistic representation of what is likely to happen over the next 100 years.

Fastest Olympic times for men and women for 100 metres (seconds). Will women catch-up?



Men's data: http://trackandfield.about.com/od/sprintsandrelays/qt/olym100medals.htm
Women's data: http://trackandfield.about.com/od/sprintsandrelays/qt/olym100women.htm

10 A
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exercise 14.5 challenge

Temperature and altitude data

13 The following table shows the external temperature in degrees Celsius at different altitudes after a plane took off from Singapore in July.

a Use a spreadsheet to graph the data as an XY scatter plot (with separate points). Use the Chart menu to add a linear trendline. Choose the options **Display equation on chart** and **Forecast: Backwards** for 1547 units (to extend the trendline back to zero altitude).

b Rewrite the equation in terms of A and T, where A metres is the altitude (instead of x) and T degrees Celsius is the temperature (instead of y). Round the values of m and b to a sensible number of decimal places.

c Use the equation to predict the temperature at ground level.

Altitude (m)	Temperature (°C)
1547	19
2481	12
3391	6
4321	0
5333	−5
6100	−9
6900	−15
7600	−19
8100	−23
8500	−27
10 700	−37

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14.6 Statistics in the media

Whenever we see the use of statistics in the media, we need to be aware that statistics can be misused. Media articles can misrepresent the facts in many ways.

- the survey questions on which claims are made were biased
- there may not be data to back up the claims that are being made
- the data may not be representative of the population it is claiming to represent
- graphs are drawn to mislead the reader

Were the survey questions biased?

A survey question may be worded so that it is biased. For example, suppose students were asked: 'Most students would rather have a shorter lunch time. What do you think?' Although some students may not be influenced by the wording of the question, others may not wish to be seen as different in their beliefs from others, so they may agree that a shorter lunch time would be desirable.

Are there valid data to support claims?

Opinions are often expressed in advertising and media articles that have no valid data to support the claims. Cigarette manufacturers, for example, use brand names and packaging to encourage smokers to believe that some types of cigarette are safer than others. Smokers often believe for example, that 'light' cigarettes and cigarettes in white or silver packets are safe. Scientific evidence may show that the claims of the cigarette manufacturers are untrue.

The journal *Addiction* claims that smokers have been misled by cigarette advertising to believe that some brands are safer than others.

> **Current packs mislead smokers: study**
>
> 20% of smokers wrongly believe some tobacco brands are safer than others, says 2011 worldwide study including 2000 Australians. Over 40% still wrongly believe 'light' brands (suggested by pack colours) are less harmful.

The Quit website states that "Around 65% of Australian smokers report smoking 'light' or 'mild' cigarettes. The majority of these smokers believe that 'light' cigarettes are better for their health than regular cigarettes.". Quit quotes a research study by Borland, Yong, King, Cummings, Fong, Elton-Marshall, 'Use of and beliefs about light cigarettes in four countries: findings from the International Tobacco Control Policy Evaluation Survey'. *Nicotine and Tobacco Research,* 2004; 6 (Supplement 3), 11–21.

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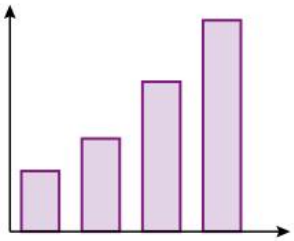
Does the data represent the population it claims to represent?

Many reports are based on sample data. Frequently the samples may not be representative of the whole population the report is claiming to represent. If an on-line opinion poll is conducted by a newspaper, then only those people who access the newspaper's website will be represented. Similarly, data collected from talk-back radio programs will represent only those people who listen to the program. It is not unusual, however, for claims to be made on the basis of biased samples such as these.

Misleading graphs

Some graphs are deliberately misleading, usually to make a change or difference look more significant than it is. Sometimes a graph is misleading simply because the person designing the graph wanted it to look more interesting and attractive.

Unless told otherwise, we tend to assume that the axes of a graph are number lines, with values steadily increasing. However, if the numbers are omitted and the axes are not labelled, then there is no way to check what the graph actually shows. For example, what could the graph on the right mean?

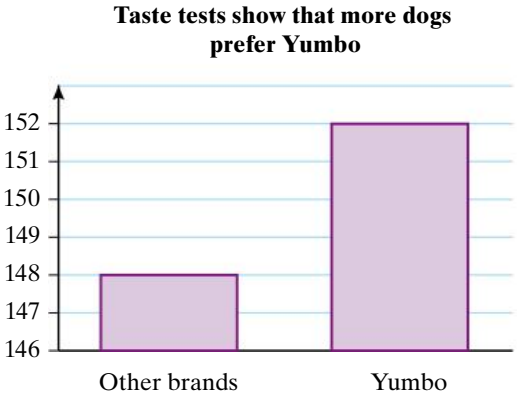


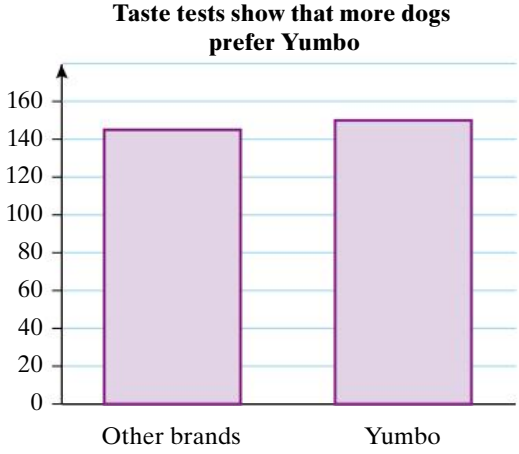
Following are examples of two common ways in which graphs can be misleading.

Vertical axis does not start from zero

People tend to assume that the vertical axis starts at zero and goes up in regular steps.

At a glance, the graph on the left appears to show that Yumbo dog food is about three times as popular as other brands. It is not clear what the numbers along the vertical axis mean, because the axis is not labelled. But if we assume that, in a taste test of some kind, 152 dogs preferred Yumbo and 148 preferred other brands, then the actual difference is only four dogs—a very small percentage of the sample. If the vertical axis started from zero as in the graph on the right, the graph would give quite a different overall impression.





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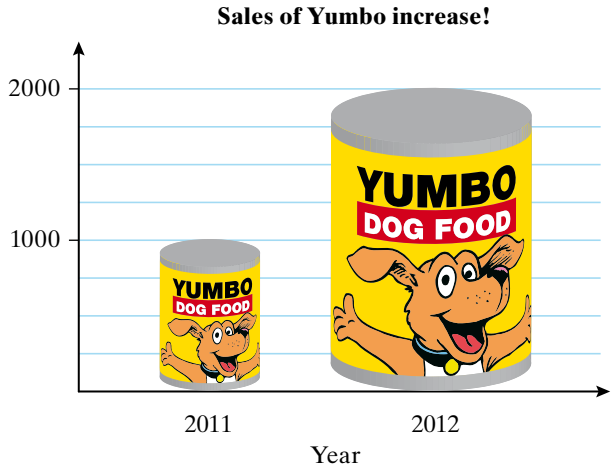
Comparing 2D or 3D pictures

Pictures can make a graph look more attractive and interesting. However, we have seen that when the side lengths of a two-dimensional shape are doubled, the area becomes four times as large. When the edges of a three-dimensional object are doubled, the volume becomes eight times as large. This can make it difficult to compare the 'size' of pictures.



At a glance, this graph appears to show that the sales of Yumbo have increased dramatically. Again, the vertical axis has not been labelled, but if we assume it shows the number of cans sold, then sales have doubled while the drawing of the can suggests that sales have increased about eight times.

10 A

Example 21

Explain what is misleading about this graph, if the height of each $1 coin represents its buying power.

Working

The horizontal scale is not divided into equal intervals.

Reasoning

The distance between 1991 and 2001 is more than between 2001 and 2012 on the horizontal scale.

continued

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Example 21 continued

Working

The area of the coin—rather than its height—appears to represent its buying power. For example, the height of the 2001 coin has dropped to about 44% of the height of the 1991 coin but the area of the 2001 coin is only about 20% of the area of the 1991 coin. This exaggerates the reduction in the first 10 years.

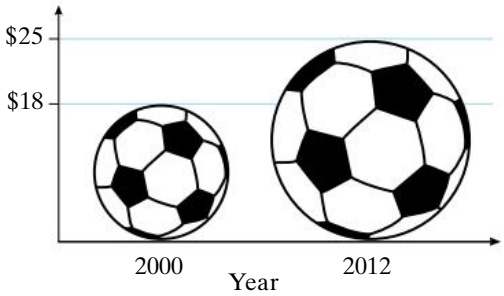
Reasoning

It is the height of the coin that represents the buying power of the dollar in that year, not its area.

exercise 14.6

LINKS TO Example 21

1 The graph shows the price of tickets for a rural soccer final in 2000 and 2010. Explain what is misleading about this graph.



2 The following Conservation Council of South Australia 2011 report suggests that misleading claims were made about the benefits of marine parks. Read the article then answer the questions below.

Evidence of marine park benefits overshadow misleading claims

Misleading claims about proposed new marine parks has led to unnecessary alarm in coastal communities, a national alliance of conservation groups said today. The Save Our Marine Life alliance of 11 conservation groups said there is already overwhelming evidence of the benefits of marine parks in Australia and overseas in protecting fish and the marine environment. The Save Our Marine Life alliance called on opponents to marine parks to also examine the evidence of the economic benefits of marine parks, such as the lucrative tourism industry the Great Barrier Reef Marine Parks has created.

Michelle Grady, spokesperson for Save Our Marine Life said communities should know the facts behind marine parks. Ms Grady is a former principal policy adviser at the SA Department of Environment and Natural Resources, and served as an adviser to Senator Janine Haines. "There is now overwhelming evidence across Australia and overseas that sanctuary or no-take areas within marine parks increase fish populations," Ms Grady said.

"The University of Adelaide recently published a report outlining 63 science-based studies that provide evidence that the healthier environments found in marine sanctuaries led to an increase in fish numbers and a decrease in marine predators, such as the Crown of Thorns Starfish." The University of Adelaide analysis includes evidence from studies conducted in Tasmania, the Great Barrier Reef and Rottnest Island in WA, and from marine sanctuaries in New Zealand, the Philippines, the Caribbean and Chile.

continued

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"An independent economic study of the south west waters of Australia less than two years ago, which included SA waters across to Kangaroo Island, found that creating marine sanctuaries would lead to a 20 per cent increase in tourism revenue."

Marine parks also provide safe havens for marine life unique to SA. "Up to 95 per cent of fish species in SA are unique. There are almost 8,000 marine species found in these waters, including more than 700 species of fish," Tim Kelly, the CEO of the Conservation Council of SA said. "The head of the Great Australian Bight is one of the most important breeding and calving sites in the world for the southern right whale. Populations of these whales are still slowly recovering from whaling, so they need safe havens for their young to grow and thrive."

In NSW, the Batemans Bay Marine Park has been in place since 2005. The manager of the marine park has confirmed that fish populations have increased by 210 per cent, including the popular snapper. "On the south coast of NSW, local newspapers have had headlines such as 'Get on the snapper revival' and 'The best snapper fishing in a decade'," Ms Grady said.

Source: Social Research Centre

a List the evidence that is used in the article to support the argument that marine parks are beneficial.

b What are the claims made by opponents of marine parks?

c Does the article present any evidence to support the claims by opponents of marine parks?

d Try to find media articles that include evidence to support the claims by opponents of marine parks.

3 Use the table of data at the right to create two line graphs, one with the title 'Salaries up!' and the other with the title 'Salaries stable'.

Year	Average salary ($)
1	32 500
2	32 600
3	33 200
4	33 000
5	33 400
6	33 700

4 Darren Pennay and Nicole Bishop from the Social Research Centre carried out a study of 'mobile phone only' households. Read the following article then answer the questions.

a What concern led to the study of 'mobile phone only' households?

b What were the findings?

c Which sub-group of the population do the researchers believe could currently be misrepresented in surveys that collect data by using landline phones?

A study of the mobile phone only population in Australia May 2009

A growing proportion of Australians live in households that do not have a landline telephone and rely solely on a mobile phone for their telecommunications. The rapid emergence of these 'mobile phone only' households has led survey researchers to question whether current approaches to undertaking telephone surveys are reaching a representative sample of the population.

continued

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It would seem that, generally speaking, the exclusion of mobile phone only households from general community survey estimates does not, as yet, have a significant bearing on the overall results. This finding is consistent with recent US findings where recent studies concluded that non-coverage due to cell-only "is currently not damaging estimates for the entire population." It would seem that this finding will hold true only for as long as the size of the mobile phone only population is relatively small and only if the attitudes and behaviours of those living in mobile phone only households do not differ from others of the same age group.

For surveys with a focus on particular population sub groups, however, more consideration will need to be given as to whether or not RDD-based telephone surveys provide a sufficiently representative sample. This is seemingly already the case for surveys of young people with this research showing that 26.6% of persons aged 16 to 24 years live in mobile phone only households.

Given that the proportion of 'mobile phone only' households is predicted to grow quite rapidly there is an obvious need for survey researchers to continue to monitor the impact that this may have on the representativeness of the estimates produced from traditional telephone survey methods.

Source: Social Research Centre

exercise 14.6 challenge

5 Graphs often use areas to represent quantities. The areas representing each data value should be proportional to the size of the data value.

Consider the picture below that compares the carbon dioxide emissions for different countries.

a Express the ratio of emissions for China and Australia as ___:1.

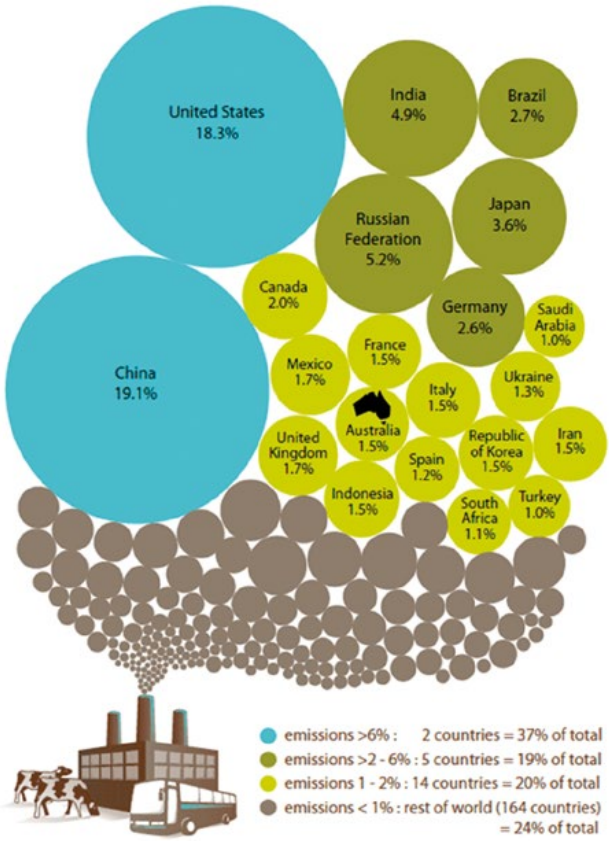
b Carefully measure the diameters of the circles representing Australia and China. Express the ratio of diameters for China and Australia as ___:1.

c Calculate the areas of the two circles. Express the ratio of areas for China and Australia as ___:1.

d Repeat steps **b** and **c** to compare the circles for Japan and the United States.

e Do you think the picture has been drawn accurately?

f Do you think the picture is a graph? Give reasons.



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Analysis task

Fuel consumption and carbon emissions

Fuel consumption data

Fuel consumption and carbon dioxide emissions data for the best-selling two-wheel drive cars and four-wheel drive vehicles is provided below. The data can be analysed using the techniques included in this chapter.

Fuel consumption (L/100 km)		Carbon dioxide emissions (gCO_2/km)	
2WD	4WD	2WD	4WD
0	6.4	0	150
0	7.3	0	169
3.9	7.6	89	178
4.1	8.2	95	190
4.4	8.2	100	190
4.4	8.3	100	193
4.6	8.3	109	193
4.6	8.3	109	193
4.7	7.2	110	169
4.7	7.4	110	173
4.9	7.5	115	174
4.9	7.5	115	179
5.5	7.5	126	179
5.3	7.7	126	179
5.3	7.6	126	180
5.4	8.5	127	201
5.6	9	128	210
5.6	9.2	128	219
5.5	9.2	128	219
5.5	4.9	132	129
5.5	4.9	132	130
5.5	7.7	132	183

Fuel consumption (L/100 km)		Carbon dioxide emissions (gCO_2/km)	
5.8	7.9	138	184
5.9	7.8	138	185
6	8	139	187
6	8.5	139	197
6	8.7	139	197
6	8.7	141	197
6	8.3	141	198
6	9.2	142	221
6	9.5	142	227
6	9.6	142	228
6.2	9.8	143	232
6.1	5.9	144	155
6.1	5.9	144	155
6.2	6	144	158
6.1	6	145	158
6.1	6	145	158
6.1	6	145	158
6.1	6	145	159
6.1	6.1	145	160
6.2	6.1	148	160
6.2	6.1	148	160
6.5	6.1	151	160
6.4	6.1	153	160

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Fuel consumption (L/100 km)		Carbon dioxide emissions (gCO_2/km)	
6.4	7.1	153	168
6.5	7.3	154	171
6.7	7.4	155	175
6.5	8.7	155	201
6.7	8.7	159	201
6.7	8.7	159	204
5.3	8.8	122	205
5.3	8.8	122	205
5.3	9.2	124	214
5.3	9	124	215
5.5	9	128	215
5.8	9.3	134	217
5.8	9.4	135	219
5.9	10.5	136	243
5.8	10.5	136	243
5.8	10.6	137	249
5.9	10.6	138	249
5.9	10.6	139	254
6	6.4	139	168
6.1	6.4	140	168
6.1	6.5	140	170
6	6.6	140	174
6.9	6.6	164	174
6.9	6.6	164	174
6.9	6.6	164	175
6.9	6.8	164	179
6.9	6.8	165	179
6.9	7.9	165	184

Fuel consumption (L/100 km)		Carbon dioxide emissions (gCO_2/km)	
6.9	8	165	188
7	8.1	166	189
7	8.1	166	189
7	8.1	167	192
7	8.1	167	192
7	8.3	167	197
7.3	8.5	168	199
7.2	8.4	170	199
7.2	9.3	170	221
7.2	9.5	170	222
7.4	9.5	170	226
7.3	9.7	170	227
7.3	9.7	170	227
7.2	9.7	171	227
7.2	9.9	171	231
7.2	10	173	233
7.3	10.1	174	234
7.3	10.1	174	236
7.3	10.3	174	240
7.4	11.3	175	269
7.4	11.5	175	272
7.4	11.6	175	276
7.4	11.7	175	279
7.7	11.8	176	280
7.7	5.6	178	147
7.5	5.6	179	149
7.5	5.8	179	153

http://www.greenvehicleguide.gov.au/GVGPublicUI/home.aspx

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a Use a spreadsheet to calculate the five-number summary for the fuel consumption data for

i 2WD cars **ii** 4WD vehicles

b Are there any outliers?

c By hand, draw parallel box plotsfor the fuel consumption data. Show any outliers.

d Discuss any differences between the two box plots.

e–h Repeat parts i–l for the carbon dioxide emissions data.

i Use the spreadsheet to construct a scatter plot of carbon dioxide emissions against fuel consumption for 2WD cars. Describe the strength of the association. Is this what you would have expected? Explain.

j Repeat for the 4WD vehicles.

Extension

k Print your two scatter plots and draw lines of good fit passing through (0, 0). Determine the gradient of each line and interpret what this gradient means.

l Use a spreadsheet to calculate the mean and the standard deviation for fuel consumption for

i 2WD cars **ii** 4WD cars

m Between which values would you expect 95% of 2WD cars to lie?

n Between which values would you expect 95% of the 4WD vehicles to lie?

o Comment on the differences between 2WD and 4WD cars with respect to fuel consumption.

p–s Repeat parts a–d for carbon dioxide emissions.

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Review Statistics

Summary

Measures of spread

- Spread describes the variability in a distribution of data values.
- Range = maximum value − minimum value.
- Interquartile range IQR = $Q_3 - Q_1$.
- An outlier is any value that is below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.
- Variance $s^2 = \dfrac{\sum (x - \bar{x})^2}{n - 1}$
- Standard deviation $s = \sqrt{\dfrac{\sum (x - \bar{x})^2}{n - 1}}$.
- For symmetrical data.
  - Two-thirds of the data is expected to fall within $\bar{x} \pm s$
  - 95% of the data is expected to fall within $\bar{x} \pm 2s$
  - Nearly all the data is expected to fall within $\bar{x} \pm 3s$.

Box plots: a visual summary

- Box plots present a visual summary of a distribution, using the five-number summary: Q_0 (minimum value), Q_1, Q_2 (median), Q_3, Q_4 (maximum value).
- Box plots can be drawn with outliers shown; then the relevant whisker stops at the last value which is not an outlier.

Analysing relationships between statistical variables

- The methods used to display bivariate data depend on the types of variables.

One categorical variable, one numerical variable

- Back-to-back stem plots (for two values of the categorical variable).
- Parallel box plots (for two or more values of the categorical variable).

Two numerical variables

- Scatter plots.

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Visual map

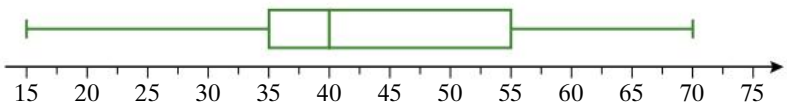
Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

back-to-back stem plot	line of good fit	range
bivariate data	lower quartile	scatter plot
box plot	mean	skewed distribution
categorical data	median	spread
centre	nominal categorical data	*standard deviation
continuous numerical data	numerical data	stem plot
discrete numerical data	ordinal categorical data	symmetrical distribution
five-number summary	outlier	univariate data
interquartile range	quartile	upper quartile
	random sample	*variance

Revision

Multiple-choice questions

1



For the box plot shown above, which of the following is incorrect?

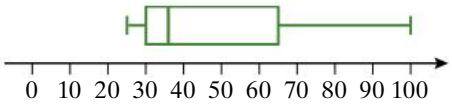
A The median is 40.
B The interquartile range is 20.
C Less than a quarter of the data values are greater than 50.
D Less than a quarter of the data values are less than 30.
E Three quarters of the data values are less than 55.

2 Two statistical variables x and y are known to have a strong negative association. Which one of the following statements is true about these variables?

A As the x-values increase, the y-values become negative.
B As the x-values decrease, the y-values tend to decrease.
C As the x-values increase, the y-values tend to decrease.
D Each x-value will be the negative of the corresponding y-value.
E Negative x-values are associated with negative y-values.

3 In this box plot, the interquartile range is

A 16 **B** 35 **C** 38
D 71 **E** 75



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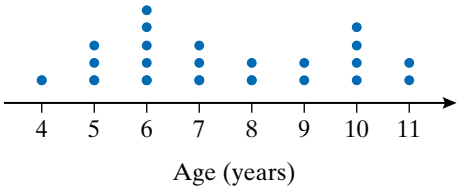
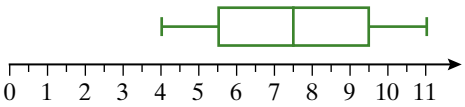
4 This dot plot shows the ages of a randomly selected group of 22 primary school children.



The corresponding box plot for this data is

A



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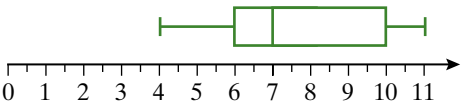

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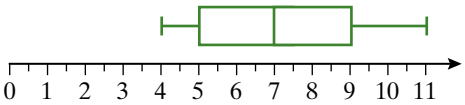

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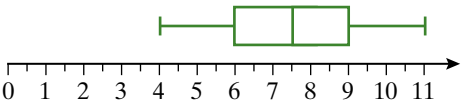
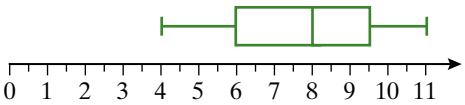


E



5 Scores on a mathematics test are known to have a mean of 66 marks and a standard deviation of 4 marks. About 95% of students who sit for the test will be expected to have scores

A below 62 **B** in the interval 62 – 70 **C** in the interval 58 – 74
D in the interval 66 – 74 **E** above 74

Short-answer questions

6 This stem plot shows class results for a maths test.

a What is the mode of the data?
b What is the mean test score?
c What is the median test score?
d What is the range?
e What is the interquartile range?
f Construct a box plot that displays this data.

Test scores out of 60

Stem	Leaf
2	8
3	2 5 8 8 9
4	0 0 2 3 3 4 5 5 5 6 7 8
5	0 2 4 5 5 6 6 7 7 9 9
6	0 0

(Key: 2|8 means 28)

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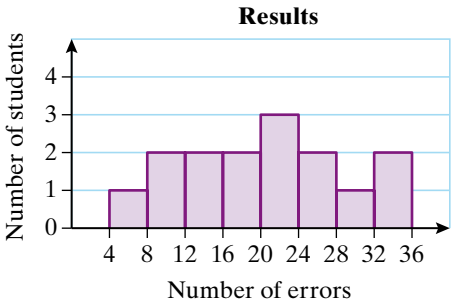
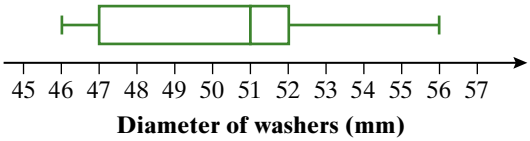
7 This histogram shows the distribution of the number of errors made by students while attempting an online test.

a How many students were tested?

b Students pass the test if they make less than 12 errors. How many students failed the test?



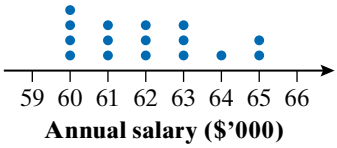
8 This box plot shows the diameter (in mm) of washers in a sample.



a Between which two values does the middle 50% of the sample lie?

b What proportion of the washers have a diameter less than 52 mm?

9 This dot plot shows the salaries (in thousands of dollars) of 16 experienced teachers. Construct a box plot of the data. Which plot shows more useful information? Why?



10 Gaetano owns a coffee shop. He makes a table to show the sales for the last six months.

a Construct two bar charts representing this data, one with the vertical scale starting at zero, the other with the vertical scale starting at 28 000.

b Gaetano wants to sell his shop. Which graph is he likely to show a potential buyer? Why?

Month	Sales (thousands of dollars)
January	32.0
February	34.5
March	36.0
April	36.5
May	37.4
June	38.2

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Extended-response questions

11 Blood potassium levels, to the nearest tenth of a milligram per litre (mg/L), are obtained for 40 women. The results are shown in this ordered stem plot.

a Describe the shape of the distribution of blood potassium levels.

b Find the five-number summary for the data.

c Give the value of the interquartile range (IQR) and interpret this value within the context of the problem.

Blood potassium levels (mg/L)

Stem	Leaf
2	
2	7
3	2 4
3	6 6 7 7 8 8 9 9
4	0 1 2 2 2 2 3 3 3 3 4
4	5 5 6 7 7 9 9 9
5	0 1 1 2 3
5	6 8 8
6	0
6	6

(Key: 2|7 means 2.7)

12 The wind speed of the strong sea breeze known as the Fremantle Doctor was recorded at the same time each afternoon for 40 days. The speeds (to the nearest knot) were:
15, 22, 14, 12, 21, 34, 19, 11, 13, 0, 16, 4, 23, 8, 12, 18, 24, 17, 14, 3,
10, 12, 9, 15, 20, 5, 19, 13, 17, 11, 16, 19, 24, 12, 7, 14, 17, 10, 14, 23.

a What type of variable is wind speed?

b Choose appropriate class intervals and present the data in a histogram. Hint: first tally the data.

c Which class interval occurs the most frequently?

d Prepare an ordered stem and leaf plot for the data.

e Calculate the five-number summary for the data.

f Are there any outliers?

g Construct a box plot of the data showing any outliers.

h Discuss the distribution of wind speeds.

13 Marcia has a mixture of classical CDs and modern CDs in her music collection. She decides to compare the track lengths of the two types of music. She lists the track length (in minutes) for 20 tracks of each type.

Modern music: 3.2, 3.5, 4.9, 3.2, 3.5, 3.7, 3.2, 3.1, 3.8, 3.9,
4.2, 3.9, 3.2, 3.2, 3.2, 3.8, 3.7, 4.9, 3.3, 5.1

Classical music: 3.3, 2.1, 3.6, 5.2, 2.1, 2.5, 5.1, 3.3, 1.4, 3.2,
3.1, 2.5, 2.8, 2.5, 3.1, 1.1, 4.2, 7.8, 0.7, 5.9

a Find the five-figure summary for each set of data.

b Determine if there are any outliers.

c Construct a box plot for each set of data, showing any outliers.

d Use the box plots to compare the track lengths for each type of music.

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10 A

BLM Football team date

14 This scatter plot shows the heights and weights (masses) of 24 players from a football team.

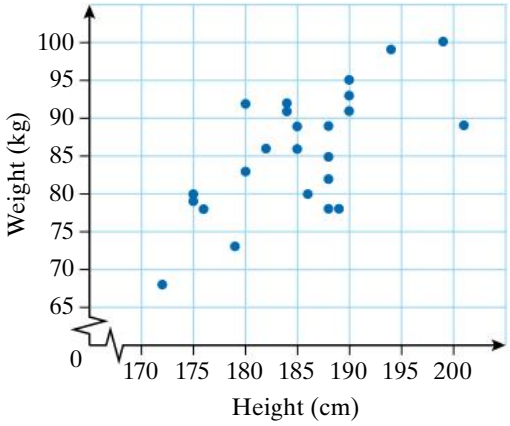
a Describe the association between the variables.

b Using a ruler and pencil, construct a line of good fit on a copy of the graph provided by your teacher or printed from the *OneStopDigital* ebook.

c Use the trend line to predict the weight of a player whose height is 195 cm.

d How confident are you of this prediction? Give an indication of the possible error in your prediction.

Heights and weights of 24 football players



15 Humans and other mammals are able to maintain a fairly constant body temperature even though the temperature of the surroundings changes. However, the temperature of a reptile's body is affected by the temperature of its surroundings. The following table shows the body temperature of a lizard as the temperature of its surroundings increased.

Temperature of surroundings (°C)	10	15	20	25	30	35	40
Body temperature of lizard (°C)	9	13	19	23	29	34	38

a Which is the independent variable?

b Construct a scatter plot of the data, using graph paper.

c Describe the association.

d Place your ruler along the points and draw a line of good fit.

e Predict the body temperature of the lizard if the temperature of the surroundings is 17 °C.

10 A

16 Use the wind speed data in question **12** for this question.

a Calculate

i the mean wind speed

ii the variance

iii the standard deviation

b Between which wind speeds would you expect two-thirds of the data to lie? How much does this compare with the actual percentage?

c Between which wind speeds would you expect 95% of the data to lie? How much does this compare with the actual percentage?

Practice quiz Chapter 14

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Exponential functions and logarithms 15

Pre-test

Warm-up

Typically, exponential functions describe growth or decay over time. For example, in a favourable environment where there are no predators, the number of individuals such as penguins in a population can grow exponentially. An increase in one generation of penguins leads to a greater increase in the next as the now larger population breeds. In chapter 5 compound interest was also seen to involve exponential growth. By contrast, compound depreciation is associated with exponential decrease.

15.1 Reviewing indices

In Year 9 you will have worked with integer indices and the index laws. This section briefly revises this previous work.

- In the expression a^n, a is the base and n is the index.
- $a^0 = 1, a \neq 0$
- $a^{-m} = \dfrac{1}{a^m}$

The index laws	
Multiplication index law	$a^m \times a^n = a^{m+n}$
Division index law	$\dfrac{a^m}{a^n} = a^{m-n}$
Power of a power index law	$(a^m)^n = a^{m \times n}$
Power of a product index law	$(ab)^m = a^m b^m$
Power of a quotient index law	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Example 1

Simplify and evaluate each of the following.

a 3^0 **b** $5^5 \times 5^{-3}$ **c** $\dfrac{10^9 \times 10^{-2}}{10^3}$ **d** $(3^2)^3$

Working	Reasoning
a $3^0 = 1$	Any number or pronumeral raised to the power zero is equal to 1.
b $5^5 \times 5^{-3} = 5^{+(-3)}$ $= 5^2$ $= 25$	To multiply numbers that are powers of the same base, add the indices. Evaluate 5^2.
c $\dfrac{10^9 \times 10^{-2}}{10^3} = \dfrac{10^{9+(-2)}}{10^3}$ $= 10^{7-3}$ $= 10^4$ $= 10\,000$	Add the indices to simplify the numerator. To divide numbers that are powers of the same base, subtract the indices. Evaluate 10^4.
d $(3^2)^3 = 3^{2 \times 3}$ $= 3^6$ $= 729$	When a power is raised to another power, multiply the indices. Evaluate 3^6.

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Example 2

Evaluate each of the following.

a 2^{-4} **b** $\dfrac{1}{5^{-2}}$ **c** $\left(\dfrac{3}{4}\right)^{-2}$ **d** 300×0.8^6

Working	Reasoning
a $2^{-4} = \dfrac{1}{2^4}$ $= \dfrac{1}{16}$	2 raised to the power -4 is the reciprocal of 2 raised to the power 4. 2^4 is then evaluated as 16.
b $\dfrac{1}{5^{-2}} = 5^2$ $= 25$	The reciprocal of 5^{-2} is 5^2.
c $\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$ $= \dfrac{16}{9}$	The reciprocal of $\left(\dfrac{3}{4}\right)^{-2}$ is $\left(\dfrac{4}{3}\right)^2$.
d $300 \times 0.8^6 = 78.6432$	Use a calculator. 300×0.8^6 means that 0.8 is raised to the power 6 and the result is multiplied by 300.

exercise 15.1

LINKS TO Example 1

1 Simplify and evaluate each of the following.

a 5^0 **b** 10^0 **c** $7^6 \times 7^2$ **d** $2^6 \times 2^{-2}$

e $10^4 \times 10^3$ **f** 6×3^7 **g** $\dfrac{10^7 \times 10^{-4}}{10^{-3}}$ **h** $(2^3)^2$

LINKS TO Examples 2a, b, c

2 Evaluate each of the following.

a 3^{-2} **b** 2^{-6} **c** 10^{-5} **d** $\dfrac{1}{3^{-2}}$

e $\dfrac{1}{2^{-4}}$ **f** $\left(\dfrac{3}{4}\right)^3$ **g** $\left(\dfrac{2}{3}\right)^4$ **h** $\left(\dfrac{1}{2}\right)^{-5}$

LINKS TO Example 2d

3 Use a calculator to evaluate each of the following, rounding to three decimal places.

a 0.9^{20} **b** 1.3^6 **c** 0.75^{10} **d** 1.2^8

e 2×3.4^2 **f** 3×0.8^3 **g** 4×1.5^{10} **h** 500×1.25^6

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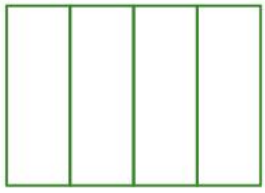
15.2 Exponential functions

In this section we extend our study of indices to look at **exponential functions**, that is, functions where the variable is an index or exponent.

To illustrate the nature of exponential functions we use the example of a sheet of A4 paper being folded in half several times.

When a sheet of paper is folded in half, and then unfolded, it has two 'columns'.

When the sheet of paper is folded in half and half again, and then unfolded, there are four columns.



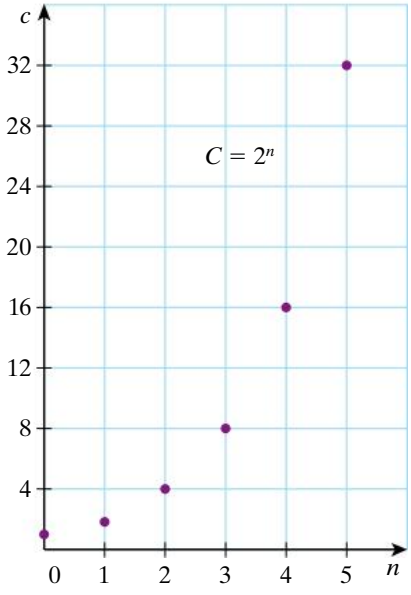
Each time the paper is folded in half, the number of columns doubles.

Using n to represent the number of times the paper is folded and C to represent the number of columns, we can construct a table of values to show the relationship between C and n. We can see that each time the paper is folded, the number of columns is being multiplied by 2.

+ 1 + 1 + 1 + 1 + 1

n	0	1	2	3	4	5
C	1	2	4	8	16	32

× 2 × 2 × 2 × 2 × 2



The rule connecting C and n is $C = 2^n$. This type of function is called an **exponential function** because the independent variable, in this case n, occurs as an index or exponent.

A graph of the values shows the graph becoming steeper and steeper as n increases.

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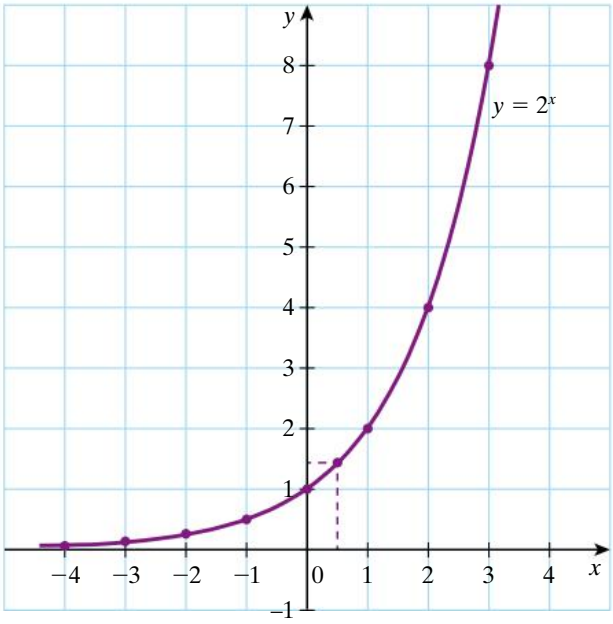
The graph of $y = 2^x$

Exponential functions

In the previous example, the independent variable, n, could have only positive integer values.

We now look at a table of values and graph for the function $y = 2^x$, where we include negative values for x.

x	-4	-3	-2	-1	0	1	2	3
y	$2^{-4} = \frac{1}{2^4} = 0.0625$	$2^{-3} = \frac{1}{2^3} = 0.125$	$2^{-2} = \frac{1}{2^2} = 0.25$	$2^{-1} = \frac{1}{2^1} = 0.5$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$



Note the following features of the graph of $y = 2^x$.

- The y-intercept is at $y = 1$.
- As x increases, y increases at a faster rate.
- $y = 2^x$ is an increasing exponential function.
- 2^x tends to infinity as x tends to infinity.
- 2^x can never be negative.
- For negative values of x, the x-axis is an asymptote of the graph.
- We can use the graph to find rational approximations for y for values of x between the integer values. For example, $2^{0.5}$, the square root of 2, is irrational but is approximately equal to 1.414 (see dotted lines on graph).

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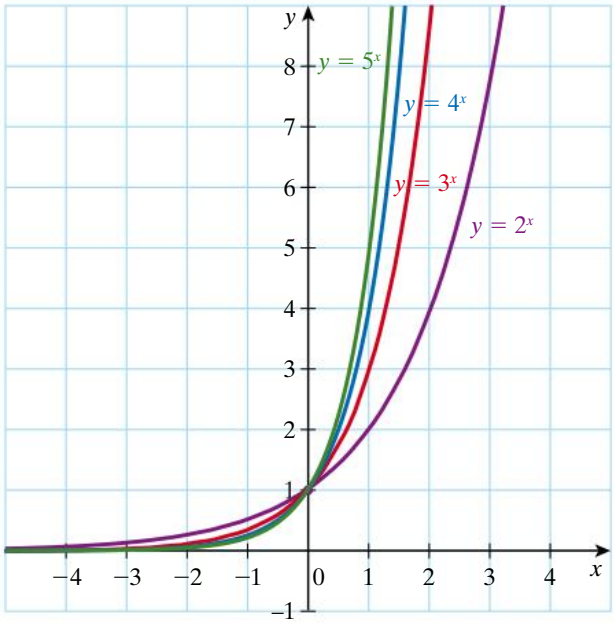
When the graphs of $y = 2^x$, $y = 3^x$, $y = 4^x$ and $y = 5^x$ are compared, we see that all the graphs have similar features. All the graphs cross the y-axis at $y = 1$. This is because any number raised to the power zero is equal to 1. The x-axis is an asymptote of each of the graphs. As the base increases, the graph rises more steeply. For example, the graph of $y = 3^x$ is steeper than the graph of $y = 2^x$.



Example 3

Consider the function $y = 2.5^x$.

a Construct a table of values for $-3 \le x \le 3$, rounding the y-values to two decimal places where appropriate.

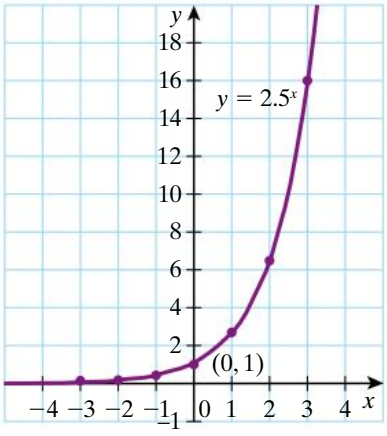
b Plot the graph, joining the points with a smooth curve.

Working

a

x	−3	−2	−1	0	1	2	3
y	0.06	0.16	0.4	1	2.5	6.25	15.63

b



Reasoning

When $x = 0$, $y = 2.5^0 = 1$
Substitute each x-value into 2.5^x and use a calculator to evaluate.

Plot the points and join with a smooth curve.
Label the y-intercept.

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Comparing the graphs of $y = 2^x$ and $y = 0.5^x$

We note the following features of the graphs.

- The graph of $y = 0.5^x$ is a reflection in the y-axis of the graph of $y = 2^x$.
- The x-axis is an asymptote to both graphs.
- The graphs intersect at their y-intercept, $y = 1$.

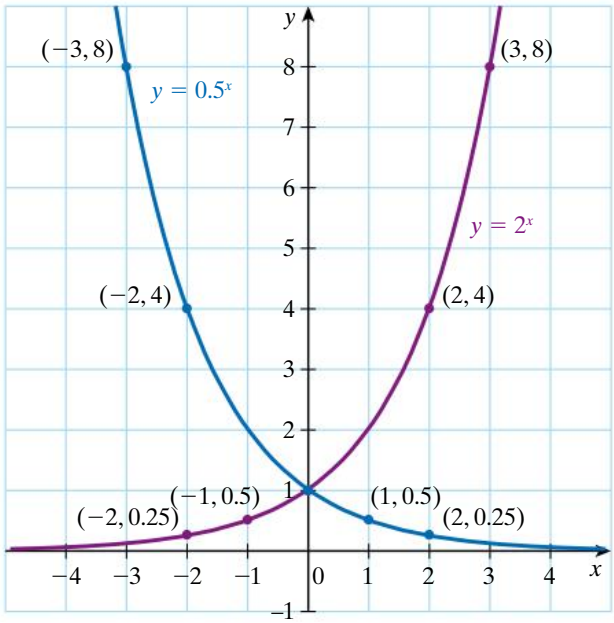
To explain why the graph of $y = 0.5^x$ is a reflection of $y = 2^x$ we write 0.5 as $\frac{1}{2}$ then express $\frac{1}{2}$ as a power of 2.

$$y = \left(\frac{1}{2}\right)^x$$

$$y = (2^{-1})^x$$

$$y = 2^{-x}$$

So $y = 0.5^x$ is the same function as $y = 2^{-x}$. This means that each x-value for the function $y = 2^x$ has been replaced with $-x$ for the function $y = 2^{-x}$. For example, the point $(2, 4)$ on the graph of $y = 2^x$ becomes the point $(-2, 4)$ on the graph of $y = 0.5^x$.



Increasing and decreasing exponential functions

The function $y = 2^x$ is an increasing exponential function because y increases as x increases. The function $y = 0.5^x$ is a decreasing exponential function because y decreases as x increases.

The general exponential equation $y = a^x$

If $a > 1$, $y = a^x$ is an increasing exponential function.

If $0 < a < 1$, $y = a^x$ is a decreasing exponential function.

We now consider the exponential function $y = k \times a^x$, where a^x is multiplied by a constant, k. From experience with the graphs of other functions such as $y = ax$ and $y = ax^2$, we expect that k will affect the steepness of the graph, that is, for $k > 1$, the graph of $y = a^x$ is vertically stretched. For $0 < k < 1$, the graph of $y = a^x$ is vertically squashed. The constant k also determines the y-intercept of the graph.

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Substituting $x = 0$ in $y = k \times a^x$ gives

$y = k \times a^0$

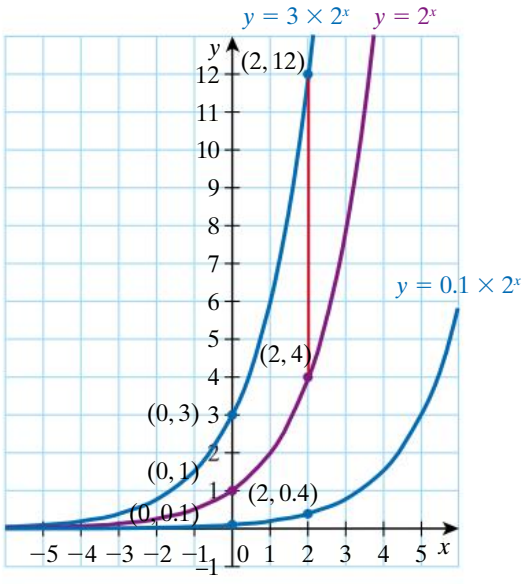
$\quad = k \times 1$

$y = k$

So the y-intercept is $(0, k)$.

The tables of values and graphs below compare the functions $y = 2^x$, $y = 3 \times 2^x$ and $y = 0.1 \times 2^x$. The graph of $y = 3 \times 2^x$ is steeper than the graph of $y = 2^x$ but the graph of $y = 0.1 \times 2^x$ is less steep.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 3 \times 2^x$	0.375	0.75	1.5	3	6	12	24
$y = 0.1 \times 2^x$	0.0125	0.025	0.05	0.1	0.2	0.4	0.8



Example 4

Consider the function $y = 2 \times 3^x$.

a Construct a table of values for $-3 \le x \le 3$, rounding the y-values to two decimal places where appropriate.

b Plot the graph, joining the points with a smooth curve.

continued

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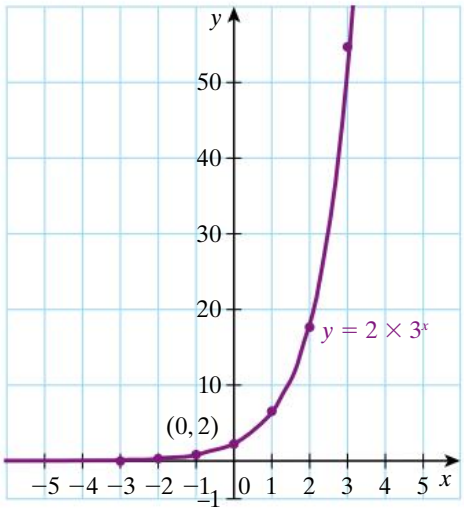
Example 4 continued

Working

a

x	-3	-2	-1	0	1	2	3
y	0.07	0.22	0.67	2	6	18	54

b



Reasoning

Substitute each x value into $y = 2 \times 3^x$.
For example, when $x = -3$,
$y = 2 \times 3^{-3}$
$= \frac{2}{3^3}$
$= \frac{2}{27}$
≈ 0.07
Alternatively, use a calculator to evaluate 2×3^{-3}.
Choose a suitable scale.
Label the axes.
Plot the points.
Join the points with a smooth curve.
Label the y-intercept.

Finding exponential rules from tables of values

To find an exponential rule, $y = k \times a^x$, we must find the values of k and a. If we look at a table of values for the exponential function $y = 5 \times 3^x$ we see that each value of y is obtained by multiplying the previous by 3.

+ 1 + 1 + 1 + 1 + 1 + 1

x	-3	-2	-1	0	1	2	3
y	5×3^{-3} $= \frac{5}{27}$	5×3^{-2} $= \frac{5}{9}$	5×3^{-1} $= \frac{5}{3}$	5×3^0 $= 5$	5×3^1 $= 15$	5×3^2 $= 45$	5×3^3 $= 135$

× 3 × 3 × 3 × 3 × 3 × 3

As we saw with the paper folding there is a **constant ratio** between successive y-values as the x-values increase in steps of 1, that is each y-value is multiplied by a constant number to produce the next y-value. This constant ratio is represented by a in the general exponential function $y = k \times a^x$. Each y-value is found by multiplying the previous y-value by a. So to find the value of a from a table of values, we divide each y-value by the previous y-value.

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Example 5

For each of these tables of values find the constant ratio, a, between the y-values given that the rule is exponential.

a

x	-2	-1	0	1	2
y	6.25	2.5	1	0.4	0.16

b

x	-2	-1	0	1	2
y	0.163	0.571	2	7	24.5

Working

a $\frac{2.5}{6.25} = 0.4, \frac{1}{2.5} = 0.4, \frac{0.4}{1} = 0.4, \frac{0.16}{0.4} = 0.4$

$a = 0.4$

b $\frac{0.571}{0.163} \approx 3.50, \frac{2}{0.571} \approx 3.50, \frac{7}{2} = 3.5, \frac{24.5}{7} = 3.5$

$a = 3.5$

Reasoning

For an exponential function, each y-value is obtained by multiplying the previous y-value by the same number, a. So to find a we divide each y-value by the preceding y-value.

To find a we divide each y-value by the preceding y-value. Due to rounding of the y-values 0.163 and 0.571 to three decimal places in the table of values, the first two calculated values for a are not exactly 3.5.

To find the value of k in the exponential rule $y = k \times a^x$, we look at the value of y when $x = 0$. For example, in example **5** part **a**, $k = 1$ and in example **5** part **b**, $k = 2$.

Finding an exponential rule $y = k \times a^x$ from a table of values

Find a by dividing each term by the term before it.

Find the value of k by looking at the value of y when $x = 0$.

Example 6

The following tables of values each fit an exponential rule.

- **i** Find the value of k.
- **ii** Find the constant ratio, a, between the y-values.
- **iii** Find the exponential rule.

continued

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Example 6 continued

a

x	-3	-2	-1	0	1	2	3
y	0.75	1.5	3	6	12	24	48

b

x	0	1	2	3	4
y	10	8	6.4	5.12	4.096

Working	Reasoning
a i When $x = 0$, $y = 6$ So $k = 6$	For the rule $y = k \times a^x$, when $x = 0$, $y = k \times a^0$ $= k \times 1$ $y = k$
ii $\frac{1.5}{0.75} = 2$, $\frac{3}{1.5} = 2$, $\frac{6}{3} = 2$ and so on. $a = 2$	To find a we divide each y-value by the preceding y-value.
iii The rule is $y = 6 \times 2^x$	The rule is of the form $y = k \times a^x$.
b i When $x = 0$, $y = 10$ So $k = 10$	For the rule $y = k \times a^x$, when $x = 0$, $y = k \times a^0$ $= k \times 1$ $y = k$
ii $\frac{8}{10} = 0.8$, $\frac{6.4}{8} = 0.8$, $\frac{5.12}{6.4} = 0.8$ and so on. $a = 0.8$	To find a we divide each y-value by the preceding y-value.
iii The rule is $y = 10 \times 0.8^x$.	The rule is of the form $y = k \times a^x$.

Sketching graphs of exponential functions

To sketch the graph of an exponential function

- decide if the function is an increasing or decreasing exponential function, that is, is the base greater than 1 or less than 1
- substitute $x = 0$ to find the y-intercept
- label one other point on the graph.

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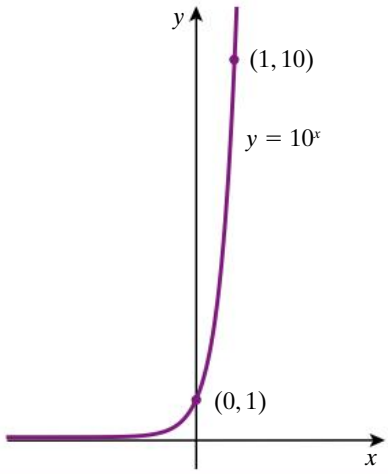
Example 7

Sketch the graph of $y = 10^x$ showing the y-intercept, and the value of y when $x = 1$.

Working

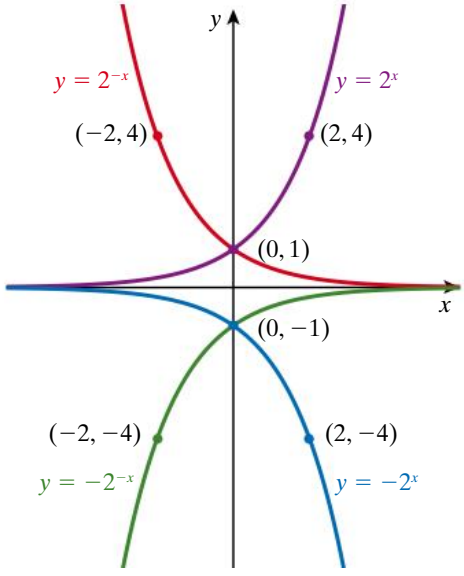


Reasoning

$y = 10^x$ is an increasing exponential function because the base, 10, is greater than 1.
When $x = 0$, $y = 10^0 = 1$
When $x = 1$, $y = 10^1 = 10$

Comparing the graphs of $y = 2^x$, $y = -2^x$, $y = 2^{-x}$ and $y = -2^{-x}$

- the x-axis is an asymptote for the four graphs
- the graphs of $y = 2^x$ and $y = 2^{-x}$ intersect at their y-intercept, (0, 1)
- the graphs of $y = -2^x$ and $y = -2^{-x}$ intersect at their y-intercept, (0, −1).
- the graph of $y = 2^{-x}$ is a reflection in the y-axis of the graph of $y = 2^x$
- the graph of $y = -2^x$ is a reflection in the x-axis of the graph of $y = 2^x$
- the graph of $y = -2^{-x}$ is a reflection in the x-axis of the graph of $y = 2^{-x}$



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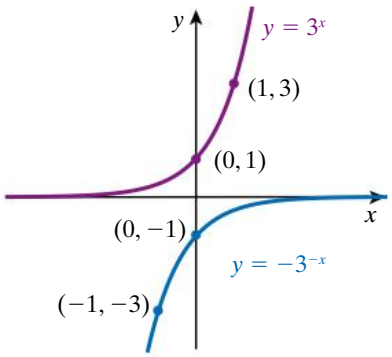
Example 8

Sketch the graphs of $y = 3^x$ and $y = -3^{-x}$ on the same axes.

Working

$y = 3^x$
y-intercept: $y = 1$
When $x = 1$, $y = 3$
$y = -3^{-x}$
y-intercept: $y = -1$
When $x = -1$, $y = -3$

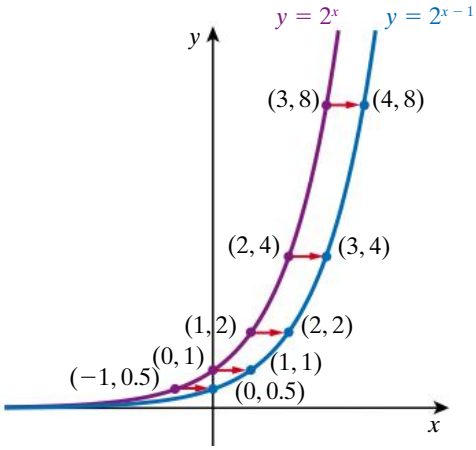


Reasoning

The index $-x$ in $y = -3^{-x}$ means that the graph of $y = 3^x$ is reflected in the y-axis. The negative sign before 3 means that the graph of $y = 3^x$ is reflected in the x-axis. So the graph of $y = -3^{-x}$ is a reflection in both axes of the graph of $y = 3^x$.
The x-axis is an asymptote to both graphs.

Horizontal translation of $y = a^x$

Based on our knowledge of other functions, we might expect that the graph of $y = 2^{x-1}$ would be obtained by translating the graph of $y = 2^x$ one unit to the right. The graphs of $y = 2^x$ and $y = 2^{x-1}$ show that this is the case. Notice that the x-axis is an asymptote to the graph, as it is for the graph of $y = 2^x$.



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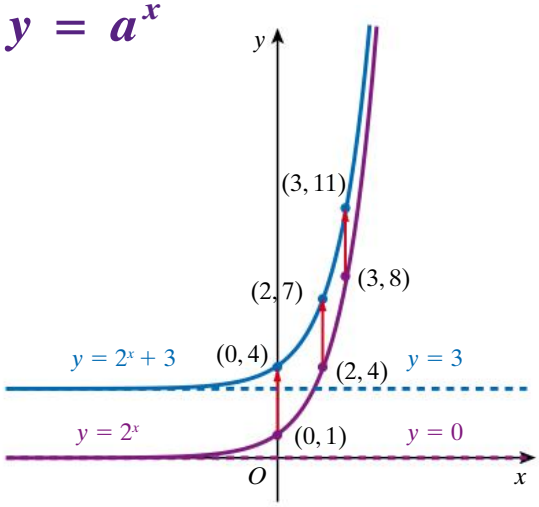
Vertical translation of $y = a^x$

Based on our knowledge of other functions we would expect that the graph of $y = 2^x + 3$ would be obtained by translating the graph of $y = 2^x$ three units upwards. The asymptote is also translated three units upwards, so that the asymptote is the line $y = 3$.



Example 9

Sketch the graph of

a $y = 3^{x+2}$

b $y = 3^x - 2$

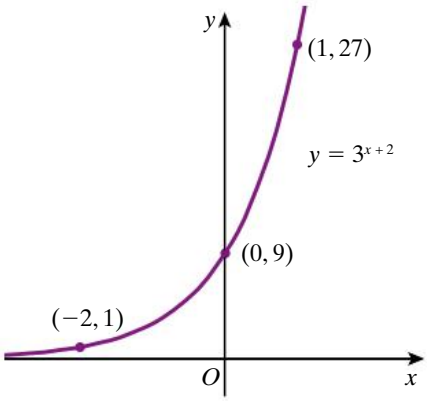
Working

a $y = 3^{x+2}$

When $x = 0$,
$y = 3^{0+2}$
$= 9$

When $x = 1$,
$y = 3^{1+2}$
$= 3^3$
$= 27$

When $x = -2$,
$y = 3^{-2+2}$
$= 3^0$
$= 1$



Reasoning

The graph of $y = 3^x$ is translated 2 units to the left. The point $(0, 1)$ on $y = 3^x$ becomes the point $(-2, 1)$.
Substitute $x = 0$ to find the y-intercept.

Substitute two more values for x to find two more points on the graph, e.g. $x = 1$, $x = -2$.

Draw and label the axes.
Mark and label the points and join with a smooth curve, with the x-axis as asymptote.

continued

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Example 9 continued

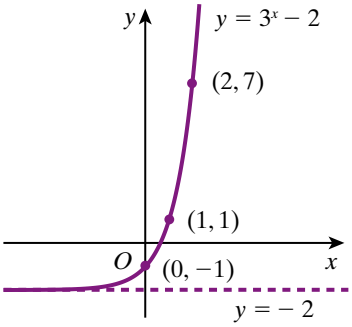
Working

b $y = 3^x - 2$

When $x = 0$,
$y = 3^0 - 2$
$= 1 - 2$
$y = -1$ $(0, -1)$

When $x = 1$,
$y = 3^1 - 2$
$= 3 - 2$
$y = 1$ $(1,1)$

When $x = 2$,
$y = 3^2 - 2$
$= 9 - 2$
$y = 7$ $(2,7)$



Reasoning

The graph of $y = 3^x$ is translated 2 units down.
The point (0,1) on $y = 3^x$ becomes the point $(0, -1)$ so the y-intercept is at $(0, -1)$.

Substitute two more values for x to find two more points on the graph.

The asymptote is also translated two units down so the asymptote is the line $y = -2$. Draw and label the axes. Mark and label the points and join with a smooth curve.

exercise 15.2

LINKS TO Example 3

1 For each of the following functions

i construct a table of values for $-3 \le x \le 3$, rounding the y-values to two decimal places where appropriate

ii choose a suitable scale and plot the graph, joining the points with a smooth curve

a $y = 0.6^x$ **b** $y = 1.4^x$ **c** $y = 8^x$

d $y = 0.2^x$ **e** $y = 2.5^{-x}$ **f** $y = 0.125^{-x}$

LINKS TO Example 4

2 For each of the following functions

i construct a table of values for $-3 \le x \le 3$, rounding the y-values to two decimal places where appropriate

ii choose a suitable scale and plot the graph, joining the points with a smooth curve

a $y = 2 \times 5^x$ **b** $y = 4 \times 3^x$ **c** $y = 3 \times 4^x$

d $y = 2.5 \times 2^x$ **e** $y = 0.4 \times 5^x$ **f** $y = 4 \times 0.5^x$

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3 Use a graphing or CAS calculator or graphing software such as GeoGebra to graph $y = 3 \times 0.5^x$ and $y = 3 \times 2^{-x}$ on the same set of axes.

a What do you notice, and why does this occur?

b Write the rule for $y = 4 \times 3^{-x}$ with a positive power.

c Write the rule for $y = 5 \times \left(\frac{1}{4}\right)^x$ with a whole-number base.

4 Which of the two graphs in question **1** are the same? Can you explain why?

LINKS TO Examples 5, 6

5 The following tables of values each fit an exponential rule.

i Find the value of k.

ii Find the constant ratio, a, between the y-values.

iii Find the exponential rule.

a

x	0	1	2	3	4
y	5	15	45	135	405

b

x	0	1	2	3	4
y	32	16	8	4	2

c

x	0	1	2	3	4
y	3	6	12	24	48

d

x	0	1	2	3	4
y	16	24	36	54	81

e

x	0	1	2	3	4
y	64	48	36	27	20.25

f

x	0	1	2	3	4
y	240	216	194	175	157

LINKS TO Example 7

6 Sketch the graph of $y = 6^x$, showing the y-intercept and the value of y when $x = 1$.

LINKS TO Example 8

7 Sketch the following pairs of graphs on the same set of axes, labelling the y-intercept and one other point on the each graph.

a $y = 3^x$ and $y = -3^x$

b $y = 3^x$ and $y = 3^{-x}$

LINKS TO Example 9

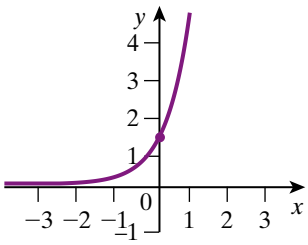
8 Sketch the following graphs, labelling the y-intercept and two other points on the graph. For parts **e** to **h**, also draw and label the asymptote.

a $y = 2^{x+4}$ **b** $y = 2^{x+1}$ **c** $y = 3^{x+3}$ **d** $y = 3^{x-1}$

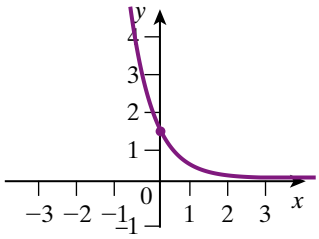
e $y = 2^x + 1$ **f** $y = 3^x - 3$ **g** $y = 2^x - 4$ **h** $y = 3^x + 5$

9 Match each of the graphs (**a** to **d**) with the correct rule below (A to D).

a



b



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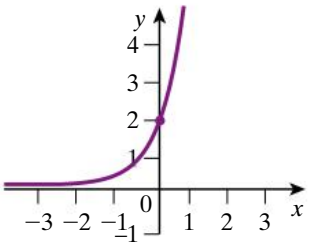
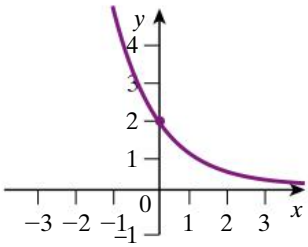
c



d

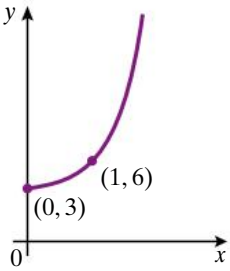


A $y = 1.5 \times 5^x$ **B** $y = 2 \times 6^x$ **C** $y = 1.5 \times 0.2^x$ **D** $y = 2 \times 0.6^x$

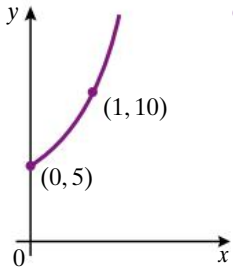
10 Match each of the graphs **a** to **d** with the correct rule A to D.

A $y = 3 \times \left(\frac{1}{2}\right)^x$ **B** $y = 3 \times 2^x$ **C** $y = 5 \times \left(\frac{1}{2}\right)^x$ **D** $y = 5 \times 2^x$

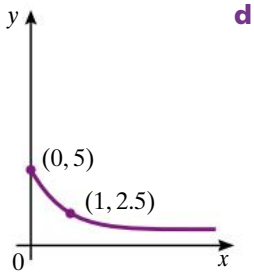
a



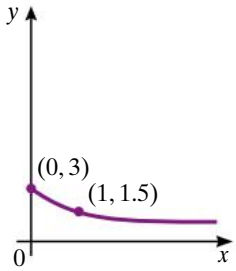
b



c

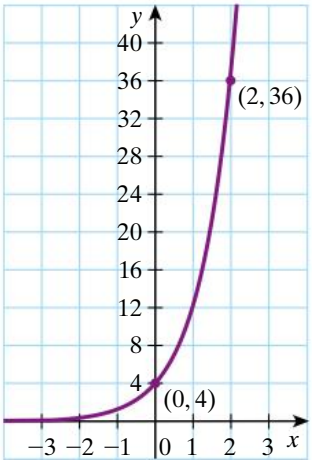


d



exercise 15.2 challenge

11 Find the rule for this exponential function.



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10 A

15.3 Applications of exponential functions

We saw in chapter 6 that compound interest and compound depreciation were exponential functions. Many changes in quantities over time can be modelled as exponential growth or decay. Exponential growth of a population usually slows down and levels off eventually, but in the initial stages of growth an exponential curve often fits the data well. Example 10 considers the growth of bacteria, which at first increase in number exponentially. Shortage of nutrients and accumulation of waste materials will eventually cause the bacteria population to decrease.

Example 10

A laboratory experiment starts with a culture of 50 bacteria. Each of the single-celled bacteria reproduces by dividing into two every hour.

a Construct a table of values to show the number of bacteria for the first three hours.

b Choose a suitable scale and plot the points.

c Write a rule for the population in terms of time.

d Use the rule to calculate the number of bacteria after 10 hours.

Working

a Let t be the time in hours and let b be the number of bacteria.

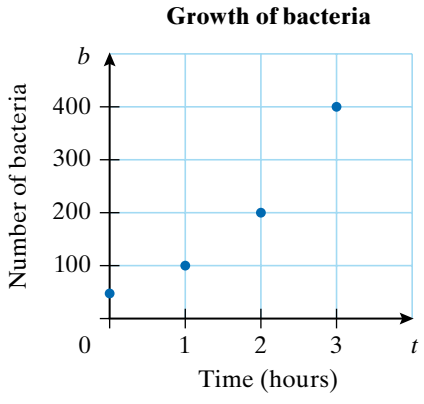
t	b
0	50
1	100
2	200
3	400

Reasoning

Every hour the number of bacteria doubles. Initially (when $t = 0$) there are 50 bacteria ($b = 50$).

We often use a vertical table of values to represent an exponential function, as the values can increase rapidly to become very large numbers.

b



The points show an increasing exponential function.

continued

Example 10 continued

Working	Reasoning
c The rule is b = Initial b-value × (Constant ratio)^t so $b = 50 \times 2^t$.	The initial or starting number of bacteria is 50. The number of cells doubles every hour so the constant ratio is 2.
d When $t = 10$, $b = 50 \times 2^{10}$ so $b = 51\,200$ There will be 51 200 bacteria after 10 hours.	To find the number of bacteria after a particular number of hours, substitute the number of hours into the rule. Substitute $t = 10$ into $b = 50 \times 2^t$.

Percentage change and exponential increase or decrease

Any quantity that increases by a constant percentage will display exponential growth.

In chapter 6 we saw that compound interest results in exponential growth of the principal invested.

If we compare the general exponential rule $y = k \times a^x$ with the compound interest formula, $A_n = P(1 + R)^n$, we see that k is the principal, P. In this case the constant ratio, a, is expressed as a percentage, $(1 + R)$ where R is the interest rate per annum. For example, if the rate of compound interest is 5% per annum, $a = 1.05$.

Example 11

$15 000 is invested at 4.8% compound interest p.a., compounded each year.

a What is the common ratio, a?

b Write a rule for the accumulated amount $\$A$ after n years.

c Calculate the accumulated amount after 10 years.

Working	Reasoning
a $a = 1.048$	$100\% + 4.8\% = 104.8\%$ $= 1.048$
b $A = 15\,000 \times 1.048^n$	General rule for an exponential function is $y = k \times a^x$ where k is the starting value, that is, when $x = 0$.
c When $n = 10$ $A = 15\,000 \times 1.048^{10}$ $= 23\,971.99$ After 10 years, the accumulated amount will be $23 971.99.	Substitute $n = 10$.

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Quantities that decrease by a constant percentage will decrease exponentially. Examples are radioactive decay and depreciation.

Example 12

A car bought for \$38 000 depreciates at the rate of 25% per annum.

a What is the common ratio, a?

b Write a rule for the value $\$V$ after n years.

c What is the value, to the nearest dollar, after 5 years?

Working	Reasoning
a $a = 0.75$	$100\% - 25\% = 75\%$ $= 0.75$
b $V = 38\,000 \times 0.75^n$	General rule for an exponential function is $y = k \times a^x$ where k is the starting value, that is, when $x = 0$.
c When $n = 5$, $V = 38\,000 \times 0.75^5$ $= 9017.58$ After 5 years, the depreciated value will be \$9018.	Substitute $n = 5$.

There are many other real world situations such as, population growth, where exponential functions based on percentage increase or decrease can be used to develop mathematical models.

Example 13

The population of a particular country was 37 million in 2005. The growth rate of the population is 2.3% per year. Let t be the number of years since 2005, and let P be the population in millions.

a What type of function links t and P? What piece of information tells us this?

b Construct a table of values showing the years up to and including 2010.

c Sketch a graph of the function.

d Find the rule for the function.

e Predict the population for 2025 if the present growth rate continues correct to the nearest million.

Working	Reasoning
a There is a constant percentage growth rate so the function is an exponential function.	A constant percentage rate of change is an indicator for an exponential function.

continued

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Example 13 continued

Working

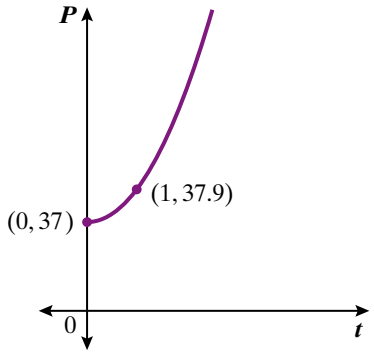
b

t	P
0	37
1	37.9
2	38.7
3	39.6
4	40.5
5	41.5

Reasoning

An increase of 2.3% means that in each successive year the population will be 102.3% of the previous year. This tells us that the constant ratio is 1.023. Multiply each P-value by 1.023 to obtain the next P-value.
When using a calculator to perform a series of calculations, keep the accurate (unrounded) result on the screen, but write the rounded result in the table.

c



The population is growing, so the sketch should show an increasing exponential function through two points.

d The rule is

$$P = (\text{initial } P\text{-value}) \times (\text{constant ratio})^t$$

so $P = 37 \times 1.023^t$

The initial or starting number of people is 37 million.
The number of people is multiplied by 1.023 every year.
Note: We could write $P(t) = 37 \times 1.023^t$

e When $t = 20$,

$$P = 37 \times 1.023^{20}$$
$$\approx 58.3$$

The predicted population in 2025 is 58 million.

To find the population after a particular number of years, substitute the number of years into the rule.
$t = 0$ in 2005 so $t = 20$ in 2025.
Substitute $t = 20$ into the rule. Give the population to the nearest million.
We could write $P(20) = 37 \times 1.023^{20}$.

When developing exponential models for data, the data values are unlikely to follow an exact exponential increase or decrease. We can estimate the common ratio a by dividing each data value by the previous data value and choosing a suitable 'average' value.

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Example 14

A painting bought for \$50 in 2005 increased in value each year as shown in the table.

a Plot a graph of V against n using the five pairs of values in the table.

b Find an approximate value for the common ratio.

c Write an exponential function rule to fit the data.

d Use the exponential model to calculate V for values of n from 0 to 4. Does your model fit the data well?

e Use your model to predict the value of the painting in

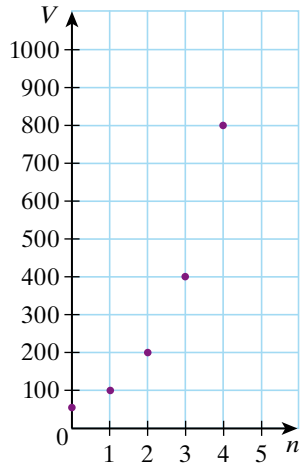
i 2011 **ii** 2020

f Comment on whether each of the predictions sounds reasonable. What does this mean for the model?

Year	Number of years (n) after 2005	Value in dollars (V)
2005	0	50
2006	1	95
2007	2	200
2008	3	410
2009	4	800

Working

a



Reasoning

b $\frac{95}{50} = 1.90$

$\frac{200}{95} \approx 2.11$

$\frac{410}{200} = 2.05$

$\frac{800}{410} \approx 1.95$

Calculate the common ratio by dividing each value of n by the previous value.

c The constant ratio is approximately 2.

A suitable value for the common ratio a is 2.

The value of V when $n = 0$ is 50.

$\therefore\ V = 50 \times 2^n$

The independent variable is n so replace x with n. The dependent variable is V so replace y with V. In the rule $y = k \times a^x$, k is the value of y when $x = 0$.

continued

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Example 14 continued

Working

d

n	V
0	50
1	100
2	200
3	400
4	800

The exponential model fits the data well.

Reasoning

Substitute each value of n into the rule $V = 50 \times 2^n$.

e i When $n = 6$,
$V = 50 \times 2^6$
$= 3200$
The model predicts that the painting would be worth \$3200 in 2011.

Substitute $n = 6$ in the rule $V = 50 \times 2^n$.

ii When $n = 15$,
$V = 50 \times 2^{15}$
$= 1\,638\,400$
The model predicts the painting would be worth \$1 638 400 in 2025.

Substitute $n = 15$ in the rule $V = 50 \times 2^n$.

f It is possible that the painting had increased in value to \$3200 by 2011, but the prediction for 2020 sounds unlikely. The model seems to fit the data for several years, but after that, the increase in value of the painting may slow down so the model $V = 50 \times 2^n$ is no longer useful.

We can often be confident using a model to make predictions a short distance ahead, but predictions further ahead may be quite unreliable.

exercise 15.3

LINKS TO Example 10

1 Alice was 1.5 m in height. When she visited Wonderland she ate a piece of magic cake. She suddenly grew to double her previous height. She had always wanted to be taller, so she ate several more pieces and her height doubled each time.

a Construct a table of values for Alice's height, H m, after eating n pieces of cake using values of n from 1 to 5.

b Sketch a graph showing the y-intercept and one other point.

c Write a rule for H in terms of n

d Calculate Alice's height after eating 10 pieces of cake.

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LINKS TO Example 11

2 $6000 is invested at 4.2% compound interest p.a., compounded each year.

a What is the common ratio, a?

b Write a rule for the accumulated amount $\$A$ after n years.

c Calculate the accumulated amount after 3 years.

LINKS TO Example 12

3 A printer bought for $990 depreciates at the rate of 18.75% per annum.

a What is the common ratio, a?

b Write a rule for the value $\$V$ after n years.

c What is the value to the nearest dollar after 4 years.

LINKS TO Example 13

4 The people of a town with population 30 000 were asked in a survey what they thought was a suitable yearly percentage increase in the town's population. Most people suggested a 10% yearly increase would be appropriate.

a If there is an annual increase of 10% what percentage will each year's population be of the previous population?

b Write this percentage as a decimal.

c Copy and complete the table to show the population each year for the next 8 years if the population grows at 10% per year.

Year	0	1	2	3	4	5	6	7	8
Population	30 000								

d Write a rule for the population P after n years.

e Use the rule to calculate the population after 20 years if this growth rate continued.

f Do you think this is what the people of the town intended?

5 In the 1920s, 18 koalas were introduced to Kangaroo Island in South Australia. It has been estimated that the population grew at the rate of about 9.6% per annum until it reached an unsustainable size by 2000 (Several thousand koalas have been moved from the island in the past decade). Estimate the number of koalas on Kangaroo Island in 2000.

6 In June 2010, Australia's population was approximately 22 400 000. Assuming that the rate of increase is 1.5% per annum, estimate Australia's population in 2050.

7 A Petri dish contains bacteria that increase in number at the rate of 70% per hour. If there were 25 bacteria at the start of the experiment, how many would there be after 24 hours, assuming they continued to increase at the same rate?

8 The population of kangaroos in a national park is increasing at the rate of 2.3% per annum. If the population has been estimated at 1200, predict how many kangaroos there will be in 10 years' time.

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LINKS TO Example 14

9 The table shows the increase in value of a rare postage stamp that was bought by a collector for $25 in 2000.

Year	Number of years since 2000 (n)	Value in dollars (V)
2000	0	25
2001	1	30
2002	2	36
2003	3	42
2004	4	48
2005	5	60
2006	6	78
2007	7	96
2008	8	125
2009	9	160

a Find an approximate value for the common ratio by dividing each value by the value in the previous year. Choose a suitable 'average' value.

b Write an exponential function rule to fit the data.

c Enter the data into a GeoGebra spreadsheet or a graphing or CAS calculator and construct a scatter plot.

d Enter your rule into the GeoGebra Input bar or into your calculator and graph it together with the scatter plot. Does your exponential function graph fit the data well?

e Use your function rule to predict the value of the stamp in 2020.

Population data Australia

10 This table summarises Australia's population in millions at the start of each decade from 1900 until 2010.

Year	Number, n, of years since 1900	Population, P (millions)
1900	0	3.8
1910	10	4.5
1920	20	5.4
1930	30	6.4
1940	40	7.3
1950	50	8.2
1960	60	10.3
1970	70	12.7
1980	80	14.7
1990	90	17.1
2000	100	19.2
2010	110	22.5

Based on Australian Bureau of Statistics data

a Using GeoGebra or a graphing or CAS calculator, enter the values of n and P into lists then construct a scatter plot.

b Starting with the 1900 population of 3.8 million, if the population increased by 2% every year, the rule for the exponential function would be $P = 3.8 \times 1.02^n$. On your scatter plot, add this function graph.

Notice that the graph of $P = 3.8 \times 1.02^n$ does not fit the points well. The points are below the curve, so the percentage population increase must be less than 2%. Adjust the base number 1.02 until you find a rule that does fit the data well. Use four decimal places of accuracy.

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c What is the annual percentage growth rate according to the exponential graph that fits the data?

d Use your exponential model to predict the population in 2010, that is, for $n = 110$. Compare this prediction with the actual 2010 population of 22.5 million.

e Use your model to predict Australia's population in 2050.

f What is the assumption on which your prediction is based?

Population data USA

11 The data shows the growth in population of the United States of America since 1815.

a Using GeoGebra or a graphing or CAS calculator, enter the values of n and P into lists then construct a scatter plot.

b Find an exponential model that fits the data.

c Use your model to predict the US population in 2010 and then compare your prediction with the actual population of the US in 2010.

Year	Number of years, n, after 1815	US population, P (millions)
1815	0	8.3
1825	10	11.0
1835	20	14.7
1845	30	19.7
1855	40	26.7
1865	50	35.2
1875	60	44.4
1885	70	55.9
1895	80	68.9
1905	90	83.2
1915	100	98.8
1925	110	114.2
1935	120	127.1
1945	130	140.1
1955	140	164.0
1965	150	190.9
1975	160	214.3

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Harry Potter

12 The table shows the US box office data (rounded to the nearest thousand US dollars) for the opening weekend and the following 11 weekends of *Harry Potter and the prisoner of Azkaban*.

Weeks	Box office takings
0	$93 687 000
1	$34 910 000
2	$18 027 000
3	$11 247 000
4	$5 938 000
5	$4 052 000
6	$2 758 000
7	$1 677 000
8	$989 000
9	$728 000
10	$451 000
11	$354 000

a Define pronumerals for the variables represented in the table. Assume that the box office takings are decreasing exponentially. Find a rule that links these variables. Hint: you may need to use an average of the ratios.

b Graph the original data and graph your rule on the same set of axes. Do they match well? If not, adjust the rule to get a better fit.

c Use the rule that you have found to predict the box office takings for weekend 10.

13 The data shows the US box office sales (rounded to the nearest thousand US dollars) for the opening weekend and the three following weekends for *Harry Potter and the Deathly Hallows: Part 2*.

Weeks	Box office takings
0	$226 117 000
1	$70 417 000
2	$34 107 000
3	$19 444 000

http://boxofficemojo.com/movies/?page=weekend&id=harrypotter3.htm

a Define pronumerals for the variables represented in the table. Assume that the box office takings are decreasing exponentially. Find a rule that links these variables. Hint: you may need to use an average of the ratios.

b Graph the original data and graph your rule on the same set of axes. Do they match well? If not, adjust the rule to get a better fit.

c Use the rule that you have found to predict the box office takings for weekend 10.

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exercise 15.3 challenge

14 When a hot liquid cools, it is known that there is a decreasing exponential relationship between the time for which the liquid has been cooling and the difference in temperature between the hot liquid and the room temperature. The following table shows the number of degrees (T°C) above room temperature for a cup of coffee m minutes after pouring.

m minutes	T°C
0	56.0
1	55.2
2	54.4
3	53.7
4	52.9
5	52.1
6	51.5
7	50.7
8	50.0
9	49.4
10	48.6
11	48.0
12	47.3
13	46.6
14	46.1
15	45.3

a Divide 55.2 by 56.0 to find the common ratio, rounded to three decimal places.. Repeat for several other pairs of temperatures.

b Using an appropriate value for the common ratio from part **a** (to three decimal places), write an exponential rule in terms of m and T for the cooling of the coffee.

c Use your rule to find the temperature after 15 minutes. How close is your predicted value to the actual value, 45.3°?

d Use your rule to predict the number of degrees above room temperature after 60 minutes.

e If the room temperature was 22°C, use your rule to calculate how long it would take the coffee to cool to 1° above room temperature, that is, $T = 1$. Hint: use trial and error with different values of m until you obtain a value for T very close to 1.

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15.4 Logarithms

Logarithms were invented by a Scottish mathematician, John Napier, in the early 17th century. At that time, there were, of course, no electronic calculators. Multiplication, division, and roots of numbers with five or more digits were lengthy and tedious tasks. Logarithms greatly simplified these operations. Logarithms are still widely used but not for the sort of calculations for which they were invented. As you will see in the following sections, logarithms are closely related to exponential functions and are important in analysing real-world exponential relationships and solving exponential equations.

The **logarithm** of a number to any positive base is the index when the number is expressed as a power of the base. We abbreviate 'logarithm' to **log**.

Logarithm form

$y = a^x$ can be expressed as $\log_a y = x$, where $a > 0$. This means 'the index of y when expressed to base a is x'.

For example, $16 = 2^4$ can be written as $\log_2 16 = 4$.
We read $\log_2 16 = 4$ as 'log to base 2 of 16'. This means 'the index of 16 when written to base 2 is 4'.

We say 'log to base a of y'.

Example 15

Write the following in index form.

a $\log_5 125 = 3$

b $\log_2 1024 = 10$

Working

a $\log_5 125 = 3$ is equivalent to $5^3 = 125$.

b $\log_2 1024 = 10$ is equivalent to $2^{10} = 1024$

Reasoning

5 is the base.
3 is the index or logarithm.
The index of 125 when written to base 5 is 3.

2 is the base.
10 is the index or logarithm.
The index of 1024 when written to base 2 is 10.

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Example 16

Write the following in logarithm form.

a $3^2 = 9$

b $10^3 = 1000$

Working

a $3^2 = 9$ is equivalent to $\log_3 9 = 2$

b $10^3 = 1000$ is equivalent to $\log_{10} 1000 = 3$.

Reasoning

3 is the base.
2 is the index or logarithm.

10 is the base.
3 is the index or logarithm.

Numbers written in index form with negative or fractional powers can also be expressed in logarithm form.

Example 17

Write the following in index form.

a $\log_2\left(\frac{1}{8}\right) = -3$

b $\log_{10} 0.0001 = -4$

Working

a $\log_2\left(\frac{1}{8}\right) = -3$ is equivalent to $2^{-3} = \frac{1}{8}$.

b $\log_{10} 0.0001 = -4$ is equivalent to $10^{-4} = 0.0001$

Reasoning

2 is the base.
-3 is the index or logarithm.
The index of $\frac{1}{8}$ when written to base 2 is -3.

10 is the base.
-4 is the index or logarithm.
The index of 0.0001 when written to base 10 is -4.

Example 18

Write the following in logarithm form.

a $\frac{1}{5} = 5^{-1}$

b $4 = 8^{\frac{2}{3}}$

Working

a $\frac{1}{5} = 5^{-1}$ is equivalent to $\log_5\left(\frac{1}{5}\right) = -1$.

b $4 = 8^{\frac{2}{3}}$ is equivalent to $\log_8 4 = \frac{2}{3}$.

Reasoning

5 is the base.
-1 is the index or logarithm.

8 is the base.
$\frac{2}{3}$ is the index or logarithm.

Each of the logarithm properties can be derived from the corresponding index property.

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$\log_a 1 = 0$ and $\log_a a = 1$

To demonstrate that $\log_a 1 = 0$, start with the related index property.

Proof:

$a^0 = 1$

Taking the logarithm to base a of both sides,

$\log_a a^0 = \log_a 1$

$0 = \log_a 1$

$\log_a 1$

$\log_a 1 = 0$

To demonstrate that $\log_a a = 1$, start with the related index property.

Proof:

$a^1 = a$

Taking the logarithm to base a of both sides,

$\log_a a^1 = \log_a a$

$1 = \log_a a$

log to base a of a

$\log_a a = 1$

$a^0 = 1$ and $\log_a 1 = 0$ are equivalent.

$a^1 = a$ and $\log_a a = 1$ are equivalent.

Example 19

Simplify each of the following.

a $\log_3 1$

b $\log_7 7$

Working	Reasoning
a $\log_3 1 = 0$	$\log_a 1 = 0$ $\log_3 1 = 0$ can be written in index form as $3^0 = 1$
b $\log_7 7 = 1$	$\log_a a = 1$ $\log_7 7 = 1$ can be written in index form as $7^1 = 7$

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Logarithm laws

We can obtain each of the laws of logarithms from the related index law.

log of a product

$\log_a(xy) = \log_a x + \log_a y$

Proof:

Let $p = \log_a x$ and $q = \log_a y$

$x = a^p$ and $y = a^q$

$xy = a^p \times a^q$

$xy = a^{p+q}$ (1st index law)

Writing in logarithm form,

$\log_a(xy) = p + q$

$\log_a(xy) = \log_a x + \log_a y$

Note that we can only apply this product law if the bases are the same. This is equivalent to being able to add indices only if the bases are the same.

Example 20

Simplify each of the following.

a $\log_5 3 + \log_5 4$

b $\log_2 15 + \log_2 3$

Working	Reasoning
a $\log_5 3 + \log_5 4$ $= \log_5(3 \times 4)$ $= \log_5 12$	$\log_a(xy) = \log_a x + \log_a y$
b $\log_2 15 + \log_2 3$ $= \log_2(15 \times 3)$ $= \log_2 45$	$\log_a(xy) = \log_a x + \log_a y$

log of a quotient

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Proof:

Let $p = \log_a x$ and $q = \log_a y$

$x = a^p$ and $y = a^q$

$\frac{x}{y} = \frac{a^p}{a^q}$

$\frac{x}{y} = a^{p-q}$ (2nd index law)

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Writing in logarithm form,

$$\log_a\left(\frac{x}{y}\right) = p - q$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

As previously, note that we can only apply this product law if the bases are the same. This is equivalent to being able to subtract indices only if the bases are the same.

Example 21

Simplify each of the following.

a $\log_2 35 - \log_2 7$

b $\log_3 24 - \log_3 8$

Working	Reasoning
a $\log_2 35 - \log_2 7$	
$= \log_2\left(\frac{35}{7}\right)$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$= \log_2 5$	
b $\log_3 24 - \log_3 8$	
$= \log_3\left(\frac{24}{8}\right)$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$= \log_3 3$	$\log_a a = 1$
$= 1$	

log of a power

$\log_a x^n = n\log_a x$

We read $\log_a x^n$ as 'log to base a of x to the power n'.

Proof:

Let $\log_a x = p$, so $x = a^p$,

$x^n = (a^p)^n$

$x^n = a^{pn}$ (power of a power index law)

In logarithm form:

$\log_a x^n = pn$

$\log_a x^n = n\log_a x$

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Example 22

Simplify each of the following.

a $\log_2 3^5$

b $\log_2 2^{-5}$

Working

a $\log_2 3^5 = 5\log_2 3$

b $\log_2 2^{-5} = -5\log_2 2$
$= -5 \times 1$
$= -5$

Reasoning

$\log_a x^n = n\log_a x$

$\log_a a^n = n\log_a a = n$

We can use our understanding of numbers in index form to evaluate logarithmic expressions. For example, we can evaluate $\log_2 64$ by expressing 64 in index form to base 2. We then apply the logarithm law, $\log_a a^n = n$, to simplify the expression.

$\log_2 64 = \log_2 2^6$
$= 6\log_2 2$
$= 6$

Summary of properties of indices and logarithms

Indices	Logarithms
$y = a^x$	$\log_a y = x$
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^x \times a^y = a^{x+y}$	$\log_a(xy) = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^n = a^{x \times n}$	$\log_a x^n = n\log_a x$

Example 23

Evaluate each of the following.

a $\log_3 9$

b $\log_9 3$

c $\log_5\left(\frac{1}{25}\right)$

d $\log_8 2$

Working

a $\log_3 9 = \log_3 3^2$
$= 2$

Reasoning

$\log_a a^n = n$

It is helpful to think

$3^? = 9$

$< 9^? = 3$

continued

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Example 23 continued

Working	Reasoning
b $\log_9 3 = \log_9\sqrt{9}$ $= \log_9 9^{0.5}$ $= 0.5\log_9 9$ $= 0.5$	Express 3 in index form with base 9. Use the logarithm law: $\log_a a^n = n$
c $\log_5\left(\frac{1}{25}\right) = \log_5 5^{-2}$ $= -2\log_5 5$ $= -2$	Express $\frac{1}{25}$ in index form with base 5. Use the logarithm law: $\log_a a^n = n\log_a a = n$ It is helpful to think $5^? = \frac{1}{25}$ $\left(\frac{1}{25}\right)^? = 5$
d $\log_8 2 = \log_8 8^{\frac{1}{3}}$ $= \frac{1}{3}$	Express 2 in index form with base 8. Use the logarithm law: $\log_a a^n = n$ It is helpful to think $8^? = 2$ $2^? = 8$

Example 24

Evaluate each of the following.

a $2\log_{10}5 + \log_{10}4$ **b** $\log_5 12.5 + \log_5 10$ **c** $\log_2 80 - \log_2 5$ **d** $\log_3 6 - \log_3 54$

Working	Reasoning
a $2\log_{10}5 + \log_{10}4 = \log_{10}5^2 + \log_{10}4$ $= \log_{10}(25 \times 4)$ $= \log_{10}100$ $= \log_{10}10^2$ $= 2$	Express $2\log_{10}5$ as $\log_{10}5^2$. Both expressions have the same base so simplify using $\log_a(xy) = \log_a x + \log_a y$. Express 100 in index form with base 10. Simplify.
b $\log_5 12.5 + \log_5 10 = \log_5(12.5 \times 10)$ $= \log_5 125$ $= \log_5 5^3$ $= 3$	Both expressions have the same base so simplify using $\log_a(xy) = \log_a x + \log_a y$. Express 125 in index form with base 5. Simplify.
c $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ $= \log_2 16$ $= \log_2 2^4$ $= 4$	Both expressions have the same base so simplify using $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$. Express 16 in index form with base 2. Simplify.

continued

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Example 24 continued

Working	Reasoning
d $\log_3 6 - \log_3 54 = \log_3\left(\frac{6}{54}\right)$	Both expressions have the same base so simplify using $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
$= \log_3\left(\frac{1}{9}\right)$	Express $\frac{1}{9}$ in index form with base 3.
$= \log_3\left(\frac{1}{3^2}\right)$	
$= \log_3 3^{-2}$	
$= -2$	Simplify.

exercise 15.4

LINKS TO Example 15

1 Write the following in index form.

a $\log_2 64 = 6$ **b** $\log_7 49 = 2$ **c** $\log_{10} 10\,000 = 4$
d $\log_5 125 = 3$ **e** $\log_3 27 = 3$ **f** $\log_3 243 = 5$
g $\log_7 343 = 3$ **h** $\log_6 216 = 3$ **i** $\log_{10} 1\,000\,000 = 6$

LINKS TO Example 16

2 Write the following in logarithm form.

a $5^3 = 125$ **b** $10^5 = 100\,000$ **c** $4^2 = 16$
d $2^4 = 16$ **e** $7^4 = 2401$ **f** $4^3 = 64$
g $49^{0.5} = 7$ **h** $3^3 = 27$ **i** $5^1 = 5$

LINKS TO Example 17

3 Write the following in index form.

a $\log_{10} 0.0001 = -4$ **b** $\log_2 \frac{1}{2} = -1$ **c** $\log_2 \frac{1}{4} = -2$
d $\log_3\left(\frac{1}{81}\right) = -4$ **e** $\log_{10} 0.0000001 = -7$ **f** $\log_6\left(\frac{1}{36}\right) = -2$
g $\log_2\left(\frac{1}{32}\right) = -5$ **h** $\log_2\left(\frac{1}{16}\right) = -4$ **i** $\log_2 0.25 = -2$
j $\log_7\left(\frac{1}{7}\right) = -1$ **k** $\log_5\left(\frac{1}{5}\right) = -1$ **l** $\log_{10}\left(\frac{1}{1000}\right) = -3$
m $\log_{10} 0.01 = -2$ **n** $\log_5\left(\frac{1}{25}\right) = -2$ **o** $\log_3\left(\frac{1}{27}\right) = -3$

LINKS TO Example 18

4 Write the following in logarithm form.

a $5^{-4} = \frac{1}{625}$ **b** $9^{-2} = \frac{1}{81}$ **c** $2^{-4} = \frac{1}{16}$ **d** $10^{-3} = 0.001$ **e** $7^{-3} = \frac{1}{343}$
f $3^{-6} = \frac{1}{729}$ **g** $2^{-6} = \frac{1}{64}$ **h** $4^{-3} = \frac{1}{64}$ **i** $8^{-2} = \frac{1}{64}$ **j** $10^{-5} = 0.00001$
k $6^{-2} = \frac{1}{36}$ **l** $25^{-\frac{1}{2}} = \frac{1}{5}$ **m** $10^{-2} = 0.01$ **n** $2^{-3} = \frac{1}{8}$ **o** $3^{-3} = \frac{1}{27}$

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LINKS TO Example 19

5 Simplify each of the following.

a $\log_5 1$ **b** $\log_{10} 1$ **c** $\log_2 1$
d $\log_7 7$ **e** $\log_{10} 10$ **f** $\log_4 4$

LINKS TO Example 20

6 Simplify each of the following.

a $\log_3 2 + \log_3 4$ **b** $\log_2 9 + \log_2 8$ **c** $\log_5 3 + \log_5 10$
d $\log_7 3 + \log_7 5$ **e** $\log_3 9 + \log_3 5$ **f** $\log_{10} 7 + \log_{10} 6$
g $\log_2 2.5 + \log_2 3$ **h** $\log_3 1.6 + \log_3 5$ **i** $\log_3 \frac{1}{2} + \log_3 8$
j $\log_5 4 + \log_5 7$ **k** $\log_2 \frac{1}{3} + \log_2 9$ **l** $\log_5 6 + \log_5 3.4$

LINKS TO Example 21

7 Simplify each of the following.

a $\log_2 48 - \log_2 6$ **b** $\log_5 60 - \log_5 4$ **c** $\log_2 35 - \log_2 7$
d $\log_3 36 - \log_3 6$ **e** $\log_4 42 - \log_4 6$ **f** $\log_5 72 - \log_5 18$
g $\log_2 12 + \log_2 5 - \log_2 6$ **h** $\log_3 8 + \log_3 2 - \log_3 4$ **i** $\log_3 21 + \log_3 2 - \log_3 7$
j $\log_{10} 27 + \log_{10} 4 - \log_{10} 18$ **k** $\log_2 24 + \log_2 5 - \log_2 40$ **l** $\log_5 160 - \log_5 4 - \log_5 8$

LINKS TO Example 22

8 Simplify each of the following.

a $\log_3 2^3$ **b** $\log_4 3^8$ **c** $\log_{10} 4^5$
d $\log_3 5^4$ **e** $\log_3 2^{-3}$ **f** $\log_{10} 4^{-1}$
g $\log_3 3^5$ **h** $\log_2 2^{-5}$ **i** $\log_{10} 10^5$
j $\log_{10} 10^{-1.6}$ **k** $\log_5 5^{2.4}$ **l** $\log_4 4^{0.5}$

LINKS TO Example 23

9 Evaluate each of the following.

a $\log_2 32$ **b** $\log_3 27$ **c** $\log_2\left(\frac{1}{8}\right)$
d $\log_9 3$ **e** $\log_7 343$ **f** $\log_3 81$
g $\log_5\left(\frac{1}{5}\right)$ **h** $\log_7\left(\frac{1}{49}\right)$ **i** $\log_3 \sqrt{3}$
j $\log_5 \sqrt{5}$ **k** $\log_3 \sqrt[4]{3}$ **l** $\log_{10} \sqrt{1000}$

LINKS TO Example 24

10 Evaluate each of the following.

a $3\log_{10} 2 + 3\log_{10} 5$ **b** $\log_3 1.5 + \log_3 18$ **c** $\log_2 12.8 + \log_2 2.5$
d $\log_3 5.4 + \log_3 5$ **e** $\log_{10} 50 + \log_{10} 20$ **f** $\log_5 12.5 - \log_5 0.5$
g $\log_2 5 - \log_2 80$ **h** $\log_3 4 - \log_3 18 - \log_3 6$ **i** $2\log_{10} 5 + \log_{10} 0.4$
j $3\log_2 4 - 2\log_2 8$ **k** $-\log_{10} 16 - 2\log_{10} 25$ **l** $\log_3 7.2 - \log_3 8 + \log_3 10$

exercise 15.4 challenge

11 Simplify

a $\log_5\left(\frac{1}{5^{-2}\sqrt{5}}\right)^3$ **b** $\log_2 32^{\frac{2}{3}} + \log_2 8\sqrt{2}$

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15.5 Exponential and logarithmic equations

Exponential equations

If the unknown in an equation occurs as an index, we call the equation an **exponential equation**. If both sides of the equation can be expressed in index form with the same base, then we can solve the equation by equating the indices. For example, if $2^x = 32$, we first express 32 in index form as 2^5. It can then be seen that $x = 5$.

Example 25

Solve the following equations.

a $5^x = 125$ **b** $2^{-x} = 2\sqrt{2}$ **c** $16^x = 64$ **d** $3^{2x-1} = 27$

Working	Reasoning
a $5^x = 125$ $5^x = 5^3$ $x = 3$	Express 125 in index form with base 5 so that the expressions on both sides have the same base. Equate the indices.
b $2^{-x} = 2\sqrt{2}$ $2^{-x} = 2^1 \times 2^{0.5}$ $2^{-x} = 2^{1.5}$ $x = -1.5$	Express $2\sqrt{2}$ in index form with base 2 so that the expressions on both sides have the same base. Equate the indices.
c $16^x = 64$ $(2^4)^x = 2^6$ $2^{4x} = 2^6$ $4x = 6$ $x = 1.5$	Express both sides with the same base. Equate the indices. Note that we could also use base 4. $16^x = 64$ $(4^2)^x = 4^3$ $4^{2x} = 4^3$ $2x = 3$ $x = 1.5$
d $3^{2x-1} = 27$ $3^{2x-1} = 3^3$ $2x - 1 = 3$ $2x = 4$ $x = 2$	Express 27 in index form with base 3 so that the expressions on both sides have the same base. Equate the indices. Solve the linear equation.

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Using guess, check and improve to solve exponential equations

We saw that the exponential equation $2^x = 32$ could be solved by expressing 32 in index form with base 2, and then equating the indices on each side of the equation. Consider now the equation $2^x = 20$. We cannot express 20 as a power of 2 with a rational index so we must find an alternative means of solving this equation. One method is to use guess, check and improve with the aid of a calculator. We know that $2^4 = 16$ and $2^5 = 32$. This tells us that x must lie between 4 and 5. Using a calculator we find that $2^{4.5} \approx 22.6$, $2^{4.4} \approx 21.1$ and $2^{4.3} \approx 19.7$. So we now know that x must lie between 4.3 and 4.4. We could continue in this way to find x to a required number of decimal places.

Example 26

Use guess, check and improve to solve the following equations, correct to three decimal places.

a $3^x = 45$ **b** $5^x = 70$

Working	Reasoning
a $3^x = 45$ $3^3 = 27$ $3^4 = 81$ x is between 3 and 4. $3^{3.5} \approx 46.8$ $3^{3.4} \approx 41.9$ $3^{3.48} \approx 45.7$ $3^{3.47} \approx 45.2$ $3^{3.46} \approx 44.8$ $3^{3.465} \approx 45.001$ $3^{3.4649} \approx 44.996$ $x = 3.465$ correct to three decimal places.	Find the two powers between which 3^x lies. Repeat the process until the required level of accuracy is reached.
b $5^x = 70$ $5^2 = 25$ $5^3 = 125$ x is between 2 and 3. $5^{2.5} \approx 55.9$ $5^{2.6} \approx 65.7$ $5^{2.7} \approx 77.1$ $5^{2.65} \approx 71.2$ $5^{2.64} \approx 70.03$ $5^{2.639} \approx 69.92$ $5^{2.6398} \approx 70.006$ $x = 2.640$ correct to three decimal places.	Find the two powers between which 5^x lies. Repeat the process until the required level of accuracy is reached.

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Solving logarithmic equations

The method used to solve a logarithmic equation depends on the position of the unknown in the equation, for example, the unknown may be the base or it may be the logarithm.

Example 27

Solve the following equations.

a $\log_5 x = -4$

b $\log_3(x - 2) = 4$

Working	Reasoning
a $\log_5 x = -4$	
$x = 5^{-4}$	Express in index form.
$x = \frac{1}{5^4}$	Express 5^{-4} with a positive index.
$x = \frac{1}{625}$	Evaluate 5^4.
b $\log_3(x - 2) = 4$	Express in index form.
$x - 2 = 3^4$	Evaluate 3^4.
$x - 2 = 81$	
$x = 83$	Solve the linear equation.

Example 28

Solve the following equations, where $x > 0$.

a $\log_x 81 = 4$

b $\log_x 3 = \frac{1}{2}$

Working	Reasoning
a $\log_x 81 = 4$	Express $\log_x 81 = 4$ in index form.
$x^4 = 81$	
$x = 81^{\frac{1}{4}}$	Take the fourth root of both sides.
$x = (3^4)^{\frac{1}{4}}$	Express 81 as 3^4 then use the index law: $(a^m)^n = a^{mn}$
$x = 3$	
b $\log_x 3 = \frac{1}{2}$	Express $\log_x 3 = \frac{1}{2}$ in index form.
$x^{\frac{1}{2}} = 3$	Square both sides.
$\left(x^{\frac{1}{2}}\right)^2 = 3^2$	
$x = 9$	

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Exponential equations, where the unknown is the exponent, can be solved by taking the log to base 10 of both sides. A calculator can then be used to find a rational approximation for the unknown. For example, the equation $2^x = 20$ can be solved as shown:

$$2^x = 20$$
$$\log_{10}2^x = \log_{10}20$$
$$x\log_{10}2 = \log_{10}20$$
$$x = \frac{\log_{10}20}{\log_{10}2}$$
$$\approx 4.32$$

This method is used in example **29** to solve the two equations that were solved by guess, check and improve in example **26**.

Example 29

Solve the following equations, correct to three decimal places.

a $3^x = 45$ **b** $5^x = 70$

Working	Reasoning
a $3^x = 45$	
$\log_{10}3^x = \log_{10}45$	Take the log to base 10 of both sides.
$x\log_{10}3 = \log_{10}45$	$\log_a x^n = n\log_a x$
$x = \dfrac{\log_{10}45}{\log_{10}3}$	Divide both sides by $\log_{10}3$.
$\approx 3.4649\ldots$	Use a calculator to evaluate $\dfrac{\log_{10}45}{\log_{10}3}$.
$x = 3.465$ correct to three decimal places.	
b $5^x = 70$	
$\log_{10}5^x = \log_{10}70$	
$x\log_{10}5 = \log_{10}70$	$\log_a x^n = n\log_a x$
$x = \dfrac{\log_{10}70}{\log_{10}5}$	Divide both sides by $\log_{10}5$.
$\approx 2.6397\ldots$	Use a calculator to evaluate $\dfrac{\log_{10}70}{\log_{10}5}$.
$x = 2.640$ correct to three decimal places.	

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10 A

exercise 15.5

LINKS TO Example 25

1 Solve the following exponential equations.

a $5^x = 125$ **b** $3^x = 81$ **c** $2^x = 512$ **d** $7^x = \frac{1}{49}$

e $3^x = 3\sqrt{3}$ **f** $2^{-x} = \frac{1}{32}$ **g** $2^{-x} = 2\sqrt{2}$ **h** $10^{1-x} = 0.0001$.

i $3^{x+1} = 243$ **j** $8^{x-2} = \sqrt{2}$ **k** $16^{2x+3} = 64$ **l** $3^{2x-1} = 27$

LINKS TO Example 26

2 Use guess, check and improve to solve the following exponential equations, correct to three decimal places.

a $2^x = 10$ **b** $3^x = 20$ **c** $10^x = 35$ **d** $5^x = 40$

e $7^x = 60$ **f** $2^x = 84$ **g** $1.5^x = 6$ **h** $2.4^x = 8$

LINKS TO Example 27

3 Solve the following logarithmic equations.

a $\log_4 x = 2$ **b** $\log_5 x = 3$ **c** $\log_2 x = 7$ **d** $\log_{10} x = -2$

e $\log_2 x = -4$ **f** $\log_7 x = -1$ **g** $\log_3 x = 0$ **h** $\log_4 x = -0.5$

i $\log_9 x = \frac{3}{2}$ **j** $\log_5(x - 3) = 2$ **k** $\log_2(x - 3) = 5$ **l** $\log_3(2x + 1) = 3$

LINKS TO Example 28

4 Solve the following logarithmic equations.

a $\log_x 64 = 3$ **b** $\log_x 5 = 1$ **c** $\log_x 3 = 0.5$ **d** $\log_x \frac{1}{8} = -3$

e $\log_x \frac{1}{25} = -2$ **f** $\log_x 729 = 6$ **g** $\log_x 81 = 2$ **h** $\log_x 9 = 0.5$

i $\log_x 125 = 3$ **j** $\log_x 121 = 2$ **k** $\log_x 64 = 2$ **l** $\log_x 64 = 6$

LINKS TO Example 29

5 Solve each of the equations in question **2** by taking the log to base 10 of both sides.

exercise 15.5 challenge

6 Solve the exponential equation $2^{1+\log_3 x} = 1$.

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15.6 Logarithmic graphs

Logarithmic graphs

We have seen the relationship between exponential and logarithmic expressions so we would expect there to be a relationship between the graph of a logarithmic function and the related exponential function. Consider, for example, the graphs of the logarithmic function $y = \log_2 x$ and the exponential function $y = 2^x$.

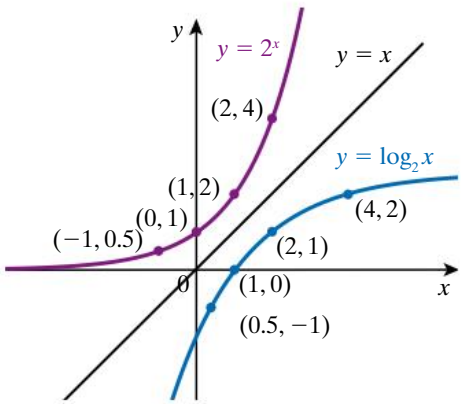
If the graph of $y = x$ is included, we see that the graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$.

$y = 2^x$	$y = \log_2 x$
x-axis is an asymptote	x-axis is an asymptote
y-intercept is at $(0, 1)$	x-intercept is at $(1, 0)$
The graph becomes steeper and steeper as x increases.	The graph becomes less and less steep as x increases.

For each of the labelled points on the graph of $y = 2^x$, there is a corresponding point on the graph of $y = \log_2 x$ with the coordinates reversed.

To understand the relationship between the graphs of $y = 2^x$ and $y = \log_2 x$, we express $y = \log_2 x$ in index form.

From the definition of logarithms, $y = \log_2 x$ means that $x = 2^y$. Note that this is of the same form as $y = 2^x$ but with x and y in reversed positions. We say that $y = 2^x$ and $x = 2^y$ (and therefore $y = \log_2 x$) are **inverse functions**. This explains why the coordinates of each point on the graph of $y = 2^x$ are reversed on the graph of $y = \log_2 x$ and why the graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$.



Example 30

Consider the functions $y = \log_3 x$ and $y = 3^x$.

a Sketch the graphs on the same set of axes, labelling any intercepts and showing the coordinates of two other points on each graph (it is helpful to include the graph of $y = x$).

b Compare the intercepts, asymptotes and coordinates of the labelled points for the two graphs.

continued

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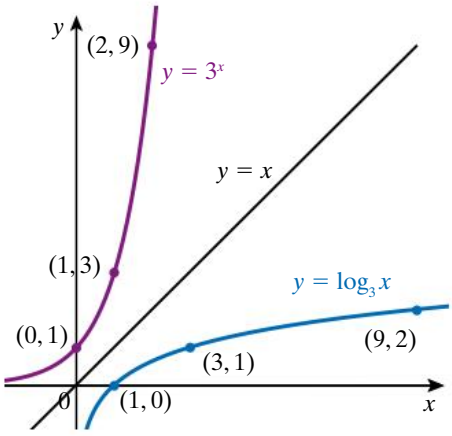
Example 30 continued

Working

a



Reasoning

For $y = 3^x$,
When $x = 0$, $y = 3^0 = 1$ so the y-intercept is at $(0, 1)$.
When $x = 1$, $y = 3^1 = 3$
When $x = 2$, $y = 3^2 = 9$
So two points on the graph are $(1, 3)$ and $(2, 9)$.
For $y = \log_3 x$,
The x-intercept is at $(1, 0)$.
When $x = 3$, $y = \log_3 3 = 1$
When $x = 9$,
$y = \log_3 9$
$= \log_3(3^2)$
$= 2$
The two points on the graph are $(3, 1)$ and $(9, 2)$.

b For $y = 3^x$, the x-axis is an asymptote.
For $y = \log_3 x$, the y-axis is an asymptote.
For $y = 3^x$, the y-intercept is $(0, 1)$.
For $y = \log_3 x$, the x-intercept is $(1, 0)$.
The points $(1, 3)$ and $(2, 9)$ are on the graph of $y = 3^x$. This means that the points $(3, 1)$ and $(9, 2)$ are on the graph of $y = \log_3 x$.

$y = \log_3 x$ can be expressed in index form as $x = 3^y$ so $y = \log_3 x$ is the inverse function of $y = 3^x$.
The coordinates of each point on the graph of $y = 3^x$ are reversed on the graph of $y = \log_3 x$. The graph of $y = \log_3 x$ is the reflection of the graph of $y = 3^x$ in the line $y = x$.

exercise 15.6

LINKS TO Example 30

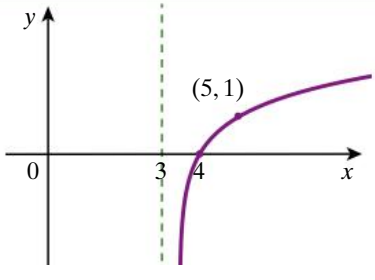
1 Sketch each of the following graphs on the same set of axes, labelling any intercepts and showing the following coordinates of the two other points on each graph (it is helpful to include the graph of $y = x$).

a $y = 4^x$ and $y = \log_4 x$

b $y = 10^x$ and $y = \log_{10} x$

exercise 15.6 challenge

2 Find the rule for the logarithmic function shown in this graph.



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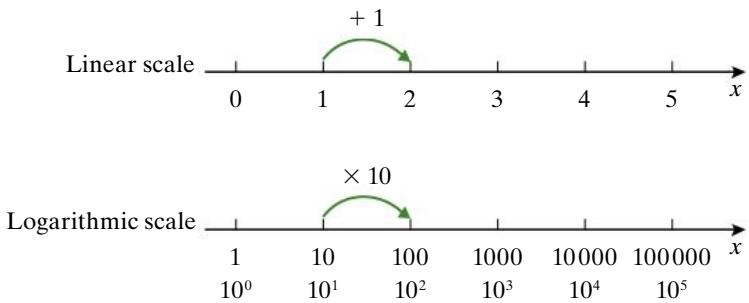
15.7 Logarithmic scales

When data covers a large range of values, it is always difficult displaying very small and very large values on a graph. For this reason a logarithmic scale can be helpful as the logarithms of the data values will have a smaller range.

A linear scale, such as a ruler used to measure distances, increases by addition. A logarithmic scale increases in steps by multiplication. The diagram illustrates this.



Several common measurements are expressed as logarithms. Three of these logarithmic scales are described below.

pH

The pH scale is a logarithmic scale used to measure how acidic or alkaline a solution is. You may work with this in science this year or in chemistry in later years. The acidity or alkalinity of a solution depends on the concentration of hydrogen ions in the solution.

The hydrogen ion concentration is written as $[H^+]$ and is measured in moles/litre.

$$pH = -\log_{10}[H^+]$$

If the solution is neutral (neither acidic or alkaline), the concentration of hydrogen ions in the solution is 10^{-7} moles/litre.

$$\begin{aligned} pH &= -\log_{10}10^{-7} \\ &= -(-7) \\ &= 7 \end{aligned}$$

So the pH of a neutral solution is 7.

Acidic solutions have pH less than 7. This means that they have hydrogen ion concentrations greater than 10^{-7} moles per litre; for example, 10^{-6} moles per lire or 10^{-3} moles per litre. Alkaline solutions have pH greater than 7, up to 14. Each one unit decrease in pH means that the acidity is increasing by 10 times.

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Example 31

What is the pH correct to one decimal place of a solution that has a hydrogen ion concentration of 2.6×10^{-7} moles per litre? Is the solution acidic or alkaline?

Working

$\text{pH} = -\log_{10}(\text{hydrogen ion concentration})$
$\text{pH} = -\log_{10}(2.6 \times 10^{-7})$
$= 6.58\ldots$
The pH is approximately 6.6.
The solution is acidic.

Reasoning

Substitute 2.6×10^{-7} for the hydrogen ion concentration.
Use a calculator to evaluate $-\log_{10}(2.6 \times 10^{-7})$
The pH is less than 7.

Decibels

Sound intensity is a measure of sound energy passing through a given area per second. The level of sound, B decibels, is related to the sound wave intensity, I watts/m^2 by the equation $B = 10\log_{10}\dfrac{I}{I_0}$ where I_0 watts/m^2 $= 10^{-12}$ watts/m^2 and is the threshold of hearing.

Example 32

For a jet aeroplane on take off, $I = 10$.

a Use the equation $B = 10\log_{10}\left(\dfrac{I}{I_0}\right)$ to calculate the number of decibels.

b For busy traffic, $B = 70$. How many times louder is a jet on take off than busy traffic?

Working

a $B = 10\log_{10}\left(\dfrac{I}{I_0}\right)$

$= 10\log_{10}\left(\dfrac{10^1}{10^{-12}}\right)$

$= 10\log_{10}10^{13}$

$= 10 \times 13$

$B = 130$

A jet aeroplane taking off has a sound level of 130 dB.

b $10\log_{10}\dfrac{I_{\text{jet}}}{I_{\text{traffic}}} = B_{\text{jet}} - B_{\text{traffic}}$

$\log_{10}\dfrac{I_{\text{jet}}}{I_{\text{traffic}}} = \dfrac{130 - 70}{10}$

$= 6$

$\dfrac{I_{\text{jet}}}{I_{\text{traffic}}} = 10^6$

The jet aeroplane on take off is 10^6 times louder than busy traffic.

Reasoning

$I_0 = 10^{-12}$

Subtract the decibel values and divide by 10.
This gives the power of 10.

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Richter scale of earthquake magnitude

The huge range in the strength of earthquakes means that it is very difficult to express the magnitude of an earthquake on a linear scale. For this reason seismologists use a logarithmic scale to measure the amount of seismic energy released by an earthquake. The Richter scale is a logarithmic scale used to compare the magnitudes of earthquakes. The magnitude, R, of an earthquake of intensity I is given by

$R = \log_{10} I$ where R is a measure of the seismic wave energy per unit area.

If an earthquake measures 5 on the Richter scale, this means that the magnitude is 10^5 times the magnitude of an earthquake measuring 10^0 or 1.

The earthquake in Japan on 11 March 2011 that caused a tsunami and killed over 22 000 people measured 9.0 on the Richter scale. The Indian ocean earthquake on 26 December 2004 measured between 9.1 and 9.3 on the Richter scale, causing a massive tsunami that killed over 300 000 people.

The following table shows the potential damage caused by earthquakes of different magnitudes on the Richter scale.

Richter scale measure	Effects
Less than 3.5	Usually not felt, but recorded by seismographs.
3.5–5.4	Often felt, but rarely causes damage.
5.5–6.0	May cause only slight damage to well-designed buildings but, over small areas, can cause major damage to poorly-constructed buildings.
6.1–6.9	Can cause considerable damage and loss of life over areas up to 100 kilometers across.
7.0–7.9	A major earthquake that can cause serious damage over large areas.
8 or greater	A great earthquake that can cause serious damage in areas several hundred kilometers across.

Example 33

How much stronger is an earthquake that measures 7 on the Richter scale than an earthquake that measures 5 on the Richter scale?

Working	Reasoning
When $R = 7$,	Substitute $R = 7$ into $R = \log_{10} I$.
$7 = \log_{10} I$	
$I = 10^7$	
When $R = 5$,	
$5 = \log_{10} I$	Substitute $R = 5$ into $R = \log_{10} I$.
$I = 10^5$	

continued

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Example 33 continued

Working

$$\frac{10^7}{10^5} = 10^2$$
$$= 100$$

An earthquake that measures 7 on the Richter scale is 100 times stronger than an earthquake that measures 5 on the Richter scale.

Reasoning

Divide 10^7 by 10^5 to compare the intensities by subtracting the indices. Evaluate 10^2.

Example 34

How much stronger is an earthquake that measures 7.2 on the Richter scale than an earthquake that measures 5.4 on the Richter scale?

Working

When $R = 7.2$,

$7.2 = \log_{10} I$

$I = 10^{7.2}$

When $R = 5.4$,

$5.4 = \log_{10} I$

$I = 10^{5.4}$

$$\frac{10^{7.2}}{10^{5.4}} = 10^{1.8}$$
$$\approx 63$$

An earthquake that measures 7.2 on the Richter scale is approximately 63 times stronger than an earthquake that measures 5.4 on the Richter scale.

Reasoning

Substitute $R = 7.2$ into $R = \log_{10} I$

Substitute $R = 5.4$ into $R = \log_{10} I$

Divide $10^{7.2}$ by $10^{5.4}$ by subtracting the indices.
Use a calculator to evaluate $10^{1.8}$.

exercise 15.7

LINKS TO Example 31

1 Find the pH of solutions with the following hydrogen ion concentrations (measured in moles per litre) and state whether the solution is acidic, alkaline or neutral.

a 10^{-3} **b** 10^{-9} **c** 0.0000001 **d** 0.00001
e 10^{-13} **f** 10^{-2} **g** 0.0000001 **h** 10^{-14}

2 Find the pH of solutions with the following hydrogen ion concentrations measured in moles per litre.

a 0.00003 **b** 0.0005 **c** 1.4×10^{-8} **d** 5.4×10^{-11}
e 0.00000007 **f** 4×10^{-6} **g** 1.8×10^{-9} **h** 2.5×10^{-4}

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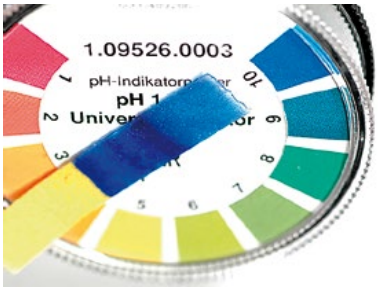
3 A dilute solution of hydrochloric acid had a pH of 4. Express its hydrogen ion concentration in index form.

4 The pH indicator paper shown here has been dipped into a solution.

a What was the pH of the solution?

b Write this in index form.



LINKS TO Example 32

5 The following table shows the intensities for some common sounds.

a Use the equation

$$B = 10 \log_{10} \frac{I}{I_0}$$

where B decibels is the sound level, I W/m^2 is the sound intensity and I_0 W/m^2 is the threshold of hearing to calculate the sound level in decibels for each of the sound wave intensities.

	Sound wave intensity (watts/m^2)	Sound level (decibels)
Threshold of hearing	10^{-12}	
Whisper	10^{-10}	
Normal conversation	10^{-6}	
Busy traffic	10^{-5}	
Loud radio	10^{-4}	
Jackhammer	10^{-2}	
Rock concert	10^{-1}	
Threshold of pain	10^{1}	

b Do the decibel measures mean that normal conversation is three times as loud as a whisper? Explain.

c How many times louder is

i busy traffic noise than normal conversation?

ii a rock concert than normal conversation?

iii a rock concert than a loud radio?

d Find out the Health and Safety regulations for the wearing of earplugs by workers. At what sound intensity level are earplugs recommended?

LINKS TO Examples 33, 34

6 **a** How much stronger is an earthquake measuring 8 on the Richter scale than an earthquake measuring 5 on the Richter scale?

b The Japanese earthquake in 2011 measured 9.0 on the Richter scale. The New Zealand earthquake that devastated Christchurch and killed 181 people on February 22, 2011 measured 6.3 on the Richter scale. How many times stronger was the Japanese earthquake than the New Zealand earthquake?

c An earthquake measuring 5.6 on the Richter scale struck Newcastle in New South Wales at 10:27 am on Thursday 28 December 1989. Thirteen people were killed and $4 billion worth of damage was caused. How many times stronger was the Japanese earthquake than the Newcastle earthquake?

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Analysis task 1

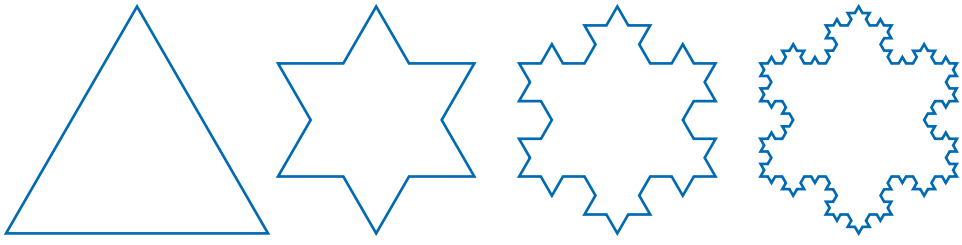
Fractals

A fractal is a shape that is made up of smaller copies of itself. This analysis task explores two different fractal shapes, one involving an increasing perimeter and the other involving a decreasing area.

Koch snowflake fractal

The first fractal is known as the Koch snowflake fractal, which is an example of exponential growth.

The following diagrams show how the Koch snowflake fractal is created, by repeating a pattern on a smaller and smaller scale.



0 1 2 3

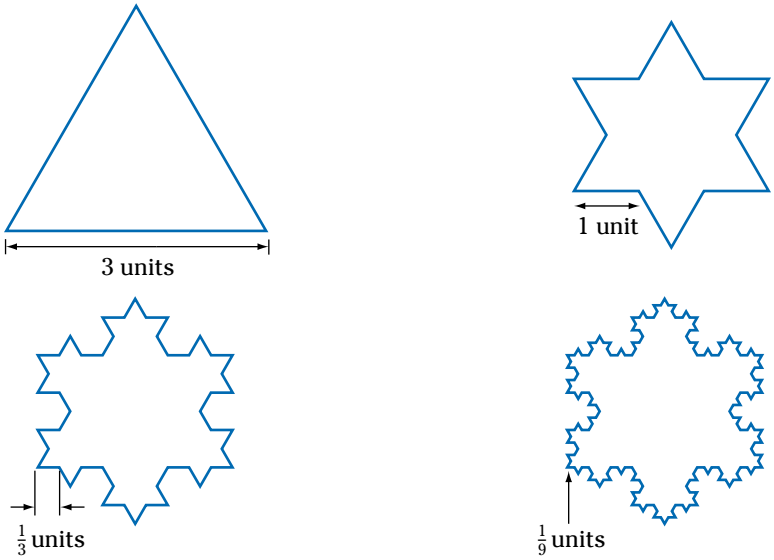
Let n be the number of the fractal in the sequence and let P be the perimeter of the fractal. We will assume that the perimeter of the starting equilateral triangle is 9 units so that the length of each side is 3 units.

a When $n = 1$ how many sides does the fractal have?

b What is the perimeter?

c By what fraction has the perimeter of the equilateral triangle been multiplied? Write this as an improper fraction.

d Repeat parts **a** to **c** for $n = 2$



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e Copy and complete:

Each time a new fractal is constructed, the side length is divided by __ and there are __ times as many sides. This means that each perimeter is $\frac{__}{__}$ times the previous perimeter. This fraction is the common ratio.

f Write a rule for the perimeter P of fractal number n, leaving the common ratio in improper fraction form.

g Construct a table of values for n and P for values of n from 0 to 5, giving the values of P as decimals correct to two decimal places.

h Plot a graph of the points.

i Use your rule from part **f** to find the perimeter of the following fractals

i $n = 10$ **ii** $n = 100$

j Copy and complete:

As n approaches infinity, the perimeter approaches ___.

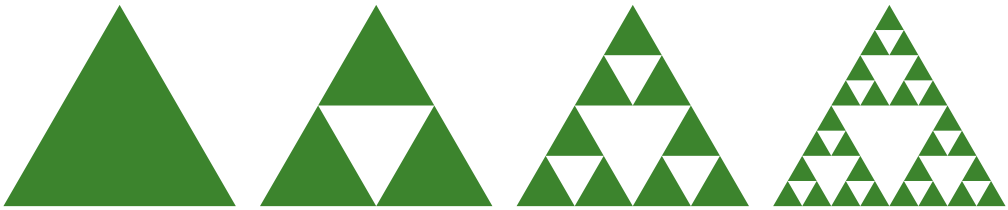
Sierpinski triangle fractal

The next diagrams show the Sierpinski triangle fractal, where a pattern is again repeated on a smaller and smaller scale. This fractal is an example of exponential decay.

Starting with an equilateral triangle, the triangle is divided into four congruent equilateral triangles and the centre triangle is cut out and removed. As more and more triangles are cut out from the original green triangle, the remaining green area is decreasing.

Let n be the number of the fractal in the sequence and let A be the area of the fractal.

We will start with the first triangle having an area of 64 square units.



$n = 0$ $n = 1$ $n = 2$ $n = 3$

k Copy and complete:

Each time a new fractal is constructed, the green area is $\frac{__}{__}$ of the previous green area.

l Write a rule for the area A of fractal number n, leaving the common ratio in fraction form.

m Construct a table of values for n and A for values of n from 0 to 5, giving the values of A as decimals correct to two decimal places where appropriate.

n Plot a graph of the points.

o Use your rule from part **l** to find the area of the following fractals

i $n = 10$ **ii** $n = 100$

p Copy and complete:

As n approaches infinity, the remaining area approaches ___.

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Analysis task 2

How your grandparents calculated

In section 15.4, we saw that 'logarithm' is another name for an index or exponent. Electronic calculators were not introduced into schools until about 1970. When your grandparents were at school, they would have had a book of logarithm tables rather than an electronic calculator. To multiply two numbers together, they looked up the numbers in their log tables. This would tell them the index to base 10 of each number. The two logarithms were added, and then a table of antilogarithms was consulted to turn the power of 10 back into an ordinary number. This analysis task shows how log tables were used. We start by looking at some powers of 10.

a Copy and complete this table of values.

Number	$1 = 10^0$	$10 = 10^1$	$100 = 10^2$	$1000 = 10^3$
Logarithm	0	1		

b Use a calculator to evaluate each of the following, correct to three decimal places.

i $10^{0.3010}$ **ii** $10^{0.4771}$ **iii** $10^{0.6021}$ **iv** $10^{0.6990}$ **v** $10^{0.7782}$

c To multiply 2 by 3 using powers of 10, copy and complete the following steps, using the appropriate values from part **b**.

2×3

$= 10^{0.3010} \times 10^{___}$ (add the indices)

$= 10^{___}$

$= ____$

d To divide 10 by 5 using powers of 10, copy and complete the following steps, using the appropriate values from parts **a** and **b**.

$10 \div 5$

$= 10^{___} \div 10^{___}$

$= 10^{___}$ (subtract the indices)

$= ____$

BLM Logarithm table

Logarithm tables are simply tables of indices to base 10 (see following page and *OneStopDigital* ebook). School students would use four-figure logarithm tables, but ten-figure tables were available for more accurate calculations. Part of a four-figure logarithm table is shown over the page. The entry circled in blue shows that the logarithm to base 10 of 2 is 0.3010, correct to four decimal places.

e Rewrite this in index form.

$\log_{10} 2 = 0.3010$

If we need to find the logarithm for 20, instead of 2, we use

$$\begin{aligned} 20 &= 2 \times 10 \\ &= 10^{0.3010} \times 10^1 \\ &= 10^{1.3010} \end{aligned}$$

Therefore, the logarithm of 20 is 1.3010.

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BLM Logarithm table

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	·0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
1.1	·0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	30	33
1.2	·0792	0828	0864	0899	0934	0969					4	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
1.3	·1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	6	9	13	16	19	22	25	28
1.4	·1461	1492	1523	1553							3	6	9	12	15	18	21	24	27
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1.5	·1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
1.6	·2041	2068	2095	2122	2148						3	5	8	11	13	16	19	21	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
1.7	·2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1.8	·2553	2577	2601	2625	2648						2	5	7	10	12	14	17	19	21
						2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1.9	·2788	2810	2833	2856	2878						2	5	7	9	11	14	16	18	20
						2900	2923	2945	2967	2989	2	4	7	9	11	13	15	18	20
2.0	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2.1	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18

You have seen that we can use the logarithm of 2 to find the logarithm of 20, 200, and so on. Therefore, when using logarithm tables, at first we just look at the digits in the number and ignore the position of the decimal point. To find the logarithm of 1.659, for example, we use the following steps.

- Look down the left-hand column until we come to 1.6.
- Look along the row horizontally from 1.6 until we come to the first column headed 5, where we see 2175.
- Look further along the row to the 9 in the second set of columns headed 1 to 9. The 23 in this column tells us to add 23 to 2175 to make 2198.

So the logarithm of 1.659 is 0.2198.

f Write $\log_{10} 1.659 = 0.2198$ in index form.

To find the logarithm of 1659 instead of 1.659, we first write 1659 as 1.659×10^3.

$$\begin{aligned} 1659 &= 1.659 \times 10^3 \\ &= 10^{0.2198} \times 10^3 \\ &= 10^{3.2198} \end{aligned}$$

Similarly, if we want the logarithm of 0.1659, we first write 0.1659 as 1.659×10^{-1}.

$$\begin{aligned} 0.1659 &= 1.659 \times 10^{-1} \\ &= 10^{0.2198} \times 10^{-1} \\ &= 10^{0.2198 - 1.0000} \\ &= 10^{-0.7802} \end{aligned}$$

g Given that $1.659 = 10^{0.2198}$, express the following numbers in index form with base 10.

i 16.59 **ii** 165.90 **iii** 0.01659 **iv** 165 900

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h Use the logarithm tables in the *OneStopDigital* ebook to express the following numbers in index form with base 10.

i 13.8 **ii** 199 **iii** 2562 **iv** 483.6

i Using your answers to part **h**, express each of the following products in index form. Then use the first index law to simplify each expression.

i 13.8×199 **ii** 13.8×2562 **iii** 199×483.6 **iv** 2562×483.8

When the answer to a calculation was found as a power of 10, a table of antilogarithms (see below and *OneStopDigital* ebook) was then used to convert the power of 10 to an ordinary number. For example, to convert $10^{2.6985}$ to an ordinary decimal number, we first ignore the whole number part and use the following steps to look up 6985 in the antilogs table.

- Look down the left-hand column to .69.
- Look along the row to the first column headed 8, where we see 4989.
- Look further along the row to the 5 in the second set of columns headed 1 to 9. The 6 in this column tells us that we add 6 to 4989 to make 4995.

We now need to decide where to put the decimal point. Notice that $10^{2.6985}$ is between 10^2 and 10^3, that is, between 100 and 1000. Therefore $10^{2.6985} = 499.5$.

BLM Antilogarithm table

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	7	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	8	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11

j Given that $10^{2.6985} = 499.5$, write the following powers of 10 as ordinary decimals.

i $10^{1.6985}$ **ii** $10^{3.6985}$ **iii** $10^{4.6985}$

k Use the antilogarithm tables to express each of your answers in part **i** as ordinary decimal numbers.

l Use a calculator to check whether the products in part **i** match your answers in **k**.

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Review *Exponential functions and logarithms*

Summary

Indices and the index laws

- In the expression a^n, a is the base and n is the index.
- $a^0 = 1, a \neq 0$

The index laws	
1st index law	$a^m \times a^n = a^{m+n}$
2nd index law	$\frac{a^m}{a^n} = a^{m-n}$
3rd index law	$(a^m)^n = a^{m \times n}$
4th index law	$(ab)^m = a^m b^m$
5th index law	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
6th index law	$a^{-m} = \frac{1}{a^m}$

The general exponential function $y = k \times a^x$

- If $a > 1$, $y = a^x$ is an increasing exponential function.
- If $0 < a < 1$, $y = a^x$ is a decreasing exponential function.
- The y-intercept is $(0, k)$.
- The x-axis is an asymptote to the graph.

Solving simple exponential equations

- Express both sides of the equation in index form to the same base then equate the indices.

Logarithms

- The **logarithm** of a number to any positive base is the index when the number is expressed as a power of the base.
- $y = a^x$ can be expressed as $\log_a y = x$. This means 'the index of y when expressed to base a is x'.

Summary of properties of indices and logarithms

- $y = \log_2 x$ and $y = 2^x$ are inverse functions.
- The graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$.

Indices	Logarithms
$y = a^x$	$\log_a y = x$
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^x \times a^y = a^{x+y}$	$\log_a(xy) = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^n = a^{x \times n}$	$\log_a x^n = n\log_a x$

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Logarithmic scales

- In cases where data increases or decreases exponentially it is useful to use the logarithm of the dependent variable. This avoids working with extremely small or extremely large numbers.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

asymptote	exponential equation	*logarithm
base	exponential function	*logarithmic equation
constant ratio	increasing exponential function	*logarithmic function
decreasing exponential function	index	*logarithmic scale
exponent	*inverse functions	power

Revision

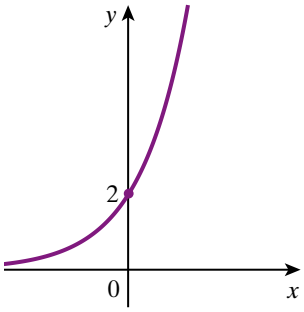
Multiple-choice questions

1 The rule for the exponential function represented in the table of values is

x	-3	-2	-1	0	1	2	3
y	140.625	56.25	22.5	9	3.6	1.44	0.576

A $y = 9 \times 0.4^x$ **B** $y = 0.4 \times 9^x$ **C** $y = 9 \times x^{0.4}$
D $y = 0.4 \times x^9$ **E** $y = 0.4 \times 3.6^x$

2 The rule for this graph could be



A $y = 4 \times 2^x$ **B** $y = 4x^2$
C $y = 2 \times 4^x$ **D** $y = 4^x + 2$
E $y = 4^x - 2$

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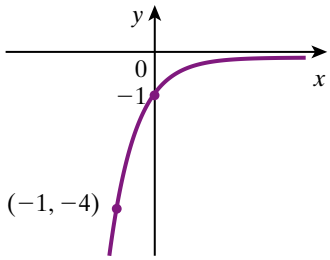
3 The rule for this graph could be

A $y = -2^{-x}$
B $y = 4^{-x}$
C $y = -4^{-x}$
D $y = -4^{x}$
E $y = -2^{x}$



4 A new computer costing \$1800 depreciates at the rate of 15% per annum. The value to the nearest dollar after 4 years is

A \$1740 **B** \$1080 **C** \$720 **D** \$940 **E** \$1

5 $\log_a 3 + \log_a 2$ can be simplified to

A $\log_a 1.5$ **B** $\log_a 5$ **C** $\log_a 6$ **D** $\log_a 8$ **E** $\log_a 9$

Short-answer questions

6 For the exponential rule $P = 2400 \times 1.35^t$, state

a the initial value **b** the percentage growth rate

7 Sketch the graphs of $y = 3^x$ and $y = -3^{-x}$ on the same set of axes, labelling the intercepts and one other point on each graph.

8 A population of 20 rabbits released on an island increased exponentially at the rate of 95% per year.

a Write the rule for the population P after n years.
b Predict the population after 10 years (assuming ample food supply, breeding space and no predators).

9 Express in index form.

a $\log_2 256 = 8$ **b** $\log_{10} 0.001 = -3$ **c** $\log_5 \frac{1}{125} = -3$ **d** $\log_a b = c$

10 Express in logarithmic form.

a $2^{10} = 1024$ **b** $7^{-3} = \frac{1}{343}$ **c** $4^{-0.5} = 0.5$ **d** $m^n = p$

11 Evaluate.

a $\log_2 128$ **b** $\log_6 \sqrt{6}$ **c** $\log_5\left(\frac{1}{25}\right)$ **d** $\log_{27} 3$

12 Simplify.

a $\log_a 5 + \log_a 4$ **b** $3\log_{10} 4 - 4\log_{10} 2$

10 A

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10 A

13 Evaluate

a $\log_3 5 + \log_3 9 - \log_3 15$

b $\log_2 10 - \log_2 4 - \log_2 5$

14 Solve these exponential equations.

a $\left(\frac{1}{9}\right)^x = 243$

b $4^{x-1} = \frac{1}{64}$

15 Solve these logarithmic equations.

a $\log_x \frac{1}{32} = -5$

b $\log_3(x - 3) = 4$

Extended-response questions

16 For each of the following functions

i construct a table of values for $-3 \le x \le 3$, rounding the y-values to two decimal places where appropriate.

ii choose a suitable scale and plot the graph, joining the points with a smooth curve.

a $y = 0.3^x$

b $y = -2^{-x}$

10 A

17 The table shows the profits in thousands of dollars of a company from 2003 until 2010.

Year	Number of years since 2003 (n)	Profit (P) in thousands of dollars
2003	0	105
2004	1	185
2005	2	330
2006	3	603
2007	4	1104
2008	5	1979
2009	6	3650
2010	7	6440

a Find an approximate value for the common ratio by dividing each value by the value in the previous year. Choose a suitable 'average'.

b Write an exponential function rule to fit the data.

c Using a GeoGebra or a CAS calculator, enter the values of n and P into lists then construct a scatter plot.

d On your scatter plot, draw the graph of your exponential function from part **b**. Does your exponential function graph fit the data well?

e Use your rule to predict the profit in 2015.

f What assumption have you made in predicting the 2015 profit?

Practice quiz
Chapter 15

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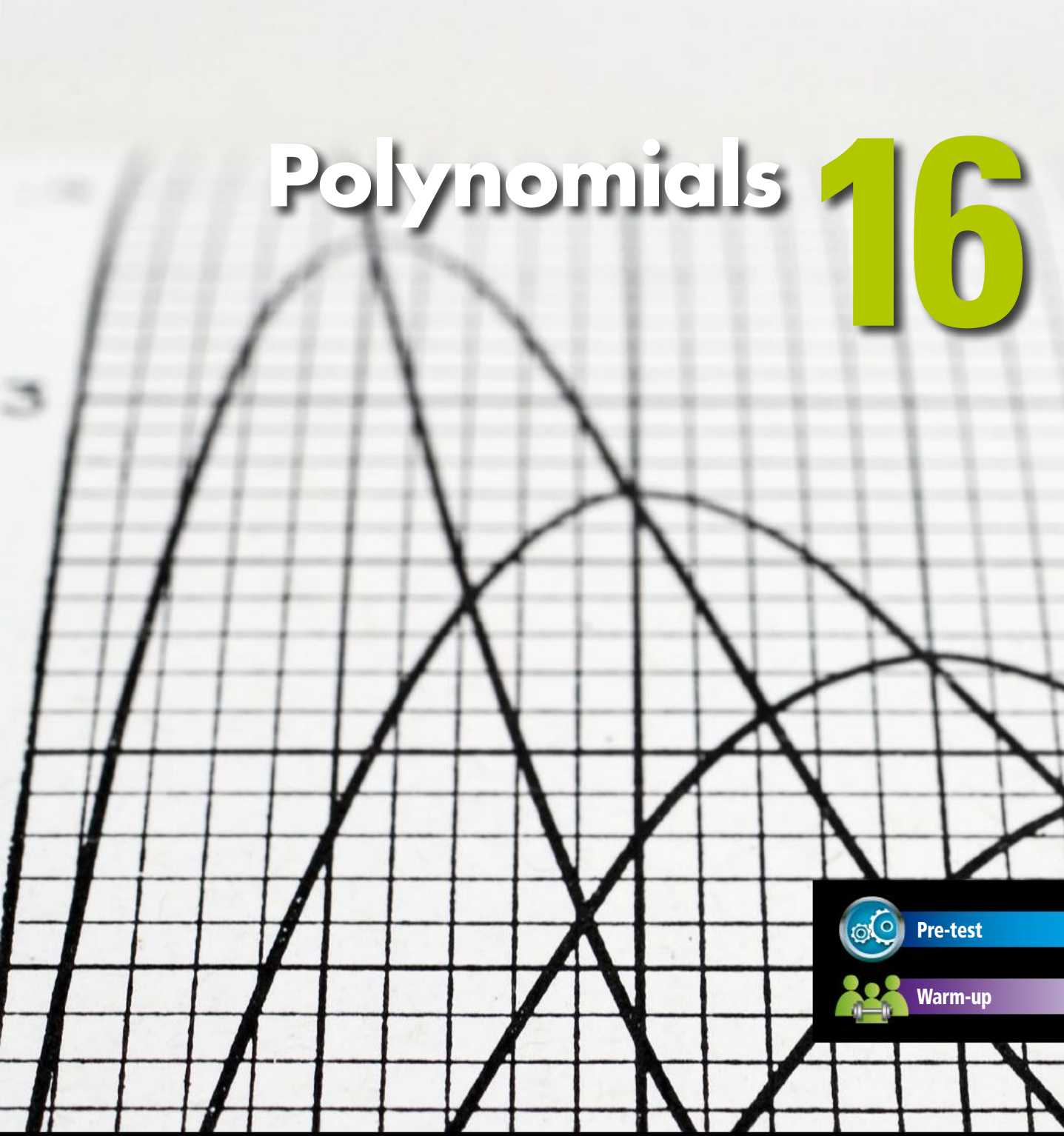
Polynomials 16



Pre-test

Warm-up

Chapters 1 and 3 covered linear expressions and functions. In chapter 7 quadratic expressions were expanded and factorised and in chapter 12 the graphs of quadratic functions were explored. This chapter focuses on factorising, solving and graphing of general polynomial expressions and equations. Just as for quadratic functions, factorising a polynomial allows us to find the x-intercepts when sketching the graph of the polynomial. The parabola is a symmetrical graph but this is not the case for the graphs of all polynomials.

16.1 Introducing polynomials

10 A

In Years 9 and 10, you have worked extensively with linear and quadratic expressions. Recall that a quadratic trinomial expression has three terms and has the form $ax^2 + bx + c$. A **polynomial** may have any number of terms but each term is a multiple of a positive whole number power of x.

Poly means many. A polynomial may have many terms.

The coefficients must be real numbers such as $-3, \frac{1}{2}, -0.4, \sqrt{3}$.

Some examples of polynomials are $2x^3 + x^2 + 4x + 7$, $-0.8x^5 - 2x^2 + x$.

Using the polynomial $2x^3 + x^2 + 4x + 7$ as an example, the following table shows how polynomials are described.

Leading term	The term with the highest index.	$2x^3$
Leading coefficient	The coefficient of the leading term.	2
Degree	The index of the leading term.	3
Constant term	A term that does not include the pronumeral x.	7

A **monic polynomial** is a polynomial that has a leading coefficient of one. For example, $x^4 - 5x^2 + 7$ and $x^2 - 7x + 6$ are monic polynomials. An example of a non-monic polynomial is $3x^4 - 5x^3 + 2x - 6$.

Note that not all powers of x less than the power of the leading term needs to be present. For example, in the polynomial $x^4 - 5x^2 + 7$ there is no x^3 term and no x term.

Example 1

Which of the following expressions are polynomials?

a $\frac{3}{x} + 2x - 5$

b $x^3 + 4x - 6$

c $3\sqrt{x} + 2x^2 - 5x + 3$

d $2x - 5x^2 + 6 + 4x^4$

Working

a $\frac{3}{x} + 2x - 5$ is not a polynomial.

b $x^3 + 4x - 6$ is a polynomial.

Reasoning

$\frac{3}{x} = 3x^{-1}$

The powers of x must all be positive whole numbers.

The powers of x are all positive whole numbers.

continued

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Example 1 continued

Working	Reasoning
c $3\sqrt{x} + 2x^2 - 5x + 3$ is not a polynomial.	$3\sqrt{x} = 3x^{\frac{1}{2}}$ The powers of x must all be positive whole numbers.
d $2x - 5x^2 + 6 + 4x^4$ is a polynomial.	The powers of x are all positive whole numbers.

Example 2

For the polynomial $-5x^3 + 3x^4 + 2x - 6$ state

a the degree
b the leading term
c the leading coefficient
d the constant term

Working	Reasoning
a $3x^4 - 5x^3 + 2x - 6$ degree 4	Rearrange the terms so that the powers of x are decreasing. The highest power of x is x^4.
b leading term is $3x^4$	The term with the highest power is $3x^4$.
c leading coefficient is 3	The coefficient of the leading term is 3.
d constant term is -6	The constant term is the term that does not include x.

Polynomials with degree 1, 2, 3, 4 or 5 are given names. Beyond 5 we refer to the polynomial by its degree. For example, $x^6 - 3x^4 + 2x^3 - 7x + 8$ is a polynomial of degree 6.

Degree	Name of polynomial	Example
1	Linear	$3x - 5$
2	Quadratic	$2x^2 - 3x + 5$
3	Cubic	$x^3 + 0.5x^2 + x$
4	Quartic	$x^4 - 3x^3 + x^2 - 4x + 11$
5	Quintic	$4x^5 - 6x^4 + 3x^3 - x + 5$

Example 3

Classify each of these polynomials as linear, quadratic, cubic or quartic.

a $3x^2 - 7x - 15$ **b** $2x + 5$ **c** $3x^4 - 6x + 5$ **d** $x^3 - 27$

continued

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Example 3 continued

Working		Reasoning
a $3x^2 - 7x - 15$	quadratic	The polynomial is degree 2, that is, the highest power of x is x^2.
b $2x + 5$	linear	The polynomial is degree 1, that is, the highest power of x is x^1.
c $3x^4 - 6x + 5$	quartic	The polynomial is degree 4, that is, the highest power of x is x^4.
d $x^3 - 27$	cubic	The polynomial is degree 3, that is, the highest power of x is x^3.

General polynomial

We express a general polynomial as $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where n is a positive whole number, the coefficients $a_0, a_1, \ldots a_n$ are real numbers and $a_n \neq 0$.

Note that any of the coefficients except for a_n may be zero.

Function notation with polynomials

We use the notation $P(x)$ to name a polynomial that has x as the variable. We say that the polynomial is a function of x.

For example, $P(x) = 2x^3 + x^2 + 4x + 7$. We can use this notation to express the value of the polynomial for any given value of x. If $x = 3$, for example, we write

$$\begin{aligned} P(3) &= 2 \times 3^3 + 3^2 + 4 \times 3 + 7 \\ &= 54 + 9 + 12 + 7 \\ &= 82 \end{aligned}$$

Example 4

For each of the following polynomials find the value indicated.

a $P(x) = 3x^3 + x^2 - 7x + 1$, $P(2)$

b $P(x) = -x^4 + 3x^3 - 2x^2 - 4x + 6$, $P(-1)$

Working	Reasoning
a $\begin{aligned} P(2) &= 3 \times 2^3 + 2^2 - 7 \times 2 + 1 \\ &= 24 + 4 - 14 + 1 \\ &= 15 \end{aligned}$	Substitute $x = 2$ in the polynomial and evaluate.
b $\begin{aligned} P(-1) &= -(-1)^4 + 3(-1)^3 - 2(-1)^2 - 4(-1) + 6 \\ &= -1 - 3 - 2 + 4 + 6 \\ &= 4 \end{aligned}$	Substitute $x = -1$ in the polynomial and evaluate.

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Adding and subtracting polynomials

In Years 7 to 10 you have seen how like terms can be collected when algebraic expressions are added or subtracted. We can apply this to the addition and subtraction of polynomials. For example, to add the polynomials $2x^3 - x^2 + 3x + 5$ and $x^3 + 5x^2 - 7x - 4$, we add the like terms.

$$2x^3 - x^2 + 3x + 5 + x^3 + 5x^2 - 7x - 4 = 3x^3 + 4x^2 - 4x + 1$$

We can also write the second polynomial under the first polynomial, aligning the powers of x before adding the like terms.

$$\begin{array}{rl} & 2x^3 - x^2 + 3x + 5 \\ \text{Add} & \underline{x^3 + 5x^2 - 7x - 4} \\ & \underline{3x^3 + 4x^2 - 4x + 1} \end{array}$$

Example 5

Simplify the following:

a Add the polynomials $4x^2 - 6x + 7$ and $2x^3 + x^2 - 3$.

b Subtract $2x^2 - 7x$ from $-6x^2 - 3x$.

Working

a $4x^2 - 6x + 7 + 2x^3 + x^2 - 3$
$= 2x^3 + 5x^2 - 6x + 4$

b $-6x^2 - 3x - (2x^2 - 7x)$
$= -6x^2 - 3x - 2x^2 + 7x$
$= -8x^2 + 4x$

Reasoning

Add like terms.

$$\begin{array}{rl} & 4x^2 - 6x + 7 \\ \text{Add} & \underline{2x^3 + x^2 \qquad - 3} \\ & \underline{2x^3 + 5x^2 - 6x + 4} \end{array}$$

Subtract like terms.
The brackets around $2x^2 - 7x$ are important to indicate that $-7x$ is to be subtracted and that $-(-7x) = +7x$

$$\begin{array}{rl} & -6x^2 - 3x \\ \text{Subtract} & \underline{(2x^2 - 7x)} \\ & \underline{-8x^2 + 4x} \end{array}$$

exercise 16.1

LINKS TO Example 1

1 Which of the following expressions are polynomials? Give reasons.

a $2x^3 - \dfrac{x^2}{3} - 4x + 6$ **b** $x^2 - 5x + 3\sqrt{5}$ **c** $3x^2 + x^{0.5} + 1$

d $\dfrac{1}{x^3} + \dfrac{2}{x^2} + 3x$ **e** $3 - 2x$ **f** $x^4 + 4x - 8$

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10 A

LINKS TO Example 2

2 For each of these polynomials state

i the degree **ii** the leading term

iii the leading coefficient **iv** the constant term

a $3x^4 - 2x^3 + x^2 - 4x + 7$ **b** $x^3 - 4x^2 + 6x - 10$

c $3x^2 + 6x - 5x^3 + 4$ **d** $-2x^4 + 3x^2 - 5x$

e $11 - 7x$ **f** $4x - 8x^3$

LINKS TO Example 3

3 Classify each of the polynomials in question **2** as linear, quadratic, cubic or quartic.

LINKS TO Example 4

4 For each of the following polynomials find the value indicated.

a $P(x) = 2x^2 - 5x + 6; P(-2)$

b $P(x) = x^4 + 5x^3 - x^2 + 10; P(2)$

c $P(x) = 2x^3 + 3x^2 + 5x + 6; P(-1)$

d $P(x) = 5x^3 + 2x^2 - 7x + 12; P(3)$

e $P(x) = x^4 - 2x^3 + 5x^2 - 8x - 16; P(-1)$

f $P(x) = x^4 - 3x^3 + 7x; P(2)$

g $P(x) = 3x^3 - 8x + 11; P(-1)$

h $P(x) = x^3 + 5x^2 - 6x + 11; P(-1)$

LINKS TO Example 5a

5 Find the sum of the following polynomials.

a $2x + 7$ and $-3x + 5$

b $-4x - 11$ and $-4x + 4$

c $3x - 8$ and $-3x + 6$

d $2x^2 + 5x$ and $-2x^2 - 12x$

e $x^2 + 5x + 4$ and $x^2 - 7x - 12$

f $4x^2 - 3x - 15$ and $-2x^2 + 4x - 3$

g $x^3 + 3x^2 - 2x + 3$ and $x^3 - 5x^2 + 8x + 4$

h $x^3 + 4x^2 - 7x + 6$ and $3x^2 + x - 6$

i $x^3 + 3x - 1$ and $x^3 + 11x^2 - 8$

j $x^3 - 3x^2 + 4$ and $3x^2 - 8x + 2$

k $x^4 + 2x^3 - 5x$ and $x^3 - x^2 - 11x$

l $x^4 - 2x^2 + 5$ and $x^3 - 3x^2 - 5$

LINKS TO Example 5b

6 Subtract the second polynomial from the first polynomial for each of the pairs of polynomials in question **5**.

exercise 16.1 challenge

7 The sum of the polynomials $3x^3 - 5x^2 + 7x - 4$ and $ax^3 + bx^2 + cx + d$ is $-2x^3 + 11x - 18$. Find the values of a, b, c and d.

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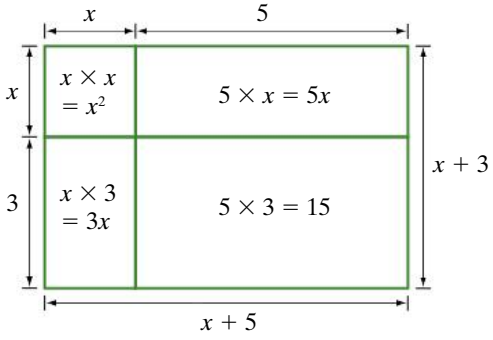
16.2 Multiplying polynomials

Chapter 3 included the multiplication of two linear factors to obtain a quadratic trinomial. Area diagrams and the mnemonic FOIL were used in to demonstrate the distributive law. Each term in one factor must be multiplied by each term in the other factor.

$$\begin{aligned}(x+3)(x+5) &= x(x+5)+3(x+5)\\ &= x^2+5x+3x+15\\ &= x^2+8x+15\end{aligned}$$



The distributive law is used in the same way to multiply any two polynomials. Each term of one polynomial must be multiplied by each term of the other polynomial. Like terms are collected. When two or more polynomials are multiplied, the product is another polynomial.

$$\begin{aligned}(x-3)(x^2-2x-10) &= x(x^2-2x-10)-3(x^2-2x-10)\\ &= x^3-2x^2-10x-3x^2+6x+30\\ &= x^3-5x^2-4x+30\end{aligned}$$

Example 6

Find the product of the following polynomials.

a x^2-2x-3 and $2x^2$

b $2x^3+3x^2-x+1$ and $x-1$

Working

a $2x^2(x^2-2x-3)$
$= 2x^4-4x^3-6x^2$

b $(x-1)(2x^3+3x^2-x+1)$
$= x(2x^3+3x^2-x+1)-1(2x^3+3x^2-x+1)$
$= 2x^4+3x^3-x^2+x-2x^3-3x^2+x-1$
$= 2x^4+x^3-4x^2+2x-1$

Reasoning

Multiply each term of x^2-2x-3 by $2x^2$.

Multiply each term of $2x^3+3x^2-x+1$ by x then each term of $2x^3+3x^2-x+1$ by -1. Collect like terms.

When multiplying three polynomials, we first multiply two of them then multiply the product by the third polynomial. In the following example, note how the constant term in the product is equal to the product of the constant terms in each of the factors. We will make use of this in section 16.4 when we use the factor theorem to find the factors of polynomials.

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Example 7

Find the following products.

a $(x-3)(x-2)(x^2+2x+5)$ **b** $(x-3)^2(x+4)$

Working

a $(x-3)(x-2)(x^2+2x+5)$
$= (x^2-5x+6)(x^2+2x+5)$
$= x^2(x^2+2x+5) - 5x(x^2+2x+5) + 6(x^2+2x+5)$
$= x^4+2x^3+5x^2-5x^3-10x^2-25x+6x^2+12x+30$
$= x^4-3x^3+x^2-13x+30$

Reasoning

Multiply the two linear factors $(x-3)(x-2)$. Multiply each term of (x^2+2x+5) by each term of (x^2-5x+6). Collect like terms.

b $(x-3)^2(x+4)$
$= (x^2-6x+9)(x+4)$
$= x(x^2-6x+9) + 4(x^2-6x+9)$
$= x^3-6x^2+9x+4x^2-24x+36$
$= x^3-2x^2-15x+36$

Expand the squared factor. Multiply each term of (x^2-6x+9) by each term of $(x+4)$.

exercise 16.2

LINKS TO Example 6

1 Find the product of the following polynomials.

a $-3x$ and x^2-2x+3 **b** $5x$ and $x^3-5x^2+7x-12$
c $3x^2$ and x^2-5x+6 **d** $-6x$ and x^2+2x-8
e $-3x^2$ and x^3-4x^2+x-3 **f** $x-3$ and x^2-2x+7
g $2x+5$ and $x^2-7x+12$ **h** x^2-8 and x^2+3x
i $2x-1$ and x^3+x^2-3x+1 **j** x^2-2x-3 and x^2-4x
k x^2-4 and x^2+4 **l** x^2+3x and x^2-3x-5

LINKS TO Example 7a

2 Find the following products.

a $(x+1)(x-1)(x-5)$ **b** $(x+2)(x-2)(x+1)$ **c** $(x+3)(x+7)(x-3)$
d $(x+1)(x-2)(x+4)$ **e** $(x-3)(x+2)(x+5)$ **f** $(x+3)(x^2-3x+9)$
g $(x-3)(x^2+3x+9)$ **h** $x(x-2)(x^2+2x+4)$ **i** $(x+3)(x-1)(x^2+2)$

LINKS TO Example 7b

3 Expand the following.

a $(x-3)^2(x+2)$ **b** $(x-1)^2(x+5)$ **c** $(x+2)(x+3)^2$
d $(x-1)^3$ **e** $(x+1)^3$ **f** $(x-2)^2(x+2)$
g $(x-1)^2(x+2)^2$ **h** $(x-3)^2(x+1)^2$ **i** $(2x+3)^2(2x-3)^2$
j $(x-1)(x^2+x-1)$ **k** $(x+2)(x^2-2x-4)$ **l** $(x-2)(x+2)(x^2+4)$

exercise 16.2 challenge

4 The product of $x+2$ and ax^2+bx+c is $3x^3+5x^2+3x+10$. Find a, b and c.

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16.3 Dividing polynomials

Any number N can be written as the product of a divisor D and a quotient Q, plus a remainder R.

$N = D \times Q + R$

For example, if 23 is divided by the divisor 8, the quotient is 2 with remainder 7. So we can write $23 = 8 \times 2 + 7$

Alternatively, we can write $\dfrac{N}{D} = Q + \dfrac{R}{D}$

For example $\dfrac{23}{8} = 2 + \dfrac{7}{8}$

If we divide a polynomial $P(x)$ by a linear polynomial $x - a$, the quotient is another polynomial $Q(x)$ and the remainder is R.

$$P(x) = (x - a)Q(x) + R$$

Remainder Quotient Divisor

Using long division to find quotients

In earlier years you may have learnt how to do long division. To refresh your memory, the process of long division is shown here for $1842 \div 37$. Check that you understand the procedure, which uses the steps:

- divide
- multiply
- subtract
- bring down.

To divide 1842 by 37, we start by seeing how many times 37 will divide into 184.

49		
37)1842	**Divide**:	$184 \div 37 \approx 4$
148	**Multiply**:	$37 \times 4 = 148$
362	**Subtract**:	$184 - 148 = 36$; **Bring down** 2; **Divide**: $362 \div 37 \approx 9$
333	**Multiply**:	$37 \times 9 = 333$
29	**Subtract**:	$362 - 333 = 29$

So we can write: $1842 = 37 \times 49 + 29$ or $\dfrac{1842}{37} = 49 + \dfrac{29}{37}$

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Long division with polynomials

Long division can also be performed with algebraic expressions, using the same sequence of divide, multiply, subtract, bring down. The following example shows the division of $x^2 - 7x + 15$ by $x - 4$.

We work with the leading terms of $x^2 - 7x + 15$ and $x - 4$.

Step 1: We decide how many times x divides into x^2. As $x^2 \div x = x$, we write x above $-7x$.

We then multiply $x - 4$ by x and write the product $x^2 - 4x$ below $x^2 - 7x$.

$$\begin{array}{r} x \\ x - 4\overline{)x^2 - 7x + 15} \\ \underline{x^2 - 4x} \end{array}$$

Divide: $x^2 \div x = x$
Multiply: $(x - 4) \times x = x^2 - 4x$

Step 2: We now subtract $x^2 - 4x$ from $x^2 - 7x$ to give $-3x$, and then bring down $+ 15$, writing it alongside $-3x$.

The process of dividing, multiplying and subtracting is then repeated. As before we work with the leading terms and see how many times x divides into $-3x$. It divides -3 times so write -3 beside x in the quotient line. Then multiply $x - 4$ by -3 and write the product $-3x + 12$ beneath $-3x + 15$. Subtracting gives the remainder, 3.

$$\begin{array}{r} x - 3 \\ x - 4\overline{)x^2 - 7x + 15} \\ \underline{x^2 - 4x} \\ -3x + 15 \\ \underline{-3x + 12} \\ 3 \end{array}$$

Divide: $x^2 \div x = x$

Multiply: $(x - 4) \times x = x^2 - 4x$
Subtract: $-7x - (-4x) = -3x$; **bring down** $+15$;
Divide: $-3x \div x = -3$
Multiply: $(x - 4) \times (-3) = -3x + 12$
Subtract: $-3x + 15 - (-3x + 12) = 3$

We can now write:

$$x^2 - 7x + 15 = (x - 3)(x - 4) + 3$$

P(x) — Quotient Q(x) — Divisor D(x) — Remainder R

or $\dfrac{x^2 - 7x + 15}{x - 4} = x - 3 + \dfrac{3}{x - 4}$

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Example 8

Use long division to find the quotient and remainder when $5x^3 - 2x^2 + 8x + 9$ is divided by $x - 1$.

Working

$$
\begin{array}{r}
5x^2 + 3x + 11 \\
x - 1\overline{)5x^3 - 2x^2 + 8x + 9} \\
\underline{5x^3 - 5x^2} \qquad\qquad \\
3x^2 + 8x \qquad \\
\underline{3x^2 - 3x} \qquad \\
11x + 9 \\
\underline{11x - 11} \\
20
\end{array}
$$

Reasoning

$5x^3 \div x = 5x^2$

$3x^2 \div x = 3x$

$11x \div x = 11$

The quotient is $5x^2 + 3x + 11$
The remainder is 20.

When a power of x lower than the degree of the polynomial is missing from the polynomial, the power is inserted with coefficient 0. For example, if a polynomial has degree 4 and there is no x^2 term, we insert $+\ 0x^2$ in the polynomial before starting the long division.

Example 9

Use long division to find the quotient and remainder when $3x^4 - 7x^2 + 11x + 18$ is divided by $x + 2$.

Working

$$
\begin{array}{r}
3x^3 - 6x^2 + 5x + 1 \\
x + 2\overline{)3x^4 + 0x^3 - 7x^2 + 11x + 18} \\
\underline{3x^4 + 6x^3} \qquad\qquad\qquad \\
-6x^3 - 7x^2 \qquad\qquad \\
\underline{-6x^3 - 12x^2} \qquad\qquad \\
5x^2 + 11x \qquad \\
\underline{5x^2 + 10x} \qquad \\
x + 18 \\
\underline{x + 2} \\
16
\end{array}
$$

Reasoning

There is no x^3 term so insert $0x^3$.

$3x^4 \div x = 3x^3$

$-6x^3 \div x = -6x^2$

$5x^2 \div x = 5x$

$x \div x = 1$

The quotient is $3x^3 - 6x^2 + 5x + 1$.
The remainder is 16.

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Remainders and factors

We know from working with numbers that a remainder of zero means that the divisor must be a factor of the number. For example, if 24 is divided by 8, the remainder is zero, so we can say 8 is a factor of 24.

Similarly, if we divide a polynomial by another polynomial and the remainder is zero, then we know the second polynomial must be a factor of the first polynomial.

In this chapter, we deal only with division by a linear polynomial of the form $(x - a)$.

Factors of $(x - a)$

If the remainder is zero when a polynomial is divided by $(x - a)$, then $(x - a)$ must be a factor of the polynomial.

Conversely, if the remainder is not zero when a polynomial is divided by $(x - a)$, then $(x - a)$ is not a factor of the polynomial.

Example 10

Show that $x - 4$ is a factor of $2x^3 - 11x^2 + 17x - 20$.

Working

$$
\begin{array}{r}
2x^2 - 3x + 5 \\
x - 4\overline{)2x^3 - 11x^2 + 17x - 20} \\
\underline{2x^3 - 8x^2} \qquad\qquad \\
-3x^2 + 17x \qquad \\
\underline{-3x^2 + 12x} \qquad \\
5x - 20 \\
\underline{5x - 20} \\
0
\end{array}
$$

The remainder is zero, so $x - 4$ is a factor of $2x^3 - 11x^2 + 17x - 20$.

Reasoning

$2x^3 \div x = 2x^2$

$-3x^2 \div x = -3x$

$5x \div x = 5$

$x - 4$ divides exactly into $2x^3 - 11x^2 + 17x - 20$.

exercise 16.3

LINKS TO Example 8

1 Use long division to find the quotient and remainder for each of the following divisions.

a $x^2 - 9x - 22$ is divided by $x + 2$

b $x^2 + 13x - 48$ is divided by $x - 2$

c $2x^2 + x - 15$ is divided by $x - 3$

d $3x^2 + 7x - 20$ is divided by $x + 4$

e $x^3 - 17x^2 + 66x - 60$ is divided by $x - 1$

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f $x^3 - 12x^2 + 32x - 12$ is divided by $x + 1$
g $x^3 + 3x^2 - 7x - 21$ is divided by $x + 3$
h $x^3 - 4x^2 + 6x - 12$ is divided by $x + 2$
i $x^4 + 3x^3 - 9x^2 + 7x + 4$ is divided by $x + 3$
j $x^4 - 2x^3 - 5x^2 + 8x - 15$ is divided by $x - 2$

2 Use long division to divide $x^2 - 7x + 12$ by $x - 4$. What is the remainder and what does this tell us?

LINKS TO Example 9

3 Use long division to find the quotient and remainder for each of the following divisions.
a $4x^3 - 5x^2 + 7$ is divided by $x - 1$
b $3x^4 + x^3 - 5x - 11$ is divided by $x + 2$
c $2x^4 - 5x^2 - 7x + 4$ is divided by $x - 3$
d $x^4 - 3x^3 + 2x^2 - 5$ is divided by $x + 1$

LINKS TO Example 10

4 For each of the following,
i find the quotient when the cubic polynomial is divided by the linear expression
ii show that the linear expression is a factor of the polynomial

a $x^3 - 2x^2 - 5x + 6$; $x - 1$
b $x^3 - 4x^2 - 9x + 36$; $x + 3$
c $x^3 + 7x^2 + 2x - 40$; $x - 2$
d $x^3 + 6x^2 - x - 30$; $x - 2$
e $x^3 + 9x^2 + 8x - 60$; $x - 2$
f $x^3 - x^2 - 10x - 8$; $x + 1$
g $x^3 - 3x^2 - 16x - 12$; $x + 2$
h $x^3 - 3x^2 - 4x + 12$; $x - 3$
i $x^3 + 9x^2 - 4x - 96$; $x + 4$
j $x^3 + 4x^2 - 20x - 48$; $x - 4$
k $x^3 - 13x - 12$; $x + 3$
l $x^3 + 13x^2 - 144$; $x - 3$

exercise 16.3 challenge

5 When $x^4 + 3x^3 - 39x^2 - 47x + a$ is divided by $x - 2$, the remainder is zero. Find the value of a.

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16.4 The remainder theorem and the factor theorem

The following notation was introduced in section 16.3,

$$P(x) = (x - a)Q(x) + R$$

Divisor Quotient Remainder

If $x = a$, then we can write

$$\begin{aligned} P(a) &= (a - a)Q(a) + R \\ &= 0 \times Q(a) + R \\ P(a) &= R \end{aligned}$$

This provides a quick and easy way of finding the remainder R when a polynomial is divided by $(x - a)$ without having to carry out the long division. For example, if we wish to know the remainder when a polynomial is divided by $(x - 2)$, we calculate $P(2)$, that is, we substitute $x = 2$ in the polynomial and evaluate it.

The remainder theorem

To find the remainder when a polynomial $P(x)$ is divided by $(x - a)$, evaluate $P(a)$.

Example 11

Find the remainder when

a the polynomial $3x^3 - 5x^2 + 2x - 1$ is divided by $x - 3$

b the polynomial $x^4 - x^3 + 3x^2 - 1$ is divided by $x + 2$

Working	Reasoning
a $P(x) = 3x^3 - 5x^2 + 2x - 1$ $P(3) = 3 \times 3^3 - 5 \times 3^2 + 2 \times 3 - 1$ $= 81 - 45 + 6 - 1$ $P(3) = 41$ The remainder when $3x^3 - 5x^2 + 2x - 1$ is divided by $x - 3$ is 41.	Calculate $P(a)$ to find the remainder when $P(x)$ is divided by $(x - a)$. Substitute $x = 3$ in the polynomial.
b $P(x) = x^4 - x^3 + 3x^2 - 1$ $P(-2) = (-2)^4 - (-2)^3 + 3 \times (-2)^2 - 1$ $= 16 + 8 + 12 - 1$ $P(-2) = 35$ The remainder when $x^4 - x^3 + 3x^2 - 1$ is divided by $x + 2$ is 35.	Calculate $P(a)$ to find the remainder when $P(x)$ is divided by $(x - a)$. Substitute $x = -2$ in the polynomial.

We can now take the remainder theorem a further step to help us find linear factors of polynomials.

The factor theorem

$P(a) = 0$, then $(x - a)$ is a factor of the polynomial $P(x)$.

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Example 12

Show that $x - 1$ is a factor of $x^4 - 3x^3 + 6x^2 - 7x + 3$.

Working

$P(x) = x^4 - 3x^3 + 6x^2 - 7x + 3$
$P(1) = 1^4 - 3 \times 1^3 + 6 \times 1^2 - 7 \times 1 + 3$
$= 1 - 3 + 6 - 7 + 3$
$P(1) = 0$
The remainder is zero so $x - 1$ is a factor of $x^4 - 3x^3 + 6x^2 - 7x + 3$.

Reasoning

Calculate $P(a)$ to find the remainder when $P(x)$ is divided by $(x - a)$.
Substitute $x = 1$ in the polynomial.
The remainder when the polynomial is divided by $x - 1$ is 0, so $x - 1$ is a factor of the polynomial.

Factorising polynomials

Long division is not usually the most efficient method for factorising quadratic trinomial expressions. As you saw in chapter 3, many quadratic trinomials can be factorised easily using a guess-and-check method. However, polynomials such as $x^3 + 2x^2 - 5x - 6$ are harder to factorise using guess and check. Not all polynomials can be factorised and many polynomials have irrational factors. For those polynomials that *do* have factors, the factor theorem and long division enable us to find the factors. We will work only with monic polynomials.

Recall that in section 16.2 we observed that the constant term in a polynomial is equal to the product of the constant terms in the factors. For example, the factors of $x^2 - x - 12$ are $(x - 4)(x + 3)$. The constant term, -12, in $x^2 - x - 12$ is equal to the product of -4 and $+3$. We make use of this fact when applying the factor theorem to monic polynomials.

When choosing suitable values of a that will give $P(a) = 0$, we know that a must be one of the factors of the constant term of $P(x)$.

A polynomial may also have a common factor which should be taken out first .

Using $x^4 + 2x^3 - 5x^2 - 6x$ as an example, the steps are shown below.

Step 1: Look for a common factor

$x^4 + 2x^3 - 5x^2 - 6x$ has a common factor x.

$x^4 + 2x^3 - 5x^2 - 6x = x(x^3 + 2x^2 - 5x - 6)$

Step 2: Use the factor theorem to find a linear factor of $x^3 + 2x^2 - 5x - 6$.

The constant term -6 gives us a clue. The product of the constant terms of the factors must be -6. We could try substituting any of the factors of -6 in $P(x)$: $1, -1, 2, -2, 3, -3, 6, -6$ until we find a zero value for $P(a)$. We will start by calculating $P(1)$:

$$\begin{aligned} P(1) &= 1^3 + 2 \times 1^2 - 5 \times 1 - 6 \\ &= 1 + 2 - 5 - 6 \\ &= -8 \end{aligned}$$

This tells us that $x - 1$ is not a factor of $x^3 + 2x^2 - 5x - 6$

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Now try $P(-1)$:

$$\begin{aligned} P(-1) &= (-1)^3 + 2 \times (-1)^2 - 5 \times (-1) - 6 \\ &= -1 + 2 + 5 - 6 \\ &= 0 \end{aligned}$$

So $x + 1$ is a factor of $x^3 + 2x^2 - 5x - 6$.

Step 3: Long division

Divide the polynomial by the factor $x + 1$.

$$\begin{array}{r} x^2 + x - 6 \\ x + 1 \overline{\big) x^3 + 2x^2 - 5x - 6} \\ \underline{x^3 + x^2} \\ x^2 - 5x \\ \underline{x^2 + x} \\ -6x - 6 \\ \underline{-6x - 6} \\ 0 \end{array}$$

Step 4: Look for factors of the quotient.

$x^2 + x - 6 = (x + 3)(x - 2)$

Step 5: Write the polynomial as the product of its factors.

$x^4 + 2x^3 - 5x^2 - 6x = x(x + 1)(x + 3)(x - 2)$

Factorising a polynomial

- Look for a common factor
- Use the factor theorem to find the first factor. Look at the factors of the constant term. Choose one of these factors to calculate $P(a)$. If $P(a) = 0$, then $x - a$ is a factor. If $P(a) \neq 0$ systematically continue testing other factors of the constant term for a until a value is found for which $P(a) = 0$.
- Use long division to find a polynomial factor.
- For a cubic polynomial, long division gives a quadratic factor that may be able to be further factorised.

Example 13

Factorise the polynomial $x^3 - 3x^2 - 10x + 24$.

Working

$$\begin{aligned} P(1) &= 1^3 - 3 \times 1^2 - 10 \times 1 + 24 \\ &= 1 - 3 - 10 + 24 \\ &= 12 \end{aligned}$$

Reasoning

Step 1:
There is no common factor.

continued

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Example 13 continued

Working

$$P(-1) = (-1)^3 - 3 \times (-1)^2 - 10 \times (-1) + 24$$
$$= -1 - 3 + 10 + 24$$
$$= 30$$

Reasoning

Step 2:
The factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24. Start by calculating $P(1)$ and continue through the list of factors of 24, including negative values, until a zero is found for $P(a)$.

$$P(2) = 2^3 - 3 \times 2^2 - 10 \times 2 + 24$$
$$= 8 - 12 - 20 + 24$$
$$= 0$$

So, $x - 2$ is a factor.

If $P(a) = 0$ then $x - a$ is a factor.

$$\begin{array}{r}
x^2 - x - 12 \\
x - 2\overline{\big)\,x^3 - 3x^2 - 10x + 24} \\
\underline{x^3 - 2x^2 } \\
-x^2 - 10x \\
\underline{-x^2 + 2x } \\
-12x + 24 \\
\underline{-12x + 24} \\
0
\end{array}$$

Step 3:
Use long division to find the quadratic factor.

Step 4:
If possible, factorise the quotient.

$$x^2 - x - 12 = (x - 4)(x + 3)$$
$$x^3 - 3x^2 - 10x + 24 = (x - 2)(x - 4)(x + 3)$$

Step 5:
Write the polynomial as the product of its factors.

Some polynomials have a repeated factor. For example, the perfect square quadratic expression $x^2 - 6x + 9 = (x - 3)^2$ has the repeated factor $x - 3$.

Example 14

Factorise the polynomial $x^4 + 4x^3 - 5x^2 - 36x - 36$.

Working

$$x^4 + 4x^3 - 5x^2 - 36x - 36$$

$$P(1) = 1^4 + 4 \times 1^3 - 5 \times 1^2 - 36 \times 1 - 36$$
$$= 1 + 4 - 5 - 36 - 36$$
$$= -72$$

Reasoning

Step 1:
There is no common factor.

Step 2:
The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36. Start by calculating $P(1)$ and continue through the list of factors of 36, including negative values, until a zero is found for $P(a)$.

continued

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Example 14 continued

Working

$P(-1)$
$= (-1)^4 + 4 \times (-1)^3 - 5 \times (-1)^2 - 36 \times (-1) - 36$
$= 1 - 4 - 5 + 36 - 36$
$= -8$

$P(2)$
$= 2^4 + 4 \times 2^3 - 5 \times 2^2 - 36 \times 2 - 36$
$= 16 + 32 - 20 - 72 - 36$
$= -80$

$P(-2)$
$= (-2)^4 + 4 \times (-2)^3 - 5 \times (-2)^2 - 36 \times (-2) - 36$
$= 16 - 32 - 20 + 72 - 36$
$= 0$

So, $x + 2$ is a factor.

Reasoning

If $P(a) = 0$ then $x - a$ is a factor.

$$
\begin{array}{r}
x^3 + 2x^2 - 9x - 18 \\
x + 2 \overline{\big) x^4 + 4x^3 - 5x^2 - 36x - 36} \\
\underline{x^4 + 2x^3} \qquad\qquad\qquad \\
2x^3 - 5x^2 \qquad\qquad \\
\underline{2x^3 + 4x^2} \qquad\qquad \\
-9x^2 - 36x \qquad \\
\underline{-9x^2 - 18x} \qquad \\
-18x - 36 \\
\underline{-18x - 36} \\
0
\end{array}
$$

Step 3:
Use long division to find the quotient when the polynomial is divided by $x + 2$.

Find the factors of $x^3 + 2x^2 - 9x - 18$.
$P(-2) = (-2)^3 + 2 \times (-2)^2 - 9 \times (-2) - 18$
$= -8 + 8 + 18 - 18$
$= 0$

So, $x + 2$ is a factor.

$$
\begin{array}{r}
x^2 - 9 \\
x + 2 \overline{\big) x^3 + 2x^2 - 9x - 18} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
0 - 9x - 18 \\
\underline{-9x - 18} \\
0
\end{array}
$$

$x^2 - 9 = (x + 3)(x - 3)$

Step 4:
If possible, factorise the quotient.
The quotient is a cubic polynomial.
There is no need to try $P(1)$, $P(-1)$ or $P(2)$ again.
If $x + 1$ or $x - 1$ or $x - 2$ are not factors of $x^4 + 4x^3 - 5x^2 - 36x - 36$ then they cannot be factors of $x^3 + 2x^2 - 9x - 18$
However, there could be a repeated factor, so try $P(-2)$. $x + 2$ is a repeated factor of the polynomial.

$x^4 + 4x^3 - 5x^2 - 36x - 36$
$= (x + 2)(x + 2)(x + 3)(x - 3)$
$= (x + 2)^2(x + 3)(x - 3)$

Step 5:
Write the polynomial as the product of its factors.

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exercise 16.4

LINKS TO Example 11

1 Find the remainder in each of these divisions.

a $(x^2 - 5x - 14) \div (x + 2)$ **b** $(x^2 + 11x - 37) \div (x - 3)$

c $(2x^2 + 7x - 21) \div (x + 3)$ **d** $(5x^2 + 8x - 12) \div (x + 4)$

2 Find the remainder when $x^3 - 17x^2 + 66x - 60$ is divided by $x - 3$.

LINKS TO Example 12

3 For each of the following

i show that the linear polynomial is a factor of the cubic polynomial

ii use long division to find the quotient

iii factorise the quotient

iv express the cubic polynomial as the product of three factors

a $x^3 - 4x^2 - 9x + 36 \div (x + 3)$ **b** $x^3 + 4x^2 - 20x - 48 \div (x - 4)$

c $x^3 + 9x^2 - 4x - 96 \div (x + 4)$ **d** $x^3 - 9x^2 - 16x + 60 \div (x + 2)$

LINKS TO Example 13

4 Using the quotients from the long divisions in question **4** in exercise **16.3**, factorise each of the polynomials as the product of three linear factors.

5 Factorise the following cubic polynomials as the product of three linear factors.

a $x^3 - x^2 - 34x - 56$ **b** $x^3 - x^2 - 22x + 40$

c $2x^3 - 21x^2 + 70x - 75$ **d** $2x^3 + 5x^2 - 6x - 9$

e $2x^3 + 15x^2 + 31x + 12$ **f** $3x^3 + 10x^2 - 27x - 10$

g $3x^3 - 7x^2 - 14x + 24$ **h** $4x^3 + 17x^2 - 44x - 12$

LINKS TO Example 14

6 Factorise the following polynomials.

a $x^4 - 7x^3 + 17x^2 - 17x + 6$ **b** $x^4 + 3x^3 - 15x^2 - 19x + 30$

c $x^4 - 6x^3 - 16x^2 + 54x + 63$ **d** $x^4 + 5x^3 - 28x^2 - 20x + 96$

exercise 16.4 challenge

7 Find the value of m that makes $5x^3 - 4x^2 + x + m$ divisible by $x + 1.5$.

8 Find the value of m that makes $5x^3 - 2x^2 + mx - 7$ divisible by $x + 1$.

9 When the polynomial $x^3 + mx^2 + 10x - 24$ is divided by $x + 2$, the remainder is 32. Find the value of m.

10 Find the value of m that makes $5x^3 + mx^2 + 37x + 10$ divisible by $x + 5$.

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16.5 Solving cubic and quartic equations

When solving quadratic equations, we used the null factor law: if the product of two factors is zero, then one or both of the factors must be equal to zero. We can now extend this to the solving of cubic and quartic equations. If the left side of the equation is in factorised form and the right side is equal to zero, then the solutions can be found by putting each of the factors equal to zero.

Consider, for example, the cubic equation $(x + 2)(x - 1)(x - 4) = 0$.

Using the null factor law

$x + 2 = 0$ or $x - 1 = 0$ or $x - 4 = 0$

$\therefore x = -2$ or $x = 1$ or $x = 4$

Example 15

Solve the cubic equation $(x + 4)(x + 1)(x - 7) = 0$.

Working

$(x + 4)(x + 1)(x - 7) = 0$

$x + 4 = 0$ or $x + 1 = 0$ or $x - 7 = 0$

$\therefore x = -4$ or $x = -1$ or $x = 7$

Reasoning

Put each of the factors equal to zero then solve each linear equation for x.

The polynomial expression in an equation may have a common factor. Once any common factor is taken out, the factor theorem and long division are then used.

Example 16

Solve the quartic equation $x^4 - 2x^3 - 31x^2 - 28x = 0$.

Working

Factorise $x^4 - 2x^3 - 31x^2 - 28x$.

$x^4 - 2x^3 - 31x^2 - 28x$

$= x(x^3 - 2x^2 - 31x - 28)$

Look for factors of $x^3 - 2x^2 - 31x - 28$.

Reasoning

If the left side can be factorised, the null factor law can be used to solve the equation.

The left side has a common factor, x.

The factor theorem is used to find a linear factor of $x^3 - 2x^2 - 31x - 28$.

continued

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Example 16 continued

Working	Reasoning
$P(1) = 1^3 - 2 \times 1^2 - 31 \times 1 - 28$ $= 1 - 2 - 31 - 28$ $= -60$ $x - 1$ is not a factor.	Start with $P(1)$ and continue with $P(-1), P(2), P(-2)$ and so on until a zero is found.
$P(-1) = (-1)^3 - 2 \times (-1)^2 - 31 \times (-1) - 28$ $= -1 - 2 + 31 - 28$ $= 0$ $x + 1$ is a factor.	
$\begin{array}{r} x^2 - 3x - 28 \\ x+1\overline{)x^3 - 2x^2 - 31x - 28} \\ \underline{x^3 + x^2} \\ -3x^2 - 31x \\ \underline{-3x^2 - 3x} \\ -28x - 28 \\ \underline{-28x - 28} \\ 0 \end{array}$	Long division is used to find the quotient when $x^3 - 2x^2 - 31x - 28$ is divided by $x + 1$.
$x^2 - 3x - 28 = (x + 4)(x - 7)$ $x^4 - 2x^3 - 31x^2 - 28x = 0$ $x(x + 1)(x + 4)(x - 7) = 0$ $x = 0$ or $x + 1 = 0$ or $x + 4 = 0$ or $x - 7 = 0$ $\therefore x = 0, x = -1, x = -4$ or $x = 7$	The quotient is a quadratic trinomial that can be factorised. Express the left side of the equation as the product of its four factors. Use the null factor law to solve the equation.

The number of solutions of a polynomial equation

The maximum number of solutions to a polynomial equation is equal to the degree of the polynomial. For example, the maximum number of solutions to a cubic equation is three, although there are cubic equations that have one or two solutions. Just as for quadratic equations, a polynomial will have no solutions if the polynomial expression cannot be factorised. A polynomial that has repeated factors has fewer solutions than the degree of the polynomial. For example, the cubic equation $(x - 2)^2(x - 1) = 0$ has only two solutions: $x = 2$ or $x = 1$. The cubic equation $(x + 4)^3 = 0$ has only one solution: $x = -4$.

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Example 17

State the number of solutions for each of these polynomial equations.

a $2x(x-2)(x+3)=0$ **b** $(x-2)^3(x+1)=0$ **c** $(x-1)(x^2+7)=0$

Working	Reasoning
a $2x(x-2)(x+3)=0$ three solutions	There are three different linear factors. Each factor gives a solution.
b $(x-2)^3(x+1)=0$ two solutions.	$x-2$ is a repeated factor. There are only two different linear factors.
c $(x-1)(x^2+7)=0$ one solution.	x^2+7 is a sum of squares so $x^2+7=0$ has no real solutions. So only the factor $x-1$ gives a solution.

exercise 16.5

LINKS TO Example 15

1 Solve the following cubic equations.

a $(x+3)(x-2)(x+4)=0$ **b** $(x-1)(x+1)(x+2)=0$
c $(x-3)(x+2)(x-4)=0$ **d** $x(x+2)(x-2)=0$
e $(x+5)(x+3)(x-6)=0$ **f** $x(x+4)(x-8)=0$

LINKS TO Example 16

2 Solve the following cubic equations.

a $x^3+4x^2-12x=0$ **b** $x^3-6x^2-7x=0$
c $x^3-x^2-20x=0$ **d** $2x^3-20x^2-48x=0$

3 Solve the following cubic equations.

a $x^3-2x^2-23x+60=0$ **b** $x^3-8x^2+17x-10=0$
c $x^3-2x^2-11x+12=0$ **d** $x^3-2x^2-29x+30=0$
e $x^3+6x^2+3x-10=0$ **f** $x^3+3x^2-10x-24=0$
g $x^3+12x^2+47x+60=0$ **h** $x^3+8x^2+17x+10=0$
i $x^3+6x^2+5x-12=0$ **j** $x^3+5x^2-2x-24=0$
k $x^3-8x^2+19x-12=0$ **l** $x^3-3x^2-10x+24=0$

4 Solve the equation $x^4-4x^3-27x^2+34x+56=0$

LINKS TO Example 17

5 State the number of solutions for each of these polynomial equations.

a $(x-2)(x+4)(x-5)=0$ **b** $(x-1)^2(x-2)^2=0$
c $(x-2)^2(x+3)(x-5)=0$ **d** $x(x+1)(x-5)(x-7)=0$
e $(x-4)^3=0$ **f** $x(x-2)^3=0$

exercise 16.5 challenge

6 Solve the quartic equation $x^4+3.5x^3-6x^2-14x+8=0$.

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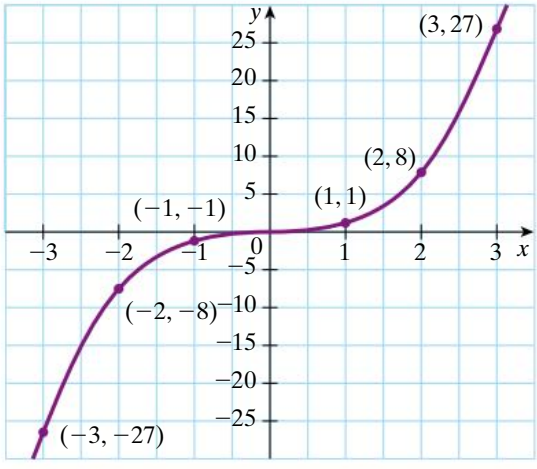
16.6 Sketching graphs of cubic polynomials

The graph of $y = x^3$

Before seeing how we sketch graphs of cubic polynomials, we begin by considering the graph of $y = x^3$. Unlike the parabola, the graph of $y = x^3$ is not symmetrical about the y-axis.

As shown in the table of values for $y = x^3$, x and y have the same sign. This means that the graph of $y = x^3$ is in the first and third quadrants, crossing the axes at $(0, 0)$.

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27



To understand the shape of the graph of $y = x^3$, we now consider the graphs of $y = x$, $y = x^2$ and $y = x^3$ in the region of $x = 1$ and $x = -1$. For values of x between 0 and 1, the square of a number is smaller than the number; for example, $0.5^2 = 0.25$. Cubing the x-value makes it even smaller. So for values of x between 0 and 1, the graphs of $y = x^2$ and $y = x^3$ are below the graph of $y = x$. At $x = 1$, y has the value 1 for all three graphs. For values of x greater than 1, squaring the x-value gives a larger number, whereas cubing the x-value gives an even larger number. So for values of x greater than 1, the graphs of $y = x^2$ and $y = x^3$ are above the graph of $y = x$ and rise more steeply as x increases.

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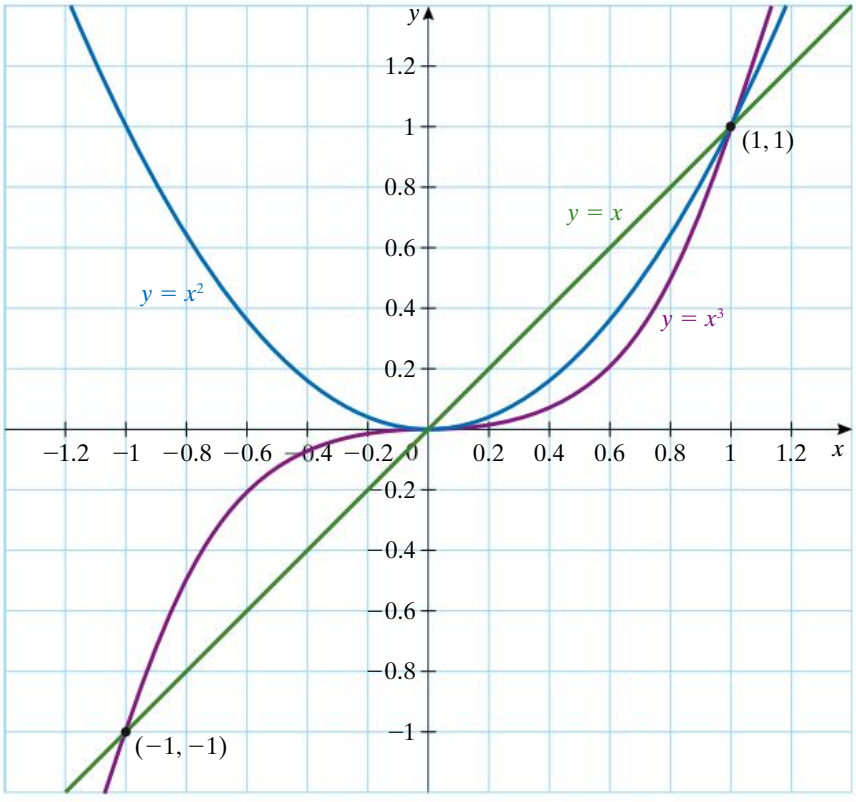
For negative values of x, the graph of $y = x^3$ is above the line $y = x$. This is because the cube of a negative number is greater than the number; for example, $(-0.5)^3$.

Whereas the graph of $y = x^2$ has a turning point at $x = 0$, the graph of $y = x^3$ becomes flat at $x = 0$.



Transformations of the graph of $y = x^3$

In chapters **11**, **12** and **15** we have seen how graphs can be reflected, translated and stretched. Example **18** shows transformations of the graph of $y = x^3$.

Example 18

Sketch graphs of the following cubic functions. For each graph show the graph of $y = x^3$ for comparison.

a $y = 2x^3$ **b** $y = -x^3$ **c** $y = (x - 2)^3$ **d** $y = x^3 - 2$

continued

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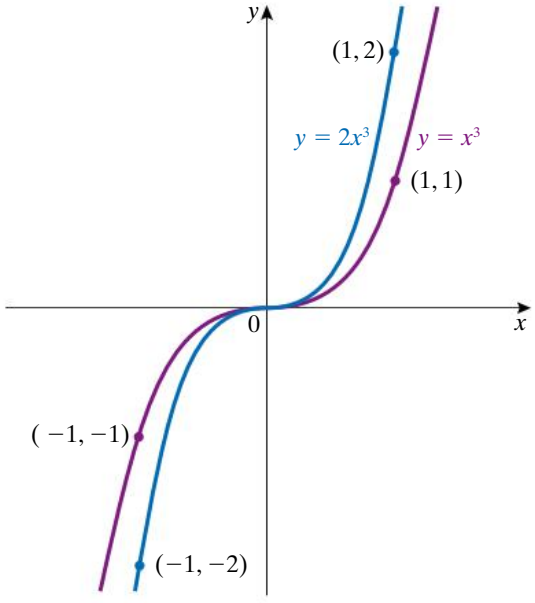
Example 18 continued

Working

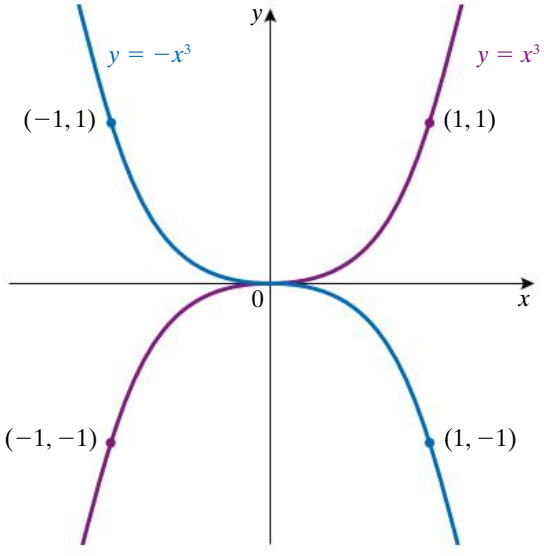
a $y = 2x^3$



Reasoning

We have seen with linear graphs $y = ax$ and parabolas $y = ax^2$ that the coefficient a affects the steepness of the graph. The graph of $y = 2x^3$ rises more steeply than the graph of $y = x^3$, that is, the graph has been stretched vertically.

b $y = -x^3$



The graph of $y = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis.

continued

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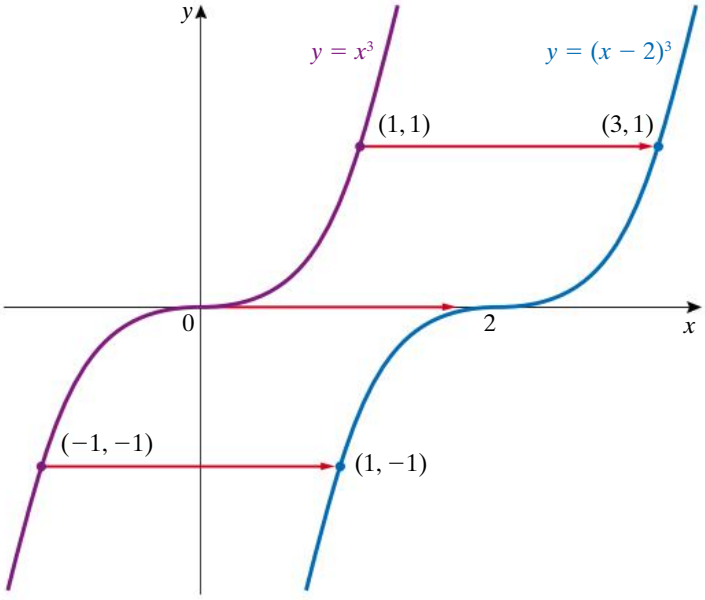
Example 18 continued

Working

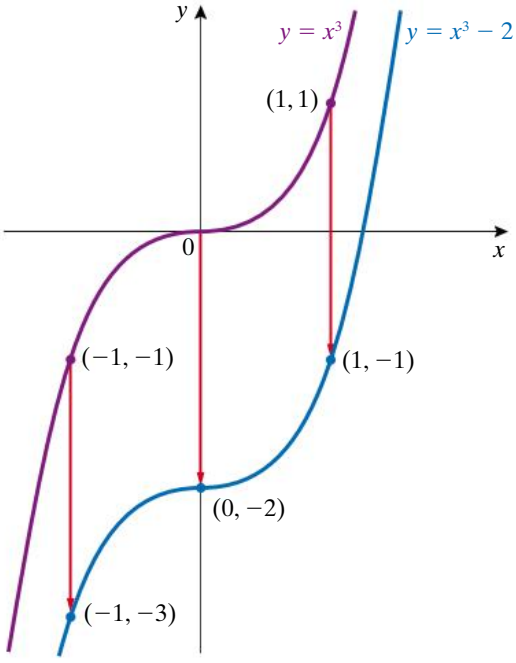
c $y = (x - 2)^3$



Reasoning

The graph of $y = (x - 2)^3$ is the graph of $y = x^3$ translated 2 units to the right. The graph becomes flat at $x = 2$.

d $y = x^3 - 2$



The graph of $y = x^3 - 2$ is the graph of $y = x^3$ translated 2 units down. The graph is flat at the point $(0, -2)$.

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Example 19

Sketch graphs of the following cubic functions showing the intercepts and two other points.

a $y = (x + 3)^3$

b $y = x^3 - 8$

Working

a $y = (x + 3)^3$

***x*-intercept:**

When $y = 0$,

$(x + 3)^3 = 0$

$x + 3 = 0$

$x = -3$

***y*-intercept:**

When $x = 0$,

$y = (0 + 3)^3$

$= 27$

When $x = -5$,

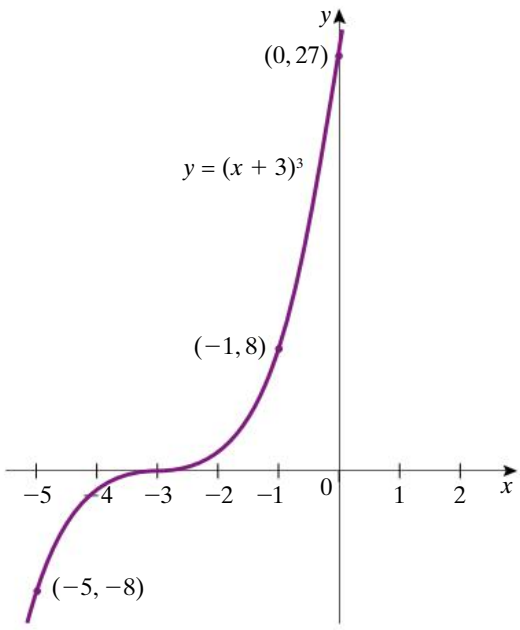
$y = (-5 + 3)^3$

$= -8$

When $x = -1$,

$y = (-1 + 3)^3$

$= 8$



Reasoning

Substitute $y = 0$ to find the x-intercept.

Substitute $x = 0$ to find the y-intercept. Choose two values for x and calculate the corresponding y-values.

The graph of $y = (x + 3)^3$ is the graph of $y = x^3$ translated 3 units to the left.

continued

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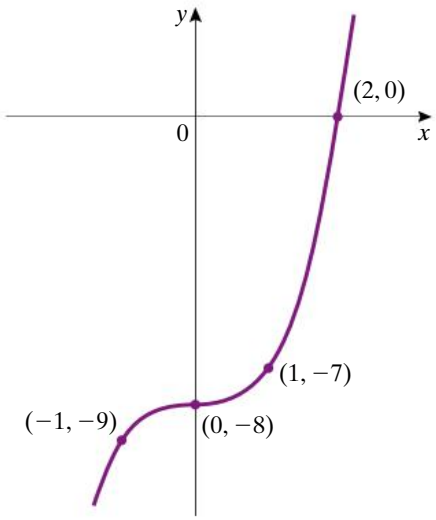
Example 19 continued

Working	Reasoning
b $y = x^3 - 8$	
***x*-intercept:**	
When $y = 0$,	Substitute $y = 0$ to find the x-intercept.
$x^3 - 8 = 0$	
$x^3 = 8$	
$x = 2$	
***y*-intercept:**	
When $x = 0$,	Substitute $x = 0$ to find the y-intercept.
$y = -8$	
When $x = -1$,	Choose two values for x and calculate the corresponding y-values.
$y = (-1)^3 - 8$	
$= -9$	
When $x = 1$,	
$y = 1^3 - 8$	
$= -7$	
	The graph of $y = x^3 - 8$ is the graph of $y = x^3$ translated 8 units down.

Graphing other cubic equations

In chapter 11, the x-intercepts of a parabola were found by substituting $y = 0$ and solving the quadratic equation. The y-intercept was found by substituting $x = 0$. The same techniques are used for sketching the graphs of cubic equations. However, we must also work out where the graph is above or below the x-axis.

Consider the cubic function $y = (x + 2)(x - 1)(x - 4)$.

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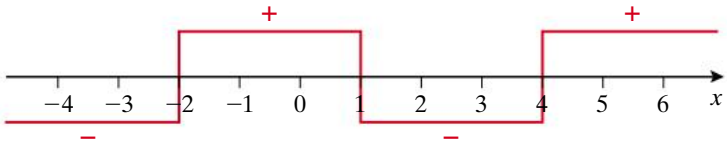
- To find the y-intercept we substitute $x = 0$.
  $y = (0 + 2)(0 - 1)(0 - 4) = 8$
  The y-intercept is at $(0, 8)$.
- To find the x-intercepts we substitute $y = 0$. Solving the equation $(x + 2)(x - 1)(x - 4) = 0$ we find that the x-intercepts are $x = -2$, $x = 1$ and $x = 4$.
- These intercepts divide the graph into four intervals as shown below and show where the graph crosses the x-axis.

- To work out where the graph is above or below the x-axis, we determine the sign of y (whether positive or negative) in each of the four intervals of the graph. If, for example, $x < -2$, each of the three factors will be negative and the product of these three negative values is negative. So in the interval $x < -2$, the graph is below the x-axis.

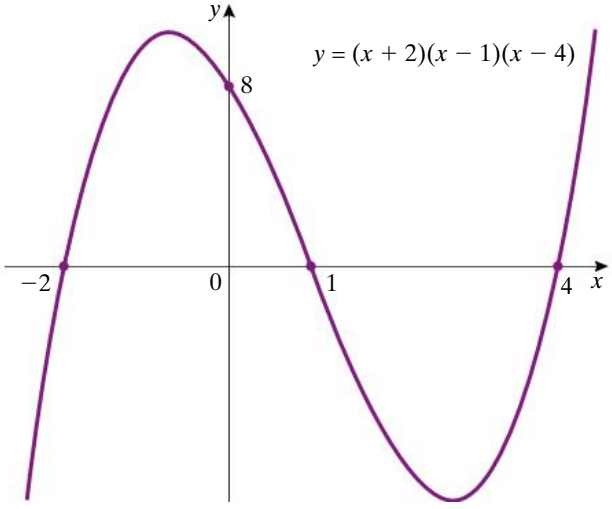
x	< -2 e.g. -3	-2	Between -2 and 1 e.g. 0	1	Between 1 and 4 e.g. 3	4	> 4 e.g. 5
y	$- \times - \times - = -$	0	$+ \times - \times - = +$	0	$+ \times + \times - = -$	0	$+ \times + \times + = +$

We can now show the position of the graph in relation to the x-axis. This diagram is called a **sign diagram**.



We now have sufficient information to sketch the graph.

Note that we do not know the coordinates of the turning points. The techniques for finding the turning points are included in senior mathematics in a branch of mathematics known as calculus. At this stage, sketching the graphs of cubic polynomials focuses on locating the intercepts and determining where the graph is above or below the x-axis.



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Finding the zeros

The values of a for which $P(a) = 0$ are referred to as the **zeroes** of the polynomial. These values of a are the x-intercepts of the graph.

Example 20

Sketch the graph of each of the following cubic polynomials.

a $y = (x + 3)(x + 1)(x - 2)$ **b** $y = -(x + 3)(x + 1)(x - 2)$

Working

a $y = (x + 3)(x + 1)(x - 2)$

Substitute $y = 0$ to find the x-intercepts.

$(x + 3)(x + 1)(x - 2) = 0$

$x + 3 = 0$ or $x + 1 = 0$ or $x - 2 = 0$

$\therefore x = -3$ or $x = -1$ or $x = 2$

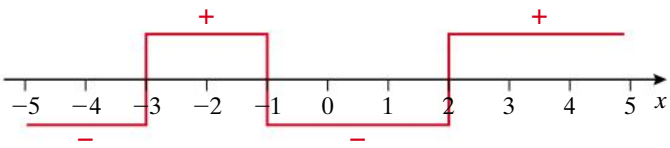
Substitute $x = 0$ to find the y-intercept

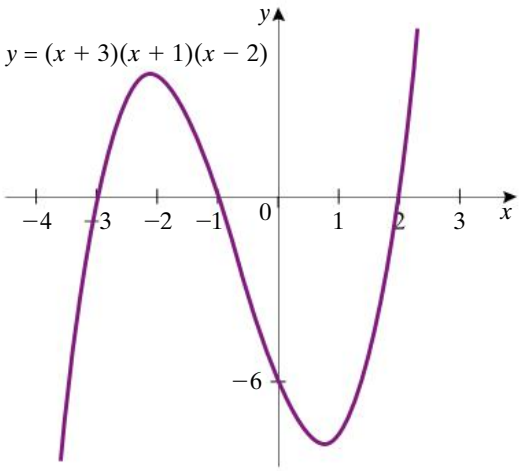
$y = (0 + 3)(0 + 1)(0 - 2)$

$y = -6$

x	-4	-3	-2	-1	1	2	3
y	$-$	0	$+$	0	$-$	0	$+$

Sign diagram:





Reasoning

Solve the equation for x. The graph has three x-intercepts.

The intercepts divide the graph into four intervals: $x < -3$, $-3 < x < -1$, $-1 < x < 2$ and $x > 2$. Choose a suitable value for x in each of the intervals and determine the sign of the product of the three factors.

$x = -4$: $- \times - \times - \rightarrow -$

$x = -2$: $+ \times - \times - \rightarrow +$

$x = 1$: $+ \times + \times - \rightarrow -$

$x = 3$: $+ \times + \times + \rightarrow +$

continued

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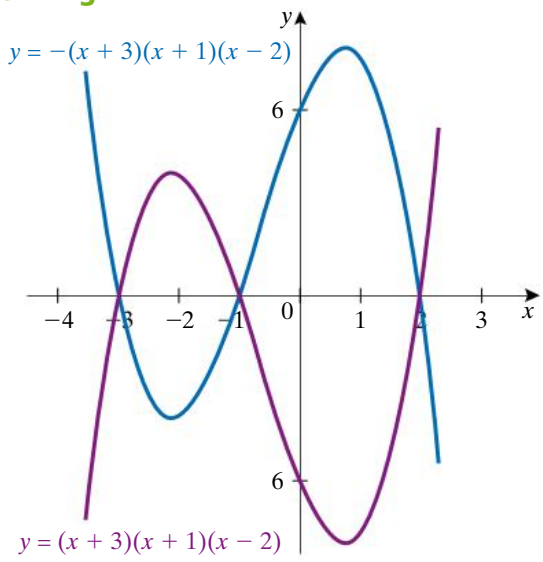
Example 20 continued

Working

b



Reasoning

The graph of $y = -(x+3)(x+1)(x-2)$ is the reflection in the x-axis of the graph of $y = (x+3)(x+1)(x-2)$.

If the cubic polynomial is not given in factorised form, the factor theorem and long division can be used to find the factors before carrying out the steps shown in the previous example.

Example 21

Sketch the graph of $y = x^3 + 2x^2 - 13x + 10$.

Working

$y = x^3 + 2x^2 - 13x + 10$

y-intercept:

When $x = 0$,

$y = 10$

x-intercept:

When $y = 0$,

$x^3 + 2x^2 - 13x + 10 = 0$

$P(1) = 1^3 + 2 \times 1^2 - 13 \times 1 + 10$

$= 1 + 2 - 13 + 10$

$= 0$

$x - 1$ is a factor

Reasoning

Use the factor theorem to find a factor of the polynomial.

continued

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Example 21 continued

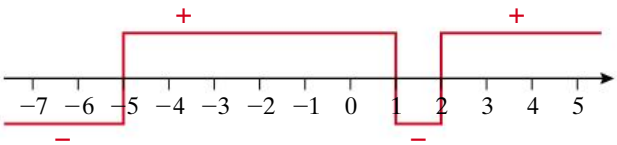
Working

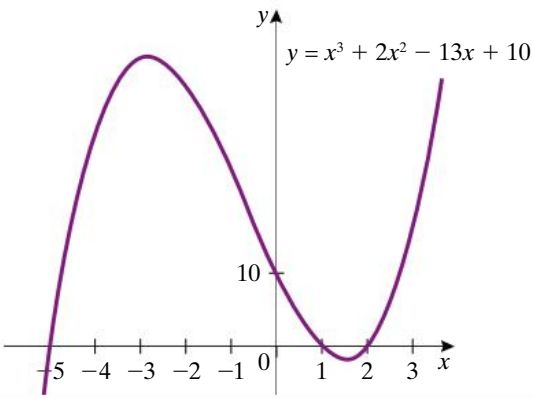
$$\begin{array}{r} x^2 + 3x - 10 \\ x - 1\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - x^2} \\ 3x^2 - 13x \\ \underline{3x^2 - 3x} \\ -10x + 10 \\ \underline{-10x + 10} \\ 0 \end{array}$$

So $x^2 + 3x - 10 = (x - 2)(x + 5)$
$\therefore (x - 1)(x - 2)(x + 5) = 0$
x-intercepts are $x = 1$, $x = 2$ and $x = -5$.

x	-6	-5	-1	1	1.5	2	3
y	$-$	0	$+$	0	$-$	0	$+$

Sign diagram:





Reasoning

Use long division to divide the polynomial by the linear factor.

Write the polynomial as the product of its factors.
Find the x-intercepts.

The intercepts divide the graph into four intervals:
$x < -5$, $-5 < x < 1$, $1 < x < 2$ and $x > 2$. Choose a suitable value for x in each of the intervals and determine the sign of the product of the three factors.

$x = -6$: $- \times - \times - \rightarrow -$
$x = -1$: $- \times - \times + \rightarrow +$
$x = 1.5$: $+ \times - \times + \rightarrow -$
$x = 3$: $+ \times + \times + \rightarrow +$

Example 22

A polynomial of degree 3 has leading coefficient 1 or -1. Its graph passes through the point $(1, -12)$ and has x-intercepts -3, 0 and 4.

a Find the equation.

b Sketch a graph of the polynomial.

continued

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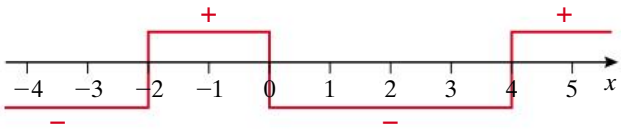
Example 22 continued

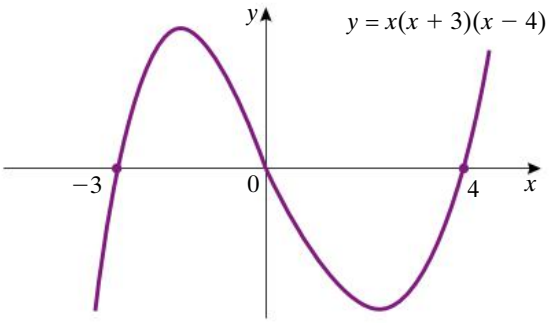
Working

a The factors are x, $(x + 3)$ and $(x - 4)$.
The equation could be $y = x(x + 3)(x - 4)$
or $y = -x(x + 3)(x - 4)$

Substitute $x = 1$, into
$y = x(x + 3)(x - 4)$
$y = 1 \times 3 \times -4 = -12$
The point $(1, -12)$ satisfies the equation $y = x(x + 3)(x - 4)$ so this is the correct equation.

b





Reasoning

$x = 0$ shows that x is a factor.
$x = -3$ shows that $x + 3$ is a factor.
$x = 4$ shows that $x - 4$ is a factor.
A monic polynomial of degree 3 can have no more than 3 factors.

The graph of $y = -x(x + 3)(x - 4)$ is a reflection in the x-axis of the graph of $y = x(x + 3)(x - 4)$ so the two graphs would share the same x-intercepts.
To identify the correct equation, substitute $x = 1$ to see if $y = -12$.

$x = -4$: $- \times - \times - \rightarrow -$
$x = -2$: $- \times + \times - \rightarrow +$
$x = 2$: $+ \times + \times - \rightarrow -$
$x = 6$: $+ \times + \times + \rightarrow +$

Repeated factors

In chapter **11**, we saw that the graph of a quadratic equation that had a repeated factor had only one x-intercept and just touched the x-axis at this point. For example, the parabola with equation $y = (x - 2)^2$ just touches the x-axis at $x = 2$.

In example **18** part **c**, we saw that the graph of $y = (x - 2)^3$ crossed the x-axis at $x = 2$ but the graph was flat at this point.

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Odd and even powers of repeated factors

The graph of $y = (x - a)^n$ is flat at $x = a$.

The graph of $y = (x - a)^n$ touches the x-axis at $x = a$ for even values of n.

The graph of $y = (x - a)^n$ crosses the x-axis at $x = a$ for odd values of n.

Example 23

Sketch the graph of $y = (x - 2)^2(x - 5)$.

Working

$y = (x - 2)^2(x - 5)$

***x*-intercepts:**

When $y = 0$,

$(x - 2)^2(x - 5) = 0$

$x - 2 = 0$ or $x - 5 = 0$

$\therefore x = 2$ or $x = 5$

***y*-intercept:**

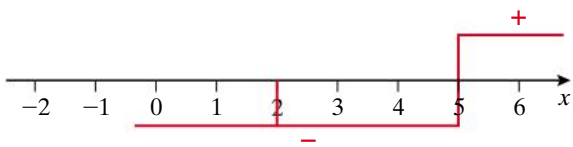
When $x = 0$,

$y = (0 - 2)^2(0 - 5)$

$y = -20$

x	1	2	3	5	6
y	−	0	−	0	+

Sign diagram:



Reasoning

The intercepts divide the graph into three intervals: $x < 2$, $2 < x < 5$ and $x > 5$. Choose a suitable value for x in each of the intervals and determine the sign of the product of the factors.

$x = 1$: $- \times - \times - \rightarrow -$

$x = 3$: $+ \times + \times - \rightarrow -$

$x = 6$: $+ \times + \times + \rightarrow +$

We need to consider the sign of each of the three factors.

There is an intercept at $x = 2$ but the graph is below the axis on both sides of 2. The graph touches the x-axis at $x = 2$.

continued

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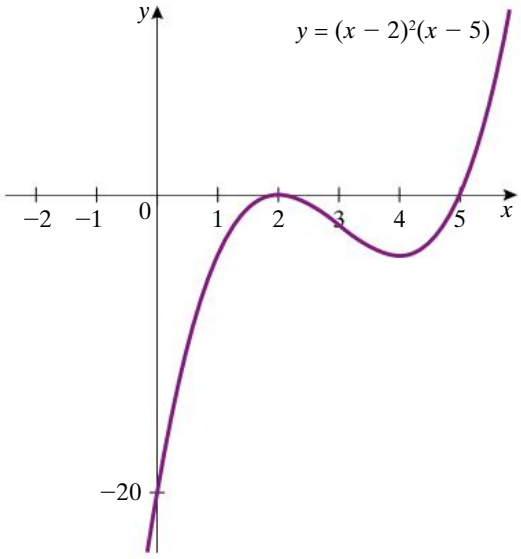
Example 23 continued

Working



Reasoning

exercise 16.6

LINKS TO Examples 18, 19

1 Sketch graphs of the following cubic functions. For each graph

- **i** show the graph of $y = x^3$ for comparison
- **ii** label the intercepts
- **iii** label two points on the graph

a $y = 3x^3$ **b** $y = -3x^3$ **c** $y = (x - 3)^3$ **d** $y = x^3 + 8$

e $y = x^3 + 64$ **f** $y = x^3 - 27$ **g** $y = (x + 3)^3$ **h** $y = (x + 2)^3$

LINKS TO Example 20

2 For each of these factorised cubic functions

- **i** state the x-intercepts of the graph
- **ii** find the y-intercept
- **iii** draw a sign diagram
- **iv** sketch the graph, labelling the intercepts

a $y = (x + 4)(x - 1)(x - 3)$ **b** $y = x(x - 2)(x - 3)$

c $y = x(x + 3)(x - 3)$ **d** $y = -(x + 1)(x - 1)(x - 5)$

e $y = (x + 6)(x + 3)(x - 5)$ **f** $y = x(x + 5)(x - 5)$

g $y = -(x + 2)(x - 3)(x - 8)$ **h** $y = x(x - 4)^2$

i $y = (x - 1)^2(x - 4)$ **j** $y = (x + 2)^2(x - 3)$

k $y = x^2(x + 4)$ **l** $y = (x - 5)^2(x - 9)$

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10 A

LINKS TO Example 21

3 For each of the following cubic functions

- **i** use your solutions to the equations in exercise **16.5** question **2** to state the x-intercepts
- **ii** find the y-intercept
- **iii** draw a sign diagram
- **iv** sketch the graph, labelling the intercepts

a $y = x^3 + 4x^2 - 12x$ **b** $y = x^3 - 6x^2 - 7x$

c $y = x^3 - x^2 - 20x$ **d** $y = 2x^3 - 20x^2 - 48x$

LINKS TO Example 22

4 The graph of a monic polynomial of degree 3 has x-intercepts at -4, -1, and 5. The y-intercept is at -20.

a What is the equation?

b Draw a sign diagram for the graph.

c Sketch the graph, labelling the intercepts.

d Use GeoGebra to check your graph.

LINKS TO Example 23

5 For each of the following cubic functions

- **i** use your solutions to the equations in exercise **16.5** question **3** to state the x-intercepts
- **ii** find the y-intercept
- **iii** draw a sign diagram
- **iv** sketch the graph, labelling the intercepts

a $y = x^3 - 2x^2 - 23x + 60$ **b** $y = x^3 - 8x^2 + 17x - 10$

c $y = x^3 - 2x^2 - 11x + 12$ **d** $y = x^3 - 2x^2 - 29x + 30$

e $y = x^3 + 6x^2 + 3x - 10$ **f** $y = x^3 + 3x^2 - 10x - 24$

g $y = x^3 + 12x^2 + 47x + 60$ **h** $y = x^3 + 8x^2 + 17x + 10$

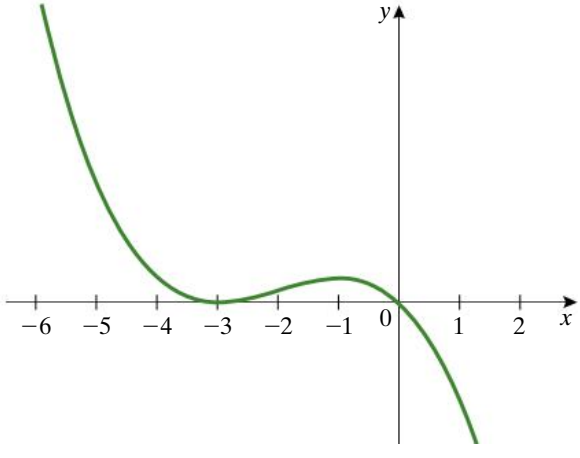
i $y = x^3 + 6x^2 + 5x - 12$ **j** $y = x^3 + 5x^2 - 2x - 24$

k $y = x^3 - 8x^2 + 19x - 12$ **l** $y = x^3 - 3x^2 - 10x + 24$

exercise 16.6 challenge

6 **a** Find three possible equations for the following monic cubic graph.

b Explain why more than one equation is possible.



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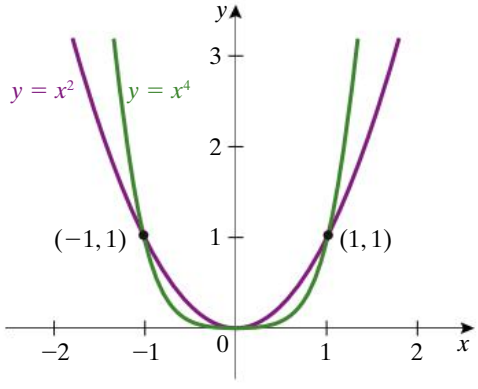
16.7 Sketching graphs of quartic polynomials

The graph of the quartic function $y = x^4$ is shown below, together with the graph of $y = x^2$ for comparison. Note that the graph of $y = x^4$ is 'flatter' at the bottom. This is because for x between 0 and 1, x^4 increases more slowly than x^2. For values of x greater than 1, x^4 increases more rapidly than x^2. Like the graph of $y = x^2$, the graph of $y = x^4$ is symmetrical about the y-axis.



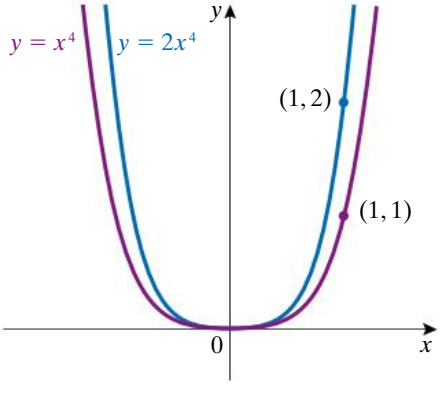
Example 24

Sketch graphs of the following quartic functions. For each graph show the graph of $y = x^4$ for comparison.

a $y = 2x^4$ **b** $y = -x^4$ **c** $y = (x - 1)^4$ **d** $y = x^4 + 3$

Working

a $y = 2x^4$



Reasoning

$y = 2x^4$ rises more steeply than the graph of $y = x^4$, that is, the graph is stretched vertically.

continued

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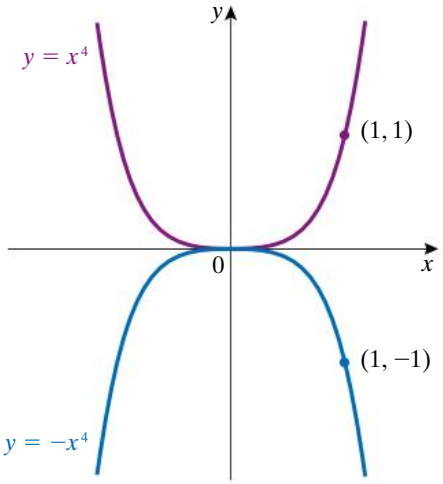
Example 24 continued

Working

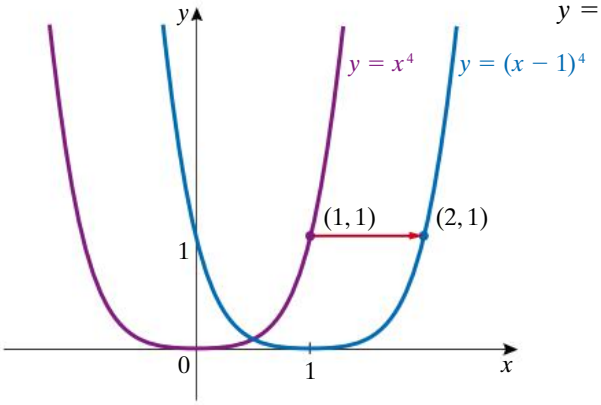
b $y = -x^4$



Reasoning

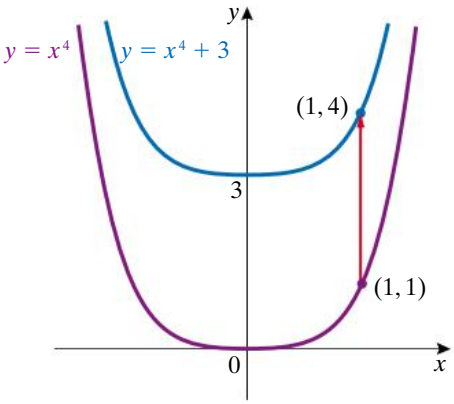
The graph of $y = -x^4$ is the reflection in the x-axis of the graph of $y = x^4$.

c $y = (x - 1)^4$



The graph of $y = (x - 1)^4$ is the graph of $y = x^4$ translated 1 unit to the right.

d $y = x^4 + 3$



The graph of $y = x^4 + 3$ is the graph of $y = x^4$ translated 3 units up.

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Example 25

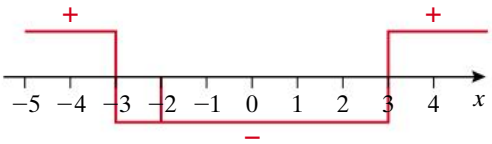
In example 14, the quartic polynomial $x^4 + 4x^3 - 5x^2 - 36x - 36$ was factorised as $(x + 2)^2(x + 3)(x - 3)$. Sketch the graph of $y = x^4 + 4x^3 - 5x^2 - 36x - 36$.

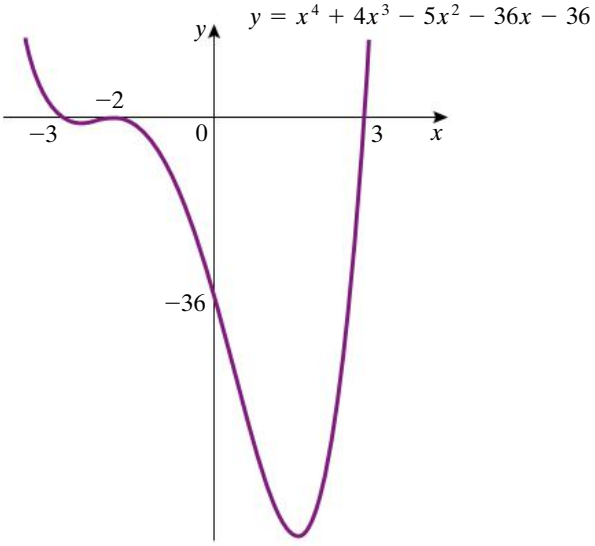
Working

x-intercepts:
When $x = -2$, $x = -3$ and $x = 3$,
The graph touches the x-axis at $x = -2$.
y-intercept:
When $x = 0$, $y = -36$

x	-4	-3	-2.5	-2	0	3	4
y	$+$	0	$-$	0	$-$	0	$+$

Sign diagram:





Reasoning

If a factor $(x - a)$ is raised to an even power, the graph just touches the axis at $x = a$.

The x-intercepts divide the graph into four intervals:
$x < -3$, $-3 < x < -2$, $-2 < x < 3$ and $x > 3$. Choose a suitable value for x in each of the intervals and determine the sign of the product of the three factors.

$x = -4$: $- \times - \times - \times - \rightarrow +$
$x = -2.5$: $- \times - \times + \times - \rightarrow -$
$x = 0$: $+ \times + \times + \times - \rightarrow -$
$x = 4$: $+ \times + \times + \times + \rightarrow +$

The graph is below the x-axis either side of $x = -2$.

Example 26

In example 16, the quartic equation $x^4 - 2x^3 - 31x^2 - 28x = 0$ was solved. The solutions were $x = 0$, $x = -1$, $x = -4$ or $x = 7$. Sketch the graph of $y = x^4 - 2x^3 - 31x^2 - 28x$.

continued

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Example 26 continued

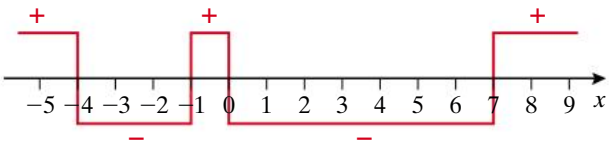
Working

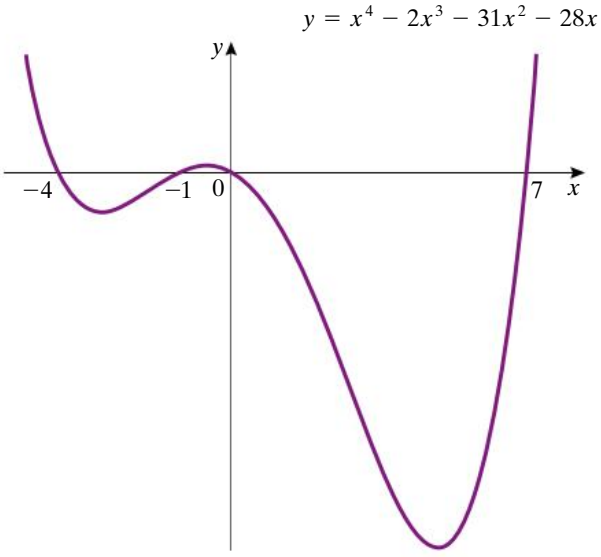
x-intercepts:
$x = 0$, $x = -1$, $x = -4$ and $x = 7$
y-intercept:
When $x = 0$, $y = 0$

Factors:
$x(x + 1)(x + 4)(x - 7)$

x	-5	-4	-2	-1	-1.5	0	3	7	8
y	$+$	0	$-$	0	$+$	0	$-$	0	$+$

Sign diagram:





Reasoning

The solutions to the equation $x^4 - 2x^3 - 31x^2 - 28x = 0$ give the x-intercepts.
x is a factor of the polynomial so the graph passes through $(0, 0)$

The intercepts divide the graph into five intervals:
$x < -4$, $-4 < x < -1$, $-1 < x < 0$, $0 < x < 7$ and $x > 7$.
Choose a suitable value for x in each of the intervals and determine the sign of the product of the three factors.

$x = -5$: $- \times - \times - \times - \rightarrow +$
$x = -2$: $- \times - \times + \times - \rightarrow -$
$x = -0.5$: $- \times + \times + \times - \rightarrow +$
$x = 3$: $+ \times + \times + \times - \rightarrow -$
$x = 8$: $+ \times + \times + \times + \rightarrow +$

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exercise 16.7

LINKS TO Example 24

1 Sketch graphs of the following quartic functions. For each graph show the graph of $y = x^4$ for comparison. Label two points on the graph.

a $y = -2x^4$ **b** $y = -x^4 + 1$ **c** $y = (x - 2)^4$ **d** $y = x^4 - 4$

LINKS TO Example 25

2 For each of the following quartic functions

i state the x-intercepts

ii find the y-intercept

iii draw a sign diagram

iv sketch the graph, labelling the intercepts

a $y = (x + 4)(x + 2)(x - 2)(x - 5)$ **b** $y = (x + 5)(x + 3)(x + 1)(x - 4)$

c $y = (x + 1)(x - 3)(x - 5)(x - 8)$ **d** $y = x(x - 3)(x - 5)(x - 6)$

e $y = x(x + 1)^2(x - 3)$ **f** $y = x^2(x - 4)(x - 6)$

g $y = x(x - 2)^2(x - 6)$ **h** $y = x(x - 1)(x - 2)^2$

LINKS TO Example 26

3 For each of the following quartic functions

i use the factors you found in exercise **16.4** question **6** to state the x-intercepts

ii find the y-intercept

iii draw a sign diagram

iv sketch the graph, labelling the intercepts

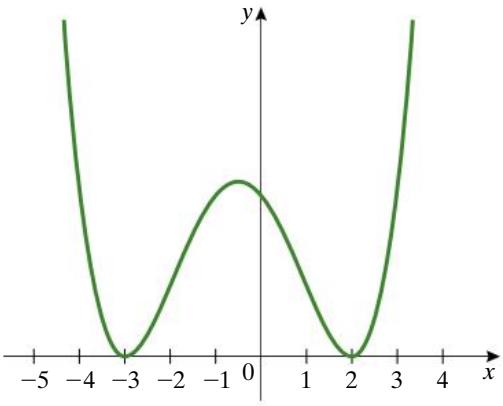
a $y = x^4 - 7x^3 + 17x^2 - 17x + 6$ **b** $y = x^4 + 3x^3 - 15x^2 - 19x + 30$

c $y = x^4 - 6x^3 - 16x^2 + 54x + 63$ **d** $y = x^4 + 5x^3 - 28x^2 - 20x + 96$

exercise 16.7 challenge

4 **a** Find the equation for the following quartic graph, if it is monic, and state the y-intercept.

b Find a possible equation if the quartic is not monic. Explain why many equations are possible.



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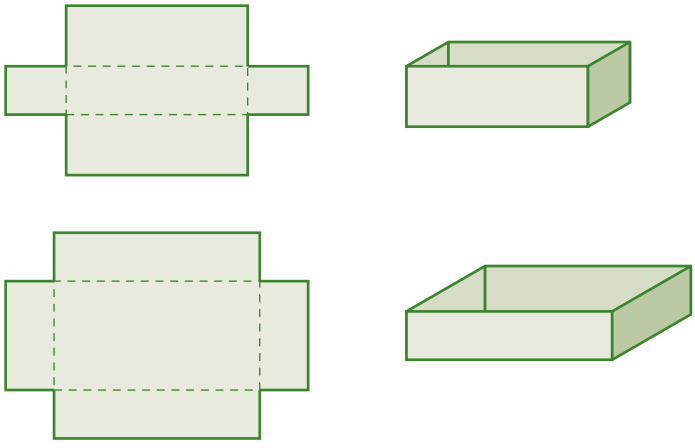
Analysis task

Cardboard cut-outs

A piece of cardboard measures 16 cm by 12 cm. Squares of side length x cm are cut from each corner and the cardboard is folded into an open box shape.



a Write expressions in terms of x for the length and width of the box.

b Write an expression in factorised form in terms of x for the volume, V cm^3, of the box.

c Expand the expression.

d If the volume function, $V(x)$, is graphed, state the x-intercepts.

e Determine the y-intercept.

f Draw a sign diagram for the function.

g Explain the sign diagram in terms of the context of the box.

h Sketch a graph of the volume function.

i Set up a spreadsheet to calculate the volume for $0 < x < 6$, using steps of 0.01 for x.

j Use the spreadsheet values to estimate the maximum volume of the box.

k For what value of x does this maximum volume occur?

l Use GeoGebra to graph the volume function. In the Input bar type
$y = 4x(6 - x)(8 - x)$.
To see the shape of the graph, you will need to change the scale on the axes. Right click on the screen and select Graphics in the Graphics list. Select the Grid tab and set the Distance for x to 1 and for y to 10. Now select the Basic tab and set y Max at 250. The other settings should take care of themselves.

m In the Input bar type: TurningPoint[$4x(6 - x)(8 - x)$, 0, 6]
A point will then appear at the turning point of the graph between $x = 0$ and $x = 6$. Right click on the point, go to Object Properties then beside Show Label choose Value. The coordinates of the turning point will then be displayed beside the point. Compare the maximum value with the value that you estimated in part **j**.

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Review Polynomials

Summary

Describing polynomials

Leading term	The term with the highest index.
Leading coefficient	The coefficient of the leading term. Polynomials with a leading coefficient of 1 are called monic polynomials.
Degree	The index of the leading term. Degree 2: Quadratic Degree 3: Cubic Degree 4: Quartic
Constant term	A term that does not include the pronumeral x.

The remainder theorem

To find the remainder when a polynomial $P(x)$ is divided by $(x - a)$, evaluate $P(a)$.

The factor theorem

If $P(a) = 0$, then $(x - a)$ is a factor of the polynomial $P(x)$.

Factorising polynomials

- Look for a common factor
- Use the factor theorem to find the first factor. Look at the factors of the constant term. Choose one of these factors to calculate $P(a)$. If $P(a) = 0$, then $x - a$ is a factor. If $P(a) \neq 0$ systematically continue testing other factors of the constant term for a until a value is found for which $P(a) = 0$.
- Use long division to find a polynomial factor.
- For a cubic polynomial, long division gives a quadratic factor that may be able to be further factorised.

Solving polynomial equations

- Factorise the polynomial expression if not already in factorised form
- Use the null factor law

Sketching graphs of polynomials

- Factorise the polynomial expression if not already in factorised form.
- Find the x-intercepts by putting $y = 0$ and solving the equation.

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- Find the y-intercept by putting $x = 0$.
- Draw a sign diagram to determine where the graph is above or below the x-axis, that is, where the polynomial expression is positive or negative.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

constant term	leading coefficient	remainder
cubic polynomial	leading term	remainder theorem
degree	long division	repeated factor
divisor	monic polynomial	sign diagram
factor	polynomial	zeros
factor theorem	quartic polynomial	
intercept	quotient	

Revision

Multiple-choice questions

1 If $P(x) = x^4 - 3x^2 + 5x - 7$ then $P(-3)$ equals

A -14 **B** 32 **C** 62 **D** 86 **E** 116

2 If the cubic equation $x^3 + 2x^2 - 3x - 5$ is divided by $x + 2$, the remainder is

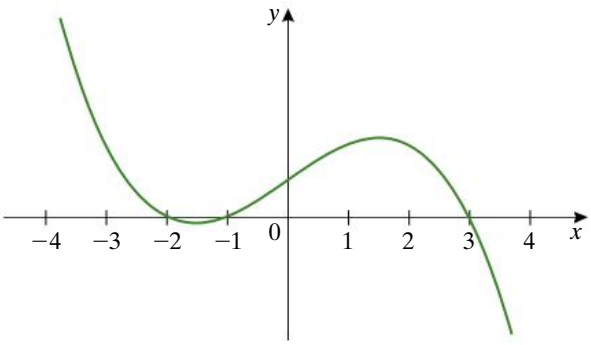
A 0 **B** 1 **C** 5 **D** 7 **E** 15

3 The cubic equation $x^3 - mx + 2$ is exactly divisible by $x - 2$. The value of m is

A 2 **B** 3 **C** -3 **D** 5 **E** -5

4 The graph of a monic polynomial of degree 3 is shown.
The equation that matches the graph is

A $y = (x + 1)(x + 2)(x + 3)$
B $y = (x - 1)(x - 2)(x + 3)$
C $y = -(x + 1)(x + 2)(x + 3)$
D $y = -(x - 1)(x - 2)(x + 3)$
E $y = -(x + 1)(x + 2)(x - 3)$



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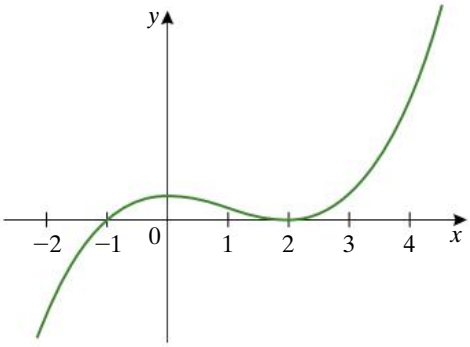
5 Which of the following is the graph of $y = (x + 2)^2(x - 1)$?

A



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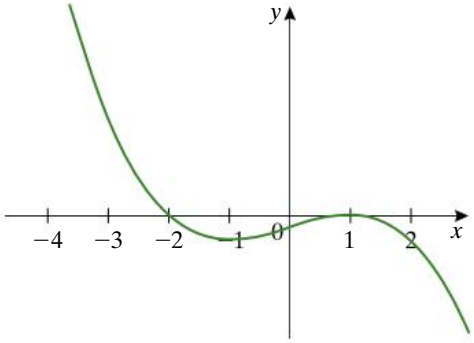

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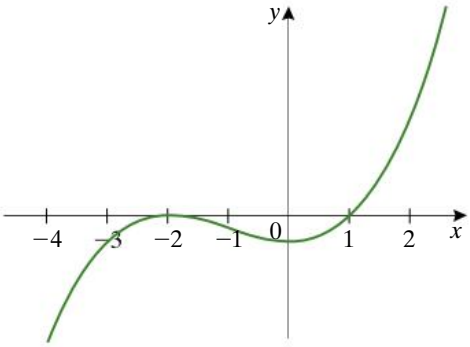

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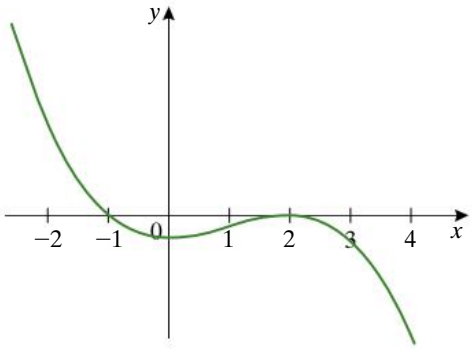
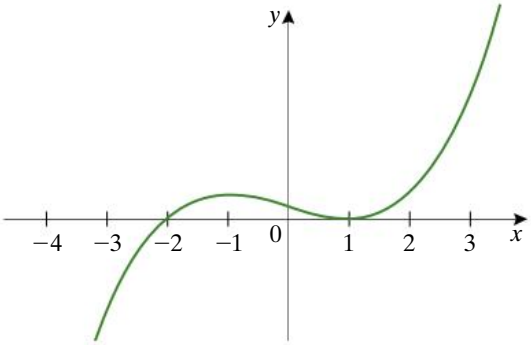


E



Short-answer questions

6 Which of the following expressions are polynomials? Give reasons.

a $\dfrac{2x^3}{5} - 4x + 6$ **b** $\dfrac{1}{x^3} - \dfrac{3}{x^2} - \dfrac{4}{x}$ **c** $x^2 - x^{0.5}$ **d** $2x^2 - 7x + \sqrt{7}$

7 For each of these polynomials

i the degree **ii** the leading term

iii the leading coefficient **iv** the constant term

a $-2x^4 + 5x^3 - x^2 - 7x + 15$ **b** $3x - 7x^2 + 6x^3 - 2$

c $-2x + 5$ **d** $-x^4 + 8x^2$

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8 Classify each of the polynomials in question **7** as linear, quadratic, cubic or quartic.

9 For each of these polynomials, calculate

i $P(-1)$

ii $P(2)$

a $x^3 - 2x^2 - 5x - 4$

b $x^4 - 4x^3 + 3x^2 + 4$

10 For each of these pairs of polynomials

i find the sum

ii subtract the second polynomial from the first

a $2x^2 - 7x + 5$ and $x^2 - 9$

b $3x^3 - 7x^2 + 4$ and $-2x^3 - 11x + 8$

11 Find the product of these polynomials

a $x - 1$ and $x^2 - 3x + 4$

b $x^2 - 4$ and $2x^2 + 5x - 3$

12 Expand.

a $(x + 1)(x + 4)(x - 5)$

b $(x - 2)^2(2x + 5)$

13 Use long division to find the quotient and remainder when these polynomials are divided by

i $x - 3$

ii $x + 2$

a $x^3 - 3x^2 + 7x - 11$

b $x^4 - 5x^2 - 8x - 23$

14 Solve these polynomial equations

a $(x + 3)(x - 2)(x - 4) = 0$

b $(x + 1)(x - 2)^2 = 0$

c $x(x + 4)(x - 6) = 0$

d $(x + 1)^3 = 0$

Extended-response questions

15 Factorise the following polynomials.

a $x^3 + 3x^2 - 10x$

b $x^3 - 10x^2 + 28x - 24$

c $x^4 - 15x^2 + 10x + 24$

d $x^4 + 2x^3 - 8x^2 - 18x - 9$

16 For each of these polynomial functions

i use the factors you found in question **15** to state the x-intercepts

ii find the y-intercept

iii draw a sign diagram

iv sketch the graph, labelling the intercepts

a $y = x^3 + 3x^2 - 10x$

b $y = x^3 - 10x^2 + 28x - 24$

c $y = x^4 - 15x^2 + 10x + 24$

d $y = x^4 + 2x^3 - 8x^2 - 18x - 9$

Practice quiz
Chapter 16

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Answers

Chapter 1

exercise 1.1

1 **a** $x = 14$ **b** $x = 32$ **c** $x = 7$
d $x = -2$ **e** $x = -9$ **f** $x = -42$
g $x = 7$ **h** $x = -8$ **i** $x = 6$
j $x = 6$ **k** $x = 300$ **l** $x = -16$

2 C

3 **a** $x = 7$ **b** $x = 3$ **c** $x = 5$
d $x = 9$ **e** $x = -4\frac{1}{2}$ **f** $x = -4\frac{1}{5}$
g $x = -3$ **h** $x = -9$ **i** $x = -8$
j $x = -9$ **k** $x = -3\frac{5}{9}$ **l** $x = -1.4$
m $x = 5$ **n** $x = -2$ **o** $x = -2$

4 **a** yes
b Subtract 8 from both sides: $4x = 3$
Divide both sides by 4: $x = \frac{3}{4}$

5 D

6 **a** $a = 56$ **b** $x = 15$ **c** $x = -20$
d $x = -2\frac{2}{5}$ **e** $a = 22$ **f** $m = 20$
g $x = 21$ **h** $x = 66$ **i** $x = 64$
j $x = -16$ **k** $x = 3\frac{1}{2}$ **l** $x = -2\frac{4}{5}$
m $a = 9\frac{4}{5}$ **n** $a = -4\frac{1}{2}$ **o** $x = 40$
p $x = -6$ **q** $x = 15$ **r** $x = 7\frac{1}{2}$

7 **a** $a = -6$ **b** $x = \frac{3}{4}$ **c** $x = 6$
d $x = -39\frac{1}{2}$ **e** $x = 3\frac{1}{2}$ **f** $x = 12$
g $x = 6\frac{1}{2}$ **h** $x = 11$ **i** $x = 23$
j $x = -2$ **k** $x = 2$ **l** $x = 5$

8 **a** $x = 5$ **b** $x = 2$ **c** $x = \frac{1}{2}$
d $x = 6$ **e** $x = 1\frac{1}{9}$ **f** $x = 1$
g $x = 1\frac{3}{4}$ **h** $a = 4\frac{1}{6}$ **i** $x = 2\frac{2}{5}$
j $x = 5\frac{4}{5}$ **k** $x = -26$ **l** $x = -18\frac{1}{5}$

9 **a** no **b** yes **c** no
d yes **e** yes **f** yes

10 **a** $x = 30$ **b** $x = 1\frac{1}{5}$ **c** $x = 40$
d $x = 45$ **e** $x = \frac{1}{2}$ **f** $x = 52\frac{1}{2}$
g $x = \frac{1}{5}$ **h** $x = 24$ **i** $x = 105$
j $x = 80$ **k** $x = 7\frac{1}{2}$ **l** $x = 120$

11 **a** $a = 3$ **b** $x = 9$ **c** $x = 7$
d $k = 10$ **e** $x = 13$ **f** $x = 3$

12 **a** $a = 4$ **b** $e = \frac{3}{5}$ **c** $d = 1$
d $m = 0$ **e** $x = -\frac{1}{2}$ **f** $m = -\frac{1}{5}$
g $k = -2\frac{1}{2}$ **h** $x = -1\frac{1}{3}$ **i** $x = -\frac{2}{5}$
j $a = \frac{3}{5}$ **k** $y = 1$ **l** $j = \frac{4}{9}$
m $e = \frac{4}{5}$ **n** $a = 8$ **o** $b = -2\frac{1}{3}$
p $a = \frac{5}{6}$

13 D

14 **a** $x = -\frac{1}{5}$ **b** $a = -1\frac{3}{5}$
c $x = 1$ **d** $x = -26$
e $x = \frac{2}{3}$ **f** $m = 3$
g $x = -10$ **h** $x = 27$
i $x = -5.8\left(-5\frac{4}{5}\right)$ **j** $x = 1\frac{1}{7}$
k $x = 8$ **l** $x = 0$
m $x = -6\frac{1}{2}$ **n** $x = -6.1\left(-6\frac{1}{10}\right)$

15 **a** $x = -6$ **b** $m = 1$ **c** $p = -4$ **d** $x = -4\frac{1}{2}$

16 **a** $x = 1\frac{3}{4}$ **b** $x = -1\frac{4}{7}$ **c** $x = 4$
d $a = 31$ **e** $x = \frac{1}{4}$ **f** $m = \frac{11}{13}$
g $p = -22$ **h** $x = 8$ **i** $y = 4\frac{1}{4}$
j $x = 8$ **k** $m = 5\frac{1}{2}$ **l** $p = 2$
m $b = 4\frac{1}{3}$ **n** $x = 3\frac{5}{7}$

17 $h = 4$ **18** $b = 47$ **19** $c = 21\frac{1}{9}$

20 $R = 4.8$ **21** $r = 3.7$

22 **a** yes
b Expand brackets: $3x - 9 = 4x$
Subtract $3x$ from both sides: $-9 = 4x - 3x$
$-9 = x$
$x = -9$
c Avoid fractions by first expanding the brackets; then isolate terms in x to one side.

23 **a** $x = -4\frac{2}{5}$ **b** $x = 2\frac{13}{16}$

24 For any value we try for x, the left side is always three less than this value and the right side is always one more, so the two sides can never be equal; or: if we subtract x from both sides, we get $-3 = 1$, which is not true.

25 Multiplying by two and then expanding the right side gives $6x + 2 = 6x - 6$. This has no solution, as with question **18**.

26 $x = 2\frac{3}{4}$

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exercise 1.2

1 **a** $a - 9 = 12; a = 21$ **b** $2b + 5 = 19; b = 7$
c $3c - 7 = 20; c = 9$ **d** $\frac{d}{2} + 6 = 10; d = 8$
e $4e + 8 = 40; e = 8$ **f** $5(f + 3) = 25; f = 2$
g $2(g + 8) = -6; g = -11$
h $3(h - 4) = 12; h = 8$

2 **a** $\frac{1}{2}(n + 3) + 2 = 7$ **b** 7 pot plants

3 23 yellow lollies

4 lowest grade is 48

5 lengths are 17 cm and 23 cm

6 30 students

7 100 cars

8 \$672

9 21 and 22

10 34 and 36

11 83, 85 and 87

12 10 m

13 length 52 m, width 20 m

14 **a** $j + 12$ **b** $j + 12 = 3(j - 28)$
c 48 **d** in 22 years

15 Anna is 7, Bella is 25.

16 100 kilometres

17 15 litres

18 **a** $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4$
b $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x; x = 84$
c 84 years old

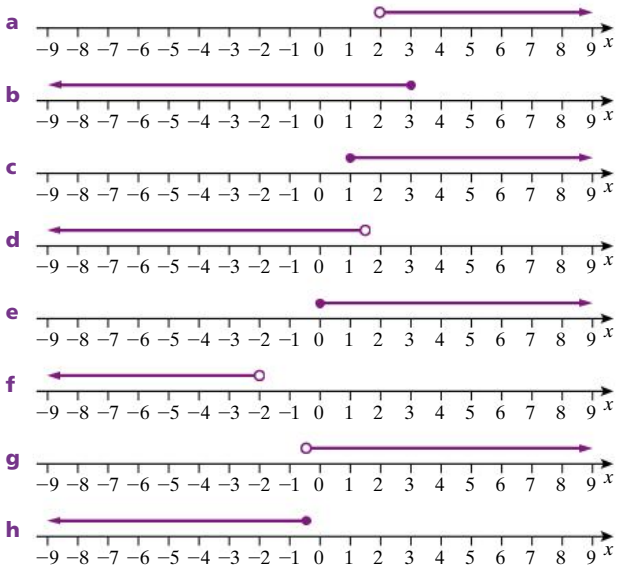
exercise 1.3

1 **a** $b = a - x$ **b** $a = \frac{F}{m}$ **c** $T = \frac{D}{S}$
d $T = \frac{I}{PR}$ **e** $r = \frac{C}{2\pi}$ **f** $m = \frac{E}{c^2}$
g $w = \frac{3V}{lh}$ **h** $b = 2M - a$
i $h = \frac{1}{2}P - l$ or $H = \frac{P - 2l}{2}$ **j** $r = \sqrt{\frac{A}{\pi}}$
k $t = \frac{v - u}{a}$ **l** $V = \frac{k}{P}$
m $b = \frac{2A}{h} - a$ or $b = \frac{2A - ah}{h}$
n $h = \frac{S}{b} - 2b$ or $h = \frac{S - 2b^2}{b}$
o $C = \frac{5}{9}(F - 32)$ **p** $a = \frac{2(s - ut)}{t^2}$
q $h = \frac{3V}{\pi r^2}$ **r** $r = \sqrt{\frac{3V}{\pi h}}$

2 **a** $x = c - a - b$ **b** $x = \frac{bc}{a}$
c $x = \frac{cd - b}{a}$ **d** $x = \frac{de - ac}{ab}$
e $a = \frac{ac}{b - a}$ **f** $c = \frac{bde}{ad + bc}$

exercise 1.4

1



2 **a** $x \geq -3$ **b** $-4 < x < 4$
c $0 < x \leq 4$ **d** $x < 5$

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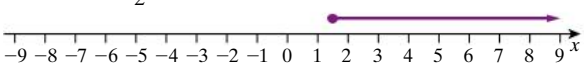
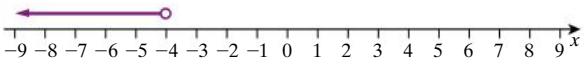
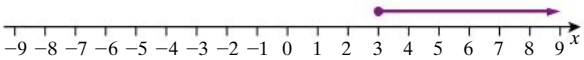
3 **a** $x \geq 1\frac{1}{2}$;



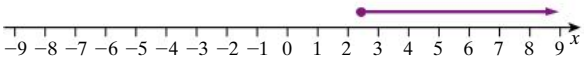
b $x < -4$;



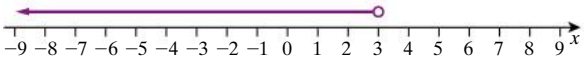
c $x \geq 3$;



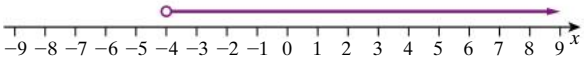
d $x \geq 2\frac{2}{5}$;



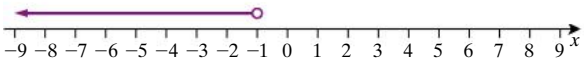
e $x < 3$;



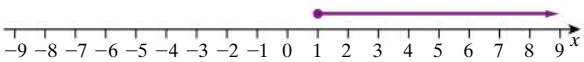
f $x > -4$;



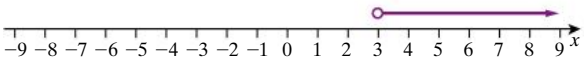
g $x < -1$;



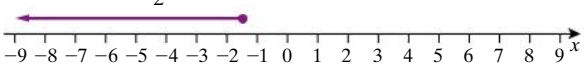
h $x \geq 1$;



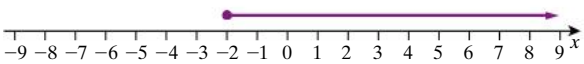
i $x > 3$;



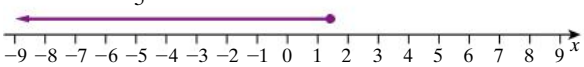
j $x \leq -1\frac{1}{2}$;



k $x \geq -2$;



l $x \leq 1\frac{2}{5}$;



4 B

5 **a** $x \leq 4$ **b** $x > 1\frac{1}{2}$ **c** $x < -1$
d $x \leq -18$ **e** $x > 6$ **f** $x > -4$
g $x < -3\frac{1}{3}$ **h** $x > -2\frac{2}{3}$ **i** $x < -5$
j $x \geq 5\frac{2}{3}$ **k** $x \leq -8$ **l** $x < -12$

6 **a** $x < 7$ **b** $m \geq 5$ **c** $p \geq -1$
d $w > 2$ **e** $x > \frac{3}{4}$ **f** $x \geq 11$
g $z \geq 7\frac{1}{2}$ **h** $x > -1\frac{4}{7}$
i $y \leq 4\frac{1}{3}$ **j** $y < 3$

7 C

8 Raphael is at least 13 and at most 18.

9 **a** $x \geq 2$ **b** $x > \frac{9}{11}$ **c** $y \leq 58$ **d** $x \leq 2\frac{1}{3}$

10 $p \leq -3$

11 Nick needs to score 83 or more.

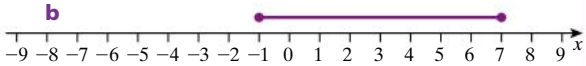
12 Plan A is cheaper if the bike is hired for less than 6 hours.

13 The first plan is cheaper if the bike is used for more than 60 kilometres.

14 Erryn's weekly sales would need to exceed $8000.

15 Answers will vary.

16 **a** $x \geq -1$ and $x \leq 7$
b



revision

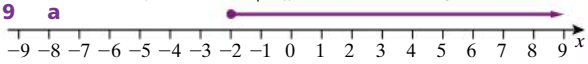
1 A **2** A **3** E **4** C **5** A

6 **a** $y = 11$ **b** $b = 8$

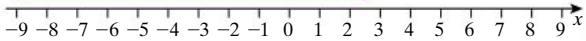
7 **a** $x = -2$ **b** $x = 6$ **c** $x = 20$
d $x = 12$ **e** $x = 3$ **f** $x = 2$
g $x = -5$ **h** $x = -\frac{1}{2}$ **i** $x = 31$
j $x = 4$ **k** $x = 36$ **l** $x = 40$
m $x = -2\frac{3}{8}$ **n** $x = -10$ **o** $x = 3\frac{1}{4}$
p $x = 9$

8 **a** $a = \frac{v - u}{t}$ **b** $r\sqrt{\frac{S}{4\pi}}$ **c** $P = \frac{3W}{5T^2}$

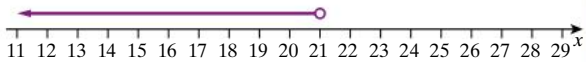
9 **a**



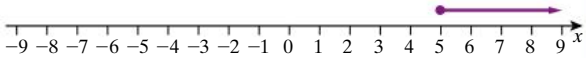
b



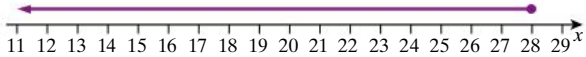
10 **a** $x < 21$



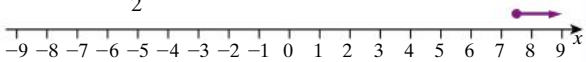
b $x \geq 5$



c $x \leq 28$



d $x \geq 7\frac{1}{2}$



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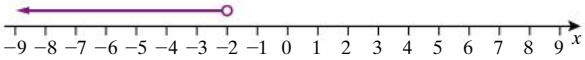
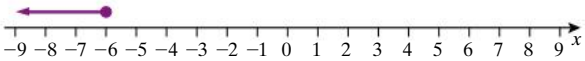
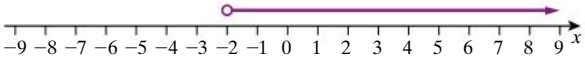
e $x < -2$



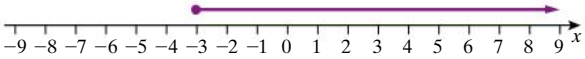
f $x \leq -6$



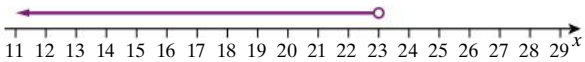
g $x > -2$



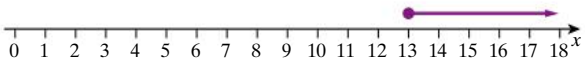
h $x \geq -3$



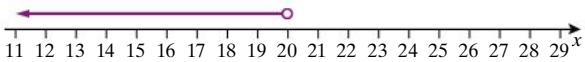
i $x < 23$



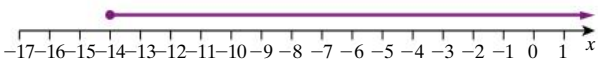
j $x \geq 13$



k $x < 20$



l $x \geq -14$



11 **a** 77°F **b** $c = \frac{5}{9}(f - 32)$
c 37.8°C **d** −40 degrees

12 **a** $48x + 7$ ($) **b** $48x + 7 = 535$
c 11 tickets

Chapter 2

exercise 2.1

1 **a** **i** $\sin\alpha° = \frac{12}{13}$ **ii** $\cos\alpha° = \frac{5}{13}$ **iii** $\tan\alpha° = \frac{12}{5}$
iv $\sin\beta° = \frac{5}{13}$ **v** $\cos\beta° = \frac{12}{13}$ **vi** $\tan\beta° = \frac{5}{12}$
b **i** $\sin\alpha° = \frac{4}{5}$ **ii** $\cos\alpha° = \frac{3}{5}$ **iii** $\tan\alpha° = \frac{4}{3}$
iv $\sin\beta° = \frac{3}{5}$ **v** $\cos\beta° = \frac{4}{5}$ **vi** $\tan\beta° = \frac{3}{4}$

2 **a** $\sin\alpha° = \cos\beta°$; $\cos\alpha° = \sin\beta°$
b $\tan\alpha°$ and $\tan\beta°$ are reciprocals, i.e.
$\tan\alpha° = \frac{1}{\tan\beta°}$
c The sine of an angle is equal to the cosine of the complementary angle. The tangent of an angle is equal to the reciprocal of the tangent of the complementary angle.

3 $\sin 7° = \cos 83°$; $\sin 36° = \cos 54°$; $\sin 49° = \cos 41°$; $\sin 54° = \cos 36°$; $\sin 72° = \cos 18°$

4 **a** 0.9877 **b** 0.9703 **c** 3.7321
d 0.5000 **e** 0.5370 **f** 0.3576
g 0.8783 **h** 1.0000 **i** 572.9572
j 1.0000 **k** 0.0017 **l** 0.9999

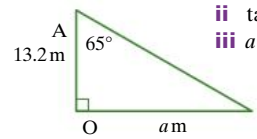
5 **a** 4.6 **b** 33.4 **c** 8.9
d 24.9 **e** 72.4 **f** 16.6
g 44.3 **h** 1004.6 **i** 254.1
j 212.6 **k** 12.8 **l** 233.4

6 **a** $x = 5.1$ **b** $x = 11.9$ **c** $x = 297.2$
d $x = 6.5$ **e** $x = 4.8$ **f** $x = 18.0$
g $x = 37.3$ **h** $x = 29.8$

7 **a** $x = 142.0$ **b** $x = 36.1$ **c** $x = 20.0$
d $x = 116.3$ **e** $x = 376.4$ **f** $x = 61.6$
g $x = 35.9$ **h** $x = 2.8$ **i** $x = 120.0$
j $x = 7.8$ **k** $x = 124.3$ **l** $x = 5.3$

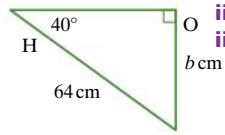
exercise 2.2

1 **a** **i**



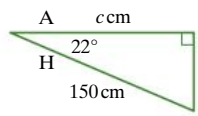
ii $\tan 65° = \frac{a}{13.2}$
iii $a \approx 28.3$

b **i**



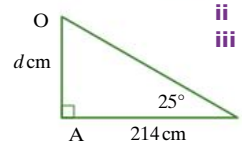
ii $\sin 40° = \frac{b}{64}$
iii $b \approx 41$

c **i**



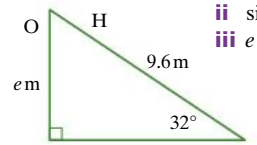
ii $\cos 22° = \frac{c}{150}$
iii $c \approx 139$

d **i**



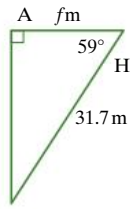
ii $\tan 25° = \frac{d}{214}$
iii $d \approx 100$

e **i**



ii $\sin 32° = \frac{e}{9.6}$
iii $e \approx 5.1$

f **i**



ii $\cos 59° = \frac{f}{31.7}$
iii $f \approx 16.3$

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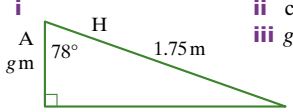
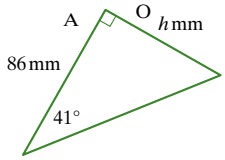
g i



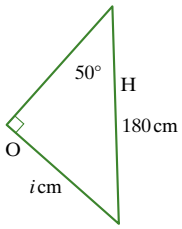
ii $\cos 78° = \frac{g}{1.75}$
iii $g \approx 0.36$

h i



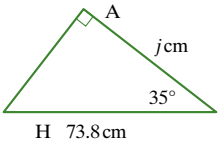
ii $\tan 41° = \frac{h}{86}$
iii $h \approx 75$

i i



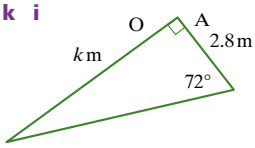
ii $\sin 50° = \frac{i}{180}$
iii $i \approx 138$

j i



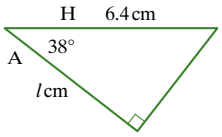
ii $\cos 35° = \frac{j}{73.8}$
iii $j \approx 60.5$

k i



ii $\tan 72° = \frac{k}{2.8}$
iii $k \approx 8.6$

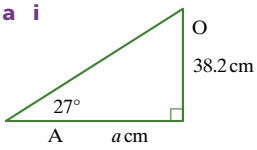
l i



ii $\cos 38° = \frac{l}{6.4}$
iii $l \approx 5.0$

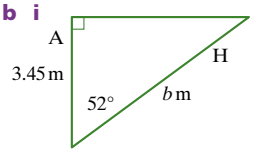
2 C

3 a i



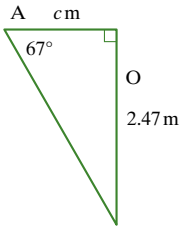
ii $\tan 27° = \frac{38.2}{a}$
iii $a \approx 75.0$

b i



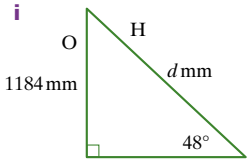
ii $\cos 52° = \frac{3.45}{b}$
iii $b \approx 5.60$

c i



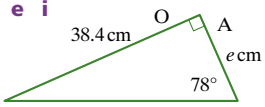
ii $\tan 67° = \frac{2.47}{c}$
iii $c \approx 1.05$

d i



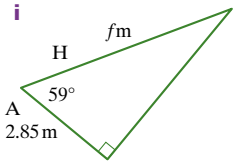
ii $\sin 48° = \frac{1184}{d}$
iii $d \approx 1593$

e i



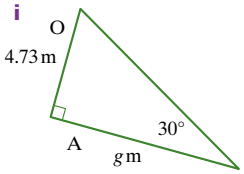
ii $\tan 78° = \frac{38.4}{e}$
iii $e \approx 8.2$

f i



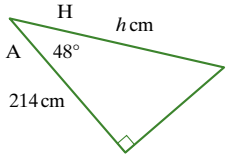
ii $\cos 59° = \frac{2.85}{f}$
iii $f \approx 5.53$

g i



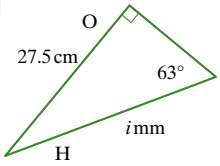
ii $\tan 30° = \frac{4.73}{g}$
iii $g \approx 8.19$

h i



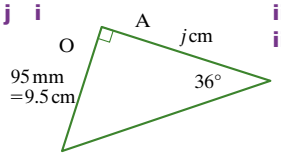
ii $\cos 48° = \frac{214}{h}$
iii $h \approx 320$

i i



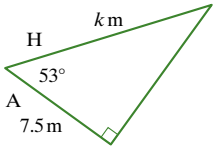
ii $\sin 63° = \frac{275}{i}$
iii $i \approx 309$

j i



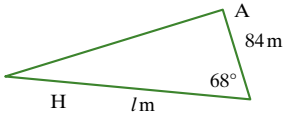
ii $\tan 36° = \frac{9.5}{j}$
iii $j \approx 13.1$

k i



ii $\cos 53° = \frac{7.5}{k}$
iii $k \approx 12.5$

l i



ii $\cos 68° = \frac{84}{l}$
iii $l \approx 224$

4 D

5 a i $\tan 53° = \frac{a}{45}$ **ii** $a \approx 60$

b i $\cos 24° = \frac{b}{65}$ **ii** $b \approx 59$

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c **i** $\sin 33° = \frac{4.2}{c}$ **ii** $c \approx 7.7$

d **i** $\cos 65° = \frac{d}{10.6}$ **ii** $d \approx 4.5$

e **i** $\tan 15° = \frac{2.8}{e}$ **ii** $e \approx 10.4$

f **i** $\cos 58° = \frac{f}{8.8}$ **ii** $f \approx 4.7$

g **i** $\sin 41° = \frac{3.4}{g}$ **ii** $g \approx 5.2$

h **i** $\tan 29° = \frac{h}{17.6}$ **ii** $h \approx 9.8$

i **i** $\cos 23° = \frac{2.7}{i}$ **ii** $i \approx 2.9$

j **i** $\tan 32° = \frac{3.6}{j}$ **ii** $j \approx 5.8$

k **i** $\sin 57° = \frac{k}{5.9}$ **ii** $k \approx 4.9$

l **i** $\sin 40° = \frac{1.4}{l}$ **ii** $l \approx 2.2$

6 **a** 58°; 75.0 m; 141.5 m **b** 69°; 19.5 m; 20.9 m

7 101°; 18.1 m; 33.6 m

exercise 2.3

1 **a** 54° **b** 41° **c** 16° **d** 48°
e 61° **f** 67° **g** 86° **h** 57°

2 **a** 21° **b** 63° **c** 57° **d** 33°
e 87° **f** 87° **g** 84° **h** 89°
i 60° **j** 45° **k** 46° **l** 31°

3 **a** **i** 9° **ii** 75.4 m
b **i** 54° **ii** 583.0 mm
c **i** 23° **ii** 115.0 cm
d **i** 58° **ii** 43.7 m
e **i** 55° **ii** 43.9 m
f **i** 62° **ii** 193.3 mm
g **i** 58° **ii** 44.8 m
h **i** 76° **ii** 191.4 cm

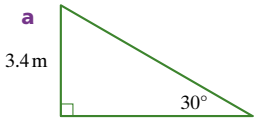
4 **a** **i** $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$
ii 28° **iii** 62°
b **i** $\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$
ii 74° **iii** 16°

5 62°; 86°; 20.3 m

exercise 2.4

1 9 m

2 **a**



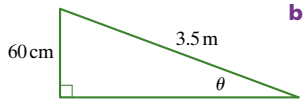
b 5.89 m

3 **a** 25.6° **b** 13.9 m

4 **a** 32° **b** 341 cm

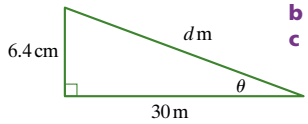
5 **a** 19° **b** 296 cm

6 **a**



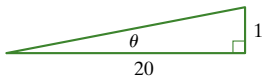
b 10°

7 **a**



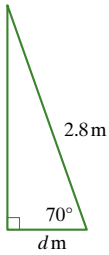
b 12°
c 30.7 m

8



a 2.9°
b 4.8°

9



a $d \approx 0.96$, so minimum distance is 96 cm, correct to the nearest centimetre.
b further away

10 D

11 **a** 16° **b** 20°

12 A

13 **a** **i** $AB = 7981$ mm
ii $DE = 2520$ mm
iii $AE = 3563$ mm
b 43.1 m

14 **a** 3.4° **b** 2.9°

exercise 2.5

1 **a** $a \approx 84.5, \theta \approx 50.3°$ **b** $b \approx 3.9, \theta = 53.1°$
c $c \approx 8.6, \theta \approx 69.0°$ **d** $d \approx 32.3, \theta \approx 58.2°$

2 **a** $a = 150, b = 170, \theta \approx 61.9°$
b $x \approx 52, y \approx 61, \theta \approx 58.6°$
c $a \approx 149, b \approx 156, \theta \approx 72.9°$
d $a \approx 45, b \approx 51, \theta \approx 36.9°$
e $a \approx 93, b \approx 117, \theta \approx 35.7°$
f $c \approx 32, d \approx 98, \theta \approx 27.5°$
g $m \approx 40, n \approx 55.0, \theta \approx 51.4°$
h $x \approx 22, y \approx 68.2, \theta \approx 23.6°$

3 D

4 **a** **i** $h = 80$ **ii** $\theta \approx 67.4°$ **iii** 6000 m^2
b **i** $x = 70$ **ii** $\theta \approx 61.9°$ **iii** 12 600 m^2
c **i** $x = 70, y = 70$
ii $\theta \approx 53.1°$ **iii** 2800 m^2
d **i** $a \approx 49.4, b \approx 59$
ii $\theta \approx 59.6°$ **iii** 1826 m^2
e **i** $a \approx 95, b \approx 115$
ii $\theta \approx 66.8°$ **iii** 7981 m^2
f **i** $d \approx 54$ **ii** $\theta \approx 55.8°$ **iii** 1658 m^2

5 2001 m^2

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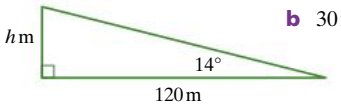
exercise 2.6

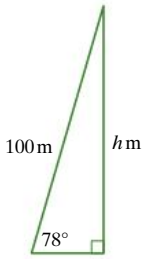
1 21 m **2** 400 m **3** 8.4 m

4 **a**



b 30 m

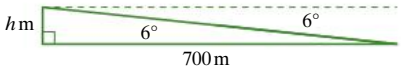
5 **a**



b 98 m

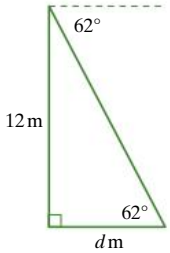
6 ≈ 63 m

7 **a**



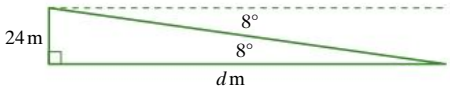
b ≈ 74 m

8 **a**



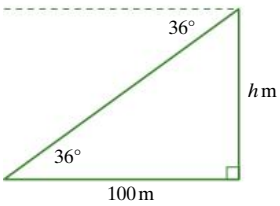
b 6.4 m

9 **a**



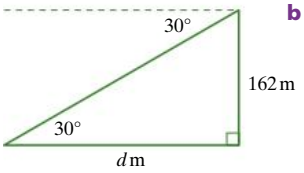
b 170 m

10 **a**



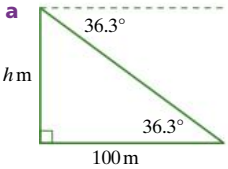
b 73 m

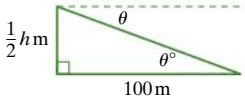
11 **a**



b 280 m

12 **a**



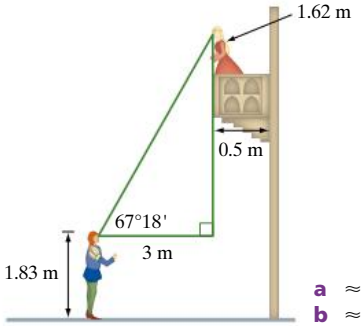


b 20.2° (height of building is 73.5 m)

13 **a** $d \approx 53.0$ **b** 31 m

14 **a** $d \approx 666$ **b** ≈ 522 m
c It would be less, ie ≈ 486 m
d no; error in height calculation is little affected by Joel's height
e If angle measurements could be out by ±0.5°, calculation with each combination of pairs of possible angles shows that the hill's height could be any value between about 456 m and about 610 m.

15



a ≈ 7.8 m
b ≈ 7.4 m

16 **a** 1180 m **b** 22.5°

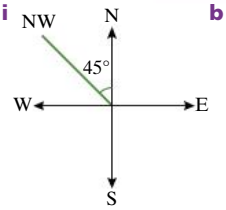
exercise 2.7

1 **a** $a = 14$ **b** $b = 2.5$ **c** $c = 1.5$ **d** $d = 13$
e $e \approx 14.3$ **f** $f \approx 46.8$ **g** $g \approx 63.2$ **h** $h \approx 2.19$

2 ≈ 14.2 cm

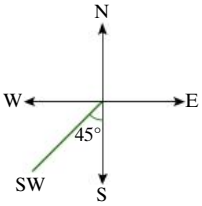
exercise 2.8

1 **a** **i**



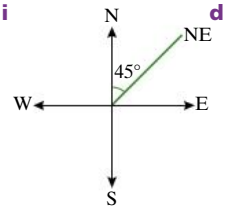
ii 315°

b **i**



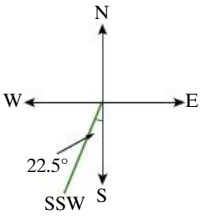
ii 225°

c **i**



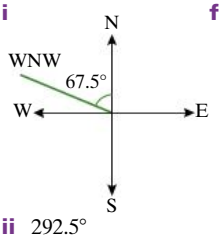
ii 045°

d **i**



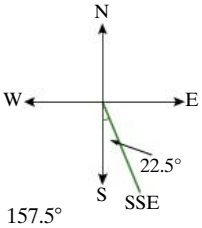
ii 202.5°

e **i**



ii 292.5°

f **i**



ii 157.5°

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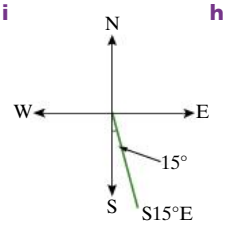
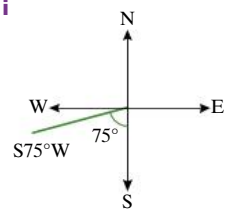
g i



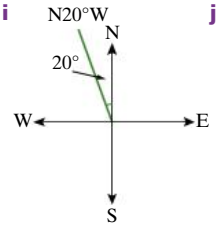
ii 165°

h i



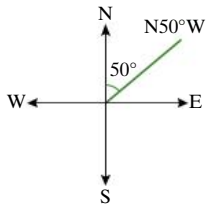
ii 255°

i i



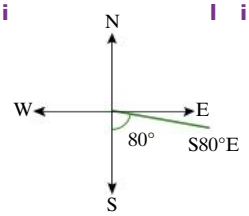
ii 340°

j i



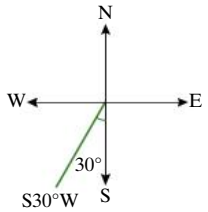
ii 050°

k i



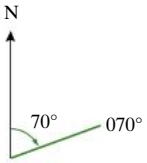
ii 100°

l i



ii 210°

2 a



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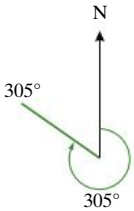

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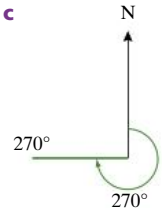

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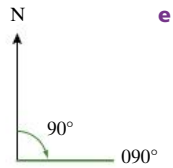
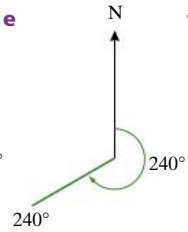
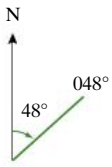


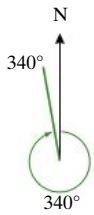
e



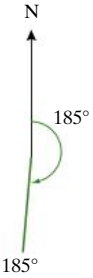
f



g



h



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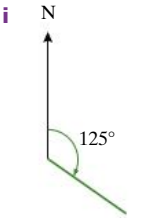

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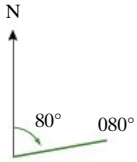

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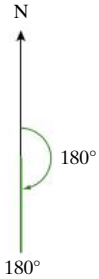
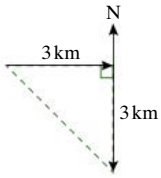


l

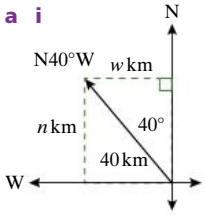
3 a



b NW

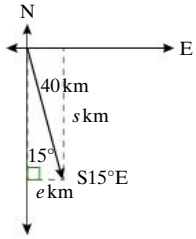
c 315°

4 a i



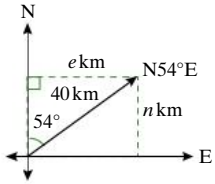
ii 30.6 km N; 25.7 km W

b i



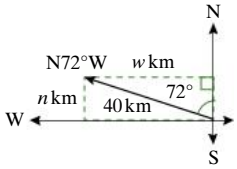
ii 38.6 km S; 10.4 km E

c i



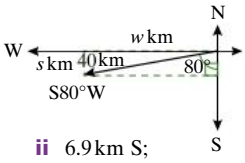
ii 23.5 km N; 32.4 km E

d i



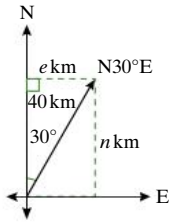
ii 12.4 km N; 38.0 km W

e i



ii 6.9 km S; 39.4 km W

f i



ii 34.6 km N; 20.0 km E

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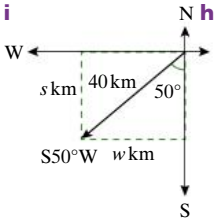
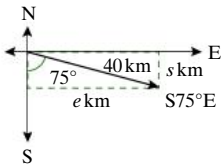
g i



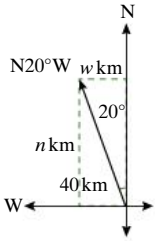
ii 25.7 km S; 30.6 km W

h i



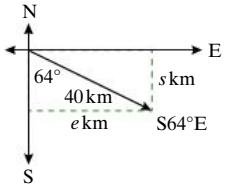
ii 10.4 km S; 38.6 km E

i i



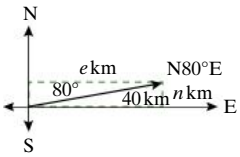
ii 37.6 km N; 13.7 km W

j i



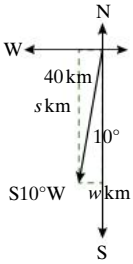
ii 17.5 km S; 36.0 km E

k i



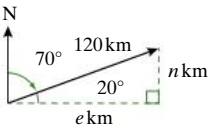
ii 6.9 km N; 39.4 km E

l i



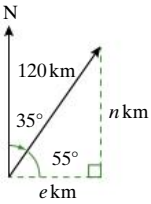
ii 39.4 km S; 6.9 km W

5 a i



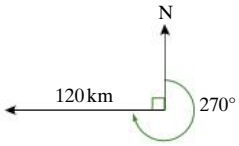
ii 41.0 km N; 112.8 km E

b i



ii 98.3 km N; 68.8 km E

c i

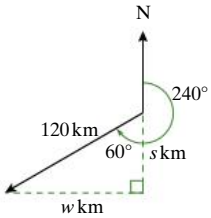


ii 120 km W

d i

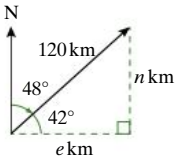
ii 120 km E

e i



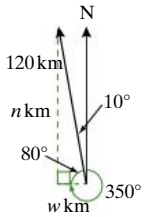
ii 60.0 km S; 103.9 km W

f i



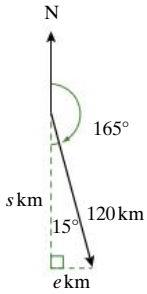
ii 80.3 km N; 89.2 km E

g i



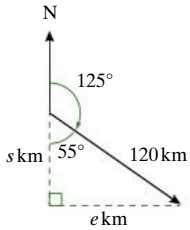
ii 118.2 km N; 20.8 km W

h i



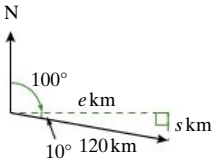
ii 115.9 km S; 31.1 km E

i i



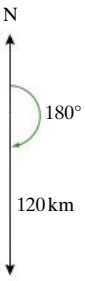
ii 68.8 km S; 98.3 km E

j i



ii 20.8 km S; 118.2 km E

k i



ii 120 km S

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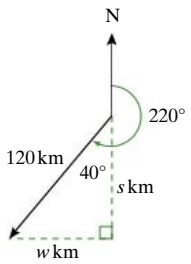
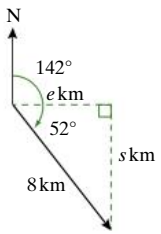
1 i



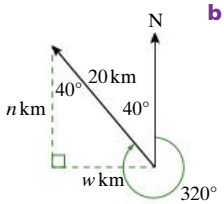
ii 91.9 km S; 77.1 km W

6 a



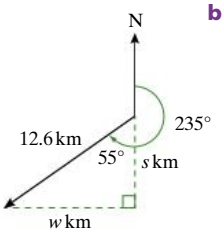
b 4.9 km E; 6.3 km S

7 a



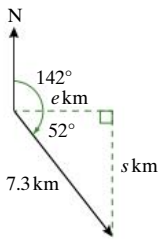
b 12.9 km W; 15.3 km N

8 a



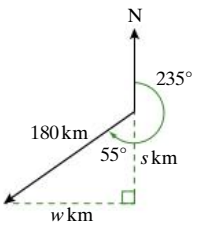
b i 10.3 km W
ii 7.2 km S

9 a



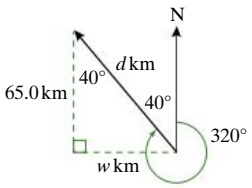
b i 4.5 km E
ii 5.8 km S

10 a



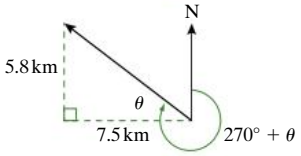
b 103 km S; 147 km W

11 a



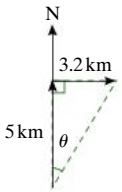
b 54.5 km
c 84.9 km

12 a



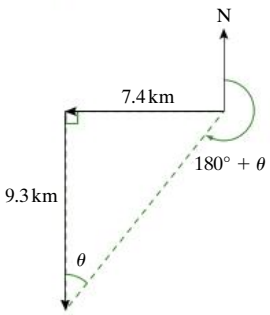
b 308° (to the nearest degree)

13 a



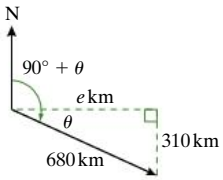
b 033°

14 a



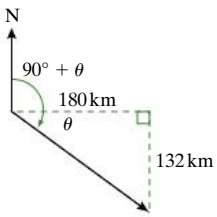
b 219°

15 a



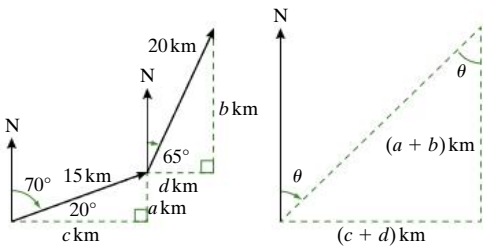
b 605 km
c 117°

16 a



b 126°

17 a



b 23.3 km N; 22.5 km E

c 044°

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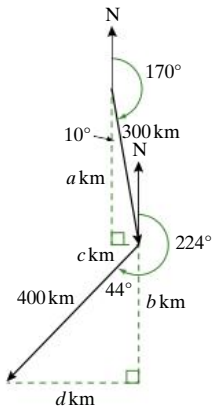
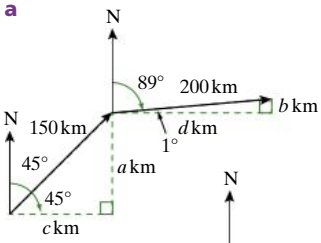
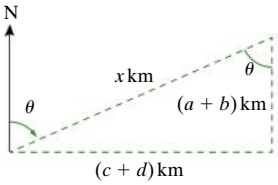
18 a



b 583 km S
c 226 km W

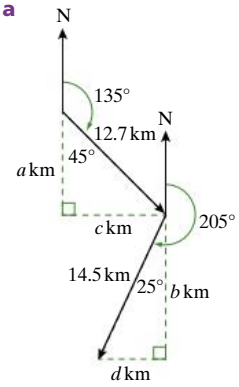
19 a



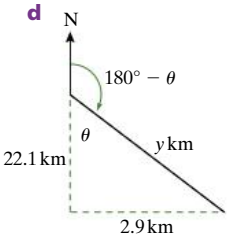


b 306 km E
c 110 km N
d 325 km
e 070°

20 a

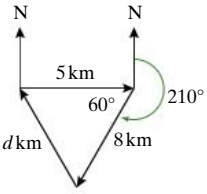


b 22.1 km S
c 2.9 km E
d



e 22.3 km
f 173°

21 a



b 7 km
c 351.8°

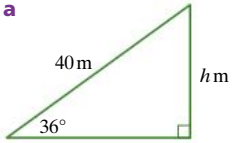
revision

1 A **2** E **3** B **4** E **5** B

6 a $a = 65.1$ **b** $b = 9.1$
c $c = 5.2$ **d** $d = 35.5$

7 a $\theta = 38.7°$ **b** $\theta = 33.6°$
c $\theta = 54.3°$ **d** $\theta = 63.4°$

8 a



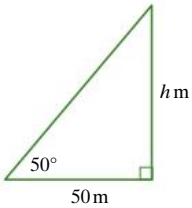
b 24 m

9 a 52.9° **b i, ii** 5.6 m

10 a $d \approx 98.9, \theta \approx 64.0°$
b $a \approx 84.1, b \approx 100, \theta \approx 72.8°$

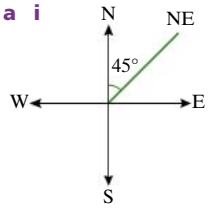
11 31 m

12 a



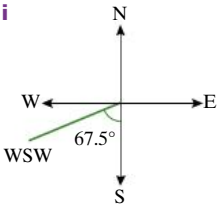
b 60 m

13 a i



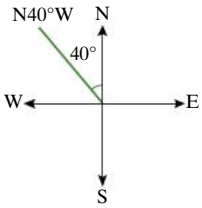
ii 045°

b i



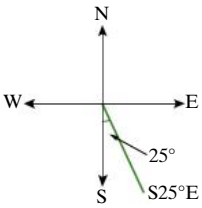
ii 247.5°

c i



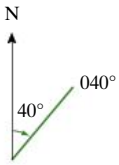
ii 320°

d i

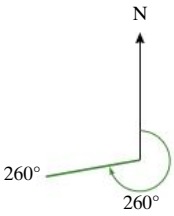


ii 155°

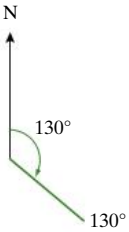
14 a



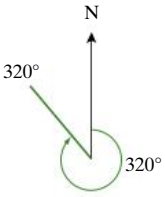
b



c



d



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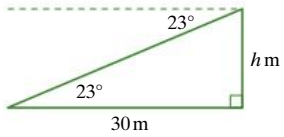
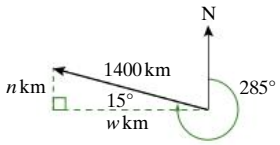
15 a



b 12.7 m
c 19°

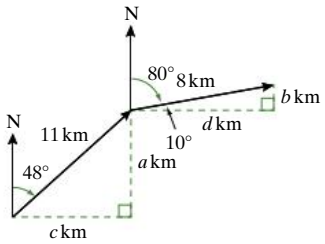
16 53 m

17 a



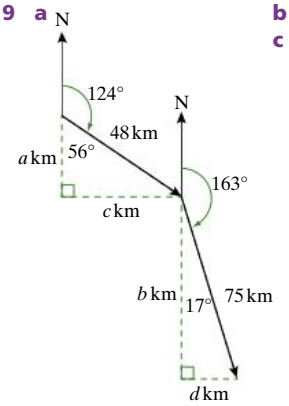
b i 1352.3 km W **ii** 362.3 km N

18 a



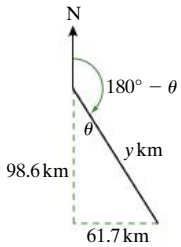
b 16.1 km E **c** 8.7 km N

19 a



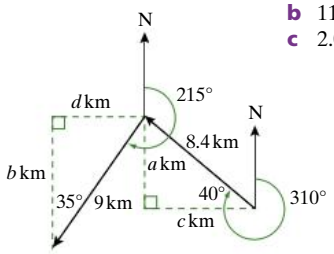
b 61.7 km E
c 98.6 km S

d



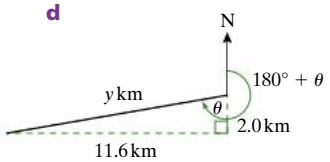
e 116.3 km
f 148°

20 a



b 11.6 km W
c 2.0 km S

d



e 11.8 km
f 260°

Chapter 3

exercise 3.1

1 **a** $\frac{3}{a^2}$ **b** $\frac{-4}{y^5}$ **c** $5x^2y^3$ **d** $\frac{n^2}{m^4}$

2 **a** $-6x^7$ **b** $\frac{-12}{m^2}$ **c** $-\frac{15}{x}$ **d** $\frac{3}{a^3}$
e $4b^{11}$ **f** $4a^6$ **g** $-\frac{5}{9x^2}$ **h** $-\frac{6x^6}{5}$

3 **a** -2 **b** $\frac{4}{5}$ **c** -12 **d** $\frac{-5}{2}$

4 **a** a^8 **b** $\frac{1}{x^6}$ **c** $\frac{1}{y^8}$ **d** $-\frac{1}{x^{12}}$

5 **a** a^6b^3 **b** $\frac{d^4}{c^6}$ **c** $\frac{27y^6}{x^9}$ **d** $-\frac{a^{12}}{8b^6}$
e $\frac{8a^3}{b^6}$ **f** $4x^2y^6$ **g** $81m^4n^8$ **h** $\frac{x^6}{125y^9}$

6 **a** $\frac{4a}{b}$ **b** 4 **c** $-\frac{2x}{3y}$
d $\frac{3b}{4c}$ **e** $-\frac{3a}{2c}$ **f** $\frac{10b}{7}$
g $\frac{5d}{4fg}$ **h** $\frac{3x}{4}$ **i** $\frac{10y^2}{3z}$
j $-\frac{5y^2}{3}$ **k** $\frac{5xy^2}{6z}$ **l** $\frac{8x^2y}{15}$
m $-\frac{5abc}{3d}$ **n** $\frac{21f^2}{4}$ **o** $\frac{3nx}{mp}$

7 B

8 **a** $\frac{n^3}{5m^5}$ **b** $-\frac{1}{108x^8y^4}$ **c** $\frac{a^4b^{10}}{72}$

9 **a** $\frac{5x}{7}$ **b** $\frac{5x}{11}$ **c** $\frac{5x}{9}$
d $-\frac{a}{2}$ **e** $\frac{14a}{15}$ **f** $\frac{17x}{16}$
g $-\frac{x}{12}$ **h** $\frac{a}{4}$ **i** $\frac{31x}{20}$

10 **a** $\frac{4x+7}{10}$ **b** $\frac{7x-6}{6}$
c $\frac{7x-21}{12}$ **d** $\frac{7x+1}{6}$
e $\frac{x+10}{12}$ **f** $-\frac{7}{12}$
g $\frac{x-18}{12}$ **h** $\frac{6x+17}{20}$
i $\frac{-x+27}{15}$

11 $\frac{23x+131}{30}$

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exercise 3.2

1 **a** **i** $2(x + 3)$ **ii** $2x + 6$
b **i** $3(2x + 4)$ **ii** $6x + 12$
c **i** $x(3x + 5)$ **ii** $3x^2 + 5x$
d **i** $x(2y + 6)$ **ii** $2xy + 6x$

2 **a** $2x + 8$ **b** $3a - 15$
c $8b - 4$ **d** $18m + 12$
e $-3x - 6$ **f** $-a^2 - 3a$
g $-4b + 20$ **h** $-14p + 2p^3$
i $5a + 8$ **j** $9x - 29$
k $17 - 2x$ **l** $2a - 9$
m $k^2 + 9k + 4$ **n** $x^2 + 8x - 20$
o $m^2 + 4m - 15$

3 **a** $d^2 - 8d - 15$ **b** $3g^2 - 10g + 8$
c $2p^2 + 8p - 12$ **d** $2x^2 - 2x + 20$
e $2x^2 - 29x + 6$ **f** $3x^2 - 22x - 15$

4 **a** $2a$ **b** $5x$ **c** $2b$
d $-8mn^2$ **e** $-14xy$ **f** $-9a$

5 **a** $6(b + 1)$ **b** $5(a - 5b)$
c $4b(a - 3b)$ **d** $pq(1 - pq)$
e $2xy(1 - xy)$ **f** $2ab(3 - 4a)$
g $4x(y - 3x)$ **h** $2xy(3x - 5y)$
i $7mn(mn + 4)$ **j** $4a^2b(3 + 4b)$
k $5xy(1 - 2xy)$ **l** $3pq(3p - 4)$
m $2(5b^3 - 3a^2)$ **n** $5x^2(1 + 3x^2)$
o $m^2n(1 + n)$ **p** $10ab^2(2a - b)$
q $-4(a + 4b)$ **r** $-7a^2b^2(b + 5a^2)$
s $-4xy(2x + 3y)$ **t** $6x(5x^2 - 3y^3)$
u $6mn^2(9n - 8m)$

6 $5xy^2(x - 5y)(2x + 3y)$; $10x^3y^2 - 35x^2y^3 - 75xy^4$

exercise 3.3

1 **a** $(x + 3)(x + 1)$; $x^2 + 4x + 3$
b $(x + y)(y + 2)$; $xy + 2x + y^2 + 2y$

2 **a** $a^2 + 6a + 8$ **b** $b^2 + 8a + 15$
c $x^2 + 4c - 21$ **d** $d^2 - 3d - 4$
e $e^2 - 8e + 12$ **f** $f^2 - 8f + 15$
g $2g^2 + 13g + 15$ **h** $6h^2 - 7h + 2$
i $2x^2 + 7x + 3$ **j** $3a^2 - 20a - 32$
k $4m^2 - 27m - 7$ **l** $6y^2 - 5y - 6$
m $3a^2 + 2ab + 8b^2$ **n** $5m^2 - 21mn + 18n^2$
o $6x^2 + 7x - 5$ **p** $-5x^2 + 18x + 8$
q $3a^2 - 25a + 28$ **r** $-8a^2 + 26ab - 15b^2$

3 **a** $x + 16, x + 12$ **b** $x^2 + 28x + 192$
c $2192\,\text{cm}^2$

4 **a** $x^3 + 6x^2 + 11x + 6$ **b** $2x^3 - 10x^2 + 4x + 16$

exercise 3.4

1 **a** $(x + 1)(a + 3)$ **b** $(x - 7)(4 + x)$
c $(2p + 3)(p - 2)$ **d** $(y - 6)(y - 1)$
e $2(3w - 5)(w + 3)$ **f** $(3 - 4z)(5 - 8z)$
g $(b - 2c)(a^2 + 1)$ **h** $(y - z)(x - 1)$
i $(a + 3)(a + 1)$ **j** $(p - 1)^2$
k $(x + 5)^2$ **l** $(2m + 3)^2$

2 **a** $ax + ay + 3x + 3y$
$= a(x + y) + 3(x + y)$
$= (x + y)(a + 3)$
b $x^2 - x + yx - y$
$= x(x - 1) + y(x - 1)$
$= (x - 1)(x + y)$
c $3m + 18 + am + 6a$
$= 3(m + 6) + a(m + 6)$
$= (m + 6)(3 + a)$
d $yz + 7z - 2y - 14$
$= z(y + 7) - 2(y + 7)$
$= (y + 7)(z - 2)$
e $3ab + 6b + 2a + 4$
$= 3b(a + 2) + 2(a + 2)$
$= (a + 2)(3b + 2)$
f $4k - 16 + km - 4m$
$= 4(k - 4) + m(k - 4)$
$= (k - 4)(4 + m)$

3 **a** $(x + 1)(x + 5)$ **b** $(a + 4)(a - 3)$
c $(x - 5)(7 + x)$ **d** $(a - 3)(a - 4)$
e $(2m - 7)(m - 1)$ **f** $(x - 9)(2x + 3)$
g $(a - 4)(3a - 2)$ **h** $(6 - 5m)(5m + 3)$
i $(x - 3)(3x - 2)$ **j** $(2a - 5)(4a - 3)$
k $(3p - 4)(2p - 5)$ **l** $(7 - 4x)(3x - 2)$

4 **a** $(b + c)(a + e)$ **b** $(p + q)(t + 3)$
c $(x + y)(5 + k)$ **d** $3(s + 2)(2t + 1)$
e $(a - 3b)(a + 4)$ **f** $(x + 3)(3 - 2y)$
g $(p + 2)(5 - 3q)$ **h** $2(a + 2)(3 - b)$
i $(a - 3b)(a - 3)$ **j** $2(4a - 1)(1 - b)$
k $(a - 4)(a + 2)$ **l** $(x + 7)(x - 2)$

5 **a** $(x + y)(x + 6)$ **b** $(b + 4)(a - 3)$
c $(p + 2)(2a + 3)$ **d** $(y - 1)(x + 8)$
e $(q + r)(p - 16r)$ **f** $(p - r)(a + p)$
g $(m - 3)(n + 6)$ **h** $(x + 1)(1 - y)$
i $(x + z)(y + 5)$ **j** $(b + c)(a + d)$
k $(n + 5)(m + 3)$ **l** $(x - 3)(y + 7)$

6 **a** $(x + 2)(x + 7)$ **b** $(c + 3)(c + 4)$
c $(a - 4)(a - 6)$ **d** $(r + 7)(r + 10)$
e $(x - 4)(x - 9)$ **f** $(y + 8)(y + 13)$
g $(c + 2)(7 - c)$ **h** $(p - 4)(7 - p)$
i $(x + 1)(2x + 1)$ **j** $3(y - 4)(9 - 2y)$
k $(m + n)(2m + 2n - 1)$
l $(x - 2)(2x - 7)$

exercise 3.5

1 **a** no; $x^2 - 12x + 36$ **b** yes
c no; $h^2 + 14h + 49$ **d** no; $y^2 - 8y + 16$
e no; $m^2 + 2m + 1$ or $m^2 + 4m + 4$
f yes

2 Expanded forms are: $x^2 + 2x + 1$, $x^2 + 4x + 4$, $x^2 + 6x + 9$, $x^2 + 8x + 16$, $x^2 + 10x + 25$, $x^2 + 12x + 36$, $x^2 + 14x + 49$, $x^2 + 16x + 64$, $x^2 + 18x + 81$, $x^2 + 20x + 100$, $x^2 + 22x + 121$, $x^2 + 24x + 144$.

3 $x^2 + 6x + 9$

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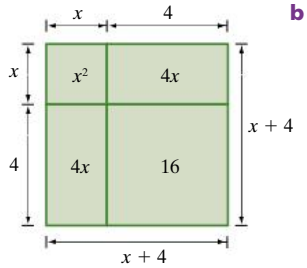
4 **a**



b $x^2 + 8x + 16$

5
a $a^2 + 12a + 36$ **b** $m^2 - 10m + 25$
c $x^2 + 6x + 9$ **d** $c^2 - 8c + 16$
e $p^2 + 10p + 25$ **f** $m^2 - 12m + 36$
g $a^2 + 4a + 4$ **h** $b^2 - 6b + 9$
i $4x^2 + 12x + 9$ **j** $16x^2 - 24x + 9$
k $x^2 + 6xy + 9y^2$ **l** $25a^2 - 20ab + 4b^2$
m $4k^2 - 28k + 49$ **n** $9x^2 - 48x + 64$
o $9x^2 + 6x + 1$ **p** $9x^2 + 12x + 4$
q $4a^2 + 20ab + 25b^2$ **r** $49y^2 - 28y + 4$

6
a $(x + 4)^2$ **b** $(y + 3)^2$ **c** $(p - 2)^2$
d $(m - 5)^2$ **e** $(k + 10)^2$ **f** $(x + 7)^2$
g $(a - 6)^2$ **h** $(x + 6)^2$ **i** $(p + 2q)^2$
j $(m - 4)^2$ **k** $(x - 1)^2$ **l** $(x - 16)^2$

7
a $(x + 7)^2$ **b** $(y + 4)^2$ **c** $(p - 3)^2$
d $(k - 5)^2$ **e** $(z + 11)^2$ **f** $(x + 8)^2$
g $(m + 5)^2$ **h** $(x - 9)^2$ **i** $(x + 2)^2$
j $(x + 12)^2$ **k** $(y - 13)^2$ **l** $(a + 15)^2$
m $(x - 20)^2$ **n** $(m - 6)^2$ **o** $(x + 1.5)^2$

8
a $2(x + 4)^2$ **b** $3(x + 1)^2$ **c** $3(x - 5)^2$
d $-(x + 2)^2$ **e** $-(x - 3)^2$ **f** $-(x - 7)^2$
g $-2(x - 9)^2$ **h** $-5(x + 3)^2$ **i** $3x(x + 1)^2$

9
a $(x - 3)^2m^2$ **b** $(6x - a)\,m^2$

exercise 3.6

1
a $p^2 - 16$ **b** $x^2 - 25$ **c** $m^2 - n^2$
d $81 - m^2$ **e** $4x^2 - 1$ **f** $9 - 4x^2$
g $81 - 25y^2$ **h** $4a^2 - 49b^2$ **i** $a^2b^2 - 16$
j $p^2 - 49$ **k** $64 - m^2$ **l** $a^2 - c^2$
m $4x^2 - 9$ **n** $49 - 25k^2$ **o** $25 - 4a^2$
p $4m^2 - 49$ **q** $25 - 9x^2$ **r** $9 - 16a^2$

2

Factorised form	Expanded form
$(x + y)(x - y)$	$x^2 - y^2$
$(m + n)(m - n)$	$m^2 - n^2$
$(y + 3)(y - 3)$	$y^2 - 9$
$(x + 2)(x - 2)$	$x^2 - 4$
$(a + 2b)(a - 2b)$	$a^2 - 4b^2$
$(3x + 2y)(3x - 2y)$	$9x^2 - 4y^2$
$(y + 5)(y - 5)$	$y^2 - 25$
$(7 + a)(7 - a)$	$49 - a^2$
$(2y - 5x)(2y + 5x)$	$4y^2 - 25x^2$
$(r + t)(r - t)$	$r^2 - t^2$
$(y + 8)(y - 8)$	$y^2 - 64$
$(5x + 2y)(5x - 2y)$	$25x^2 - 4y^2$

3
a $(x + 10)(x - 10)$ **b** $(p + 5)(p - 5)$
c $(9 + y)(9 - y)$ **d** $(6 + x)(6 - x)$
e $(x + 7)(x - 7)$ **f** $(1 + a)(1 - a)$
g $(y + 12)(y - 12)$ **h** $(x + 8)(x - 8)$
i $(2a + 3)(2a - 3)$ **j** $(4y + 7)(4y - 7)$
k $(k + 6m)(k - 6m)$ **l** $(2p + 9q)(2p - 9q)$
m $(10 + y)(10 - y)$ **n** $(5 + x)(5 - x)$
o $(3a + 2)(3a - 2)$ **p** $(7b + 4)(7b - 4)$
q $(10a + 7)(10a - 7)$ **r** $(5x + 1)(5x - 1)$
s $(4b + 5)(4b - 5)$ **t** $(4 + 3x)(4 - 3x)$
u $(5 + 7m)(5 - 7m)$ **v** $(8 + 9a)(8 - 9a)$
w $(3x + 13)(3x - 13)$ **x** $(11x + 7)(11x - 7)$

4
a 280 **b** 2600 **c** 1800 **d** 31 600

5
a $(a + 5)(a - 1)$ **b** $(b + 8)(b - 2)$
c $(c + 17)(c - 1)$ **d** $(d + 1)(d - 11)$
e $(e + 11)(e + 3)$ **f** $(f - 2)(f - 22)$
g $(g + 3)(g - 11)$ **h** $(h + 10)(h - 6)$
i $-(i + 9)(i + 1)$ **j** $-j(j + 4)$
k $(2 + k)(8 - k)$ **l** $(1 + l)(5 - l)$

6
a $4x + 3y)(x - 3y)$ **b** $3(a + 4b)(a - 4b)$
c $2(x + 7y)(x - 7y)$ **d** $7(x + 2)(x - 2)$
e $-3(x + 5)(x - 5)$ **f** $-2(m + 5n)(m - 5n)$
g $4(x + 10y)(x - 10y)$ **h** $6(b + 3)(b - 3)$

7
a $x^2\text{m}^2$
b $(x + 1)(x - 1)\,\text{m}^2$
c smaller, as $(x + 1)(x - 1) = x^2 - 1$, so area is $1\,\text{m}^2$ less than the original garden

8
a $(x + \sqrt{5})(x - \sqrt{5})$
b $(y + \sqrt{2})(y - \sqrt{2})$
c $(\sqrt{7} + a)(\sqrt{7} - a)$
d $(\sqrt{19} + c)(\sqrt{19} - c)$
e $(p + 2\sqrt{3})(p - 2\sqrt{3})$
f $(z + 4\sqrt{2})(z - 4\sqrt{2})$
g $3(x + 2\sqrt{2})(x - 2\sqrt{2})$
h $5(x + 2\sqrt{3})(x - 2\sqrt{3})$
i $5(\sqrt{5} + r)(\sqrt{7} - r)$
j $7(\sqrt{7} + x)(\sqrt{7} - x)$

9
a $(x + 4 + \sqrt{13})(x + 4 - \sqrt{13})$
b $(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$
c $(x - 2 + \sqrt{5})(x - 2 - \sqrt{5})$
d $(x - 7 + \sqrt{10})(x - 7 - \sqrt{10})$
e $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$
f $(x - 1 + 3\sqrt{2})(x - 1 - 3\sqrt{2})$
g $(x + 1 + \sqrt{3})(x + 1 - \sqrt{3})$
h $(y + 5 + \sqrt{7})(y + 5 - \sqrt{7})$

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i $(p - 3 + \sqrt{11})(p - 3 - \sqrt{11})$
j $(x + 7 + 2\sqrt{2})(x + 7 - 2\sqrt{2})$
k $(x - 1 + 3\sqrt{3})(x - 1 - 3\sqrt{3})$
l $(\sqrt{15} + x + 2)(\sqrt{15} - x - 2)$
m $(3\sqrt{2} + k - 2)(3\sqrt{2} - k + 2)$
n $(m - 5 + \sqrt{5})(m - 5 - \sqrt{5})$
o $3(a + 2 + \sqrt{5})(a + 2 - \sqrt{5})$

10 **a** $(7b + 3)(b + 3)$ **b** $-(6x - 1)(2x + 1)$

exercise 3.7

1 **a** $(x + 1)(x + 2)$ **b** $(x + 1)(x + 3)$
c $(x + 6)^2$ **d** $(x + 4)(x + 10)$
e $(x - 5)(x - 4)$ **f** $(x - 2)(x + 6)$
g $(x - 3)(x + 21)$ **h** $(x - 9)(x - 4)$
i $(x - 18)(x - 4)$ **j** $(x - 3)(x + 15)$
k $(x - 8)(x - 7)$ **l** $(x - 4)(x + 14)$
m $(a - 9)(a - 2)$ **n** $(a - 8)(a + 3)$
o $(b - 4)(b + 18)$ **p** $(d - 15)(d + 4)$
q $(m - 24)(m + 2)$ **r** $(y + 3)(y + 15)$

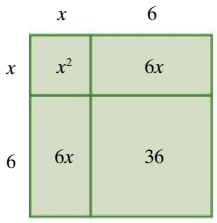
2 **a** $5(x + 1)(x + 2)$ **b** $3(x + 1)(x + 3)$
c $5(x - 4)(x + 2)$ **d** $5(x - 4)(x - 2)$
e $4(x - 5)(x + 2)$ **f** $2(x - 3)(x - 2)$
g $3(x - 9)(x + 4)$ **h** $5(x - 7)(x + 10)$
i $2(x - 18)(x - 4)$ **j** $2(x + 1)(x + 3)$
k $3(x + 3)(x + 4)$ **l** $8(x - 3)(x - 2)$
m $-(x - 8)(x + 1)$ **n** $-(x - 5)(x + 2)$
o $-(x + 4)(x + 5)$

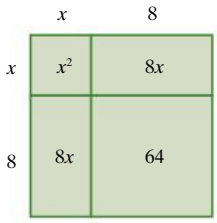
3 **a** $(x - 17)(x + 13)$ **b** $(x + 7)(x + 19)$
c $(x - 8)(x + 13)$ **d** $(x - 21)(x - 9)$

exercise 3.8

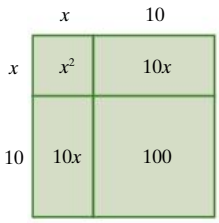
1 **a** 5 units are needed to complete the square, so $x^2 + 6x + 4$ is 5 units less than $(x + 3)^2$, so $x^2 + 6x + 4 = (x + 3)^2 - 5$.
b 13 units are needed to complete the square, so $x^2 + 10x + 12$ is 13 units less than $(x + 5)^2$, so $x^2 + 10x + 12 = (x + 5)^2 - 13$.

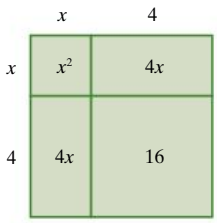
2 **a** $(x + 6)^2 - 36$ **b** $(x + 8)^2 - 64$



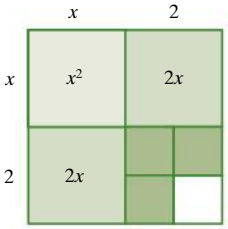


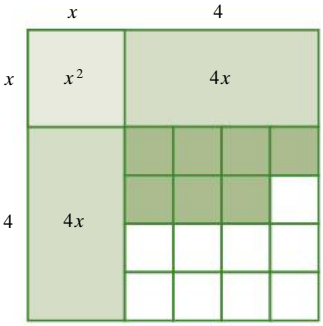
c $(x + 10)^2 - 100$ **d** $(x - 4)^2 - 16$



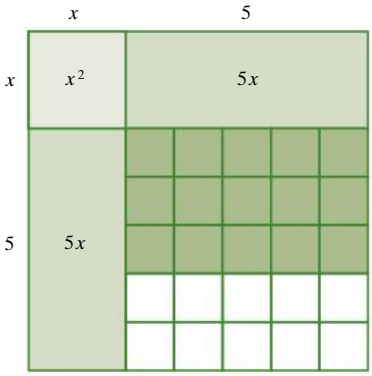


3 **a** $(x + 2)^2 - 1$ **b** $(x + 4)^2 - 9$

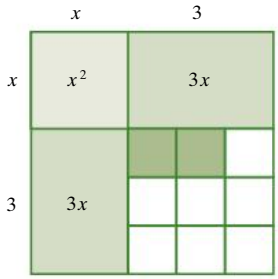




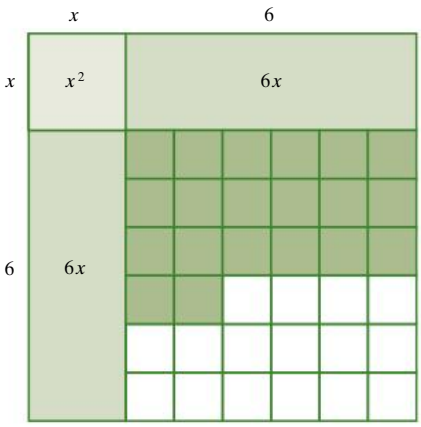
c $(x + 5)^2 - 10$



d $(x + 3)^2 - 7$



e $(x + 6)^2 - 16$



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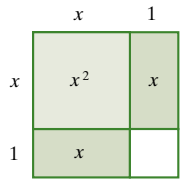
f $(x + 1)^2 - 1$



4 **a** $(x + 5 + \sqrt{6})(x + 5 - \sqrt{6})$
b $(x - 8 + \sqrt{11})(x - 8 - \sqrt{11})$
c $(x + 1 + 2\sqrt{5})(x + 1 - 2\sqrt{5})$
d $(x - 12 + 4\sqrt{3})(x - 12 - 4\sqrt{3})$
e $(x + 3 + \sqrt{3})(x + 3 - \sqrt{3})$
f $(x - 2 + \sqrt{5})(x - 2 - \sqrt{5})$
g $(x + 4 + \sqrt{41})(x + 4 - \sqrt{41})$
h $(x - 7 + 4\sqrt{2})(x - 7 - 4\sqrt{2})$
i $(x + 2 + \sqrt{29})(x + 2 - \sqrt{29})$
j $(x + 1 + \sqrt{41})(x + 1 - \sqrt{41})$
k $(x - 1 + \sqrt{51})(x - 1 - \sqrt{51})$
l $(x + 6 + \sqrt{23})(x + 6 - \ - \sqrt{23})$

5 **a** $x^2 + 6x + 9 = (x + 3)^2$
b $x^2 + 16x + 64 = (x + 8)^2$
c $x^2 + 18x + 81 = (x + 9)^2$
d $x^2 + 22x + 121 = (x + 11)^2$

6 **a** $(x + 3)^2 + 3$ **b** $(x + 1)^2 + 4$

7 **a** $(x + 6 + \sqrt{6})(x + 6 - \sqrt{6})$
b $(x - 5 + \sqrt{5})(x - 5 - \sqrt{5})$
c $(x + 7 + \sqrt{17})(x + 7 - \sqrt{17})$
d $(x + 2 + \sqrt{5})(x + 2 - \sqrt{5})$
e $(x + 11 + \sqrt{21})(x + 11 - \sqrt{21})$
f $(x - 3 + \sqrt{11})(x - 3 - \sqrt{11})$
g $(x + 9 + \sqrt{41})(x + 9 - \sqrt{41})$
h $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$
i $(x + 4 + \sqrt{19})(x + 4 - \sqrt{19})$
j $(x - 8 + \sqrt{23})(x - 8 - \sqrt{23})$
k $(x + 12 + \sqrt{23})(x + 12 - \sqrt{23})$
l $(x + 4 + \sqrt{11})(x + 4 - \sqrt{11})$
m $(x + 4 + \sqrt{29})(x + 4 - \sqrt{29})$
n $(x - 1 + \sqrt{6})(x - 1 - \sqrt{6})$
o $(x + 2 + \sqrt{11})(x + 2 - \sqrt{11})$
p $(x + 7 + \sqrt{51})(x + 7 - \sqrt{51})$

8 **a** $\left(x + \frac{5}{2} + \frac{\sqrt{21}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{21}}{2}\right)$
b $\left(x + \frac{9}{2} + \frac{\sqrt{57}}{2}\right)\left(x + \frac{9}{2} - \frac{\sqrt{57}}{2}\right)$

exercise 3.9

1 **a** $3x(x + 2)$ **b** $3b(2a - b)$
c $q(4p + 3q)$ **d** $(x + 3)(5x - 9)$
e $(x - y)(x + y + 2)$ **f** $(2x + 3y)(2x - 3y + 7)$

2 **a** $(2x + 3y)^2$ **b** $(2a + 7b)^2$ **c** $(3x - 5y)^2$
d $(4x - 3y)^2$ **e** $(5m - 7n)^2$ **f** $(7x - 8y)^2$
g $\left(x - \frac{1}{8}\right)^2$ **h** $\left(a - \frac{1}{4}\right)^2$ **i** $\left(x + \frac{2}{5}\right)^2$

3 **a** $(x + 3)(2x + 1)$ **b** $(x + 3)(3x + 1)$
c $(x + 2)(2x + 3)$ **d** $(x + 1)(3x + 2)$
e $(x + 5)(2x + 1)$ **f** $(x + 1)(7x + 3)$
g $(x + 7)(3x + 1)$ **h** $(x - 3)(2x + 1)$
i $(x + 1)(5x + 2)$ **j** $(x + 1)(3x - 2)$
k $(x - 1)(2x - 3)$ **l** $(x + 1)(3x - 5)$

4 **a** $(3x + 1)(4x + 1)$ **b** $(2x + 5)(4x - 1)$
c $(2x - 3)(4x + 5)$ **d** $(3x + 4)(5x + 2)$
e $(3x - 2)(7x - 2)$ **f** $(5x - 2)(7x + 1)$
g $-(2x - 3)(4x + 5)$ **h** $-(x - 1)(x + 4)$
i $-(x + 3)(3x - 2)$ **j** $(2x - 1)(5x + 2)$
k $(x - 2)(4x - 3)$ **l** $(x - 3)(3x + 2)$
m $(x + 2)(5x + 1)$ **n** $(x - 3)(5x - 7)$
o $(x + 3)(3x - 10)$ **p** $(x - 1)(7x + 5)$
q $(x + 2)(4x - 9)$ **r** $(2x + 1)(5x - 2)$
s $(2x + 7)(4x + 5)$ **t** $(x - 2)(6x - 1)$
u $(x - 2)(7x + 3)$

5 **a** $4(x + 3)(2x - 3)$ **b** $5(2x - 1)(3x + 7)$
c $3(x + 3)(2x - 3)$ **d** $3(2x + 1)(2x + 3)$
e $5(2x + 1)(5x - 1)$ **f** $-2(x + 1)(5x - 2)$
g $3(x + 1)(2x - 5)$ **h** $4(x - 5)(3x - 2)$
i $5(2x + 2)(3x - 2)$

6 **a** $x(x + 1)(x + 4)$ **b** $x(x + 2)(3x - 2)$
c $x(2x + 5)(4x - 1)$ **d** $x^2(2x + 1)(3x + 1)$
e $x^2(2x - 1)(5x + 2)$ **f** $3x^2(x - 8)(x + 3)$

7 **a** $(x - 2)(x - 1)(x + 1)(x + 2)$
b $(x - 5)(x - 2)(x + 2)(x + 5)$
c $(2x - 3)(2x - 1)(2x + 1)(2x + 3)$
d $(x + 1)(2x + 9)$
e $(x + 1)(3x - 1)$
f $(x + 6)(3x + 4)$
g $(x - 2)(x + 1)(x + 2)(x + 5)$
h $(x - 3)(x - 2)(x + 1)(x + 2)$

8 **a** $2x^2 + 8x + 2$
$= 2[x^2 + 4x + 1]$
$= 2[x^2 + 4x + 2^2 + 1 - 2^2]$
$= 2[(x + 2)^2 - 3]$
$= 2(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$

b $3x^2 + 12x + 6$
$= 3[x^2 + 4x + 2]$
$= 3[x^2 + 4x + 2^2 + 2 - 2^2]$
$= 3[(x + 2)^2 - 2]$
$= 3(x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$

c $2x^2 + 10x + 4$
$= 2[x^2 + 5x + 2]$
$= 2\left[x^2 + 5x + \left(\frac{5}{2}\right)^2 + 2 - \left(\frac{5}{2}\right)^2\right]$
$= 2\left[\left(x + \frac{5}{2}\right)^2 - \frac{17}{4}\right]$
$= 2\left(x + \frac{5}{2} + \frac{\sqrt{17}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{17}}{2}\right)$

d $5x^2 + 10x + 2$
$= 5\left[x^2 + 2x + \frac{2}{5}\right]$
$= 5\left[x^2 + 2x + 1^2 + \frac{2}{5} - 1^2\right]$
$= 5\left[(x = 1)^2 - \frac{3}{5}\right]$
$= 5\left(x + 1 + \frac{\sqrt{15}}{5}\right)\left(x + 1 - \frac{\sqrt{15}}{5}\right)$

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9 **a** $2(x - 4 + \sqrt{14})(x - 4 - \sqrt{14})$
b $6(x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$
c $4(x + 4 + \sqrt{5})(x + 4 - \sqrt{5})$
d $5(x + 3 + \sqrt{3})(x + 3 - \sqrt{3})$
e $3\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)$
f $2(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

10 **a** $\left(x + \frac{11}{2} + \frac{\sqrt{41}}{2}\right)\left(x + \frac{11}{2} - \frac{\sqrt{41}}{2}\right)$
b $\left(x - \frac{9}{2} + \frac{\sqrt{33}}{2}\right)\left(x + \frac{9}{2} - \frac{\sqrt{33}}{2}\right)$
c $\left(x + \frac{15}{2} + \frac{\sqrt{97}}{7}\right)\left(x + \frac{15}{2} - \frac{\sqrt{97}}{2}\right)$
d $\left(x + \frac{3}{2} + \frac{\sqrt{13}}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{13}}{2}\right)$
e $\left(x + \frac{13}{2} + \frac{\sqrt{369}}{2}\right)\left(x + \frac{13}{2} - \frac{\sqrt{369}}{2}\right)$
f $\left(x - \frac{7}{2} + \frac{\sqrt{57}}{2}\right)\left(x - \frac{7}{2} - \frac{\sqrt{57}}{2}\right)$

11 If the radius is r, then $A = \pi r^2$. Also, $A = p(4x^2 + 20x + 25) = p(2x + 5)^2$, so equating gives $r = 2x + 5$.

revision

1 E **2** C **3** E **4** E **5** C

6 **a** $x^2 + 8x + 16 = (x + 4)^2$, so i
b $a^2 - 1 = a^2 - 1^2$, so ii
c $4b^2 - 12b + 9 = (2b - 3)^2$, so i
d $(w - 5)(w + 5) = w^2 - 5^2$, so ii
e $-z^2 + 25 = 5^2 - z^2$, so ii
f $(2x + 3)(2x - 3) = 4x^2 - 9$, so ii

7 **a** $2a^2 + ay - 3y^2$ **b** $4x^2 - 4xy + y^2$
c $x^2 - 33x + 11$

8 **a** $3ab^5(4a - b^3)$ **b** $(2x + 3)(1 - 2x)$
c $(3y - 1)(5y + 2)$ **d** $(2x + y)(2x + y + 3)$

9 **a** $(x - 7)^2$ **b** $(x + 3y)^2$

10 **a** $(12 + c)(12 - c)$ **b** $(3x + 11)(3x - 11)$
c $(x - 1)(x + 7)$ **d** $(a + 1)(9 - a)$
e $(3x - 10)(5x - 14)$ **f** $(2x + \sqrt{3})(2x - \sqrt{3})$

11 **a** $(e - 3)(e + 8)$ **b** $(3p - 4)(5p + 2)$
c $(x + 7)(x + 8)$ **d** $(x - 7)(x + 9)$
e $(a - 4)(a + 12)$ **f** $(m + 3)(m + 15)$
g $(x - 3)(x + 7)$ **h** $(x - 7)(x - 3)$

12 **a** $(x + 4 + \sqrt{15})(x + 4 - \sqrt{15})$
b $(m - 3 + \sqrt{6})(m - 3 + \sqrt{6})$

13 **a** $5(x + 1)^2$ **b** $2(a + 6)^2$
c $6(x - 2)^2$ **d** $3(m - 7)^2$
e $20(x + 3)^2$ **f** $-3(x - 5)^2$
g $-4(x + 11)^2$ **h** $-5(x - 9)^2$
i $5(3x + 3)^2$ **j** $3(a + 2)(a - 2)$
k $4(2 + x)(2 - x)$ **l** $4(y + 5z)(y - 5z)$
m $16(2d + e)(2d - e)$ **n** $5(x + 3)(x - 3)$
o $3(2x + 5y)(2x - 5y)$

14 **a** $(3x + 4)(5x - 2$ **b** $2(m + 3)(m - 5)$
c $(10m + 3)(20m + 11)$ **d** $(5x - 3y)^2$
e $\left(y - \frac{1}{5}\right)^2$ **f** $(3a - 2b)^2$
g $5(w - 5)(2w + 3)$ **h** $2x(x - 7)$

15 **a** $6x^3 + 10x^2\,\text{m}^3$ **b** $16x^2 + 20x\,\text{m}^2$
c \$958.80

Chapter 4

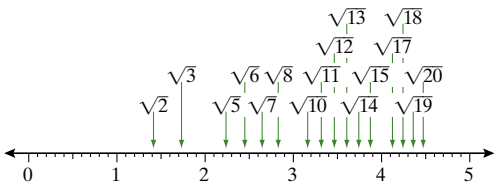
exercise 4.1

1 Rational: $\sqrt[3]{8}$, -6, $3\frac{1}{2}$, $-5.242424\ldots$, 0, $-\frac{5}{9}$;
irrational: 3π, $\sqrt{10}$, $\sqrt[3]{7}$, $3.1703402\ldots$

2 E

3 **a** $\sqrt{1} = 1, \sqrt{2}, \sqrt{3}, \sqrt{4} = 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9} = 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}, \sqrt{16} = 4, \sqrt{17}, \sqrt{18}, \sqrt{19}, \sqrt{20}$
b Surds: $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}, \sqrt{17}, \sqrt{18}, \sqrt{19}, \sqrt{20}$; not surds: $\sqrt{1}, \sqrt{4}, \sqrt{9}, \sqrt{16}$
c More surds than non-surds; only square roots of perfect squares are non-surds and there are just four perfect squares between 1 and 20.

4



5 **a** 1.732 **b** 3.606 **c** 4.899 **d** 2.646
e 7.071 **f** 11.180 **g** 14.697 **h** 12.124

6 $\sqrt{18} = 3\sqrt{2}$, $\sqrt{28} = 2\sqrt{7}$, $\sqrt{300} = 10\sqrt{3}$, $\sqrt{108} = 6\sqrt{3}$

7 D

8 **a** $2\sqrt{3}$ **b** $3\sqrt{7}$ **c** $2\sqrt{6}$ **d** $3\sqrt{2}$
e $5\sqrt{2}$ **f** $3\sqrt{3}$ **g** $2\sqrt{10}$ **h** $3\sqrt{6}$
i $10\sqrt{7}$ **j** $-3\sqrt{10}$ **k** $4\sqrt{2}$ **l** $3\sqrt{11}$
m $2\sqrt{5}$ **n** $6\sqrt{2}$ **o** $5\sqrt{3}$ **p** $2\sqrt{7}$

9 **a** $6\sqrt{2}$ **b** $6\sqrt{5}$ **c** $15\sqrt{3}$ **d** $15\sqrt{2}$
e $6\sqrt{11}$ **f** $-10\sqrt{7}$ **g** $20\sqrt{6}$ **h** $-20\sqrt{3}$
i $6\sqrt{5}$ **j** $-21\sqrt{2}$ **k** $-12\sqrt{10}$ **l** $12\sqrt{2}$
m $4\sqrt{3}$ **n** $12\sqrt{11}$ **o** $-6\sqrt{7}$ **p** $6\sqrt{7}$

10 D

11 **a** $\sqrt{45}$ **b** $\sqrt{8}$ **c** $\sqrt{75}$ **d** $\sqrt{63}$
e $\sqrt{44}$ **f** $-\sqrt{24}$ **g** $\sqrt{500}$ **h** $-\sqrt{27}$
i $\sqrt{48}$ **j** $-\sqrt{20}$ **k** $-\sqrt{99}$ **l** $\sqrt{72}$

12 **a** $4\sqrt{3} = \sqrt{48}$, $5\sqrt{2} = \sqrt{50}$, $3\sqrt{6} = \sqrt{54}$, $2\sqrt{10} = \sqrt{40}$, $2\sqrt{11} = \sqrt{44}$; descending order: $3\sqrt{6}, 5\sqrt{2}, 4\sqrt{3}, 2\sqrt{11}, 2\sqrt{10}$.
b $3\sqrt{6} \approx 7.35$, $5\sqrt{2} \approx 7.07$, $4\sqrt{3} \approx 6.93$, $2\sqrt{11} \approx 6.63$, $2\sqrt{10} \approx 6.32$

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13

Number	Product of prime factors	Surd	Simplified surd
12	$2^2 \times 3$	$\sqrt{12}$	$2\sqrt{3}$
18	2×3^2	$\sqrt{18}$	$3\sqrt{2}$
50	2×5^2	$\sqrt{50}$	$5\sqrt{2}$
28	$2^2 \times 7$	$\sqrt{28}$	$2\sqrt{7}$
90	$2 \times 3^2 \times 5$	$\sqrt{90}$	$3\sqrt{10}$

14 **a** $2\sqrt{3}$ m
b **i** $3\sqrt{3}$ m **ii** $4\sqrt{3}$ m **iii** $8\sqrt{3}$ m **iv** $x\sqrt{3}$ m

exercise 4.2

1 **a** 2 **b** 3 **c** 4 **d** 5
e 1.732 **f** 2 **g** 3.684

2 A

3 **a** Factor ladder shows three 2s, two 3s and one 5.
b $360 = 2^3 \times 3^2 \times 5$
c $\sqrt{360} = \sqrt{2^3 \times 3^2 \times 5}$
$= \sqrt{2^2 \times 3^2} \times \sqrt{2 \times 5}$
d $6\sqrt{10}$

4 **a** **i** $576 = 2^6 \times 3^2$ **ii** 24
b **i** $180 = 2^2 \times 3^2 \times 5$ **ii** $6\sqrt{5}$
c **i** $1568 = 2^5 \times 7^2$ **ii** $28\sqrt{2}$
d **i** $864 = 2^5 \times 3^3$ **ii** $12\sqrt{6}$

5 **a** $x = \frac{1}{3}$ **b** $x = \frac{1}{2}$ **c** $x = \frac{1}{6}$ **d** $x = 2\frac{1}{2}$
e $x = 4$ **f** $x = 4$ **g** $x = 7$ **h** $x = 9$

exercise 4.3

1 **a** 27 **b** 4 **c** 32 **d** 100 000
e 125 **f** 100 000 **g** 27 **h** 8
i 81 **j** 16 **k** 32 **l** 4
m $\frac{1}{125}$ **n** $\frac{1}{4}$ **o** $\frac{1}{32}$ **p** 0.001

2 E

3 B

4 **a** $2^{\frac{4}{3}}$ **b** $5^{\frac{3}{2}}$ **c** $5^{\frac{4}{3}}$ **d** $2^{\frac{5}{2}}$
e $7^{\frac{5}{2}}$ **f** $3^{\frac{2}{5}}$ **g** $7^{\frac{2}{3}}$ **h** $3^{\frac{3}{5}}$

5 **a** $5^{\frac{3}{2}}$ **b** $2^{\frac{7}{2}}$ **c** $6^{\frac{5}{2}}$ **d** $2^{\frac{9}{2}}$
e $7^{\frac{5}{4}}$ **f** $3^{\frac{11}{5}}$ **g** $2^{\frac{4}{3}}$ **h** $2^{\frac{11}{3}}$

6 **a** $x = \frac{3}{2}$ **b** $x = \frac{5}{2}$ **c** $x = \frac{2}{3}$ **d** $x = \frac{3}{4}$
e $x = \frac{2}{3}$ **f** $x = \frac{3}{8}$ **g** $x = \frac{1}{4}$ **h** $x = \frac{8}{3}$

7 **a** $\frac{1}{3}$ **b** $\frac{1}{27}$ **c** 27 **d** $\frac{27}{8}$
e m^4 **f** m^9 **g** $\frac{t^6}{u^9}$ **h** $\frac{w^3}{v^{12}}$

exercise 4.4

1 **a** $4 \times a = 4a$ **b** $4 \times \sqrt{2} = 4\sqrt{2}$
c $6 \times \sqrt{5} = 6\sqrt{5}$ **d** $3 \times \sqrt{6} = 3\sqrt{6}$

2 **a** $3\sqrt{5} + 2\sqrt{5} = 5\sqrt{5}$ **b** $4\sqrt{2} - 3\sqrt{2} = \sqrt{2}$
c $2\sqrt{3} - 4\sqrt{3} + 2\sqrt{3} = 0$
d $-2\sqrt{6} + 3\sqrt{5} + 4\sqrt{6} = 2\sqrt{6} + 3\sqrt{5}$
e $3\sqrt{10} + 3 - 4\sqrt{10} + 2 = 5 - \sqrt{10}$

3 **a** $7\sqrt{7}$ **b** $3\sqrt{2}$ **c** $7\sqrt{6}$ **d** $8\sqrt{3}$
e $3\sqrt{10}$ **f** 0 **g** $11\sqrt{5}$ **h** $8\sqrt{6}$
i $6\sqrt{3}$ **j** $8\sqrt{5}$ **k** 0 **l** $-4\sqrt{11}$
m $2\sqrt{2}$ **n** $-7\sqrt{3}$ **o** $-\sqrt{10}$ **p** 0

4 **a** $4\sqrt{7}$ **b** $-\sqrt{5}$ **c** $-\sqrt{3}$ **d** $2\sqrt{2}$
e 0 **f** $-3\sqrt{11}$ **g** $6\sqrt{5}$ **h** $\sqrt{10}$
i $4\sqrt{7}$ **j** 0 **k** $-4\sqrt{2}$ **l** $-4\sqrt{10}$
m $\sqrt{11}$ **n** $-2\sqrt{3}$ **o** $3\sqrt{5}$ **p** $5\sqrt{6}$
q $4\sqrt{7}$ **r** 0

5 **a** $4\sqrt{2} + 4\sqrt{5}$ **b** $5\sqrt{3} + 5\sqrt{7}$
c $3\sqrt{6} + 4\sqrt{3}$ **d** $10\sqrt{2} + 4\sqrt{6}$
e $5\sqrt{7}$ **f** $6\sqrt{5} - 2\sqrt{11}$
g $5\sqrt{5} - 2\sqrt{3}$ **h** $8\sqrt{6} - 3\sqrt{10}$
i $6\sqrt{3} + 4\sqrt{6}$ **j** $3\sqrt{5} + 3\sqrt{3}$
k $3\sqrt{3} + 10\sqrt{7}$ **l** $5\sqrt{7}$
m $\sqrt{11} + 3\sqrt{2}$ **n** $-\sqrt{10} + 4\sqrt{5}$
o $-5\sqrt{3} + 6\sqrt{2}$ **p** $3\sqrt{3}$

6 **a** $3\sqrt{6}$ **b** $\sqrt{2}$ **c** $5\sqrt{3}$ **d** $\sqrt{5}$
e $5\sqrt{7}$ **f** $-\sqrt{11}$ **g** $2\sqrt{10}$ **h** $-3\sqrt{2}$
i $7\sqrt{3}$ **j** $\sqrt{5}$ **k** $7\sqrt{6}$ **l** $5\sqrt{3}$
m 0 **n** $11\sqrt{2}$ **o** $6\sqrt{6}$ **p** 0

7 **a** $4\sqrt{7}$ **b** $9\sqrt{2}$
c $13\sqrt{3} + 3\sqrt{2}$ **d** $\sqrt{11} - 2\sqrt{7}$
e 0 **f** $5\sqrt{6} + 2\sqrt{5}$
g $8\sqrt{10}$ **h** $5\sqrt{3} - 10\sqrt{7}$
i $3\sqrt{2}$ **j** $-11\sqrt{6}$
k $\sqrt{11} + \sqrt{2}$ **l** $8\sqrt{2} - \sqrt{3}$
m $3\sqrt{2}$ **n** $-\sqrt{10} - \sqrt{5}$
o $16\sqrt{2} - 7\sqrt{3}$ **p** $9\sqrt{3} + 4\sqrt{6}$

8 **a** $3\sqrt{2}$ cm **b** $12\sqrt{2}$ cm

9 **a** $(2\sqrt{3} + 6)$ cm **b** $2\sqrt{3}$ cm^2

10 $6\sqrt{6} + 14$ cm; 28.70 cm

11 **a** $8\sqrt{2}$ m **b** $18\sqrt{2}$ m

exercise 4.5

1 **a** $\sqrt{22}$ **b** $\sqrt{30}$ **c** $\sqrt{30}$ **d** $\sqrt{10}$
e $\sqrt{14}$ **f** $\sqrt{6}$ **g** $\sqrt{21}$ **h** 2
i 5 **j** 3 **k** $2\sqrt{5}$ **l** $3\sqrt{2}$
m $2\sqrt{15}$ **n** $3\sqrt{10}$ **o** $7\sqrt{2}$

2 B

3 **a** $5\sqrt{21}$ **b** $7\sqrt{22}$ **c** $8\sqrt{15}$ **d** $20\sqrt{6}$
e $12\sqrt{10}$ **f** $2\sqrt{30}$ **g** $10\sqrt{33}$ **h** $4\sqrt{30}$
i $10\sqrt{66}$ **j** $9\sqrt{14}$ **k** $12\sqrt{35}$ **l** $15\sqrt{70}$
m $30\sqrt{15}$ **n** $14\sqrt{30}$ **o** $2\sqrt{110}$ **p** $20\sqrt{33}$

4 **a** 3 **b** 7 **c** 10 **d** 6
e 30 **f** 6 **g** 33 **h** 30
i 30 **j** 24 **k** 30 **l** 63
m 360 **n** 70 **o** 18 **p** 220

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5 **a** $15\sqrt{2}$ **b** $18\sqrt{5}$ **c** $30\sqrt{3}$ **d** $60\sqrt{2}$
e 24 **f** $6\sqrt{5}$ **g** $10\sqrt{15}$ **h** $30\sqrt{7}$
i $20\sqrt{6}$ **j** $6\sqrt{11}$ **k** $20\sqrt{5}$ **l** $24\sqrt{3}$
m $40\sqrt{5}$ **n** $30\sqrt{3}$ **o** $60\sqrt{11}$ **p** $210\sqrt{3}$

6 B

7 **a** $\sqrt{21}$ **b** 42 **c** $18\sqrt{11}$ **d** $30\sqrt{14}$
e $2\sqrt{77}$ **f** $40\sqrt{2}$ **g** $\sqrt{70}$ **h** 60
i 5 **j** $\sqrt{22}$ **k** $8\sqrt{70}$ **l** $12\sqrt{15}$
m $24\sqrt{5}$ **n** $6\sqrt{15}$ **o** 20 **p** $\sqrt{30}$

8 **a** $4\sqrt{2} + 12$ **b** $10 - 5\sqrt{7}$
c $6\sqrt{5} + 3$ **d** $\sqrt{10} + 3\sqrt{2}$
e $\sqrt{6} - \sqrt{15}$ **f** $\sqrt{35} + \sqrt{10}$
g $10\sqrt{3} + 2\sqrt{7}$ **h** $3\sqrt{2} - 21\sqrt{11}$
i $3 + 5\sqrt{3}$ **j** $\sqrt{10} - 5$
k $3\sqrt{2} - \sqrt{30}$ **l** $30 + 20\sqrt{2}$
m $6\sqrt{3} - 18$ **n** $6\sqrt{5} + 10\sqrt{2}$
o $6 + 6\sqrt{2} + 4\sqrt{6}$

9 D

10 **a** $\sqrt{6} + 3\sqrt{3} + 4\sqrt{2} + 12$
b $\sqrt{10} + \sqrt{5} + 6\sqrt{2} + 6$
c $7\sqrt{3} + 13$
d $11\sqrt{2} + 30$
e $\sqrt{6} + 3\sqrt{3} - 4\sqrt{2} - 12$
f $38 - 11\sqrt{10}$
g $\sqrt{6} + 5\sqrt{3} + 5\sqrt{2} + 3$
h $4\sqrt{6} + 6\sqrt{2}$
i $9\sqrt{3} + 16$
j $5\sqrt{10} - 23$
k $\sqrt{10} + 8$
l $19 - 8\sqrt{3}$
m $65 - 6\sqrt{14}$
n 5
o 37
p $101 + 8\sqrt{30}$

11 **a** $9\,\text{cm}^2$ **b** 3 cm

12 **a** $\sqrt{2}$ and $3\sqrt{2}$; $\sqrt{3}$ and $2\sqrt{3}$; $\sqrt{6}$ and $\sqrt{6}$
b $\sqrt{2}$ and $6\sqrt{2}$; $2\sqrt{2}$ and $3\sqrt{2}$; $\sqrt{3}$ and $4\sqrt{3}$; $2\sqrt{3}$ and $2\sqrt{3}$; $\sqrt{6}$ and $2\sqrt{6}$
c $(2\sqrt{3} + 3)(2\sqrt{3} - 3) = 3$

13 3; reject negative solution since the surd must be positive.

14 **a** 4 **b** 5 **c** 7 **d** 10 **e** 2 **f** 6

15 **a** $6 = 2 \times 3$, $12 = 3 \times 4$, $20 = 4 \times 5$ etc., i.e. the number takes the form $n(n + 1)$.
b For example, $\sqrt{56 + \sqrt{56 + \sqrt{56 + \ldots}}} = 8$
c Yes; $\sqrt{6 - \sqrt{6 - \sqrt{6 - \ldots}}} = 2$

$$\text{Let } x = \sqrt{6 - \sqrt{6 - \sqrt{6 - \ldots}}}$$
$$x^2 = 6 - \sqrt{\sqrt{6} - \sqrt{6} - \sqrt{6}}$$
$$x^2 = 6 - x$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x = -3 \text{ or } x = 2$$

Discard negative solution so
$\sqrt{6 - \sqrt{6} - \sqrt{6}} = 2$

exercise 4.6

1 **a** $\sqrt{2}$ **b** $\sqrt{3}$ **c** $\sqrt{7}$ **d** $\sqrt{11}$
e $\sqrt{2}$ **f** $\sqrt{10}$ **g** $\sqrt{3}$ **h** $\sqrt{5}$
i $2\sqrt{2}$ **j** $\sqrt{3}$ **k** 5 **l** $2\sqrt{3}$
m $3\sqrt{2}$ **n** 3 **o** $2\sqrt{5}$ **p** 2

2 **a** $\sqrt{3}$ **b** $\sqrt{\frac{2}{3}}$ **c** 2 **d** $\frac{2}{\sqrt{3}}$
e $\sqrt{\frac{3}{5}}$ **f** $2\sqrt{3}$ **g** $\frac{\sqrt{3}}{2}$ **h** 2
i $\frac{1}{\sqrt{5}}$ **j** $\frac{\sqrt{7}}{2}$ **k** $\sqrt{2}$ **l** $\frac{1}{3}$
m $\sqrt{\frac{6}{13}}$ **n** $3\sqrt{2}$ **o** $\sqrt{7}$ **p** 5

3 B

4 **a** $2\sqrt{7}$ **b** $2\sqrt{11}$ **c** $\frac{2\sqrt{3}}{3\sqrt{2}}$
d $\frac{2\sqrt{5}}{3\sqrt{2}}$ **e** 1 **f** $\frac{1}{2}$
g $\frac{6\sqrt{3}}{5\sqrt{5}}$ **h** $\frac{7\sqrt{7}}{5\sqrt{11}}$ **i** $\frac{7\sqrt{2}}{2\sqrt{5}}$
j $\frac{4}{\sqrt{7}}$ **k** $\frac{6}{\sqrt{2}}$ **l** $\frac{3\sqrt{3}}{\sqrt{5}}$ or $\frac{3\sqrt{15}}{5}$

5 D

6 $3\sqrt{7}$ cm

7 $x = \sqrt{2} + \sqrt{3}$ $\left(\text{which is the same as } x = \frac{2 + \sqrt{6}}{\sqrt{2}}\right)$

exercise 4.7

1 **a** 3 **b** 7 **c** 15 **d** 18

2 **a** $\frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{5}}{5}$ **c** $\frac{\sqrt{7}}{7}$ **d** $\frac{\sqrt{10}}{10}$
e $\frac{3\sqrt{2}}{2}$ **f** $\frac{4\sqrt{3}}{3}$ **g** $\frac{\sqrt{14}}{14}$ **h** $\frac{6\sqrt{7}}{7}$
i $\frac{\sqrt{6}}{6}$ **j** $\frac{5\sqrt{2}}{2}$ **k** $\frac{\sqrt{15}}{15}$ **l** $\frac{7\sqrt{10}}{10}$
m $\sqrt{3}$ **n** $2\sqrt{2}$ **o** $\frac{\sqrt{6}}{2}$ **p** $2\sqrt{15}$

3 **a** $\frac{\sqrt{30}}{6}$ **b** $\frac{\sqrt{33}}{11}$ **c** $\frac{\sqrt{10}}{2}$ **d** $\frac{\sqrt{6}}{3}$
e $\frac{\sqrt{14}}{7}$ **f** $\frac{\sqrt{70}}{10}$ **g** $\frac{\sqrt{42}}{7}$ **h** $\frac{\sqrt{30}}{10}$
i $\frac{\sqrt{3}}{3}$ **j** $\sqrt{2}$ **k** $\frac{\sqrt{6}}{3}$ **l** $\frac{\sqrt{77}}{11}$
m $\frac{1}{2}$ **n** $\frac{\sqrt{30}}{10}$ **o** $\frac{\sqrt{2}}{2}$ **p** $\frac{\sqrt{3}}{2}$

4 D

5 **a** $\frac{\sqrt{2}}{10}$ **b** $\frac{\sqrt{3}}{6}$ **c** $\frac{3\sqrt{5}}{10}$ **d** $\frac{7\sqrt{3}}{6}$
e $\frac{5\sqrt{2}}{6}$ **f** $\frac{3\sqrt{10}}{20}$ **g** $\frac{2\sqrt{2}}{3}$ **h** $\frac{\sqrt{3}}{5}$
i $\frac{\sqrt{6}}{9}$ **j** $\frac{3\sqrt{5}}{2}$ **k** $\frac{2\sqrt{15}}{9}$ **l** $\frac{2\sqrt{3}}{7}$
m $\frac{2\sqrt{2}}{5}$ **n** $\frac{\sqrt{6}}{4}$ **o** $\frac{\sqrt{3}}{4}$ **p** $\frac{3\sqrt{10}}{4}$

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6 **a** $\frac{\sqrt{6}}{10}$ **b** $\frac{\sqrt{14}}{21}$ **c** $\frac{\sqrt{77}}{22}$ **d** $\frac{\sqrt{6}}{21}$
e $\frac{\sqrt{10}}{10}$ **f** $\frac{\sqrt{15}}{10}$ **g** $\frac{2\sqrt{42}}{21}$ **h** $\frac{\sqrt{30}}{5}$
i $\frac{3\sqrt{14}}{7}$ **j** $\frac{\sqrt{70}}{15}$ **k** $\frac{\sqrt{10}}{3}$ **l** $\frac{2\sqrt{33}}{33}$
m $\frac{2\sqrt{10}}{3}$ **n** $\frac{2\sqrt{6}}{5}$ **o** $\frac{\sqrt{15}}{5}$ **p** $\frac{\sqrt{14}}{3}$

7 E

8 **a** **i** $\sqrt{11} - 3$ **ii** 2
b **i** $3 + \sqrt{5}$ **ii** 4
c **i** $2\sqrt{5} - \sqrt{3}$ **ii** 17
d **i** $2\sqrt{3} + \sqrt{7}$ **ii** 5
e **i** $\sqrt{7} + \sqrt{3}$ **ii** 4
f **i** $2\sqrt{2} + \sqrt{5}$ **ii** 3

9 **a** $\frac{\sqrt{7} + 2}{3}$ **b** $\frac{4 - \sqrt{10}}{6}$
c $3\sqrt{3} - 3\sqrt{2}$ **d** $-2\sqrt{3} - 2$
e $3 + \sqrt{6}$ **f** $\frac{\sqrt{15} - 3}{2}$
g $3\sqrt{2} + 4$ **h** $\frac{6\sqrt{5} + 9}{11}$

10 **a** $\frac{5\sqrt{3} + 9}{3}$ **b** $5\sqrt{5} - 11$
c $\frac{2\sqrt{14} - 1}{11}$ **d** $3 - 2\sqrt{2}$

11 $\frac{21\sqrt{2} - 10\sqrt{3}}{6}$ **12** $\frac{6\sqrt{6} - 9\sqrt{2} - 2\sqrt{3}}{6}$

13 $\sqrt{6} + 4$ **14** $x = \frac{2 - \sqrt{3}}{3}$

15 $\frac{5\sqrt{5} - 7}{4}$

revision

1 D **2** C **3** B **4** A **5** D

6 **a** 2 **b** 27 **c** $\frac{1}{25}$

7 **a** $\sqrt{50}$ **b** $-\sqrt{63}$

8 **a** $2\sqrt{5}$ **b** $-\sqrt{7}$

9 **a** $3\sqrt{6}$ **b** $6\sqrt{2}$ **c** $3\sqrt{5}$ **d** $4\sqrt{3}$

10 **a** $13\sqrt{3}$ **b** $12\sqrt{5} + \sqrt{2}$

11 **a** $3\sqrt{2}$ **b** 6 **c** $4\sqrt{10}$ **d** $5\sqrt{3}$

12 **a** $2\sqrt{5} + 6\sqrt{7}$ **b** $3\sqrt{2} + 4\sqrt{6}$
c $16 + 5\sqrt{10}$ **d** $23\sqrt{3} - 51$

13 **a** $5 - 3\sqrt{2}$ **b** 7

14 **a** $5\sqrt{2}$ **b** $\frac{\sqrt{7} - \sqrt{5}}{2}$ **c** $\frac{6 + \sqrt{3}}{11}$
d $\frac{\sqrt{2}}{3}$ **e** $\frac{5 + \sqrt{10}}{3}$ **f** $\frac{11\sqrt{2} - 14}{23}$

15 **a** $(8\sqrt{2} + 18)$ cm; 29.31 cm
b $(17\sqrt{2} + 26)$ cm^2; 50.04 cm^2

16 **a** $(360 + 30\sqrt{2} + 30\sqrt{3})$ cm **b** 454 cm

Chapter 5

exercise 5.1

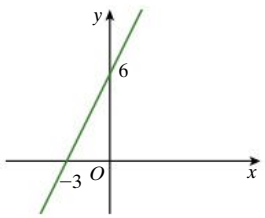
1 **a** 1 **b** $\frac{3}{2}$ **c** -2 **d** $-\frac{1}{2}$ **e** $\frac{3}{2}$ **f** $-\frac{3}{5}$

2 **a** $y = 2x - 6$ **b** $y = -3x + 4$
c $y = \frac{2}{3}x - 1$ **d** $y = -\frac{4}{5}x + 9$
e $y = \frac{5}{7}x - 3$ **f** $y = -3x + \frac{1}{2}$

3 **a** $y = 2x + 7$ **b** $y = -5x + 38$
c $y = -3x - 2$ **d** $y = 4x - 13$
e $y = -\frac{5}{4}x + \frac{47}{4}$ (or $4y = -5x + 47$)
f $y = \frac{2}{3}x - \frac{7}{3}$ (or $3y = 2x - 7$)
g $y = \frac{3}{4}x - \frac{11}{4}$ (or $4y = 3x - 11$)
h $y = -\frac{1}{2}x + \frac{5}{2}$ (or $2y = -x + 5$)

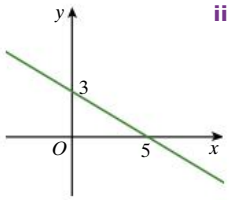
4 **a** $y = -11x + 16$
b $y = \frac{3}{2}x + 2$ (or $2y = 3x + 4$)
c $y = 2x - 5$
d $y = -3x - 7$
e $y = x + 6$
f $y = \frac{-4x}{3}$ (or $3y = -4x$)
g $y = \frac{3}{4}x + \frac{7}{4}$ (or $4y = 3x + 7$)
h $y = -x + 5$
i $y = \frac{3}{2}x - \frac{11}{2}$ (or $2y = 3x - 11$)
j $y = \frac{11}{8}x + \frac{23}{8}$ (or $8y = 11x + 23$)
k $y = 3x + 4$
l $y = \frac{2}{3}x + \frac{4}{3}$ (or $3y = 2x + 4$)

5 **a** **i**



ii $y = 2x + 6$

b **i**



ii $y = -\frac{3}{5}x + 3$
(or $5y = -3x + 15$)

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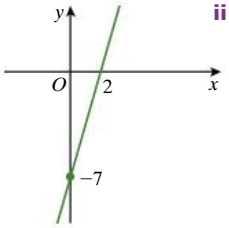
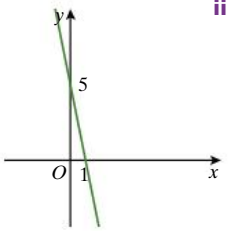
c i



ii $y = \frac{7}{2}x - 7$

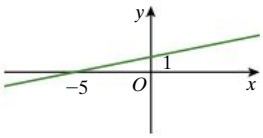
(or $2y = 7x - 14$)

d i



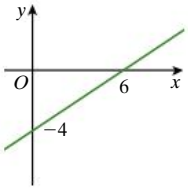
ii $y = -5x + 5$

e i



ii $y = \frac{1}{5}x + 1$ (or $5y = x + 5$)

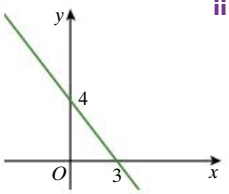
f i



ii $y = \frac{2}{3}x - 4$

(or $3y = 2x - 12$)

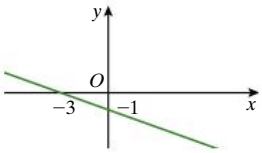
g i



ii $y = -\frac{4}{3}x + 4$

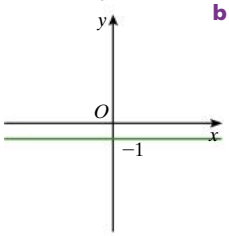
(or $3y = -4x + 12$)

h i

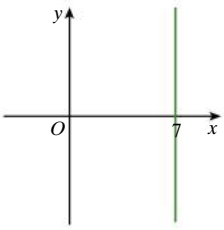


ii $y = -\frac{1}{3}x - 1$ (or $3y = -x - 3$)

6 a



b



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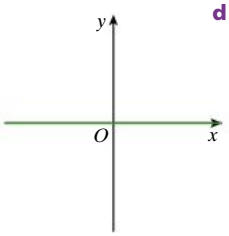

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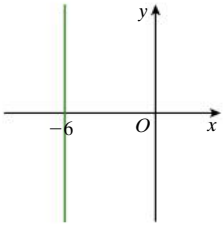

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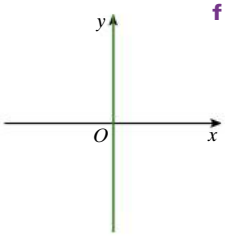

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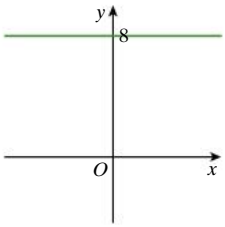


7 **a** $y = 5$ **b** $y = -3$ **c** $x = -4$
d $y = -6$ **e** $x = -2$ **f** $x = 6$

8 **a** $7x - 3y - 24 = 0$ **b** $2x - 5y - 35 = 0$
c $2x + 11y + 11 = 0$ **d** $3x - 2y + 5 = 0$
e $4x + 5y + 9 = 0$ **f** $3x + 4y + 10 = 0$

9 **a** $y = -\frac{1}{2}x + \frac{5}{4}$; $m = -\frac{1}{2}$, $b = \frac{5}{4}$

b $y = \frac{5}{2}x - 5$; $m = \frac{5}{2}$, $b = -5$

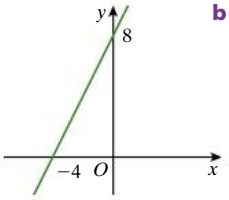
c $y = \frac{4}{3}x + 5$; $m = \frac{4}{3}$, $b = 5$

d $y = \frac{5}{8}x - \frac{5}{2}$; $m = \frac{5}{8}$, $b = -\frac{5}{2}$

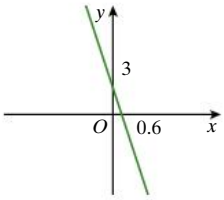
e $y = \frac{3}{2}x - \frac{25}{4}$; $m = \frac{3}{2}$, $b = -\frac{25}{4}$

f $y = -\frac{7}{4}x + \frac{5}{2}$; $m = -\frac{7}{4}$, $b = \frac{5}{2}$

10 a



b



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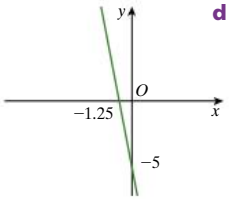

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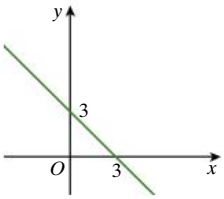

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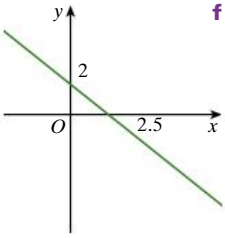

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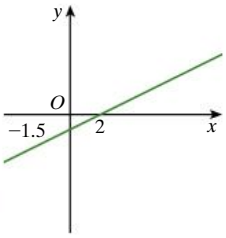

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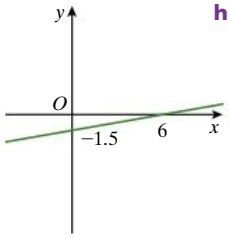

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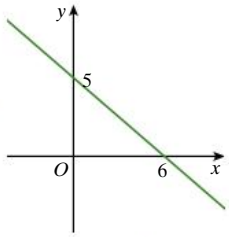

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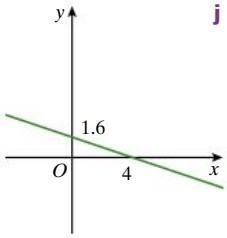

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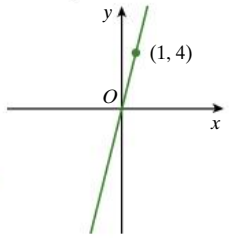

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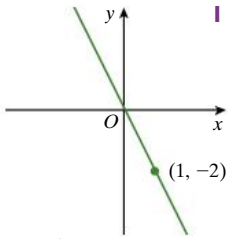

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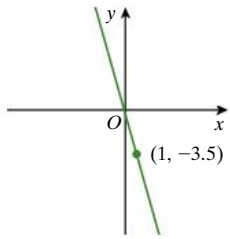


11 **a** $y = \frac{5}{2}x + \frac{3}{2}$ **b** $y = -2x + 3$

c $y = -x + 4$ **d** $y = 2x - 1$

e $y = -\frac{1}{2}x + 3$ **f** $y = \frac{1}{2}x - 2$

g $y = 3x$ **h** $y = -5x$

i $y = -4$ **j** $x = 5$

12 $y = -\frac{1}{2}x + \frac{9}{2}$

exercise 5.2

1 $y = -4x - 1$

2 $2y = 5x - 34 \left(\text{or } y = \frac{5}{2}x - 17\right)$

3 1st line: $m_1 = \frac{7 - (-2)}{-3 - (-6)} = \frac{9}{3} = 3$;

2nd line: $m_2 = 3$; $m_1 = m_2$, so the lines are parallel.

4 **a** $m = 1$ **b** $m = -\frac{1}{4}$ **c** $m = -2$ **d** $m = \frac{4}{5}$

e $m = 10$ **f** $m = -\frac{1}{3}$ **g** $m = -\frac{3}{2}$ **h** $m = \frac{1}{2}$

5 1st line: $m_1 = 4$; 2nd line: equation can be written as $y = -\frac{1}{4}x + \frac{7}{4}$, so $m_2 = -\frac{1}{4}$; $m_1 \times m_2 = -$, so the lines are perpendicular.

6 **a** $y = -\frac{1}{3}x - \frac{2}{3}$ **b** $y = \frac{4}{3}x - \frac{11}{3}$

c $y = x + 3$ **d** $y = -\frac{1}{2}x + 10$

e $y = -\frac{5}{3}x - \frac{35}{3}$ **f** $y = -4x + 17$

g $y = -\frac{3}{4}x - 2$ **h** $y = -3x + 8$

i $y = -x - 3$ **j** $y = \frac{2}{3}x + \frac{13}{3}$

k $y = \frac{1}{3}x - 6$ **l** $y = \frac{7}{2}x$

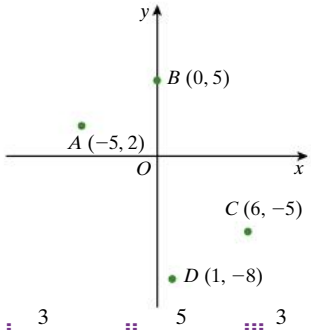
7 **a** **i** $m = -\frac{1}{2}$ **ii** $y = -\frac{1}{2}x - 3$

b **i** $m = 2$ **ii** $y = 2x + 2$

c **i** $m = \frac{1}{2}$ **ii** $y = \frac{1}{2}x - 4$

d **i** $m = -1$ **ii** $y = -x + 4$

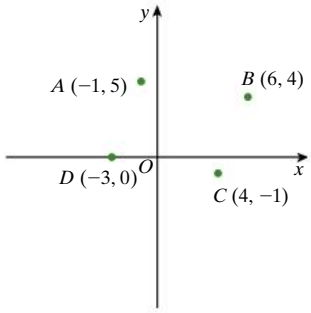
8 **a**



b **i** $\frac{3}{5}$ **ii** $-\frac{5}{3}$ **iii** $\frac{3}{5}$ **iv** $-\frac{5}{3}$

c Opposite sides have the same gradient and adjacent sides are perpendicular, so $AB \parallel CD$, $BC \parallel DA$, $AB \parallel BC$, and thus $ABCD$ is a rectangle.

9 **a**



b AB and CD each have gradient $-\frac{1}{7}$; BC and DA each have gradient $\frac{5}{2}$; opposite sides have the same gradient, so $AB \parallel CD$, $BC \parallel DA$, and thus $ABCD$ is a parallelogram.

10 For diagonal AC: $m_1 = \frac{3 - 5}{2 - 0} = \frac{-2}{2} = -1$; for diagonal BD: $m_2 = \frac{7 - 1}{4 - (-2)} = \frac{6}{6} = 1$;

$m_1 \times m_2 = -1$, so the diagonals of the rhombus are perpendicular.

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exercise 5.3

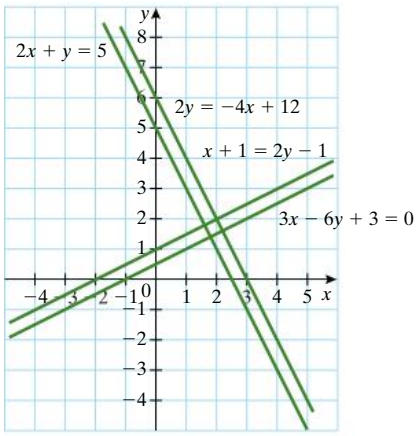
1 **a** $x = 3, y = 1$ **b** $x = 2, y = -1$
c $x = 3, y = 0$ **d** $x = 3\frac{1}{2}, y = 3\frac{1}{2}$
e $x = 2, y = 5$ **f** $x = 2, y = 1$

2 **a** $x = 6, y = 3$ **b** $x = 5, y = -1$
c $x = 4, y = 2$ **d** $x = 2, y = 5$
e $x = 7, y = 5$ **f** $x = 2\frac{1}{2}, y = 5\frac{1}{2}$

3 **a** $x = 2, y = 1$
b no solutions; parallel lines
c infinite number of solutions; lines coincide

4 $(-2, 3)$

5 **a** 6
b 1st and 3rd: parallel lines; 2nd and 4th: parallel lines
c $y = -2x + 5; y = \frac{1}{2}x + 1; y = -2x + 6;$ $y = \frac{1}{2}x + \frac{1}{2}$



d four in total
e 1st and 2nd: $\left(\frac{8}{5}, \frac{9}{5}\right)$, or (1.6, 1.8); 1st and 4th: $\left(\frac{9}{5}, \frac{7}{5}\right)$, or (1.8, 1.4);2nd and 3rd:(2, 2); 3rd and 4th: $\left(\frac{11}{5}, \frac{8}{5}\right)$, or (2.2, 1.6)

6 **a** one (different gradients)
b none (parallel lines)
c infinite (lines coincide)
d none (parallel lines)

7 **a** $T = 40h + 40$ **b** $T = 30h + 60$
c Pete; \$10 cheaper **d** Brad; \$20 cheaper
e 2 hours (both charge \$120)

8 **a** $x = -1, y = -6$
b lines very steep and close together

9 1st and 3rd: parallel lines

10 answers will vary

11 **a** **i** $a = 2$ **ii** any $a \neq 2$
b No. For $a = 2$, the corresponding lines coincide. For $a \neq 2$, the corresponding lines have different gradients and so have a unique point of intersection.

12 **a** **i** $k = -3$ **ii** any $k \neq -3$
b No, the corresponding lines either coincide or are parallel.

13 **a** $m = 3$ and $p = 9$ **b** $m = 3$ and $p \neq 9$
c $m \neq 3$ and any p

exercise 5.4

1 **a** $x = 2, y = 3$ **b** $x = -2, y = 17$
c $x = 3, y = 1$ **d** $x = 2\frac{1}{2}, y = \frac{1}{2}$
e $x = 1, y = 0$ **f** $x = 3, y = 4$
g $x = 3, y = 1$ **h** $x = 3, y = 2$

2 **a** $x = 1, y = 0$ **b** $x = 3, y = 1$
c $x = 3, y = 2$ **d** $x = 3, y = 4$
e $x = 4, y = 1$ **f** $x = 3, y = -2$
g $x = 2, y = 3$ **h** $x = -2, y = -1$
i $x = 5, y = 4$ **j** $x = 4, y = -2$
k $x = 4, y = 3$ **l** $x = -1, y = 3$

3 **a** $m = 2, n = -1$ **b** $j = 3, k = 1$
c $q = \frac{3}{2}, r = \frac{5}{6}$ **d** $x = 1, y = 1$

4 **a** $x = 1, y = 4$ **b** $b = 5, c = -4$
c $y = \frac{3}{2}, x = \frac{5}{6}$ **d** $x = 5, y = -3$
e $a = 2, b = 1$ **f** $a = 1, b = 2$
g $x = 4, y = -1$ **h** $x = 1, y = -1$
i $x = -2, y = 4$ **j** $x = 1, y = -4$
k $x = 4, y = 2$ **l** $x = 3, y = -1$

5 **a** $x = 2, y = 1$ **b** $x = 5, y = 2$
c $x = 2, y = -1$ **d** $x = -19, y = 16$
e $x = 3, y = 3$ **f** $x = 2\frac{1}{2}, y = 3\frac{1}{2}$
g $x = 5, y = -2$ **h** $x = 6, y = -2$
i $x = 2, y = 1$

6 **a** $y = 3 - x$ and $y = 2x - 1$
b $x = 1\frac{1}{3}, y = 1\frac{2}{3}$

7 **a** $x = \frac{2}{3}, y = \frac{1}{3}$ **b** $x = -2\frac{1}{5}, y = -1\frac{2}{5}$
c $x = 2\frac{3}{4}, y = 2\frac{1}{4}$
d $x = 0.8, y = 0.4$ (or $x = \frac{4}{5}, y = \frac{2}{5}$)

8 **a** $x = 2\frac{2}{3}, y = -7$ **b** $x = 2\frac{1}{2}, y = 3$

9 **a** equations to solve: $a + b = -2$ and $3a + b = 4$; solution is $a = 3, b = -5$. Equation of line is $y = 3x - 5$.
b From part **a**, the line has equation $y = 3x - 5$; its graph is as follows (observe that it passes through the points (1, −2) and (3, 4)).

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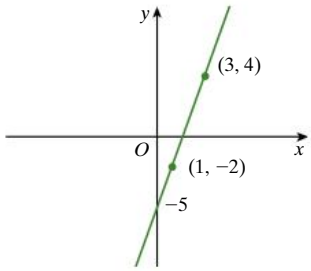


exercise 5.5

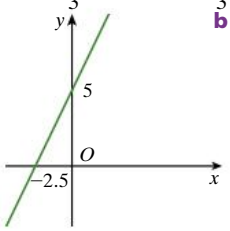
1 27 and 21
2 19 and 8
3 8 boys, 16 girls
4 Julie's son is 6 years old.
5 100 km (cost is then \$390 for each company)
6 12 minutes (cost is then 42 cents for each phone card)
7 250 adults
8 48 runs in the first innings, 96 runs in the second innings
9 Alannah is 8, Oscar is 5.
10 180 adults, 120 children
11 17 ants, 8 spiders
12 10 chickens, 8 cows
13 72 10-cent coins, 180 20-cent coins
14 Maria is 40, Kate is 10.

revision

1 D **2** E **3** D **4** A **5** B

6 **a** **i** -3 **ii** 7
b **i** $\frac{2}{5}$ **ii** $-3\frac{3}{5}$
c **i** $1\frac{2}{3}$ **ii** 0
d **i** $-1\frac{1}{3}$ **ii** $-\frac{2}{3}$

7 **a**



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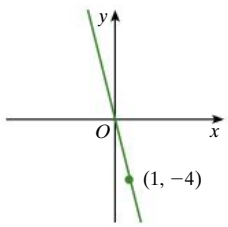

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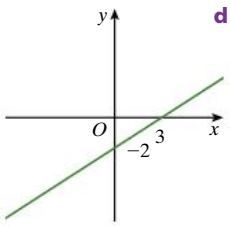

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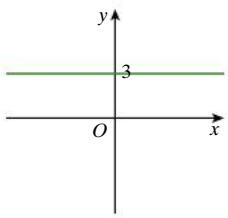

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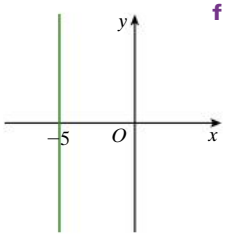

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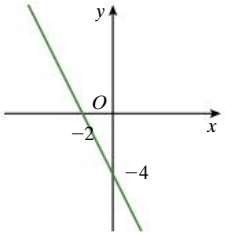

8 **a** $m = \frac{1}{2}$ **b** $m = -1$
c $m = -\frac{5}{2}$ **d** $m = -2$

9 1st line: $m_1 = \frac{2}{3}$; 2nd line: equation can be written as, $y = -\frac{3}{2}x + 2\frac{3}{4}$ so $m_2 = -\frac{3}{2}$; $m_1 \times m_2 = -1$, so the lines are perpendicular.

10 $y = -4x - 22$

11 **a** $y = -3x + 5$ **b** $y = \frac{2}{5}x - 1$

12 **a** $y = 3 - x$ and $y = 2x + 1$ **b** $x = \frac{2}{3}$, $y = 2\frac{1}{3}$

13 **a** $x = 0, y = 3$ **b** $x = 2, y = 8$
c $x = 0, y = 0$ **d** $x = 3, y = 0$
e $x = -\frac{1}{2}, y = 1$ **f** $x = 3, y = 3$
g $x = 7, y = -3$ **h** $x = 10, y = 2$
i $x = 3, y = 8$

14 **a** none (parallel lines)
b infinite (lines coincide)
c one (different gradients)
d none (parallel lines)

15 \$6 per adult, \$5 per child

Chapter 6

exercise 6.1

1 **a** \$98.60 **b** \$52.20 **c** \$1600 **d** \$6000
2 \$828
3 **a** \$71.25 **b** 5 years
4 10 years
5 6 years
6 **a** \$3000 **b** \$6250 **c** \$1000 **d** \$1000
7 **a** 4% **b** 4.5% **c** 5.1% **d** 1.8%
8 **a** \$1080 **b** \$7080
9 \$637.50
10 **a** \$5100 **b** \$335
11 **a** \$311.04 **b** \$48.64
12 **a** \$125 **b** \$1445
c \$195 **d** 7.8%
13 **a** \$432 **b** \$3492
c \$612 **d** 7.08%

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14 **a** $108 **b** $376.03 **c** $91.23

15 $13.49

16 **a** 0.049773% **b** 18.17%

exercise 6.2

1 **a**

After	Interest ($)	Total($)
0 years		5000
1 year	300	5300
2 years	318	5618
3 years	337.08	5955.08

b $5955.08 **c** $955.08

2 **a**

After	Interest ($)	Total ($)
0 years		12000
1 year	600	12600
2 years	630	13230
3 years	661.50	13891.50
4 years	694.575	14586.08

b $14586.08 **c** $2586.08

3 **a** **i** $1215.51 **ii** $215.51
b **i** $4899.17 **ii** $899.17
c **i** $9612.31 **ii** $2112.31
d **i** $3618.82 **ii** $618.82
e **i** $12744.29 **ii** $2744.29
f **i** $7517.85 **ii** $1517.85

4 **a** **i** $200 **ii** $15.51
b **i** $832 **ii** $67.17
c **i** $1920 **ii** $192.31
d **i** $576 **ii** $42.82
e **i** $2500 **ii** $244.29
f **i** $1392 **ii** $125.85

exercise 6.3

1 **a** **i** $1215.51 **ii** $215.51
b **i** $3374.59 **ii** $374.59
c **i** $10227.50 **ii** $2727.50
d **i** $3618.82 **ii** $618.82
e **i** $13540.81 **ii** $3540.81
f **i** $10544.06 **ii** $4544.06
g **i** $42316.53 **ii** $17316.53
h **i** $5239.02 **ii** $1239.02
i **i** $3221.21 **ii** $721.21
j **i** $3618.82 **ii** $618.82
k **i** $19846.34 **ii** $4846.34
l **i** $11001.21 **ii** $5001.21

2 **a** 5 years **b** 5 years **c** 6 years **d** 3 years
e 4 years **f** 6 years **g** 11 years **h** 15 years

3 $4631.93

4 $9000

exercise 6.4

1 **a** **i** $2676.45 **ii** $2687.83
iii $2693.71 **iv** $2697.70
b **i** $7348.76 **ii** $7367.67
iii $7377.38 **iv** $7383.96
c **i** $3272.80 **ii** $3288.58
iii $3296.75 **iv** $3302.30
d **i** $16426.05 **ii** $16552.94
iii $16618.76 **iv** $16663.56
e **i** $19116.44 **ii** $19186.82
iii $19223.19 **iv** $19247.89
f **i** $10519.02 **ii** $10569.29
iii $10595.23 **iv** $10612.84

2

Interest compounded	Interest rate	Effective Interest rate
Monthly	6.43%	6.62%
Quarterly	6.48%	6.64%
Half-yearly	6.54%	6.65%
Yearly	6.65%	6.65%

3

Interest compounded	Interest rate	Effective Interest rate
Monthly	7.00%	7.23%
Quarterly	7.05%	7.24%
Half-yearly	7.12%	7.25%
Yearly	7.25%	7.25%

exercise 6.5

1 **a**

After	Depreciation ($)	Value ($)
0 years		14000
1 year	2800	11200
2 years	2240	8960
3 years	1792	7168

b $7168 **c** $6832

2 **a**

After	Depreciation ($)	Value ($)
0 years		1200
1 year	240	960
2 years	192	768
3 years	153.60	614.40

b $614.40 **c** $585.60

3 **a**

After	Depreciation ($)	Value ($)
0 years		45000
1 year	8100	36900
2 years	6642	30258
3 years	5446.44	24811.56

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b \$24 812 **c** \$20 188

4 **a** \$29 520 **b** \$7359

5 \$47 250 **6** \$1584

7 \$41 675 **8** \$9480

9 \$28 480 **10** \$22 641

exercise 6.6

1 **a** \$790 **b** \$210 **c** 4 years 4 months
d \$400 **e** Answers may vary

2 **a** \$14 695.20 **b** \$4695.20
c \$303.32 **d** \$11 398.28
e \$11 337.56; saving of \$60.72

3 **a** \$423 819.50 **b** \$223 819.50
c \$224 067.52, an extra \$244.78 interest over that of the fortnightly scheme.
d \$223 717.82, a saving of \$104.92 interest on the fortnightly scheme.
e \$650 per week
f \$676 000
g \$326 000 (online calculator gives \$325 998.82)
h \$497 744; interest is \$147 744 (online calculator gives \$147 745.32)

4 **a** $10000\left(1+\frac{7}{1200}\right)^{120} = 20096.613\ldots$

b Fortnightly: \$20 118.59 (check:
$10000\left(1+\frac{7}{2600}\right)^{260} = 20118.594\ldots$)
Weekly: \$20 128.05 (check:
$10000\left(1+\frac{7}{5200}\right)^{520} = 20128.049\ldots$)

c \$19 318.14

5 **a** \$775.30
b Interest: \$583.33; principal: \$191.97
c Interest: \$4.50; principal: \$770.80
d \$781.31 **e** \$6.01 **f** \$898.83 **g** \$123.53

6 **a** \$16 429.92 **b** \$1429.92 **c** 11 000
d \$310.65 **e** \$18 639 **f** \$3639

revision

1 C **2** B **3** D **4** C **5** C

6 \$448

7 **a** \$48 **b** \$54 per month

8 **a**

After	Interest (\$)	Total (\$)
0 years		4000
1 year	300	4300
2 years	322.50	4622.50
3 years	346.69	4969.19

b \$4969.19 **c** \$969.19

9 **a** **i** \$2031.12 **ii** \\$531.12
b **i** \$5660.48 **ii** \$660.48
c **i** \$3075.51 **ii** \$575.51
d **i** \$18 262.70 **ii** \$6262.70

10 **a** **i** \$2040.47 **ii** \$540.47
b **i** \$5671.38 **ii** \$671.38
c **i** \$3086.52 **ii** \$586.52
d **i** \$18 397.37 **ii** \$6397.37

11 **a** **i** \$2045.31 **ii** \$545.31
b **i** \$5677.01 **ii** \$677.01
c **i** \$3092.24 **ii** \$592.24
d **i** \$18 467.48 **ii** \$6467.48

12 **a** **i** \$2048.59 **ii** \$548.59
b **i** \$5680.83 **ii** \$680.83
c **i** \$3096.13 **ii** \$596.13
d **i** \$18 515.31 **ii** \$6515.31

13 **a** 5.06% **b** 6.17% **c** 6.40% **d** 6.59%

14 **a**

After	Depreciation (\$)	Value (\$)
0 years		165 000
1 year	33 000	132 000
2 years	26 400	105 600
3 years	21 120	84 480

b \$84 480 **c** \$80 520

15 \$7027

16 **a** \$42 392 **b** \$5392 **c** 3.3%
d \$31 450 **e** \$16 417

Chapter 7

exercise 7.1

1 **a** 31.2 m^2 **b** 970 mm^2 **c** 30 cm^2
d 735 cm^2 **e** 73.9 cm^2 **f** 89.6 cm^2

2 **a** **i** 47.85 cm^2 **ii** 287.1 cm^2 **iii** 3214.2 cm^2
b **i** 60.5 cm^2 **ii** 484 cm^2 **iii** 2968 cm^2
c **i** 34.5 cm^2 **ii** 172.5 cm^2 **iii** 1845 cm^2
d **i** 62.4 cm^2 **ii** 374.4 cm^2 **iii** 2908.8 cm^2

3 227 cm^2

4 **a** **i** 314 cm^2 **ii** 471 cm^2
b **i** 1282 cm^2 **ii** 2187 cm^2
c **i** 1232 cm^2 **ii** 1539 cm^2
d **i** 118 cm^2 **ii** 157 cm^2
e **i** 19.4 m^2 **ii** 27.0 m^2
f **i** 21.1 m^2 **ii** 57.3 m^2

5 **a** 53 cm^2 **b** 792 cm^2
c 580.5 cm^2 **d** 527.8 cm^2

6 2267.2 cm^2

exercise 7.2

1 **a** 10.2 cm^3 **b** 1500 mm^3 **c** 7.5 cm^3
d 945 cm^3 **e** 1.536 m^3 **f** 34.6 cm^3
g 7740 cm^3 **h** 189 000 cm^3 **i** 0.63 m^3

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2 **a** 11 484 cm^3 **b** 12 100 cm^3
c 5175 cm^3 **d** 11 232 cm^3

3 **a** 785 cm^3 **b** 7690 cm^3 **c** 4310 cm^3
d 147 cm^3 **e** 11 m^3 **f** 25 m^3

4 **a** **i** 0.6 m^3 **ii** 0.6 kL
b **i** 52 m^3 **ii** 52 kL

5 **a** 19 cm^3 **b** 54 cm^3 **c** 795 cm^3 **d** 600 cm^3

6 8445 cm^3

7 6728 cm^3

exercise 7.3

1 **a** **i** 4.05 m^3 **ii** 4.05 kL
b 11.65 m^2

2 **a** 1.496 m^3 **b** 1496 L **c** 6.30 m^2

3 **a** 500 m^3 **b** 500 kL

4 6 m

5 **a** 183 mL **b** 11 drinks

6 19.9 cm

7 **a** 8544 cm^2 **b** 34 560 cm^3

8 4107 cm^2

9 **a** 41.4 m^3 **b** 43.2 m^3

10 **a** Cylinder: 49 359 mm^2; equilateral triangle prism: 65 785 mm^2
b The cylinder is more economical.

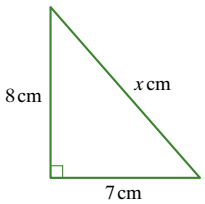
11 **a** 5.1 cm^3 **b** 39.6 g

exercise 7.4

1 **a** ≈ 12.6 cm **b** ≈ 21.6 cm
c ≈ 32.6 cm **d** ≈ 29.1 cm

2 E

3 **a**



b 5.4 cm

4 **a** 15 cm **b** 12 cm **c** 8 cm **d** 4.8 cm

5 **a** 11.2 cm **b** 15.3 cm **c** 28.3 cm **d** 3.5 m

6 **a** 12.6 cm **b** 19.2 cm **c** 4.6 cm **d** 4.0 cm

7 **a** $p = 48$ **b** $p = 30$ **c** $p = 34$ **d** $h = 24$
e $h = 75$ **f** $p = 52$ **g** $p = 58$ **h** $a = 90$

8 C

9 **a** 28 cm **b** 47 cm **c** 14 cm **d** 9 cm

10 **a** 3.7 cm **b** 43.2 cm^2 **c** 8.6 cm **d** 232 cm^2

exercise 7.5

1 **a** **i** 60 cm^2 **ii** 85 cm^2
b **i** 56 cm^2 **ii** 72 cm^2
c **i** 48 cm^2 **ii** 84 cm^2
d **i** 5120 cm^2 **ii** 9216 cm^2
e **i** 128 cm^2 **ii** 192 cm^2
f **i** 2304 cm^2 **ii** 3600 cm^2
g **i** 4160 cm^2 **ii** 5760 cm^2
h **i** 2176 cm^2 **ii** 3200 cm^2

2 C

3 **a** **i** 240 cm^2 **ii** 384 cm^2
b **i** 4160 cm^2 **ii** 5760 cm^2
c **i** 2000 cm^2 **ii** 3600 cm^2
d **i** 700 cm^2 **ii** 896 cm^2
e **i** 1625 cm^2 **ii** 2250 cm^2
f **i** 375 m^2 **ii** 600 m^2
g **i** 260 cm^2 **ii** 360 cm^2
h **i** 544 m^2 **ii** 800 m^2
i **i** 480 cm^2 **ii** 736 cm^2
j **i** 960 cm^2 **ii** 1360 cm^2
k **i** 672 cm^2 **ii** 868 cm^2
l **i** 1440 cm^2 **ii** 1764 cm^2

4 **a** $16\sqrt{48}\,cm^2 = 64\sqrt{3}\,cm^2$ **b** 110.9 cm^2

5 **a** **i** 628 cm^2 **ii** 829 cm^2
b **i** 188 cm^2 **ii** 302 cm^2
c **i** 1508 cm^2 **ii** 1960 cm^2
d **i** 1571 cm^2 **ii** 2827 cm^2
e **i** 427 cm^2 **ii** 628 cm^2
f **i** 204 cm^2 **ii** 283 cm^2
g **i** 188 cm^2 **ii** 302 cm^2
h **i** 22 cm^2 **ii** 28 cm^2
i **i** 1822 cm^2 **ii** 3079 cm^2
j **i** 5105 cm^2 **ii** 7069 cm^2
k **i** 1838 cm^2 **ii** 2545 cm^2
l **i** 309 cm^2 **ii** 396 cm^2

6 **a** 28 952.9 cm^2 **b** 706.9 cm^2 **c** 3217.0 cm^2
d 24.6 m^2 **e** 7238.2 mm^2 **f** 289.5 cm^2

7 **a** 314.2 cm^2 **b** 5026.5 cm^2 **c** 201.1 cm^2
d 12.6 m^2 **e** 7854.0 mm^2 **f** 452.4 cm^2

8 **a** 1.3 cm **b** 1.7 cm **c** 15.0 cm
d 35.0 mm **e** 0.6 m **f** 50 cm

9 Perpendicular height of each triangular face $= \dfrac{\sqrt{3}a}{2}$

Area of each triangular face $= \dfrac{\sqrt{3}a^2}{4}$

Total area of four triangular faces $= \sqrt{3}a^2$

Total surface area $= a^2 + \sqrt{3}a^2 = a^2(1 + \sqrt{3})$

10 **a** **i** $16\pi\,cm^3$ **ii** $64\pi\,cm^2$
iii $256\pi\,cm^2$ **iv** $1024\pi\,cm^2$
b The surface area increases by four times.

11 Increase by a factor of four.

exercise 7.6

1 **a** 1600 cm^3 **b** 0.48 m^3 **c** 6250 cm^3
d 937.5 m^3 **e** 400 cm^3 **f** 1280 m^3
g 7800 cm^3 **h** 23.98 m^3 **i** 384 cm^3
j 25 600 m^3 **k** 8000 cm^3 **l** 87 120 cm^3

2 **a** **i** 30 cm **ii** 10 240 cm^3
b **i** 75 cm **ii** 160 000 cm^3
c **i** 24 cm **ii** 32 768 cm^3

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d **i** 21 cm **ii** 11 200 cm^3
e **i** 24 cm **ii** 64 800 cm^3
f **i** 24 cm **ii** 1568 cm^3

3 **a** **i** 3 m **ii** 144 m^3
b **i** 12 cm **ii** 9216 cm^3
c **i** 14 m **ii** 2688 m^3
d **i** 6.9 cm **ii** 82.8 cm^3

4 **a** $2160\sqrt{3}\,cm^3$ **b** 3741 cm^3

5 **a** 260 cm^3 **b** 140 cm^3 **c** 8400 cm^3
d 7540 cm^3 **e** 4222 cm^3 **f** 25 130 cm^3
g 268 cm^3 **h** 101 cm^3 **i** 94 cm^3
j 151 cm^3 **k** 2827 cm^3 **l** 1885 cm^3

6 C

7 **a** 1005 cm^3 **b** 314 cm^3 **c** 302 cm^3 **d** 10 cm^3

8 **a** Angle at A is common, angles at B and D are equal (right angles), so angles at C and E are equal; hence triangles are similar.
b $\frac{x}{12} = \frac{24 + x}{20}$
c $x = 36$
d Subtract the volume of the small cone from that of the large cone.
e 19 704 cm^3

9 **a** **i** 14 cm **ii** 2268 cm^3
iii 672 cm^3 **iv** 1596 cm^3
b **i** 3.6 m **ii** 37.5 m^3
iii 19.2 m^3 **iv** 18.3 m^3

10 **a** 463 247 cm^3 **b** 1767 cm^3 **c** 17 157 cm^3
d 11 m^3 **e** 57 906 mm^3 **f** 463 cm^3

11 **a** 524 cm^3 **b** 33 510 cm^3 **c** 268 cm^3
d 4 m^3 **e** 65 450 mm^3 **f** 905 cm^3

12 **a** 16 755.2 cm^3 **b** 190 851.8 cm^3
c 45 976.2 mm^3 **d** 5.7 m^3
e 134 041.3 mm^3 **f** 56 548.7 cm^3

13 **a** 3.6 cm **b** 0.8 m **c** 2.1 cm
d 7.5 mm **e** 2.5 cm **f** 4.3 cm

14 Perpendicular height of each triangular face $= \frac{\sqrt{3}a}{2}$

Height of pyramid $= \sqrt{\left(\frac{\sqrt{3}a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = \frac{\sqrt{2}a}{2}$

Volume of pyramid $= \frac{1}{3}a^2 \times \frac{\sqrt{2}a}{2} = \frac{\sqrt{2}a^3}{6}$

15 **a** **i** 20 cm **ii** 2094 cm^3
iii 452 cm^3 **iv** 1642 cm^3
b **i** 42 cm **ii** 59 376 cm^3
iii 17 593 cm^3 **iv** 41 783 cm^3

exercise 7.7

1 866 cm^2 (including the plastic base)

2 35.6 m^2

3 **a** 5.29 hectares **b** 2 583 000 m^3
c 1.03 m^3 **d** 7.5 million tonnes
e 41 749 m^3

4 77 mL

5 571 cm^2

6 **a** 7917 cm^2 **b** 58.2 mL **c** 82.0 mL

7 **a** 85 cm^3 (85 mL) **b** 8.5 litres

8 8 times

9 **a** 4.9 m **b** 304.6 m^2 **c** 6.6 m **d** 355.4 m^2
e Cylindrical tank has a greater surface area by about 50 m^2.
f Spherical tank uses less material to construct, but would be harder to make.

10 **a** 3.9 cm **b** 248 cm^3 **c** 3.5 cm
d 179 cm^3 **e** 89% **f** 64%

11 **a** 1200 cm^3 **b** 328 cm^3 **c** 452 cm^3
d 780 cm^3 **e** 420 cm^3 **f** 140 cm^3
g 3 candles

12 86 m^3

13 **a** 510 million km^2 **b** 362 million km^2
c about 6.9 m **d** about 14.5%

14 **a** 1433 cm^2 **b** 36.1 cm

15 **a** $72\sqrt{2}\,mm^2$ **b** $72\sqrt{2}\,mm^3$

revision

1 D **2** E **3** C **4** A **5** E
6 C **7** A **8** C **9** C **10** B

11 **a** 1130 cm^3 **b** 40 m^3

12 **a** 603 cm^2 **b** 77 m^2

13 **a** 4130 cm^3 **b** 4 L **c** 970 cm^2 **d** 1430 cm^2

14 **a** **i** 905 cm^3 **ii** 905 mL
b 3595 cm^3

15 **a** 1280 m^3 **b** 400 cm^3 **c** 2413 cm^3
d 8042 cm^3 **e** 905 cm^3 **f** 4189 cm^3

16 **a** 800 m^2 **b** 360 cm^2 **c** 1206 cm^2
d 2513 cm^2 **e** 452 cm^2 **f** 1257 cm^2

17 **a** 361.911 cm^3 **b** 326.49 cm^2

18 **a** **i** 17 241 cm^3 **ii** 17 L
b **i** 7069 cm^3 **ii** 7 L

19 264 cm^3

20 **a** 300 L **b** 1.38 m^2 **c** 300 kg

21 **a** 21.2 m^2 **b** 7.07 m^3 **c** 7.07 kL

Chapter 8

exercise 8.1

1 a, c, d, e, g are quadratic equations

2 **a** $x = 0$ **b** $x = 3$ **c** $a = 6$ **d** $y = -7$
e $x = 0$ **f** $x = 11$ **g** $b = \frac{5}{3}$ **h** $x = \frac{15}{2}$

3 **a** $x = 0$ or $x = 7$ **b** $a = 0$ or $a = -1$
c $m = 0$ or $m = 2$ **d** $k = 0$ or $k = -6$
e $d = 0$ or $d = 25$ **f** $h = 0$ or $h = 9$
g $x = 4$ or $x = 1$ **h** $a = 7$ or $a = 5$
i $p = -4$ or $p = 3$ **j** $y = 6$ or $y = 7$

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k $n = -3$ or $n = 11$ **l** $x = 2$ or $x = -1$
m $a = -3$ or $a = 20$ **n** $d = 5$ or $d = 16$
o $x = 2$ or $x = -14$ **p** $x = 15$ or $x = 4$

4 **a** $x = 0$ or $x = \frac{5}{2}$ **b** $b = 0$ or $b = \frac{1}{2}$
c $y = 0$ or $y = \frac{8}{3}$ **d** $a = 0$ or $a = \frac{15}{4}$
e $x = 0$ or $x = \frac{25}{2}$ **f** $k = 0$ or $k = \frac{7}{3}$
g $m = 4$ or $m = \frac{1}{2}$ **h** $x = 2$ or $x = \frac{9}{2}$
i $x = -1$ or $x = \frac{8}{5}$ **j** $n = -2$ or $n = \frac{9}{4}$
k $x = -\frac{1}{2}$ or $x = 5$ **l** $n = \frac{11}{2}$ or $n = 1$
m $a = \frac{4}{3}$ or $a = -7$ **n** $x = -\frac{1}{2}$ or $x = \frac{5}{2}$
o $x = \frac{5}{3}$ or $x = \frac{3}{2}$ **p** $x = \frac{7}{4}$ or $x = \frac{1}{2}$

5 D

6 **a** $x = 4$ **b** $x = -2$ **c** $x = 7$
d $x = 11$ **e** $x = -5$ **f** $x = \frac{1}{2}$
g $x = -\frac{5}{2}$ **h** $x = \frac{4}{3}$ **i** $x = \frac{5}{4}$
j $x = -\frac{7}{2}$ **k** $x = \frac{8}{5}$ **l** $x = -\frac{1}{4}$

7 **a** $a = 3$ or $a = -7$ **b** $h = 8$
c $x = 12$ or $x = -12$ **d** $m = -2$
e $x = 5$ or $x = -9$ **f** $x = -\frac{3}{2}$
g $b = 0$ or $b = 5$ **h** $x = -\frac{8}{3}$
i $x = \frac{5}{3}$ or $x = -1$ **j** $x = -\frac{3}{2}$ or $x = \frac{2}{3}$
k $x = 0$ or $x = \frac{11}{4}$ **l** $x = -\frac{2}{5}$ or $x = \frac{7}{2}$

8 C

9 **a** $x(x + 6) = 0$ **b** $(x - 1)(4x - 3) = 0$
c $x(7x + 6) = 0$ **d** $(3x + 2)(4x - 11) = 0$

exercise 8.2

1 **a** $x = 0$ or $x = 6$ **b** $x = 0$ or $x = -5$
c $a = 0$ or $a = 12$ **d** $k = 0$ or $k = 0.5$
e $m = 0$ or $m = 8$ **f** $x = 0$ or $x = -\frac{1}{4}$
g $y = 0$ or $y = 11$ **h** $d = 0$ or $d = 7.5$
i $x = 0$ or $x = -0.1$ **j** $x = 0$ or $x = 14$
j $b = 0$ or $b = 2.5$ **l** $x = 0$ or $x = 2.4$

2 **a** $x = 0$ or $x = 2$ **b** $x = 0$ or $x = 3$
c $x = 0$ or $x = -3$ **d** $x = 0$ or $x = -6$
e $x = 0$ or $x = 3$ **f** $x = 0$ or $x = -4$
g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -7$
i $x = 0$ or $x = 8$ **j** $x = 0$ or $x = -6$
k $x = 0$ or $x = 5$ **l** $x = 0$ or $x = 6$

3 E

4 **a** $x = 3$ or $x = -3$ **b** $x = 8$ or $x = -8$
c $a = 7$ or $a = -7$ **d** $k = 9$ or $k = -9$
e $x = 6$ or $x = -6$ **f** $x = 11$ or $x = -11$
g $x = 3$ or $x = -3$ **h** $x = 5$ or $x = -5$
i $x = 2$ or $x = -2$ **j** $x = 4$ or $x = -4$
k $x = 1$ or $x = -1$ **l** $x = 5$ or $x = -5$

5 **a** $x = 1$ **b** $x = -3$ **c** $x = -2$ **d** $x = 5$
e $x = 8$ **f** $x = -7$ **g** $x = 10$ **h** $x = 4$
i $x = 6$ **j** $x = \frac{1}{2}$ **k** $x = \frac{5}{2}$ **l** $x = \frac{2}{3}$

6 **a** $x = 1$ or $x = 6$ **b** $x = 2$ or $x = 4$
c $x = 2$ or $x = 7$ **d** $x = -2$ or $x = 7$
e $x = 2$ or $x = 5$ **f** $x = -2$ or $x = 5$
g $x = -2$ or $x = 8$ **h** $x = -2$ or $x = 15$
i $x = -2$ or $x = 9$ **j** $x = -2$ or $x = 6$
k $x = 2$ or $x = 3$ **l** $x = 6$ or $x = 7$
m $x = 4$ or $x = 9$ **n** $x = -3$ or $x = 8$
o $x = 3$ or $x = -9$

7 D

8 **a** $x = 0$ or $x = 1$ **b** $x = 2$ or $x = -2$
c $x = -3$ **d** $x = 2$ or $x = 6$
e $x = -1$ or $x = 11$ **f** $x = 6$ or $x = -8$
g $x = 2$ **h** $x = 2$ or $x = 4$
i $x = 2$ or $x = -6$ **j** $x = 3$ or $x = -4$
k $x = 0$ or $x = -6$ **l** $x = 4$ or $x = -4$
m $x = 1$ or $x = -15$ **n** $x = 0$ or $x = 8$
o $x = 1$ or $x = 2$ **p** $x = 2$ or $x = -2$
q $x = 4$ **r** $x = 0$ or $x = 4$
s $x = -2$ or $x = -3$ **t** $x = 1$ or $x = -6$
u $x = 0$ or $x = -4$ **v** $x = 4$ or $x = -7$
w $x = -4$ or $x = -6$ **x** $x = 2$ or $x = -12$

9 **a** $x = \frac{7}{5}$ or $x = 3$ **b** $x = \frac{2}{5}$ or $x = \frac{4}{3}$
c $x = -\frac{1}{7}$ or $x = \frac{2}{5}$ **d** $x = 1\frac{1}{4}$ or $x = \frac{7}{2}$

exercise 8.3

1 **a** $x = 0$ or $x = 8$ **b** $x = 0$ or $x = -2$
c $x = 0$ or $x = -2.5$ **d** $x = 0$ or $x = 1.4$
e $x = 0$ or $x = \frac{1}{2}$ **f** $x = 4$ or $x = -1$
g $x = 1$ or $x = -6$ **h** $x = 5$ or $x = -2$
i $x = 7$ or $x = -2$ **j** $x = 1$ or $x = 8$
k $x = 3$ or $x = 5$ **l** $x = 8$ or $x = -2$
m $x = 3$ or $x = -9$ **n** $x = 3$ or $x = -8$
o $x = 6$ or $x = -5$

2 A

3 B

4 **a** $x = 3$ or $x = -4$
b Less chance of error with substitution.

5 **a** $x = 2$ **b** $x = 3$ or $x = 4$
c $x = \frac{2}{3}$ or $x = -\frac{4}{3}$ **d** $x = 1$ or $x = -1$
e $x = 1$ or $x = 7$ **f** $x = 1$ or $x = 8$

6 **a** $x = \pm 1$ or $x = \pm 3$ **b** $x = \pm 2$ or $x = \pm 3$
c $x = \pm 1$ or $x = \pm 4$ **d** $x = \pm 3$

7 **a** $x = -4$ or $x = 1$ **b** $x = 3$ or $x = 5$
c $x = 8$ or $x = -3$ **d** $x = 12$ or $x = -1$
e $x = 5$ or $x = -1$ **f** $x = 3$ or $x = -15$

8 13 sides

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exercise 8.4

1 **a** $x = \sqrt{6}$ or $x = -\sqrt{6}$
b $x = \sqrt{11}$ or $x = -\sqrt{11}$
c $x = \sqrt{5}$ or $x = -\sqrt{5}$
d $x = \sqrt{2}$ or $x = -\sqrt{2}$
e $x = \sqrt{17}$ or $x = -\sqrt{17}$
f $x = 3\sqrt{2}$ or $x = -3\sqrt{2}$
g $x = 2\sqrt{5}$ or $x = -2\sqrt{5}$
h $x = 2\sqrt{2}$ or $x = -2\sqrt{2}$
i $x = 2\sqrt{3}$ or $x = -2\sqrt{3}$
j $x = \sqrt{19}$ or $x = -\sqrt{19}$

2 **a** $x = 1 + \sqrt{10}$ or $x = 1 - \sqrt{10}$
b $x = 2$ or $x = 4$
c $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$
d $x = -3 + \sqrt{13}$ or $x = -3 - \sqrt{13}$
e $x = 1$ or $x = 3$
f $x = -3 + \sqrt{5}$ or $x = -3 - \sqrt{5}$
g $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$
h $x = 1 + \sqrt{6}$ or $x = 1 - \sqrt{6}$
i $x = 8 + \sqrt{69}$ or $x = 8 - \sqrt{69}$
j $x = 5 + \sqrt{35}$ or $x = 5 - \sqrt{35}$
k $x = 10$ or $x = -2$
l $x = 4$ or $x = -6$
m $x = 2 + \sqrt{13}$ or $x = 2 - \sqrt{13}$
n $x = 5$ or $x = -3$
o $x = 4 + 2\sqrt{5}$ or $x = 4 - 2\sqrt{5}$
p $x = 1 + 2\sqrt{2}$ or $x = 1 - 2\sqrt{2}$
q $x = 5 + 4\sqrt{2}$ or $x = 5 - 4\sqrt{2}$
r $x = 6 + 4\sqrt{2}$ or $x = 6 - 4\sqrt{2}$

3 b, e, k, l, n; each has integer answers

4 **a** $x = \dfrac{3 + \sqrt{17}}{2}$ or $x = \dfrac{3 - \sqrt{17}}{2}$
b $x = 1$ or $x = 4$
c $x = \dfrac{1 + \sqrt{13}}{2}$ or $x = \dfrac{1 - \sqrt{13}}{2}$
d $x = 2$ or $x = -1$
e $x = \dfrac{5 + \sqrt{33}}{2}$ or $x = \dfrac{5 - \sqrt{33}}{2}$
f $x = 2$ or $x = -3$
g $x = \dfrac{3 + \sqrt{37}}{2}$ or $x = \dfrac{3 - \sqrt{37}}{2}$
h $x = 6$ or $x = -1$
i $x = \dfrac{3 + \sqrt{29}}{2}$ or $x = \dfrac{3 - \sqrt{29}}{2}$
j $x = 1$ or $x = 2$
k $x = 5$ or $x = -4$
l $x = \dfrac{1 + \sqrt{21}}{2}$ or $x = \dfrac{1 - \sqrt{21}}{2}$

5 b, d, f, h, j, k; each has integer answers

6 **a** Harry's solutions are incorrect. In the fourth line he shouldn't have written the 2 at the front and the square brackets. It should read:

$$x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 - \frac{1}{16} - \frac{3}{2} = 0$$

In the second last line. It should read:

$$x = \frac{5}{4} + \frac{1}{4} \text{ or } x = -\frac{5}{4} + \frac{1}{4}$$

This makes the last line: $x = \dfrac{3}{2}$ or $x = -1$.

b Harry could simply have used factorisation: $2x^2 - x - 3 = (2x - 3)(x + 1)$.

7 **a** $\dfrac{x}{1} = \dfrac{x + 1}{x}$
b $x^2 - x - 1 = 0$
c $x = \dfrac{1 + \sqrt{5}}{2}$ or $x = \dfrac{1 - \sqrt{5}}{2}$
d no; $x > 0$ as its length
e $x = \dfrac{1 + \sqrt{5}}{2}$ is the golden ratio.
f 1.618

exercise 8.5

1 **a** $x = 3$ or $x = -\dfrac{5}{2}$ **b** $x = 3$ or $x = -\dfrac{7}{2}$
c $x = 11$ or $x = -\dfrac{1}{2}$ **d** $x = \dfrac{3}{2}$ or $x = \dfrac{1}{2}$
e $x = \dfrac{5}{2}$ or $x = -\dfrac{3}{2}$ **f** $x = \dfrac{5}{2}$ or $x = -\dfrac{2}{3}$
g $x = \dfrac{1}{2}$ or $x = -7$ **h** $x = \dfrac{1}{2}$ or $x = \dfrac{1}{3}$
i $x = 1$ or $x = -\dfrac{4}{3}$ **j** $x = \dfrac{1}{2}$ or $x = -\dfrac{5}{3}$
k $x = \dfrac{1}{3}$ or $x = -\dfrac{1}{4}$ **l** $x = \dfrac{3}{2}$ or $x = \dfrac{2}{3}$
m $x = \dfrac{2}{3}$ or $x = -6$ **n** $x = \dfrac{4}{3}$ or $x = -2$
o $x = \dfrac{1}{4}$ or $x = -3$ **p** $x = \dfrac{5}{3}$ or $x = -2$
q $x = \dfrac{3}{5}$ or $x = -2$ **r** $x = 1$ or $x = -\dfrac{3}{4}$
s $x = \dfrac{1}{4}$ or $x = -\dfrac{1}{2}$ **t** $x = -\dfrac{1}{3}$ or $x = -\dfrac{1}{4}$
u $x = 1$ or $x = \dfrac{3}{7}$ **v** $x = 3$ or $x = \dfrac{2}{3}$
w $x = 5$ or $x = \dfrac{7}{2}$ **x** $x = \dfrac{4}{3}$ or $x = -\dfrac{1}{2}$

2 **a** $x = \dfrac{1}{3}$ **b** $x = -\dfrac{1}{2}$ **c** $x = \dfrac{5}{4}$
d $x = \dfrac{5}{3}$ **e** $x = \dfrac{2}{5}$ **f** $x = \dfrac{5}{2}$
g $x = -\dfrac{1}{5}$ **h** $x = \dfrac{7}{2}$ **i** $x = -\dfrac{2}{3}$
j $x = \dfrac{3}{8}$ **k** $x = \dfrac{10}{3}$ **l** $x = \dfrac{3}{4}$

3 **a** $x = -1$ or $x = -\dfrac{1}{2}$ **b** $x = 3$ or $x = -\dfrac{1}{2}$
c $x = \dfrac{3}{2}$ or $x = 1$ **d** $x = 3$ or $x = -\dfrac{4}{3}$
e $x = \dfrac{1}{2}$ or $x = -\dfrac{5}{2}$ **f** $x = \dfrac{1}{3}$ or $x = 7$

4 **a** $x = 3$ or $x = \dfrac{7}{5}$ **b** $x = \dfrac{2}{5}$ or $x = \dfrac{4}{3}$

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c $x = \frac{2}{5}$ or $x = -\frac{1}{7}$ **d** $x = \frac{5}{4}$ or $x = \frac{7}{2}$

e $x = \frac{8}{3}$ or $x = -\frac{3}{2}$ **f** $x = \frac{9}{2}$ or $x = -\frac{4}{3}$

5 **a** $x = \frac{1}{2}$ or $x = -\frac{2}{3}$ **b** $x = -\frac{1}{6}$ or $x = -\frac{8}{3}$

c $x = 2$ or $x = \frac{9}{4}$ **d** $x = -\frac{3}{2}$ or $x = -\frac{11}{8}$

exercise 8.6

1 **a** $x = \frac{3 + \sqrt{17}}{2}$ or $x = \frac{3 - \sqrt{17}}{2}$

b $x = \frac{5 + \sqrt{13}}{2}$ or $x = \frac{5 - \sqrt{13}}{2}$

c $x = \frac{2}{5}$ or $x = 1$

d $x = \frac{9 + \sqrt{33}}{8}$ or $x = \frac{9 - \sqrt{33}}{8}$

e $x = \frac{-1 + \sqrt{19}}{3}$ or $x = \frac{-1 - \sqrt{19}}{3}$

f $x = \frac{1 + \sqrt{61}}{10}$ or $x = \frac{1 - \sqrt{61}}{10}$

g $x = \frac{-4 + \sqrt{6}}{5}$ or $x = \frac{-4 - \sqrt{6}}{5}$

h $x = \frac{1 + \sqrt{21}}{4}$ or $x = \frac{1 - \sqrt{21}}{4}$

i $x = 3 + 2\sqrt{3}$ or $x = 3 - 2\sqrt{3}$

j $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$

k $x = \frac{3 + \sqrt{23}}{2}$ or $x = \frac{3 - \sqrt{23}}{2}$

l $x = \frac{5 + \sqrt{201}}{8}$ or $x = \frac{5 - \sqrt{201}}{8}$

m $x = \frac{-5 + \sqrt{73}}{4}$ or $x = \frac{-5 - \sqrt{73}}{4}$

n $x = \frac{1 + \sqrt{13}}{4}$ or $x = \frac{1 - \sqrt{13}}{4}$

o $x = 1$ or $x = -\frac{7}{3}$

p $x = \frac{7 + \sqrt{41}}{4}$ or $x = \frac{7 - \sqrt{41}}{4}$

q $x = -1$ or $x = -\frac{2}{3}$

r $x = \frac{-1 + \sqrt{37}}{6}$ or $x = \frac{-1 - \sqrt{37}}{6}$

2 **a** $x \approx 1.61$ or $x \approx -5.61$
b $x \approx 1.54$ or $x \approx -4.54$
c $a \approx -0.27$ or $a \approx -3.73$
d $y \approx 2.40$ or $y \approx -1.73$
e $d \approx 1.55$ or $d \approx -3.88$
f $k \approx -0.39$ or $k \approx -7.61$
g $x \approx 2.11$ or $x \approx -7.11$
h $h \approx 6.08$ or $h \approx -0.58$
i $a \approx 1.19$ or $a \approx -3.19$
j $x \approx 2.28$ or $x \approx -26.28$
k $y \approx 2.64$ or $y \approx -1.57$
l $x \approx 2.64$ or $x \approx -1.89$

3 **a** $x \approx 3.55$ or $x \approx -1.55$
b $m \approx 1.35$ or $x \approx -1.85$
c $x \approx 2.39$ or $x \approx -13.39$
d $x \approx 6.36$ or $x \approx -2.36$
e $h \approx 6.27$ or $h \approx -1.27$
f $x \approx 1.11$ or $x \approx -2.51$
g $x \approx 1.76$ or $x \approx -4.26$
h $m \approx 1.30$ or $m \approx -2.55$
i $a \approx 452.65$ or $a \approx -2.65$
j $x \approx 4.24$ or $x \approx -0.24$
k $x \approx 14.35$ or $x \approx -0.35$
l $x \approx 4.82$ or $x \approx -0.48$

4 C

5 **a** **i** 0.2 min, 3.8 min **ii** 0.8 min, 3.2 min
iii 1.2 min, 2.8 min
b The train passes through each speed as it goes faster, then slows down to a stop at B passing through each speed again.
c After 2 minutes; 60 km/h
d After 4 minutes; speed is zero km/h

6 **a** 40 sec **b** 7.8 sec, 32.2 sec

7 **a** **i** 1 m, 199 m **ii** 13 m, 187 m
iii 50 m, 150 m
b The golf ball passes through each height on its way up, then passes through each height on its way down again.
c At 100 m; max height 20 m
d 200 m; height is zero metres

8 Many answers, for example: $(3x - 2)(5x + 6) = 0$; $15x^2 + 8x - 12 = 0$; $x(15x + 8) = 12$.

exercise 8.7

1 **a** $n^2 - 23n + 120 = 0$ **b** $n = 8$ or $n = 15$
c The integers are 8 and 15.

2 **a** $n^2 - 25n + 136 = 0$ **b** $n = 8$ or $n = 17$
c The integers are 8 and 17.

3 **a** $n^2 + n - 156 = 0$ **b** $n = 12$ or $n = -13$
c The integers are 12 and 13.

4 **a** $n^2 - 2n - 8 = 0$ **b** $n = 4$ or $n = -2$
c The integers are 4 and 5.

5 **a** $n^2 + 2n - 168 = 0$ **b** $n = 12$ or $n = -14$
c The integers are 12 and 14 or -12 and -14.

6 **a** $n^2 + 2n - 143 = 0$ **b** $n = 11$ or $n = -13$
c The integers are 11 and 13 or -13 and -11.

7 **a** $n^2 - 2n - 3 = 0$ **b** $n = 3$ or $n = -1$
c The integers are 3 and 5 or -1 and 1.

8 **a** $n^2 - n - 12 = 0$ **b** $n = 4$ or $n = -3$.
c The integers are 4 and 6.

9 **a** $n^2 - 9n - 10 = 0$ **b** $n = 10$ or $n = -1$
c The integers are 10, 11 and 12.

10 **a** $2w^2 - 72 = 0$ **b** $w = 6$ or $w = -6$
c The width is 6 m and the length is 12 m.

11 **a** $w^2 + 4w - 60 = 0$ **b** $w = 6$ or $w = -10$
c The width is 6 m and the length is 10 m.

12 **a** $w^2 + 11w - 102 = 0$ **b** $w = 6$ or $w = -17$
c The width is 6 m and the length is 17 m.

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13 **a** $w^2 + 20w - 69 = 0$ **b** $w = 3$ or $w = -23$
c The width is 3 m and the length is 23 m.

14 **a** $w^2 - 36w + 308 = 0$ **b** $w = 14$ or $w = 22$
c The width is 14 m and the length is 22 m.

15 **a** $a^2 - 2a - 15 = 0$ **b** $a = 5$ or $a = -3$
c The lengths of the sides are 5 cm, 12 cm and 13 cm.

16 **a** $a^2 - 16a - 80 = 0$ **b** $a = 20$ or $a = -4$
c The lengths of the sides are 20 cm, 21 cm and 29 cm.

17 **a** $6a^2 - 42a = 0$ **b** $a = 0$ or $a = 7$
c The lengths of the sides are 7 cm, 24 cm and 25 cm.

18 The width of the path is 2 m.

exercise 8.8

1 **a** $\Delta = 17$; 2 solutions
b $\Delta = 20$; 2 solutions
c $\Delta = -56$; no solutions
d $\Delta = 65$; 2 solutions
e $\Delta = 0$; 1 solution
f $\Delta = -15$; no solutions

2 **a** Real, rational, equal
b Real, rational, unequal
c Real, rational, unequal
d Real, irrational, unequal
e Unreal
f Real, irrational, unequal
g Real, rational, unequal
h Real, rational, equal
i Unreal

3 A

4 C

5 **a** 2 solutions **b** 2 solutions **c** 2 solutions

6 **a** No real solutions
b $a \approx 0.56$ or $a \approx -12.56$
c $y \approx -0.30$ or $y \approx -6.70$
d $x \approx 3.11$ or $x \approx -1.61$
e No real solutions
f $y \approx 0.32$ or $y \approx -6.32$
g $x \approx 0.39$ or $x \approx -0.64$
h No real solutions
i No real solutions
j $x \approx 3.59$ or $x \approx -2.09$
k No real solutions
l $x \approx -0.65$ or $x \approx -3.85$

7 **a** $\Delta = 4 - 12k$ **b** $\Delta = m^2 + 32$
c $\Delta = 4n^2 - 12$ **d** $\Delta = 16 - 4k$

8 $k > -\frac{9}{4}$

9 $m > \frac{49}{20}$

10 $n = 1$ or $n = -3$

11 $\Delta = k^2 + 24 > 0$ for all values of k, so two real solutions for all values of k.

12 $\Delta = 4k^2 \geq 0$ for all values of k, so real solutions for all values of k (if $k = 0$, $\Delta = 0$ and there is 1 solution; if $k = 3$, the equation is linear and there is 1 solution; in all other cases, $\Delta > 0$ and there are 2 solutions).

13 $\Delta = k^2 - 4 = (k - 2)(k + 2) > 0$ if $k < -2$ or $k > 2$; so Rachel and Min found part of the answer, but Simone was wrong.

revision

1 D **2** D **3** A **4** E **5** A

6 **a** $m = 4$ **b** $x = \frac{4}{3}$

7 **a** $x = 8$ or $x = -5$ **b** $b = 7$ or $b = -\frac{5}{2}$
c $b = 2.5$ **d** $x = \frac{7}{2}$

8 **a** $x^2 - 2x - 15 = 0$ **b** $3x^2 - 8x + 4 = 0$

9 **a** $a = 0$ or $a = 6$ **b** $x = 8$ or $x = -1$
c $x = 0$ or $x = \frac{2}{3}$ **d** $k = 13$ or $k = -13$
e $t = 3$ or $t = -3$ **f** $x = 7$
g $x = 3$ or $x = 5$ **h** $a = 7$ or $a = -6$
i $x = 7$ or $x = 9$ **j** $x = 0$ or $x = 1.6$
k $b = 12$ or $b = -2$ **l** $x = \frac{5}{2}$

10 **a** $x = 5$ or $x = 7$ **b** $y = 8$ or $y = -3$
c $x = 7$ or $x = -4$ **d** $x = 9$ or $x = -6$
e $x = 1$ or $x = 4$ **f** $x = 9$ or $x = -2$

11 **a** $x = 1$ or $x = -5$ **b** $x = 2$ or $x = -\frac{3}{2}$
c $x = \pm 1$ or $x = \pm 5$ **d** $x = \pm 2$ or $x = \pm 5$

12 **a** **i** $x = -2 + \sqrt{17}$ or $x = -2 - \sqrt{17}$
ii $x \approx 2.12$ or $x \approx -6.12$
b **i** $x = 4 + 2\sqrt{3}$ or $x = 4 - 2\sqrt{3}$
ii $x \approx 7.46$ or $x \approx 0.54$
c **i** $x = \frac{-1 + \sqrt{13}}{2}$ or $x = \frac{-1 - \sqrt{13}}{2}$
ii $x \approx 1.30$ or $x \approx -2.30$
d **i** $x = \frac{5 + \sqrt{97}}{2}$ or $x = \frac{5 - \sqrt{97}}{2}$
ii $x \approx 7.42$ or $x \approx -2.42$

13 **a** **i** $x = -3 + \sqrt{26}$ or $x = -3 - \sqrt{26}$
ii $x \approx 2.10$ or $x \approx -8.10$
b **i** $x = \frac{-1 + \sqrt{22}}{3}$ or $x = \frac{-1 - \sqrt{22}}{3}$
ii $x \approx 1.23$ or $x \approx -1.90$
c **i** $x = \frac{7 + \sqrt{65}}{8}$ or $x = \frac{7 - \sqrt{65}}{8}$
ii $x \approx 1.88$ or $x \approx -0.13$
d **i** $x = \frac{4 + \sqrt{10}}{3}$ or $x = \frac{4 - \sqrt{10}}{3}$
ii $x \approx 2.39$ or $x \approx 0.28$

14 **a** **i** $\Delta = 64$ **ii** 2 rational solutions
b **i** $\Delta = -31$ **ii** no real solutions
c **i** $\Delta = -31$ **ii** no real solutions
d **i** $\Delta = 0$ **ii** 1 rational solution

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15 $n = 10$ or $n = -2$

16 **a** **i** $n^2 + n - 12 = 0$
ii $n = 3$ or $n = -4$
iii The integers are 3 and 5.
b **i** $w^2 + 13w - 48 = 0$
ii $w = 3$ or $w = -16$
iii The width is 3 m and the length is 16 m.
c **i** $x^2 - 8x - 9 = 0$
ii $x = 9$ or $x = -1$
iii The lengths of the sides are 9 cm, 40 cm and 41 cm.

Chapter 9

exercise 9.1

1 Outline of proof: $\triangle ADC \equiv \triangle CBA$(SSS), so $\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$. As AC is a transversal and $\angle DAC = \angle BCA$ are alternate angles, then $AD \parallel BC$. Similarly, as AC is a transversal and $\angle DCA = \angle BAC$ are alternate angles, then $AB \parallel DC$. Hence, a quadrilateral with both pairs of opposite sides equal is a parallelogram.

2 $\angle BAC = a°$ ($\angle ABC$ isosceles) and $\angle PAC = b°$ ($\angle PAC$ isosceles), so $\angle PAB = a° + b°$. The sum of the angles in $\triangle PAB = 2a° + 2b° = 180°$ and hence $\angle PAB = a° + b° = 90°$.

3 Outline of proof: $\triangle ABC \equiv \triangle ADC$ (SSS), so $\angle ABC = \angle ADC$ (matching angles).

4 **a** From question **3**, $\triangle ABC \equiv \triangle ADC$, so $\angle BAC = \angle DAC$ (matching angles) and $\angle BCA = \angle DCA$ (matching angles), i.e. AC bisects $\angle DAB$ and $\angle DCB$.
b Outline of proof: $\triangle ADE \equiv \triangle ABE$ (SAS), so $\angle AED = \angle AEB$ and these must be right angles since they form a straight angle. Similarly $\angle DEC$ and $\angle BEC$ are right angles, so AC and BD intersect at right angles.

5 Outline of proof: $\triangle AOB \equiv \triangle COD$ (ASA), so $AO = CO$ and $BO = DO$ (matching sides) and the diagonals bisect each other.

6 From question **5**, the diagonals of the rhombus bisect each other, so the four triangles that make up the rhombus are congruent (SAS). Then the four angles at M are equal (matching angles in congruent triangles) and since they add to 360°, each angle must be 90°.

7 **a** Outline of proof: $\triangle OCE \equiv \triangle ODE$ (SSS), so $\angle COE = \angle DOE$(matching angles).
b To ensure the triangles formed are congruent.
c No; the triangles formed do not need to be isosceles to be congruent.

8 **a** 2α. Reason: $\angle APO = \alpha$ (base angles of isosceles triangle) so $\angle PAB = \alpha + \alpha = 2\alpha$ (exterior angle of triangle is sum of two interior opposite angles).
b 2α. Reason: $\angle ABP = \angle BAP$ (base angles of isosceles triangle).
c 3α. Reason: $\angle APB = 180° - 4\alpha$ (third angle in $\angle ABP$) and angles on a straight line add to 180°.
So $\angle BPQ = 180 - \angle APB - \angle APO$
$= 180 - (180 - 4\alpha) - \alpha$
$= 180 - 180 + 4\alpha - \alpha$
$= 3\alpha$
d No

9 **a** An isosceles triangle with angles of 45°, 45° and 90°.
b Example: a scalene triangle with angles of 125°, 15° and 40°.
c Provide the other two angles add to 120°, they do not have to be 60°. Example: 60°, 73° and 47°.

10 In $\triangle EBG$ and $\triangle FDG$; $\angle EBG = \angle FDG$(alternate angles); $\angle EGB = \angle FGD$ (vertically opposite angles). So $\angle BEG = \angle DFG$ (if two angles of a pair of triangles are equal, the third angles must be equal).
$AB = DC$ (opposite sides of parallelogram);
$AE = CF$ (given).
So $EB = FD$ ($AB - AE = DC - CF$),
so $\triangle EBG \equiv \triangle FDG$(ASA)

11 In $\triangle ABE \equiv \triangle CDG$, $AB = CD$ (opposite sides of parallelogram);
$\angle AEB = \angle CGD$ (both angles are 90°, given).
So $\angle EAB = \angle GCD$ (if two angles of a pair of triangles are equal, the third angles must be equal);
$\angle ABE = \angle DCG$ (alternate angles between parallel sides of $ABCD$). So $\triangle ABE \equiv \triangle CDG$ (ASA)

12 In $\triangle AED$ and $\triangle CFB$, $AE = CF$(given);
$AD = BC$ (opposite sides of parallelogram $ABCD$); $\angle DAE = \angle BCF$ (opposite angles of parallelogram $ABCD$). So $\triangle AED \equiv \triangle CFB$ (SAS), so $ED = BF$(matching sides of congruent triangles); $EB = DF$ ($AB - AE = DC - CF$). So $DEBF$ is a parallelogram (both pairs of opposite sides equal)

13 In $\triangle ABC$ and $\triangle EBD$, $\angle BAC = \angle BED$ (given);
$AB = BE$ (given); $\angle ABC = \angle EBD$ (same angle).
So $\triangle ABC \equiv \triangle EBD$ (ASA). So $BD = BC$.
(matching sides of congruent triangles).
So $AD = EC$($AB - DB = BE - BC$)

14 In $\triangle ABD$ and $\triangle CBE$, $\angle BAD = \angle BCE$ (given);
$\angle BAD = \angle BCE$; $\angle ABC = \angle FBE$ (given);
$\angle ABD = \angle CBE$ ($\angle ABD = \angle ABC + \angle CBD$ and $\angle CBE = \angle FBE + \angle CBD$). So $\angle BEC = \angle BDA$ (if two angles of a pair of triangles are equal, the third angles must be equal)

15 **a** Four
b Since the figure is a parallelogram, adjacent angles are supplementary, so the two angles adjacent to the right angle are right angles. The remaining angle must then be a right angle.

16 **a** Four
b Since the figure is a rectangle, opposite sides are equal. If also two adjacent sides are equal, then all four must be equal.

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17 **a** Many answers **b** Many answers
c Any point along the perpendicular bisector of AB.

18 Drop a perpendicular from P to the line meeting it at point A. Let B be any other point on the line. Then $\triangle PAB$ is a right-angled triangle with right angle at A. By Pythagoras' theorem, $PA^2 + AB^2 = PB^2$, so $PA^2 < PB^2$ and hence $PA < PB$. Since B is any point on the line other than A, PA is the shortest distance to the line.

exercise 9.2

1 **a** $\angle AXB = \angle CXD$(vertically opposite angles), $\angle BAC = \angle DCA$ and $\angle ABD = \angle CDB$ (alternate angles, $AB \parallel DC$) so the triangles are equiangular and hence similar.
b CX **c** $d = 28, e = 24$

2 $\angle DAE = \angle BAC$ (shared angle), $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$ (corresponding angles, $DE \parallel BC$) so the triangles are equiangular and hence similar.

3 **a** True (AAA) **b** False
c False **d** True (AAA)

4 **a** The corresponding lengths of the given sides in each triangle are in the same proportion (2) and the angles contained between each pair are equal vertically opposite angles, so the triangles are similar (SAS).
b $\angle BAD = \angle CDA$ and $\angle ABC = \angle DCB$ (matching angles in similar triangles). But these also form pairs of alternate angles, so as they are equal, $AB \parallel CD$.

5 $BE = 8$ cm, $CE = 12.5$ cm (using the fact that $\triangle ABE$ is similar to $\triangle CDE$).

6 **a** $\triangle PBA$ and $\triangle ABC$ each have a right angle and $\angle ABP = \angle ABC$ (shared angle), so the remaining angles in each triangle are equal; so the triangles are equiangular and hence similar. A similar argument shows that $\triangle PAC$ and $\triangle ABC$ are similar.
b $AP = 4.8$ cm, $PB = 3.6$
c $DQ = 6.72$ cm, $EQ = 23.04$ cm

exercise 9.3

1 Outline of proof: $\triangle AOP$ is isosceles (radii of circle) so $\angle APO = \angle PAO = a°$ (base angles of isosceles triangle are equal). Then the exterior angle of $\triangle AOP$ at O is $2a°$.
Similarly $\angle BPO = \angle PBO = b°$ and the exterior angle of $\triangle BOP$ at O is $2b°$. So $\angle APB = a° + b°$ and $\angle AOB = 2a° + 2b° = 2(a° + b°) = 2\angle APB$.

2 Outline of proof: obtuse angle $\angle DOB = 2\angle DAB$; reflex angle $\angle DOB = 2\angle DCB$. The sum of the obtuse angle $\angle DOB$ and the reflex angle $\angle DOB$ is 360° and their sum is also equal to $2\angle DAB + 2\angle DCB = 2(\angle DAB + \angle DCB)$.
So: $2(\angle DAB + \angle DCB) = 360°$ and hence $\angle DAB + \angle DCB = 180°$.
Angle sum of any quadrilateral is 360°.
So $\angle ADB\ \angle ABC$ must also be 180°.

3 **a** $a = 114$ **b** $b = 49$
c $c = 57$ **d** $x = 53, y = 41$
e $e = 322$ **f** $f = 105$
g $g = 102.5$ **h** $h = 63$
i $i = 27$ **j** $j = 56$
k $k = 44$ **l** $l = 87$
m $a = 104, b = 92$ **n** $x = 116, y = 134$
o $e = 104, m = 101$

4 $\angle BAD + \angle BCD = 180°$ (opposite angles of a cyclic quadrilateral are supplementary).
$\angle BCP + \angle BCD = 180°$ (angles on a straight line).
Hence $\angle BCP = \angle BAD$.

5 Outline of proof: call the cyclic quadrilateral $ABCD$. With respect to the diameter AC, angles B and D are both angles in a semicircle, so they are each right angles. Similarly with respect to the diameter BD, angles A and C are both angles in a semicircle, so they are each right angles. Hence in the cyclic quadrilateral $ABCD$, all four angles are right angles. Thus the quadrilateral is a rectangle.

exercise 9.4

1 Outline of proof: $\angle ACD = \angle DBA$, $\angle CAB = \angle BDC$ (angles on the same arc) so triangles PAC and PDB are similar.
Then $\dfrac{AP}{PD} = \dfrac{CP}{PB}$, and multiplying through by $PB \times PD$ gives $AP \times PB = CP \times PD$.

2 Outline of proof: consider triangles OAP and OBP. $OA = OB$ (equal radii), OP is shared and $\angle OAP$ and $\angle OBP = 90°$ (tangent is perpendicular to the radius). Then the triangles are congruent (RHS) so $AP = BP$ matching sides.

3 **a** $a = 90, b = 136$ **b** $d = 119$
c $c = 90$ **d** $k = 64, p = 64$

4 **a** $a = 8$ **b** $x = 7.5$ **c** $b = 4$ **d** $n = 7.7$

5 $r = 5.25$, i.e. the radius of the ball is 5.25 cm.

6 For each chord, construct its perpendicular bisector. Where the two perpendicular bisectors meet is the centre of the circle.

7 Draw two chords on the picture of the plate going between the intact circumference on the plate. Using the method outlined in answer to question **6**, the radius of the plate is about 12 cm.

8 **a** The perpendicular bisector of a chord passes through the centre of the circle.
b $r^2 = (r - a)^2 + \left(\dfrac{c}{2}\right)^2$
c $r \approx 129.75$, i.e. the radius of the circle is about 129.75 m.
d $s \approx 30.5$, i.e. the speed of the car is about 30.5 m/s.
e The car was travelling at about 110 km/h.

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9 Each of the sides of the triangle is a chord of the required circle. For any two of these sides, construct their perpendicular bisectors; where the perpendicular bisectors meet is the centre O of the circle. The circle can then be drawn using any of OA, OB or OC as the radius.

10 Outline of proof: in exercise **9.4**, question **4**, we proved that the exterior angle of a cyclic quadrilateral equals the opposite interior angle. In the given diagram, $\angle PDB = \angle PAC$, $\angle PBD = \angle PCA$(exterior angles of a cyclic quadrilateral equal opposite interior angles) so triangles PAC and PDB are similar. Then $\frac{AP}{PD} = \frac{CP}{PB}$, and multiplying through by $PB \times PD$ gives $AP \times PB = CP \times PD$.

exercise 9.5

1 **a** $AB + CD = AD + BC$.
b $AB + CD = (AP + BP) + (CR + DR)$ $= (AS + BQ) + (CQ + DS)$ (two tangents from a point to a circle are equal in length, so $AP = AS$ etc). Rearranging gives: $(AS + DS) + (CQ + BQ)$ $= AD + BC$.

2 **a** $OABC$ is a rhombus.
b $\angle PAB = \angle P'CB$ (given)
c Triangles OAB and OCB are congruent triangles (SSS) so $\angle OAB = \angle OCB$.
d Triangles OAP and OCP' are congruent triangles (SAS) and they are also isosceles, so their base angles are equal and hence $\angle AOP = \angle APO = \angle COP' = \angle CP'O$.
e $\angle OAP = \angle OCP'$, so the remaining angles on straight lines must be equal.
f In triangle OAP, the sum of the three angles is 180°, i.e. $a + 2b + c = 180$ (1). In rhombus $OABC$, the sum of two adjacent angles is 180°, i.e. $d + 2b + c = 180$ (2). Comparing (1) and (2), we see that $a = d$, i.e. $\angle POP' = \angle PAB$.
g Triangles OAP and OCP' are congruent triangles (SAS) so $OP = OP'$.

3 **a** $CEDP$ is a rhombus, since $CE = DE = CP = DP$.
b $\triangle EAB$ is isosceles with $AE = BE$, as triangles ACE and BDE are congruent (SAS).
c Triangles ACP and BDP are congruent isosceles triangles (SAS).
d Triangles ACE and BDE are isosceles right-angled triangles, so their base angles are each 45°.
e In triangle ACE, $\angle ACE = a° + b° = 90°$, i.e. $a + b = 90$ (1). In rhombus $CEDP$, the sum of two adjacent angles is 180°, i.e. $b + 45 + c + 45 = 180$, so $b + c = 90$ (2). Comparing (1) and (2), we see that $a = c$, i.e. $\angle AEB = \angle ACP$. Then since triangles AEB and ACP are isosceles, their three angles must be equal and the triangles are similar.
f From part **e**, isosceles triangles AEB and ACP are similar, so their base angles are equal. Hence $45 + e = e + d$, giving $d = 45$. So triangle APB is an isosceles triangle with base angles of 45°, so $\angle APB = 90°$ (sum of triangle).

revision

1 B **2** A **3** A **4** E **5** D

6 $\angle ECF = 30°$ (sum of 3 angles in $\triangle ECF$ is 180°). $\angle ADB = 60°$(sum of 3 angles in $\triangle ADB$ is 180°). $\angle CGD = 180° - 30° - 60° = 90°$, i.e. it is a right angle (sum of 3 angles in $\triangle CGD$ is 180°).

7 $a = 72$, $b = 50$, $c = 58$, $d = 95$, $e = 27$, $f = 115$, $g = 38$, $h = 142$

8 126°

9 **a** To ensure that the two arcs intersect above the line.
b Outline of proof: $AP = BP$, $AC = BC$ and PC is shared, so $\triangle APC \equiv \triangle BCP$ (SSS). Then$\angle APC = \angle BPC$ and since these angles form a straight angle, each must be a right angle, i.e. CP is perpendicular to AB.
c To ensure that the triangles APC and BPC are congruent.

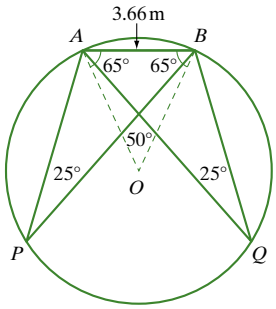
10 Outline of proof: since AM is the perpendicular bisector of OB, $\triangle AOM \equiv \triangle ABM$ (SAS). Then $OA = AB$. But OA is a radius of the circle, as is OB, so $OA = AB = OB$. Thus $\triangle AOB$ is equilateral.

11 **a** $x = 38$, $y = 61$ **b** $b = 64$
c $c = 39$ **d** $d = 44$
e $e = 322$ **f** $a = 82$
g $d = 25$ **h** $h = 120.5$
i $x = 90$

12 Outline of proof: $\angle DAB = \angle DCB$, $\angle ADC = \angle ABC$ (angles on the same arc) so triangles ADX and CBX are equiangular and hence similar.

13 $AX = 9.6$ cm

14 **a** The arc AB will subtend an angle of 25° at any point on the major arc of the circle.
b 50°
c Isosceles, because AO and BO are radii of the circle
d 65° **e, f**



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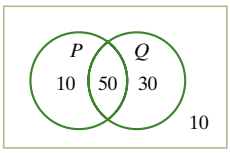
Chapter 10

exercise 10.1

1 **a** 0.62 **b** 0.30 **c** 0.18 **d** 0.82

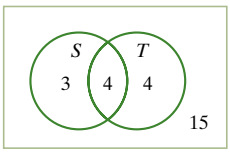
2 **a**



$n = 100$

b 0.9

3 **a**



$n = 26$

b **i** $\frac{4}{13}$ **ii** $\frac{7}{26}$ **iii** $\frac{2}{13}$ **iv** $\frac{11}{26}$ **v** $\frac{15}{26}$ **vi** $\frac{3}{26}$

4 **a** $\frac{7}{30}$ **b** $\frac{1}{10}$ **c** $\frac{3}{10}$ **d** $\frac{1}{15}$

5 **a** $\frac{11}{15}$ **b** $\frac{8}{15}$ **c** $\frac{2}{5}$ **d** 26 **e** $\frac{13}{15}$

f $\frac{13}{15}$; same as part **e** answer

6 0.91

7 **a** **i** $\frac{1}{2}$ **ii** $\frac{5}{6}$ **iii** $\frac{2}{3}$ **iv** $\frac{5}{6}$

b iii and iv

8 **a** $\frac{1}{2}$ **b** $\frac{1}{13}$ **c** $\frac{1}{26}$ **d** $\frac{7}{13}$

9 **a** 0.59 **b** 0.41

10 0.48

11 **a** $\frac{17}{30}$ **b** $\frac{1}{6}$ **c** $\frac{7}{30}$ **d** $\frac{2}{5}$

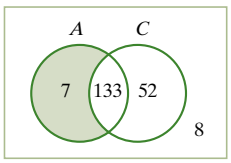
12 **a**

	A	*A′*	Total
B	30	20	50
B′	32	18	50
Total	62	38	100

b

	A	*A′*	Total
B	0.30	0.20	0.50
B′	0.32	0.18	0.50
Total	0.62	0.38	1

13 **a** **i**



$n = 200$

ii

	A	*A′*	Total
C	133	52	185
C′	7	8	15
Total	140	60	200

iii

	A	*A′*	Total
C	0.665	0.26	0.925
C′	0.035	0.04	0.075
Total	0.7	0.3	1

b **i** $0.665\left(\frac{133}{200}\right)$ **ii** $0.035\left(\frac{7}{200}\right)$

14 **a**

	X	*X′*	Total
Y	0.55	0.33	0.88
Y′	0.03	0.09	0.12
Total	0.58	0.42	1

b **i** 0.03 **ii** 0.09

15 **a**

	S	*S′*	Total
F	0.42	0.18	0.60
F′	0.28	0.12	0.40
Total	0.70	0.30	1

b 0.12

16

	P	*P′*	Total
Q	0.35	0.07	0.42
Q′	0.40	0.18	0.58
Total	0.75	0.25	1

17

	A	*A′*	Total
B	0.21	0.31	0.52
B′	0.44	0.04	0.48
Total	0.65	0.35	1

18

	Y	*Y′*	Total
X	0.35	0.10	0.45
X′	0.45	0.10	0.55
Total	0.80	0.20	1

19 0.81

20 $\frac{7}{15}$

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exercise 10.2

1 Events in part **c** are most likely not independent; all other pairs of events are independent.

2 **a**

	A	A'	Total
B	0.35	0.15	0.5
B'	0.29	0.21	0.5
Total	0.64	0.36	1

$\Pr(A) \times \Pr(B) = 0.32$ and $\Pr(A \text{ and } B) = 0.35$, so A and B are not independent.

b

	A	A'	Total
B	0.28	0.52	0.8
B'	0.07	0.13	0.2
Total	0.35	0.65	1

$\Pr(A) \times \Pr(B) = 0.28 = \Pr(A \text{ and } B)$, so A and B are independent.

c

	A	A'	Total
B	0.55	0.14	0.69
B'	0.25	0.06	0.31
Total	0.80	0.20	1

$\Pr(A) \times \Pr(B) = 0.552$ and $\Pr(A \text{ and } B) = 0.55$, so A and B are not independent.

d

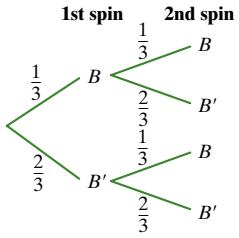
	A	A'	Total
B	0.21	0.63	0.84
B'	0.04	0.12	0.16
Total	0.25	0.75	1

$\Pr(A) \times \Pr(B) = 0.21 = \Pr(A \text{ and } B)$, so A and B are independent.

3

	P	P'	Total
Q	0.05	0.15	0.2
Q'	0.20	0.60	0.8
Total	0.25	0.75	1

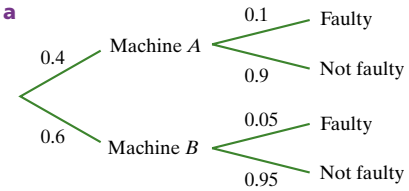
4 **a**



b **i** $\frac{1}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{4}{9}$ **iv** $\frac{2}{9}$

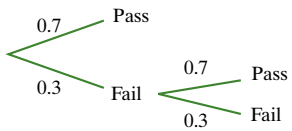
5 $\frac{1}{16}$

6 **a**



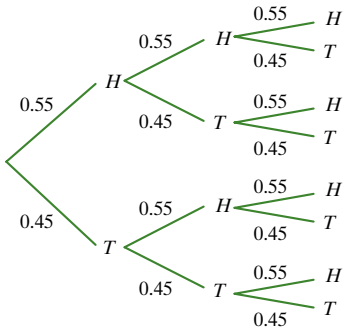
b 0.36 **c** 0.07

7 **a**



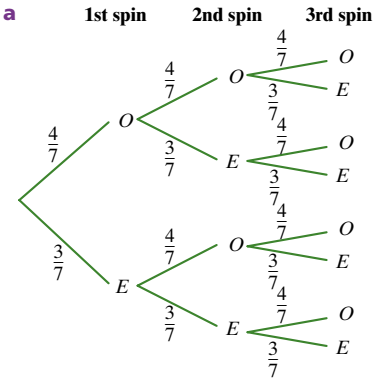
b **i** 0.09 **ii** 0.91

8 **a**



b **i** 0.0911 **ii** 0.4084 **iii** 0.8336

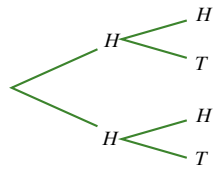
9 **a**



b **i** $\frac{64}{343}$ **ii** $\frac{144}{343}$ **iii** $\frac{36}{343}$

10 **a** $\frac{1}{12}$ **b** $\frac{5}{12}$ **c** $\frac{1}{8}$ **d** $\frac{7}{8}$

11 **a**



b $\frac{1}{2}$

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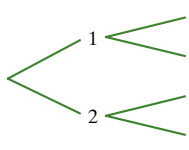
12 **a**



b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{4}$

13 **a** $\frac{1}{33}$ **b** $\frac{3}{22}$

14 $\frac{1}{879\,138}$

15 **a** ≈ 0.5177, so a bit better than even chance of winning.

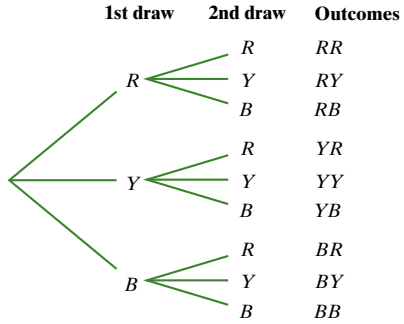
b ≈ 0.4914, so a little less than even chance of winning.

exercise 10.3

1 **a** $\frac{4}{17}$ **b** $\frac{1}{17}$ **c** $\frac{13}{51}$ **d** $\frac{1}{51}$

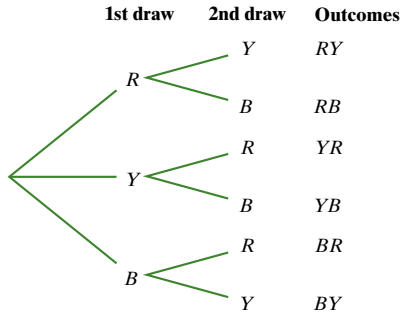
2 **a** $\frac{1}{3}$ **b** $\frac{5}{12}$ **c** $\frac{3}{11}$

3 **a**



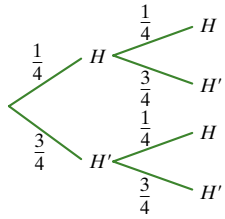
b 9 **c** $\frac{1}{3}$

4 **a**

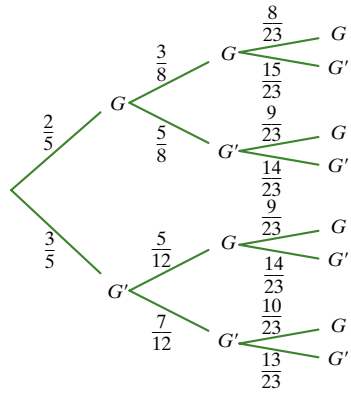


b 6

5 $\frac{3}{8}$



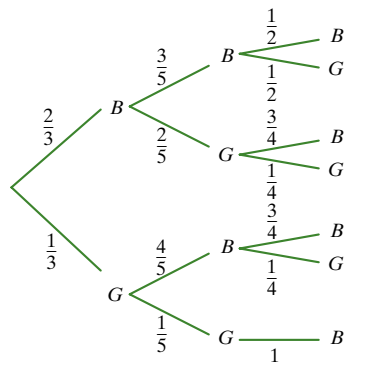
6



a $\frac{6}{115}$ **b** $\frac{21}{46}$ **c** $\frac{2}{5}$

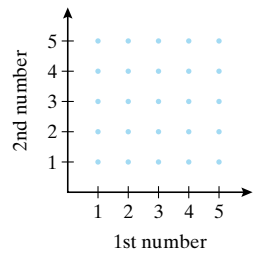
7 **a** $\frac{2}{7}$ **b** $\frac{2}{7}$

8 **a**



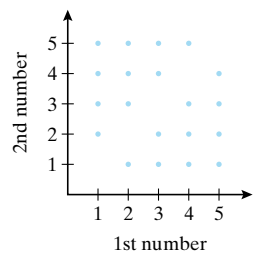
b **i** $\frac{1}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{3}{5}$ **iv** $\frac{1}{5}$

9 **a**



b $\frac{1}{5}$

c



d $\frac{2}{5}$

e $\frac{3}{5}$

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exercise 10.4

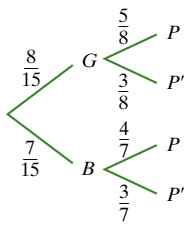
1 **a** The probability of event M occurring given that we know event P has occurred.
b The probability of event P occurring given that we know event M has occurred.

2 **a** $\Pr(H|R)$ **b** $\Pr(R|H)$
c $\Pr(T|P)$ **d** $\Pr(L|M)$

3 **a** 0.6 **b** 0.6

4 **a** 0.71 **b** 0.33 **c** 0.67

5 **a**

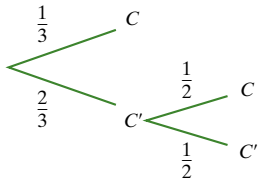


b **i** $\frac{1}{3}$ **ii** $\frac{2}{5}$
c $\frac{5}{8}$
d $\frac{4}{9}$

6 0.8 **7** 0.7 **8** 0.2

9 0.625 **10** 0.9375

11 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$ **c**



revision

1 C **2** E **3** D **4** D **5** C

6 **a** $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{2}$

7 **a** (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4),(4, 1), (4, 2), (4, 3), (4, 4)
b **i** $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{4}$

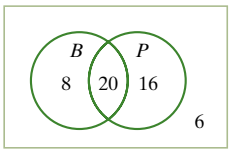
8 **a** Yes: the ticket can be either red or yellow but not both at once.
b **i** 0.82 **ii** 0.18

9 **a** 0.39 **b** 0.63

10 0.1008

11 **a** $\frac{3}{25}$ **b** $\frac{3}{5}$ **c** $\frac{43}{125}$ **d** $\frac{8}{125}$

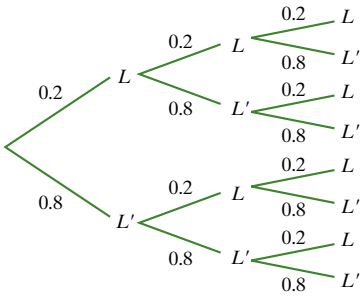
12 **a**



$n = 50$

b **i** $\frac{14}{25}$ **ii** $\frac{2}{5}$ **c** $\frac{5}{7}$

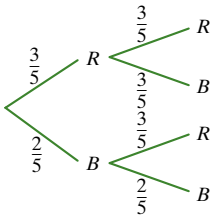
13 **a**



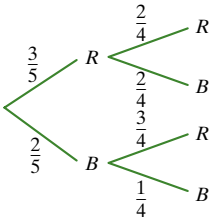
b **i** 0.512 **ii** 0.384 **iii** 0.128 **iv** 0.2

14 **a** **i** $\frac{11}{12}$ **ii** $\frac{1}{12}$ **iii** 0
b **i** 10 **ii** $\frac{1}{20}$

15 **a** **i** $\frac{3}{5}$ **ii** $\frac{2}{5}$
b No **c** Yes **d**



e $\frac{4}{25}$ **f** $\frac{12}{25}$ **g** No
h



i $\frac{1}{10}$ **j** $\frac{3}{5}$

Chapter 11

exercise 11.1

1 **a** **i** $y = -2x^2$

x	−3	−2	−1	0	1	2	3
y	−18	−8	−2	0	−2	−8	−18

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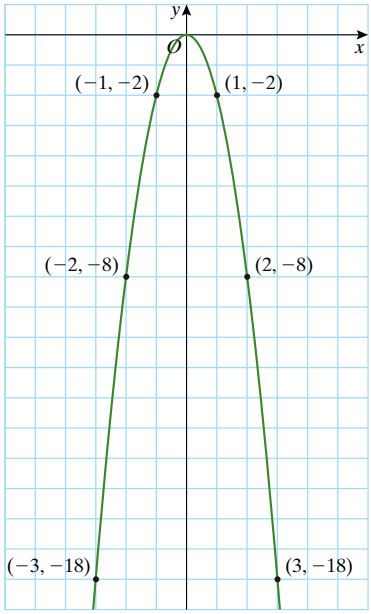
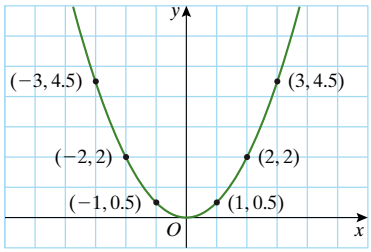
ii



b i $y = 0.5x^2$

x	−3	−2	−1	0	1	2	3
y	4.5	2	0.5	0	0.5	2	4.5

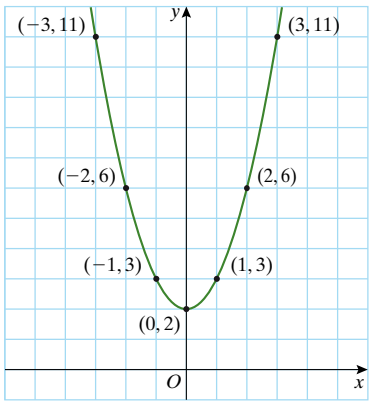
ii



c i $y = x^2 + 2$

x	−3	−2	−1	0	1	2	3
y	11	6	3	2	3	6	11

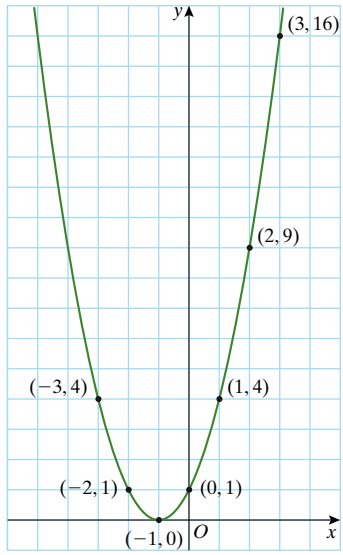
ii



d i $y = (x + 1)^2$

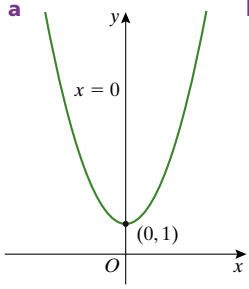
x	−3	−2	−1	0	1	2	3
y	4	1	0	1	4	9	16

ii

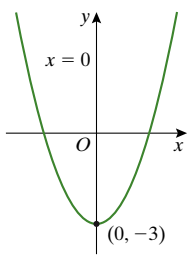


2 **a** upright **b** inverted **c** inverted **d** upright

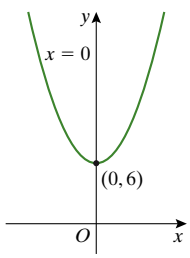
3 **a**



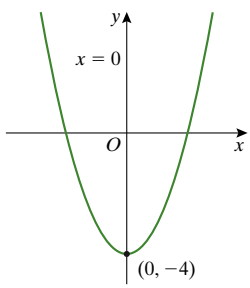
b



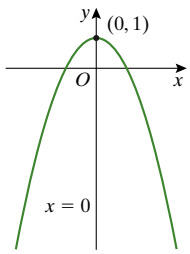
c



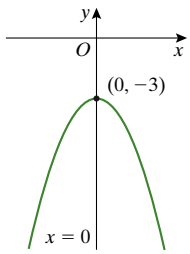
d



e



f



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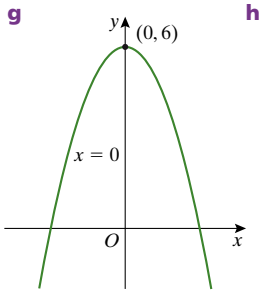
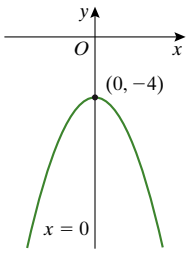
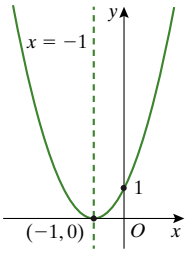
g



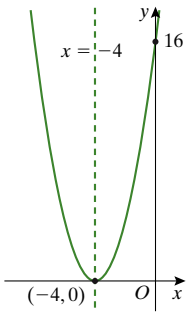
h



4 **a**



b



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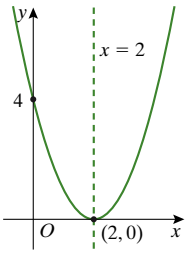

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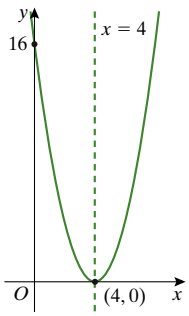

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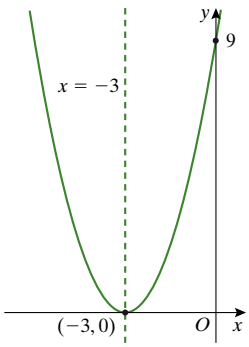
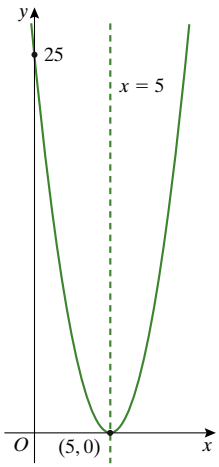


f



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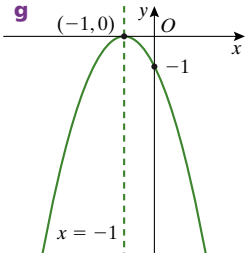

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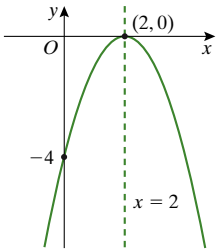
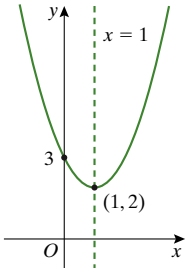


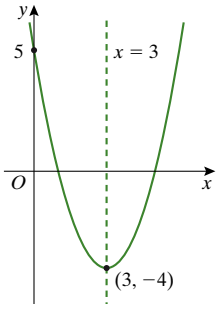
5 **a** **i** (0, 3) **ii** (1, 2) **iii** $x = 1$

iv



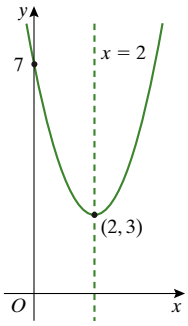
b **i** (0, 5) **ii** (3, −4) **iii** $x = 3$

iv



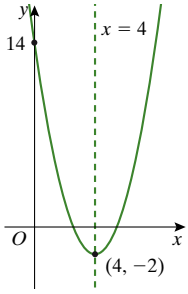
c **i** (0, 7) **ii** (2, 3) **iii** $x = 2$

iv



d **i** (0, 14) **ii** (4, −2) **iii** $x = 4$

iv



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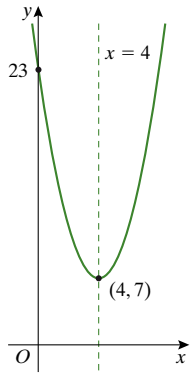
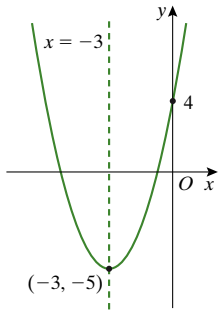
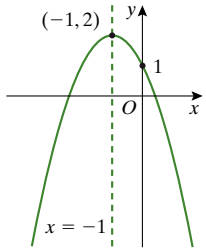
e **i** (0, 23) **ii** (4, 7)
iii $x = 4$
iv



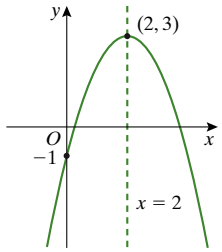
f **i** (0, 4) **ii** (−3, −5)
iii $x = -3$
iv



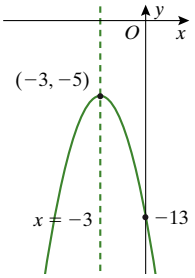
g **i** (0, 1) **ii** (−1, 2)
iii $x = -1$
iv



h **i** (0, −1) **ii** (2, 3)
iii $x = 2$
iv



i **i** (0, −13)
ii (−3, −5)
iii $x = 3$
iv



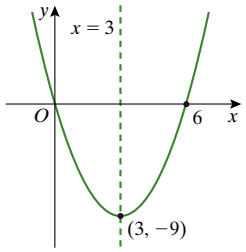
6 **a** $y = (x - 4)^2 - 3$
b $y = (x + 6)^2 + 2$
c $y = x^2 + 4$
d $y = (x + 3)^2$
e $y = (x - 2)^2$
f $y = x^2 - 7$

7 **a** $y = (x - 4)^2$
b $y = x^2 - 5$
c $y = (x + 2)^2 + 6$
d $y = (x - 8)^2 + 1$

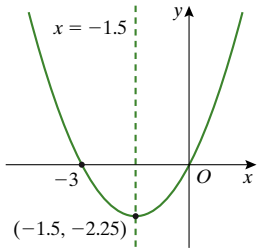
8 **a** $y = -(x - 2)^2 + 3$
b $y = -(x + 2)^2 + 3$

exercise 11.2

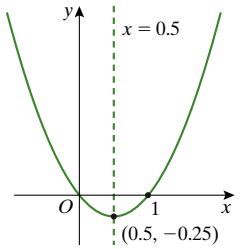
1 **a** **i** $x = 0, 6$
ii $x = 3$
iii $y = -9$
iv $x = 3$
v $y = 0$
vi



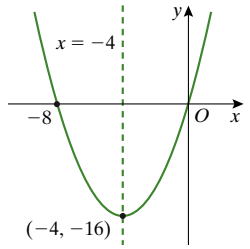
b **i** $x = -3, 0$
ii $x = -1.5$
iii $y = -2.25$
iv $x = -1.5$
v $y = 0$
vi



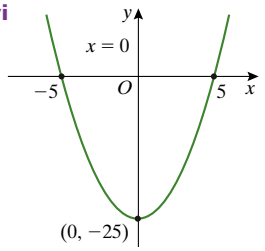
c **i** $x = 0, 1$ **ii** $x = 0.5$
iii $y = -0.25$ **iv** $x = 0.5$
v $y = 0$
vi



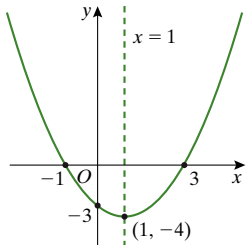
d **i** $x = 0, -8$ **ii** $x = -4$
iii $y = -16$ **iv** $x = -4$
v $y = 0$
vi



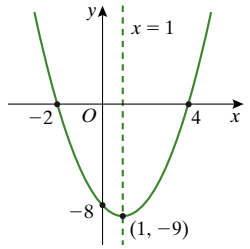
e **i** $x = -5, 5$ **ii** $x = 0$
iii $y = -25$ **iv** $x = 0$
v $y = -25$
vi



f **i** $x = -1, 3$ **ii** $x = 1$
iii $y = -4$ **iv** $x = 1$
v $y = -3$
vi



g **i** $x = -2, 4$ **ii** $x = 1$
iii $y = -9$ **iv** $x = 1$
v $y = -8$
vi



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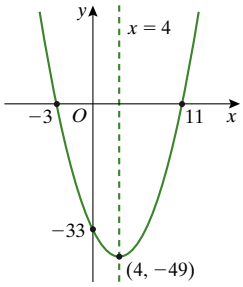
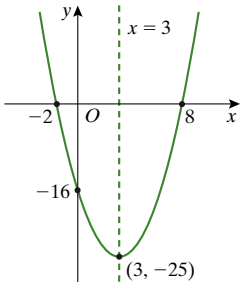
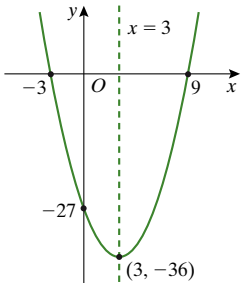
h **i** $x = -3, 11$
ii $x = 4$
iii $y = -49$
iv $x = 4$
v $y = -33$
vi



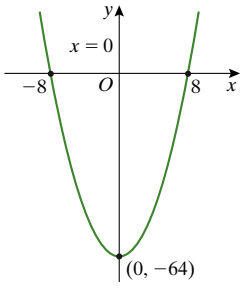
i **i** $x = -2, 8$ **ii** $x = 3$
iii $y = -25$ **iv** $x = 3$
v $y = -16$
vi



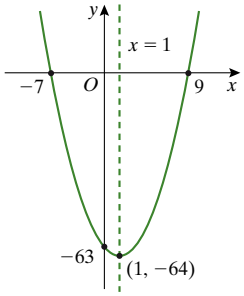
j **i** $x = -3, 9$ **ii** $x = 3$
iii $y = -36$ **iv** $x = 3$
v $y = -27$
vi



k **i** $x = -8, 8$ **ii** $x = 0$
iii $y = -64$ **iv** $x = 0$
v $y = -64$
vi



l **i** $x = -7, 9$ **ii** $x = 1$
iii $y = -64$ **iv** $x = 1$
v $y = -63$
vi



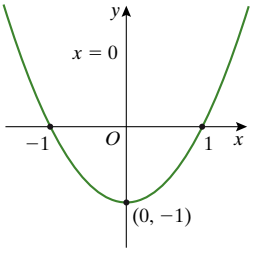
2 **a** $y = (x - 5)(x + 1)$
b $y = (x + 4)(x - 4)$
c $y = (x + 3)(x + 1)$
d $y = (x - 4)(x - 7)$
e $y = (x + 8)(x + 4)$
f $y = (x - 3)(x + 3)$
g $y = (x - 7)(x + 6)$
h $y = x(x + 2)$
i $y = x(x - 8)$
j $y = (x - 1.5)(x + 1)$
k $y = (x - 2.5)(x + 3.5)$
l $y = (x - \sqrt{3})(x + \sqrt{3})$

exercise 11.3

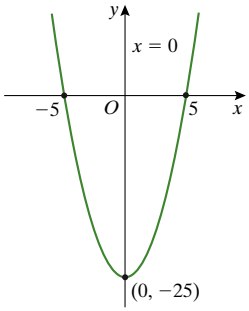
1 **a** Two intercepts: $x = -1, 1$
b Two intercepts: $x = -5, 5$
c Two intercepts: $x = -6, 6$
d One intercept: $x = -3$
e One intercept: $x = 4$
f One intercept: $x = -2$
g One intercept: $x = 2$
h Two intercepts: $x = -5, 3$
i Two intercepts: $x = -2, 8$
j Two intercepts: $x = -9, 3$
k Two intercepts: $x = -4, 14$
l Two intercepts: $x = -7, -1$

2 a, b, d, e have no x-intercepts

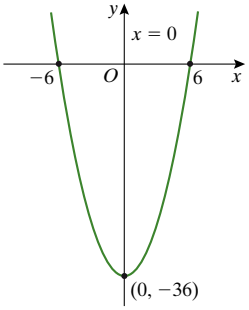
3 **a** **i** $(0, -1)$
ii $y = -1$
iii $x = 0$
iv



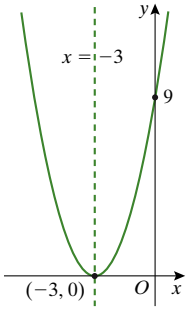
b **i** $(0, -25)$ **ii** $y = -25$
iii $x = 0$
iv



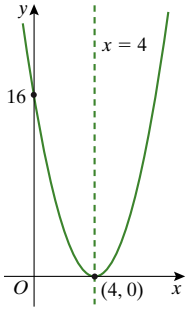
c **i** $(0, -36)$ **ii** $y = -36$
iii $x = 0$
iv



d **i** $(-3, 0)$ **ii** $y = 9$
iii $x = -3$
iv



e **i** $(4, 0)$ **ii** $y = 16$
iii $x = 4$
iv



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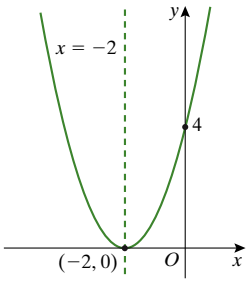
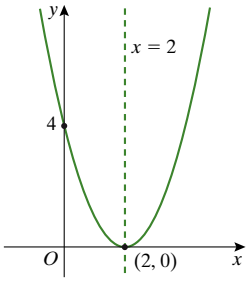
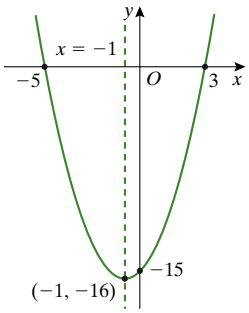
f **i** $(-2, 0)$ **ii** $y = 4$
iii $x = -2$
iv



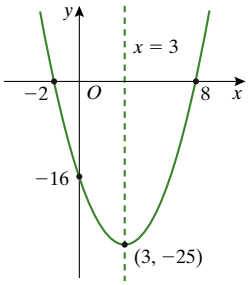
g **i** $(2, 0)$ **ii** $y = 4$
iii $x = 2$
iv



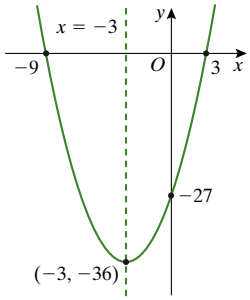
h **i** $(-1, -16)$ **ii** $y = -15$
iii $x = -1$
iv



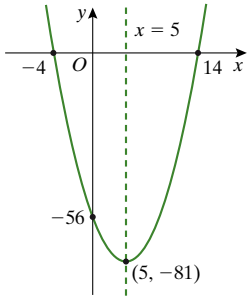
i **i** $(3, -25)$ **ii** $y = -16$
iii $x = 3$
iv



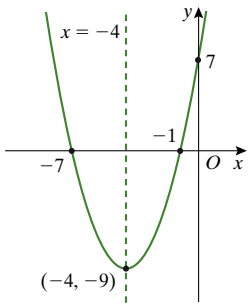
j **i** $(-3, -36)$ **ii** $y = -27$
iii $x = -3$
iv



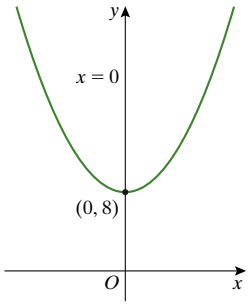
k **i** $(5, -81)$ **ii** $y = -56$
iii $x = 5$
iv



l **i** $(-4, -9)$ **ii** $y = 7$
iii $x = -4$
iv



4 **a**



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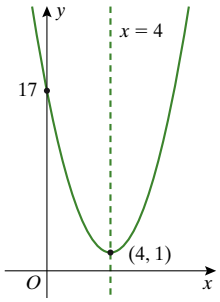

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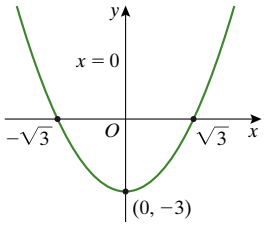

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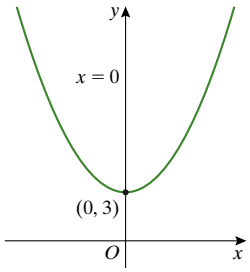

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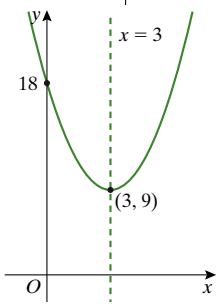

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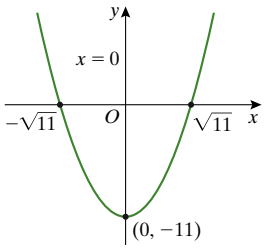
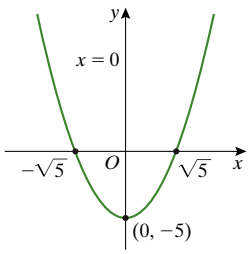


5 **a**



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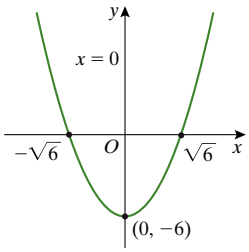

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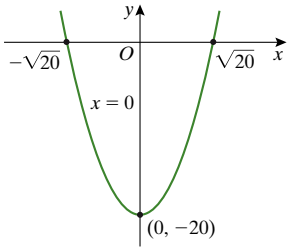

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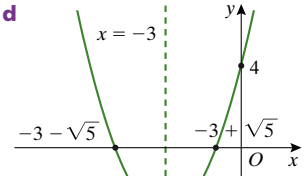

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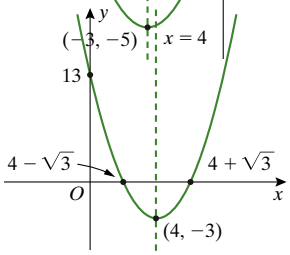

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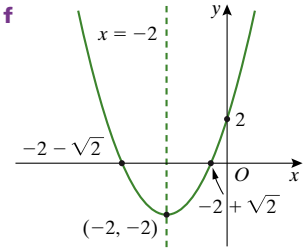

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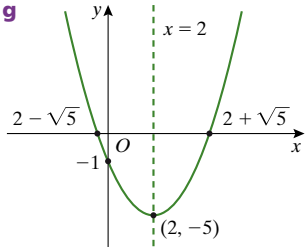

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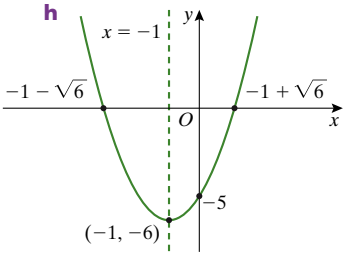

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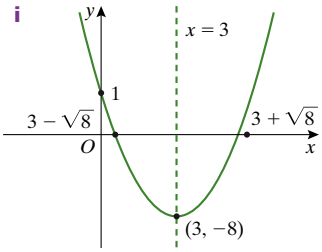

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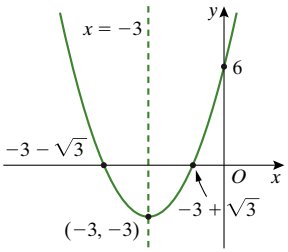

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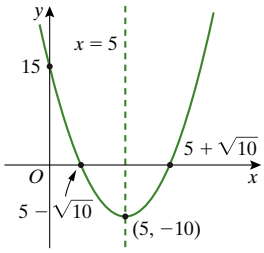

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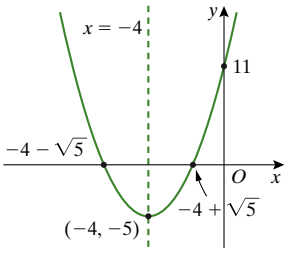
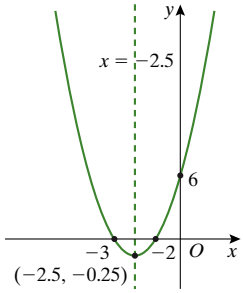


6 **a** C **b** A **c** D **d** B

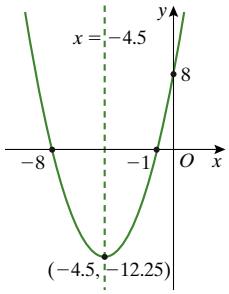
7 $y = (x - 2)^2 - 36$

exercise 11.4

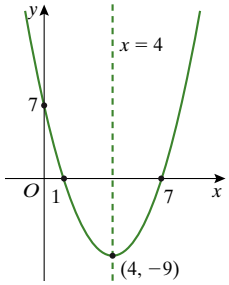
1 **a** **i** $y = 6$
ii $x = -3, -2$
iii $(-2.5, -0.25)$
iv $x = -2.5$
v



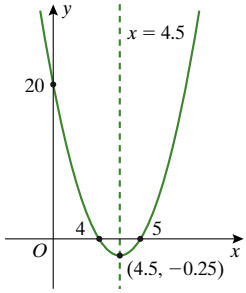
b **i** $y = 8$
ii $x = -8, -1$
iii $(-4.5, -12.25)$
iv $x = -4.5$
v



c **i** $y = 7$ **ii** $x = 1, 7$
iii $(4, -9)$ **iv** $x = 4$
v



d **i** $y = 20$ **ii** $x = 4, 5$
iii $(4.5, -0.25)$ **iv** $x = 4.5$
v



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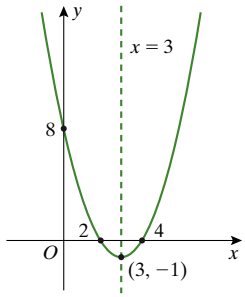
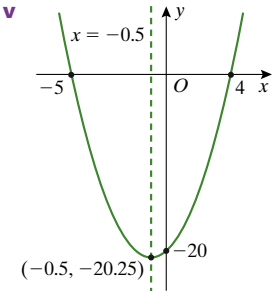
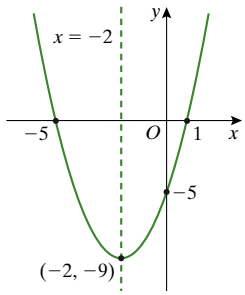
e **i** $y = 8$ **ii** $x = 2, 4$
iii $(3, -1)$ **iv** $x = 3$
v



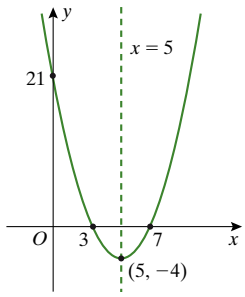
f **i** $y = -20$
ii $x = -5, 4$
iii $(-0.5, -20.25)$
iv $x = -0.5$
v



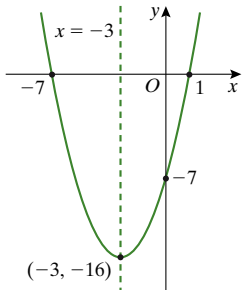
g **i** $y = -5$
ii $x = -5, 1$
iii $(-2, -9)$
iv $x = -2$
v



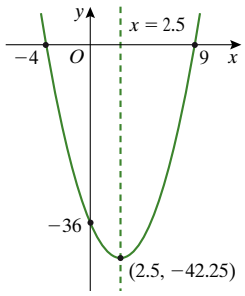
h **i** $y = 21$ **ii** $x = 3, 7$
iii $(5, -4)$ **iv** $x = 5$
v



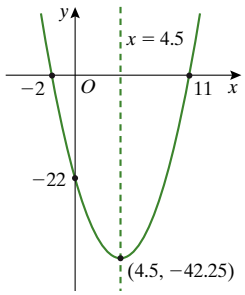
i **i** $y = -7$
ii $x = -7, 1$
iii $(-3, -16)$ **iv** $x = -3$
v



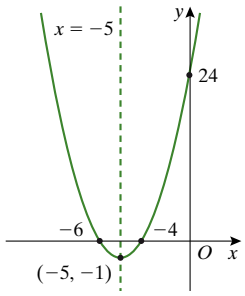
j **i** $y = -36$
ii $x = -4, 9$
iii $(2.5, -42.25)$
iv $x = 2.5$
v



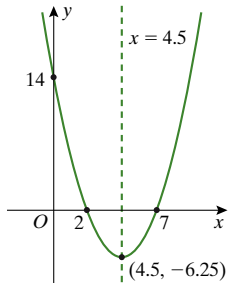
k **i** $y = -22$
ii $x = -2, 11$
iii $(4.5, -42.25)$
iv $x = 4.5$
v



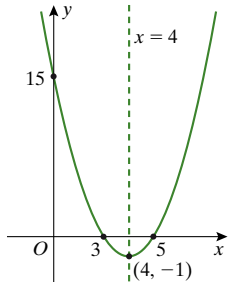
l **i** $y = 24$
ii $x = -6, -4$
iii $(-5, -1)$ **iv** $x = -5$
v



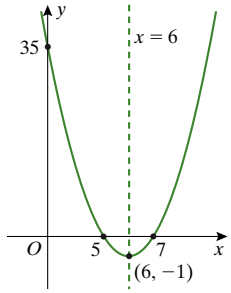
m **i** $y = 14$ **ii** $x = 2, 7$
iii $(4.5, -6.25)$ **iv** $x = 4.5$
v



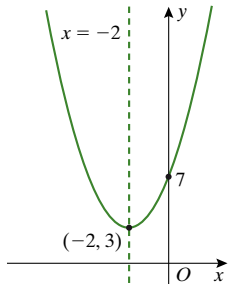
n **i** $y = 15$ **ii** $x = 3, 5$
iii $(4, -1)$ **iv** $x = 4$
v



o **i** $y = 35$ **ii** $x = 5, 7$
iii $(6, -1)$ **iv** $x = 6$
v



2 **a** **i** $y = 7$ **ii** $(-2, 3)$
iii no x-intercept
iv no x-intercept
v $x = -2$
vi



b **i** $y = -5$ **ii** $(-1, 6)$
iii $x = -1 \pm \sqrt{6}$
iv $x \approx -3.4, 1.4$
v $x = -1$

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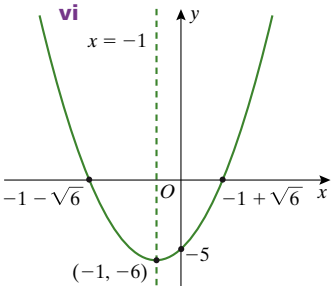
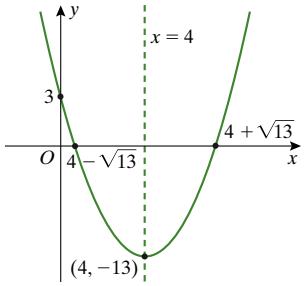
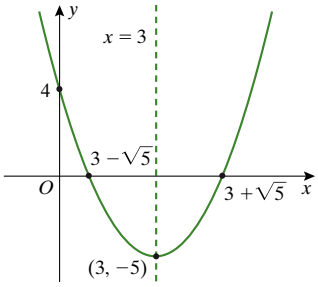
vi



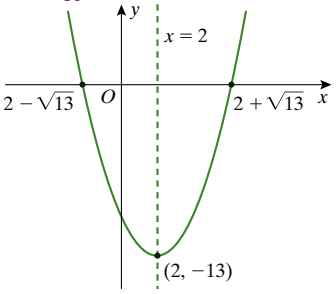
c **i** $y = 3$ **ii** $(4, -13)$
iii $x = 4 \pm \sqrt{13}$
iv $x \approx 0.4, 7.6$
v $x = 4$
vi



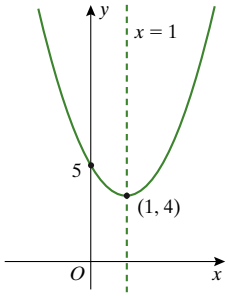
d **i** $y = 4$ **ii** $(3, -5)$
iii $x = 3 \pm \sqrt{5}$
iv $x \approx 0.8, 5.2$
v $x = 3$
vi



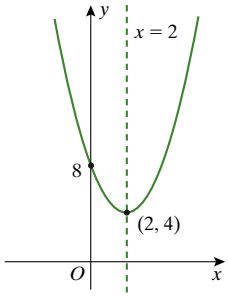
e **i** $y = -9$ **ii** $(2, -13)$
iii $x = 2 \pm \sqrt{13}$
iv $x \approx -1.6, 5.6$
v $x = 2$
vi



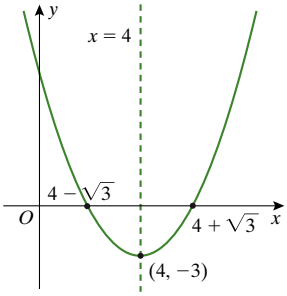
f **i** $y = 5$
ii $(1, 4)$
iii no x-intercept
iv no x-intercept
v $x = 1$
vi



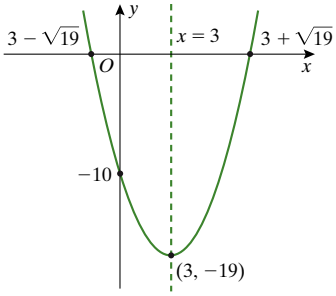
g **i** $y = 8$
ii $(2, 4)$
iii no x-intercept
iv no x-intercept
v $x = 2$
vi



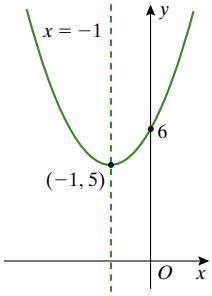
h **i** $y = 13$
ii $(4, -3)$
iii $x = 4 \pm \sqrt{3}$
iv $x \approx 2.3, 5.7$
v $x = 4$
vi



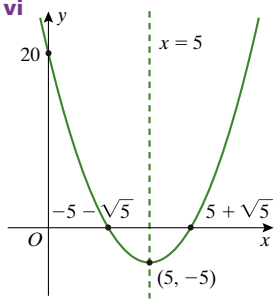
i **i** $y = -10$ **ii** $(3, -19)$
iii $x = 3 \pm \sqrt{19}$
iv $x \approx -1.4, 7.4$
v $x = 3$
vi



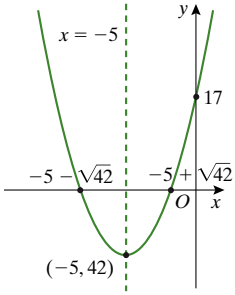
j **i** $y = 6$ **ii** $(-1, 5)$
iii no x-intercept
iv no x-intercept
v $x = -1$
vi



k **i** $y = 20$ **ii** $(5, -5)$
iii $x = 5 \pm \sqrt{5}$
iv $x \approx 2.8, 7.2$
v $x = 5$
vi



l **i** $y = 17$
ii $(-5, 42)$
iii $x = -5 = \sqrt{42}$
iv 2.8 and 7.2
v $x = -5$
vi



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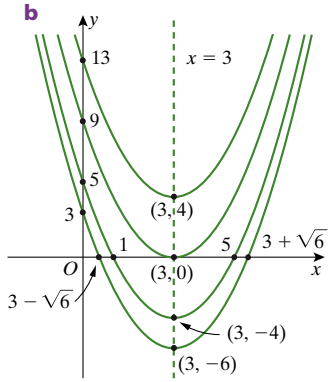
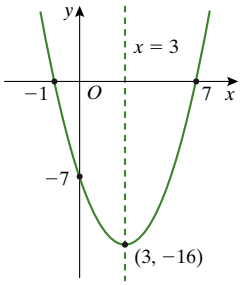
3 **a**

Quadratic function	Number of x-intercepts	x-intercepts	Turning point
$y = x^2 - 6x + 13$	zero	-	$(3, 4)$
$y = x^2 - 6x + 9$	one	$x = 3$	$(3, 0)$
$y = x^2 - 6x + 5$	two	$x = 1, 5$	$(3, -4)$
$y = x^2 - 6x + 3$	two	$3 \pm \sqrt{6}$	$(3, -6)$

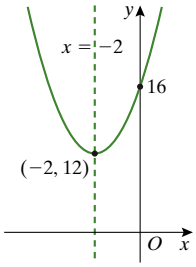
b



4 **a** $y = (x - 3)^2 - 16$
b $y = -7$
c $(3, -16)$
d



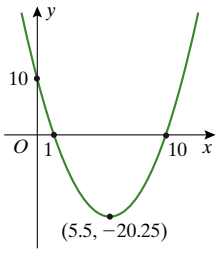
5 **a** $y = (x + 2)^2 + 12$
b no x-intercepts
c $(-2, 12)$
d



exercise 11.5

1 **a** $x = 1, 10$ **b** $x = -1, 1.5$
c $x = \pm 3$ **d** $x = -4, 3$
e $x = -5, 1$ **f** $x = 1, 2$

2 **a**



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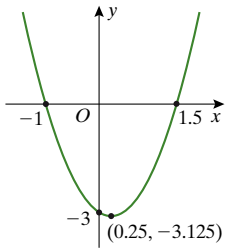
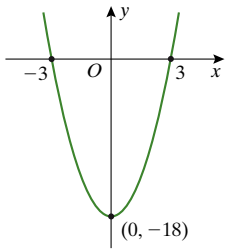
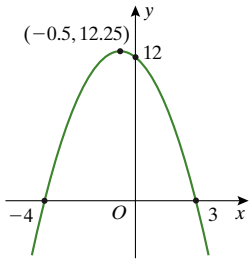


c



d



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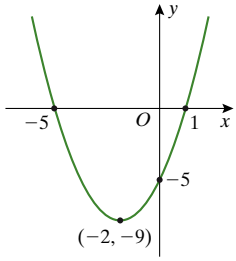

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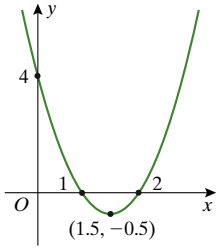
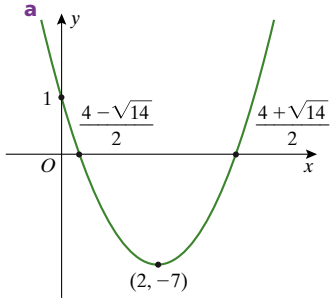
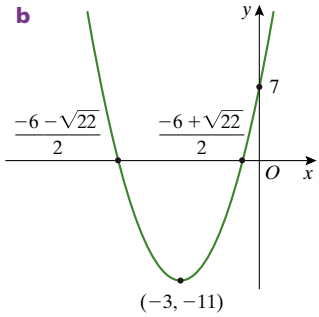


3 **a**



b



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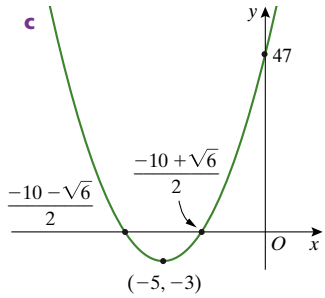

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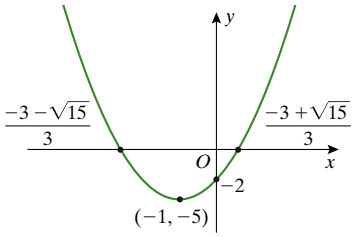
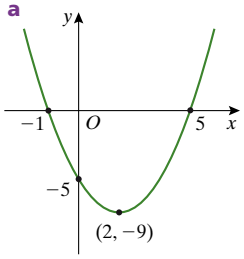
d

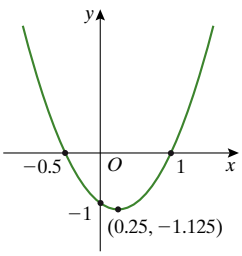


4 **a** two intercepts **b** two intercepts
c two intercepts **d** no intercepts
e two intercepts **f** one intercept

5 **a**



b



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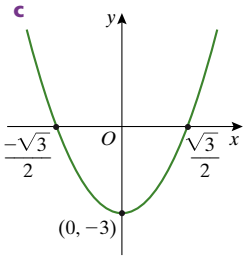
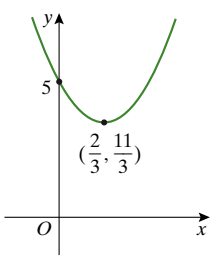
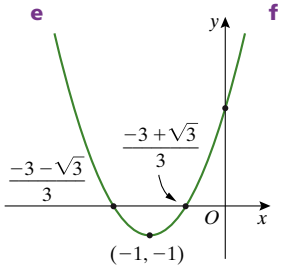


d



e



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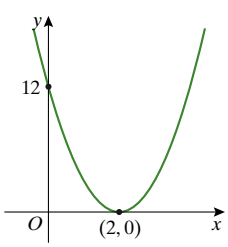

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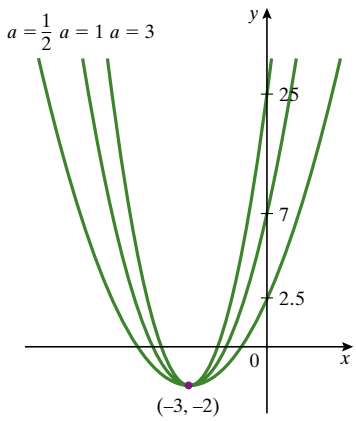
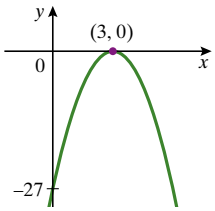


All the graphs have the same turning point at $(-3, 2)$.

7 Answers will vary, but each graph must have the same turning point at $(3, -2)$.

8 **a** $y = a(x - 3)^2$ **b** $y = -3(x - 3)^2$

c



9 $m < \frac{21}{4}$

10 $n < -\frac{25}{8}$

11 $k = -3, 5$

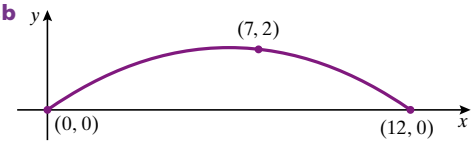
12 $\Delta = (k + 2)^2 + 24 \geq 24$, i.e. $\Delta > 0$, for all values of k, so there must be two intercepts. Hence it has x-intercepts for all values of k.

13 $k = 1$ only

14 $\Delta = 16$, so the parabola cuts the x-axis twice for any value of k except $k = -3$, where there would be no x^2 term. Using the quadratic formula or otherwise gives $x = 1$ and
$x = \frac{k - 1}{k + 3}$ for the x-intercepts, where $k \neq -3$.

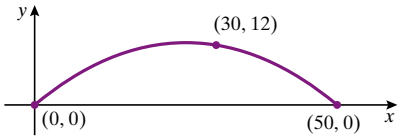
exercise 11.6

1 **a,b**



c $y = -\frac{2}{35}x(x - 12)$

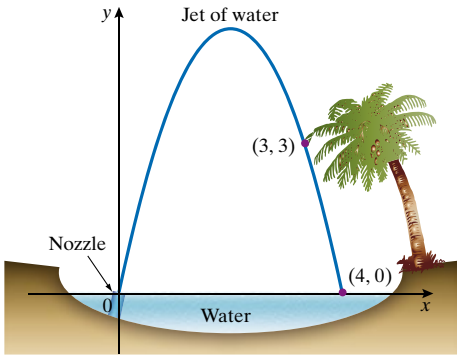
2 **a**



b Points include the x-intercepts

c $y = -\frac{1}{50}x(x - 50)$

3 **a**



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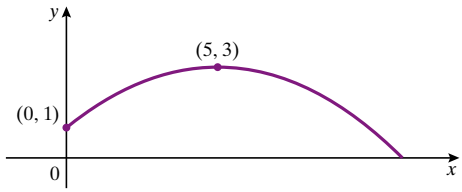
b $y = -x(x - 4)$ **c** 4 metres

4 $y = -\frac{3}{800}x(x - 8)$

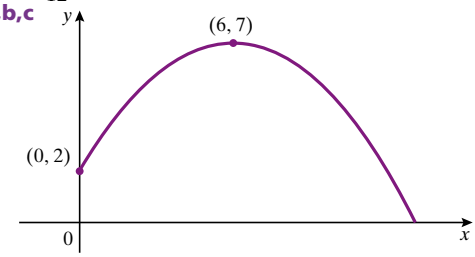
5 **a**



b $y = -\frac{2}{25}(x - 5)^2 + 3$

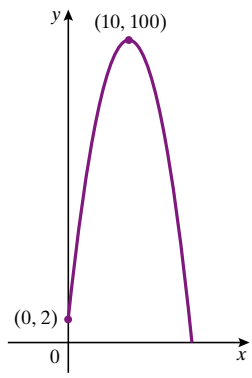
6 $y = -\frac{1}{12}x(x - 8)$

7 **a,b,c**



d $y = -\frac{5}{36}(x - 6)^2 + 7$ **e** 13.1 metres

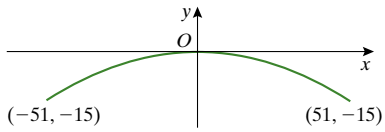
8 **a,b**



c $y = -\frac{49}{50}(x - 10)^2 + 100$ **d** 20.1 metres

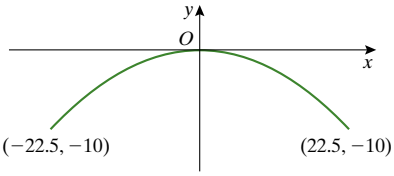
9 **a** $y = -\frac{5}{867}x^2$ $(y = -0.00577x^2)$

b



10 **a** $y = -\frac{8}{405}x^2$ $(y = -0.0198x^2)$

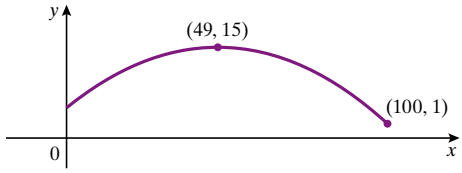
b



11 **a** $h = -\frac{1}{120}x(x - 160)$ **b** 80 metres

c $53\frac{1}{3}$ metres

12 **a,b,c**



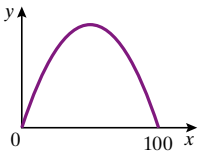
d $y = -0.00548(x - 49)^2 + 15$

e 2.1 metres

13 **a** $y = -\frac{1}{50}x^2$ **b** -128

14 **a** $(100 - x)$ m **b** $y = x(100 - x)$

c

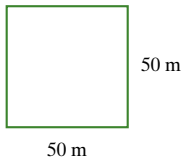


d $x = 50$

e $y = 2500$

f 2500 m^2

g



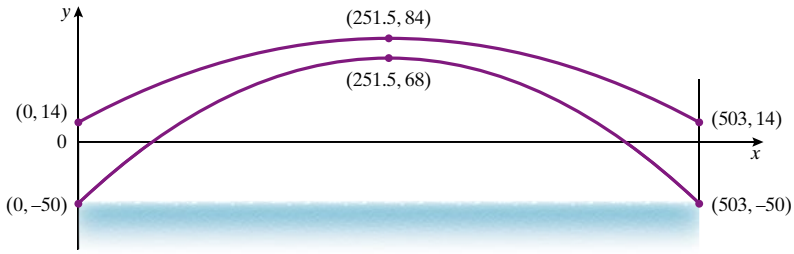
15 **a, d** see below

b $y = -0.0019(x - 251.5)^2 + 68$

c $y = -0.0011(x - 251.5)^2 + 84$

e 109 metres

15 **a**



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revision

1 D **2** E **3** E **4** B **5** A

6 **a** -15 **b** $x = -5, 3$ **c** $(-1, -16)$

7 **a** $y = (x + 2)^2$ **b** $y = (x + 1)^2$ **c** $y = x^2$
d $y = (x - 1)^2$ **e** $y = (x - 2)^2$

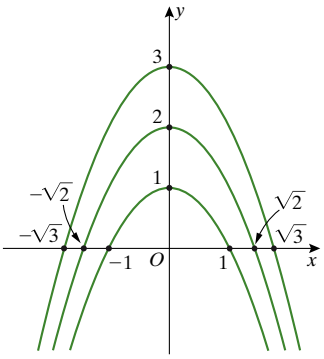
8 **a** B **b** C **c** A **d** D

9 Many answers possible, but equations must all take the form $y = (x - 1)^2 + k$ for three values of k.

10 **a** $b = 1$ **b** $a = 4$ **c** $x = \pm 0.4$

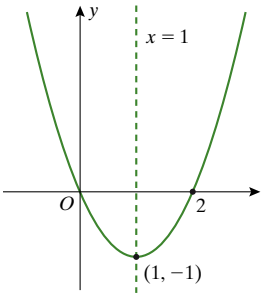
11 **a** $y = x^2 - 3$
b No x-intercepts are shown, so any rule of the form $y = ax^2 - 3$, where $a > 0$ is possible.

12

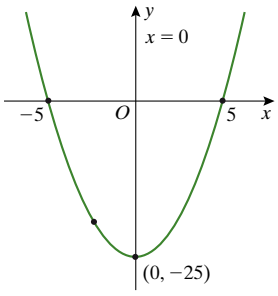


13 $y = -\dfrac{1}{100}x^2$

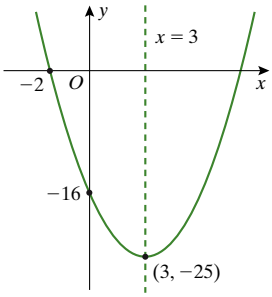
14 **a**



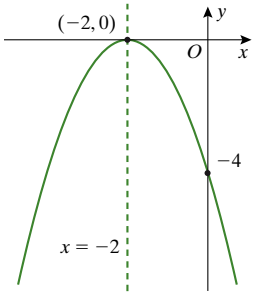
b



c



d



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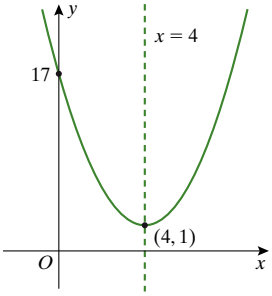

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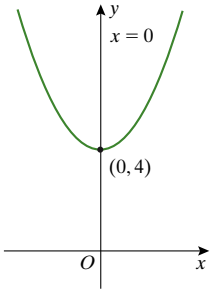

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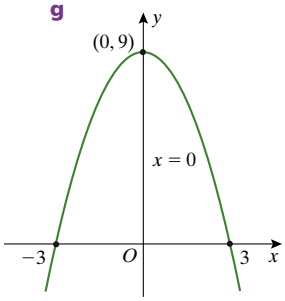

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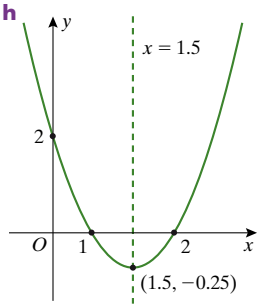
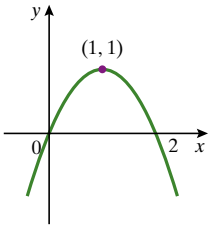


15 **a**



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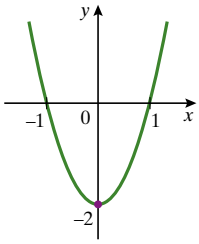

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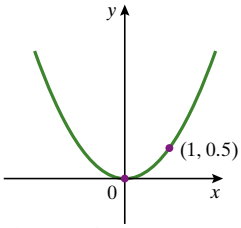
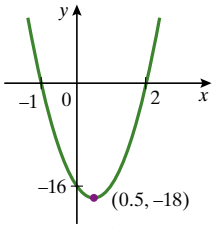
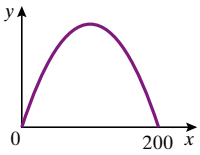


d



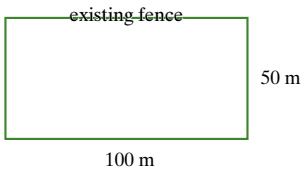
16 **a** $\left(100 - \dfrac{x}{2}\right)$ m **b** $y = x\left(100 - \dfrac{x}{2}\right)$

c



d $x = 100$
e $y = 5000$
f $5000\,\text{m}^2$

g



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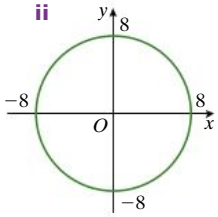
Chapter 12

exercise 12.1

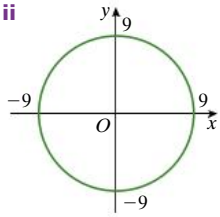
1 **a** $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 49$
c $x^2 + y^2 = 36$ **d** $x^2 + y^2 = 100$
e $x^2 + y^2 = 12$ **f** $x^2 + y^2 = 50$

2 **a** $(2, 5)$: $2^2 + 5^2 = 4 + 25 = 29$;
$(5, 2)$: $5^2 + 2^2 = 25 + 4 = 29$
b $(-1, 4)$: $(-1)^2 + 4^2 = 1 + 16 = 17$;
$(-4, -1)$: $(-4)^2 + (-1)^2 = 16 + 1 = 17$
c LS $= 4^2 + (-1)^2 = 17$
RS $= 15$
So LS $\neq$ RS

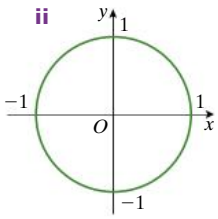
3 **a** **i** $r = 8$
ii



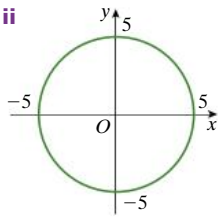
b **i** $r = 9$
ii



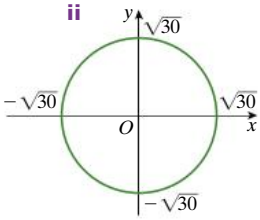
c **i** $r = 1$
ii



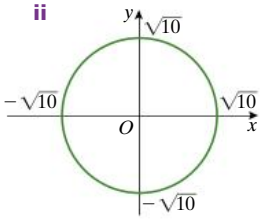
d **i** $r = 5$
ii



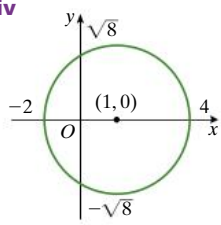
e **i** $r = \sqrt{30}$
ii



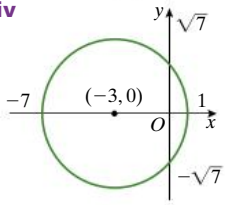
f **i** $r = \sqrt{10}$
ii



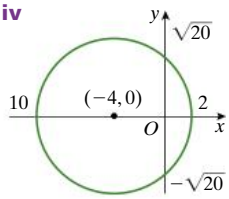
4 **a** **i** $(x - 1)^2 + y^2 = 9$
ii $x = -2, 4$
iii $y = \pm\sqrt{8}$
iv



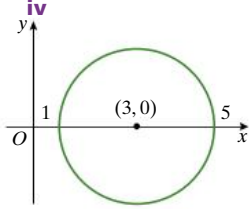
b **i** $(x+3)^2 + y^2 = 16$
ii $x = -7, 1$
iii $y = \pm\sqrt{7}$
iv



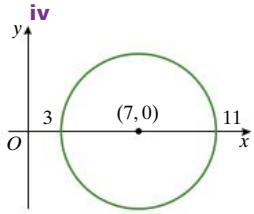
c **i** $(x+ 4)^2 + y^2 = 36$ **ii** $x = -10, 2$
iii $y = \pm\sqrt{20}$ **iv**



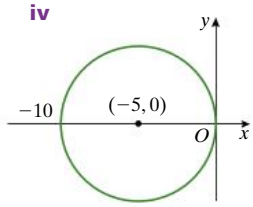
d **i** $(x - 3)^2 + y^2 = 4$ **ii** $x = 1, 5$
iii no y-intercept **iv**



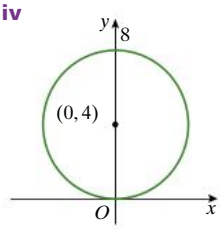
e **i** $(x - 7)^2 + y^2 = 16$ **ii** $x = 3, 11$
iii no y-intercept **iv**



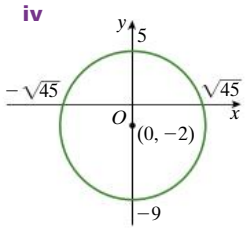
f **i** $(x+ 5)^2 + y^2 = 25$ **ii** $x = -10, 0$
iii $y = 0$ **iv**



5 **a** **i** $x^2 + (y - 4)^2 = 16$ **ii** $x = 0$
iii $y = 0, 8$ **iv**



b **i** $x^2 + (y+ 2)^2 = 49$ **ii** $x = \pm\sqrt{45}$
iii $y = -9, 5$ **iv**



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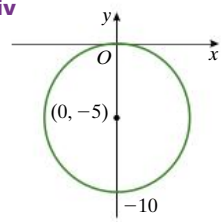
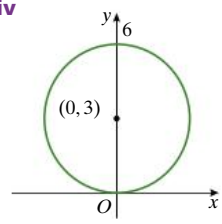
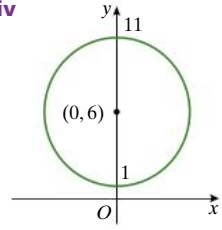
c **i** $x^2 + (y+5)^2 = 25$ **ii** $x = 0$
iii $y = -10, 0$ **iv**



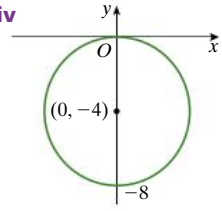
d **i** $x^2 + (y-3)^2 = 9$ **ii** $x = 0$
iii $y = 0, 6$ **iv**



e **i** $x^2 + (y-6)^2 = 25$ **ii** no x-intercept
iii $y = 1, 11$ **iv**



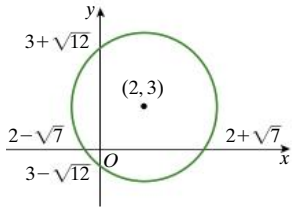
f **i** $x^2 + (y+4)^2 = 16$ **ii** $x = 0$
iii $y = -8, 0$ **iv**



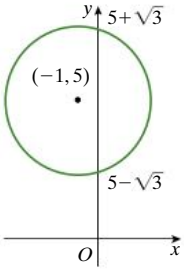
6 D

7 **a** **i** $(2, 0)$ **ii** $r = 3$ **iii** $(x-2)^2 + y^2 = 9$
b **i** $(3, 0)$ **ii** $r = 6$ **iii** $(x-3)^2 + y^2 = 36$
c **i** $(-3, 0)$ **ii** $r = 5$ **iii** $(x+3)^2 + y^2 = 25$
d **i** $(7, 0)$ **ii** $r = 7$ **iii** $(x-7)^2 + y^2 = 49$
e **i** $(0, 2)$ **ii** $r = 4$ **iii** $x^2 + (y-2)^2 = 16$
f **i** $(0, 5)$ **ii** $r = 5$ **iii** $x^2 + (y-5)^2 = 25$
g **i** $(0, 3)$ **ii** $r = 1$ **iii** $x^2 + (y-3)^2 = 1$
h **i** $(0, -1)$ **ii** $r = 2$ **iii** $x^2 + (y+1)^2 = 4$

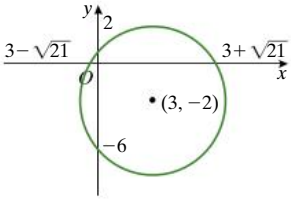
8 **a** **i** $(x-2)^2 + (y-3)^2 = 16$
ii $x = 2 \pm \sqrt{7}$; $y = 3 \pm \sqrt{12}$
iii



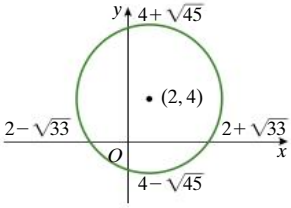
b **i** $(x+1)^2 + (y-5)^2 = 4$
ii no x-intercept; $y = 5 \pm \sqrt{3}$
iii



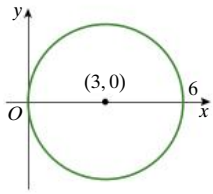
c **i** $(x-3)^2 + (y+2)^2 = 25$
ii $x = 3 \pm \sqrt{21}$; $y = -6, 2$
iii



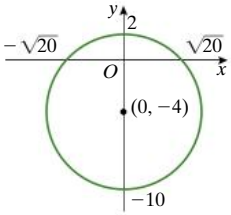
d **i** $(x-2)^2 + (y-4)^2 = 49$
ii $x = 2 \pm \sqrt{33}$; $y = 4 \pm \sqrt{45}$
iii



9 **a** **i** centre $(3, 0)$, $r = 3$
ii $x = 0, 6$; $y = 0$
iii



b **i** centre $(0, -4)$, $r = 6$
ii $x = \pm\sqrt{20}$; $y = -10, 2$
iii



c **i** centre $(0, -1)$, $r = 5$
ii $x = \pm\sqrt{24}$; $y = -6, 4$

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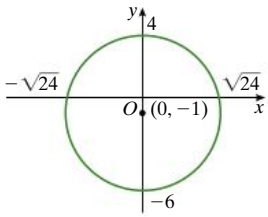
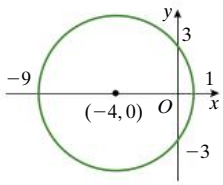
iii



d **i** centre $(-4, 0)$, $r = 5$ **ii** $x = -9, 1$; $y = \pm 3$

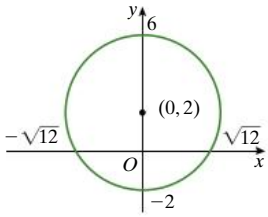
iii



e **i** centre $(0, 2)$, $r = 4$

ii $x = \pm\sqrt{12}$; $y = -2, 6$

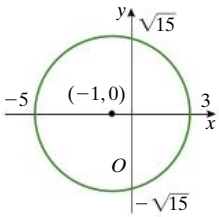
iii



f **i** centre $(-1, 0)$, $r = 4$

ii $x = -5, 3$; $y = \pm\sqrt{15}$

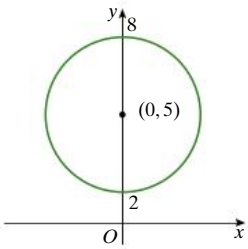
iii



g **i** centre $(0, 5)$, $r = 3$

ii no x-intercept; $y = 2, 8$

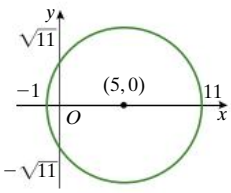
iii



h **i** centre $(5, 0)$, $r = 6$

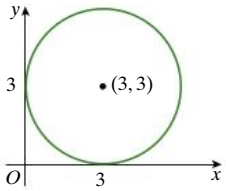
ii $x = -1, 11$; $y = \pm\sqrt{11}$

iii



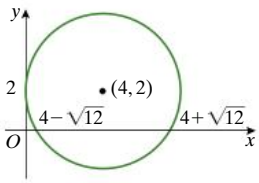
i **i** centre $(3, 3)$, $r = 3$ **ii** $x = 3$; $y = 3$

iii



j **i** centre $(4, 2)$, $r = 4$ **ii** $x = 4 \pm \sqrt{12}$; $y = 2$

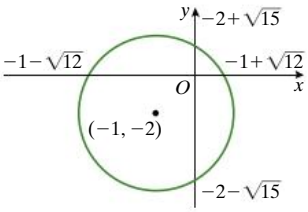
iii



k **i** centre $(-1, -2)$, $r = 4$

ii $x = -1 \pm \sqrt{12}$; $y = -2 \pm \sqrt{15}$

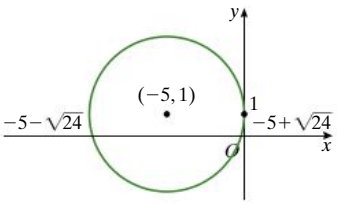
iii



l **i** centre $(-5, 1)$, $r = 5$

ii $x = -5 \pm \sqrt{24}$; $y = 1$

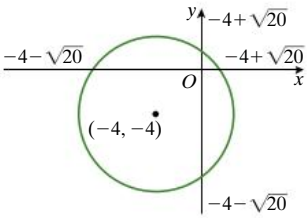
iii



m **i** centre $(-4, -4)$, $r = 6$

ii $x = -4 \pm \sqrt{20}$; $y = -4 \pm \sqrt{20}$

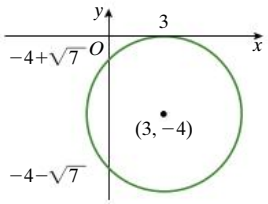
iii



n **i** centre $(3, -4)$, $r = 4$

ii $x = 3$; $y = -4 \pm \sqrt{7}$

iii



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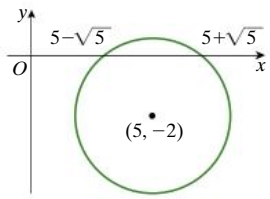
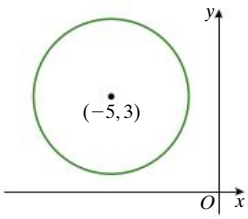
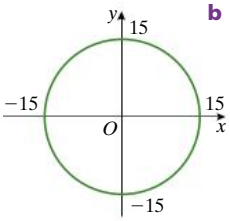
o **i** centre $(5, -2)$, $r = 3$
ii $x = 5 \pm \sqrt{5}$; no y-intercept
iii



p **i** centre $(-5, 3)$, $r = 2$
ii no x-intercept;no y-intercept
iii



10 **a**



b $x^2 + y^2 = 225$

c 11 metres and 29 metres

11 E

12 **a** **i** $(3, 3)$ **ii** $r = 3$
iii $(x - 3)^2 + (y - 3)^2 = 9$
b **i** $(4, -4)$ **ii** $r = 4$
iii $(x - 4)^2 + (y + 4)^2 = 16$
c **i** $(2, -2)$ **ii** $r = 4$
iii $(x - 2)^2 + (y + 2)^2 = 16$
d **i** $(-1, 2)$ **ii** $r = 2$
iii $(x + 1)^2 + (y - 2)^2 = 4$
e **i** $(1, -2)$ **ii** $r = 3$
iii $(x - 1)^2 + (y + 2)^2 = 9$
f **i** $(-2, -5)$ **ii** $r = 5$
iii $(x + 2)^2 + (y + 5)^2 = 25$
g **i** $(4, -2)$ **ii** $r = 4$
iii $(x - 4)^2 + (y + 2)^2 = 16$
h **i** $(-1, -3)$ **ii** $r = 3$
iii $(x + 1)^2 + (y + 3)^2 = 9$

13 $y = 0$: $(x + 4)^2 + 36 = 9$, i.e. $(x + 4)^2 + 27 = 0$; no solution so no x-intercepts.
$x = 0$: $16 + (y - 6)^2 = 9$, i.e. $7 + (y - 6)^2 = 0$; no solution so no y-intercepts.

14 **a** **i** $(x + 3)^2 + (y + 1)^2 = 16$
ii centre $(-3, -1)$, $r = 4$
b **i** $(x - 4)^2 + (y + 2)^2 = 49$
ii centre $(4, -2)$, $r = 7$
c **i** $(x + 1)^2 + (y + 7)^2 = 64$
ii centre $(-1, 7)$, $r = 8$
d **i** $(x - 2)^2 + (y + 3)^2 = 25$
ii centre $(2, -3)$, $r = 5$
e **i** $(x + 5)^2 + (y - 6)^2 = 64$
ii centre $(-5, 6)$, $r = 8$
f **i** $(x - 4)^2 + (y - 1)^2 = 36$
ii centre $(4, 1)$, $r = 6$

15 **a** **i** $(x + 3)^2 + (y - 3)^2 = 9$
ii $(x - 3)^2 + (y + 3)^2 = 9$
b **i** $(x + 4)^2 + (y + 4)^2 = 16$
ii $(x - 4)^2 + (y - 4)^2 = 16$
c **i** $(x + 2)^2 + (y + 2)^2 = 16$
ii $(x - 2)^2 + (y - 2)^2 = 16$
d **i** $(x - 1)^2 + (y - 2)^2 = 4$
ii $(x + 1)^2 + (y + 2)^2 = 4$
e **i** $(x + 1)^2 + (y + 2)^2 = 9$
ii $(x - 1)^2 + (y - 2)^2 = 9$
f **i** $(x - 2)^2 + (y + 5)^2 = 25$
ii $(x + 2)^2 + (y - 5)^2 = 25$
g **i** $(x + 4)^2 + (y + 2)^2 = 16$
ii $(x - 4)^2 + (y - 2)^2 = 16$
h **i** $(x - 1)^2 + (y + 3)^2 = 9$
ii $(x + 1)^2 + (y - 3)^2 = 9$

16 **a** $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **b** $\frac{x^2}{9} + \frac{y^2}{49} = 1$

17 Outline of proof: when $y = 0$, $x = -4, 8$ so A and B have coordinates $(-4, 0)$ and $(8, 0)$. Now show that the product of the gradients is:

$$m_{AP} \times m_{BP} = \frac{b^2}{(a - 2)^2 - 36}.$$

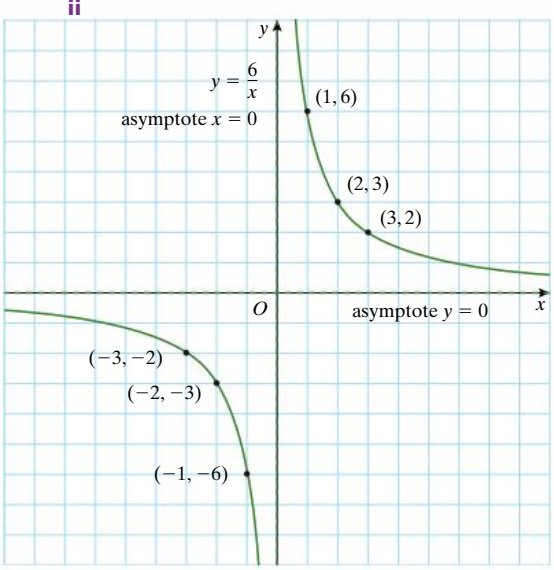
Next use the fact that $P(a, b)$ lies on the circle to show that $(a - 2)^2 - 36 = -b^2$ and hence that $m_{AP} \times m_{BP} = -1$, so that $\angle APB$ is a right angle.

exercise 12.2

1 **a** **i** $y = \frac{6}{x}$

x	−3	−2	−1	0	1	2	3
y	−2	−3	−6	-	6	3	2

ii



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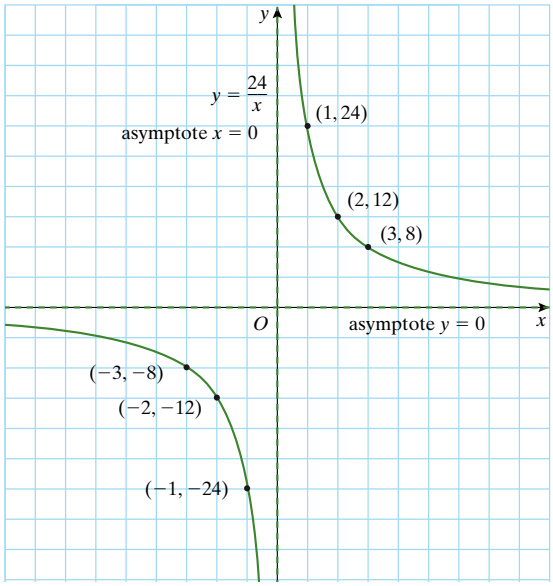
b i $y = \frac{24}{x}$

x	−3	−2	−1	0	1	2	3
y	−8	−12	−24	-	24	12	8

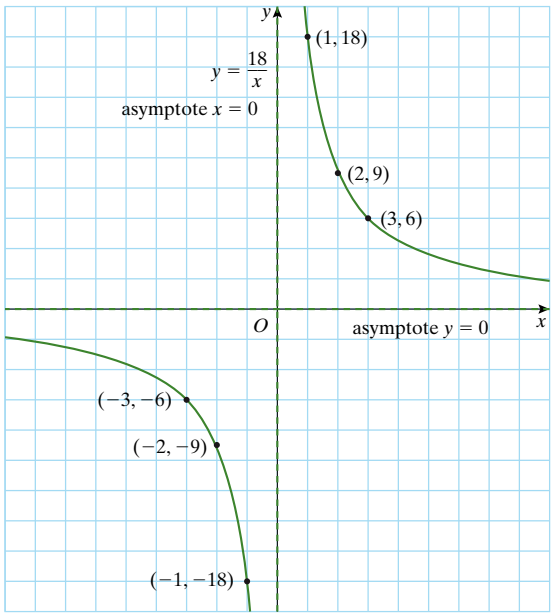
ii



c i $y = \frac{18}{x}$

x	−3	−2	−1	0	1	2	3
y	−6	−9	−18	-	18	9	6

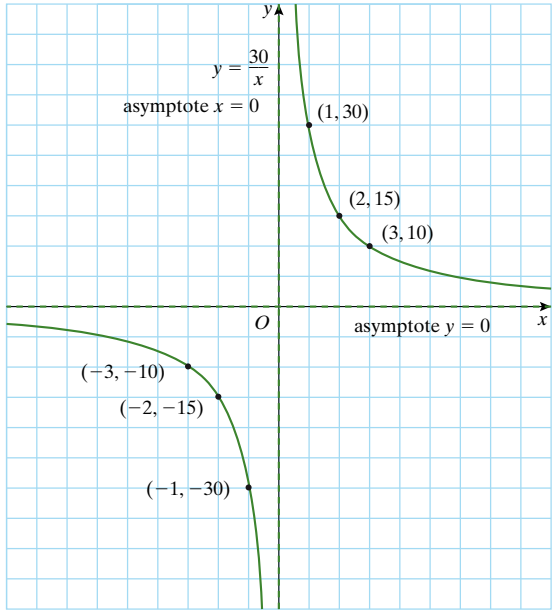
ii



d i $y = \frac{30}{x}$

x	−3	−2	−1	0	1	2	3
y	−10	−15	−30	-	30	15	10

ii



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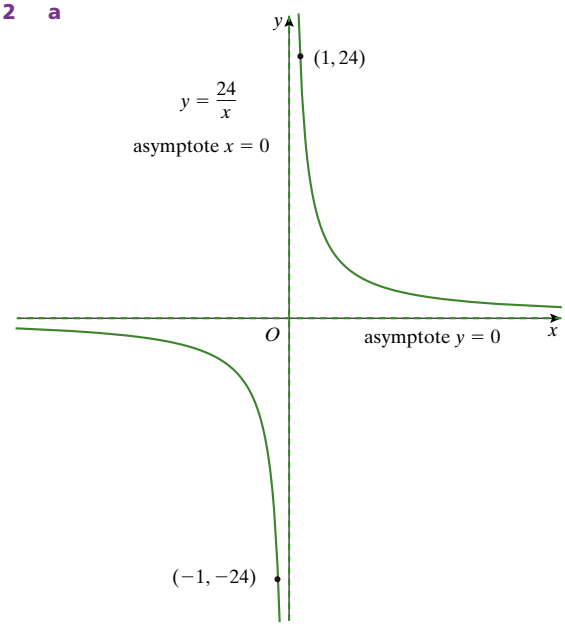

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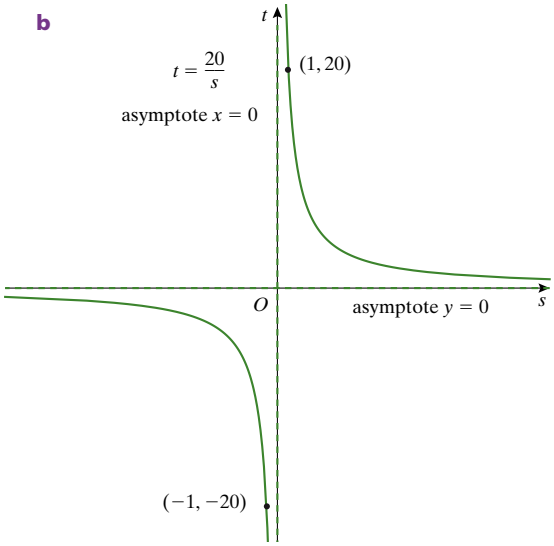

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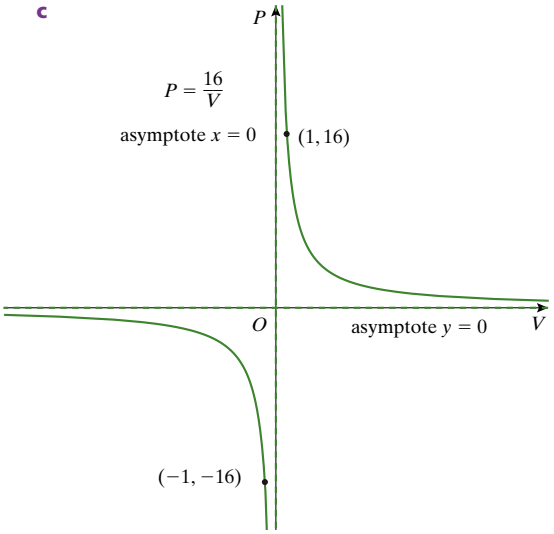

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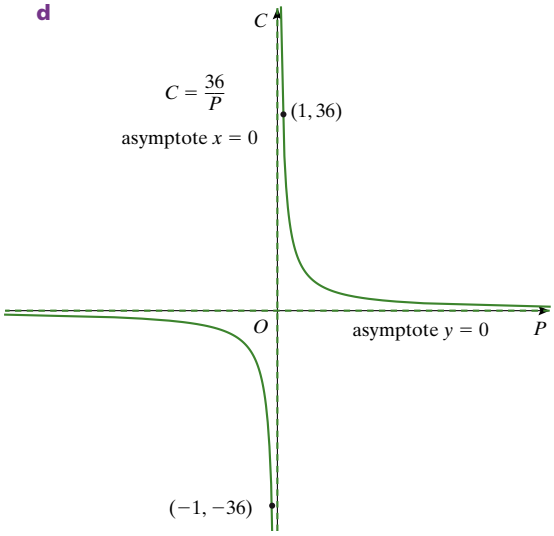

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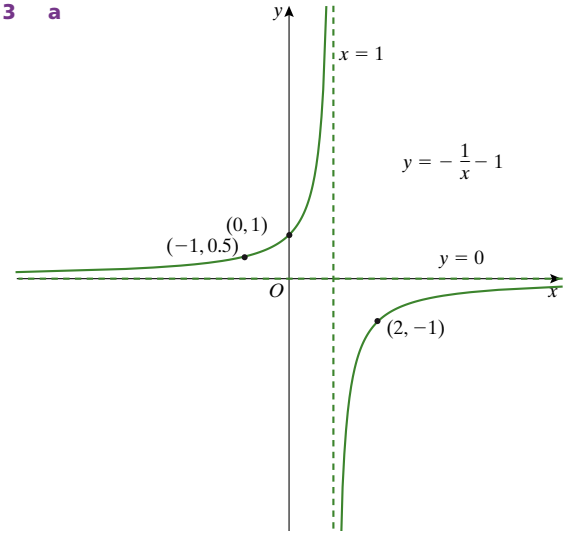

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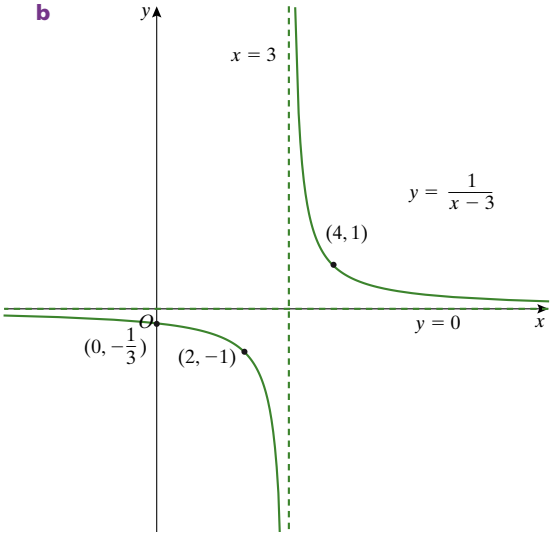

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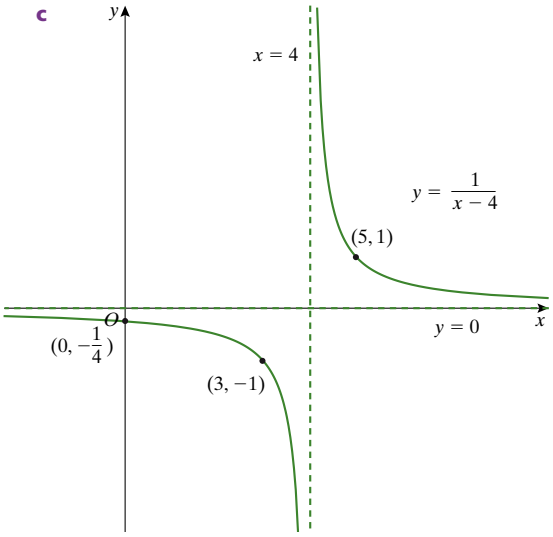

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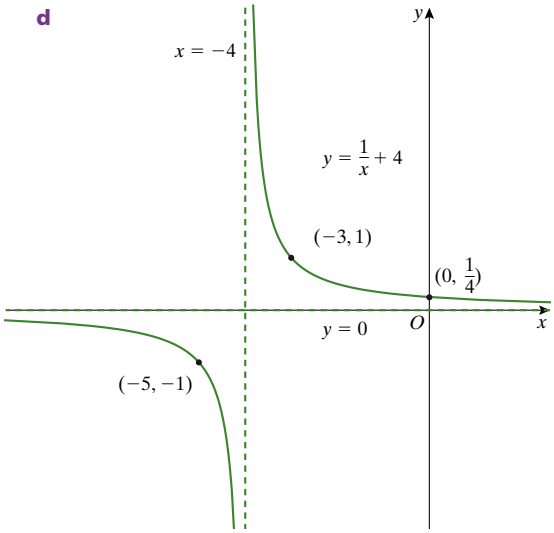

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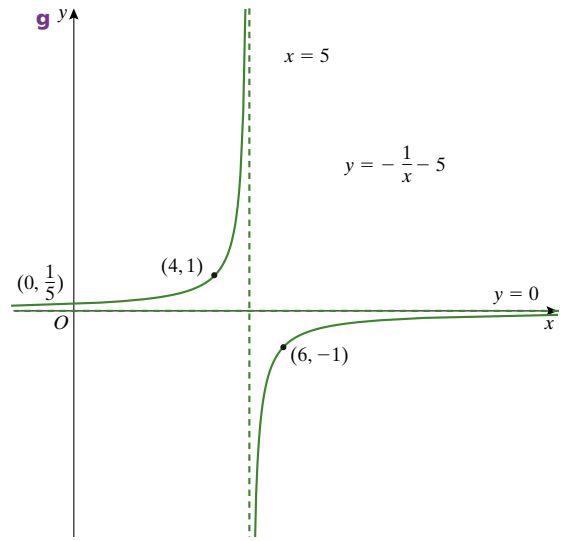

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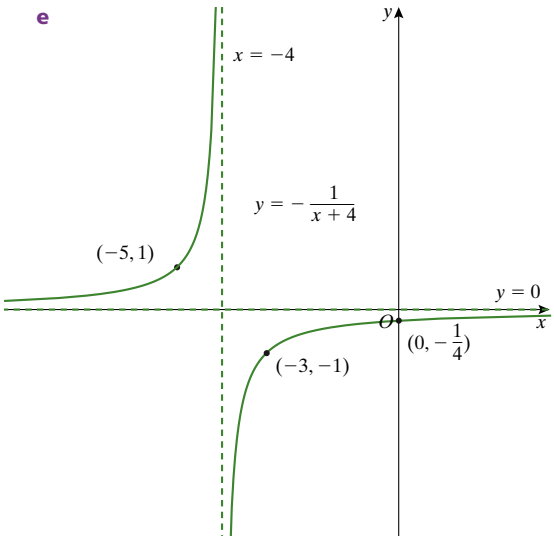

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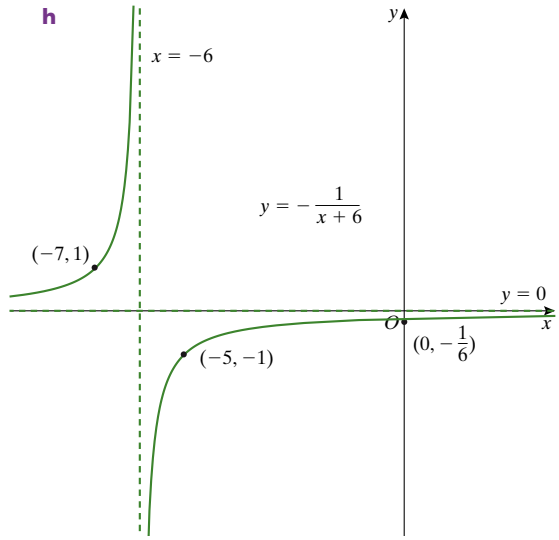

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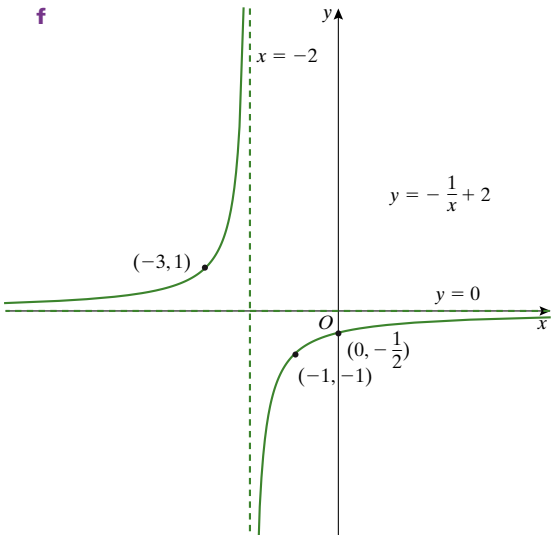

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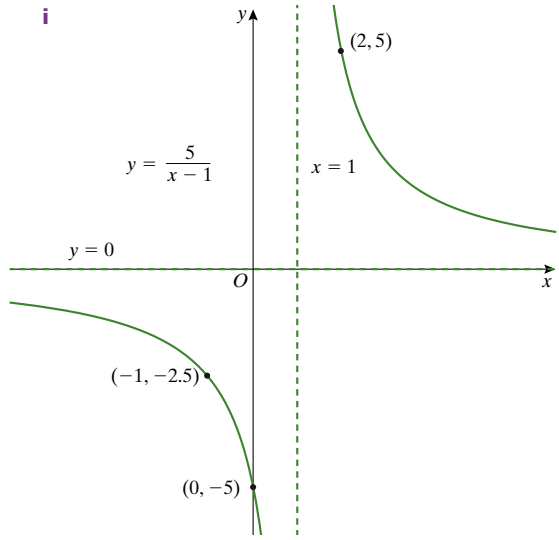

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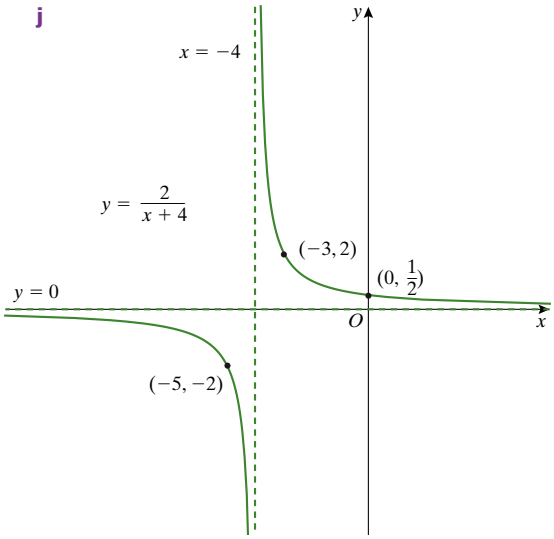

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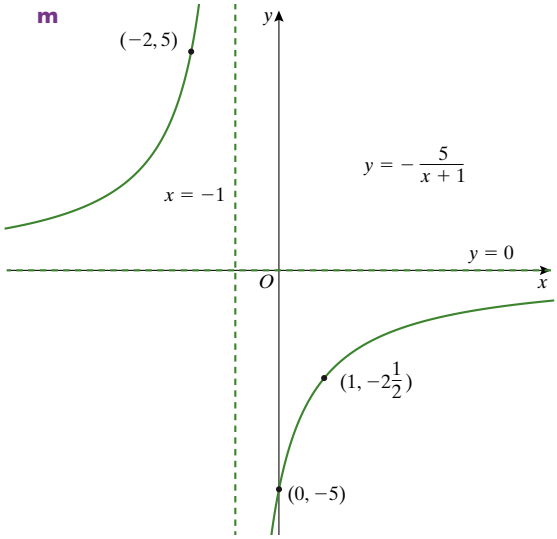

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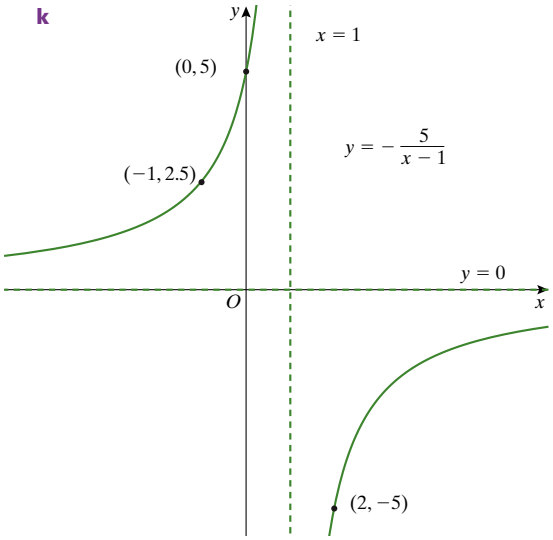

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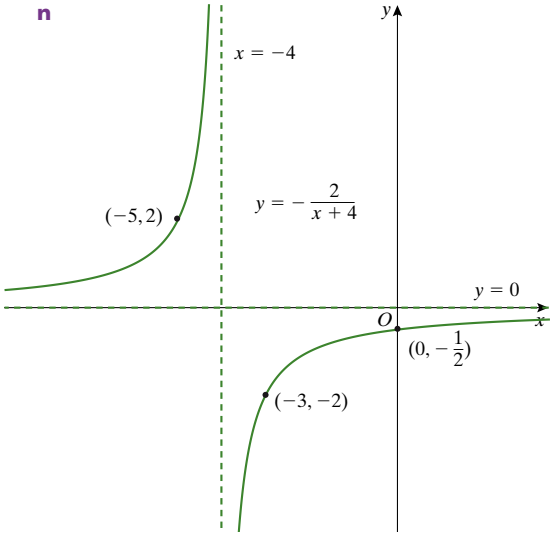

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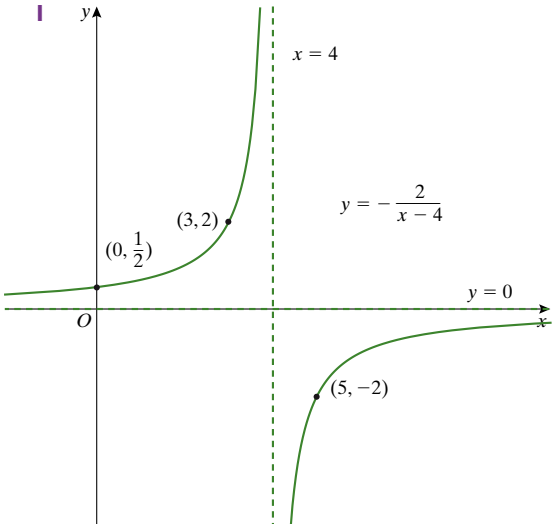

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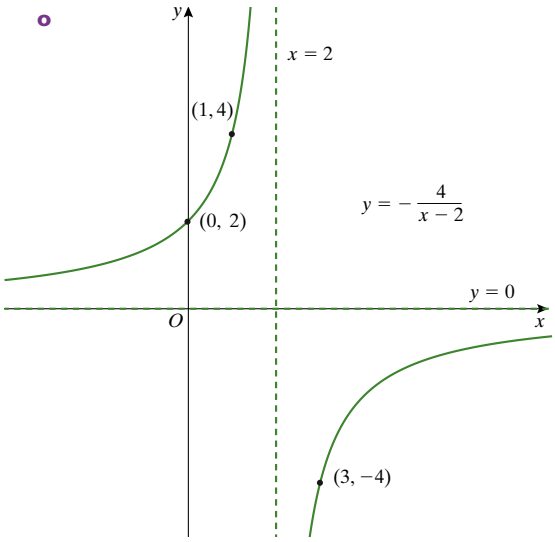

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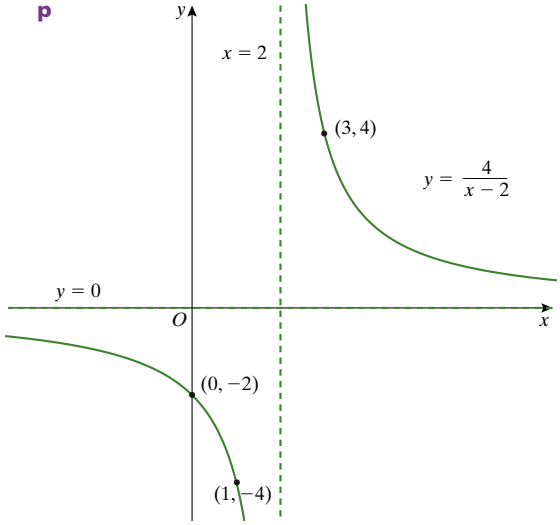

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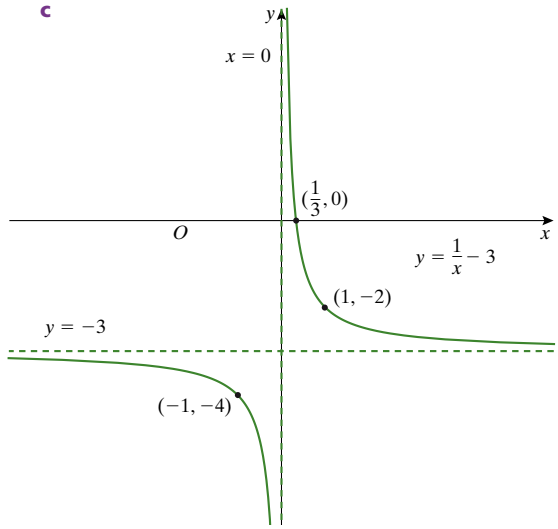

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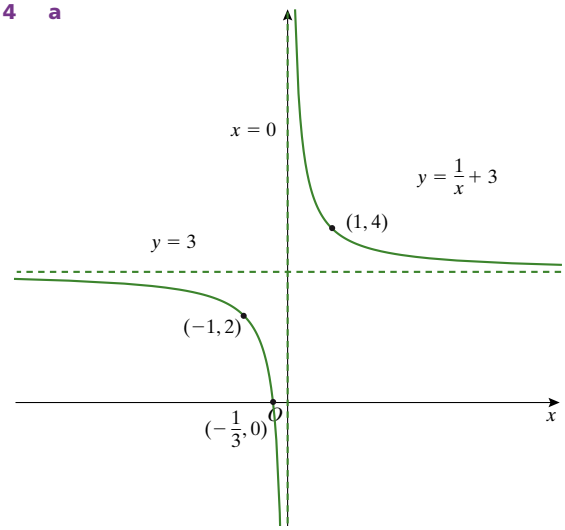

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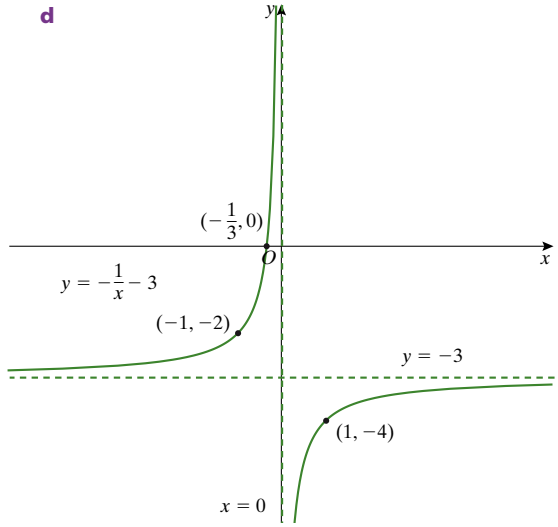

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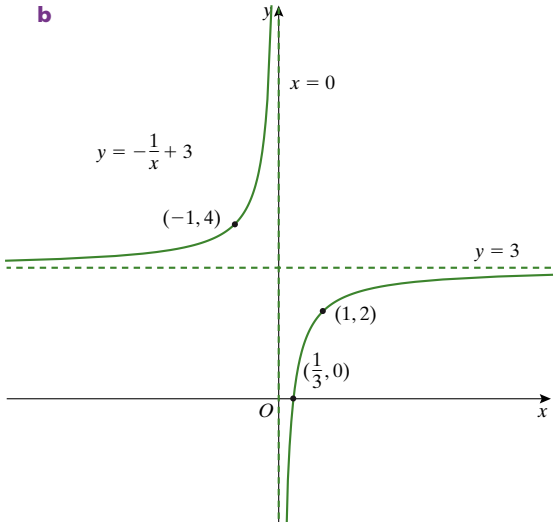

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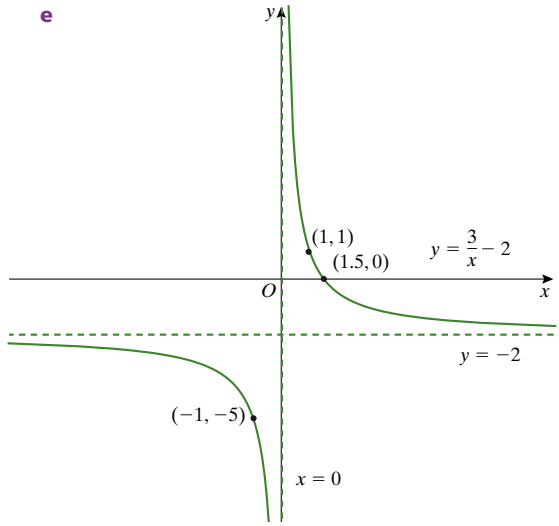

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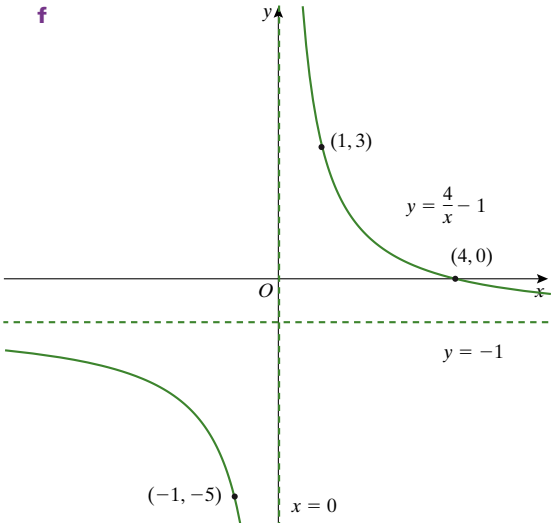

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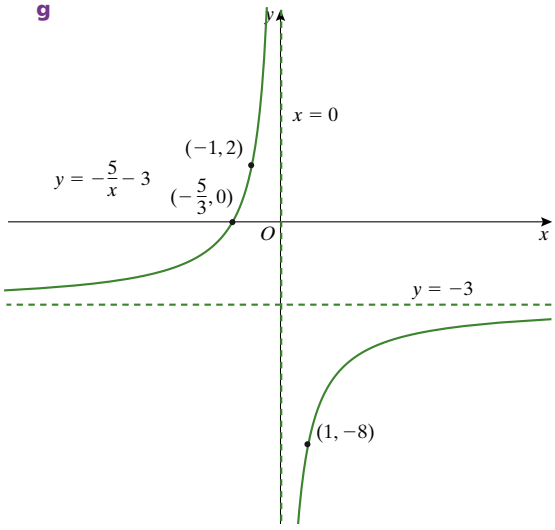

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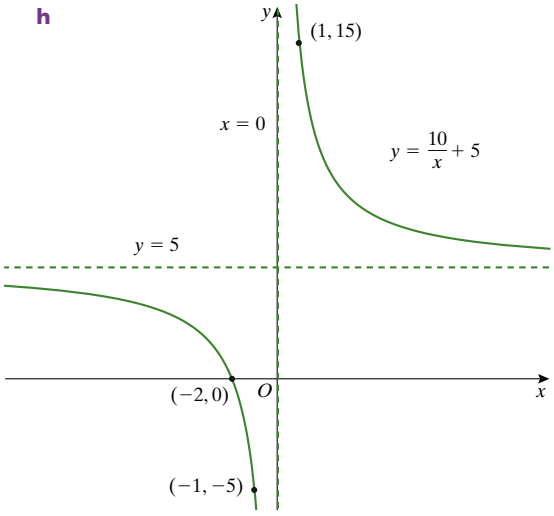


5 **a** G **b** E **c** A **d** F
e H **f** B **g** D **h** C

6 **a** **i** $y = -\frac{8}{x}$ **ii** $y = -\frac{8}{x}$

b **i** $y = -\frac{3}{x} + 5$ **ii** $y = -\frac{3}{x} - 5$

c **i** $y = \frac{2}{x} + 1$ **ii** $y = \frac{2}{x} - 1$

d **i** $y = -\frac{8}{x+1}$ **ii** $y = -\frac{8}{x-1}$

e **i** $y = \frac{3}{x-1}$ **ii** $y = \frac{3}{x+1}$

f **i** $y = \frac{6}{x+2}$ **ii** $y = \frac{6}{x-2}$

g **i** $y = -\frac{3}{x} - 5$ **ii** $y = -\frac{3}{x-5}$

h **i** $y = \frac{3}{x} + 1$ **ii** $y = \frac{3}{x} - 1$

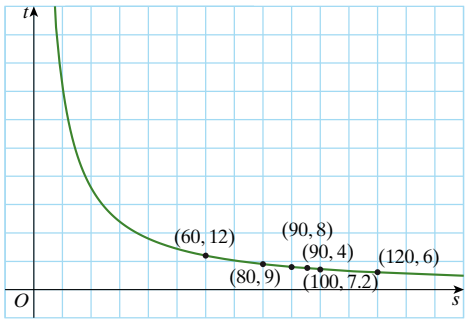
7 **a** 7.2 hours 12 minutes **b** 72 km/h

c

s (km/h)	60	80	90	100	120
t (h)	12	9	8	7.2	6

d $t = \frac{720}{s}$

e



8 **a** 8 cm **b** 12 cm **c** $y = \frac{48}{x}$

d

x	1	2	3	4	6	8	12	16	24	48
y	48	24	16	12	8	6	4	3	2	1

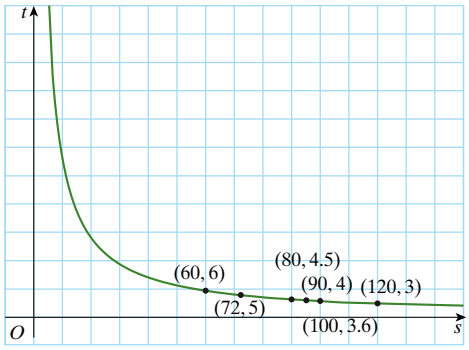
e see page 856

9 **a**

S	60	72	80	90	100	120
t	6	5	4.5	4	3.6	3

b $t = \frac{360}{S}$

c



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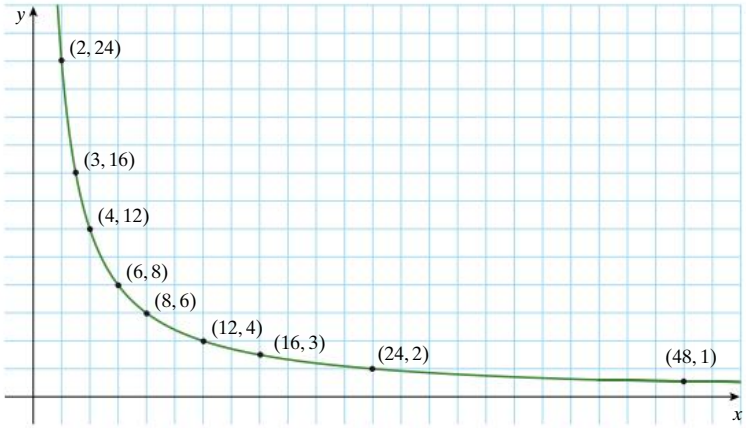
8 **e**

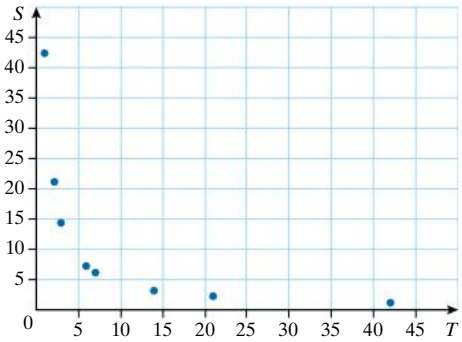


d 3.75 hours 45 minutes

10 **a** (1, 42), (2, 21), (3, 14), (6, 7), (7, 6), (14, 3), (21, 2), (42, 1)

b $S = \dfrac{42}{T}$

c Since only whole numbers make sense, it is not appropriate to join the points.



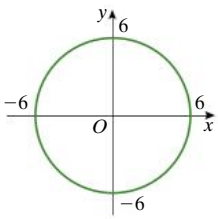
d No, as we cannot have $\dfrac{42}{T} = 0$.

revision

1 B **2** D **3** C **4** B **5** C

6 **a** **i** centre (0, 0), $r = 6$ **ii** $x = \pm 6$; $y = \pm 6$

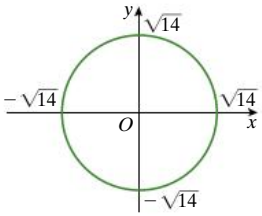
iii



b **i** centre (0, 0), $r = \sqrt{14}$

ii $x = \pm\sqrt{14}$; $y = \pm\sqrt{14}$

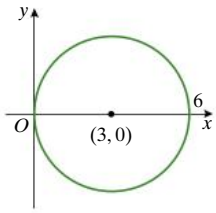
iii



c **i** centre (3, 0), $r = 3$

ii $x = 0, 6$; $y = 0$

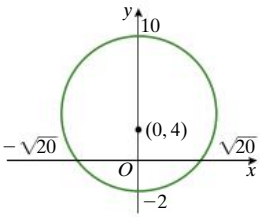
iii



d **i** centre (0, 4), $r = 6$

ii $x = \pm\sqrt{20}$; $y = -2, 10$

iii



7 **a** **i** $x^2 + y^2 = 9$

ii $x = \pm 3$; $y = \pm 3$

b **i** $x^2 + y^2 = 18$

ii $x = \pm\sqrt{18}$; $y = \pm\sqrt{18}$

c **i** $(x + 5)^2 + y^2 = 36$

ii $x = -11, 1$; $y = \pm\sqrt{11}$

d **i** $x^2 + (y - 2)^2 = 25$

ii $x = \pm\sqrt{21}$; $y = -3, 7$

8 **a** **i** centre (4, 4), $r = 4$

ii $x = 4$; $y = 4$

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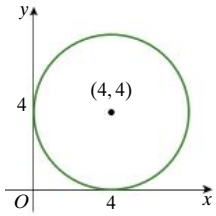
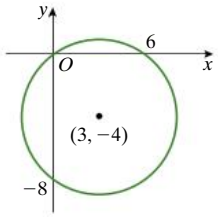
iii



b i centre $(3, -4)$, $r = 5$
ii $x = 0, 6$; $y = -8, 0$
iii



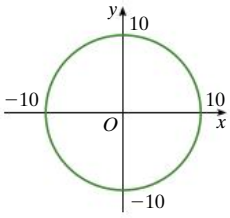
9 $x^2 + (y - 3)^2 = 25$

10 **a** $x = 0, y = 0$ **b** $x = -2, y = 0$
c $x = 0, y = 3$ **d** $x = 0, y = -5$

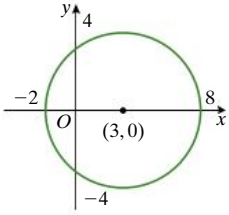
11 **a** $x = -4$ **b** $x = 6$

12 **a** $y = 2$ **b** $y = -4$

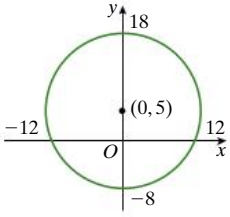
13 **a i** centre $(0, 0)$, $r = 10$
ii $x = \pm 10$; $y = \pm 10$
iii



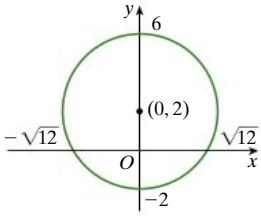
b i centre $(3, 0)$, $r = 5$
ii $x = -2, 8$; $y = \pm 4$
iii



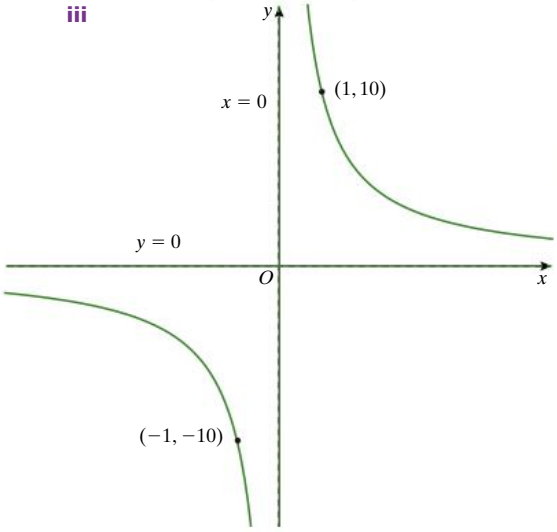
c i centre $(0, 5)$, $r = 13$
ii $x = \pm 12$; $y = -8, 18$
iii



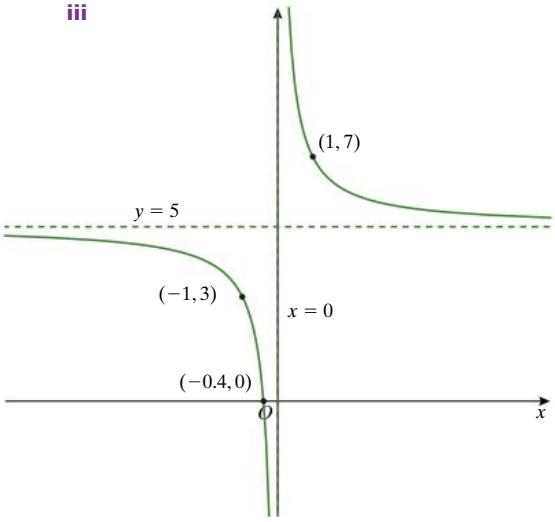
d i centre $(0, 2)$, $r = 4$
ii $x = \pm\sqrt{12}$; $y = -2, 6$
iii



14 **a i** $x = 0, y = 0$
ii no x-intercept; no y-intercept
iii



b i $x = 0, y = 5$ **ii** $x = -0.4$
iii



c i $x = 4, y = 0$ **ii** $y = -2$

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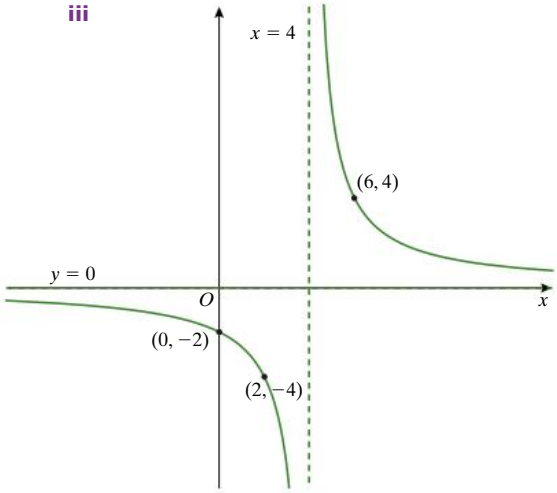
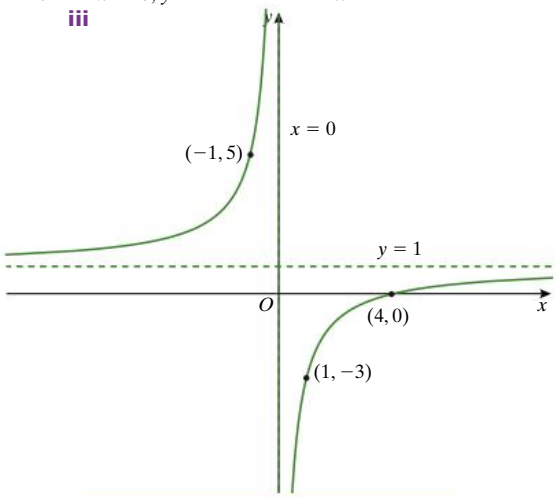
iii



d **i** $x = 0, y = 1$ **ii** $x = 4$

iii

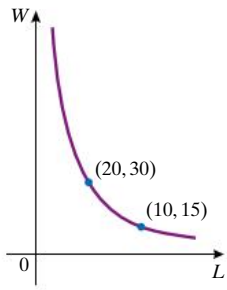


15 **a**

Length (l m)	10	20	30	40
Width (w m)	60	30	20	15

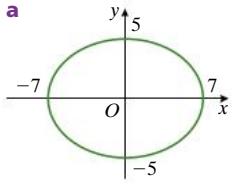
b Reciprocal relationship; product of the numbers is the constant 600.

c $w = \dfrac{600}{l}$ **d**

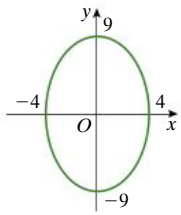


16 **a** **i** $(x - 3)^2 + (y - 7)^2 = 64$
ii centre $(3, 7)$, $r = 8$
b **i** $(x - 2)^2 + (y + 5)^2 = 36$
ii centre $(2, -5)$, $r = 6$

17 **a**



b



Chapter 13

exercise 13.1

1 **a** 2nd **b** 3rd **c** 1st **d** 4th
e 4th **f** 2nd **g** 3rd **h** 3rd

2

Quadrant	sin θ	cos θ	tan θ
First	+	+	+
Second	+	−	−
Third	−	−	+
Fourth	−	+	−

3 **a** positive **b** positive **c** positive **d** negative
e negative **f** negative **g** positive **h** negative
i positive **j** negative **k** negative **l** negative

4 **a** 0.6018 **b** 0.7431 **c** 8.1443 **d** −0.1045
e −0.2679 **f** −0.9135 **g** 0.9998 **h** −0.9877
i 28.6363 **j** −0.9703 **k** −0.3640 **l** −0.3640

5 **a** 37°, 143° **b** 56°, 124° **c** 76°, 104° **d** 29°, 151°

6 **a** 51°, 309° **b** 78°, 282° **c** 23°, 337° **d** 7°, 353°

7 **a** 58°, 238° **b** 35°, 215° **c** 18°, 198° **d** 85°, 265°

8 **a** $\dfrac{1}{2}$ **b** $-\dfrac{1}{2}$ **c** $-\dfrac{1}{2}$

9 **a** 0.848 **b** −0.848 **c** −0.848 **d** 0.848

10 **a** −0.951 **b** −0.951 **c** 0.951 **d** 0.951

11 **a** −1.804 **b** 1.804 **c** −1.804

12 **a** 75°, 285° **b** 11°, 169° **c** 78°, 258° **d** 53°, 127°
e 36°, 324° **f** 71°, 109° **g** 89°, 269° **h** 81°, 279°

13 **a** 113°, 247° **b** 194°, 346°
c 101°, 281° **d** 220°, 320°
e 148°, 212° **f** 244°, 296°
g 91°, 271° **h** 97°, 263°

14 $\pm$ 0.96

15 $\dfrac{7}{24}$

16 $\dfrac{21}{29}$

17 **a** $-\dfrac{1}{\sqrt{5}}$ **b** $-\dfrac{1}{2}$

exercise 13.2

1 C

2 **a** $180° - \angle B$ **b** $\sin\angle ABP = \sin B$

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c **i** $h = c\sin\angle ABP = c\sin B$
ii $h = b\sin C$
d $b\sin C = c\sin B$, so $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

3 **a** $a = 59.7$ **b** $b = 34.4$ **c** $c = 16.3$ **d** $d = 21.3$
e $e = 74.8$ **f** $f = 95.8$ **g** $g = 97.2$ **h** $h = 86.1$
i $i = 47.9$ **j** $j = 78.8$ **k** $k = 91.4$ **l** $l = 19.3$

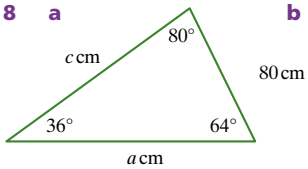
4 **a** $a = 10.8$ **b** $c = 33.8$ **c** $b = 38.8$ **d** $a = 2.9$

5 A

6 **a** 43° **b** 35° **c** 43° **d** 41°, 139°
e 86°, 94° **f** 27° **g** 65°, 115° **h** 70°, 110°
i 33° **j** 32° **k** 49° **l** 50°

7 **a** 21° **b** 37°, 143° **c** 20° **d** 42°

8 **a**



b 134.0 cm, 122.3 cm

9 **a** The given angle is less than 90° and $BC < AC$, so there are two possible positions for B and hence two possible triangles.
b $\angle B = 135.6°$, $\angle C = 14.4°$ or $\angle B = 44.4°$, $\angle C = 105.6°$
c $AB = 12.5$ cm or 48.2 cm

exercise 13.3

1 B

2 **a** **i** $x = b\cos C$ **ii** $h^2 = b^2 - x^2$
iii $x - a$ **iv** $h^2 = c^2 - (x - a)^2$
b $b^2 - x^2 = c^2 - (x - a)^2$
c $b^2 = c^2 + 2ax - a^2$
d $b^2 = c^2 + 2ab\cos C - a^2$; rearranging gives $c^2 = a^2 + b^2 - 2ab\cos C$

3 **a** $a = 6.5$ **b** $b = 8.4$ **c** $c = 13.8$ **d** $d = 13.8$
e $e = 15.3$ **f** $f = 12.4$ **g** $g = 13.3$ **h** $h = 8.8$
i $i = 23.9$ **j** $j = 5.7$ **k** $k = 14.0$ **l** $l = 9.8$

4 **a** $a = 10.6$ **b** $c = 11.3$ **c** $b = 25.3$ **d** $c = 34.5$

5 A

6 **a** 35.7° **b** 44.4° **c** 67.4° **d** 93.8°
e 50.1° **f** 38.0° **g** 98.2° **h** 59.0°
i 26.9° **j** 42.0° **k** 101.4° **l** 56.2°

7 **a** 29.9° **b** 106.0° **c** 82.8° **d** 80.3°

8 94.9°

exercise 13.4

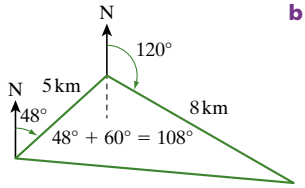
1 **a** $x = 35.7$ **b** $\theta = 111°$ **c** $x = 10.8$
d $x = 5.0$ **e** $x = 3.0$ **f** $\theta = 93°$
g $\theta = 49.1°$ **h** $x = 52.1$ **i** $x = 107.2$
j $\theta = 69°$ **k** $\theta = 41°$ **l** $x = 66.4$

2 **a** 31.7 cm **b** 52.3 cm **c** 79.1°

3 37.2 cm, 66.9 cm

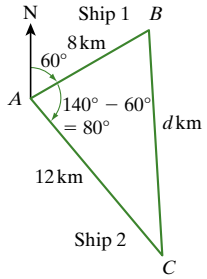
4 41.0 cm

5 **a**



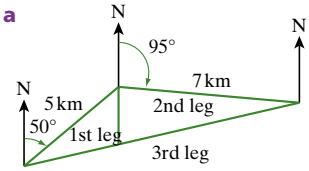
b 10.7 km

6 **a**



b 13.2 km

7 **a**

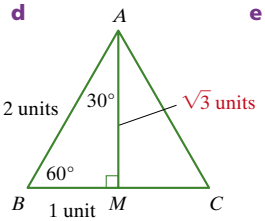


b 11.1 km
c 256.4°

exercise 13.5

1 **a** 30° **b** 60° **c** $\sqrt{3}$ units

d



e

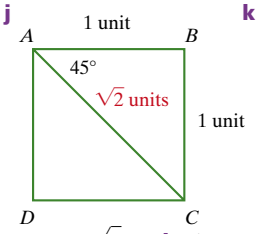
	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

f same

g The sine of an angle is equal to the cosine of the complementary angle.

h 45° **i** $\sqrt{2}$ units

j



k

	sin	cos	tan
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

2 **a** $a = 4\sqrt{3}$ **b** $b = 20$ **c** $c = 80$
d $d = 8\sqrt{3}$ **e** $e = 6\sqrt{2}$ **f** $f = 9$

3 $\dfrac{\sin 30°}{\cos 30°} = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}} = \tan 30°$

4 **a** $\sin^2 30° + \cos^2 30° = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2$

$= \dfrac{1}{4} + \dfrac{3}{4} = 1$

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b $\sin^2 60° + \cos^2 60° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$
$= \frac{3}{4} + \frac{1}{4} = 1$

c $\sin^2 45° + \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$
$= \frac{1}{2} + \frac{1}{2} = 1$

5 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{2}$ **d** -1
e $-\frac{1}{2}$ **f** $-\frac{1}{2}$ **g** -1 **h** $-\frac{\sqrt{3}}{2}$
i $-\frac{\sqrt{3}}{2}$ **j** $\sqrt{3}$ **k** $\frac{\sqrt{3}}{2}$ **l** $-\frac{\sqrt{3}}{2}$

exercise 13.6

1 **a** $30.8\,\text{cm}^2$ **b** $47.8\,\text{cm}^2$ **c** $38.9\,\text{cm}^2$
d $1497.9\,\text{cm}^2$ **e** $59.1\,\text{cm}^2$ **f** $125.2\,\text{cm}^2$
g $40.1\,\text{cm}^2$ **h** $14.6\,\text{cm}^2$ **i** $29.7\,\text{cm}^2$
j $1018.4\,\text{cm}^2$ **k** $1327.0\,\text{cm}^2$ **l** $522.9\,\text{cm}^2$

2 **a** $7.5\,\text{cm}^2$ **b** $9.8\,\text{cm}^2$ **c** $34.6\,\text{cm}^2$
d $42.4\,\text{cm}^2$ **e** $14.7\,\text{cm}^2$ **f** $39.2\,\text{cm}^2$
g $50.1\,\text{cm}^2$ **h** $52.2\,\text{cm}^2$ **i** $188.0\,\text{cm}^2$
j $37.9\,\text{cm}^2$ **k** $17.4\,\text{cm}^2$ **l** $12.0\,\text{cm}^2$

3 **a** $880\,\text{cm}^2$ **b** $7.8\,\text{m}^2$ **c** $1247.2\,\text{cm}^2$
d $1522\,\text{cm}^2$ **e** $2887\,\text{cm}^2$ **f** $1.5\,\text{m}^2$
g $37.1\,\text{m}^2$ **h** $241\,\text{m}^2$ **i** $15.6\,\text{m}^2$
j $6.1\,\text{m}^2$

4 **a** **i** $\sqrt{9(9-6)(9-6)(9-6)}$
$= \sqrt{243} = 9\sqrt{3}\,\text{cm}^2$
ii $\frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}\,\text{cm}^2$
iii $\frac{1}{2} \times 6 \times 6 \times \sin 60° = \frac{1}{2} \times 6 \times 6 \times \frac{\sqrt{3}}{2}$
$= 9\sqrt{3}\,\text{cm}^2$
b $15.6\,\text{cm}^2$

5 $7135\,\text{m}^2$

6 **a** $5734\,\text{m}^2$ **b** $157.1\,\text{m}$

exercise 13.7

1 210°, 330°

2 30°, 330°

3 **a** 60°, 300° **b** 240°, 300°
c 45°, 135°, 225°, 315° **d** 45°, 135°, 225°, 315°
e 30°, 150°, 210°, 330° **f** 30°, 150°, 210°, 330°

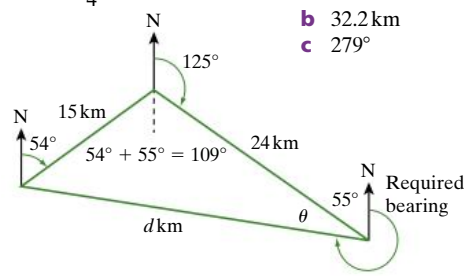
4 **a** 1 **b** $\sin\theta$ **c** $\cos\theta$
d $1 + \sin\theta$ **e** 1 **f** $\sin\theta$

5 2

6 $\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta} \times \frac{1-\cos\theta}{1-\cos\theta}$
$= \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)}$
$= \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}$
$= \frac{\sin\theta}{1-\cos\theta} = \text{RS}$

revision

1 B **2** D **3** B **4** D **5** D
6 **a** 3rd **b** 1st **c** 4th **d** 2nd **e** 3rd **f** 2nd
7 **a** 40°, 140° **b** 152°, 208° **c** 65°, 245°
8 1
9 **a** $a = 22.8$ **b** $b = 38.6$
10 **a** 52° **b** 57°
11 $C = 41°, b = 5.3, c = 5.3$
12 $AC = 11.5\,\text{cm}, BD = 34.4\,\text{cm}$
13 **a** $217\,\text{cm}^2$ **b** $126\,\text{cm}^2$
14 **a** $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** -1 **d** $-\frac{1}{2}$
15 **a** $a = 4\sqrt{2}$ **b** $b = 6\sqrt{3}$
16 60°, 120°
17 $\sin\theta = \pm\frac{1}{4}$
18 **a**

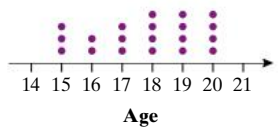


b 32.2 km
c 279°

Chapter 14

exercise 14.1

1 **a**



b Dot plot shows some slight skewness to higher ages with 12 ages of 18–20 and 8 ages of 15–17.

2 **a** **i**

stem	leaf
0	3 5 5 5 8
1	1 3 3 4 5 6 9
2	1 1 7 7 8
3	2 2 4 5 7 8 8
4	1 2 3 8 9
5	
6	2

Key 1|1 means 11

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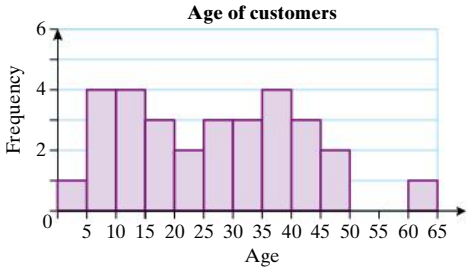
ii

stem	leaf
0	3
0	5 5 5 8
1	1 3 3 4
1	5 6 9
2	1 1
2	7 8
3	2 2 4
3	5 7 8 8
4	1 2 3
4	8 9
5	
5	
6	2

Key 1|1 means 11

b Both stem plots are useful, but the split stem plot gives more detail regarding the age of the customers.

3 a



b The distribution has two peaks, one around ages 5–15 and another around ages 35–40. There is a clear outlier in the interval 60–65 (the original data shows a 62 year old). The probable explanation is that among the people at the fast food outlet, there are a number of children and a number of parents.

4 a

Reaction times in seconds for 25 Year 7 boys and 25 Year 7 girls

Leaves Boys	Stem	Leaves Girls
	2	6
4 4 4 4 4 2 2 1	3	1 2 2 3 4 4
8 7 5	3	5 5 5 6 7 7 9 9 9
3 3 2 2 0 0 0	4	0 0 0 3 3
8	4	9
30	5	4
6	5	9
4	6	2
	6	
	7	
	7	
	8	
9	8	

Key: 3|1| means 0.31 and |2|6 means 0.26

b Median time: boys 0.40 sec; girls 0.37 sec

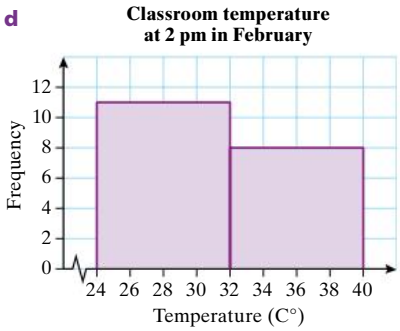
c Range: boys 0.67 sec; girls 0.36 sec

d The distributions for boys and for girls are positively skewed, with the greater times more spread out. The median times are similar. The range for boys is greater, largely due to the one boy with a reaction time of 0.89 seconds. This value could be regarded as an outlier.

5 a 2 degrees **b** 19

c $\frac{8}{19}$ (about 42%)

d

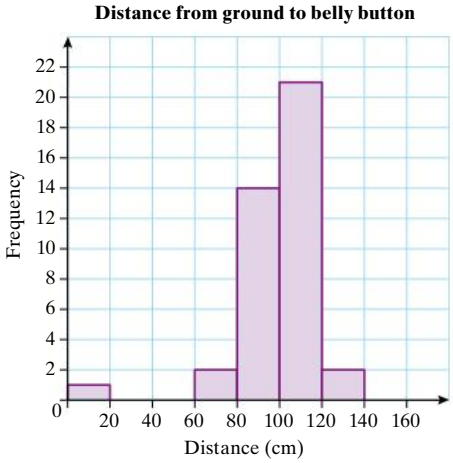


6 a Distances can be any value in a range (though they may be rounded)

b $\frac{23}{40}$ (57.5%)

c Misinterpretation of the question; or a measurement like 1.2 m recorded as 1.2 cm.

d i



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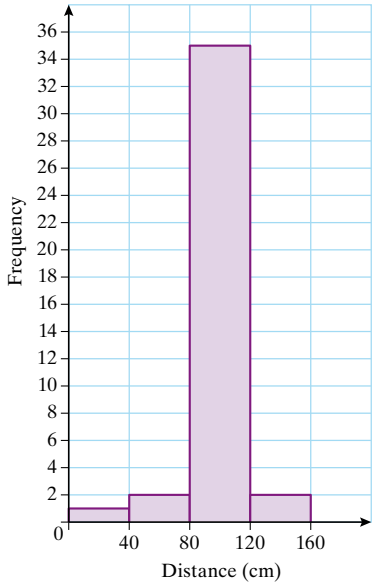
ii



e Less detail, but easier to see most students have measurements between 80 cm and 120 cm.

f 5 cm is not a multiple of the interval in the given display (need to have the original data).

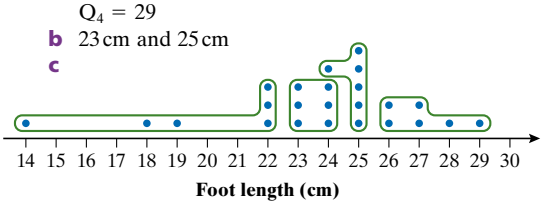
exercise 14.2

1 **a** **i** $Q_1 = 5$, $Q_3 = 10$ **ii** IQR = 5
b **i** $Q_1 = 16$, $Q_3 = 23$ **ii** IQR = 7
c **i** $Q_1 = 48$, $Q_3 = 71$ **ii** IQR = 23
d **i** $Q_1 = 1.6$, $Q_3 = 2.0$ **ii** IQR = 0.4
e **i** $Q_1 = 4$, $Q_3 = 8.5$ **ii** IQR = 4.5
f **i** $Q_1 = 26.5$, $Q_3 = 67$ **ii** IQR = 40.5
g **i** $Q_1 = 16.5$, $Q_3 = 19$ **ii** IQR = 2.5
h **i** $Q_1 = 97$, $Q_3 = 101.5$ **ii** IQR = 4.5

2 **a** $Q_0 = 2$, $Q_1 = 20$, $Q_2 = 47$, $Q_3 = 67$, $Q_4 = 92$
b $Q_0 = 23.21$, $Q_1 = 24.535$, $Q_2 = 25.54$, $Q_3 = 26.635$, $Q_4 = 26.98$

3 **a** Company A: median = 50.25, IQR = 1.4; Company B: median = 51.0, IQR = 0.6
b Company A weights centred on 50.25 g, closer to the labelled value of 50 g than those of Company B that are centred on 51.0 g; Company A weights show more variation around their centre than those of Company B; some Company A weights are less than 50 g, but all of Company B's weights are above 50 g.
This suggests that Company B may have better quality control.

4 **a** $Q_0 = 14$, $Q_1 = 22.5$, $Q_2 = 24$, $Q_3 = 25.5$, $Q_4 = 29$
b 23 cm and 25 cm
c



The lowest quartile has the longer tail, so the spread is greater in this quartile.
d $Q_1 - 1.5 \times \text{IQR}$
$= 18$ and $Q_3 + 1.5 \times \text{IQR} = 30$,
so only outlier is 14 cm.

5 **a** $Q_0 = 10$, $Q_1 = 15$, $Q_2 = 23$, $Q_3 = 27$, $Q_4 = 40$
b IQR = 12 **c** 15 and 27
d The paper would not suit people with a low level of reading ability. However the distribution is positively skewed, with fewer very long sentences.

6 **a** 59 g **b** 55 g and 65 g **c** No

7 **a** **i** 600 g **ii** 840 g **iii** 920 g
iv 980 g **v** 1200 g
b 140 g

exercise 14.3

1 **a** **i** 7.8 **ii** 9.74 **iii** 3.12
b **i** 19.2 **ii** 13.74 **iii** 3.71
c **i** 59.8 **ii** 154.74 **iii** 12.44
d **i** 1.75 **ii** 0.074 **iii** 0.27
e **i** 6.81 **ii** 12.96 **iii** 3.60
f **i** 48 **ii** 517.87 **iii** 22.76
g **i** 17.8 **ii** 3.01 **iii** 1.74
h **i** 98.95 **ii** 6.37 **iii** 2.52

2 **a** 22.08 **b** 71.30 **c** 8.44

3 **a** 53.27 **b** 3.78 **c** 1.94

4 **a** Girls' heights: mean 163.92 cm, sd 4.66 cm; boys heights: mean 174.48 cm, sd 8.96 cm
b On average, the boys are approximately 10 cm taller than the girls, but the boys' heights have a greater spread.

5 **a** Mean 0.4727 sec, sd 0.4859 sec
b 2.96 sec; possible misreading of stopwatch or misunderstanding of instructions
c Mean 0.3869 sec, sd 0.1265 sec
d Expect about two thirds or 19 reaction times out of 29 to lie between 0.2604 sec and 0.5133 sec; actual number is 23 out of 29. (Data fairly skewed, so not surprising.)
e Expect about 95% or 28 reaction times out of 29 to lie between 0.1340 sec and 0.6398 sec; actual number is 26 out of 29. (Data fairly skewed, so not surprising.)

6 **a** Yr 10 heights: mean 168.06 cm, sd 8.47 cm
Yr 7 heights: mean 159.32 cm, sd 8.62 cm
b Yr 10 heights: expect about two thirds or 67% of heights to lie between 159.6 cm and 176.5 cm; actual number is 37 out of 50 or 74%.
c Yr 7 heights: expect about two thirds or 67% of heights to lie between 150.7 cm and 167.9 cm; actual number is 37 out of 50 or 74%.
d On average, the Year 10 students are nearly 9 cm taller than the Year 7 students. The distribution of heights about the mean is very similar for both groups.

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7 **a** 22.4 kg/day and 31.6 kg/day
b 17.8 kg/day and 36.2 kg/day

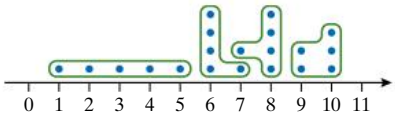
8 **a** **i** 6.3 and 8.7 **ii** 5.1 and 9.9
b **i** 0.69 and 0.99 **ii** 0.54 and 1.14
c **i** 34.7 and 38.9 **ii** 32.6 and 41.0
d **i** 34.1 and 34.5 **ii** 33.9 and 34.7

9 **a** 2720 g and 3760 g
b Expect about 2.5%, i.e. 5 pumpkins

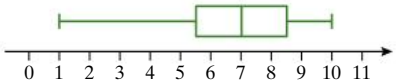
exercise 14.4

1 **a** 7 **b** 10 **c** 5
d 10 **e** 7 and 14 **f** 5 and 10
g 4 and 5

2



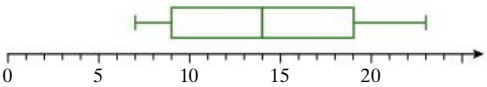
a 7
b 6 and 8
c The first 25% between Q_0 and Q_1
d



3 **a** 6 hours **b** 0 hours
c 12 hours **d** 12 hours
e $Q_1 = 3$, $Q_3 = 8$ **f** 5 hours
g Half the students spend 6 to 12 hours per week on homework, and a quarter spend 8 to 12 hours; another quarter spend less than 3 hours; the middle 50% spend between 3 hours and 8 hours.

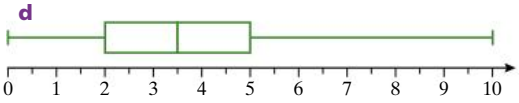
4 **a** 77 kg **b** 39 kg
c 70 kg and 83 kg **d** 25%

5 **a** $Q_0 = 7$, $Q_1 = 9$, $Q_2 = 14$, $Q_3 = 19$, $Q_4 = 23$
b

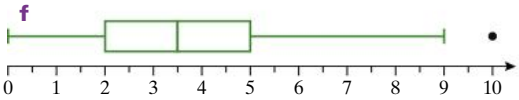


6 Many possible sets of 14 values chosen that $Q_0 = 2$, $Q_1 = 4$, $Q_2 = 7$, $Q_3 = 13$, $Q_4 = 18$; for example, **2**, 3, 3, **4**, 5, 6, **6**, **8**, 11, 11, **13**, 15, 17, **18** (note that the average of the middle two values is 7, the median).

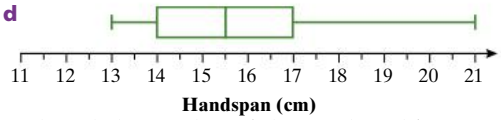
7 **a** $Q_0 = 0$, $Q_1 = 2$, $Q_2 = 3.5$, $Q_3 = 5$, $Q_4 = 10$
b 10 **c** 3
d



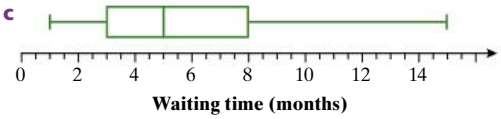
e $Q_1 - 1.5 \times \text{IQR} = -2.5$ and $Q_3 + 1.5 \times \text{IQR} = 9.5$, so only outlier is 10.
f



8 **a** $Q_0 = 13$, $Q_1 = 14$, $Q_2 = 15.5$, $Q_3 = 17$, $Q_4 = 21$
b 8 cm **c** 3 cm
d

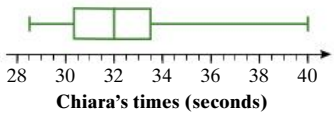


9 **a** $Q_0 = 1$, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 8$, $Q_4 = 15$
b 5 hours
c

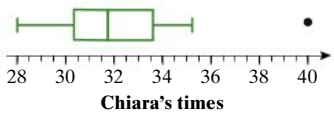


d $Q_1 - 1.5 \times \text{IQR} = -4.5$ and $Q_3 + 1.5 \times \text{IQR} = 15.5$, so there are no outliers.

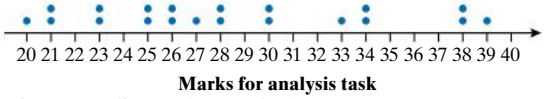
10 **a** $Q_0 = 28.4$, $Q_1 = 30.2$, $Q_2 = 31.65$, $Q_3 = 33.3$, $Q_4 = 39.9$
b range = 11.5 sec, IQR = 3.1 sec
c The small value suggests Chiara's times are fairly consistent.
d



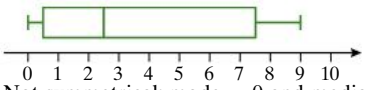
e $Q_1 - 1.5 \times \text{IQR} = 25.55$ and $Q_3 + 1.5 \times \text{IQR} = 37.95$, so only outlier is 39.9 sec.
f



11 **a** $Q_0 = 20$, $Q_1 = 24$, $Q_2 = 27.5$, $Q_3 = 33.5$, $Q_4 = 39$
b **i** 5 **ii** 10
c 27.5
d

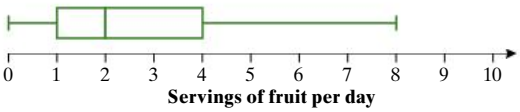


12 **a** Median = $Q_2 = 2.5$, IQR = 7
b



c Not symmetrical; mode = 0 and median = 2.5, but some cases of 8 or 9 errors.

13 **a** **i** $Q_0 = 0$, $Q_1 = 1$, $Q_2 = 2$, $Q_3 = 4$, $Q_4 = 8$ (take the first bar to represent 0 to less than 1 serving, i.e. 0 servings etc., so the last bar represents 8 to less than 9 servings, i.e 8 servings)
ii



b The location of each 25% block of data values.

14 Class A had a higher median and a higher top score; also 25% of Class A outperformed every student in Class B; so although Class A had some lower scores, overall it appears that they performed better.

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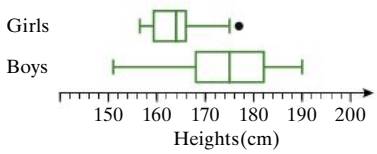
15 **a** The median number of species was greater in the state forest (9.5) than in replanted forest (7).
b The interquartile range was greater in the state forest (3.5) than in the re-planted forest (2).
c The range was the same (~7) in the state forest and re-planted forest.
d In the upper 25% of nights, there were more species in the state forest (10.5–11) than in the re-planted forest (8–11).
e On average, greater numbers of species seem to be present in state forests. So the planting has overall not yet restored the rat habitats. (Although it may well be that it is too early to expect them to be the same.)

16 **a** Day 1: $Q_0 = 0$, $Q_1 = 2$, $Q_2 = 5$, $Q_3 = 9.5$, $Q_4 = 25$
Day 2: $Q_0 = 0$, $Q_1 = 2$, $Q_2 = 4$, $Q_3 = 7.5$, $Q_4 = 29$
b Day 1: only outlier is 25 min; day 2: 26 min and 29 min are outliers.
c see below
d Apart from the two outliers, delays on day 2 were shorter than on day 1, though there were more flights with no delay on day 1.

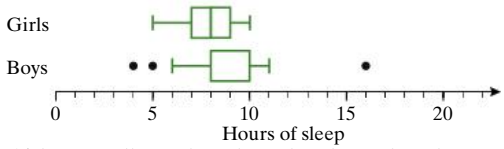
17 **a** **i** Girls: $Q_0 = 157$, $Q_1 = 160$, $Q_2 = 165$, $Q_3 = 166$, $Q_4 = 176$
Boys: $Q_0 = 151$, $Q_1 = 167.5$, $Q_2 = 175$, $Q_3 = 182.5$, $Q_4 = 190$
ii Girls: 176 cm is an outlier; boys: no outliers
b



c Girls heights show much less spread than those of boys; on average, boys are taller than girls with 75% of the boys taller than the upper quartile of 166 cm of the girls.

18 **a** $Q_0 = 2$, $Q_1 = 7$, $Q_2 = 8$, $Q_3 = 9$, $Q_4 = 16$
b Outliers: 2, 3 and 16 hours.
c $Q_0 = 6$, $Q_1 = 9$, $Q_2 = 10$, $Q_3 = 10$, $Q_4 = 16$; outliers: 6, 6, 6, 6, 7, 12, 12, 12, 13 and 16 hours.
d see below
e Year 7 students generally get more hours of sleep than year 10 students; the median is 10 hours for year 7 students but just 8 hours for year 10 students.

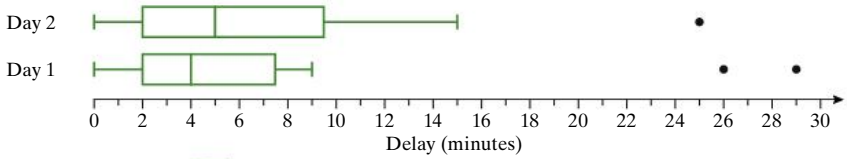
19 **a** **i** Year 10 girls: $Q_0 = 5$, $Q_1 = 7$, $Q_2 = 8$, $Q_3 = 9$, $Q_4 = 10$
Year 10 boys: $Q_0 = 4$, $Q_1 = 8$, $Q_2 = 8$, $Q_3 = 9.5$, $Q_4 = 16$
ii Year 10 girls: no outliers; Year 10 boys: 4, 5 and 16 hours are outliers.
b



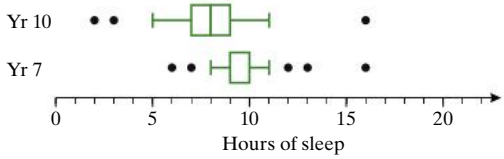
c Girls generally get less sleep than boys, though the median number of hours of sleep is the same, 8 hours, for each group.

20 **a** **i** Year 10: $Q_0 = 151$, $Q_1 = 162$, $Q_2 = 166$, $Q_3 = 174$, $Q_4 = 190$
Year 7: $Q_0 = 142$, $Q_1 = 152$, $Q_2 = 160$, $Q_3 = 164$, $Q_4 = 186$
ii Year 10: no outliers; year 7: 186 cm is an outlier
b see below

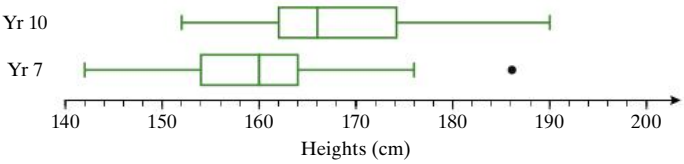
16 **c**



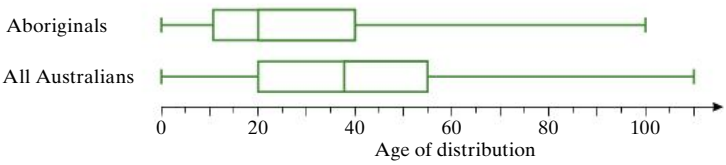
18 **d**



20 **b**



21 **a**



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c Both groups show similar spread, but year 10s are generally taller than year 9s with 75% of the year 10s taller than the median height of 160 cm of the year 9s.

21 **a** see page 864

b The median age for all Australians is 37 years, 16 years more than that of Aboriginal and Torres Strait Islanders.

c The IQR for all Australians is 35 years, compared with 30 years for Aboriginal and Torres Strait Islanders.

d Approximately 25% of all Australians are under the age of 20.
Approximately 50% of Aboriginal and Torres Strait Islanders are under the age of 20.
Approximately 50% of all Australians are over the age of 40.
Approximately 25% of Aboriginal and Torres Strait Islanders are over the age of 40.

e Aboriginal and Torres Strait Islanders have a significantly lower life expectancy than that of the general Australian population.

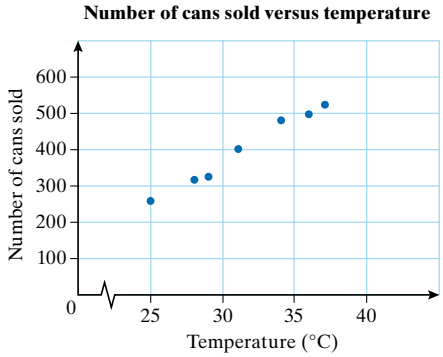
f Answers will vary.

22 Many answers possible, but for no right-hand whisker, the upper quartile and maximum value must be equal. This requires at least the 6 largest values to be identical.

exercise 14.5

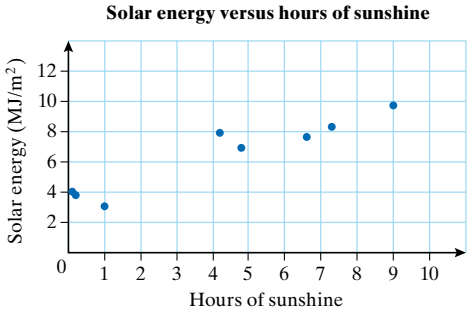
1 **a** Temperature; the number of cans sold depends on the temperature.

b



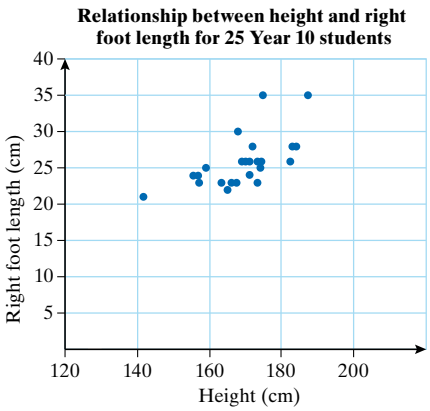
c Strong, positive association between temperature and sales. It shows that as the temperature increases, so does the number of drink cans sold.

2 **a**



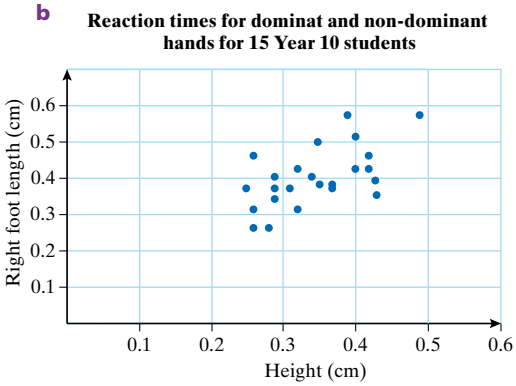
b Strong, positive association between hours of sunshine and solar energy.

3 **a**



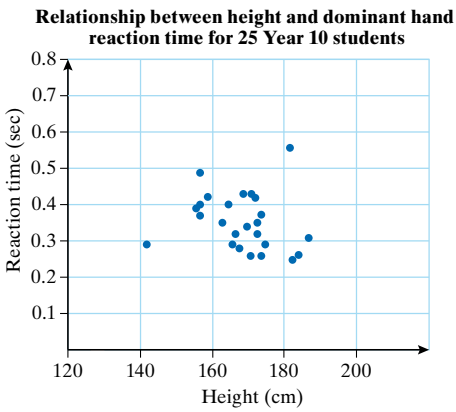
There is a moderately strong positive association between height and length of right foot.

b



There is a weak positive relationship between reaction times for dominant and non-dominant hands.

c



There does not seem to be a relationship between height and reaction time.

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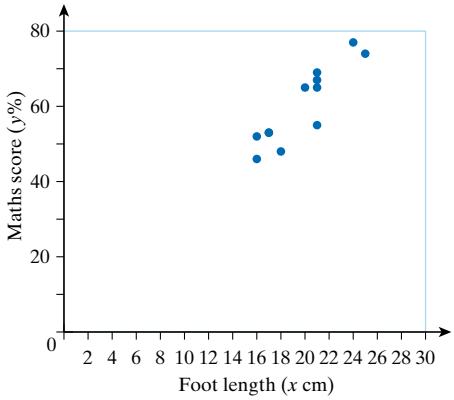
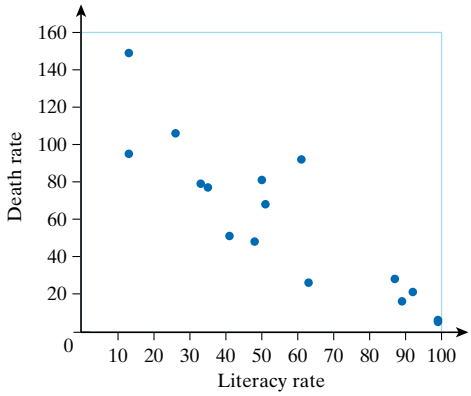
4 **a**



b Strong, positive association.

c The students may vary in age; older students will tend to have longer foot length and they will also tend to score higher on the test. The data does *not* tell us that longer foot length *causes* higher test results.

5 **a**



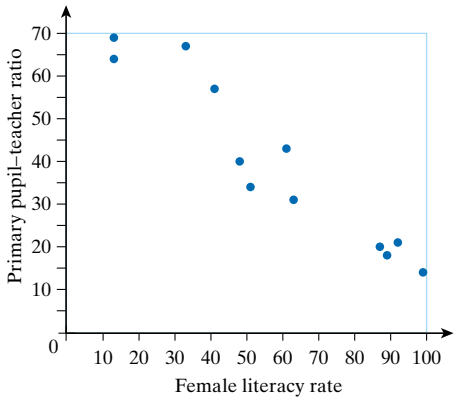
b Fairly strong, negative association.

c The two variables are likely to be linked due to other variables.

d For example, limited access to education for females in poorer countries, where infant death rates are higher.

e About 85 deaths per 1000 live births; estimate should be reasonable since we are interpolating and the association is strong.

6 **a**

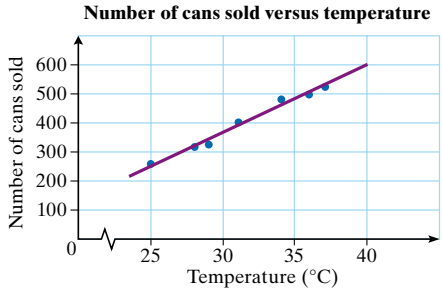


b Fairly strong, negative association.

c Higher pupil-teacher ratio might have some effect on general literacy, but the variables are also likely to be linked due to other variables.

d For example, limited access to education for females in poorer countries, or leaving school early to help support the family.

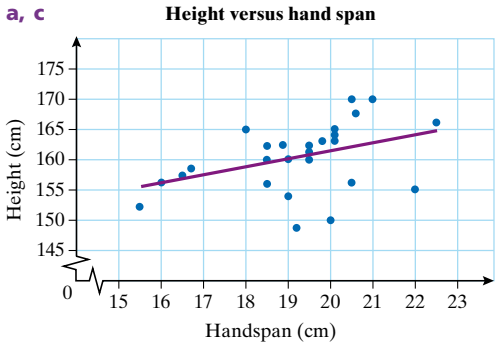
7



a **i** about 140 **ii** about 600

b The points lie very close to the line of good fit and this means the predictions may be reasonable. However, both temperatures lie outside the given data range, so we need to be cautious in our prediction due to the dangers of extrapolation. The observed linear trend may not extend in either direction.

8 **a, c**

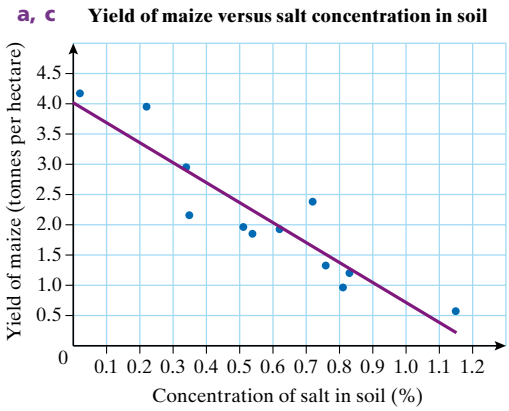


b Weak, positive association between hand span and height.

d About 163 cm. Not very confident as the weak association means there may be a large error in the prediction.

e No, as the relationship is quite weak.

9 **a, c**



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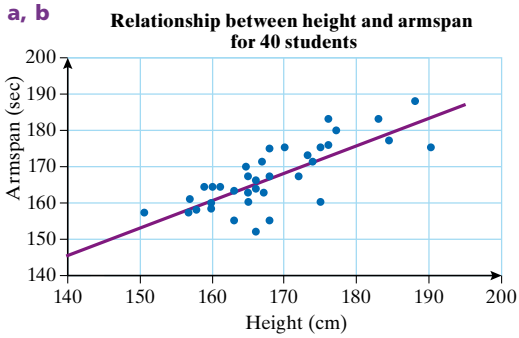
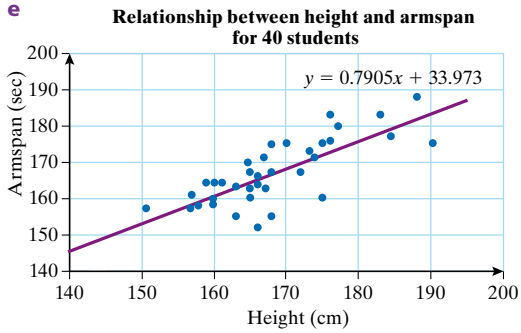
b Moderately strong, negative association between yield and salt concentration.
d $y = -3.3x + 4$ (answers may vary)
e The yield of maize is 4 tonnes per hectare when the salt concentration is zero.
f For each 0.1% increase in salt concentration, there is an estimated decrease of 0.33 tonnes of maize per hectare.

10 a, b

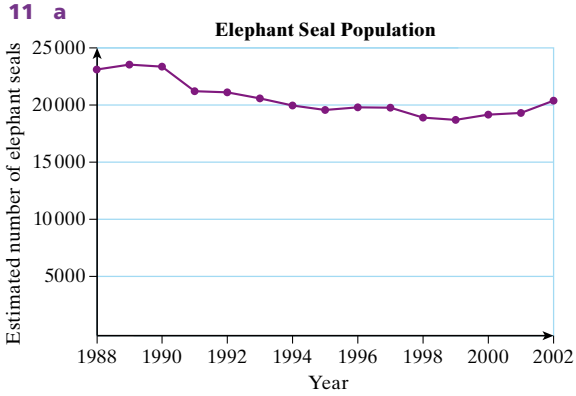


c **i** 183 cm **ii** 187 cm **iii** 146 cm
d Reasonably confident but the answers will vary due to placing the line of good fit in a slightly different position.
e



f **i** 183 cm **ii** 188 cm **iii** 145 cm
g The predictions for the three armspans are similar for the hand-drawn line of good fit and the Excel trendline.

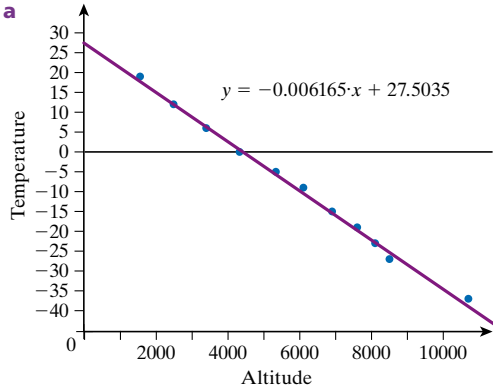
11 a



b There is a decreasing trend over time in the numbers of elephant seals, although there has been a small upswing in the last 3 years.

12 a In each case, the trend is for the times to decrease.
b The graph is trying to show that eventually (around 2150) women will catch up to men, i.e. women's times are decreasing at a faster rate than men's times, so they will eventually run faster than men.
c Linear trend lines may be useful within the given year range, but they cannot be extrapolated as shown, since they will eventually meet the horizontal axis to suggest that both men and women could perform the sprint in zero time!
d Graphs should show a gentle curve that starts to level out (there must be some times below which it is impossible to reach).

13 a



b $T = -0.0062A + 27.5$ **c** 27.5° C

exercise 14.6

1 The diameter of the soccer balls correctly shows the price of tickets. However, visually it is the area of the circles that we tend to compare. This misleads us into thinking that there has been a greater price increase.

2 **a** Increase in fish size and numbers; better fishing conditions; increase in tourism revenue.
b Unclear, but may be suggesting economic decline.
c No evidence given.
d NA

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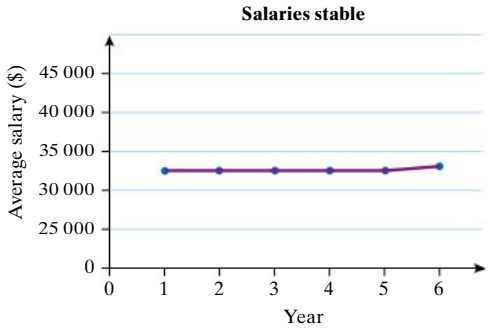
3



4 **a** Telephone surveys, which use landline phones, may not reach a representative sample of the entire population.
b No significant bearing on estimates for the entire population, since number of mobile-phone only households is small at present.
c Age 16–24 year olds may be misrepresented since just over one quarter live in mobile only households.

5 **a** 12.7:1 **b** about 3.7.1 **c** about 13.6.1.
d Diameter ratio about 2.3:1; area ratio about 5.3:1. Actual ratio from data is 5.1:1.
e The picture is reasonably accurate, since area ratios are quite close to numerical ratios (and area ratios are approximate).
f Not a typical graph since there are no axes with scales of any sort. An alternative would be to draw a more conventional bar graph.

revision

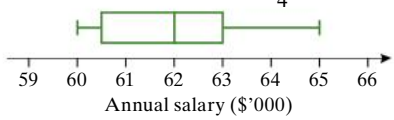
1 C **2** C **3** B **4** B **5** C

6 **a** 45 **b** 47.4 **c** 46 **d** 32 **e** 16
f see below

7 **a** 15 **b** 12

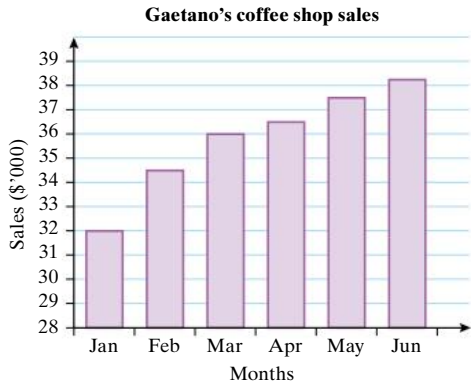
8 **a** 47 and 52 **b** $\frac{3}{4}$ (75%)

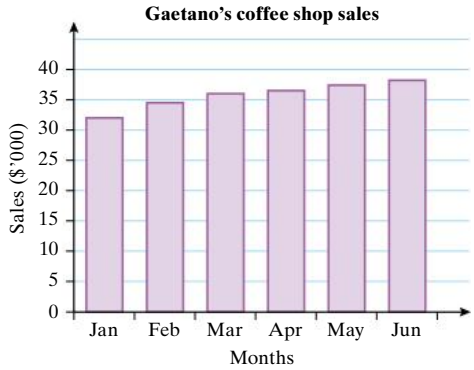
9



The dot plot is more useful when there are a small number of values as here, since it enables all the data values to be read and gives a reasonably clear view of the centre and spread of the data.

10 **a**

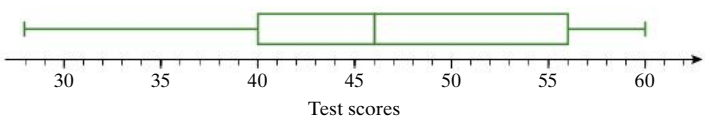




b He would show the graph with the vertical scale starting at 28 000, since it gives the impression of a strong rise in sales figures over the six months.

11 **a** The distribution is quite symmetrical, ranging from 2.7 mg/L to 6.6 mg/L. Most of the data lies between 3.6 mg/L and 5.3 mg/L.
b $Q_0 = 2.7$, $Q_1 = 3.9$, $Q_2 = 4.3$, $Q_3 = 4.95$, $Q_4 = 6.6$
c IQR = 1.05; the middle 50% of the data has a spread of 1.05 mg/L, and the rule for outliers shows that the 6.6 mg/L is an outlier for this data set.

6 **f**



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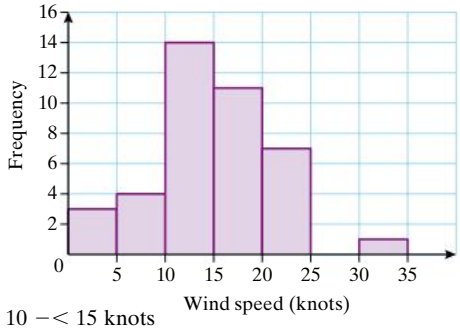
12 **a** Continuous numerical

b



c 10 −< 15 knots

d

Wind speed (knots)

Stem	Leaf
0	0 3 4
0	5 7 8 9
1	0 0 1 1 2 2 2 2 3 3 4 4 4 4
1	5 5 6 6 7 7 7 8 9 9 9
2	0 1 2 3 3 4 4
2	
3	4

(Key: 0|3 means 3)

e $Q_0 = 0$, $Q_1 = 11$, $Q_2 = 14$, $Q_3 = 19$, $Q_4 = 34$

f $Q_1 - 1.5 \times IQR = -1$ and $Q_3 + 1.5 \times IQR = 31$, so only outlier is 34 knots.

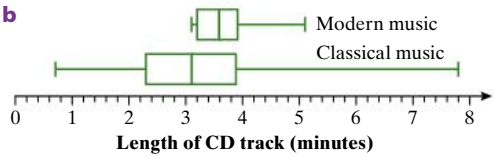
g see below

h Reasonably symmetrical with one peak in the modal interval 10−<15 knots. The middle 50% of the data has a spread of 8 knots. The distribution's centre could be taken as the median (14 knots) or the mean (which is approximately 14.9 knots).

13 **a** Modern music: $Q_0 = 3.1$, $Q_1 = 3.2$, $Q_2 = 3.6$, $Q_3 = 3.9$, $Q_4 = 5.1$

Classical music: $Q_0 = 0.7$, $Q_1 = 2.3$, $Q_2 = 3.1$, $Q_3 = 3.9$, $Q_4 = 7.8$

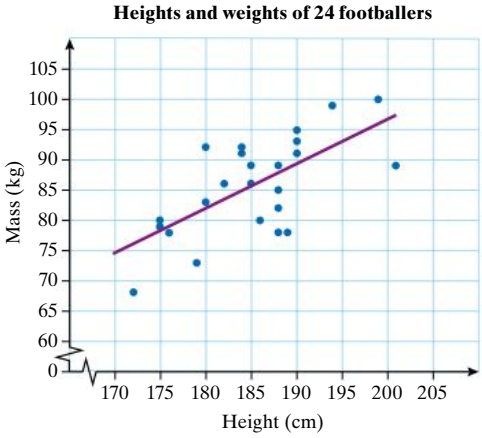
b



c Track length for modern music has a significantly smaller range of values, and the middle 50% of values has a spread of 0.7 minutes compared with a spread of 1.6 minutes for the middle 50% of values for classical music. Overall, the track lengths for modern music are more similar in length; there is more variation in track length with classical music.

14 **a** Weak to moderate positive association

b

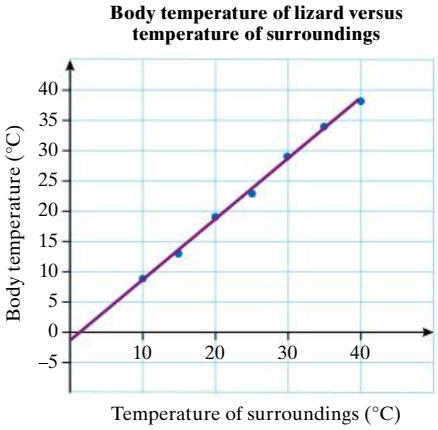


c Approximately 93 kg

d This point is within the range of data values so you would expect the prediction to be reliable. But the points are scattered in the order of 10 kg either side of the line of good fit, so we might expect the player's weight to be out by as much as 10 kg from the predicted value.

15 **a** Temperature of the surroundings

b, d



c Very strong, positive association

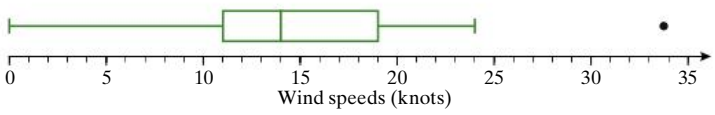
e About 15° C

16 **a** **i** 14.7 **ii** 43.5 **iii** 6.6

b 8.1 knots and 21.3 knots; actual percentage between these values is 70%, so quite close to expected percentage.

c 1.5 knots and 27.9 knots; actual percentage between these values is 95%, so exactly as expected.

12 **g**



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Chapter 15

exercise 15.1

1 **a** 1 **b** 1 **c** 5 764 801
d 16 **e** 10 000 000 **f** 13 122
g 1 000 000 **h** 64

2 **a** $\frac{1}{9}$ **b** $\frac{1}{64}$ **c** $\frac{1}{100\,000}$ **d** 9
e 16 **f** $\frac{27}{64}$ **g** $\frac{16}{81}$ **h** 32

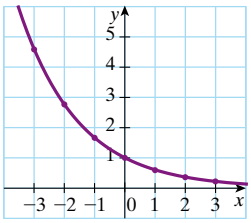
3 **a** 0.122 **b** 4.827 **c** 0.056 **d** 4.300
e 23.120 **f** 1.536 **g** 230.660 **h** 1907.349

exercise 15.2

1 **a** **i** $y = 0.6^x$

x	−3	−2	−1	0	1	2	3
y	4.63	2.78	1.67	1	0.6	0.36	0.22

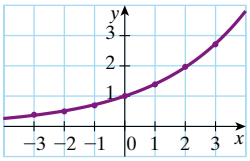
ii



b **i** $y = 1.4^x$

x	−3	−2	−1	0	1	2	3
y	0.36	0.51	0.71	1	1.4	1.96	2.74

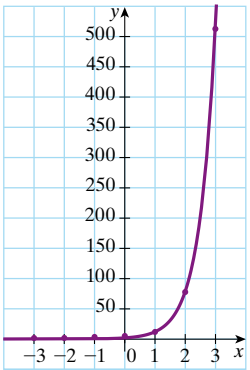
ii



c **i** $y = 8^x$

x	−3	−2	−1	0	1	2	3
y	0.00	0.02	0.13	1	8	64	512

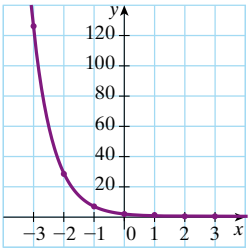
ii



d **i** $y = 0.2^x$

x	−3	−2	−1	0	1	2	3
y	125	25	5	1	0.2	0.04	0.01

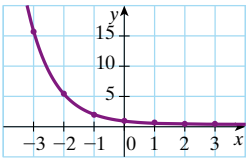
ii



e **i** $y = 2.5^{-x}$

x	−3	−2	−1	0	1	2	3
y	15.63	6.25	2.5	1	0.4	0.16	0.06

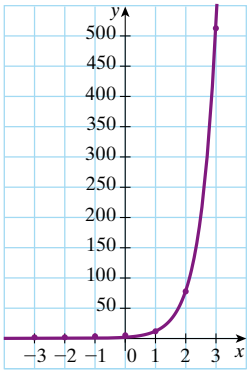
ii



f **i** $y = 0.125^{-x}$

x	−3	−2	−1	0	1	2	3
y	0.00	0.02	0.13	1	8	64	512

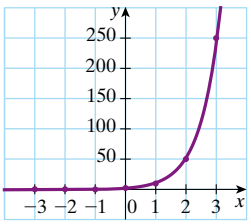
ii



2 **a** **i** $y = 2 \times 5^x$

x	−3	−2	−1	0	1	2	3
y	0.02	0.08	0.4	2	10	50	250

ii



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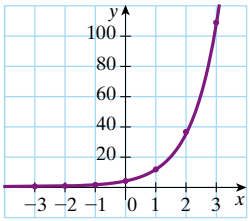
b i $y = 4 \times 3^x$

x	−3	−2	−1	0	1	2	3
y	0.15	0.44	1.33	4	12	36	108

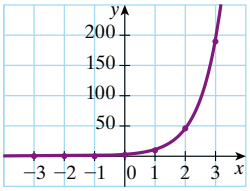
ii



c i $y = 3 \times 4^x$

x	−3	−2	−1	0	1	2	3
y	0.05	0.19	0.75	3	12	48	192

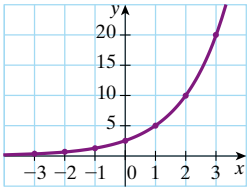
ii



d i $y = 2.5 \times 2^x$

x	−3	−2	−1	0	1	2	3
y	0.31	0.63	1.25	2.5	5	10	20

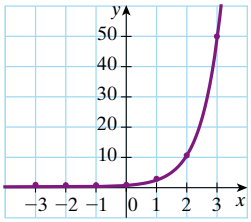
ii



e i $y = 0.4 \times 5^x$

x	−3	−2	−1	0	1	2	3
y	0.00	0.02	0.08	0.4	2	10	50

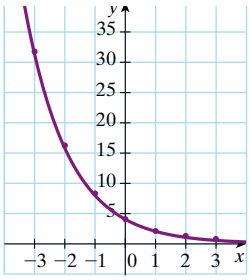
ii



f i $y = 4 \times 0.5^x$

x	−3	−2	−1	0	1	2	3
y	32	16	8	4	2	1	0.5

ii



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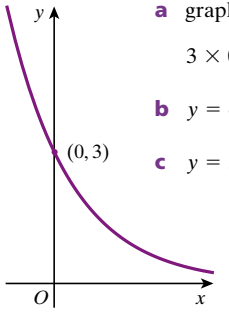


a graphs are identical;

$3 \times 0.5^x = 3 \times \left(\frac{1}{2}\right)^x = 3 \times 2^{-x}$

b $y = 4 \times \left(\frac{1}{3}\right)^x$

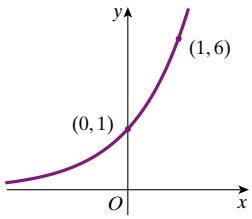
c $y = 5 \times 4^{-x}$

4 c and f; $8^x = \left(\frac{1}{8}\right)^{-x} = 0.125^{-x}$

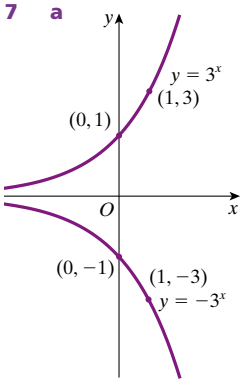
5

a	**i** 5	**ii** 3	**iii** $y = 5 \times 3^x$
b	**i** 32	**ii** $\frac{1}{2}$	**iii** $y = 32 \times \left(\frac{1}{2}\right)^x$
c	**i** 3	**ii** 2	**iii** $y = 3 \times 2^x$
d	**i** 16	**ii** $\frac{3}{2}$	**iii** $y = 16 \times \left(\frac{3}{2}\right)^x$
e	**i** 64	**ii** $\frac{3}{4}$	**iii** $y = 64 \times \left(\frac{3}{4}\right)^x$
f	**i** 240	**ii** 0.9	**iii** $y = 240 \times 0.9^x$

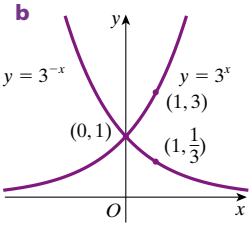
6



7 a



b



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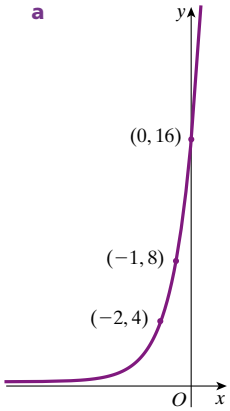
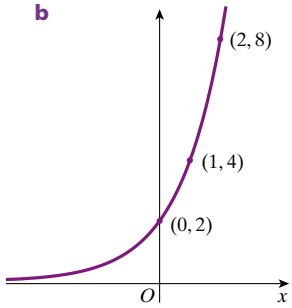
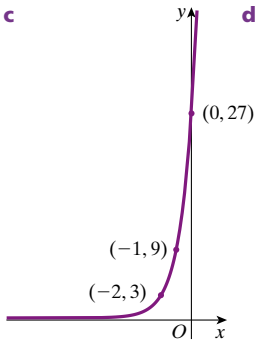
8 **a**



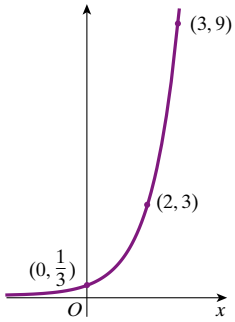
b



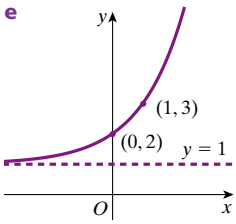
c



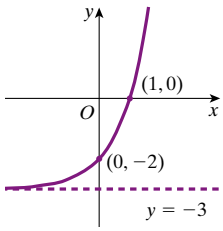
d



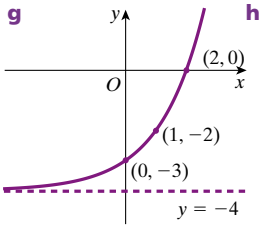
e



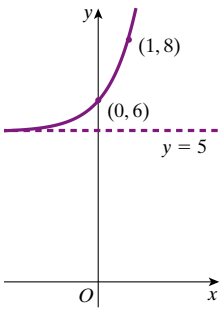
f



g



h



9 **a** A **b** C **c** B **d** D

10 **a** B **b** D **c** C **d** A

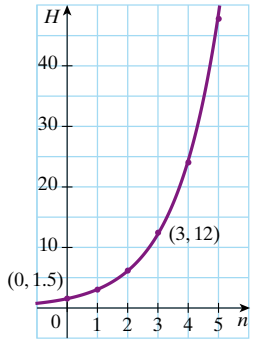
11 $y = 4 \times 3^x$

exercise 15.3

1 **a**

n	1	2	3	4	5
H	3	6	12	24	48

b



c $H = 1.5 \times 2^n$ **d** 1536 m

2 **a** 1.042 **b** $A = 6000 \times 1.042^n$
c \$6788.20

3 **a** 0.8125 **b** $V = 990 \times 0.8125^n$
c \$431

4 **a** 110% **b** 1.1
c see below **d** $P = 30000 \times 1.1^n$
e 201 825 **f** unlikely

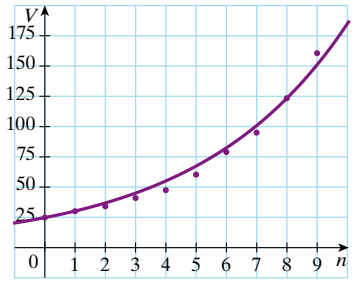
5 27 550

6 About 40 634 000

7 About 8 486 200

8 About 1506 kangaroos

9 **a** 1.22
b $V = 25 \times 1.22^n$
c, d



The function is a reasonably good fit.

e \$1334

4 **c**

Year	0	1	2	3	4	5	6	7	8
Population	30 000	33 000	36 300	39 930	43 923	48 315	53 147	58 462	64 308

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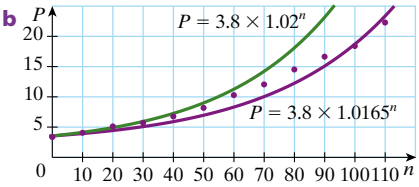
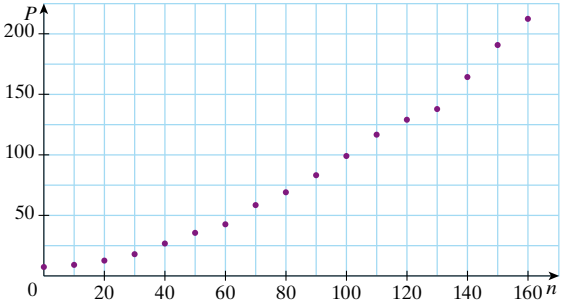
10 **a, b**



c 1.65%
d 23.0 million; slightly higher than actual
e 44.2 million
f assumes exponential growth continues indefinitely

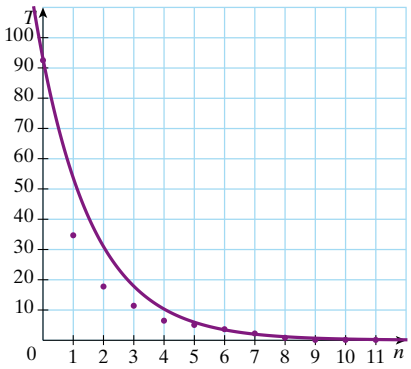
11 **a**



b $P = 8.3 \times 1.21^n$ (answers may vary)
c 477.6 million; actual is about 310 million; scatter plot suggests that an exponential growth model over this period of time is not a very good fit.

12 **a** $T = 93.7 \times 0.613^n$, where T, in millions of dollars, is the takings and n is the number of weekends after the opening weekend (answers will vary).

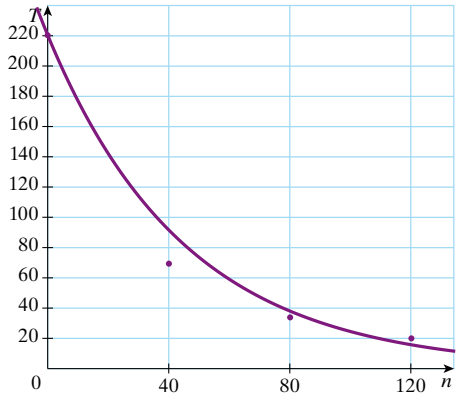
b



c about \$702 000 (model is not a good fit)

13 **a** $T = 226 \times 0.4^n$, where T, in millions of dollars, is the takings and n is the number of weekends after the opening weekend (answers will vary).

b



c about \$24 000

14 **a** 0.986 (ratios vary from 0.983 to 0.988)
b $T = 56.0 \times 0.986^m$
c 45.3°; predicted value, correct to 3 decimal places, is the same as the actual value
d 24.0°
e 4 hours 46 minutes

exercise 15.4

1 **a** $2^6 = 64$ **b** $7^2 = 49$ **c** $10^4 = 10\,000$
d $5^3 = 125$ **e** $3^3 = 27$ **f** $3^5 = 243$
g $7^3 = 343$ **h** $6^3 = 216$ **i** $10^6 = 1\,000\,000$

2 **a** $\log_5 125 = 3$ **b** $\log_{10} 100\,000 = 5$
c $\log_4 16 = 2$ **d** $\log_2 16 = 4$
e $\log_7 2401 = 4$ **f** $\log_4 64 = 3$
g $\log_{49} 7 = 0.5$ **h** $\log_3 27 = 3$
i $\log_5 5 = 1$

3 **a** $10^{-4} = 0.0001$ **b** $2^{-1} = \frac{1}{2}$
c $2^{-2} = \frac{1}{4}$ **d** $3^{-4} = \frac{1}{81}$
e $10^{-7} = 0.0000001$ **f** $6^{-2} = \frac{1}{36}$
g $2^{-5} = \frac{1}{32}$ **h** $2^{-4} = \frac{1}{16}$
i $2^{-2} = 0.25$ **j** $7^{-1} = \frac{1}{7}$
k $5^{-1} = \frac{1}{5}$ **l** $10^{-3} = \frac{1}{1000}$
m $10^{-2} = 0.01$ **n** $5^{-2} = \frac{1}{25}$
o $3^{-3} = \frac{1}{27}$

4 **a** $\log_5 \frac{1}{625} = -4$ **b** $\log_9 \frac{1}{81} = -2$
c $\log_2 \frac{1}{16} = -4$ **d** $\log_{10} 0.001 = -3$
e $\log_7 \frac{1}{343} = -3$ **f** $\log_3 \frac{1}{729} = -6$
g $\log_2 \frac{1}{64} = -6$ **h** $\log_4 \frac{1}{64} = -3$
i $\log_8 \frac{1}{64} = -2$ **j** $\log_{10} 0.00001 = -5$

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k $\log_6\frac{1}{36} = -2$ **l** $\log_{25}\frac{1}{5} = -\frac{1}{2}$

m $\log_{10}0.01 = -2$ **n** $\log_2\frac{1}{8} = -3$

o $\log_3\frac{1}{27} = -3$

5 **a** 0 **b** 0 **c** 0 **d** 1 **e** 1 **f** 1

6 **a** $\log_3 8$ **b** $\log_2 72$ **c** $\log_5 30$
d $\log_7 15$ **e** $\log_3 45$ **f** $\log_{10} 42$
g $\log_2 7.5$ **h** $\log_3 8$ **i** $\log_3 4$
j $\log_5 28$ **k** $\log_2 3$ **l** $\log_5 20.4$

7 **a** $\log_2 9$ **b** $\log_5 15$ **c** $\log_2 5$
d $\log_3 6$ **e** $\log_4 7$ **f** $\log_5 4$
g $\log_2 10$ **h** $\log_3 4$ **i** $\log_3 6$
j $\log_{10} 13$ **k** $\log_2 3$ **l** 1

8 **a** $3\log_3 2$ **b** $8\log_4 3$ **c** $5\log_{10} 4$
d $4\log_3 5$ **e** $-3\log_3 2$ **f** $-\log_{10} 4$
g 5 **h** -5 **i** 5
j -1.6 **k** 2.4 **l** 0.5

9 **a** 5 **b** 3 **c** -3
d $\frac{1}{2}$ **e** 3 **f** 4
g -1 **h** -2 **i** $\frac{1}{2}$
j $\frac{1}{2}$ **k** $\frac{1}{4}$ **l** $\frac{3}{2}$

10 **a** 3 **b** 3 **c** 5 **d** 3
e 3 **f** 2 **g** -4 **h** -3
i 1 **j** 0 **k** -4 **l** 2

11 **a** $\frac{9}{2}$ **b** $\frac{41}{6}$

exercise 15.5

1 **a** 3 **b** 4 **c** 9 **d** -2
e $\frac{3}{2}$ **f** 5 **g** $-\frac{3}{2}$ **h** 5
i 4 **j** $\frac{13}{6}$ **k** $-\frac{3}{4}$ **l** 2

2 **a** 3.322 **b** 2.727 **c** 1.544 **d** 2.292
e 2.104 **f** 6.392 **g** 4.419 **h** 2.375

3 **a** 16 **b** 125 **c** 128 **d** $\frac{1}{100}$
e $\frac{1}{16}$ **f** $\frac{1}{7}$ **g** 1 **h** $\frac{1}{2}$
i 27 **j** 28 **k** 35 **l** 13

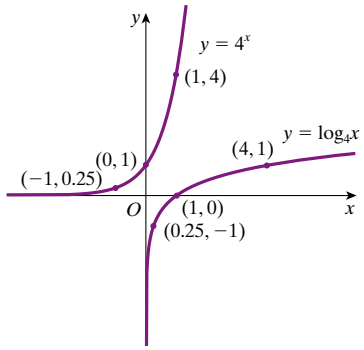
4 **a** 4 **b** 5 **c** 9 **d** 2
e 5 **f** 3 **g** 9 **h** 81
i 5 **j** 11 **k** 8 **l** 2

5 See question **2** answers.

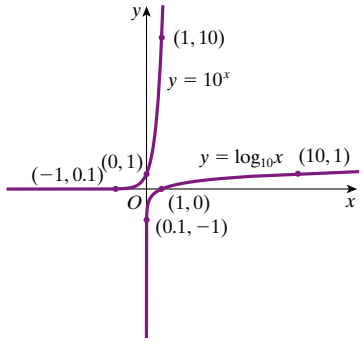
6 $\frac{1}{3}$

exercise 15.6

1 **a**



b



2 $y = \log_2(x - 3)$

exercise 15.7

1 **a** 3; acidic **b** 9; alkaline **c** 7; neutral
d 5; acidic **e** 13; alkaline **f** 2; acidic
g 6; acidic **h** 14; alkaline

2 **a** 4.52 **b** 3.30 **c** 7.85 **d** 10.27
e 7.15 **f** 5.40 **g** 8.74 **h** 3.60

3 10^{-4}(moles per litre)

4 **a** 10 **b** 10^{-10}(moles per litre)

5 **a** 0, 20, 60, 70, 80, 100, 110, 130
b No; 10^4 times the intensity.
c **i** 10 times **ii** 10^5 times **iii** 10^3 times
d Rules vary, but most require earplugs for continuous exposure to decibel levels of 80–90 decibels or higher.

6 **a** 1000 times stronger
b about 500 times stronger
c about 2500 times stronger

revision

1 A **2** C **3** C **4** D **5** C

6 **a** 2400 **b** 351

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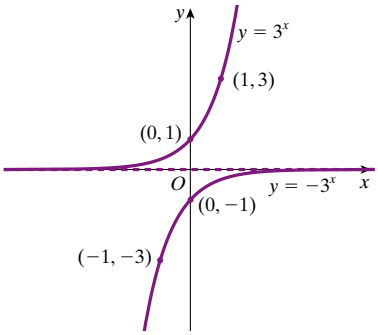
8



9 **a** $2^8 = 256$ **b** $10^{-3} = 0.001$
c $5^{-3} = \frac{1}{125}$ **d** $a^c = b$

10 **a** $\log_2 1024 = 10$ **b** $\log_7 \frac{1}{343} = -3$
c $\log_4 0.5 = -0.5$ **d** $\log_m p = n$

11 **a** 7 **b** $\frac{1}{2}$ **c** -2 **d** $\frac{1}{3}$

12 **a** $\log_a = 20$ **b** $2\log_{10}2$ $(= \log_{10}4)$

13 **a** 1 **b** -1

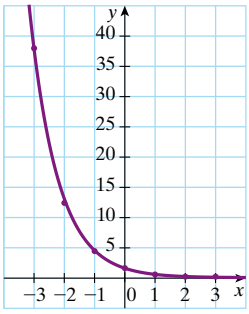
14 **a** $-\frac{5}{2}$ **b** -2

15 **a** 2 **b** 84

16 **a** **i** $y = 0.3^x$

x	-3	-2	-1	0	1	2	3
y	37.04	11.11	3.33	1	0.3	0.09	0.03

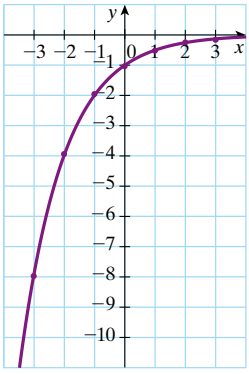
ii



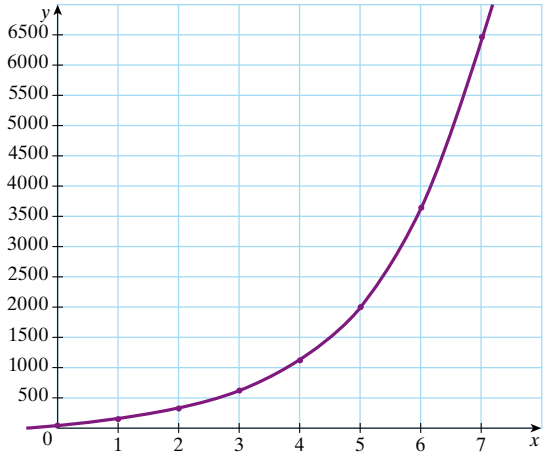
b **i** $y = -2^{-x}$

x	-3	-2	-1	0	1	2	3
y	-8	-4	-2	-1	-0.5	-0.25	-0.13

ii



17 **a** 1.8
b $P = 105 \times 1.8^n$
c, d



Exponential function graph is a very good fit.
e \$121 500 (to the nearest hundred dollars)
f Assumes that the growth rate continues over the next 5 years.

Chapter 16

exercise 16.1

1 **a** A polynomial; powers of x are positive whole numbers.
b A polynomial; powers of x are positive whole numbers.
c Not a polynomial; the term $x^{0.5}$ has a power that is not a whole number.
d Not a polynomial; each of the terms $\frac{1}{x^3} = x^{-3}$ and $\frac{2}{x^2} = 2x^{-2}$ has a power that is not a whole number.
e A polynomial; powers of x are positive whole numbers.
f A polynomial; powers of x are positive whole numbers.

2 **a** **i** 4 **ii** $3x^4$ **iii** 3 **iv** 7
b **i** 3 **ii** x^3 **iii** 1 **iv** -10
c **i** 3 **ii** $-5x^3$ **iii** -5 **iv** 4
d **i** 4 **ii** $-2x^4$ **iii** -2 **iv** 0
e **i** 1 **ii** $-7x$ **iii** -7 **iv** 11
f **i** 3 **ii** $-8x^3$ **iii** -8 **iv** 0

3 a, d are quartic; b, c, f are cubic; e is linear

4 **a** 24 **b** 62 **c** 2 **d** 144
e 0 **f** 6 **g** 16 **h** 21

5 **a** $-x + 12$ **b** $-8x - 7$
c -2 **d** $-7x$
e $2x^2 - 2x - 8$ **f** $2x^2 + x - 18$
g $2x^3 - 2x^2 + 6x + 7$ **h** $x^3 + 7x^2 - 6x$

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i $2x^3 + 11x^2 + 3x - 9$ j $x^3 - 8x + 6$
k $x^4 + 3x^3 - x^2 - 16x$ l $x^4 + x^3 - 5x^2$

6 a $5x + 2$ b -15
c $6x - 14$ d $4x^2 + 17x$
e $12x + 16$ f $6x^2 - 7x - 12$
g $8x^2 - 10x - 1$ h $x^3 + x^2 - 8x + 12$
i $-11x^2 + 3x + 7$ j $x^3 - 6x^2 + 8x + 2$
k $x^4 + x^3 + x^2 + 6x$ l $x^4 - x^3 + x^2 + 10$

7 $a = -5, b = 5, c = 4, d = -14$

exercise 16.2

1 a $-3x^3 + 6x^2 - 9x$
b $5x^4 - 25x^3 + 35x^2 - 60x$
c $3x^4 - 15x^3 + 18x^2$
d $-6x^3 - 12x^2 + 48x$
e $-3x^5 + 12x^4 - 3x^3 + 9x^2$
f $x^3 - 5x^2 + 13x - 21$
g $2x^3 - 9x^2 - 11x + 60$
h $x^4 + 3x^3 - 8x^2 - 24x$
i $2x^4 + x^3 - 7x^2 + 5x - 1$
j $x^4 - 6x^3 + 5x^2 + 12x$
k $x^4 - 16$
l $x^4 - 14x^2 - 15x$

2 a $x^3 - 5x^2 - x + 5$ b $x^3 + x^2 - 4x - 4$
c $x^3 + 7x^2 - 9x - 63$ d $x^3 + 3x^2 - 6x - 8$
e $x^3 + 4x^2 - 11x - 30$ f $x^3 + 27$
g $x^3 - 27$ h $x^4 - 8x$
i $x^5 + 2x^4 - x^3 + 4x^2 - 6x$

3 a $x^3 - 4x^2 - 3x + 18$
b $x^3 + 3x^2 - 9x + 5$
c $x^3 + 8x^2 + 21x + 18$
d $x^3 - 3x^2 + 3x - 1$
e $x^3 + 3x^2 + 3x + 1$
f $x^3 - 2x^2 - 4x + 8$
g $x^4 + 2x^3 - 3x^2 - 4x + 4$
h $x^4 - 4x^3 - 2x^2 + 12x + 9$
i $16x^4 - 72x^2 + 81$
j $x^3 - 2x + 1$
k $x^3 - 8x - 8$
l $x^4 - 16$

4 $a = 3, b = -1, c = 5$

exercise 16.3

1 a $x - 11$, remainder 0
b $x + 15$, remainder -18
c $2x + 7$, remainder 6
d $3x - 5$, remainder 0
e $x^2 - 16x + 50$, remainder -10
f $x^2 - 13x + 45$, remainder -57
g $x^2 - 7$, remainder 0
h $x^2 - 6x + 18$, remainder -48
i $x^3 - 9x + 34$, remainder -98
j $x^3 - 5x - 2$, remainder -19

2 Remainder is 0, so $x - 4$ is a factor of $x^2 - 7x + 12$.

3 a $4x^2 - x - 1$, remainder 6
b $3x^3 - 5x^2 + 10x - 25$, remainder 39
c $2x^3 + 6x^2 + 13x + 32$, remainder 100
d $x^3 - 4x^2 + 6x - 6$, remainder 1

4 a i $x^2 - x - 6$
ii remainder zero so $(x - 1)$ is a factor.
b i $x^2 - 7x + 12$
ii remainder zero so $(x + 3)$ is a factor.
c i $x^2 + 9x + 20$
ii remainder zero so $(x - 2)$ is a factor.
d i $x^2 + 8x + 15$
ii remainder zero so $(x - 2)$ is a factor.
e i $x^2 + 11x + 30$
ii remainder zero so $(x - 2)$ is a factor.
f i $x^2 - 2x - 8$
ii remainder zero so $(x + 1)$ is a factor.
g i $x^2 - 5x - 6$
ii remainder zero so $(x + 2)$ is a factor.
h i $x^2 - 4$
ii remainder zero so $(x - 3)$ is a factor.
i i $x^2 + 5x - 24$
ii remainder zero so $(x + 4)$ is a factor.
j i $x^2 + 8x + 12$
ii remainder zero so $(x - 4)$ is a factor.
k i $x^2 - 3x - 4$
ii remainder zero so $(x + 3)$ is a factor.
l i $x^2 + 16x + 48$
ii remainder zero so $(x - 3)$ is a factor.

5 $a = 210$

exercise 16.4

1 a 0 b 5 c -24 d 36

2 12

3 a i $P(-3) = 0$ ii $x^2 - 7x + 12$
iii $(x - 4)(x - 3)$
iv $(x - 4)(x - 3)(x + 3)$
b i $P(4) = 0$ ii $x^2 + 8x + 12$
iii $(x + 2)(x + 6)$
iv $(x - 4)(x + 2)(x + 6)$
c i $P(-4) = 0$ ii $x^2 + 5x - 24$
iii $(x - 3)(x + 8)$
iv $(x - 3)(x + 4)(x + 8)$
d i $P(2) = 0$ ii $x - 7x - 30$
iii $(x - 10)(x + 3)$
iv $(x - 10)(x - 2)(x + 3)$

4 a $(x - 3)(x - 1)(x + 2)$
b $(x - 4)(x - 3)(x + 3)$
c $(x - 2)(x + 4)(x + 5)$
d $(x - 2)(x + 3)(x + 5)$
e $(x - 2)(x + 5)(x + 6)$
f $(x - 4)(x + 1)(x + 2)$
g $(x - 6)(x + 1)(x + 2)$
h $(x - 3)(x - 2)(x + 2)$
i $(x - 3)(x + 4)(x + 8)$
j $(x - 4)(x + 2)(x + 6)$
k $(x - 4)(x + 1)(x + 3)$
l $(x - 3)(x + 4)(x + 12)$

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5 **a** $(x - 7)(x + 2)(x + 4)$
b $(x - 4)(x - 2)(x + 5)$
c $(x - 5)(x - 3)(2x - 5)$
d $(2x - 3)(x + 1)(x + 3)$
e $(x + 3)(x + 4)(2x + 1)$
f $(x - 2)(x + 5)(3x + 1)$
g $(x - 3)(3x - 4)(x + 2)$
h $(x - 2)(x + 6)(4x + 1)$

6 **a** $(x - 3)(x - 2)(x - 1)^2$
b $(x - 3)(x - 1)(x + 2)(x + 5)$
c $(x - 7)(x - 3)(x + 1)(x + 3)$
d $(x - 3)(x - 2)(x + 2)(x + 8)$

7 $m = 27.375$

8 $m = -14$

9 $m = 21$

10 $m = 32$

exercise 16.5

1 **a** $x = -4, -3, 2$ **b** $x = -2, -1, 1$
c $x = -2, 3, 4$ **d** $x = -2, 0, 2$
e $x = -5, -3, 6$ **f** $x = -4, 0, 8$

2 **a** $x = -6, 0, 2$ **b** $x = -1, 0, 7$
c $x = -4, 0, 5$ **d** $x = -2, 0, 12$

3 **a** $x = -5, 3, 4$ **b** $x = 1, 2, 5$
c $x = -3, 1, 4$ **d** $x = -5, 1, 6$
e $x = -5, -2, 1$ **f** $x = -4, -2, 3$
g $x = -5, -4, -3$ **h** $x = -5, -2, -1$
i $x = -4, -3, 1$ **j** $x = -4, -3, 2$
k $x = 1, 3, 4$ **l** $x = -3, 2, 4$

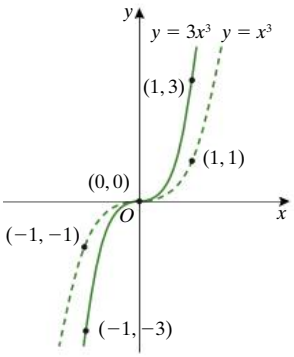
4 $x = -4, -1, 2, 7$

5 **a** 3 **b** 2 **c** 3 **d** 4 **e** 1 **f** 2

6 $x = -4, -2, 0.5, 2$

exercise 16.6

1 **a**



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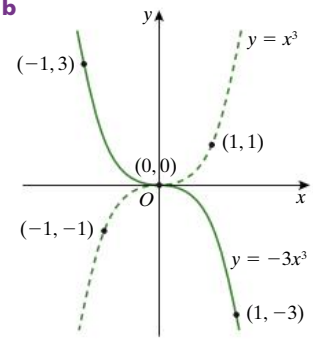

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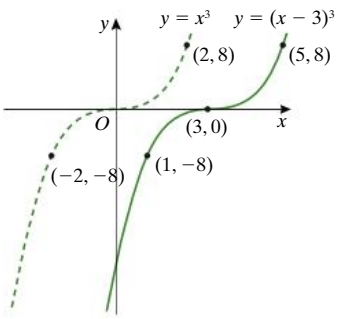

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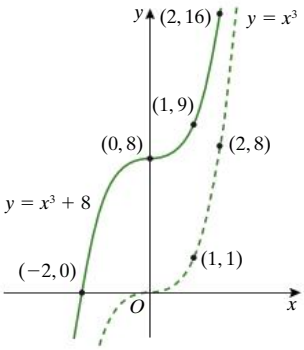
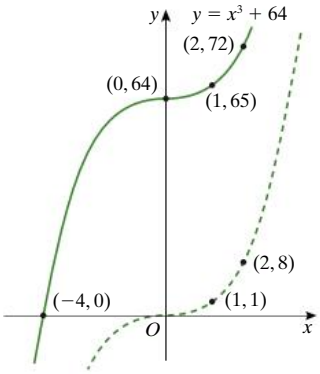


e



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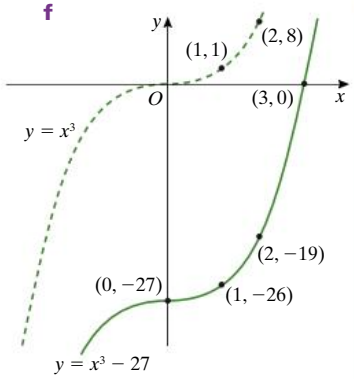
f



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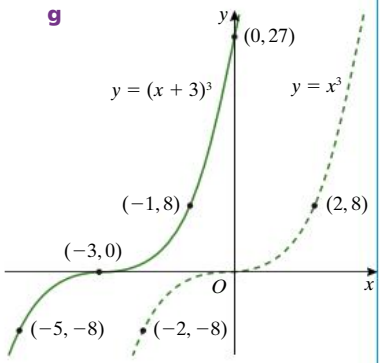

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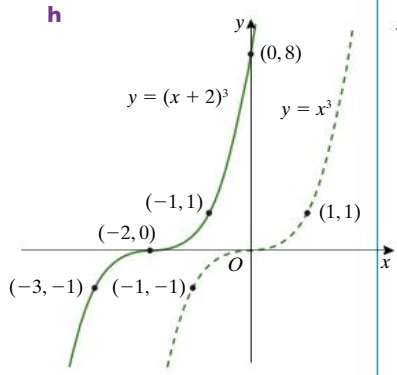
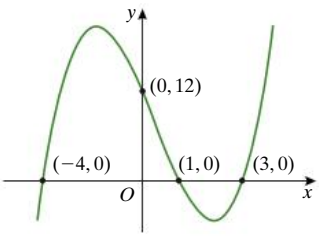
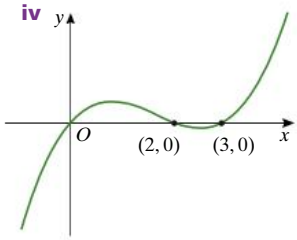


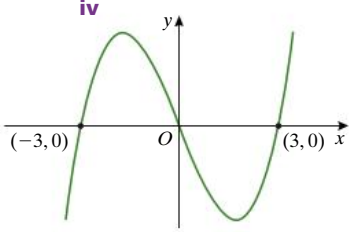
2 **a** **i** $x = -4, 1, 3$ **ii** $y = 12$
iii see page 879
iv



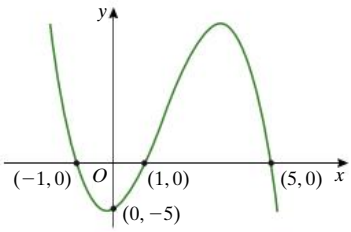
b **i** $x = 0, 2, 3$ **ii** $y = 0$
iii see page 879
iv



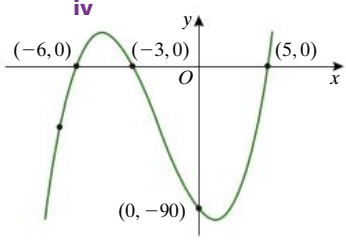
c **i** $x = -3, 0, 3$ **ii** $y = 0$
iii see page 879
iv



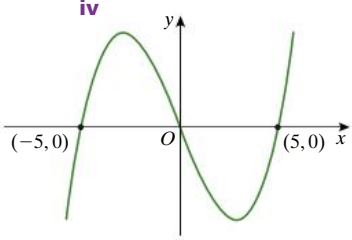
d **i** $x = -1, 1, 5$ **ii** $y = -5$
iii see page 879
iv



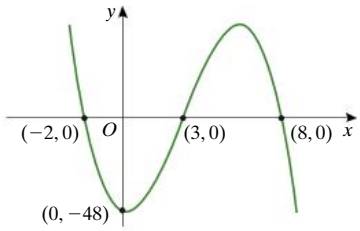
e **i** $x = -6, -3, 5$
ii $y = -90$
iii see page 879
iv



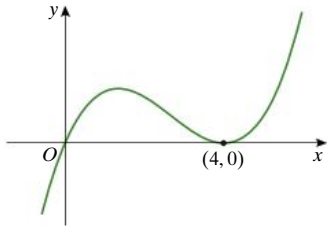
f **i** $x = -5, 0, 5$ **ii** $y = 0$
iii see page 879
iv



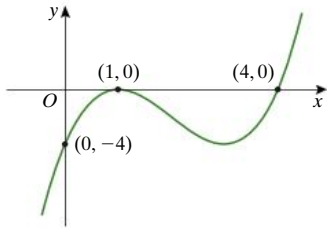
g **i** $x = -2, 3, 8$ **ii** $y = -48$
iii see page 879
iv



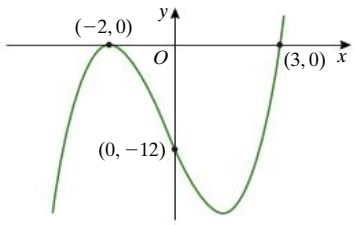
h **i** $x = 0, 4$ **ii** $y = 0$
iii see page 879
iv



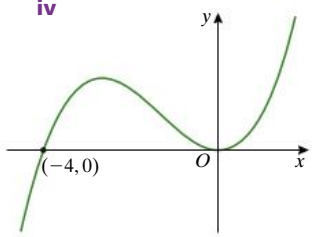
i **i** $x = 1, 4$ **ii** $y = -4$
iii see page 879
iv



j **i** $x = -2, 3$ **ii** $y = -12$
iii see page 879
iv



k **i** $x = -4, 0$ **ii** $y = 0$
iii see page 879
iv



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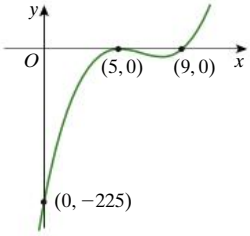
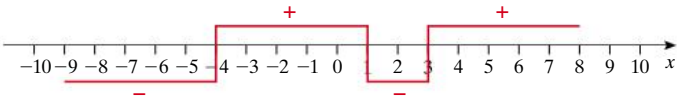
l **i** $x = 5, 9$ **ii** $y = -225$

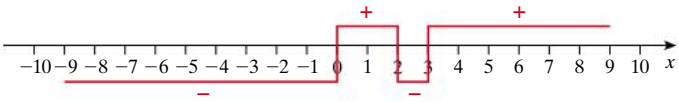
iii see below **iv**



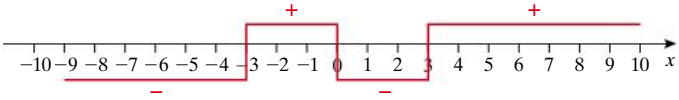
2 **a** **iii**



b **iii**



c **iii**



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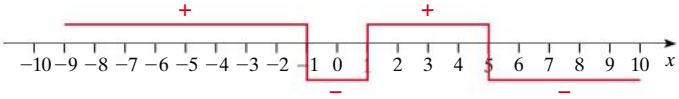

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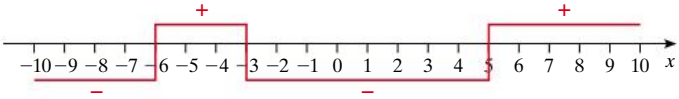
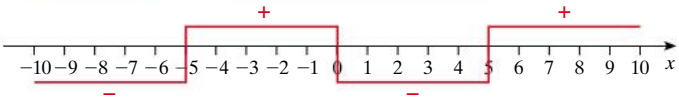
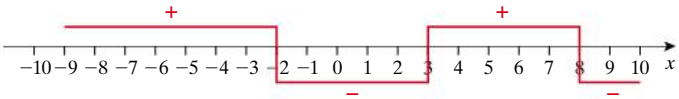


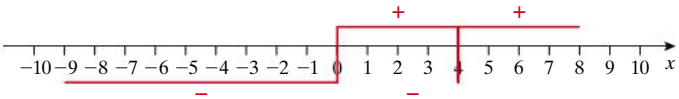
f **iii**



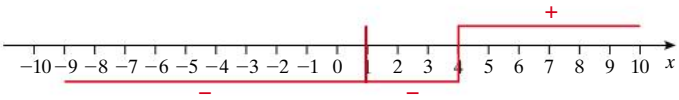
g **iii**



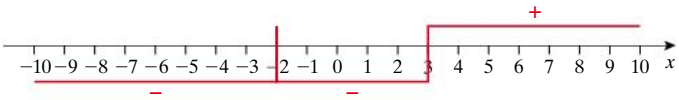
h **iii**



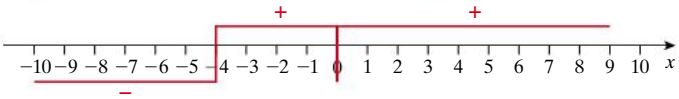
i **iii**



j **iii**



k **iii**



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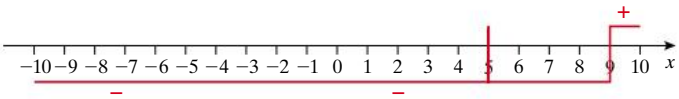

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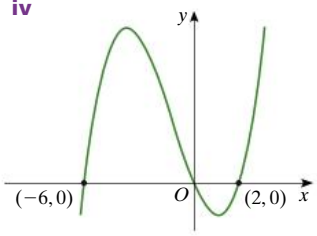
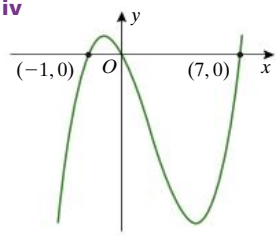
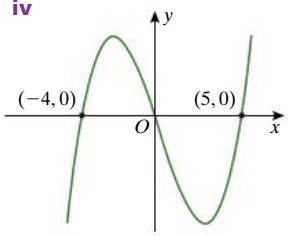
3 **a** **i** $x = -6, 0, 2$ **ii** $y = 0$
iii see bottom of page
iv



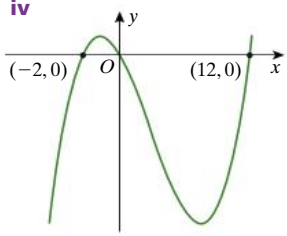
b **i** $x = -1, 0, 7$ **ii** $y = 0$
iii see bottom of page
iv



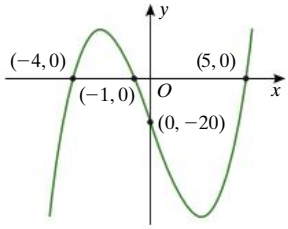
c **i** $x = -4, 0, 5$ **ii** $y = 0$
iii see bottom of page
iv



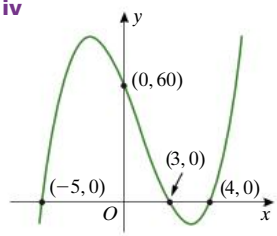
d **i** $x = -2, 0, 12$
ii $y = 0$
iii see bottom of page
iv



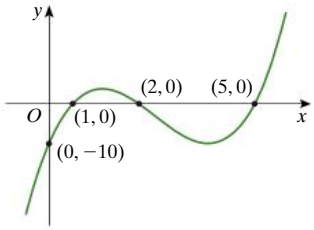
4 **a** $y = (x + 4)(x + 1)(x - 5)$
b see bottom of page
c



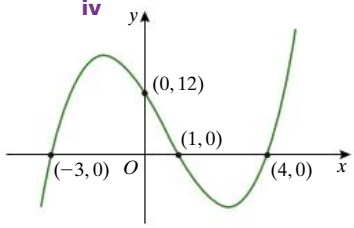
5 **a** **i** $x = -5, 3, 4$ **ii** $y = 60$
iii see page 881
iv



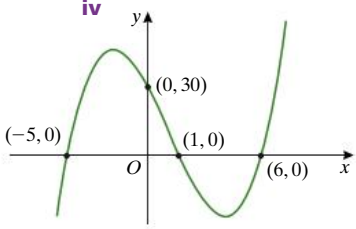
b **i** $x = 1, 2, 5$ **ii** $y = -10$
iii see page 881
iv



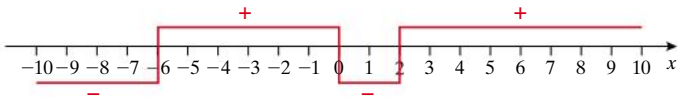
c **i** $x = -3, 1, 4$ **ii** $y = 12$
iii see page 881
iv



d **i** $x = -5, 1, 6$ **ii** $y = 30$
iii see page 881
iv



3 **a** **iii**



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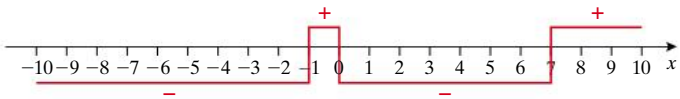

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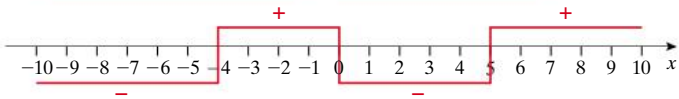

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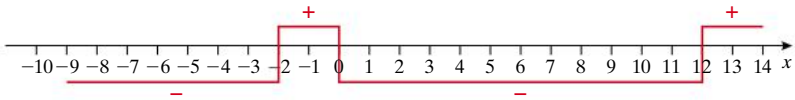

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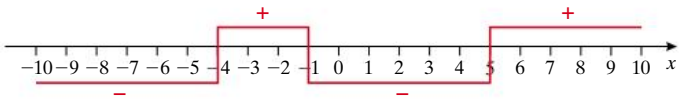

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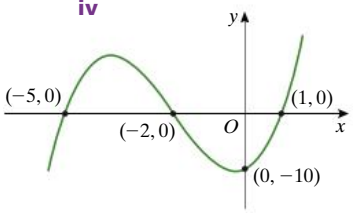
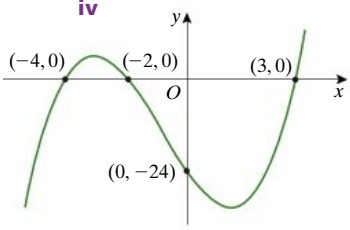
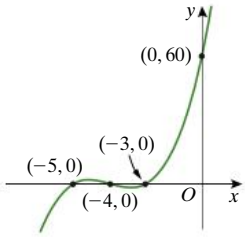
e **i** $x = -5, -2, 1$
ii $y = -10$
iii see this page
iv



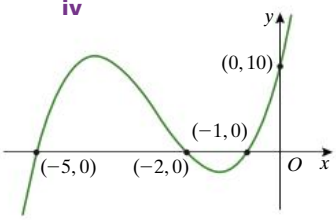
f **i** $x = -4, -2, 3$
ii $y = -24$
iii see this page
iv



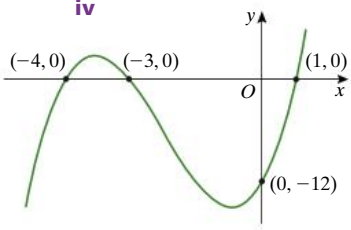
g **i** $x = -5, -4, -3$
ii $y = 60$
iii see this page
iv



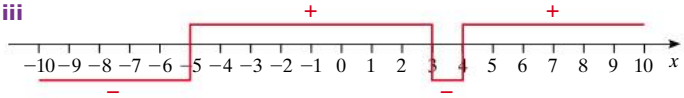
h **i** $x = -5, -2, -1$
ii $y = 10$
iii see this page
iv



i **i** $x = -4, -3, 1$
ii $y = -12$
iii see this page
iv



5 **a** **iii**



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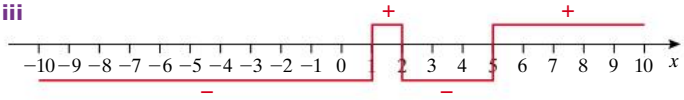

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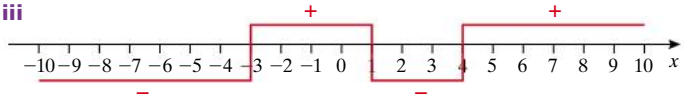

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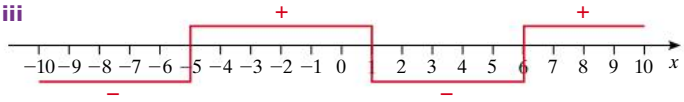

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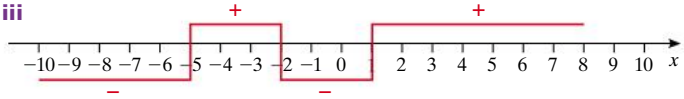

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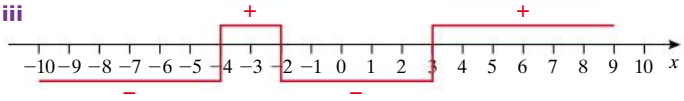

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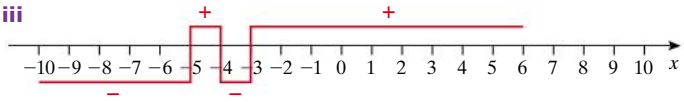

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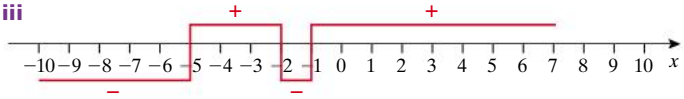

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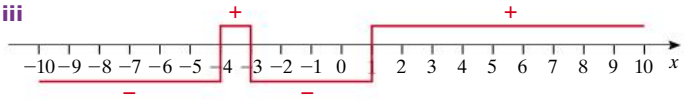

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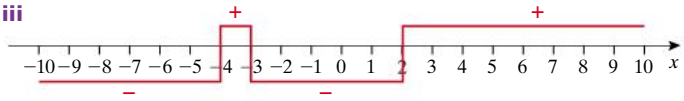

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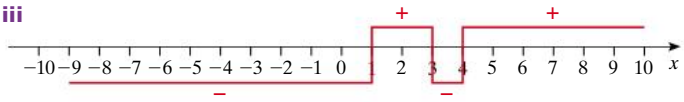

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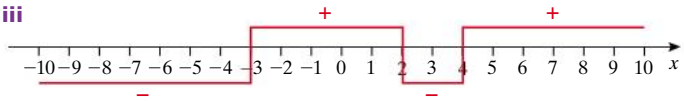

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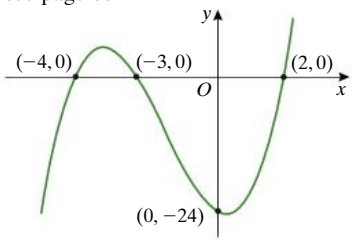
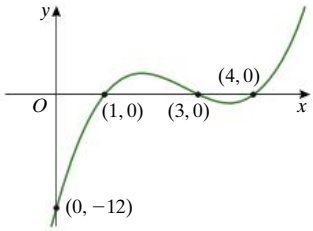
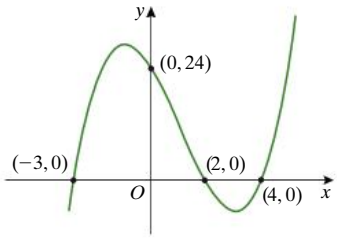
j **i** $x = -4, -3, 2$ **ii** $y = -24$
iii see page 881
iv



k **i** $x = 1, 3, 4$ **ii** $y = -12$
iii see page 881
iv



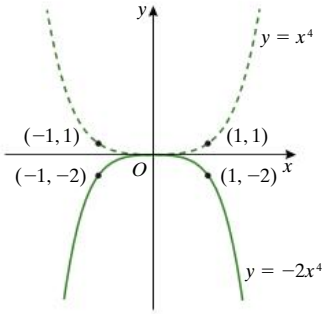
l **i** $x = -3, 2, 4$ **ii** $y = 24$
iii see page 881
iv



6 **a** For example, $y = -x(x + 3)^2$, $y = -2x(x + 3)^2$, $y = -3x(x + 3)^2$.
b Any rule of the form $y = ax(x + 3)^2$, where $a < 0$ is possible, since this has intercepts $(-3, 0)$ and $(0, 0)$.

exercise 16.7

1 **a**



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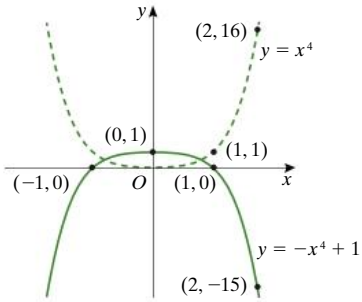

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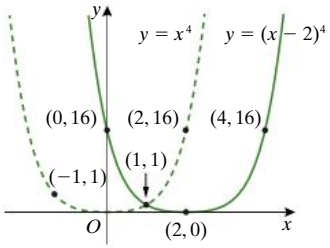

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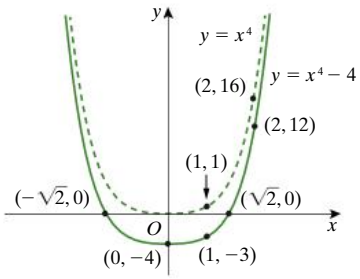
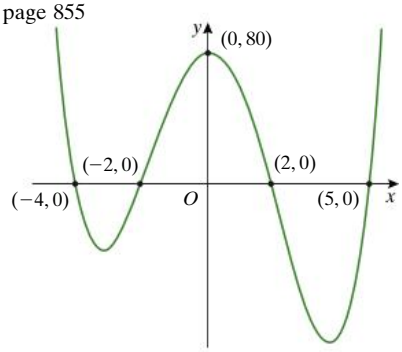
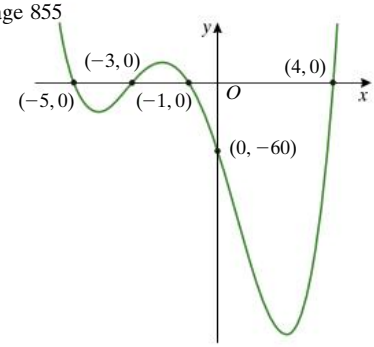


2 **a** **i** $x = -4, -2, 2, 5$ **ii** $y = 80$
iii see page 855
iv



b **i** $x = -5, -3, -1, 4$ **ii** $y = -60$
iii see page 855
iv



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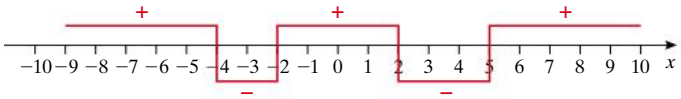
2 a iii



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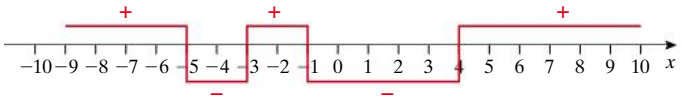

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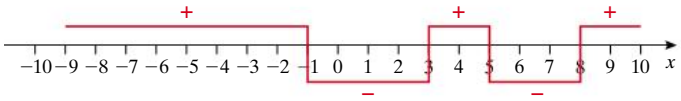

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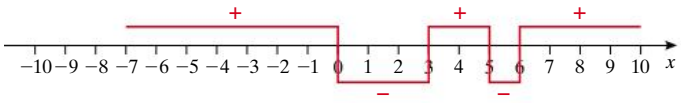

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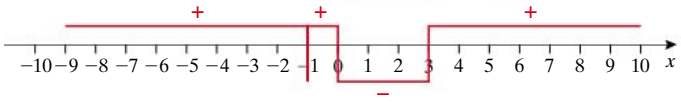

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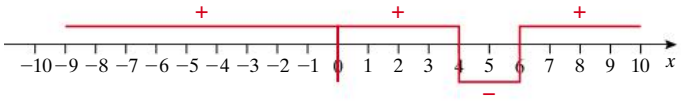

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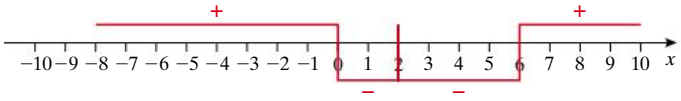

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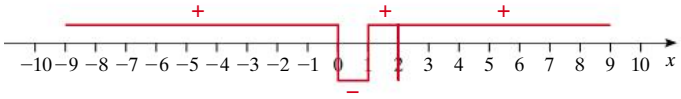

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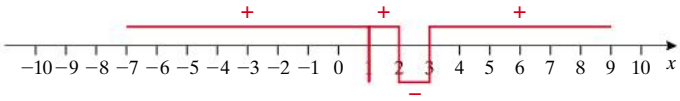

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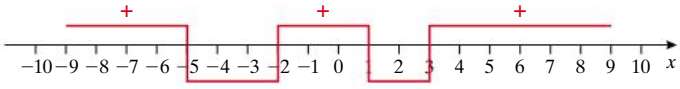

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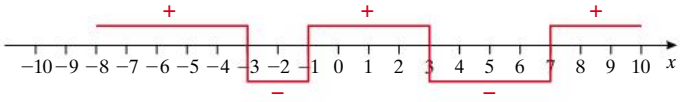

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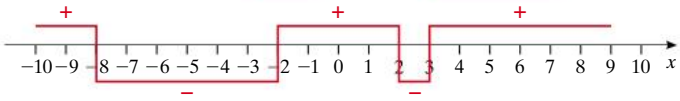

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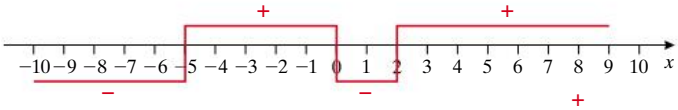

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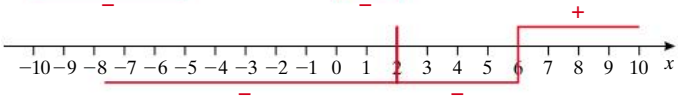

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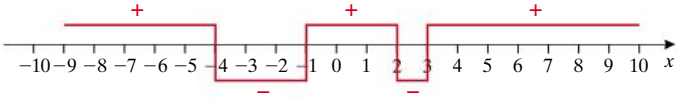

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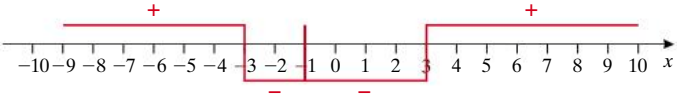

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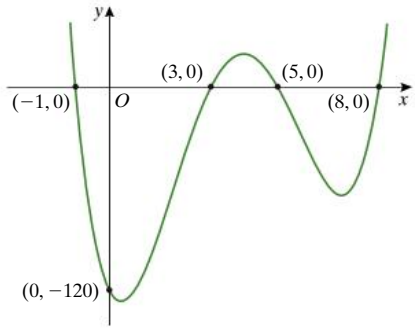
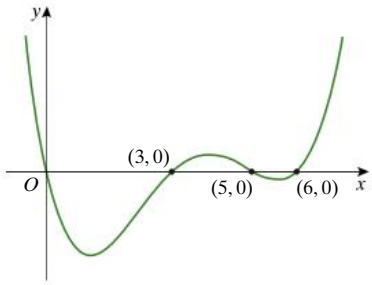
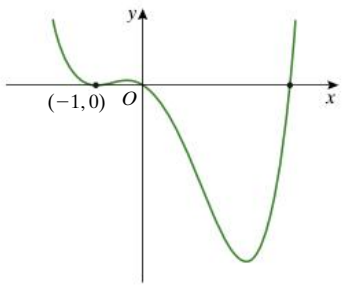
c **i** $x = -1, 3, 5, 8$ **ii** $y = -120$
iii see page 883
iv



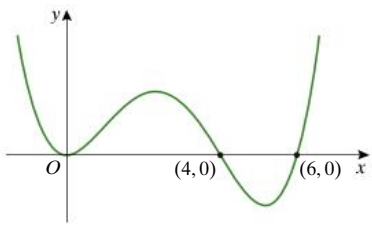
d **i** $x = 0, 3, 5, 6$ **ii** $y = 0$
iii see page 883
iv



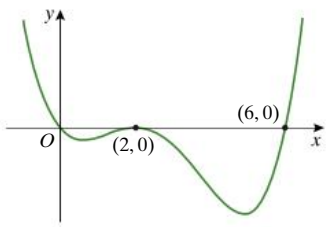
e **i** $x = -1, 0, 3$ **ii** $y = 0$
iii see page 883
iv



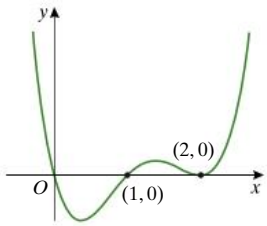
f **i** $x = 0, 4, 6$ **ii** $y = 0$
iii see page 883
iv



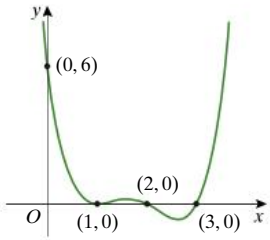
g **i** $x = 0, 2, 6$ **ii** $y = 0$
iii see page 883
iv



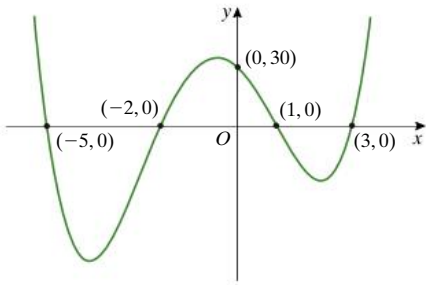
h **i** $x = 0, 1, 2$ **ii** $y = 0$
iii see page 883
iv



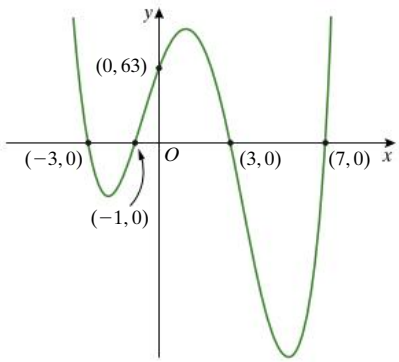
3 **a** **i** $x = 1, 2, 3$ **ii** $y = 6$
iii see page 883
iv



b **i** $x = -5, -2, 1, 3$ **ii** $y = 30$
iii see page 883
iv



c **i** $x = -3, -1, 3, 7$ **ii** $y = 63$
iii see page 883
iv



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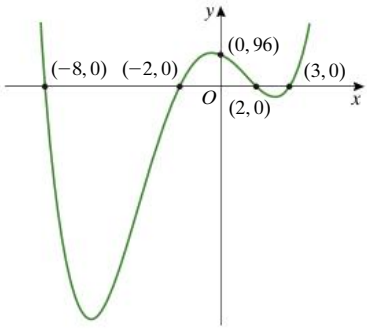
d **i** $x = -8, -2, 2, 3$ **ii** $y = 96$
iii see page 883
iv



4 **a** $y = (x + 3)^2(x - 2)^2$; y-intercept $= 36$
b For example, $y = 2(x + 3)^2(x - 2)^2$. Any rule of the form $y = a(x + 3)^2(x - 2)^2$, where $a > 0$ is possible, since this has intercepts $(-3, 0)$ and $(2, 0)$, and no y-intercept is given.

revision

1 B **2** B **3** D **4** E **5** C

6 **a** A polynomial; powers of x are positive whole numbers.
b Not a polynomial; each of the terms $\frac{1}{x^3} = x^{-3}$, $\frac{3}{x^2} = 3x^{-2}$ and $\frac{4}{x} = 4x^{-1}$ has a power that is not a whole number.
c Not a polynomial; $x^{0.5}$ has a power that is not a whole number.
d A polynomial; powers of x are positive whole numbers.

7 **a** **i** 4 **ii** $-2x^4$ **iii** -2 **iv** 15
b **i** 3 **ii** $6x^3$ **iii** 6 **iv** -2
c **i** 1 **ii** $-2x$ **iii** -2 **iv** 5
d **i** 4 **ii** $-x^4$ **iii** -1 **iv** 0

8 a, d are quartic; b is cubic; c is linear

9 **a** **i** -2 **ii** -14
b **i** 12 **ii** 0

10 **a** **i** $3x^2 - 7x - 4$
ii $x^2 - 7x + 14$
b **i** $x^3 - 7x^2 - 11x + 12$
ii $5x^3 + 7x^2 + 11x - 4$

11 **a** $x^3 - 4x^2 + 7x - 4$
b $2x^4 + 5x^3 - 11x^2 - 20x + 12$

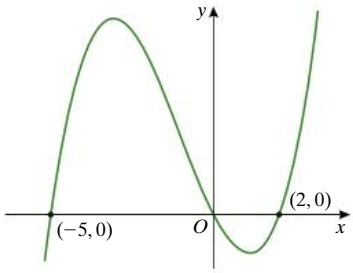
12 **a** $x^3 - 21x - 20$ **b** $2x^3 - 3x^2 - 12x + 20$

13 **a** **i** $x^2 + 7$, remainder 10
ii $x^2 - 5x + 17$, remainder -45
b **i** $x^3 + 3x^2 + 4x + 4$, remainder -11
ii $x^3 - 2x^2 - x - 6$, remainder -11

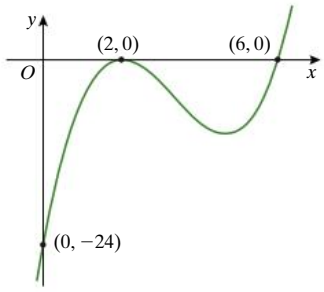
14 **a** $x = -3, 2, 4$ **b** $x = -1, 2$
c $x = -4, 0, 6$ **d** $x = -1$

15 **a** $x(x + 5)(x - 2)$ **b** $y = (x - 2)^2(x - 6)$
c $(x + 4)(x + 1)(x - 2)(x - 3)$
d $y = (x + 3)(x + 1)^2(x - 3)$

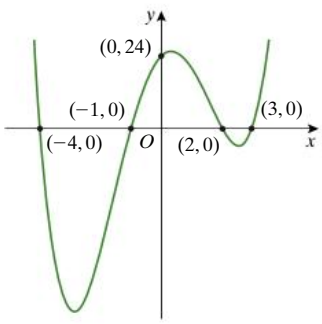
16 **a** **i** $x = -5, 0, 2$ **ii** $y = 0$
iii see page 883
iv



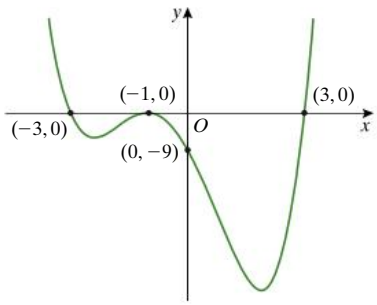
b **i** $x = 2, 6$ **ii** $y = -24$
iii see page 883
iv



c **i** $x = -4, -1, 2, 3$ **ii** $y = 24$
iii see page 883
iv



d **i** $x = -3, -1, 3$ **ii** $y = -9$
iii see page 883
iv



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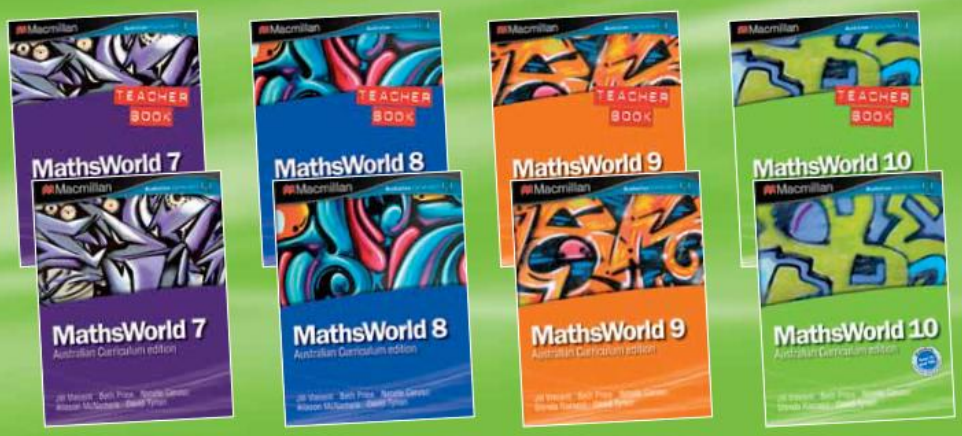

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