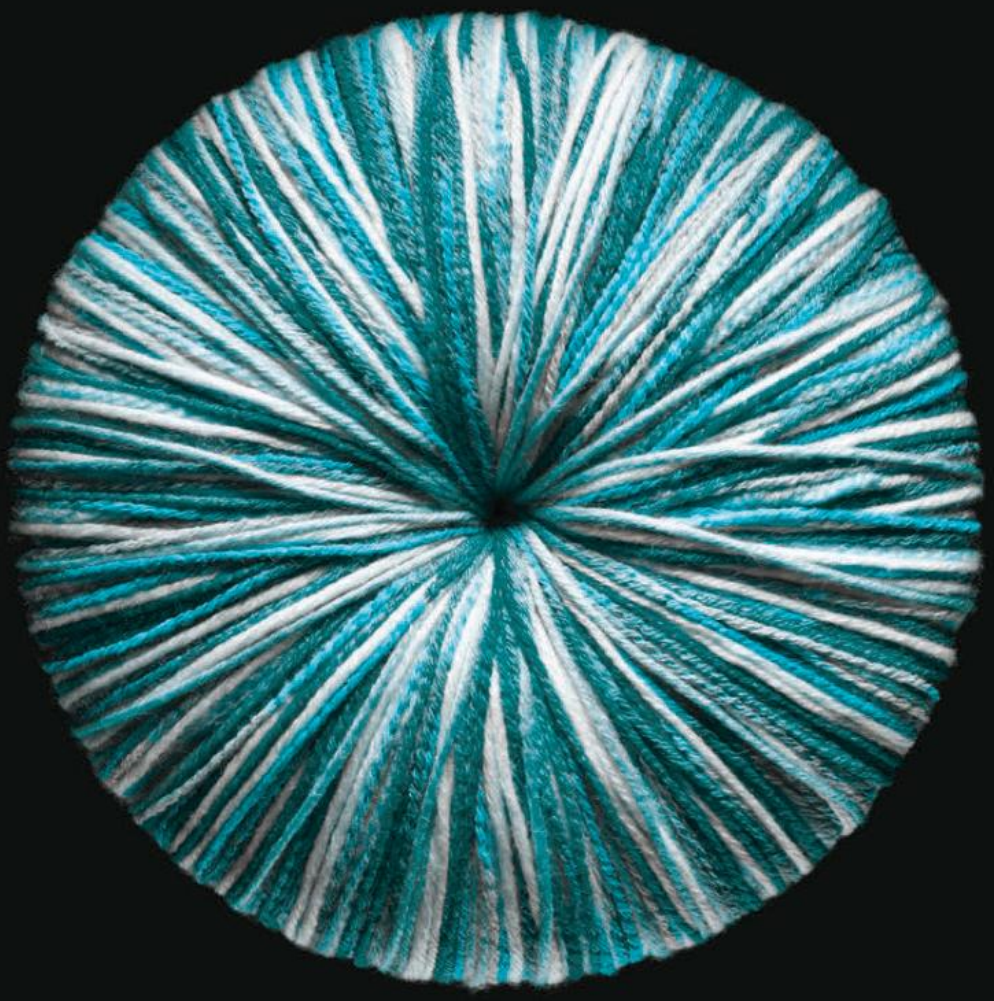
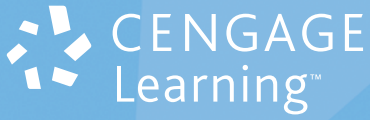
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# NELSON SENIOR MATHS

FOR THE AUSTRALIAN CURRICULUM | **METHODS 11**




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# NELSON SENIOR MATHS

FOR THE AUSTRALIAN CURRICULUM | **METHODS 11**

NELSON
CENGAGE Learning

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**1st Edition**
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There are 8 books for this series currently. These cover the subjects Essential Mathematics, General Mathematics, Mathematical Methods and Specialist Mathematics.

These books have all been written specifically, from scratch, by a national author team for the Senior Australian Curriculum.

There are worked solutions available for purchase, which accompany each of the General, Methods and Specialist Student Books.

There are also ExamView question banks which can be purchased separately. These will contain a large number of multiple choice questions covering each course; General, Methods and Specialist. An ExamView test generator is also provided, so that teachers can construct their own tests for each topic.

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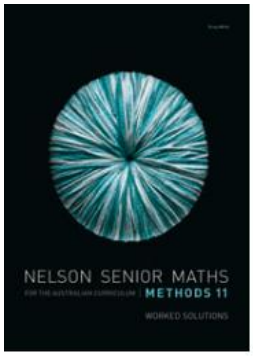
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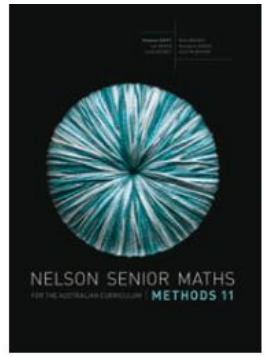
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The tools on the horizontal toolbar allow you to navigate around your eBook and change settings.






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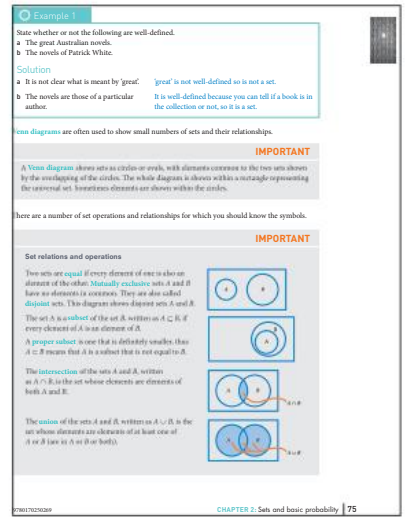
# ABOUT THIS BOOK




A clear outline of chapter contents is provided.

Links to curriculum content descriptions are included.

A Prior learning worksheet is provided, enabling teachers to assess pre-requisite understanding.


Example 1

State whether or not the following are well-defined.
a The great Australian novels.
b The novels of Patrick White.

Solution

a It is not clear what is meant by 'great'. 'great' is not well-defined so is not a set.

b The novels are those of a particular author. It is well-defined because you can tell if a book is in the collection or not, so it is a set.

Venn diagrams are often used to show small numbers of sets and their relationships.

IMPORTANT

A Venn diagram shows sets as circles or ovals, with elements common to the two sets shown by the overlapping of the circles. The whole diagram is shown within a rectangle representing the universal set. Sometimes elements are shown within the circles.

There are a number of set operations and relationships for which you should know the symbols.

IMPORTANT

Set relations and operations

Two sets are equal if every element of one is also an element of the other. Mutually exclusive sets A and B have no elements in common. They are also called disjoint sets. This diagram shows disjoint sets A and B.

The set A is a subset of the set B, written as $A \subseteq B$, if every element of A is an element of B.

A proper subset is one that is definitely smaller, thus $A \subseteq B$ means that A is a subset that is not equal to B.

The intersection of the sets A and B, written as $A \cap B$, is the set whose elements are elements of both A and B.

The union of the sets A and B, written as $A \cup B$, is the set whose elements are elements of at least one of A or B (see it's A or B or both).

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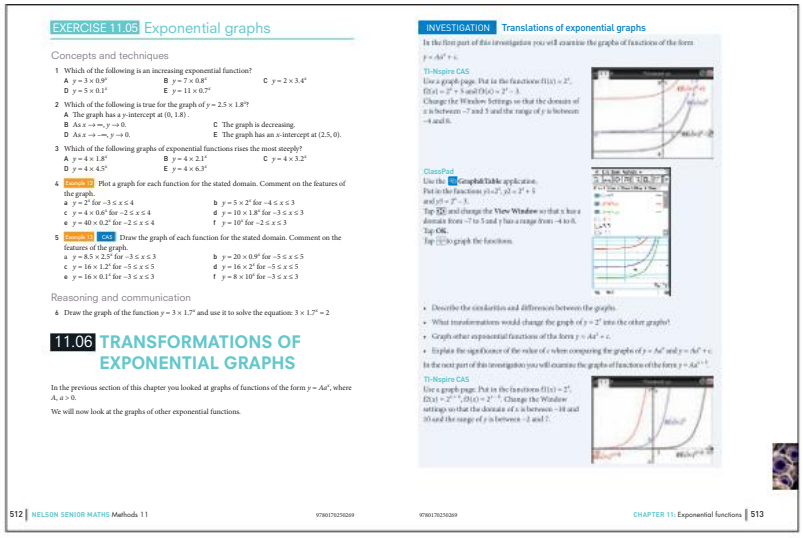
CHAPTER 2: Sets and basic probability 75

Examples are clearly set out with reasoning (in black) and writing (in blue) in separate columns.

Examples are sequenced in a logical order.

There are generally three examples leading to an exercise.

Examples show solutions and steps which guide students through the use of the TI-Nspire and the ClassPad CAS calculators.



EXERCISE 11.05 Exponential graphs

Concepts and techniques

1 Which of the following is an increasing exponential function?
A $y = 3 \times 0.9^x$ B $y = 7 \times 0.8^x$ C $y = 2 \times 3.4^x$
D $y = 5 \times 0.1^x$ E $y = 11 \times 0.7^x$

2 Which of the following is true for the graph of $y = 2.5 \times 1.8^x$?
A The graph has a $y$-intercept at (0, 1.8) .
B As $x \to \infty$, $y \to 0$. C The graph is decreasing.
D As $x \to -\infty$, $y \to 0$. E The graph has an $x$-intercept at (2.5, 0).

3 Which of the following graphs of exponential functions rises the most steeply?
A $y = 4 \times 1.8^x$ B $y = 4 \times 2.1^x$ C $y = 4 \times 3.2^x$
D $y = 4 \times 4.5^x$ E $y = 4 \times 6.3^x$

4 Example 11 Plot a graph for each function for the stated domain. Comment on the features of the graph.
a $y = 2^x$ for $-3 \le x \le 4$ b $y = 5 \times 2^x$ for $-4 \le x \le 3$
c $y = 4 \times 0.6^x$ for $-2 \le x \le 4$ d $y = 10 \times 1.8^x$ for $-3 \le x \le 3$
e $y = 40 \times 0.2^x$ for $-2 \le x \le 4$ f $y = 10^x$ for $-2 \le x \le 3$

5 Example 12 CAS Draw the graph of each function for the stated domain. Comment on the features of the graph.
a $y = 8.5 \times 2.5^x$ for $-3 \le x \le 3$ b $y = 20 \times 0.9^x$ for $-5 \le x \le 5$
c $y = 16 \times 1.2^x$ for $-5 \le x \le 5$ d $y = 16 \times 2^x$ for $-5 \le x \le 5$
e $y = 16 \times 0.1^x$ for $-3 \le x \le 3$ f $y = 8 \times 10^x$ for $-3 \le x \le 3$

Reasoning and communication

6 Draw the graph of the function $y = 3 \times 1.7^x$ and use it to solve the equation: $3 \times 1.7^x = 2$

11.06 TRANSFORMATIONS OF EXPONENTIAL GRAPHS

In the previous section of this chapter you looked at graphs of functions of the form $y = Aa^x$, where $A, a > 0$.

We will now look at the graphs of other exponential functions.

INVESTIGATION Translations of exponential graphs

In the first part of this investigation you will examine the graphs of functions of the form $y = Aa^x + c$.

TI-Nspire CAS

ClassPad

- Describe the similarities and differences between the graphs.
- What transformations would change the graph of $y = 2^x$ into the other graphs?
- Graph other exponential functions of the form $y = Aa^x + c$.
- Explain the significance of the value of $c$ when comparing the graphs of $y = Aa^x$ and $y = Aa^x + c$.

In the next part of this investigation you will examine the graphs of functions of the form $y = Aa^{x+b}$.

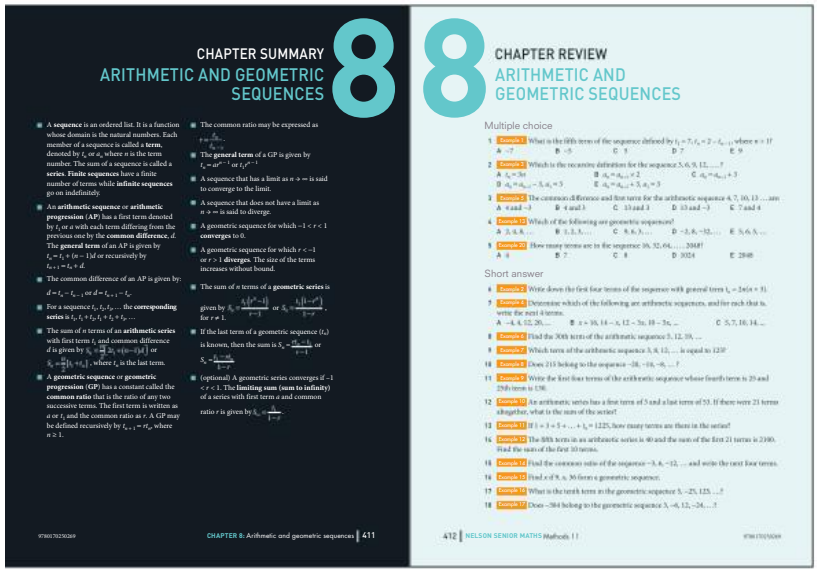
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Exercise questions are clearly divided into two categories: concepts and techniques and reasoning and communication.

Where appropriate, worksheets are provided for additional practice and consolidation of key concepts.

Each chapter contains at least one investigation, providing students with the opportunity to apply their understanding to a practical application.



CHAPTER SUMMARY 8
ARITHMETIC AND GEOMETRIC SEQUENCES

- A **sequence** is an ordered list. It is a function whose domain is the natural numbers. Each member of a sequence is called a **term**, denoted by $t_n$ or $u_n$ where $n$ is the term number. The sum of a sequence is called a **series**. **Finite sequences** have a finite number of terms while **infinite sequences** go on indefinitely.
- An **arithmetic sequence** or **arithmetic progression** (**AP**) has a first term denoted by $t_1$ or $a$ with each term differing from the previous one by the **common difference**, $d$. The **general term** of an AP is given by $t_n = t_1 + (n-1)d$ or recursively by $t_{n+1} = t_n + d$.
- The common difference of an AP is given by: $d = t_n - t_{n-1}$ or $d = t_{n+1} - t_n$.
- For a sequence $t_1, t_2, t_3, \ldots$ the **corresponding series** is $t_1 + t_2 + t_3 + t_4 \ldots$
- The sum of $n$ terms of an **arithmetic series** with first term $t_1$ and common difference $d$ is given by $S_n = \frac{n}{2}[2t_1 + (n-1)d]$ or $S_n = \frac{n}{2}[t_1 + t_n]$, where $t_n$ is the last term.
- A **geometric sequence** or **geometric progression** (**GP**) has a constant called the **common ratio** that is the ratio of any two successive terms. The first term is written as $a$ or $t_1$ and the common ratio as $r$. A GP may be defined recursively by $t_{n+1} = rt_n$, where $n \ge 1$.
- The common ratio may be expressed as $r = \frac{t_{n+1}}{t_n}$.
- The **general term** of a GP is given by $t_n = ar^{n-1}$ or $t_n r^{n-1}$.
- A sequence that has a limit as $n \to \infty$ is said to converge to the limit.
- A sequence that does not have a limit as $n \to \infty$ is said to diverge.
- A geometric sequence for which $-1 < r < 1$ **converges** to 0.
- A geometric sequence for which $r < -1$ or $r > 1$ **diverges**. The size of the terms increases without bound.
- The sum of $n$ terms of a **geometric series** is given by $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_1(1 - r^n)}{1 - r}$, for $r \ne 1$.
- If the last term of a geometric sequence ($t_n$) is known, then the sum is $S_n = \frac{rt_n - t_1}{r - 1}$ or $S_n = \frac{t_1 - rt_n}{1 - r}$.
- (optional) A geometric series converges if $-1 < r < 1$. The **limiting sum (sum to infinity)** of a series with first term $a$ and common ratio $r$ is given by $S_\infty = \frac{a}{1 - r}$.

CHAPTER 8: Arithmetic and geometric sequences 411

8 CHAPTER REVIEW
ARITHMETIC AND GEOMETRIC SEQUENCES

Multiple choice

Short answer

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Each chapter concludes with a detailed chapter summary and a comprehensive review of concepts.

At the end of every three chapters a comprehensive mixed revision set is provided.

# ICONS IN THE TEXTBOOK

Example 1 — Link from question to worked example




Prior learning

CAS — CAS exercise and example




Worksheet




Interactive spreadsheet




Practice quiz

## GREEK ALPHABET

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A, | α | alpha | I, | ι | iota | P, | ρ | rho |
| B, | β | beta | K, | κ | kappa | Σ, | σ | sigma |
| Γ, | γ | gamma | Λ, | λ | lambda | T, | τ | tau |
| Δ, | δ | delta | M, | μ | mu | Y, | υ | upsilon |
| E, | ε | epsilon | N, | ν | nu | Φ, | ϕ | phi |
| Z, | ζ | zeta | Ξ, | ξ | xi | X, | χ | chi |
| H, | η | eta | O, | ο | omicron | Ψ, | ψ | psi |
| Θ, | θ | theta | Π, | π | pi | Ω, | ω | omega |


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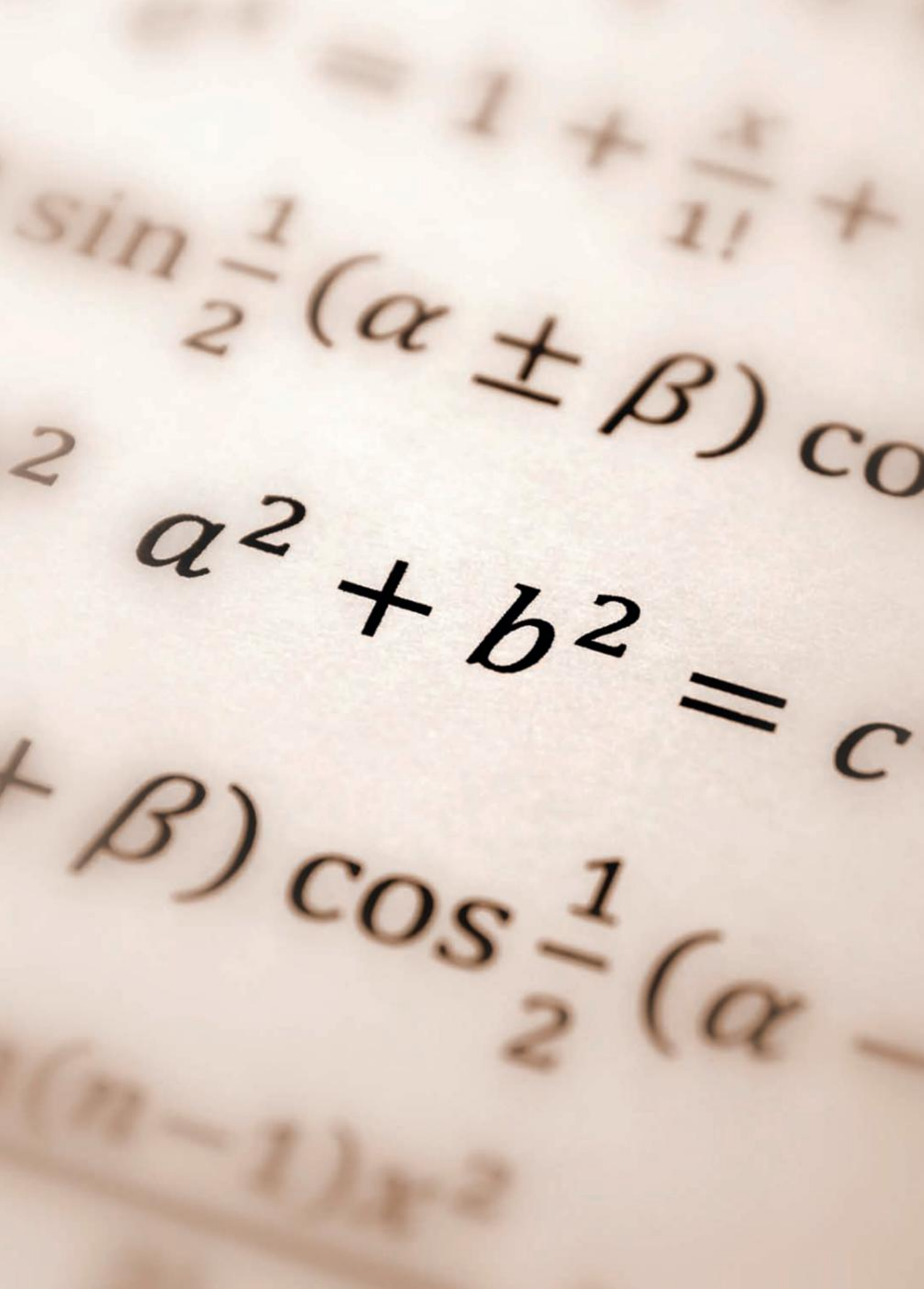
**Anne McMahon**
Anne has been teaching Mathematics for 36 years. She holds a Bachelor of Secondary Mathematics Teaching, with Distinction. Anne has taught in many different schools ranging from small country schools to large metropolitan schools and has taught all levels of Mathematics from year 7 to year 12. For the past 23 years she has been teaching at Gunnedah High School, the past eleven of those years as Head Teacher of Mathematics.

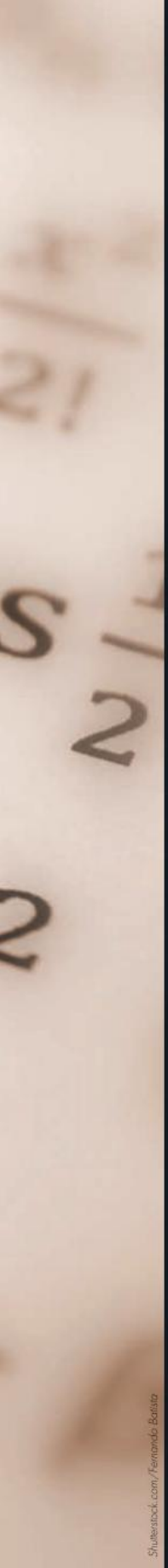
The **Digital Resources Team** includes Greg Neal, Margaret Denham, Issam Kanaan, Jason Warwick and Roger Walter.

# ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| $\approx$ | approximately equal to |
| $=$ | equal to |
| $\equiv$ | identically equal to, congruent |
| $\neq$ | not equal to |
| $<$ | less than |
| $>$ | greater than |
| $\leq$ | less than or equal to |
| $\geq$ | greater than or equal to |
| LHS | left-hand side |
| RHS | right-hand side |
| $\Delta$ | the discriminant of a quadratic equation, given by $\Delta = b^2 - 4ac$ |
| $\pm$ | plus or minus |
| $\in$ | is a member of |
| $\notin$ | is not a member of |
| $\subseteq$ | is a subset of |
| $\subset$ | is a proper subset of |
| $\cap$ | intersection |
| $\cup$ | union |
| $\{\}, \varnothing$ | null set (empty set) |
| $U$ | universal set |
| $\binom{n}{r}, {}^nC_r$ | the binomial coefficients, the number of combinations of $r$ objects from $n$ objects |
| $\overline{A}\ A'$ | the complement of the set or event $A$ |
| $A = \{2, 4, 6, 8\}$ | the set $A$ consisting of the numbers 2, 4, 6 and 8 |
| $A = \{x: 0 < x < 10$ and $x$ an even number$\}$ | the set $A$ of elements $x$ such that $0 < x < 10$ and $x$ is an even number |
| $\lvert A \rvert$ | the cardinality of the set $A$, for finite sets, the number of elements |
| $n(A)$ | for finite sets, the number of elements in the set $A$ |
| $P(A)$ | the probability of the event $A$ |
| $P(A\lvert B)$ | the conditional probability of event $A$ given event $B$ |
| $\boldsymbol{N}, \mathbb{N}$ | the set of natural numbers, $\{1, 2, 3, 4, \ldots.\}$ |
| $\boldsymbol{Z}, \mathbb{Z}$ or $\boldsymbol{J}$ | the set of integers, $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ |
| $\boldsymbol{Q}, \mathbb{Q}$ | the set of rational numbers |
| $\boldsymbol{Q}'$ or $\overline{\boldsymbol{Q}}$ | the set of irrational numbers |
| $\boldsymbol{Q}^+$ | the set of positive rational numbers |
| $\boldsymbol{R}, \mathbb{R}$ | the set of real numbers |
| $\boldsymbol{R}/0$ or $\boldsymbol{R}/\{0\}$ | means the real numbers except 0 |
| $36° 27' 15''$ | 36 degrees 27 minutes and 15 seconds |
| $\sin(\theta)$ | the sine of the angle $\theta$ |
| $\cos(\theta)$ | the cosine of the angle $\theta$ |
| $\tan(\theta)$ | the trigonometric tangent of the angle $\theta$ |
| 053° | a bearing of 53° |
| $\pi$ (pi) | about 3.14159… |
| $\angle$ | angle |
| $\triangle$ | triangle |
| $1.27^c$ | 1.27 rad, 1.27 radians |
| $x \to \infty$ | $x$ becomes large and positive ($x$ approaches infinity) |
| $x \to -\infty$ | $x$ becomes large and negative ($x$ approaches negative infinity) |
| $f(x)$ | the function $f$ of the variable $x$ |
| $\Delta x, \Delta y$ | change in $x$, change in $y$ |
| w.r.t. | with respect to |
| $\delta x, \delta y$ | small change in $x$, small change in $y$ |
| $t_n, a_n$ | the $n$th term of a sequence |
| $S_n$ | the sum of the first $n$ terms of a sequence |
| AP | arithmetic progression |
| $d$ | the common difference of an arithmetic sequence |

| | | | |
|---|---|---|---|
| GP | geometric progression | $f'(a)$ | the value of the derivative of $f$ at $x = a$ |
| $r$ | the common ratio of a geometric sequence | $\left.\frac{dy}{dx}\right\|_{x=a}$ | the derivative of $y$ with respect to $x$ at $x = a$ |
| $S_\infty$ | the limiting sum of a (geometric) series, the infinite sum | $F(x), \int fx\ dx$ | the antiderivative (primitive or indefinite integral) of $f(x)$ |
| $\lim_{x \to a} f(x)$ | the limit of the function $f(x)$ as $x$ gets close to $a$ | $m$ | gradient |
| $\frac{d}{dx}$ | derivative with respect to $x$ | $\sqrt{}$ | square root of |
| $f'(x)$ | derivative of the function $f$ | | |


$a^2 + b^2 = c$




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# 1

## TERMINOLOGY

binomial
coefficient
constant
cubic
direct proportion
disciminant
factor theorem
formula
highest common factor
inverse proportion
leading term
linear
monic
null factor law
polynomial
quadratic
quartic
relationship
remainder theorem
root
simultaneous equations
subject
trinomial
variable

Prior learning

FUNCTIONS AND GRAPHS

# EQUATIONS AND POLYNOMIALS

1.01 Algebra review
1.02 Binomial factorisation
1.03 Linear equations
1.04 Linear relationships
1.05 Graphical solution of simultaneous equations
1.06 Algebraic solution of simultaneous equations
1.07 Quadratic equations
1.08 The quadratic formula and the discriminant
1.09 Direct and inverse proportion
1.10 Polynomials
1.11 Polynomial factorisation
1.12 Polynomial equations
Chapter summary
Chapter review

## LINES AND LINEAR RELATIONSHIPS

- examine examples of direct proportion and linearly related variables (ACMMM002)
- solve linear equations (ACMMM005)

## REVIEW OF QUADRATIC RELATIONSHIPS

- examples of quadratically related variables (ACMMM006)
- solve quadratic equations using the quadratic formula and by completing the square (ACMMM008)

## INVERSE PROPORTION

- examine examples of inverse proportion (ACMMM012)

## POWERS AND POLYNOMIALS

- identify the coefficients and the degree of a polynomial (ACMMM015)
- expand quadratic and cubic polynomials from factors (ACMMM016)
- factorise cubic polynomials in cases where a linear factor is easily obtained (ACMMM018)
- solve cubic equations using technology and algebraically in cases where a linear factor is easily obtained (ACMMM019) AC

# 1.01 ALGEBRA REVIEW

Many of the algebraic techniques you encountered in junior mathematics will be used and applied in the topics of this course.

**IMPORTANT**

The distributive law states that $a(b + c) = ab + ac$ for any real numbers $a$, $b$ and $c$.

The distributive law is used to expand single and multiple brackets. Terms may be collected to simplify the expression obtained. In the case of **binomial products** like $(3x - 5)(x + 3)$, the FOIL shortcut may be used to save time.

**IMPORTANT**

**FOIL method**

**F**irst terms, **O**uter terms, **I**nner terms, **L**ast terms

$(a + b)(c + d) = ac + ad + bc + bd$

Some binomial products have special results

**IMPORTANT**

**Perfect squares**

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

**Difference of two squares**

$(a + b)(a - b) = a^2 - b^2$


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To multiply expressions with more than two terms you should use the distributive law, perhaps applying it several times.

## Example 1

Expand and simplify

**a** $5(2x-7)-(3x-9)$ **b** $(2x-3)(3x-8)$

**c** $(m-5n)^2$ **d** $(7p+4)(7p-4)$

**e** $(x+2)(2x-3y+7)$

### Solution

**a** Write $-(3x-9)$ as $-1(3x-9)$, then multiply the brackets out.

$$5(2x-7)-1(3x-9)=5\times 2x+5\times(-7)+(-1)\times 3x+(-1)\times(-9)$$
$$=10x-35-3x+9$$

Collect like terms.

$$=7x-26$$

**b** Method 1: Distributive law.

$$(2x-3)(3x-8)=2x(3x-8)-3(3x-8)$$
$$=6x^2-16x-9x+24$$

Method 2: FOIL.

$$(2x-3)(3x-8)=2x\times 3x+2x\times(-8)+(-3)\times 3x+(-3)\times(-8)$$
$$=6x^2-16x-9x+24$$

Collect like terms from either method.

$$=6x^2-25x+24$$

**c** Use $(a-b)^2=a^2-2ab+b^2$.

$$(m-5n)^2=m^2-2\times m\times 5n+(5n)^2$$
$$=m^2-10mn+25n^2$$

**d** Use $(a+b)(a-b)=a^2-b^2$

$$(7p+4)(7p-4)=(7p)^2-4^2$$
$$=49p^2-16$$

**e** Use the distributive law.

$$(x+2)(2x-3y+7)=x(2x-3y+7)+2(2x-3y+7)$$

Use it again.

$$=2x^2-3xy+7x+4x-6y+14$$

Collect like terms.

$$=2x^2-3xy+11x-6y+14$$

Factorisation is the reverse of expanding brackets. It uses the distributive law in reverse. Always look for the highest common factor first. You place this outside the brackets, leaving the remaining factors of each term inside.

WS Expanding algebraic expressions

WS Factorising algebraic expressions

## Example 2

Factorise

**a** $4mn+6m^2$ **b** $4(y+2)-3y(y+2)$ **c** $5x-3y+xy-15$

### Solution

**a** The highest common factor is $2m$. Take it outside the brackets.

$$4mn+6m^2=2m(2n+3m)$$

**b** The highest common factor is $y+2$. Take it outside the brackets.

$$4(y+2)-3y(y+2)=(y+2)(4-3y)$$

**c** There is no common factor for all terms, so we group the terms into pairs with common factors.

$$5x-3y+xy-15=5x-15+xy-3y$$

The highest common factor of $5x - 15$ is 5 and the highest common factor of $xy - 3y$ is $y$. You take both of these factors outside the brackets.

$= 5(x - 3) + y(x - 3)$

Now $x - 3$ is a common factor that can be taken outside the brackets.

$= (x - 3)(5 + y)$

When operating on algebraic fractions you factorise where possible and remember to use the same rules as you would for numeric fractions.

## Example 3

Simplify

**a** $\dfrac{7a-21}{35}$ **b** $\dfrac{m}{10}+\dfrac{3(m+5)}{4}$

**c** $\dfrac{2}{x-5}-\dfrac{1}{x+2}$ **d** $\dfrac{3x+6y-7ax-14ay}{2x+y}\times\dfrac{10x^2+5xy}{7x+14y}$

### Solution

**a** Factorise the numerator and then cancel.

$$\frac{7a-21}{35}=\frac{7^{1}(a-3)}{35^{5}}$$
$$=\frac{a-3}{5}$$

**b** Change each fraction so that they have a common denominator.

$$\frac{m}{10}+\frac{3(m+5)}{4}=\frac{2\times m}{2\times 10}+\frac{5\times 3(m+5)}{5\times 4}$$
$$=\frac{2m+15(m+5)}{20}$$

Expand the brackets.

$$=\frac{2m+15m+75}{20}$$

Simplify.

$$=\frac{17m+75}{20}$$

**c** Use the common denominator $(x - 5)(x + 2)$. Change both fractions so that they have a common denominator.

$$\frac{2}{x-5}-\frac{1}{x+2}=\frac{2(x+2)}{(x-5)(x+2)}-\frac{1(x-5)}{(x-5)(x+2)}$$
$$=\frac{2(x+2)-1(x-5)}{(x-5)(x+2)}$$

Expand the brackets and collect like terms.

$$=\frac{2x+4-x+5}{(x-5)(x+2)}$$
$$=\frac{x+9}{(x-5)(x+2)}$$

**d** Factorise numerators and denominators.

$$\frac{3x+6y-7ax-14ay}{2x+y}\times\frac{10x^2+5xy}{7x+14y}$$
$$=\frac{3(x+2y)-7a(x+2y)}{2x+y}\times\frac{5x(2x+y)}{7(x+2y)}$$

Cancel common factors.

$$=\frac{\cancel{(x+2y)^1}(3-7a)}{\cancel{2x+y}^1}\times\frac{5x\cancel{(2x+y)}^1}{7\cancel{(x+2y)}^1}$$

Multiply numerators together and denominators together.

$$=\frac{5x(3-7a)}{7}$$

# EXERCISE 1.01 Algebra review

## Concepts and techniques

**1** Example 1 Expand and simplify the following.

**a** $9-5(b+3)$ **b** $5(3-2m)+7(m-2)$

**c** $2(h+4)+3(2h-9)$ **d** $a(2a+1)-(a^2+3a-4)$

**e** $2ab(3-a)-b(4a-1)$ **f** $2(3t-4)-(t+1)+3$

**2** Expand and simplify the following.

**a** $(a+5)(a+2)$ **b** $(2y-3)(y+5)$ **c** $(x+4)(x+3)$

**d** $(2x-3)(x+2)$ **e** $(x+5)(x-5)$ **f** $(x-4)(y+7)$

**3** Expand and simplify the following.

**a** $(x^2+3)(x-2)$ **b** $(x+9)(x-2y+2)$ **c** $(b-3)(2a+2b-1)$

**d** $(x+2)(x^2-2x+4)$ **e** $(a-3)(a^2+3a+9)$

**4** Expand and simplify the following.

**a** $(y+8)^2$ **b** $(2b+5)^2$ **c** $(3-x)^2$ **d** $(3y-1)^2$

**e** $(3a-b)^2$ **f** $(4d+5e)^2$ **g** $(t+4)(t-4)$ **h** $(x-3)(x+3)$

**i** $(2a+3)(2a-3)$ **j** $(x-5y)(x+5y)$

**5** Expand and simplify the following.

**a** $(x^2+2)(x^2-2)$ **b** $(x^2+5)^2$

**c** $(3ab-4c)(3ab+4c)$ **d** $\left(x+\frac{2}{x}\right)^2$

**e** $\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)$ **f** $(a+3)^2-(a-3)^2$

**g** $16-(z-4)(z+4)$ **h** $\left(x-\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^2+2$

**6** Example 2 Factorise the following.

**a** $24-18y$ **b** $2y^2+4y$ **c** $ab^2+ab$

**d** $3mn^3+9mn$ **e** $8x^2z-2xz^2$ **f** $3q^5-2q^2$

**7** Factorise the following.

**a** $x(m+5)+7(m+5)$ **b** $2(y-1)-y(y-1)$

**c** $(x-3)^2+5(x-3)$ **d** $5ab-3b+10a-6$

**e** $x^4-4x^3-5x+20$ **f** $3x-6y-x^2+2xy$

8 Example 3 Simplify the following.

a $\frac{5a+10}{5}$ b $\frac{6t-3}{3}$ c $\frac{8y+2}{6}$ d $\frac{8}{4d-2}$

e $\frac{x^2}{5x^2-2x}$ f $\frac{2a^2b-4ab}{a^2-2a}$ g $\frac{x-2}{4}+\frac{2x-1}{5}$ h $\frac{3a+2b}{3}-\frac{2a-5b}{4}$

9 Simplify the following.

a $\frac{2}{x}+\frac{3}{x}$ b $\frac{1}{x-1}-\frac{2}{x}$ c $x-\frac{x^2}{x+2}$

d $\frac{1}{x+1}+\frac{1}{x-3}$ e $\frac{15m-10n}{12}\times\frac{6m+18n}{3m^2-2mn}$ f $\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}$

g $\frac{2}{p^2+pq}+\frac{3}{pq-q^2}$ h $\left(\frac{8x-3y}{2}-\frac{12x+4y}{4}\right)\div\frac{2x-5y}{6}$

# 1.02 BINOMIAL FACTORISATION

A **binomial product** has two binomial expressions multiplied together. When they are expanded they produce an expression with three terms such as $x^2 + 2x - 3$. This is called a **quadratic** (**trinomial**) expression. Factorising a trinomial (usually) gives a binomial product. A monic quadratic is one where the coefficient of $x^2$ is 1. When the **coefficient** of $x^2$ is 1, we can factorise the quadratic using a special property.

The coefficient of $x$ is the sum of the numbers in the brackets and the **constant** is given by the product of these two numbers.

If you recognise a perfect square or a difference of squares, you should use it as a short cut to factorise a quadratic.

**IMPORTANT**

**Monic quadratic factorisation**

$x^2 + (a+b)x + ab = (x+a)(x+b)$

## Example 4

Factorise

a $a^2 + 5a + 6$ b $m^2 - 4m - 21$ c $y^2 - 8y + 16$ d $x^2 - 36$

### Solution

a Find two numbers that add to +5 and multiply to +6. $a^2 + 5a + 6$

$2 + 3 = 5$ and $2 \times 3 = 6$

Factorise by writing the binomial product. $a^2 + 5a + 6 = (a+2)(a+3)$

b Find two numbers that add to −4 and multiply to −21. $m^2 - 4m - 21$

$3 - 7 = -4$ and $3 \times (-7) = -21$

Factorise by writing the binomial product. $m^2 - 4m - 21 = (m+3)(m-7)$

c This is a perfect square. $y^2 - 8y + 16 = y^2 - 2 \times y \times 4 + 4^2$

Use $(a-b)^2 = a^2 - 2ab + b^2$ in reverse. $= (y-4)^2$

d This is a difference of two squares. $x^2 - 36 = x^2 - 6^2$

Use $(a+b)(a-b) = a^2 - b^2$ in reverse. $= (x+6)(x-6)$

When the coefficient of $x^2$ is not 1, for example in the expression $5x^2 - 10x - 4$, we need to use a different method to factorise the quadratic.

You can use the cross method or the fraction method to factorise quadratics of the form $ax^2 + bx + c$.

You use these steps for the cross method.

- identify all of the factors of $ax^2$ and of $c$
- choose a pair of factors, from each, as a combination to test out
- find the 'cross' product of the pairs
- add the products to find which pair gives the middle term, $bx$

The factors that give the correct middle term are your answer.

The fraction method is now used more because you don't have to guess. The fraction method steps are as follows:

- write a fraction of the form $\frac{(ax \ldots\ldots)(ax \ldots\ldots)}{a}$
- find the product of $a$ and $c$
- identify which factors of $a$ and $c$ generate $b$ when added
- place the identified factors into the fraction, one in each bracket
- cancel to get the answer (The denominator will always cancel completely)

## Example 5

Factorise:

**a** $5y^2 - 13y + 6$ **b** $4y^2 + 4y - 3$

### Solution

**a** **Cross method**

Since $c = 6$ is positive, both of its factors have the same sign.

Since $b = -13$ is negative, both the factors of 6 must be negative.

Identify the factors of $5y^2$ and 6:

$5y$, $y$ and $-1$, $-6$ or $-2$, $-3$

Possible combinations are:

$5y$ $-1$ / $y$ $-6$ $\quad$ $5y$ $-6$ / $y$ $-1$

$5y$ $-2$ / $y$ $-3$ $\quad$ $5y$ $-3$ / $y$ $-2$

Choose a combination to test.

Work out the cross products and write them beside your cross. The sum must give the middle term of $-13y$.

$5y$ $-1$ / $y$ $-6$ $\quad$ $y \times -1 = -y$, $5y \times -6 = -30y$, $-31y$ ✗

Choose another combination to test.

$$\begin{array}{cc} 5y & -6 \\ y & -1 \end{array} \qquad \begin{array}{r} -6y \\ -5y \\ \hline -11y \end{array} \times$$

Again the sum of the products is not $-13y$, so try another combination.

$$\begin{array}{cc} 5y & -2 \\ y & -3 \end{array} \qquad \begin{array}{r} -2y \\ -15y \\ \hline -17y \end{array} \times$$

By guessing and checking, you finally get the correct combination.

$$\begin{array}{cc} 5y & -3 \\ y & -2 \end{array} \qquad \begin{array}{r} -3y \\ -10y \\ \hline -13y \end{array} \checkmark$$

Write the answer.
The first bracket is from the top line of the cross, the second bracket is from the bottom line of the cross.

$5y^2 - 13y + 6 = (5y - 3)(y - 2)$

**Fraction method**

Multiply the first and last coefficient to get $ac$.

$5 \times 6 = 30$

Find two numbers that multiply to give $ac$ and add to give $b$.

$-3 \times (-10) = 30$ and $-3 + (-10) = -13$

Write the quadratic.

$5y^2 - 13y + 6$

Write it in the form $\frac{(ax.....)(ax.....)}{a}$.

$= \frac{(5y\quad)(5y\quad)}{5}$

Put $-3$ and $-10$ in the brackets.

$= \frac{(5y-3)(5y-10)}{5}$

Now cancel out any common factors from each bracket with the denominator.

$= \frac{(5y-3)\cancel{(5y-10)}^{(y-2)}}{\cancel{5}_{1}}$

Write your answer.

$= (5y - 3)(y - 2)$

**b Cross method**

Since $c = -3$, is negative its factors have different signs.

Identify the factors of $4y^2$ and 3.

$4y, y$ or $2y, 2y$ and $1, 3$

Choose a combination to test. The sum must be the middle term $4y$.

$4y^2 + 4y - 3$

$$\begin{array}{cc} 4y & 3 \\ y & -1 \end{array} \qquad \begin{array}{r} 3y \\ -4y \\ \hline -y \end{array} \times$$


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Choose another combination to test.

$4y \times -1 \quad -y$
$y \times 3 \quad 12y$
$11y$ ✗

Again the sum of the products is not $4y$, so try another combination. This one is right!

$2y \times 3 \quad -2y$
$2y \times -1 \quad 6y$
$4y$ ✓

Write the answer.

$4y^2 + 4y - 3 = (2y + 3)(2y - 1)$

**Fraction method**

Find the value of $ac$.

$4 \times (-3) = -12$

Write the quadratic.

$4y^2 + 4y - 3$

Write the quadratic in the fraction form.

$= \dfrac{(4y \quad)(4y \quad)}{4}$

The factors of $-12$ that add to 4 are $6, -2$.
Place each factor into a bracket and cancel completely.

$= \dfrac{(2y+3)(2y-1)}{(4y+6)(4y-2)}$
$2_1$

Write your answer.

$4y^2 + 4y - 3 = (2y + 3)(2y - 1)$

The other main method of factorisation of non-monic quadratics is called the PSF method. It is also called the *ac* method, the grouping method or the decomposition method. This method involves finding the product, sum and factors as we did in the fraction method, but after decomposing the middle term, you need to factorise in pairs. This method uses the same principles as the fraction method.

In the quadratic expression, $ax^2 + bx + c$, if there are no factors of the product of $a$ and $c$ that will add to give the middle coefficient $b$, then the quadratic expression will not factorise.

Your teacher may prefer a particular method of quadratic factorisation.

# EXERCISE 1.02 Binomial factorisation

## Concepts and techniques

1 Example 4 Factorise the following.

a $y^2 + 7y + 12$
b $m^2 + 2m + 1$
c $z^2 + z - 6$
d $x^2 - 5x - 6$
e $v^2 - 8v + 15$
f $x^2 + 9x - 10$
g $y^2 - 10y + 21$
h $y^2 + 9y - 36$
i $x^2 - 5x - 24$
j $y^2 - 5y - 36$
k $n^2 - 10n + 24$
l $q^2 + 12q + 20$

**2** Factorise the following.

| | | | |
|---|---|---|---|
| **a** $y^2 - 2y + 1$ | **b** $x^2 + 6x + 9$ | **c** $t^2 - 4t + 4$ | **d** $x^2 - 12x + 36$ |
| **e** $16b^2 - 8b + 1$ | **f** $a^2 - 4$ | **g** $x^2 - 9$ | **h** $y^2 - 1$ |
| **i** $x^2 - 25$ | **j** $1 - 4z^2$ | **k** $x^2 - 4y^2$ | **l** $x^4 - 1$ |

**3** Example 5 Factorise the following.

| | | |
|---|---|---|
| **a** $2a^2 + 11a + 5$ | **b** $5y^2 + 7y + 2$ | **c** $3x^2 + 10x + 7$ |
| **d** $3x^2 + 8x + 4$ | **e** $2b^2 - 5b + 3$ | **f** $2x^2 + 11x + 12$ |
| **g** $2y^2 - 11y - 6$ | **h** $8t^2 - 14t + 3$ | **i** $6x^2 - x - 12$ |
| **j** $6y^2 + 47y - 8$ | **k** $4n^2 - 11n + 6$ | **l** $12q^2 + 23q + 10$ |

**4** Factorise the following.

| | | |
|---|---|---|
| **a** $8r^2 + 22r - 6$ | **b** $6p^2 - 5p - 6$ | **c** $12b^2 - 43b + 36$ |
| **d** $49y^2 + 14y + 1$ | **e** $16k^2 - 24k + 9$ | **f** $25k^2 - 20 + \frac{4}{k^2}$ |
| **g** $4x^2 - 49$ | **h** $36x^2 - y^2$ | **i** $4a^2 - 81b^2$ |
| **j** $(a-1)^2 - (b-2)^2$ | **k** $x^2 - \frac{1}{4}$ | **l** $9x^6 - 4y^2$ |

## Reasoning and communication

**5** You may need to use more than one method of factorisation to completely factorise each of the following expressions. (*Hint*: Always look for a common factor first.)

| | |
|---|---|
| **a** $2x^2 - 18$ | **b** $3p^2 - 3p - 36$ |
| **c** $4a^3b + 8a^2b^2 - 4ab^2 - 2a^2b$ | **d** $5a^2 - 10a + 5$ |
| **e** $-2x^2 + 11x - 12$ | **f** $3z^3 + 27z^2 + 60z$ |
| **g** $9ab - 4a^3b^3$ | **h** $x^3 - x$ |
| **i** $6x^2 + 8x - 8$ | **j** $3m - 15 - 5n + mn$ |
| **k** $(x-3)^2 - (x+4)^2$ | **l** $y^2(y+5) - 16(y+5)$ |

**6** Completely factorise each of the following expressions.

| | | |
|---|---|---|
| **a** $x^3 - 3x^2 - 10x$ | **b** $x^3 - 3x^2 - 9x + 27$ | **c** $4x^2y^3 - y$ |
| **d** $18x^2 + 33x - 30$ | **e** $3x^2 - 6x + 3$ | **f** $x^3 + 2x^2 - 25x - 50$ |
| **g** $z^3 + 6z^2 + 9z$ | **h** $4x^4 - 13x^2 + 9$ | **i** $2x^5 + 2x^2y^3 - 8x^3 - 8y^3$ |
| **j** $4a^3 - 36a$ | **k** $4k^3 + 40k^2 + 100k$ | **l** $3x^3 + 9x^2 - 3x - 9$ |

# 1.03 LINEAR EQUATIONS

An equation is a mathematical sentence which contains an equals sign with an expression on both sides. The simplest kind of equation is a **linear** equation in one variable.

**IMPORTANT**

The terms in a **linear equation** are either constants or simple variable terms with a constant. That is, any variables have only a power of 1. A value that makes an equation true is called a **root** or **solution** of the equation.

To **solve** a linear equation, rearrange it using inverse operations to isolate the variable.

Whatever you do to one side of an equation, you must do the same to the other side of the equation.

If there are brackets in an equation, get rid of them by expanding.

If there are fractions, get rid of them by multiplying both sides of the equation by the lowest common denominator.

## Example 6

Solve

**a** $4y - 3 = 8y - 21$

**b** $2(3x + 7) = 6 - (x - 1)$

**c** $\frac{y+1}{5} - \frac{y-2}{3} = \frac{5}{6}$

**d** $\frac{11}{2n} - \frac{4}{3n} = 5$

### Solution

**a** Write the equation.

$$4y - 3 = 8y - 21$$

Subtract $4y$ from both sides to put the variables on one side so that the coefficient is positive.

$$4y - 4y - 3 = 8y - 4y - 21$$

Simplify both sides.

$$-3 = 4y - 21$$

Add 21 to isolate the variable.

$$-3 + 21 = 4y - 21 + 21$$

Simplify both sides.

$$18 = 4y$$

Undo the multiplication by dividing both sides by 4.

$$\frac{18}{4} = \frac{4y}{4}$$

Simplify and write the variable on the left hand side.

$$y = 4\frac{1}{2}$$

**b** Write the equation.

$$2(3x + 7) = 6 - (x - 1)$$

Expand the brackets.

$$6x + 14 = 6 - x + 1$$

Simplify.

$$6x + 14 = 7 - x$$

Get the variables on one side and the constants on the other.

$$6x + 14 + x - 14 = 7 - x + x - 14$$

Simplify.

$$7x = -7$$

Divide both sides by 7 to get the answer.

$$x = -1$$

**c** Write the equation.

$$\frac{y+1}{5} - \frac{y-2}{3} = \frac{5}{6}$$

Multiply every term by the lowest common denominator, which is 30.

$$30 \times \left(\frac{y+1}{5}\right) - 30 \times \left(\frac{y-2}{3}\right) = 30 \times \left(\frac{5}{6}\right)$$

Cancel and simplify.

$$6(y + 1) - 10(y - 2) = 25$$

Expand the brackets and simplify.

$$6y + 6 - 10y + 20 = 25$$
$$-4y + 26 = 25$$

Subtract 26 from both sides and simplify.

$$-4y + 26 - 26 = 25 - 26$$
$$-4y = -1$$

Divide both sides by –4 and simplify to give the answer.

$$\frac{-4y}{-4} = \frac{-1}{-4}$$
$$y = \frac{1}{4}$$

**d** Write the equation.

$$\frac{11}{2n} - \frac{4}{3n} = 5$$

Multiply every term by $6n$, which is the lowest common denominator.

$$6n \times \left(\frac{11}{2n}\right) - 6n \times \left(\frac{4}{3n}\right) = 6n \times (5)$$

Cancel and simplify.

$$3 \times 11 - 2 \times 4 = 30n$$
$$33 - 8 = 30n$$
$$25 = 30n$$

Divide both sides by 30 and simplify to give answer.

$$\frac{25}{30} = \frac{30n}{30}$$
$$n = \frac{5}{6}$$

A CAS calculator can be used to solve any linear equation.

## Example 7

CAS Solve each of the following.

**a** $2(3x + 7) = 7 - x$ **b** $\frac{3}{x+1} = \frac{2}{3x-4}$

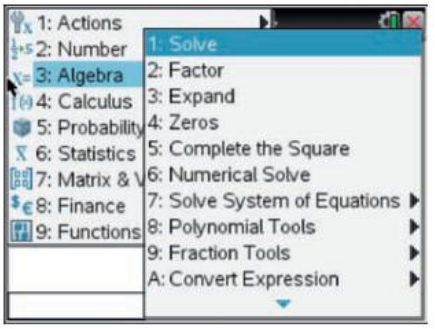
### Solution

**TI-Nspire CAS**

Use a Calculator page.

Press

- menu
- 3:Algebra
- 1: Solve.




**a** Type into the entry line:
$2(3x + 7) = 7 - x$, $x$ and press enter





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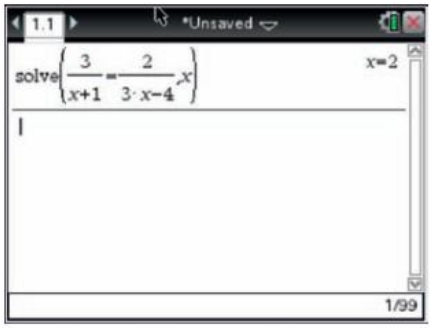
Press enter.




Write the answer.

$x = -1$

**b** Use 'solve(' again and press [ctrl] to get $\frac{\square}{\square}$. Then complete the equation:
$\frac{3}{x+1} = \frac{2}{3x-4}, x$
and press [enter].



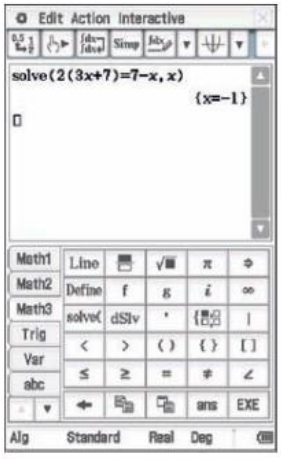

Write the answer.

$x = 2$

### ClassPad

Before starting, make sure that the calculator is set to **Standard** at the bottom so the answer is expressed as a fraction rather than a decimal.
Use the Main application and **solve**, which is found by tapping **Action** then **Equation/Inequality.**
Alternatively you can press [Keyboard], tap [Math3] and tap [solve(].

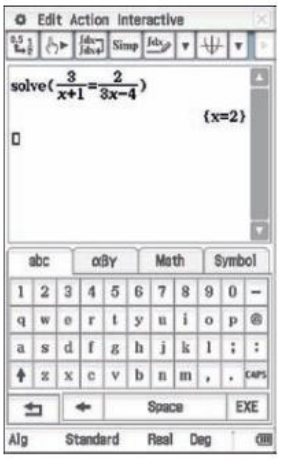
**a** Enter **solve(** and type 2 [(] 3 [x] [+] 7 [)] [=] 7 [(] [−] [x] [,] [x] and close the brackets.
Tap [EXE] or press the [EXE] key.
The solution appears.




Write your answer.

$x = -1$

**b** Use the same method as part **a**.
Press Keyboard. Tap any of the tabs Math1, Math2, Math3 or Trig to use ▣ in conjunction with the arrows to enter the fractions.
Enter: solve $\left(\frac{3}{x+1}=\frac{2}{3x-4}, x\right)$ EXE




Write the answer. $x = 2$

A **formula** is also an equation. You can rearrange a formula to isolate a particular variable. This is called making the variable the **subject**. You use inverse operations to do this in a similar way to solving linear equations.

## Example 8

**a** CAS Make $f$ the subject of $c = \frac{5}{9}(f - 32)$

**b** Make $h$ the subject of $V = \frac{1}{3}\pi r^2 h$

**c** The formula for the volume $V$ of a cylinder is $V = \pi r^2 h$, where $r$ is the radius of the base and $h$ is the height. Find the radius, correct to 2 decimal places, when the volume is 350 and the height is 6.5.

### Solution

**a** Write the equation. $c = \frac{5}{9}(f - 32)$

Multiply both sides by 9 to get rid of the fraction. $9 \times c = 9 \times \frac{5}{9}(f - 32)$

Cancel and simplify. $9c = 5(f - 32)$

Expand the bracket. $9c = 5f - 160$

Add 160 to both sides and simplify. $9c + 160 = 5f - 160 + 160$
$9c + 160 = 5f$

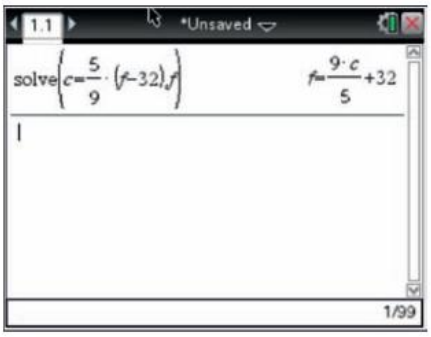
Divide both sides by 5 and rewrite with the subject on the left-hand side. $\frac{9c + 160}{5} = \frac{5f}{5}$
$f = \frac{9}{5}c + 32$


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### TI-Nspire CAS

Press

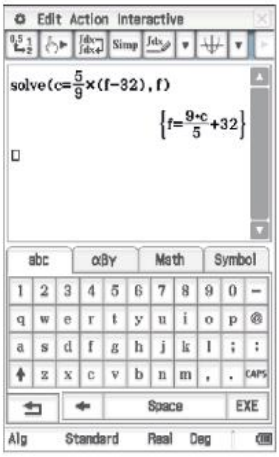
- menu
- 3:Algebra
- 1: Solve.
- Type the formula, pressing $x=$ and $\frac{\square}{\square}$ to enter the $\frac{5}{9}$.
- $c=\frac{5}{9}(f-32), f$ and press enter




### ClassPad

Before starting, make sure that the calculator is set to **Standard** at the bottom so the answer is expressed in fractions rather than decimals.
Use the $\sqrt{\alpha}$ menu.
Tap **Action** then **Equation/Inequality**, then **solve**.
To get variables, press Keyboard and tap the abc tab to display the alphabetic keyboard.

Tap any of the tabs Math1, Math2, Math3 or Trig to use $\frac{\blacksquare}{\blacksquare}$ with the arrow keys to enter the fraction in the expression solve$(c=\frac{5}{9}(f-32), f)$
Tap EXE or press the EXE key.




**b** Write the equation. $V=\frac{1}{3}\pi r^2 h$

Multiply both sides by 3. $3\times V=3\times\frac{1}{3}\pi r^2 h$

$3V=\pi r^2 h$

Divide both sides by $\pi r^2$. $\frac{3V}{\pi r^2}=\frac{\pi r^2 h}{\pi r^2}$

Cancel and rewrite the equation with $h$ as the subject on the left-hand side. $h=\frac{3V}{\pi r^2}$

**c** Write the equation. $V=\pi r^2 h$

Rewrite the equation with the given values. Substitute $V=350$ and $h=6$. $350=\pi r^2\times 6.5$

Divide both sides by $6.5\pi$ and simplify. $\frac{350}{6.5\pi}=\frac{6.5\pi r^2}{6.5\pi}$

Simplify and put $r^2$ on the LHS. $r^2=\frac{350}{6.5\pi}$

Find the square root. $r=\sqrt{\frac{350}{6.5\pi}}$

Use your calculator. $r=4.1400...$

Round to the specified accuracy. $r\approx 4.14$ correct to 2 decimal places

# EXERCISE 1.03 Linear equations

## Concepts and techniques

**1** Example 6 Solve the following.

**a** $z + 1.7 = -3.9$ **b** $5 = x - 7$ **c** $-2 = \frac{n}{8}$

**d** $33 = 4k + 9$ **e** $-2 = 5x - 27$ **f** $\frac{x}{2} - 3 = 7$

**g** $3x + 5 = 17$ **h** $7y - 1 = 20$ **i** $3(x + 2) = 15$

**j** $7t + 4 = 3t - 12$ **k** $2(a - 2) = 4 - 3a$ **l** $3(t + 7) = 2(2t - 9)$

**m** $3.7x + 1.2 = 5.4x - 6.3$

**2** Solve the following.

**a** $\frac{b}{5} = \frac{2}{3}$ **b** $\frac{4}{y} = \frac{9}{10}$ **c** $\frac{4}{5} = \frac{9}{2k}$

**d** $\frac{5t}{4} = \frac{3}{4}$ **e** $\frac{y}{2} = -\frac{3}{5}$ **f** $\frac{w-3}{2} = 5$

**g** $\frac{x}{4} + \frac{1}{2} = 4$ **h** $\frac{x+4}{3} + \frac{x}{2} = 1$ **i** $\frac{t+3}{7} + \frac{t-1}{3} = 4$

**j** $\frac{q-1}{3} - \frac{q-2}{4} = 2$ **k** $\frac{3b}{4} - \frac{1}{5} = \frac{b}{2}$ **l** $\frac{5}{x+2} = \frac{3}{x}$

**m** $\frac{2}{t-3} + \frac{1}{t+4} = 0$

**3** Example 7 CAS Solve each of the following.

**a** $\frac{7}{x} = \frac{1}{5} \quad (x \neq 0)$ **b** $\frac{5x}{4} = \frac{11}{7}$ **c** $\frac{x}{3} - 4 = 8$

**d** $\frac{5+x}{7} = \frac{2}{7}$ **e** $\frac{x}{9} - \frac{2}{3} = 7$ **f** $\frac{2t}{5} - \frac{t}{3} = 2$

**g** $\frac{x}{5} - \frac{x}{2} = \frac{3}{10}$ **h** $\frac{p-3}{2} + \frac{2p}{3} = 2$ **i** $\frac{x+5}{9} - \frac{x+2}{5} = 1$

**j** $\frac{x+3}{5} + 2 = \frac{x+7}{2}$ **k** $\frac{a}{3} + \frac{3}{4} = \frac{5}{8}$ **l** $\frac{1}{y+1} = \frac{1}{3y-1}$

**4** Example 8 Change the subject of the formula to the pronumeral given in brackets.

**a** $x = \frac{x_1 + x_2}{2} \quad (x_2)$ **b** $V = \frac{1}{3}Ah \quad (A)$

**c** $S = \frac{n}{2}(2a + (n-1)d) \quad (a)$ **d** $v^2 = u^2 + 2as \quad (s)$

**e** Given $b \geq 0$, $c^2 = a^2 + b^2 \quad (b)$

**5** **a** Given that $v = u + at$ is the formula for the velocity of a particle at time $t$, find the value of $t$ when $u = 17.3$, $v = 100.6$ and $a = 9.8$.

**b** The formula for finding the area of a triangle is $A = \frac{1}{2}bh$. Find $b$ when $A = 36$ and $h = 9$.

**c** Find the value of $y$ when $x = 3$, given the straight line equation $5x - 2y - 7 = 0$.


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**d** The area of a rhombus is given by the formula $A = \frac{1}{2}xy$, where $x$ and $y$ are its diagonals. Find the value of $x$, correct to 2 decimal places, when $y = 7.8$ and $A = 25.1$.

**e** The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find $y_1$ when $m = -\frac{5}{6}$, $y_2 = 7$, $x_2 = -3$ and $x_1 = 1$.

**f** A formula for depreciation is $D = P(1 - r)^n$. Find $r$ if $D = 12\,000$, $P = 15\,000$ and $n = 3$.

**g** Given that the height of a particle at time $t$ is $h = 5t^2$, evaluate $t$ when $h = 23$.

**h** If the surface area of a sphere is $S = 4\pi r^2$, evaluate $r$, correct to 3 significant figures, when $S = 56.3$.

**i** If $y = \frac{2}{x^3 - 1}$, find the value of $x$ when $y = 3$.

**j** The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Evaluate $r$, correct to 1 decimal place, when $V = 150$.

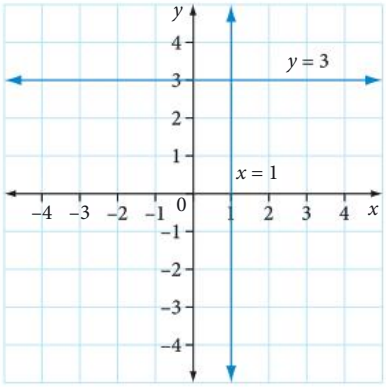
# 1.04 LINEAR RELATIONSHIPS

You studied **linear relationships** in junior mathematics.

**IMPORTANT**

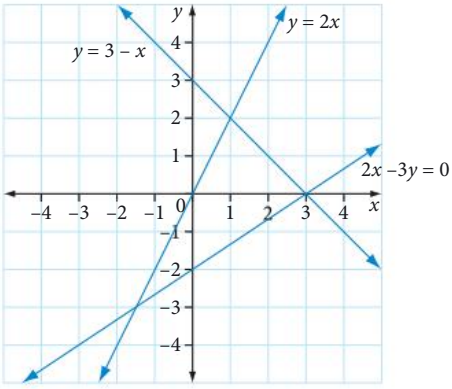
The power of any variable in a linear relationship is one. A linear relationship involving one or two variables can be plotted on the Cartesian plane and will give a straight line graph.

When you draw the graph of a linear relationship involving only one variable you will get a horizontal or vertical line. The graphs of $y = 3$ and $x = 1$ are shown on the right.




All other straight lines are generated by a linear relationship involving two variables. You can write these in the form $y = mx + c$. When you write the equation like this, the value of $y$ depends upon the value of $x$, so you call $y$ the **dependent** variable and $x$ the **independent** variable.

In the graph below, the equation $2x - 3y - 6 = 0$ is written in the general form. You could rearrange the equation of this line to $y = \frac{2}{3}x - 2$.



## Example 9

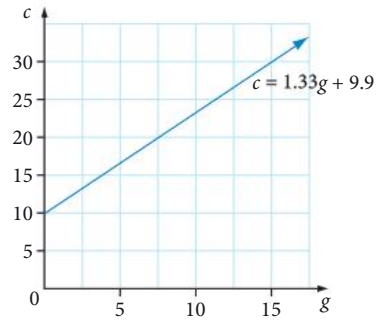
The monthly cost of Bill's internet connection was \$9.90 plus \$1.33 per GB. His monthly cost, $c$ (\$), for downloading $g$ (gigabytes) is found from the equation $c = 1.33g + 9.9$. Show this on a graph and comment on the relationship between $g$ and $c$.

### Solution

Construct a table of values with the independent variable in the first row.

| $g$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $c$ | 9.9 | 11.23 | 12.56 | 13.89 | 15.22 |

Draw a number plane with the independent variable on the horizontal axes. Plot the points from the table of values and join them. You get the straight line on the right. The arrow on the end shows that the line continues for higher values of $g$.




Use the fact that it is a straight line.

The graph shows it is a linear relationship.

For quantities $x$ and $y$ in direct proportion, as $x$ increases or decreases $y$ increases or decreases in the same ratio.

**IMPORTANT**

The graph of quantities in **direct proportion** is a straight line through the origin, so it is a linear relationship. The equation is of the form $y = kx$, where $k$ is a constant.

When you compare $y = mx + c$ to $y = kx$, you can see that direct proportion is a linear relationship with $c = 0$.

## Example 10

Stefan works as a salesperson who earns 5% commission on his sales for the week. His weekly wage, \$$w$, for sales of \$$s$ is represented by $w = 0.05s$. Show the relationship on a graph and comment on the shape.

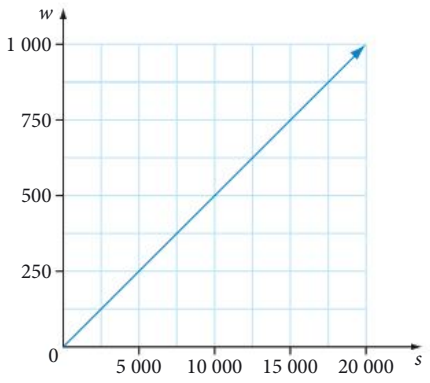
### Solution

Construct a table of values.

| $s$ | 0 | 5000 | 10 000 |
|---|---|---|---|
| $w$ | 0 | 250 | 500 |


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Draw a number plane. Plot the points from the table of values and join. You get a straight line through the origin.




Make a comment.

The graph shows the relationship is direct proportion.

# EXERCISE 1.04 Linear relationships

## Concepts and techniques

1 State which of the following is a linear relationship.

a $y = 4x + 5$ b $\sqrt{x} + y = 3$ c $y = x^2 - 3$

d $5b^2 + 3 = a$ e $5^x + y = 11$ f $\frac{1}{x} + 3 = y$

g $(v + 4)(v - 4) = w + 2$ h $3(m + 2) - 5(n - 2) = 4(n + 4)$

2 Example 9 The linear relationship between the distance, $d$ km, travelled by a truck and the running costs, $\$C$, for the trip is given by $C = 0.7d + 8.4$. Draw a graph of this relationship on a number plane.

3 A criminologist discovered that there was a linear relationship between the number of road accidents per month, $A$, in a given area and the number of police cars, $C$, patrolling the area. The relationship was shown by $A = -4C + 32$.

a Which is the independent variable?

b Copy and complete this table for the given relationship.

| Number of police cars, $C$. | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Number of accidents per month, $A$. | | | | | |

c Draw a graph of the relationship.

4 Example 10 The number of teachers, $t$, at Smallville High School is directly proportional to the number of students, $s$, who attend the school. The staffing is calculated using the formula $t = 0.09s$. Draw a graph of this relationship. Is this a linear relationship?

5 The mass, $M$ (in grams), of a metal is directly proportional to its volume, $V$ (in cubic centimetres). This relationship for liquid bronze is $M = 8V$. Draw its graph and name the dependent variable.

## Reasoning and communication

6 During the summer, crickets chirp faster at night if the temperature is higher. Folklore says that if you count the number of chirps per minute then you can work out the temperature using this equation: $T = \frac{63n + 2160}{486}$. $T$ is the temperature in degrees Celsius and $n$ is the number of chirps per minute.

a Draw a graph of this relationship for up to 250 chirps per minute.

b Identify the dependent and independent variables.

c If the temperature rises from 18°C to 20°C, what happens to the cricket's chirp rate?

7 The number of trees in a forest is decreasing each year. The equation $n = -7200t + 850\,000$ shows the number of trees, $n$, that will be in the forest after $t$ years.

a Copy and complete this table.

| **Time in years, *t*.** | 0 | 5 | 10 | 20 | 50 |
|---|---|---|---|---|---|
| **Number of trees, *n*.** | | | | | |

b Draw a graph of this equation.

c Comment on the shape of the graph.

8 The weight of an astronaut on the Moon, $M$ (in kilograms), compared to his weight on Earth, $E$ (in kilograms) can be calculated by using the formula $M = 0.165E$.

a Draw a graph of this equation.

b Is this a linear relationship? Why?

c Is this an example of direct proportion? Justify your answer.

d From your graph find the weight on Mars of an object weighing 200 kg on Earth.

# 1.05 GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

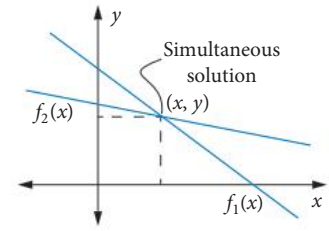
Linear equations may involve more than one variable. To solve equations with more than one variable, more than one equation is needed.

**IMPORTANT**

**Simultaneous equations** are sets of equations with more than one variable. Values that make all the equations true are called **simultaneous solutions**.

You can solve simultaneous linear equations in two variables by graphing the equations. The point of intersection of the graphs is the solution. The graphs of two linear equations can intersect only once, and so they can have only one simultaneous solution.

Sometimes, the graphs of linear equations don't intersect. This can only happen if the lines are parallel, in which case there is no simultaneous solution.





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## Example 11

Use graphs to solve the simultaneous equations $y = 3x - 2$ and $x + 2y - 10 = 0$.

### Solution

Write down the first equation.

$y = 3x - 2$

Construct a table of values. Three points will be enough.

| $x$ | $-2$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-8$ | $-2$ | $7$ |

Write down the ordered pairs.

$(-2, -8)$, $(0, -2)$, $(3, 7)$

Write down the second equation.

$x + 2y - 10 = 0$

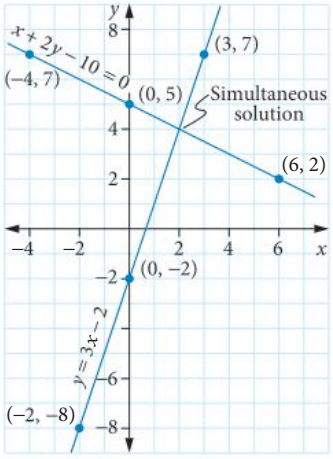
Susbstitute some values for $x$ that give easy (whole number) answers for $y$.

| $x$ | $-4$ | $0$ | $6$ |
|---|---|---|---|
| $y$ | $7$ | $5$ | $2$ |

Write down the ordered pairs.

$(-4, 7)$, $(0, 5)$, $(6, 2)$

Draw the graphs on the same axes.




Find the intersection.

The lines intersect at $(2, 4)$.

Check the solution by substitution in the original equations.

$4 = 3 \times 2 - 2$ OK

$2 + 2 \times 4 - 10 = 0$ OK

Write the solution.

The simultaneous solution is $(2, 4)$ or $x = 2, y = 4$.

It is difficult to get accurate answers from hand-drawn graphs. The answers obtained from CAS calculator graphs will be more accurate.

## Example 12

CAS Solve the simultaneous equations $y = 2x + 1$ and $4x + y + 5 = 0$.

### Solution

Make sure both equations are in the form $y = mx + c$. Rearrange the second equation so $y$ is the subject.

You could do this by entering solve($4x + y + 5 = 0$, $y$) on your CAS calculator, but it's a bit of overkill.

$4x + y + 5 = 0$
$4x + y + 5 - 4x - 5 = 0 - 4x - 5$
$y = -4x - 5$

#### TI-Nspire CAS

Choose a Graphs page .

Complete the function entry line as:

f1 = $2x + 1$ enter

Then press menu, 3 Graph Entry/Edit and 1: Function.

Complete as follows:

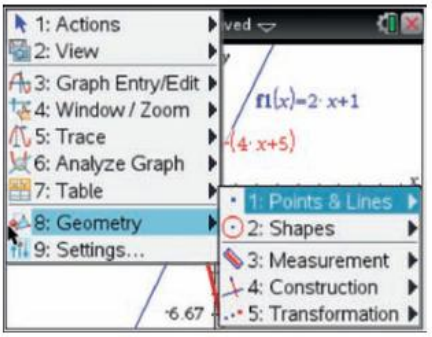
f2 = (−) $4x - 5$

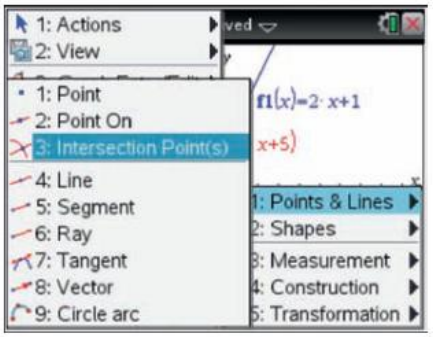
Press enter.

To find the point of intersection of the two lines:

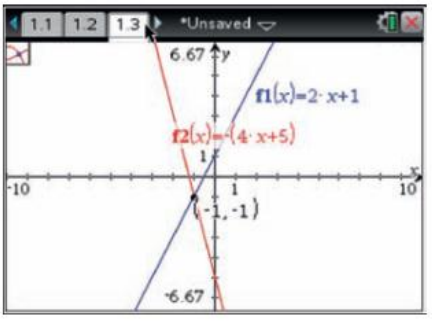
Press

- menu
- 8: Geometry
- 1: Points & Lines
- 3: Intersection Point(s)




Move the cursor and click on each line. The point of intersection will appear.





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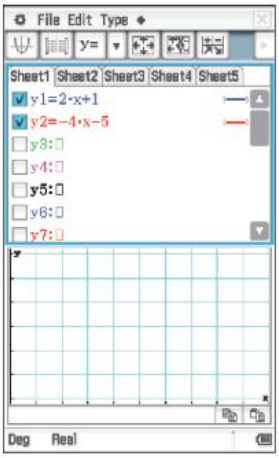
Write the solution.

The simultaneous solution is $(-1, -1)$ or $x = -1, y = -1$.

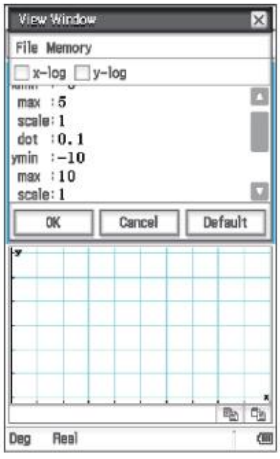
**ClassPad**

Go to the **Graphs&Table** application.
Clear the screen by tapping **Edit** and then **Clear All**.

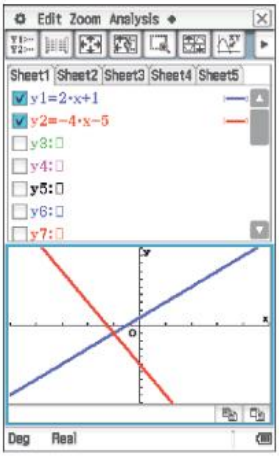
Make $y1 = 2x + 1$ and press EXE.
Make $y2 = -4x - 5$ and press EXE.
Make sure both $y1$ and $y2$ are ticked.



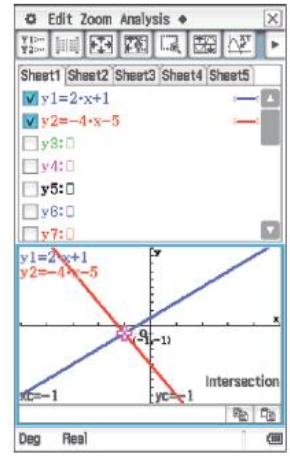

Tap View Window. Set reasonable limits for $x$ and $y$, e.g. $-5 \le x \le 5$ and $-10 \le y \le 10$.
If these prove to be unsuitable, you can always go back to **View Window** and set new limits.




Tap **OK** on the **View Window** window.
Tap .
Both graphs appear and seem to intersect at $(-1, -1)$.



To find the point of intersection, tap **Analysis**, then **G-Solve**, then **Intersection**.
The intersection point is shown below as $xc = -1$, $yc = -1$.




Write the solution.

The simultaneous solution is $(-1, -1)$ or $x = -1$, $y = -1$.

## EXERCISE 1.05 Graphical solution of simultaneous equations

### Concepts and techniques

1 Example 11 Find the simultaneous solutions to the following linear equations using graphs.

a $y = x + 4$ and $y = -x - 6$
b $y = 2x + 1$ and $y = -x - 2$
c $y = 3x + 4$ and $y = x + 1$
d $y = -2x + 7$ and $y = -x - 4$
e $x = 1 - y$ and $3 = y + 2x$
f $x = y + 3$ and $x + 2y = 9$
g $4x - y = 10$ and $x + 3y = 9$
h $4x - y = 6$ and $3x + 2y = -1$
i $5x + y = 12$ and $3x + 2y = 3$
j $2x + 3y + 1 = 0$ and $x + 4y = 2$

2 Example 12 CAS Find simultaneous solutions of the following.

a $3x + 2y = 14$ and $4x - 2y = 7$
b $-5x + 5y = 25$ and $3x - 5y = -21$
c $6x + 2y = 22$ and $3x + 2y = 16$
d $-2x + 5y = 20$ and $-2x + 2y = 5$
e $3x - 2y = 10$ and $-3x - 2y = -14$
f $2x - 3y = 13$ and $3y - x = -8$

# 1.06 ALGEBRAIC SOLUTION OF SIMULTANEOUS EQUATIONS

You have seen that even though it is possible to solve simultaneous equations graphically, the process can be tedious and lack accuracy. You can solve simultaneous equations using algebraic methods.

The elimination method adds the equations.


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**IMPORTANT**

**Steps for the elimination method**

1. Choose which variable to eliminate.
2. Make the coefficients of that variable the same size, but opposite in sign, by multiplying by suitable numbers.
3. Add the equations to eliminate the variable.
4. Solve the new equation.
5. Substitute the solution into either of the original equations and solve for the other variable *or* repeat the elimination method for the other variable.
6. Check the values in *both* of the original equations.

## Example 13

Solve by elimination: $2m + n = 12$

$3m - 4n = 7$

### Solution

| | |
|---|---|
| Write down and label the equations. | $2m + n = 12$ [1]<br>$3m - 4n = 7$ [2] |
| Eliminate $n$ first. You need the same coefficient but opposite signs for $n$ in both equations. Multiply [1] by 4 and relabel it. Show the equation numbers and changes on the right of the equations. | $3m - 4n = 7$ [2]<br>$8m + 4n = 48$ [1] × 4 = [1a] |
| Eliminate $n$ by adding [1a] and [2]. | $11m = 55$ [2] + [1a] |
| Solve for $m$. | $m = 5$ |
| Substitute for $m$ in [2]. | Substitute $m = 5$ into [2].<br>$3 \times 5 - 4n = 7$ |
| Simplify. | $15 - 4n = 7$ |
| Subtract 15 from both sides. | $-4n = -8$ |
| Solve for $n$. | $n = 2$ |
| Check the values of $m$ and $n$ in the original equations by substituting $m = 5$ and $n = 2$. | $2m + n = 2 \times 5 + 2 = 10 + 2 = 12$ [1] OK<br>$3m - 4n = 3 \times 5 - 4 \times 2 = 15 - 8 = 7$ [2] OK |
| Write the solution. | The simultaneous solution is $m = 5$ and $n = 2$. |

It may be necessary to simplify equations first. Get rid of fractions and brackets before you start. You need to collect all variables on the left-hand side and the constant on the right-hand side of each equation.

The other method is called the substitution method. You substitute an expression from one equation into the other. You need to arrange one of the equations so that one of the variables is the subject.

**IMPORTANT**

**Steps for the substitution method**

1. Use one equation to work out a formula for one variable.
2. Substitute the formula for the variable into the other equation.
3. Solve the new equation.
4. Substitute the solution into the formula or one of the original equations.
5. Check the values in *both* of the original equations.

## Example 14

Solve by substitution: $2a + b = 3$

$-5a + 2b = -12$

### Solution

| | |
|---|---|
| Write down and label the equations. | $2a + b = 3$ [1]<br>$-5a + 2b = -12$ [2] |
| Rearrange [1] to make $b$ the subject. | $b = 3 - 2a$ [1a] |
| Now substitute for $b$ in [2]. | Substitute $b = 3 - 2a$ in [2].<br>$-5a + 2(3 - 2a) = -12$ [2a] |
| Expand the brackets. | $-5a + 6 - 4a = -12$ |
| Regroup and simplify terms. | $-9a = -18$ |
| Divide both sides by $-9$ to solve for $a$. | $a = 2$ |
| Substitute for $a$ in [1a]. | Substitute $a = 2$ into [1a]<br>$b = 3 - 2 \times 2$ |
| Simplify. | $b = -1$ |
| Check the values of $a$ and $b$ in the original equations. | $2a + b = 2 \times 2 + (-1) = 4 - 1 = 3$ [1] OK<br>$-5a + 2b = -5 \times 2 + 2 \times (-1) = -10 - 2 = -12$ [2] OK |
| Write the solution. | The simultaneous solution is $a = 2$ and $b = -1$. |

Use whichever method or combination is best for a particular problem. If the equations have a variable with a coefficient of 1, the substitution method may be easier. If one variable has a fractional coefficient, the elimination method is usually easier.

You can use a CAS calculator to solve simultaneous equations. You do not need to change the equations in any way.


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## Example 15

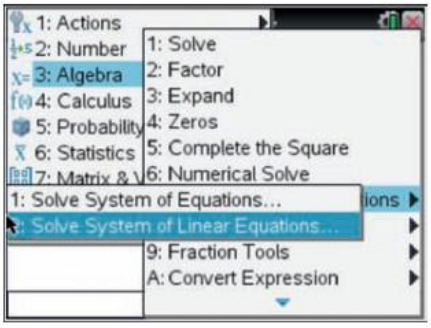
CAS Solve $5p = -2(q + 9)$ and $4q + \frac{13 - 2p}{3} = -1 - 3p$.

### Solution

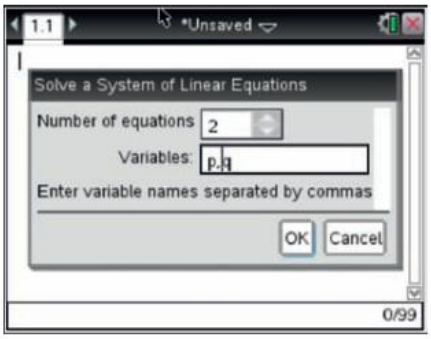
**TI-Nspire CAS**

Choose a Calculator page .
Press

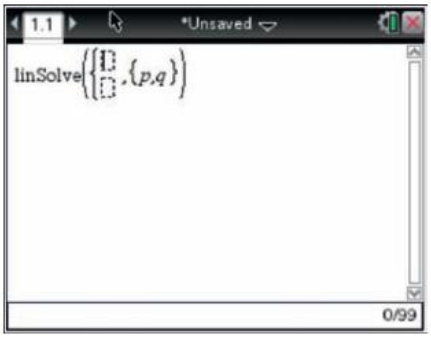
- menu
- 3:Algebra
- 7: Solve System of Equations
- 2: Solve System of Linear Equations



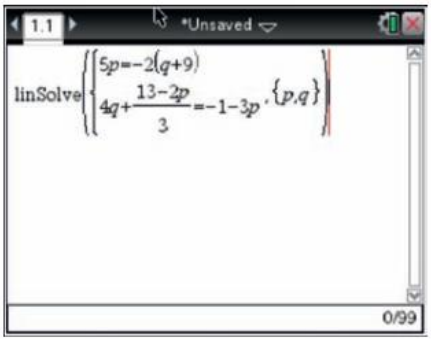

Change the variables to those in your equations, $p$ and $q$.
Then press OK.



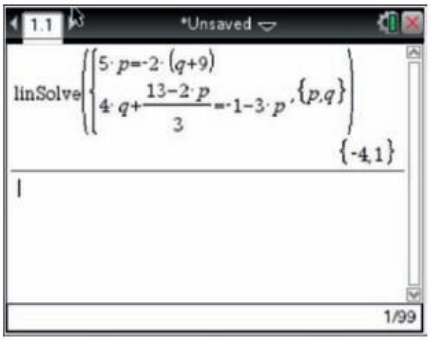

The screen will give you two lines to type in your two equations.




Type them in exactly as given in the question.
Press the key and choose the fraction template ($\frac{\square}{\square}$) to do $\frac{13 - 2p}{3}$.



Then press enter.

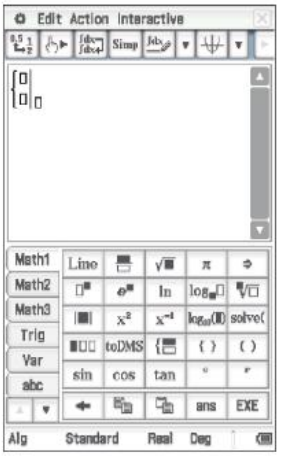



Write the solution.

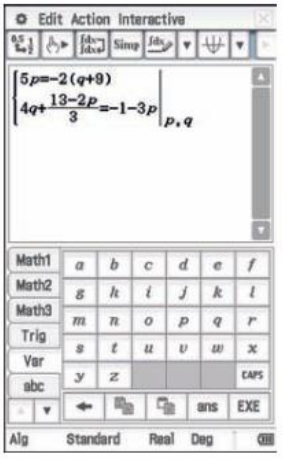
The simultaneous solution is $p = -4$ and $q = 1$.

### ClassPad

Use the Main $\sqrt{\alpha}$ menu.
Press Keyboard, then tap the Math1 tab.
Tap the simultaneous equations symbol $\{\square$.



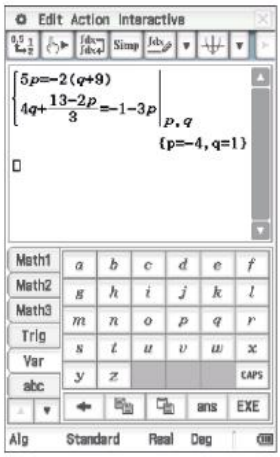

Enter the first equation.
Tap the Var tab to use the variables. Do not use the abc tab.
Tap the down arrow and enter the second equation.
Tap Math1 and use $\square$ with the arrows for fractions.
Tap the right arrow and enter the variables you want to find as $p$, $q$.





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Tap EXE or press the EXE key.
The values of $p$ and $q$ are shown.




Write the solution.

The simultaneous solution is $p = -4$ and $q = 1$.

# EXERCISE 1.06 Algebraic solution of simultaneous equations

## Concepts and techniques

Simultaneous equations

**1** Example 13 Solve the following simultaneous equations by elimination.

**a** $3a + 2b = 12$
$4a - 2b = 2$

**b** $3x - 4y = 8$
$2x + y = 9$

**c** $2k + 3j = 5$
$5j + 3k = 9$

**d** $3g + 2f = -6$
$3f + g = 5$

**e** $p + 2q = 3$
$2p + q = 9$

**f** $f + g = 3$
$3f - g = -15$

**2** Solve the following simultaneous equations by elimination. Rearrange them first.

$4(15 - 2m) = 10m + 6n$ and $m + \dfrac{16 - n}{5} = 2m$

**3** Example 14 Solve the following simultaneous equations by substitution.

**a** $k + 2m = 11$
$3m - 2k = 6$

**b** $k - 2j = -3$
$2k + 3j = 22$

**c** $2p - 3q = 11$
$2p - q = 1$

**d** $2b - 3a = 22$
$a + 5b = 4$

**e** $x + y = -4$
$2x - 3y = 7$

**f** $3v + w = 0$
$2v + 3w = 14$

**4** Solve the following simultaneous equations.

**a** $2b + 3u = 11$
$5b - 2u = -1$

**b** $2a + 3r = 10$
$4a - 2r = -4$

**c** $4e - 3m = -4$
$6m - 4e = 9$

**d** $y - t = 2$
$3y - 4t = 12$

**e** $3a - 2n = 3$
$a + 3n = 23$

**f** $4i + 7s = 10$
$2i - 5s = -46$

**g** $3b + 4y = -3$
$b - y = 3$

**h** $2u - 9p = -7$
$3u + 2p = 36$

**5** Solve the following.

**a** $5(2e-f)+4f=3(3e+1)$ and $5(2e-1)=5e-2f+38$

**b** $\frac{x}{2}+g=5$ and $\frac{3x}{4}+\frac{g}{3}=4$

**c** $\frac{2k}{3}=13+m$ and $3k-\frac{3m}{2}=7$

**d** $\frac{2}{b}+\frac{5}{a}=6$ and $\frac{5}{a}-\frac{2}{b}=4$ (Hint: Put $x=\frac{1}{a}$ and $y=\frac{1}{b}$.)

**6** Example 15 CAS Solve these equations.

**a** $3h+i=3$
$2h-i=7$

**b** $4c-3d=11$
$3c-4d=17$

**c** $2t+3u=4$
$2u-t=12$

**d** $3f-g=2$
$f+2g=17$

**e** $\frac{4a+3b}{2}=3a-\frac{5a}{2}-3$
$\frac{b-7a}{18}=\frac{5b}{12}-\frac{a-1}{2}$

## Reasoning and communication

**7** A theme park charges \$78 for adults and \$48 for children. A party of 21 people paid \$1188 altogether. How many adults and how many children were there?

**8** Petra is four times as old as her daughter Philippa. In 4 years' time she will only be three times as old as Philippa. How old will Petra be when Philippa is half her age?

**9** The materials to make 25 kg of an alloy of copper and zinc cost \$124. If the copper costs \$6.40/kg and the zinc costs \$2.80/kg, find the composition of the alloy.

**10** A tourist boat makes trips both upriver and downriver from its mooring. The downriver trip is 36 km each way and takes 8 hours for the round trip. The upriver destination is 60 km away and it takes 10 hours to get there. Find the speed of the boat in still water and the speed of the river.

# 1.07 QUADRATIC EQUATIONS

An equation where the variable has a power of 2 is called a **quadratic** equation. It can also have a term with a power of 1 and/or a constant term. You can use the factorisation methods from Section **1.02** to solve some quadratic equations.

**IMPORTANT**

The general form of a **quadratic equation** is $ax^2+bx+c=0$ where $a\neq 0$

Quadratic equations where you can factorise the expression $ax^2+bx+c=0$ are solved using the **null factor law**.


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**IMPORTANT**

**Null factor law**

If $ab = 0$, then either $a = 0$, $b = 0$ or $a = b = 0$.

## Example 16

Solve the following quadratic equations.

a $(x-3)(x+4)=0$ b $5p^2+2p-70=2p^2+14p-7$

c $x+5=-\frac{6}{x}$ d $3a^2-14a=-8$

### Solution

| | | |
|---|---|---|
| a | Write down the equation. | $(x-3)(x+4)=0$ |
| | Apply the null factor law. | $(x-3)=0$ or $(x+4)=0$ |
| | Solve the resulting equations. | $x=3$ or $x=-4$ |
| | State the solution. | $x=3$ or $x=-4$ |
| b | Write down the equation. | $5p^2+2p-70=2p^2+14p-7$ |
| | Express in general form. | $3p^2-12p-63=0$ |
| | Take out the common factor. | $3(p^2-4p-21)=0$ |
| | Factorise the quadratic. | $3(p-7)(p+3)=0$ |
| | Apply the null factor law. | $p-7=0$ or $p+3=0$ |
| | Solve the resulting equations. | $p=7$ or $p=-3$ |
| | State the solution. | $p=7$ or $p=-3$ |
| c | Write down the equation. | $x+5=-\frac{6}{x}$ |
| | Multiply by the denominator to clear the fraction. | $x^2+5x=-6$ |
| | Express in general form. | $x^2+5x+6=0$ |
| | Factorise. | $(x+2)(x+3)=0$ |
| | Apply the null factor law. | $x+2=0$ or $x+3=0$ |
| | Solve the resulting equations. | $x=-2$ or $x=-3$ |
| | State the solution. | $x=-2$ or $x=-3$ |

| | | |
|---|---|---|
| d | Write down the equation. | $3a^2 - 14a = -8$ |
| | Express in general form. | $3a^2 - 14a + 8 = 0$ |
| | Factorise. | $(3a - 2)(a - 4) = 0$ |
| | Solve using the null factor law. | $3a - 2 = 0$ or $a - 4 = 0$ |
| | Solve the resulting equations. | $a = \frac{2}{3}$ or $a = 4$ |
| | State solution. | $a = \frac{2}{3}$ or $a = 4$ |

You can use other methods to solve quadratic equations.

**Completing the square** is a technique that relies on the perfect square factorisation $x^2 + 2ax + a^2 = (x + a)^2$. The two solutions are obtained using the fact that there are two numbers that square to give a positive number. For example, $6^2 = 36 = (-6)^2$. The solution to $x^2 = 36$ is thus $\pm 6$, which means 6 or $-6$.

**IMPORTANT**

**Completing the square**

1. Write the equation in general form.
2. Divide each term by the coefficient of $x^2$, unless it is 1.
3. Put the constant term on the right-hand side of the equation.
4. Halve the coefficient of $x$ to find the value of $a$ in $x^2 + 2ax + a^2 = (x + a)^2$.
5. Square the value of $a$ and add it to both sides.
6. Factorise the left side as a perfect square.
7. Take the square root of both sides, remembering to do both the positive and negative roots of the number on the RHS.
8. Solve the two equations to find both solutions.

## Example 17

Solve by completing the square where necessary.

a $(x + 3)^2 = 11$

b $x^2 + 6x - 16 = 0$

c $3p^2 = 12 - 15p$, correct to 2 decimal places

### Solution

| | | |
|---|---|---|
| a | Write the equation. | $(x + 3)^2 = 11$ |
| | Take square root of both sides. | $x + 3 = \pm\sqrt{11}$ |
| | Solve the equations by subtracting 3 from both sides. | $x = -3 \pm \sqrt{11}$ |
| | Write your solutions. | $x = -3 \pm \sqrt{11}$ |

**b** Write down the equation. $x^2 + 6x - 16 = 0$

Move the constant term to the RHS. $x^2 + 6x = 16$

Halve the coefficient of $x$, which is 6, to give $a$. So $a = 3$. Add $a^2$ to both sides. $x^2 + 6x + 9 = 16 + 9$

Use the pattern $x^2 + 2ax + a^2 = (x + a)^2$ to write the LHS as a perfect square. $(x + 3)^2 = 25$

Take the square root of both sides. $x + 3 = \pm 5$

Solve the resulting equations. $x = -3 + 5$ or $x = -3 - 5$

Write the solution. $x = 2$ or $x = -8$

**c** Write the equation. $3p^2 = 12 - 15p$

Express with constant term only on the RHS. $3p^2 + 15p = 12$

Divide throughout by the common factor 3. $p^2 + 5p = 4$

Now $a = 5 \div 2 = \frac{5}{2}$, so $a^2 = \frac{25}{4}$

Add $a^2$ to both sides. $p^2 + 5p + \frac{25}{4} = 4 + \frac{25}{4}$

Use $x^2 + 2ax + a^2 = (x + a)^2$ to factorise the LHS, and simplify the RHS. $\left(p + \frac{5}{2}\right)^2 = \frac{41}{4}$

Take the square roots. $p + \frac{5}{2} = \pm\frac{\sqrt{41}}{2}$

Solve the resulting equations. $p = \frac{\sqrt{41} - 5}{2}$ or $\frac{-\sqrt{41} - 5}{2}$

Write as approximations. $p = 0.7015...$ or $p = -5.7015...$

Write the solutions correct to 2 decimal places. $p \approx 0.70$ or $p \approx -5.70$ to 2 decimal places.

# EXERCISE 1.07 Quadratic equations

## Concepts and techniques



Completing the square

**1** Example 16 Solve the following using factorisation where necessary.

**a** $2x(x - 12) = 0$
**b** $(x + 2)(x - 5) = 0$
**c** $x^2 - 8x = 0$
**d** $4x^2 + 6x = 0$
**e** $x^2 + 9x + 14 = 0$
**f** $x^2 - 8x + 15 = 0$

**2** Solve by factorisation.

**a** $3x^2 + 5x - 12 = 0$
**b** $2x^2 - 24x + 22 = 0$
**c** $3x^2 + 15x + 12 = 0$
**d** $4x^2 - 28x - 32 = 0$
**e** $2x^2 + 14x + 20 = 0$
**f** $x^2 - 2x = 3$

3 Solve by factorisation.

a $x-4=\frac{5}{x}$ b $x^2-x=6$ c $x^2-x=12$

d $x^2-6x=-9$ e $2x^2+5x+2=0$ f $3x^2+10x+3=0$

4 Solve by factorisation.

a $5a^2+8a-4=0$ b $2v^2-19v+42=0$

c $n^2-64=0$ d $4c^2-28c+49=0$

e $14e^2+33e+18=0$ f $9p^2+10p-16=0$

5 Solve by factorisation.

a $2b^2=11b-14$ b $10d^2+10d+10=3d^2-9d$

c $6x^2-5x-50=5x^2-5x-1$ d $5a^2=4(a^2+2a-4)$

e $4g(g+6)=g-15$ f $16(x^2-1)=14x-19$

6 Solve by factorisation.

a $x^2-2x+2=\frac{1+3x^2}{4}$ b $\frac{x^2+9x}{4}+5=\frac{4x+1}{2}-2x$

c $\frac{3x^2}{2}+2x=1\frac{1}{3}-\frac{x}{3}$ d $\frac{x^2}{4}+\frac{x}{3}=\frac{x^2}{6}+\frac{x+2}{4}$

e $16x=\frac{2x-54}{x-4}$ f $4x-\frac{48}{x-3}=x+2$

7 Example 17 Solve by completing the square. Express irrational answers correct to 2 decimal places.

a $k^2+4k-5=0$ b $v^2+2v-35=0$

c $j^2=9j-8$ d $x^2=x+132$

e $m^2=\frac{8}{9}-\frac{2m}{3}$ f $b^2+4b-6=0$

# 1.08 THE QUADRATIC FORMULA AND THE DISCRIMINANT

For most quadratics, completing the square is a tedious method. You can use the **quadratic formula instead**.

**IMPORTANT**

**The quadratic formula**

For any quadratic equation $ax^2+bx+c=0$ where $a\neq 0$, the solutions are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The quadratic formula can be proved by completing the square for the general quadratic equation $ax^2+bx+c=0$. Your teacher might want you to do this as an exercise.


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## Example 18

Solve $3x^2 = 4x + 12$ using the quadratic formula. Leave your answer in exact form.

### Solution

| | |
|---|---|
| Write the equation. | $3x^2 = 4x + 12$ |
| Express in general form. | $3x^2 - 4x - 12 = 0$ |
| Compare with $ax^2 + bx + c = 0$. | $a = 3, b = -4, c = -12$ |
| Write the quadratic formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Substitute for $a$, $b$ and $c$. | $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-12)}}{2 \times 3}$ |
| Evaluate. | $= \dfrac{4 \pm \sqrt{16 + 144}}{6}$ |
| Simplify. | $= \dfrac{4 \pm \sqrt{160}}{6}$ |
| Simplify the surd. | $= \dfrac{4 \pm \sqrt{16} \times \sqrt{10}}{6}$ <br> $= \dfrac{4 \pm 4\sqrt{10}}{6}$ |
| Factorise. | $= \dfrac{4(1 \pm \sqrt{10})}{6}$ |
| Simplify. | $= \dfrac{2(1 \pm \sqrt{10})}{3}$ |
| Write the solution. | $x = \dfrac{2(1 + \sqrt{10})}{3}$ or $\dfrac{2(1 - \sqrt{10})}{3}$ |

You can obtain the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ by using the quadratic formula. The expression $b^2 - 4ac$ is called the **discriminant**. You can use it to work out the number of solutions the equation has and whether any solutions are rational or irrational.



Quadratic formula

Remember that rational numbers can be written in the form $\frac{a}{b}$, where $a$ and $b$ are integers. Irrational numbers cannot. For example, $\sqrt{2}$, $\sqrt{11}$ and $\sqrt{20}$ are irrational. Written as decimals, rational numbers are terminating or recurring decimals, but irrational numbers do not terminate or recur.

The expression $3x^2 - 6x - 8$ cannot be factorised so the equation $3x^2 - 6x - 8 = 0$ has irrational solutions. Using the quadratic formula, you get the solutions $x = \frac{3+\sqrt{33}}{3}$ or $x = \frac{3-\sqrt{33}}{3}$. Notice that for this equation the discriminant is $(-6)^2 - 4 \times 3 \times (-8)^2 = 132$, which is not a perfect square.

The expression $3x^2 - 4x - 4$ can be factorised, so the equation $3x^2 - 4x - 4 = 0$ has rational solutions. From the formula, we get $x = \frac{4+\sqrt{64}}{6}$ or $x = \frac{4-\sqrt{64}}{6}$, which gives $x = 2$ or $x = -\frac{2}{3}$. Notice for this equation that the discriminant is 64, which is a perfect square.

The discriminant in the equation $16x^2 - 40x + 25 = 0$ is zero, so the equation has only one rational solution, $x = \frac{5}{4}$.

The discriminant of the equation $3x^2 - 9x + 8 = 0$ is $-15$. Using the formula gives you $x = \frac{9 \pm \sqrt{-15}}{6}$. There is no real number with a square of $-15$, so there are no real solutions to the equation.

The number of solutions and types of solutions are determined by the value of the expression under the square root sign, $b^2 - 4ac$. The symbol for the discriminant is $\Delta$.

**IMPORTANT**

The **discriminant** for any quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

- For $\Delta < 0$, there are *no* real solutions.
- For $\Delta = 0$, there is only *one* real solution, which is given by $x = -\frac{b}{2a}$. If the coefficients of the equation are rational, then this solution will be rational.
- For $\Delta > 0$, there are *two* real solutions. For equations with rational coefficients, the solutions are:
  - *rational* if $\Delta$ is a perfect square, but
  - *irrational* if $\Delta$ is not a perfect square.

### Example 19

Use the discriminant to determine the number and nature of the solutions of $10x^2 - 9x - 7 = 0$.

#### Solution

| | |
|---|---|
| Write the equation. | $10x^2 - 9x - 7 = 0$ |
| Compare with $ax^2 + bx + c = 0$. | $a = 10, b = -9, c = -7$ |
| Write the formula for the discriminant. | $\Delta = b^2 - 4ac$ |
| Substitute for $a$, $b$ and $c$. | $\Delta = (-9)^2 - 4 \times 10 \times (-7)$ |
| Evaluate. | $\Delta = 361$ |
| $\Delta = 361 > 0$, so there are 2 solutions. 361 is a perfect square ($19^2 = 361$), so the solutions are rational. State your answer. | There are two rational solutions. |

Your CAS calculator can be used to solve quadratic equations using the same steps as linear equations. The equation does not have to be written in general form first.


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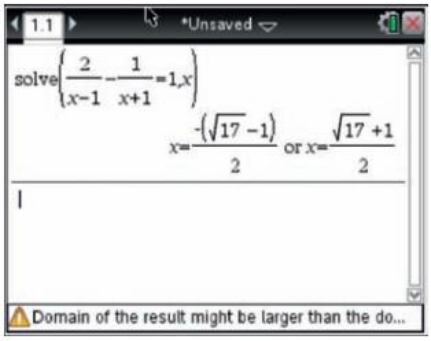
## Example 20

CAS Solve $\frac{2}{x-1}-\frac{1}{x+1}=1$

### Solution

**TI-Nspire CAS**

Use solve as you did for linear equations.




Write your solutions.

Solution: $x=\frac{-\sqrt{17}+1}{2}$ or $x=\frac{\sqrt{17}+1}{2}$

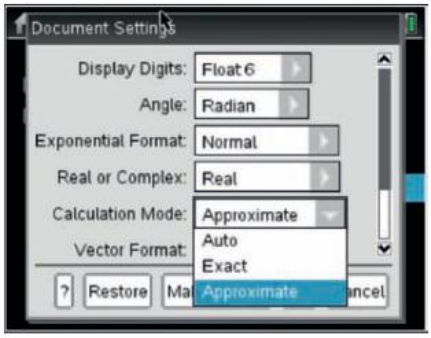
If approximate answers are needed, before going to a Calculator page, you can change your settings.

From the right-hand side of Home Page, choose

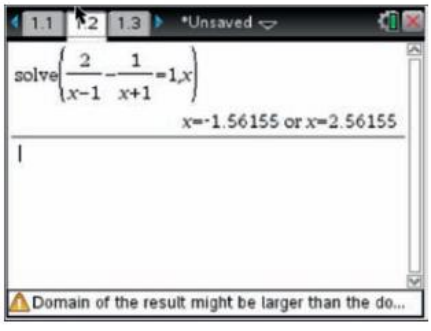
- 5 Settings
- 2: Document Settings
- Calculation Mode
- Approximate

This sets the mode so that all answers will be shown as decimals, which may be approximations.



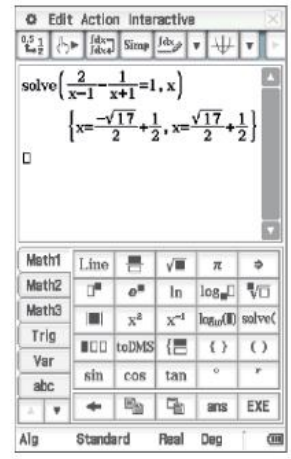




Then perform the same calculation again. Alternatively, you can just enter one of the numbers in the equation as a decimal to force decimal answers.



### ClassPad

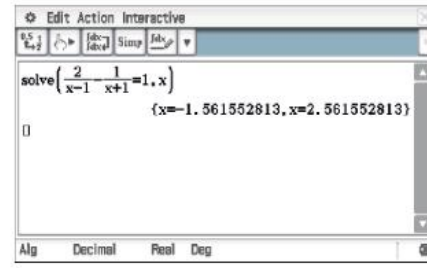
Ensure the calculator is in **Standard** mode. (If not, tap **Decimal**.) Use the same procedure as for linear equations.




Write your two solutions.

Solution: $x = \dfrac{-\sqrt{17}+1}{2}$ or $x = \dfrac{\sqrt{17}+1}{2}$

If you want approximate answers, change the mode at the bottom to **Decimal** instead of **Standard**.




# EXERCISE 1.08 The quadratic formula and the discriminant

## Concepts and techniques

**1** Example 18 Solve the following equations using the quadratic formula. Leave your answers in exact form.

**a** $x^2 + 14x + 45 = 0$ **b** $x^2 + 20x + 51 = 0$ **c** $x^2 = 7x + 294$

**d** $x^2 + 4x = 7$ **e** $x^2 - 6x = 21$ **f** $x^2 = 13x + 140$

**2** Solve the following using the quadratic formula, correct to 2 decimal places if necessary.

**a** $3f^2 + f - 14 = 0$ **b** $t^2 = 9t + 18$ **c** $6x^2 = 5x + 1$

**d** $x^2 - x - 5 = 0$ **e** $9p^2 = 15p + 5$ **f** $2d^2 + 3d = 6$

**3** Solve the following equations for $x$, using the most appropriate method.

**a** $x^2 + 6x - 10 = 0$ **b** $18x^2 + 15x = 12$

**c** $x^2 - 4x - 13 = 0$ **d** $x^2 - 6.146x + 9.328 = 0$

**e** $1 - \dfrac{x}{2} + \dfrac{x^2}{24} = 0$ **f** $\dfrac{1}{x+1} - \dfrac{4}{x} = \dfrac{1}{x+3}$

**g** $x^2 + 29x + 6 = 0$ **h** $3x^2 - 5.6x + 0.56 = 0$


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4 Example 19 Use the discriminant to determine the number and nature of the solutions of the following quadratic equations.

a $x^2 - 4x - 1 = 0$
b $2x^2 + 3x + 7 = 0$
c $-4x^2 + 2x - 1 = 0$
d $6x^2 - x - 2 = 0$
e $-x^2 - 3x = 0$
f $x^2 + 4 = 0$
g $x^2 - 2x + 1 = 0$
h $-3x^2 - 2x + 5 = 0$
i $-2x^2 + x + 2 = 0$
j $-x^2 + 4x - 4 = 0$

5 CAS Example 20 Solve the following quadratic equations. Leave your answers in exact form.

a $x^2 + x - 4 = 0$
b $3x^2 - 5x + 1 = 0$
c $q^2 - 4q - 3 = 0$
d $4h^2 + 12h + 1 = 0$
e $3s^2 - 8s + 2 = 0$
f $x^2 + 11x - 3 = 0$
g $6d^2 + 5d - 2 = 0$
h $x^2 - 2x = 7$
i $t^2 = t + 1$
j $2x^2 + 1 = 7x$

6 CAS Solve the following quadratic equations. Write your answers correct to 3 significant figures.

a $y^2 + 6y + 2 = 0$
b $2x^2 - 5x + 3 = 0$
c $b^2 - b - 9 = 0$
d $2x^2 - x - 1 = 0$
e $-8x^2 + x + 3 = 0$
f $n^2 + 8n - 2 = 0$
g $m^2 + 7m + 10 = 0$
h $x^2 - 7x = 0$
i $x^2 + 5x = 6$
j $y^2 = 3y - 1$

# 1.09 DIRECT AND INVERSE PROPORTION

You already know the equation of quantities $x$ and $y$ in direct linear proportion is of the form $y = kx$, where $k$ is a constant. However, not all relationships involving proportion are linear. The distance a stone falls ($d$) when dropped is directly proportional to the square of the time ($t$) that it has been falling, i.e. $m = kt^2$. This is an example of non-linear proportion.

You need to read carefully how variables are related.

| Direct proportion relationship | Equations |
|---|---|
| $y$ is directly proportional to $x$. | $y = kx$ |
| $y$ is directly proportional to the square of $x$. | $y = kx^2$ |
| $y$ is directly proportional to the cube of $x$. | $y = kx^3$ |
| $y$ is directly proportional to the square root of $x$. | $y = k\sqrt{x}$ |

The steps you use to solve a direct proportion problem are as follows.

**IMPORTANT**

Steps for solving proportion problems.

- Identify the two variables and write a proportion equation.
- Substitute known values for the two variables to find the value of $k$, the constant.
- Rewrite the proportion equation using the value of $k$.
- Substitute a value for one variable to find the second variable.
- Express the solution in terms of the problem.

## Example 21

**a** A 14 metre tall tree casts a 10 metre long shadow. What is the height of a tree that casts a 15 metre long shadow at the same time?



**b** The distance a stone falls, in metres, when dropped from the top of a cliff is directly proportional to the square of the time, in seconds, that it has been falling. A stone falls 80 metres in 4 seconds. How long would it take the stone to fall 245 metres?

### Solution

**a**

| | |
|---|---|
| The two variables are the height of the tree ($h$) and the length of the shadow ($s$). Write a proportion equation. | $h = ks$ |
| Substitute given values in for the two variables. A 14 m tree casts a 10 m shadow so $h = 14$, $s = 10$. | When $h = 14$, $s = 10$.<br>$14 = 10k$ |
| Solve to find $k$. | $k = 1.4$ |
| Rewrite the proportion equation using the value of $k$. | $h = 1.4s$ |
| Substitute the value of $s$ into the equation. | When $s = 15$, $h = 1.4 \times 15$ |
| Solve the equation. | $h = 21$ |
| Answer the question. | The height of the tree is 21 m. |


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| | | |
|---|---|---|
| **b** | Identify the variables and write a proportion equation.<br>The variables are distance ($d$) and time ($t$). | $d = kt^2$ |
| | Write the values for the variables.<br>Substitue values in the equation and solve to find $k$. | When $d = 80$, $t = 4$<br>$80 = k \times 4^2$<br>$80 = 16k$<br>$k = 5$ |
| | Rewrite the proportion equation using the value of $k$. | $d = 5t^2$ |
| | Substitute the value $d = 245$ into the equation. | $245 = 5t^2$ |
| | Solve the equation by dividing by 5, and taking the square root of both sides. (Note: as time falling is positive, you only take the positive square root.) | $49 = t^2$<br>$t = 7$, since $t \geq 0$ |
| | Answer the question. | The stone falls for 7 seconds. |

We can also use the unitary method to solve proportion questions.

## Example 22

Luka had 200 Ubeaut shares. His annual dividend was $568. How much dividend did his friend Shayne receive for her 360 shares?

### Solution

| | |
|---|---|
| Write down the relationship for the total number of parts. | Dividend for 200 shares = $568 |
| Find the value of 1 part. | Dividend for 1 share = $568 ÷ 200<br>= $2.84 |
| Find the value of the required number of parts. | Dividend for 360 shares = $2.84 × 360<br>= $1022.40 |
| Write your answer. | Shayne receives a dividend of $1022.40. |

Boyle's Law in Science states that the relationship between the absolute pressure (P) and volume ($V$) of a confined gas is given by $PV = c$, where $c$ is a constant. As P increases, $V$ decreases and as P decreases, $V$ increases. This is the reverse of what happens in direct proportion and is called **inverse proportion**. This relationship could be rewritten as $V = \frac{c}{P}$ which, again, is non-linear.

**IMPORTANT**

In **inverse proportion,** two variables are related so that as one increases the other decreases by the same factor. The equation of variables $x$ and $y$ in inverse proportion is of the form $y = \frac{k}{x}$, where $k$ is a constant.

'Inverse' means 'the opposite'. In direct proportion you multiply the constant by the variable, but in inverse proportion you divide the constant by the variable.

There are many examples of inverse proportion in our everyday life. For example:

- The more workers you have, the less time it takes to build a house.
- The faster you travel, the less time it takes to get there.
- The higher the cost, the less you can buy for the same money.

You can follow the same steps when solving an inverse proportion problem as in direct proportion problems.

### Example 23

Three workers build a shed in 12 hours. How long would it take 8 equally productive workers?

#### Solution

| | |
|---|---|
| This is inverse proportion. Identify the variables and write the equation. The variables are the number of workers ($w$) and the time taken ($t$). | $t = \frac{k}{w}$ |
| Substitute the given values into the two variables. 3 workers take 12 hours so $w = 3$ and $t = 12$. | When $w = 3, t = 12$<br>$12 = \frac{k}{3}$ |
| Solve to find $k$. | $k = 12 \times 3$<br>$= 36$ |
| Rewrite the proportion equation using the value of $k$. | $t = \frac{36}{w}$ |
| Substitute the value of $w$ into the equation. | When $w = 8, t = \frac{36}{8}$ |
| Solve the equation. | $t = 4.5$ |
| Answer the question. | It takes 8 workers 4.5 hours to build the shed. |

## EXERCISE 1.09 Direct and inverse proportion

### Concepts and techniques

1 For each of the following, write the equation that relates the variables.

a $v$ is directly proportional to $t$ and when $t = 10$, $v = 98$

b $p$ is directly proportional to $q$ and when $q = 6$, $p = 19.2$

c $E$ is directly proportional to the square of $v$ and when $v = 4.5$, $E = 72.9$

d $d$ is inversely proportional to $a$ and when $a = 5$, $d = 60$

e $C$ is inversely proportional to $V$ and when $V = 200$, $C = 20$


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## Modelling and problem solving

2 Example 21 The distance a car travels is directly proportional to its speed. In a given time, a car can travel 765 km at an average speed of 85 km/h.
   a How far can the car travel in the same time, if the average speed is increased to 90 km/h?
   b What is the average speed of the car if it only travels 657 km in the same time?

3 The extension (extra length) of a spring is directly proportional to the mass on the end of it. A spring is 54 mm longer with a weight of 63 g on it.
   a How much longer would the spring be if the weight was increased to 105 g?
   b What mass would you need to put on the spring to increase its length by 21 mm?

4 The area of metal used to fabricate a container is directly proportional to the square of its height. 14.4 $m^2$ of metal is needed to fabricate a container 6 m high.
   a How much metal is needed to built a container 8 m high?
   b How tall is a container that has been made from 36.1 $m^2$?

5 The time taken to upload data is directly proportional to the size of the file. A computer takes 5.5 minutes to upload 825 megabytes of data.
   a How long would the computer take to upload 1000 megabytes of the data?
   b How much of the data could be uploaded by the computer in 2 minutes?

6 Example 22 16 horses eat 250 kg of hay each day. How much hay would be needed to feed 36 horses?

7 A production line can manufacture 4500 metres of material in a 12-hour shift. How much material would be produced in 27 hours?

8 Corie paid $694.50 to buy tickets to the rock concert for herself and five friends. How much did the friends owe her altogether?

9 Example 23 The time that a car takes to travel a distance is inversely proportional to its speed. A car takes 2 hours 15 minutes to reach its destination travelling at an average speed of 92 km/h. How long would it take the car to travel the same distance at an average speed of 69 km/h?

10 Boyle's gas law states that, when there is no change in temperature, the volume of a gas is inversely proportional to the pressure of the gas. A balloon with a volume of 15 L is filled with a gas at 3 atmospheres. If the pressure is reduced to 0.5 atmospheres without a change in temperature, what would be the volume of the balloon?

11 A life raft designed for 8 people has sufficient food and water to last 10 days. The ship it was from sank so quickly that only 5 people made it to the raft. How long will the provisions last them?

12 A swimming pool can be filled in 36 hours using a garden hose and in 24 hours using a large diameter hose. Assuming that the flow rates through each hose are unaffected by the use of the other, how long will it take to fill using both hoses?

Polynomials

# 1.10 POLYNOMIALS

You have examined linear and quadratic expressions. These are part of the family of **polynomials**. Polynomials are sums of non-negative integer powers of a variable, such as $3x^5 + 7x - 5$ or $x^3 + 2x^2 - 3x^7 - 2x^2$.

**IMPORTANT**

### Polynomial definitions

A polynomial is a sum of terms containing non-negative integer powers of the variable(s). In this course you will deal only with polynomials in one variable.. A polynomial in $x$ is of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

where $n$ is a positive whole number and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.

The highest power of $x$ is called the **degree** of the polynomial. The term that contains the highest power is called the **leading term**. The degree of the polynomial above is $n$, the leading term is $a_n x^n$ and the coefficient of the leading term, $a_n$, is called the **leading coefficient**.

When the leading coefficient, $a_n$, is 1, the polynomial is called a **monic** polynomial.
The numbers $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are called the **coefficients**.

The **constant** is $a_0$ and is also known as the **constant term**.

Examples of polynomials are:

| | |
|---|---|
| $2$ | A polynomial of degree 0. (It could be written as $2x^0$.) The leading term is 2. |
| $5x - 7$ | A polynomial of degree 1, with leading term $5x$ and a constant of $-7$. It is a linear polynomial. |
| $4x^2 + 3\sqrt{2}x + 9$ | A polynomial of degree 2. The leading term is $4x^2$, the constant is 9 and the coefficient of $x$ is $3\sqrt{2}$. This is a quadratic polynomial. |
| $x^3 - 3x^2 + 6x$ | A monic polynomial of degree 3, with leading term $x^3$, leading coefficient 1 and constant term 0. This is a cubic polynomial. |
| $\frac{2}{3}x^4 + 2x^2 - 5x + 0.28$ | A polynomial of degree 4. The leading term is $\frac{2}{3}x^4$, the leading coefficient is $\frac{2}{3}$, the constant term is 0.28 and the coefficient of $x^3$ is 0. This is a quartic polynomial. |

## Example 24

Determine whether each of the following is a polynomial, giving reasons if it is not. State the degree, leading term and coefficients of any polynomial found.

**a** $3x - 2x^2 + 7$

**b** $x^4 + \frac{x^2}{4} - \frac{3}{x} + 7x^3$

**c** $\frac{2}{5}\sqrt{x} + 3x^2 - 9$

**d** $\sqrt{3}x + 7x^3 - \frac{x^5}{2}$


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## Solution

**a** Write the expression with descending powers. All of the powers are non-negative whole numbers

$-2x^2 + 3x + 7$

This is a polynomial.

Write the degree, leading term and coefficients.

Degree = 2

Leading term = $-2x^2$

Coefficients = $-2, 3, 7$

**b** Write the expression with descending powers. Not all of the powers are non-negative whole numbers.

$x^4 + 7x^3 + \frac{1}{4}x^2 - 3x^{-1}$

This is not a polynomial as $3x^{-1}$ has a negative power.

**c** Write the expression with descending powers. Not all of the powers are non-negative whole numbers.

$3x^2 + \frac{2}{5}x^{\frac{1}{2}} - 9$

This is not a polynomial as $\frac{2}{5}x^{\frac{1}{2}}$ has a fractional power.

**d** Write the expression with descending powers. All of the powers are non-negative whole numbers.

$-\frac{x^5}{2} + 7x^3 + \sqrt{3}x$

Write the degree, leading term and coefficients.

Degree = 5

Leading term = $-\frac{1}{2}x^5$

Coefficients = $-\frac{1}{2}, 7, \sqrt{3}$

When you expand

- 2 linear factors, you get a **quadratic** polynomial (a polynomial of degree 2)
- 3 linear factors, you get a **cubic** polynomial (a polynomial of degree 3)
- 4 linear factors, you get a **quartic** polynomial (a polynomial of degree 4)
- 5 linear factors, you get a **quintic** polynomial (a polynomial of degree 5).

When expanding 3 linear factors, you expand two first and simplify before multiplying by the remaining factor.

**IMPORTANT**

When expanding a perfect cube, you can use one of the rules below.

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

## Example 25

Expand the following factors.

**a** $(x+4)(x+2)(x-7)$ **b** $3(x+2)^2(5x+2)$

**c** $2x(x+4)(x+5)(x-1)$ **d** $(4x-3y)^3$

**e** CAS $(2x-1)(x-5)(3x+4)$

### Solution

**a** Write the expression. $(x+4)(x+2)(x-7)$

Expand two linear factors and simplify.
$= (x+4)(x^2-7x+2x-14)$
$= (x+4)(x^2-5x-14)$

Multiply by the remaining factor and simplify.
$= x^3-5x^2-14x+4x^2-20x-56$
$= x^3-x^2-34x-56$

**b** Write the expression. $3(x+2)^2(5x+2)$

Expand by using the expansion of a perfect square.
$= 3(5x+2)(x^2+4x+4)$

Multiply by the remaining factor and simplify.
$= 3(5x^3+20x^2+20x+2x^2+8x+8)$
$= 3(5x^3+22x^2+28x+8)$
$= 15x^3+66x^2+84x+24$

**c** Write the expression $2x(x+4)(x+5)(x-1)$

Expand two linear factors and simplify.
$= 2x(x+4)(x^2-x+5x-5)$
$= 2x(x+4)(x^2+4x-5)$

Multiply by the next factor and simplify.
$= 2x(x^3+4x^2-5x+4x^2+16x-20)$
$= 2x(x^3+8x^2+11x-20)$

Multiply by the remaining factor.
$= 2x^4+16x^3+22x^2-40x$

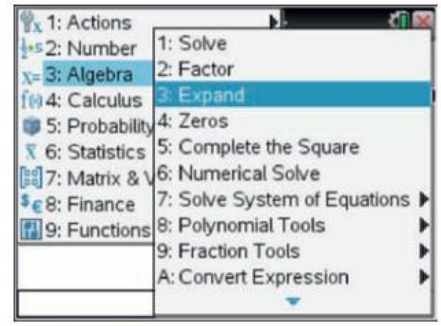
**d** Write the expression. $(4x-3y)^3$

Use the perfect cube expansion and simplify.
$(4x-3y)^3 = (4x)^3-3(4x)^2(3y)+3(4x)(3y)^2-(3y)^3$
$= 64x^3-144x^2y+108xy^2-27y^3$

**e** **TI-Nspire CAS**

Choose a Calculator page  . Then press:

- menu
- 3:Algebra
- 3: Expand

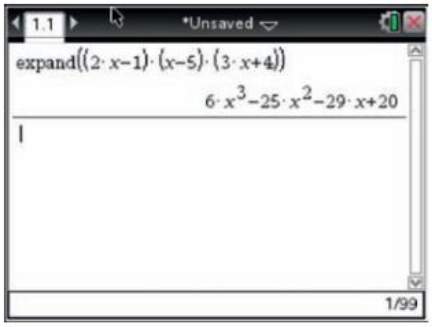




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Type
expand$((2x-1)(x-5)(3x+4))$

Press enter.

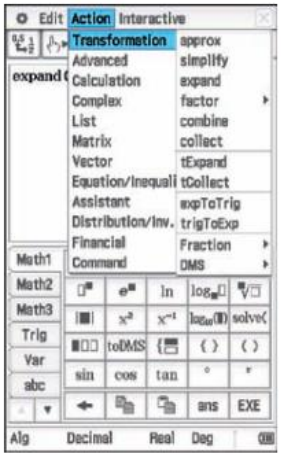


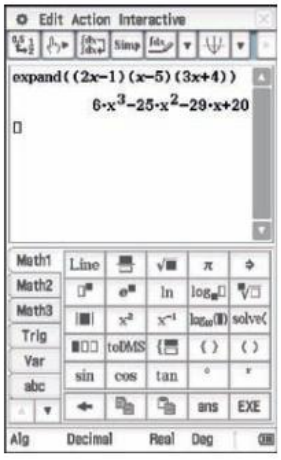

State the answer.

$(2x-1)(x-5)(3x+4) = 6x^3 - 25x^2 - 29x + 20$

### ClassPad

Use the Main $\sqrt{\alpha}$ menu and **expand.**
Tap **Action** then **Transformation** and finally **expand.**
Type the factors and press EXE.




State the answer.

$(2x-1)(x-5)(3x+4) = 6x^3 - 25x^2 - 29x + 20$

The value of a polynomial is dependent on the value of the variable. Given the polynomial $P(x) = 2x^3 - 5x^2 - x + 9$, you can work out the value of the polynomial for different values of $x$ using substitution. For example, if you wanted to find the value when $x = 3$ and when $x = -1$, you write:

$P(3) = 2(3)^3 - 5(3)^2 - (3) + 9$
$= 2 \times 27 - 5 \times 9 - 3 + 9$
$= 15$

and

$P(-1) = 2(-1)^3 - 5(-1)^2 - (-1) + 9$
$= 2 \times (-1) - 5 \times 1 + 1 + 9$
$= 3$

Your CAS calculator can also be used to find the value of a polynomial.

## Example 26

**a** Given $g(x) = 2x^3 - 6x^2 + 5$, evaluate $g(-3)$.
**b** If $f(x) = x^3 - 4x^2 + 8x - 2$, find $f(a + 2)$
**c** CAS Evaluate $P(-2)$ when $P(x) = -2x^4 + 5x^3 + 3x^2 + 11$.

### Solution

**a** Write down $g(x)$. $g(x) = 2x^3 - 6x^2 + 5$

Substitute $x = -3$. $g(-3) = 2 \times (-27) - 6 \times 9 + 5$

Evaluate your answer. $g(-3) = -103$

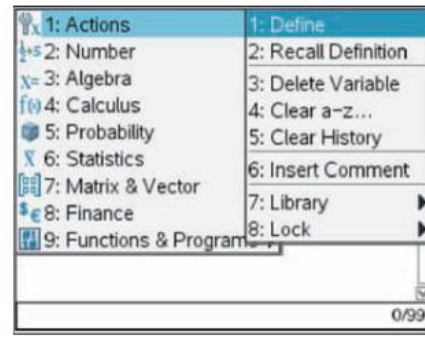
**b** Write down $f(x)$. $f(x) = x^3 - 4x^2 + 8x - 2$

Substitute $x = a + 2$. $f(a + 2) = (a + 2)^3 - 4(a + 2)^2 + 8(a + 2) - 2$

Expand and simplify. $= a^3 + 6a^2 + 12a + 8 - 4a^2 - 16a - 16 + 8a + 16 - 2$
$= a^3 + 2a^2 + 4a + 6$

**c** **TI-Nspire CAS**
Choose a Calculator page  .
Press

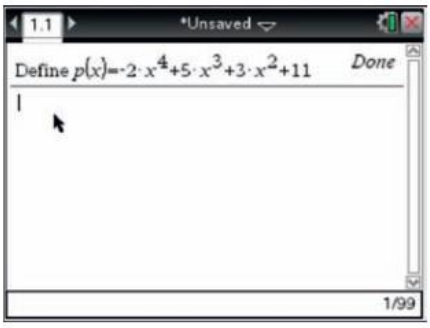
- menu
- 1:Actions
- 1: Define



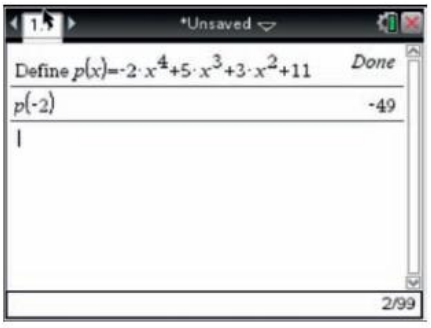

Type in
$P(x) = -2x^4 + 5x^3 + 3x^2 + 11$.


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Press enter and the word '*Done*' will appear.




To evaluate $P(-2)$, type into the entry line $P(-2)$ and press enter

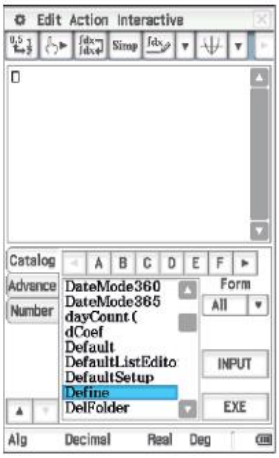


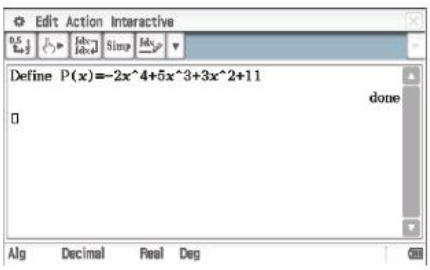

State your answer.

$P(-2) = -49$

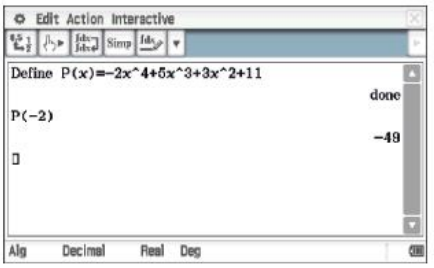
### ClassPad

Use the Main $\sqrt{a}$ menu and press Keyboard for the soft keys.
Tap ▼ to access the functions under the Catalog tab. **Tap D** and select **Define** from the list.
Tap ▲ then abc to get the 'P'. (Do NOT get P from the Var tab!)
If you want a capital *P*, first tap ⇧.
The P should be upright and not in heavy type.
Use the keyboard for $x$ or press Keyboard **and** Var then tap ***x***. It should be in italics. This shows that it is a variable.
Type the polynomial, using ^ for powers. Tap EXE or press EXE.
The response 'done' means the polynomial has been recorded.



Use the abc tab in the soft keyboard and the hard keyboard to enter $P(-2)$.
Tap EXE or press EXE.
The calculator will calculate the value of $P(x)$ for $x = -2$.




State your answer.

$P(-2) = -49$

# EXERCISE 1.10 Polynomials

## Concepts and techniques

1 Example 24 Which of the following are not polynomials?

a $5x^4 - 3x^2 + x + \frac{1}{x}$
b $x^2 + 3^x$
c $x^2 + 3x - 7$
d $3x + 5$
e $6x^3 + 4\sqrt{x-8}$
f $3x^2 - \frac{1}{2}x + 1$
g $4x^3 + 7x^{-2} + 5$

2 Write down the degree and leading term of each polynomial and identify any monic polynomials.

a $5x^7 - 3x^5 + 2x^3 - 3x + 1$
b $3 + x + x^2 - x^3 + 2x^4$
c $3x + 5$
d $x^{11} - 5x^8 + 4$
e $2 - x - 5x^2 + 3x^3$
f $3$
g $2x^4 - x$

3 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$ find

a the degree of $P(x)$
b the constant term
c the coefficient of $x^4$
d the coefficient of $x^2$

4 For the polynomial $P(x) = (a+1)x^3 + (b-7)x^2 + c + 5$ find values for $a$, $b$ or $c$ if

a $P(x)$ is monic
b the coefficient of $x^2$ is 3
c the constant is $-1$
d $P(x)$ has degree 2
e the leading term has a coefficient of 5

5 Example 25 Expand and simplify the following.

a $a(a+5)(a+2)$
b $m(m-4)(m+5)$
c $-3y(y-2)(y-4)$
d $-5n(n-4)^2$
e $x^2(x-1)(x+1)$
f $a(a-3)(a+2)$
g $(4-x)(2x+9)(x-3)$
h $(3x+9)(5-2x)(x+1)$
i $(x-3)^2(2x+7)$
j $(x+8)(x-8)(x+1)$
k $-(x-3)(2x+3)^2$
l $-(6-x)(x+7)(x+6)$


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**6** Expand and simplify the following.

**a** $(x+2)^3$ **b** $(5-x)^3$ **c** $(2x+3)^3$

**d** $-(3x-1)^3$ **e** $(4x-5y)^3$

**7** CAS Expand and simplify, then use your CAS calculator to check your answers.

**a** $(x-4)^2(x+4)^2$ **b** $x(2x-9)(x+2)^2$ **c** $x(2x+5)^3$

**d** $-2x(3x+1)^2(3x-1)$ **e** $7x^2(6-5x)(2x-1)$

**8** Example 26 For the polynomial $P(x) = x^3 - 7x^2 + x - 1$ find

**a** $P(2)$ **b** $P(-1)$ **c** $P(x+2)$

**9** **a** Given $f(x) = x + 3$, find $f(1)$ and $f(-3)$

**b** If $h(x) = x^2 - 2$ find $h(0)$, $h(2)$ and $h(-4)$

**c** If $f(x) = -x^2$, find $f(5)$, $f(-1)$, $f(3)$ and $f(-2)$.

**d** Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

**e** Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$.

**f** If $f(x) = 2x - 5$, find $x$ when $f(x) = 13$.

**g** Given $f(x) = x^2 + 3$, find any values of $x$ for which $f(x) = 28$.

**h** If $f(x) = 2x - 9$ find $f(p)$ and $f(x + h)$.

**i** Find g$(x - 1)$ when $g(x) = x^2 + 2x + 3$.

**j** Given $f(t) = t^2 + 2t + 1$, find $t$ when $f(t) = 0$. Also, find any values of $t$ for which $f(t) = 9$.

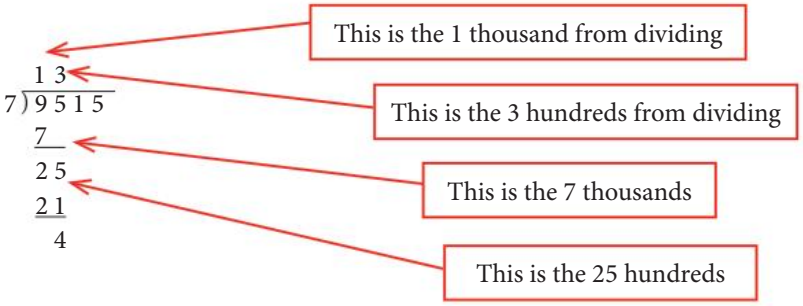
# 1.11 POLYNOMIAL FACTORISATION

You use the method of long division to divide polynomials.

When you work out 9515 ÷ 7 using long division, you actually separate the division into place values, so in the thousands you divide 9 thousand by 7 first, getting 1 thousand. 7 × 1 thousand = 7 thousand and 9 thousand − 7 thousand gives 2 thousand left over.

This is set out in the thousands place underneath.

You change the 2 thousand left over into 20 hundreds and add the 5 hundreds to get 25 hundreds. Then you divide the hundred by the 7 and repeat the process, so 25 hundreds divided by 7 gives 3 hundreds and 7 times 3 hundreds means that 21 hundreds have been used and there are 4 hundreds left over. The setting out is shown below.




Polynomial division is done the same way, but you separate the polynomial into descending powers instead of descending place values.

## Example 27

**a** Perform the following division. $(3x^3 - 8x^2 + 7x - 1) \div (x - 2)$

**b** CAS Perform the division. $(3x^3 - 8x^2 + 7x - 1) \div (x - 2)$

### Solution

**a** Divide the highest power of $x$ from $(x - 2)$ into the highest power of $(3x^3 - 8x^2 + 7x - 1)$.
This is $3x^3 \div x$, which gives $3x^2$.
Put this up the top in the $x^2$ place.
Work out how much of $(3x^3 - 8x^2)$ has been accounted for by doing $(x - 2) \times 3x^2 = 3x^3 - 6x^2$.
Put this under the $(3x^3 - 8x^2)$ and subtract to work out how many $x^2$ is left $(-2x^2)$.
When you do the next part of the division you will go as far as $x^1$, so add the $+ 7x$.

$$\begin{array}{r} 3x^2 \\ x-2\overline{)\,3x^3-8x^2+7x-1} \\ 3x^3-6x^2 \\ \hline -2x^2+7x \end{array}$$

$x$ into $-2x^2$ goes $-2x$ times. Put the $-2x$ above the $-8x^2$.
$(x - 2) \times (-2x) = -2x^2 + 4x$. Put the answer $-2x^2 + 4x$ underneath the $-2x^2 + 7x$ and subtract to get $3x$.
Bring down the $-1$.

$$\begin{array}{r} 3x^2-2x \\ x-2\overline{)\,3x^3-8x^2+\ 7x-1} \\ 3x^3-6x^2 \\ \hline -2x^2+7x \\ -2x^2+4x \\ \hline 3x-1 \end{array}$$

$x$ into $3x$ goes 3 times. Put the 3 above the $7x$.
$(x - 2) \times 3 = 3x - 6$. Put the answer $3x - 6$ underneath the $3x - 1$ and subtract to get 5.

$$\begin{array}{r} 3x^2-2x+3 \\ x-2\overline{)\,3x^3-8x^2+7x-1} \\ 3x^3-6x^2 \\ \hline -2x^2+7x \\ -2x^2+4x \\ \hline 3x-1 \\ 3x-6 \\ \hline 5 \end{array}$$

Write the answer.

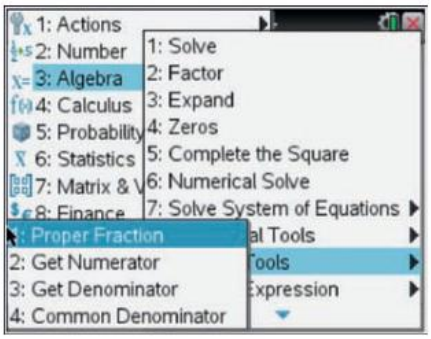
$(3x^3 - 8x^2 + 7x - 1) \div (x - 2)$
$= 3x^2 - 2x + 3$ with remainder 5
$(3x^3 - 8x^2 + 7x - 1) = (x - 2)(3x^2 - 2x + 3) + 5$


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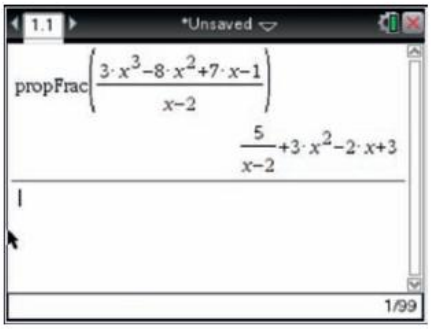
**b** **TI-Nspire CAS**

Choose a Calculator page . Then press:

- menu
- 3: Algebra
- 9: Fraction Tools
- 1: Proper Fractions




Type
propFrac$((3x^3 - 8x^2 + 7x - 1)/(x - 2))$ and press enter.

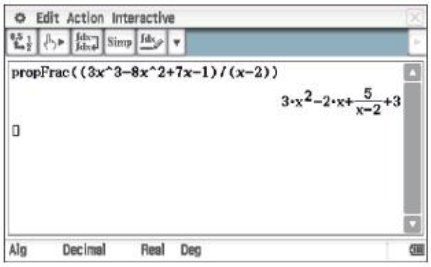



Write the answer. The CAS calculator presents the answer as a fraction, with the remainder first. So when writing your answer, remember to reverse the order and place the remainder last.

$(3x^3 - 8x^2 + 7x - 1) \div (x - 2)$
$= 3x^2 - 2x + 3$ remainder 5
$(3x^3 - 8x^2 + 7x - 1) = (x - 2)(3x^2 - 2x + 3) + 5$

**ClassPad**

Press Keyboard. Use **propFrac** which is found in the Catalog of functions or found by first tapping **Action**, **Transformation**, then **Fraction**.
Both expressions should be in brackets.
Enter as follows.
propFrac$((3x^3 - 8x^2 + 7x - 1) \div (x - 2))$
Tap EXE or press the EXE key.
Notice that the order of terms has changed slightly.




Write the result.

$(3x^3 - 8x^2 + 7x - 1) \div (x - 2)$
$= 3x^2 - 2x + 3$ remainder 5
$(3x^3 - 8x^2 + 7x - 1) = (x - 2)(3x^2 - 2x + 3) + 5$

In general, if $P(x)$ is divided by $x - a$, the quotient is denoted by $Q(x)$ and the remainder is $R$. It is usual to represent a polynomial division in the form $P(x) = (x - a)Q(x) + R$

Dividing higher degree polynomials by smaller degree polynomials can help factorise them.

**IMPORTANT**

**The remainder theorem**

When a polynomial $P(x)$is divided by $x - a$, the remainder is $P(a)$.

From Example **27** you know that when $P(x) = 3x^3 - 8x^2 + 7x - 1$ is divided by $x - 2$, the remainder is 5. You could use the remainder theorem to give the same result.

$P(x) = 3x^3 - 8x^2 + 7x - 1$

$P(2) = 3(2)^3 - 8(2)^2 + 7(2) - 1$

$= 3 \times 8 - 8 \times 4 + 7 \times 2 - 1$

$= 5$

## Example 28

**a** Find the remainder when $3x^4 - 2x^2 + 5x + 1$ is divided by $x - 2$.

**b** Evaluate $m$ if the remainder is 4 when dividing $2x^4 + mx + 5$ by $x + 3$.

**c** CAS Find the remainder when $-2x^3 + 8x^2 + 9x - 15$ is divided by $x + 4$.

### Solution

**a** Write down $P(x)$.

$P(x) = 3x^4 - 2x^2 + 5x + 1$

The remainder when $P(x)$ is divided by $x - 2$ is $P(2)$. Evaluate $P(2)$.

$P(2) = 3(2)^4 - 2(2)^2 + 5(2) + 1$
$= 3 \times 16 - 2 \times 4 + 10 + 1$
$= 51$

Write the answer.

The remainder is 51.

**b** Write down $P(x)$.

$P(x) = 2x^4 + mx + 5$

The remainder when $P(x)$ is divided by $x + 3$ is $P(-3)$.

$P(-3) = 2(-3)^4 + m(-3) + 5$
$= 2 \times 81 - 3m + 5$
$= 167 - 3m$

The remainder is 4. So, $P(-3) = 4$.

$167 - 3m = 4$

Solve the equation by subtracting 167 from both sides then dividing by $-3$.

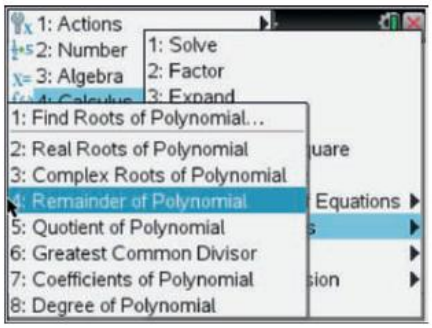
$-3m = 4 - 167$
$-3m = -163$
$m = 54\frac{1}{3}$


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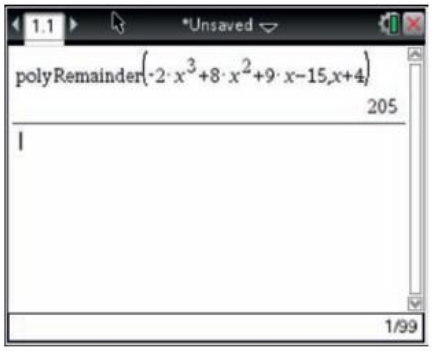
c **TI-Nspire CAS**

Choose a Calculator page . Then press:

- menu
- 3:Algebra
- 8: Polynomial Tools
- 4: Remainder of Polynomials




Type
polyRemainder$(-2x^3 + 8x^2 + 9x - 15, x + 4)$ and press enter

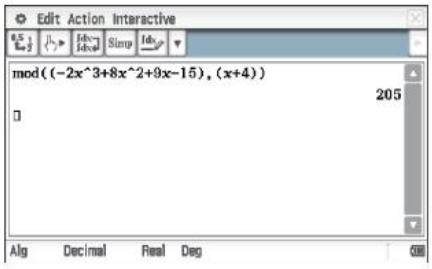



State your answer.

The remainder is 205.

**ClassPad**

Tap Main $\sqrt{\alpha}$ and press Keyboard. Use **mod**, which is found after tapping ▼ to access the Catalog of functions.
Both expressions should be in brackets.
Enter as follows.
mod$((-2x^3 + 8x^2 + 9x - 15), (x + 4))$
Tap EXE or press the EXE key.




State your answer.

The remainder is 205.

The **factor theorem** is a direct result of the remainder theorem. If a polynomial is divided by one of its factors then there is no remainder.

**IMPORTANT**

**The factor theorem**

For a polynomial $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial.

The converse of this theorem is also true.

**IMPORTANT**

For a polynomial $P(x)$, if $x - a$ is a factor of the polynomial, then $P(a) = 0$.

Obviously, if $x - a$ is a factor of the polynomial $P(x)$, then $a$ is a factor of the constant of the polynomial. Although the reverse is not true, this can help narrow down the search for factors. It is always worth trying 1 and –1.

## Example 29

**a** Show that $x - 1$ is a factor of $P(x) = 2x^3 - x^2 - 7x + 6$.
**b** Divide $P(x)$ by $x - 1$.
**c** Factorise $P(x)$ completely.
**d** Find all factors of $f(x) = x^3 + 3x^2 - 4x - 12$
**e** CAS Factorise $P(x) = x^3 - 8x^2 + 19x - 12$

### Solution

**a** Write the polynomial.

$P(x) = 2x^3 - x^2 - 7x + 6$

For $x - 1$ to be a factor of $P(x)$, $P(1) = 0$.
Evaluate $P(1)$.

$$\begin{aligned} P(1) &= 2(1)^3 - (1)^2 - 7(1) + 6 \\ &= 2 - 1 - 7 + 6 \\ &= 0 \end{aligned}$$

State your answer.

$\therefore x - 1$ is a factor of $P(x)$.

**b** Use the process of polynomial long division.

$$\begin{array}{r} 2x^2 + x - 6 \\ x-1 \overline{)\,2x^3 - x^2 - 7x + 6} \\ 2x^3 - 2x^2 \\ \hline x^2 - 7x \\ x^2 - x \\ \hline -6x + 6 \\ -6x + 6 \\ \hline 0 \end{array}$$

State your answer in the form $P(x) = (x - a)Q(x)$

$2x^3 - x^2 - 7x + 6 = (x - 1)(2x^2 + x - 6)$

**c** A cubic polynomial (of degree 3) has at most 3 linear factors. We have one of those, $x - 1$. To find the other two factors you have to factorise the quotient $2x^2 + x - 6$. This is a quadratic expression.

$$\begin{aligned} 2x^2 + x - 6 &= \frac{\overset{x+2}{\cancel{(2x+4}}\,{}_{1}\cancel{)}(2x-3)}{\cancel{2}^{1}} \\ &= (x+2)(2x-3) \end{aligned}$$

$2x^3 - x^2 - 7x + 6 = (x - 1)(x + 2)(2x - 3)$


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**d** Write the polynomial.

$f(x) = x^3 + 3x^2 - 4x - 12$

Try factors of 12 (i.e. ±1, ±2, ±3, ±4, ±6, ±12) until a remainder of 0 is found.

$f(1) = (1)^3 + 3(1)^2 - 4(1) - 12$
$= -12$
$\neq 0$
So $x - 1$ is not a factor.

$f(-1) = (-1)^3 + 3(-1)^2 - 4(-1) - 12$
$= -6$
$\neq 0$
So $x - (-1) = x + 1$ is not a factor.

$f(2) = (2)^3 + 3(2)^2 - 4(2) - 12$
$= 0$
So $x - 2$ is a factor.

Use the process of polynomial long division to divide $x - 2$ into $f(x)$.

$$
\begin{array}{r}
x^2 + 5x + 6 \\
x - 2 \overline{)\, x^3 + 3x^2 - 4x - 12} \\
\underline{x^3 - 2x^2} \\
5x^2 - 4x \\
\underline{5x^2 - 10x} \\
6x - 12 \\
\underline{6x - 12} \\
0
\end{array}
$$

Now factorise $x^2 + 5x + 6$.

$x^2 + 5x + 6 = (x + 2)(x + 3)$

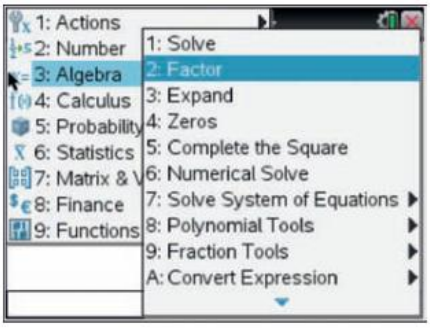
State your answer.

$x^3 + 3x^2 - 4x - 12 = (x - 2)(x + 2)(x + 3)$

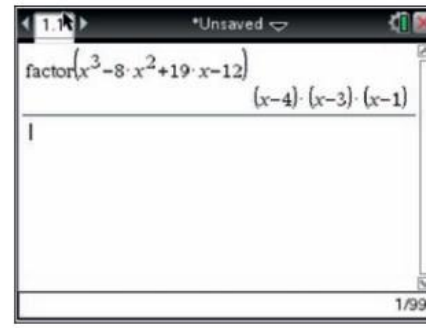
**e** **TI-Nspire CAS**

Choose a Calculator page  . Then press:

- menu
- 3:Algebra
- 2: Factor



Type factor($x^3 - 8x^2 + 19x - 12$) enter.

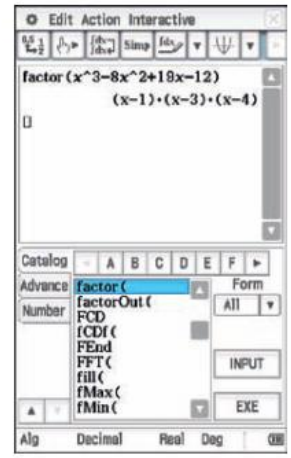



State the answer.

$x^3 - 8x^2 + 19x - 12 = (x-4)(x-3)(x-1)$

**ClassPad**

Use the Main $\sqrt{\alpha}$ menu and **factor**.
Press Keyboard for the soft keyboard.
Tap ▼ to access the functions under Catalog and then tap **factor**. (Alternatively, **factor** is obtained from the menu that appears after tapping **Action**, **Transformation** and **factor**.
Type the expression and tap EXE or press the EXE key.




State the answer.

$x^3 - 8x^2 + 19x - 12 = (x-1)(x-3)(x-4)$

# EXERCISE 1.11 Polynomial factorisation

## Concepts and techniques

**1** Example 27 Work out each of the following divisions.

a $(3x^2 + 2x + 5) \div (x + 4)$
b $(x^2 - 7x + 4) \div (x - 1)$
c $(x^3 + x^2 + 2x - 1) \div (x - 3)$
d $(4x^2 + 2x - 3) \div (2x + 3)$
e $(x^3 + x^2 - x - 3) \div (x - 2)$
f $(x^4 - x^3 - 2x^2 + x - 3) \div (x + 4)$
g $(4x^3 - 2x^2 + 6x - 1) \div (2x + 1)$
h $(6x^2 - 3x + 1) \div (3x - 2)$
i $(x^2 + 5x - 2) \div (x + 1)$
j $(x^4 - 2x^2 + 5x + 4) \div (x - 3)$

**2** CAS Work out the remainders for each of the following divisions.

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$
b $(x^2 + 5x + 3) \div (x + 2)$
c $(2x^3 - 4x - 1) \div (x + 3)$
d $(3x^5 + 2x^2 - x + 4) \div (x - 5)$
e $(5x^3 + 2x^2 + 2x - 9) \div (x - 1)$
f $(x^4 - x^3 + 3x^2 - x - 1) \div (x + 2)$
g $(2x^2 + 7x - 2) \div (x + 7)$
h $(x^7 + 5x^3 - 1) \div (x - 3)$
i $(2x^6 - 3x^2 + x + 4) \div (x + 5)$
j $(3x^4 - x^3 - x^2 - x - 7) \div (x + 1)$


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**3** Example 28 Find the remainders when each of the following polynomials is divided by each of the factors given.

**a** $3x^4 - 2x^3 + x^2 + 2$ by $x + 2$, $x - 1$ or $x - 2$

**b** $x^4 - 2x^2 - 2x + 1$ by $x + 1$, $x - 2$ or $x - 3$

**c** $-x^4 + 3x^3 + x^2 + 2x - 9$ by $x + 1$, $x - 1$ or $x - 3$

**d** CAS $x^4 - 5x^3 - 4x^2 + 2x - 8$ by $x - 2$, $x + 2$ or $x - 1$

**e** CAS $2x^5 - 3x^4 + 2x^2 - x - 1$ by $x - 3$, $x - 1$ or $x + 2$

**4** Find the value of $k$ if

**a** the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$

**b** the remainder is $-4$ when $x^3 - (k - 1)x^2 + 5kx + 4$ is divided by $x + 2$

**c** the remainder is 0 when $2x^5 + 7x^2 + 1 + k$ is divided by $x + 6$

**d** $2x^4 - kx^3 + 3x^2 - x$ is divisible by $x - 3$

**e** the remainder is 25 when $2x^4 - 3x^2 + 5$ is divided by $x - k$

**5** Example 29 **a** Find the remainder when $f(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 2$.

**b** Is $x - 2$ a factor of $f(x)$?

**c** Divide $x^3 - 4x^2 + x + 6$ by $x - 2$.

**d** Factorise $f(x)$ fully and write $f(x)$ as a product of its factors.

**6** **a** Show that $x + 3$ is a factor of $P(x) = x^4 + 3x^3 - 9x^2 - 27x$

**b** Divide $P(x)$ by $x + 3$ and write $P(x)$ as a product of its factors.

**7** Write each polynomial as a product of its factors.

**a** $x^3 + 27$ **b** $1 - x^3$ **c** $1000 + 8t^3$

**d** $\frac{x^3}{8} - 27$ **e** $y^3 + (3 + x)^3$ **f** $x^2 - 2x - 8$

**g** $x^3 + x^2 - 2x$ **h** $x^3 + x^2 - 10x + 8$ **i** $x^3 + 4x^2 - 11x - 30$

**j** $x^3 - 11x^2 + 31x - 21$ **k** $x^3 - 12x^2 + 17x + 90$ **l** $x^3 - 7x^2 + 16x - 12$

**m** $x^4 + 6x^3 + 9x^2 + 4x$ **n** $x^3 + 3x^2 - 4$ **o** $x^3 - 7x - 6$

**8** CAS Factorise each of the following polynomials.

**a** $f(x) = x^4 + 10x^3 + 23x^2 - 34x - 120$

**b** $p(x) = x^4 + 3x^3 - 13x^2 - 51x - 36$

**c** $f(x) = x^3 - 3x^2 - 34x + 120$

## Reasoning and communication

**9** The remainder is 5 when $P(x) = ax^3 - 4bx^2 + x - 4$ is divided by $x - 3$ and the remainder is 2 when $P(x)$ is divided by $x + 1$. Find the values of $a$ and $b$.

**10** When $f(x) = ax^2 - 3x + 1$ and $g(x) = x^3 - 3x^2 + 2$ are divided by $x + 1$ they leave the same remainder. Find the value of $a$.

**11** **a** Show that $x - 3$ is not a factor of $P(x) = x^5 - 2x^4 + 7x^2 - 3x + 5$.

**b** Find a value of $k$ such that $x - 3$ is a factor of $Q(x) = 2x^3 - 5x + k$

**12** The polynomial $P(x) = x^3 + ax^2 + bx + 2$ has factors $x + 1$ and $x - 2$. Find the values of $a$ and $b$.

**13** **a** The remainder, when $f(x) = ax^4 + bx^3 + 15x^2 + 9x + 2$ is divided by $x - 2$, is 216, and $x + 1$ is a factor of $f(x)$. Find $a$ and $b$.
**b** Divide $f(x)$ by $x + 1$ and write the polynomial in the form $f(x) = (x + 1)g(x)$.
**c** Show that $x + 1$ is a factor of $g(x)$.
**d** Write $f(x)$ as a product of its factors.

**14** **a** Write $P(x) = x^3 - 7x + 6$ as a product of its factors.
**b** What are the zeros of $P(x)$?
**c** Is $(x - 2)(x + 3)$ a factor of $P(x)$?

**15** **a** Write the polynomial $P(u) = u^3 - 4u^2 + 5u - 2$ as a product of its factors.
**b** Hence or otherwise, solve $(x - 1)^3 - 4(x - 1)^2 + 5(x - 1) - 2 = 0$.

**16** **a** Write the polynomial $f(p) = p^3 - 2p^2 - 5p + 6$ as a product of its factors.
**b** Hence or otherwise, solve $(2x + 1)^3 - 2(2x + 1)^2 - 5(2x + 1) + 6 = 0$.

**17** Evaluate $a$, $b$, $c$ and $d$ if
**a** $x^3 + 3x^2 - 2x + 1 = ax^3 + b(x - 1)^2 + cx + d$
**b** $x^3 - x^2 + 4x = ax^3 + b(x + 2)^2 + c(x + 2) + d$
**c** $2x^3 - x + 7 = ax^3 + b(x + 1)^2 + c(x + 1) + d + 2$
**d** $x^3 + x^2 + 5x - 3 = ax^3 + b(x - 3)^2 + cx + d$
**e** $4x^3 - x + 3 = (a + 1)x^3 + b(x + 4)^2 + c(x + 4) + d - 1$

**18** A monic polynomial of degree 3 has zeros −3, 0 and 4. Find the polynomial.

**19** The polynomial $P(x) = ax^3 - bx^2 + cx - 8$ has zeros 2 and −1 and $P(3) = 28$. What are the values of $a$, $b$ and $c$?

**20** A polynomial with leading term $2x^4$ has zeros −2, 0, 1 and 3. Find the polynomial.

**21** Show that a polynomial of degree 3 cannot have 4 zeros.

# 1.12 POLYNOMIAL EQUATIONS

A polynomial equation such as $x^3 + 3x + 5 = 0$ is not easily solved. In fact, only a small proportion of cubic and higher equations can be solved easily, and this is usually by using the factor theorem.

You use the null factor law to solve cubic and quartic equations, as you did with quadratic equations.

**IMPORTANT**

**Null factor law**

If $abc = 0$, then at least one of $a = 0$, $b = 0$, $c = 0$ is true.

If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions are $x = a$, $x = b$, $x = c$.

This is easily extended to any product of numbers.



Solving algebraic equations

Always look for a common factor, a perfect cube, sum or difference of cubes before trying other factorisation methods for cubic equations.


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## Example 30

Solve

a $x^3 + 6x^2 + 12x + 8 = 0$

b $6x^3 + 15x^2 - 12x - 9 = 0$

c $x^4 + 4x^3 + x^2 - 6x = 0$

d CAS Solve $2x^3 - 3x^2 - 39x + 20 = 0$

### Solution

**a** Write the equation. $x^3 + 6x^2 + 12x + 8 = 0$

Use the perfect cube: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$x^3 + 3 \times 2x^2 + 3 \times 2^2x + 2^3 = 0$

$(x + 2)^3 = 0$

Use the null factor law. $x + 2 = 0$

Solve for $x$. $x = -2$

State your solution. $x = -2$

**b** Write the equation. $6x^3 + 15x^2 - 12x - 9 = 0$

Divide through by the common factor, 3. $2x^3 + 5x^2 - 4x - 3 = 0$

Write the simplified polynomial. $P(x) = 6x^3 + 15x^2 - 12x - 9$

Try factors of –3 (i.e. ±1, ±2, ±3). $P(1) = 6(1)^3 + 15(1)^2 - 12(1) - 9$

$= 0$

So $x - 1$ is a factor of $P(x)$.

Divide $P(x)$ by $x - 1$.

$$
\begin{array}{r}
2x^2 + 7x + 3 \\
x - 1 \overline{)\,2x^3 + 5x^2 - 4x - 3} \\
\underline{2x^3 - 2x^2} \\
7x^2 - 4x \\
\underline{7x^2 - 7x} \\
3x - 3 \\
\underline{3x - 3} \\
0
\end{array}
$$

Write the equation as a product of the factors found so far. $(x - 1)(2x^2 + 7x + 3) = 0$

Factorise the quadratic. $(x - 1)(2x + 1)(x + 3) = 0$

Use the null factor law. $x - 1 = 0$ or $2x + 1 = 0$ or $x + 3 = 0$

Solve each equation. $x = 1$ or $x = -0.5$ or $x = -3$

State your solution. Solution: $x = -3, -0.5$ or $2$

**c** Write the equation. $x^4 + 4x^3 + x^2 - 6x = 0$

Take $x$ out as a common factor. $x(x^3 + 4x^2 + x - 6) = 0$

Try factors of 6 (i.e. ±1, ±2, ±3, ±6). $P(x) = x^3 + 4x^2 + x - 6$

$P(1) = (1)^3 + 4(1)^2 + (1) - 6$

$= 0$

So $x - 1$ is a factor of $P(x)$.

Divide $P(x)$ by $x - 1$.

$$\begin{array}{r} x^2+5x+6 \\ x-1\overline{)\,x^3+4x^2+x-6} \\ x^3-x^2 \\ \hline 5x^2+x \\ 5x^2-5x \\ \hline 6x-6 \\ 6x-6 \\ \hline 0 \end{array}$$

State the equation as a product of the factors found so far. $x(x-1)(x^2+5x+6) = 0$

Factorise the quadratic. $x(x-1)(x+2)(x+3) = 0$

Use the null factor law. $x = 0$ or $x - 1 = 0$ or $x + 2 = 0$ or $x + 3 = 0$

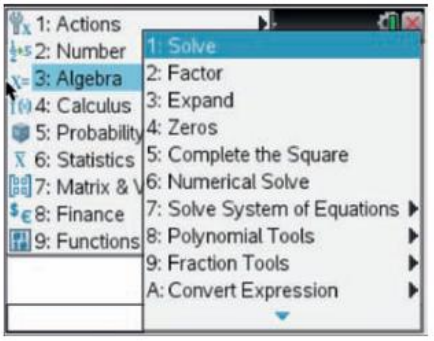
Solve each equation. $x = 0$ or $x = 1$ or $x = -2$ or $x = -3$

State your solution. Solution: $x = 0, 1, -2$ or $-3$

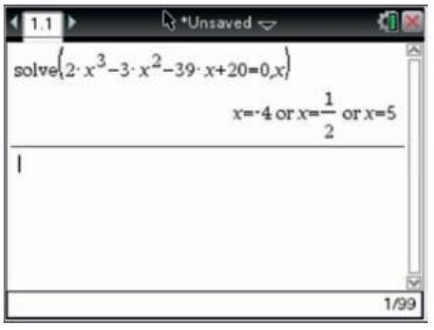
**d** **TI-Nspire CAS**

From a calculator page, press:

- menu
- 3:Algebra,
- 1: Solve




Type solve $(2x^3 - 3x^2 - 39x + 20 = 0, x)$ and press enter.





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Write the solution.

$x = -4, \frac{1}{2}$, or 5

**ClassPad**

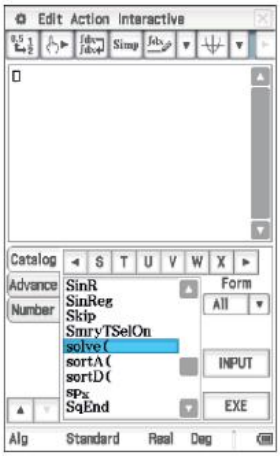
Use the Main $\sqrt{a}$ menu. Make sure the calculator is set to **Standard**, not **Decimal**.

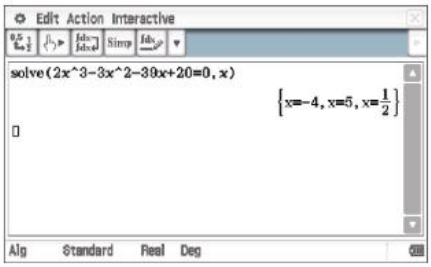
Press Keyboard and choose **solve** from the Catalog.

Type $2x^3 - 3x^2 - 39x + 20 = 0$, $x$ and close the brackets.

Tap EXE or press the EXE key.

The solution appears.




Write the answer.

$x = -4, \frac{1}{2}$, or 5

## INVESTIGATION General formula for solving cubic equations

In this section you have looked at solving cubic and some quartic equations which can be reduced to simple linear factors. To solve a quadratic equation which won't factorise easily, we use the quadratic formula. Investigate the existence of a formula for solving cubic equations.

If it exists:

- what is it?
- why is it not generally known or used?

Is there a formula for solving quartic equations or solving quintic equations? If they exist, how practical are they to use?

# 1.12 Polynomial equations

## Concepts and techniques

1 Example 30 Solve the following.

a $x^3 + 3x^2 + 3x + 1 = 0$
b $8x^3 - 12x^2 + 6x - 1 = 0$
c $x^3 + 12x^2 + 48x + 64 = 0$
d $-x^3 - 4x^2 + 5x = 0$
e $2x^3 + x^2 - 15x = 0$
f $x^3 + 3x^2 - 10x - 24 = 0$
g $x^3 + x^2 - 9x - 9 = 0$
h $12 - 19x + 8x^2 - x^3 = 0$
i $x^3 - 13x + 12 = 0$
j $-x^3 + 2x^2 + 9x - 18 = 0$
k $x^3 + 2x^2 - 4x - 8 = 0$
l $x^3 - 5x^2 + 8x - 4 = 0$

2 Solve

a $x(x + 2)(x - 3)(x - 9) = 0$
b $x^4 + 3x^3 + 2x^2 = 0$
c $-x^4 + 2x^3 + 3x^2 = 0$
d $x^4 - 3x^3 - 4x^2 + 12x = 0$
e $4x^4 + 8x^3 - x^2 - 2x = 0$

3 CAS Solve the following equations.

a $42x^3 - 41x^2 - 37x + 6 = 0$
b $108x^3 - 39x^2 - 5x + 2 = 0$
c $2x^3 - 5x^2 - 24x + 63 = 0$
d $4x^3 - 27x - 27 = 0$
e $x^4 + 7x^3 - 36x^2 - 252x = 0$


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# CHAPTER SUMMARY
# EQUATIONS AND POLYNOMIALS

1

- The **distributive law** states that $a(b + c) = ab + ac$ for any real numbers $a$, $b$ and $c$. It is used to remove brackets.
- A **binomial product** is a product of two expressions, each with two terms**.** You can expand a binomial product by repeated use of the distributive law, multiplying each term in the second bracket by each term in the first bracket.
- You can use the FOIL method to expand a binomial product, multiplying the: **F**irst terms, **O**uter terms, **I**nner terms, **L**ast terms

  $(a + b)(c + d) = ac + ad + bc + bd$
- The perfect squares are $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.
- The difference of two squares is $(a+b)(a-b) = a^2 - b^2$.
- Factorisation is the process of changing an expression to a product of factors. The **highest common factor** is placed outside the brackets, leaving the remaining factors of each term inside.
- When you simplify algebraic fractions, factorise if possible and then use the same rules as for numeric fractions.
- A **quadratic expression** is of the form $ax^2 + bx + c$, where $a$, $b$ and $c$ are real numbers. It is often referred to as a **trinomial** as it has three terms.
- Factorising a trinomial usually gives a binomial product. For $x^2 + (a + b)x + ab = (x + a)(x + b)$, the coefficient of $x$ is the sum of the numbers in the brackets and the **constant** is their product.
- The **cross method** of factorising a trinomial involves checking different pairs of factors.
- To use the **fraction method** of factorisation of a trinomial $ax^2 + bx + c$, write it as the product of $(ax + m)(ax + n)$ on a denominator of $a$, where the numbers $m$ and $n$ add to give $b$ and multiply to give the product $ac$. The fraction will cancel to give the factored expression.
- An **equation** is a mathematical sentence that contains an equal sign with an expression on each side.
- The simplest equation is a **linear** equation. Any variables have only a power of 1 and other terms are constants.
- You solve a linear equation using inverse operations to find a number that makes the equation true. This is called a **root** or **solution** of the equation**.**
- A **formula** is an equation that shows a relationship between variables.
- To make a variable the **subject** of a formula you use inverse operations to isolate that variable.
- The **power** of any variable in a **linear relationship** is one.
- The graph of a linear relationship involving one or two variables is **a straight line** in the form $y = mx + c$. The **dependent variable** is $y$ and the **independent variable** is $x$.
- The graph of a **direct proportion** relationship is a straight line through the origin of the form $y = kx$, where $k$ is a constant. In direct proportion, as $x$ increases or decreases $y$ also increases or decreases.
- **Simultaneous equations** are sets of equations with more than one variable. Values that make all the equations true are called **simultaneous solutions**.
- The solution of two simultaneous equations is the point of intersection of their graphs.

- Solving simultaneous equations algebraically uses either: the **elimination method**, which adds multiples of the equations to eliminate one of the variables; or the **substitution method**, which substitutes one equation into the other.
- A **quadratic equation** has a highest power of 2 and has the general form $ax^2 + bx + c = 0$, where $a \neq 0$.
- The **null factor law** states that if a product is zero, then one or more of the factors must be zero. For two factors, if $ab = 0$ then $a = 0$, $b = 0$ or $a = b = 0$.
- **Completing the square** is a technique that relies on the perfect square factorisation of $x^2 + 2ax + a^2 = (x + a)^2$ to solve a quadratic equation.
- The **quadratic formula** is used to solve any quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. The solutions are given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- The **discriminant** for any quadratic equation, $ax^2 + bx + c = 0$, is $\Delta = b^2 - 4ac$.

  For $\Delta < 0$, there are *no* real solutions.

  For $\Delta = 0$, there is only *one* real solution that is given by $x = -\frac{b}{2a}$. If the coefficients of the equation are rational, then this solution will be rational.

  For $\Delta > 0$, there are *two* real solutions. For equations with rational coefficients, the solutions are:

  – *rational* if $\Delta$ is a perfect square, but

  – *irrational* if $\Delta$ is not a perfect square.
- The equation of any **inverse proportion** is of the form $y = \frac{k}{x}$, where $k$ is a constant. As one quantity increases the other decreases.
- A **polynomial** is a sum of terms containing non-negative integer powers of the variable(s). A polynomial in $x$ is of the form: $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ where $n$ is a positive whole number and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.
- The highest power of $x$ is called the **degree** of the polynomial.
- The term that contains the highest power is called the **leading term**.
- The leading term of a **monic** polynomial has a coefficient of 1.
- A **cubic** polynomial has degree 3.
- A **quartic** polynomial has degree 4.
- The **value of a polynomial** for a particular value of $x$ is found by substitution.
- **Division of polynomials** can be performed using long division.
- The **remainder theorem** states: When a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.
- The **factor theorem** states: For a polynomial, $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial.
- The **converse** of the factor theorem states: For a polynomial, $P(x)$, if $x - a$ is a factor of the polynomial then $P(a) = 0$.
- To **factorise polynomials** try the factors of the constant term to find $a$ so $P(a) = 0$.
- The expansion of a perfect cube is $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ or $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.


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# CHAPTER REVIEW
# EQUATIONS AND POLYNOMIALS

1

## Multiple choice

1 Example 1 Expand and simplify $(3x-2y)^2$.

A $3x^2-12xy-2y^2$ B $9x^2-12xy-4y^2$ C $3x^2-6xy-2y^2$

D $9x^2-12xy+4y^2$ E $3x^2-12xy+2y^2$

2 Example 3 Simplify $\frac{3}{x^2-4}+\frac{2}{x-2}-\frac{1}{x+2}$.

A $\frac{x-3}{(x+2)(x-2)}$ B $\frac{x+1}{(x+2)(x-2)}$ C $\frac{x+4}{(x+2)(x-2)}$

D $\frac{x+5}{(x+2)(x-2)}$ E $\frac{x+9}{(x+2)(x-2)}$

3 Example 8 If $V=\pi r^2h$ is the volume of a cylinder, find the exact value of $r$ when $V=9$ and $h=16$.

A $\frac{4}{3\sqrt{\pi}}$ B $\frac{3}{4\sqrt{\pi}}$ C $\frac{4\sqrt{\pi}}{3}$ D $\frac{3\sqrt{\pi}}{4}$ E $\frac{16\pi}{9}$

4 Examples 13, 14 The simultaneous solution of $2x+y=1$ and $3x-y=14$ is:

A $(5,-3)$ B $(5,3)$ C $(3,-5)$ D $(-3,5)$ E $(4,-7)$

5 Example 17 Completion of the square changes $2x^2+4x-5=0$ to:

A $(x+2)^2=9$ B $(x+1)^2=6$ C $(x+1)^2=3.5$

D $(x+2)^2=6.5$ E $(x+1)^2=-3.5$

6 Example 18 The solutions of $3x^2+4x-5=0$, correct to 2 decimal places, are:

A 1.77 and $-0.44$ B $-1.77$ and 0.44

C 2.12 and $-0.79$ D $-2.12$ and 0.79

E $-6.36$ and 2.36

7 Example 19 $x^2-2x-2=0$ has

A no real roots B two rational roots C 1 real root

D 1 rational root E 2 irrational roots

8 Example 21 The quantity, $Q$, is directly proportional to the square of $d$. The equation of this relationship is given by:

A $Q=k^2d$ B $Q^2=kd$ C $d=kQ$ D $Q=kd^2$ E $Q=kd$

9 Example 29 Factorise $f(x)=x^3+4x^2-x-4$

A $f(x)=(x-1)(x+2)(x+2)$ B $f(x)=(x-1)(x+1)(x+4)$

C $f(x)=(x-2)(x-1)(x-1)$ D $f(x)=(x-1)^2(x-4)$

E $f(x)=(x-2)(x+1)(x+2)$

10 Example 29 Find the value of $a$ if $(x+1)$is a factor of $2x^3-x^2+ax-2$

A $-8$ B $-5$ C $-1$ D 1 E 5

## Short answer

11 Example 1 Expand and simplify $5(a+3)-(a-2)$.

12 Example 1 Expand and simplify $(2a-1)(3a-2b+5)$.

13 Example 1 Expand and simplify the following.

a $(3p+1)^2$ b $(4d-3)^2$ c $(2x+11)(2x-11)$

14 Example 2 Factorise the following.

a $mn-3m+5n-15$ b $3t(2t+5)-8(2t+5)$ c $4xy^2-8xy$

15 Example 3 Simplify the following.

a $\frac{2x-3}{5}+\frac{1-x}{4}$ b $\frac{a-3}{2}-\frac{a-2}{5}$

c $\frac{4x-12}{5y^3}\times\frac{10y}{2x^2-6x}$ d $\frac{5m+10}{m^2-3m}\div\frac{15m+30}{8mn}$

16 Example 4 Factorise the following.

a $m^2+9m+20$ b $p^2-12p+36$ c $t^2-49$

17 Example 5 Factorise the following.

a $2x^2+17x+36$ b $12x^2-5x-2$ c $9a^2-12a+4$

18 Example 6 Solve the following.

a $8=3x-22$ b $4(3y+1)=11y-3$

19 Example 6 Solve $\frac{n}{4}-\frac{n+2}{3}=9$.

20 Example 7 CAS Solve $\frac{3}{x+1}=\frac{4}{x}$.

21 Example 8 The area of a trapezium is given by $A=\frac{1}{2}h(a+b)$.

a Rewrite with $a$ as the subject.

b Find $A$ when $h=6$, $a=5$ and $b=7$.

c Find $b$ when $A=40$, $h=5$ and $a=4$.

d Use your CAS calculator to check your answers to parts **a**, **b** and **c**.

22 Example 11 Solve these simultaneous equations graphically.

$3x+2y=5$ and $y=2x+6$

23 Example 13 Solve these simultaneously using the elimination method.

$5m-3n=19$ and $2m-4n=16$

24 Example 13 Solve these simultaneously using the elimination method

$\frac{4x}{5}=y+2$ and $\frac{5y}{4}-3x=9$

25 Example 14 Solve these simultaneously using the substitution method

$3x+2y=4$ and $y-2x=9$

26 Example 16 Solve the following.

a $(2x+9)(x-5)=0$ b $x^2+6x-16=0$

c $4x^2-13x-12=0$ d $x-9=4x-\frac{48}{x-3}$

27 Example 18 Use the quadratic formula to find the exact solution to $3x^2+9x=-5$



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28 Example 19 Use the discriminant to determine the number and nature of the solutions of $3x^2 = 4x + 12$.

29 Example 20 Use your CAS calculator to find the exact solutions of the quadratic equation $2x^2 + 4x - 5 = 0$

30 Example 24 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find

a the degree
b the coefficient of $x$
c the zeros
d the leading term.

31 Example 25 Expand $(2x - 9)(3 - x)(x + 5)$.

32 Example 26 If $f(x) = 3x - 4$, find

a $f(2)$
b $x$ when $f(x) = 7$
c $x$ when $f(x) = 0$

33 Example 27 Divide $p(x) = 3x^5 - 7x^3 + 8x^2 - 5$ by $x - 2$, and write $p(x)$ in the form $p(x) = (x - 2)Q(x) + R(x)$.

34 Example 28 Find the remainder when $P(x) = x^3 + 7x^2 + 5x - 4$ is divided by $x + 7$.

35 Example 29 Factorise $P(x) = x^3 + 7x^2 - 5x - 75$.

36 Example 30 Solve $x^3 + 6x^2 - 9x - 14 = 0$.

## Application

37 The amount of seed, $s$, (in kilograms) used to sow a paddock is directly proportional to the size of the paddock, $p$ (in hectares). A variety of dryland cotton was to be planted using the formula $s = 7.5p$.

a Draw a graph of this equation.
b Is this a linear relationship? Why?
c Is this an example of direct proportion? Justify your answer.
d From your graph, find the weight of seed needed to plant an 80 ha paddock.

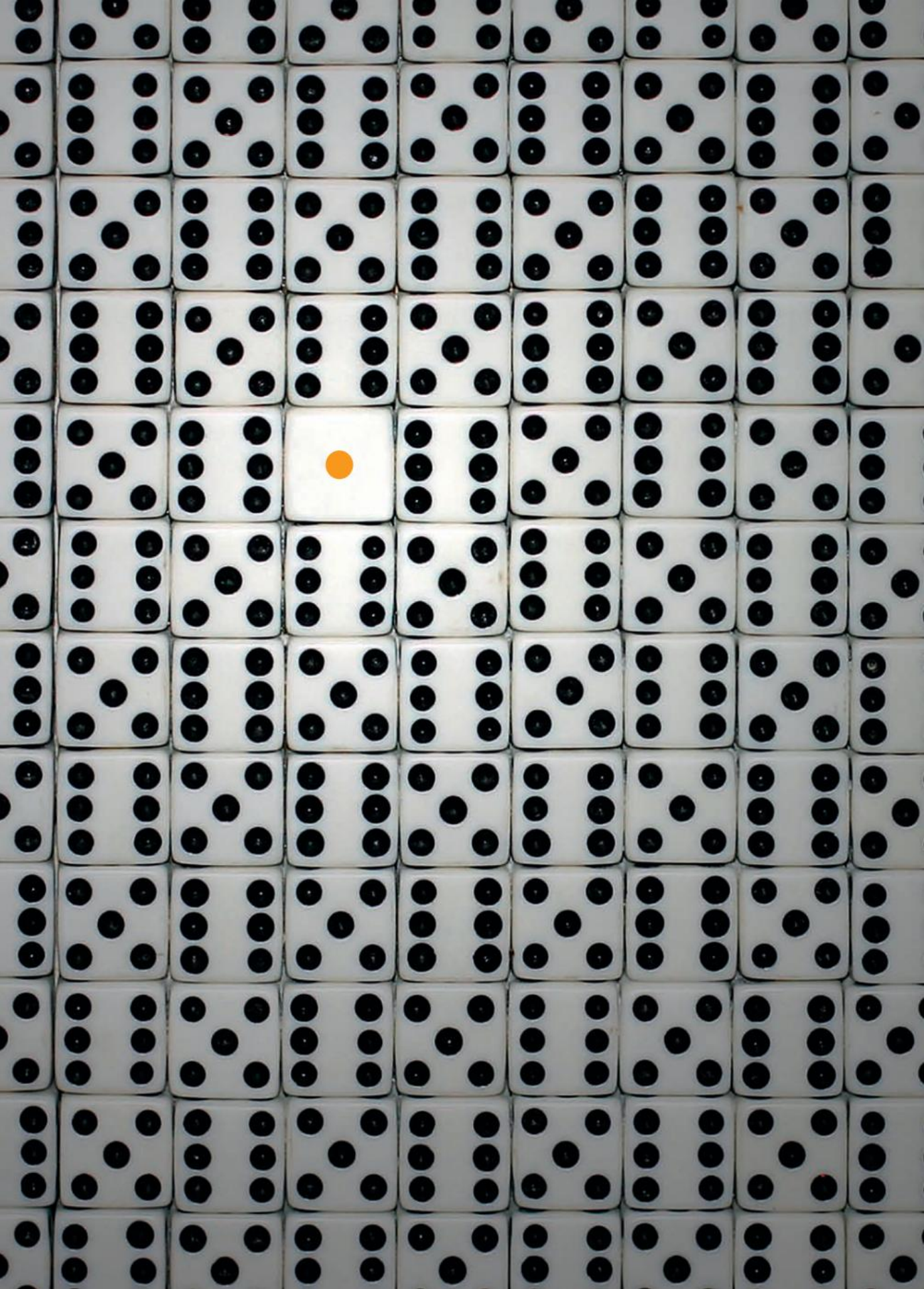
38 A shipping container holds 5 pallets of washing powder and 3 pallets of sugar. The weight of the contents of the shipping container is 1880 kg. The contents of another shipping container, holding 3 pallets of washing powder and 5 pallets of sugar, weigh 1640 kg. Find the weight of a pallet of washing powder and a pallet of sugar. Use your CAS calculator to check your answer.

39 Find the quantity of milk that would be produced by 150 dairy cows, if a herd of 120 cows produces 5400 litres of milk per day. Assume that the average daily production rate per cow remains the same.

Practice quiz

40 At Avefun Theme Park, the number of tourists who can travel on a chair lift at any time is inversely proportional to the distance between the chairs. When the chairs are 8 metres apart, the lift can carry 1125 people in one hour. How many people could the lift carry if the chairs were 7.5 metres apart?

41 When $8x^3 - 5kx + 9$ is divided by $x - 2$, the remainder is $-1$. Evaluate $k$.





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# 2

## TERMINOLOGY

binomial coefficient
binomial expansion
binomial theorem
cardinality
certain
combination
complement
disjoint
element
elementary outcomes
event
expected frequency
impossible
intersection
mutually exclusive
null set
Pascal's triangle
probability
proper subset
random
relative frequency
sample space
set
subset
trial
uniform probability distribution
union
universal set
Venn diagram
well-defined

Prior learning

PROBABILITY

# SETS AND BASIC PROBABILITY


2.01 Set notation
2.02 The binomial theorem
2.03 Counting combinations
2.04 Sample spaces and events
2.05 Definition of probability
2.06 Probability in finite sample spaces
2.07 Probabilities using combinations
2.08 Relative frequency
2.09 Probabilities of combined events
Chapter summary
Chapter review

## COMBINATIONS

- understand the notion of a combination as an unordered set of $r$ objects taken from a set of $n$ distinct objects (ACMMM044)
- use the notation $\binom{n}{r}$ and the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for the number of combinations of $r$ objects taken from a set of $n$ distinct objects (ACMMM045)
- expand $(x+y)^n$ for small positive integers $n$ (ACMMM046)
- recognise the numbers $\binom{n}{r}$ as binomial coefficients, (as coefficients in the expansion of $(x+y)^n$) (ACMMM047)
- use Pascal's triangle and its properties. (ACMMM048)

## LANGUAGE OF EVENTS AND SETS

- review the concepts and language of outcomes, sample spaces and events as sets of outcomes (ACMMM049)
- use set language and notation for events, including $\overline{A}$ (or $A'$) for the complement of an event $A$, $A \cap B$ for the intersection of the events $A$ and $B$, and $A \cup B$ for the union, and recognise mutually exclusive events (ACMMM050)
- use everyday occurrences to illustrate set descriptions and representations of events, and set operations (ACMMM051)

## REVIEW OF THE FUNDAMENTALS OF PROBABILITY

- review probability as a measure of 'the likelihood of occurrence' of an event (ACMMM052)
- review the probability scale: $0 \leq P(A) \leq 1$ for each event $A$, with $P(A) = 0$ if $A$ is an impossibility and $P(A) = 1$ if $A$ is a certainty (ACMMM053)
- review the rules: $P(\overline{A}) = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (ACMMM054)
- use relative frequencies obtained from data as point estimates of probabilities (ACMMM055) AC

# 2.01 SET NOTATION

You have already used the concept of a **set** in junior mathematics, but it now needs to be defined more carefully. About 100 years ago it was realised that simply defining a set as a collection of objects or symbols led to contradictions. Sets are now always limited by the context and must be **well-defined** to avoid problems.

### IMPORTANT

A **well-defined** collection is one in which there is no ambiguity as to whether or not an object belongs to the collection. A **set** is a well-defined collection of objects, usually denoted by a capital letter. Each member of a set is called an **element**. The elements are listed in braces such as $A = \{1, 3, 5, 7, \ldots\}$ or defined by a property like 'a counting number not divisible by 2'. This is written formally as $A = \{x: x \text{ is a counting number not divisible by } 2\}$, which is read as 'the set $A$ with elements $x$ such that $x$ is a counting number not divisible by 2'. It doesn't matter what order you write the elements of a set in, so {a, e, i, o, u} is the same as {e, a, o, i, u}.

The **universal set** is a fixed set of all the elements used in a particular problem, theory, context, or discussion. The universal set is shown as $U$ or $\xi$ (the Greek letter xi). The universal set could be implied by the context. In some cases it might not be formally specified but is still clearly understood.

The **null set** is the set with no elements, shown as {} or $\varnothing$.

Sets are usually shown using capital letters and the symbol $\in$ means 'is a member of', so $11 \in A$ means 11 is a member of $A$ and $14 \notin A$ means 14 is not a member of $A$.


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## Example 1

State whether or not the following are well-defined.

a The great Australian novels.

b The novels of Patrick White.

### Solution

| | |
|---|---|
| a It is not clear what is meant by 'great'. | 'great' is not well-defined so is not a set. |
| b The novels are those of a particular author. | It is well-defined because you can tell if a book is in the collection or not, so it is a set. |

**Venn diagrams** are often used to show small numbers of sets and their relationships.

**IMPORTANT**

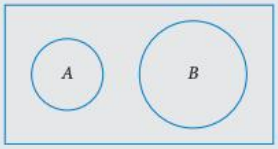
A **Venn diagram** shows sets as circles or ovals, with elements common to the two sets shown by the overlapping of the circles. The whole diagram is shown within a rectangle representing the universal set. Sometimes elements are shown within the circles.

There are a number of set operations and relationships for which you should know the symbols.

**IMPORTANT**

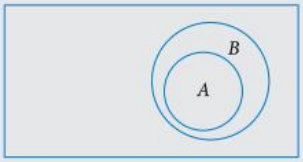
**Set relations and operations**

Two sets are **equal** if every element of one is also an element of the other. **Mutually exclusive** sets $A$ and $B$ have no elements in common. They are also called **disjoint** sets. This diagram shows disjoint sets $A$ and $B$.

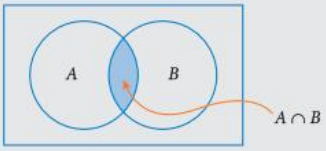



The set $A$ is a **subset** of the set $B$, written as $A \subseteq B$, if every element of $A$ is an element of $B$.

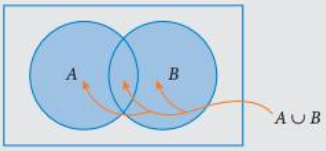
A **proper subset** is one that is definitely smaller, thus $A \subset B$ means that $A$ is a subset that is not equal to $B$.




The **intersection** of the sets $A$ and $B$, written as $A \cap B$, is the set whose elements are elements of both $A$ and $B$.



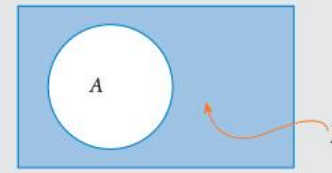

The **union** of the sets $A$ and $B$, written as $A \cup B$, is the set whose elements are elements of at least one of $A$ or $B$ (are in $A$ or $B$ or both).



The **complement** of the set $A$, written as $A'$ or $\overline{A}$, is the set consisting of all elements not in $A$: $\overline{A} = \{x: x \notin A\}$.

Clearly, $A \cap \overline{A} = \varnothing$, $A \cup \overline{A} = U$, $\overline{\varnothing} = U$ and $\overline{U} = \varnothing$.

The **cardinality** of a set $A$ measures its size and is shown as $|A|$. For finite sets it is the number of elements in the set and is often written as $n(A)$.




## Example 2

In the universal set of digits $\{0, 1, 2, \ldots, 9\}$, $A = \{3, 4, 5, 6, 8\}$, $B = \{2, 4, 9\}$ and $C = \{1, 2, 3, 4\}$. Show the elements in a Venn diagram and write down

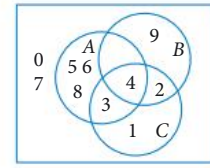
**a** $A \cup B$ **b** $B \cap C$ **c** $\overline{A}$

**d** $\overline{C \cap A}$ **e** $\overline{C} \cup \overline{A}$

**f** Compare the answers to parts **d** and **e**.

### Solution

Show the digits in each set in overlapping circles. Remember to put the 0 and 7 outside the circles to complete the universal set.




**a** Write the elements in either $A$ or $B$.

$A \cup B = \{2, 3, 4, 5, 6, 8, 9\}$

**b** Write the elements in both $B$ and $C$.

$B \cap C = \{2, 4\}$

**c** Write the elements that are not in $A$.

$\overline{A} = \{0, 1, 2, 7, 9\}$

**d** Find $C \cap A$ first.

$C \cap A = \{3, 4\}$

Now write the elements that are not in $C \cap A$.

$\overline{C \cap A} = \{0, 1, 2, 5, 6, 7, 8, 9\}$

**e** Write down $\overline{C}$ and $\overline{A}$.

$\overline{C} = \{0, 5, 6, 7, 8, 9\}$ and $\overline{A} = \{0, 1, 2, 7, 9\}$

Now write their union.

$\overline{C} \cup \overline{A} = \{0, 1, 2, 5, 6, 7, 8, 9\}$

**f** The answers are the same.

$\overline{C \cap A} = \overline{C} \cup \overline{A}$


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## Example 3

$A = \{a, b, c, d, f, g, h, k, m, n\}$, $B = \{e, i, p, q, r\}$ and $C = \{a, b, c, p, q, u, x, y, z\}$. Show the elements of these sets in a Venn diagram.

### Solution

The universal set is implied.

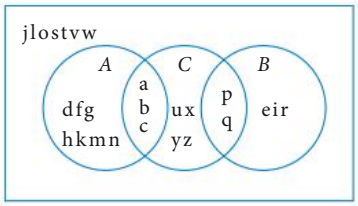
The universal set is the English alphabet.

Look for intersections.

$A$ and $B$ are disjoint, but both have elements in common with $C$.

Draw the diagram and fill in the letters of the alphabet.

Make sure that elements belonging to two sets are shown in the overlap.




Some sets have special symbols.

**IMPORTANT**

**Symbols for important sets**

The set of **natural numbers** is shown as $\boldsymbol{N}$. $\boldsymbol{N} = \{1, 2, 3, 4, \ldots\}$

The set of **integers** is shown as $\boldsymbol{Z}$ or $\boldsymbol{J}$. $\boldsymbol{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of **rational numbers** is shown as $\boldsymbol{Q}$.

The set of **real numbers** is shown as $\boldsymbol{R}$.

A superscript of + or − is used to show only positive or negative cases. $\boldsymbol{Q}^+$ means the positive rational numbers.

A back slash can be used to show an omission. $\boldsymbol{R}\backslash 0$ or $\boldsymbol{R}\backslash\{0\}$ means the real numbers except 0.

Whole numbers are often shown as $\boldsymbol{W} = \{0, 1, 2, 3, \ldots\}$ and some mathematicians use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ for the natural numbers, integers, rational and real numbers.

WS
Set operations

# EXERCISE 2.01 Set notation

## Concepts and techniques

**1** Example 1 State whether of not each of the following sets is well-defined.

**a** Rational numbers
**b** Self-governing states of the world
**c** Letters of the Greek alphabet
**d** The set of powers of 5
**e** {21, 63, 147, 189, 441, 567, 1323, 1701, …}

**2** While quadratics and third-power equations have formulas for their solution, it has been proven that 5th or higher degree equations do not have formulas for their solution. Does this necessarily mean that the solutions of the equation $x^6 - 10x^3 - 9 = 0$ are not well-defined?

**3** Is the set of numbers exactly divisible by 3 well-defined?

**4** List the elements of the following sets.

**a** $A = \{x: x \text{ is a prime number less than } 15\}$
**b** $B = \{y: 4y = 12\}$
**c** $C = \{n: n \text{ is an even natural number}\}$
**d** $D = \{x \in \boldsymbol{N}: x^2 = 5\}$
**e** $E = \{x \in \boldsymbol{J}: -5.4 \leq x \leq 3.7\}$

**5** Example 2 If the universal set is the rational numbers from 0 to 1 inclusive, and $A = \left\{x : x = \frac{1}{n}, n \in \boldsymbol{N}\right\}$, $B = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, $C = \left\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$ and $D = \left\{x : x = \frac{1}{n}, n \in \boldsymbol{N}, n \leq 10\right\}$, find the following.

**a** $A \cap B$
**b** $A \cup B$
**c** $\{x: x > 0.5, x \in A\}$
**d** $B \cap C$
**e** $\overline{C} \cap \overline{B}$
**f** $\overline{C \cap D}$
**g** $\overline{\overline{C}} \cup D$
**h** $\overline{\overline{B} \cup \overline{C}}$
**i** $A \cap (D \cup B) \cap C$

**6** List all the subsets of $M = \{x, y, z\}$.

**7** State whether each of the following is true or false.

**a** $2 \in \{\text{even numbers}\}$
**b** $2 \in \{3, 5, 7, 9\}$
**c** $2 \in \{x: 0 \leq x \leq 10\}$

**8** Write down $|A|$ for each of the following sets.

**a** $A = \varnothing$
**b** $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
**c** $A = \{x \in \boldsymbol{Z}: x^2 = 3\}$
**d** $A = \{x \in \boldsymbol{Q}^+: x^2 = 3\}$
**e** $A = \{x \in \boldsymbol{Q}: x^2 = 3\}$

**9** Example 3 Given $A = \{3, 5, 7, 9\}$, $B = \{4, 5, 6, 7\}$ and $C = \{2, 8\}$, draw a Venn diagram to show $A$, $B$ and $C$ and shade $A \cap B$.

**10** Draw a Venn diagram to show each of the following.

**a** $\overline{A \cup B}$
**b** $\overline{A \cap \overline{B}}$
**c** $(A \cap B) \cup C$
**d** $\overline{A \cap B \cap C}$
**e** $\overline{A \cup B \cup C}$
**f** $\overline{A \cup \overline{B} \cap C}$
**g** $(A \cup B) \cap \overline{C}$
**h** $(A \cup B) \cap C$
**i** $\overline{(A \cap \overline{B}) \cap C}$


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# 2.02 THE BINOMIAL THEOREM

You should recall that the expansion of a perfect square is $(x+y)^2 = x^2 + 2xy + y^2$.

What happens when you expand $(x+y)^3$?

Since $(x+y)^3 = (x+y)^2(x+y)$, you can just multiply $(x^2 + 2xy + y^2)$ by $(x+y)$.

This can be shown as

$$\begin{aligned}(1x^2 + 2xy + 1y^2)(x+y) &= 1x^2(x+y) + 2xy(x+y) + 1y^2(x+y)\\ &= 1x^3 + 1x^2y + 2x^2y + 2xy^2 + 1xy^2 + 1y^3\\ &= 1x^3 + (1+2)x^2y + (2+1)xy^2 + 1y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

To find $(x+y)^4 = (x+y)^3(x+y)$, you can just multiply $(x^3 + 3x^2y + 3xy^2 + y^3)$ by $(x+y)$.

This can be shown as

$$\begin{aligned}(1x^3 + 3x^2y + 3xy^2 + 1y^3)(x+y) &= 1x^3(x+y) + 3x^2y(x+y) + 3xy^2(x+y) + 1y^3(x+y)\\ &= 1x^4 + 1x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + 1xy^3 + 1y^4\\ &= 1x^4 + (1+3)x^3y + (3+3)x^2y^2 + (3+1)xy^3 + 1y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

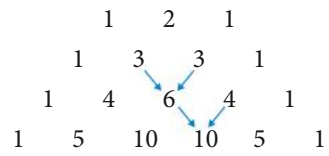
To find $(x+y)^5 = (x+y)^4(x+y)$, you can just multiply $(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4)$ by $(x+y)$.

This can be shown as

$$\begin{aligned}(1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4)(x+y) &= 1x^4(x+y) + 4x^3y(x+y) + 6x^2y^2(x+y) + 4xy^3(x+y) + 1y^4(x+y)\\ &= 1x^5 + 1x^4y + 4x^4y + 4x^3y^2 + 6x^3y^2 + 6x^2y^3 + 4x^2y^3 + 4xy^4 + 1xy^4 + 1y^5\\ &= 1x^5 + (1+4)x^4y + (4+6)x^3y^2 + (6+4)x^2y^3 + (4+1)xy^4 + 1y^5\\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\end{aligned}$$

The coefficients for each expansion are shown in red above and are the sums of the successive coefficients of the previous expansion, apart from the 1s at either end.

The coefficients for the expansions from $(x+y)^2$ to $(x+y)^5$ can be set out in a triangle as shown below. Some of the additions above are also shown.




The expansions of $(x+y)^0$ and $(x+y)^1$ can be added to the top of the triangle. Remember that $a^0 = 1$ for any $a \neq 0$, so $(x+y)^0 = 1$ and $(x+y)^1 = 1x + 1y$. Numbering each row by the power and putting these rows on top gives the complete triangle below.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 1 | | | | Row 0 |
| | 1 | | 1 | | | Row 1 |
| | 1 | 2 | 1 | | | Row 2 |
| 1 | 3 | 3 | 1 | | | Row 3 |
| 1 | 4 | 6 | 4 | 1 | | Row 4 |
| 1 | 5 | 10 | 10 | 5 | 1 | Row 5 |

This triangle of numbers is called Pascal's triangle. It is constructed in exactly the same way as the coefficients in the expansions of $(x+y)^n$: each number is the sum of the two numbers diagonally above it. The numbers are the coefficients of the expansion of $(x+y)^n$ when the expansion is written out, with decreasing powers of $x$ in each successive term. The terms in the expansion start with $x^n$, then $x^{n-1}y$, then $x^{n-2}y^2$, and so on until you get to $xy^{n-1}$ and finally $y^n$. Notice that the powers of $x$ and $y$ in each term add up to $n$.

## INVESTIGATION Pascal's triangle

Look at the numbers in the diagonals of Pascal's triangle.

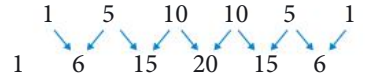
1. What are the numbers in the second diagonal?
2. What are the numbers in the third diagonal?
3. What do the numbers in the fourth row add up to?
4. What do the numbers in the fifth row add up to?
5. Can you find some of the other patterns in Pascal's triangle?

### Example 4

Use Pascal's triangle to write the expansion of $(x+y)^6$.

#### Solution

Write row 5 of Pascal's triangle.
Work out row 6. Start with 1.
Add the numbers diagonally.




Write the term powers. $x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6$

Put the coefficients in place $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Expansions of the type $(x+y)^n$ are called **binomial expansions** because there are two numbers in the brackets.

There is a formula for the values of the numbers in Pascal's triangle so you don't have to write out the values of row 15 to find the expansion of $(x+y)^{15}$.

**IMPORTANT**

The numbers in Pascal's triangle are written as $\binom{n}{r}$ and have the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

where $n! = n \times (n-1) \times (n-2) \cdots 3 \times 2 \times 1$ and $0! = 1$.

Calculators show $\binom{n}{r}$ as $nCr$ or ${}^nC_r$.


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## Example 5

CAS Calculate the value of $\binom{6}{4}$.

### Solution

Write the formula.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Find the last term of the numerator.

$$n-r+1 = 6-4+1 = 3$$

Substitute values.

$$\binom{6}{4} = \frac{6\times5\times4\times3}{1\times2\times3\times4}$$

Cancel where possible and evaluate.

$$= \frac{{}^{3}\cancel{6}\times5\times\cancel{4}\times\cancel{3}}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}} = 15$$

**TI-Nspire CAS**

Choose a calculator page, press [doc] to open the catalogue and use the nCr( function. You can press N to save scrolling through the whole catalogue.

You can also use [menu], 5: Probability and 3: Combinations or just type nCr(.

Fill in the values and press [enter].



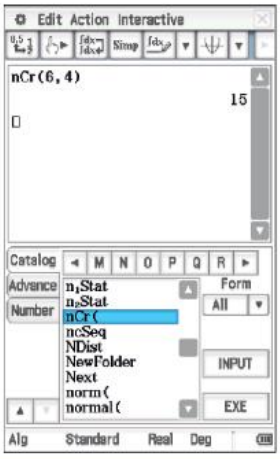

**ClassPad**

Use the Main $\sqrt{\alpha}$ menu and press [Keyboard] for the 'soft' keyboard.

Tap [▼] and from the [Catalog] tab scroll along the letters at the top of the keyboard, tap N and scroll down to **nCr**.

Enter the value of $n$ first, then $r$.

For $\binom{6}{4}$ enter nCr(6,4) [EXE].




Many scientific calculators also have the *nCr* function.

When you put together all of the information covered so far in this section you get the following.

**IMPORTANT**

**The binomial theorem**

The expansion of $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$,

where the values ${}^nC_r$ for $0 \le r \le n$ are called the **binomial coefficients**.

The binomial expansion is symmetrical with respect to $x$ and $y$, so you can reverse the coefficients to get

$$(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \cdots + \binom{n}{n-r}x^{n-r}y^r + \cdots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n.$$

**IMPORTANT**

The values in Pascal's triangle are symmetrical, so $\binom{n}{r} = \binom{n}{n-r}$ for $0 \leq r \leq n$.

### Example 6

Find the fourth term of the expansion of $(2a-1)^{15}$.

#### Solution

| | |
|---|---|
| State the binomial theorem. | $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots$ |
| Write the fourth term. | Fourth term $= \binom{n}{3}x^{n-3}y^3$ |
| State the values. | For $(2a-1)^{15}$, $n=15$, $x=2a$ and $y=-1$ |
| Substitute values. | Fourth term of $(2a-1)^{15} = \binom{15}{3}(2a)^{12}(-1)^3$ |
| Simplify. | $= 455 \times 4096x^{12} \times (-1)$<br>$= -1\,863\,680x^{12}$ |
| Write the answer. | The fourth term of $(2a-1)^{15}$ is $-1\,863\,680x^{12}$. |

## EXERCISE 2.02 The binomial theorem

### Concepts and techniques

1 Example 4 Use Pascal's triangle to write the expansion of $(x+y)^4$.

2 Use Pascal's triangle to write the expansion of $(x+y)^7$.

3 Example 5 Calculate the value of each of the following.

a ${}^8C_3$ b ${}^9C_2$ c ${}^{11}C_4$ d ${}^7C_5$ e ${}^8C_6$

4 CAS Find the value of each of the following.

a ${}^{12}C_2$ b ${}^{16}C_3$ c ${}^{11}C_5$ d ${}^{13}C_7$ e ${}^{14}C_6$

5 Example 6 Find the third term of the expansion of $(a+2)^{12}$.

6 Find the sixth term of the expansion of $(a+b)^{15}$.

7 Find the fifth term of the expansion of $(b-3)^{14}$.

8 Find the eighteenth term of the expansion of $(2v-w)^{22}$.

## Reasoning and communication

9 Write down the expansion of $(2x + 3y)^7$.

10 Write down the expansion of $(3x - 2y)^9$.

# 2.03 COUNTING COMBINATIONS

There is another way to think about the coefficients in binomial expansions that is particularly relevant to probability.

You can find the terms in the expansion of $(x + y)^4$ by looking at the expansion as the selection of possible products from $(x + y)(x + y)(x + y)(x + y)$.

In how many ways can you choose an $x$ from each bracket?

There is only 1 way: take an $x$ from each bracket.

Number the brackets from I to IV. To get a product of $x^3y$ you can take an $x$ from each of brackets I, II and III and a $y$ from bracket 4. You can also get the $x$s from I, II and IV, I, III and IV and I, II and IV so there are 4 lots of $x^3y$ and the second term is $4x^3y$.

What about terms in $(x + y)^5$?

Choose an $x$ from each of two of the brackets $(x + y)(x + y)(x + y)(x + y)(x + y)$ to make $x^2y^3$.

You can choose from brackets I and II, I and III, I and IV, I and V, II and III, II and IV, II and V, III and IV, III and V and last of all IV and V. This makes 10 lots altogether, so the term with $x^2$ is $10x^2y^3$.

The numbers in Pascal's triangle are also the numbers of ways of choosing $r$ things from $n$ things when the order doesn't matter. A choice like this is called a **combination**. This is why the numbers are shown as ${}^nC_r$ as well as $\binom{n}{r}$. The $C$ stands for combination.

**IMPORTANT**

A **combination** is an un-ordered selection of some or all of the symbols from a group.

The number of combinations of $r$ objects from $n$ distinct objects is given by

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

## Example 7

How many combinations of 4 things from 10 things are possible?

### Solution

| | |
|---|---|
| Write out the formula. | $\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ |
| Work out the last value. | $n-r+1 = 10-4+1 = 7$ |
| Substitute into the formula. | $\binom{10}{4} = \frac{10\times9\times8\times7}{4\times3\times2\times1}$ |
| Evaluate. | $= 210$ |
| Write the answer. | There are 210 ways to choose 4 things from 10. |

Notice that there are 4 numbers on the top of $\binom{10}{4}$, starting from 10 and going down, and the numbers on the bottom go from 1 up to 4, so you don't really have to work out $n-r+1$.

## Example 8

There are 8 dominos left in a game. In how many different ways could you choose 3 of them?

### Solution

| | |
|---|---|
| You are choosing 3 from 8 things. | Number of ways $= \binom{8}{3} = {}^8C_3$ |
| Write out the formula. | $= \frac{8\times7\times6}{1\times2\times3}$ |
| Evaluate. | $= 56$ |
| Write the answer. | There are 56 ways to choose 3 dominos from 8. |

Rather than writing out the formula $\binom{n}{r} = {}^nC_r$ for large values of $r$, you should use its symmetry to change to a small number.

## Example 9

18 players must be chosen from a squad of 24 for a game of football. In how many ways could you do this?

### Solution

| | |
|---|---|
| You are choosing 18 from 24 things. | Number of ways $= \binom{24}{18} = {}^{24}C_{18}$ |
| Use the symmetry of $\binom{n}{r} = {}^nC_r$, which is the same as *leaving* 6. | $= {}^{24}C_6$ |


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| | |
|---|---|
| Write out the formula with 6 numbers on the top and bottom. | $= \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$ |
| Cancel where possible. | $= 23 \times 22 \times 7 \times 2 \times 19$ |
| Evaluate. | $= 134\,596$ |
| Write the answer. | There are 134 596 possible teams. |

Many people prefer to use the ${}^{n}C_{r}$ notation when they are working out combinations, especially in probability calculations.

# EXERCISE 2.03 Counting combinations

## Concepts and techniques

**1** Example 7 Find the values of each of the following.

**a** ${}^{9}C_{3}$ **b** ${}^{7}C_{2}$ **c** ${}^{8}C_{4}$ **d** ${}^{20}C_{3}$ **e** ${}^{15}C_{2}$

**2** Find the values of each of the following

**a** ${}^{9}C_{7}$ **b** ${}^{12}C_{10}$ **c** ${}^{15}C_{12}$ **d** ${}^{10}C_{8}$ **e** ${}^{8}C_{5}$

**3** Find the values of each of the following

**a** ${}^{12}C_{4}$ **b** ${}^{24}C_{3}$ **c** ${}^{17}C_{2}$ **d** ${}^{11}C_{7}$ **e** ${}^{14}C_{5}$

## Reasoning and communication

**4** Example 8 A soccer team usually has 2 strikers in a team of eleven players. In how many different ways could you choose the strikers in a team?

**5** There are ten dropper bottles on a tray in a chemistry laboratory that contain dilute sulphuric acid. In how many different ways could you choose 3 of them?

**6** Example 9 In how many ways could you choose a team of 6 players from a squad of 10 basketballers?

**7** A torch needs 4 batteries. How many different selections of batteries could you make from a container with only 6 left in it?

**8** You are allowed three guesses at a number between 1 and 10 inclusive. In how many different ways could you choose your three numbers?

**9** Four people are to be chosen from the first nine people entering a room. How many different combinations are possible (without regard for order)?

**10** Two different 'Lotto' games have players select six numbers from 40 and seven numbers from 45. Which has the smallest number of combinations?

# 2.04 SAMPLE SPACES AND EVENTS

In junior mathematics you regarded experimental probability as being based on data and theoretical probability as being based on a list of possibilities. The fact that the list of possibilities in theoretical probability was actually based on experience was ignored. For example, we assume that there are two possible outcomes when a coin is thrown: heads and tails. This is based on experience. We also assume, based on experience, that heads and tails are equally likely.

**IMPORTANT**

A **sample space** $S$ is a set of individual possible occurrences. The elements of the set $S$ are called **elementary outcomes** (or **elements**) and subsets of $S$ are called **events**. The set $S$ is the universal set for the situation considered.

There are 36 different ways that two dice can fall because for every possible number shown on the first, there are 6 possible numbers on the other. This means that there are 36 individual occurrences, even if you are only interested in the totals.

Most of the sample spaces you deal with are finite, which means that you do not need to consider some of the complexities of infinite sample spaces.

You *choose* a sample space so that it is most useful for the problem or situation under consideration.

The set $S$ = {heads, tails} or {H, T} associated with tossing a coin is a simple example of a mathematical sample space. Would this sample space still be an appropriate model if the coins were thicker?

Each element of a sample space should correspond to a single separate possible occurrence in a real situation like the dice mentioned above.

### Example 10

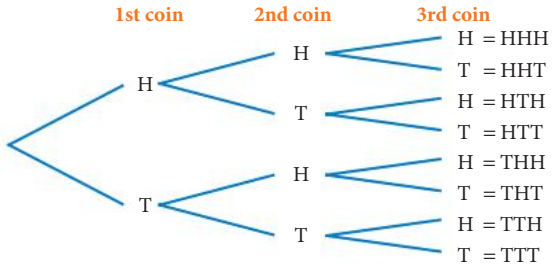
List the sample space for tossing three coins, using H and T for heads and tails.

#### Solution

| | |
|---|---|
| The first, second and third coins could all show heads. | $HHH \in S$ |
| The first and second could be heads and the other tails. | $HHT \in S$ |
| The first could be heads, the second tails and the third heads. | $HTH \in S$ |
| Continue the pattern to get the whole sample space. | $S$ = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} |


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You may recall using a tree diagram to show sample spaces like that in Example **10**. The tree diagram for Example **10** is shown below.




The number of elements in an event is important in probability for finite sample spaces. The event $E$ = 'exactly two heads' from the sample space above is {HHT, HTH, THH} so $|E| = 3$. This is often written as $n(E)$. You can also use a table to show a sample space.

## Example 11

Two dice are tossed. One is red and one is blue. How many elements are in the event 'a total of 5'?

### Solution

Show the sample space as a table.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **1** | 1 & 1 | 1 & 2 | 1 & 3 | 1 & 4 | 1 & 5 | 1 & 6 |
| **2** | 2 & 1 | 2 & 2 | 2 & 3 | 2 & 4 | 2 & 5 | 2 & 6 |
| **3** | 3 & 1 | 3 & 2 | 3 & 3 | 3 & 4 | 3 & 5 | 3 & 6 |
| **4** | 4 & 1 | 4 & 2 | 4 & 3 | 4 & 4 | 4 & 5 | 4 & 6 |
| **5** | 5 & 1 | 5 & 2 | 5 & 3 | 5 & 4 | 5 & 5 | 5 & 6 |
| **6** | 6 & 1 | 6 & 2 | 6 & 3 | 6 & 4 | 6 & 5 | 6 & 6 |

Write down the event. {1 & 4, 2 & 3, 3 & 2, 4 & 1}

Write the answer. There are 4 elements in the event.

You can combine events in the same ways as other sets.

**IMPORTANT**

$A$ **and** $B$ means the intersection of the events $A$ and $B$, $A \cap B$.

$A$ **or** $B$ means the union of the events $A$ and $B$, $A \cup B$.

**Not** $A$ means the complement of the event $A$, $\overline{A}$ or $A'$.

### Example 12

Show the following events from throwing two normal dice.

**a** A double (both dice the same).

**b** A total of 8.

**c** A double or a total of 8.

#### Solution

**a** A double has both dice the same. $D = \{1 \& 1, 2 \& 2, 3 \& 3, 4 \& 4, 5 \& 5, 6 \& 6\}$

**b** The total has to be 8. $T_8 = \{2 \& 6, 3 \& 5, 4 \& 4, 5 \& 3, 6 \& 2\}$

**c** Write the union. $D \cup T_8 = \{1 \& 1, 2 \& 2, 3 \& 3, 4 \& 4, 5 \& 5, 6 \& 6, 2 \& 6, 3 \& 5, 5 \& 3, 6 \& 2\}$

## EXERCISE 2.04 Sample spaces and events

### Concepts and techniques

**1** Example 10 List the sample space for tossing 4 coins.

**2** What is the sample space for throwing a normal die?

**3** Write the sample space for picking a student from a class with 11 boys and 15 girls.

**4** Example 11 Show the sample space for tossing a coin and rolling a normal die as a grid.

**5** How many elements are in the event 'a total of 7' from tossing two normal dice?

**6** How many elements are in the event 'one of the dice shows a 2' from tossing two dice?

**7** How many elements are in the event 'at least 2 tails' from tossing 4 coins?

**8** Example 12 A bag contains 9 marbles, of which 4 are red, 3 are white and 2 are black. Two marbles are taken out together. $A$ = 'at least one of the marbles is red', $B$ = 'the marbles are the same colour' and $C$ = 'at least one of the marbles is white'. Show the following events using R1, R2, R3 and so on for the marbles. The order of the marbles is not important.

**a** $A$ and $B$ **b** $B$ or $C$

**c** The marbles are not the same colour and neither is red.

**9** A coin is tossed 5 times. Show the event 'there are at least two heads in a row and at least one tail'.

### Reasoning and communication

**10** Show that $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$ for events $A$ and $B$ in a finite sample space.

**11** Show that for disjoint events $A$ and $B$, $\overline{A \text{ and } B} = S = \overline{A} \text{ or } \overline{B}$.

**12** Show that $n(A \text{ or } B) = n(A) + n(B)$ for mutually exclusive events $A$ and $B$ in a finite sample space.


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# 2.05 DEFINITION OF PROBABILITY

The mathematical definition of **probability** has been chosen to make it useful for working out the likelihood of something occurring. Different mathematicians have worked out different definitions and sets of rules, but they are really very much the same. The definition below is a simplified version of these *mathematical* definitions.

**IMPORTANT**

The **probability** of an event $A$ in a sample space $S$ is a real number $P(A)$ such that

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$ and $P(\varnothing) = 0$
3. The probability of a union of disjoint events is the sum of the probabilities of the events.

They are the rules that apply to sample spaces like the total of two dice. In that case, the elements are the different ways that the dice can fall, and every element has the same probability. Of course, if one of the dice were biased, then the elements could have different probabilities rather than being all the same. This is one of the reasons that mathematicians have defined probability like this.

The third property of probability means that probability has to add sensibly. The probability of an event made by combining separate (disjoint) events has to be the total of the probabilities of the separate events. In Year 10 you used the number of items in a finite sample space to work out the probability of an event. This easily satisfies the definition.

## Example 13

Write the sample space and the probability rule for working out the probability of getting each number from rolling a normal six-sided die. Check that it follows the probability rules.

### Solution

| | |
|---|---|
| The sample space is the possible numbers. | $S = \{1, 2, 3, 4, 5, 6\}$ |
| What is the probability of an event in the sample space? | $P(A) = \dfrac{n(A)}{6}$ as $n(S) = 6$ |
| Consider $P(A)$ for $A \subseteq S$. | $0 \leq n(A) \leq 6$ for $A \subseteq S$ so $0 \leq P(A) \leq 1$ |
| Consider $P(S)$. | $P(S) = \dfrac{n(S)}{6} = \dfrac{6}{6} = 1$ |
| Consider $P(\varnothing)$. | $P(\varnothing) = \dfrac{n(\varnothing)}{6} = \dfrac{0}{6} = 0$ |
| Consider disjoint unions. | For disjoint sets $A$ and $B$, it is clear that $\lvert A \cup B \rvert = \lvert A \rvert + \lvert B \rvert$ so $P(A \cup B) = P(A) + P(B)$ and this extends to more events. |
| State your answer. | The sample space and probability rule satisfy the conditions so the probability of rolling any number is $\frac{1}{6}$. |

What about probabilities involving areas?

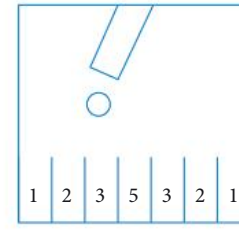
You usually assume that the probability of 'hitting' a particular area in a 2D problem is proportional to the area, and this satisfies the mathematical requirements for probability. However, the probability of hitting a single point in an infinite sample space is usually zero, otherwise the total probability of all the points would be more than 1.

## Example 14



Shutterstock.com/Ronald Sumners

A laughing clown game in sideshow alley has 7 chutes for balls to run down. The chutes are all the same size, but the ball delivery tube is in front of them for different times. The centre chute scores 5 and the tube takes 1 second to cross it. As it goes out it slows down, so it takes 2 seconds to cross the 3s either side of the middle, 3 seconds to cross the 2s and then 4 seconds to cross the 1s at the edge (each way). What are the sample space and probability rule? Work out the probability of getting a score of 5 from a single ball.




### Solution

Make the sample space the possible scores in a complete cycle forward and back.

$S = \{1, 2, 3, 5\}$

Find the time for a complete cycle.

$$\text{Time for 1 cycle} = 4 \times 4 + 4 \times 3 + 4 \times 2 + 2 \times 1 = 38 \text{ s}$$

The probability of getting a score is going to be the proportion of time the ball can roll to that score.

$$P(\text{score}) = \frac{\text{Time to pass score}}{\text{Cycle time}}$$

Consider $P(A)$ for $A \subseteq S$.

$0 \leq \text{total time for } A \leq 38 \text{ so } 0 \leq P(A) \leq 1$

Consider $P(S)$ and $P(\varnothing)$.

$$P(S) = \frac{38}{38} = 1 \text{ and } P(\varnothing) = \frac{0}{38} = 0$$


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| | |
|---|---|
| Consider disjoint unions. | The time for a disjoint union is obviously the sum of the individual times, so the condition is satisfied. |
| State the result. | The probability rule satisfies the conditions. |
| Calculate $P(5)$. | $P(5) = \frac{2}{38} = \frac{1}{19}$ |
| State the answer. | The probability of scoring 5 is $\frac{1}{19}$. |

# EXERCISE 2.05 Definition of probability

## Reasoning and communication

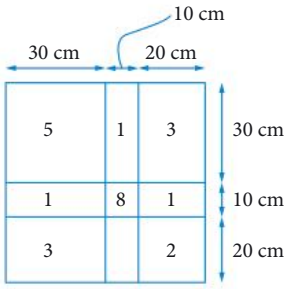
WS Dartboard probabilities

1 Example 13 Write the sample space and the rule for finding the probability of getting each colour from a set containing one of each of the following coloured pencils: blue, red, green, yellow, black, white, purple, orange and grey. Check that it follows the probability rules.

2 Write the sample space and the rule for finding the probability of getting each card from a set of 20 flash cards.

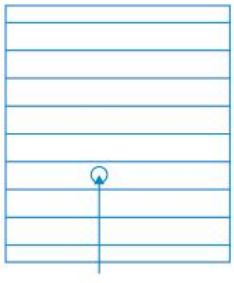
3 Write the sample space and the rule for finding the probability of getting a particular card by taking a single card from a normal deck of cards.

4 Example 14 A square board is divided into scoring areas by lines 30, 10 and 20 cm wide in both the vertical and horizontal directions as shown on the right. Small pieces of sticky coloured plasticine are thrown at the board from a distance of 2 m. Write the sample space and the rule for finding the probability of each score. Assume that the plasticine hits and sticks to the board.




5 An archery target has 10 concentric circles with radii from 5 cm to 50 cm. Write the sample space and the rule for finding the probability of landing in each of the ten regions from the inside to the outside.

6 In the game 'shove halfpenny', coins are shoved from the edge of the board across the board. There are nine lines across the board separated by a distance of one-and-a-half times the diameter of the coins. The first line and last lines are a distance from the edge equal to the diameter of the coin. To score a point, a player's coin must land completely between two lines. Write the sample space and probability rule for scoring, assuming that a player's coin remains on the board.



# 2.06 PROBABILITY IN FINITE SAMPLE SPACES

In probability situations for finite sample spaces, the elementary outcomes often have equal probabilities. In this case, the probability of an event is determined by simply counting the number of elements in the event.

You will often see the expression 'choose at random'. What does this mean mathematically?

If you chose a marble at random from a bag that contained 10 red marbles and 4 green marbles, every marble would have an equal probability of being chosen. This obviously does not mean that the probability of getting a red marble is the same as getting a green marble because there are different numbers of each colour.

For finite sample spaces, it is more common to use the counting notation $n(E)$ than the set notation $|E|$.

If you use the sample space {2 heads, tail and head, 2 tails} with 3 elements for throwing two coins at the same time, the formula doesn't give useful results. The sample space {HH, HT, TH, TT} with 4 elements has to be used so that the probabilities of each element are the same.

**IMPORTANT**

The **probability** of an event $E$ in a finite sample space $S$, whose elementary outcomes have equal probabilities, is given by

$$P(E) = \frac{n(E)}{n(S)} \left( = \frac{|E|}{|S|} \right).$$

This is often called a **uniform probability distribution**.

**IMPORTANT**

A **random** choice means that all the possible choices have equal probabilities. The possible choices make a uniform probability distribution.

### Example 15

What is the probability that a person's birthday this year is on a weekend?

#### Solution

| | |
|---|---|
| The sample space is the days of the week. | $S = \{\text{Mon, Tue, Wed, Thurs, Fri, Sat, Sun}\}$ |
| All outcomes are equally likely. | $P(\text{weekend}) = \frac{n(\text{weekend})}{n(S)} = \frac{2}{7}$ |
| Write the answer. | The probability that the person's birthday is on a weekend is $\frac{2}{7}$. |

The words **impossible** and **certain** have particular meanings in probability.

It is often easier to find the probability of something not occurring than to find the probability of it occurring. For example, it is easier to find the probability of getting the same number on two dice than it is to work out the probability of different numbers. 'The same number' and 'different numbers' are separate events but make the whole sample space. They are **complementary events**. You can use the complement of an event to find a probability.

**IMPORTANT**

An **impossible event** is the empty set, so its probability is 0.

A **certain event** is the sample space, so its probability is 1.

For an event $A$, the probability of its **complement** $\overline{A}$ or $A'$ is given by $P(\overline{A}) = 1 - P(A)$.


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$A \cup \overline{A} = S$ so $P(S) = P(A \cup \overline{A})$

But $A$ and $\overline{A}$ are mutually exclusive, so $P(A \cup \overline{A}) = P(A) + P(\overline{A})$.

This gives $P(A) + P(\overline{A}) = 1$ so the property is proved.

## Example 16

A normal die is thrown. What is the probability of not getting a 3?

### Solution

| | |
|---|---|
| Find the probability of getting a 3. | $P(3) = \frac{1}{6}$ |
| Now use the complement. | $P(\text{not } 3) = 1 - \frac{1}{6} = \frac{5}{6}$ |
| Write the answer. | The probability of not getting a 3 is $\frac{5}{6}$. |

If you take a card at random from a normal pack of cards and put it back 1000 times, how many times would you expect to get a spade? Because there are equal numbers of spades, hearts, diamonds and clubs, you would expect to get spades about 250 times. The probability of an event shows you the proportion of times the event can be expected to happen in the long run.

**IMPORTANT**

The **expected frequency** of an event from $n$ experiments is given by:

Expected frequency = probability $\times n$

Of course, the expected frequency would only occur sometimes. It is more correct to say it is the average number expected. After all, if you toss a coin 50 times, you would expect 25 heads, even though you wouldn't get this exact number particularly often.

Some finite sample spaces will not be uniform. In these cases you need to know the probabilities of the elementary outcomes to solve problems.

## Example 17

Traffic lights on a 90 second cycle are set up to be green for 40 seconds, amber for 8 seconds and red for 42 seconds in each direction, including all-red for 2 seconds. The all-red condition is when the lights are red in every direction as a safety measure. Lights on shorter or longer cycles have the times set proportionally, although the all-red is normally kept at 2 seconds. A taxi goes through 192 intersections with traffic lights. How many green lights would you expect the taxi to get?

### Solution

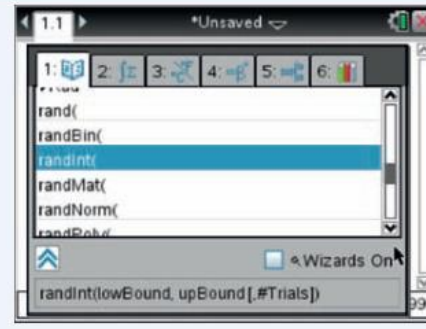
| | |
|---|---|
| Write the sample space for one direction. | $S = \{R, G, A\}$ |
| Write the probabilities for green. | $P(G) = \frac{40}{90} = \frac{4}{9}$ |
| Use the formula. | Expected green $= 192 \times \frac{4}{9} \approx 85.3$ |
| Write the answer. | The taxi would get about 85 green lights. |

## INVESTIGATION Traffic lights

You can simulate arriving at traffic lights using a computer spreadsheet or a CAS calculator. Calculators work with integer random numbers. The cycle involves a whole number of seconds and lasts for 90 seconds. You can simulate this situation by generating a random number from 1 to 90. The first 40 numbers (from 1 to 40) represent the lights being green. The next 8 numbers (41–48) represent amber and the last 42 numbers (49–90) represent red.

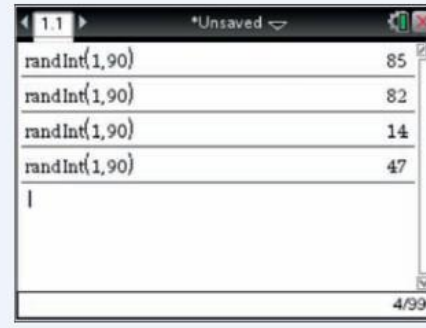
### TI-nspire CAS

In a calculator page, press for the catalogue and scroll down to randint(. You can press R to save scrolling down the whole list. You can also type randint or use menu 5: Probability, 4: Random and 2: Integer.

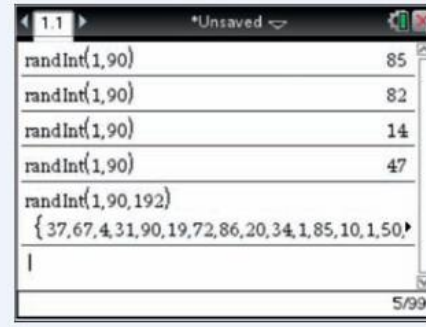



Enter 1, 90 and enter to get a random number from 1 to 90. Each time you press enter you will get another random number from 1 to 90.

The calculator shows two red (85, 82), a green (14) and an amber light (47) in this case.

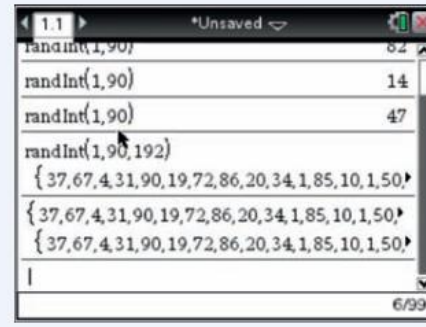



You can get as many numbers as you like in a list by specifying how many you want after the limits. Make a list of 192 as shown.




You can sort the list as follows.

First store the list in the variable *a*, say, by pressing ctrl var a.

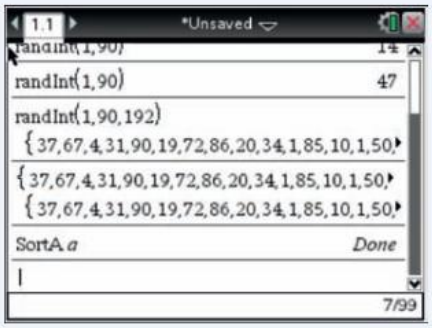




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Go back to the catalogue ( ) and scroll down to SortA. This will sort in ascending order.
You can also type SortA or use menu 6: Statistics, 4: List Operations and 1: Sort Ascending.

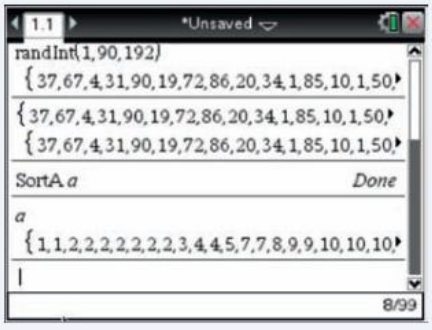
Sort the list you have stored in the variable *a*.




Now you can look at your sorted list of 192 random numbers from 1 to 90 by entering a.

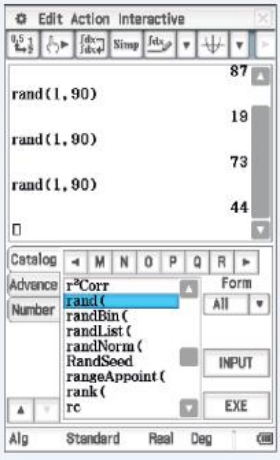
Use the arrow keys to go back up and scroll across the list.

You can easily count the number of red, green and amber lights.

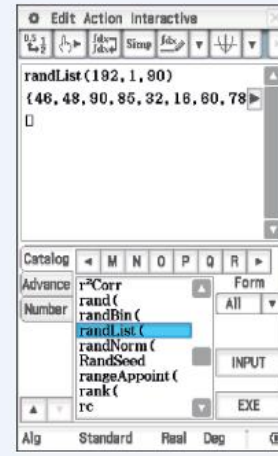



**ClassPad**

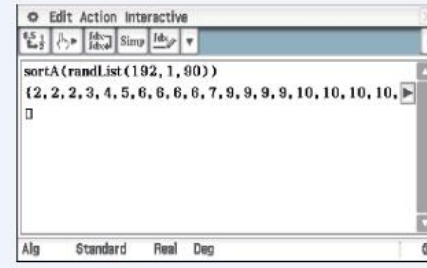
Use the Main $\sqrt{\alpha}$ menu and press Keyboard.
Use the Catalog **tab** to enter rand(1,90).
Tap EXE or press EXE.
rand(1,90) will cause the calculator to display a random number from 1 to 90.
The calculator shows this has been done four times giving 87 (red), 19 (green), 73 (red) and 44 (amber) for four sets of traffic lights.



This can be done 192 times by using randList. randList(192,1,90) will produce a 192 numbers between 1 and 90. You will then need to count how many times you get a red, green and amber light. Tap the right arrow ▶ to move through the list. This will take time and there is a better way.




If you scroll down the list of commands, you will find **sortA** and **sortD**. These are short for 'Sort in ascending order' and 'Sort in descending order'.
If we choose **sortA** first and then **randList(192,1,90)** the calculator will select 192 random numbers from 1 to 90 and then sort them in ascending order.
Be prepared to wait for up to 30 s.
It is then easy to count how many of each type there are.




# EXERCISE 2.06 Probability in finite sample spaces

## Concepts and techniques

1 Example 15 In a lottery, 200 000 tickets are sold. Lucia buys 10 tickets. What is the probability of her winning first prize?

2 A bag contains 6 red balls and 8 white balls. One ball is drawn out of the bag at random. Find the probability that it will be:
a White b Red.

3 A shoe shop orders in 19 pairs of black, 14 pairs of navy and 3 pairs of brown school shoes. The boxes lose their labels. What is the probability that a box selected at random will contain brown shoes?


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4 Example 16 A baby has a 0.2% chance of being born with a disability. What is the probability of a baby being born without any disabilities?

5 The probability of selecting a card with the number 5 on it is 0.27. What is the probability of selecting a card without a 5 on it?

6 Example 17 A light globe has a 5% probability of being defective. If 160 are checked, how many would be expected to be defective?

7 The probability of breeding a white budgerigar is $\frac{2}{9}$. How many of the 153 budgerigars that Mr Seed bred would you expect to be white?

## Reasoning and communication

8 A biased coin is weighted so that heads comes up twice as often as tails. Find the probability of tossing a tail.

9 A normal six-sided die has the centre dot painted out of the 5 so that it becomes a 4. Find the probability of throwing:

a 2 b 4 c a number less than 5.

10 Buses run past the corner near David's house every 24 minutes, but he has lost his timetable. What is the probability that if he goes to the stop at a random time, he will have to wait for 2 minutes or less?

11 The probabilities of 3 new cars passing a quality control check are as follows.

$P(3 \text{ passing}) = \frac{1}{16}$, $P(1 \text{ passing}) = \frac{3}{16}$, $P(0 \text{ passing}) = \frac{5}{16}$. Find each of the following.

a The probability that 2 pass b The probability that at least 1 passes.

12 When 4 seeds are planted, the probability of all 4 germinating is $\frac{6}{25}$. The probability that none germinate is $\frac{2}{75}$, that 2 germinate is $\frac{1}{4}$ and that only one germinates is $\frac{1}{12}$. What is the probability that only one does not germinate?

13 A bag contains red, green and yellow marbles. The probability of drawing a red marble is $\frac{4}{9}$ and the probability of drawing a green marble is $\frac{1}{3}$. There are 4 yellow marbles in the bag. How many red marbles are there?

14 The probability of a new spark plug being defective is 2%. From a production run of spark plugs it is found that 15 are defective. How many would you expect to be OK?

# 2.07 PROBABILITIES USING COMBINATIONS

If you pick two cards from the alphabet, what is the likelihood that they will be the same as those chosen by someone else? You can use combinations to work this out as well as many other similar problems.

## Example 18

As part of a test of psychic ability, a subject has to guess which two letters of the alphabet have been written on a card. What is the probability of guessing correctly?

### Solution

This is a combination of 2 letters selected from 26.

$n(\text{2 letter combinations}) = {}^{26}C_2$

Substitute in the formula.

$= \frac{26 \times 25}{1 \times 2}$

Simplify.

$= 325$

There is only one event that is correct. All combinations are equally likely.

Write the answer.

The probability of guessing correctly is $\frac{1}{325}$.

## Example 19

CAS A bag of jelly beans labelled 'lots of black ones' has 20 black jelly beans and 5 of each of the other 6 colours. What is the probability that when you take out 4 at random, they will all be black?

### Solution

At random means that all the jelly beans are equally likely.
You are choosing 4 from 50 things.

$\text{Number of possibilities} = {}^{50}C_4$

How many ways could they all be black?

$\text{Number of all black possibilities} = {}^{20}C_4$

Find the probability.

$\text{Probability of all black} = \frac{{}^{20}C_4}{{}^{50}C_4}$

Substitute in the formulas.

$= \frac{20 \times 19 \times 18 \times 17}{1 \times 2 \times 3 \times 4} \div \frac{50 \times 49 \times 48 \times 47}{1 \times 2 \times 3 \times 4}$

Change to multiplication.

$= \frac{20 \times 19 \times 18 \times 17}{1 \times 2 \times 3 \times 4} \times \frac{1 \times 2 \times 3 \times 4}{50 \times 49 \times 48 \times 47}$

Cancel where possible

$= \frac{19 \times 3 \times 17}{5 \times 49 \times 4 \times 47}$

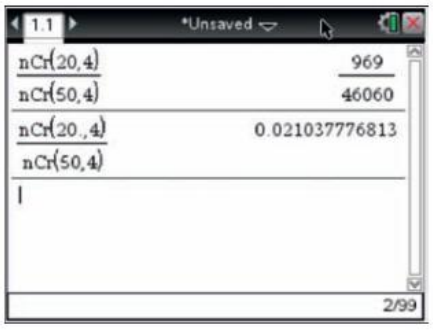
Simplify.

$= \frac{969}{46\,060} \approx 0.021$


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### TI-Nspire CAS

Choose a calculator page . Use the nCr( function (see page 81) and divide by the second one. If you want a decimal answer, insert one of the numbers with a decimal point. You can copy from the previous line by using the up-arrow twice to highlight the previous formula. Then use ctrl-C to copy, move back down and use ctrl-V to paste.




### ClassPad

Use the Main $\sqrt{\alpha}$ menu, press Keyboard and tap ▼ to access the Catalog **tab**.
Select **nCr(** from the list and set out as shown on the right.
Tap EXE or press EXE.
If the bottom is set to **Standard** you will get a fraction. If it is set to **Decimal** you will get an approximate decimal answer.







Write the answer.

The probability that they are all black is about 0.021 or 2.1%.

Make sure you that think carefully about combinations when you use them. In some cases you will have to use several combinations to get a required number.

### Example 20

A box has 4 faulty and 8 good electronic components. What is the probability that 4 components taken at random will have 2 faulty and 2 good ones?

#### Solution

| | |
|---|---|
| You are taking 4 from 12. | Number of possibilities $= {}^{12}C_4$ |
| For faulty ones, it's 2 from 4. | $n(2 \text{ faulty}) = {}^{4}C_2$ |
| For good ones, it's 2 from 8. | $n(2 \text{ good}) = {}^{8}C_2$ |
| Find the number of ways to get 2 faulty and 2 good. | $n(2 \text{ faulty and } 2 \text{ good}) = {}^{4}C_2 \times {}^{8}C_2$ |
| Write the probability. | $P(2 \text{ faulty and } 2 \text{ good}) = \dfrac{{}^{4}C_2 \times {}^{8}C_2}{{}^{12}C_4}$ |
| Evaluate. | $= \dfrac{56}{165} \approx 0.3394$ |
| Write the answer. | The probability of 2 faulty and 2 good is about 33.94%. |

## EXERCISE 2.07 Probabilities using combinations

### Concepts and techniques

1 Example 18 What is the probability of correctly guessing two numbers from the numbers 1 to 10?

2 A bag has 3 red, 4 green and 5 black marbles in it. What is the probability that three marbles shaken out at random will all be red?

3 A game of musical chairs has two chairs removed at each round instead of the usual one, while the players are moving around. When the music stops, the two people who could not find chairs to sit on are out. After a few rounds there are 8 people left. If the chances of each player getting a chair are the same, what is the probability that the next two players out will be Kevin and Josie?

### Reasoning and communication

4 Example 19 What is the probability that 5 cards dealt at random from a normal pack of cards will all be hearts? What is the probability of a flush (all cards the same suit)?

5 A bag of lollies has 10 toffees and 10 jubes. What is the probability that 3 lollies taken at random will all be toffees?

6 A table at a sale has 12 size XL jumpers that are identical except that 6 are red and the others are green. Ahmed takes three jumpers at random.

a What is the probability thay they are all red?

b What is the probability that they are all the same colour?

c What is the probability that they are not all the same colour?

d What is the probability that 2 are red and one is green?

7 Example 20 A child's game has animal cards with lions, tigers, bears, rabbits, donkeys and pigs on them. There are 4 of each card. What is the probability that 4 cards dealt from the pack will be 2 donkeys and 2 pigs?

8 Four representatives are supposed to be chosen by a class for the school student council. Since no one volunteers, everybody's name is put into a hat and four are drawn out at random. If there are 8 boys and 10 girls in the class, what is the probability of getting an even representation of boys and girls?

9 There are 5 jubes, 3 toffees and 4 mint chews left in a lolly bag. What is the probability that when Andy shakes out four lollies, he gets 2 jubes and 2 mint chews?

10 Akua is inviting friends for her eighteenth birthday party. Her parents are paying for 5 of her friends to go to a silver-service restaurant with her family. She really wants to invite 3 boys and 6 girls and can't decide which ones, so draws 5 names at random from a hat. What is the probability that she draws 3 girls and 2 boys?

# 2.08 RELATIVE FREQUENCY

In many practical situations, the information to set up a sample space and probabilities as described in previous sections is not available. However, there could be statistical information available that allows you to estimate the probability of a particular event.

**IMPORTANT**

The **relative frequency** of an event is an estimate of its probability. It is defined as Relative frequency $= \frac{\text{frequency of event}}{\text{total number of observations}} = \frac{\text{frequency of event}}{\text{total frequency}}$.

The observations are usually called **trials**.

Suppose the probability of an event is $P(E)$. If you do $n$ experiments, you expect the frequency of the event to be $f(E) \approx P(E) \times n$. Dividing by $n$ gives $P(E) \approx \frac{f(E)}{n}$.

## Example 21

The quality control department checked the masses of fifty 200 g chocolate bars from the production line. Thirty were found to be from 200 to 201 g, eight from 201 to 202 g and two from 199 to 200 g. Estimate the probability of underweight bars.

### Solution

Use the formula. Relative frequency $= \frac{2}{50} = 0.04$

Write the answer. The probability of underweight bars is about 4%.

## Example 22

Weather Bureau records show that it rains an average of 12.7 days in September in Mt Gambier. How many wet days would you expect in a school week in Mt Gambier in September?

### Solution

| | |
|---|---|
| Estimate the probability. | Probability of rain $\approx \frac{12.7}{31}$ |
| A school week has 5 days. | Expected rainy days $= \frac{12.7}{31} \times 5 \approx 2.05$ |
| Write the answer. | Mt Gambier can expect about 2 rainy days in a school week in September. |

A large number of trials will make things more accurate, but then you need to be careful about the way you use probabilities based on statistical information.

## Example 23

73 500 males and 70 000 females died in Australia in 2010. 2920 men and 6083 women died with dementia, including Alzheimer's disease.

a What is the probability for any given man that when he dies it will be with dementia?
b What is the probability for any given woman that when she dies it will be with dementia?
c Is dementia more prevalent among women than men?

### Solution

| | |
|---|---|
| a Use the formula. | Relative frequency (men) $= \frac{2920}{73\,500} \approx 3.97\%$ |
| b Use the formula. | Relative frequency (women) $= \frac{6083}{70\,000} = 8.69\%$ |
| c Dementia is a disease that generally affects older people. | From this information alone, it does appear to be more prevalent among women. However, women tend to live longer than men so there could be other factors involved. |

## INVESTIGATION Relative frequency and probability

Work in pairs for the first part of this investigation. Each pair needs a normal pack of cards.

1 Deal out 12 cards and count the number of each suit. Record your answers in a table such as the one below:

| Deal | Spades | Hearts | Diamonds | Clubs |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |


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2 After doing and recording 10 deals, find the average number of each suit.
3 Share the results with your neighbours.
4 Pool the results for the whole class and do a super-average.
5 What do you find about the numbers of each suit?
6 How much variation is there in the numbers of each suit?

# EXERCISE 2.08 Relative frequency

## Concepts and techniques

1 Example 21 A packaging machine fills 100 g packets of potato chips. As part of a quality assurance program, the contents of 40 packets are weighed. The results are shown in the table.

| Weight of contents (g) | 90–94 | 95–99 | 100–104 | 105–109 | 110–114 |
|---|---|---|---|---|---|
| Number of packets | 3 | 8 | 21 | 7 | 1 |

Estimate the probability that the next packet filled by the machine will:
a contain less than 95 g of chips
b contain at least 110 g of chips
c be underweight (< 100 g)

2 The reasons that Queensland drivers had their licences cancelled or were disqualified from holding a driver's licence during a particular year are shown in the table.

| Reason for cancellation | Number of drivers |
|---|---|
| Alcohol related: | |
| • blood alcohol > 0.05% | 3 357 |
| • refusal to provide breath sample | 417 |
| • other | 525 |
| Dangerous driving | 1589 |
| Speeding | 3792 |
| Driving under disqualification | 1516 |
| Driving without a licence | 1413 |
| Demerit points suspension | 3159 |
| Other | 2846 |
| Total | 18 614 |



Fairfax Photos/Ruth Hartmann

Use this information to estimate the probability that the next cancellation or disqualification results from:
a speeding
b the accumulation of demerit points
c an alcohol-related cause.

3 Example 22 The fire department in a large city found that 210 house fires had been caused by electrical faults, 240 had been caused by heating appliances (including open fires) and 35 had been deliberately lit.

a What is the probability that a house fire is caused by an electrical fault?
b What is the probability that a house fire is caused by arson?
c Of the next 80 fires, how many could be expected to be caused by heating appliances?
d From the next 300 fires, how many could the police suspect to be caused by arson?
e In 20 fires, how many would you expect to be caused by electrical faults?

4 In the dice game 'First Down the Mountain', for up to 11 players, each player has a score from 2 to 12 inclusive. The number of times the score needs to be rolled to win is as follows.

| Score | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of times to win | 1 | 2 | 3 | 4 | 5 | 6 | 5 | 4 | 3 | 2 | 1 |

The first player to roll their score this number of times wins the game.
The following results were obtained from a series of games held over a weekend.

| Score | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Wins | 24 | 20 | 15 | 16 | 13 | 21 | 17 | 18 | 22 | 22 | 12 |

a How many trials were there?
b Calculate the relative frequency of wins for each score (2 to 12).
c On these results, how many wins would you expect for a score of 5 from 500 trials?
d How many wins for a score of 9 would you expect from the next 50 trials?
e How many wins for a score of 7 would you expect from the next 150 trials?

5 The watches worn by a group of students at a school were checked against the actual time. The approximate amounts they were out (in seconds) were as follows.
+10, 0, 0, +20, 0, −10, −50, +40, +30, 0, 0, +20, 0, 0, 0, 0, −30,
−10, 0, 0, +30, −20, +50, +50, −50, 0, −30, 0, +20, 0, −10, −30,
+10, 0, 0, −10, −20, +20, 0, 0, +10, 0, −10, −20, 0, 0, +20, 0

a What was the relative frequency of the correct time?
b Could you use this information to work out how many students had the right time on their watches?

6 Ten identical pieces of cardboard were tossed into a wastepaper basket and the number of cards that made it into the bin were counted. They were retrieved and the same thing was done 100 times. The numbers of pieces of cardboard that went into the bin were as follows.

0 1 1 1 1 2 3 3 3 3 1 1 1 1 2 2 2 2 2 2 1 1 1 1 1
4 4 5 5 2 2 2 6 5 1 2 2 5 5 2 2 2 2 2 2 2 2 4 6 6
2 2 3 3 3 3 2 2 2 2 2 0 0 0 3 3 4 4 4 4 0 2 2 3 3
3 3 3 3 0 0 2 2 6 7 2 3 3 3 3 3 3 3 4 4 4 4 6 6 9

Estimate the probability that 2 pieces will make it into the bin if another 10 are thrown.


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## Reasoning and communication

**7** Example 23 In measles cases, complications are relatively common. From 4000 cases in children in a recent outbreak, 54 developed complications of some sort, resulting in death for 3 children. In 5 cases where children survived, they suffered serious brain damage, and in 2 other cases the children became deaf. Complications from the measles vaccination also occur. From 50 000 vaccinations, 80 children had some complications. In 75 of these, it was a fever that subsided after a few days. In 1 case, the child suffered some loss of hearing, and the other 4 cases developed mild cases of measles that did not result in any permanent loss of hearing.

- **a** What is the relative frequency of complications from measles?
- **b** What is the probability of complications from the vaccine?
- **c** What is the probability of serious complications from measles?
- **d** What is the probability of serious complications from the vaccine?
- **e** Does this information suggest that vaccinations are a good thing for individuals?

**8** Of 4647 house fires in NSW, 1059 were caused by heating appliances, 568 were suspicious, 540 were electrical faults, 592 had no determined cause, and the rest had other causes. Estimate the probability of a house fire being caused by heating appliances and comment the relevance of the results in other states such as Victoria or Queensland.

# 2.09 PROBABILITIES OF COMBINED EVENTS

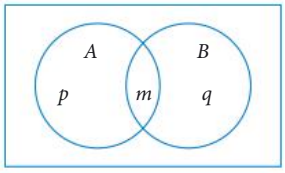
Look at the Venn diagram below for a finite sample space with two events.

WS Venn diagrams

$n(A \cap B) = m$

$n(A) = p + m$

$n(B) = m + q$




It is obvious that $n(A \cup B) = p + m + q$

so $n(A \cup B) = p + m + m + q - m$

and $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Dividing by $n(S)$ gives $\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$

so $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

**IMPORTANT**

For any events $A$ and $B$ in a sample space $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

This can also be written as $P(A \text{ and } B) = P(A) + P(B) - P(A \text{ or } B)$.

For mutually exclusive events $P(A \text{ and } B) = 0$, so $P(A \text{ or } B) = P(A) + P(B)$.

The third probability rule means that this also applies to infinite sample spaces.

### Example 24

In a particular group of people, half of them have blue eyes or fair hair. The probability of having blue eyes is 30% and the probability of fair hair is 40%. What is the probability of having blue eyes and fair hair in this group?

#### Solution

Use $P(A \text{ and } B)$.
$P(\text{blue and fair}) = P(\text{blue}) + P(\text{fair}) - P(\text{blue or fair})$

Substitute values.
$= 30\% + 40\% - 50\%$
$= 20\%$

Write the answer.
The probability of having both blue eyes and fair hair in this group is 20%.

### Example 25

A poker player keeps the 7, 8, 9 and 10 of spades. All discards are shuffled before players get their replacement cards. The player draws one card. Find the probability that it is:

**a** a jack or a 6
**b** another spade
**c** the jack of spades or the 6 of spades
**d** a jack, 6 or spade

#### Solution

**a** There are 8 jacks and 6s in the remaining cards.
$P(\text{J or 6}) = \frac{8}{48} = \frac{1}{6}$

**b** There are 9 spades left.
$P(\text{spade}) = \frac{9}{48} = \frac{3}{16}$

**c** There are 48 cards left.
$P(\text{J♠ or 6♠}) = \frac{2}{48} = \frac{1}{24}$

**d** Use $P(A \text{ or } B)$.
$P(\text{J or 6 or spade}) = P(\text{J or 6}) + P(\text{spade}) - P(\text{J♠ or 6♠})$

Substitute values.
$= \frac{1}{6} + \frac{3}{16} - \frac{1}{24}$

Calculate the result.
$= \frac{15}{48} =$

## EXERCISE 2.09 Probabilities of combined events

### Concepts and techniques

1 Example 24 Developers of new plant varieties plant thousands of seeds and select the ones they want. A grower estimates that the probability of getting a good flower is 5% and the probability of strong plants is 10%. The probability of having both is only 2%. What is the probability of good flowers or strong plants?


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2 A survey of 80 people with dark hair or brown eyes showed that 63 had dark hair and 59 had brown eyes. If one of the people surveyed is chosen at random, find the probability that the person will have both brown eyes and dark hair.

3 Example 25 From 100 cards, numbered from 1 to 100, one is selected at random. Find the probability that the card selected is even or less than 20.

4 A number is chosen at random from the numbers 1 to 20. Find the probability that the number chosen will be:
- a divisible by 3
- b less than 10 or divisible by 3
- c a composite number
- d a composite number or a number greater than 12.

5 A set of 50 cards is labelled from 1 to 50. One card is drawn out at random. Find the probability that the card will be:
- a a multiple of 5
- b an odd number
- c a multiple of 5 or an odd number
- d a number greater than 40 or an even number

6 A set of 26 cards, each with a different letter of the alphabet on it, is placed in a box and one is drawn out at random. Find the probability that the letter on the card drawn will be:
- a a vowel
- b a vowel or one of the letters in the word 'random'
- c a consonant or one of the letters in the word 'movies'.

7 A class of 28 students puts on a concert with all class members singing or dancing (or both). 15 dance and 19 sing in the performance. Find each of the following probabilities for a student chosen at random.
- a The student will both sing and dance.
- b The student will only sing.
- c The student will only dance.

## Reasoning and communication

8 A list is made up of people with professional experience of either computers or digital cameras. Of the list, 65% have computer experience while 45% have experience with a digital camera. If one name is chosen at random from the list, find the probability that the person will have experience with only digital cameras.

9 In a group of students who study either History or Geography, 72% take History and 41% take geography. What is the probability that a student selected at random takes History but not Geography?

10 In a group of 20 dogs at obedience school that have been taught to walk to heel or stay, 14 dogs will walk to heel and 12 will stay when told. One dog is chosen at random. What is the probability that the dog will walk to heel but not stay?

# 2 CHAPTER SUMMARY
# PROBABILITY

- A **set** is a well-defined collection of objects. Each member of a set is called an **element**. The elements are listed in braces like $\{2, 4, 6, 8\}$ or defined by a property in braces like $\{x \text{ an even number: } 1 < x < 9\}$
- The **universal set** shown as $U$ is a fixed set of all elements used in the particular situation
- The null set shown as $\{\}$ or $\varnothing$ is the set with no elements
- Two sets are **equal** if every element of one is also an element of the other
- **Mutually exclusive** (**disjoint**) sets have no elements in common
- A **subset** $A$ of the set $B$ is a set whose elements are all elements of $B$. This is shown as $A \subseteq B$. If $A$ is definitely smaller than $B$ it is called a proper subset, written as $A \subset B$.
- The **intersection** of the sets $A$ and $B$, written as $A \cap B$, is the set whose elements are elements of both $A$ and $B$.
- The **union** of the sets $A$ and $B$, $A \cup B$, is the set whose elements are in $A$ or $B$ or both.
- The **complement** of the set $A$, $A'$ or $\overline{A}$, is the set of all elements not in $A$. $\overline{A} = \{x: x \notin A\}$, $A \cap \overline{A} = \varnothing$ and $A \cup \overline{A} = U, \overline{\varnothing} = U$ and $\overline{U} = \varnothing$
- The **cardinality** of a set $A$ measures its size and is shown as $|A|$. For finite sets it is the number of elements in the set
- A **Venn diagram** shows sets as circles or ovals, with intersections shown by overlapping of them. The universal set is shown as a rectangle around the sets.
- The set of **natural numbers** is shown as $\boldsymbol{N}$. $\boldsymbol{N} = \{1, 2, 3, 4, \ldots\}$
- The set of **integers** is shown as $\boldsymbol{Z}$ or $\boldsymbol{J}$. $\boldsymbol{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- The set of **rational numbers** is shown as $\boldsymbol{Q}$
- The set of **real numbers** is shown as $\boldsymbol{R}$
- A superscript of + or − is used to show only positive or negative cases. $\boldsymbol{Q}^+$ means the positive rational numbers
- A back slash can be used to show an omission. $\boldsymbol{R}\backslash 0$ or $\boldsymbol{R}\backslash\{0\}$ means the real numbers except 0
- Expansions of $(x + y)^n$ are called **binomial expansions**
- The **binomial theorem** states that the expansion of

$$(x+y)^n = \binom{n}{0}{}^nC_0x^n + \binom{n}{1}{}^nC_1x^{n-1}y + \cdots + \binom{n}{r}{}^nC_rx^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

  where the values ${}^nC_r$ for $0 \le r \le n$ are called the **binomial coefficients**
- A **combination** is an unordered selection of some or all of the symbols from a group.

  The number of combinations of $r$ objects from $n$ distinct objects is given by

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

- A **sample space** $S$ is a set normally associated with individual occurrences in a real situation. The elements of the set $S$ are called **elementary outcomes** and subsets of $S$ are called **events**. $S$ is the universal set for the situation.
- $A$ **and** $B$ means the intersection of the events $A$ and $B$, $A \cap B$.


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- $A$ **or** $B$ means the union of the events $A$ and $B$, $A \cup B$.
- **Not** $A$ means the complement of the event $A$, $\overline{A}$ or $A'$.
- The **probability** of an event $A$ in a sample space $S$ is a real number $P(A)$ such that:
  1. $0 \leq P(A) \leq 1$
  2. $P(S) = 1$ and $P(\varnothing) = 0$
  3. The probability of a union of disjoint events is the sum of the probabilities of the events.
- The **probability** of an event $E$ in a finite sample space $S$ whose elementary outcomes have equal probabilities is given by $P(E) = \dfrac{n(E)}{n(S)}$. This is a **uniform probability distribution.**
- A **random** choice means that all the possible choices have equal probabilities. The sample space has a uniform probability distribution.
- An **impossible event** is the empty set, so its probability is 0.
- A **certain event** is the sample spaces, so its probability is 1.
- For an event and its complement, $P(\overline{A}) = 1 - P(A)$.
- **Expected frequency** from $n$ experiments = probability $\times\, n$.
- The probability of an event in a practical situation can be estimated by **relative frequency** = $\dfrac{\text{frequency of event}}{\text{total number of observations}} = \dfrac{\text{frequency of event}}{\text{total frequency}}$.
- Each observation is called a **trial.**
- For events $A$ and $B$,

  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  or $P(A \text{ and } B) = P(A) + P(B) - P(A \text{ or } B)$
- For **mutually exclusive** events, $P(A \text{ and } B) = 0$ and $P(A \text{ or } B) = P(A) + P(B)$.

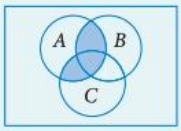
# 2 CHAPTER REVIEW
# PROBABILITY

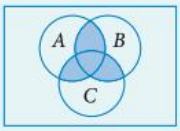
## Multiple choice

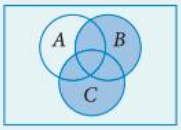
1 Example 1 Which of the following is not a well-defined set?

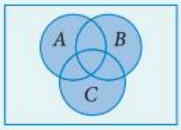
A The prime numbers
B The even numbers between 101 and 199
C Diseases caused by hormonal imbalance, deficiency or overproduction
D Rare genetic diseases
E Native plants of Victoria

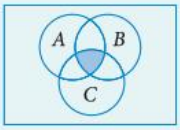
2 Example 2 Which of the following best shows the Venn diagram for the set $A \cap (B \cup C)$?

A 


B 


C 


D 


E 


3 Example 5 For the sample space for throwing two normal dice, $n(\text{total} = 6) =$

A 3 B 4 C 5 D 6 E 7

4 Example 10 A 12-sided die has the numbers from 1 to 9, an extra 2 and two extra 5s. The probability of it landing on 6 is:

A $\frac{1}{12}$ B $\frac{1}{10}$ C $\frac{1}{9}$ D $\frac{1}{6}$ E $\frac{1}{5}$

5 Example 11 A card is drawn at random from a normal pack of 52 playing cards. The probability that it is not a heart is

A $\frac{1}{52}$ B $\frac{1}{13}$ C $\frac{1}{4}$ D $\frac{3}{4}$ E $\frac{12}{13}$

6 Example 14 From 40 Australian students, 12 reported that they had blue eyes, 16 said they had brown eyes, 4 said they had green eyes, 1 had grey eyes and 7 said they had hazel eyes. From 70 Australian students, how many would you expect to have brown eyes?

A 9 B 16 C 21 D 28 E 30

7 Example 16 The probability of a student studying Art or Music is $\frac{3}{10}$. $\frac{1}{8}$ of students study Music and $\frac{1}{4}$ study Art. What is the probability of a student studying both Art and Music?

A $\frac{3}{40}$ B $\frac{1}{8}$ C $\frac{7}{40}$ D $\frac{9}{20}$ E $\frac{17}{40}$


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## Short answer

8 Example 3 For $A = \{3, 6, 9, 12, 15, 18\}$, $B = \{n \text{ even: } 1 \leq n \leq 19\}$ write down

a $A \cup B$ b $A \cap B$

9 Example 4 Write down the sample space for throwing two four-sided dice with sides labelled Δ, □, × and +.

10 Example 6 Two eight-sided dice numbered 1–8 are thrown. $A$ = 'the total is 7' and $B$ = 'one of the dice shows a 2'. List the following events.

a $A$ b $B$ c $A$ and $B$ d $A$ and $\overline{B}$

11 Example 12 Kam has a 35% chance of beating Peter at lightning chess and a 5% chance of drawing. How many times would you expect him to lose if they played 12 games?

12 Example 13 From 65 people interviewed in a city mall, 35 said they liked Thai food. Estimate the probability of people liking Thai food and the number from the next 20 people who would not like Thai food.

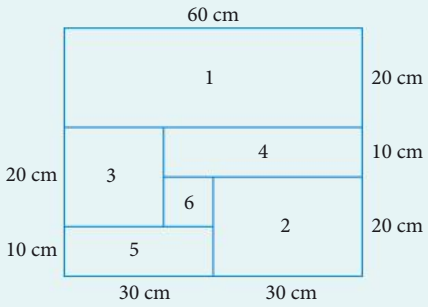
13 Example 14 In 2009, 471 men and 107 women died from mesothelioma. Altogether, 72 300 men and 68 400 women died that year. Estimate the probabilities of men and women dying from mesothelioma and explain whether this means mesothelioma is a more dangerous disease for men than women.

14 Example 17 The 74 members of an amateur theatrical group take on various roles during the year. All members perform on stage or do front-of-house or backstage roles. No one performs both front-of-house and backstage roles. The probability of performing a stage role is 0.6 and the probability of doing back-stage or front-of house is 0.55. How many people perform both stage and front-of-house or backstage roles?

## Application

15 What are the sample space and the probability of each possible outcome if two numbers are chosen at random from the numbers 3, 4, 5, 6 and 7?

16 A fairground game has a card table set up with scoring areas as shown in this diagram. Players throw a counter towards the table from 2.5 metres away. The numbers in the spaces that the counters land on are the scores. Write the sample space and probabilities for each score, assuming that the counter lands on the table.




Practice quiz

17 A prawn farm has 5 ponds of the same size, but one pond is always being cleaned while the others are in production. The pond scheduled to be cleaned next has the greatest loss, with 15% of the prawns from the pond dying. The pond just cleaned suffers only a 2% loss and the other two ponds suffer 6% and 10% losses. What is the probability that a particular prawn comes from the pond that has a 6% loss, assuming that each pond starts with the same number of prawns?

# 1 MIXED REVISION
## CHAPTERS 1 • 2 • 3

### Multiple choice

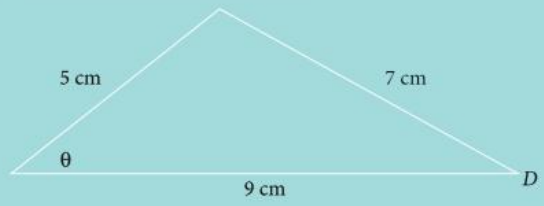
1 Completion of the square changes $2x^2 + 4x - 5 = 0$ to:

A $(x + 2)^2 = 9$
B $(x + 1)^2 = 6$
C $(x + 1)^2 = 3.5$
D $(x + 2)^2 = 6.5$
E $(x + 1)^2 = -3.5$

2 One bag contains a red, a blue and a green marble and a second bag contains a purple, an orange and a yellow marble. The sample space for taking two marbles from the first bag and two from the second, without regard to order, is as follows. Note that R = red, B = blue, G = green, P = purple, O = orange and Y = yellow.

A {RGBP, RGBO, RGBY, RGPO, RGPY, RGOY, RBPO, RBPY, RBOY, BGPO, BGPY, BGOY, RPOY, BPOY, GPOY}
B {RGPO, RGPY, RGOY, RBPO, RBPY, RBOY, BGPO, BGPY, BGOY}
C {RG, RB, RP, RO, RY, GB, GP, GY, GO, BP, BO, BY, PO, PY, OY}
D {RG, PO, RB, PY, GB, OY}
E {RG, PO, PY, OY, RB, PO, PY, OY, GB, PO, PY, OY}

3 Which is the true statement?




A $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 7 \times 9}$
B $\cos(\theta) = \dfrac{9^2 + 7^2 - 5^2}{2 \times 9 \times 7}$
C $\cos(\theta) = \dfrac{5^2 + 9^2 - 7^2}{2 \times 5 \times 9}$
D $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 5 \times 9}$
E $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 5 \times 7}$

4 Expand $(x + 2)(x - 4)(3 - x)$

A $-x^3 + 5x^2 - 2x - 24$
B $-x^3 + 5x^2 + 2x - 24$
C $x^3 + 5x^2 + 2x - 24$
D $x^3 - 5x^2 - 2x + 24$
E $-x^3 - 5x^2 + 2x - 24$


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5 The probability of getting a number card or a spade when one card is chosen at random from a normal pack is:

A $\frac{1}{4}$ B $\frac{11}{26}$ C $\frac{9}{13}$ D $\frac{10}{13}$ E $\frac{49}{52}$

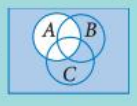
6 The exact value of $\cos\left(\frac{11\pi}{6}\right)$ is

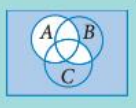
A $\frac{\sqrt{3}}{2}$ B $\frac{1}{\sqrt{2}}$ C $-\frac{\sqrt{3}}{2}$ D $-\frac{1}{2}$ E $\frac{1}{2}$

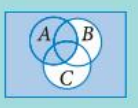
7 The quadratic equation $6x^2 - x - 15 = 0$ has:

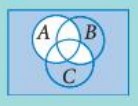
A no real roots
B 1 real, rational root
C 1 real, irrational root
D 2 real, rational roots
E 2 real, irrational roots

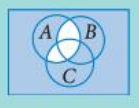
8 The Venn diagram for the set $\overline{A} \cap (B \cup C)$ is:

A 

B 

C 

D 

E 


9 A compass bearing of N25°W is the same as a true bearing of

A 025° B 115° C 155° D 205° E 335°

## Short answer

1 Divide $p(x) = 3x^3 - 5x^2 + 4x - 7$ by $x + 3$.

2 Set up a sample space and probability rule for finding the difference of two numbers chosen at random from 1, 3, 5, and 8.

3 Convert

a 35° to radians
b $\frac{4\pi}{5}$ to degrees
c 29.3° to radians, correct to 2 decimal places
d 1.06 to degrees, to 1 decimal place
e 410° to radians

4 A market gardener's research has shown that the number of lettuces he has for sale is inversely proportional to the number of rabbits. He has 186 lettuces when there are 8 rabbits. How many lettuces will he have if there are 12 rabbits?

5 In a particular school, the probability that a chemistry student does not do geography is 0.8, while the probability that one does do history is 0.25. 30% of chemistry students do history or geography and there are 6 who do both history and geography. How many chemistry students are there in this school?

6 Find the length of $AB$ in $\triangle ABC$ if $BC = 4.9$ cm, $\angle BAC = 52°$ and $\angle BCA = 41°$

## Application

1 A furniture manufacturer must compete against local firms, so cannot fully recover the cost of delivery of furniture to retailers. The cost of delivery of a truckload of furniture is about $300 plus about $20/km. The profit on that quantity of furniture sold from the factory is about $3000. Work out the profit on loads of furniture delivered to retailers at distances of 10 km, 50 km, 100 km and 150 km, and draw a graph.

a Use the graph to find the profit on loads that must be delivered:

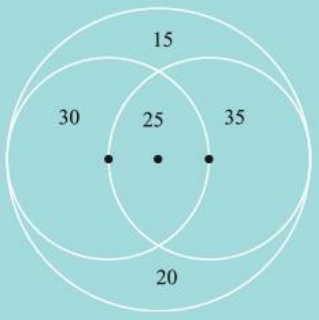
i 60 km ii 30 km iii 120 km iv 90 km.

b At what distance should the manufacturer not accept orders of furniture?

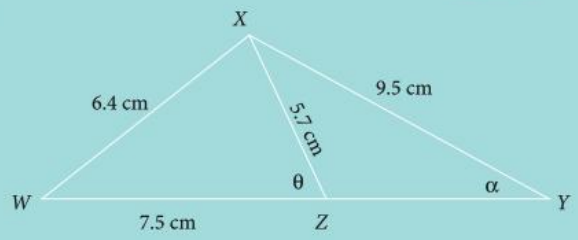
2 The number of leaves on a certain plant is directly proportional to the square of its height. The height of the plant triples in one month. What happens to the number of leaves on the plant?

3 Prove that for mutually exclusive events $A$ and $B$, $P(A \text{ or } (B \text{ and } C)) = P(A) + P(B \text{ and } C)$.

4 A circular target is divided into 5 regions by two overlapping circles whose radii are both two-thirds of the radius of the target. The centres of the circles are concurrent and the circles touch the perimeter of the target. Arrows sticking into the target get a score from 15 to 35 as shown in the diagram. Set up a sample space and probabilities for the scores. Assume that all arrows stick into the target.




5




In the diagram, $WX = 6.4$ cm, $XZ = 5.7$ cm, $XY = 9.5$ cm and $WZ = 7.5$ cm. Find the value (to the nearest degree) of

a $\theta$ b $\alpha$


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6 The gradient of a ramp from a path up to a doorway is to be $\frac{2}{5}$. The doorway is 1.2 m above ground level.

a What is the angle of inclination of the ramp to the horizontal?

b Find the length $l$ needed for the ramp.

c The triangular space on either side of the ramp is to be filled in with lattice. Find the area of lattice needed.

d After the ramp is built, it is discovered that the ground is not flat, but slopes downwards towards the doorway. The angle between the doorway and the ramp is 63°.




Find the area of lattice needed to fill in both sides of the ramp.





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# 3

## TERMINOLOGY

angle of depression
angle of elevation
arc
circular measure
cosine rule
included angle
radian
sector
segment
sine rule

Prior learning

## TRIGONOMETRIC FUNCTIONS

# TRIGONOMETRY


3.01 Review of right-angled triangles
3.02 Angles of any magnitude
3.03 Area of a triangle
3.04 The sine rule
3.05 The cosine rule
3.06 Applications of trigonometry
3.07 Circular measure
3.08 Radians and the circle
Chapter summary
Chapter review

## COSINE AND SINE RULES

- review sine, cosine and tangent as ratios of side lengths in right-angled triangles (ACMMM028)
- understand the unit circle definition of cos θ, sin θ and tan θ and periodicity using degrees (ACMMM029)
- examine the relationship between the angle of inclination of a line and the gradient of that line (ACMMM030)
- establish and use the sine and cosine rules and the formula Area = $\frac{1}{2}bc\sin A$ for the area of a triangle (ACMMM031)

## CIRCULAR MEASURE AND RADIAN MEASURE

- define and use radian measure and understand its relationship with degree measure (ACMMM032)
- calculate lengths of arcs and areas of sectors in circles. (ACMMM033)

## TRIGONOMETRIC FUNCTIONS

- recognise the exact values of sin θ, cos θ and tan θ at integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (ACMMM035).

# 3.01 REVIEW OF RIGHT-ANGLED TRIANGLES

In this section you will revise the work you have done previously on right-angled triangles using the basic trigonometric ratios.

Angles are measured in degrees in this section. Later on in this chapter, you will look at a different way to measure angles.

Angles can be whole degrees, such as 39°, or include decimals, for example, 56.3°, or in degrees and minutes, such as 78°12′. Calculators use seconds as well, but these are too small to be significant in this course.

Scientific and CAS calculators usually convert between decimals and degrees and minutes, or you can use the fact that a minute is $\frac{1}{60}$ th of a degree.

For example, to change 56.3° into degrees and minutes, you use $56\frac{3}{10} = 56\frac{18}{60}$ to get 56°18′.

To change 78°12′ into a decimal, use $78\frac{12}{60}$ and divide 12 by 60 to get 0.2.

So 78°12′ = 78.2°.

**IMPORTANT**

60 minutes = 1 degree

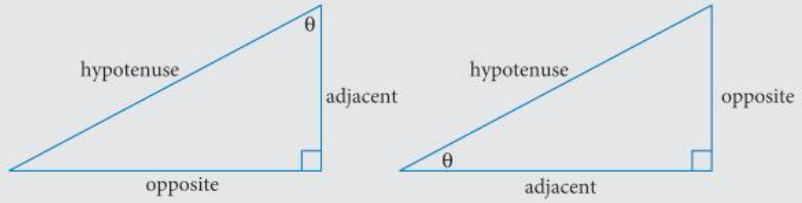
60′ = 1°

60 seconds = 1 minute

60″ = 1′

You should remember the trigonometric definitions shown below.

**IMPORTANT**



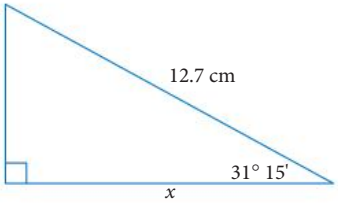

Sine: $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ Cosine: $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$ Tangent: $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$


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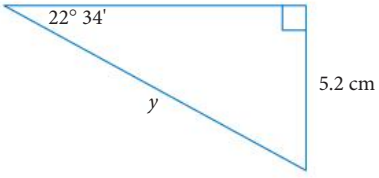
You should state solutions to the same accuracy as figures given in the question unless asked to do otherwise.

## Example 1

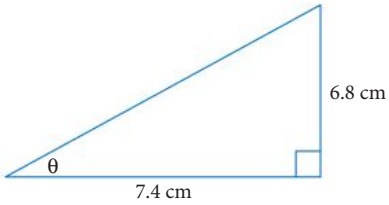
**a** Find the value of $x$.




**b** Evaluate $y$.




**c** Evaluate θ to the nearest minute.




### Solution

**a** $x$ is the adjacent side and 12.7 is the hypotenuse, so use cosine.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Substitute in the information.

$$\cos(31°15') = \frac{x}{12.7}$$

Multiply both sides by 12.7.

$$12.7\cos(31°15') = x$$

Evaluate and round to three significant figures.

$$x \approx 10.9$$

State the result.

$$x \approx 10.9 \text{ cm}$$

**b** $y$ is the hypotenuse and 5.2 is the opposite side, so use sine.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Substitute in the information.

$$\sin(22°34') = \frac{5.2}{y}$$

Multiply both sides by $y$.

$$y\sin(22°34') = 5.2$$

Divide both sides by sin (22°34′).

$$y = \frac{5.2}{\sin(22°34')}$$

Use your calculator to evaluate $y$.

$$y = 13.5502\ldots$$

Round to 2 significant figures and state the result.

$$y \approx 14 \text{ cm}$$

**c** 6.8 is the opposite side and 7.4 is the adjacent side, so use tangent.

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.$$

Substitute in the information.

$$\tan(\theta) = \frac{6.8}{7.4}$$

Use $\tan^{-1}$ on your calculator to evaluate θ.

$$\theta = 42.5804\ldots$$

Change to degrees and minutes and round to the nearest minute.

$$\theta \approx 42°35'$$

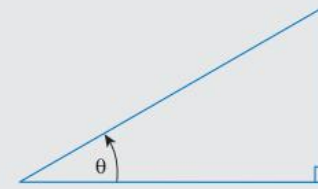
Notice in Example 1 that when the unknown side was on the top of the ratio, you multiplied the known side by the trigonometric ratio. When it was on the bottom, you divided the known side by the trigonometric ratio. You can use trigonometry to solve problems such as finding the height of objects that you cannot measure directly.



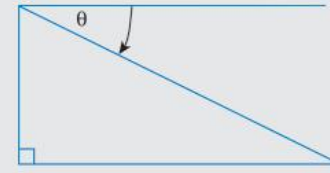
Photo courtesy Margaret Grove

**IMPORTANT**

- The **angle of elevation** of an object is the angle measured from the horizontal (ground) when looking up to the top of an object.




- The **angle of depression** of an object is the angle measured from the horizontal when looking down to an object below.





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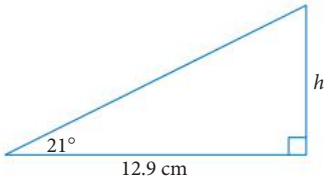
## Example 2

**a** Find the height of a tree that makes an angle of elevation of 21° from a point 12.9 m out from its base.

**b** A boat is 34.7 m out from the foot of a 25.3 m high cliff. What is the angle of depression from the top of the cliff to the boat, to the nearest degree?

### Solution

**a** Draw a diagram showing the information.




$h$ is the opposite side and 12.9 is the adjacent side, so use tangent.

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.$$

Substitute the information.

$$\tan(21°) = \frac{h}{12.9}$$

Multiply both sides by 12.9.

$$12.9 \tan(21°) = h$$

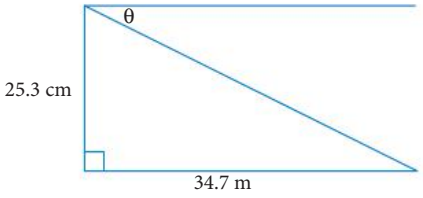
Use your calculator to evaluate $h$.

$$h = 4.9518\ldots$$

Round the result to three figures.

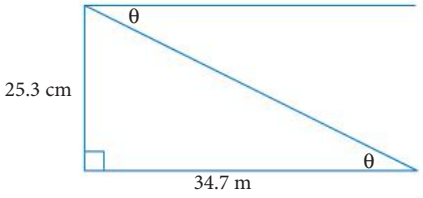
The height of the tree is about 4.95 m.

**b** Draw a diagram showing the information.




Relate θ to the inside of the triangle.

Using alternate angles, θ is equal to the angle at the base of the triangle.




25.3 is the opposite side and 34.7 is the adjacent side, so use tangent.

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.$$

Substitute in the information.

$$\tan(\theta) = \frac{25.3}{34.7}$$

Use your calculator to evaluate θ.

$$\theta = 36.0960\ldots$$

Round to the nearest degree for the result.

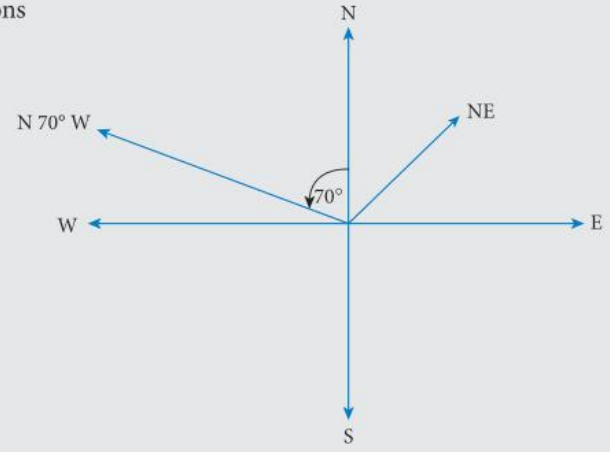
The angle of depression is about 36°.

You can apply trigonometry to problems using bearings. Bearings can be described in different ways.

**IMPORTANT**

### Bearings

**Compass bearings** describe positions like south (S), north east (NE) or N 70° W.

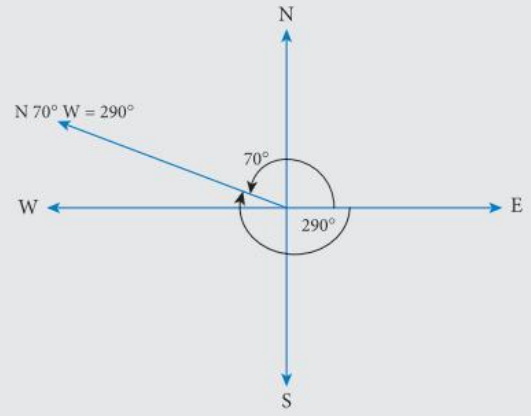



**Bearings** measure angles clockwise from north.

Bearings are sometimes called **three-figure bearings**.

Bearings have 3 digits and can range between 0° and 360°.

The compass bearing N 70° W is the same direction as a bearing of 360° – 70° or 290°.




## Example 3

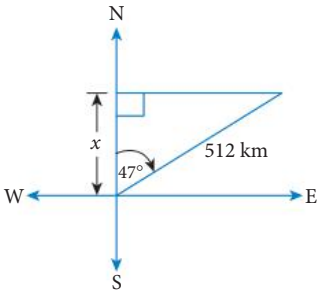
**a** A plane flies from Melbourne on a bearing of 047° for 512 km. How far north of Melbourne is the plane?

**b** Andrea leaves her house and rides her bike south for 3.7 km. She then turns and rides east for 4.3 km. What is Andrea's bearing from her house to the nearest degree?


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## Solution

**a** Draw a diagram showing the information.




$x$ is the adjacent side and 512 is the hypotenuse, so use cosine.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

Substitute in the information.

$$\cos(47°) = \frac{x}{512}$$

Multiply both sides by 512.

$$512\cos(47°) = x$$

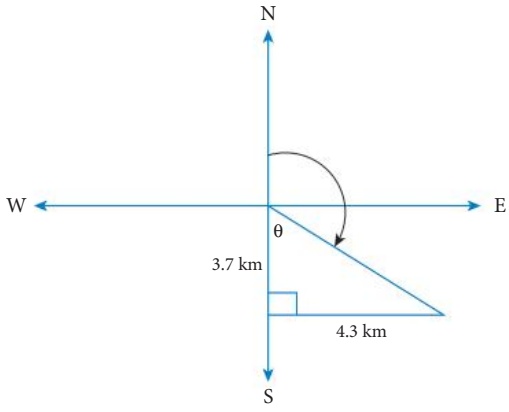
Use your calculator to evaluate $x$.

$$x = 349.1831\ldots$$

Round to 3 figures and state the result.

The plane is approximately 349 km north of Melbourne.

**b** Draw a diagram showing the information.




4.3 is the opposite side and 3.7 is the adjacent side, so use tangent.

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.$$

Substitute in the information.

$$\tan(\theta) = \frac{4.3}{3.7}$$

Use $\tan^{-1}$ on your calculator to evaluate $\theta$.

$$\theta = 49.2891\ldots$$

Round $\theta$ to the nearest degree and find the bearing.

$$180° - 49° = 131°$$

State the result.

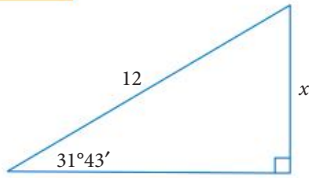
Andrea is on a bearing of about 131° from her house.

# EXERCISE 3.01 Review of right-angled triangles

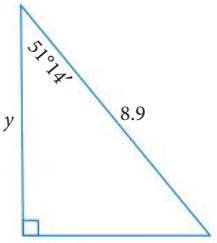
## Concepts and techniques

**1** Example 1 Find the value of each unknown to appropriate accuracy.

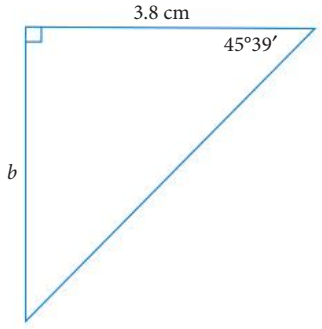
**a**



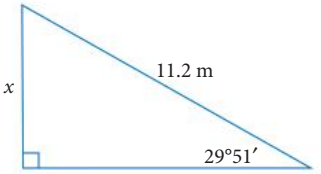

**b**



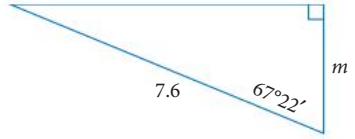

**c**



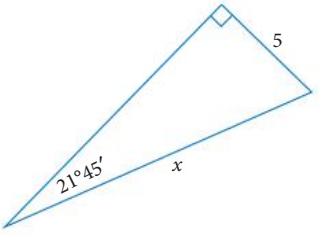

**d**



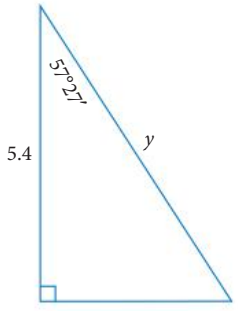

**e**



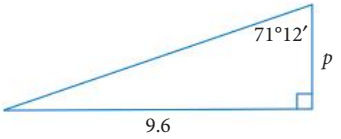

**f**




**g**



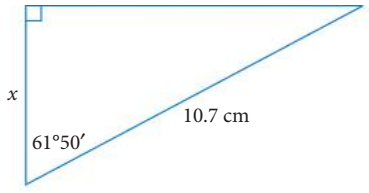

**h**



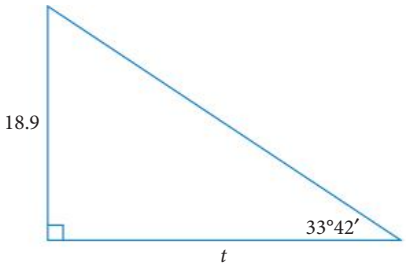


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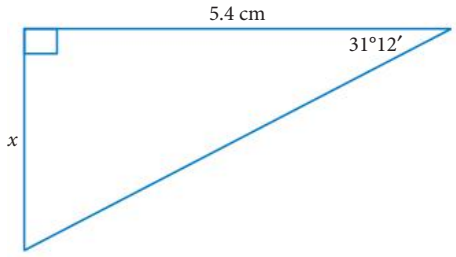
i


x
10.7 cm
61°50′

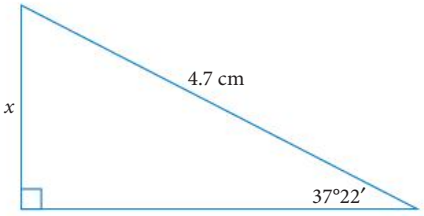
j


18.9
33°42′
t

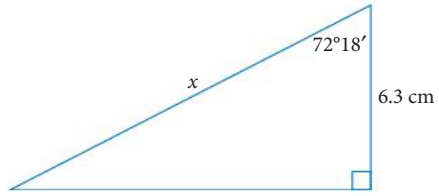
k


5.4 cm
31°12′
x

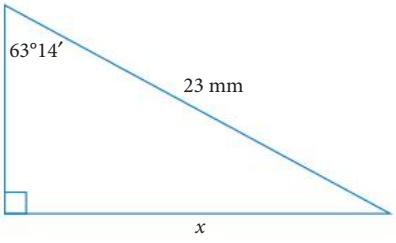
l


4.7 cm
x
37°22′

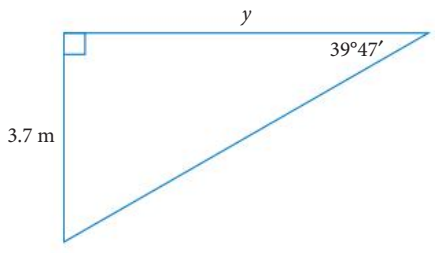
m


72°18′
x
6.3 cm

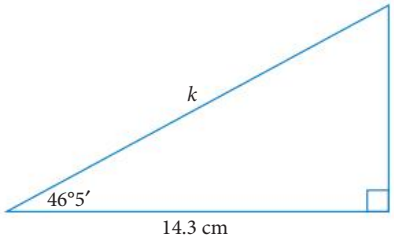
n


63°14′
23 mm
x

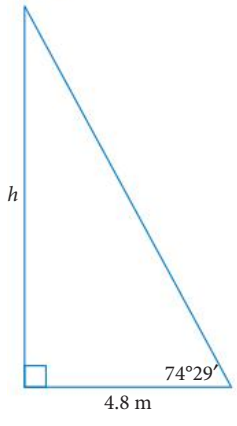
o


y
39°47′
3.7 m

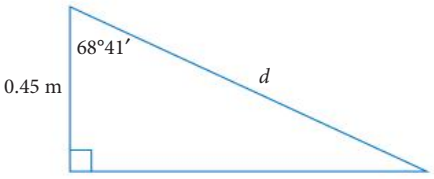
p


k
46°5′
14.3 cm

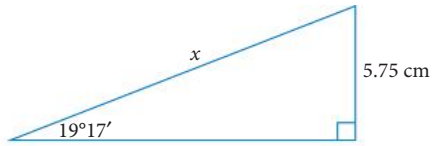
q


h
74°29′
4.8 m

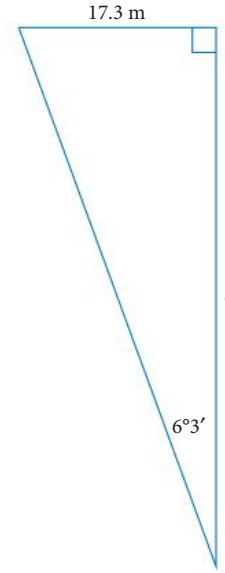
r


68°41′
0.45 m
d

**s**

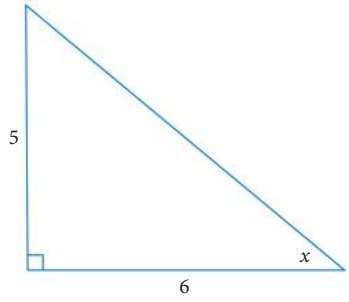



**t**

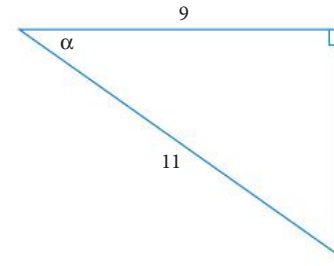



**2** Find the value of the angles indicated, in degrees and minutes.

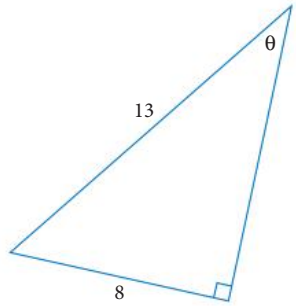
**a**



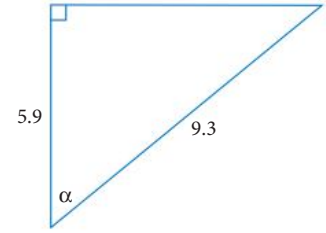

**b**



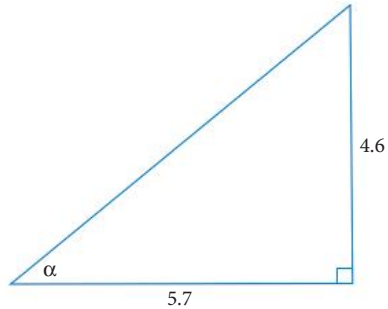

**c**



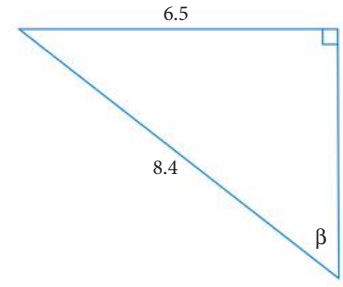

**d**




**e**



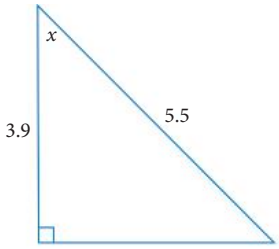

**f**



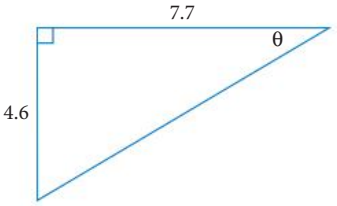


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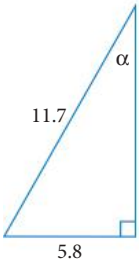
g


x
3.9
5.5

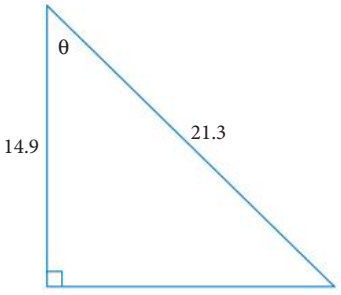
h


7.7
θ
4.6

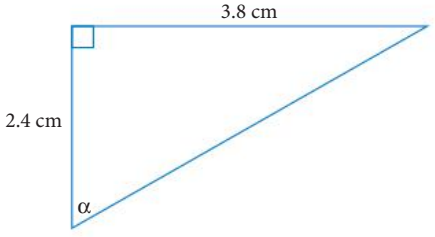
i


α
11.7
5.8

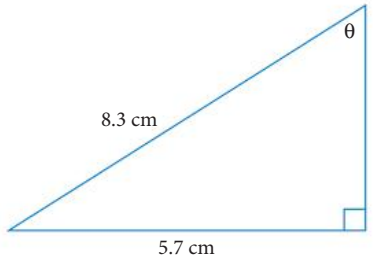
j


θ
14.9
21.3

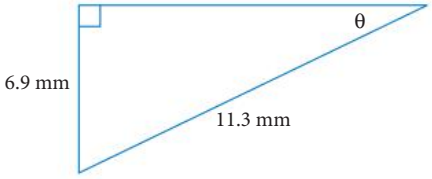
k


3.8 cm
2.4 cm
α

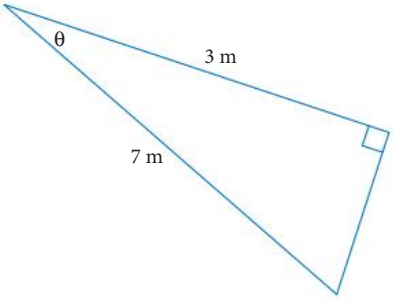
l


θ
8.3 cm
5.7 cm

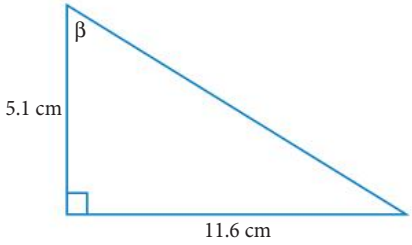
m


θ
6.9 mm
11.3 mm

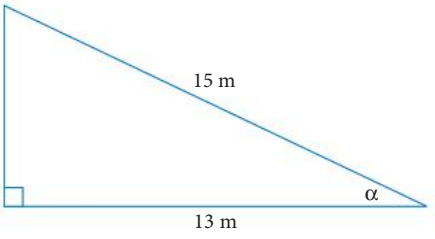
n


θ
3 m
7 m

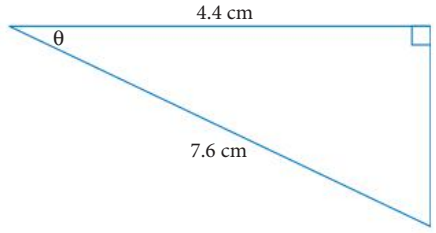
o


β
5.1 cm
11.6 cm

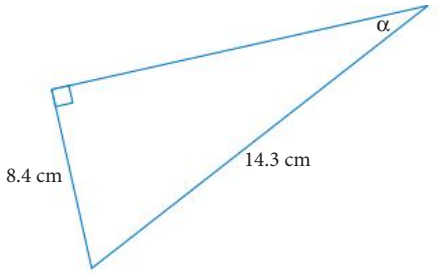
p


15 m
α
13 m

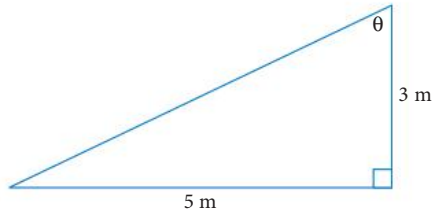
q



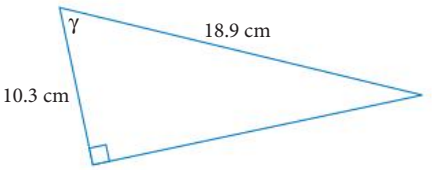

r




s




t




3 Example 2 **a** The angle of elevation from a point 11.5 m away from the base of a tree up to the top of the tree is 42°12′. Find the height of the tree.

**b** Geoff stands 25.8 m away from the base of a tower and measures the angle of elevation as 39°20′. Find the height of the tower.

**c** A wire is suspended from the top of a 100 m tall bridge tower down to the bridge at an angle of elevation of 52°. How long is the wire, correct to 1 decimal place?

**d** The angle of elevation of a tower is 39°44′ when measured at a point 100 m from its base. Find the height of the tower, correct to 1 decimal place.

**e** A 10 m rope is tied to the top of a pole. It is stretched out straight to reach the ground, making an angle of elevation of 67°13′ with the ground. Find the height of a pole, correct to 1 decimal place.



Getty Images/AFP/Aris Messinis

4 **a** A cat crouches at the top of a 4.2 m high cliff and looks down at a mouse 1.3 m out from the foot (base) of the cliff. What is the angle of depression, correct to the nearest minute?

**b** The angle of depression from the top of an 8 m tree down to a rabbit is 43°52′. If an eagle is perched in the top of the tree, how far does it need to fly to reach the rabbit, correct to the nearest metre?

**c** The angle of depression from the top of a cliff down to a boat 100 m out from the foot of the cliff is 59°42′. How high is the cliff, correct to the nearest metre?

**d** The angle of depression from the top of a 15 m tree down to a pond is 25°41′. If a bird is perched in the top of the tree, how far does it need to fly to reach the pond, correct to the nearest metre?

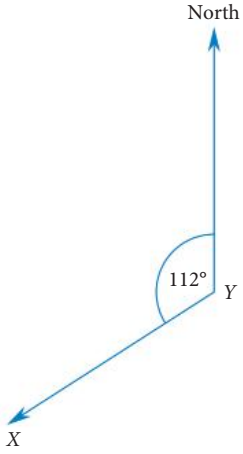
**e** The angle of depression from the top of a tower down to a car 250 m out from the foot of the tower is 38°19′. How high is the tower, correct to the nearest metre?

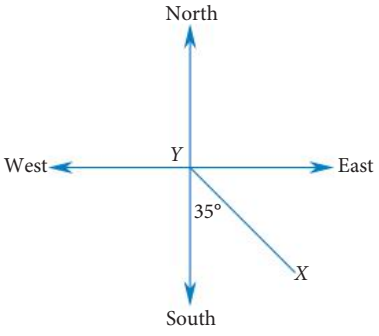

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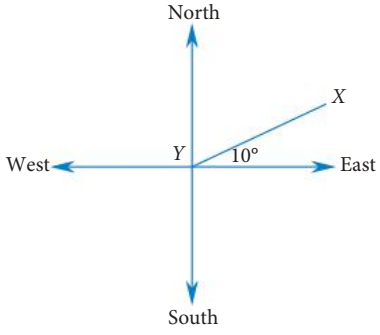
5 Example 3 Draw a diagram to show the bearing in each question.

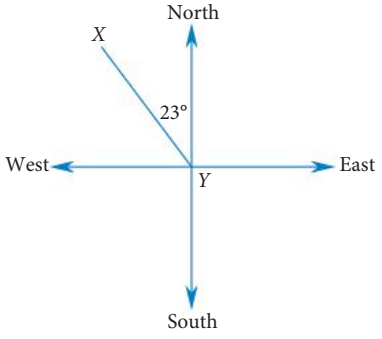
a A boat is on a bearing of 100° from a beach house.
b Jamie is on a bearing of 320° from a campsite.
c A seagull is on a bearing of 200° from a jetty.
d Alistair is on a bearing of 050° from the bus stop.
e A plane is on a bearing of 285° from Broken Hill.
f A farmhouse is on a bearing of 012° from a dam.
g Mohammed is on a bearing of 160° from his house.
h A mine shaft is on a bearing of 080° from a town.
i Yvonne is on a bearing of 349° from her school.
j A boat ramp is on a bearing of 280° from an island.

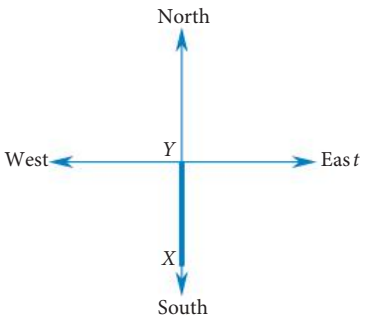
6 Find the bearing of $X$ from $Y$ in each question in 3-figure bearings.

a



b



c


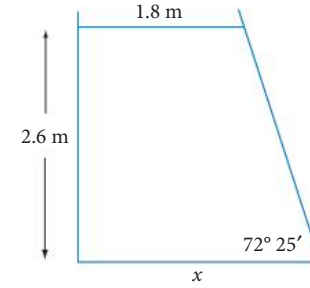

d



e


7 a Jack is on a bearing of 260° from Jill. What is Jill's bearing from Jack?
b A tower is on a bearing of 030° from a house. What is the bearing of the house from the tower?
c Tamworth is on a bearing of 340° from Newcastle. What is the bearing of Newcastle from Tamworth?
d A plane leaves Melbourne and flies on a bearing of 065° for 2500 km.
i How far north of Melbourne is the plane?
ii How far east of Melbourne is it?
iii What is the bearing of Melbourne from the plane?
e Kim leaves his house and walks for 2 km on a bearing of 155°. How far south is Kim from his house now, correct to 1 decimal place?
f A plane flies north from Sydney for 560 km, then turns and flies east for 390 km. What is its bearing from Sydney, correct to the nearest degree?
g A group of students are bushwalking. They walk north from their camp for 7.5 km, then walk west until their bearing from camp is 320°. How far are they from camp, correct to 1 decimal place?
h A plane leaves Sydney and flies for 2000 km on a bearing of 195°. How far due south of Sydney is it?
i A girl starting at her house, walks south for 2.7 km, then walks east for 1.6 km. What is her bearing from the house, correct to the nearest degree?
j A hot air balloon flies south for 3.6 km then turns and flies east until it is on a bearing of 127° from where it started. How far east does it fly?

## Reasoning and communication

8 A flat verandah roof 1.8 m deep is 2.6 m up from the ground. At a certain time of day, the sun makes an angle of elevation of 72°25′. How much shade is provided on the ground by the verandah roof at that time?




9 A train depot has train tracks running north for 7.8 km where they meet another set of tracks going east for 5.8 km into a station. What is the bearing of the depot from the station, to the nearest degree?

10 Jessica leaves home and walks for 4.7 km on a bearing of 075°. She then turns and walks for 2.9 km on a bearing of 115° and she is then due east of her home.
a How far north does Jessica walk?
b How far is she from home?

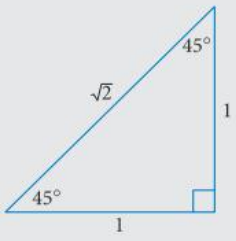

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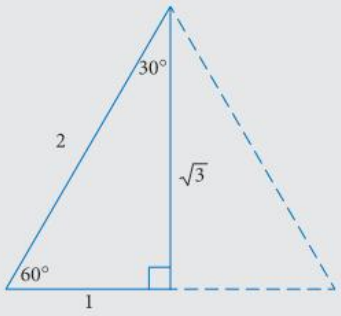
# 3.02 ANGLES OF ANY MAGNITUDE

You may have already noticed that sin(30°) is exactly a half. There are other angles that have exact trigonometric ratios.

**IMPORTANT**

There are two special triangles that give exact ratios.




$\sin(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan(45°) = 1$

$\sin(30°) = \frac{1}{2}$

$\cos(30°) = \frac{\sqrt{3}}{2}$

$\tan(30°) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin(60°) = \frac{\sqrt{3}}{2}$

$\cos(60°) = \frac{1}{2}$

$\tan(60°) = \sqrt{3}$

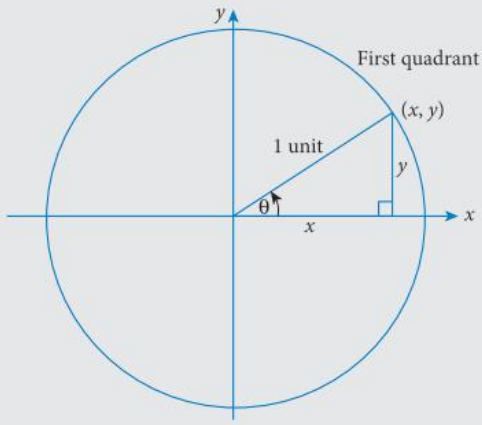
## INVESTIGATION Exact ratios

Notice that some of the above ratios are the same, for example, sin (30°) and cos (60°). Others are reciprocals, such as tan (30°) and tan (60°).

Explore the right-angled triangles and see if you can find out why this happens.

By measuring angles around a circle, you can find trigonometric ratios for any angles, not just angles less than 90°. Remember that in mathematics, angles turn anticlockwise.

**IMPORTANT**




**Trigonometric ratios in a unit circle**

In the first quadrant, $x$ and $y$ are both positive.

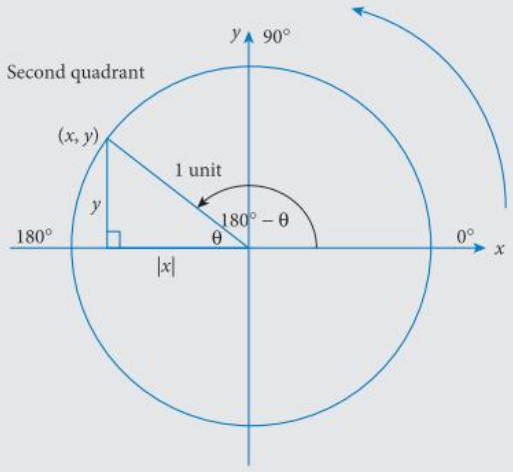
$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = \frac{y}{x}$$

As $x$ and $y$ values become negative in different quadrants, sin (θ), cos (θ) and tan (θ) will become negative.

**IMPORTANT**




**Trigonometric ratios in a unit circle**

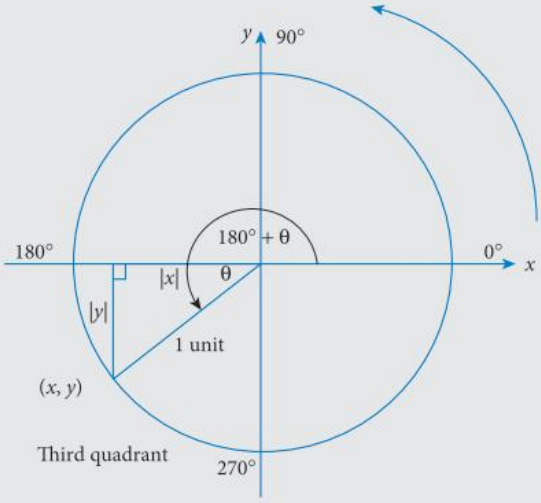
In the second quadrant, $y$ is positive and $x$ is negative.

$$\sin(\theta) > 0$$

$$\cos(\theta) < 0$$

$$\tan(\theta) < 0$$


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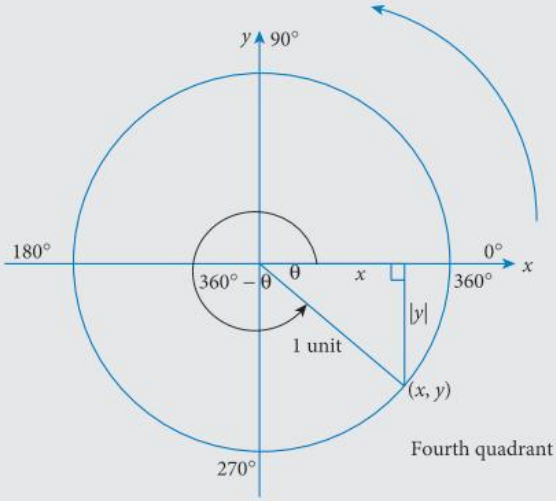

**Trigonometric ratios in a unit circle**

In the third quadrant, $x$ and $y$ are both negative.

$\sin(\theta) < 0$

$\cos(\theta) < 0$

$\tan(\theta) > 0$




In the fourth quadrant, $x$ is positive and $y$ is negative.

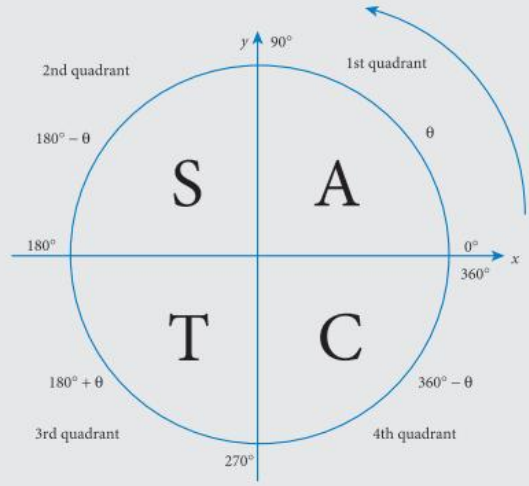
$\sin(\theta) < 0$

$\cos(\theta) > 0$

$\tan(\theta) < 0$

Putting this information together gives the general rules for the angles of any magnitude.

**IMPORTANT**




| Second quadrant | Third quadrant | Fourth quadrant |
|---|---|---|
| $\sin(180° - \theta) = \sin(\theta)$ | $\sin(180° + \theta) = -\sin(\theta)$ | $\sin(360° - \theta) = -\sin(\theta)$ |
| $\cos(180° - \theta) = -\cos(\theta)$ | $\cos(180° + \theta) = -\cos(\theta)$ | $\cos(360° - \theta) = \cos(\theta)$ |
| $\tan(180° - \theta) = -\tan(\theta)$ | $\tan(180° + \theta) = \tan(\theta)$ | $\tan(360° - \theta) = -\tan(\theta)$ |

This is called the ASTC rule and you can learn it by making up a phrase that has these letters in it. For example, 'All stations to Central' or 'All students to class'.

- **A** stands for 'all', which means that all trig ratios are positive in the first quadrant.
- **S** stands for sine, which means that only sin (θ) is positive in the second quadrant and the others are negative.
- **T** stands for tangent, which means that only tan (θ) is positive in the third quadrant and the others are negative.
- **C** stands for cosine, which means that only cos (θ) is positive in the fourth quadrant and the others are negative.

## Example 4

**a** Complete each statement.

i $\tan(180° + \alpha) =$ ii $\sin(360° - x) =$

**b** Find the exact value of

i $\tan(330°)$ ii $\cos(225°)$


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## Solution

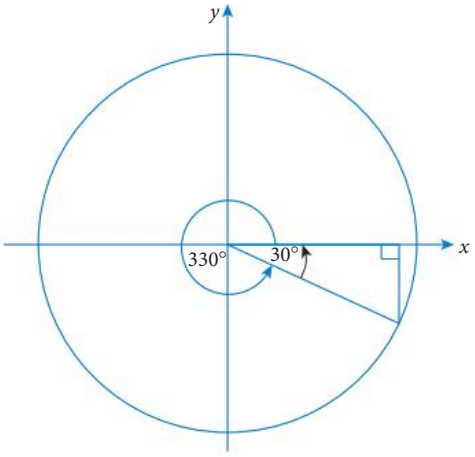
**a i** State the quadrant. $180° + \alpha$ is in the 3rd quadrant.

Complete the statement. $\tan(180° + \alpha) = \tan(\alpha)$

**ii** State the quadrant. $360° - x$ is in the 4th quadrant.

Complete the statement. $\sin(360° - x) = -\sin(x)$

**b i** Draw 330° in a unit circle and draw an acute angle using the ray for 330°.

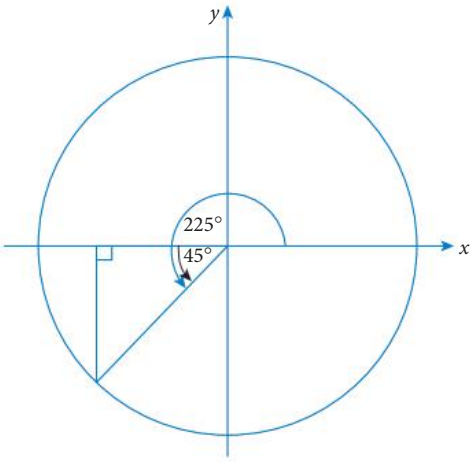



Write tan (330°) using the acute angle and 360° or 180°. $\tan(330°) = \tan(360° - 30°)$

Use the rule. $= -\tan(30°)$

Substitute the exact value. $= -\frac{\sqrt{3}}{3}$

**ii** Draw 225° in a unit circle and draw an acute angle using the ray for 225°.




Write cos (225°) using the acute angle and 360° or 180°. $\cos(225°) = \cos(180° + 45°)$

Use the rule. $= -\cos(45°)$

Substitute the exact value. $= -\frac{\sqrt{2}}{2}$

You can find trigonometric ratios of angles greater than 360° by going around the circle more than once.

## Example 5

**a** Which quadrant is 800° in?
**b** Find the exact value of sin (510°).

### Solution

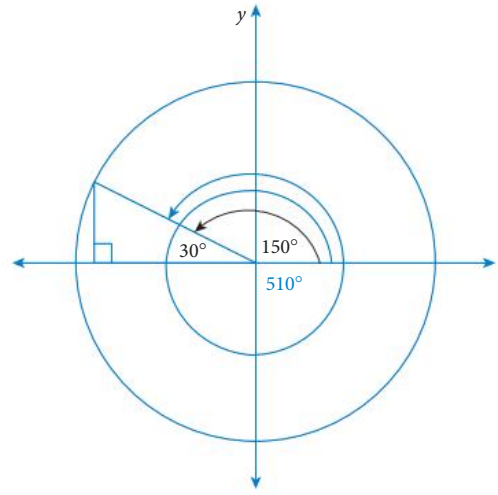
**a** Find the number of times 800° revolves around the circle.

360° is one revolution.
720° is two revolutions.
800° – 720° = 80°

State the result.

80° is in the 1st quadrant, so 800° is in the 1st quadrant.

**b** Draw 510° in a unit circle by going round as much as necessary and draw an acute angle using the ray for 510°.




Write sin (510°) as a sum of 360° and an angle less than 360°.
Simplify by omitting the period 360°.

$\sin(510°) = \sin(360° + 150°)$
$= \sin(150°)$

Now write sin (150°) using the acute angle and 360° or 180°.
Use the rule.

$= \sin(180° - 30°)$
$= \sin(30°)$

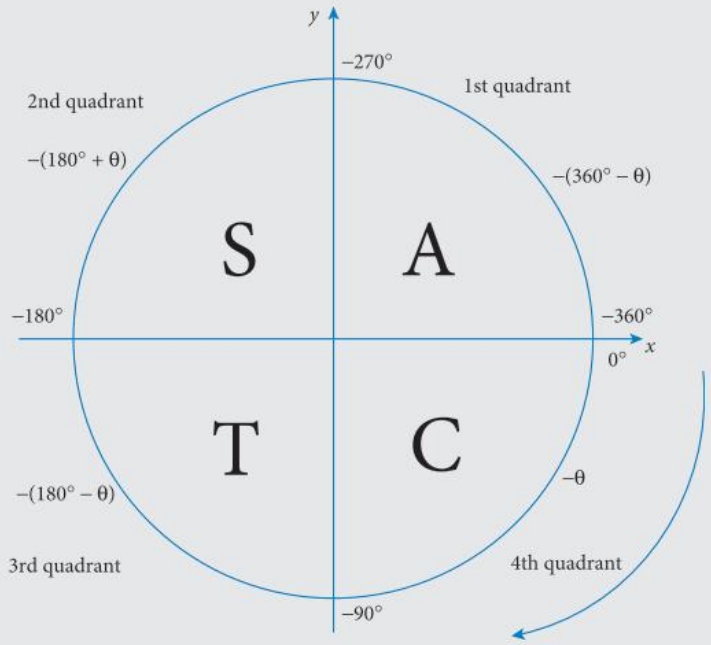
Substitute the exact value.

$= \frac{1}{2}$


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Angles that turn in the opposite direction (clockwise from the $x$-axis) are negative.

**IMPORTANT**




**Fourth quadrant**

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\tan(-\theta) = -\tan(\theta)$

**Second quadrant**

$\sin(-(180° + \theta)) = \sin(-180° - \theta) = \sin(\theta)$

$\cos(-(180° + \theta)) = \cos(-180° - \theta) = -\cos(\theta)$

$\tan(-(180° + \theta)) = \tan(-180° - \theta) = -\tan(\theta)$

**Third quadrant**

$\sin(-(180° - \theta)) = \sin(-180° + \theta) = -\sin(\theta)$

$\cos(-(180° - \theta)) = \cos(-180° + \theta) = -\cos(\theta)$

$\tan(-(180° - \theta)) = \tan(-180° + \theta) = \tan(\theta)$

**First quadrant**

$\sin(-(360° - \theta)) = \sin(-360° + \theta) = \sin(\theta)$

$\cos(-(360° - \theta)) = \cos(-360° + \theta) = \cos(\theta)$

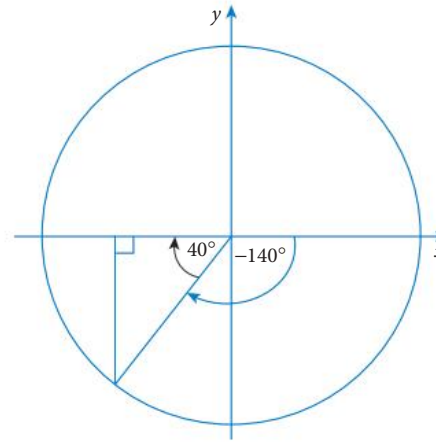
$\tan(-(360° - \theta)) = \tan(-360° + \theta) = \tan(\theta)$

## Example 6

**a** In which quadrant does −140° lie?
**b** Find the exact value of tan (−240°).

### Solution

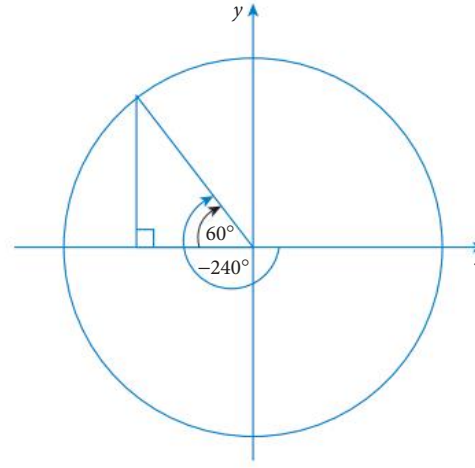
**a** Turn clockwise from the $x$-axis.




State the quadrant.

−140° is in the third quadrant.

**b** Draw −240° clockwise in a unit circle and draw an acute angle in the terminal quadrant using the ray for −240°.




Write tan (−240°) using the acute angle and 360° or 180°.

$\tan(-240°) = \tan(180° - 60°)$

Use the rule.

$= -\tan(60°)$

Substitute the exact value.

$= -\sqrt{3}$


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# EXERCISE 3.02 Angles of any magnitude

## Concepts and techniques

**1** Example 4 Complete each statement.

**a** $\sin(180° - \theta) =$ **b** $\cos(180° + x) =$ **c** $\cos(360° - \theta) =$

**d** $\tan(360° - \phi) =$ **e** $\sin(360° - \beta) =$ **f** $\tan(180° - t) =$

**g** $\cos(180° - x) =$ **h** $\sin(180° + \alpha) =$ **i** $\tan(180° + \theta) =$

**2** Find the exact value of

**a** $\sin(120°)$ **b** $\cos(315°)$ **c** $\tan(150°)$

**d** $\tan(225°)$ **e** $\sin(330°)$ **f** $\cos(135°)$

**g** $\tan(300°)$ **h** $\cos(120°)$ **i** $\sin(135°)$

**j** $\sin(315°)$

**3** Example 5 Find the exact value of

**a** $\cos(420°)$ **b** $\tan(480°)$ **c** $\sin(600°)$

**d** $\tan(675°)$ **e** $\cos(390°)$ **f** $\tan(405°)$

**g** $\sin(840°)$

**4** Example 6 In which quadrant are each of the following angles?

**a** $-50°$ **b** $-190°$ **c** $-100°$ **d** $-310°$ **e** $-240°$

**5** Complete each statement.

**a** $\sin(360° + x) =$ **b** $\cos(-\theta) =$ **c** $\tan(-180° - \theta) =$

**d** $\cos(-180° + x) =$ **e** $\sin(-t) =$

**6** Find the exact value of each of the following.

**a** $\cos(-45°)$ **b** $\sin(-240°)$ **c** $\sin(-315°)$

**d** $\tan(-210°)$ **e** $\cos(-300°)$ **f** $\sin(-135°)$

**g** $\tan(-120°)$

## Reasoning and communication

**7** Find the exact value of each of the following.

**a** $\tan(225°) - \cos(225°)$ **b** $\sin(210°) + \sin(120°)$

**c** $\cos(210°) + \sin(300°)$ **d** $\tan^2(300°)$

**e** $\sin^2(120°) + \cos^2(120°)$

**8** Simplify

**a** $\cos(-\theta) - \cos(180° - \theta)$ **b** $\cos(180° + x) - 2\cos(x)$

**c** $3\tan(180° - \theta) + 2\tan[-(180° - \theta)]$ **d** $\sin(360° - \theta) + 2\sin(180° - \theta)$

**e** $2\tan(-x) + \tan(180° + x)$

**9** **a** Draw a right angled triangle that shows $\sin(\theta) = \frac{3}{4}$.

**b** Find the exact length of the other side.

**c** If the triangle is in the second quadrant, find the exact value of

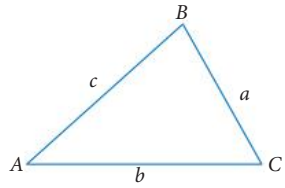
**i** $\tan(\theta)$ **ii** $\cos(\theta)$

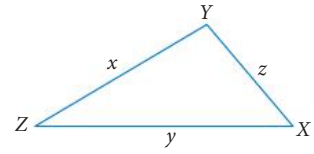
**10** If $\tan(\theta) = \frac{5}{8}$ and $\theta$ is in the third quadrant, find the exact value of

**a** $\sin(\theta)$ **b** $\cos(\theta)$

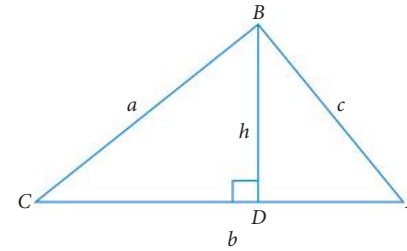
# 3.03 AREA OF A TRIANGLE

The angles and their opposite sides in a non-right-angled triangle are named using the same letters. Upper case letters are used for the angles and lower case ones are used for the sides.




You can use trigonometry to find the area of a triangle when you do not know its height.




In $\triangle BCD$

$$\sin(C) = \frac{h}{a}$$

$$h = a \sin(C)$$

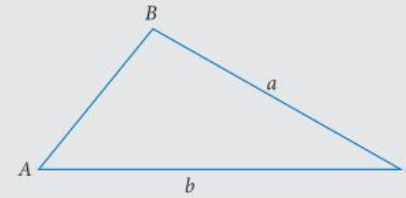
$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2}b(a \sin(C))$$

$$= \frac{1}{2}ab \sin(C)$$

**IMPORTANT**

The area of $\triangle ABC$ is given by $A = \frac{1}{2}ab \sin(C)$ where $C$ is the **included angle** between sides $a$ and $b$.




Other forms of the area formula are

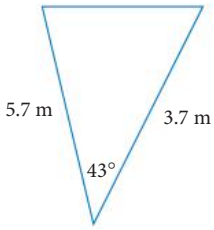
$$A = \frac{1}{2}bc \sin(A)$$

$$A = \frac{1}{2}ac \sin(B)$$


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## Example 7

Find the area of the triangle below, correct to 1 decimal place.




### Solution

| | |
|---|---|
| Write the formula. | $A = \frac{1}{2}\ ab \sin(C)$. |
| Substitute values. | $= \frac{1}{2} \times 5.7 \times 3.7 \sin(43°)$ |
| Evaluate. | $= 7.1916\ldots$ |
| State the result. | The area is about 7.2 $m^2$. |

You can find the area of other figures by using the triangle area formula.

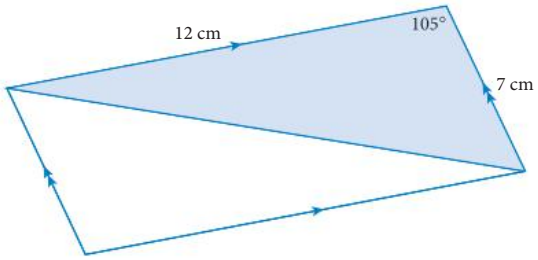
WS
Areas of triangles

## Example 8

Find the area of a parallelogram with sides of 12 cm and 7 cm in length, if one of its interior angles is 105°.

### Solution

Make a sketch of the parallelogram showing the information and draw in a diagonal.




| | |
|---|---|
| Write the formula. | $A = \frac{1}{2}\ ab \sin(C)$. |
| Substitute values. | $= \frac{1}{2} \times 7 \times 12 \sin(105°)$ |
| Evaluate. | $= 40.5688\ldots$ |
| Find the area of the parallelogram. | The two triangles are congruent (equal).<br>$A = 2 \times 40.5688\ldots$<br>$= 81.1377\ldots.$ |
| State the result. | The area of the parallelogram is 81.1 $cm^2$. |

### Example 9

A house has a garden bed in the shape of an equilateral triangle with sides 12.5 m. Find the amount of topsoil needed to go in this garden if it needs to be 200 mm deep.

#### Solution

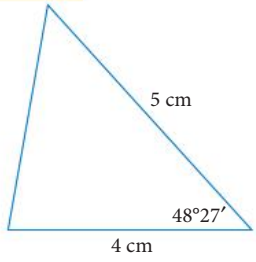
| | |
|---|---|
| State the angle. | An equilateral triangle has 60° angles |
| Write the formula. | $A = \frac{1}{2} ab \sin(C)$. |
| Substitute values. | $= \frac{1}{2} \times 12.5 \times 12.5 \sin(60°)$ |
| Evaluate. | $= 67.6582...$ |
| Find the volume using $V = Ah$. | The depth is 200 mm or 0.2 m.<br>$V = 67.6582... \times 0.2$ |
| Evaluate. | $= 13.5316...$ |
| State the result. | 13.5 $m^3$ of topsoil is needed for the garden. |

## EXERCISE 3.03 Area of a triangle

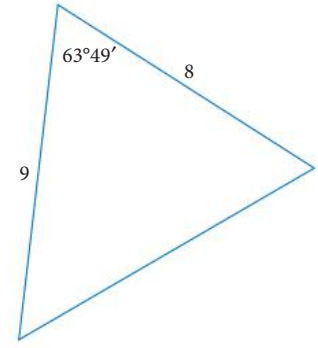
### Concepts and techniques

1 Example 7 Find the area of each triangle correct to 1 decimal place.

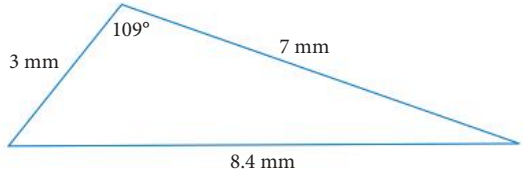
a



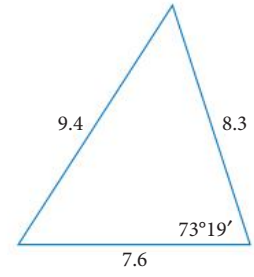

b



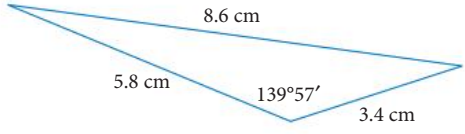

c




d



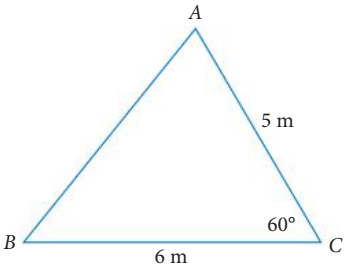

e



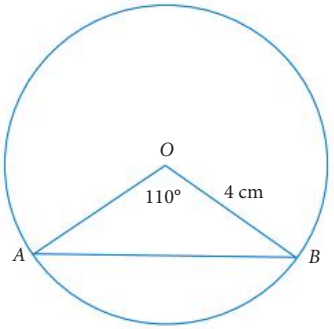


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**2** Calculate the exact area of $\triangle ABC$.

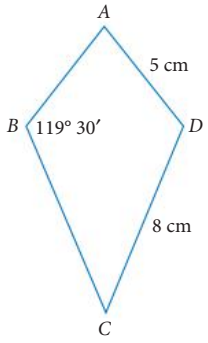



**3** Find the area of $\triangle OAB$ correct to 1 decimal place ($O$ is the centre of the circle).




**4** Example 8 Find the area of a parallelogram with sides 3.5 cm and 4.8 cm, if one of its internal angles is 67°13′, correct to 1 decimal place.

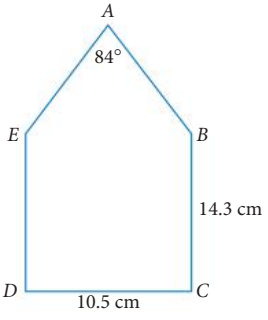
**5** Find the area of kite $ABCD$, correct to 3 significant figures.



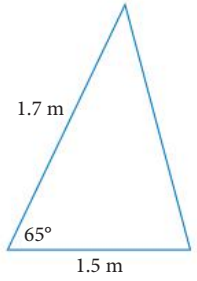

**6** The figure below is made from a rectangle and isosceles triangle with $AE = AB$ as shown.

**a** Find the length of $AE$.

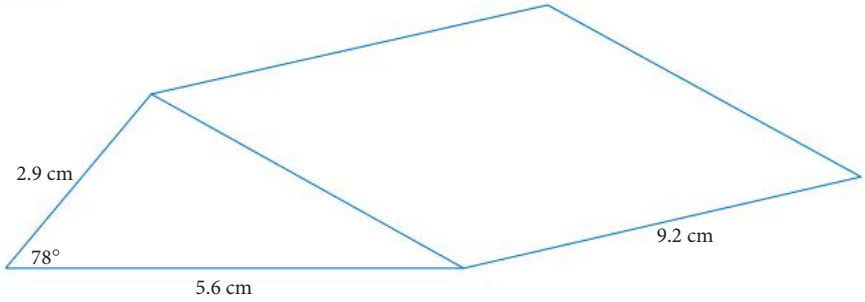
**b** Find the area of the figure.




**7** Find the area of the sail shown below.



8 Example 9 Find the volume of the prism below.




## Reasoning and communication

9 An equilateral triangle has an area of 200 $cm^2$. What is the length of its side, correct to 1 decimal place?

10 An aviary in the shape of a triangular pyramid is to be constructed. The concrete for the base is to be 100 mm thick and the triangle is equilateral with a side length of 8.2 m. Find the volume of concrete needed to build the base of the pyramid.

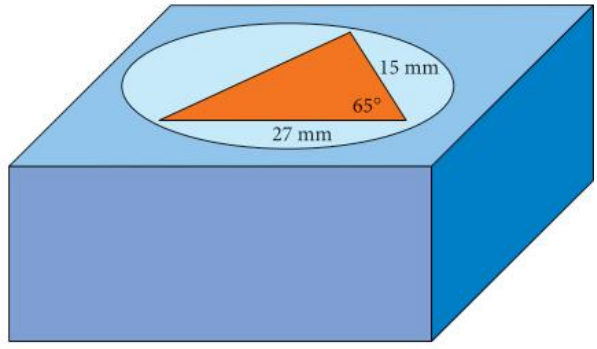


Photo courtesy Margaret Grove

11 Trang designed a lid for a jewellery box that has a triangle of jade as shown.

a What volume of jade does Trang need if it is 5 mm thick?

b If Trang can buy jade for 3 cents per $mm^3$, how much does the jade for this box cost?



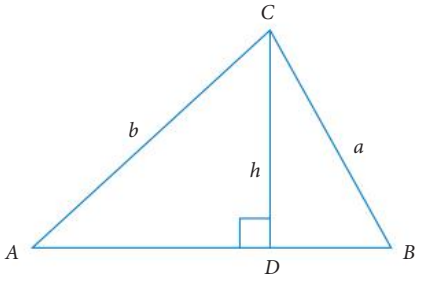


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# 3.04 THE SINE RULE

The **sine rule** is used to find unknown sides or angles in non-right-angled triangles.

You can demonstrate the sine rule by drawing a right-angled triangle inside a non-right-angled triangle.




From $\triangle CAD$, $\sin(A) = \frac{h}{b}$

So $h = b \sin(A)$

From $\triangle CBD$, $\sin(B) = \frac{h}{a}$

So $h = a \sin(B)$

Combining $h = b \sin(A)$ and $h = a \sin(B)$ gives $b \sin(A) = a \sin(B)$

This gives $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$.

Starting with the altitude $BD$ instead of $CD$ gives $\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$.

A full proof of the sine rule has to show that it is also true for obtuse-angled triangles.

You can use the sine rule to find an unknown side in a non-right-angled triangle when you know another side and two angles.

**IMPORTANT**

In any $\triangle ABC$

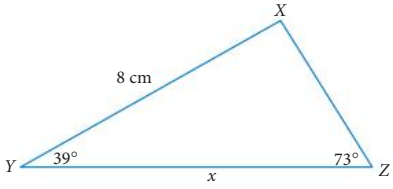
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

or

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

## Example 10

CAS Evaluate $x$ correct to one decimal place.




### Solution

Find $\angle X$.

$\angle X = 180° - (39° + 73°)$
$= 68°$

To find a side, write the sine rule for $\triangle XYZ$ with the sides on top, using the unknown and known sides.

$$\frac{x}{\sin(X)} = \frac{z}{\sin(Z)}$$

Substitute the values.

$$\frac{x}{\sin(68°)} = \frac{8}{\sin(73°)}$$

Multiply both sides by $\sin(68°)$.

$$x = \frac{8\sin(68°)}{\sin(73°)}$$

Use your calculator to evaluate $x$.

$= 7.7563\ldots$

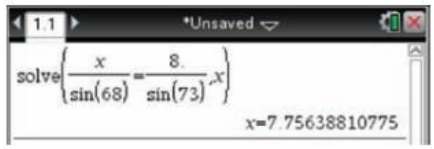
Round to 1 decimal place and state the answer.

$x \approx 7.8$ cm

You can use the solve( function on your CAS calculator. Make sure it is in degrees.

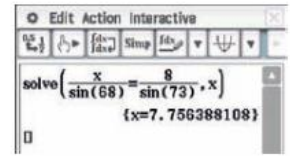
**TI-Nspire CAS**

Use a calculator page ( ).




**ClassPad**

Use the Main menu ( ).

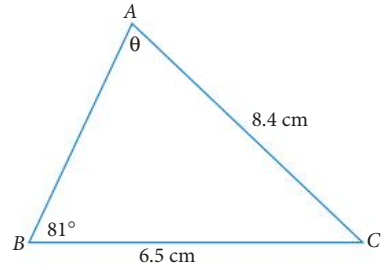



You can also use the sine rule to find an unknown angle in a non-right-angled triangle when you know another angle and two sides.

Since sin (θ) = sin (180°– θ), there are sometimes two possible answers when finding the angle using the sine rule. Your calculator will give the acute angle, but it could also be obtuse, depending on the shape of the triangle.

## Example 11

CAS Evaluate θ correct to the nearest degree in $\triangle ABC$ (the triangle is not drawn to scale).




### Solution

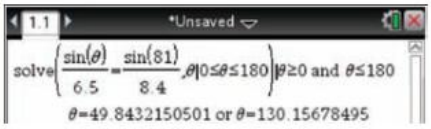
| | |
|---|---|
| Since you are finding an angle, write the sine rule for $\triangle ABC$ using the unknown angle $A$ and the known angle $B$ with the angles on top. | $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$ |
| Substitute the values. | $\frac{\sin(\theta)}{6.5} = \frac{\sin(81°)}{8.4}$ |
| Multiply both sides by 6.5. | $\sin(\theta) = \frac{6.5\sin(81°)}{8.4}$ |
| | $= 0.7642\ldots$ |
| Use $\sin^{-1}$ on your calculator to evaluate θ. | $\theta = 49.8432\ldots$ |
| Round to the nearest degree and state the answer. | $\approx 50°$ |
| Find the corresponding obtuse angle. | $\theta = 180° - 50° = 130°$ |
| Check the obtuse angle. | Sum of angles = 130° + 81° > 180° |
| State the result. | The angle could only be 50°. |


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You can use the solve( function on your CAS calculator. Make sure it is in degrees and restrict the range of the angle to $0 < \theta < 180°$.

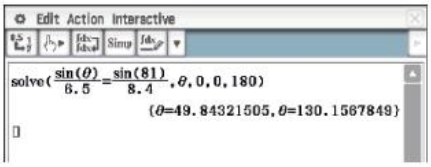
### TI-Nspire CAS

Use a calculator page ( ). The range restriction $\theta \geq 0$ and $\theta \leq 180$ is placed after solve(). You need to put a vertical line before it.




### ClassPad

Use the Main menu ( ). The range restriction [0, 180] is placed inside solve. You need to put an extra value (0) in front of it.

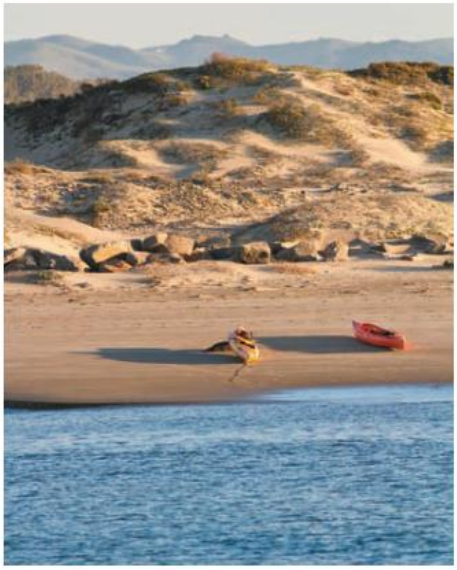



You can use the sine rule to solve problems.

WS
The sine rule

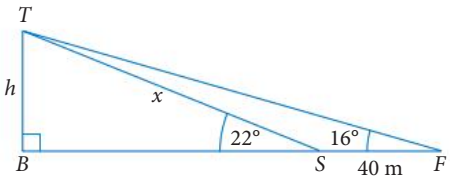
## Example 12

Jermaine sees that the top of a sand dune at the end of a beach is at an angle of elevation of 16°. On walking 40 m closer, she finds that the angle of elevation increases to 22°. What is the height of the sand dune?


Shutterstock.com/Jean-Edouard Rozey

### Solution

Make a sketch of the information. Label sides and angles as needed. The height is side *TB*. Side *TS* is in between the known and unknown sides. Call it *x* and find it first.




Use the exterior angle of $\triangle SFT$ to find $\angle STF$.

$22° = 16° + \angle STF$

$\angle STF = 6°$

Use the sine rule in $\triangle SFT$ to find $x$.

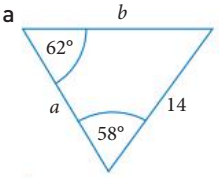
$$\frac{x}{\sin(16°)} = \frac{40}{\sin(6°)}$$

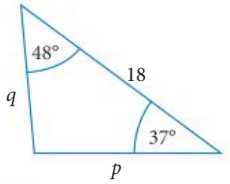
| | |
|---|---|
| Multiply both sides by sin (16°). Leave $x$ as a fraction. You do not need to evaluate it. | $x = \frac{40 \sin(16°)}{\sin(6°)}$ |
| Now use $\triangle BST$ and the sine ratio to find $h$. | $\sin(22°) = \frac{h}{x}$ |
| Multiply both sides by $x$. | $h = x \sin(22°)$ |
| Evaluate using the uncalculated value of $x$. | $= \frac{40 \sin(16°) \sin(22°)}{\sin(6°)}$ |
| | $= 39.5129...$ |
| Round and write the answer. | The dune is about 39.5 m high. |

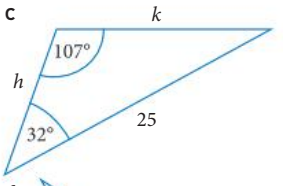
# EXERCISE 3.04 The sine rule

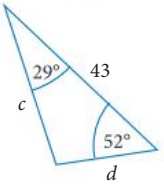
## Concepts and techniques

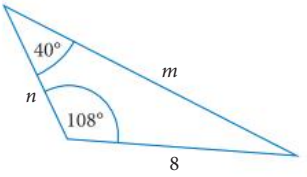
1 Example 10 Calculate the lengths of the unknown sides in these triangles, correct to 1 decimal place.

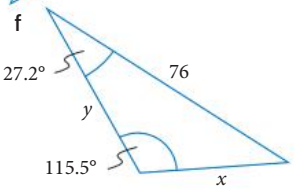
a 


b 

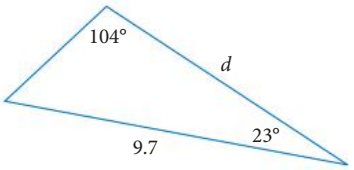

c 

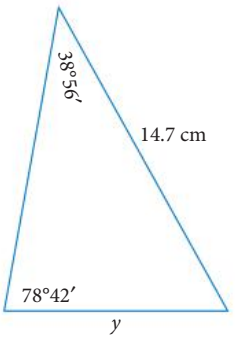

d 


e 


f 


2 Evaluate all pronumerals, correct to 1 decimal place.

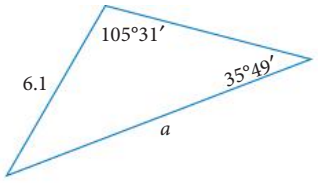
a 


b 

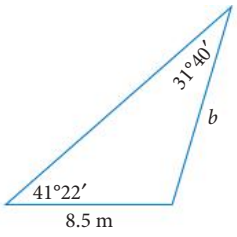


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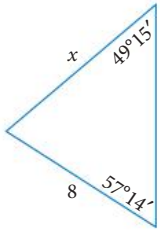
c




d

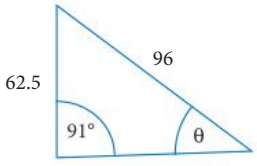



e

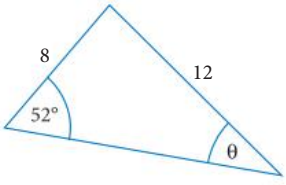



**3** Example 11 Calculate the value of θ in each of the following triangles, correct to 1 decimal place.

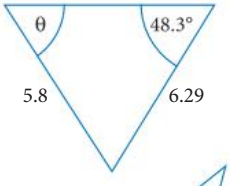
a



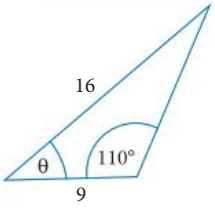

b



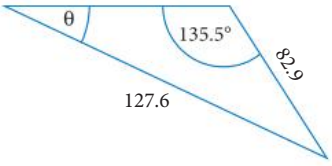

c



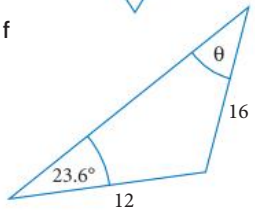

d




e




f




**4** CAS Solve each of the following triangles. Calculate the sizes of all unknown angles and sides, correct to one decimal place. Include both possibilities in ambiguous cases.

a $\triangle XYZ$, where $Z = 65°$, $y = 11$ m and $z = 16.2$ m

b $\triangle PQR$, where $P = 35.3°$, $Q = 52.8°$ and $q = 67.5$ cm

c $\triangle DEF$, where $F = 111°$, $f = 12.5$ km and $d = 8.96$ km

d $\triangle ABC$, where $B = 124.1°$, $C = 18.7°$ and $c = 94.6$ cm

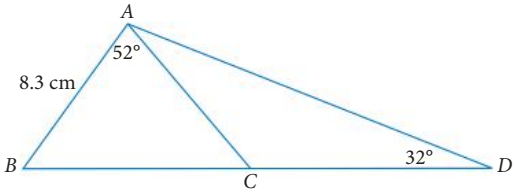
e $\triangle XYZ$, where $X = 68.41°$, $Y = 54.23°$ and $x = 12.75$ m

f $\triangle KLM$, where $K = 71.83°$, $M = 42.57°$ and $l = 2.614$ km

**5** Example 12 Triangle $ABC$ is isosceles with $AB = AC$. $BC$ is produced to $D$ as shown. If $AB = 8.3$ cm, $\angle BAC = 52°$ and $\angle ADC = 32°$ find the length of

a $AD$

b $BD$.



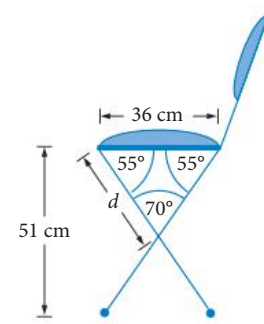

**6** Triangle $ABC$ is equilateral with sides of length 63 mm. A line is drawn from $A$ to $BC$ where it meets $BC$ at $D$ and $\angle DAB = 26°15'$. Find the length of

a $AD$

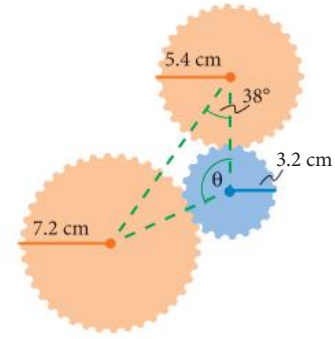
b $DC$.

## Reasoning and communication

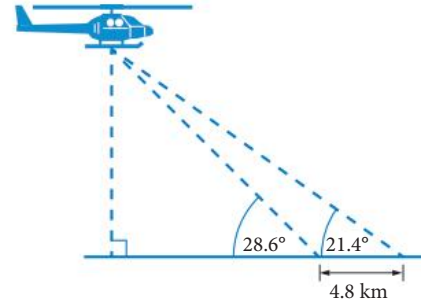
**7** The folding chair shown on the right has a seat that is 36 cm deep. The legs of the chair are inclined at an angle of 55° to the seat and join at an angle of 70°. The seat is 51 cm above the floor. Find the distance from the seat, *d*, at which the legs are joined.




**8** Three gears are arranged as shown on the right. The radii of the gears are 7.2 cm, 5.4 cm and 3.2 cm. Calculate θ.




**9** A helicopter is sighted at the same time by two ground observers who are 4.8 km apart, as shown in the diagram on the right. The angles from the observers to the helicopter (measured to the horizontal) are 21.4° and 28.6°. How high is the helicopter?




**10** Anne is flying from Brisbane to Dalby at a speed of 150 km/h. She thinks her altimeter, which shows a height of 15 000 feet (4600 m), is faulty. Looking ahead, she sees the top of a hill marked as 405 m on her map at an angle of depression of 13°. Two minutes later, the angle of depression has increased to 20.6°.

- **a** Find her height above the hill and thus her altitude,
- **b** What should the radio operator tell her about her altimeter?

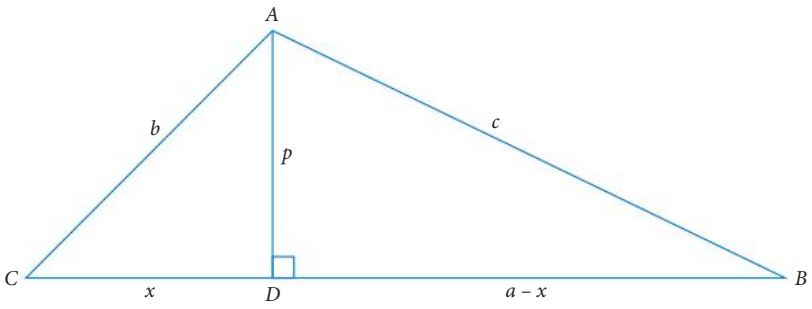


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# 3.05 THE COSINE RULE

The **cosine rule** is a generalisation of Pythagoras' theorem to cases where the triangle is not right angled.




In $\triangle ABC$, draw the perpendicular $AD$ with length $p$ and let $CD = x$.

Since $BC = a$, $BD = a - x$

From $ACD$

$b^2 = x^2 + p^2$ [1]

$\cos(C) = \frac{x}{b}$

$b\cos(C) = x$ [2]

From $\triangle DAB$

$c^2 = p^2 + (a - x)^2$

$= p^2 + a^2 - 2ax + x^2$

$= p^2 + x^2 + a^2 - 2ax$ [3]

Substitute [1] into [3].

$c^2 = b^2 + a^2 - 2ax$ [4]

Substituting [2] into [4]:

$c^2 = b^2 + a^2 - 2a(b\cos(C))$

$= a^2 + b^2 - 2ab\cos(C)$

**IMPORTANT**

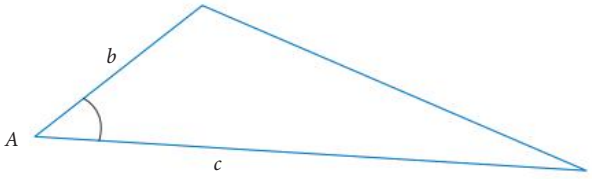
In any $\triangle ABC$

$c^2 = a^2 + b^2 - 2ab\cos(C)$

The cosine rule could also be written as

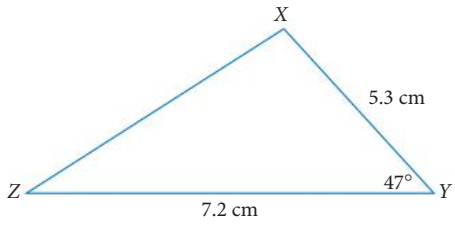
$a^2 = b^2 + c^2 - 2bc\cos(A)$ or $b^2 = a^2 + c^2 - 2ac\cos(B)$

You can use the cosine rule to find an unknown side when given the other two sides and their **included angle**. The included angle is the angle formed by the two sides as shown below.



## Example 13

CAS Find the length of $XZ$, correct to 2 decimal places, in the triangle below.



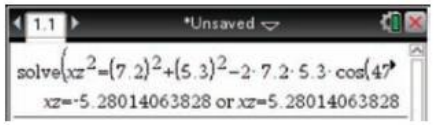

### Solution

| | |
|---|---|
| State the cosine rule for $\triangle XYZ$ with $XZ$ ($y$) as the unknown side. | $y^2 = x^2 + z^2 - 2xz\cos(Y)$ |
| Substitute in the information. | $XZ^2 = 7.2^2 + 5.3^2 - 2(7.2)(5.3)\cos(47°)$ |
| Evaluate. | $= 27.8798\ldots$ |
| Find the square root and round to 2 decimal places. | $XZ = \sqrt{27.8798\ldots}$<br>$\approx 5.28$ |
| State the result. | $XZ \approx 5.28$ cm |

You can use the solve( function on your CAS calculator. Make sure it is in degrees.

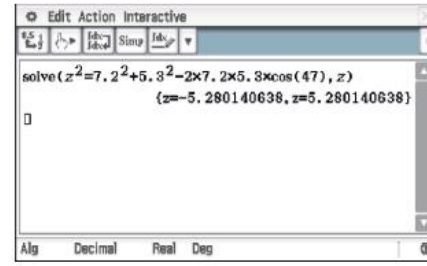
**TI-Nspire CAS**

Use a calculator page ( ).




**ClassPad**

Use the Main menu ( ). The ClassPad will not accept xz as a variable, so we call it z instead.




You can also use the cosine rule to find an unknown angle when given 3 sides. You can rearrange the cosine rule to make the formula easier to use.

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$2ab\cos(C) = a^2 + b^2 - c^2$$

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine rule could also be written as

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{or} \quad \cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$

**IMPORTANT**

In any $\triangle ABC$

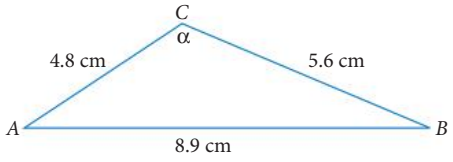
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

Since $\cos(\theta)$ is negative when $\theta$ is obtuse, there are no ambiguous cases with the cosine rule.


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## Example 14

CAS Find α to the nearest degree.




### Solution

State the cosine rule for $\triangle ABC$ with the unknown angle at $C$.

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

Substitute the information and evaluate.

$$\cos(\alpha) = \frac{5.6^2 + 4.8^2 - 8.9^2}{2 \times 4.8 \times 5.6}$$
$$\approx -0.4614\ldots$$

Use $\cos^{-1}$ to find α.

$$\alpha = 117.4836\ldots$$

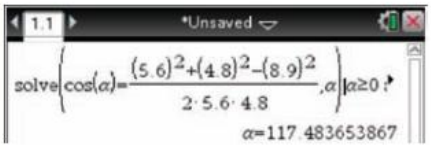
Round to the nearest degree and state the result.

$$\alpha \approx 117°$$

You can use the solve( function on your CAS calculator. Make sure it is in degrees and restrict the range of the angle to $0 < \theta < 180°$.

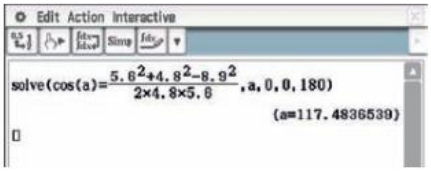
**TI-Nspire CAS**

Use a calculator page ( ). The range restriction $\theta \geq 0$ and $\theta \leq 180$ is placed after solve(). You need to put a vertical line before it.




**ClassPad**

Use the Main menu ( ). The range restriction [0, 180] is placed inside solve. You need to put an extra value (0) in front of it.



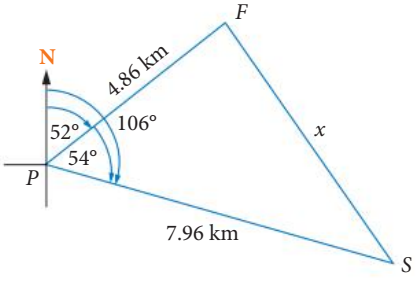

You can use the cosine rule to help solve practical problems.

## Example 15

A tower in a direction N 52° E is found to be at a range of 4.86 km. The range finder shows that another tower in a direction 106° is at a range of 7.96 km. Find the distance between the towers.

### Solution

Make a sketch and name the points.
Name the observation point *P*.
Name the tower positions *F* and *S*.
Draw in the bearings.
The angle between the towers must be 54°.
We want to find the distance between the towers, *FS*. Label *FS* as *x*.




WS
The cosine rule

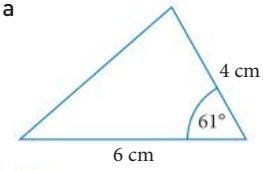
| | |
|---|---|
| Write the cosine rule for $x$. | $FS^2 = PF^2 + PS^2 - 2(PF)(PS)\cos(P)$ |
| Substitute the information and evaluate. | $= 4.86^2 + 7.96^2 - 2(4.86)(7.96)\cos(54°)$<br>$= 41.5035\ldots$ |
| Find the square root. | $FS = \sqrt{41.5035\ldots}$<br>$= 6.442\ldots$ |
| Round off and state the result. | The distance between the towers is about 6.44 km. |

# EXERCISE 3.05 The cosine rule

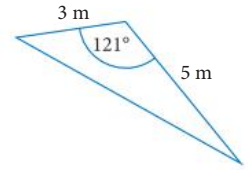
## Concepts and techniques

1 Example 13 Calculate the unknown side lengths, correct to 1 decimal place, in the following triangles.

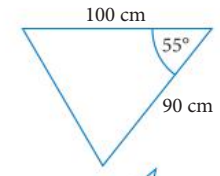
a



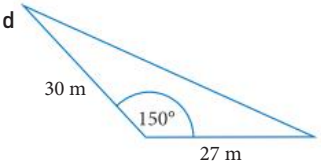

b



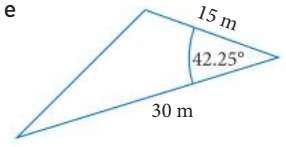

c



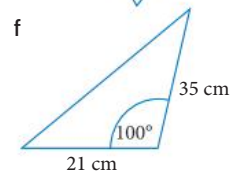

d




e

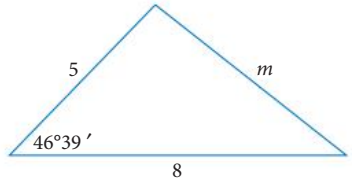



f

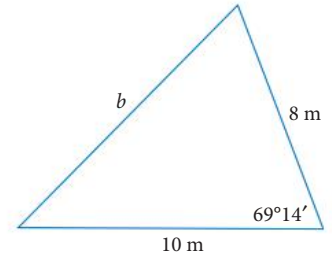



2 Find the value of all pronumerals, correct to 1 decimal place.

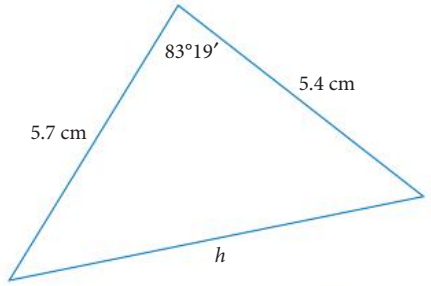
a



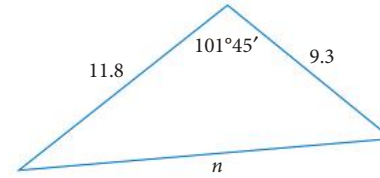

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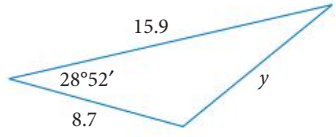

c




d




e





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3 CAS Find the third side of each of the following triangles.

a $\triangle MNO$, where $M = 41.1°$, $n = 27.8$ and $o = 39.2$

b $\triangle XYZ$, where $Z = 28.3°$, $y = 571$ and $x = 421$

c $\triangle DEF$, where $F = 45.6°$, $d = 72.3$ and $e = 89.4$

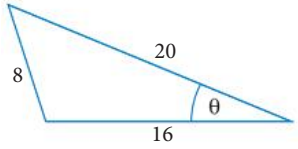
d $\triangle GHI$, where $G = 67.3°$, $h = 37.9$ and $i = 40.8$

e $\triangle STU$, where $S = 112.8°$, $t = 62.8$ and $u = 122$

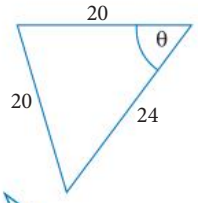
f $\triangle PQR$, where $Q = 74.8°$, $p = 891.9$ and $r = 642.7$

4 Example 14 Find θ in each of the following triangles.

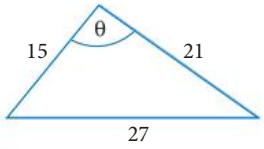
a



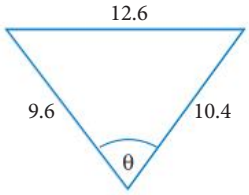

b



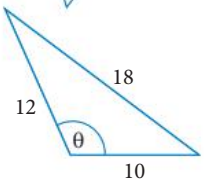

c



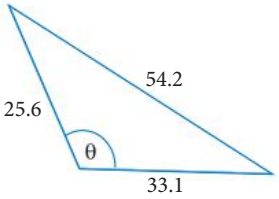

d




e

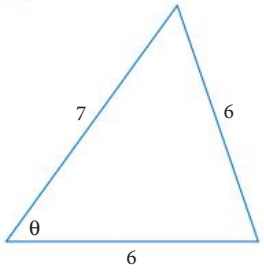



f

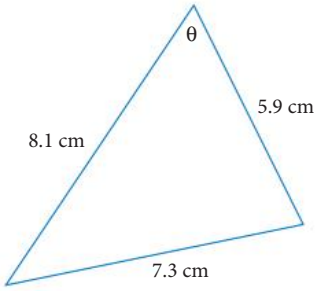



5 CAS Evaluate marked angles, correct to the nearest minute.

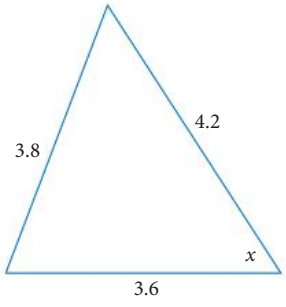
a



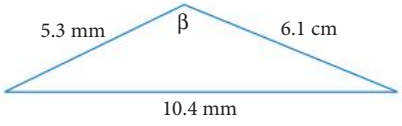

b



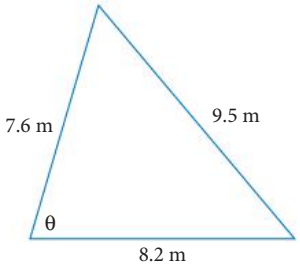

c




d




e



6 Example 15 The isosceles trapezium $MNOP$ has $MP = NO = 12$ mm, $MN = 8.9$ mm and $OP = 15.6$ mm.

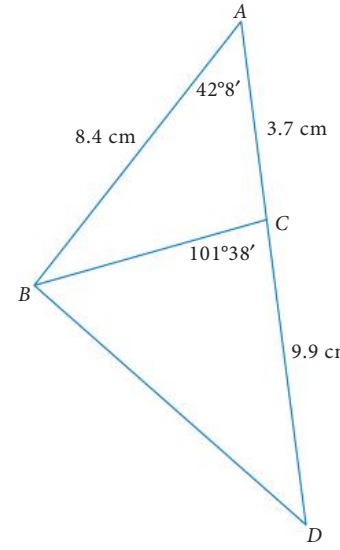
a Find $\angle NOP$.

b Find the length of diagonal $NP$.

7 Given the figure on the right, find the length of

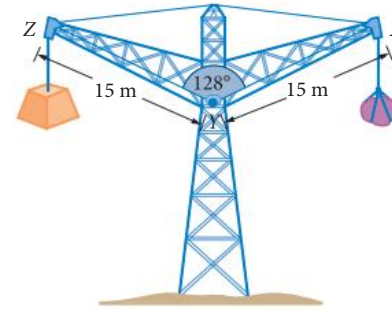
a $AC$

b $AD$.

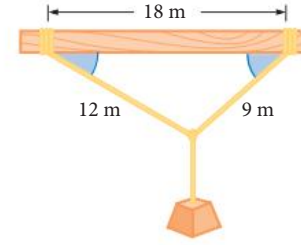



## Reasoning and communication

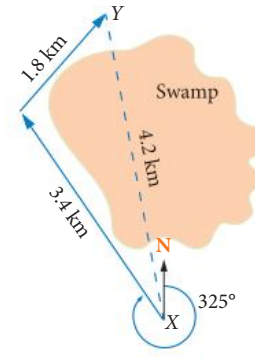
8 The diagram shows a crane with two arms, each 15 m long. The arms of the crane are separated by an angle of 128°.
The load is attached at point $X$ and the counterweight is attached at point $Z$.
Find $XZ$.




9 A weight is supported by cables attached to a horizontal beam as shown in the diagram.
Find the angle that each cable makes with the beam.




10 A hiker encounters a swamp at point $X$ as shown on the right.
To travel around the swamp, the hiker walks 3.4 km from $X$ on a bearing of 325° and then walks 1.8 km to point $Y$ on the other side of the swamp. Find the bearing of $X$ from $Y$.





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# 3.06 APPLICATIONS OF TRIGONOMETRY

You have already seen some applications of trigonometry in this chapter, including angles of elevation, angles of depression and bearings. This section continues with more applications, using a mixture of right- and non-right-angled triangles.

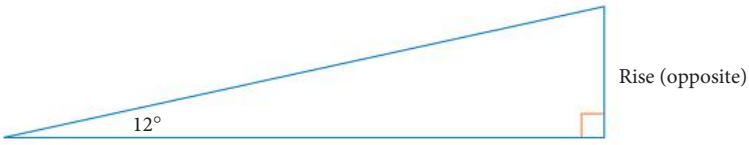
## Example 16

Find the gradient of a ramp that is inclined at 12° as shown below.




### Solution

Use gradient $= \frac{\text{rise}}{\text{run}}$.




$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{opposite}}{\text{adjacent}}$$
$$= \tan(12°)$$

Evaluate. $= 0.2125\ldots$

State the result. The gradient is approximately 0.21.

You may sometimes need a mixture of rules to solve a problem.

**IMPORTANT**

**Solving triangle problems**

- When given a right-angled triangle, use simple trig ratios rather than the sine and cosine rules.
- Use the sine rule when you are given 2 sides and an opposite angle, or 2 angles and a side.
- Use the cosine rule when you are given 2 sides and the included angle, or three sides (that don't make a Pythagorean triple) and need to find an angle.
- If there is more than one triangle, you may need to find the common side by using the triangle you know the most about first.

## Example 17

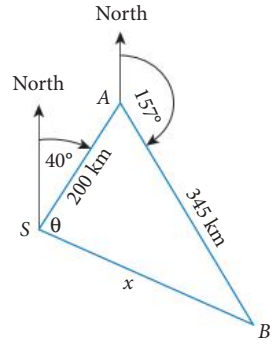
A plane flies from Sydney for 200 km on a bearing of 040° then flies for 345 km on a bearing of 157°.

**a** How far from Sydney, to the nearest km, is the plane?

**b** What is the bearing of the plane from Sydney, to the nearest degree?

### Solution

Draw a diagram to show the information.




**a** Find $\angle SAN$ using co-interior angles.

$\angle SAN = 180° - 40°$
$= 140°$

Now find $\angle SAB$.

$\angle SAB = 360° - (140° + 157°) = 63°$

Substitute the information into the cosine rule $c^2 = a^2 + b^2 - 2ab\cos(C)$.

$x^2 = 200^2 + 345^2 - 2 \times 200 \times 345 \cos(63°)$

Evaluate $x^2$.

$= 96\,374.31104\ldots$

Evaluate $x$.

$x = \sqrt{96\,374.31104}$
$= 310.4421\ldots$

State the result.

The plane is about 310 km from Sydney.

**b** Substitute the information into the sine rule $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$.

$\frac{\sin(\theta)}{345} = \frac{\sin(63°)}{310.4421...}$

Simplify.

$\sin(\theta) = \frac{345\sin(63°)}{310.4421...}$

Evaluate $\sin(\theta)$.

$= 0.9901\ldots$

Use $\sin^{-1}$ to evaluate $\theta$ and round to the nearest degree.

$\theta \approx 82°$

Find the bearing from Sydney by measuring clockwise from north.

Bearing $\approx 40° + 82°$
$= 122°$

State the result.

The bearing of the plane from Sydney is 122°.

Sometimes you can use trigonometry for 3-dimensional problems.


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## Example 18

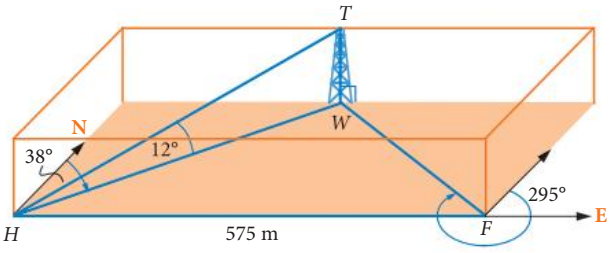
Hassan observes that the top of a transmission tower at a bearing of 038° is at an angle of elevation of 12°. Fatima is 575 m due east of Hassan, and she says the bearing of the tower is 295°. Find the height of the tower.



Shutterstock.com/chungking

### Solution

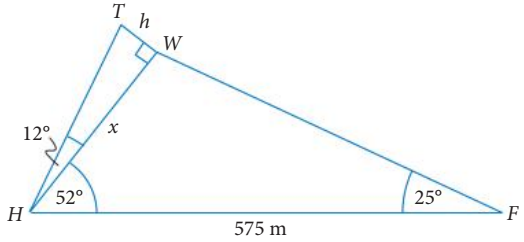
Make a sketch inside a box with Hassan's position at $H$ and Fatima's position at $F$. Show north and east. Call the bottom of the tower $W$ and the top of the tower $T$.




Evaluate angles $FHW$ and $WFH$.

$\angle FHW = 90° - 38°$
$= 52°$
$\angle WFH = 295° - 270°$
$= 25°$

Unfold and draw the triangles flat. Let the height of $TW$ be $h$ and call the common side $x$.



| | |
|---|---|
| Evaluate $\angle HWF$. | From $\triangle WHF$,<br>$\angle HWF = 180° - (52° + 25°)$<br>$= 103°$ |
| Use the sine rule in $\triangle WHF$ to find $x$. | $\frac{x}{\sin(25°)} = \frac{575}{\sin(103°)}$ |
| Rearrange but don't calculate. | $x = \frac{575\sin(25°)}{\sin(103°)}$ |
| Now use tan in $\Delta THW$ to find $h$. | $\tan(12°) = \frac{h}{x}$ |
| Rearrange. | $h = x\tan(12°)$ |
| Substitute $x$ and calculate. | $= \frac{575\sin(25°)\tan(12°)}{\sin(103°)}$<br>$= 53.0110...$ |
| Round and write the answer. | The transmission tower is about 53 m high. |

# EXERCISE 3.06 Applications of trigonometry

## Concepts and techniques

1 Example 16 Find the gradient of the line whose inclination with the horizontal makes an angle of

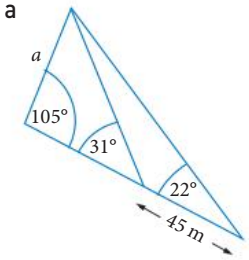
**a** 25° **b** 54° **c** 75.74° **d** 14° 23′ **e** 68° 54′

2 Find the inclination to the horizontal, correct to 1′, of lines with gradient

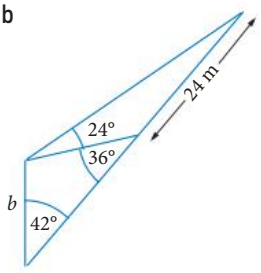
a 0.5 **b** −1 **c** 2.59 **d** −0.2 **e** 1.6

3 Example 17 Find the marked sides in the following.

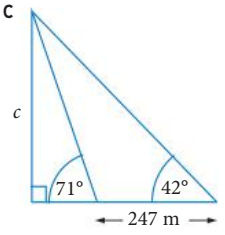
**a**



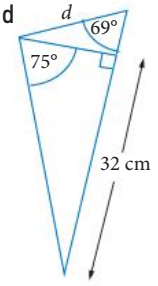

**b**



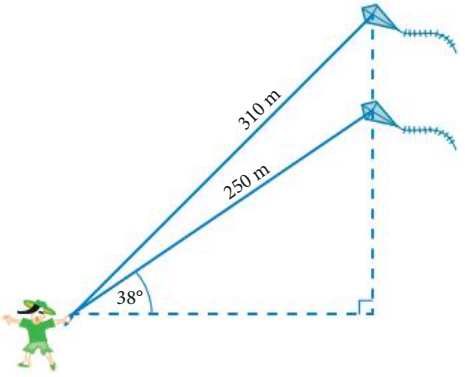

**c**




**d**



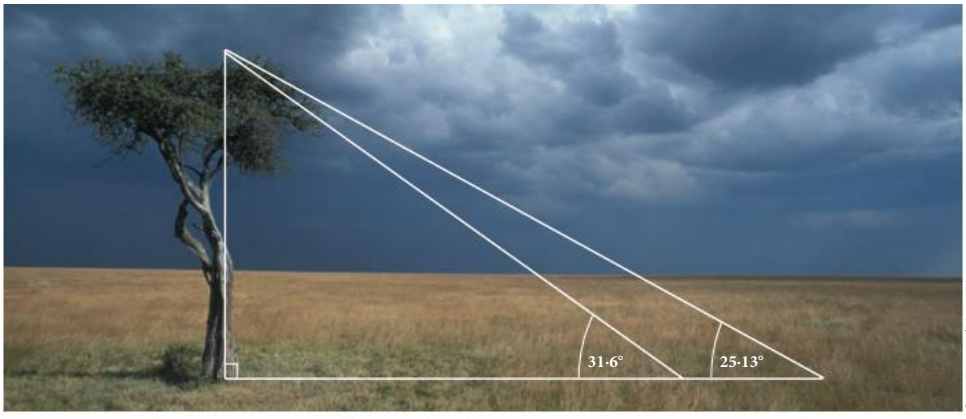

4 A girl is flying a kite and has extended 250 m of string when an updraft makes the kite suddenly rise vertically and she has to let out another 60 m of string. If the angle of elevation to the kite was 38° before the updraft, how far has the kite been driven vertically?





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**5** From a distance, the angle of elevation to the top of a tree is 25.13°. Ten metres closer, the angle of elevation is 31.6°. How high is the tree?



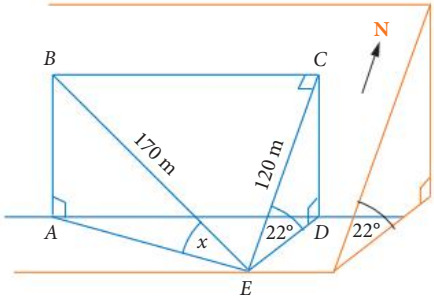
Getty Images/Martin Harvey

**6** Example 18 From a lighthouse 150 m above sea level, the lighthouse keeper observes a boat due east at an angle of depression of 25° and another boat due south at an angle of depression of 32°.

**a** Find the distance that separates the boats.

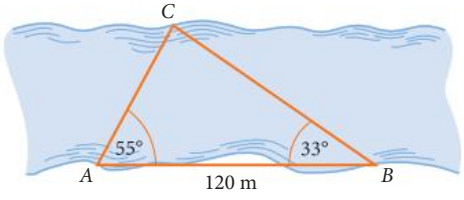
**b** Find the bearing of the second boat from the first.

**7** A hillside is considered to have a uniform slope of 22° with the horizontal as shown in the diagram on the right. A hiker, starting from point $E$, walks due north up the hillside for 120 m to point $C$, then walks due west to point $B$. The hiker then walks 170 m back to the starting point. Calculate the angle of inclination of $BE$ to the horizontal and the total distance covered by the hiker.




## Reasoning and communication

**8** Two points, $A$ and $B$, on the same bank of a river are 120 m apart. Point $C$ is on the opposite bank. $\angle CAB$ and $\angle CBA$ are measured to be 55° and 33° respectively. Find the width of the river.




**9** Sophie and Will set out from point $A$ at the same time. Sophie travels at 30 km/h along a straight road in the direction 042°. Will travels at 24 km/h along another straight road in the direction 142°. Find their distance apart after 4 hours.

**10** Miguel is 90 m due west of Consuella on a level road. He sees that the angle of elevation of a kite flown by their son is 30°, while Consuella sees that it is due north at an angle of elevation of 38°. What is the height of the kite?

# 3.07 CIRCULAR MEASURE

Angles in triangles are normally measured in degrees.

**Circular measure** uses **radians** to measure angles instead of degrees. Radians are easier to use with circles, trigonometric graphs and calculus, which you will study later in this course.

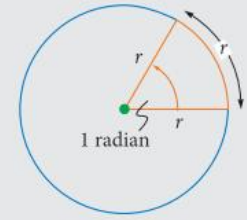


Photo courtesy Margaret Grove

**IMPORTANT**

One radian ($1^c$ or 1 rad) is the angle at the centre of a circle produced by an arc that is equal in length to the radius of the circle.

While the symbol ° shows degrees, it is not *necessary* to use the symbol $^c$ for radians.




Since the circumference of a circle is $2\pi r$, one complete revolution in radians is $2\pi^c$, so $2\pi^c = 360°$ and $\pi^c = 180°$. You can use this for conversions.

**IMPORTANT**

**Conversions between degrees and radians**

To convert degrees to radians, angle in radians $= \left(\text{angle in degrees} \times \frac{\pi}{180}\right)^c$

To convert radians to degrees, angle in degrees $= \left(\text{angle in radians} \times \frac{180}{\pi}\right)^\circ$

$\frac{\pi}{2} = 90°$, $\frac{\pi}{3} = 60°$, $\frac{\pi}{6} = 30°$ and $\frac{\pi}{4} = 45°$


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## Example 19

Convert

a $\frac{3\pi}{2}$ to degrees

b 1.145 to degrees (to 1 decimal place)

c 210° to radians (exact value)

d 50° to radians

i in terms of π ii to 2 decimal places.

### Solution

a Substitute 180° for π and simplify.

$$\frac{3\pi^c}{2} = \frac{3\times 180^\circ}{2}$$
$$= 270^\circ$$

b Use the conversion

angle in degrees $= \left(\text{angle in radians} \times \frac{180}{\pi}\right)^\circ$ and simplify.

$$1.145^c = 1.145 \times \frac{180^\circ}{\pi}$$
$$\approx 65.6^\circ$$

c Use the conversion

angle in radians $= \left(\text{angle in degrees} \times \frac{\pi}{180}\right)^c$ and simplify.

$$210^\circ = 210 \times \frac{\pi^c}{180}$$
$$= \frac{210\pi^c}{180}$$
$$= \frac{7\pi^c}{6}$$

d Use the conversion

angle in radians $= \left(\text{angle in degrees} \times \frac{\pi}{180}\right)^c$.

$$50^\circ = 50 \times \frac{\pi^c}{180}$$

i Simplify the fraction.

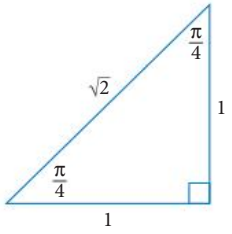
$$= \frac{50\pi^c}{180}$$
$$= \frac{5\pi^c}{18}$$

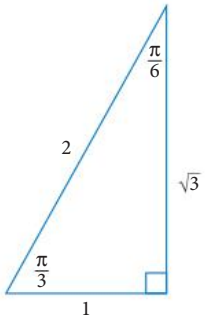
ii Evaluate to 2 decimal places.

$$\approx 0.87^c$$

You can find trigonometric ratios on your calculator when angles are in radians by changing the mode from degrees to radians. For example, $\cos 0.7^c = 0.764\,842\,18728...$

You can also find exact ratios from the special triangles that you investigated earlier in the chapter, using radians rather than degrees.




WS Converting degrees and radians

**IMPORTANT**

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{4}\right) = 1 \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

It is better to use the rationalised forms of the exact values ($\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{3}$) than those with the square roots in the denominator.

## Example 20

**a** Find the exact value of

i $\tan\left(\frac{\pi}{3}\right)$ ii $\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$

**b** CAS Evaluate to 2 decimal places.

i $\cos\left(\frac{\pi}{11}\right)$ ii $\sin(1.03)$

### Solution

**a i** Use exact trigonometric ratios to evaluate the expression.

$$\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

**ii** Use exact trigonometric ratios to evaluate the expression.

$$\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$$
$$=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}+\frac{1}{2}\times\frac{\sqrt{2}}{2}$$

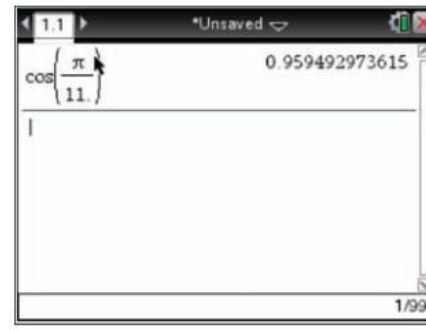
Simplify.

$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$

**b i** **TI-Nspire CAS**

Choose a calculator page, press doc, choose 7:Settings and status and 2: Document settings and check that the angle is set to radians. Then press trig, choose cos, press π. choose π and press ÷11.enter.

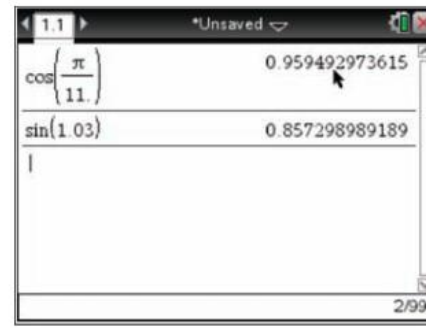
The decimal point after the 11 will force a decimal answer.




Write the answer.

$$\cos\left(\frac{\pi}{11}\right)\approx 0.96$$

**ii** Do as above but select sin. Since the angle includes a decimal point it automatically produces a decimal answer.



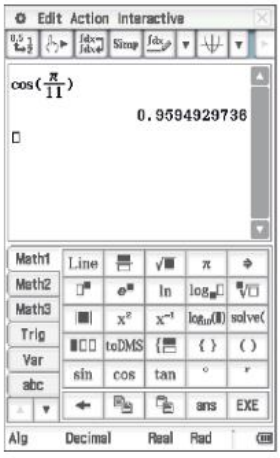

Write the answer.

$$\sin(1.03)\approx 0.86$$


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**b i** **ClassPad**

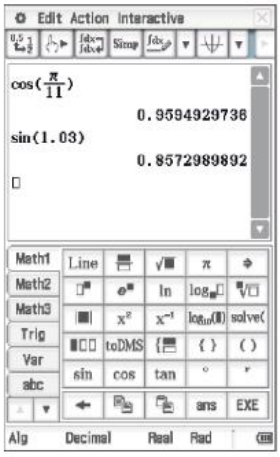
Use the √α Main menu and make sure that under the screen it shows **Decimal** (not **Standard**) and **Rad** (not **Deg**) bottom right. Press Keyboard. Either tap cos( from the Catalog tab or tap cos from the Math1 or Math2 or Trig tab.
From any tab listed above, tap ▣, π is also available from the associated keyboards.
Tap and use the arrows to enter $\frac{\pi}{11}$ and close the brackets.
Tap EXE or press EXE.




Write the answer.

$\cos\left(\frac{\pi}{11}\right) \approx 0.96$

**ii** Do as above but select or type sin.




Write the answer.

$\sin(1.03) \approx 0.86$

The ASTC rule for angles in degrees works the same way for radians.

**IMPORTANT**

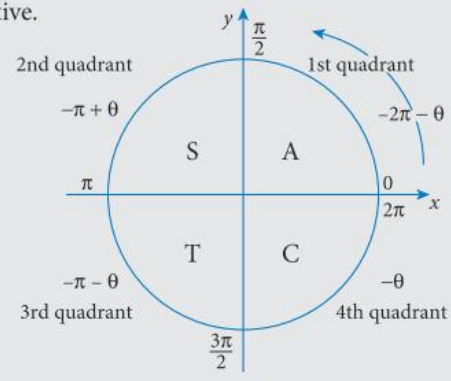
In the first quadrant All trigonometric ratios are positive.

In the second quadrant only Sine is positive.

In the third quadrant only Tangent is positive.

In the fourth quadrant only Cosine is positive.

The sign of the result depends on the quadrant of the original angle



## Example 21

Find the exact value of

**a** $\cos\left(\frac{11\pi}{6}\right)$ **b** $\sin\left(\frac{4\pi}{3}\right)$ **c** $\tan\left(\frac{-3\pi}{4}\right)$

### Solution

**a** State the quadrant of $\frac{11\pi}{6}$ and the rule for cosine in that quadrant.

$\frac{11\pi}{6}$ is in the 4th quadrant.

$\cos(2\pi - \theta) = \cos(\theta)$

Write in terms of an acute angle.

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{6}\right)$$

Evaluate.

$$= \frac{\sqrt{3}}{2}$$

**b** State the quadrant of $\frac{4\pi}{3}$ and the rule for sine in that quadrant.

$\frac{4\pi}{3}$ is in the 3rd quadrant.

$\sin(\pi + \theta) = -\sin(\theta)$

Write in terms of an acute angle.

$$\sin\left(\frac{4\pi}{3}\right) = \sin\left(\pi + \frac{\pi}{3}\right)$$

$$= -\sin\left(\frac{\pi}{3}\right)$$

Evaluate.

$$= -\frac{\sqrt{3}}{2}$$

**c** State the quadrant of $\frac{-3\pi}{4}$ and the rule for tangent in that quadrant.

$\frac{-3\pi}{4}$ is in the 3rd quadrant.

$\tan(-\pi + \theta) = \tan(\theta)$

Write in terms of an acute angle.

$$\tan\left(\frac{-3\pi}{4}\right) = \tan\left(-\pi + \frac{\pi}{4}\right)$$

$$= \tan\left(\frac{\pi}{4}\right)$$

Evaluate.

$$= 1$$


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# EXERCISE 3.07 Circular measure

## Concepts and techniques

**1** Example 19 Change into degrees, rounding to 1 decimal place where necessary.

**a** $\frac{\pi}{5}$ **b** $\frac{2\pi}{3}$ **c** $\frac{5\pi}{4}$ **d** $\frac{7\pi}{6}$

**e** $3\pi$ **f** $\frac{7\pi}{9}$ **g** $\frac{4\pi}{3}$ **h** $\frac{7\pi}{3}$

**i** $\frac{\pi}{9}$ **j** $\frac{5\pi}{18}$ **k** 1.09 **l** 0.768

**m** 1.16 **n** 0.99 **o** 0.32 **p** 3.2

**q** 2.7 **r** 4.31 **s** 5.6 **t** 0.11

**2** Convert into radians in terms of $\pi$.

**a** 135° **b** 30° **c** 150° **d** 240° **e** 300°

**f** 63° **g** 15° **h** 450° **i** 225° **j** 120°

**3** Convert to radians, rounding to 2 decimal places.

**a** 56° **b** 68° **c** 127° **d** 289°

**e** 312° **f** 18°34′ **g** 35°12′ **h** 101°56′

**i** 88°29′ **j** 50°39′

**4** Example 20 Find the following, correct to 2 decimal places.

**a** sin (0.342) **b** cos (1.5) **c** tan (0.056)

**d** cos (0.589) **e** tan (2.29)

**5** Example 21 Find the exact value of each ratio.

**a** $\tan\left(\frac{3\pi}{4}\right)$ **b** $\cos\left(\frac{4\pi}{3}\right)$ **c** $\tan\left(\frac{2\pi}{3}\right)$ **d** $\sin\left(\frac{5\pi}{4}\right)$ **e** $\tan\left(\frac{7\pi}{6}\right)$

**f** $\sin\left(\frac{9\pi}{4}\right)$ **g** $\tan\left(\frac{7\pi}{3}\right)$ **h** $\cos\left(\frac{11\pi}{4}\right)$ **i** $\tan\left(\frac{19\pi}{6}\right)$ **j** $\sin\left(\frac{10\pi}{3}\right)$

**6** CAS Find the value of each of the following, correct to 3 decimal places.

**a** $\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right)$ **b** $\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right)$

**c** $\frac{2\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{4}\right)}$ **d** $1 + \tan^2\left(\frac{\pi}{6}\right)$

**e** $\frac{1-\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)}$ **f** $\sin^2\left(\frac{\pi}{4}\right) - 1$

## Reasoning and communication

**7** Simplify

**a** $\sin(\pi - t) + 2\sin(-(2\pi - t))$

**b** $4\cos(2\pi - x) - 3\cos(-x)$

**c** $\tan(\pi - \theta) + \tan(\pi + \theta)$

**d** $\cos(\pi + \theta) - 2\cos(-\pi - \theta)$

**e** $\sin(2\pi - p) + 3\sin(-p)$

**8** Show that $\sin\left(\pi + \frac{\pi}{4}\right) = \sin(\pi)\cos\left(\frac{\pi}{4}\right) + \cos(\pi)\sin\left(\frac{\pi}{4}\right)$.

**9** Show that $\cos\left(\frac{2\pi}{3}\right) = \cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$.

**10** Prove that $\tan\left(\pi + \frac{\pi}{3}\right) = \frac{\tan(\pi) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{3}\right)}$.

# 3.08 RADIANS AND THE CIRCLE

When radians are used, finding the length of an **arc** in a circle is simple.

Compare the ratios of arc length to circumference, and angle size to angle of revolution.

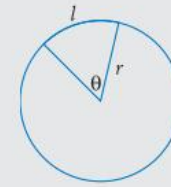
$$\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle } \theta}{\text{angle of revolution}}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

This gives $l = r\theta$

**IMPORTANT**

The length of an arc $l$ that subtends an angle of $\theta$ at the centre of a circle, radius $r$, is given by $l = r\theta$.




### Example 22

**a** Find the exact length of the arc formed by an angle of $\frac{\pi}{3}$ in a circle of radius 6 cm.

**b** Find the angle subtended at the centre of a circle of radius 5.4 cm if it cuts off a portion of arc 3.6 cm long (to 2 decimal places).

**c** Find the length of the arc formed by an angle of 45° in a circle of radius 2 cm.

### Solution

**a** Substitute the information into the arc formula and simplify.

$l = r\theta$
$= 6 \times \frac{\pi}{3}$
$= 2\pi$

State the result.

The length of the arc is $2\pi$ cm.


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**b** Substitute the information into the arc formula.

$l = r\theta$

$3.6 = 5.4\theta$

Divide both sides by 5.4.

$\theta = 0.6666\ldots$

State the result.

The angle at the centre of the circle is 0.67.

**c** Convert the angle into radians.

$45° = 45° \times \frac{\pi}{180°} = \frac{\pi}{4}$

Substitute the information into the arc formula and simplify.

$l = r\theta$

$= 2 \times \frac{\pi}{4}$

$= \frac{\pi}{2}$

State the result.

The length of the arc is $\frac{\pi}{2}$ cm.

Comparing the ratios of an area of a **sector** and the angle it makes inside the circle gives:

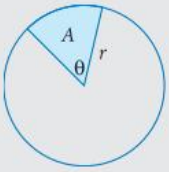
$$\frac{\text{area of sector } A}{\text{area of circle}} = \frac{\text{angle } \theta}{\text{angle of revolution}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

This gives $A = \frac{1}{2}r^2\theta$

**IMPORTANT**

The area of a sector $A$ that subtends an angle of $\theta$ at the centre of a circle, radius $r$, is given by $A = \frac{1}{2}r^2\theta$.




## Example 23

**a** Find the area of a sector formed by an angle of $\frac{\pi}{3}$ in a circle of radius 6 cm.

**b** Find the radius of a circle if the area of the sector is $\frac{49\pi}{2}$ cm$^2$ and the angle subtended at the centre of the circle is $\frac{\pi}{16}$.

### Solution

**a** Substitute the information into the area of a sector formula and simplify.

$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 6^2 \times \frac{\pi}{3}$

$= 6\pi$

State the result.

The area of the sector is $6\pi$ cm$^2$.

**b** Substitute the information into the area of a sector formula and simplify.

$$A = \frac{1}{2}r^2\theta$$
$$\frac{49\pi}{2} = \frac{1}{2}r^2 \times \frac{\pi}{16}$$
$$= \frac{\pi r^2}{32}$$

Multiply both sides by 32 and divide by π.

$$r^2 = 784$$

Evaluate $r$.

$$r = \sqrt{784}$$
$$= 28$$

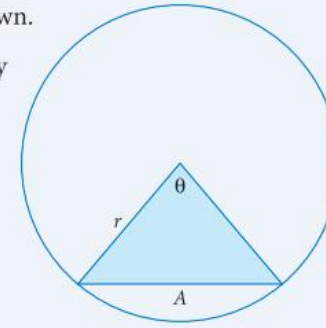
State the result.

The radius of the circle is 28 cm.

You can use the formulas for arc length and area of a sector to solve circle problems.

## INVESTIGATION Area of a segment

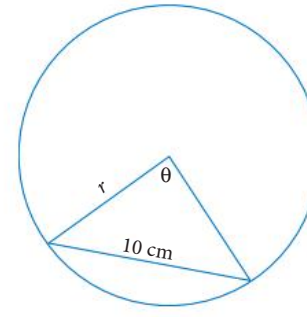
1. Find a formula for the area of the triangle in the circle shown.
2. Write down the formula for the area of the sector cut off by the angle θ in the circle
3. By considering these two formulas, find a formula for the area of the minor segment $A$ in the circle.




When you have to use intermediate values in calculations, keep the full values stored in your calculator memories and use these in the calculations. You should round *answers*, not the intermediate values used to find an answer.

### Example 24

An 80 cm piece of wire is bent into the shape of a circle and a 10 cm piece of wire is joined to it to form a chord as shown.




Find

**a** the length of the radius
**b** the angle θ subtended by this chord at the centre of the circle
**c** the area of the sector
**d** the area of the segment between the chord and the circumference of the circle.


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## Solution

**a** Substitute the given information into the circumference of the circle formula $C = 2\pi r$.

$80 = 2\pi r$

Divide both sides by $2\pi$ and evaluate.

$r = 12.732\ldots$

State the result and store the full answer in your calculator.

The radius of the circle is about 12.7 cm.

**b** Write the cosine rule.

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

Substitute the information.

$$\cos(\theta) = \frac{12.7...^2 + 12.7...^2 - 10^2}{2 \times 12.7... \times 12.7...}$$

Evaluate cos (θ). (Use the full value for $r$ from your calculator.)

$= 0.691\ldots$

Evaluate θ by using $\cos^{-1}$ on the calculator in radian mode.

$\theta = 0.807\ldots$

State the result and store the full value in your calculator.

The angle at the centre of the circle is about 0.81.

**c** Write the formula for the area of a sector. Substitute values using the full value from your calculator.

$$A = \frac{1}{2}r^2\theta$$

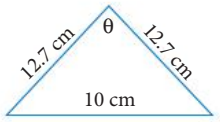
$$= \frac{1}{2} \times 12.7...^2 \times 0.807...$$

$= 65.423\ldots$

State the result.

The area of the sector is about 65.4 $cm^2$.

**d** To find the area of the segment, subtract the area of the triangle from the area of the sector.




Write the area of a triangle formula.

$$A = \frac{1}{2}ab\sin(C)$$

Substitute the exact values from your calculator.

$$= \frac{1}{2} \times 12.7... \times 12.7... \times \sin(0.807...)$$

Evaluate.

$= 58.547\ldots$

Subtract the area of the triangle from the area of the sector.

$A = 65.423... - 58.547...$

$= 6.875...$

State the rounded result.

The area of the segment is about 6.88 $cm^2$.

# EXERCISE 3.08 Radians and the circle

## Concepts and techniques

**1** Example 22 Find the lengths of circular arcs (correct to 2 decimal places where relevant) with the following radii and angles subtended at the centre.

**a** 4 cm, $\pi$ **b** 3 m, $\frac{\pi}{3}$ **c** 10 cm, $\frac{5\pi}{6}$ **d** 3 cm, 30°

**e** 7 mm, 45° **f** 1.5 m, 0.43 **g** 3.21 cm, 1.22 **h** 7.2 mm, 55°

**i** 5.9 cm, 23° 12′ **j** 2.1 m, 82° 35′

**2** Find the radii of circular arcs (correct to 2 decimal places where relevant) with the following lengths and angles subtended at the centre

**a** 27 cm, 3.1 **b** $\frac{3\pi}{2}$ m, $\frac{\pi}{5}$ **c** 18 m, 0.8 **d** $4\pi$ mm, $\frac{\pi}{4}$

**e** 22 m, 4.2 **f** $\frac{5\pi}{7}$, $\frac{2\pi}{7}$ **g** 194 cm, 3.9 **h** 16.4 m, 0.93

**3** Find the angles subtended at the centre (correct to 2 decimal places where relevant) by circular arcs with radii and lengths

**a** 16 m, 28 m **b** 3 cm, $\frac{2\pi}{7}$ cm **c** $\frac{\pi}{3}$ m, $\frac{\pi}{2}$ m **d** 18 cm, 28 cm

**e** 194 mm, 218 mm **f** 9 m, 6 m **g** 19.2 cm, 12 cm **h** $\pi$, $2\pi$

**4** Example 23 Find the areas of sectors (correct to 2 decimal places where relevant) with the following radii and angles subtended at the centre.

**a** 4 cm, $\pi$ **b** 3 m, $\frac{\pi}{3}$ **c** 10 cm, $\frac{5\pi}{6}$ **d** 3 cm, 30°

**e** 7 mm, 45° **f** 1.5 m, 0.43 **g** 3.21 cm, 1.22 **h** 7.2 mm, 55°

**i** 5.9 cm, 23° 12′ **j** 2.1 m, 82° 35′

**5** **a** The area of the sector of a circle is 20 cm$^2$. If the radius is 3 cm, find the angle subtended at the centre of the circle by this sector.

**b** The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is $6\pi$ m$^2$. Find the radius of the circle.

## Reasoning and communication

**6** **a** Example 24 The circumference of a circle is 300 mm. Find the length of the arc that is formed by an angle of $\frac{\pi}{6}$ subtended at the centre of the circle.

**b** A circle with area 60 cm$^2$ has an arc 8 cm long. Find the angle that is subtended at the centre of the circle by the arc.

**c** A circle with circumference 124 mm has a chord cut off it that subtends an angle of 40° at the centre. Find the length of the arc cut off by the chord.

**d** A circle has a circumference of 185 mm. Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre.

**e** If the area of a circle is 200 cm$^2$ and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre, find the area of the sector.


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f Find the area of the sector of a circle with radius 5.7 cm if the length of the arc formed by this sector is 4.2 cm.

g A circle with radius 7 cm has a sector cut off by an angle of 30° subtended at the centre of the circle. Find the following

i arc length of the sector ii area of the sector

**7** The area of a sector is $\frac{3\pi}{10}$ cm$^2$ and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the angle subtended at the centre of the circle and find the radius of the circle.

**8** An angle of $\frac{\pi}{7}$ is subtended at the centre of a circle with radius 3 cm. Find the

a exact arc length

b exact area of the sector

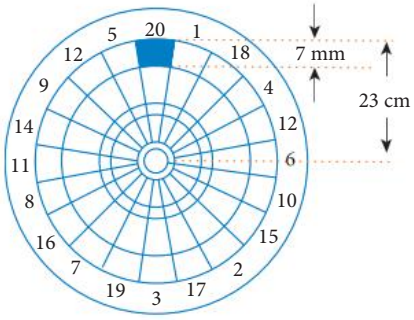
c area of the minor segment, correct to 2 decimal places.

**9** The outer circle on the dartboard on the right has a radius of 23 cm. The shaded area has a height of 7 mm. If a player must hit the shaded area to win, find

a the area of the shaded part, to the nearest square centimetre

b the percentage of the shaded part in relation to the whole area of the dartboard, correct to 1 decimal place

c the perimeter of the shaded area.




**10** The hour hand of a clock is 12 cm long. Find the exact

a length of the arc through which the hand would turn in 5 hours

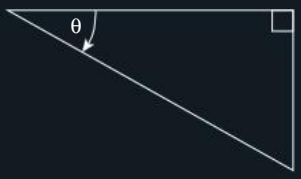
b area through which the hand would pass in 2 hours.

# 3 CHAPTER SUMMARY
# TRIGONOMETRY

- Degrees can be divided into **minutes** and **seconds**. One minute is $\frac{1}{60}$th of a degree.
- Trigonometric ratios: $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$
- The **angle of elevation** of an object from the horizontal (ground) is the angle measured when looking up to the top of an object.



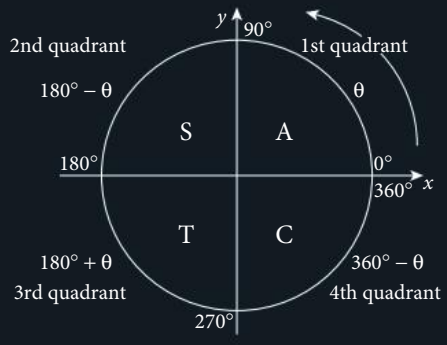

- The **angle of depression** of an object is the angle formed when looking down from a high place to an object below.




- Bearings measure angles clockwise from north.
- Exact angles

| θ | $30° = \frac{\pi}{6}$ | $45° = \frac{\pi}{4}$ | $60° = \frac{\pi}{3}$ |
|---|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

- Angles of any magnitude




Angles of any magnitude can be specified in terms of the unit circle as

$\sin(\theta) = y$, $\cos(\theta) = x$ and $\tan(\theta) = \frac{y}{x}$

In the first quadrant **A**ll trigonometric ratios are positive.
In the second quadrant only **S**ine is positive.
In the third quadrant only **T**angent is positive.
In the fourth quadrant only **C**osine is positive.
The sign of the result depends on the quadrant of the original angle.

- One **radian** is the angle at the centre of a circle produced by an arc equal in length to the radius of the circle. Radians are called **circular measure** of angles and may optionally be shown by a superscript c.




- $\pi^c = 180°$
- To convert degrees to radians, angle in radians $= \left(\text{angle in degrees} \times \frac{\pi}{180}\right)^c$


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- To convert radians to degrees, angle in degrees $= \left(\text{angle in radians} \times \frac{180}{\pi}\right)^\circ$
- The area of a triangle is $A = \frac{1}{2}ab \sin C$.
- Sine rule: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ or $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$
- Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos(C)$ or $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$
- Arc length: $l = r\theta$




- Area of a sector $A = \frac{1}{2}r^2\theta$



# 3 CHAPTER REVIEW
# TRIGONOMETRY

## Multiple choice

1 Example 3 The bearing of a plane from Melbourne is 295°. What is the bearing of Melbourne from the plane?

A 025° B 065° C 115° D 205° E 335°

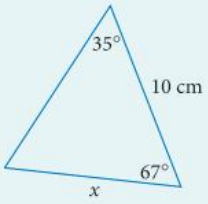
2 Example 4 Which statement is correct?

A $\sin(360° - x) = \sin(x)$ B $\tan(180° - x) = \tan(x)$

C $\cos(360° - x) = -\cos(x)$ D $\tan(180° + x) = \tan(x)$

E $\sin(180° - x) = -\sin(x)$

3 Example 10 Which is the correct statement for the diagram below?




A $\frac{x}{\sin(35°)} = \frac{10}{\sin(67°)}$ B $\frac{x}{\sin(35°)} = \frac{10}{\sin(78°)}$

C $\frac{x}{\sin(67°)} = \frac{10}{\sin(78°)}$ D $\frac{x}{\sin(67°)} = \frac{10}{\sin(35°)}$

E $\frac{x}{\sin(78°)} = \frac{10}{\sin(35°)}$

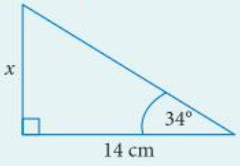
4 Example 21 The exact value of $\sin\left(\frac{5\pi}{6}\right)$ is

A $-\frac{1}{2}$ B $-\frac{\sqrt{3}}{2}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{2}$ E $\frac{\sqrt{2}}{2}$

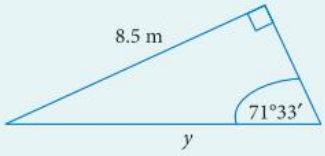
## Short answer

5 Example 1 Evaluate θ to the nearest degree, and the marked sides to 2 significant figures.

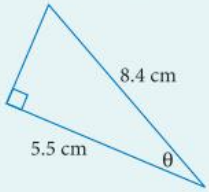
a




b




c




6 Example 2 The angle of depression from the top of a cliff down to a boat that is 25.8 m out from its base is 21°. How high is the cliff?

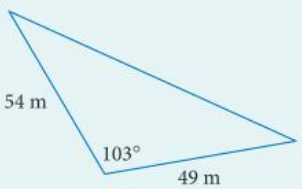
7 Example 3 Jamie drives home from work. He drives 8.7 km west along one road and then turns and drives south for 11.3 km along another road until he reaches his home. What is the bearing of his home from work?

8 Examples 4, 5 and 6 Find the exact value of

a $\tan(150°)$ b $\sin(-60°)$ c $\cos(210°)$

d $\sin(495°)$ e $\cos(-225°)$

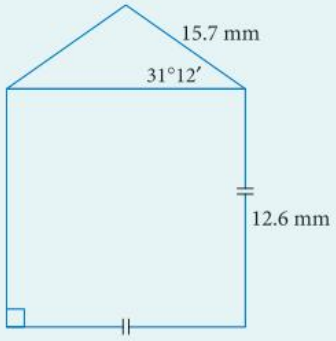
9 Example 7 Find the area of the triangle to the nearest $m^2$.





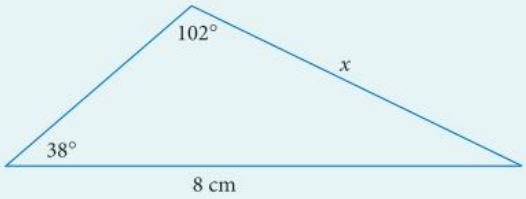
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10 Example 8 Find the area of the figure below.




11 Example 9 A vase is in the shape of a triangular prism with an equilateral triangle on it base. The sides of the base are 8 cm. Find the volume of water in the vase if it is filled to a depth of 18 cm.

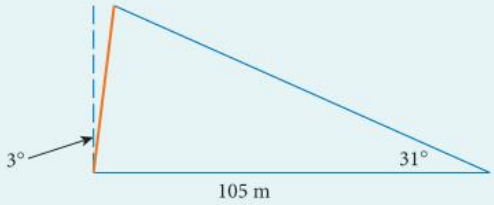
12 Example 10 Evaluate $x$.



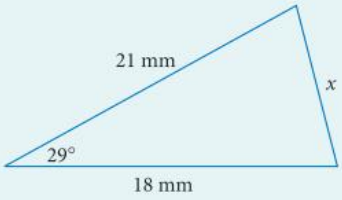

13 Example 11 Evaluate θ correct to the nearest minute.



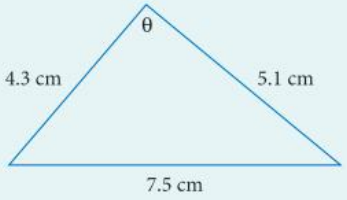

14 Example 12 A pole is 3° from the vertical. The angle of elevation of the top from a point 105 m away is 31°. Find the height of the pole.




15 Example 13 Evaluate $x$ to 1 decimal place.



16 Example 14 Find the value of θ to 1 decimal place.



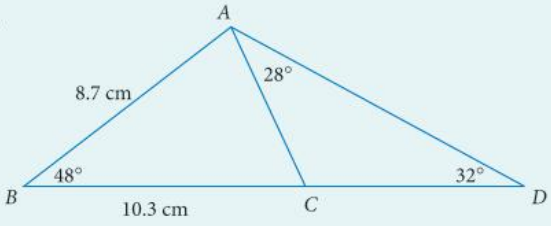

17 Example 15 Two goal posts are 7.32 m apart. A football player stands 15.8 m away from one post and 11.7 m from the other. Through what angle would the player have to kick the ball to score a goal?

18 Example 16 Find the gradient of a ramp that has an angle of inclination of 12°.

19 Example 17 In the figure below, $AB = 8.7$ cm, $BC = 10.3$ cm, $\angle ABC = 48°$, $\angle CAD = 28°$ and $\angle CDA = 32°$.

a Find the length of $AC$ to 1 decimal place.

b Find the length of $CD$ to 1 decimal place.




20 Example 17 A farmer measures the angle of elevation to the top of a tree from where she is standing as 20°. She walks 5 m closer to the tree and measures the angle of elevation as 26°. Find the height of the tree to the nearest metre.

21 Example 19 a Convert to radians, giving the exact value.

i 15° ii 80° iii 170° iv 315° v 270°

b Convert to radians, rounding to 2 decimal places.

i 57° ii 108° iii 73° iv 59.7° v 102° 23′

c Convert to degrees, rounding to the nearest degree where necessary.

i $\frac{\pi}{7}$ ii $\frac{3\pi}{10}$ iii 0.74 iv 3.8 v 4.9

22 Example 20 Evaluate to 2 decimal places.

a $\sin\left(\frac{\pi}{8}\right)$ b $\tan\left(\frac{8\pi}{7}\right)$ c $\cos(1.6)$

23 Example 21 Find the exact value of

a $\sin\left(\frac{8\pi}{3}\right)$ b $\cos\left(\frac{3\pi}{4}\right)$ c $\sin\left(-\frac{5\pi}{4}\right)$

d $\tan\left(\frac{9\pi}{4}\right)$ e $\cos\left(\frac{11\pi}{6}\right)$


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**24** Examples 23, 24 The area of a circle is $\frac{25\pi}{9}$ cm$^2$. An arc and sector are cut off by an angle of $\frac{2\pi}{3}$ subtended at the centre of the circle. Find:

**a** the length of the arc **b** the area of the sector

**25** Example 22 An arc 11.7 cm long is cut off by an angle of 61° at the centre of a circle. Find the radius of the circle.

**26** Example 23 Find the area of a sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre of a circle with radius 2 cm.

## Application

**27** **a** By considering the area of a sector in a circle and the area of a triangle, show that the formula for the area of a segment is given by $A = \frac{1}{2}r^2(\theta - \sin\theta)$ where $r$ is the radius of the circle and $\theta$ is the angle in radians subtended at the centre.

**b** Find the exact area of the segment cut off by an angle of $\frac{\pi}{3}$ in a circle of radius 4 cm.

**28** From the top of a pole, the angle of depression of a man standing out from the foot of the pole is 43°. On the other side of the pole is another man, and the angle of depression from the top of the pole to this man is 52°. The men are standing 58 m apart. Find the height of the pole to the nearest metre.

**29** Sharon's eyes are 150 cm above the ground. When she was standing on the beach, the edge of the water was at an angle of depression of 10°. After walking down the 5° slope of the beach 10 m towards the water, the angle of depression increased to 17°. How much further would she have to walk to reach the water?

**30** Tim was lying on the edge of one side of the roof top of a building in the city looking down to the base of a building on the other side of the street. With his eyes at the edge, he observed that the angle of depression of the point directly opposite was 31°. He carefully moved to the other end of the 80 m wide building and looked at an angle to the same point. The angle of depression this time had decreased to 19°. How wide was the street (the distance between the buildings)?

Practice quiz





# 4

## TERMINOLOGY

asymptote
axis of symmetry
codomain
concave
dependent variable
dilation
discontinuous
discriminant
domain
even
function
gradient
hyperbola
independent variable
intercept
linear function
midpoint
odd
parabola
point of inflection
quadratic function
range
relation
translation
turning point
zeros

Prior learning

ALGEBRA, FUNCTIONS AND GRAPHS 1

# FUNCTIONS AND GRAPHS


4.01 Functions and relations
4.02 Further properties of functions
4.03 Properties of linear functions
4.04 Equations of linear functions
4.05 Quadratic functions
4.06 Quadratic equations and the quadratic function
4.07 Graphs of functions
4.08 Graphs of relations
Chapter summary
Chapter review

## LINES AND LINEAR RELATIONSHIPS

- determine the coordinates of the midpoint of two points (ACMMM001)
- recognise features of the graph of $y = mx + c$ including its linear shape, its intercepts and its slope or gradient (ACMMM003)
- find the equation of a straight line given sufficient information; parallel and perpendicular lines (ACMMM004)

## REVIEW OF QUADRATIC RELATIONSHIPS

- recognise features of the graphs of $y = x^2$, $y = a(x - b)^2 + c$, and $y = a(x - b)(x - c)$, including their parabolic shapes, turning points, axes of symmetry and intercepts (ACMMM007)
- find the equation of a quadratic given sufficient information (ACMMM009)
- turning points and zeros of quadratics and the role of the discriminant (ACMMM010)
- recognise features of the graph of the general quadratic $y = ax^2 + bx + c$. (ACMMM011)

## INVERSE PROPORTION

- features of the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x - b}$, including their hyperbolic shapes, and their asymptotes. (ACMMM013)

## POWERS AND POLYNOMIALS

- recognise features of the graphs of $y = x^n$ for $n \in N$, $n = -1$ and $n = \frac{1}{2}$, including shape, and behaviour as $x \to \infty$ and $x \to -\infty$ (ACMMM014)
- recognise features of the graphs of $y = x^3$, $y = a(x - b)^3 + c$, and $y = k(x - a)(x - b)(x - c)$, including shape, intercepts and behaviour as $x \to \infty$ and $x \to -\infty$ (ACMMM017)

## GRAPHS OF RELATIONS

- recognise features of the graphs of $x^2 + y^2 = r^2$ and $(x - a)^2 + (y - b)^2 = r^2$, including their circular shapes, their centres and their radii (ACMMM020)
- recognise features of the graph of $y^2 = x$ including its parabolic shape and its axis of symmetry. (ACMMM021)

## FUNCTIONS

- understand the concept of a function as a mapping between sets, and as a rule or formula that defines one variable quantity in terms of another (ACMMM022)
- use function notation, domain and range, independent and dependent variable (ACMMM023)
- understand the concept of the graph of a function (ACMMM024)
- examine translations and the graphs of $y = f(x) + a$ and $y = f(x + b)$ (ACMMM025)
- examine dilations and the graphs of $y = cf(x)$ and $y = f(kx)$ (ACMMM026)
- recognise the distinction between functions and relations and the vertical line test. (ACMMM027)


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# 4.01 FUNCTIONS AND RELATIONS

Many mathematical rules can be expressed as relationships between **variables**. You have seen many of these already. A **relation** is the most general kind of relationship between variables. You often write a relation or function as $y = R(x)$ or $y = f(x)$. The first variable in the ordered pair $(x, y)$ is used to work out the value of the second according to a rule. In this case, the second variable depends on the first one.

**IMPORTANT**

A **relation** $R$ is a set of ordered pairs $(x, y)$, where $x \in X$ and $y \in Y$. $X$ and $Y$ are often both the set of real numbers. The first element of an ordered pair $(x, y)$ in a relation is called the **independent variable** and the second element, $y$, is called the **dependent variable**.

You can show that an ordered pair belongs to a relation $R$ by writing $(x, y) \in R$ or $y = R(x)$.

The set of values of the independent variable $x$ is called the **domain** and is often the set $X$.

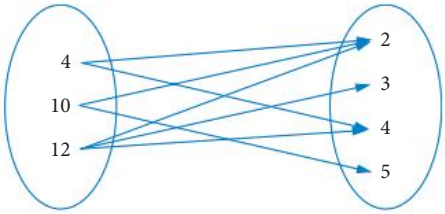
The set $Y$ often includes values that are not in the ordered pairs and is called the **codomain.** The set of values of the dependent variable $y$ is called the **range**.

Most of the relations you study in mathematics are ordered pairs of numbers and many are shown by a rule such as $y = x^2 - 3$ or $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

The second rule could be written as $y = \pm\frac{3}{2}\sqrt{4 - x^2}$ but the first form is more common.

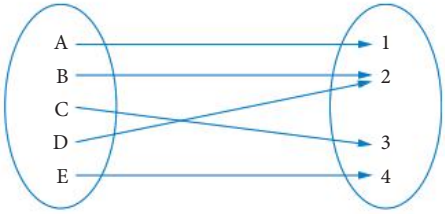
You can also show a relation with a mapping. The diagram below shows the mapping of the ordered pairs (4, 2), (4, 4), (10, 2), (10, 5), (12, 2), (12, 3) and (12, 4).

The sets $X = \{4, 10, 12\}$ and $Y$ are sets of natural numbers. The domain is $\{4, 10, 12\}$, the codomain could be the natural numbers and the range is $\{2, 3, 4, 5\}$.




Relations are also shown as graphs on the Cartesian plane. When drawing a graph from a rule, you usually choose values of $x$ to substitute into an equation. These values are independent since you could choose any value. The $y$ values depend on what we have chosen for $x$.

The mapping opposite shows another relation.




In the second example, the domain is $\{A, B, C, D \text{ and } E\}$ and the range is $\{1, 2, 3, 4\}$.

To find the domain and range of numeric relations, you look for real numbers that will fit the rule. For example, for $y = x^2 - 3$, you could substitute any real number for $x$. So the domain is the set of all real numbers.

Since $x^2 \geq 0$ for all real $x$, $x^2 - 3 \geq -3$.

Since $y = x^2 - 3$, the range is $\{y: y \geq -3\}$.

## Example 1

State the domain and range of each relation.

**a** $(1, 3), (2, 5), (2, 3), (5, 7), (6, 7), (6, 5), (8, -1)$

**b** $y = \sqrt{x-4}$

**c** CAS $y = \dfrac{3}{x^2-1}$

### Solution

**a** The domain is the set of first values and the range is the set of second values of the ordered pairs.

Domain is $\{1, 2, 5, 6, 8\}$.
Range is $\{-1, 3, 5, 7\}$.

**b** The domain and range consist of real numbers, but you can't find the square root of a negative number.

$x - 4 \geq 0$
So $x \geq 4$
The domain is $\{x: x \geq 4\}$.

The square root is always positive or zero.

The range is $\{y: y \geq 0\}$

**c** You can't divide by zero, so $y = \dfrac{3}{x^2-1}$ is undefined if $x^2 = 1$.

$x^2 \neq 1$
$x \neq \pm 1$
So the domain is $\{x: x \neq \pm 1\}$.

The value of $y$ (3 divided by $x^2 - 1$) is never zero.

$y = 0$ is not in the range.

Look at values of $y$ for which $x > 1$.

$x^2 > 1$ so $\dfrac{3}{x^2-1} > 0$
So $y > 0$

Look at values of $y$ for which $x < -1$

$x^2 > 1$
So $y > 0$

Look at values of $y$ for which $-1 < x < 1$

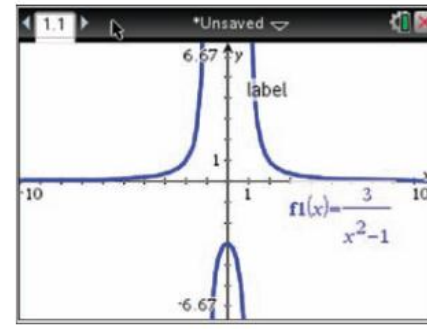
$0 \leq x^2 < 1$ so $\dfrac{1}{x^2-1} \leq -1$

Thus $\dfrac{3}{x^2-1} \leq -3$
So $y \leq -3$

Write the whole range of the function.

The range is $\{y: y \leq -3 \text{ or } y > 0\}$

You can draw the graph on your CAS calculator to find the domain and range more easily.




**TI-Nspire CAS**

Use a graph page.
Enter as f1(x)=3÷(x²−1). The calculator will automatically change this to fraction form.
If you need to trace the function to find the domain and range do the following.
Press menu, then choose 5: Trace and 1: Graph Trace. You can reset the steps using 3: Trace Step.


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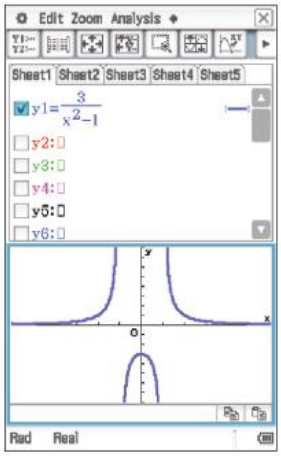
**ClassPad**

Use the **Graph & Table** application, set y1=3÷(x^2−1).

Tap **View Window**. Set $-6 \leq x \leq 6$ and $-8 \leq y \leq 8$.

Tap to sketch the graph.

Tap Resize to see the full graph. You may wish to set new values for $x$ and $y$ using **View Window**.




Use the graph to find the domain and range.

The domain is $\{x: x \neq \pm 1\}$ and the range is $\{y: y \leq -3 \text{ or } y > 0\}$

You have already studied **functions** in previous years. All functions have the rule that each $x$ value gives only one value for the dependent variable $y$.

In Example **1**, parts **b** and **c** are functions, but **a** is not since $x = 2$ has two values for $y$. However, a function can have more than one $y$ value. In Example **1** part **c**, the pairs (2, 1) and (−2, 1) both have the same $y$ value, but different $x$ values.

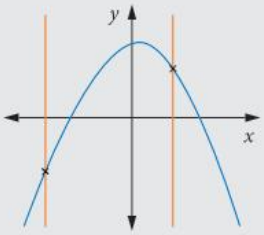
There is a simple test that shows if a graph is a function.

**IMPORTANT**

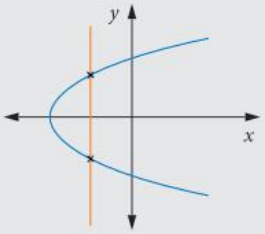
**Functions and the vertical line test**

A **function** is a relation $f$ such that each value in the domain has only one ordered pair.

If a vertical line cuts a graph at most once anywhere along the graph, the graph is a function.




If a vertical line cuts a graph in more than one place anywhere along the graph, the graph is not a function.

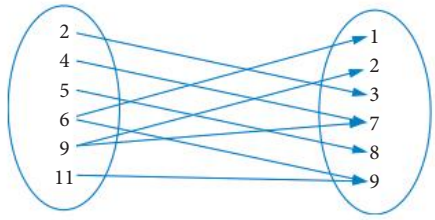



Some functions are not defined for all values of $x$, so a vertical line might not cut the graph at all. To be a function, a vertical line cannot cut the graph more than once.

## Example 2

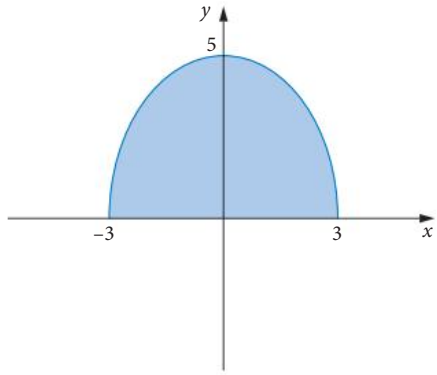
Find the domain and range of each relation and state whether it is a function.

**a**

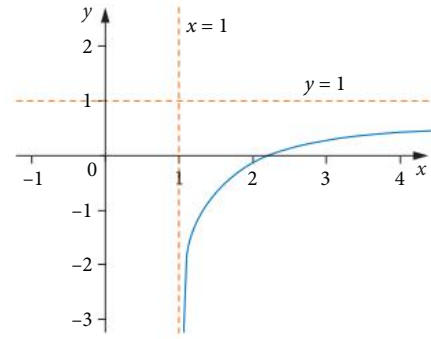



**b** CAS $y = \dfrac{x}{x^2 - 1}$

**c** The shaded region below.




**d**




### Solution

**a** State the domain and range.

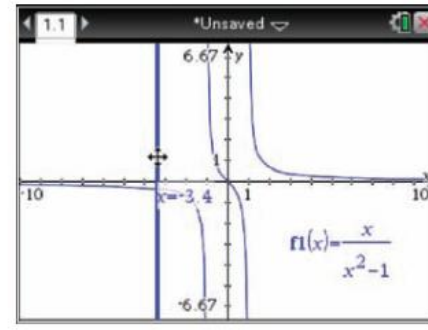
Domain is {2, 4, 5, 6, 9, 11}.
Range is {1, 2, 3, 7, 8, 9}.

Check the number of $y$ values for each $x$ value.

The $x$ values 6 and 9 have two $y$ values, so this mapping is not a function.

**b** **TI-Nspire CAS**

Use a graph page and enter the relation.
Press menu, 3: Graph Entry/Edit, 2: Equation, 1: Line and choose 2: $x=c$.
Put in a value, say −2, then 'grab' the vertical line using the cursor controls and move it back and forth to do the vertical line test.

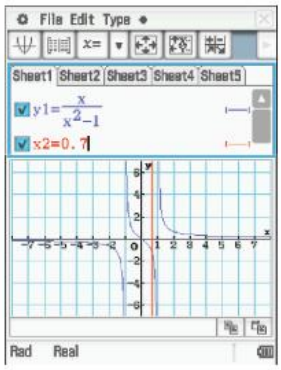




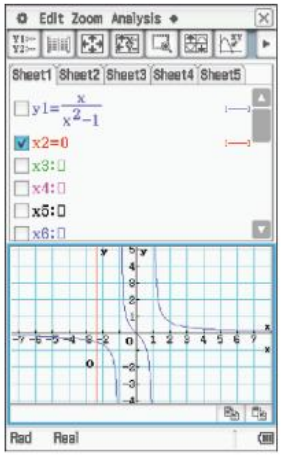
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**ClassPad**

Use the Graph&Table menu.
Use thin lines by tapping [——] and following the prompts.
Enter the equation in y1 and tick the box.
Set View Window suitably (see the screen on the right).
Tap [⊥] to draw the graph.
The vertical grid lines automatically do the vertical line test.
You can test in between points.
Tap ◆ and then Save Picture. Choose a file name to save the picture and Save the current screen.




Clear the graph (Edit, Clear All).
Tap ◆ and Open Picture. Select your file. This will be a permanent background.
Tap next to y2. Tap [▼] (next to [y=]) and select [x=]. Enter x2=0 with a thin line.
Remove the tick from the original graph. Tap Settings ⚙, then Graph Format. Turn Axes and Grid both off. Tap [⊥].
Remember, the original graph is now a permanent background. You can now drag the red vertical line around to do the vertical line test.
When you have finished, set Axes to Number and Grid to Line. Remove the background by tapping ◆ and then Clear Picture.




State the domain and range.

Domain is $\{x: x \neq \pm 1\}$.
Range is $\{y:$ all real $y\}$ or $\{y: y \in \boldsymbol{R}\}$ or $\boldsymbol{R}$.

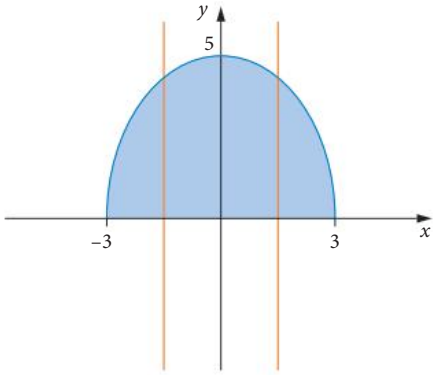
Check if it is a function.

The rule shows only one y-value for each x-value, so it is a function.

**c** State the domain and range.

Domain is $\{-3 \leq x \leq 3\}$.
Range is $\{0 \leq y \leq 5\}$.

Use the vertical line test.

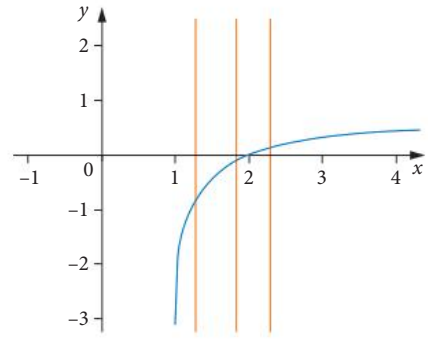



The vertical line cuts the shaded region in more than one place, so it is not a function.

**d** State the domain and range.

Use the vertical line test.

Domain is $\{x: x > 1\}$.
Range is $R$.




The vertical line cuts the graph in at most one place, so it is a function.

WS
Functions and relations

Most graphs that you sketch are functions of **continuous** variables, where $x$ can be any value. Some have **discrete** variables where the $x$ values go in steps such as 1, 2, 3 … and are called functions of a discrete variable.

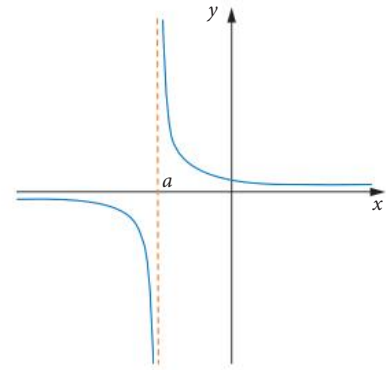
**IMPORTANT**

A **discontinuous** function has a break in the line of its graph.

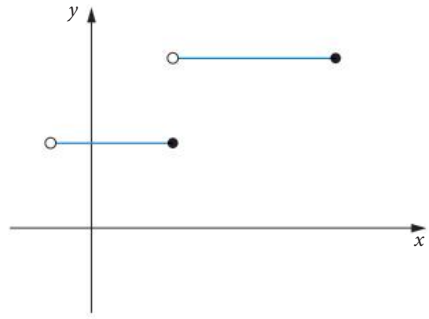
The function has a **discontinuity** at the break. You would have to lift your pencil off the paper when you draw the graph.

The **hyperbola** on the right is an example of a discontinuous function.

The vertical line $x = a$ is called a vertical **asymptote**. It is a line that the curve approaches as $x$ gets close to $a$, but never reaches it. Many graphs have vertical asymptotes at discontinuities.




There are other types of discontinuous functions such as the step graph below.




Remember that an open circle means that the point at that end of the line is not included, but a closed circle means the point is included.

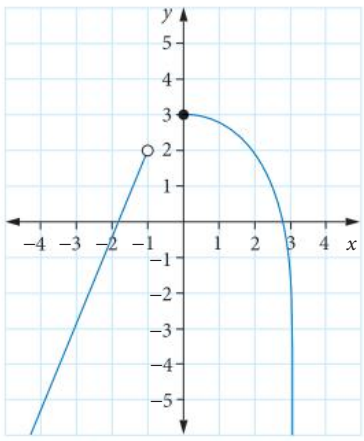
Even though there is a $y$ value for every $x$ value on the graph, there are discontinuities where the graph jumps from one line to another.


9780170250269

## Example 3

**a** A shoe shop checks the sizes that are available for a certain brand of shoe before it orders them. Is the variable of shoe size continuous or discrete?

**b** State the values for which the function shown on the right is discontinuous.




### Solution

**a** Examples of shoe sizes are 4, $4\frac{1}{2}$, 5 and so on.

The variables are discrete.

**b** Look at the domain to find $x$ values for which the function is discontinuous. It cannot be continuous at the breaks.

From the graph, the line has domain $\{x: x < -1\}$ and the curve has domain $\{x: x \geq 0\}$. So the values where the function is discontinuous are $\{x: -1 \leq x < 0\}$

# EXERCISE 4.01 Functions and relations

## Concepts and techniques

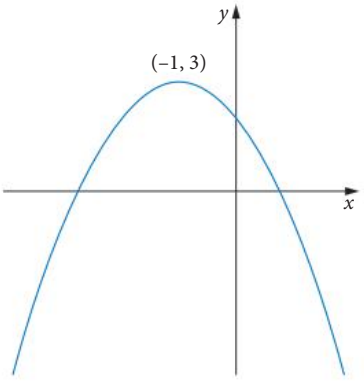
**1** Example 1 State the domain and range of each of the following.

**a** (3, 5), (4, 7), (5, 5), (6, 10), (7, 6)

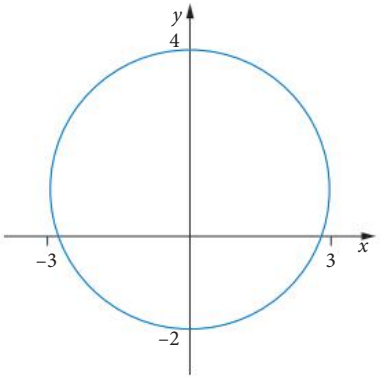
**b** $(A, 3), (B, 2), (C, 7), (D, 5), (E, 2)$

**c** $y = x^2$

**d**




**e**



2 State the domain and range of each of the following.

a $y = 2x + 3$

b $y = 2x + 3$ for $0 \leq x \leq 3$

c $y = \dfrac{1}{x-2}$

d $y = \sqrt{x}$

e $y = 2^x$ where $x$ is a positive integer from 1 to 5

f $y = x^2 - 2x + 1$ where $x$ is an integer between $-3$ and 3 inclusive

g $y = \dfrac{1}{x}$ for $1 \leq x \leq 4$

h $y = x^4 - x$ where $x$ is an integer between 0 and 4 inclusive

i $y = \sqrt{x-1}$

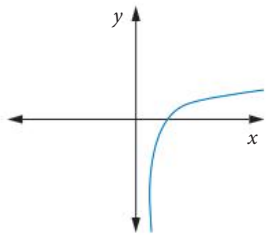
j $y = -\sqrt{x+5}$

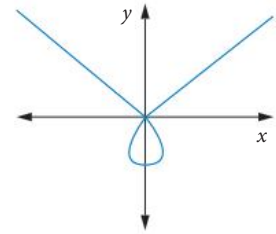
3 CAS Sketch the graphs for each of the following and state the domain and range.

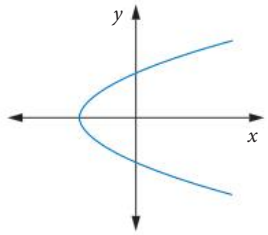
a $y = x^3 - 1$ b $y = 3$ c $y = \dfrac{3}{x^2+1}$

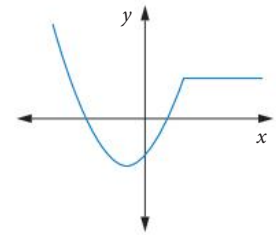
d $y = -\sqrt{9-x^2}$ e $y = \dfrac{2}{x^2-4}$

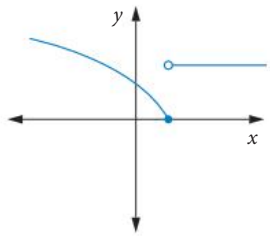
4 Example 2 Which of the following are functions?

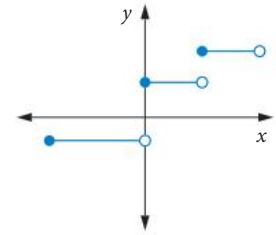
a 

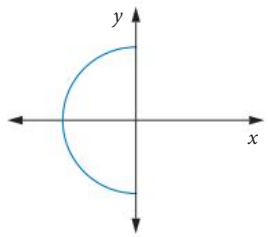

b 

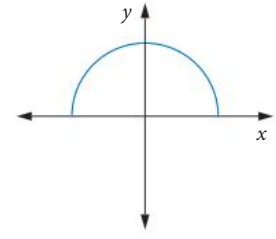

c 


d 


e 


f 


g 

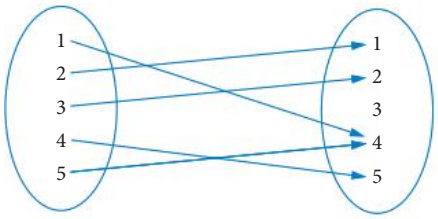

h 


i (1, 3), (2, –1), (3, 3), (4, 0)

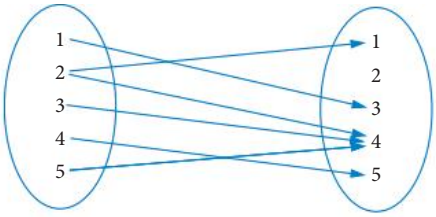
j (1, 3), (2, –1), (2, 7), (4, 0)


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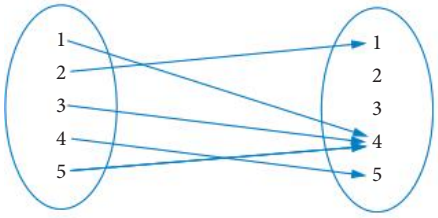
k




l




m




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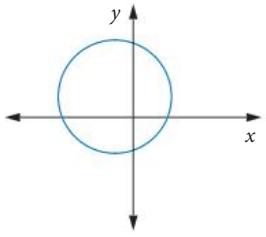
| Name | Ben | Paul | Pierre | Hamish | Jacob | Lee | Pierre | Lien |
|---|---|---|---|---|---|---|---|---|
| Sport | Tennis | Football | Tennis | Football | Football | Badminton | Football | Badminton |

o

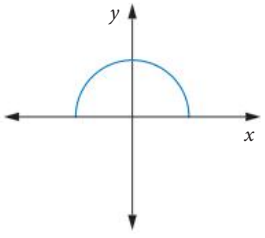
| | |
|---|---|
| A | 3 |
| B | 4 |
| C | 7 |
| D | 3 |
| E | 5 |
| F | 7 |
| G | 4 |

**5** Which of the following are functions?

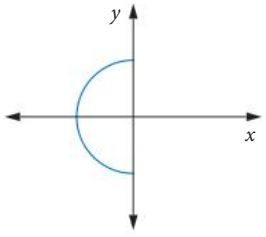
a



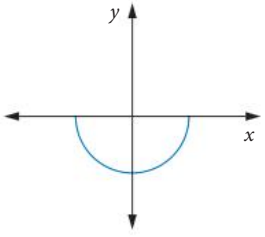

b



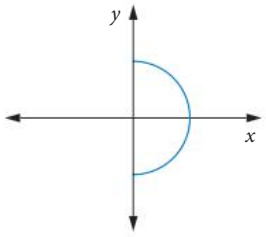

c




d




e



6 Example 3 State whether the following variables are discrete or continuous

a The distance between cities on a map.

b The number of people at the Australian Open tennis championship each day.

c The pulse rate of patients in a hospital.

d The surface area of a hot air balloon as it is blown up.



Photo courtesy Margaret Grove

e Shirt sizes.

f Weight of rocks in a quarry.

g Average monthly temperatures.

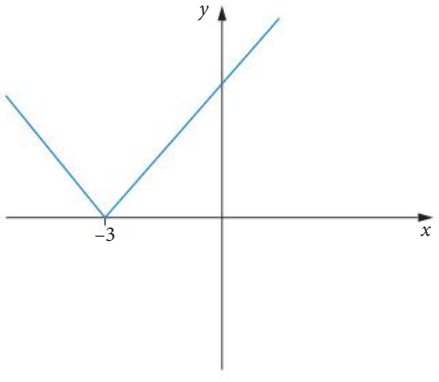
h Daily ticket sales at a cinema.

i Height of students in a class.

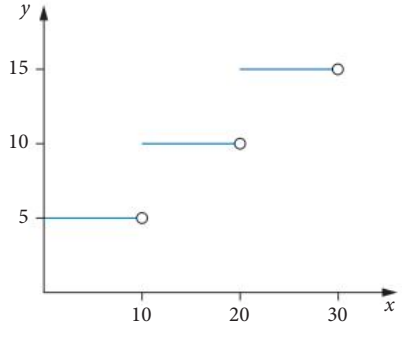
j Number of television shows on pay TV.

7 For each graph, state whether it is continuous or discontinuous. If it is discontinuous, state the $x$ values for which it is discontinuous.

a



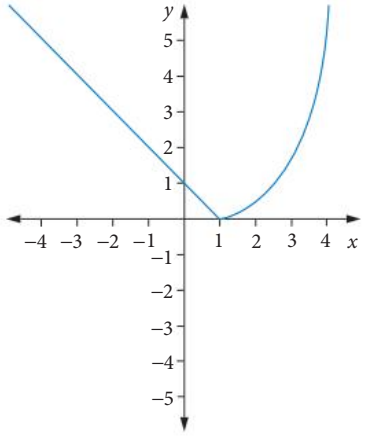

b



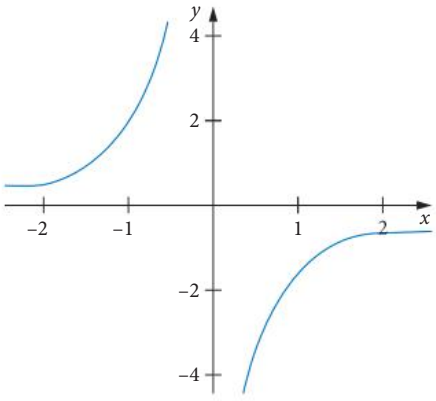


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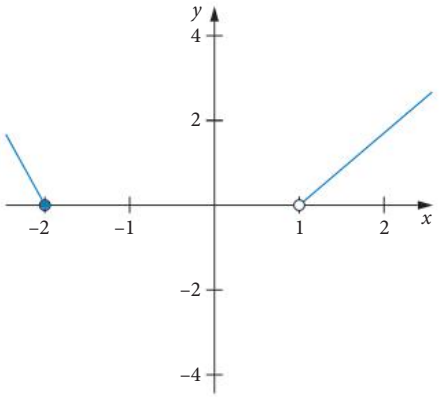
c




d




e




**8** State whether each of the following is a function of a discrete or a continuous variable.



Alamy/Jeremy Graham/dbimages

**a** The number of people entering a sports stadium per minute as a function of the number of times the turnstiles have rotated.

**b** The volume of water passing through a pipe as a function of the time in hours after midnight.

**c** $y = 5x + 7$ where $x$ is a real number ($\boldsymbol{R}$).

**d** $y = 2x^2 + 3x - 9$ where $x$ is an integer ($\boldsymbol{J}$).

e The height of a ball as a function of the time in seconds after it is dropped.
f The distance a runner has completed as a function of the number of steps taken since the start of a race.

## Reasoning and communication

9 a Draw up a table with the name of each person in your class and how many pets each person has.
b Draw a graph of this information
c Is the graph a function?
d Does the graph have continuous or discrete variables?



Photo courtesy Margaret Grove

10 A courier service has the following fees to deliver parcels.
Up to 2 kg: $5.50
Over 2 kg and up to 5 kg: $11.25
Over 5 kg and up to 10 kg: $24.95
Over 10 kg and up to 15 kg: $38.50
a Draw a graph showing this information
b Is the graph a function?
c Is the graph continuous or discontinuous?
d What would be the price for delivering a parcel that weighs
i 8.7 kg? ii 3.1 kg? iii 10 kg?
e State the domain and range for this graph.

# 4.02 FURTHER PROPERTIES OF FUNCTIONS

We usually write functions of single variables using the notation $f(x)$ or $y = f(x)$. You could write $f(x) = 4x^3 - 5$, $g(x) = 2x + 3$ to define the functions $f$ and $g$ in terms of the variable $x$. $g(-6)$ would mean the value of $g$ when $-6$ is substituted for the variable $x$.

## Example 4

Given $f(x) = x^2 - 2x + 1$, find

**a** $f(-1)$ **b** $f(m + h)$ **c** values of $x$ for which $f(x) = 0$

### Solution

**a** Substitute and evaluate.

$$f(-1) = (-1)^2 - 2 \times (-1) + 1$$
$$= 4$$

**b** Substitute and simplify.

$$f(m + h) = (m + h)^2 - 2(m + h) + 1$$
$$= m^2 + 2mh + h^2 - 2m - 2h + 1$$

**c** Put $f(x) = 0$

$$x^2 - 2x + 1 = 0$$

Factorise.

$$(x - 1)^2 = 0$$

Solve the equation.

$$x - 1 = 0$$
$$x = 1$$

State the result

$f(x) = 0$ when $x = 1$

## INVESTIGATION Translations, dilations and reflections

Use your CAS calculator or a computer with graphing software to sketch graphs and explore what effect a parameter (a variable $k$) has on each type of graph below.

Use the functions $f(x) = x, f(x) = x^2, f(x) = x^3, f(x) = \frac{1}{x}$. Your teacher may want you to use particular ones.

You can change the value of $k$ in (say) $f(x) = x^2 + k$. Try values such as $\pm 1, \pm 2, \pm 3, \pm \frac{1}{2}$.

1 Sketch families of graphs of $y = f(x) + k$ where $k$ is a parameter. What effect does the parameter $k$ have on $y = f(x)$?

2 Sketch graphs of $y = f(x + k)$. What effect does $k$ have on $y = f(x)$ in this case?

3 Sketch graphs of $y = kf(x)$. What effect does $k$ have on $y = f(x)$ in this case?

4 Sketch graphs of $y = f(kx)$. What effect does $k$ have on $y = f(x)$ in this case?

5 Compare the graphs of $y = f(x)$, $y = f(-x)$ and $y = -f(x)$ and describe what effect the negative sign has on the function in each case.

### TI-Nspire CAS

Use a graph page.

Enter f1$(x) = x^2$

Go to the vertex and grab the graph when you see the translation symbol ✣.

You can now move the graph up and down or sideways and see how the equation of the graph changes.

Go to another part of the graph and grab it when you see the dilation symbol ⇲. You can now move the graph to see how its equation changes.

### ClassPad

Use the **Graph&Table** application.

The example below is for investigating $y = f(x) + k$

Set y1= f(x)

Set y2= f(x)+1

Set y3= f(x)−1, etc.

Press EXE after entering each function. Do not type f(x). If, for example, $f(x) = x^2$, then enter y2=x^2+1, etc.

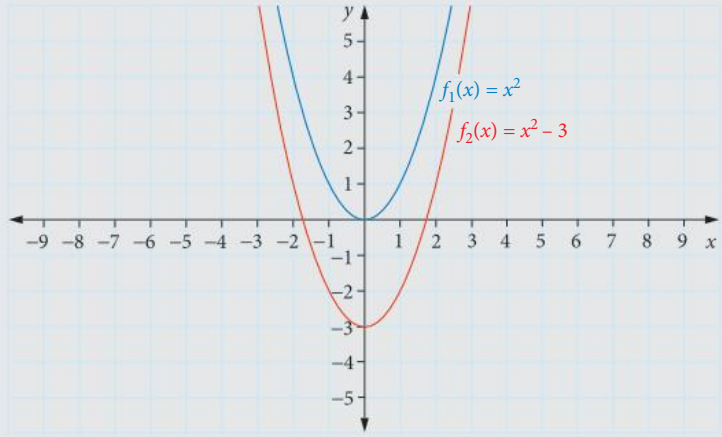
Use **View Window** to set the domain and range for all graphs.

Tap to see all the graphs displayed.

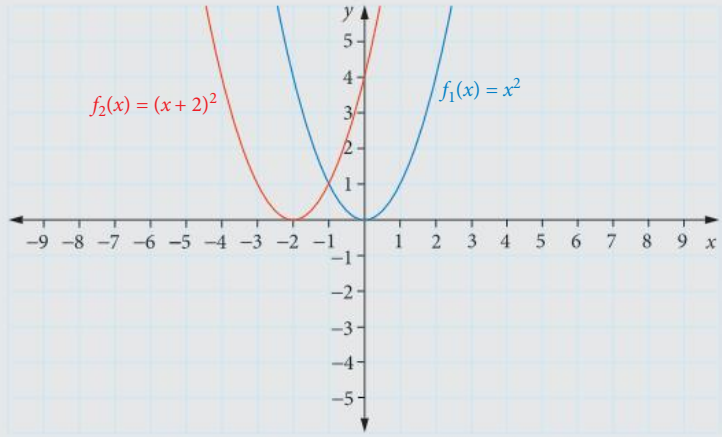
If you don't want to see all the graphs at once, just tick the ones you want displayed.

## IMPORTANT

For the family of functions $y = f(x) + a$, as $a$ varies, the graph moves up or down (**vertical translation**).



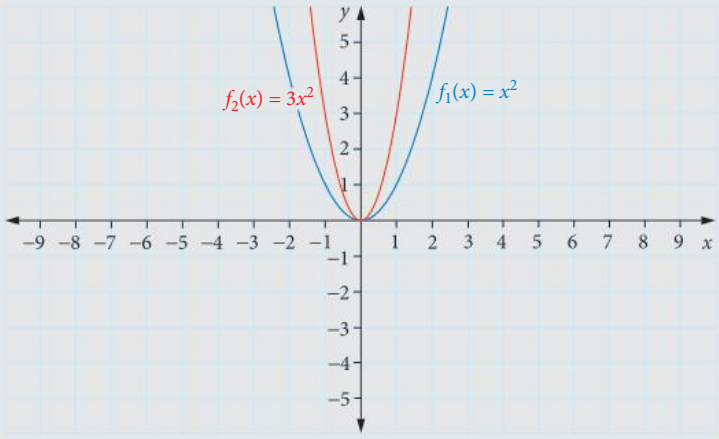

For the family of functions $y = f(x + b)$, as $b$ varies, the graph moves left or right (**horizontal translation**).



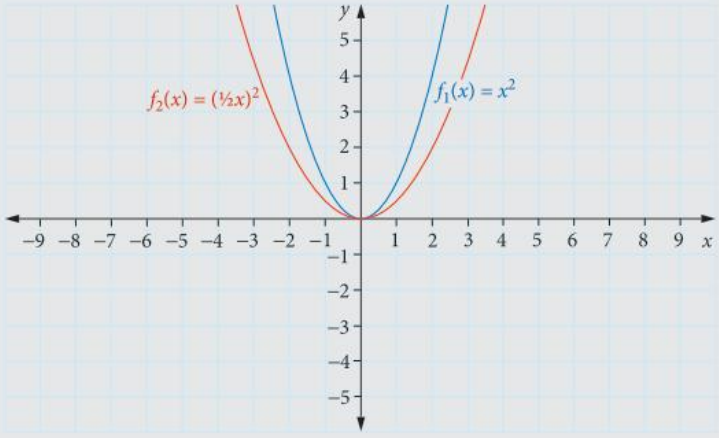


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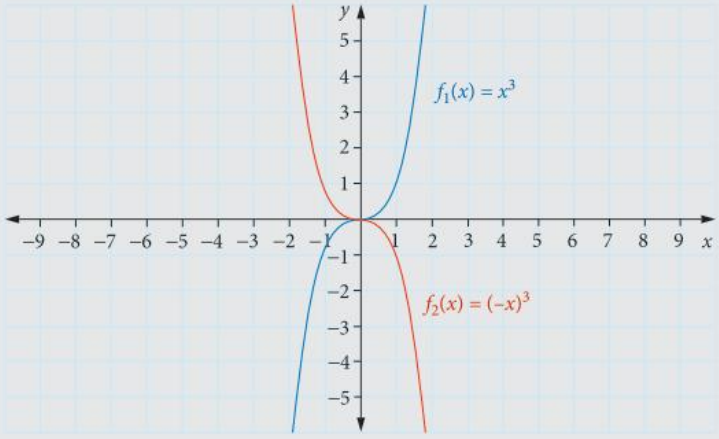
For the family of functions $y = cf(x)$ as $c$ varies, the function stretches or shrinks vertically. This is called (vertical) **dilation**. If $c > 1$, it is stretched vertically so that it looks thinner and if $c < 1$ it is compressed vertically so that it looks fatter.



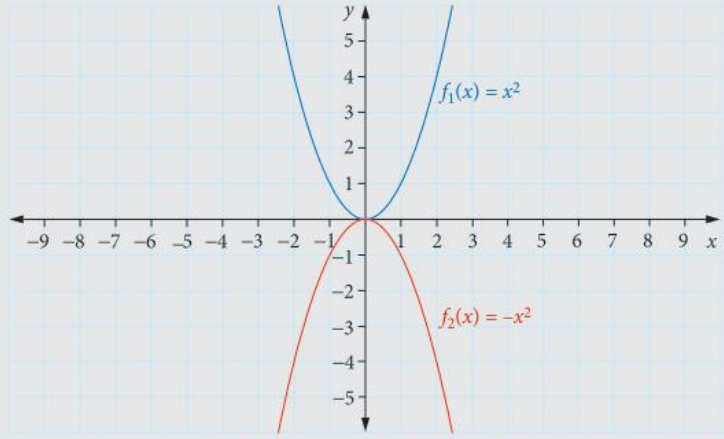

Horizontal dilation occurs with functions $y = f(dx)$. If $d > 1$, it compresses horizontally and if $d < 1$ it expands horizontally.




The graph of $y = f(-x)$ is a reflection of $y = f(x)$ in the $y$-axis.



The graph of $y = -f(x)$ is a reflection of $y = f(x)$ in the $x$-axis.




## Example 5

**a** Describe how the graph of $g(x) = x^3 - 2x + 7$ differs from $f(x) = x^3 - 2x$

**b** Describe how the graph of $m(x) = -3(x^3 - 2x)$ differs from $f(x) = x^3 - 2x$

### Solution

**a** Compare $g(x) = x^3 - 2x + 7$ with $f(x) = x^3 - 2x$.

$g(x) = f(x) + 7$

State the result.

The function $g(x) = x^3 - 2x + 7$ is a vertical translation of $f(x) = x^3 - 2x$. It moves up 7 units.

**b** Compare $m(x) = -3(x^3 - 2x)$ with $f(x) = x^3 - 2x$.

$m(x) = -3f(x)$

State the result.

The function $m(x) = -3(x^3 - 2x)$ is a dilation and reflection of $f(x) = x^3 - 2x$. The 3 narrows the function and the negative sign reflects the function in the $x$-axis (turns it upside down).

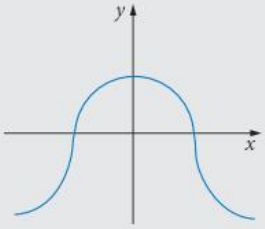

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## Even and odd functions

**IMPORTANT**

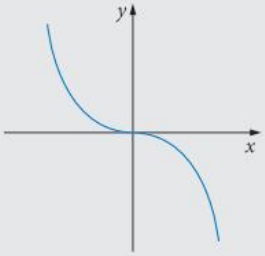
If $f(-x) = f(x)$ for all real values of $x$, the function is **even**.

An even function has line symmetry with the $y$-axis as the axis of symmetry.




If $f(-x) = -f(x)$ for all real values of $x$, the \function is **odd**.

An odd function has rotational symmetry, with the origin (0, 0) as the point of symmetry.




### Example 6

Show that

**a** $f(x) = x^3 - 2x$ is an odd function

**b** $f(x) = \dfrac{x^2}{x^4 - 1}$ is an even function.

### Solution

**a** Write $f(-x)$ in terms of $f(x)$. $f(-x) = (-x)^3 - 2 \times (-x)$

Simplify. $= -x^3 + 2x$

Factorise. $= -(x^3 - 2x)$

Substitute $f(x) = x^3 - 2x$. $= -f(x)$

State the result. Since $f(-x) = -f(x)$, the function is odd

**b** Write $f(-x)$ in terms of $f(x)$. $f(-x) = \dfrac{(-x)^2}{(-x)^4 - 1}$

Simplify. $= \dfrac{x^2}{x^4 - 1}$

Substitute $f(x) = \dfrac{x^2}{x^4 - 1}$. $= f(x)$

State the result. Since $f(-x) = f(x)$, the function is even.

# EXERCISE 4.02 Further properties of functions

## Concepts and techniques

**1** Example 4 **a** Given $f(x) = x + 3$, find $f(1)$ and $f(-3)$.

**b** If $h(x) = x^2 - 2$, find $h(0)$, $h(2)$ and $h(-4)$.

**c** If $f(x) = -x^2$, find $f(5)$, $f(-1)$, $f(3)$ and $f(-2)$.

**d** Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

**e** Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$.

**f** If $f(x) = 2x - 5$, find $x$ when $f(x) = 13$.

**g** Given $f(x) = x^2 + 3$, find any values of $x$ such that $f(x) = 28$.

**h** If $f(x) = 3^x$, find $x$ when $f(x) = \frac{1}{27}$.

**i** If $f(x) = 2x - 9$, find $f(p)$ and $f(x + h)$.

**2** Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

**3** **a** If $f(x) = x^3 - 1$, find $f(k)$ as a product of factors.

**b** Given $f(t) = t^2 + 2t + 1$, find $t$ when $f(t) = 0$. Also find any values of $t$ for which $f(t) = 9$.

**c** Given $f(t) = t^4 + t^2 - 5$, find the value of $f(b) - f(-b)$.

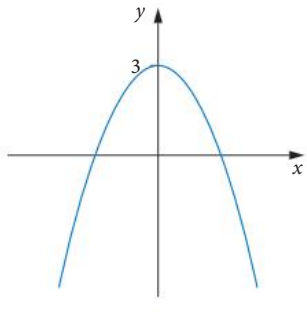
**4** Example 5 Match each equation with its graph.

**a** $f(x) = x^2 + 3$ **b** $f(x) = 3x^2$ **c** $f(x) = x^2 - 3$

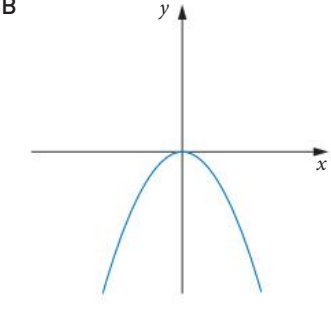
**d** $f(x) = -x^2$ **e** $f(x) = -x^2 + 3$ **f** $f(x) = -x^2 - 3$

**g** $f(x) = (x + 3)^2$ **h** $f(x) = (x - 3)^2$

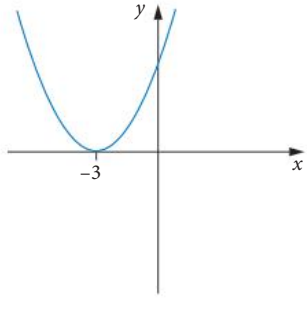
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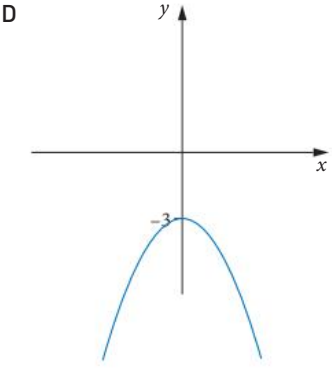

B




C



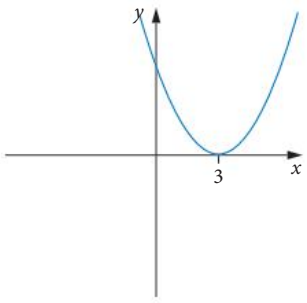

D



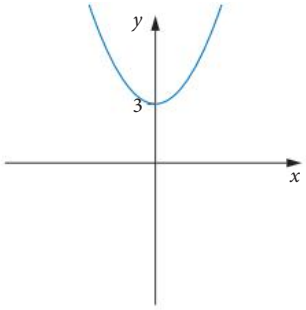


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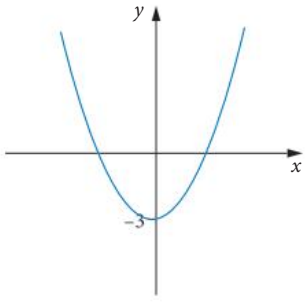
E



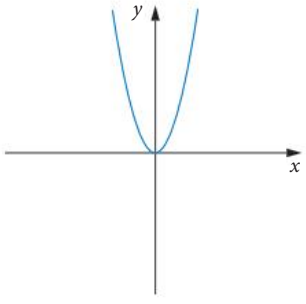

F




G




H




**5** If $f(x) = x^3 + 1$, find

**a** $f(x^2)$ **b** $[f(x)]^2$ **c** $f(-x)$

**6** Example 6 **a** Show that $g(x) = x^8 + 3x^4 - 2x^2$ is an even function.

**b** Show that $f(x)$ is odd, where $f(x) = x$.

**c** Show that $f(x) = x^2 - 1$ is an even function.

**d** Show that $f(x) = 4x - x^3$ is an odd function.

**e** Prove that $f(x) = x^4 + x^2$ is an even function and hence find $f(x) - f(-x)$.

## Reasoning and communication

**7** **a** $f(x) = \begin{cases} x^3 & \text{for } x > 1 \\ x & \text{for } x \leq 1 \end{cases}$ Find $f(5), f(1)$ and $f(-1)$.

**b** $f(x) = \begin{cases} 2x - 4 & \text{if } x \geq 1 \\ x + 3 & \text{if } -1 < x < 1 \\ x^2 & \text{if } x \leq -1 \end{cases}$ Find the value of $f(2) - f(-2) + f(-1)$.

**c** Find $g(3) + g(0) + g(-2)$ if $g(x) = \begin{cases} x + 1 & \text{when } x \geq 0 \\ -2x + 1 & \text{when } x < 0 \end{cases}$

**d** Find the value of $f(3) - f(2) + 2f(-3)$ when $f(x) = \begin{cases} x & \text{for } x > 2 \\ x^2 & \text{for } -2 \leq x \leq 2 \\ 4 & \text{for } x < -2 \end{cases}$

**e** Find the value of $f(1) - f(3)$ if $f(x) = \begin{cases} x^3 - 1 & \text{for } x \geq 2 \\ 2x^2 + 3x - 1 & \text{for } x < 2 \end{cases}$

**8** If $n$ is a positive integer, for what values of $n$ is the function $f(x) = x^n$

**a** even? **b** odd?

**9** Can the function $f(x) = x^n + x$ ever be

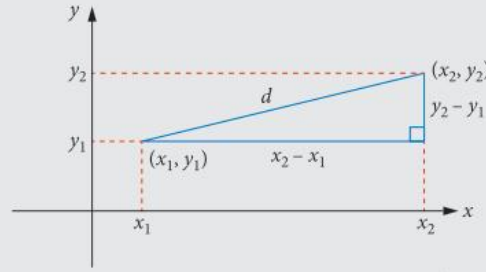
**a** even? **b** odd?

# 4.03 PROPERTIES OF LINEAR FUNCTIONS

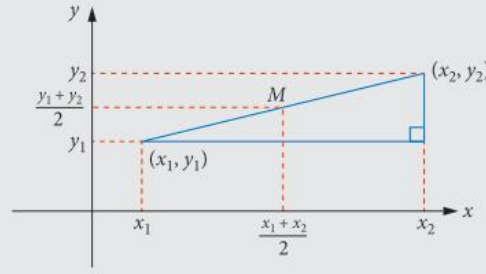
In Chapter 1 you looked at direct proportion and completed some practical work on graphs of straight lines. In this section you will continue to revise **linear functions** and their properties.

**IMPORTANT**

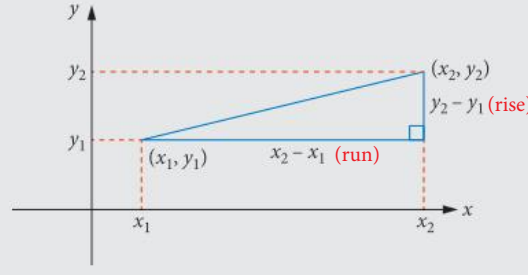
The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$




The **midpoint** of two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$



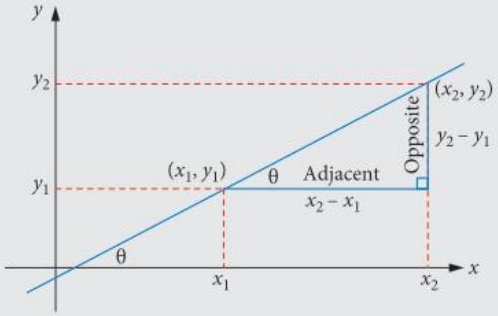

The **gradient** of the straight line between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$





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The gradient of a straight line is given by $m = \tan(\theta)$, where $\theta$ is the angle that the line makes with the positive $x$-axis.




## Example 7

a For the points (6, –3) and (–2, –1), find
   i the distance between the points (correct to 1 decimal place)
   ii the midpoint of the points
   iii the gradient of the straight line passing through the points

b Evaluate the exact gradient of the line that makes an angle of 120° with the $x$-axis in the positive direction.

c Find the angle that a linear function makes with the $x$-axis in the positive direction if its gradient is 2 (to the nearest degree).

### Solution

a i Substitute (6, –3) and (–2, –1) into the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(-2-6)^2 + (-1-(-3))^2}$

Simplify and evaluate.

$= \sqrt{(-8)^2 + 2^2}$
$= \sqrt{64 + 4}$
$= \sqrt{68}$
$= 8.2462\ldots$

State the result.

The distance is about 8.2 units.

ii Substitute (6, –3) and (–2, –1) into the formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$M = \left(\frac{6 + (-2)}{2}, \frac{-3 + (-1)}{2}\right)$

Simplify.

$= \left(\frac{4}{2}, \frac{-4}{2}\right)$
$= (2, -2)$

State the result.

The midpoint is (2, –2).

iii Substitute (6, –3) and (–2, –1) into the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$m = \frac{-1-(-3)}{-2-6}$

Remove brackets and simplify.

$= \frac{-1+3}{-2-6}$

$= \frac{2}{-8}$

$= -\frac{1}{4}$

State the result.

The gradient is $-\frac{1}{4}$.

**b** Substitute 120° into the formula $m = \tan(\theta)$.

$m = \tan(120°)$

Use 180° – 60° = 120° to evaluate.

$= -\tan(60°)$ (second quadrant)

$= -\sqrt{3}$

State the result.

The gradient is $-\sqrt{3}$.

**c** Substitute $m = 2$ into the formula $m = \tan(\theta)$

$2 = \tan(\theta)$

Evaluate θ using $\tan^{-1}$.

$\theta = 63.4349\ldots°$

State the result.

The line makes an angle of about 63° with the $x$-axis.

WS

Finding the gradient, distance and midpoint

You can use the distance, midpoint and gradient formulas to solve problems and to explore geometry and properties of shapes.

## Example 8

**a** Find the values of $x$ and $y$ if the midpoint of $(x, 3)$ and $(-1, y)$ is $(2, -4)$.

**b** The points $A(0, 4)$, $B(3, 4)$, $C(3, 0)$ and $D(0, 0)$ form the vertices of a rectangle. Show that the diagonals $AC$ and $BD$ are

i equal ii bisect each other

### Solution

**a** Substitute $(x, 3)$ and $(-1, y)$ into the formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$(2, -4) = \left(\frac{x + (-1)}{2}, \frac{3 + y}{2}\right)$

Find $x$ by equating the $x$ values.

$2 = \frac{x - 1}{2}$

Multiply both sides by 2.

$4 = x - 1$

Add 1 to both sides.

$5 = x$

Find $y$ by equating $y$ values.

$-4 = \frac{3 + y}{2}$

Multiply both sides by 2.

$-8 = 3 + y$

Subtract 3 from both sides.

$-11 = y$

State the result.

$x = 5$ and $y = -11$

**b i** Find the length of the diagonals by using the distance formula.
For $AC$ substitute (0, 4) and (3, 0) into the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$AC = \sqrt{(3-0)^2 + (0-4)^2}$

Simplify and evaluate.

$= \sqrt{3^2 + (-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

For $BD$ substitute (3, 4) and (0, 0) into the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$BD = \sqrt{(0-3)^2 + (0-4)^2}$

Simplify and evaluate.

$= \sqrt{(-3)^2 + (-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

State the result.

$AC = BD = 5$ units so the diagonals are equal.

**ii** Find the midpoint of the diagonals. For $AC$ substitute (0, 4) and (3, 0) into the formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and simplify.

$M = \left(\frac{0+3}{2}, \frac{4+0}{2}\right)$

$= \left(1\frac{1}{2}, 2\right)$

For $BD$ substitute (3, 4) and (0, 0) into the formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and simplify.

$M = \left(\frac{3+0}{2}, \frac{4+0}{2}\right)$

$= \left(1\frac{1}{2}, 2\right)$

State the result.

Since $AC$ and $BD$ have the same midpoint, they bisect each other.



Photo courtesy Margaret Grove

You may have noticed that a gradient can be either positive or negative.

The Greek letter Δ is used to show changes in variables.

**IMPORTANT**

The gradient of a straight line is given by $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$

where $\Delta y$ is the change in $y$ values and $\Delta x$ is the change in $x$ values.

A line increasing in height from left to right has a **positive gradient**.

A line decreasing in height from left to right has a **negative gradient**.

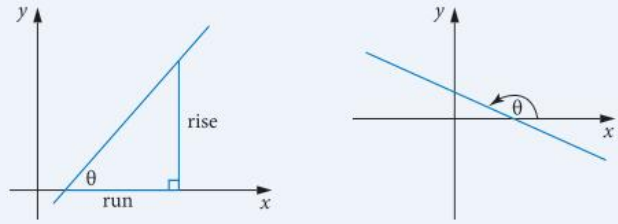



In the next section you will look at different ways of writing the equation of a linear function. The simplest of these is the form $y = mx + c$.

**IMPORTANT**

A linear function with the equation $y = mx + c$. has a gradient of $m$ and a $y$-intercept of $c$

## INVESTIGATION




1. Which angles give a positive gradient?
2. Which angles give a negative gradient? Why?
3. What is the gradient of a horizontal line? What angle does it make with the $x$-axis?
4. What angle does a vertical line make with the $x$-axis? Can you find its gradient?


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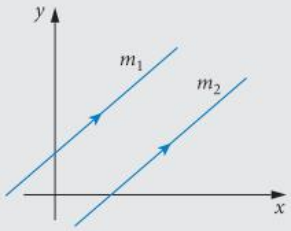
**IMPORTANT**

If two lines are parallel, then they have the same gradient. That is, $m_1 = m_2$

If two lines with gradients $m_1$ and $m_2$ respectively are perpendicular, then

$m_1 m_2 = -1$

i.e. $m_2 = -\dfrac{1}{m_1}$




Parallel and perpendicular lines have closely related equations.

Parallel lines can have their equations written so only the constant term differs. For example, $2x + 3y - 5 = 0$ and $2x + 3y + 1 = 0$ are parallel. They both have gradient $-\frac{2}{3}$.

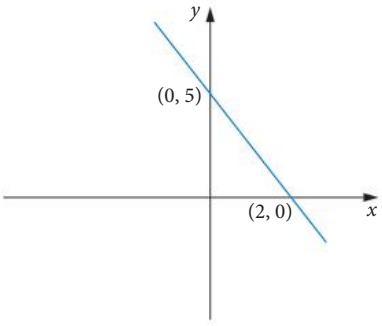
Perpendicular lines can have their equations written as $ax + by + c_1 = 0$ and $bx - ay + c_2 = 0$.

For example, $2x + 3y = 5$ and $6x - 4y + 7 = 0$ are perpendicular as they can be written as $2x + 3y - 5 = 0$ and $3x - 2y + 3\frac{1}{2} = 0$. Their gradients are $-\frac{2}{3}$ and $\frac{3}{2}$.

## Example 9

Find the gradient of each of the following lines.

**a**




**b** The line with equation $2x - 3y - 12 = 0$

**c** The line perpendicular to the line $3x - y + 2 = 0$

### Solution

**a** The line is decreasing so has negative gradient. Use $m = \dfrac{\text{rise}}{\text{run}}$ where rise is $-5$ and run is 2.

$m = -\frac{5}{2}$

You could also use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and substitute (0, 5) and (2, 0).

$m = \dfrac{0-5}{2-0}$

$= \dfrac{-5}{2}$

$= -\dfrac{5}{2}$

State the result.

The gradient is $-\frac{5}{2}$.

| | | |
|---|---|---|
| **b** | Write the equation of the line and rearrange into the form $y = mx + c$. | $2x - 3y - 12 = 0$ |
| | Add $3y$ to both sides. | $2x - 12 = 3y$ |
| | Divide both sides by 3. | $\frac{2x}{3} - 4 = y$ |
| | Reverse the equation. | $y = \frac{2x}{3} - 4$ |
| | Compare with $y = mx + c$. | $m = \frac{2}{3}$ |
| **c** | Write the equation of the first line and rearrange into the form $y = mx + c$. | $3x - y + 2 = 0$ |
| | Add $y$ to both sides. | $3x + 2 = y$ |
| | Reverse the equation. | $y = 3x + 2$ |
| | Compare with $y = mx + c$. | $m_1 = 3$ |
| | Write the rule for perpendicular lines (the gradient of the perpendicular line is the negative reciprocal). | $m_2 = -\frac{1}{m_1}$ $= -\frac{1}{3}$ |
| | State the result. | The gradient of the perpendicular line is $-\frac{1}{3}$. |

WS
Parallel and perpendicular lines

# EXERCISE 4.03 Properties of linear functions

## Concepts and techniques

**1** Example 7 For the line passing through the following points, find
**i** the distance to 2 decimal places **ii** the midpoint **iii** the gradient.
**a** (2, 3) and (6, 0) **b** (0, 5) and (−5, −7) **c** (−1, 2) and (2, 5)
**d** (4, −2) and (2, 2) **e** (−8, 3) and (−2, 1)

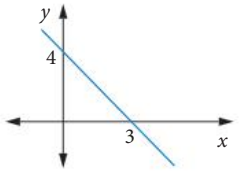
**2** **a** Find the gradient, correct to two decimal places where necessary, of the line that makes an angle with the positive direction of the $x$-axis
**i** 40° **ii** 60° (exact value) **iii** 45°
**iv** 129° **v** 150° (exact value)
**b** Find the angle, to the nearest degree, that each line makes with the positive $x$-axis given that the gradient of the line is
**i** 0.5 **ii** 1.34 **iii** −0.26 **iv** $\frac{\sqrt{3}}{3}$ **v** −2.7

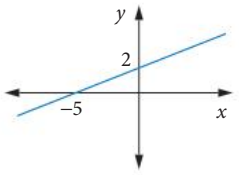
**3** Example 8 If the distance between $(a, -1)$ and $(3, 4)$ is 5, find the value of $a$.

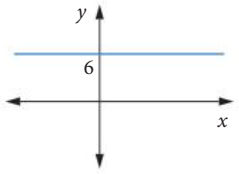
**4** Show that triangle $ABC$, where $A = (3, 4)$, $B = (-2, 7)$ and $C = (6, -1)$ is isosceles.

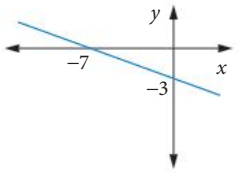
**5** A circle has centre $(-2, 5)$. One end of a diameter is at $(4, -3)$. Find the coordinates of the point at the other end of the diameter.

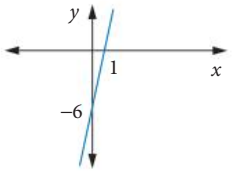
6 Example 9 Find the gradient of each straight line.

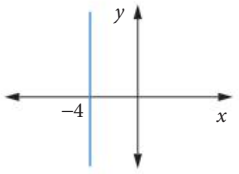
a 

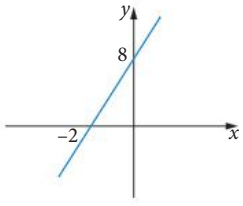

b 

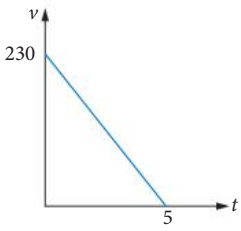

c 


d 


e 


f 


g 


h 


i $y = -2x + 1$
j $x + y - 3 = 0$
k $x + 4y - 9 = 0$
l the line parallel to $3x - y - 1 = 0$
m the line parallel to $x + 5y + 3 = 0$
n the line perpendicular to $y = 5x - 2$
o the line perpendicular to $x + y - 2 = 0$
p the line perpendicular to $3x - 2y = 0$

## Reasoning and communication

7 What type of triangle has vertices $A(-2, 3)$, $B(1, 2)$ and $C(-1, -4)$?

8 What type of quadrilateral has vertices $A(-3, -7)$, $B(-7, 1)$, $C(5, 3)$ and $D(9, -5)$?

9 A ramp leads up to the front doors of an office block that are 520 mm higher than the ground. Workplace health and safety regulations state that the maximum safe inclination of the ramp in this situation is $\frac{1}{12}$.


Shutterstock.com/Christian Colista

a What is the shortest distance that the ramp should be from the base of the platform?

b If the ramp had a slope of 6° would it be considered safe?

c If there is only room for a ramp to be built 8 m from the base of the platform, what is the greatest height that the platform can be for the ramp to be safe?

10 A straight line has $x$-intercept $A(a, 0)$ and $y$-intercept $B(0, b)$, where $a$ and $b$ are positive integers. The gradient of line $AB$ is $-1$. Find $\angle OBA$ where $O$ is the origin and hence prove that $a = b$.

# 4.04 EQUATIONS OF LINEAR FUNCTIONS

You have already looked at the equation $y = mx + c$.

**IMPORTANT**

The equation $y = mx + c$ is called the **gradient–intercept** form of a linear function.

It is a continuous function with domain and range the real numbers. Its graph is a straight line.

$ax + by + c = 0$ is the **standard form** (or general form) of a linear function. When expressed like this, the equation is changed so that rational constants are integers and all terms are on the LHS with a positive coefficient of $x$.

## INVESTIGATION Gradient–intercept form of a linear function

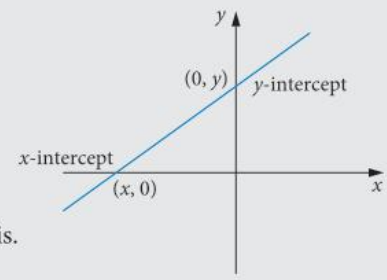
1 Using a CAS calculator, compare $y = x$ with $y = kx$ where $k$ is a parameter that can be positive, negative or a fraction. How does $k$ affect the function?

2 Compare $y = x$ with $y = x + k$ where $k$ can be positive or negative. How does $k$ affect the function in this case?

**IMPORTANT**

You can draw the graph of a linear function by finding its intercepts.

An ***x*-intercept** is the point where the graph of a function crosses the $x$-axis of the Cartesian plane. This is a value of $x$ that makes the function zero ($y = 0$). An $x$-intercept is also called a **zero**.




A ***y*-intercept** is the point where the graph crosses the $y$-axis. It is the value of the function when $x = 0$

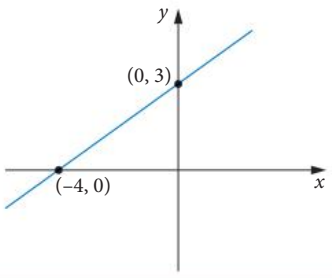
Intercepts are also the distances from the origin to the point where the graph meets the axes.


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## Example 10

Find the $x$- and $y$-intercepts of $3x - 4y + 12 = 0$ and sketch its graph.

### Solution

| | |
|---|---|
| Write the equation. | $3x - 4y + 12 = 0$ |
| Let $y = 0$ to find the $x$-intercept. | $3x - 4 \times 0 + 12 = 0$ |
| Solve for $x$. | $3x + 12 = 0$ |
| Subtract 12 from each side. | $3x = -12$ |
| Divide both sides by 3. | $x = -4$ |
| State the result. | The $x$-intercept is $(-4, 0)$. |
| Let $x = 0$ to find the $y$-intercept. | $3 \times 0 - 4y + 12 = 0$ |
| Solve for $y$. | $-4y + 12 = 0$ |
| Subtract 12 from each side. | $-4y = -12$ |
| Divide both sides by $-4$. | $y = 3$ |
| State the result. | The $y$-intercept is $(0, 3)$. |
| Plot the intercepts and draw a straight line through them. |  |



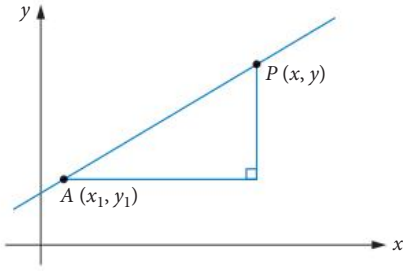
You can find the equation of a linear function given certain information.

Given a point $(x_1, y_1)$ on the line with gradient $m$ and letting $P = (x, y)$, the line $AP$ has the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1)$$









x- and y-intercepts

**IMPORTANT**

The gradient–point form is $y - y_1 = m(x - x_1)$, where $m$ is the gradient and $(x_1, y_1)$ is a point on the line.

The intercept form is $\frac{x}{a} + \frac{y}{b} = 1$, where $a$ and $b$ are the $x$- and $y$-intercepts respectively.

The two-point form is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$, where $(x_1, y_1)$ and $(x_2, y_2)$ are points on the line.

## Example 11

Find the equation of the straight lines

a passing through (–4, 3) and (1, –3)

b with $x$-intercept –2 and $y$-intercept 3

c parallel to the line $4x + 3y - 7 = 0$ and passing through the point (4, –5).

### Solution

**a** Write the formula. $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute (–4, 3) as $(x_1, y_1)$ and (1, –3) as $(x_2, y_2)$. $\frac{y - 3}{x - (-4)} = \frac{-3 - 3}{1 - (-4)}$

Simplify. $\frac{y - 3}{x + 4} = \frac{-3 - 3}{1 + 4} = \frac{-6}{5}$

Cross multiply. $5(y - 3) = -6(x + 4)$

Expand out of the brackets and simplify. $5y - 15 = -6x - 24$

Rearrange into general form by adding $6x$ and 24 to both sides (you can do this in one or two steps). $6x + 5y + 9 = 0$

State the result. The equation is $6x + 5y + 9 = 0$

**b** Substitute $a = -2$ and $b = 3$ into the formula $\frac{x}{a} + \frac{y}{b} = 1$. $\frac{x}{-2} + \frac{y}{3} = 1$

Multiply both sides by 6 (the common denominator) and simplify. $6\left(\frac{x}{-2} + \frac{y}{3}\right) = 6 \times 1$

$-3x + 2y = 6$

Multiply by –1. $3x - 2y + 6 = 0$

State the result. The equation is $3x - 2y + 6 = 0$


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**c** Write the equation of the first line and rearrange into the form $y = mx + c$.

$4x + 3y - 7 = 0$

Subtract $4x$ and add 7 to both sides.

$3y = -4x + 7$

Divide both sides by 3.

$y = -\frac{4x}{3} + \frac{7}{3}$

Compare with $y = mx + c$.

$m_1 = -\frac{4}{3}$

Write the rule for parallel lines.

$m_1 = m_2$

Find the gradient of the second line.

$m_2 = -\frac{4}{3}$

Substitute $m_2 = -\frac{4}{3}$ and $(4, -5)$ into the formula $y - y_1 = m(x - x_1)$ and simplify.

$y - (-5) = -\frac{4}{3}(x - 4)$

$y + 5 = -\frac{4}{3}(x - 4)$

Multiply both sides by 3.

$3(y + 5) = -4(x - 4)$

Remove brackets and simplify by adding $4x$ and subtracting 16 from both sides.

$3y + 15 = -4x + 16$

$4x + 3y - 1 = 0$

Write the result.

The equation is $4x + 3y - 1 = 0$

If $a = 0$ or $b = 0$, the equation $ax + by + c = 0$ becomes a vertical or horizontal line.

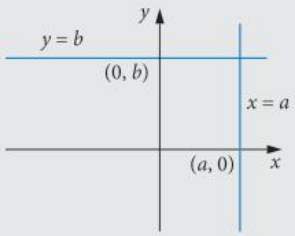
WS Equations of lines

**IMPORTANT**

### Horizontal and vertical lines

$x = a$ is a vertical line with $x$-intercept $a$. It is not a function, but a relation with domain $\{x: x = a\}$ and range $\{y: y \in \boldsymbol{R}\}$. Its gradient is undefined.

$y = b$ is a horizontal line with $y$-intercept $b$. It is a function with domain $\{x: x \in \boldsymbol{R}\}$ and range $\{y: y = b\}$. It has a zero gradient.




## Example 12

Find the equations of the straight lines described below and state their domain and range.

**a** Parallel to the $x$-axis with $y$-intercept 1.

**b** Parallel to the $y$-axis and passing through (4, 0).

### Solution

**a** The line parallel to the $x$-axis is horizontal with its equation in the form $y = b$. Substitute $b = 1$ into the formula.

$y = 1$

State the result.

The equation is $y = 1$. The domain is $\{x: x \in \boldsymbol{R}\}$ and the range is $\{y: y = 1\}$.

**b** The line parallel to the $y$-axis is vertical with its equation in the form $x = a$. Substitute $a = 4$ into the formula.

$x = 4$

State the result.

The equation is $x = 4$. The domain is $\{x: x = 4\}$ and the range is $\{y: y \in \boldsymbol{R}\}$.

# EXERCISE 4.04 Equations of linear functions

## Concepts and techniques

WS
Equations of lines parallel to axes

**1** Example 10 Find the $x$- and $y$-intercepts of each linear function and sketch its graph.

**a** $y = x - 2$ **b** $y = 2x + 3$ **c** $2x + y - 1 = 0$

**d** $x - y + 3 = 0$ **e** $3x - 6y - 2 = 0$

**2** Find the equation of each linear function.

**a** with gradient 4 and $y$-intercept $-1$

**b** with gradient $-3$ and passing through $(0, 4)$

**c** passing through the origin with gradient 5

**d** with gradient 4 and $x$-intercept $-5$

**e** with $x$-intercept 1 and $y$-intercept 3

**f** with $x$-intercept 3, $y$-intercept $-4$

**g** with $y$-intercept $-1$ and making an angle of 45° with the positive $x$-axis

**3** Example 11 Find the equation of each linear function in general form.

**a** passing through $(2, 3)$ and parallel to the line $y = x + 6$

**b** through $(-1, 5)$ and parallel to the line $x - 3y - 7 = 0$

**c** with $x$-intercept 5 and parallel to the line $y = 4 - x$

**d** through $(3, -4)$ and perpendicular to the line $y = 2x$

**e** through $(-2, 1)$ and perpendicular to the line $2x + y + 3 = 0$

**f** through $(7, -2)$ and perpendicular to the line $3x - y - 5 = 0$

**g** through $(-3, -1)$ and perpendicular to the line $4x - 3y + 2 = 0$

**4** Find the equation of the straight line with gradient $-2$ that passes through the midpoint of $(5, -2)$ and $(-3, 4)$.

**5** Example 12 Find the equation of the line

**a** parallel to the $x$-axis and passing through $(2, 3)$

**b** parallel to the $y$-axis and passing through $(-1, 2)$.

**6** Find the equation of the straight line that has zero gradient and passes through the point $(-1, 7)$.

**7** Find the equation of the line perpendicular to the line $y = -3$ and passing through the point $(4, 5)$.


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## Reasoning and communication

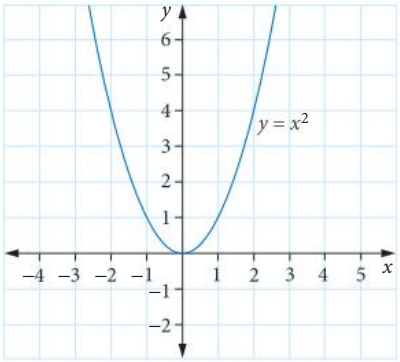
8 A circle has centre $C(1, 3)$ and a radius of 5 units.
- a Find the value of $a$ if the point $A(a, -1)$ lies on the circumference and $a > 2$.
- b Find the equation of $AC$.
- c Find the equation of the line through $C$ perpendicular to $AC$.

9 a Find the gradient of the linear function $l_1$ with equation $ax + by + c = 0$
- b Show that the linear function $l_2$ with equation $bx - ay + d = 0$ is perpendicular to $l_1$.
- c Find the equation of the linear function perpendicular to $l_1$ and with $x$-intercept $p$.

10 Show that the equation of a line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

11 Prove that the intercept form of the equation of a straight line is correct.

# 4.05 QUADRATIC FUNCTIONS




You have seen **quadratic functions** such as $y = x^2$ in Years 9 and 10. The graph is called a **parabola**, a symmetrical curve with an **axis of symmetry** at the $y$-axis. It has a **turning point** at $(0, 0)$ which is the **minimum** point of the graph.

The graph of any quadratic function $y = ax^2 + bx + c$ is always a parabola.

## INVESTIGATION Shape of the quadratic function

On your CAS calculator or computer, explore the effect that a parameter $k$ has on the quadratic function $y = x^2$. The parameter $k$ can be positive or negative, integer or a fraction.

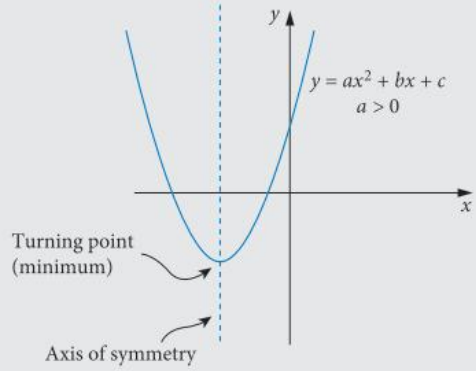
1 Sketch the following graphs and see the effect that $k$ has on the function $y = x^2$ in each case.
- a $y = kx^2$
- b $y = x^2 + k$
- c $y = (x + k)^2$
- d $y = (kx)^2$
- e $y = x^2 + kx$

2 Sketch the graph of $y = (x - a)(x - b)$ for different values of $a$ and $b$ and explore the effect that $a$ and $b$ have on the graph.

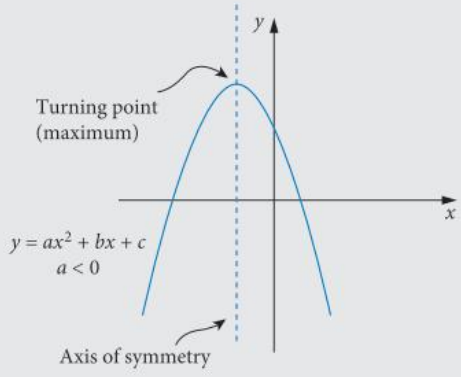
3 Sketch the graph of $y = a(x - b)^2$ for different values of $a$ and $b$ and explore the effect that $a$ and $b$ have on the graph.

**IMPORTANT**

For $a > 0$, $y = ax^2 + bx + c$ is **concave upwards**, with a minimum turning point.




For $a < 0$, $y = ax^2 + bx + c$ is **concave downwards**, with a maximum turning point.




The $x$-intercepts of a straight line are where $y = 0$. This applies to other functions including quadratics.

**IMPORTANT**

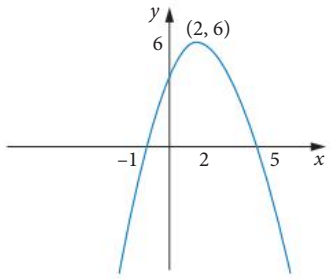
The **zeros** (roots) of the quadratic equation $ax^2 + bx + c = 0$ give the $x$-intercepts of the quadratic function $y = ax^2 + bx + c$.


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## Example 13

**a** Find the $x$- and $y$-intercepts of the quadratic function $y = x^2 + 2x - 3$ and sketch its graph.

**b** For the parabola on the right, state where the quadratic function is increasing and decreasing, and where its gradient is positive, negative and zero.




### Solution

**a** State the quadratic function. $y = x^2 + 2x - 3$

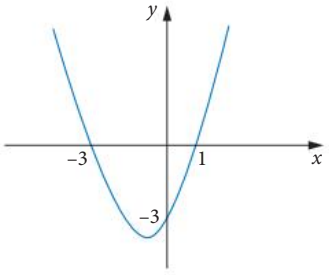
For the $x$-intercepts, let $y = 0$. $0 = x^2 + 2x - 3$

Reverse, factorise and solve the equation.
$x^2 + 2x - 3 = 0$
$(x + 3)(x - 1) = 0$
$x + 3 = 0, x - 1 = 0$
$x = -3, x = 1$

For the $y$-intercept, let $x = 0$.
$y = 0^2 + 2 \times 0 - 3$
$= -3$

State the result. The quadratic function has $x$-intercepts $-3$ and 1, and $y$-intercept $-3$.

Sketch the graph as a parabola passing through the intercepts.



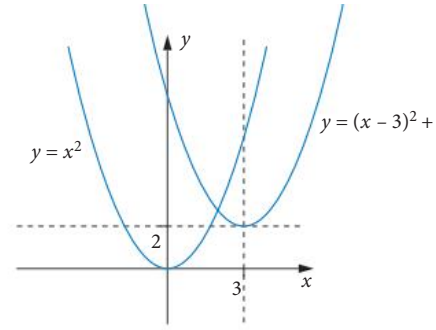

**b** The graph is increasing on the left of the vertex and decreasing on the right. The curve is increasing for $x < 2$ and decreasing for $x > 2$.

Where the graph is increasing, it has a positive gradient and where it is decreasing it has a negative gradient. The gradient is zero at the vertex (turning point). The gradient is positive for $x < 2$, negative for $x > 2$ and zero at $x = 2$.

You can use translations to find different features of the quadratic function. For example, $y = (x - 3)^2$ is a **horizontal translation** of $y = x^2$, moving 3 units to the right and $y = x^2 + 2$ is a **vertical translation**, moving 2 units upwards.

So $y = (x - 3)^2 + 2$ is a translation of $y = x^2$ moving 3 units to the right and 2 units up.

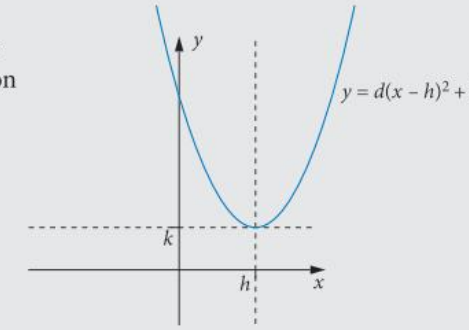
Notice that the equation of the **axis of symmetry** is $x = 3$ and the **vertex** (turning point) is the point (3, 2).




You can change the equation of any quadratic function into the form $y = a(x - h)^2 + k$ by completing the square.

**IMPORTANT**

For the quadratic function $y = d(x - h)^2 + k$, the equation of the **axis of symmetry** is $x = h$ and the **turning point** has coordinates $(h, k)$. It is a dilation of $y = x^2$ from the $x$-axis by the factor $d$ from $y = x^2$ followed by a translation $h$ to the right and $k$ up.




## Example 14

For the quadratic function $y = x^2 - 6x - 1$:

a complete the square and write the function in the form $y = d(x - h)^2 + k$

b find the equation of the axis of symmetry and the coordinates of the turning point.

### Solution

a Write the equation. $y = x^2 - 6x - 1$

Complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$ on the right-hand side and simplifying. $y = x^2 - 6x + 9 - 9 - 1$
$= x^2 - 6x + 9 - 10$

Write in the form $y = d(x - h)^2 + k$. $y = (x - 3)^2 - 10$

b Write equation of the axis of symmetry $x = h$. The equation of the axis of symmetry is $x = 3$

Substitute $h = 3$ and $k = -10$ into $(h, k)$. The coordinates of the turning point are $(3, -10)$.


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You can also find the equation of the axis of symmetry from a formula.

From the quadratic formula, the zeros are the roots of $ax^2 + bx + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. So the $x$-intercepts of the general parabola $y = ax^2 + bx + c$ are at $\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right)$ and $\left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$.

The axis of symmetry will be halfway between these points. So its equation will be the average of the two roots. The turning point is on the axis of symmetry.

$$x = \frac{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}{2}$$

$$= \frac{\dfrac{-2b}{2a}}{2}$$

$$= -\frac{b}{2a}$$

**IMPORTANT**

The axis of symmetry of the parabola $y = ax^2 + bx + c$ has the equation $x = -\dfrac{b}{2a}$

The coordinates of the turning point are $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

## Example 15

Find

a the equation of the axis of symmetry for $y = x^2 - 6x - 1$

b the coordinates of the turning point

c the minimum value of the quadratic function.

### Solution

**a** $y = x^2 - 6x - 1$ has $a = 1$ and $b = -6$. Substitute into the formula $x = -\dfrac{b}{2a}$.

$x = -\dfrac{-6}{2 \times 1}$

$= 3$

State the result.

The equation of the axis of symmetry is $x = 3$

**b** Find $f(3)$.

$y = 3^2 - 6 \times 3 - 1$

$= -10$

State the result.

The coordinates of the turning point are $(3, -10)$.

**c** State the result (notice that since $a = 1$, $a > 0$ so the function will be concave upwards and have a minimum value).

The minimum value of the function is $-10$.

# EXERCISE 4.05 Quadratic functions

WS
Sketching quadratic functions

## Concepts and techniques

**1** Example 13 Find the $x$-and $y$-intercepts of each of the following quadratic functions.

**a** $y = x^2 - 6x + 9$ **b** $f(x) = 4 - x^2$ **c** $f(x) = -x^2 + x + 2$

**d** $y = x^2 + 5x$ **e** $f(x) = 3x^2 - 13x + 12$

**2** For each quadratic function,

**i** sketch the graph of the function, showing all intercepts and use the symmetry of the parabola to mark in the axis of symmetry

**ii** find the $x$ values where the curve is increasing and decreasing

**iii** find the $x$ values where the gradient of the curve is positive, negative and zero.

**a** $f(x) = x^2 - 2x$ **b** $y = x^2 - 6x + 8$ **c** $f(x) = -x^2 + 4x - 4$

**3** Example 14 For each quadratic function,

**i** find the equation of the axis of symmetry

**ii** find the coordinates of the turning point and state what type of turning point it is.

**a** $y = (x - 2)^2 + 1$ **b** $f(x) = -3(x + 1)^2 + 5$ **c** $y = 4(x - 1)^2 + 2$

**d** $y = -(x + 3)^2 - 2$ **e** $f(x) = -2(x - 5)^2 - 4$ **f** $y = (x + 1)^2 - 2$

**g** $y = 3(x - 2)^2 + 3$ **h** $y = 9 + (x - 1)^2$ **i** $y = 1 - 2(x + 3)^2$

**j** $y = 5[12 - 4(x + 2)^2]$

**4** Find the equation of the axis of symmetry and the coordinates of the turning point for each parabola.

**a** $y = x^2 + 6x - 3$ **b** $y = -x^2 - 8x + 1$ **c** $y = -2x^2 + 5x$

**d** $y = 4x^2 + 10x - 7$ **e** $y = 3x^2 + 18x + 4$

**5** Example 15 Find

**i** the equation of the axis of symmetry

**ii** the minimum or maximum value and

**iii** the turning point of the parabola.

**a** $y = x^2 + 2x - 2$ **b** $y = -2x^2 + 4x - 1$

**6** For each quadratic function

**i** find any $x$-intercepts using the quadratic formula

**ii** state whether the function has a maximum or minimum value and find this value

**iii** sketch the function on a number plane.

**a** $f(x) = x^2 + 4x + 4$ **b** $f(x) = x^2 - 2x - 3$ **c** $y = x^2 - 6x + 1$

**d** $f(x) = x^2 + 2x$ **e** $y = 2x^2 - 18$ **f** $y = 3x^2 + x - 2$

**g** $f(x) = -x^2 - 2x + 6$ **h** $f(x) = -x^2 - x + 3$ **i** $y = -x^2 - 3x + 2$

**j** $y = -2x^2 + 4x + 5$


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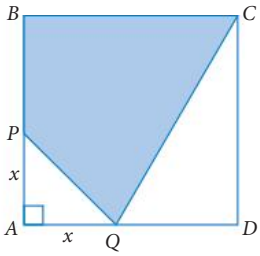
## Reasoning and communication

**7** **a** Sketch the parabola $y = x^2 - 5x + 6$.

**b** From the graph, find values of $x$ for which $x^2 - 5x + 6 > 0$.

**c** Find the domain over which $x^2 - 5x + 6 \leq 0$.

**8** The diagram shows a square plot of ground with a side length of 40 m. Two areas of the plot, $\triangle PAQ$ and $\triangle CDQ$, are found to be unsuitable for planting crops. The plot owner wants to use the remaining (shaded) area for planting vegetables.




Show that the area, $A$, available for planting vegetables is given by $A = 800 + 20x - \frac{1}{2}x^2$ and find the value of $x$ for which the area is a maximum.

**9** A company sells a special piece of equipment. The profit of a company can be modelled as a function of its daily sales given by

$P(x) = -4x(x - 100) - 6400$

where $P(x)$ is the profit in dollars and $x$ is the number of pieces sold in a day.

**a** How much will the company lose if there are no sales in a day?

**b** What is the minimum number of daily sales required to show a profit for the day?

**c** What is the maximum daily profit that can be achieved?

**10** The height of a ball thrown upwards at 25 m/s is approximately given by the equation

$H(t) = 2 - 5t^2 + 25t$

Find the turning point and intercepts, and hence the time taken to reach the maximum height and the time taken to reach the ground.



Getty Images/Scott Barbour

# 4.06 QUADRATIC EQUATIONS AND THE QUADRATIC FUNCTION

## INVESTIGATION Some uses of the parabola in society

The parabolic shape is used in many different applications as it has special properties that are very useful.

If a light is placed inside a mirrored parabola at a particular place called the focus, then all rays coming from it will radiate out parallel to each other, giving a strong light.

Similarly, parallel light or sound rays coming in to a parabolic object will be reflected onto the focus.

1 Research some of the commonly used items that use this property of a parabola.

2 Some bridges look as if they have a parabolic shape, but instead they are built in the shape of a catenary. Research the catenary and see how it is different from the parabola.



Photo courtesy Margaret Grove

You can draw a parabola using the zeros and turning point or $y$-intercept. A parabola is also uniquely determined by three points.

The equation of a quadratic function in the form $y = d(x - h)^2 + k$ may also be found if you know its turning point and one other point.


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## Example 16

**a** Find the equation of the quadratic function that passes through the points (–1, –3), (0, 3) and (2, 21).

**b** Find the equation of a quadratic function that has a turning point at (1, 4) and $x$-intercept –1.

### Solution

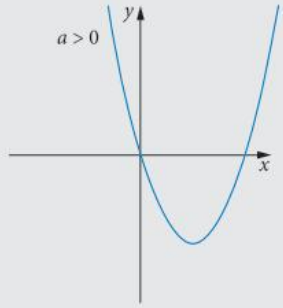
| | | |
|---|---|---|
| **a** Substitute (–1, –3) into $y = ax^2 + bx + c$. | $-3 = a(-1)^2 + b(-1) + c$ | |
| Simplify the equation. | $a - b + c = -3$ | [1] |
| Substitute (0, 3) into $y = ax^2 + bx + c$. | $3 = a(0)^2 + b(0) + c$ | |
| Simplify the equation. | $c = 3$ | [2] |
| Substitute (2, 21) into $y = ax^2 + bx + c$. | $21 = a(2)^2 + b(2) + c$ | |
| Simplify the equation. | $4a + 2b + c = 21$ | [3] |
| Substitute [2] into [1]. | $a - b + 3 = -3$ | |
| Simplify. | $a - b = -6$ | [4] |
| Substitute [2] into [3]. | $4a + 2b + 3 = 21$ | |
| Simplify. | $2a + b = 9$ | [5] |
| Add equations [4] and [5]. | $a - b = -6$<br>$2a + b = 9$<br>$3a = 3$ | |
| Divide both sides by 3. | $a = 1$ | |
| Substitute $a = 1$ into [5]. | $2 \times 1 + b = 9$ | |
| Simplify. | $b = 7$ | |
| State the result. | The equation is $y = x^2 + 7x + 3$ | |
| **b** Write the turning point form of the quadratic function for the turning point $(h, k)$. | $y = d(x - h)^2 + k$<br>$y = d(x - 1)^2 + 4$ | |
| Substitute the $x$-intercept, (–1, 0). | $0 = d(-1 - 1)^2 + 4$ | |
| Solve for $d$. | $d = -1$ | |
| Substitute all values. | $y = -1(x - 1)^2 + 4$ | |
| Simplify. | $y = -x^2 + 2x + 3$ | |
| State the result. | The equation is $y = -x^2 + 2x + 3$ | |

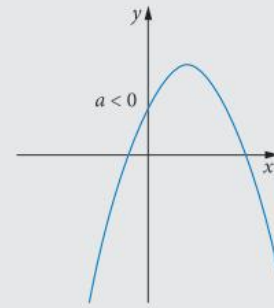
In Chapter 1, you used the **discriminant** to find if a quadratic equation had 0, 1 or 2 roots. The zeros of a quadratic function are the roots of the corresponding equation, so the number of roots of the equation is the same as the number of zeros of the function.

## IMPORTANT

The **discriminant** of the quadratic function $y = ax^2 + bx + c$ is $\Delta = b^2 - 4ac$.

**1** If $\Delta > 0$ then the quadratic equation $ax^2 + bx + c = 0$ has 2 real unequal (different) roots.

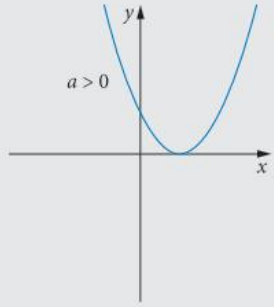


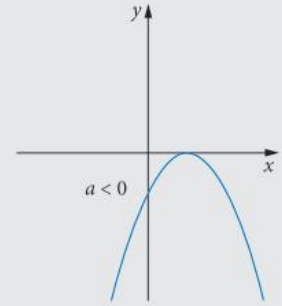




If $\Delta$ is a perfect square, the roots are rational.

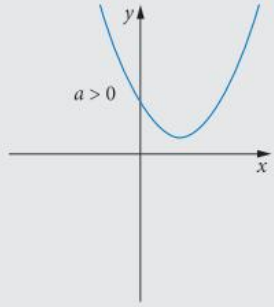
If $\Delta$ is not a perfect square, the roots are irrational.

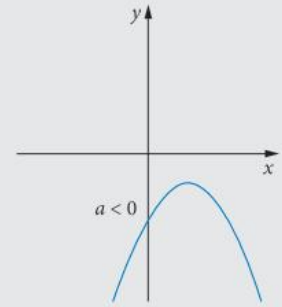
**2** If $\Delta = 0$, then the quadratic equation $ax^2 + bx + c = 0$ has 1 real root or 2 equal roots.




**3** If $\Delta < 0$, then the quadratic equation $ax^2 + bx + c = 0$ has no real roots.




If $\Delta < 0$ and $a > 0$, $ax^2 + bx + c > 0$ for all $x$.

If $\Delta < 0$ and $a < 0$, $ax^2 + bx + c < 0$ for all $x$.


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## Example 17

**a** Show that the function $y = 2x^2 + x + 4$ has no real zeros.
**b** Find the values of $k$ for which the function $y = 5x^2 - 2x + k$ has real zeros.

### Solution

**a** Write the formula for the discriminant. $\Delta = b^2 - 4ac$

Substitute $a = 2$, $b = 1$ and $c = 4$ and simplify. $= 1^2 - 4 \times 2 \times 4 = -31$

State the result. Since $\Delta < 0$, $y = 2x^2 + x + 4$ has no real zeros.

**b** Write the discriminant condition. For real zeros $\Delta \geq 0$

Use $a = 5$, $b = -2$ and $c = k$ to find the discriminant.

$$\Delta = b^2 - 4ac$$
$$= (-2)^2 - 4 \times 5 \times k$$
$$= 4 - 20k$$

Put $\Delta \geq 0$ and solve.

$$4 - 20k \geq 0$$
$$4 \geq 20k$$
$$k \leq \frac{1}{5}$$

State the result. $y = 5x^2 - 2x + k$ has real zeros for $k \leq \frac{1}{5}$

You can use the discriminant when you want to find out how many zeros a quadratic function has without needing to solve it.

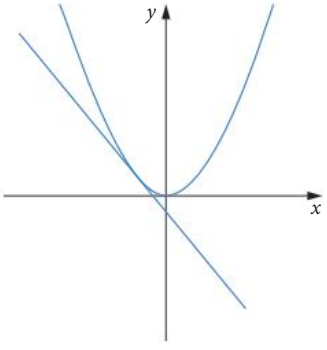
## Example 18

Show that the line $4x + y + 4 = 0$ is a tangent to the parabola $y = x^2$.

### Solution

State the condition for $4x + y + 4 = 0$ to be a tangent to $y = x^2$.

A line is a tangent to a parabola if they intersect at exactly one point.




Solve simultaneous equations to find any points of intersection.

$y = x^2$ [1]
$4x + y + 4 = 0$ [2]

Substitute [1] into [2]. $4x + x^2 + 4 = 0$

Rearrange. $x^2 + 4x + 4 = 0$

| | |
|---|---|
| Use the discriminant to find the number of zeros (roots) the equation has using $a = 1$, $b = 4$ and $c = 4$. | $\Delta = b^2 - 4ac$<br>$= 4^2 - 4 \times 1 \times 4$<br>$= 0$ |
| State the result. | Since $\Delta = 0$, the quadratic equation has 1 root and so the line is a tangent to the parabola. |

# EXERCISE 4.06 Quadratic equations and the quadratic function

## Concepts and techniques

**1** Example 16 Find the equation of the parabola that passes through the points

**a** (0, –5), (2, –3) and (–3, 7)
**b** (1, –2), (3, 0) and (–2, 10)
**c** (–2, 21), (1, 6) and (–1, 12)
**d** (2, 3), (1, –4) and (–1, –12)
**e** (0, 1), (–2, 1) and (2, –7)

**2** Find the equation of the quadratic function that has

**a** $y$-intercept –6 and turning point (2, –2)
**b** $x$-intercept 2 and turning point (–1, 18)
**c** a turning point of (–2, –3) and passes through the point (1, 6)
**d** turning point (2, 5) and $x$-intercept 3
**e** turning point (–1, 3) and $y$-intercept 6

**3** Example 17 Find the discriminant of each quadratic function and state whether the zeros are real or not, and if they are real, whether they are equal or unequal, rational or irrational.

**a** $y = x^2 - x - 4$
**b** $y = 2x^2 + 3x + 6$
**c** $f(x) = x^2 - 9x + 20$
**d** $y = x^2 + 6x + 9$
**e** $f(x) = 2x^2 - 5x - 1$
**f** $f(x) = -x^2 + 2x - 5$
**g** $y = -2x^2 - 5x + 3$
**h** $f(x) = -5x^2 + 2x - 6$
**i** $y = -x^2 + x$
**j** $y = -2x^2 + 8x - 2$

**4** For the quadratic function $y = x^2 + 2x + p$, find values of $p$ for which the function has

**a** 2 real zeros
**b** 1 real zero
**c** no real zeros

**5** Example 18 Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ at 2 points.

**6** Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$.

**7** The line $y = 3x - p + 1$ is a tangent to the parabola $y = x^2$. Evaluate $p$.

## Reasoning and communication

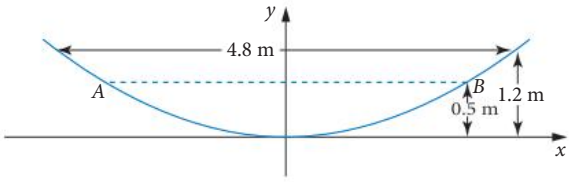
**8** **a** Show that the quadratic function $y = kx^2 + px + k$ has one $x$-intercept when $p = \pm 2k$.
**b** Show that the $x$-intercept is –1 when $p = 2k$ and 1 when $p = -2k$.

**9** In a science experiment, a student throws a ball into the air from the top of a 5 m cliff and measures its height as 8 m after 1 s and 5 m after 4 s. Its height is a quadratic function of time.

**a** Write an equation for the height ($h$) in metres over time $t$ seconds.
**b** Find the maximum height that the ball reaches.
**c** How many seconds does it take for the ball to reach the ground below the cliff?


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10 A satellite dish is in the shape of a parabola with depth 1.2 m and width 4.8 m. When it is drawn on a number plane, the parabola has a turning point at (0, 0).




a Find the equation of the parabola.
b Find the width $AB$ of the parabola, correct to 1 decimal place, when the depth is 0.5 m.

# 4.07 GRAPHS OF FUNCTIONS

In Chapter 1 you looked at inverse proportion. The hyperbola $y = \frac{k}{x}$ is the graph of an inverse proportion function. The **hyperbola** is a discontinuous function with vertical and horizontal asymptotes. It is called a **rectangular hyperbola** since the asymptotes are perpendicular (at right angles) to each other.

## INVESTIGATION Exploring the hyperbola

Is the hyperbola always a function? Can it be odd or even?

Use your CAS calculator or computer software to explore the effect of parameter $k$ on the hyperbola $y = \frac{k}{x}$.

Sketch the following graphs where $k$ can be positive or negative integers or fractions.

1 $y = \frac{k}{x}$

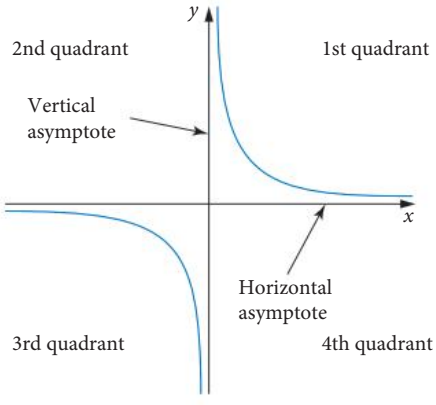
2 $y = \frac{1}{x} + k$

3 $y = \frac{1}{x+k}$

What effect does $k$ have on the graph in each case?

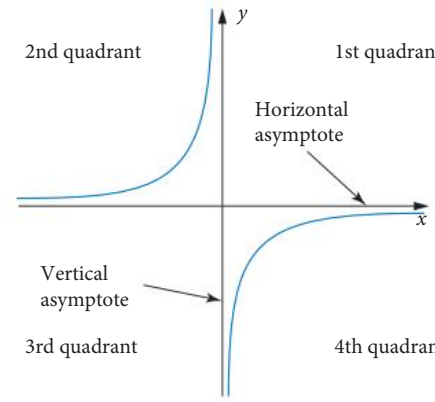
When $k > 0$ the hyperbola is in the first and third quadrants.

Both parts of the curve are decreasing, so the gradient is negative except at the vertical asymptote where the gradient does not exist.



When $k < 0$ the hyperbola is in the second and fourth quadrants.

Both parts of the curve are increasing, so the gradient is always positive except at the vertical asymptote.




You can use the domain and range to sketch a hyperbola.

## Example 19

Sketch the graphs of

**a** $y = \dfrac{4}{x+2}$

**b** $f(x) = -\dfrac{1}{x-1}$

### Solution

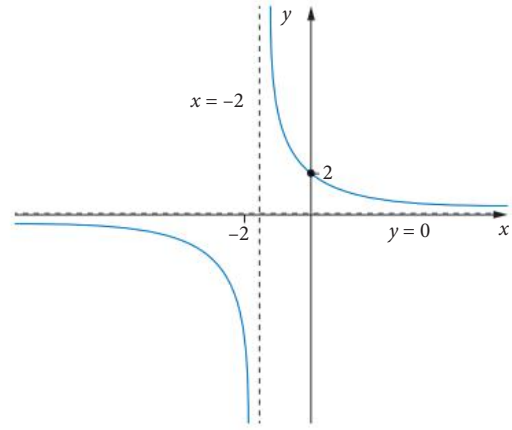
**a** State the domain and range.

$x + 2 \neq 0$
$x \neq -2$
Domain is $\{x: x \neq -2\}$.
Range is $\{y: y \neq 0\}$.

Find the $y$-intercept by substituting $x = 0$.

$y = \dfrac{4}{0+2}$
$= 2$

Sketch the hyperbola in the first and third quadrants with a vertical asymptote at $x = -2$, a horizontal asymptote at $y = 0$ and $y$-intercept 2.




**b** State the domain and range.

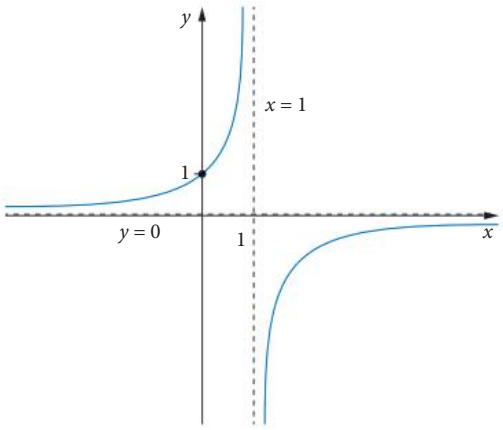
$x - 1 \neq 0$ so $x \neq 1$
Domain is $\{x: x \neq 1\}$.
Range is $\{y: y \neq 0\}$.

Find the $y$-intercept by substituting $x = 0$.

$f(0) = -\dfrac{1}{0-1} = 1$


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Since the hyperbola is in the form $f(x) = -\frac{a}{x-b}$ it will be in the 2nd and 4th quadrants. Sketch it with a vertical asymptote at $x = 1$, a horizontal asymptote at $y = 0$ and $y$-intercept 1.




# Technology

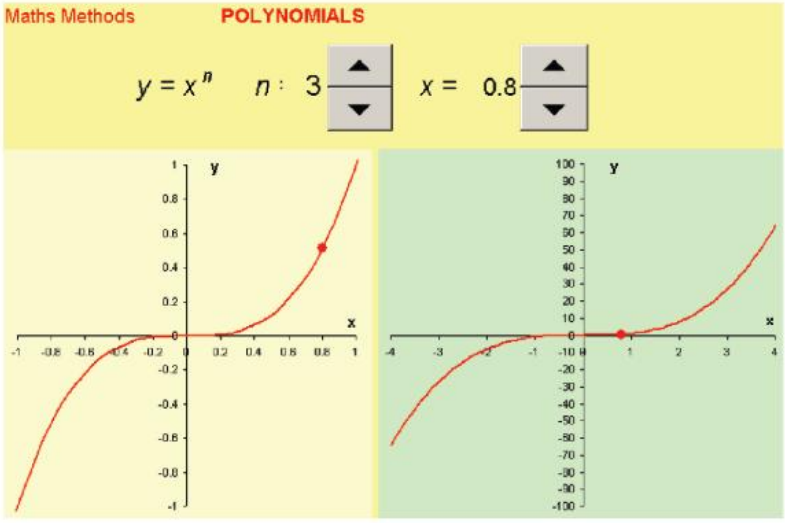
## Polynomials of the form $y = x^n$

Different types of polynomial functions have distinctive shapes. If you are familiar with these shapes, it will help you to draw the graphs of other polynomial functions.

NelsonNet has a spreadsheet named 'Polynomials' that lets you see how the graph of $y = x^n$ changes as the value of $n$ changes.

Use the spreadsheet to see how the value of $n$ affects the shape of the graph.

Write down any conclusions that you make.

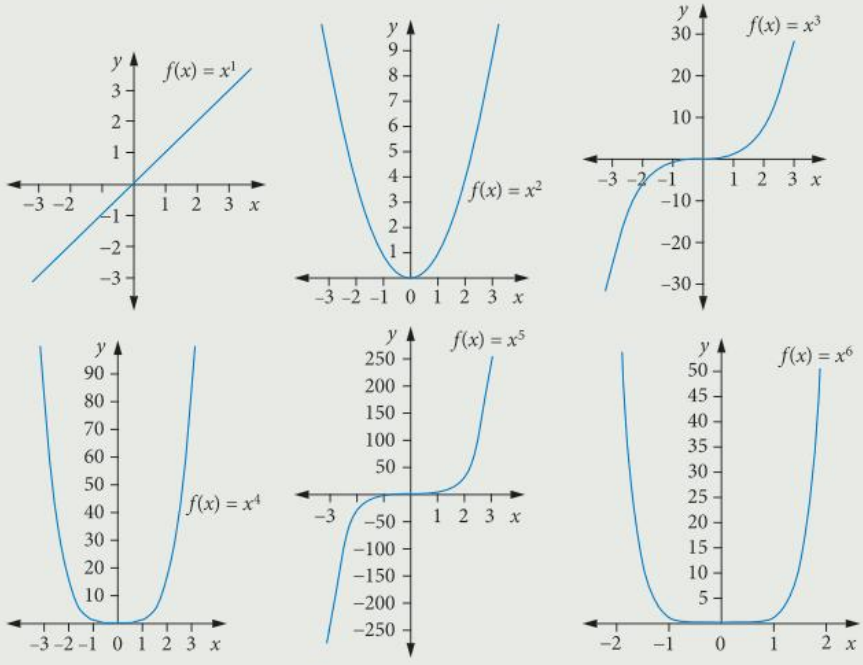



What graph would you get if $n = -1$?

What graph would you get if $n = \frac{1}{2}$?

**IMPORTANT**

The graph of $y = x^n$ varies for different values of $n$. Here are the graphs of polynomials with $n$ from 1 to 6.




Notice from the graphs that the function $y = x^n$ is even for even values of $n$. As $x$ becomes large ($x \to \infty$) $y$ also becomes large ($y \to \infty$) and as $x \to -\infty$, $y \to \infty$.

For odd values of $n$, the function is odd. As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Polynomials with degree 2 or more have turning points at (0, 0). For odd values of $n$ the turning point is called a **point of inflection**.

The negatives of the functions are the same graphs turned upside down.

Once you know the general shape of a polynomial, you can use translations and dilations to sketch the graphs of related polynomials.

## Example 20

Sketch the graph of $y = (x + 2)^3 - 1$

### Solution

Compare $y = (x + 2)^3 - 1$ with $y = x^3$.

$y = (x + 2)^3 - 1$ is a horizontal translation 2 units to the left and a vertical translation 1 unit below $y = x^3$.


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Find the $x$-intercept(s).

Substitute $y = 0$

$0 = (x + 2)^3 - 1$

$1 = (x + 2)^3$

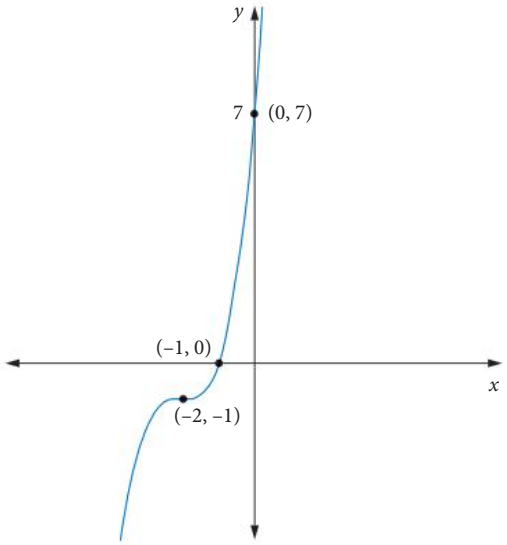
$\sqrt[3]{1} = x + 2$

$x = -1$

Find the $y$-intercept.

$y = (0 + 2)^3 - 1 = 7$

Sketch the graph. It will have the same shape as $y = x^3$.




The cubic function $y = x^3$ is an example of a polynomial. The general form of a cubic function is $y = ax^3 + bx^2 + cx + d$. This makes it difficult to compare with the graph of $y = x^3$. However, you can sketch polynomials in this form by finding their intercepts.

## Example 21

Sketch the graph of $y = -x^3 + x^2 + 2x$ showing all intercepts.

### Solution

Factorise $y = -x^3 + x^2 + 2x$

$y = -x^3 + x^2 + 2x$

$= -x(x^2 - x - 2)$

$= -x(x - 2)(x + 1)$

Find the $x$-intercepts of the function.

Substitute $y = 0$.

$0 = -x(x - 2)(x + 1)$

$x = 0, x - 2 = 0, x + 1 = 0$

$x = 0, 2, -1$

Find the $y$-intercept.

$f(0) = -0(0 - 2)(0 + 1) = 0$

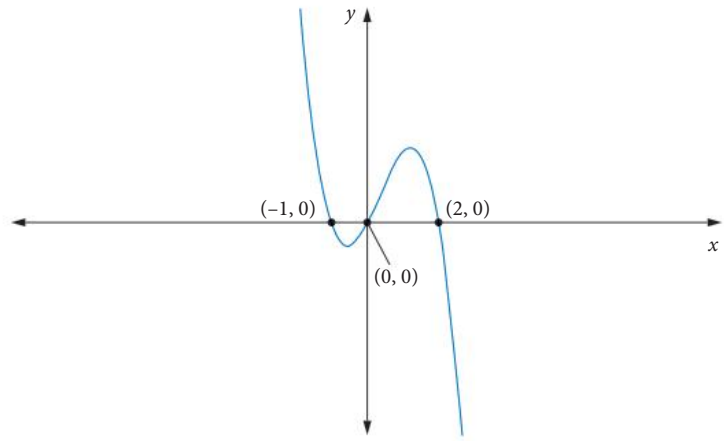
Find the behaviour of the function as $x$ becomes large. You can find some values between the zeros if you like.

As $x \to \infty$, $-x^3$ dominates and $y \to -\infty$.

As $x \to -\infty$, $-x^3$ dominates and $y \to \infty$.

$f(1) = 2$.

Sketch the graph.




If you use a CAS calculator to draw the graph above you will find that the local minimum between −1 and 0 is *not* at $x = -\frac{1}{2}$. Similarly, the local maximum is *not halfway* between 0 and 2.

# EXERCISE 4.07 Graphs of functions

## Concepts and techniques

1 Example 19 For each graph:
- i state the domain and range
- ii find the $y$-intercept if it exists
- iii sketch the graph.

a $y = \frac{2}{x}$ b $y = -\frac{1}{x}$ c $f(x) = \frac{1}{x+1}$ d $f(x) = \frac{3}{x-2}$

e $y = \frac{1}{3x+6}$ f $f(x) = -\frac{2}{x-3}$ g $f(x) = \frac{4}{x-1}$ h $y = -\frac{2}{x+1}$

i $f(x) = \frac{2}{6x-3}$ j $y = -\frac{6}{x+2}$

2 Show that $f(x) = \frac{2}{x}$ is an odd function.

3 Find the range of each function over the given domain.

a $f(x) = \frac{1}{2x+5}$ for $-2 \le x \le 2$ b $y = \frac{1}{x+3}$ for $-2 \le x \le 0$

c $f(x) = \frac{5}{2x-4}$ for $-3 \le x \le 1$ d $f(x) = -\frac{3}{x-4}$ for $-3 \le x \le 3$

e $y = -\frac{2}{3x+1}$ for $0 \le x \le 5$

4 Example 20 Explain how the graph of $y = x^3$ can be transformed to obtain a graph of each of the following. Sketching the graphs of the functions may be helpful.

a $y = x^3 + 4$ b $y = x^3 - 3$ c $y = 2x^3$

d $y = \frac{1}{2}x^3$ e $y = (x+4)^3$ f $y = (x-5)^3$


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**5** Sketch the graphs of the following functions.

**a** $f(x) = (x + 1)^3$ **b** $f(x) = -(x - 2)^3$ **c** $f(x) = x^3 - 3$

**d** $f(x) = 2 - x^3$ **e** $f(x) = -1 - x^3$ **f** $f(x) = x^3 + 1$

**g** $f(x) = (x - 1)^3 + 3$ **h** $f(x) = 4 - (x + 2)^3$ **i** $f(x) = (x - 3)^3 - 2$

**j** $f(x) = 3 - (x - 1)^3$ **k** $f(x) = -3 - 0.4(x + 2)^3$ **l** $f(x) = 2(x - 4)^3 + 1$

**6** Example 21 Sketch the graphs of the following functions.

**a** $f(x) = (x + 1)(x - 2)(x - 3)$ **b** $P(x) = x(x + 4)(x - 2)$

**c** $P(x) = -x(x - 1)(x - 3)$ **d** $f(x) = x(x + 2)^2$

**e** $g(x) = (5 - x)(x + 2)(x + 5)$

**7** **i** Write each polynomial as a product of its factors.

**ii** Sketch the graph of the polynomial.

**a** $P(x) = x^3 - 2x^2 - 8x$ **b** $f(x) = -x^3 - 4x^2 + 5x$

**c** $P(x) = x^4 + 3x^3 + 2x^2$ **d** $A(x) = 2x^3 + x^2 - 15x$

**e** $P(x) = -x^4 + 2x^3 + 3x^2$

## Reasoning and communication

**8** A small town is affected by a new strain of flu, causing a minor epidemic. The percentage of people with no immunity to the disease follows the relation

$$P(d) = \frac{120}{0.2d + 1}$$

where $d \geq 1$ is the number of days from the first reported case. When the percentage with no immunity drops below 5%, the epidemic ceases to spread.

**a** Sketch a graph of $P(d)$.

**b** What percentage of people have no immunity after 5 days?

**c** What percentage of people have no immunity after 10 days?

**d** After how many days do 50% have some immunity?

**e** After how many days does the disease cease to spread epidemically?

At a nearby town that has not yet reported the disease, a vaccination campaign is carried out, so by the time the first case is reported there, 25% of the population have been vaccinated and have acquired some immunity.

**f** Modify the function to take account of the initial immunity of this town.

**g** Find the time taken for the disease to cease spreading epidemically in the second town.

**9** CAS The composition of live algae after $d$ days in a reservoir, where the water has a high phosphate content, is modelled by

$$m(d) = 0.39d^2 - 0.01d^3 + 0.75d$$

where $m$ is measured in grams/litre.

**a** Use your CAS calculator to sketch $m(d)$.

**b** Find the $x$-intercepts of $m(d)$.

**c** What mass of algae would be present after 10, 20 and 30 days?

**d** Use a CAS calculator to find the maximum mass of algae present. How would this be related to the initial amount of nutrient present in the water?

**e** From your graph, suggest the domain of $d$ for which the model is accurate.

10 A specimen of bacteria is smeared on a round agar plate of diameter 12 cm and left for testing. The area of the bacterial colony is given by
$A = 6t^2 - t^3 + 16t + 0.5$ cm$^2$
where $t$ is the number of weeks after the culture is started.

a What is the initial size of the colony?
b Sketch the graph for the first 10 weeks.
c When does the model of growth become inadequate?
d What percentage of the plate is covered after 4 weeks?
e What is the maximum percentage of the plate that is covered?



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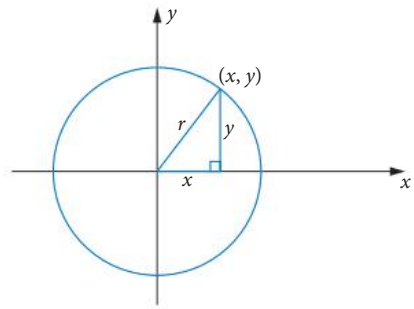
# 4.08 GRAPHS OF RELATIONS

A circle is not a function. You can find its equation by using Pythagoras' rule.

**IMPORTANT**

**Circle with centre at (0, 0)**

The equation of a circle with centre (0, 0) and radius $r$ is $x^2 + y^2 = r^2$.




The circle has centre (0, 0) and radius $r$.

Let $(x, y)$ be a general point on the circle, with distances from the origin $x$ on the $x$-axis and $y$ on the $y$-axis as shown.

By Pythagoras' theorem:

$a^2 + b^2 = c^2$

$\therefore x^2 + y^2 = r^2$

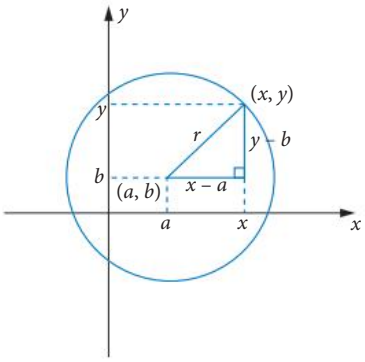

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**IMPORTANT**

**Circle with centre at (*a*, *b*)**

The equation of a circle, centre $(a, b)$ and radius $r$ is $(x - a)^2 + (y - b)^2 = r^2$.

Take a general point on the circle, $(x, y)$ and draw a right-angled triangle as shown.




Notice that the smaller sides of the triangle are $x - a$ and $y - b$ and the hypotenuse is $r$, the radius.

By Pythagoras' theorem:

$a^2 + b^2 = c^2$

$(x - a)^2 + (y - b)^2 = r^2$

## Example 22

**a** Sketch the graph of $x^2 + y^2 = 81$ and state its domain and range.

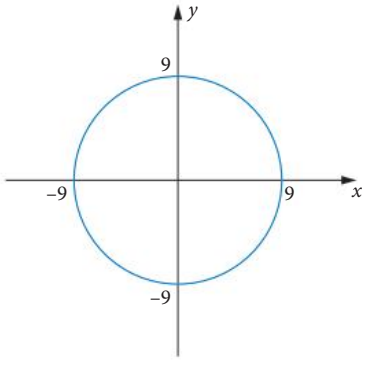
**b** Sketch the circle $(x - 1)^2 + (y + 2)^2 = 4$ and state its domain and range.

### Solution

**a** State the centre and radius of the circle.

The graph is a circle with centre (0, 0) and radius $\sqrt{81} = 9$.

Draw the graph



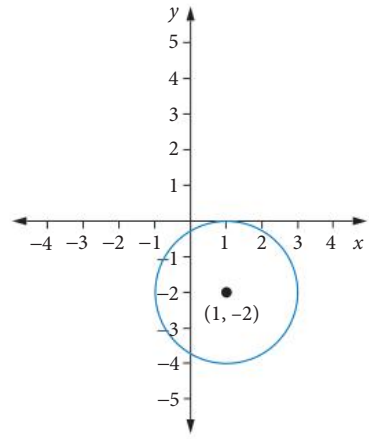

State the domain and range.

Domain is $\{x: -9 \leq x \leq 9\}$.
Range is $\{y: -9 \leq y \leq 9\}$.

**b** Compare with the general equation. $(x-1)^2 + (y+2)^2 = 4$ is of the form $(x-a)^2 + (y-b)^2 = r^2$

Write the centre and radius. The centre is $(1, -2)$ and the radius is $\sqrt{4} = 2$.

Draw the graph.




State the domain and range. Domain is $\{x: -1 \leq x \leq 3\}$.
Range is $\{y: -4 \leq y \leq 0\}$.

Sometimes you need to put the equation of a circle into the right form before you can find its radius and centre.

## Example 23

**a** Write the equation of the circle with centre $(2, -3)$ and radius 5 in expanded form.
**b** Find the radius and centre of the circle given by $x^2 + y^2 + 6x - 2y - 6 = 0$.

### Solution

**a** Write the formula version of the equation. $(x-a)^2 + (y-b)^2 = r^2$
Substitute values. $(x-2)^2 + (y+3)^2 = 5^2$
Expand brackets. $x^2 - 4x + 4 + y^2 + 6y + 9 = 25$
Simplify. $x^2 + y^2 - 4x + 6y - 12 = 0$

**b** Write the equation. $x^2 + y^2 + 6x - 2y - 6 = 0$
Rearrange and complete the square. $x^2 + 6x + 9 + y^2 - 2y + 1 = 6 + 9 + 1$
Write in the factored form. $(x+3)^2 + (y-1)^2 = 4^2$
Write the answer. The radius is 4 and the centre $(-3, 1)$.

WS
Equations of circles

## INVESTIGATION Semicircles

The circle is not a function. Could you break the circle up into two functions?

Change the subject of the equation $x^2 + y^2 = r^2$ to $y$. Can you find the domain and range of this result?

How could you draw the graph of a circle on a CAS calculator?

You know what the graph of the parabola $y = x^2$ looks like. The parabola $y^2 = x$ is the inverse relation with the $x$ and $y$ values reversed.

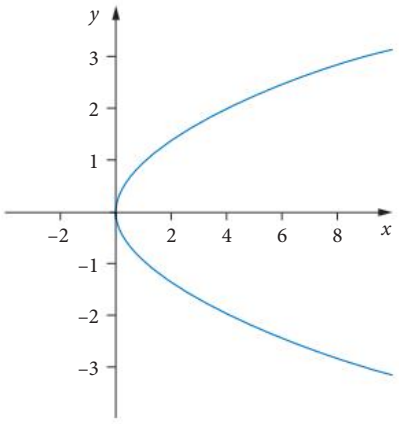
Here is a table of values for $y^2 = x$ where you can use $y$ values of $-3$ to 3 to find $x$.

| $x$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
|---|---|---|---|---|---|---|---|
| $y$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

Drawing the graph gives us a parabola with turning point at (0, 0) and a horizontal axis of symmetry.

Notice that this parabola is not a function. You can change the equation to $y = \pm\sqrt{x}$.

Drawing both the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ together will give you the parabola.




## INVESTIGATION Exploring the parabola with a horizontal axis

Find the effect of the parameter $k$ on the graph of $y^2 = x$ in each of the following. You can change the subject of the equation and break it up into two functions to graph it on your CAS calculator or you can use the conics section to enter the equation directly as in Example **24**.

1 $y^2 = kx$

2 $y^2 = x + k$

3 $(y + k)^2 = x$

Where is the vertex and axis of symmetry for each graph?

## Example 24

**a** CAS Draw the graph of $y^2 = 4x + 8$

**b** State the coordinates of its vertex (turning point).

**c** What is the equation of the axis of symmetry?

**d** State the domain and range.

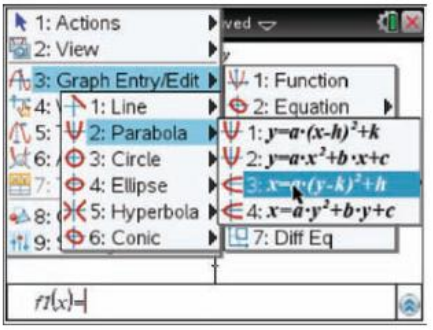
### Solution

**a** Change the equation to the form $x = \ldots$

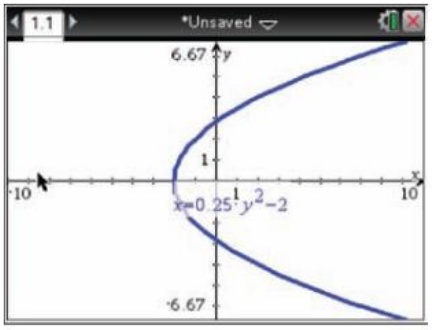
$$y^2 = 4x + 8$$
$$4x = y^2 - 8$$
$$x = \frac{1}{4}y^2 - 2$$

### TI-Nspire CAS

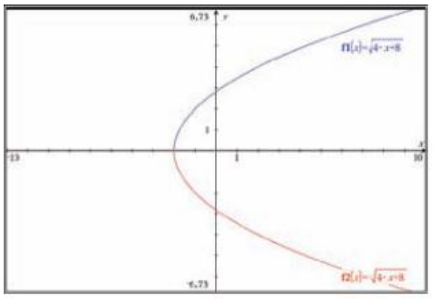
Use a graph page . Press menu, click on 3: Graph Entry/Edit, click on 2: Equation, click on 2: Parabola and click on 3: $x = a \cdot (y-k)^2 + h$.




Enter as $x = 0.25 \cdot (y - 0)^2 - 2$.

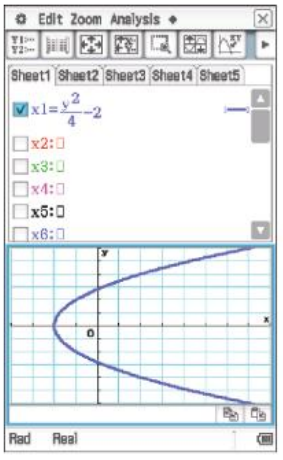



Alternatively, sketch the graph using the two functions $f1(x) = \sqrt{4x+8}$ and $f2(x) = -\sqrt{4x+8}$



### ClassPad

Select the Graph & Table application menu and clear the screen if necessary by tapping **Edit** then **Clear All**.
Tap the arrow next to y= at the top and select x=. Make $x1=y^2\div4-2$. Press EXE.
Tap to open View Window.
Set $-4 \le x \le 8$ and $-6 \le y \le 6$ and tap **OK**.
Tap to view the graph.
Tap Resize to get the full graph. You may then wish to reset **View Window**.





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| | | |
|---|---|---|
| b | Use the graph. | The vertex is $(-2, 0)$. |
| c | Use the graph. | The axis of symmetry is the line $y = 0$ (the $x$-axis). |
| d | Use the graph. | The domain is $\{x: x \geq -2\}$.<br>The range is $\{y: y \text{ is real}\}$. |

# EXERCISE 4.08 Graphs of relations

## Concepts and techniques

**1** Example 22 Find the length of the radius and the coordinates of the centre of each circle.

**a** $x^2 + y^2 = 100$ **b** $x^2 + y^2 = 5$ **c** $(x - 4)^2 + (y - 5)^2 = 16$

**d** $(x - 5)^2 + (y + 6)^2 = 49$ **e** $x^2 + (y - 3)^2 = 81$

**2** For each of the following

**i** sketch the graph **ii** state the domain and range.

**a** $x^2 + y^2 = 9$ **b** $x^2 + y^2 - 16 = 0$ **c** $(x - 2)^2 + (y - 1)^2 = 4$

**d** $(x + 1)^2 + y^2 = 9$ **e** $(x + 2)^2 + (y - 1)^2 = 1$

**3** Example 23 Find the coordinates of the centre and the length of the radius of each circle.

**a** $x^2 - 4x + y^2 - 2y - 4 = 0$ **b** $x^2 + 8x + y^2 - 4y - 5 = 0$

**c** $x^2 + y^2 - 2y = 0$ **d** $x^2 - 10x + y^2 + 6y - 2 = 0$

**e** $x^2 + 2x + y^2 - 2y + 1 = 0$ **f** $x^2 - 12x + y^2 = 0$

**g** $x^2 + 6x + y^2 - 8y = 0$ **h** $x^2 + 20x + y^2 - 4y + 40 = 0$

**i** $x^2 - 14x + y^2 + 2y + 25 = 0$ **j** $x^2 + 2x + y^2 + 4y - 5 = 0$

**4** Find the equation of each circle in expanded form (without grouping symbols).

**a** Centre (0, 0) and radius 4 **b** Centre (3, 2) and radius 5

**c** Centre (−1, 5)and radius 3 **d** Centre (2, 3) and radius 6

**e** Centre (−4, 2)and radius 5 **f** Centre (0, −2)and radius 1

**g** Centre (4, 2) and radius 7 **h** Centre (−3, −4)and radius 9

**i** Centre (−2, 0)and radius $\sqrt{5}$ **j** Centre (−4, −7)and radius $\sqrt{3}$

**5** Example 24 CAS Sketch the graph of $y^2 = 3x$ and state its domain and range.

**6** State the two functions that combine to form the graph of

**a** $y^2 = 5x$ **b** $(y + 1)^2 = x$ **c** $(y - 3)^2 = x + 1$

**7** CAS Draw the following graphs

**a** $(y - 2)^2 = x$ **b** $y^2 = -3x + 6$ **c** $(y + 1)^2 = x - 2$

## Reasoning and communication

**8** Show that the circles $x^2 + 2x + y^2 - 4y - 20 = 0$ and $x^2 + 2x + y^2 - 4y - 4 = 0$ are concentric (have the same centre).

**9** A circle has centre $C(-1, 3)$ and radius 5 units.

- **a** Find the equation of the circle.
- **b** The line $3x - y + 1 = 0$ meets the circle at two points. Find their coordinates.
- **c** Let these points be $X$ and $Y$. Find the coordinates of point $Z$ such that $YZ$ is a diameter of the circle.
- **d** Show $\angle ZXY = 90°$.

**10** A parabola with $x$-intercept 2 has its vertex at $(1, 2)$. Its axis of symmetry is the line $y = 2$ and it passes through the point $(2, 4)$. Find the equation of the parabola.


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# CHAPTER SUMMARY
# FUNCTIONS AND GRAPHS

4

- A **relation** is a set of ordered pairs $(x, y)$ that maps a set $X \rightarrow Y$, where $x \in X$ and $y \in Y$.
- The **domain** of a relation is the set of values of the **independent variable** $x$ and is often identical to the set $X$.
- The **codomain** is the set $Y$ into which the relation maps elements of the domain.
- The **range** of a relation is the set of values of the **dependent variable** $y$.
- A **function** is a relation where each value in the domain is in only one ordered pair.
- If a vertical line cuts a graph at more than one point, the graph is not a function.
- A **discontinuous** function has a break in the line of its graph.
- For the family of functions, $y = f(x) + a$, as $a$ varies, the graph moves up or down (vertical **translation**).
- For the family of functions, $y = f(x + b)$, as $b$ varies, the graph moves left or right (horizontal translation).
- For the family of functions, $y = cf(x)$, as $c$ varies, the graph narrows or expands (vertical **dilation**).
- For the family of functions, $y = f(dx)$, as $d$ varies, the graph narrows or expands (horizontal dilation).
- The graph of $y = f(-x)$ is a reflection of $y = f(x)$ in the $y$-axis.
- The graph of $y = -f(x)$ is a reflection of $y = f(x)$ in the $x$-axis.
- If $f(-x) = f(x)$ for all values of $x$, the function is **even.**
- If $f(-x) = -f(x)$ for all values of $x$, the function is **odd.**
- An $x$-intercept is a value of $x$ where $y = 0$.
- The $y$-intercept is the value of $y$ where $x = 0$.
- The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint of two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The **gradient** of a **linear function** is
  - $m$ given the equation $y = mx + c$.
  - $m = \frac{y_2 - y_1}{x_2 - x_1}$ given $(x_1, y_1)$ and $(x_2, y_2)$ are points on the line.
  - $m = \tan(\theta)$, where $\theta$ is the angle between the positive direction of the $x$-axis and the line.
- Two straight lines are parallel if $m_1 = m_2$.
- Two straight lines are perpendicular if $m_1 m_2 = -1$.
- The equation of a linear function is
  - $y - y_1 = m(x - x_1)$ given gradient $m$ and point $(x_1, y_1)$ on the line.
  - $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ given that $(x_1, y_1)$ and $(x_2, y_2)$ are points on the line.
  - $\frac{x}{a} + \frac{y}{b} = 1$ where $a$ is the $x$-intercept and $b$ is the $y$-intercept of the line.
- The graph of **quadratic function** $y = ax^2 + bx + c$ is a **parabola.**
  - if $a > 0$, the parabola is **concave** upwards with a minimum **turning point.**
  - if $a < 0$, the parabola is concave downwards with a maximum turning point.

- For the quadratic equation $y = d(x - h)^2 + k$, the equation of the **axis of symmetry** is $x = h$ and the coordinates of the turning point are $(h, k)$.
- For the quadratic equation $y = ax^2 + bx + c$, the equation of the axis of symmetry is $x = -\frac{b}{2a}$ and the coordinates of the turning point are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- The **discriminant** of the quadratic function $y = ax^2 + bx + c$ is $\Delta = b^2 - 4ac$.
  - if $\Delta > 0$, the function has 2 real zeros. The **zeros** are rational if $\Delta$ is a perfect square and irrational if $\Delta$ is not a perfect square.
  - if $\Delta = 0$, the quadratic has 1 real zero (or 2 equal zeros).
  - if $\Delta < 0$, the quadratic has no real zeros.
- The function $f(x) = \frac{a}{x - b}$ is a **hyperbola** with a vertical **asymptote** at $x = b$ and a horizontal asymptote at $y = 0$.
- For the polynomial function $y = x^n$.
  - the function is even when $n$ is even and $y \to \infty$ when $x \to \pm\infty$.
  - the function is odd when $n$ is odd, the turning point is a **point of inflection** and $y \to \infty$ when $x \to \infty$ and $y \to -\infty$ when $x \to -\infty$.
- The equation of a circle with radius $r$ is.
  - $x^2 + y^2 = r^2$ with centre $(0, 0)$.
  - $(x - a)^2 + (y - b)^2 = r^2$ with centre $(a, b)$.
- The relation $y^2 = ax$ is a parabola with turning point $(0, 0)$ and a horizontal axis of symmetry at $y = 0$.

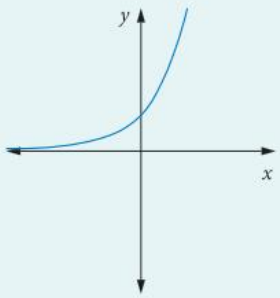

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# CHAPTER REVIEW
# FUNCTIONS AND GRAPHS
# 4

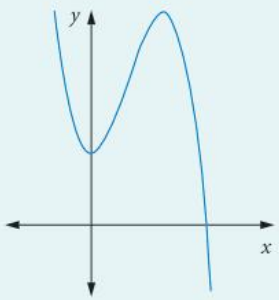
## Multiple choice

1 Example 2 Which graph is not a function?

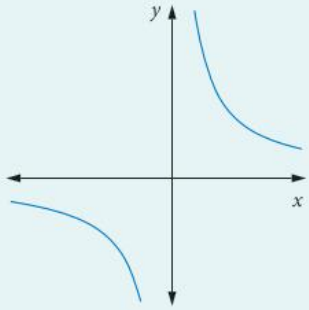
A



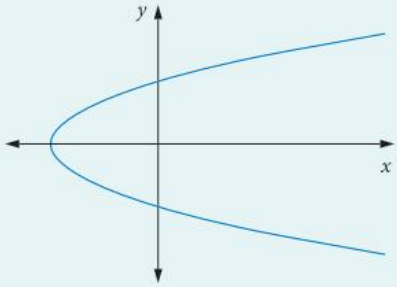

B



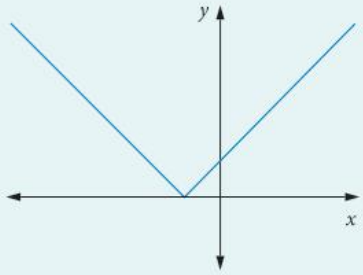

C




D

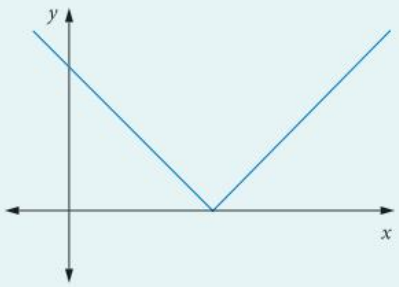



E

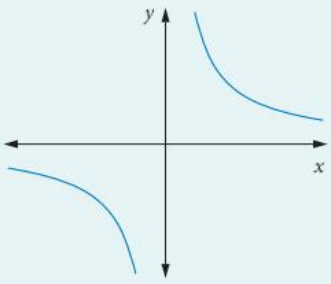


2 Example 3 Which function is discontinuous?

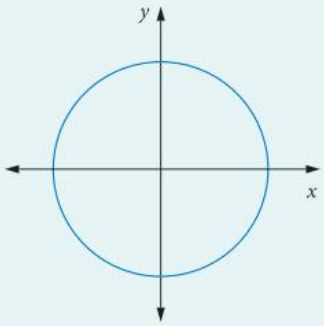
A



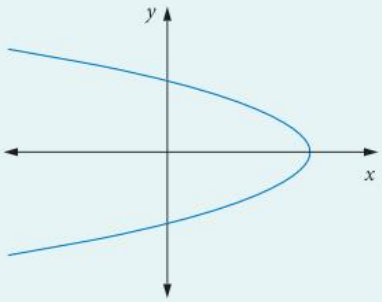

B



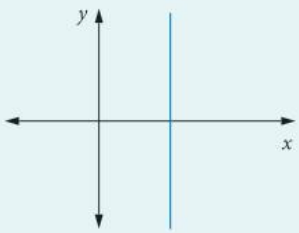

C



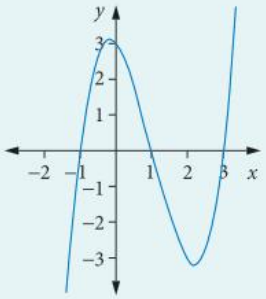

D




E



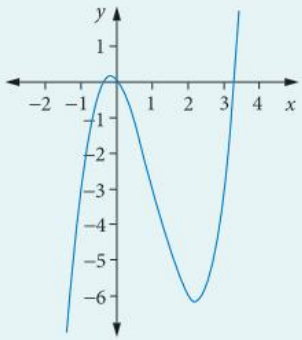

3 Example 5 Below is the graph of $y = f(x)$. Which graph represents $y = f(x) - 3$?



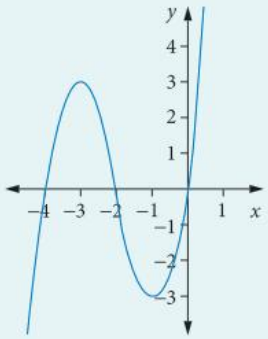


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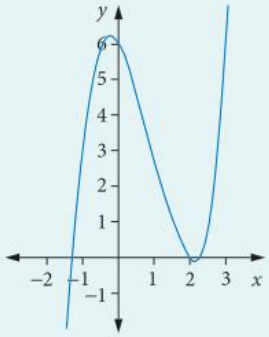
A



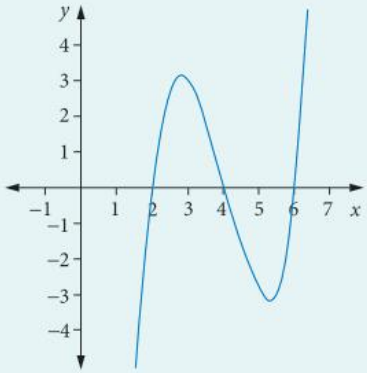

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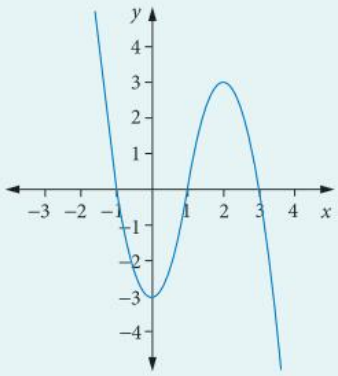

C




D




E




4 Example 11 Find the equation of the linear function

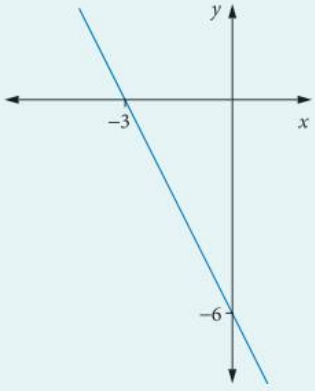
A $2x - y - 6 = 0$

B $x - 2y + 6 = 0$

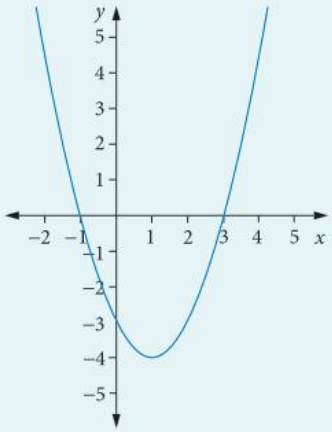
C $2x + y - 6 = 0$

D $x + 2y - 6 = 0$

E $2x + y + 6 = 0$

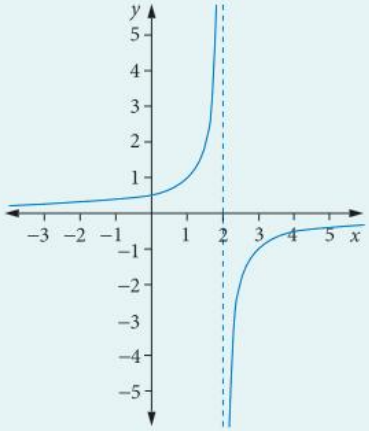


5 Example 13 The graph below is a quadratic function. Which statement is true?




A The curve is increasing and has a negative gradient for $x < 1$.
B The curve is increasing and has a positive gradient for $x > 1$.
C The curve is decreasing and has a negative gradient for $x > 1$.
D The curve is decreasing and has a positive gradient for $x > 1$.
E The curve is decreasing and has a positive gradient for $x < 1$.

6 Example 19 The equation of the graph shown is




A $y = \frac{2}{x-1}$
B $y = -\frac{1}{x+2}$
C $y = -\frac{1}{x-2}$
D $y = \frac{1}{x-2}$
E $y = -\frac{2}{x-1}$

## Short answer

7 Example 1 State the domain and range of
a $(-3, 2), (2, 1), (0, 0), (1, 5), (4, -7), (2, 5)$
b $f(x) = \sqrt{3x-6}$
c $y = -\frac{2}{x+5}$

8 Example 2 State whether each graph is a function or not.
a $y = x^2 - 5x + 3$
b $y^2 = 4x$
c $x^2 + y^2 = 1$
d $\frac{x}{3} - \frac{y}{2} = 1$
e $y = \frac{1}{1+2x}$


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9 Example 3 State whether each graph is continuous or discontinuous.

a $y = \dfrac{1}{3x}$ b $f(x) = -\dfrac{2}{x^2+1}$ c $y = \sqrt{1-x}$

d $f(x) = \dfrac{2}{x^2-4}$ e $y = \dfrac{1}{x^2+1}$

10 Example 4 If $f(x) = x^3 - 3$, find

a $f(0)$ b $f(-2)$ c $f(3x)$

d $f(x-1)$ e $x$ when $f(x) = 5$

11 Example 5 Given the function $f(x) = x^3$, describe transformations that produce each of the following functions.

a $F(x) = x^3 + 2$ b $G(x) = 2x^3$ c $H(x) = (x-2)^3$

12 Example 6 Show that

a $f(x) = \dfrac{1}{x^4+3}$ is even

b $f(x) = x^5 - x$ is odd

c $f(x) = x^2 - 2x + 3$ is neither even nor odd

13 Example 7 Given the points $(-3, 1)$ and $(4, 2)$, find

a the distance between them

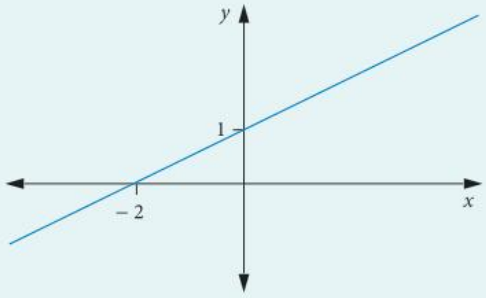
b their midpoint

c the gradient of the line between them

14 Example 8 The gradient of the straight line between $(3, y)$ and $(-2, 5)$ is $-\frac{3}{5}$. Find the value of $y$.

15 Examples 7, 9 Find the gradient in each of the following

a




b the straight line that passes through $(1, -1)$ and $(3, 7)$

c the line with equation $5x - 3y - 1 = 0$

d the line parallel to the line $x - y - 3 = 0$

e the line perpendicular to the line $6x + 2y + 7 = 0$

f the line that makes an angle of 45° with the positive $x$-axis

16 Example 10 a Find the $x$- and $y$-intercepts of $2x - y + 4 = 0$.

b Sketch the graph of $2x - y + 4 = 0$.

17 Examples 11, 12 Find the equation of each linear function
- a with gradient $-2$ and $x$-intercept 3
- b passing through $(2, -3)$ and $(4, 1)$
- c with $x$-intercept 4 and $y$-intercept $-1$
- d making an angle of 135° with the positive $x$-axis, with $y$-intercept 4
- e parallel to the line $y = 3x - 1$ and passing through $(2, 1)$
- f perpendicular to the line $2x + y - 3 = 0$ and with $x$-intercept $-2$
- g parallel to the $x$-axis and passing through $(3, -5)$
- h parallel to the $y$-axis and passing through $(-2, -1)$

18 Example 13 a Sketch the graph of $y = (x - 2)^2 - 1$ showing all intercepts.
b State its domain and range.

19 Example 14 a Write the quadratic function $f(x) = x^2 + 6x - 2$ in the form $f(x) = d(x - h)^2 + k$
b Hence find the equation of the axis of symmetry and the coordinates of the turning point.

20 Example 15 a Find the equation of the axis of symmetry and the coordinates of the turning point of $y = x^2 + 2x - 4$
b State the minimum value of the quadratic function.

21 Example 16 Find the equation of each quadratic function
- a with $x$-intercepts 2 and $-3$, and passing through $(1, 4)$
- b with turning point $(0, 1)$ and passing through $(-2, 5)$.

22 Example 17 Find the discriminant of each quadratic equation and state what this means in terms of the zeros.
- a $2x^2 - 3x + 1 = 0$
- b $4x^2 + x + 5 = 0$
- c $x^2 - 6x + 9 = 0$
- d $x^2 + 5x - 3 = 0$

23 Examples 15, 17 The equation of a quadratic function is $y = -2x^2 + x - 5$
- a Find the equation of its axis of symmetry and the coordinates of the turning point.
- b Is the turning point a minimum or maximum? What does this say about the $x$-intercepts of the graph?
- c Find the discriminant of the quadratic equation $-2x^2 + x - 5 = 0$ and state what this means about the zeros of the equation.
- d Sketch the graph of $y = -2x^2 + x - 5$

24 Example 17 Find values of $p$ for which the quadratic equation $x^2 + x + p - 3 = 0$ has real roots.

25 Example 18 a By using the discriminant, show that the parabolas $y = x^2 - 2x + 3$ and $y = -2x^2 + 5x + 1$ intersect at two points.
b Find these points of intersection.

26 Example 19 Sketch the graph of $y = \dfrac{3}{2x + 1}$

27 Examples 20, 21 Sketch the graph of
- a $f(x) = -2(x - 3)(x + 1)(x + 2)$
- b $y = 3x^3 - 7$

28 Example 22 Sketch the graph of each of the following.
- a $x^2 + y^2 = 4$
- b $(x - 2)^2 + (y + 1)^2 = 9$


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29 Example 23 Find the radius and centre of the circle $x^2 + 8x + y^2 - 2y + 1 = 0$

30 Example 23 Find the equation of the circle with

a radius 7 and centre (0, 0)

b radius 2 and centre (−5, 3)

31 Examples 2, 24 CAS a Sketch the graph $(y + 1)^2 = 2x - 1$

b Find the equation of the axis of symmetry.

c Find the coordinates of its vertex.

d State its domain and range.

e Is this graph a function?

## Application

32 a Factorise $\dfrac{x^2 - 2x}{x - 2}$

b State the domain of $y = \dfrac{x^2 - 2x}{x - 2}$

c What would the graph of $y = \dfrac{x^2 - 2x}{x - 2}$ look like?

33 The discriminant of a quadratic equation is 49.

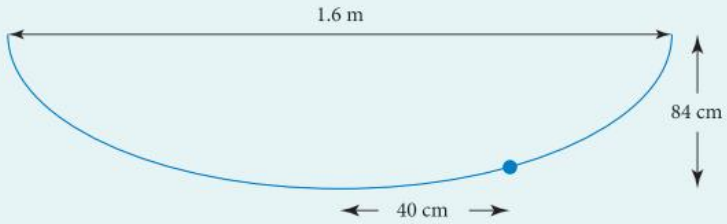
a Describe the zeros of the quadratic equation.

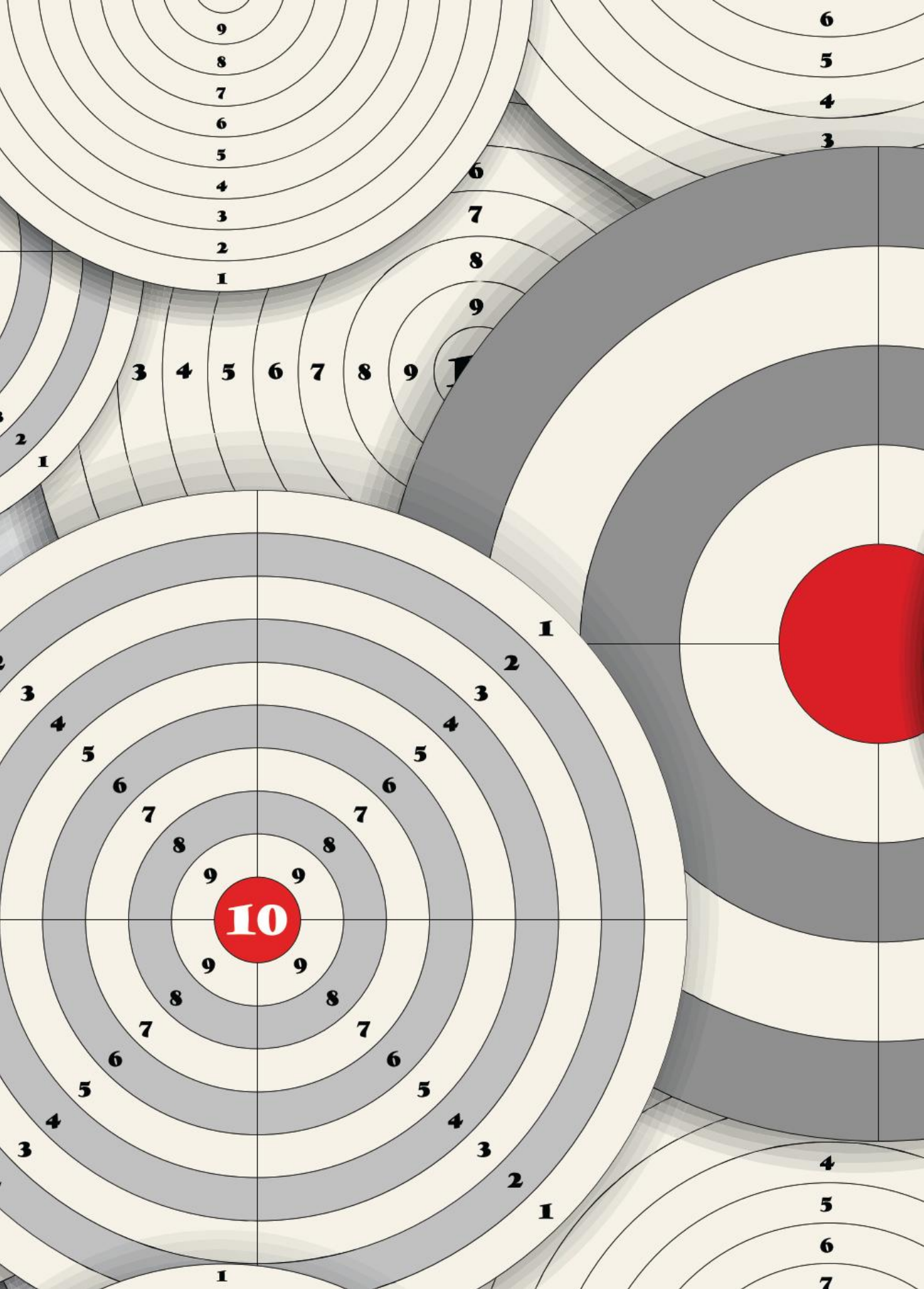
b Evaluate $k$ if the equation is $kx^2 + 5x - 3 = 0$

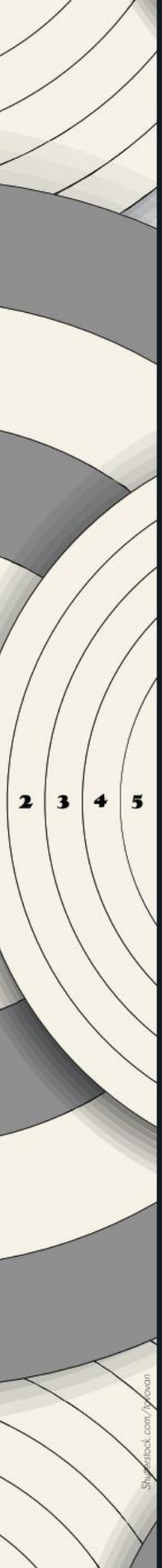
34 Sketch the graph of $y = \sqrt{1 - x^2}$ and state its domain and range.

Practice quiz

35 A parabolic satellite dish is 84 cm deep and 1.6 m wide. How deep is a point 40 cm out from its vertex?




10




# 5

## TERMINOLOGY

combination
complement
conditional probability
dependent events
independent events
intersection
mutually exclusive
tree diagram
two-way table
union
Venn diagram

PROBABILITY

# CONDITIONAL PROBABILITY

5.01 Tree diagrams
5.02 Two-way tables
5.03 Venn diagrams
5.04 Conditional probability
5.05 Dependent and independent events
5.06 Calculating probabilities
5.07 Applications to data
Chapter summary
Chapter review

Prior learning

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## CONDITIONAL PROBABILITY AND INDEPENDENCE

- understand the notion of a conditional probability and use language that indicates conditionality (ACMMM056)
- use the notation $P(A|B)$ and the formula $P(A \cap B) = P(A|B)P(B)$ (ACMMM057)
- understand the notion of independence of an event $A$ from an event $B$, as defined by $P(A|B) = P(A)$ (ACMMM058)
- establish and use the formula $P(A \cap B) = P(A)P(B)$ for independent events $A$ and $B$, and recognise the symmetry of independence (ACMMM059)
- use relative frequencies obtained from data as point estimates of conditional probabilities and as indications of possible independence of events (ACMMM060) AC

# 5.01 TREE DIAGRAMS

In the early sixties, doctors and medical researchers concluded that smoking increased the chances of a person contracting lung cancer. At the time, cigarette companies strenuously denied that this was true, correctly stating that even overwhelming statistical evidence did not constitute proof. It was not until much later that a definite mechanism that linked smoking to cancer was found. However, the statistical evidence that showed a higher probability of cancer amongst smokers did drive the research that found the link.

How do you work out a probability of an event if the chances of it happening depend on whether something else occurs? How do you know if the probability of something happening is affected by something else? The mathematical answers to these questions are the subject of this chapter.

### Example 1

A box of batteries used for calculators originally had 24 new batteries in it. Unfortunately, 10 flat batteries were put into the box when the batteries in a class calculator set were changed, meaning there were only 14 new batteries in the box. The next time batteries were needed, any flat batteries found had to be thrown away.

a If the first battery taken out was flat, what is the probability that the next one will also be flat?

b If the first 2 batteries taken out were OK, what is the probability that the next will be OK?

c If the first 5 taken out were flat (Murphy's law!), what is the probability that the next will also be flat?

### Solution

a There would be 9 flat ones out of the 23 left. $P(\text{Second flat, given first flat}) = \frac{9}{23}$

b There would be 12 useable ones out of the 22 left. $P(\text{Third OK, given first 2 OK}) = \frac{12}{22} = \frac{6}{11}$

c There would be 5 flat ones out of the 19 left. $P(\text{6th flat, given first 5 flat}) = \frac{5}{19}$


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You have previously seen how tree diagrams are used to help define a sample space. They are also useful in situations where the occurrence of one event changes the sample space for another.

**IMPORTANT**

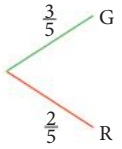
A **tree diagram** of a probability situation shows successive stages of a sample space as branching. It is usually convenient to combine branches with the same outcome and label the combined branch according to the number of possible outcomes. If there are 16 possible outcomes of a stage and 5 of them are the same, this combined branch is labelled as $\frac{5}{16}$.

## Example 2

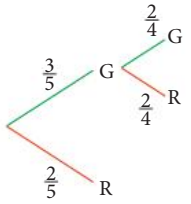
A bag contains 3 green and 2 red marbles. Two marbles are randomly withdrawn. What is the probability that they are of different colours?

### Solution

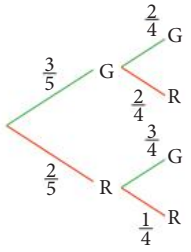
Think of the marbles as being drawn one after the other. In the first stage there are 5 possible outcomes, 3 green and 2 red. Show these as combined branches labelled $\frac{3}{5}$ and $\frac{2}{5}$.




Now do the second stage of the green branch. The sample space for this branch is different because one green marble is gone. There are 4 marbles left, 2 green and 2 red. Show these as combined branches labelled $\frac{2}{4}$ and $\frac{2}{4}$.



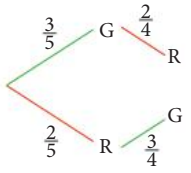

Now do the second stage of the red branch. The sample space for this branch is different because one red marble is gone. There are 4 marbles left, 3 green and 1 red. Show these as combined branches labelled $\frac{3}{4}$ and $\frac{1}{4}$.




The denominators show that the first stage has 5 possible outcomes. Every outcome from the second stage has 4 possible outcomes. Multiply the denominators to get the total.

$n(\text{Sample space}) = 5 \times 4 = 20$

The branch paths that show different colours are the GR and the RG paths. Concentrate on these because that is what you want the probability for.




The numerators show the number of outcomes on each branch. The GR path has 3 on the first part and 2 on the second. Multiply to get the total.

$n(\text{GR}) = 3 \times 2 = 6$

| | |
|---|---|
| Multiply the numerators on the RG path to get the total number of outcomes. | $n(\text{RG}) = 2 \times 3 = 6$ |
| State the number of favourable outcomes. | $n(\text{different colours}) = 12$ |
| Find the probability. | $P(\text{different colours}) = \frac{12}{20} = \frac{3}{5}$ |
| State the answer. | The probability that the marbles are different colours is $\frac{3}{5}$ (60%). |

In Example **2**, you could obtain the same result by multiplying the fractions on the branches to get the probabilities.

$P(\text{GR}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$ and $P(\text{RG}) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$, so

$P(\text{different colours}) = P(\text{GR}) + P(\text{RG}) = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$.

You can add the probabilities because the GR and RG events are mutually exclusive.

This is always true for different paths on tree diagrams because they always show mutually exclusive events. This is a property of their construction.

## Example 3

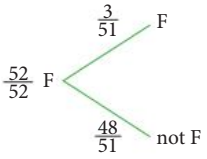
Use a tree diagram to find the probability of being dealt three of a kind as your first three cards from a normal well-shuffled pack of 52 cards.

### Solution

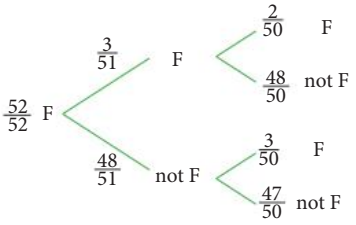
In the first stage, it doesn't matter what the first card is, so this is a 'trunk'. Call the first card F.




In the second stage it only matters if the first card is the same rank as the first (F) or not the same. There are only 3 Fs left out of the 51 cards remaining. Draw the branches for the second stage.




In the third stage it still only matters if the first card is the same as the first (F) or not the same. Now there are only 2 Fs left out of the 50 cards remaining in the F branch. There are 3 out of the 50 in the not F branch. Draw the branches for the third stage.




To get the probability, multiply the fractions on the path you want (F-F-F).

$P(\text{3 of a kind}) = \frac{52}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{1}{425}$

$\approx 0.002\,353\ldots$

Write the answer.

The probability of being dealt 3 of a kind in 3 cards is 1 in 425 or about 0.2353%.


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You could also do Example **3** as a combination problem, in which case the probability would be $\frac{13\times {}^{4}C_{3}}{{}^{52}C_{3}}$, which naturally gives the same answer. Tree diagrams are particularly useful when you can classify the possible outcomes into two or three types at each branch.

In some cases, such as rolling a fair die, the probability of obtaining the desired outcome is the same at each stage.

# EXERCISE 5.01 Tree diagrams

WS
Tree diagrams

## Concepts and techniques

Use tree diagrams whenever appropriate in this exercise.

**1** Example 1 A bag of lollies has 10 milk shakes, 10 toffees and 10 sherbets in it. Lollies are taken out at random.

- **a** If the first 2 are sherbets, what is the probability that the next will also be a sherbet?
- **b** If the first 2 are sherbets, what is the probability that the next will be a toffee?
- **c** The first person to pick keeps putting them back when no one is looking until he has 5 toffees. What is the probability that the next lolly taken out at random is a toffee?
- **d** If the first 5 taken out were toffees, what is the probability that the next will be a milk shake?
- **e** Given that the first 3 taken out were two milk shakes and a sherbet, what is the probability that the next will also be a sherbet?

**2** A PE class of 24 students is divided into two teams (with the 2 left as substitutes) to play hockey after several weeks of learning the game. There are 27 sticks in the cupboard, and 15 of them are nearly new. The others are quite worn. Students are handed the sticks at random by a teacher to save arguments.

- **a** If the first 3 given out are worn, what is the probability that the next will also be worn?
- **b** If the first 5 are nearly new, what is the probability that the next will be worn?
- **c** Given that the first 7 handed out are 3 newish and 4 worn ones, what is the probability that the next will be worn?
- **d** Given that the first 7 are worn, what is the probability that the next will be nearly new?
- **e** Given that the first 3 are nearly new, what is the probability that the next will be nearly new?

**3** Example 2 A bag contains 8 yellow and 6 green lollies. If I choose 2 lollies at random, find the probability that they will both be green.

**4** Two cards are drawn from a deck of 20 red and 25 blue cards (without replacement). Find the probability that they will both be red.

**5** A bag contains 5 red marbles and 8 blue marbles. If 2 marbles are chosen at random, find the probability of getting

- **a** 2 red marbles
- **b** 2 blue marbles
- **c** a red and a blue marble.

**6** Example 3 In a batch of 100 cars, past experience suggests that 3 will be faulty. If 3 cars are selected at random from a batch, find the probability that

- **a** all 3 cars are faulty
- **b** exactly 2 are faulty
- **c** none are faulty.

7 A bag contains 5 red, 6 blue, 2 white and 7 green balls. If 2 are selected at random, find the probability of getting 2 red balls.

8 Three coins are thrown. Find the probability of getting 3 tails.

9 If 2 dice are thrown, find the probability of throwing two 6s.

### Reasoning and communication

10 In Yahtzee, 5 dice are rolled. Find the probability of rolling
a five 6s b no 6s c at least one 6.

11 An archer normally hits the target 9 out of every 10 shots. Under normal circumstances, what is the probability that at least one of her 3 shots hits the target?

12 A coin is thrown 6 times. Find the probability of getting at least 5 heads.

## 5.02 TWO-WAY TABLES

**Two-way tables** are used in calculating probabilities in sample spaces such as throwing two dice.

**IMPORTANT**

A **two-way table** has frequencies or elements of a sample space with two characteristics listed in rows and columns. One characteristic is listed across the columns and the other is listed down the rows. The other cells contain the frequency or element that has the characteristics shown in the corresponding row and column headings.

### Example 4

Find the probability that the total of two normal six-sided dice is more than 4, given that at least one of the dice shows a 3.

#### Solution

Show the totals as a two-way table.
Shade the part for one dice showing a 3.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

How many are in the sample space for 'given that one of the dice shows a 3'?

$n(\text{Sample space}) = 11$


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| | |
|---|---|
| How many of these outcomes are more than 4? | $n(>4) = 9$ |
| Find the probability. | $P(>4, \text{given a } 3) = \frac{9}{11} \approx 0.8182$ |
| State the answer. | The probability that the total is more than 4, given that one of the dice shows a 3, is about 81.2%. |

Two-way tables are often used with data to look for patterns indicating that the probability of one thing depends on another. In medical research data is often collected to determine if a particular indicator could influence other conditions. For example, data could be collected on age at death and BMI to find the extent to which high BMI influences morbidity.

## Example 5

The two-way table below shows the results of a survey of students' eye and hair colours.

| | | Eye colour | | | |
|---|---|---|---|---|---|
| | | Brown | Hazel | Blue | Green |
| Hair colour | Blond | 7 | 10 | 94 | 16 |
| | Black | 68 | 15 | 20 | 5 |
| | Red | 26 | 14 | 17 | 14 |
| | Brown | 119 | 54 | 74 | 29 |

**a** Find the probability of a student having green eyes.
**b** Find the probability of a student with red hair having green eyes.
**c** Compare the probabilities from parts **a** and **b** and comment.

### Solution

| | |
|---|---|
| **a** 64 students from the 582 have green eyes. | $P(\text{Green eyes}) = \frac{64}{582} \approx 11\%$ |
| **b** 14 of the 71 students with red hair have green eyes. | $P(\text{Green eyes, given red hair}) = \frac{14}{71} \approx 19.7\%$ |
| **c** There appears to be some relationship. | Students with red hair are nearly twice as likely to have green eyes than students generally. |

When you are looking for relationships in two-way tables, it is common practice to find the totals of the columns and rows to simplify comparisons. You can also express the values as percentages. In this case there are three choices for the base of the percentages: the table total, the row totals or the column totals. The choice you make depends on what you are investigating.

## Example 6

The table below shows the results of a survey of the kinds of jobs done by a group of workers.

| | Males | Females |
|---|---|---|
| **Blue-collar** | 16 | 7 |
| **White-collar** | 28 | 25 |
| **Service** | 22 | 32 |

a Change the table to a table of percentage of the table total.
b What is the probability that a worker chosen at random from this group is a female white-collar worker?
c What is the probability that a white-collar worker in this group is a woman?
d What is the probability that a woman in this group is a white-collar worker?

### Solution

a The table total is 130.
Change each frequency to a percentage.
Show the totals of the rows and columns.

| | Male | Female | Total |
|---|---|---|---|
| **Blue-collar** | 12.3% | 5.4% | 17.7% |
| **White-collar** | 21.5% | 19.2% | 40.8% |
| **Service** | 16.9% | 24.6% | 41.5% |
| **Total** | 50.8% | 49.2% | 100% |

b Read from the table.
The probability that a random worker is a female white-collar worker is 19.2%.

c 25 of the 53 white-collar workers are women.
$P(\text{Woman, given white-collar}) = \frac{25}{53} \approx 47.2\%$

d 25 of the 64 women were white-collar workers.
$P(\text{White-collar, given woman}) = \frac{25}{64} \approx 39.1\%$

WS Two-way tables

In Example **6**, you could also calculate the answer to part **c** using the percentages $(\frac{19.2}{40.8})$. This gives a slightly different answer (47.1%), and is clearly a little less accurate because it is done after rounding to a percentage. The three different probabilities concerning white-collar women workers show how important it is to be clear about what you intend.

# EXERCISE 5.02 Two-way tables

## Concepts and techniques

1 Example 4 Two normal six-sided dice are thrown and the total of the dice is calculated.
a What is the probability that, given a total of 6, one of the dice shows a 2?
b What is the probability that the total is less than 8, given that one of the dice shows a 5?
c What is the probability that the dice show the same number if the total is 6?
d What is the probability that the dice show the same number?
e What is the probability that the dice are the same number if the total is 4?

2 A six-sided die and a ten-sided die are rolled together and the total is found.
   a What is the probability that the total is more than 5 if the ten-sided die shows a 4?
   b What is the probability that the total is more than 5 if the six-sided die shows a 4?
   c What is the probability that the total is more than 5 if one of the dice shows a 4?
   d What is the probability that one of the dice shows a 4 if the total is more than 5?
   e What is the probability that the six-sided die shows a 4 if the total is more than 5?
   f What is the probability that the ten-sided die shows a 4 if the total is more than 5?
   g What is the probability of a double?

## Reasoning and communication

3 Example 5 The table below shows the types of vehicles people from the city and country drive.

| | Sedan | Station wagon | 4WD | Ute |
|---|---|---|---|---|
| City | 40 | 10 | 15 | 4 |
| Country | 30 | 12 | 20 | 15 |

For this group of people:
   a What is the probability that someone driving a ute is from the country?
   b What is the probability that someone from the country drives a ute?
   c What is the probability that a driver is from the country and drives a ute?
   d What is the probability that a sedan driver is from the city?
   e What is the probability that a country driver is driving a sedan?
   f Explain why the probabilities from parts **d** and **e** do not add up to 1.

4 The table below shows the lunch-break preferences of employees at a particular company. Of course, shorter lunch breaks mean earlier finishing times.

| | Office | Workshop | On-site |
|---|---|---|---|
| Half hour | 6 | 14 | 20 |
| 45 minutes | 8 | 20 | 14 |
| One hour | 13 | 13 | 6 |

For workers in this company:
   a what is the probability that an office employee prefers an hour-long lunch?
   b what is the probability that a workshop employee prefers an hour-long break?
   c what is the probability that an on-site employee prefers an hour-long break?
   d what is the probability that an employee chosen a random prefers an hour-long break?
   e explain why the average of the answers to parts **a**, **b** and **c** does not give the answer in **d**.

5 Example 6 The table below shows the physical assaults reportedly experienced by men and women since the age of 15 in the work environment.

| | Assaults at work since 15 (thousands) | |
|---|---|---|
| | On men | On women |
| Assaults by men | 21.8 | 141.8 |
| Assaults by women | 23.8 | 7.3 |

*Source: ABS*

   a Change the table to a table of percentages of the table total.

For people who have experienced an assault since age 15 in the work environment:

b what is the probability that it was an assault by a man?

c what is the probability that an assault by a man was on another man?

d what is the probability that an assault by a woman was on a man?

e what is the probability that an assault on a man was by a woman?

f what is the probability that an assault on a woman was by another woman?

6 The table below shows the number of years of education of some blue-collar, white-collar and management employees, including school, apprenticeship, TAFE and university.

| | Years of education | | | | |
|---|---|---|---|---|---|
| Worker type | 11–12 | 13 | 14–15 | 16–17 | 18–19 |
| Blue-collar | 10 | 9 | 11 | 0 | 0 |
| Management | 0 | 0 | 5 | 7 | 8 |
| White-collar | 10 | 11 | 19 | 10 | 0 |

a Change the table to percentages of row totals.

b Change the table to percentages of column totals.

c What is the probability that an employee with 14 or 15 years of education is a blue-collar employee?

d What is the probability that an employee with more than 15 years of education is a white-collar employee?

e What is the probability that a white-collar employee has more than 15 years of education?

# 5.03 VENN DIAGRAMS

**Venn diagrams** are very useful for events that have some elements in common in a sample space. This is particularly the case for situations where the elements can be classified according to different characteristics.

**IMPORTANT**

A **Venn diagram** shows sets or events as closed shapes (usually circles or ellipses) with intersections shown as overlapping. The circles or ellipses are enclosed by a rectangle representing the universal set (sample space).

### Example 7

In a class of 24 students, 9 take Physics and 11 study Chemistry. 4 take both Physics and Chemistry.

a Draw a Venn diagram to show the information.

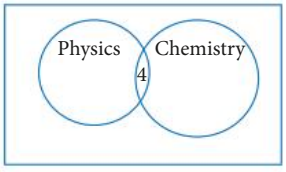
b Find the probability that a student not doing Chemistry does do Physics.

c Find he probability that a student who does Physics also does Chemistry.

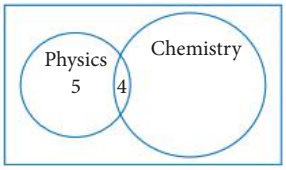

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## Solution

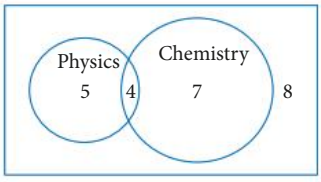
**a** Start numbering by putting 4 in the intersection.




9 – 4 = 5 must go in the rest of Physics.




Finish the rest of the numbers.
11 – 4 = 7 must go in the rest of Chemistry.
24 – (5 + 4 + 7) = 8 goes in the rest of the sample space.




**b** There are 8 + 5 = 13 students not doing Chemistry. Of these, 5 do Physics.

$P(\text{Physics, given not Chemistry}) = \frac{5}{13}$

**c** Of the 9 students doing Physics, 4 also do Chemistry.

$P(\text{Chemistry, given Physics}) = \frac{4}{9}$

Venn diagrams are even more useful in situations with three overlapping events in a sample space.

WS
Venn diagrams

## Example 8

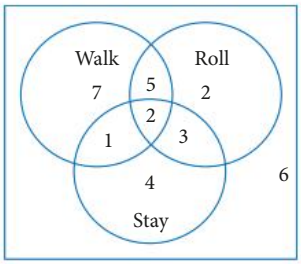
In a puppy school, dogs are taught to walk to heel, to stay and to roll over on command. Out of the 30 dogs in a particular group, 10 have learnt to stay and 12 to roll over. 2 dogs can do all three, 7 can walk to heel and roll over, 5 can stay and roll over and 3 can walk to heel and stay. 6 have not yet learnt any of the obedience tasks.

**a** Find the probability that a random dog who can walk to heel can also roll over.

**b** Find the probability that a random dog who can stay can walk to heel.

## Solution

Draw a Venn diagram showing the numbers of dogs. Start with the 2 who can do everything in the intersection of all three.
Then put in the other 5 who can both walk to heel and roll over and so on.




**a** Of the 15 dogs who can walk to heel, 7 can roll over.

$P(\text{Roll over, given walk to heel}) = \frac{7}{15}$

**b** Of the 10 who can stay, 3 can walk to heel.

$P(\text{Walk to heel, given stay}) = \frac{3}{10}$

# EXERCISE 5.03 Venn diagrams

## Concepts and techniques

1 Example 7 In a class of 27 students, 15 take neither art nor music. Of the 6 who take music, 3 also study art.
   a What is the probability that a student who takes music also takes art?
   b Given that a student does not do music, what is the probability the student takes art?
   c What is the probability that a student who does not take art does not study music either?

2 A bargain table has 20 polo-necked and 20 round-necked T-shirts. There are 6 red polonecks, 4 green polonecks, 8 green round necks and 2 red ones.
   a Given that the first shirt sold is red, what is the probability that it is a poloneck?
   b If the first shirt sold is a poloneck, what is the probability that it is green?
   c If the first two shirts sold are red, what is the probability that the next shirt sold will be neither red nor green?

3 Example 8 The numbers from 1 to 100 may be classified as being divisible by 2, 3 or 5. Draw a Venn diagram to show how many are in each classification. A number from 1 to 100 is chosen at random.
   a Given that the number is divisible by 5, what is the probability that it is also divisible by 2?
   b If the number is not divisible by 2, what is the probability that it is divisible by 3?
   c What is the probability that a number is divisible by 5, given that it is divisible by 2?

## Reasoning and communication

4 From a selection of Australian electorates, 25 include some city areas and of these 16 do not include any rural areas. 17 of the electorates include rural areas and 11 include some coastal areas. 2 of the city areas that do not include any rural areas include part of the coast. There are three electorates that include coastal, rural and city areas and 7 of the electorates including rural land also include coastal land. All the areas in electorates are classified as city, rural or coastal.
   a How many electorates are in the selection?
   b Given that an electorate includes city areas, what is the probability that it includes no coastal or rural areas?
   c If an electorate includes country areas, what is the probability that it includes city areas?
   d Given that an electorate includes coastal areas, what is the probability it does not include any city areas?

5 On a free dress day, 9 of the 12 boys in a class wear jeans, and of these 6 wear blue tops. 8 of the girls wear jeans and 5 of these wear blue tops. There are 26 students in the class, and 14 of them wear blue tops. Only one of the girls not wearing jeans wears a blue top.
   a What is the probability that someone wearing jeans is a boy?
   b What is the probability that a boy not wearing jeans has a blue top?
   c What is the probability that a girl wearing a blue top is also wearing jeans?

6 Sunscreens may include many different ingredients, including the active ingredients, oils, preservatives, colouring, scent, emulsifiers, water and others. The ingredients of 25 different 30+ sunscreens are examined and 15 have scents in them. Of these, 3 also have both preservatives and colouring. 10 of the sunscreens have colouring and the 4 that do not have colouring, preservative or scent are labelled 'non-allergenic'. 5 of those with scent also have

colouring and 7 of those with scent have preservatives. 2 of the sunscreens have colouring but no preservative or scent.

a Given that a sunscreen has scent, what is the probability that it has preservative?
b What is the probability that a sunscreen with preservative also has scent?
c What is the probability that a sunscreen without preservative is labelled 'non-allergenic'?
d What is the probability that a sunscreen with colouring also has preservative?
e What is the probability that a sunscreen has colouring, given that it has no scent?

# 5.04 CONDITIONAL PROBABILITY

In the earlier sections of this chapter, you used tree diagrams, two-way tables and Venn diagrams to find the probabilities of events in cases where other events occur as well. The events that are assumed to occur may occur simultaneously or before the event whose probability you are calculating. It is even possible that the assumed event occurs after the event whose probability you are calculating. In all these cases, where a particular event is assumed to occur and the probability of another event is calculated under this assumption, we say that the calculated probability is a **conditional probability**. Since this kind of calculation occurs so often, it is convenient to have a definition and notation for conditional probability.

**IMPORTANT**

The **conditional probability** of an event $A$ given event $B$ is written as $P(A|B)$, and is the probability that event $A$ occurs, assuming that event $B$ occurs. The sample space for $P(A|B)$ is restricted to $B$.

## Example 9

Three cards are taken from a normal 52-card pack.

What is the probability of getting three kings given that the first two cards are kings?

If the first two cards are kings, there are 2 kings in the 50 cards left.

$P(3 \text{ kings} \mid \text{first 2 kings}) = \frac{2}{50} = \frac{1}{25}$

How is the probability $P(A|B)$ related to other probabilities? Since the sample space for this is just $B$, only elements within $B$ are in its sample space.

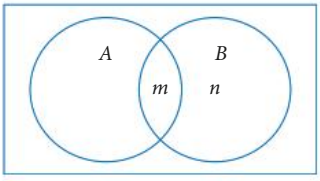
**IMPORTANT**

The conditional probability of $A$ given $B$ can be calculated using the formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$.

This can be rewritten as $P(A \cap B) = P(A \mid B)P(B)$.

The formula above is easily proven for finite sample spaces.

Suppose there are $n + m$ elements in $B$ with $m$ of them in $A \cap B$ and $u$ elements in the universal set as shown in the diagram.



Then $P(A|B) = \dfrac{m}{m+n}$

$$= \frac{m}{1} \times \frac{1}{m+n}$$
$$= \frac{m}{1} \times \frac{u}{u} \times \frac{1}{m+n}$$
$$= \frac{m}{u} \times \frac{u}{m+n}$$
$$= \frac{m}{u} \div \frac{m+n}{u}$$
$$= P(A \cap B) \div P(B)$$
$$= \frac{P(A \cap B)}{P(B)}$$

Multiplying by $P(B)$ gives the alternate form $P(A \cap B) = P(A \mid B)P(B)$.

There are often times when the use of the formula above is much more efficient that other means of calculation.

While tree diagrams are an important probability method, you can use conditional probability to solve many problems without drawing a tree diagram.

## Example 10

What is the probability of being dealt two jacks in the first two cards from a normal shuffled 52 card pack?

### Solution

| | |
|---|---|
| Think of two jacks in succession. | $P(2 \text{ jacks}) = P(2\text{nd is jack} \mid \text{first is jack}) \times P(\text{first is jack})$ |
| After one jack, there are 3 left. | $= \frac{3}{51} \times \frac{4}{52} = \frac{1}{221} \approx 0.004\,525$ |
| Write the answer. | The probability of 2 jacks is about 0.45%. |

## Example 11

In Australia, the probability of having blue eyes is about 35% and the probability of having (natural) fair hair is about 22%. The probability of having blue eyes and fair hair is about 16%.

**a** What is the probability that a fair haired person will have blue eyes?

**b** What is the probability that someone with blue eyes will have fair hair?

| | |
|---|---|
| **a** This is blue eyes given fair hair. | $P(\text{blue} \mid \text{fair}) = \dfrac{P(\text{blue and fair})}{P(\text{fair})} = \dfrac{0.16}{0.22} \approx 73\%$ |
| Write the answer. | The probability that a fair-haired person has blue eyes is about 73%. |
| **b** This is fair hair given blue eyes. | $P(\text{fair} \mid \text{blue}) = \dfrac{P(\text{blue and fair})}{P(\text{blue})} = \dfrac{0.16}{0.35} \approx 46\%$ |
| Write the answer. | The probability that a blue-eyed person has fair hair is about 46%. |


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You can use a string of conditional probabilities together. For example, the probability of getting 4 red marbles out of 4 marbles drawn from a bag containing 5 red and 5 blue ones is

$$\text{P(Red}|\text{3 reds)} \times \text{P(Red}|\text{2 reds)} \times \text{P(Red}|\text{1 red)} \times \text{P(Red)} = \frac{2}{7} \times \frac{3}{8} \times \frac{4}{9} \times \frac{5}{10} = \frac{1}{42}.$$

The string of probabilities above may make more sense to you if you read it backwards. Then you can think of it as red, then another red after the first red, then another red after the first two reds and another red after the first 3 reds.

# EXERCISE 5.04 Conditional probability

## Concepts and techniques

1 Example 9 Two cards are taken from a normal pack of cards.
   a What is $P(2 \text{ spades}|1 \text{ spade})$?
   b What is $P(2 \text{ aces}|1 \text{ ace})$?
   c What is $P(2 \text{ red cards}|1 \text{ red card})$?

2 Two counters are taken at random from a bag containing 3 green, 4 red and 5 orange counters.
   a What is $P(2 \text{ green}|1 \text{ green})$?
   b What is $P(\text{Green and orange}|1 \text{ green})$?
   c What is $P(2 \text{ orange}|1 \text{ orange})$?
   d What is $P(\text{Red and orange}|1 \text{ red})$?
   e What is $P(2 \text{ red}|1 \text{ red})$?

3 Example 10 What is the probability of being dealt 2 spades in a row from a normal pack of cards?

4 What is the probability of being dealt 5 hearts in a hand of 5 cards from a normal pack of cards?

5 A bag contains 5 red, 4 green and 3 yellow cards. What is the probability that 3 cards taken out at random are:
   a all red? b all green? c all yellow?

6 A coin tray contained 6 five-cent, 5 ten-cent, 7 twenty-cent, 3 fifty-cent, 4 one-dollar and 5 two-dollar coins mixed up together. Three coins are taken out at random.
   a What is the probability that they are all five-cent coins?
   b What is the probability that they are all fifty-cent coins?
   c What is the probability that they are all twenty-cent coins?
   d What is the probability that the first two are ten-cent coins and the last is a twenty-cent coin?
   e What is the probability that they make a total of 40 cents?

## Reasoning and communication

7 Example 11 If one engine fails, a twin-engine light aeroplane can continue flying on only the second engine, but the probability of failure of the second engine increases by 3 times due to the extra load. Pilots are trained to land as soon as possible whenever some failure occurs. On a flight from Melbourne to Shepparton the probability of failure of an engine on a twin-engine plane is 0.01%. What is the probability of both engines failing?

8 A factory has 2 stamping machines that are used to make steel washers. The probability that the die becomes misaligned on any day in the new machine is 1%, but in the older machine it is 5%. If a die is misaligned the machine produces faulty washers. The new machine works faster, making twice as many washers as the old one.
   a What is the probability that a washer is produced by the old machine?
   b What is the probability that a washer is from the old machine and is defective?
   c What is the probability that a washer is from the new machine and is defective?
   d What is the probability that a defective washer is from the old machine?

# 5.05 DEPENDENT AND INDEPENDENT EVENTS

In the previous section of this chapter you learned how to work out conditional probability. This is the formal mathematical method of calculating the probabilities of combined events where the occurrence of one event affects the probability of the other. Events such as these are called **dependent events**. The name for events that do not affect each other's probabilities is **independent events**.

**IMPORTANT**

Two events $A$ and $B$ are **independent** if and only if $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

For independent events $A$ and $B$, $P(A \cap B) = P(A) \times P(B)$.

If and only if means that the sentence works both ways. If $P(A|B) = P(A)$ and $P(B|A) = P(B)$ then the events are independent, and if the events are independent then $P(A|B) = P(A)$ and $P(B|A) = P(B)$.
In fact, if $P(A|B) = P(A)$, then $P(B|A) = P(B)$ and vice versa, as shown below.

Since $P(A \cap B) = P(A|B) \times P(B)$, if $P(A|B) = P(A)$, then substituting gives $P(A \cap B) = P(A) \times P(B)$.

Now $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$, so substituting gives $P(B|A) = \dfrac{P(A) \times P(B)}{P(A)} = P(B)$.
Thus all four conditions are equivalent: $A$ and $B$ are independent; $P(A|B) = P(A)$; $P(B|A) = P(B)$; and $P(A \cap B) = P(A) \times P(B)$ are interchangeable.

Mathematically, we define dependent events as below.

**IMPORTANT**

**Dependent** events are not independent.

$P(A|B) \neq P(A)$, $P(B|A) \neq P(B)$; and $P(A \cap B) \neq P(A) \times P(B)$.

Notice that whether events are dependent or independent, the probability of the event $A \cap B$ can be obtained by multiplying two probabilities. In the case of dependent events you need to use conditional probability, but if the events are independent you just multiply the probabilities of the separate events. This is the same as the method used for tree diagrams.

## Example 12

A fair eight-sided die is rolled twice. What is the probability of two ones?

### Solution

| | |
|---|---|
| Write the probability of getting a one. | $P(\text{one}) = \frac{1}{8}$ |
| Use the rule for independent events. | $P(\text{one} \cap \text{one}) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$ |
| Write the answer. | The probability of 2 ones is $\frac{1}{64}$. |

For multiple repetitions of independent events, you can just keep multiplying the probabilities.

## Example 13

Assuming boys and girls are equally likely, what is the probability that in a family of 5 children, all the children will be girls?

### Solution

| | |
|---|---|
| Write the probability of a girl. | $P(\text{girl}) = \frac{1}{2}$ |
| Write the probability of 5 girls. | $P(5 \text{ girls}) = P(\text{girl} \cap \text{girl} \cap \text{girl} \cap \text{girl} \cap \text{girl})$ |
| Use the rule for independent events. | $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$ |
| Write the answer. | The probability of 5 girls in a family of 5 children is $\frac{1}{32}$. |

Some families have lots of children of the same sex over several generations. This means that you cannot always assume that the birth of boys or girls is equally likely. Probability is often used in court cases. For example, DNA evidence often is given like 'the probability that another person left this blood behind is 1 in 150 million'.

## Example 14

SIDS (cot-death) in non-smoking households in Britain was claimed to occur in only 1 in 8543 cases in a number of famous court cases, although the general occurrence was about 1 in 1300. If you assume that cases of SIDS in the same family are independent, what is the probability of two such deaths in a family of non-smokers?

### Solution

| | |
|---|---|
| Write the probability of a SIDS death. | $P(\text{SIDS death}) = \frac{1}{8543}$ |
| Use the rule for independent events. | $P(2 \text{ SIDS deaths}) = \frac{1}{8543} \times \frac{1}{8543}$ |

| | |
|---|---|
| Calculate the result. | $= \frac{1}{72\,982\,849}$ |
| Write the answer. | If you make the assumptions in the question, the probability of 2 SIDS deaths in the same family is about 1 in 73 million. |

There is no basis to assume that SIDS deaths in the same family are independent of each other – medically one would immediately suspect some genetic factor. Sally Clark, an English solicitor and daughter of a policeman, was convicted in 1999 of murdering her two sons on the basis of the calculations shown in Example **14**. Among others, the Royal Statistical Society objected to this blatant misuse of statistics. Her conviction was eventually overturned and she was released after having served over 3 years in prison.

### INVESTIGATION Misuse of probability and statistics

Research the Sally Clark case and other similar cases for the misuse of probability in court.

Find other misuses of probability and statistics.

What is the 'gambler's fallacy'? Why is it wrong?

Your teacher might want you to write a report on common misconceptions about probability.

## EXERCISE 5.05 Dependent and independent events

### Concepts and techniques

1 Example 12 A die is rolled multiple times. What is the probability that:
   a The first two times are 5s?
   b The first three times are 3s?
   c The first four times are 2s?

2 A card is drawn from a well-shuffled normal pack of 52 cards. The card is checked, replaced and the pack is reshuffled.
   a What is the probability of getting two kings when this is done twice?
   b What is the probability that if this is done three times you will get 3 spades?
   c What is the probability that if you do this four times you will get 4 aces?
   d What is the probability that if you do this three times you will get 3 number cards?
   e What is the probability that if you do this 5 times you will get 5 court cards (J, Q, K)?

3 What is the probability of getting ten heads in a row when you throw a fair coin?

4 Examples 13, 14 A box of practice basketballs has 18 balls, of which 7 are slightly flat.
   a One of the players takes out a ball, and if it is flat, throws it back and tries again. What is the probability that three in a row are flat?
   b Then three balls are taken out together. What is the probability they are all flat?

5 Szechuan pepper has bits of stem attached to about 40% of the peppercorns. You usually remove the stems before using the pepper. Assuming that the probability of having stems

attached does not change, what is the probability that when you check 5 Szechuan peppercorns they are all clean (without stems)?

6 A mixture of red, green and black peppercorns has 20% red, 30% green and 50% black peppercorns. Assuming that when you take out a small quantity the proportions are unchanged, what is the probability of getting:
**a** 4 red ones in a row? **b** 5 green ones in a row? **c** 6 black ones in a row?

7 A box of dice has 5 red, 5 green and 5 white dice in it. Work out each of the following probabilities when dice are taken out at random.
**a** 3 green in a row if the die is replaced before the next is taken.
**b** 3 green if a row if they are not replaced.
**c** 2 red in a row with replacement.
**d** 2 red in a row without replacement.
**e** 4 white in a row with replacement
**f** 4 white in a row without replacement.

8 A cutlery drawer contains knives and forks from two incomplete sets with different patterns. There are 5 knives, 2 spoons and 3 forks from one set and 3 knives, 4 spoons and 4 forks from the other. If places are set at random, what is the probability that the first place has:
**a** matching knife and fork from the first set?
**b** matching knife and fork from the second set?
**c** unmatching knife and fork?
**d** matching knife, fork and spoon?
**e** unmatching cutlery (obviously two of the three must match)?

## Reasoning and communication

9 A blood test for a relatively rare condition is correct 90% of the time. The condition is actually found in only 0.5% of the population.
For random screening tests, find each of the following probabilities.
**a** The probability that a test will be positive if the person does not have the condition (a false positive).
**b** The probability that a test will be positive if the person does have the condition (a true positive).
**c** The probability that a test will be negative but the person will have the condition (a false negative).
Doctors normally ask for blood tests only if they suspect a condition will be present.
**d** What are the implications of these results for random screen tests for rare conditions?
**e** What are the implications of these results for tests ordered by doctors?

10 Busy roads often have penetrating objects like screws, nails and pieces of wire on them. It is estimated that the probability of getting a puncture in 100 km of a trip on a busy road is about 0.05%. What is the probability of getting two punctures on a trip from Melbourne to Brisbane via the Newell highway (about 1700 km)? Hint: Find the probability of not getting a puncture on the trip.

11 A very large city has such high pollution rates that the probability of exceeding recommended levels of air pollution on any day is an average of 15% (over the year).
Assume that the pollution rates on successive days are independent and find the following.
**a** The probability of exceeding recommended levels on 2 successive days.
**b** The probability of exceeding recommended levels on 5 successive days.
**c** The probability of a two-week period when recommended levels are exceeded every day.

# 5.06 CALCULATING PROBABILITIES

In Chapter 2, you learnt the rules for counting combinations, complementary events and combining the probabilities of unions of events.

**IMPORTANT**

The number of **combinations** of $r$ objects from $n$ distinct objects is given by

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!}.$$

For an event $A$, the probability of its **complement** $\overline{A}$ is given by $P(\overline{A}) = 1 - P(A)$.

For any events $A$ and $B$ in a sample space, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

For **mutually exclusive** events, $A \cap B = \varnothing$, i.e. $P(A \text{ and } B) = 0$, so $P(A \text{ or } B) = P(A) + P(B)$.

The words 'and' and 'or' have specific meanings for events.

- $A$ and $B$ means the intersection, $A \cap B$
- $A$ or $B$ means the union, $A \cup B$

Together with the rules you have learnt in this chapter, you now have a wide set of tools for calculating probabilities.

## Example 15

CAS A bag contains 3 red, 4 green and 5 black jellybeans. What is the probability that when two jellybeans are shaken out at random neither is black?

### Solution

One way to do this to use mutually exclusive events.

$P(\text{no black}) = P(2 \text{ red}) + P(2 \text{ green}) + P(\text{red and green})$

It is easy to use conditional probability for the first two and combinations for the last.

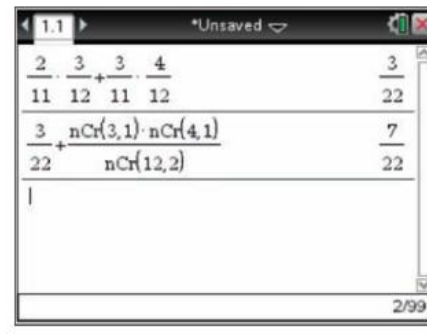
$$= \frac{2}{11} \times \frac{3}{12} + \frac{3}{11} \times \frac{4}{12} + \frac{{}^3C_1 \times {}^4C_1}{{}^{12}C_2}$$

Perform the calculations.

$$= \frac{1}{22} + \frac{1}{11} + \frac{2}{11} = \frac{7}{22}$$

**TI-Nspire CAS**

Do the first part of the calculation, then add the second part.
Use a calculator page ( ) and use $\frac{\square}{\square}$ from the templates ( ).
Get nCr from the catalogue ( ) or from menu, 5: Probability and 3: Combinations.
You can do it altogether but the screen gets a bit crowded





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**ClassPad**

Make sure that you have set **Standard** at the bottom.

Use the $\sqrt{\alpha}$ menu and press Keyboard.

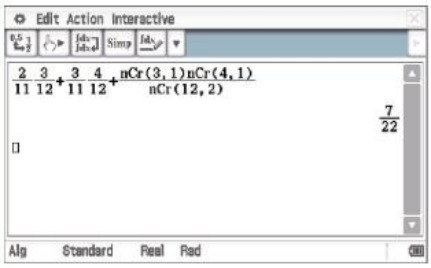
The fraction symbol $\frac{\square}{\square}$ is found in the Math1, Math2, Math3 or Trig tabs. Enter the values in the fractions using the arrows to move between the numerator and denominator.

**nCr** is found in the list of commands under the Catalog tab.

Write $\binom{3}{1}$ as nCr(3,1).

When the expression is complete, tap EXE or press EXE.

Note that the multiplication symbol is optional when multiplying fractions.




Write the answer.

The probability of getting no black ones is $\frac{7}{22}$.

There are other ways to do Example **15**, and you should already know that ${}^{n}C_1 = n$, so you would use $3 \times 4$ instead of ${}^{3}C_1 \times {}^{4}C_1$.

## Example 16

What is the probability of getting a 4 or a 5 on at least one of two 6-sided dice when they are thrown?

### Solution

You can use the rule for a union instead of drawing a two-way table for this problem.

$P(4 \text{ or } 5) = P(4 \text{ or } 5 \text{ on one die}) + P(4 \text{ or } 5 \text{ on other}) - P(4 \text{ or } 5 \text{ on both})$

(4 or 5 on both) has 4 disjoint events: $(4 \cap 4)$, $(4 \cap 5)$, $(5 \cap 4)$ and $(5 \cap 5)$, and each one has probability $\frac{1}{6} \times \frac{1}{6}$.

$= \frac{2}{6} + \frac{2}{6} - 4 \times \frac{1}{6} \times \frac{1}{6} = \frac{5}{9}$

Write the answer.

The probability of a 4 or 5 is $\frac{5}{9}$.

If you work out a probability problem using different but correct methods, you will get the same answer each time.

# EXERCISE 5.06 Calculating probabilities

## Concepts and techniques

1 CAS Example 15 A pencil case contains 4 blue biros, 3 red biros and 3 black biros. Two biros are taken out at random. What is the probability that:

a they are both blue
b they are both black
c neither is blue
d neither is black?

2 CAS Example 16 A roulette wheel has numbers from 0–36. You cannot bet on zero in casinos.
Eight of the odd numbers are black and the others are red. 10 of the even numbers are red and the others are black. The zero is green. What is the probability that when the wheel is spun the ball lands on:
a black or an odd number?
b red or an even number?


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3 CAS Two fair six-sided dice are rolled. What is the probability that the total is less than 5 or more than 9?

4 CAS One biased coin has a probability of 0.6 of landing on heads. Another has a probability of 0.4. If the two coins are thrown, what is the probability of getting a head and a tail?

## Reasoning and communication

5 CAS Two dice are biased so that one has double the normal chance of landing with the six uppermost and the other has half the normal chance of landing with the six uppermost. In both cases, the probabilities of landing with other numbers up are equal.
a Compare the probability of a total of 12 with the probability for fair dice.
b Compare the probability of a total of 2 with the probability for fair dice.
c Compare the probability of a total of 10 with the probability for fair dice.
d Compare the probability of a total of 4 with the probability for fair dice.

6 CAS In the game pontoon, you are initially dealt two cards. An ace counts as 1 or 11 and the court cards count as 10 points. You always bet against the dealer, and the object is to get a total as close to 21 as possible without going over (busting). The best possible hand is a pontoon, which has an ace and a ten or court card. What is the probability of being dealt a pontoon?

7 CAS The probability that a particular pedigree dog will develop breathing problems is 0.06. This particular breed is also subject to joint problems, with 8% developing severe arthritis. Assuming these conditions are independent, what is the probability of a dog of this breed developing breathing or joint problems?

8 CAS Mai wants to record 3 programs that are on different channels on TV one night. The first program is due to finish at 7:30 p.m., the second to start at 7:30 p.m. and finish at 8 p.m. and the third to start a 8 p.m. and finish at 9 p.m. She can program her PVR to record them and switch over at the end of each program, but knows that the programs may not start or finish on time. From experience, she knows the first program has a 10% chance of starting early and a 20% chance of starting late, the second a 15% chance of starting early and a 25% chance of starting late and the third a 5% chance of starting early and a 15% chance of starting late. If a program starts early or late the length of the program does not change. Assume that the change in starting or finishing time is the same for each channel and work out the probability that if she sets the PVR to start early and finish late by more than this amount she will be able to watch all three programs in their entirety.

# 5.07 APPLICATIONS TO DATA

Governments, businesses and other organisations such as medical research institutes collect a great deal of data. They collect the data to do better planning, to inform decisions, to guide decisions and so on. In order to make sense of the data, it is analysed in various ways. One way of doing so is to use the data to estimate the probabilities of events and to determine whether variables are likely to be dependent or independent. For example, retailers are very interested in the effectiveness of advertising campaigns, so will conduct research to determine whether their advertising expenditures produce worthwhile improvements in sales.

## Example 17

It is common practice to make 'introductory offers' on new products. A market research company collected information on the sales of a new biscuit from 50 different stores matched for a range of variables. In half the stores, the biscuits were introduced at prices of $1.99 per packet, rising to the regular price of $2.99 after two weeks. Sales were compared after 6 weeks for the entire period and the following results were obtained. Sales are in thousands of packets.

| | Sales in thousands of packets | | |
|---|---|---|---|
| | 40–59.9 | 60–79.9 | 80–99.9 |
| **Introductory offer** | 10 | 9 | 6 |
| **No introductory offer** | 7 | 9 | 8 |

a Given the introductory offer, what is the probability of total sales of 80 000 or more packets?
b Given no introductory offer, what is the probability of total sales of 80 000 or more packets?
c What conclusion can be drawn?

### Solution

a 6 out of 25 stores achieved high sales with the offer.

$P(\text{high sales}|\text{offer}) = \frac{6}{25} = 24\%$

b 8 out of 25 stores achieved high sales with no offer.

$P(\text{high sales}|\text{no offer}) = \frac{8}{25} = 32\%$

c The stores were matched, so the sample was well chosen and there were a reasonable number.

In this case, for biscuit sales, an introductory offer may have hindered long-term sales.

Example **17** is actually a simplification of the evidence obtained in a real study in the United States after a psychology professor suggested that introductory discounts would have a long-term detrimental effect on sales. The actual analysis was much more detailed, but it did validate the professor's theory. It is interesting to speculate on the reasons for this and whether it applies to other products.

## Example 18

A large-scale study of breast cancer screening was performed in which 31 000 women aged 40–64 were encouraged to undertake screening. Another group of 31 000 40–64 year-old women were not so encouraged. This was the control group. The groups were randomly selected from the members of a health fund. Of the women encouraged to undertake breast cancer screening, 10 800 chose not to do so. The study, over a period of 5 years, had the following results.

| | Sample size | Breast cancer deaths | Other deaths |
|---|---|---|---|
| **Screened** | 20 200 | 23 | 428 |
| **Refused screening** | 10 800 | 16 | 409 |
| **Control** | 31 000 | 63 | 879 |

For this group, over the five years of the study:

**a** change the numbers of deaths to numbers of deaths per thousand women for each group.
**b** what is the probability of dying of breast cancer given screening?
**c** what is the probability of dying of breast cancer given no screening?
**d** is death from breast cancer independent of screening?

### Solution

**a** Divide the numbers of deaths by the number of women and multiply by 1000 for each row.

| | Breast cancer | Other |
|---|---|---|
| **Screened** | 1.14 | 21.19 |
| **Refused** | 1.48 | 37.87 |
| **Control** | 2.03 | 28.35 |

**b** The number is for 1000 women.

$P(\text{Death from breast cancer}|\text{screening}) \approx 0.114\%$

**c** Also for 1000 women.

$P(\text{Death from breast cancer}|\text{no screening}) \approx 0.148\%$

**d** The probabilities are not equal.

Death from breast cancer appears to be dependent on screening. Screening seems to reduce the chances of dying.

Calculations like those in Example **18** are common in medical statistics. If you don't know the incidence of a particular condition in the population, it makes sense to compare the results of applying and not applying a particular procedure.


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## INVESTIGATION Deaths from cancer

The other results of the study in Example **18** are also interesting. It is obvious that merely being offered screening could not reduce the chance of getting breast cancer and increase the chance of dying of something else. How do you explain the numbers?

There is a great deal of statistical research available on cancers.

Choose a cancer and examine the statistical information available to calculate probabilities. The risk factors and screening methods for some cancers are well-known. For example, skin cancers, bowel cancer and mesothelioma.

Are the screening methods useful?

If they are useful, who should pay for them

# EXERCISE 5.07 Applications to data

## Reasoning and communication

1 Examples 17, 18 The table below shows the deaths from suicide in Australia over the 10-year period from 2001–2010 inclusive. The number of men and women over the period was an average of about 10 million each.

| | Method | | | | |
|---|---|---|---|---|---|
| | Poison | | Hanging | Firearms | Other |
| | With drugs | Other | | | |
| Males | 1391 | 2849 | 9063 | 1765 | 4229 |
| Females | 1354 | 591 | 2032 | 105 | 1017 |

*Source: ABS*

Find the following over the period shown in the table.

a The probability of suicide.
b The probability of suicide, given the person was male.
c Was the suicide independent of sex?
d The probability that a suicide was committed using firearms.
e Given that the person was male, the probability that a suicide was committed using firearms.
f Was suicide committed using firearms independent of sex?
g The probability that a suicide was committed using drugs.
h Given the person was female, the probability that a suicide was committed using drugs.
i Was suicide committed using drugs independent of sex?

2 The table below shows the number of Australian school students (15–19 years old) with part-time jobs in the middle of each year shown. The figures are in thousands. The numbers of 15–19 year old boys and girls attending school were about 400 000 each over the period.

| | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|
| **Boys** | 111.4 | 111.7 | 110.2 | 108.1 |
| **Girls** | 136.1 | 141.0 | 143.5 | 144.6 |

*Source: ABS*

Calculate the following for students in the age group shown.

a In 2009, what was the probability that a school student had a part-time job?
b In 2009, given a student was female, what was the probability that she had a part-time job?
c In 2009, was having a part-time job independent of sex?
d In 2012, what was the probability that a school student had a part-time job?
e In 2012, given a student was female, what was the probability that she had a part-time job?
f In 2012, was having a part-time job independent of sex?
g Do your results support the proposition that part-time employment of students changed over the period? How?

3 A famous Australian court case heard from a prosecution expert witness about the results of an experimental blood test. The blood test, which was 80% accurate, was used to determine whether a blood stain in a car came from a baby. A positive result shows the presence of the foetal blood factor, found only in children under 1 year old. About 0.9% of the population are under 1 year old. The result was positive and the expert witness said that this proved the blood was from a baby.

a Calculate the probability that the blood was not from a baby and the result was wrong.
b Calculate the probability that the blood was from a baby and the result was right.
c Calculate the probability that a random blood test will return a positive result.
d Calculate the probability that a positive blood sample is from a baby.
e Comment on the testimony of the expert witness in this case.



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4 A study of weight and heart disease had the following results.

| | Obese or overweight | Not obese or overweight |
|---|---|---|
| **Subjects with heart disease** | 821 | 379 |
| **Subjects without heart disease** | 287 | 913 |

The probability of having heart disease in the general population is about 30%.

a What is the probability of having heart disease and being obese or overweight? Hint: You cannot just use $\frac{821}{1108}$ or $\frac{821}{1200}$.

b What is the probability of not having heart disease and being obese or overweight?

c What is the probability of being obese or overweight?

d What is the probability, given being obese or overweight, of having heart disease?

e What is the probability, given that the person is not obese or overweight, of having heart disease?

f Compare the risk of heart disease for people who are obese or overweight and for people who are not obese or overweight.

5 A clinical study of sleep and colds was done on healthy men and women aged 21–55. The volunteers were infected with rhinovirus after their sleep patterns were established for the 14 days prior to infection. The results were as follows.

| | Cold | No cold |
|---|---|---|
| **Less than 7 hours sleep** | 37 | 32 |
| **7 to 8 hours sleep** | 23 | 40 |
| **More than eight hours sleep** | 10 | 42 |

Assuming that the clinical trials measure whether you will get a cold when exposed to the virus, calculate the following for people exposed to the virus.

a The probability of getting a cold.

b Given less than 7 hours of sleep a night preceding exposure, the probability of getting a cold.

c Given more than 8 hours of sleep a night preceding exposure, the probability of getting a cold.

d What can you conclude about sleep and susceptibility to a cold?

# 5 CHAPTER SUMMARY
# CONDITIONAL PROBABILITY

- A **tree diagram** of a probability situation shows successive stages of a sample space as branching. It is convenient to combine branches with the same outcome and label the combined branch according to the number of possible outcomes.
- A **two-way table** has frequencies or elements of a sample space with two characteristics listed in rows and columns.
- A **Venn diagram** shows sets or events as closed shapes (usually circles and ellipses) with intersections shown as overlap. They are enclosed by a rectangle that represents the universal set (sample space).
- Tree diagrams, two-way tables and Venn diagrams can be used to calculate the probability of a second event depending on the occurrence of a first.
- The **conditional probability** of an event $A$ given event $B$ is written as $P(A|B)$, and is the probability that event $A$ occurs, assuming that event $B$ occurs. The sample space for $P(A|B)$ is restricted to $B$.
- The conditional probability of $A$ given $B$ can be calculated using the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ or $P(A \cap B) = P(A|B)P(B)$.
- Two events $A$ and $B$ are **independent** if and only if $P(A|B) = P(A)$ or $P(B|A) = P(B)$.
- The statements $A$ and $B$ are independent, $P(A|B) = P(A)$, $P(B|A) = P(B)$, and $P(A \cap B) = P(A) \times P(B)$ are equivalent.
- **Dependent** events are not independent. $P(A|B) \neq P(A)$, $P(B|A) \neq P(B)$, and $P(A \cap B) \neq P(A) \times P(B)$.
- The number of **combinations** of $r$ objects from $n$ distinct objects is given by

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$$

- For an event $A$, the probability of its **complement** $\overline{A}$ is given by $P(\overline{A}) = 1 - P(A)$.
- For any events $A$ and $B$ in a sample space $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
- For **mutually exclusive** events, $A \cap B = \varnothing$ and $P(A \text{ and } B) = 0$, so $P(A \text{ or } B) = P(A) + P(B)$.
- Probability is used in research to help work out effects. The words 'and' and 'or' have specific meanings for events: $A$ and $B$ means the intersection, $A \cap B$; $A$ or $B$ means the union, $A \cup B$.
- Probability is used in research to help work out effects.


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# CHAPTER REVIEW
# CONDITIONAL PROBABILITY

5

## Multiple choice

1 Examples 1, 10 5 bags of cement at a hardware store were exposed to water and ruined. Unfortunately they were mixed up with 10 good bags. If the first 3 bags taken at random from the store were good, what is the probability that the next one will also be good?

A $\frac{2}{3}$ B $\frac{7}{15}$ C $\frac{5}{6}$ D $\frac{7}{12}$ E $\frac{1}{3}$

The following information is used in questions **2** and **3**.

After a class had been to the library researching a project on adaptations, there were 4 books about mammals, 3 about reptiles, 4 about fish and 5 about insects on the top of a trolley of reserved books.

2 Examples 2, 12 The next class came in and the first student to the trolley took two books at random. What was the probability that they were both about fish?

A $\frac{1}{20}$ B $\frac{1}{8}$ C $\frac{1}{16}$ D $\frac{1}{4}$ E $\frac{1}{40}$

3 Examples 4, 9 Given that the first four books taken off the top of the trolley were each about a different kind of animal, what was the probability that the next book would be about insects?

A $\frac{4}{15}$ B $\frac{1}{12}$ C $\frac{2}{7}$ D $\frac{1}{4}$ E $\frac{1}{3}$

4 Example 13 A six-sided die has been repainted so that each of the symbols ☺, 😐, and ☹ appears on two sides of the die. When the die is thrown, what is the probability of getting 4 smiley faces in a row?

A $\frac{4}{81}$ B $\frac{1}{216}$ C $\frac{1}{81}$ D $\frac{16}{243}$ E $\frac{1}{16}$

5 Example 16 What is the probability of getting a spade or a court card (J, Q, K) when a card is taken at random from a normal 52-card pack?

A $\frac{25}{52}$ B $\frac{3}{52}$ C $\frac{11}{26}$ D $\frac{6}{13}$ E $\frac{5}{13}$

## Short answer

6 Example 3 An esky has soft-drinks packed in ice. There are 8 raspberry, 7 cola and 9 lemonade cans in the esky. Cans are taken out at random.

a What is the probability that the first two will be cola?

b What is the probability that the first two are different colours?

7 Examples 5, 6 The table below shows the transport methods and time taken by some Australian students to get to school.

| | Transport and time in minutes for students to get to school | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0–9 | 10–19 | 20–29 | 30–39 | 40–49 | 50–59 | 60+ |
| Bike | 0 | 3 | 0 | 0 | 0 | 0 | 0 |
| Bus | 3 | 11 | 16 | 22 | 8 | 6 | 5 |
| Car | 36 | 23 | 11 | 6 | 2 | 2 | 1 |
| Train/tram | 0 | 3 | 1 | 2 | 4 | 2 | 0 |
| Walk | 16 | 5 | 5 | 1 | 0 | 0 | 0 |
| Other | 1 | 1 | 1 | 1 | 0 | 0 | 2 |

a What is the probability of taking more than 19 minutes to get to school?
b What is the probability that a student who walks takes more than 19 minutes to get to school?
c What is the probability that a student who goes to school by car takes 10–19 minutes to get to school?
d What is the probability that a student who takes 10–19 minutes to get to school goes to school by car?

8 Example 8 Thirty of the species in a section of a public garden are classed as flowering plants, 18 have small leaves and 20 are Australian. Fourteen of the small-leaved species are also flowering plants, and of these, half are Australian. Sixteen of the flowering plants are Australian. There are only 10 non-flowering species. Eight of the small-leaved species are Australian.
a How many species are there altogether?
b Find the probability that a flowering plant is not an Australian native.
c Given that a species is Australian, what is the probability that it is not a flowering plant?

9 Examples 9, 10 The music department in a school has 70 classical, 60 jazz, 50 popular, 50 folk and 90 other CDs in its collection of recorded music.
a Given that a CD taken at random is not classical, what is the probability that it is jazz?
b What is $P(\text{classical}/\text{not jazz})$?
c What is the probability of taking out two classical CDs at random?

10 Example 11 About 70% of people can roll their tongues and about 20% can fold their tongues. About 10% can both fold and roll their tongues.
a What is the probability that a tongue-roller will be a tongue-folder?
b What is the probability that a tongue-folder will be a tongue-roller?


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11 Example 14 Up until recent years, about 1 in 90 pregnancies produced twins. In Australia the incidence had risen to about 1 in 65 in recent years.
   a Assuming that twin births are independent, what is the probability of a woman having sets of twins in two pregnancies in Australia?
   b Is it reasonable to assume that a second set of twins is independent of a first set?

12 Example 15 There are three kinds of chairs in a particular classroom. Twelve are grey, 8 are blue and 10 are green. What is the probability that a student who sits at random in the classroom will sit on different coloured chairs in two successive lessons?

13 Example 16 A restaurant's dessert menu has the following dishes.
Pannacotta with crème freche and caramel
Pyramid of cream puffs
Black forest cake
Sticky date pudding
Zabaglione
Assuming that their choices are independent, what is the probability that at least one of a couple will choose the black forest cake or sticky date pudding?

## Application

14 Three coins are biased so that heads are more likely than tails. The three coins have probabilities of 0.55, 0.6 and 0.7 of getting heads. Find the probability of getting 2 heads and a tail when the three coins are thrown.

15 The following data was collected from Australian Year 10 and 11 school students and shows their height and foot sizes in cm.

| | | Height (cm) | | | |
|---|---|---|---|---|---|
| | | 150–159 | 160–169 | 170–179 | 180–189 |
| Foot length (cm) | 19–21 | 1 | 2 | 0 | 0 |
| | 22–24 | 2 | 9 | 3 | |
| | 25–27 | 0 | 3 | 3 | 6 |
| | 28–30 | 0 | 1 | 2 | 1 |

*Source: Census at schools*

   a What is the probability that a student has a foot size of 25 cm or greater?
   b What is the probability that a student of height 170 cm or more has a foot size of 25 cm or greater?
   c Does this suggest that foot size is independent of height?
   d Compare the probability of an average foot size of 25 cm or more for students under 170 cm and for students 170 cm or taller.
   e Compare the probability of a foot size of under 25 cm for students under 170 cm and students of 170 cm or more.

Practice quiz

# MIXED REVISION
# CHAPTERS 4 • 5 • 6

2

## Multiple choice

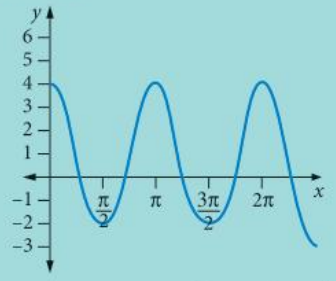
1 State the equation of the linear function with $x$-intercept $-3$ and $y$-intercept 2.

A $3x - 2y + 6 = 0$
B $2x - 3y - 6 = 0$
C $3x + 2y + 6 = 0$
D $2x - 3y + 6 = 0$
E $2x + 3y - 6 = 0$

2 3 Alizarin crimson, 4 burnt umber and 5 iridescent electric blue new acrylic paint tubes have lost their labels. What is the probability that if you open 2 at random they are the same colour?

A $\frac{1}{22}$ B $\frac{1}{11}$ C $\frac{5}{33}$ D $\frac{19}{66}$ E $\frac{7}{22}$

3 Find the equation of the function below.




A $y = 6\cos(x) + 1$
B $y = 6\cos(x) + 2$
C $y = 3\cos(2x) + 2$
D $y = 3\cos(2x) + 1$
E $y = 6\cos(2x) + 1$

4 The quadratic function $f(x) = (x + 1)^2 - 3$ has:

A turning point $(1, 3)$ and axis of symmetry $x = 1$
B turning point $(1, -3)$ and axis of symmetry $x = 1$
C turning point $(-1, 3)$ and axis of symmetry $x = -1$
D turning point $(-1, -3)$ and axis of symmetry $x = -1$
E turning point $(-1, -3)$ and axis of symmetry $x = 1$

5 A packet of mixed crimp-on auto electric cable joiners has four 2.5 mm, six 4 mm and five 6 mm joiners left in it. Given that the first 3 you take out are all 4 mm ones, what is the probability that the next one you take at random is a 2.5 mm one?

A $\frac{4}{455}$ B $\frac{4}{91}$ C $\frac{1}{4}$ D $\frac{4}{15}$ E $\frac{1}{3}$

6 Find the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

A $\frac{\sqrt{3}-1}{\sqrt{2}}$ B $\frac{\sqrt{3}+1}{2\sqrt{2}}$ C $\frac{\sqrt{3}+1}{\sqrt{2}}$ D $\frac{1-\sqrt{3}}{2\sqrt{2}}$ E $\frac{\sqrt{3}-1}{2\sqrt{2}}$

7 If $f(x) = -x^2 + 3x - 4$, evaluate $f(-2)$.

A $-14$ B $-8$ C $-6$ D $-2$ E $6$

8 In a particular area with special bait, the probability that when you catch a fish it will be a perch is 20%. The probability that a perch is over 1 kg is 20% and the probability that any other fish caught there is over 1 kg is 10%. What is the probability that a fish over 1 kg is a perch?

A 4% B 8% C 16% D 33% E 40%

9 Solve $2\sin(x) + 1 = 0$ for $0 \leq x \leq 2\pi$.

A $x = \frac{4\pi}{3}, \frac{5\pi}{3}$ B $x = \frac{\pi}{6}, \frac{5\pi}{6}$ C $x = \frac{7\pi}{6}, \frac{11\pi}{6}$ D $x = \frac{5\pi}{4}, \frac{7\pi}{4}$ E $x = \frac{\pi}{3}, \frac{2\pi}{3}$

## Short answer

1 Find the equation of the straight line that has $x$-intercept 2 and is perpendicular to the line $3x + y - 5 = 0$.

2 A bag contains 10 marbles, 2 black, 3 green and 5 white ones. What is the probability that if two are taken out at random they will be different colours?

3 Sketch the graph of $y = \tan(x)$ in the domain $-\pi \leq x \leq \pi$.

4 State the domain and range of the function $y = \frac{2}{3x-5}$.

5 The probability that a lead-acid car battery fails in the first 12 months is about 8%. What is the probability that 3 such batteries bought at the same time all fail within 12 months of purchase?

6 Solve $\tan^2(x) = 3$ for $0° \leq x \leq 360°$.

## Application

1 Bob builds a ramp up to his back door that is 11.3 m long. The door is 575 mm off the ground.




a What is the gradient of the ramp, correct to 3 decimal places?

b The maximum gradient of ramps in Australia is 1 : 14. Does this ramp comply with the Australian standards?

2 The equation of displacement $d$ cm of a particle over time $t$ seconds is given by $d = t^3 - 8t^2 + 15t$.

a Find the displacement after 2 seconds.

b Factorise the equation.

c Sketch the graph of the displacement.

d Find the times when the displacement is zero.

e Find the approximate velocity at 2 seconds.

3 12 of a group of 25 students are doing Maths Methods and 8 are doing Physics. 14 are doing Biology, and of these 5 are doing Maths Methods. Only 1 student is doing all three subjects and only one student is doing Physics but not Maths Methods or Biology. There are 6 students doing Maths Methods and Physics.

a How many students are doing Physics and Biology?

b Given that a student is not doing Physics or Biology, what is the probability the student is not doing Maths Methods either?


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c Given that a student is doing Physics, what is the probability the student is also doing Biology?

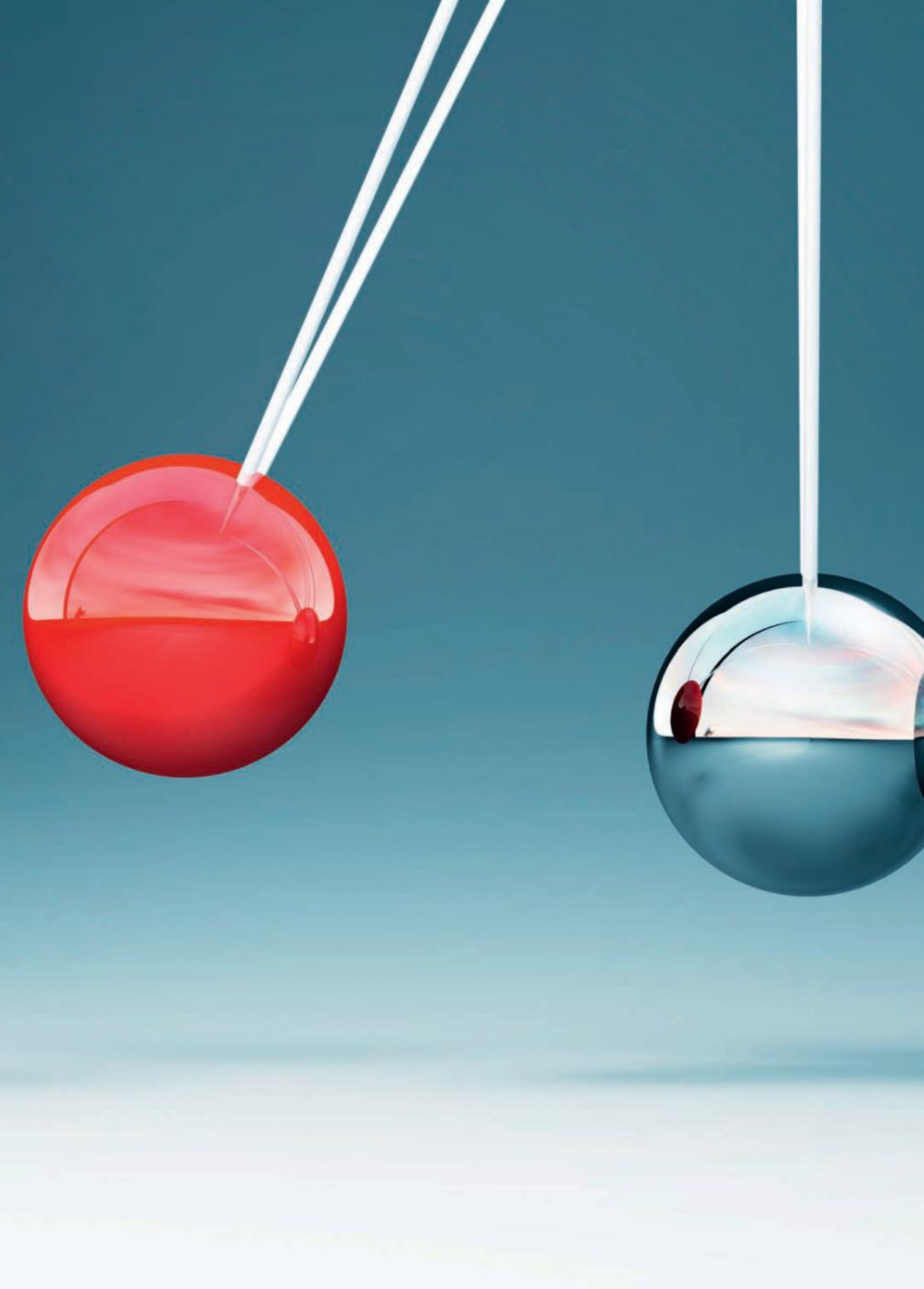
4 About 27% of high school students learn a musical instrument, some in school and some outside school (and some both). A study of school students' results in a large city and participation in instrumental music yielded the following results. School results were added together by assigning scores of 1–5 to school grades in 5 subjects, with 1 = E to 5 = A. Results were classified as high (above 18), average (12–17) and low (under 12).

| | 5–11 | 12–17 | 18–25 |
|---|---|---|---|
| Instrumental music | 59 | 235 | 206 |
| Not instrumental music | 115 | 256 | 129 |

a What is the probability of learning an instrument and having high achievement?
b Given a student is learning an instrument, what is the probability of high achievement?
c What is then probability of not learning an instrument and having high achievement?
d Given a student is not learning an instrument, what is the probability of high achievement?
e Compare the probability of high achievement for students who are learning an instrument to those who are not.

5 The depth of water, $d(t)$ m, at a pier changes according to the rule $d(t) = 5 + 2\cos\left(\frac{\pi}{6}t\right)$, where $t$ is time in hours from high tide. On 18 March, high tide was at 7:30 a.m.
a Draw a sketch showing the depth of water for the 24 hours after 7:30 a.m. on 18 March.
b When was the first low tide on that day?
c When was the water at a depth of 6 m?
d How long is it between low tide and high tide?

6 The temperature in a room varies over the day in a sinusoidal fashion, with a minimum of 8°C at 6 a.m. and a maximum of 20°C at 6 p.m.
a State the expected temperature at noon.
b At what other time(s) of the day would you expect the temperature to be the same as at noon?
c What is the period of the temperature variations?





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# 6

## TERMINOLOGY

amplitude
circular measure
cycle
period
periodic
phase
radian
sinusoidal
trigonometric functions
trigonometric identities

TRIGONOMETRIC FUNCTIONS

# TRIGONOMETRIC FUNCTIONS AND GRAPHS


6.01 Basic trigonometric graphs
6.02 Amplitude changes
6.03 Period changes
6.04 Phase changes
6.05 Angle sums and differences
6.06 Trigonometric equations
6.07 Modelling periodic phenomena
Chapter summary
Chapter review


Prior learning

## TRIGONOMETRIC FUNCTIONS

- understand the unit circle definition of cos θ, sin θ and tan θ and periodicity using radians (ACMMM034)
- recognise the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ on extended domains (ACMMM036)
- examine amplitude changes and the graphs of $y = a \sin x$ and $y = a \cos x$ (ACMMM037)
- examine period changes and the graphs of of $y = \sin bx$, $y = \cos bx$, and $y = \tan bx$ (ACMMM038)
- examine phase changes and the graphs of of $y = \sin (x + c)$, $y = \cos (x + c)$ and $y = \tan (x + c)$ (ACMMM039)
- examine the relationships $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ and $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ (ACMMM040)
- prove and apply the angle sum and difference identities (ACMMM041)
- identify contexts suitable for modelling by trigonometric functions and use them to solve practical problems (ACMMM042)
- solve equations involving trigonometric functions using technology, and algebraically in simple cases (ACMMM043). AC

# 6.01 BASIC TRIGONOMETRIC GRAPHS

In Chapter 3 you were introduced to **circular measure** and **radians**. You used the unit circle (circle with radius 1 unit) to find exact values of angles within the 4 quadrants.

In this chapter, you will look at the unit circle more closely and use it to help draw the graphs of the **trigonometric functions** $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$.

While you can draw graphs using degrees along the $x$-axis, it is more meaningful to use radians. Since radians are ratios of measures around the circumference of a circle, they represent real numbers. For example, π is approximately 3.14, $\frac{\pi}{6}$ is about 0.52 and so on.

## INVESTIGATION Trigonometric ratios of special angles

Remember from Chapter 3 that
$\sin(\theta) = y$, $\cos(\theta) = x$ and $\tan(\theta) = \frac{y}{x}$.

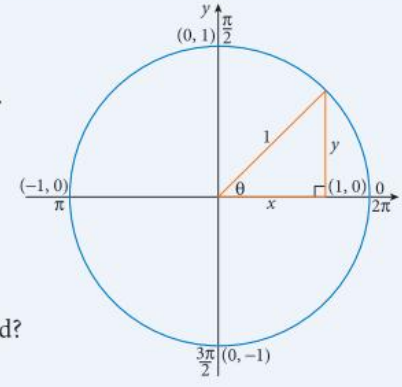
Use these to explore the exact values of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $2\pi$.

For example, since 0 radians is at (1, 0), $x = 1$ and $y = 0$.

So $\cos(0) = x = 1$ and $\sin(0) = y = 0$.

Check these values on the calculator.

Find the other ratios in the same way and check them on your calculator. Are there any values that you can't find? Can you see why?




Go around the circle again and find ratios for $2\pi + \frac{\pi}{2}$, $2\pi + \pi$ and so on. What do you notice?

By using $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $2\pi$ as values along the $x$-axis, could you draw the graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$? Use your calculator to find other values to help with the shape of the graphs.


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## Example 1

Use the unit circle to evaluate

**a** $\sin\left(\frac{3\pi}{2}\right)$ **b** $\tan(\pi)$ **c** $\cos\left(\frac{\pi}{2}\right)$

### Solution

**a** Use $\sin(\theta) = y$ at the point $(0, -1)$. $\sin\left(\frac{3\pi}{2}\right) = -1$

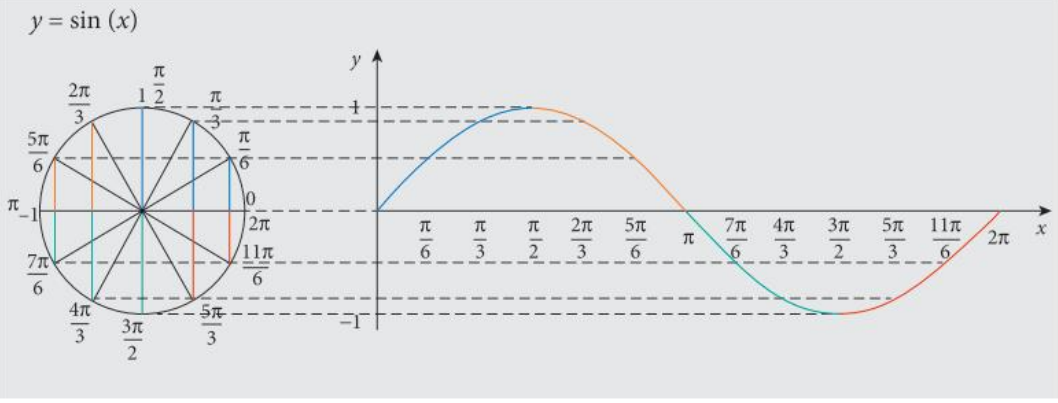
**b** Use $\tan(\theta) = \frac{y}{x}$ at the point $(-1, 0)$. $\tan(\pi) = \frac{0}{-1} = 0$

**c** Use $\cos(\theta) = x$ at the point $(0, 1)$. $\cos\left(\frac{\pi}{2}\right) = 0$

You can use the exact ratios for multiples of $\frac{\pi}{6}$ and the ratios of 0, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$ and $2\pi$ to draw the graphs of trigonometric functions.

Since $\sin(\theta) = y$, take the $y$ values around the unit circle and plot them for $\sin(\theta)$ on a graph as shown in the diagram.

**IMPORTANT**

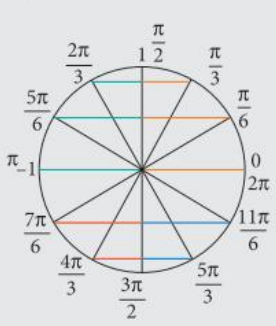


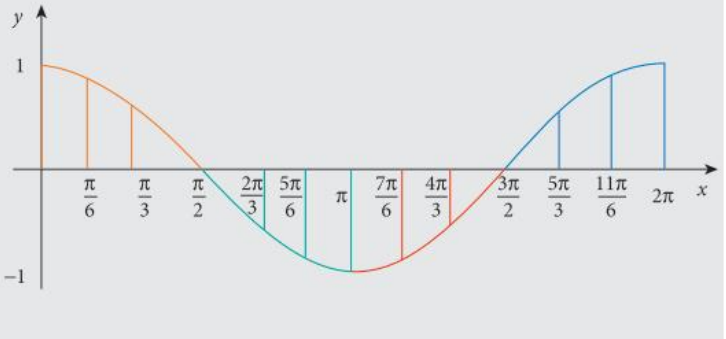

The graph of $y = \sin(x)$ fits the ASTC rule as values of $\sin(x)$ are positive in the first and second quadrants (above the $x$-axis) and negative in the third and fourth quadrants (below the $x$-axis).

Since $\cos(\theta) = x$, take the $x$ values around the unit circle and plot them for $\cos(\theta)$ on a graph as shown in the diagram. The $x$ values are horizontal lines so they will not line up directly with $y$ values. You will need to measure the horizontal lines and then plot them on the graph as vertical lines.

**IMPORTANT**

$y = \cos(x)$



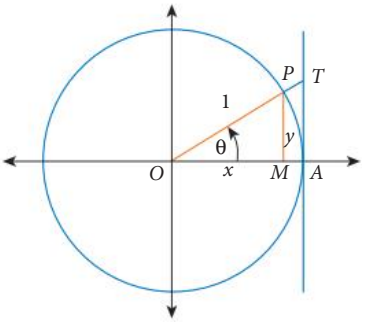

The graph of $y = \cos(x)$ fits the ASTC rule as values of cos $(x)$ are positive in the first and fourth quadrants and negative in the second and third quadrants.

$\tan(\theta) = \frac{y}{x}$ which is a ratio of two sides.

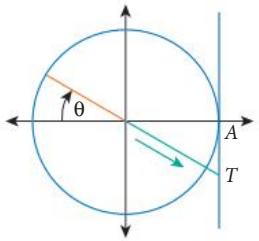
Notice that $OA = 1$ (radius) and using similar triangles $OMP$ and $OAT$:

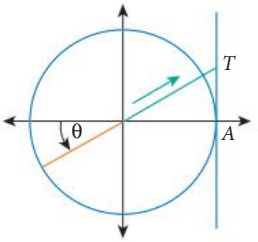
$$\frac{y}{x} = \frac{AT}{1} = AT$$

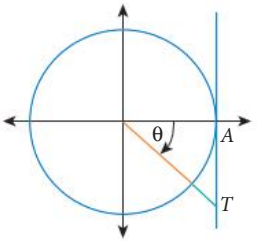
This means $\tan(\theta) = AT$ = the length of the tangent cut off by the radius extending to touch the tangent. This is why this ratio is called the tangent.




This happens in all four quadrants. The radius is extended to meet the tangent in each case.



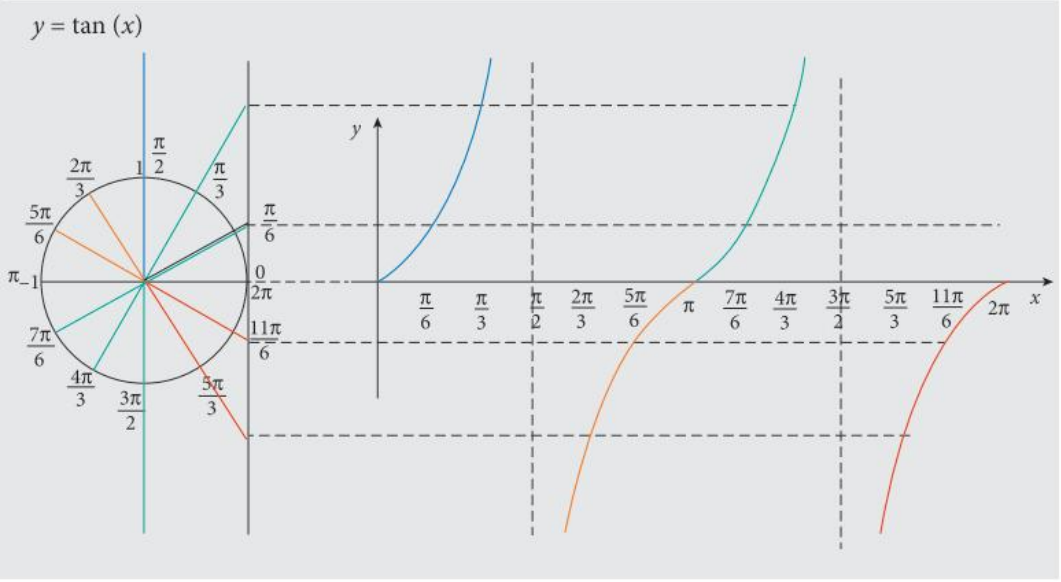





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**IMPORTANT**



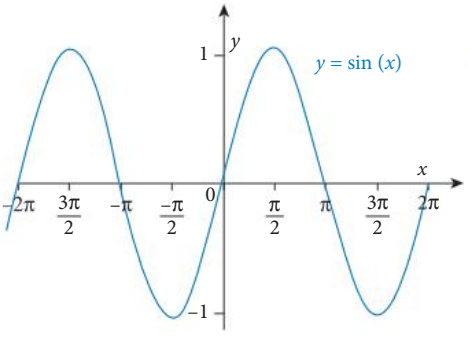

Notice on the circle that the radii at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ do not meet the tangent, so $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$ are undefined. These angles occur at points (0, 1) and (0, –1).

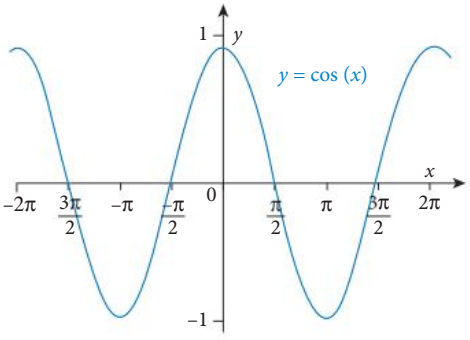
Using $\tan(\theta) = \frac{y}{x}$ gives $\tan\left(\frac{\pi}{2}\right) = \frac{1}{0}$ which is undefined and $\tan\left(\frac{3\pi}{2}\right) = \frac{-1}{0}$ which is also undefined. This means there are asymptotes at these points on the graph of $y = \tan(x)$.

The graph of $y = \tan(x)$ fits the ASTC rule as values of $\tan x$ are positive in the first and third quadrants and negative in the second and fourth quadrants.

Remember that angles can be greater than $2\pi$ and they can be negative. These values of trigonometric functions repeat, for example,

$$\sin\left(\frac{\pi}{3}\right) = \sin\left(2\pi + \frac{\pi}{3}\right) = \sin\left(4\pi + \frac{\pi}{3}\right) = \sin\left(-2\pi + \frac{\pi}{3}\right) \ldots$$



## IMPORTANT

The **period** of a function $f(x)$ is the smallest value of $t$ such that $f(x) = f(x + t)$ for all $x$. The section of the function from $f(x)$ to $f(x + t)$ is called a **cycle**. Functions with a period are called **periodic** functions.

The **amplitude** of a periodic function is half the vertical distance between the maximum and minimum values. It is the distance from the centre to the extreme values.

The period of $y = \sin(x)$ is $2\pi$ and its amplitude is 1.

The period of $y = \cos(x)$ is $2\pi$ and its amplitude is 1.

The period of $y = \tan(x)$ is $\pi$ and it has no amplitude.

### Example 2

Sketch the graph of each function.

a $y = \cos(x)$ in the domain $0 \leq x \leq 2\pi$

b $y = \sin(x)$ in the domain $-\pi \leq x \leq \pi$

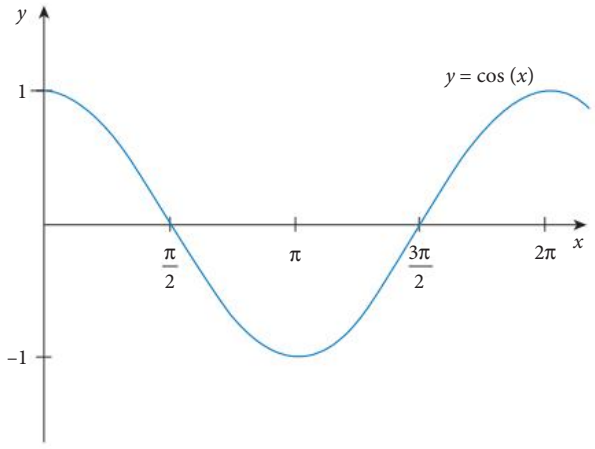
c $y = \tan(x)$ in the domain $0 \leq x \leq \pi$

### Solution

a Notice the shape of $y = \cos(x)$ between 0 and $2\pi$ from the important box above or draw up a table of values.

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $y$ | 1 | 0 | $-1$ | 0 | 1 |

Sketch the graph as a smooth curve showing the main points.

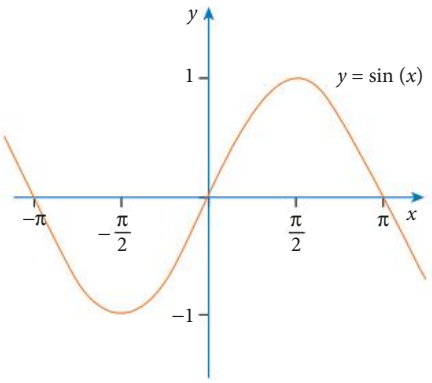



b Notice the shape of $y = \sin(x)$ in the important box above or draw up a table of values.

| $x$ | $-\pi$ | $-\frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|
| $y$ | 0 | $-1$ | 0 | 1 | 0 |


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Sketch the graph as a smooth curve showing the main points.




**c** Notice the shape of $y = \tan(x)$ between 0 and π from the important box above or draw up a table of values.

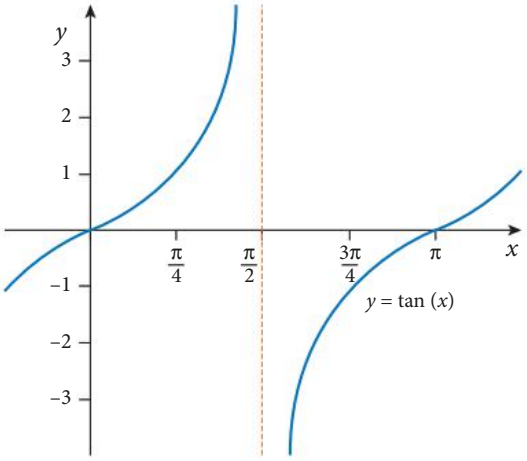
| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|
| $y$ | 0 | 1 | Undefined | −1 | 0 |

Look at values of $\tan(x)$ to the left and right of $\frac{\pi}{2}$.

For $x < \frac{\pi}{2}$, $\tan(x) > 0$ (1st quadrant)

For $x > \frac{\pi}{2}$, $\tan(x) < 0$ (2nd quadrant)

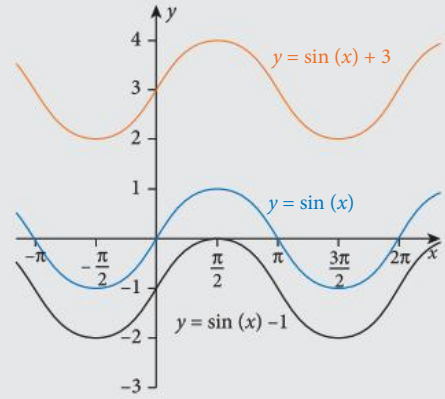
Sketch the graph as a smooth curve showing the main points.




Once you know the basic shape of the trigonometric functions, you can use translations and dilations to draw similar functions.

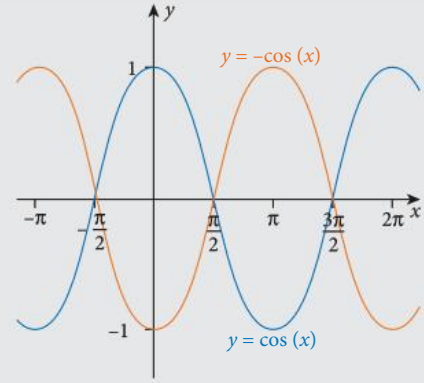
**IMPORTANT**

The graphs of $y = \sin(x) + c$, $y = \cos(x) + c$ and $y = \tan(x) + c$ are translated $c$ units vertically compared with the original graphs of $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. If $c > 0$, the graph is translated up and if $c < 0$, it is translated down. For example:




**IMPORTANT**

The graphs of $y = -\sin(x)$, $y = -\cos(x)$ and $y = -\tan(x)$ are reflected in the $x$-axis. For example:



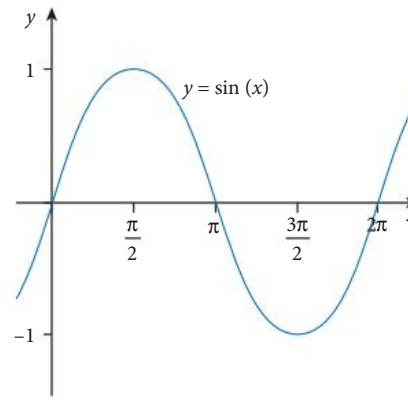

## Example 3

Sketch each graph in the domain $0 \le x \le 2\pi$.

**a** $y = \sin(x) + 1$ **b** $f(x) = -\cos(x)$ **c** $y = \tan(x) + 3$

### Solution

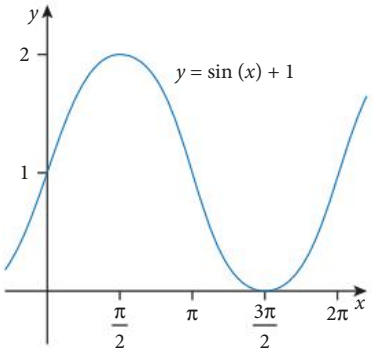
**a** Draw the graph of $y = \sin(x)$ in the domain $0 \le x \le 2\pi$.



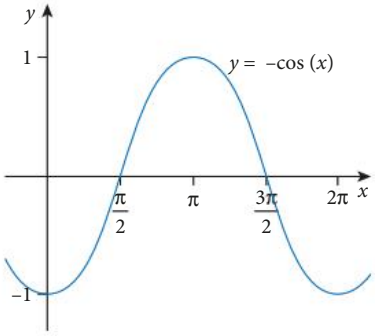


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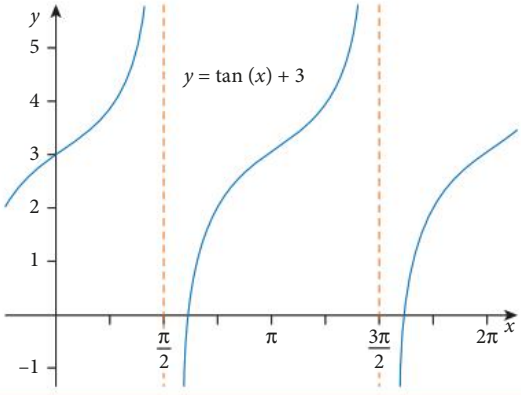
The graph $y = \sin(x) + 1$ is a vertical translation 1 unit upwards of $y = \sin(x)$. Draw the new graph.




**b** The graph of $y = -\cos(x)$ is a reflection of $y = \cos(x)$ in the $x$-axis (the graph $y = \cos(x)$ is shown in Example **2a**). Draw the graph.




**c** The graph of $y = \tan(x) + 3$ is a translation 3 units up of the graph $y = \tan(x)$.




# EXERCISE 6.01 Basic trigonometric graphs

## Concepts and techniques

**1** Example 1 Use the unit circle to find the value of

**a** $\tan(2\pi)$ **b** $\sin\left(\frac{\pi}{2}\right)$ **c** $\cos(\pi)$ **d** $\sin(\pi)$

**e** $\tan(0)$ **f** $\cos\left(\frac{3\pi}{2}\right)$ **g** $\sin(2\pi)$ **h** $\cos(0)$

**2** Example 2 Sketch the graph of each trigonometric function in the domain $0 \leq x \leq 2\pi$.

**a** $y = \sin(x)$ **b** $f(x) = \cos(x)$ **c** $f(x) = \tan(x)$

**3** Sketch the graph of each trigonometric function in the domain $-\pi \leq x \leq \pi$.

**a** $y = \cos(x)$ **b** $f(x) = \tan(x)$

4 Sketch the graph of each function in the domain $-2\pi \le x \le 2\pi$.

a $f(x) = \sin(x)$ b $y = \tan(x)$ c $f(x) = \cos(x)$

5 Example 3 Sketch each graph for $0 \le x \le 2\pi$.

a $y = \cos(x) + 2$ b $y = \sin(x) - 1$ c $f(x) = 2 + \tan(x)$

6 Sketch each graph for $-\pi \le x \le \pi$.

a $f(x) = -\sin(x)$ b $y = -\cos(x)$ c $f(x) = -\tan(x)$

7 Draw the graph of $y = 2 - \cos(x)$ in the domain $-2\pi \le x \le 2\pi$.

## Reasoning and communication

8 a Draw the graph of the function $y = 1 - \sin(t)$ in the domain $0 \le t \le 2\pi$.

b From the equation, find the exact value of $y$ when $t = \frac{\pi}{3}$.

c From the graph, find

i values of $t$ where $y = 1$

ii the approximate value of $y$ when $t = 5$.

9 The velocity of an object is given by $v = 1 + \cos(t)$ cm/s where $t$ is time in seconds.

a Find the exact velocity after

i $\frac{\pi}{2}$ seconds ii $\frac{\pi}{3}$ seconds iii $\frac{\pi}{4}$ seconds

iv $\pi$ seconds v $\frac{3\pi}{2}$ seconds.

b Sketch the graph of the velocity for $0 \le t \le 4\pi$.

c From the graph, find all the times when the velocity is 1.

10 A pendulum moves so that its displacement $x$ mm from the centre is given by the formula $x = \cos(t)$ where $t$ is time in seconds.

a Find the displacement, to 2 decimal places, after 1 second.

b Find the exact displacement of the pendulum after

i $\frac{\pi}{4}$ seconds ii $\frac{\pi}{6}$ seconds.

c What is the initial displacement (when $t = 0$)?

d Draw a graph to show the displacement of the pendulum from 0 to $2\pi$ seconds.

e From the graph, find

i the times when the pendulum is at the centre

ii when the displacement is 1 mm either side of the centre.

f If the graph was extended beyond $2\pi$, find all the times when the displacement is 1 mm either side of the centre.

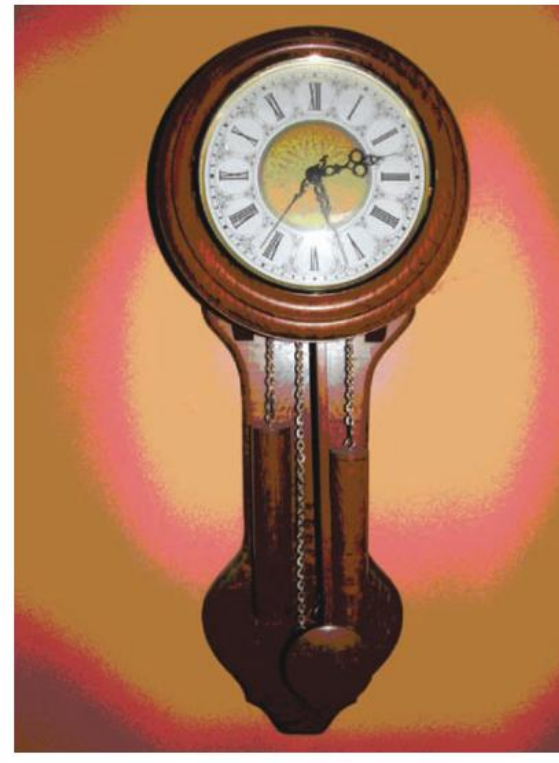


Photo courtesy of Margaret Grove

# 6.02 AMPLITUDE CHANGES

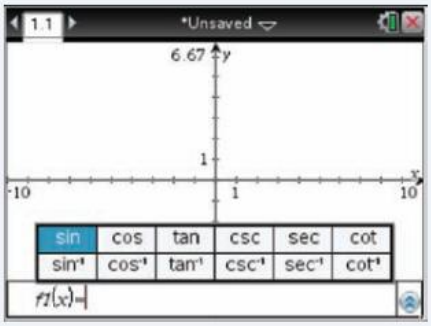
You have already studied how adding a constant into an equation can change a function in Chapter 4. In this section you will look at how $a\,f(x)$ changes $f(x)$ when $f(x)$ is a trigonometric function. Generally it is a vertical dilation of the original function.

## INVESTIGATION $y = a\sin(x)$ and $y = a\cos(x)$

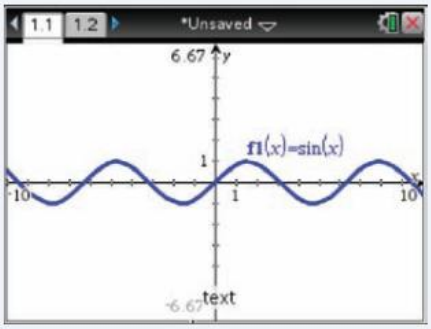
1 Use your CAS calculator to explore the functions $y = a\sin(x)$ and $y = a\cos(x)$ for different values of $a$.

### TI-Nspire CAS

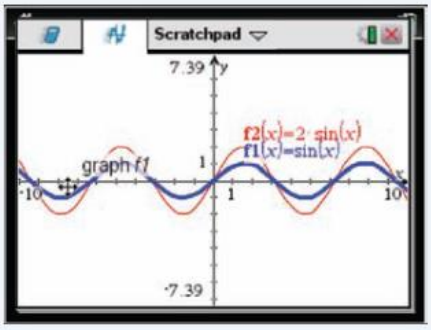
Use a Graphs page ( ).
To enter $f1(x) = \sin(x)$, press trig and choose sin by pressing the arrows and enter



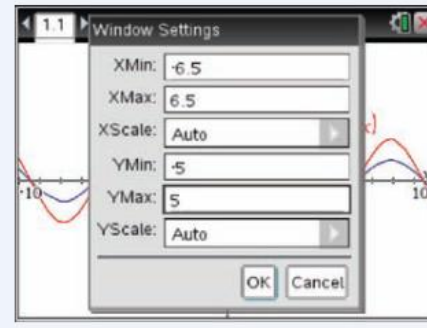

Put $x$ into the function and press enter.



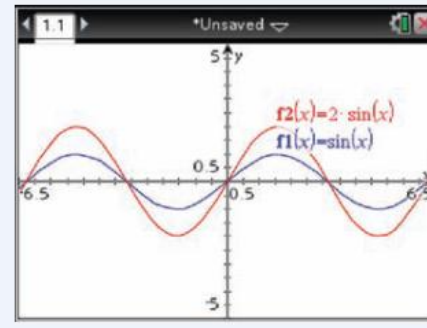

To put in another function, press menu and choose 3: Graph Entry/Edit and 1: Function. Put in $f1(x) = 2\sin(x)$ and press enter. Notice that when you move the curser over one of the graphs it highlights that graph and shows which graph is highlighted, as 'graph $f1$'.



Change the size of the screen using the Window settings obtained in the menu by selecting 4: Window/Zoom and 1: Window Settings. Make the horizontal scale to (say) $-6.5 \leq x \leq 6.5$ and the vertical scale to $-5 \leq y \leq 5$. You should use the (-) button for negative.




The screen should look like this.
Add more functions with different values of $a$.
Note that the $x$ and $y$ scales are slightly out of proportion as they are not in the ratio 3 : 2.

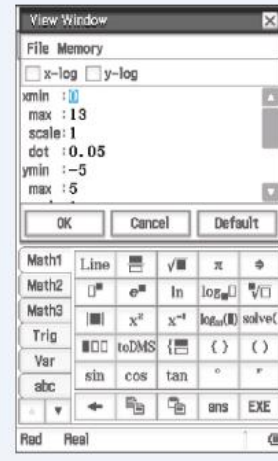



## ClassPad

Use the **Graph&Table** application and press Keyboard. Tap Trig to see sin, cos and tan. Set the bottom right to **Rad** for radians (or **Deg** for degrees).
Enter the two graphs you wish to compare as y1= and y2=.
Press EXE or tap EXE after entering each formula.



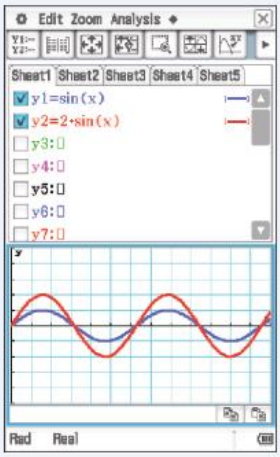

Tap **View Window** to set limits on the graph.
Since $\pi$ is a little bigger than 3, make $0 \leq x \leq 13$ to go from 0 to $4\pi$ and $-5 \leq y \leq 5$ will cover values of y for most graphs.
Tap **OK**.





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Tap ⊞ to see the graphs.
If you want to see them individually, the calculator will only draw the ones which are ticked, so if, for example, only the first graph is ticked, the calculator will only draw the first graph when you tap ⊞.

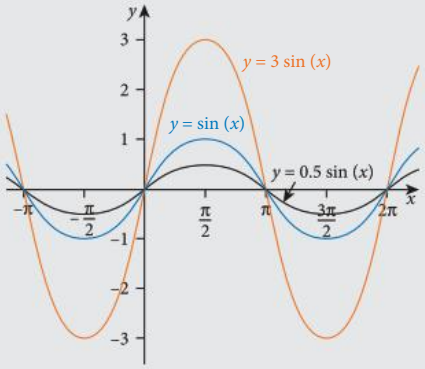



2 What effect does $a$ have on these functions (remember to try some negative values of $a$)?

3 Explore the function $y = a \tan(x)$. What effect does $a$ have on this function?

## IMPORTANT

The functions $y = a \sin(x)$ and $y = a \cos(x)$ have amplitude $a$. All the $y$ values are multiplied by $a$, so the graphs are stretched vertically by a factor of $a$ compared with the original graphs $y = \sin(x)$ and $y = \cos(x)$. For example:




The function $y = \tan(x)$ has no amplitude but will still be stretched, so it will be steeper.

## Example 4

Find the amplitude of each function

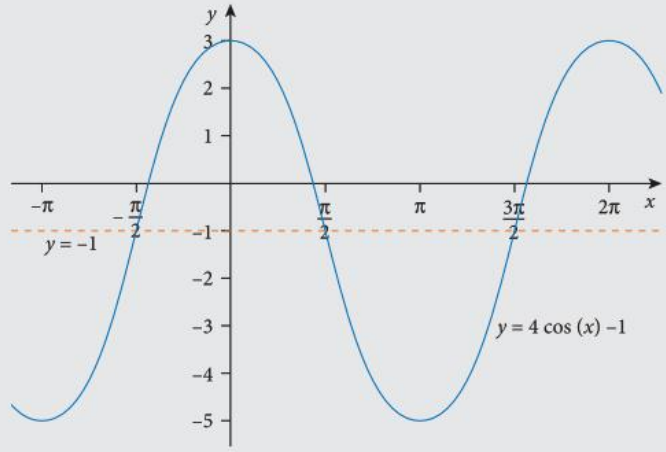
a $y = 3 \cos(x)$ b $y = 7 \sin(x)$

### Solution

| | | |
|---|---|---|
| a | Substitute into $y = a \cos(x)$. | $a = 3$ |
| | State the result. | The function has amplitude 3. |
| b | Substitute into $y = a \sin(x)$. | $a = 7$ |
| | State the result. | The function has amplitude 7. |

## IMPORTANT

Functions in the form $y = \pm a \sin(x) + b$ or $y = \pm a \cos(x) + b$ have amplitude $a$. The line $y = b$ runs through the centre of the graph. For example with $a = 4$ and $b = -1$, $y = 4 \cos(x) - 1$ is centred on $y = -1$. It has a maximum of 3 and a minimum of −5. The amplitude is the distance from the centre to the maximum or minimum.




## Example 5

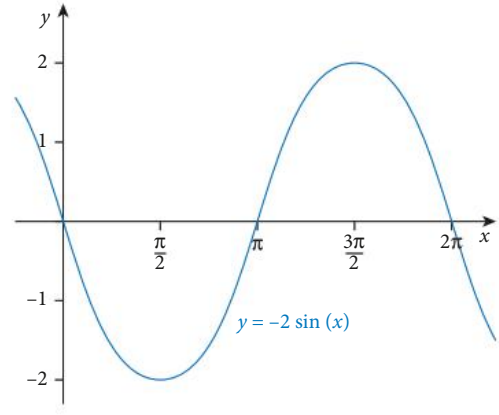
**a** Sketch the graph of $y = -2 \sin(x)$ in the domain $0 \le x \le 2\pi$ and find its amplitude.
**b** Sketch the graph of $y = 3 \sin(x) + 1$ in the domain $0 \le x \le 2\pi$ and find its amplitude.

### Solution

**a** Use a table of values if you are not sure what the graph looks like.

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $y$ | 0 | −2 | 0 | 2 | 0 |

The graph is reflected in the $x$-axis and stretched vertically by a factor of 2 compared to $y = \sin(x)$. Sketch the graph.




State the result.

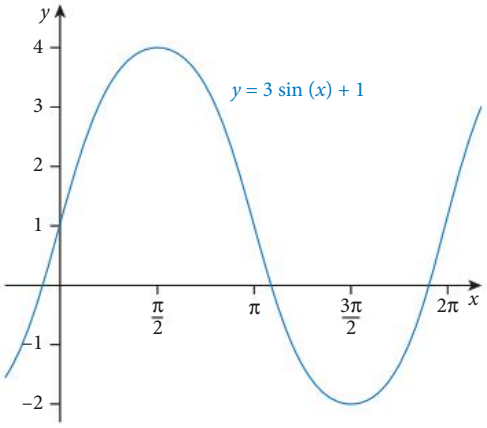
The amplitude is 2.


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**b** Use a table of values if you are not sure what the graph looks like.

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $y$ | 1 | 4 | 1 | −2 | 1 |

The graph is stretched vertically by a factor of 3 and raised 1 unit vertically compared to $y = \sin(x)$. Sketch the graph.




State the amplitude.

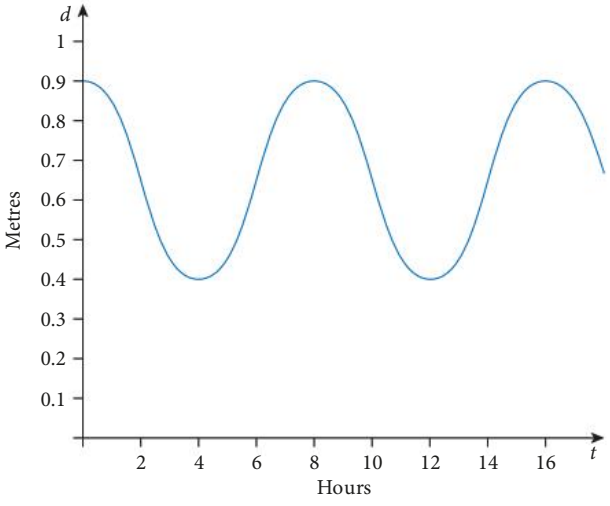
The amplitude is 3.

You can find the amplitude of a function even if you don't know its equation.

## Example 6

For the function below, find

**a** the amplitude **b** the period.




### Solution

**a** Find the top and bottom of the curve.

The top of the curve is at $d = 0.9$ and the bottom is at $d = 0.4$.

Find the centre of the curve.

The centre is halfway between 0.4 and 0.9.

$\frac{0.4 + 0.9}{2} = 0.65$

| | | |
|---|---|---|
| | Find the amplitude, which is the distance from the centre to the highest or lowest point. | $0.9 - 0.65 = 0.25$ or $0.65 - 0.4 = 0.25$ |
| | State the result. | The amplitude is 0.25 |
| | Alternative solution: | The amplitude is half the distance between the highest and lowest points.<br>Distance $= 0.9 - 0.4$<br>$= 0.5$<br>Half the distance $= \frac{0.5}{2} = 0.25$ |
| **b** | State the period. | Since the curve repeats after every 8 hours, the period is 8 hours |

# EXERCISE 6.02 Amplitude changes

## Concepts and techniques

**1** Example 4 Find the amplitude of each function below.

**a** $y = 5\cos(x)$
**b** $y = 9\sin(x)$
**c** $f(x) = 4\sin(x)$
**d** $f(x) = -\sin(x)$
**e** $y = -2\cos(x)$
**f** $y = 6\sin(x) - 1$
**g** $y = 8\cos(x) + 3$
**h** $f(x) = -11\cos(x) + 1$
**i** $y = -4\sin(x) - 5$
**j** $f(x) = 9 - \cos(x)$

**2** Example 5 Sketch each function in the domain $0 \leq x \leq 2\pi$ and state its amplitude.

**a** $y = 4\sin(x)$
**b** $y = 5\cos(x)$

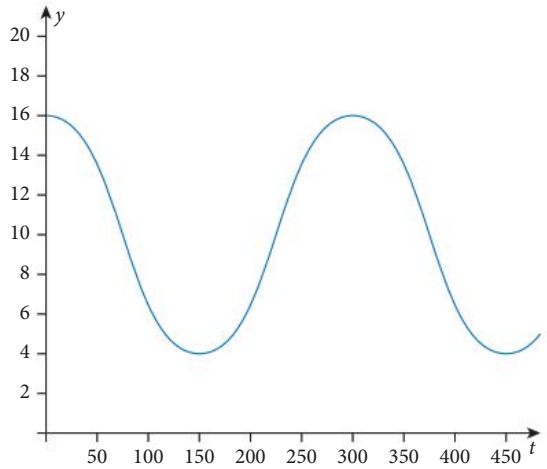
**3** Sketch each function in the domain $-\pi \leq x \leq \pi$ and and state its amplitude.

**a** $y = -3\cos(x)$
**b** $f(x) = -4\sin(x)$

**4** Sketch each function for $0 \leq x \leq 2\pi$ and state its amplitude.

**a** $y = 3\cos(x) + 1$
**b** $y = 7\sin(x) - 3$

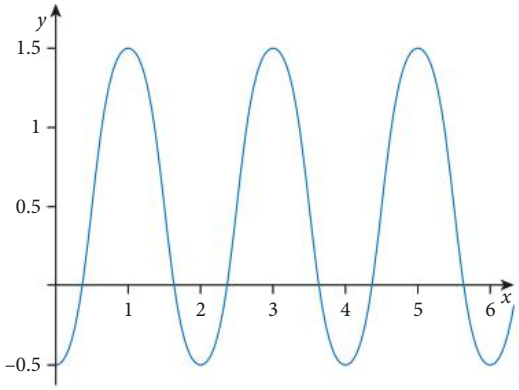
**5** Example 6 State the amplitude and period of the function below.





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**6** State the amplitude and period of the function below.




## Reasoning and communication

**7** A trigonometric function is given by the equation $y = a \sin(x) + b$.

- **a** What is its amplitude?
- **b** The function passes through the points $\left(\frac{\pi}{2}, 3\right)$ and $(\pi, 1)$. Find values for $a$ and $b$ and hence the equation of the function.
- **c** Sketch the function in the domain $0 \leq x \leq 2\pi$.

**8** Assume that in summer the average highest temperature at the beach side is 28° at midday and the average lowest temperature is 20° at midnight.



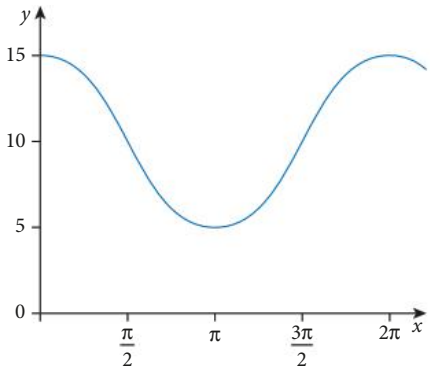
Photo courtesy Margaret Grove

- **a** What is the amplitude of the temperatures?
- **b** What is the period of the temperatures?
- **c** If the equation of the temperatures is in the form $y = a \cos(x) + b$, evaluate $a$ and $b$.

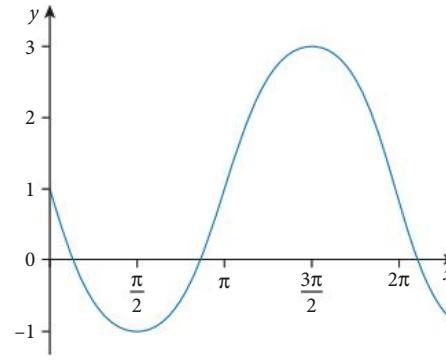
**9** CAS By sketching the function $f(x) = 3 \sin(x) + 4 \cos(x)$, find its amplitude and period.

10 Find the equation of each function.

a




b




# 6.03 PERIOD CHANGES

In this section you will look at how $f(bx)$ changes $f(x)$ in trig functions. Generally this is a horizontal dilation of the original function.

## INVESTIGATION $y = \sin(bx)$, $y = \cos(bx)$ and $y = \tan(bx)$

1 Use your CAS calculator (see page 293) to explore the functions $y = \sin(bx)$, $y = \cos(bx)$ and $y = \tan(bx)$ for different values of $b$.

2 What effect does $b$ have on these functions?

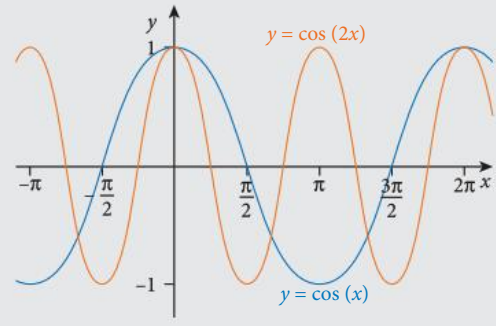
3 Use a table of values to sketch various graphs and find their period. For example, $y = \sin(2x)$.

| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

The function $y = \sin(x)$ has a period of $2\pi$. In the example above in the investigation, there are two complete cycles of $y = \sin(2x)$ in the domain $0 \leq x \leq 2\pi$. So the period is halved.

**IMPORTANT**

The functions $y = \sin(bx)$ and $y = \cos(bx)$ have period $\frac{2\pi}{b}$. This means they are compressed horizontally by a factor of $b$ so they repeat $b$ times as often as $y = \sin(x)$ and $y = \cos(x)$.



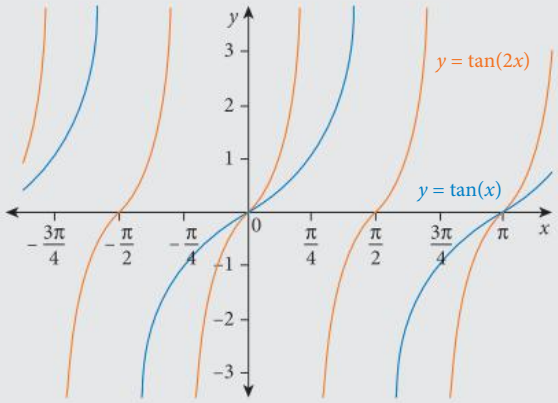


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**IMPORTANT**

The function $y = \tan(bx)$ has period $\frac{\pi}{b}$.

This means it is compressed horizontally by a factor of $b$ so it repeats $b$ times as often as $y = \tan(x)$.




## Example 7

State the period of each function

a $y = \sin(7x)$ b $y = \cos\left(\frac{x}{2}\right)$ c $y = \tan(3x)$

### Solution

a The period is $\frac{2\pi}{b}$, so substitute $b = 7$. $\frac{2\pi}{b} = \frac{2\pi}{7}$

State the result. The period is $\frac{2\pi}{7}$.

b Substitute $b = \frac{1}{2}$ into $\frac{2\pi}{b}$. $\frac{2\pi}{\frac{1}{2}} = 4\pi$

State the result. The period is $4\pi$.

c Substitute $b = 3$ into $\frac{\pi}{b}$. $\frac{\pi}{b} = \frac{\pi}{3}$

State the result. The period is $\frac{\pi}{3}$.

You can use the period to draw the graph of a trig function.

## Example 8

State the period and draw the graph of $y = \cos(3x)$ in the domain $0 \le x \le 2\pi$.

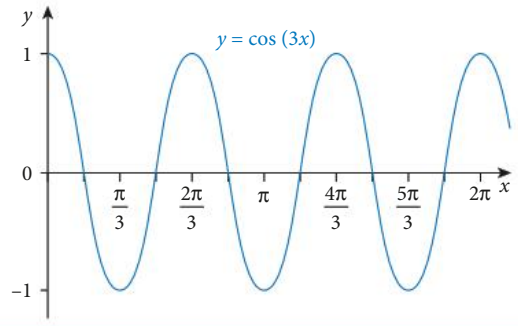
### Solution

State the period. The period is $\frac{2\pi}{3}$

Find the number of cycles of $\cos(3x)$ in the domain $0 \le x \le 2\pi$. There are 3 cycles for $0 \le x \le 2\pi$:

$0 \le x \le \frac{2\pi}{3}, \frac{2\pi}{3} \le x \le \frac{4\pi}{3}, \frac{4\pi}{3} \le x \le 2\pi$

Draw the graph.




You can use both the period and amplitude to draw the graph of a trig function.

## Example 9

State the period and amplitude and draw the graph of each function in the domain $0 \leq x \leq 2\pi$.

**a** $y = 5 \sin(4x)$ **b** $y = 2 \tan(3x)$

### Solution

**a** State the period and amplitude.

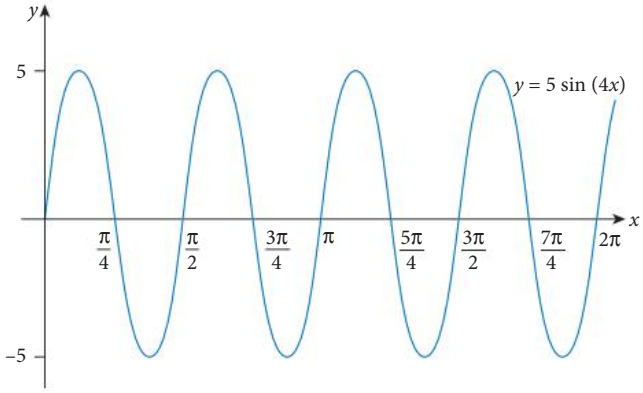
The period is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$ and the amplitude is 5.

Find the number of cycles of $\sin(4x)$ in the domain $0 \leq x \leq 2\pi$.

There are 4 cycles for $0 \leq x \leq 2\pi$:

$0 \leq x \leq \frac{\pi}{2}, \frac{\pi}{2} \leq x \leq \pi, \pi \leq x \leq \frac{3\pi}{2}, \frac{3\pi}{2} \leq x \leq 2\pi$

Draw the graph.




**b** State the period and amplitude.

The period is $\frac{\pi}{3}$ and there is no amplitude.

Find the number of cycles of $\tan(3x)$ in the domain $0 \leq x \leq 2\pi$.

There are 6 cycles for $0 \leq x \leq 2\pi$:

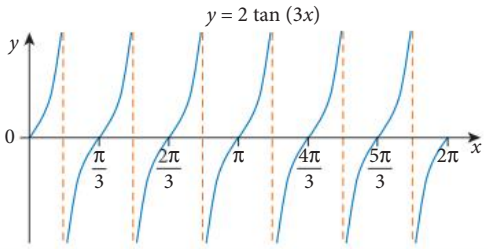
$0 \leq x \leq \frac{\pi}{3}, \frac{\pi}{3} \leq x \leq \frac{2\pi}{3},$

$\frac{2\pi}{3} \leq x \leq \pi, \pi \leq x \leq \frac{4\pi}{3},$

$\frac{4\pi}{3} \leq x \leq \frac{5\pi}{3}, \frac{5\pi}{3} \leq x \leq 2\pi.$


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Draw the graph.




# EXERCISE 6.03 Period changes

WS Sketching periodic functions – amplitude and period

## Concepts and techniques

Your teacher will direct you about using your CAS calculator in this exercise.

**1** Example 7 Find the period of each function.

**a** $y = \sin(6x)$ **b** $f(x) = \cos(9x)$ **c** $y = \cos(2x)$ **d** $y = \tan(8x)$

**e** $f(x) = 3\sin(2x)$ **f** $y = 4\cos(7x)$ **g** $f(x) = \tan\left(\frac{x}{3}\right)$ **h** $y = \cos\left(\frac{x}{5}\right)$

**i** $y = 2\sin\left(\frac{x}{4}\right)$ **j** $y = 3\cos(\pi x)$ **k** $y = 4\sin\left(\frac{\pi x}{4}\right)$ **l** $f(x) = \tan(3\pi x)$

**2** Example 8 Sketch the graph of each function in the domain $0 \le x \le 2\pi$.

**a** $y = \cos(2x)$ **b** $y = \tan(2x)$ **c** $f(x) = \sin(3x)$

**3** Draw the graph of each function in the domain $-\pi \le x \le \pi$.

**a** $y = \sin(2x)$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \cos(3x)$ **d** $y = \tan(4x)$

**4** Sketch a graph of each function showing one full cycle (one period).

**a** $y = \sin(4x)$ **b** $y = \cos(6x)$ **c** $y = \tan\left(\frac{x}{2}\right)$

**5** Example 9 State the period and amplitude of each function.

**a** $f(x) = 4\sin(5x)$ **b** $y = 2\cos(3x)$ **c** $y = 8\cos\left(\frac{x}{2}\right)$

**d** $f(x) = -3\tan(2x)$ **e** $y = -\cos(11x)$

**6** Use the period and amplitude to sketch the graph of each function in the domain $0 \le x \le 2\pi$.

**a** $y = 3\cos(4x)$ **b** $y = 2\sin(3x)$ **c** $y = 4\tan(2x)$

**d** $y = 6\cos(3x)$ **e** $y = 2\sin\left(\frac{x}{2}\right)$

**7** Sketch each function in the domain $-\pi \le x \le \pi$.

**a** $y = \sin(2x) + 3$ **b** $f(x) = 2\cos(2x)$

**c** $y = -\tan\left(\frac{x}{4}\right)$ **d** $y = -3\sin\left(\frac{x}{2}\right) + 2$

## Reasoning and communication

**8** Sketch the following graphs in the domain $-2 \le x \le 2$.

**a** $y = \sin(\pi x)$ **b** $y = 3\cos(2\pi x)$ **c** $y = 2\tan(\pi x)$

**9** The sales, $S$ in 100s of units, of a seasonal product are modelled by $S = 54.8 + 32.5\cos\left(\frac{\pi t}{6}\right)$ where $t$ is the time in months ($t = 1$ is January and $t = 12$ is December).

**a** Draw a graph of the sales for a period of 12 months.

**b** Use the graph to determine the months for which sales exceed 6800 units.

10 In an unusual meteorological investigation, the temperature, $T$°C, in a town in central Queensland was found to fluctuate approximately according to the rule $T = 25 + 6\sin(0.1\pi t)$ where $t$ is the number of hours after 10:00 a.m.

a Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.

b Use the graph to determine the maximum and minimum temperatures.

c When did they occur?

d At what time was the temperature:

i 27°C? ii 20°C?

e Why is this unusual?

# 6.04 PHASE CHANGES

In this section you will look at how $f(x + c)$ changes $f(x)$ in trig functions.

## INVESTIGATION $y = \sin(x + c)$, $y = \cos(x + c)$ and $y = \tan(x + c)$

1 Use your CAS calculator (see page 293) to explore the functions $y = \sin(x + c)$, $y = \cos(x + c)$ and $y = \tan(x + c)$ for different values of $c$.

2 What effect does $c$ have on these functions?

3 Use a table of values to sketch various graphs and find their period. For example, $y = \sin\left(x + \frac{\pi}{4}\right)$.

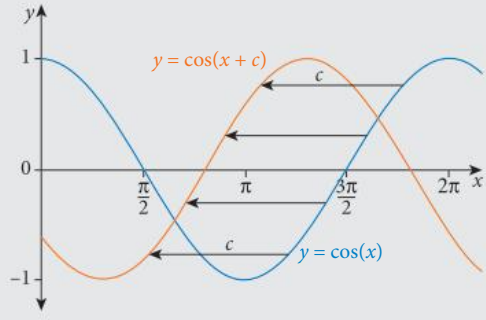
| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

Comment on the positions of points in the graph of $y = \sin\left(x + \frac{\pi}{4}\right)$, compared with points on the graph of $y = \sin(x)$.

4 Now use your calculator to draw the graphs of $\sin(x)$ and $\cos\left(x + \frac{\pi}{2}\right)$ on the same axes. What do you find?

## IMPORTANT

In general, the graphs of functions $y = \sin(x + c)$, $y = \cos(x + c)$ and \$y = \tan(x + c)$ are horizontal translations or phase changes to the left if $c > 0$ and to the right if $c < 0$.





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A sine, cosine or combination of these is called a **sinusoidal** function. The **phase** (shift) of a sinusoidal function $f(x + c)$ relative to $f(x)$ is $c$. The graph of $f(x)$ is translated horizontally by $c$ units. For $c > 0$, it shifts to the left (earlier) and for $c < 0$, to the right (later). Whenever possible, $c$ is taken relative to sin $(x)$ or cos $(x)$.

The functions $y = \sin(x)$ and $y = \cos(x)$ have the same graphs with different phases.

The phase shift of $y = \cos(x)$ relative to $y = \sin(x)$ is $\frac{\pi}{2}$, as $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$.

$\cos(x)$ is $\frac{\pi}{2}$ 'ahead' of $\sin(x)$.

The phase shift of $\sin(x)$ relative to $\cos(x)$ is $-\frac{\pi}{2}$, as $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$. $\sin(x)$ is $\frac{\pi}{2}$ 'behind' $\cos(x)$.

## Example 10

Draw a sketch of each function in the domain $0 \leq x \leq 2\pi$.

**a** $y = \cos\left(x - \frac{\pi}{3}\right)$ **b** $y = \tan\left(x + \frac{\pi}{4}\right)$

### Solution

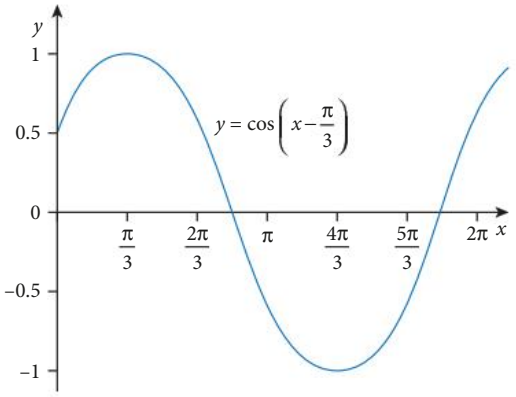
**a** Describe how $y = \cos\left(x - \frac{\pi}{3}\right)$ is different from $y = \cos(x)$.

$y = \cos\left(x - \frac{\pi}{3}\right)$ is a phase change or translation of $\frac{\pi}{3}$ units to the right.

Complete a table of values with $x$ values increasing by $\frac{\pi}{3}$.

| $x$ | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $y$ | 0.5 | 1 | 0.5 | −0.5 | −1 | −0.5 | 0.5 |

Draw the graph.



**b** Describe how $y = \tan\left(x + \frac{\pi}{4}\right)$ is different from the graph of $y = \tan(x)$.

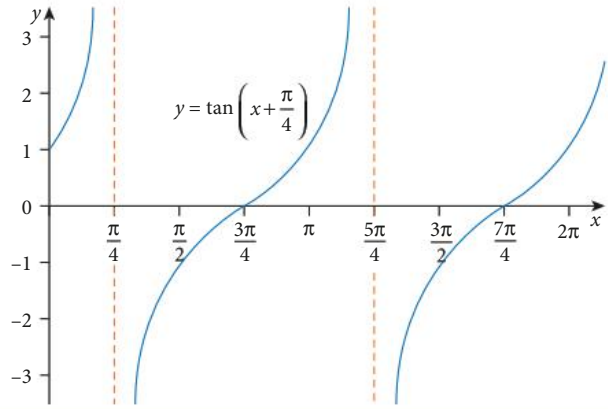
$y = \tan\left(x + \frac{\pi}{4}\right)$ is a phase change or translation of $\frac{\pi}{4}$ units to the left.

Complete a table of values with $x$ values increasing by $\frac{\pi}{4}$.

| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | U* | −1 | 0 | 1 | U* | −1 | 0 | 1 |

*U = undefined

Draw the graph.




You can put together the phase changes with the amplitude and period that you learnt in the previous sections.

## Example 11

Sketch the graph of $y = 2\sin\left(x + \frac{\pi}{6}\right) - 1$ in the domain $0 \le x \le 2\pi$.

### Solution

Find the main features.

The graph has period $2\pi$ and amplitude 2. It differs from the graph of $y = 2\sin(x)$ with a phase change of $\frac{\pi}{6} \approx 0.191$ to the left and a translation of 1 unit downwards.

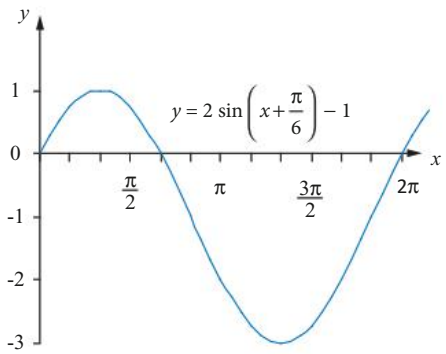
If you cannot see where the graph will go, complete a table of values with intervals of $\frac{\pi}{6}$.

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $y$ | 0 | 0.732 | 1 | 0.732 | 0 | −1 | −2 |

| $x$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|
| $y$ | −2.732 | −3 | −2.732 | −2 | −1 | 0 |


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Draw the graph.




## INVESTIGATION Phase changes

WS Sketching periodic functions – phase and vertical shift

1 Use your CAS calculator to compare the graphs of $y = \sin(2x)$ with $y = \sin\left(2x+\frac{\pi}{2}\right)$. Is the phase change $\frac{\pi}{2}$? Why, or why not?

2 Explore other trig functions with different periods and look at how phase changes affect these functions.

# EXERCISE 6.04 Phase changes

## Concepts and techniques

1 Example 10 Draw the graph of each of the following in the domain $-\pi \le x \le \pi$.

a $y = \sin\left(x+\frac{\pi}{6}\right)$

b $f(x) = \cos\left(x+\frac{\pi}{3}\right)$

c $y = \tan\left(x-\frac{\pi}{4}\right)$

d $y = \cos\left(x+\frac{\pi}{4}\right)$

2 Sketch for $0 \le x \le 2\pi$.

a $y = \sin(x+\pi)$

b $y = \tan\left(x+\frac{\pi}{2}\right)$

c $f(x) = \cos(x-\pi)$

d $y = \cos\left(x-\frac{\pi}{4}\right)$

3 a Sketch the graphs of $y = \sin(x)$ and $y = \cos\left(x-\frac{\pi}{2}\right)$ in the domain $0 \le x \le 2\pi$.

b Show that $\sin(x) = \cos\left(x-\frac{\pi}{2}\right)$.

4 CAS Sketch graphs of the following and check with a CAS calculator, showing a full cycle of each.

a $y = \sin\left(x-\frac{\pi}{2}\right)$

b $y = \sin\left(x+\frac{\pi}{2}\right)$

c $y = \cos\left(x-\frac{\pi}{6}\right)$

d $y = \cos\left(x+\frac{\pi}{6}\right)$

5 Example 11 Sketch for $0 \le x \le 2\pi$.

**a** $y = 3\sin\left(x - \frac{\pi}{2}\right)$ **b** $f(x) = 2\cos\left(x + \frac{\pi}{2}\right)$

**c** $f(x) = 2\cos\left(x + \frac{\pi}{2}\right) + 1$ **d** $y = 2 - 3\sin\left(x - \frac{\pi}{2}\right)$

6 Sketch the graph of the following in the domain $-\pi \le x \le \pi$.

**a** $y = 2\tan\left(x - \frac{\pi}{4}\right)$ **b** $y = -\sin\left(x + \frac{\pi}{6}\right)$

**c** $y = -2\cos\left(x + \frac{\pi}{2}\right)$ **d** $y = 1 - 3\tan\left(x + \frac{\pi}{4}\right)$

7 Sketch graphs of the following showing a full cycle of each.

**a** $y = 2\sin\left(x - \frac{\pi}{3}\right)$ **b** $y = 2\sin\left(x + \frac{\pi}{3}\right)$

**c** $y = 3\cos\left(x - \frac{\pi}{4}\right)$ **d** $y = 3\cos\left(x + \frac{\pi}{4}\right)$

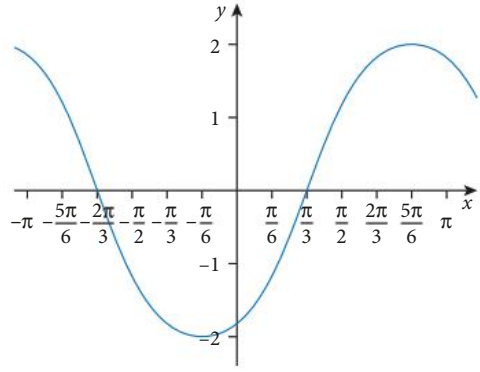
## Reasoning and communication

8 CAS **a** Sketch the graph of $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ on your CAS calculator.

**b** Sketch the graph of $y = \sin(x) - \cos(x)$ on the same screen.

**c** Find the exact amplitude of $y = \sin(x) - \cos(x)$.

9 The graph below has the equation in the form of $y = a\sin(x + c)$.




**a** Find the amplitude of the function.

**b** Find its equation.

10 Using the following table of values

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 | 5.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | −2.1 | 0.4 | 2.7 | 4.4 | 5 | 4.4 | 2.7 | 0.4 | −2.1 | −4 | −5 | −4 |

**a** Draw a graph showing this information

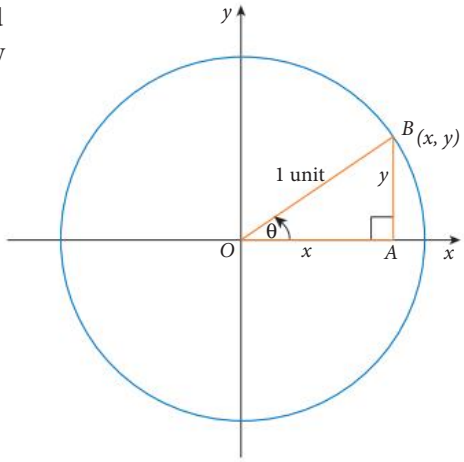
**b** Find the equation of the graph, given that it is in the form $y = a\cos(x + c)$.


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# 6.05 ANGLE SUMS AND DIFFERENCES

There are some trigonometric identities that can be used to simplify or evaluate expressions. In Chapter 3 you saw that in the unit circle, $\sin(\theta) = y$, $\cos(\theta) = x$ and $\tan(\theta) = \frac{y}{x}$.

This gives the trigonometric identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ by substituting $\sin(\theta)$ for $y$ and $\cos(\theta)$ for $x$.




In Chapter 4 you saw that the equation of a circle with centre (0, 0) and radius 1 unit is given by $x^2 + y^2 = 1$. This gives the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

In $\triangle OAB$, $\angle OBA = \frac{\pi}{2} - \theta$ (angle sum of a triangle is $2\pi$)

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{x}{1} = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = y = \sin(\theta)$$

You will need to use some of these trigonometric identities in this section.

**IMPORTANT**

**Some trigonometric identities**

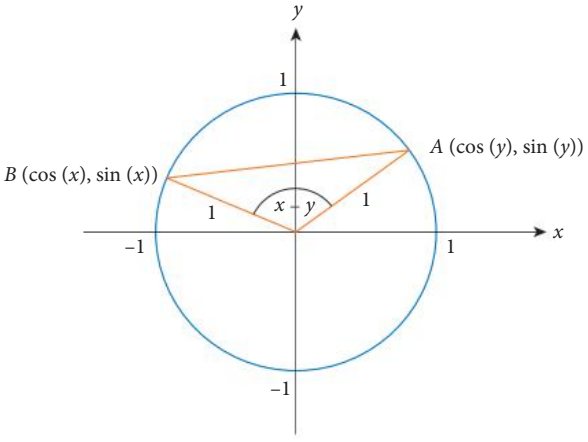
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

In this section you will look at other trigonometric identities that come from the unit circle.



By the distance formula:

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$AB^2 = [\cos(x) - \cos(y)]^2 + [\sin(x) - \sin(y)]^2$

$= \cos^2(x) - 2\cos(x)\cos(y) + \cos^2(y) + \sin^2(x) - 2\sin(x)\sin(y) + \sin^2(y)$

$= [\cos^2(x) + \sin^2(x)] + [\cos^2(y) + \sin^2(y)] - 2\cos(x)\cos(y) - 2\sin(x)\sin(y)$

$= 2 - 2[\cos(x)\cos(y) + \sin(x)\sin(y)]$ [1]

By the cosine rule:

$c^2 = a^2 + b^2 - 2ab\cos(C)$

$AB^2 = 1^2 + 1^2 - 2(1)(1)\cos(x - y)$

$= 2 - 2\cos(x - y)$ [2]

From [1] and [2]:

$2 - 2\cos(x - y) = 2 - 2[\cos(x)\cos(y) + \sin(x)\sin(y)]$

$\therefore \cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

You can prove other identities from this identity. For example, substituting $-y$ gives:

$\cos(x - (-y)) = \cos(x)\cos(-y) + \sin(x)\sin(-y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ [since $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$]

**IMPORTANT**

**Sums and differences of angles**

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x + y) = \dfrac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

## Example 12

**a** Expand

i $\cos(3a + 2b)$ ii $\tan\left(\alpha - \frac{\pi}{7}\right)$

**b** Simplify

i $\cos(2p)\cos(5q) + \sin(2p)\sin(5q)$ ii $\sin(54°)\cos(33°) - \cos(54°)\sin(33°)$

**c** CAS i $\cos\left(\frac{\pi}{2} + x\right)$ ii $\cos(x + 2y)$ iii $\cos(2x) + 2\sin^2(x)$

## Solution

**a i** Write the sum formula.
Substitute $x = 3a$ and $y = 2b$.

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
$\cos(3a + 2b) = \cos(3a)\cos(2b) - \sin(3a)\sin(2b)$

**ii** Write the difference formula.

$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}.$$

Substitute $x = \alpha$ and $y = \frac{\pi}{7}$.

$$\tan\left(\alpha - \frac{\pi}{7}\right) = \frac{\tan(\alpha) - \tan\left(\frac{\pi}{7}\right)}{1 + \tan(\alpha)\tan\left(\frac{\pi}{7}\right)}$$

**b i** Write the difference formula backwards.

$\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x - y)$.

Substitute $x = 2p$ and $y = 5q$.

$\cos(2p)\cos(5q) + \sin(2p)\sin(5q) = \cos(2p - 5q)$

**ii** Write the difference formula backwards.

$\sin(x)\cos(y) - \cos(x)\sin(y) + \sin(x - y)$

Substitute $x = 54°$ and $y = 33°$.

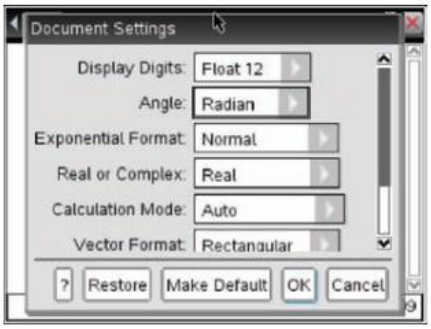
$\sin 54° \cos 33° - \cos 54° \sin 33° = \sin(54° - 33°)$

Simplify.

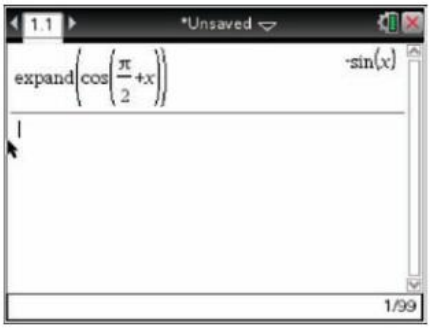
$= \sin(21°)$

### TI-Nspire CAS

Use a calculator page ( ) and make sure your calculator is set to radians.
Look at the document settings by pressing doc and choosing 7: Settings and Status and 2: Document Settings.



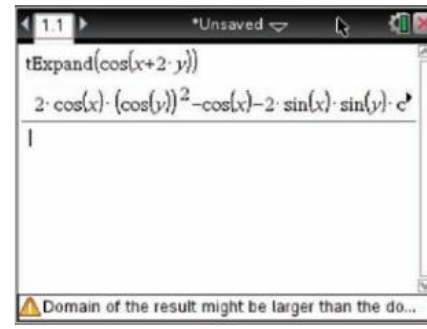

**c i** Press menu, 3: Algebra and 3: Expand, type expand() or choose expand() from the catalogue ( ).
Press trig and choose cos. Use the fraction template ( ), press π, choose π and complete the expression $\frac{\square}{\square}$.




Write the answer.

$\cos\left(\frac{\pi}{2} + x\right) = -\sin(x)$

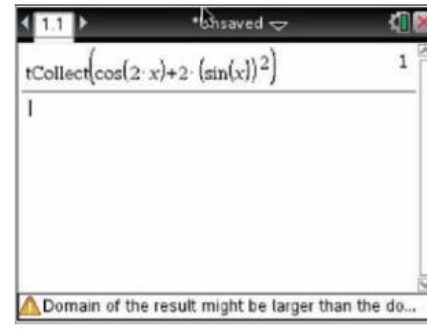
ii Press menu, 3: Algebra, B: Trigonometry and 1: Expand, type tExpand() or choose tExpand() from the catalogue and put in $\cos(x + 2y)$.
Go up and scroll across to see the rest of the answer. The answer might not be in the form you want.




Write the answer. Notice that the calculator expands $\cos(y + y)$ as well.

$\cos(x + 2y) = \cos(x)\cos^2(y) - \cos(x) - 2\sin(x)\sin(y)\cos(y)$

iii Press menu, 3: Algebra, B: Trigonometry and 1: Collect, type tCollect() or choose tCollect() from the catalogue, and put in $\cos(2x) + 2(\sin(x))^2$. You need to put $(\sin(x))^2$ to instruct the calculator to square the sine.




Write the answer.
You can check this by expanding $\cos(x + x)$.

$\cos(2x) + 2\sin^2(x) = 1$

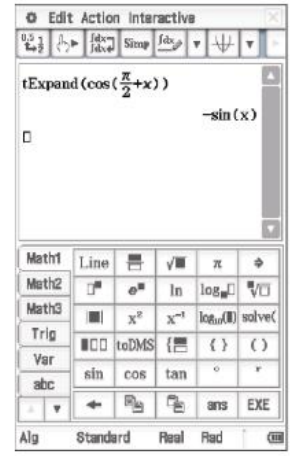
### ClassPad

Use the Main menu and press Keyboard.
Make sure the bottom right is set to **Rad** for radians in this case.

c i Enter the following:
tExpand$\left(\cos\left(\frac{\pi}{2} + x\right)\right)$ EXE.

Tap ▼, scroll down the Catalog tab and tap **tExpand(**.

Tap ▲ and use the Trig tab to get sin, cos and tan. Use the Math1 tab for ▯/▯ for fractions and π.
The calculator expands the expression given using values for sin $\left(\frac{\pi}{2}\right)$ and cos $\left(\frac{\pi}{2}\right)$.

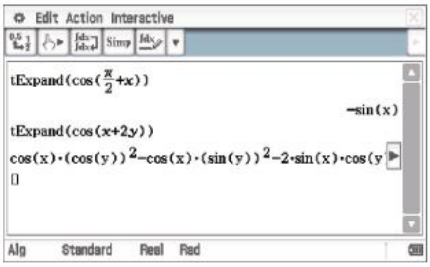



Write the answer.

$\cos\left(\frac{\pi}{2} + x\right) = -\sin(x)$


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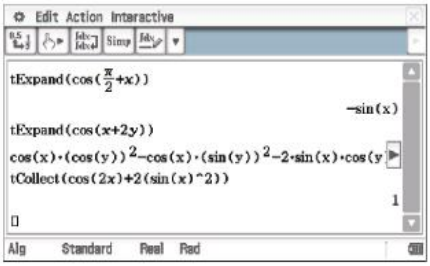
ii Enter the following in the same way as for part i.
tExpand(cos $(x + 2y)$) EXE
Notice that the calculator expands $\cos(y + y)$ as well.
Tap the right arrow to get the full expression. (Tapping Rotate will show more of the expression on the screen.) What you get from the calculator will not always be in the form you want.




Write the answer.

$\cos(x + 2y) = \cos(x)\cos^2(y) - \cos(x)\cos^2(y) - 2\sin(x)\sin(y)\cos(y)$

Use **tCollect** and enter the following in the same way as for part i.
tCollect(cos(2$x$) + 2(sin($x$))$^2$) EXE
The calculator will expand cos 2$x$ and simplify the answer.




Write the answer.
You can check this by expanding $\cos(x + x)$.

$\cos(2x) + 2\sin^2(x) = 1$

If you prefer, you can type sin, cos and tan on your CAS calculator instead of accessing the functions.

In Chapter 3 you found exact ratios for angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$. Now you can use sums and differences of angles to find exact ratios of other angles.

Multiples of $\frac{\pi}{12}$ can be written in terms of these special angles. For example:

$$\frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}$$
$$\frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}$$
$$\frac{\pi}{4} + \frac{\pi}{3} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}$$

## Example 13

Find the exact value of

**a** $\sin\left(\frac{\pi}{12}\right)$

**b** $\tan\left(\frac{7\pi}{12}\right)$

**c** CAS $\cos\left(\frac{11\pi}{12}\right)$ and $\tan\left(\frac{\pi}{8}\right)$.

**d** $\cos\left(\frac{3\pi}{16}\right)\cos\left(\frac{\pi}{16}\right) - \sin\left(\frac{3\pi}{16}\right)\sin\left(\frac{\pi}{16}\right)$

**e** $\sin(a-b)$ given $\sin(a) = \frac{5}{7}$ and $\cos(b) = \frac{3}{5}$ where $a$ and $b$ are both in the first quadrant

### Solution

**a** Write $\frac{\pi}{12}$ as the difference of two known angles.

$$\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$$

Substitute into $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.

$$\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

Substitute exact ratios and simplify.

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Rationalise the denominator.

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

State the result.

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

**b** Write $\frac{7\pi}{12}$ as the sum of two angles.

$$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

Substitute into $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$.

$$\tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}$$

Substitute exact ratios and simplify.

$$= \frac{1+\sqrt{3}}{1-1\times\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$


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Rationalise the denominator.

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{4+2\sqrt{3}}{1-3}$$

$$= -2-\sqrt{3}$$

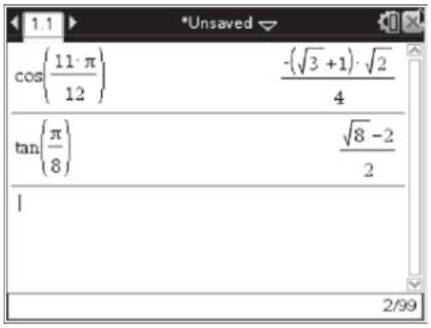
State the result.

$$\tan\left(\frac{7\pi}{12}\right) = -2-\sqrt{3}$$

**c** **TI-Nspire CAS**

Use a calculator page.
Check in the Document Settings that the angle is set to radians and the calculation mode is Auto or Exact.

Put in $\cos\left(\frac{11\pi}{12}\right)$ and then do $\tan\left(\frac{\pi}{8}\right)$.




Write the answers, but write the $\sqrt{2}$ at the front as usual and simplify the expressions.

$$\cos\left(\frac{11\pi}{12}\right) = -\frac{\sqrt{2}(\sqrt{3}+1)}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

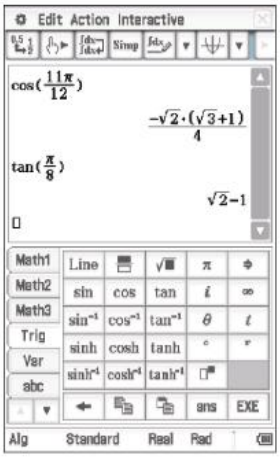
$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$$

**ClassPad**

Use the Main $\sqrt{\alpha}$ menu and press Keyboard.
First make sure the calculator is set to **Standard** to produce exact values and not decimals and **Rad** for radians.
Enter $\cos\left(\frac{11\pi}{12}\right)$

Use the Trig tab to get sin, cos and tan.
Use Math1 and the fraction template for fractions.

Repeat for $\tan\left(\frac{\pi}{8}\right)$.




Write the answers.

$$\cos\left(\frac{11\pi}{12}\right) = -\frac{\sqrt{2}(\sqrt{3}+1)}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$$

**d** Substitute into
$\cos(x+y) = \cos x \cos y - \sin x \sin y$.

$$\cos\left(\frac{3\pi}{16}\right)\cos\left(\frac{\pi}{16}\right) - \sin\left(\frac{3\pi}{16}\right)\sin\left(\frac{\pi}{16}\right)$$

Simplify.

$$= \cos\left(\frac{3\pi}{16} + \frac{\pi}{16}\right)$$

$$= \cos\left(\frac{4\pi}{16}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

Find the exact ratio.

$= \frac{\sqrt{2}}{2}$

State the result.

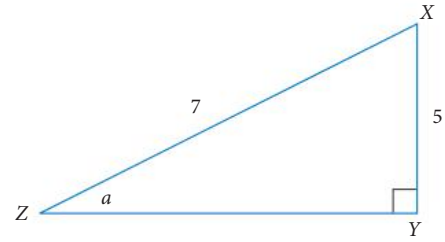
$\cos\left(\frac{3\pi}{16}\right)\cos\left(\frac{\pi}{16}\right) - \sin\left(\frac{3\pi}{16}\right)\sin\left(\frac{\pi}{16}\right) = \frac{\sqrt{2}}{2}$

**e** Substitute into
$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.

$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

Find the exact value of $\cos(a)$ from a right-angled triangle.

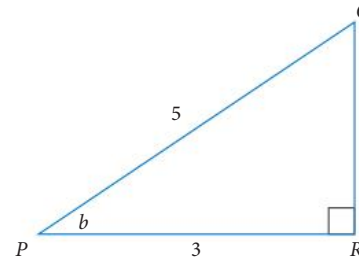
$\sin(a) = \frac{5}{7} = \frac{\text{opposite}}{\text{hypotenuse}}$




By Pythagoras:
$c^2 = a^2 + b^2$
$7^2 = YZ^2 + 5^2$
$7^2 - 5^2 = YZ^2$
$24 = YZ^2$
$\sqrt{24} = YZ$
$\cos(a) = \frac{\sqrt{24}}{7}$ (Positive since $a$ is in the first quadrant)

Find the exact value of $\sin(b)$ from a right-angled triangle.

$\cos(b) = \frac{3}{5} = \frac{\text{adjacent}}{\text{hypotenuse}}$




By Pythagoras:
$c^2 = a^2 + b^2$
$5^2 = QR^2 + 3^2$
$5^2 - 3^2 = QR^2$
$16 = QR^2$
$\sqrt{16} = QR$
$4 = QR$
$\sin(b) = \frac{4}{5}$ (Positive since $b$ is in the first quadrant)

Substitute into
$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
and simplify.

$\sin(a - b) = \frac{5}{7} \times \frac{3}{5} - \frac{\sqrt{24}}{7} \times \frac{4}{5}$
$= \frac{15 - 4\sqrt{24}}{35}$


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You may wish to use your CAS calculator to investigate what other angles have trigonometric ratios that can be expressed exactly.

You can use the sums and differences of angles to find other identities.

### Example 14

Find an expression for $\cos(2x)$ in terms of $\cos(x)$

#### Solution

Use $2x = x + x$ and substitute into $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

$$\cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x)$$
$$= \cos^2(x) - \sin^2(x)$$

Use $\sin^2(\theta) + \cos^2(\theta) = 1$ in the form $\sin^2(\theta) = 1 - \cos^2(\theta)$ to substitute for $\sin^2(x)$ and simplify.

$$= \cos^2(x) - (1 - \cos^2(x))$$
$$= \cos^2(x) - 1 + \cos^2(x)$$
$$= 2\cos^2(x) - 1$$

State the result.

$$\cos(2x) = 2\cos^2(x) - 1$$

## EXERCISE 6.05 Angle sums and differences

### Concepts and techniques

In this exercise your teacher will tell you where to use your CAS calculator.

**1** Example 12 Expand

**a** $\sin(3\alpha + \beta)$ **b** $\cos(5m - 3n)$ **c** $\tan(\theta - 3\psi)$

**d** $\cos(x + 7y)$ **e** $\tan(34° + y)$ **f** $\sin(2\theta - 3\delta)$

**2** Simplify

**a** $\sin(a)\cos(b) + \cos(a)\sin(b)$ **b** $\dfrac{\tan(36°) + \tan(29°)}{1 - \tan(36°)\tan(29°)}$

**c** $\cos(28°)\cos(27°) - \sin(28°)\sin(27°)$ **d** $\sin(2x)\cos(3y) + \cos(2x)\sin(3y)$

**e** $\dfrac{\tan(3\theta) - \tan(\theta)}{1 + \tan(3\theta)\tan(\theta)}$ **f** $\sin(74°)\cos(42°) - \cos(74°)\sin(42°)$

**g** $\sin(a + b) + \sin(a - b)$ **h** $\sin(x + y) - \sin(x - y)$

**i** $\cos(x - y) - \cos(x + y)$ **j** $\cos(m + n) + \cos(m - n)$

**3** Simplify $\dfrac{\tan(x + y) + \tan(x - y)}{1 - \tan(x + y)\tan(x - y)}$.

**4** Example 13 Find the exact value of

**a** $\cos\left(\dfrac{\pi}{12}\right)$ **b** $\sin\left(\dfrac{7\pi}{12}\right)$

**c** $\tan\left(\dfrac{5\pi}{12}\right)$ **d** $\sin(15°)$

**e** $\dfrac{\tan(74°)-\tan(44°)}{1+\tan(74°)\tan(44°)}$

**f** $\cos\left(\frac{\pi}{15}\right)\cos\left(\frac{\pi}{10}\right)-\sin\left(\frac{\pi}{15}\right)\sin\left(\frac{\pi}{10}\right)$

**g** $\dfrac{\tan\left(\frac{\pi}{12}\right)+\tan\left(\frac{\pi}{6}\right)}{1-\tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{6}\right)}$

**h** $\sin\left(\frac{7\pi}{36}\right)\cos\left(\frac{5\pi}{36}\right)+\cos\left(\frac{7\pi}{36}\right)\sin\left(\frac{5\pi}{36}\right)$

**i** $\cos\left(\frac{4\pi}{9}\right)\cos\left(\frac{7\pi}{36}\right)+\sin\left(\frac{4\pi}{9}\right)\sin\left(\frac{7\pi}{36}\right)$

**j** $\sin(250°)\cos(25°)-\cos(250°)\sin(25°)$

**5** If $\sin(x)=\frac{2}{3}$ and $\cos(y)=\frac{3}{4}$, where $x$ and $y$ are in the first quadrant, find the exact value of

**a** $\sin(x+y)$ **b** $\cos(x-y)$ **c** $\tan(x+y)$

**6** If $\sin(x)=\frac{3}{5}$ and $\cos(y)=\frac{5}{13}$, where $x$ is the second quadrant and $y$ in the fourth, find the exact value of

**a** $\cos(x)$ **b** $\sin(y)$ **c** $\sin(x-y)$

**d** $\tan(y)$ **e** $\tan(x+y)$

**7** Example 14 By taking $2x = x + x$, find an expression for:

**a** $\sin(2x)$ **b** $\tan(2x)$

## Reasoning and communication

**8** By substituting $\frac{\pi}{2}-x$ in the identity $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$, show that $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

**9** Find an expression for:

**a** $\sin(x+y)+\sin(x-y)$ **b** $\cos(x+y)-\cos(x-y)$

**c** $\sin(x+y)-\sin(x-y)$ **d** $\cos(x+y)+\sin(x-y)$

**10** Find an expression for $\sin(3x)$ in terms of $\sin(x)$.

# 6.06 TRIGONOMETRIC EQUATIONS

In Chapter 3 you solved trigonometric equations when finding an angle in a right-angled triangle. For example, $\sin(\theta) = 0.64$ gives a solution of $\theta \approx 40°$. But with extended angles, there are actually other solutions, such as $\theta = 140°$, 400° and 500°. Check with your calculator to see that the sine of these other angles is 0.64.

You can solve trigonometric equations by looking at their graphs.

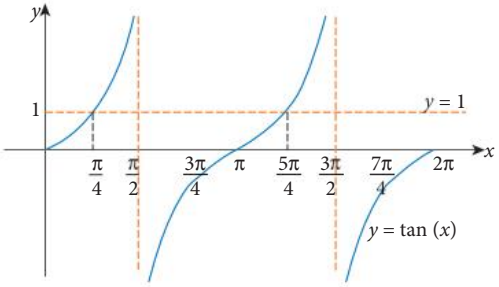
## Example 15

**a** Draw the graph of $y = \tan(x)$ in the domain $0 \leq x \leq 2\pi$ and use the graph to find all solutions to $\tan(x) = 1$ in this domain.

**b** CAS Sketch the graphs of $y = \sin(x)$ and $y = x - 1$ and use the graphs to solve $\sin(x) = x - 1$

### Solution

**a** Draw the graph of $y = \tan(x)$.

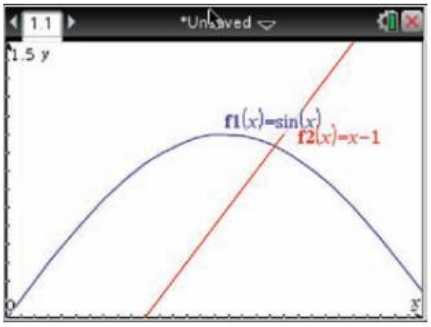



From the graph, find all $x$ values where $y = 1$.

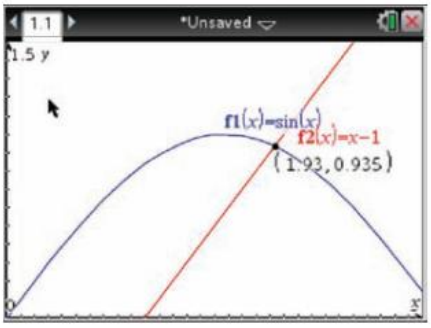
$x = \frac{\pi}{4}, \frac{5\pi}{4}$

**b** **TI-Nspire CAS**

Use a graph page and enter the functions as f1 and f2. Change the Window Settings to focus on the intersection. See page 298 for a reminder.




Find the point of intersection:
You can use menu 6: Analyse Graph and 4: Intersection. Move around the graphs and press enter for the lower bound and upper bound to set the search area.
You can also do this using menu 5: Trace and 1: Graph Trace to step along the graphs. You can set the display to a higher number of significant figures using menu 9: Settings.

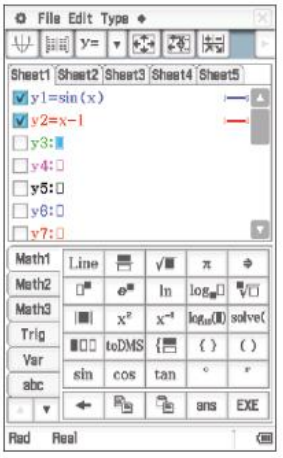



State the result.

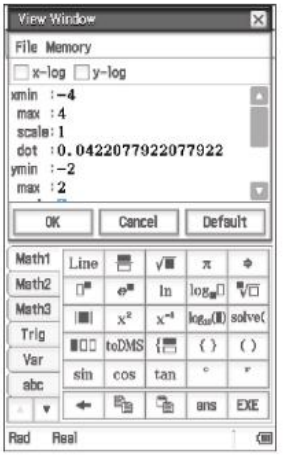
The solution to the equation $\sin(x) = x - 1$ is $x \approx 1.93$.

### ClassPad

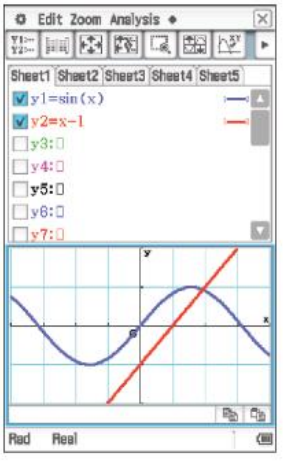
Use the **Graph&Table** application.
Set the calculator to Rad (radians).
Set y1=sin($x$) and y2=$x$–1.




Tap for **View Window**.
Think of the maximum and minimum values of sin x as a horizontal 'strip' and set $x$ and $y$ limits accordingly. A suggestion is $-4 \leq x \leq 4$ and $-2 \leq y \leq 2$.
Tap **OK**.



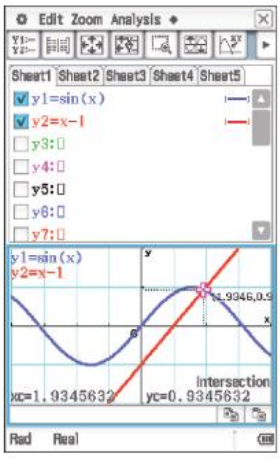

Tap to see the two graphs.





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Tap **Analysis**. From the drop down menu, tap **G-Solve** and then tap Intersection.
The answer is given as xc= and yc= .
We are only interested in the value of $x$.




Write the solution.

$x \approx 1.93$

Notice in Example **16** a that the two solutions to $\tan(x) = 1$ in the domain $0 \leq x \leq 2\pi$ are $x = \frac{\pi}{4}, \frac{5\pi}{4}$. These angles are in the first and third quadrants where $\tan(x) > 0$. The ASTC rule can help solve trig equations.

Trig functions of the angles of the form $\left(\frac{n\pi}{2} \pm x\right)$ for odd $n$ simplify to give the co-ratio, but functions of angles of the form $(n\pi + x)$ simplify to give the same ratio.

This means that you can simplify an angle of any size to the same ratio of an acute angle by expressing it as the sum or difference of an acute angle and a multiple of $\pi$ (180°). You work out the sign from the ASTC rule for the original angle.

## Example 16

**a** Solve the following equations in the domain $0 \leq x \leq 2\pi$.

**i** $\cos(x) = -\frac{\sqrt{2}}{2}$

**ii** $\tan(x) = 1.2$

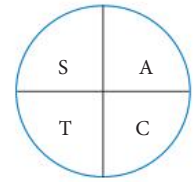
**b** Solve $\sin(x) = -\frac{\sqrt{3}}{2}$ in the domain $0° \leq x \leq 360°$.

**c** Solve $\sin(x) = 0$ for the domain $-2\pi \leq x \leq 2\pi$.

**d** CAS Solve $\sin(x) = \frac{1}{2}$ for $-2\pi \leq x \leq 2\pi$.

## Solution

**a i** The domain $0 \leq x \leq 2\pi$ is in radians, so the solutions to the equation will be in radians. Draw the ASTC diagram.




Write the acute angle for $\cos(x) = \frac{\sqrt{2}}{2}$.

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

In which quadrants is cosine negative?

Cosine is negative in the 2nd and 3rd quadrants.

Put $x$ in the 2nd and 3rd quadrants.

$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

State the result.

$x = \frac{3\pi}{4}, \frac{5\pi}{4}$

**ii** Use $\tan^{-1}$ on your calculator in radian mode to find the acute angle for $\tan(x) = 1.2$.

$x \approx 0.876$

In which quadrants is tangent positive?

Tangent is positive in the 1st and 3rd quadrants.

Put $x$ in the 1st and 3rd quadrants.

$x \approx 0.876, \pi + 0.876$

State the result.

$x \approx 0.876, 4.018$

**b** Write the acute angle for $\sin(x) = \frac{\sqrt{3}}{2}$.

$\sin(60°) = \frac{\sqrt{3}}{2}$

In which quadrants is sine negative?

Sine is negative in the 3rd and 4th quadrants.

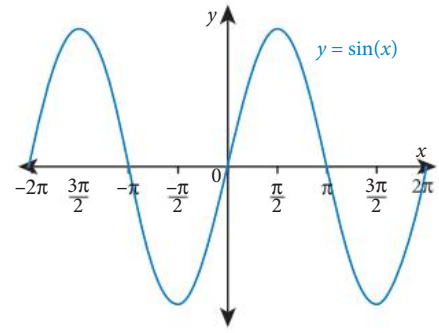
Put $x$ in the 3rd and 4th quadrants.

$x = 180° + 60°$ and $360° - 60°$

State the result.

$x = 240°, 300°$

**c** Sketch the graph over the domain required.



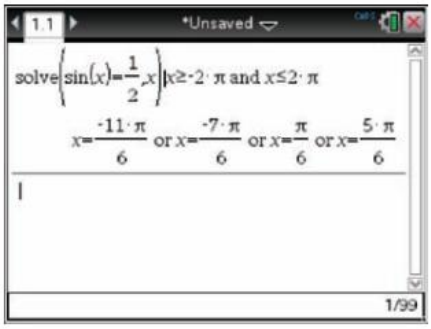

Read the solutions from the sketch.

$x = -2\pi, -\pi, 0, \pi$ or $2\pi$


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**d** **TI-Nspire CAS**

Use a calculator page [CALC].
Make sure Document Settings is radians (see page 298) and type solve(), use [menu], 3: Algebra and 1: Solve or choose solve() from the catalogue, [📖].
Put the equation into the brackets followed by a comma and $x$. To get the correct domain put a vertical line using [ctrl] = and type in the domain as $x \geq -2\pi$ and $x \leq 2\pi$.
Entering 0.5 instead of a fraction will give approximate answers.

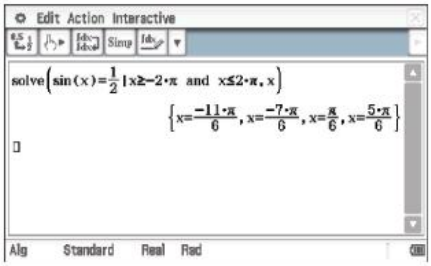



Write the solution in ascending order.

$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

**ClassPad**

Use the Main $\sqrt{\alpha}$ application and press [Keyboard].
Set the Calculator to **Standard** and **Rad**.
Use [solve(] from the [Math1] tab and [Trig] for the trigonometric ratios. For the fraction, tap [▣].
To get the correct domain, tap [Math3] then put a vertical line [|]. Also available from [Math3] are [≤] and [≥]. Type in the domain as $x \geq -2\pi$ and $x \leq 2\pi$.
Tap the right arrow [▶] to see the full solution, or tap Rotate.




Interpret the answer.

One solution is $\frac{\pi}{6}$ plus any whole number of $2\pi$s. In the given domain this gives $\frac{\pi}{6}$ and $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$.
The other solution is $-\frac{\pi}{6}$ plus any whole number of $2\pi$s. In the given domain this gives $-\frac{\pi}{6}$ and $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$.

Write the solution in ascending order.

$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Some equations need simplifying or the domain extended in order to solve them.

## Example 17

**a** Solve $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$

**b** Solve $\sin^2(\theta) = \frac{1}{2}$ for $0° \leq \theta \leq 360°$

**c** Solve $2\sin(2x) = 1$ for $0° \leq x \leq 360°$

**d** Solve $2\cos^2(x) + \cos(x) - 1 = 0$ for $0 \leq x \leq 2\pi$

## Solution

**a** Rearrange the equation.

$2\cos(x) + 1 = 0$

$2\cos(x) = -1$

$\cos(x) = -\frac{1}{2}$

Write the acute angle for which $\cos(x) = \frac{1}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

In which quadrants is cosine negative?

Cos is negative in the 2nd and 3rd quadrants.

Solve $\cos(x) = -\frac{1}{2}$ in the domain.

$x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

State the result.

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

**b** Take the square root of both sides.

$\sin^2(\theta) = \frac{1}{2}$

$\sin(\theta) = \pm\sqrt{\frac{1}{2}}$

$= \pm\frac{1}{\sqrt{2}}$

Write with a rational denominator.

$= \pm\frac{\sqrt{2}}{2}$

Write the acute angle for which $\sin(x) = \frac{\sqrt{2}}{2}$

$\sin(45°) = \frac{\sqrt{2}}{2}$

Solve $\sin(x) = \pm\frac{\sqrt{2}}{2}$ in the domain. The answer can be in any quadrant because it is + and –.

$x = 45°, 180° - 45°, 180° + 45°, 360° - 45°$

State the result.

$x = 45°, 135°, 225°, 315°$

**c** Rearrange the equation.

$2\sin(2x) = 1$

$\sin(2x) = \frac{1}{2}$

Adjust the domain for $2x$.

Since $0° \leq x \leq 360°$

$0° \leq 2x \leq 720°$

Find the acute angle in degrees that has $\sin(2x) = \frac{1}{2}$.

$2x = 30°$

In which quadrants is sine postive?

Sine is positive in the 1st and 2nd quadrants.

Solve $\sin(2x) = \frac{1}{2}$ in the domain $0° \leq 2x \leq 720°$.

$2x = 30°, 180° - 30°, 360° + 30°, 360° + 180 - 30°$

Simplify.

$2x = 30°, 150°, 390°, 510°$

Divide by 2 and state the result for $x$.

$x = 15°, 75°, 195°, 255°$


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**d** This is a quadratic equation in cos $(x)$.
Factorise the equation and solve for cos $(x)$.

$2\cos^2(x) + \cos(x) - 1 = 0$
$(2\cos(x) - 1)(\cos(x) + 1) = 0$
$2\cos(x) - 1 = 0, \cos(x) + 1 = 0$
$2\cos(x) = 1, \cos(x) = -1$
$\cos(x) = \frac{1}{2}, \cos(x) = -1$

Solve $\cos(x) = \frac{1}{2}$

$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$ [Cosine is positive in the first and fourth quadrants]
$= \frac{\pi}{3}, \frac{5\pi}{3}$

Solve $\cos(x) = -1$

$x = \pi$ [from the unit circle or the graph of $y = \cos(x)$]

State the result.

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

# EXERCISE 6.06 Trigonometric equations

## Concepts and techniques

**1** Example 15 Solve graphically for $0° \le \theta \le 360°$

**a** $\sin(\theta) = 0.35$ **b** $\cos(\theta) = -\frac{1}{2}$ **c** $\tan(\theta) = -1$

**d** $\sin(\theta) = \frac{\sqrt{3}}{2}$ **e** $\tan(\theta) = -\frac{\sqrt{3}}{3}$ **f** $\cos(\theta) = 0.72$

**2** CAS Solve graphically for $0 \le x \le 2\pi$.

**a** $\cos(x) = \frac{1}{2}$ **b** $\sin(x) = -\frac{1}{\sqrt{2}}$ **c** $\tan(x) = 1$

**d** $\tan(x) = \sqrt{3}$ **e** $\cos(x) = -\frac{\sqrt{3}}{2}$ **f** $\sin(x) = \frac{\sqrt{3}}{2}$

**3** CAS Solve graphically in the domain $0 \le x \le 2\pi$

**a** $\tan(x) = x$ **b** $\cos(x) = 2x - 3$ **c** $\sin(2x) = x$

**d** $\sin(x) = \cos(x)$ **e** $\sin(x) = x$

**4** CAS Example 16 Solve each of the following in the domain indicated.

**a** $\tan(\theta) = 1$ for $0° \le \theta \le 360°$ **b** $\sin(\theta) = \frac{1}{2}$ for $0° \le \theta \le 360°$

**c** $\cos(\theta) = -\frac{\sqrt{3}}{2}$ for $0 \le \theta \le 2\pi$ **d** $\sin(\theta) = -\frac{\sqrt{2}}{2}$ for $0 \le \theta \le 2\pi$

**e** $\tan(\theta) = \sqrt{3}$ for $0 \le \theta \le 2\pi$ **f** $\cos(x) = -\frac{1}{2}$ for $0° \le x \le 360°$

**5** CAS Solve the following trig equations in the domain $0 \le x \le 2\pi$. Check your answers with your CAS calculator.

**a** $\cos(x) = 1$ **b** $\tan(x) = 0$ **c** $\sin(x) = -1$

**d** $\cos(x) = 0$ **e** $\sin(x) = 1$

6 Example 17 Solve for $-\pi \le \theta \le \pi$

a $\tan(\theta) - \sqrt{3} = 0$ b $2\sin(\theta) - 1 = 0$ c $3\tan^2(\theta) - 1 = 0$

d CAS $\sin(\theta) + 1 = 0$ e CAS $2\cos^2(\theta) - 1 = 0$

7 Solve for $-180° \le x \le 180°$

a CAS $\tan^2(x) - 1 = 0$ b CAS $2\sin^2(x) - \sin(x) = 0$

c $\tan(2x) = 1$ d $2\cos(2x) + \sqrt{3} = 0$

## Reasoning and communication

8 Solve $\sin(x) = \sqrt{3}\cos(x)$ for $0 \le x \le 2\pi$

9 Find, correct to 2 decimal places, the two values of $x$ closest to 0 for which $5\sin\left(2x - \frac{\pi}{6}\right) - 2 = 0$

10 Solve

a $2\sin^2(x) + \cos(x) - 2 = 0$ for $0° \le x \le 360°$

b $\tan^2(x) - 3\tan(x) + 2 = 0$ for $-\pi \le \theta \le \pi$

# 6.07 MODELLING PERIODIC PHENOMENA

In this section you will look at some examples where trigonometry can model periodic situations such as tides, temperatures over the seasons and a pendulum.

If you recognise that a graph has periodic data, you can measure its amplitude and period and in some cases find its equation.

### Example 18

The table shows the highest average monthly temperatures in Sydney over 12 months.

| $x$ | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ (°C) | 26.1 | 26.1 | 25.1 | 22.8 | 19.8 | 17.4 | 16.8 | 18.0 | 20.1 | 22.2 | 23.9 | 25.6 |

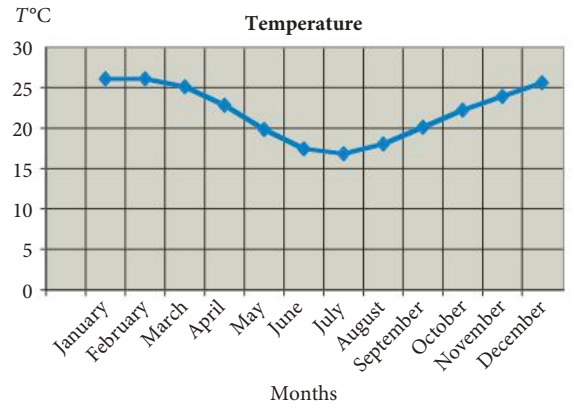
a Draw a graph of this data, by hand or on a CAS calculator.

b What is the period?

c Find the amplitude.

### Solution

a Draw the graph.



**b** State the period.

The curve is approximately a cosine curve with one full period.
The period is 12 months.

**c** Find the amplitude.

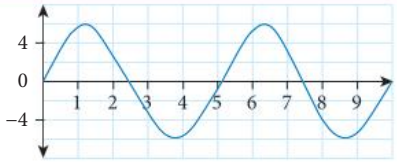
The maximum temperature is about 26° and the minimum is about 17°.

$$\text{Amplitude} = \frac{26° - 17°}{2}$$
$$= 4.5°$$

When a graph looks periodic, you can find a model for its equation by using the skills you learned earlier in this chapter and checking the model on a CAS calculator.

## Example 19

Find a model for this periodic function. Check your answer using a CAS calculator, spreadsheet or other computer program.




### Solution

Find the period and amplitude.

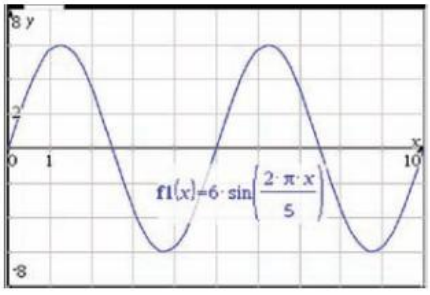
From the graph, the period is 5 and the amplitude is 6.

$$\text{Period: } \frac{2\pi}{b} = 5$$
$$2\pi = 5b$$
$$\frac{2\pi}{5} = b$$
$$a = 6$$

Substitute into the equation $y = a \sin(bt)$.

$$y = 6 \sin\left(\frac{2\pi x}{5}\right)$$

Check the model on a CAS calculator.




The graph looks the same as the one given so $y = 6 \sin\left(\frac{2\pi x}{5}\right)$ is a good model for the original graph.

When you know the equation, you can use it to find out extra information.

## Example 20

The distance, in metres, of a wave from a point on a beach can be modelled by the equation $d = 5 \sin(4\pi t) + 7$ where $t$ is time in minutes.



Photo courtesy Margaret Grove

**a** What are the maximum and minimum distances of the wave from this point?
**b** What is the initial distance of the wave from the point on the beach?
**c** How long does it take each wave to wash up on the beach?
**d** How many waves wash up onto the beach in an hour?
**e** At what times are the waves 9.5 m out from this point on the beach?

### Solution

**a** Find the centre and amplitude of the model.

The centre is 7 and the amplitude is 5.

Find the maximum and minimum distance.

$7 - 5 = 2$ and $7 + 5 = 12$.
So the maximum distance is 12 m and the minimum distance is 2 m.

**b** Substitute the initial value of $t = 0$ into the equation $d = 5 \sin(4\pi x) + 7$.

$d = 5 \sin(4\pi \times 0) + 7$
$= 7$

State the result.

Initially the wave is 7 m out from the point on the beach.

**c** Find the period.

$\frac{2\pi}{b} = \frac{2\pi}{4\pi}$
$= \frac{1}{2}$

State the result.

It takes $\frac{1}{2}$ a minute for a wave to wash up onto the beach.

**d** Use the result from **c**.

Each wave takes $\frac{1}{2}$ a minute.
This means 2 each minute or $2 \times 60$ each hour.

State the result.

120 waves wash up on the beach in an hour.


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**e** Substitute $d = 9.5$ into the equation $d = 5 \sin(4\pi t) + 7$.

$9.5 = 5 \sin(4\pi t) + 7$

Subtract 7 from both sides.

$2.5 = 5 \sin(4\pi t)$

Divide both sides by 5.

$\frac{1}{2} = \sin(4\pi t)$

Find the angle that has $\sin(4\pi t) = \frac{1}{2}$ and the quadrants where $\sin(4\pi t)$ is positive (find all solutions).

$4\pi t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \pi - \frac{\pi}{6}, \ldots$

Solve for $t$.

$t = \frac{1}{24}, \frac{5}{24}, \frac{13}{24}, \frac{17}{24}, \ldots$ minutes

Change to seconds.

$t = 2.5, 12.5, 32.5, 42.5, \ldots$ seconds

State the result.

The wave is 9.5 m out from the point on the beach after 2.5 s, 12.5 s, 32.5 s, 42.5 s, ...

# EXERCISE 6.07 Modelling periodic phenomena

## Reasoning and communication

**1** Example 18 The graph below shows the times of sunsets seen over 2 years.

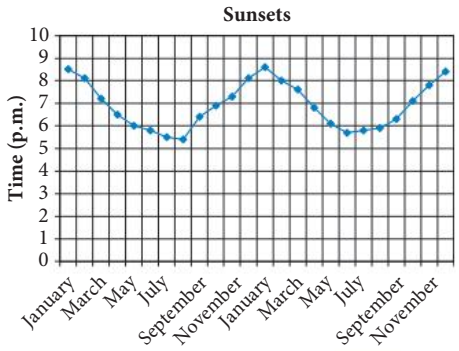


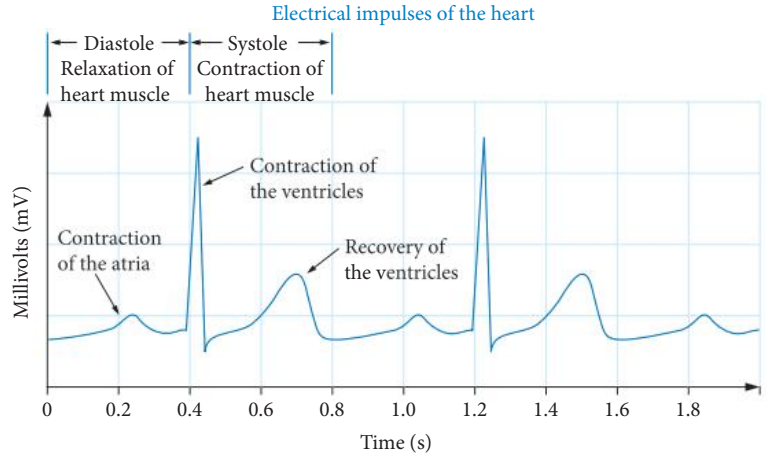



Photo courtesy Margaret Grove

**a** Find the period and amplitude of the graph.

**b** At approximately what time would you expect the sun to set in July?

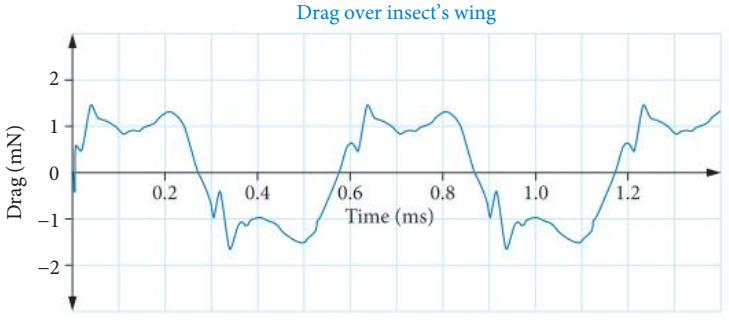
2 The electrical impulses of the heart are shown in the diagram.




a What is the period?
b What is the pulse rate in heart beats/minute?
c How are the pulse rate and period (in minutes) related?
d How long does it take for the heart to beat 1000 times?

3 The graph on the right shows how the drag over the wing of an insect in flight varies over time.

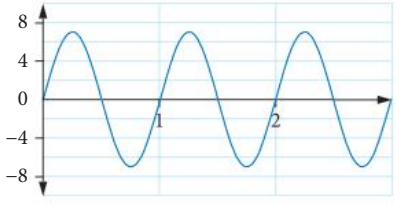
a What is the period?
b What is the amplitude of the drag?
c At what times up to 3 ms will the drag be the greatest?



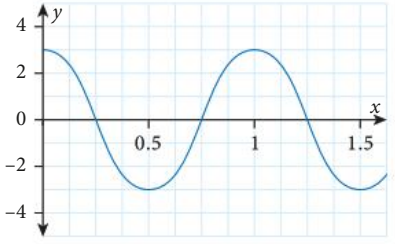

d The period of the drag represents one beat of the insect's wings. How many times will the insect's wings beat in 10 s?

4 Example 19 Find equations of each of the following.

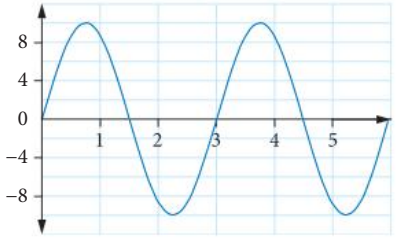
a



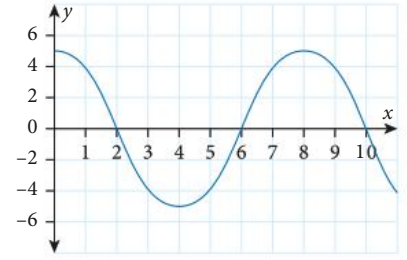

b




c




d

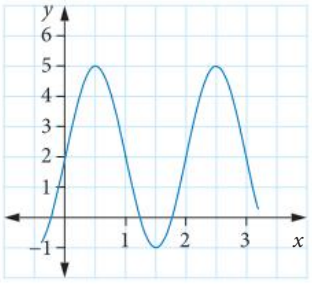




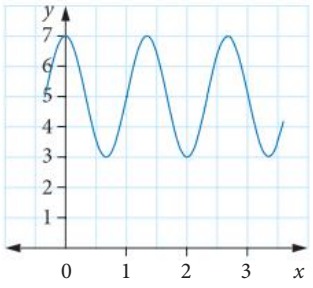
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**5** Find a model (equation) for each of the periodic functions shown below, and check your answers using a CAS calculator, spreadsheet or other computer program.

**a**




**b**




**6** Example 20 The displacement $d$ cm of a spring from a fixed point $O$ at time $t$ seconds is given by the equation $d = 5\cos(2t) + 5$.

**a** What is the spring's initial displacement from $O$?

**b** How far is the spring from $O$ after

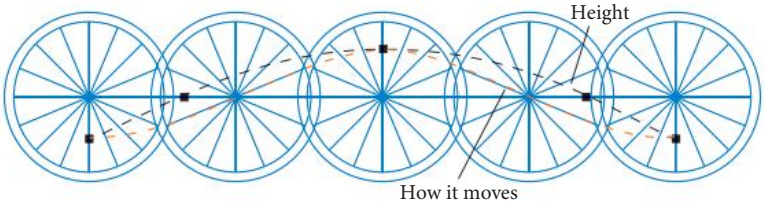
**i** 3 s? **ii** 7 s? **iii** 1 minute?

**c** At what exact times is the spring

**i** at $O$? **ii** 5 cm from $O$? **iii** 10 cm from $O$?

**d** After how long is the spring first 2 cm from $O$ (to 1 decimal place)?

**7** A reflector on a bicycle wheel is fixed at a distance of 25 cm from the centre of the wheel. The wheel has a diameter of 66 cm. For the starting position of the wheel, the reflector is at its lowest height.




**a** What is the least height of the reflector from the ground as the wheel turns?

**b** What is the greatest height of the reflector as the wheel turns?

**c** How far does the wheel roll along the ground when it makes one complete turn?

**d** Draw a graph to show the height of the reflector as the wheel rolls along.

**e** From your graph, estimate how far the wheel rolls before the reflector is first at a height of 33 cm.

**f** Estimate how far the wheel rolls before the reflector is first at a height of 40 cm.



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8 The depth of a tidal pool on a particular day could be modelled by the function $h(t) = 3 \sin\left(\frac{\pi t}{6}\right) + 5$ where $h$ is the depth in metres and $t$ gives the hours since midnight.

a What was the depth at midnight?
b What was the depth at 6:00 a.m.?
c What was the earliest time at which the depth was 2 m?
d At what times was the pool at maximum depth?

9 A cable-car ride can take tourists to an observation deck 240 m above the base level in 15 minutes without stopping. The cable cars travel in an approximately straight line from the base level to the observation deck and return, stopping for 2 minutes at either end of the trip. There are four cable cars on the cable.

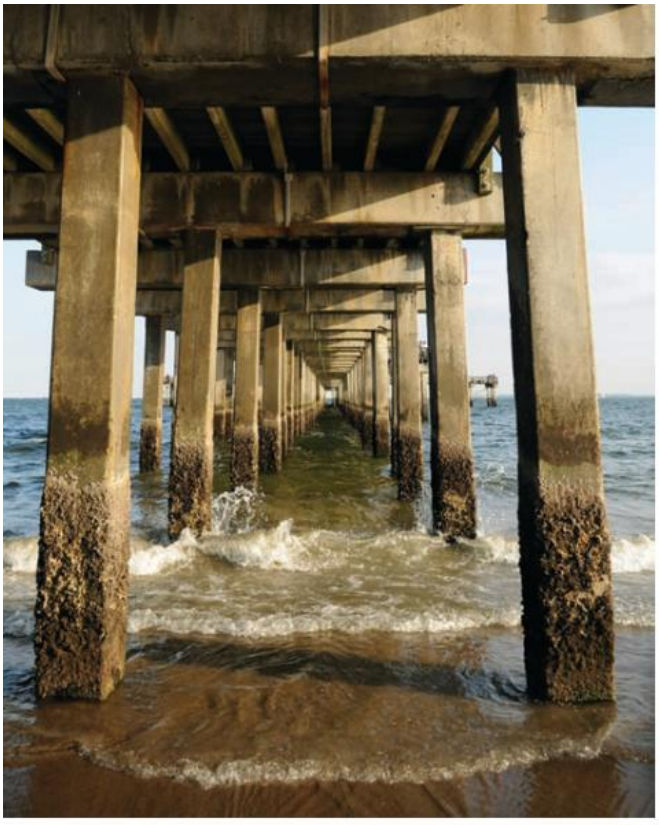
a How many times does a cable car stop between the same points on a round trip?
b How long does a round trip take?
c Draw a graph showing the height of a cable car for two round trips.
d How many round trips are made by a single cable car in 12 daylight hours?
e If each cable car can take 10 passengers, how many people can do the return trip in one 12-hour day, assuming that the cable cars going to the top are full on every trip?
f How long is it from the time a cable car is at 200 m until it is again at this height?




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10 The time between successive high tides at a pier is 12 h 20 min. The average depth of water is 4 m, but at low tide it is 3.1 m. Write an equation to model this relationship as a sine curve and use it to find the lengths of time that boats with draughts of 3.5 m and 4.5 m can use the pier. The draught of a boat is the minimum depth of water it needs.



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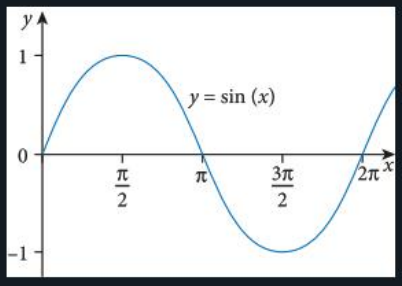
# 6 CHAPTER SUMMARY
# TRIGONOMETRIC FUNCTIONS AND GRAPHS

- For a **periodic function** $f(x)$ the smallest value $t$ such that $f(x) = f(x + t)$ for all $x$ is called the **period**. The section of the function from $f(x)$ to $f(x + t)$ is called a **cycle**.

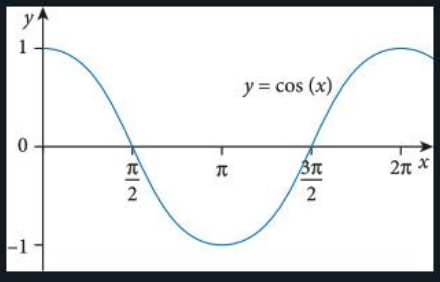
  The **amplitude** of a periodic function is half the vertical distance between the maximum and minimum values.

- Trigonometric graphs:

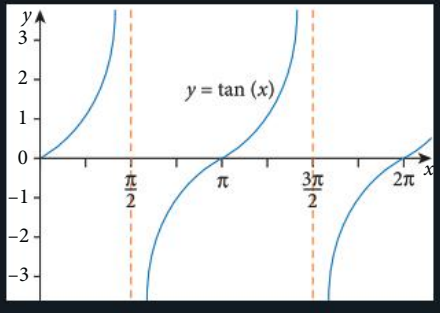
  $y = \sin(x)$ has period $2\pi$ and amplitude 1




  $y = \cos(x)$ has period $2\pi$ and amplitude 1




  $y = \tan(x)$ has period $\pi$ and no amplitude




- General amplitude and period:
  $y = a \sin(bx) + c$ has amplitude $a$, period $\frac{2\pi}{b}$ and is translated vertically by $c$ units.
  $y = a \cos(bx) + c$ has amplitude $a$, period $\frac{2\pi}{b}$ and is translated vertically by $c$ units.
  $y = a \tan(bx) + c$ is stretched or compressed by a factor of a vertically, has period $\frac{\pi}{b}$ and is translated vertically by $c$ units.

- Phase changes:
  $y = \sin(x + c)$ is a horizontal translation of $y = \sin(x)$ $c$ units to the right (later) if $c < 0$, to the left (earlier) if $c > 0$.
  $y = \cos(x + c)$ is a horizontal translation of $y = \cos(x)$ $c$ units to the right if $c < 0$, to the left if $c > 0$.
  $y = \tan(x + c)$ is a horizontal translation of $y = \tan(x)$ $c$ units to the right if $c < 0$, to the left if $c > 0$.

  $$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

  $$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

- Trigonometric identities:

  $$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

  $$\sin^2(\theta) + \cos^2(\theta) = 1$$

  $$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

  $$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

- Sums and differences of angles:

  $$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
  $$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
  $$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
  $$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

  $$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

  $$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

- Many periodic phenomena can be modelled by the **sinusoidal** functions $y = a \sin(bx) + c$ or $y = a \cos(bx) + c$.
- Trigonometric equations can be solved graphically or algebraically.


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# CHAPTER REVIEW
# TRIGONOMETRIC FUNCTIONS AND GRAPHS
# 6

## Multiple choice

1 Example 7 The period of $y = 2\tan(3\pi x)$ is

A $\frac{2}{3}$ B $\frac{1}{3}$ C $\frac{\pi}{3}$ D 3 E $3\pi$

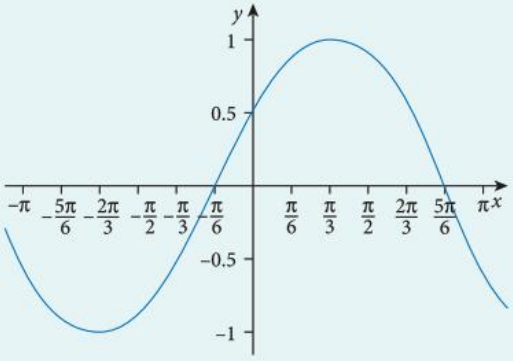
2 Example 12 $\cos(a + b) =$

A $\sin(a)\cos(b) - \sin(a)\cos(b)$
B $\cos(a)\cos(b) + \sin(a)\sin(b)$
C $\sin(a)\cos(b) + \cos(a)\sin(b)$
D $\cos(a)\cos(b) - \sin(a)\sin(b)$
E $\cos(a)\sin(b) - \sin(a)\cos(b)$

3 Example 16 Solve $\cos(x) = -1$ for $-2\pi \leq x \leq 2\pi$

A $x = \pi$ B $x = \frac{\pi}{2}$ C $x = -\frac{\pi}{2}, \frac{3\pi}{2}$
D $x = \pm\pi$ E $x = 0, \pm 2\pi$

4 Example 19 Find the equation of the following graph.




A $y = \sin\left(x + \frac{\pi}{6}\right)$ B $y = \sin\left(x - \frac{\pi}{6}\right)$ C $y = \sin(3\pi x)$
D $y = \sin\left(x + \frac{\pi}{3}\right)$ E $y = \sin\left(\frac{\pi x}{6}\right)$

## Short answer

5 Example 1 Use the unit circle to evaluate:

a $\sin 270°$ b $\cos 2\pi$ c $\tan\pi$
d $\cos 90°$ e $\sin\pi$

6 Example 2 Sketch $y = \cos(x)$ for $-\pi \leq x \leq \pi$.

7 Example 3 Sketch in the domain $0 \leq x \leq 2\pi$

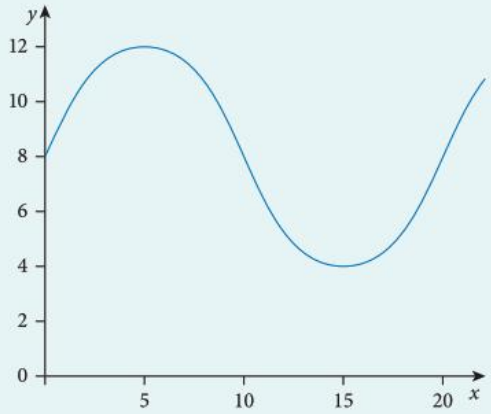
a $y = \cos(x) - 1$ b $y = -\tan(x)$

8 Example 4 State the amplitude of:

a $y = 7\cos(3x)$ b $y = -4\sin(2x)$

9 Example 5 Draw the graph of $y = -3\cos(x)$ in the domain $0 \le x \le 2\pi$

10 Example 6 Find the amplitude and period of the graph below.




11 Example 7 State the period of the following functions

a $y = \tan(4x)$ b $y = \sin\left(\frac{x}{5}\right)$ c $y = \cos(3\pi x)$

12 Example 8 Sketch the graph of the following functions in the domain $0 \le x \le 2\pi$.

a $y = \sin(2x)$ b $y = \tan\left(\frac{x}{2}\right)$

13 Example 9 State the period and amplitude and draw one full cycle of the graph of $y = 5\cos(3x)$.

14 Example 10

a Draw the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ for $-2\pi \le x \le 2\pi$.

b What other function is the same as this one?

15 Example 10 Draw the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ in the domain $0 \le x \le 2\pi$.

16 Example 11 Sketch the graph of $y = 3\sin\left(x - \frac{\pi}{6}\right) - 2$ for $-\pi \le x \le \pi$.

17 Example 12 Simplify.

a $\sin(2a)\cos(3b) + \cos(2a)\sin(3b)$ b $\frac{\tan(2\theta) - \tan(\beta)}{1 + \tan(2\theta)\tan(\beta)}$

c $\cos(3x)\cos(4y) - \sin(3x)\sin(4y)$

18 Example 13 Find the exact value of:

a $\tan\left(\frac{\pi}{12}\right)$ b $\cos\left(\frac{5\pi}{12}\right)$

19 Example 14 Show that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

20 Example 15 CAS By drawing the graphs $y = \sin(x)$ and $y = \frac{x}{2}$, solve graphically the equation $\sin(x) = \frac{x}{2}$


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21 Example 16 Solve for $0 \leq x \leq 2\pi$.

a $\cos(x) = \dfrac{\sqrt{2}}{2}$ b $\tan(x) = -\sqrt{3}$ c $\sin(x) = -0.37$

22 Example 17 Solve $2\cos^2(\alpha) - 1 = 0$ for $0° \leq \alpha \leq 360°$.

23 Example 18 The graph shows the incidence of crimes committed over 24 years in Gotham City.




a Approximately how many crimes were committed in the 10th year?

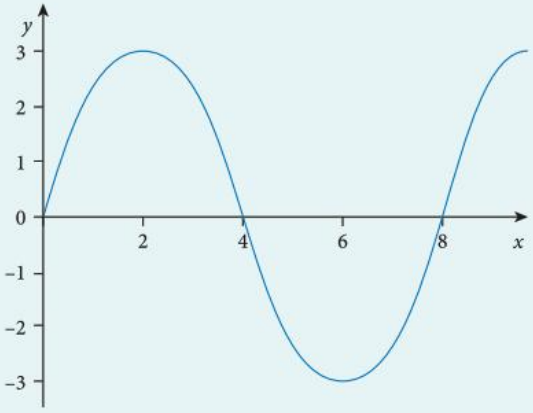
b What was the:

i highest, and

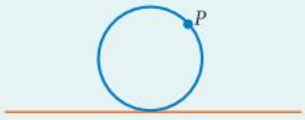
ii lowest number of crimes?

c Find the amplitude and the period of the graph.

24 Example 19 Find an equation for the periodic function below.




25 Example 20 A large hoop is rolling along the ground. The vertical distance above the ground of a Point $P$ on the rim of the hoop is given by $y = 1.25 - 1.25\cos(2\pi t)$ where $y$ is in metres and $t$ is in seconds. Find the first three times at which $P$ is 2.00 m above the ground.



## Analysis

**26** The monthly average overnight minimum temperature for Melbourne can be modelled by the function $T = 4.5 \cos\left(\frac{\pi m}{6}\right) + 10.5$, where $T$ is the temperature in degrees Celcius and $m$ is the number of months since January, assuming they can be taken as equal in length.

- **a** What is the maximum average overnight minimum, and when does it occur?
- **b** What is the mean value of the average overnight minimum, and during what months does it occur?
- **c** During what months is the average overnight minimum increasing?

**27** The depth of water in a port is given by $d = 3.8 \cos\left(\frac{\pi t}{6}\right) + 12.2$ where $d$ is in metres and $t$ is the number of hours since high tide.

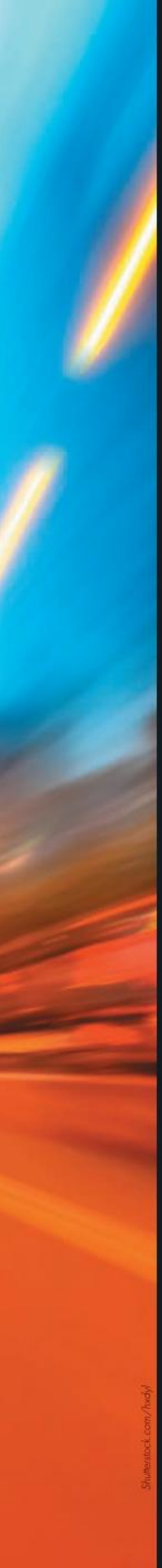
- **a** How many hours are there between high tides?
- **b** If high tide was at midnight, during what hours was the depth more than 12.2 m?
- **c** During what hours was the depth less than 10.3 m?

**28** A clock stands on a shelf. The centre of the clock, where the hands are attached, is 15 cm above the shelf, and the minute hand is 10 cm long. The distance of the tip of the hand from the shelf can be modelled by a sinusoidal function. Write an equation for the model in terms of the time, $t$, where $t$ is the number of minutes past midnight. Hence find the number of times during the day when the tip is 10 cm above the shelf.

Practice quiz


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# 7

## TERMINOLOGY

acceleration
approximate rate of change
average rate of change
average speed
chord
dependent variable
displacement
gradient of a graph
independent variable
instantaneous
instantaneous rate of change
rate of change
secant
secant from below
secant from above
secant around a point
small change
speed
tangent
velocity

Prior learning

## INTRODUCTION TO DIFFERENTIAL CALCULUS

# RATES OF CHANGE


7.01 Average rates of change
7.02 Instantaneous rates of change
7.03 The average rate of change of a function
7.04 The instantaneous rate of change of a function
7.05 Approximation of rates of change
7.06 Rate of change functions
Chapter summary
Chapter review


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## RATES OF CHANGE

- interpret the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change of a function $f$ (ACMMM077)
- use the Leibniz notation $\delta x$ and $\delta y$ for changes or increments in the variables $x$ and $y$ (ACMMM078)
- use the notation $\frac{\delta y}{\delta x}$ for the difference quotient $\frac{f(x+h)-f(x)}{h}$ where $y = f(x)$ (ACMMM079)
- interpret the ratios $\frac{f(x+h)-f(x)}{h}$ and $\frac{\delta y}{\delta x}$ as the slope or gradient of a chord or secant of the graph of $y = f(x)$ (ACMMM080) AC

# 7.01 AVERAGE RATES OF CHANGE

A **rate of change** is worked out using the changes in variables instead of the variables themselves. The symbol $\Delta$ (Greek capital delta) is used to show the change in a quantity. You can find a rate of change even if the rate is not constant.

### IMPORTANT

The **rate of change** of a quantity $y$ with respect to a quantity $x$ is the quotient of their changes:

$$\text{Rate of change} = \frac{\text{change in second quantity}}{\text{change in first quantity}} = \frac{\Delta y}{\Delta x}$$

The first variable is taken as the **independent variable**. The second variable is the **dependent variable**.

On a straight-line graph the rate of change is the gradient of the line, $m = \frac{\Delta y}{\Delta x}$.

The rate of change of a quantity, with no independent variable, means the rate of change with respect to time. The phrase 'with respect to' is often abbreviated as w.r.t.

### Example 1

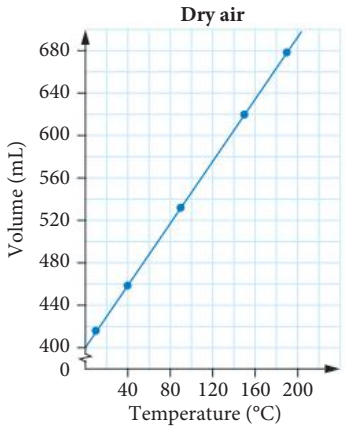
The volume of some dry air in a tube was measured at different temperatures but at the same pressure. The results are shown in the table below. Draw a graph and find the rate of change of volume with respect to temperature using a line of best fit.

| Temperature (°C) | 10 | 40 | 90 | 150 | 190 |
|---|---|---|---|---|---|
| Volume (mL) | 415 | 459 | 532 | 620 | 678 |


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## Solution

Draw the graph with temperature as the independent variable.




Find the slope of the line. This calculation uses (10, 415) and (190, 678) but other points may be used.

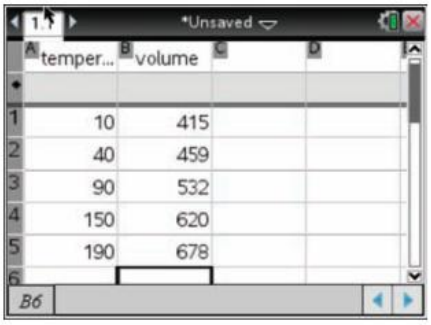
$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\approx \frac{678 - 415}{190 - 10}$$
$$\approx 1.46 \text{ mL/°C}$$

Use the gradient as the rate of change.

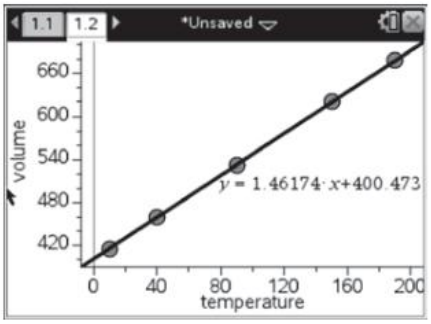
Rate of change is about 1.46 mL/C.

### TI-Nspire CAS

Use a Lists & Spreadsheet page  .
Go up to the top of the columns and type in the variable names 'temperature' and 'volume'. Then type in the data in the columns.




Insert a Data & Statistics page by pressing ctrl, doc and choosing Data and Statistics  . Use the mouse-pad to move the cursor to the bottom 'click to add variable' and temperature should be highlighted Click or press enter. Then move the cursor back to the side so that 'click to add variable' reappears at the side. Use the down arrow or the mouse to highlight volume and click or press enter. Press menu, 4: Analyse, 6: Regression and 1: Show Linear(mx + b).



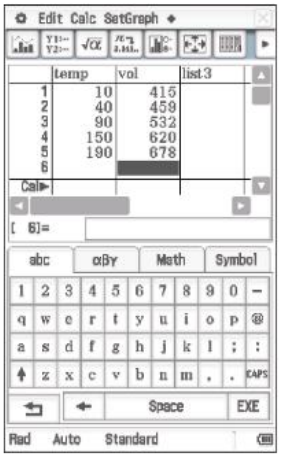

Use the gradient as the rate of change.

Rate of change is about 1.46 mL/°C.

### ClassPad

Use the **Statistics** menu.

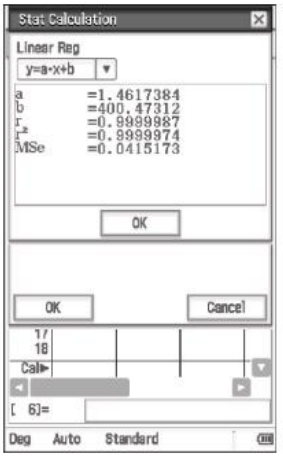
Rename the first column "temp" and the second column "vol". Enter the values for the independent variable temperature in the temp column and the volumes in the vol column.




Tap **Calc** then **Regression** and **Linear Reg** from the menu that drops down. Complete the **Set Calculation** screen as shown on the right.
Tap **OK**.




The calculator works out the equation of the straight line that fits the points best, in the form $y = ax + b$. This is called the line of best fit.
The gradient is $a$ and the $y$-intercept is $b$.




Use the gradient as the rate of change.

Rate of change is about 1.46 mL/°C.

When you go for a trip, the speeds you travel on the way are not as important as the speed for the overall trip. In this case, you would use the **average speed**. This is an example of an **average rate of change**.


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## IMPORTANT

The **average rate of change** of a variable with respect to another between two points is:

$$\frac{\text{second value of dependent variable} - \text{first value of dependent variable}}{\text{second value of independent variable} - \text{first value of independent variable}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Where the independent variable is time, average rate of change of $x = \frac{\Delta x}{\Delta t}$, assuming the change $\Delta x$ occurs over a time $\Delta t$.

**Speed** is the rate of change of the total distance travelled by a moving object.

**Velocity** is the rate of change of the displacement of a moving object. **Displacement** is the change in position from the starting point to the end point without regard for the direction travelled.

**Acceleration** is the rate of change of the velocity of a moving object. Displacement, velocity and acceleration are considered to have a direction as well as a measure.

**Flow rate** measures the speed at which a fluid travels and is the rate of change of volume.

### Example 2

What is the average rate of change of volume when a petrol tank is filled from 8 L to 53 L in 3 minutes?

#### Solution

| | |
|---|---|
| Write the formula. | $\text{Average rate of change} = \frac{\text{second value} - \text{first value}}{\text{time taken for the change}}$ |
| Substitute the values. | $= \frac{53 - 8}{3}$ |
| Evaluate. | $= 15$ L/min |
| Write the answer. | The tank fills at a rate of 15 L/min. |

The average of some rates of change is not the same as the average rate of change, as shown by the next example for speed.



## Example 3

A delivery van takes a parcel to a customer and then returns to the depot. On the way to the customer it travels at 40 km/h and on the way back it travels at 60 km/h. Calculate the average speed and average velocity for the round trip of 90 km.

### Solution

| | |
|---|---|
| The distance to the customer is 45 km. | Time going to customer $= \frac{45}{40} = 1.125$ h |
| The distance back is the same. | Time returning $= \frac{45}{60} = 0.75$ h |
| Find the total time. | Total time = 1.875 h |
| Now find the average speed. | Average speed $= \frac{90}{1.875} = 48$ km/h |
| The van returns to the starting point. | Total displacement = 0 km |
| Work out the average velocity. | Average velocity $= \frac{0}{1.875} = 0$ km/h |

## INVESTIGATION Forensic science

Forensic scientists, archaeologists and geologists use bone measurements to determine the characteristics of a person or animal. Sometimes the shape and size of a whole animal have been worked out from a single bone. It is possible for a forensic scientist or archaeologist to work out the height of a person by measuring a single bone.

1. To begin this investigation, you will need to do some forensic investigation at home. Make the following measurements of people in your household! Don't forget to include yourself. For each person:

   Measure their height.

   Note their sex.

   Measure the length from the point of their elbow to the point of their shoulder.

   Measure the length from their inside ankle knuckle to their kneecap (with the leg bent).

2. Bring the information you have collected to class and pool it with the information collected by other class members.

3. As part of your forensic analysis, make graphs of the pooled measurements and consider the relationships between height and other body measurements. Use a CAS calculator or spreadsheet to make this task easier.

4. Write a report, describing the work that you did and any conclusions you reached. You should also make some conclusions about the reliability of the estimations of height (and weight) made by scientists on the basis of bone measurements.


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# EXERCISE 7.01 Average rates of change

## Concepts and techniques

1 A car travelling at 20 m/s increased its speed to 36 m/s over a period of 4 s. What was the average acceleration?

**A** 4 m/s$^2$ **B** 5 m/s$^2$ **C** 9 m/s$^2$ **D** 14 m/s$^2$ **E** 28 m/s$^2$

2 The height of water in a dam dropped from 15 m to 13 m in 4 hours. What was the average rate of change of the level?

**A** 0.25 /h **B** 0.5 m/h **C** 1 m/h **D** 2 m/h **E** 3 m/h

3 Example 1 The table below shows how the length of a spring changes as different masses are hung on it.

| Mass (g) | 30 | 50 | 80 | 120 | 140 | 150 |
|---|---|---|---|---|---|---|
| Length (cm) | 15 | 16.8 | 18 | 19.8 | 22.2 | 23.4 |

**a** Draw a graph to show the information.
**b** Comment on the graph.
**c** What is the average rate of change of length with respect to mass?
**d** What is the length for a mass of 0 g, and what does it represent?

4 The mass of a bucket of water varies with the height of the water in the bucket as follows.

| Water height (cm) | 10 | 15 | 25 | 35 | 50 |
|---|---|---|---|---|---|
| Mass (kg) | 10 | 14 | 22 | 30 | 42 |

**a** Draw a graph to illustrate the information.
**b** Comment on the graph.
**c** What is the rate of change of mass with respect to height?
**d** What does the mass represent when the height is zero?

5 Example 2 An RACQ map shows the distance and time that it should take to drive a car between Bundaberg and Mackay via Gladstone and Rockhampton. The distance is 334 km and the time given is 3 hours 40 minutes. What average driving speed has been assumed?

6 A car travelling at 30 m/s along a straight road increases its velocity to 50 m/s in 5 s. What is the average rate of change of velocity (acceleration)?

7 A bushwalk in a national park is signed as being an easy 700 m walk. The park ranger indicates that it should take about 12 minutes to complete the walk. Find the average speed at which the park ranger has assumed people can walk on this type of path.

8 Example 3 A car travels at 100 km/h for half an hour, at 60 km/h for an hour and then at 80 km/h for the last half hour of a trip. What is the average speed?

9 A loaded truck travels at 20 km/h from an open-cut mine to the rail hopper, but at 80 km/h in the 10 km back to the mine. What was the average speed of the trip?



## Reasoning and communication

**10** The Moon orbits at an average distance of 383 000 km from the centre of the Earth. The Moon takes 27.3 days to complete an orbit. The orbit is in the same direction as the revolution of the Earth around the Sun.

**a** Find how far the Moon travels in 24 hours.

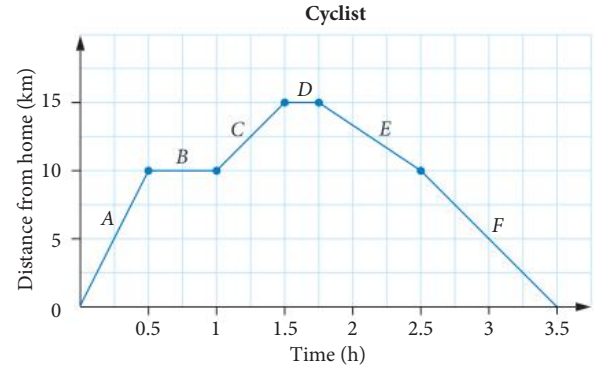
**b** What is the speed of the Moon in its orbit?

**c** To an observer on Earth, why does the Moon appear to take 28 days to orbit?



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**11** The graph shows the distance of a cyclist from her home at various times on a trip to a shop 15 km away and back again.




**a** When was she moving at the greatest speed?

**b** What was her average speed to the shop?

**c** How long did she stop at the shop?

**d** What was her average speed on section *E*?

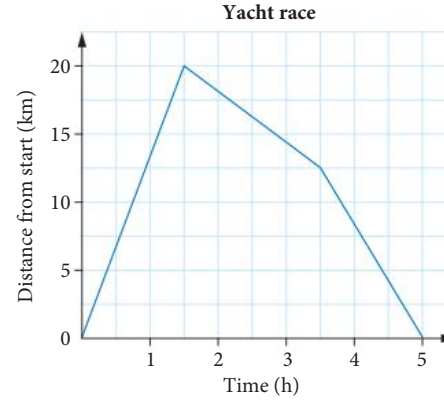
**e** What was her average speed for the whole trip?

**12** A geostationary satellite remains in orbit above the same place on the Earth throughout the day, so it orbits once a day in the same direction that the Earth spins. If the radius of the Earth is 6400 km and the height of such a satellite is 36 000 km, find:

**a** the distance that the satellite travels in a day

**b** the speed of the satellite in kilometres per hour and metres per second.

**13** The graph shows the distance of a yacht from the starting line in a three-legged race.




**a** What is the fastest speed shown on the graph?

**b** On which leg would the spinnaker most likely have been used (the fastest leg, with the wind behind)?

**c** Which leg is the beat to windward (the slowest leg, with the most tacking needed)?

**d** What is the average speed of the yacht shown on the graph?

**14** An aeroplane travels from Sydney to Brisbane (about 600 km) in $1\frac{1}{2}$ hours. It takes 1 hour to get from the Sydney central business district (CBD) to the airport at Mascot (about 20 km) and about 15 minutes to get from Brisbane airport to the Brisbane CBD (about 15 km). Check-in at the airport takes about 40 minutes, and it takes about 20 minutes to disembark and collect luggage.

**a** What is the average speed of the plane?

**b** What is the average speed for a trip between the city centres?

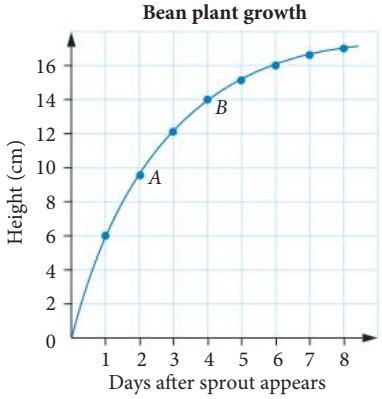
**c** If the plane can be refuelled in the time taken for check-in and disembarkation, how long would the plane take for a return trip (from takeoff to landing back at Mascot)?

**d** What would the average speed and velocity be for such a return trip?

**15** The graph shows the average height of bean plants measured as they grow. Use the graph to calculate the average rate of growth for:

**a** the first 2 days **b** the next 2 days **c** days 5 and 6.

State also how the rate of growth is changing.



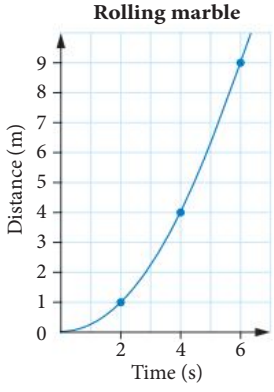

**16** An air-conditioning system in a car has a small leak. The temperature in the car, with the air-conditioning on full, gradually increases as shown below. During this time the external temperature remains at about 28°C.

| Time | 11:00 a.m. | 11:20 a.m. | 11:40 a.m. | 12:00 | 12:20 p.m. | 12:40 p.m. | 1:00 p.m. | 1:20 p.m. |
|---|---|---|---|---|---|---|---|---|
| Temp. (°C) | 18.0 | 19.8 | 21.3 | 22.5 | 23.5 | 24.3 | 25.0 | 25.5 |

**a** Draw a graph.
**b** Comment on the rate of change of the temperature in the car.
**c** What is the average rate of change of temperature during this time?
**d** Will the air-conditioning be any use at 3 p.m.?
**e** Will it be any use the next day?

**17** The distance a marble rolls down a long plank at an angle of 5° is shown in the graph. Find the average speed from:

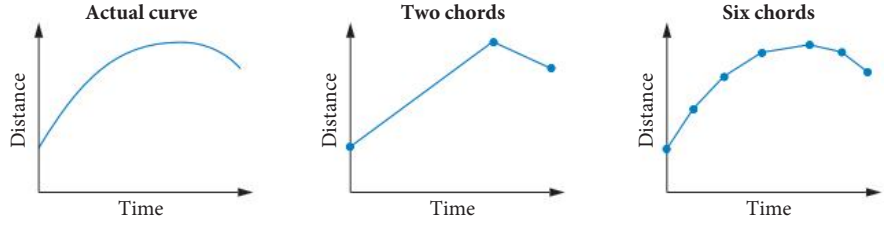
**a** 0 s to 2 s
**b** 2 s to 4 s
**c** 4 s to 6 s
**d** 0 s to 6 s




**18** Prove that the average speed for a return trip is given by the formula $v = \frac{2v_1v_2}{v_1 + v_2}$, where $v_1$ and $v_2$ are the speeds there and back.

# 7.02 INSTANTANEOUS RATES OF CHANGE

The speedometer ('speedo') on a car gives us an immediate indication of the speed *at an instant*, without having to worry about averages. When we use a number of **chords** or **secants** to work out average speeds, we replace the curve of the distance–time graph by a series of straight lines. This is shown below.




**IMPORTANT**

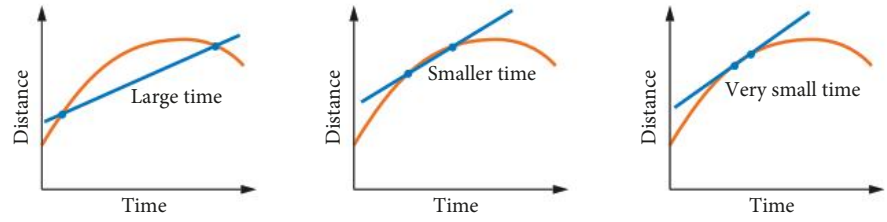
**Chords, secants and tangents**

A **secant** is a straight line that passes through two points on a curve.

A **chord** is the segment of a straight line cut off by a curve.

A **tangent** is a straight line that touches a curve.

As more chords are used, the change in time for each chord gets smaller, and the shape of the graph made by the chords gets closer to the shape of the actual graph. If the change in time for each chord is reduced to an instant, we get exactly the same shape as the actual graph. The average speeds also become **instantaneous** speeds. We can do this for a single chord, but it is easier to see what happens if we use a secant instead. This is shown in the following.




The secant cuts the curve in two places, forming the ends of a chord. As we reduce the change in time, the secant becomes more like a tangent, which touches the curve at just one point. The slope of a secant (or chord) gives the average speed, so the gradient of a tangent gives the instantaneous speed. This applies to any rate of change.

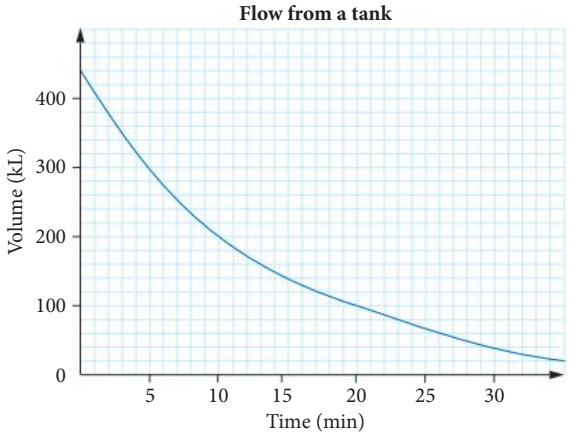
**IMPORTANT**

The **instantaneous rate of change** of a variable at a point is the gradient of the tangent to the graph at that point.


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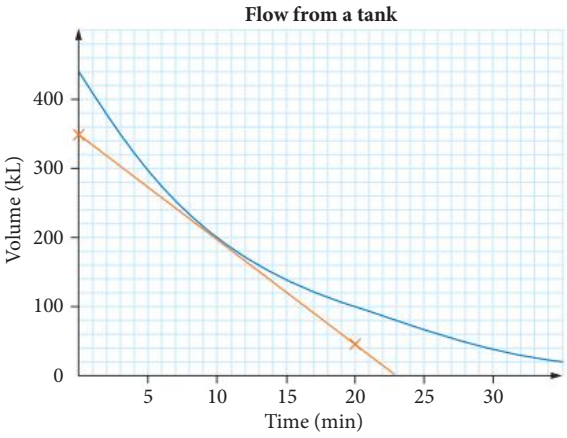
## Example 4

A valve on a tank of water is opened to drain its contents. Use a tangent to the graph shown here to find the flow rate 10 minutes after the valve is opened.




### Solution

Use a plastic ruler and pencil to draw a tangent at 10 minutes.




Find two easy points on the tangent.

(0, 350) and (20, 45) are on the tangent line.

Write the formula for the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute in the values and find the answer.

$$\frac{45 - 350}{20 - 0} \approx -15 \text{ kL/min}$$

Write the answer.

The water is flowing out at about 15 kL/min.





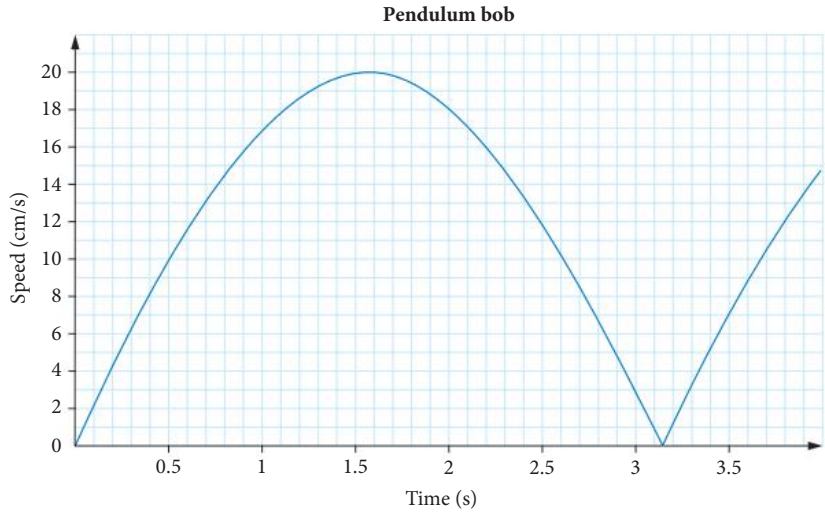

Rates of change — Gradients of secants

# EXERCISE 7.02 Instantaneous rates of change

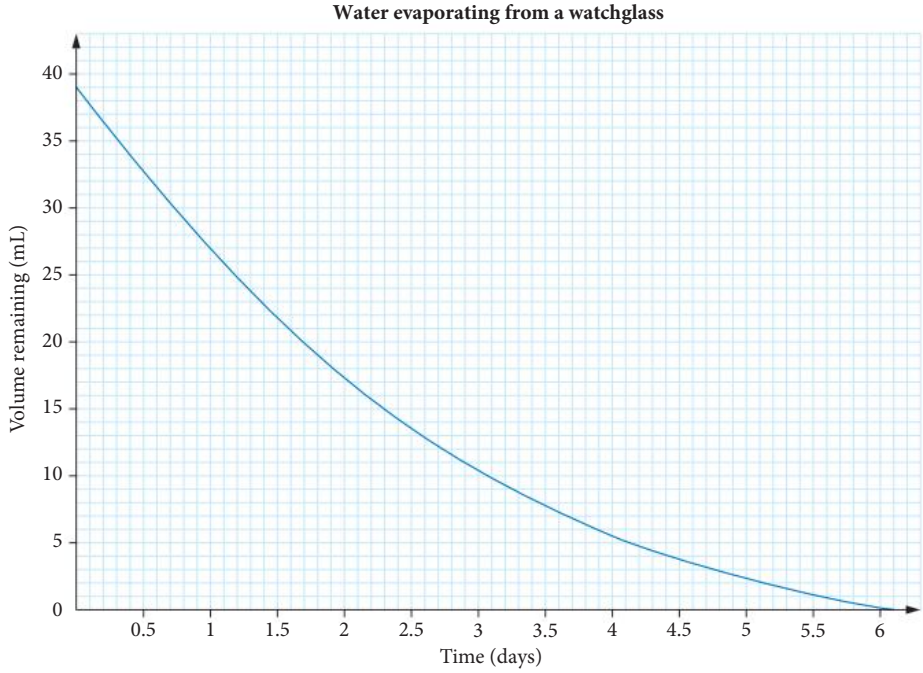
WS
Rates of change

## Concepts and techniques

1 Example 4 The following graph shows the speed of the bob of a long pendulum. Use tangents to find the acceleration of the bob at 1 s, 2 s and 3 s.



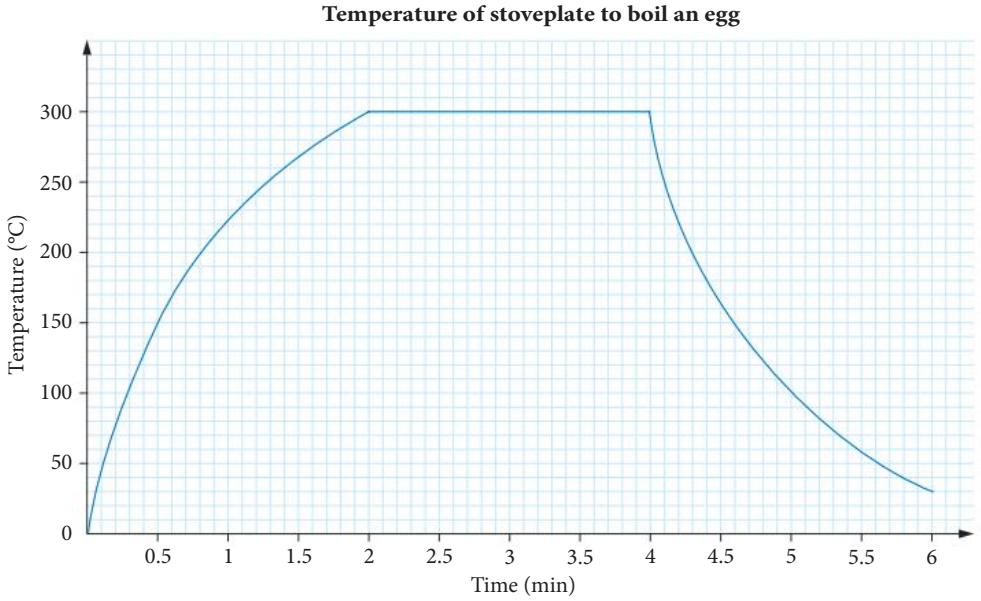

2 The amount of water left in a watchglass is shown below. Use tangents to find the rate of evaporation after 1 day and 4 days.



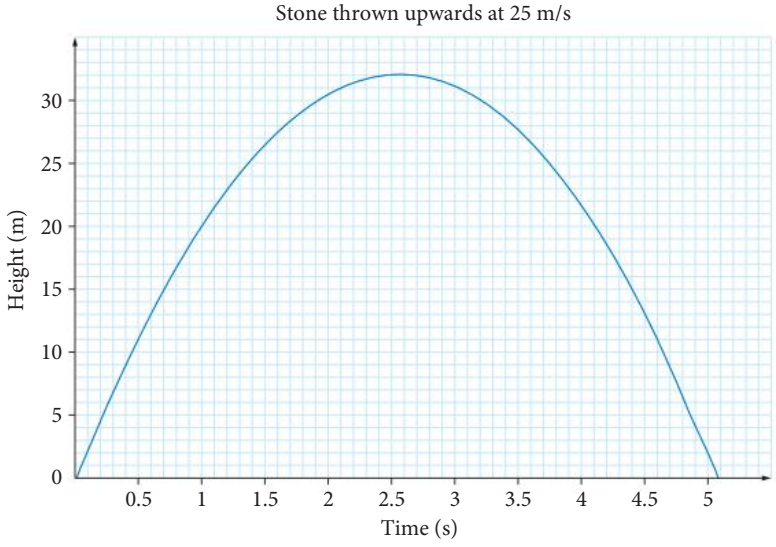


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3 The temperature of a stove plate used to boil an egg is shown below. Use tangents to find the rate of change of temperature after 1 minute, 3 minutes and 5 minutes.

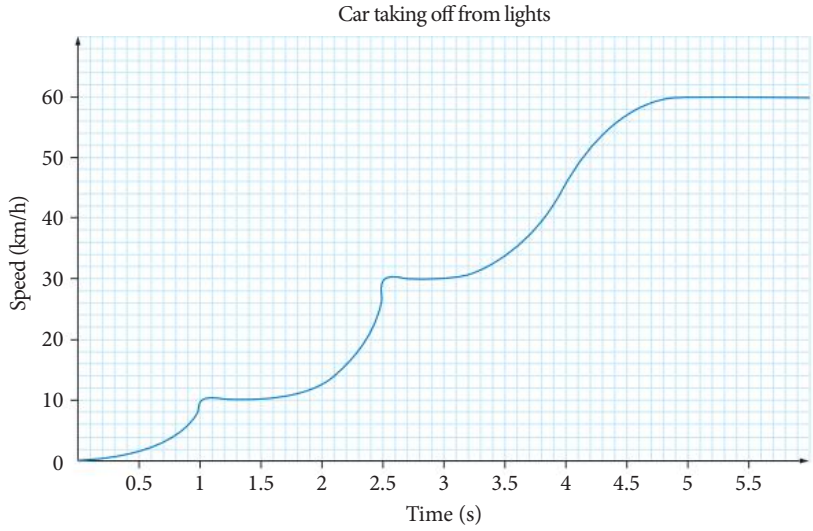



4 The following graph shows the height of a stone thrown upwards at 25 m/s. Use tangents to find the speed of the stone at 1 s, 2 s, 3 s and 4 s after it has been thrown.






5 Use tangents to find the acceleration of a car taking off from the lights, as shown in the graph below, at 2 s and 4 s.




# 7.03 THE AVERAGE RATE OF CHANGE OF A FUNCTION

You can find the average rates of change of mathematical functions in the same way as you have for variables like distance and time.

**IMPORTANT**

### Average rates of change of functions

For a function $y = f(x)$ the **average rate of change** from $x_1$ to $x_2$ is given by:

$$\text{Average rate of change (of } y \text{ with respect to } x) = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

WS Instantaneous rates of change

## Example 5

Find the average rate of change of $f(x) = 3x^2 + x - 8$ from $x = 3$ to $x = 7$.

### Solution

Find the value of the function at $x = 3$.

$f(3) = 3 \times 3^2 + 3 - 8 = 22$

Find the value of the function at $x = 7$.

$f(7) = 3 \times 7^2 + 7 - 8 = 146$

Write the formula for the average rate of change.

$$\text{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute values.

$$= \frac{146 - 22}{7 - 3}$$


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| | |
|---|---|
| Calculate the answer. | $= 31$ |
| Write the answer. | The average rate of change of $f(x)$ from $x = 3$ to $x = 7$ is 31. |

You can use your CAS calculator to draw secants to curves and calculate their gradients to get more accurate results.

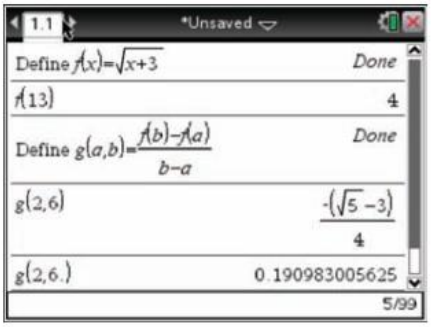
## Example 6

CAS Calculate the average rate of change of $y = \sqrt{x+3}$ from $x = 2$ to $x = 6$.

### Solution

#### TI-Nspire CAS

Use a calculator page .
Press menu, 1: Actions and 1: Define, type Define or use the catalogue to obtain Define.
Type in $f(x) = \sqrt{x+3}$, and press enter.
Try typing $f(13)$ enter.
Now define $g(a,b)=(f(b)-f(a))\div(b-a)$ in the same way as for $f(x)$.
Try $g(2,6)$.
Now type $g(2,6.)$ to force a decimal answer.




Write the answer.

The average rate of change of $y = \sqrt{x+3}$ from $x = 2$ to $x = 6$ is $\frac{3-\sqrt{5}}{4} \approx 0.191$.

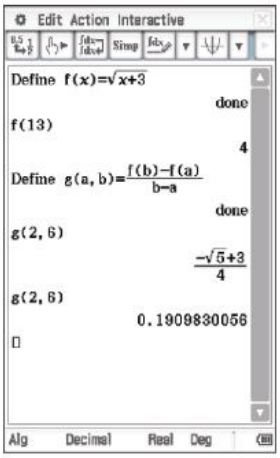
#### ClassPad

Tap Menu choose √α and press Keyboard.
Ensure that the calculator is set to **Standard**.
Tap **Interactive** at the top of the screen and tap **Define** at the bottom of the pop-up menu.
Alternatively, type Define or tap Math3 then Define, or find it under the Catalog tab.

Type in f(x)= $\sqrt{x+3}$, using Math1 to obtain √■ and press EXE.
Try putting in f(13).

Now define g(a,b)=$\frac{f(b)-f(a)}{b-a}$.

Press Math1 and choose ▣ to get the fraction.
Tap abc to get back to the alphabet.
Try g(2, 6).
Now change **Standard** to **Decimal** at the bottom of the screen and calculate g(2, 6) again.
Pressing Keyboard will hide the keyboard and show the full display.



Write the answer.

The average rate of change of $y = \sqrt{x+3}$ from $x = 2$ to $x = 6$ is $\frac{3-\sqrt{5}}{4} \approx 0.191$.

Once you have set up the gradient function $g(a, b)$ on your CAS calculator, you can use it with other functions by redefining the function $f(x)$.

# EXERCISE 7.03 The average rate of change of a function

## Concepts and techniques

**1** Example 5 Find the average rates of change of $f(x) = x^2 + x - 8$ from:

**a** $x = 2$ to 3 **b** $x = 1$ to 5 **c** $x = 4$ to 6

**d** $x = 6$ to 8 **e** $x = 4$ to 8

**2** Find the average rates of change of $f(x) = x^3 - 2x^2 + 5x - 4$ from:

**a** $x = -2$ to 2 **b** $x = -1$ to 1 **c** $x = 0$ to 3

**d** $x = 3$ to 6 **e** $x = 0$ to 6

**3** Find the average rates of change of $f(x) = (m + 5)(m - 4)$ from:

**a** $m = 5$ to 6 **b** $m = 6$ to 8 **c** $m = 5$ to 8

**d** $m = -4$ to $-2$ **e** $m = -2$ to 4

**4** Find the average rates of change of $g(x) = 3x^2 + 5x - 10$ from:

**a** $x = 3$ to 4 **b** $x = 5$ to 9 **c** $x = 3$ to 9

**d** $x = -5$ to 2 **e** $x = -7$ to $-3$

**5** Find the average rates of change of $h(z) = z^3 - z^2 + z - 1$ from:

**a** $z = 1$ to 3 **b** $z = 3$ to 5 **c** $z = 1$ to 5

**d** $z = 6$ to 8 **e** $z = 4$ to 10

**6** CAS Example 6 For $f(x) = 2x^2 - 5x + 3$, work out the following.

**a** The average rate of change from $x = 0$ to $x = 4$.

**b** The average rate of change from $x = -2$ to $x = 1$.

**7** CAS Draw $f(x) = x^3 - x^2 + 3x + 1$ and calculate:

**a** The average rate of change from $x = -2$ to $x = 2$.

**b** The average rate of change from $x = 2$ to $x = 4$.

**8** CAS For $f(x) = \frac{1}{x+2}$, work out the following.

**a** The average rate of change from $x = -1$ to $x = 1$.

**b** The average rate of change from $x = 2$ to $x = 5$.

**9** CAS For $f(x) = \sqrt{x-1}$, work out the following.

**a** The average rate of change from $x = 2$ to $x = 3$.

**b** The average rate of change from $x = 3$ to $x = 5$.


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## Reasoning and communication

10 Find the average rates of change of $f(x) = x^2 - 3x - 3$ from $x_1 = 0$ to $x_2 = 1$, $x_2 = 2$, $x_2 = 3$, $x_2 = 4$ and $x_2 = 5$ and plot a graph of the average rates of change against $x_2$. Comment on the graph you obtain.

11 Find the average rates of change of $f(x) = x^3$ from $x_1 = 0$ to $x_2 = 1$, $x_2 = 2$, $x_2 = 3$, $x_2 = 4$ and $x_2 = 5$ and plot a graph of the average rates of change against $x_2$. Comment on the graph.

12 Find the average rates of change of $f(x) = \sqrt{x}$ from $x_1 = 0$ to $x_2 = 1$, $x_2 = 2$, $x_2 = 3$, $x_2 = 4$ and $x_2 = 5$ and plot a graph of the average rates of change against $x_2$. Comment on the graph.

# 7.04 THE INSTANTANEOUS RATE OF CHANGE OF A FUNCTION

You have to use the tangent to find the instantaneous rates of change for mathematical functions. For mathematical functions, we simply call this the rate of change.

**IMPORTANT**

**Rates of change**

The **rate of change** of the function at any point is the gradient of the tangent at that point. For a function $y = f(x)$ the gradient of the tangent on the graph is called the **gradient of the graph**.

### Example 7

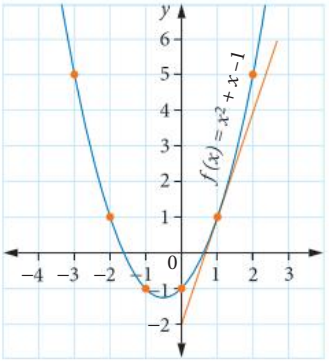
Plot the graph of $f(x) = x^2 + x - 1$ from $-3$ to $2$ and find the rate of change at $x = 1$ by drawing a tangent.

#### Solution

Draw a table of values.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|
| $y$ | $5$ | $1$ | $-1$ | $-1$ | $1$ | $5$ |

Plot the graph with a smooth line and draw the tangent at $x = 1$.



| | |
|---|---|
| Choose two points on the tangent. | (0, –2) and (2, 4) are on the tangent. |
| Find the gradient. | $m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-(-2)}{2-0}=3$ |
| Write the answer. | The rate of change at $x = 1$ is 3. |

You can use your CAS calculator to draw tangents to curves to find rates of change.

## Example 8

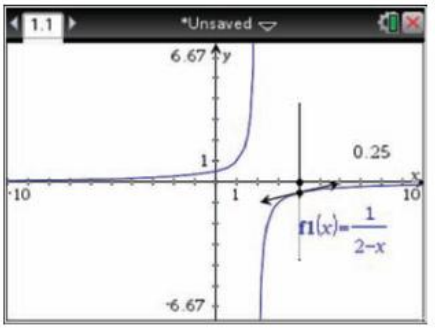
CAS Draw $f(x)=\frac{1}{2-x}$ and calculate the rate of change at $x = 4$ using the tangent.

### Solution

#### TI-Nspire CAS

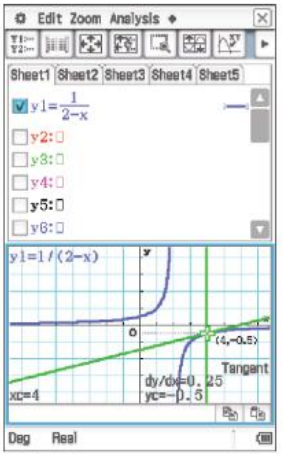
Use a graph page to draw $f(x)=\frac{1}{2-x}$.

Press menu, 8: Geometry, 1: Points & lines and 7: Tangent to draw a tangent at $x = 4$.
You might find it easier to construct a line parallel to the $y$-axis using menu, 8: Geometry, 4: Construction and 2: Parallel to obtain the precise point first.
Then press menu, 8: Geometry, 3: Measurement and 7: Slope. Use the mouse-pad to click on the tangent to get the slope.




#### ClassPad

Use **Graph&Table** to draw $f(x)=\frac{1}{2-x}$. Note that you may need to change the values on the **View Window** by tapping to ensure that you are looking at a sufficiently large area of the graph. At the top of the screen tap **Analysis**, **Sketch** and then **Tangent**. Use the arrow keys to move the crosshair to x = 4 to show the tangent to the graph at this point. You might find it easier to tap **Sketch** then **Vertical** to put a vertical line through x = 4 first to obtain the precise point.



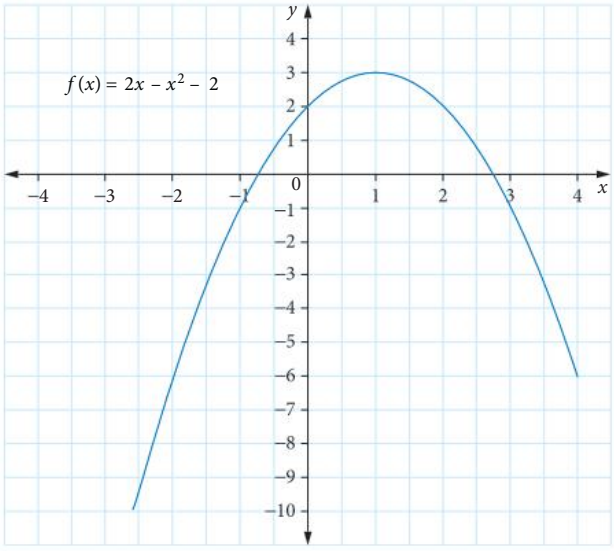

| | |
|---|---|
| Write the answer. | The rate of change of $f(x)=\frac{1}{2-x}$ at $x = 4$ is about 0.25. |


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# EXERCISE 7.04 The instantaneous rate of change of a function

## Concepts and techniques

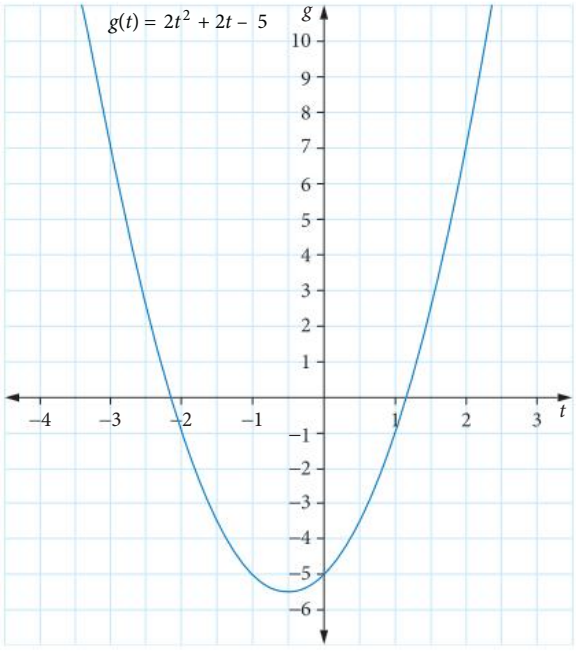
1 Example 7 Use tangents on the graph of $f(x) = 2x - x^2 + 2$ shown here to find the rates of change of $f(x)$ at:




**a** $x = -2$ **b** $x = 0$ **c** $x = 1$ **d** $x = 2$ **e** $x = 3$

2 Use tangents on the graph of $g(t) = 2t^2 + 2t - 5$ shown here to find the rates of change of $g(t)$ at:

**a** $t = -3$ **b** $t = -1$ **c** $t = 0$ **d** $t = 0.5$ **e** $t = 1$

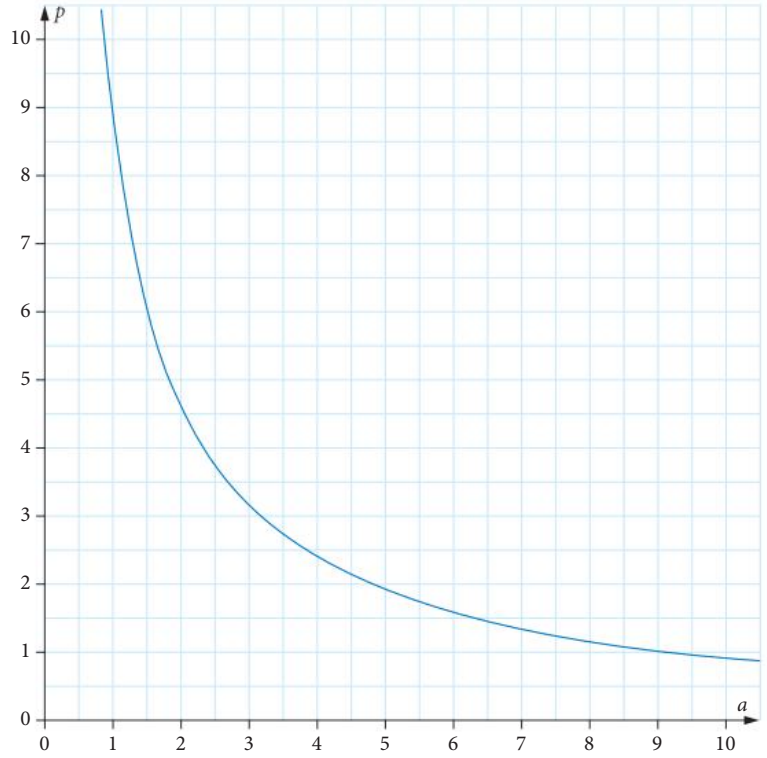





WS
Instantaneous rates of change

3 Use tangents on the graph of $p(a) = \frac{9}{a}$ shown here to find the rates of change of $p(a)$ at:

**a** $p = 1\frac{1}{2}$ **b** $p = 2$ **c** $p = 3$ **d** $p = 6$ **e** $p = \infty$




4 Plot the graph of $f(x) = x^2 - 4$ from $x = -3$ to $x = 3$ and use tangents to find the rates of change of $f(x)$ at:

**a** $x = -2$ **b** $x = -0.5$ **c** $x = 1$ **d** $x = 1.5$ **e** $x = 2.5$

5 Plot the graph of $g(n) = (n - 2)(n + 2)(n - 4)$ from $n = -2.5$ to $n = 4.5$ and use tangents to find the rates of change of $g(n)$ at:

**a** $n = -1$ **b** $n = 0$ **c** $n = 1$ **d** $n = 2$ **e** $n = 3$

6 Example 8 CAS Draw $f(x) = 2x^2 - 5x + 3$ and calculate:

**a** the rate of change at $x = 2$ using the tangent

**b** the rate of change at $x = 0$ using the tangent.

7 CAS Use your calculator to draw $f(x) = x^3 - x^2 + 3x + 1$ and calculate:

**a** the rate of change at $x = -1$ using the tangent

**b** the rate of change at $x = 3$ using the tangent.

8 CAS Use your calculator to draw $f(x) = \frac{1}{x+2}$ and calculate:

**a** the rate of change at $x = -3$ using the tangent

**b** the rate of change at $x = 3$ using the tangent.

9 CAS Use your calculator to draw $f(x) = \sqrt{x-1}$ and calculate:

**a** the rate of change at $x = 2$ using the tangent

**b** the rate of change at $x = 5$ using the tangent.


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## Reasoning and communication

10 Find the rates of change of $f(x) = x^2 - 3x - 3$ at $x = -1, 0, 1, 2$ and 3 and plot a graph of the rates of change. Comment on the graph you obtain.

# 7.05 APPROXIMATION OF RATES OF CHANGE

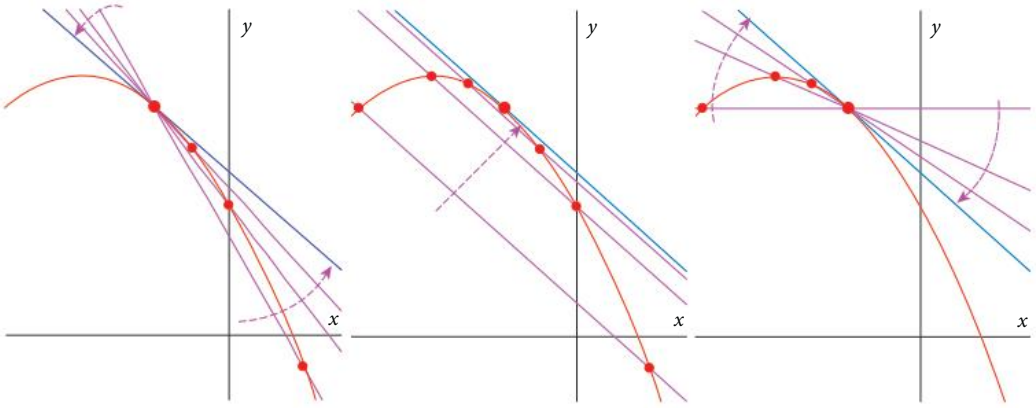
Drawing graphs and using the tangents to find approximate rates of change of functions is very tedious, even when using a CAS calculator. In this section you will learn how to find approximate rates of change using small intervals. Remember that a change in a quantity is shown by the Greek capital delta, Δ.

**IMPORTANT**

A **small change** in a quantity $x$ is shown by putting the lower case Greek letter δ in front of the variable, so $\delta x$ means a small change in the value of $x$.

You have already seen that as smaller and smaller chords are used, the secant through a chord gets closer and closer to the tangent of a curve. It doesn't matter whether the chord is taken from above, below or around the point of interest. The diagrams below show the progressions, with the tangent shown in blue and the secants in pink.

Secants from above point | Secants around point | Secants from below point

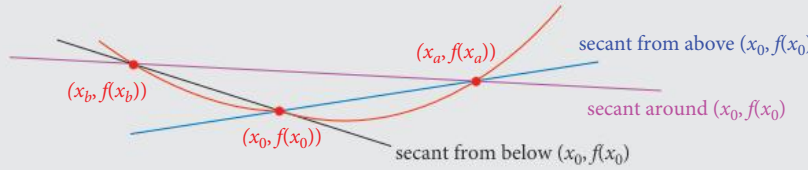



The average rate of change of the function for each secant is still given by $\frac{\Delta y}{\Delta x}$, but for small changes, we write this as $\frac{\delta y}{\delta x}$ and say that it approximates the rate of change.

**IMPORTANT**

Given a point $(x_0, f(x_0))$ and values $x_b$ and $x_a$ such that $x_b < x_0 < x_a$:

- a line through $(x_0, f(x_0))$ and $(x_a, f(x_a))$ is called a **secant from above** the point
- a line through $(x_b, f(x_b))$ and $(x_0, f(x_0))$ is called a **secant from below** the point
- a line through $(x_b, f(x_b))$ and $(x_a, f(x_a))$ is called a **secant around the point.**




## Example 9

Use $\delta x = 0.1$ to find the approximate rate of change of $f(x) = x^3 + 2x + 3$ at $x = 2$, with the secant above the point.

### Solution

| | |
|---|---|
| Find the value at the point. | $f(2) = 2^3 + 2 \times 2 + 3 = 15$ |
| Find the value above the point, at $x + \delta x$. | $f(2.1) = 2.1^3 + 2 \times 2.1 + 3 = 16.461$ |
| Find the value of $\delta y$. | $\delta y = 16.461 - 15 = 1.461$ |
| Find the approximate rate of change. | Rate of change $\approx \dfrac{\delta y}{\delta x} = \dfrac{1.461}{0.1} = 14.61$ |

You need to use $x_a = x_0 - \dfrac{\delta x}{2}$ and $x_b = x_0 + \dfrac{\delta x}{2}$ for secants around a point.

## Example 10

Use $\delta x = 0.1$ to find the approximate rate of change of $f(x) = \dfrac{1}{x+1}$ at $x = 2$, with the secant around the point.

### Solution

| | |
|---|---|
| Find the value before the point, at $x - \frac{1}{2}\delta x$. | $f(1.95) = \dfrac{1}{1.95+1} = \dfrac{20}{59}$ |
| Find the value after the point, at $x + \frac{1}{2}\delta x$. | $f(2.05) = \dfrac{1}{2.05+1} = \dfrac{20}{61}$ |
| Find the value of $\delta y$. | $\delta y = \dfrac{20}{61} - \dfrac{20}{59} = -0.011\,114\,198\ldots$ |


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Find the approximate rate of change.

$$\text{Rate of change} \approx \frac{\delta y}{\delta x}$$
$$= \frac{-0.011\,114\,198...}{0.1}$$
$$= -0.111\,141\,98...$$
$$\approx -0.111$$

You can use your CAS calculator to find approximate rates of change.

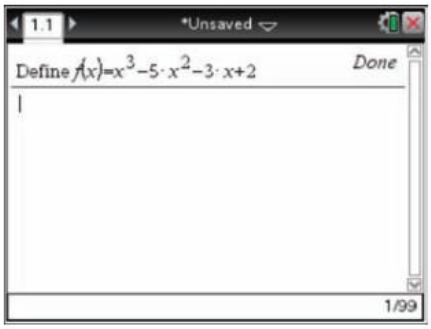
## Example 11

CAS Use $\delta x = 0.05$ to find the approximate rate of change of $f(x) = x^3 - 5x^2 - 3x + 2$ at $x = 3$, with the secant around the point.

### Solution

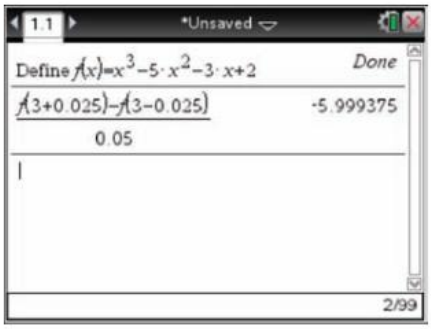
**TI-Nspire CAS**

Use a calculator page .
Press menu, 1: Actions and 1: Define, type Define or use the catalogue to obtain Define.
Then type $f(x) = x^3 - 5x^2 - 3x + 2$ and enter.




Work out $\dfrac{f(3+0.025)-f(3-0.025)}{0.05}$.

Type as $(f(3+0.025)-f(3-0.025)) \div 0.05$ enter or use the fraction template from .

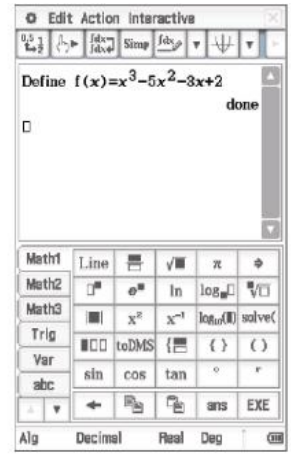



Write the answer.

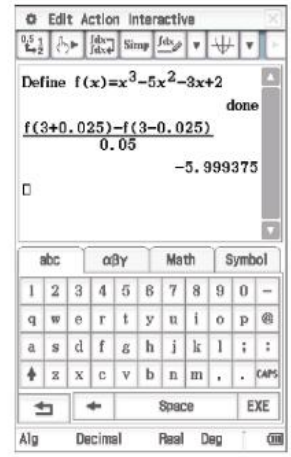
At $x = 3$, $f(x) = x^3 - 5x^2 - 3x + 2$ has a rate of change of about $-6$.

### ClassPad

Tap Main $\sqrt{\alpha}$ and press Keyboard.
Type Define or use the Math3 tab and tap Define.
(Alternatively, tapping **Interactive** at the top of the screen and **Define** will display a pop-up box for you to enter the function.)
Enter $f(x) = x^3 - 5x^2 - 3x + 2$, using abc to get letters and tap Math1 and $\square^{\square}$ for the powers. You will also need to use the arrows.




Work out $\dfrac{f(3+0.025)-f(3-0.025)}{0.05}$, using Math1 $\square$ for the fraction and abc to get f




Write the answer.

At $x = 3$, $f(x) = x^3 - 5x^2 - 3x + 2$ has a rate of change of about $-6$.

You could define a function $g(x, h) = \dfrac{f(x+h)-f(x)}{h}$ to work out the rates of change from above using your CAS calculator for different values of $x$ and $h = \delta x$. Choosing a negative value of $h$ would work out the calculation from below. If you wanted to, you could even define a function $a(x, h) = \dfrac{f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)}{h}$ to calculate the rate of change around the point. Remember from Example **6** that redefining $f$ would automatically change the calculation of $g$ or $a$.

## INVESTIGATION Variation of calculation of rates of change

In this investigation you will examine how the approximate rate of change of a function is affected when using secants from above, around and below the function. Your teacher will probably want you to use your CAS calculator.

- Use $\delta x = 0.2$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from above.
- Use $\delta x = 0.2$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant around the point.


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- Use $\delta x = 0.2$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from below.
- Use $\delta x = 0.02$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from above.
- Use $\delta x = 0.02$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant around the point.
- Use $\delta x = 0.02$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from below.
- Use $\delta x = 0.002$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from above.
- Use $\delta x = 0.002$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant around the point
- Use $\delta x = 0.002$ to find the rate of change of $f(x) = 2x^2 - x^3 + 3x - 2$ at $x = 4$ using a secant from below.

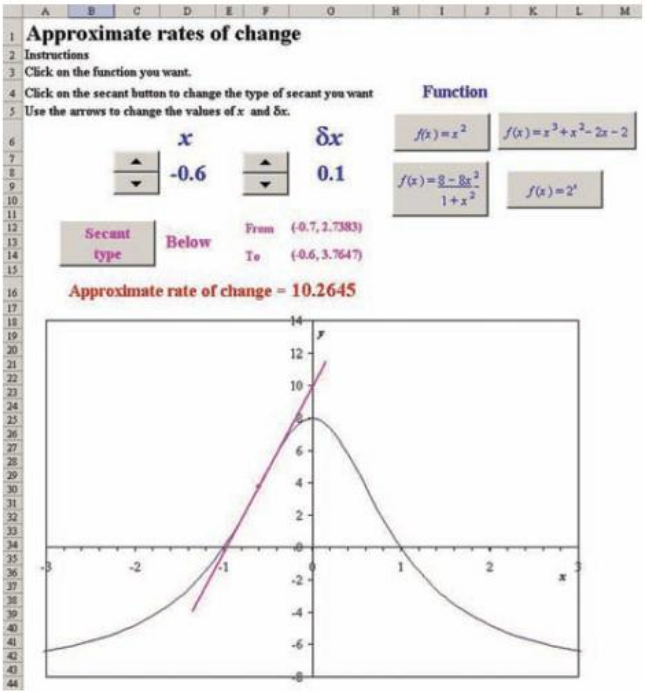
Do the values you found vary according to the placement of the secant?

What happens as you decrease the value of $\delta x$?

Check your conclusions by trying some other functions

## Technology

You can use a spreadsheet to calculate approximate rates of change. Load the spreadsheet 'Approximate rates of change' from the NelsonNet website and use it to investigate what happens to the calculation of the rate of change of a function as you change the value of $\delta x$ for different values of $x$.




Change the function and repeat your investigation.

# EXERCISE 7.05 Approximation of rates of change

## Concepts and techniques

1 Example 9 Use $\delta x = 0.1$ to find the approximate rates of change of $f(x) = x^2 + 2x - 3$ at each of the following values using secants above the points.

a $x = 2$ b $x = -3$ c $x = 0.5$ d $x = -0.1$ e $x = 6$

2 Use $\delta x = 0.2$ to find the approximate rates of change of $f(x) = 5^{x+1}$ at each of the following values using secants above the points.

a $x = 5$ b $x = -1$ c $x = 0.1$ d $x = 4$ e $x = -3$

3 Use each of the following values of $\delta x$ to find the approximate rates of change of $f(x) = \sqrt[3]{2x+5}$ at $x = 1$ using secants above the point.

a $\delta x = 1$ b $\delta x = 0.5$ c $\delta x = 0.1$ d $\delta x = 0.05$ e $\delta x = 0.01$

4 Use $\delta x = 0.1$ to find the approximate rates of change of $f(x) = (x - 3)^4$ at each of the following values using secants below the points.

a $x = -1$ b $x = 0.5$ c $x = 0$ d $x = 1.5$ e $x = 3$

5 Use $\delta x = 0.02$ to find the approximate rates of change of $f(x) = \frac{1}{x+4}$ at each of the following values using secants below the points.

a $x = 0$ b $x = 1$ c $x = 4$ d $x = -5$ e $x = -3$

6 Use each of the following values of $\delta x$ to find the approximate rates of change of $f(x) = \frac{1}{\sqrt{x}}$ at $x = 4$ using secants below the point.

a $\delta x = 1$ b $\delta x = 0.5$ c $\delta x = 0.1$ d $\delta x = 0.05$ e $\delta x = 0.01$

7 Example 10 Use $\delta x = 0.1$ to find the approximate rates of change of $f(x) = x + x^{-1}$ at each of the following values using secants around the points.

a $x = 0.5$ b $x = 1$ c $x = -2$ d $x = -0.3$ e $x = 4$

8 Use $\delta x = 0.02$ to find the approximate rates of change of $f(x) = \frac{1}{(x-3)^2}$ at each of the following values using secants around the points.

a $x = -1$ b $x = 1$ c $x = 2$ d $x = 2.2$ e $x = 2.9$

9 Use each of the following values of $\delta x$ to find the approximate rates of change of $f(x) = x^{1.5} + 2x^{0.5}$ at $x = 2$ using secants around the point.

a $\delta x = 1$ b $\delta x = 0.5$ c $\delta x = 0.1$ d $\delta x = 0.05$ e $\delta x = 0.01$

10 CAS Example 11 Use $\delta x = 0.1$ to find the approximate rates of change of $f(x) = x^{2.5}$ at each of the following values using secants around the points.

a $x = 1$ b $x = 3$ c $x = 4$ d $x = 0.4$ e $x = 0.2$

11 CAS Use $\delta x = 0.02$ to find the approximate rates of change of $f(x) = 2^{x^2-1}$ at each of the following values using secants around the points.

a $x = -2$ b $x = 1$ c $x = 0$ d $x = 4$ e $x = -4$


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**12** CAS Use each of the following values of $\delta x$ to find the approximate rates of change of $f(x) = \log(x)$ at $x = 2.5$ using secants around the point.

**a** $\delta x = 1$ **b** $\delta x = 0.5$ **c** $\delta x = 0.1$ **d** $\delta x = 0.05$ **e** $\delta x = 0.01$

## Reasoning and communication

**13** CAS Find the approximate rates of change of $f(x) = 4^{\sqrt{x}}$ at $x = 5$ using $\delta x = 0.5$ for secants:

**a** above the point **b** below the point **c** around the point.

**d** Comment on the values found.

**14** CAS Find the approximate rates of change of $f(x) = \sqrt{x^3 + 1}$ at $x = 4$ using $\delta x = 0.1$ for secants:

**a** above the point **b** below the point **c** around the point.

**d** Comment on the values found.

**15** CAS Find the approximate rates of change of $f(x) = \dfrac{1}{x+2}$ at $x = 0$ using $\delta x = 0.04$ for secants:

**a** above the point **b** below the point **c** around the point.

**d** Comment on the values found.

**16** CAS Find the approximate rates of change of $f(x) = \dfrac{1}{x-4}$ at $x = 4$ using $\delta x = 0.2$ for secants:

**a** above the point **b** below the point **c** around the point

**d** around the point with $\delta x = 0.01$.

**e** Comment on the values found. Can you explain your findings?

**17** CAS Find the approximate rates of change of $f(x) = \dfrac{1}{x^2 - 4}$ at $x = 2$ using the following values of $\delta x$ around the point.

**a** 0.5 **b** 0.1 **c** 0.01

**d** Comment on the values found. Can you explain your findings?

**18** CAS Suggest values of $x$ for which the approximate rates of change found using secants would make no sense for the function $f(x) = \dfrac{1}{x^2 - 9}$.



# 7.06 RATE OF CHANGE FUNCTIONS

The rate of change of a function is usually another function. For example, the rate of change of the distance of an object from its original position is the speed of the object. Both its distance and its speed may be considered as functions. It is easiest to use your CAS calculator for all the work in this section.

## Example 12

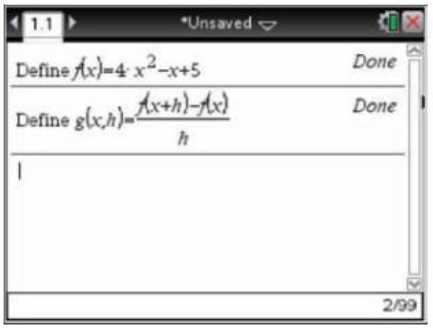
CAS Find approximate rates of change for the function $f(x) = 4x^2 - x + 5$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

### Solution

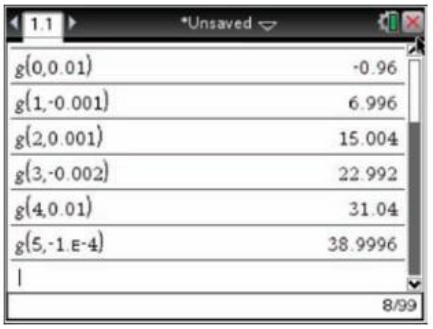
#### TI-Nspire CAS

Use a calculator page .

Define the functions $f(x) = 4x^2 - x + 5$ and $g(x, h) = \dfrac{f(x+h) - f(x)}{h}$.



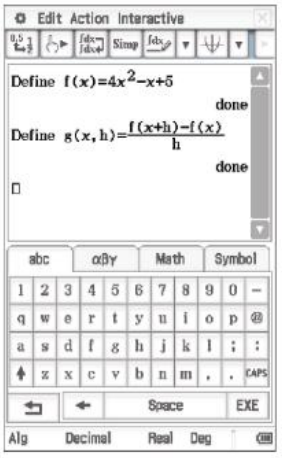

Find values of $g(x, h)$ for $x$ from 0 to 5, using $h = 0.01$ or $h = -0.01$ or less.




#### ClassPad

Tap Main $\sqrt{\alpha}$ and press Keyboard.

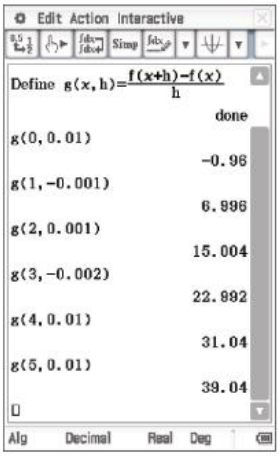
Define the functions $f(x) = 4x^2 - x + 5$ and $g(x, h) = \dfrac{f(x+h) - f(x)}{h}$.





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Make sure the calculator is set to **Decimal**. Find values of $g(x, h)$ for $x$ from 0 to 5, using $h = 0.01$ or $h = -0.01$ or less. Press Keyboard to see the whole display.




Make a table of values.

Approximate values of $g(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $g(x)$ | −1 | 7 | 15 | 23 | 31 | 39 |

The function goes up in steps of 8.

The function seems like $g(x) = 8x - 1$.

## Example 13

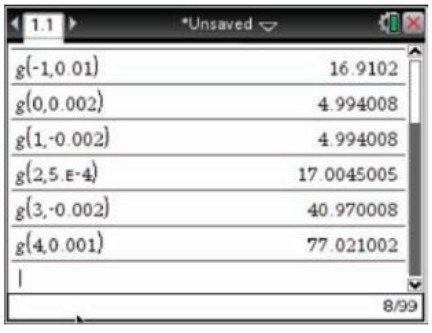
CAS Find approximate rates of change for the function $f(x) = 2x^3 - 3x^2 + 5x + 4$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

### Solution

#### TI-Nspire CAS

Define the functions $f(x) = 2x^3 - 3x^2 + 5x + 4$ and $g(x, h) = \dfrac{f(x+h) - f(x)}{h}$ like Example **12**.

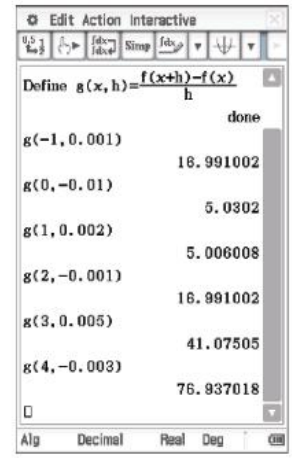
Find values of $g(x, h)$ for $x$ from −1 to 4, using $h = 0.01$ or $h = -0.01$ or less.



### ClassPad

Define the functions $f(x) = 2x^3 - 3x^2 + 5x + 4$ and $g(x, h) = \dfrac{f(x+h) - f(x)}{h}$ in a manner like that of Example **12**.

Find values of $g(x, h)$ for $x$ from $-1$ to 4, using $h = 0.01$ or $h = -0.01$ or less.




Make a table of values.

Approximate values of $g(x)$

| $x$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $g(x)$ | 17 | 5 | 5 | 17 | 41 | 77 |

The table of values suggests a quadratic.

Try $g(x) = ax^2 + bx + c$

Try values for $x = -1$, 0 and 1.

$g(0) = c$, $g(1) = a + b + c$,
$g(-1) = a - b + c$

Substitute in the table values.

$c = 5$, $a + b + c = 5$, $a - b + c = 17$

Solve for $a$, $b$ and $c$.

$a = 6$, $b = -6$ and $c = 5$

Check the values in a table.

$y = 6x^2 - 6x + 5$

| $x$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $y$ | 17 | 5 | 5 | 17 | 41 | 77 |

The values match.

The function seems like $g(x) = 6x^2 - 6x + 5$.

Examples **12** and **13** suggest a function for the rate of change of the original function, but they *do not prove* that the suggested function is the rate of change function because the values match for only a limited range of values. A proof would show that the suggested function worked for all values.


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# EXERCISE 7.06 Rate of change functions

## Reasoning and communication

1 Example 12 CAS Find approximate rates of change for the function $f(x) = 3x^2 - 4x - 7$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

2 CAS Find approximate rates of change for the function $f(x) = x^2 + 6x + 3$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

3 CAS Find approximate rates of change for the function $f(x) = 5x^2 + 7x - 8$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

4 Example 13 CAS Find approximate rates of change for the function $f(x) = 4x^3 + 8x^2 + 9x - 10$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

5 CAS Find approximate rates of change for the function $f(x) = 3x^3 - 5x^2 - 7x + 1$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

6 CAS Find approximate rates of change for the function $f(x) = 7x^4 - x^3 + 3x^2 - 2x - 5$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

7 CAS Find approximate rates of change for the function $f(x) = x^2$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

8 CAS Find approximate rates of change for the function $f(x) = x^3$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

9 CAS Find approximate rates of change for the function $f(x) = x^4$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

10 CAS Find approximate rates of change for the function $f(x) = x^5$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.



# 7 CHAPTER SUMMARY
# RATES OF CHANGE

- The symbol $\Delta$ (Greek capital delta) is used to show the change in a quantity.
- A **rate of change** of a quantity $y$ with respect to $x$ is given by
  $$\frac{\text{change in second quantity}}{\text{change in first quantity}} = \frac{\Delta y}{\Delta x}.$$
  The first variable is taken as **independent** and the second as **dependent**.

  For a straight line, the rate of change is the slope $m$.
- The **average rate of change** of a variable between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by
  $$= \frac{y_2 - y_1}{x_2 - x_1}.$$
- When the independent variable is time: Average rate of change $= \frac{\Delta x}{\Delta t}$, where the change in the variable, $\Delta x$ occurs over the time $\Delta t$.
- **Speed** is the rate of change of the total distance travelled by a moving object.
- **Displacement** is the change in position from the starting point to the end point without regard for the path travelled.
- **Velocity** is the rate of change of the displacement of a moving object.
- **Acceleration** is the rate of change of the velocity of a moving object.
- A **secant** is a straight line that passes through two points on a curve. A **chord** is the segment of a straight line cut off by a curve. A **tangent** is a straight line that touches a curve.
- The **instantaneous rate of change** of a variable is the **gradient of the tangent** to the graph at that point.
- The **rate of change of a function** at any point is the gradient of the tangent at that point. For a function $y = f(x)$ the gradient of the tangent on the graph is called the **gradient of the graph**.
- A **small change** in a quantity $x$ is shown by putting the lower case Greek letter $\delta$ in front of the variable, so $\delta x$ means a small change in the value of $x$.
- The **approximate rate of change of a function** $y = f(x)$ is given by $\frac{\delta y}{\delta x}$.
- A **secant from above** a point on a curve passes through the point and another that has a higher $x$-coordinate.
- A **secant from below** a point on a curve passes through the point and another that has a lower $x$-coordinate.
- A **secant around** a point on a curve passes through points on the curve with $x$-coordinates below and above that of the point.


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# CHAPTER REVIEW
# RATES OF CHANGE
# 7

## Multiple choice

1 Example 2 The speed of a meteor dropped from 4000 m/s to 3100 m/s in 6 seconds as it entered the atmosphere. What was the average rate of change of its speed?

A $54\ \text{m/s}^2$ B $150\ \text{m/s}^2$ C $300\ \text{m/s}^2$ D $600\ \text{m/s}^2$ E $900\ \text{m/ s}^2$

2 Example 5 What is the average rate of change of $m(x) = 4x^3 - 8x^2 + 3x - 4$ from $x = 2$ to $x = 5$?

A 3 B 103 C 309 D $310\frac{1}{3}$ E 311

3 Example 7 The average rate of change of $y$ with respect to $x$ is given by:

A $\frac{\Delta y}{\Delta t}$ B $\frac{y}{x}$ C $\frac{\Delta y}{\Delta x}$ D $\frac{\Delta x}{\Delta t}$ E $\frac{\delta y}{\delta x}$

4 Example 9 The approximate rate of change of $y$ compared to $x$ is given by:

A $\frac{\Delta y}{\Delta t}$ B $\frac{y}{x}$ C $\frac{\Delta y}{\Delta x}$ D $\frac{\Delta x}{\Delta t}$ E $\frac{\delta y}{\delta x}$

5 Example 10 Use $\delta x = 0.02$ to find the approximate rate of change of $f(x) = \frac{1}{2-x}$ at $x = 3$ using a secant around the point.

A 0.9804 B 1.000 025 C 1.0001 D 1.0004 E 1.0204

## Short answer

6 Example 1 David made the measurements below of the length of an elastic band and the mass hung from it.

| Mass (g) | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|
| Length (cm) | 9 | 12 | 14 | 17 | 19 |

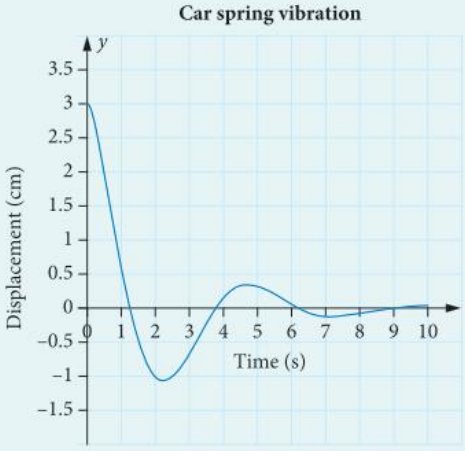
a Draw a graph to determine the average rate of change of length with respect to mass.
b What is the likely length for a mass of 0 g? What does this represent?

7 Example 2 A 60 ML backyard pool is topped up between 9 a.m. and 1 p.m. The volume of water in the pool increases from 36 ML to 42 ML.

a What is the average rate of filling?
b At this rate, at what time will it be full?

8 Example 3 A runner in training does four 500 metre laps of an oval, with lap times of 95 s, 102 s, 105 s and 103 s. Find the average speed for each lap and the average speed for the whole 2 km.

9 Example 4 A car spring has shock absorbers that slow its bouncing. The displacement of the centre of the spring from its normal position after the cars goes over a large pothole is shown in the graph below.




Find the speed of the spring (up or down) at 1, 2 and 3 s.

10 Example 5 Find the average rates of change for $f(x) = x^3 + 2x - 9$ from:

a $-1$ to 1 b 0 to 4

11 Example 6 CAS Find the rate of change for $f(x) = 7 - 2x^2 - 5$ from $x = 2$ to $x = 4$ using the secant.

12 Example 7 Draw the graph of $f(x) = x^3 + 2x - 9$ from $x = 1$ to $x = 5$ and use a tangent to find the rate of change at $x = 4$.

13 Example 8 CAS Draw $f(x) = 2 - 5x - 4x^4$ and find the rate of change at $x = -1$ using the tangent.

14 Example 9 Use $\delta x = 0.1$ to find the approximate rate of change for $f(x) = x^3 + 3x + 2$ at $x = 4.4$ using a secant above the point.

15 Example 10 Use $\delta x = 0.04$ to find the approximate rate of change for $f(x) = \sqrt{x+4}$ at $x = 5$ using a secant around the point.

16 Example 11 CAS Use $\delta x = 0.02$ to find the approximate rate of change for $f(x) = 2^{-x}$ at $x = 1$ using a secant around the point


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## Application

**17** A delivery truck leaves the depot at 9 a.m. and reaches the first address at 9:30, 20 km from the depot. Half an hour later, it travels a further 15 km to the next address, reaching it at 10:18 a.m. It takes 20 minutes to unload the second pack and then returns to the depot back along the same route, arriving at ten past eleven.

**a** Find the average speed for each part of the round trip.

**b** Find the average speed of the truck for the whole trip, excluding stops.

**c** Find the average velocity of the truck for the whole trip.

**18** Find the rates of change for $f(x) = 3x^2 - 2x + 4$ at $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$ and plot a graph of the rates of change. Comment on the graph you obtain.

**19** Use the following values of $\delta x$ to find the approximate rates of change for $f(x) = x^{-0.5x}$ at $x = 3$ using secants below the point.

**a** 0.5 **b** 0.1 **c** 0.01.

**d** Comment on the values found.

**20** CAS Find approximate rates of change for the function $f(x) = 4x^2 - 5x + 8$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.



Practice quiz





8

## TERMINOLOGY

arithmetic progression
arithmetic sequence
common difference
common ratio
general term
geometric progression
geometric sequence
limiting sum
partial sum of a sequence
sequence
series
sum to infinity
term

Prior learning

ARITHMETIC AND GEOMETRIC SEQUENCES AND SERIES

# ARITHMETIC AND GEOMETRIC SEQUENCES


8.01 Sequences
8.02 Arithmetic sequences
8.03 Arithmetic sequence formula
8.04 Sum of an arithmetic sequence
8.05 Geometric sequences
8.06 Geometric sequence formula
8.07 Sum of a geometric sequence
8.08 Infinite geometric series (optional)
Chapter summary
Chapter review

## ARITHMETIC SEQUENCES

- recognise and use the recursive definition of an arithmetic sequence: $t_{n+1} = t_n + d$ (ACMMM068)
- use the formula $t_n = t_1 + (n-1)d$ for the general term of an arithmetic sequence and recognise its linear nature (ACMMM069)
- use arithmetic sequences in contexts involving discrete linear growth or decay, such as simple interest (ACMMM070)
- establish and use the formula for the sum of the first $n$ terms of an arithmetic sequence (ACMMM071).

## GEOMETRIC SEQUENCES

- recognise and use the recursive definition of a geometric sequence: $t_{n+1} = rt_n$ (ACMMM072)
- use the formula $t_n = r^{n-1}t_1$ for the general term of a geometric sequence and recognise its exponential nature (ACMMM073)
- understand the limiting behaviour as $n \to \infty$ of the terms $t_n$ in a geometric sequence and its dependence on the value of the common ratio $r$ (ACMMM074)
- establish and use the formula $S_n = t_1 \frac{r^n - 1}{r - 1}$ for the sum of the first $n$ terms of a geometric sequence (ACMMM075)
- use geometric sequences in contexts involving geometric growth or decay, such as compound interest (ACMMM076). AC

# 8.01 SEQUENCES

You have previously studied number patterns in mathematics. The mathematical name for a number pattern is a **sequence**. A simple sequence of numbers is 5, 10, 15, 20, 25, … The numbers in a sequence are called **terms**. In the sequence above, the first term ($t_1$) is 5, the second term ($t_2$) is 10 and so on. In this case, each term can be found by adding 5 to the previous one. The pattern of the sequence 1, 2, 3, 6, 11, 20, 37, 68, … is a little harder to find. After the first 3 terms, each term is the sum of the previous 3.

### IMPORTANT

A **sequence** is an ordered list. In most cases, you can think of it as a function whose domain is the natural numbers. Each member of a sequence is called a **term**. The first term, second term, and $n$th term are often written as $t_1$, $t_2$ and $t_n$. The **term number** is written as a subscript to the term symbol. Terms are also often written as $a_1$, $a_2$, $a_3$, and so on.

A sequence may be defined by listing the terms in order, by giving the rule as a function, or by a **recursive** definition.

A recursive definition gives each term as a rule involving the previous term (or terms), except for the first term(s), which have to be specified.

The sequence 75, 58, 86, 69, 97, 80, 9, 91, 20, 3, 31, 14, … starts with 75 and uses the recursive rule of swapping the tens and units digits and adding 1.


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## Example 1

Given $a_n = 2a_{n-1} + 1$ and that $a_1 = 6$, write the first five terms of the sequence.

### Solution

| | |
|---|---|
| Write the second term. | $a_2 = 2a_1 + 1$<br>$= 2 \times 6 + 1$<br>$= 13$ |
| Write the third term. | $a_3 = 2 \times 13 + 1$<br>$= 27$ |
| Write the fourth term. | $a_4 = 2 \times 27 + 1$<br>$= 55$ |
| Write the fifth term. | $a_5 = 2 \times 55 + 1$<br>$= 111$ |
| Write the answer. | The first five terms are 6, 13, 27, 55 and 111. |

Remember that a centred dot can be used for a multiplication sign.

## Example 2

Write down the first four terms of sequence whose general term is $t_n = (-1)^n \cdot 3^n$

### Solution

| | |
|---|---|
| Write down the general term. | $t_n = (-1)^n \cdot 3^n$ |
| Let $n = 1$. | $t_1 = (-1)^1 \cdot 3^1$<br>$= -3$ |
| Let $n = 2$. | $t_2 = (-1)^2 \cdot 3^2$<br>$= 9$ |
| Let $n = 3$. | $t_3 = (-1)^3 \cdot 3^3$<br>$= -27$ |
| Let $n = 4$. | $t_4 = (-1)^4 \cdot 3^4$<br>$= 81$ |
| Write the answer. | The first four terms are $-3$, 9, $-27$, 81 |



# Technology

## Recursive sequences

EXCEL is an easy tool that can be used to help generate sequences when a recursive sequence is defined. This spreadsheet will generate the first 15 terms of the recursive sequence: $a_n = 2a_{n-1} + 1$, $a_1 = 6$ (See Example **1**).

**Step 1:** Using EXCEL put the numbers 1 to 15 in cells B3:B17.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | n | | a(n) |
| 3 | | 1 | a(1) | 6 |
| 4 | | 2 | a(2) | 13 |
| 5 | | 3 | a(3) | 27 |
| 6 | | 4 | a(4) | 55 |
| 7 | | 5 | a(5) | 111 |
| 8 | | 6 | a(6) | 223 |
| 9 | | 7 | a(7) | 447 |
| 10 | | 8 | a(8) | 895 |
| 11 | | 9 | a(9) | 1791 |
| 12 | | 10 | a(10) | 3583 |
| 13 | | 11 | a(11) | 7167 |
| 14 | | 12 | a(12) | 14335 |
| 15 | | 13 | a(13) | 28671 |
| 16 | | 14 | a(14) | 57343 |
| 17 | | 15 | a(15) | 114687 |
| 18 | | | | |

**Step 2:** Put 6 in cell D3.

**Step 3:** Put the formula =2*D3+1 in cell D4.

**Step 4:** Mark cell D4 and fill down to cell D17.

To label each term a(1), a(2), …

**Step 5:** Put = "a("&B3&")" in cell C3.

**Step 6:** Mark cell C3 and fill down to cell C17.

### Example 3

The first six terms of a sequence are 2, 4, 6, 10, 16, 26, …

Which term is equal to 178?

#### Solution

| | |
|---|---|
| Look for a pattern (rule). | Each term is $a_n = a_{n-1} + a_{n-2}$ |
| Continue the pattern. | $a_7 = 42$, $a_8 = 68$, $a_9 = 110$, $a_{10} = 178$ |
| Write the answer. | The 10th term is 178. |

In Example **3** you could use a recursive definition of the sequence: $a_n = a_{n-1} + a_{n-2}$, with $a_1 = 2$ and $a_2 = 4$. Many sequences have simpler recursive definitions, such as 3, 5, 7, 9, 11, … which you could write as $a_n = a_{n-1} + 2$ with $a_1 = 3$.

## INVESTIGATION Hailstone sequences

Sequences such as

5, 16, 8, 4, 2, 1, 4, 2, 1, …

and

11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1

are known as **hailstone sequences** because they 'bounce around' like a hailstone when it hits the ground, before coming to rest. Hailstone sequences are generated as follows:

1. Start with any positive integer $n$.
2. If $n$ is even, divide by 2 to give $n' = \frac{n}{2}$.
   If $n$ is odd, multiply it by 3 and add 1 to give $n' = 3n + 1$.
3. Take $n'$ as the new starting number and repeat the process.


Shutterstock.com/Ryszard Stelmachowicz

Try out different starting values and see how many terms are generated before the sequence eventually settles to 4, 2, 1, 4, 2, 1, …

This is known as the Collatz conjecture, named after Lothar Collatz, who first proposed in 1937 that every starting point would eventually reach 1. Today it is still one of mathematics' unsolved problems .

Could there be a sequence that never settles down to this repeating cycle?

# EXERCISE 8.01 Sequences

## Concepts and techniques

**1** Example 1 Write the first five terms in the sequences given by:

**a** $a_1 = 1, a_n = a_{n-1} + 1$

**b** $a_1 = 10, a_n = 2a_{n-1} - 3$

**c** $a_1 = -3, a_2 = 5, a_n = a_{n-1} + a_{n-2}$

**d** $a_4 = 15, a_3 = 25, a_n - a_{n+1} = 2a_{n-1}$

**2** Example 2 Find the first four terms of each of the following sequences ($n > 0$).

**a** $t_n = 2n^2 + 1$

**b** $t_n = \frac{3n+4}{n}$

**c** $t_n = 3^n + \frac{1}{n}$

**d** $t_n = 3^{2n-1}$

**e** $t_n = n(-1)^n$

**f** $t_1 = 2, t_{n+1} = 2(t_n)^n$

**3** Example 3 Find the 10th term for each of the following sequences if the first five or seven terms are:

**a** 2, 4, 6, 8, 10, …

**b** 20, 16, 12, 8, 4, …

**c** 9, 11, 13, 15, 17, …

**d** 1, 1, 2, 3, 5, 8, 13 …

## Reasoning and communication

4 A grain of rice is placed on the first square of a chess board, three grains on the second square, five grains on the third, and so on.

a Complete the following table showing the number of grains on successive squares.

| Square number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Number of rice grains | 1 | 3 | 5 | | | | | | |

b How many rice grains will be placed on the 12th square if the same pattern is followed?

c If the squares of the chess board were filled by placing one grain of rice in the first square, two in the 2nd, four in the third and so on, how many grains of rice would need to be placed on the 12th square? Would they fit?

5 A long strip of paper is folded in half, making one crease at the fold.
When the folded strip is folded again, it then has three creases.

a Complete the table for the number of creases the strip will have after successive folds.

| Number of folds | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Number of creases | 1 | 3 | | | | | |

b How many creases will the strip of paper have after 10 folds?

# 8.02 ARITHMETIC SEQUENCES

The simplest kind of sequence is one that goes up (or down) by the same amount from one term to the next. Sequences of this type are called **arithmetic progressions** or **arithmetic sequences**.

**IMPORTANT**

An **arithmetic progression** (AP) has a **common difference** $d$. It is the difference between successive terms, written as $d = t_n - t_{n-1}$ or $d = t_{n+1} - t_n$.

An AP can be defined recursively as $t_{n+1} = t_n + d$.

### Example 4

Determine if each of the following sequences are arithmetic, and if they are, write the next four terms.

a 3, 5, 7, 9, …  b 1, 2, 5, 10, …  c $1, 1+\sqrt{5}, 1+2\sqrt{5}, 1+3\sqrt{5}, \ldots$

### Solution

a Work out $t_2 - t_1$. $t_2 - t_1 = 5 - 3 = 2$

Work out $t_3 - t_2$. $t_3 - t_2 = 7 - 5 = 2$

Work out $t_4 - t_3$. $t_4 - t_3 = 9 - 7 = 2$

Write the meaning. There is a common difference, $d = 2$


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| | | |
|---|---|---|
| | Write the answer. | The sequence is arithmetic and the next four terms are 11, 13, 15 and 17. |
| b | Check $t_2 - t_1$. | $t_2 - t_1 = 2 - 1 = 1$ |
| | Check $t_3 - t_2$. | $t_3 - t_2 = 5 - 2 = 3$ |
| | Write the answer. | There is no common difference so the sequence is not arithmetic. |
| c | Check $t_2 - t_1$. | $t_2 - t_1 = \sqrt{5}$ |
| | Check $t_3 - t_2$. | $t_3 - t_2 = \sqrt{5}$ |
| | Check $t_4 - t_3$. | $t_4 - t_3 = \sqrt{5}$ |
| | Write the meaning. | There is a common difference, $d = \sqrt{5}$ |
| | Write the answer. | The sequence is arithmetic and the next four terms are $1+4\sqrt{2}, 1+5\sqrt{2}, 1+6\sqrt{2}$ and $1+7\sqrt{2}$. |

## Example 5

Determine the common differences for the following arithmetic sequences.

a 3, 5, 7, 9, 11, … b $\sqrt{50}, \sqrt{98}, \sqrt{162}, \sqrt{242}$ … c $\pi + 3, 2\pi + 5, 3\pi + 7, 4\pi + 9, \ldots$

### Solution

| | | |
|---|---|---|
| a | Find $t_3 - t_2$. | $t_3 - t_2 = 7 - 5$<br>$= 2$ |
| | Check $t_2 - t_1$. | $t_2 - t_1 = 5 - 3$<br>$= 2$ |
| | Write the common difference. | The common difference $d = 2$ |
| b | Find $t_3 - t_2$. | $t_3 - t_2 = \sqrt{162} - \sqrt{98}$<br>$= 9\sqrt{2} - 7\sqrt{2}$<br>$= 2\sqrt{2}$ |
| | Check $t_2 - t_1$. | $t_2 - t_1 = \sqrt{98} - \sqrt{50}$<br>$= 7\sqrt{2} - 5\sqrt{2}$<br>$= 2\sqrt{2}$ |
| | Write the common difference. | The common difference $d = 2\sqrt{2}$ |
| c | Find $t_3 - t_2$. | $t_3 - t_2 = (3\pi + 7) - (2\pi + 5)$<br>$= \pi + 2$ |
| | Check $t_2 - t_1$. | $t_2 - t_1 = (2\pi + 5) - (\pi + 3)$<br>$= \pi + 2$ |
| | Write the common difference. | The common difference $d = \pi + 2$ |



## EXERCISE 8.02 Arithmetic sequences

### Concepts and techniques

**1** Example 4 Which of the following are arithmetic sequences?

**a** $m, m+3, m+6, m+9, \ldots$ **b** $r, 2r+q, 3r+q, 4r+q, \ldots$ **c** $t_n = 3n-1$

**d** $t_n = n^2+1$ **e** $t_n = \frac{n}{3} - 2$ **f** $t_n = \frac{3}{n} + 4$

**2** Write the first four terms in each of the arithmetic sequences defined as follows.

**a** $a = 5, d = 3$ **b** $a = -13, d = 6$ **c** $a = 3, d = -1\frac{1}{4}$

**d** $a = m, d = 2n$ **e** $t_n = 5n-1$ **f** $u_n = 3 + u_{n-1}, u_1 = 1, n \geq 2$

**3** Example 5 Find the common difference in each of the following arithmetic sequences.

**a** 21, 17, 13, 9, ... **b** 6, 13, 20, 27, ...

**c** 8.9, 10.5, 12.1, 13.7, ... **d** $1 - \sqrt{3}, 3 + \sqrt{3}, 5 + \sqrt{12}, 7 + \sqrt{27}, \ldots$

**e** $5 - 3x, 3 - x, x + 1, 3x - 1, \ldots$ **f** $9 + 6\pi, 11 + 5\pi, 13 + 4\pi, 15 + 3\pi, \ldots$

### Reasoning and communication

**4** The following sequences are arithmetic. Evaluate all variables.

**a** $r, r+3, 2r+5, \ldots$ **b** $m-5, 3m, 3m+1, \ldots$

**c** $2t-3, 3t+1, 5t+2, \ldots$ **d** $\sqrt{3}, m, \sqrt{27}, \ldots$

**e** $\sqrt{50}, \sqrt{98}, 3n, \ldots$

**5** Pine logs waiting in storage at a woodchip plant are piled in such a way that there are 15 in the top row, 16 in the next row down, 17 in the next and so on. If there are 40 in the bottom row, how many rows are in the pile?

**6** A market gardener plants daffodil bulbs in rows, starting with a row of 45 bulbs. Each successive row has 5 more bulbs than the previous row.

**a** Calculate the number of bulbs in the 34th row.

**b** Which row would be the first to have more than 100 bulbs in it?

**7** Steve walked 5 km on 1 August and he increased this by 0.1 km each day thereafter.

**a** What distance will he be walking on 3 August?

**b** What distance will he be walking on 16 August?

**c** Is this an arithmetic sequence? Explain your reasoning.

**d** On what date would he walk 20 km?

**8** A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on.

**a** Calculate the number of cans in the 11th row down.

**b** If there are 320 cans in the display altogether, how many rows are there?

# 8.03 ARITHMETIC SEQUENCE FORMULA

For the sequence 3, 7, 11, 15, 19, … we can write each term as the previous term plus a common difference.

$t_1 = 3$

$t_2 = 3 + 4$

$t_3 = 3 + 4 + 4 = 3 + 2 \times 4$

$t_4 = 3 + 4 + 4 + 4 = 3 + 3 \times 4$

We can clearly see the pattern $t_n = 3 + (n - 1) \times 4$

For any AP with common difference $d$,

$t_2 = t_1 + d$

$t_3 = t_2 + d = (t_1 + d) + d = t_1 + 2d$

$t_4 = t_3 + d = (t_1 + 2d) + d = t_1 + 3d$

This gives us a rule for the general term $t_n$ of an arithmetic sequence.

$t_n = t_1 + (n - 1)d$

**IMPORTANT**

An arithmetic progression (AP) with common difference $d = t_n - t_{n-1}$ may be written as

$t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots, t_1 + (n - 1)d$

The **general term** $(t_n)$ of an AP is $t_n = t_1 + (n - 1)d$.

The first term can also be written as $a$.

## Example 6

**a** Find the 20th term of the arithmetic sequence 4, 11, 18, …

CAS **b** Find the 39th term of the arithmetic sequence 24, 21, 18, ...

### Solution

**a** Write the first term. $t_1 = 4$

Find the common difference. $d = 11 - 4 = 7$

Write the number of terms. $n = 20$

Write the general term formula. $t_n = t_1 + (n - 1)d$

Substitute $t_1 = 4$, $n = 20$ and $d = 7$. $t_{20} = 4 + (20 - 1) \times 7$
$= 137$

Write the answer. The 20th term of the sequence is 137.

**TI-Nspire CAS**

**b**

Use a calculator page.

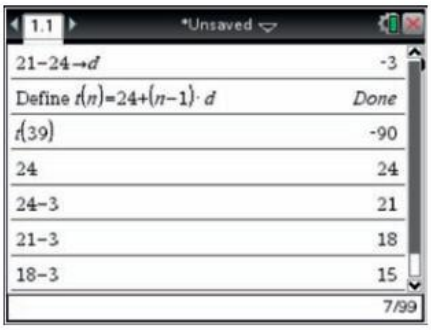
Find the common difference and store it as $d$.

Then enter the formula for the nth term as Define $t(n) = 24 + (n - 1)d$.

Type in $t(39)$.

You could also start with 24, then type +3 enter.

Pressing enter repeatedly will generate the sequence. The calculator automatically puts in the last answer.

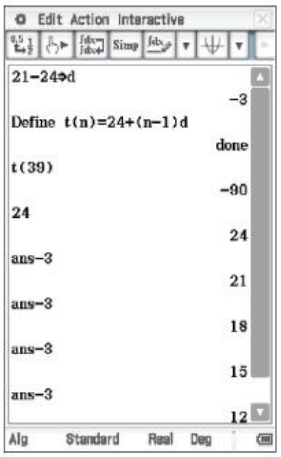



You will see on line 20 that $t_{20} = 137$.

### ClassPad

b

Use the **Main** application and press Keyboard. Find the common difference and store → as $d$. Then enter the formula for the $n$th term as
Define t($n$) = 24 + ($n$–1)$d$. Type in t(39).
Press Keyboard to show the whole screen.
You could also start with 24, then type
(−) 3 EXE.
Pressing EXE repeatedly will generate the sequence. The calculator automatically puts in the last answer.




You could enter store the value of $t_1$, in say, $a$ and enter the formula as Define $t(n) = a + (n - 1)d$.

You could then find terms of different sequences by just storing different values for $a$ and $d$.

On most calculators, both CAS and scientific, pressing enter/EXE repeats the last calculation. You might have to use a memory to make it work on some calculators.

## Example 7

Which term of the arithmetic sequence 3, 7, 11, … is equal to 111?

### Solution

| | |
|---|---|
| Write the first term. | $t_1 = 3$ |
| Write the common difference. | $d = 4$ |
| Write the $n$th term. | $t_n = 111$ |
| Write the formula for the $n$th term. | $t_n = t_1 + (n - 1)d$ |
| Substitute values and solve for $n$. | $111 = 3 + (n - 1) \times 4$<br>$4n - 4 + 3 = 111$<br>$4n = 112$<br>$n = 28$ |
| Write the answer. | The 28th term in the sequence is 111. |


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## Example 8

Does 115 belong to the sequence –23, –11, 1, … ?

### Solution

| | |
|---|---|
| Write the first term. | $t_1 = -23$ |
| Write the common difference. | $d = 12$ |
| Write the term being checked as $t_n$. | $t_n = 115$ |
| Write the formula for the $n$th term. | $t_n = t_1 + (n-1)d$ |
| Substitute the known values in. | $115 = -23 + (n-1) \times 12$ |
| Rearrange and simplify. | $115 = 12n - 35$<br>$12n = 150$<br>$n = 12.5$ |
| Write the answer. | 115 is the '12.5th term' so it is between the 12th and the 13th term of the sequence. 115 is not a term in the sequence. |

Remember that $t_1 = a$ are alternate notations for the first term.

## Example 9

Write the first four terms of the arithmetic sequence with a third term of 19 and a 20th term of 121.

### Solution

| | | |
|---|---|---|
| Write $t_3$ in the form $a + (n-1)d$. | $19 = a + 2d$ | [1] |
| Write $t_{20}$ in the form $a + (n-1)d$. | $121 = a + 19d$ | [2] |
| Subtract [1] from [2]. | $102 = 17d$ | |
| Rearrange and solve. | $d = 6$ | |
| Substitute for $d$ in [1] and solve for $a$. | $a = 7$ | |
| Use $a$ and $d$ to write the AP. | 7, 13, 19, 25 | |






Classifying sequences

## EXERCISE 8.03 Arithmetic sequence formula

**1** Example 6 Write the first four terms in each of the following arithmetic sequences.

**a** $t_1 = 3, d = 5$ **b** $t_1 = 6, d = -3$ **c** $a = -5, d = -2$

**d** $a = x, d = y$ **e** $t_1 = -15, d = 3k$

**2** Write the 10th term in each of the following arithmetic sequences.

**a** 1, 5, 9, 13, ... **b** 4, −1, −6, −11, ... **c** $1.3 + 0.9 + 0.5 + \cdots$

**d** $t_n = 3n + 2$ **e** $u_n = 5 - 8n$

**3** **a** Write the 35th term of the sequence 213, 208, 203, ...

**b** Write the 17th term of the sequence $a + 5d, a + 12d, a + 19d, \ldots$

**c** Write the 25th term of the sequence $\log_2(4), \log_2(8), \log_2(16), \ldots$ .

**d** Write the 20th term of the sequence $t_n = t_{n-1} - 4$ where $t_1 = 0$.

**e** Write the 75th term of the sequence $t_n = 3 + t_{n-1}$ where $t_1 = 6$.

**4** Example 7 Find the $n$th term in each of the following arithmetic sequences.

**a** −3, −1, 1, 3, ... **b** 19, 18, 17, 16, ... **c** 5, 6, 7, 8, ...

**d** −20, −16, −12, −8, … **e** $x, x + 5, x + 10, \ldots$

**5** Example 8 Find the number of terms in each of the following arithmetic sequences.

**a** 6, 10, 14, ..., 82 **b** 9, 15, 21, ..., 195 **c** 14, 23, 32, ..., 554

**d** 217, 211, 205, ..., −203 **e** 51, 46, 41, … , −109

**6** **a** Which term of the sequence with $n$th term, $t_n = 9n - 15$ is equal to 129?

**b** Find the value of $n$ that gives the first term of the sequence $t_n = 3n + 2$ that is greater than 100.

**c** Find the value of $n$ for which the sequence determined by $t_n = 4n - 1$, has a term greater than 350.

**d** Find the first value of $n$ for which the terms of the arithmetic sequence 100, 93, 86, ... become less than 20.

**e** Find the values of $n$ for which the terms of the sequence −86, −83, −80 , ... are positive.

**7** Example 9 Write the general term $(t_n)$, for each of the following arithmetic sequences.

**a** $d = 2, t_5 = 15$ **b** $t_1 = 7, t_8 = 35$ **c** $a = -3, t_{11} = -53$

**d** $d = -7, t_9 = -12$ **e** $a = 3, t_{12} = 30.5$ **f** $t_3 = 2, t_{10} = 23$

**g** $t_7 = 4, t_{15} = -44$ **h** $t_8 = -18, t_5 = -t_2$ **i** $t_5 = 5, t_8 = -\frac{1}{2}t_4$

**j** $t_3 = 5, t_{10} + t_2 = -t_7$

**8** **a** Is 257 a term of the sequence with $t_n = 12 + 3n$? Justify your answer.

**b** For a particular arithmetic sequence, $t_1 = 8$ and $t_{14} - t_4 = 30$. Find the value of $t_{20}$.

**c** Write

**i** the value of $n$ that gives the first negative term of the sequence $u_n = 80 - 6n$

**ii** the first negative term.

### Reasoning and communication

**9** The 7th floor of a building is 23 m above street level and the 13th floor is 41 m above street level. What is the height (above street level) of the first floor, and what is the height of one floor?


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10 At the break of the drought, the water tank was almost empty. After three hours of continuous rain, the tank had 50 litres in it. It continued to rain at a steady rate. After 7 hours, the tank contained 106 litres. At what rate was the tank collecting water, and how much was in the tank before it started raining?

# 8.04 SUM OF AN ARITHMETIC SEQUENCE



The Bridgeman Art Library/Jensen, Christian Albrecht (1792-1870)/ The Pushkin State Museum of Fine Arts, Moscow

Carl Friedrich Gauss (1777–1855) was an infant prodigy. He enjoyed numerical computation as a child and one story from his early schooling is often told as follows.

One day, in order to keep the class busy, the teacher asked the students to add up all the numbers from one to a hundred. Almost immediately Gauss placed his slate on the table, saying, 'There it is.'

The teacher was not amused and thought it was a joke. Finally, when the other students had finished, it was revealed that Gauss was indeed correct.

The ten-year-old boy had mentally computed the sum of the arithmetic progression $1 + 2 + 3 + \cdots + 99 + 100$ in next to no time without any additions.

Gauss said he solved the problem by observing that $100 + 1 = 101$, $99 + 2 = 101$, $98 + 3 = 101$, etc., and so there are 50 pairs of numbers totalling 101. Thus the answer is $50 \times 101$, or 5050.

The sum of a sequence of numbers is called a **series**. Many series occur in real life – think of the way that plants grow, or the way money accumulates as it earns interest in a bank.

For example, the sequence 3, 7, 11 … , 31 when summed becomes the series 3, 10, 21, ... 136

When you add the arithmetic sequence 3, 7, 11, …, 31 you get the series 3, 3 + 7, 3 + 7 + 11, 3 + 7 + 11 + … + 31. You can work out the sum using Gauss' method as follows.

Write $S_8 = 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31$.

Rearrange to get $S_8 = (3 + 31) + (7 + 27) + (11 + 23) + (15 + 19) = 4 \times 34 = 136$.

You can use Gauss' method for an odd number of terms as well.

Consider the arithmetic series $9 + 17 + 25 + 33 + 41 + 49 + 57$.

The middle term, $33 = \frac{1}{2} \times 66 = \frac{1}{2}(9+57)$.

For this series, $S_7 = (9 + 57) + (17 + 49) + (25 + 41) + 33 = 3\frac{1}{2} \times 66 = 231$.

The middle term of an arithmetic series with an odd number of terms is always going to be half the total of the first and last terms. Suppose an arithmetic sequence has first term $a$, common difference $d$ and there are $(2n + 1)$ terms.

Then the $(2n + 1)$th term, the last term $= a + 2nd$.

The middle term is the $(n + 1)$th term $= a + nd = \frac{1}{2}(2a+2nd) = \frac{1}{2}[a+(a+2nd)] = \frac{1}{2}[\text{first} + \text{last}]$.

To use Gauss' method for an arithmetic series, you add the first and last terms and multiply by half the number of terms.

For a general arithmetic series, let $S_n = t_1 + t_2 + t_3 + \ldots + t_n = \frac{n}{2}(t_1 + t_n) = \frac{n}{2}(a + \ell)$,

where $a$ is the first term and $\ell$ is the last term.

You can rearrange the formula as $S_n = \frac{n}{2}(t_1 + t_n) = \frac{n}{2}(t_1 + (t_1 + (n-1)d) = \frac{n}{2}(2t_1 + (n-1)d)$.

**IMPORTANT**

The sum of the **arithmetic series** $S_n$ with first term $t_1$, common difference $d$ and $n$ terms is given by

$$S_n = \frac{n}{2}(2t_1 + (n-1)d) = \frac{n}{2}(t_1 + t_n),$$

where $S_n = t_1 + t_2 + t_3 + \ldots + t_n$. $t_n$ is sometimes replaced by the letter $l$.

You can also prove the formula using $2S_n = S_n + S_n$ by writing the second sum in reverse order.

$$\begin{aligned} S_n &= t_1 + t_2 + t_3 + \ldots + t_n \\ &= t_1 + (t_1 + d) + (t_1 + 2d) + \ldots + [t_1 + (n-3)d] + [t_1 + (n-2)d] + [t_1 + (n-1)d] \\ S_n &= [t_1 + (n-1)d] + [t_1 + (n-2)d] + [t_1 + (n-3)d] + \ldots + (t_1 + 2d) + (t_1 + d) + t_1 \end{aligned}$$

Adding gives

$$\begin{aligned} 2S_n &= [2t_1 + (n-1)d] + [2t_1 + (n-1)d] + [2t_1 + (n-1)d] + \ldots + [2t_1 + (n-1)d] + [2t_1 + (n-1)d] \\ 2S_n &= n[2t_1 + (n-1)d] \end{aligned}$$

Dividing by 2, you get $S_n = \frac{n}{2}(2t_1 + (n-1)d)$.

## Example 10

Evaluate the arithmetic series $9 + 14 + 19 + \cdots + 224$.

### Solution

| | |
|---|---|
| Write down the information provided. | $t_1 = 9$, $d = 5$ and $t_n = 224$ |
| Substitute into $t_n = t_1 + (n-1)d$ to work out the number of terms. | $224 = 9 + (n-1) \times 5$<br>$5n + 4 = 224$<br>$5n = 220$<br>$n = 44$ |
| Write the appropriate formula. | $S_n = \frac{n}{2}[a + \ell]$ |
| Substitute values. | $S_{44} = \frac{44}{2}[9 + 224]$ |
| Simplify. | $= 22 \times 233$<br>$= 5126$ |
| Write the answer. | $9 + 14 + 19 + \cdots + 224 = 5126$ |


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## Example 11

If $10 + 13 + 16 + \cdots + t_n = 198$, how many terms are there in the series?

### Solution

Write down the information provided.

$t_1 = 10,\ d = 3,\ S_n = 198$

Write the formula for $S_n$.

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

Substitute values.

$$198 = \frac{n}{2}\left[2(10) + (n-1)\times 3\right]$$

Simplify to form a quadratic equation.

$$396 = 3n^2 + 17n$$

Rearrange into standard form.

$$3n^2 + 17n - 396 = 0$$

Solve the quadratic equation.

$$(n-9)(3n+44) = 0$$

$$n = 9, -\frac{44}{3}$$

Reject the false answer.

Since $n > 0$, then $n \neq -14\frac{2}{3}$

Write the answer.

There are 9 terms in the series.

## Example 12

The 6th term of an arithmetic series is 23 and the sum of the first 10 terms is 210.

Find the sum of the first 20 terms.

### Solution

Write down the information provided.

$t_6 = 23$ and $S_{10} = 210$

Write equations using the information provided.

$$t_1 + 5d = 23 \quad [1]$$

$$5[2t_1 + 9d] = 210 \quad [2]$$

Solve the equations simultaneously.

$$2t_1 + 10d = 46 \quad [1] \times 2$$

$$2t_1 + 9d = 42 \quad [2] \div 5$$

Subtract the equations.

$d = 4$

Substitute in $t_1 + 5d = 23$

Find $t_1$.

$t_1 + 5 \times 4 = 23$

$t_1 = 3$

Write the value of $S_{20}$ using $S_n$.

$S_{20} = 10[2 \times 3 + 19 \times 4]$

$= 820$



WS

Arithmetic progressions

# EXERCISE 8.04 Sum of an arithmetic sequence

## Concepts and techniques

**1** Example 10 Evaluate the following.

**a** $15 + 20 + 25 + \cdots + 535$
**b** $9 + 17 + 25 + \cdots + 225$
**c** $5 + 2 - 1 - \cdots - 91$
**d** $81 + 92 + 103 + \cdots + 378$
**e** $229 + 225 + 221 + \cdots + 25$
**f** $-2 + 6 + 14 + \cdots + 94$
**g** $0 - 9 - 18 - \cdots - 216$
**h** $79 + 81 + 83 + \cdots + 229$
**i** $14 + 11 + 8 + \cdots - 43$
**j** $11 + 23 + 35 + \cdots + 251$

**2** Find the sum of the first 15 terms of the arithmetic series below.

**a** $4 + 7 + 10 + \cdots$
**b** $2 + 7 + 12 + \cdots$
**c** $60 + 56 + 52 + \cdots$
**d** $5 - 4 - 13 - \cdots$
**e** $t_n = 4n - 1$

**3** Example 11 **a** How many terms of the series $45 + 47 + 49 + \cdots + t_n$ give a sum of 1365?

**b** How many terms of the series $80 + 73 + 66 + \cdots$ must be added to give a sum of 495?

**c** The sum of $n$ terms of the series $18 + 15 + 12 + \cdots$ is 33. Find all possible values of $n$.

**d** How many terms of the series $20 + 18 + 16 + \cdots$ give a sum of 104?

**e** How many terms of the series with $t_n = 5 - 2n$ give a sum of $-2021$?

**4** Example 12 **a** The sum of the first 5 terms of an arithmetic series is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

**b** The sum of the first 5 terms of an arithmetic series is 35 and the sum of the next 5 terms is 160. Find the first term and the common difference.

**c** The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83. Find the sum of 40 terms.

**d** The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46. Find the sum of the first 20 terms.

## Reasoning and communication

**5** Prove that $t_n = S_n - S_{n-1}$ for any AP.

**6** Find the sum of all integers between 1 and 100 that are not multiples of 6.

**7** A fireworks rocket rises vertically upwards from the ground with sufficient initial force to make it rise 110 m in the 1st second of its flight. It travels 99 m in the 2nd second, 88 m in the 3rd second and so on.

**a** Calculate how long it takes before the rocket stops going up.

**b** Calculate its greatest height above the ground.

**8** The total weight of 100 packages is 10.4 tonnes. If the weights of consecutive packages differ by 2 kg, find the weights of the heaviest and lightest packages.


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9 Pine logs waiting in storage at a woodchip plant are piled in such a way that there are 25 in the top row, 26 in the next row down, 27 in the next and so on. If there are 120 in the bottom row:




a how many rows are in the pile?

b how many logs are in the pile?

10 Mandy deposits $200 on 1 January and on the first day of each of the following months for a year. The bank pays 4% p.a. simple interest on deposits and there are no account-keeping fees. Simple interest is calculated based on the amount of time the money is deposited. How much will Mandy have in the bank at the end of the year?

# 8.05 GEOMETRIC SEQUENCES

The sequence 2, 4, 8, 16, 32, … is not an arithmetic sequence as there is not a common difference.

In this case the terms can be written 2, $2 \times 2$, $2 \times 2 \times 2$, ... or $2^1$, $2^2$, $2^3$, ... It is clear that the terms in this type of sequence are going to get bigger rapidly (exponentially).

A sequence of numbers in which each term is generated by multiplying the preceding term by a non-zero constant number is called a **geometric progression** or **geometric sequence**.

**IMPORTANT**

A **geometric progression** (GP) has a **common ratio** $r$. It is the multiplicative constant that is the ratio between successive terms, written as $r = t_n \div t_{n-1}$ or $r = \frac{t_n}{t_{n-1}}$.

A GP can be defined recursively as $t_n = r \times t_{n-1}$, for $n \geq 2$.

The first term is sometimes written as $a$.

## Example 13

Is $\frac{1}{4}, \frac{1}{6}, \frac{1}{18}, \cdots$ a geometric sequence?

### Solution

| | |
|---|---|
| Find the ratio of the first two terms. | $r = \frac{1}{6} \div \frac{1}{4} = \frac{2}{3}$ |
| Find the ratio of the second and third terms. | $r = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$ |
| Write the result. | Therefore, the sequence is not geometric as there is no common ratio. |

## Example 14

Find the common ratio of the sequence 3, 6, 12, … and write the next four terms.

### Solution

| | |
|---|---|
| Find $t_2 \div t_1$. | $6 \div 3 = 2$ |
| Check $t_3 \div t_2$. | $12 \div 6 = 2$ |
| Write the common ratio. | $r = 2$ |
| Write the next four terms in the sequence. | 24, 48, 96, 192 |

## Example 15

Find $x$ if 5, $x$, 45 forms a geometric sequence.

### Solution

| | |
|---|---|
| Find $t_2 \div t_1$. | $r = \frac{x}{5}$ |
| Find $t_3 \div t_2$. | $r = \frac{45}{x}$ |
| Write an equation. | $\frac{x}{5} = \frac{45}{x}$ |
| Solve for $x$. | $x^2 = 5 \times 45 = 225$<br>$x = \pm 15$ |
| Write a conclusion. | There are two geometric sequences that are possible:<br>5, –15, 45, ... and 5, 15, 45, ... |

WS
Arithmetic series


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# EXERCISE 8.05 Geometric sequences

## Concepts and techniques

**1** Example 13 Determine which of the following are geometric sequences.
Find the common ratio for each geometric sequence.

**a** $1, 2, 4, 8, \ldots$ **b** $-4, -12, -36, -108, \ldots$ **c** $t_n = 2(-2)^n$

**d** $t_n = 3 + n^2$ **e** $1m, 2m, 3m, 4m, \ldots$ **f** $5, 5\sqrt{2}, 10, 10\sqrt{2}, \cdots$

**g** $4, 8, 24, 96, \ldots$

**2** Example 14 Find the common ratio and the next two terms in each of these geometric sequences.

**a** $1, 3, 9, \ldots$ **b** $3, -6, 12, \ldots$ **c** $\frac{1}{2}, 2, 8, \cdots$

**d** $13, -13, 13, \ldots$ **e** $64, 16, 4, \ldots$ **f** $-2, 1, -\frac{1}{2}, \ldots$

**3** Example 15 Find all possible values of $x$ in these geometric sequences. Show your working.

**a** $2, x, 50, \ldots$ **b** $\frac{1}{2}, x, 8, \ldots$ **c** $5, 2\frac{1}{2}, x, \ldots$

**d** $81, 27, x, \ldots$ **e** $\ldots, \frac{5}{9}, \frac{1}{3}, x, \ldots$ **f** $\ldots, x, -18, 108, \ldots$

**g** $\ldots, x, \frac{3}{4}, -\frac{1}{18}, \ldots$ **h** $\ldots, 28, x, \frac{16}{7}, \ldots$

## Reasoning and communication

**4** The population of a shire was known to be 15 000 at the time of the shire council election. Researchers discover that the population is increasing at the rate of approximately 4.8% per annum due to the influx of city dwellers seeking a rural lifestyle. What will the population of the shire be in 10 years time if this trend continues?

**5** Sue has $1000 to be invested at 8% compounding annually. How much will her investment be worth after 25 years?

**6** Cars depreciate at varying rates depending on the make and how well they have been maintained by their owner. A car that was initially purchased for $39 000 depreciated at the rate of 16% p.a. Find its value 10 years after it was purchased.

**7** The green tree frog population has been growing at a constant rate of 2.5% p.a. in the local rainforest. The current population is 360 frogs.



Shutterstock.com/Awei

**a** What was the population 5 years ago?

**b** How long will it take to double its current population?

8 A southern brown bandicoot population in Western Australia is decreasing by 5% each year.
   a What percentage of the population is left after 5 years?
   b After how many years will the population be only 50%?
   c How many years will it take for the population to decrease by 80%?

9 A 100 m cliff erodes by $\frac{2}{15}$ of its height each year.
   a What will the height of the cliff be after 10 years?
   b After how many years will the cliff be less than 50 m high?

10 Geoff wants to invest enough money to pay for a $10 000 holiday in 5 years time. If interest is 8% p.a., how much does Geoff need to invest now?

# 8.06 GEOMETRIC SEQUENCE FORMULA

For the sequence 3, 6, 12, 24, … we can write each term as the previous term multiplied by a common ratio.

$t_1 = 3$

$t_2 = 3 \times 2$

$t_3 = 3 \times 2 \times 2 = 3 \times 2^2$

$t_4 = 3 \times 2 \times 2 \times 2 = 3 \times 2^3$

$t_5 = 3 \times 2 \times 2 \times 2 \times 2 = 3 \times 2^4$

We can clearly see the pattern $t_n = 3 \times 2^{n-1}$

**IMPORTANT**

The **general term** $(t_n)$ of a GP with first term $a$ and common ratio $r$ is $t_n = a \times r^{n-1}$.

## Example 16

**a** What is the seventh term in the geometric sequence 6, −30, 150, … ?

**b** CAS Find the sixth term of the GP 3, 4.5, 6.75, ...

### Solution

Find the common ratio.

$$r = \frac{150}{-30} = \frac{-30}{6} = -5$$

Write the formula.

$t_n = t_1 r^{n-1}$

Substitute values.

$t_7 = 6 \times (-5)^{7-1}$

Simplify.

$= 93\,750$

Write the answer.

Hence, the seventh term is 93 750.


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### TI-Nspire CAS

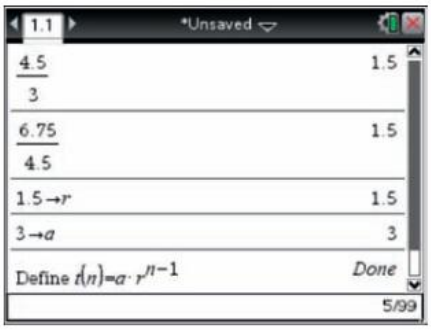
Use a calculator page .

**b** Check that there is a common ratio.

Store ctrl var the common ratio in the variable $r$.

Store the first term in the variable $a$.

Define the formula $t(n) = ar^{n-1}$.



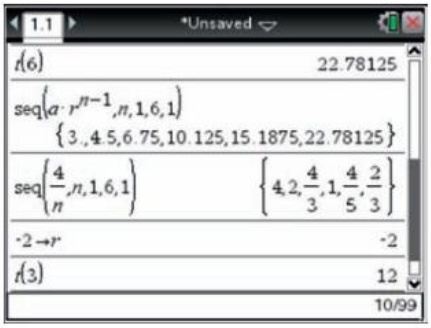

Type in $t(6)$.

You can also use the seq() function to generate a sequence. You need to put in the formula, the variable, the starting finishing values and you can also make it step more than 1.

Type in seq($a \times r^{n-1}$,$n$,1,6,1).

You can use seq() to generate different sequences.

You can store different values in $r$ and $a$ for different GPs.




### ClassPad

Use the **Main** application and press Keyboard.

Check that there is a common ratio.

Store → the common ratio in the variable $r$.

Store the first term in the variable $a$.

Define the formula $t(n)=ar^{n-1}$.

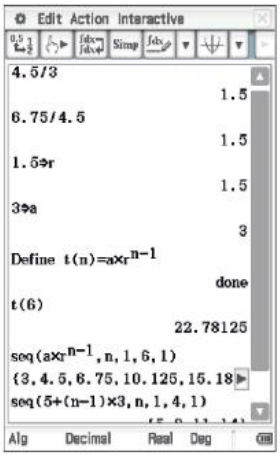
Type in t(6).

You can also use the **seq** (function to generate a sequence. You need to put in the formula, the variable, the starting and finishing values and you can also make it step more than 1.

Type in seq($a \times r^{n-1}$, $n$, 1, 6, 1). You will have to tap the right arrow ▶ to see the last values.

You can use **seq** (to generate different sequences.

Press Keyboard to see the whole screen.




Write the answer.

The sixth term is 22.781 25

The Casio also has a Sequence menu, but it is designed for more complicated sequences.

## Example 17

Does –32 768 belong to the geometric sequence 1, –2, 4, –8,…?

### Solution

| | |
|---|---|
| Write the first term and the common ratio. | $t_1 = 1, r = -2$ |
| Substitute values and solve for $n$. | $t_n = t_1 r^{n-1}$ |
| Substitute values. | $-32\,768 = 1 \times (-2)^{n-1}$ |
| Simplify. | $(-2)^{n-1} = -32\,768$ |
| What power of 2 gives 32 768? | $2^{15} = 32\,768$ |
| Check $ar^{n-1}$ for $n = 16$. | $1 \times (-2)^{16-1} = (-2)^{15} = -32\,768$ |
| Write the answer. | –32 768 is the 16th term of this geometric sequence. |

## Example 18

The second and sixth terms of a geometric sequence are 28 and 448 respectively. What is the tenth term of the sequence?

### Solution

| | |
|---|---|
| Using $t_n$ write the second term. | $t_n = t_1 \times r^{n-1}$<br>$28 = t_1 \times r^1$ [1] |
| Using $t_n$ write the sixth term. | $t_n = t_1 \times r^{n-1}$<br>$448 = t_1 \times r^5$ [2] |
| Solve these equations simultaneously. | [2] ÷ [1]<br>$\therefore 16 = r^4$<br>$\therefore r = \pm 2$<br>$\therefore t_1 = \pm 14$, the same sign as $r$ |
| Write the tenth term of the sequence. | $t_{10} = t_1 \times r^9$<br>$= (\pm 14) \times (\pm 2)^9$<br>$= 7168$ for both values of $r$ |

WS Geometric progressions 1

WS Geometric progressions 2

You could do Example 18 using a shortcut as follows:

$t_{10} = t_6 \times r^4 = t_6 \times (t_6 \div t_2) = 448 \times 448 \div 28 = 7168$


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## INVESTIGATION Terms of a GP as $n \to \infty$

Use your CAS calculator to see what happens to each of the following GPs as $n$ becomes large.

In each case, put in the initial term. Then multiply by the common ratio and keep pressing return.

a 10, 5, 2.5, …

6, −2, 0.666…, …

3, 6, 12, …

−2, 3, −4.5, …

2, 1.8, 1.62, …

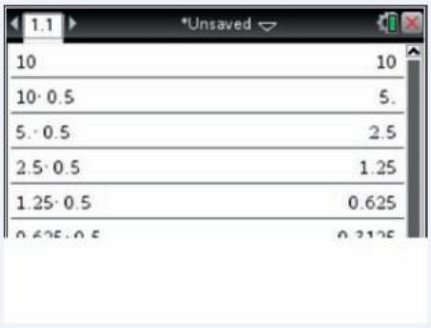
1.2, −1.32, 1.452, …

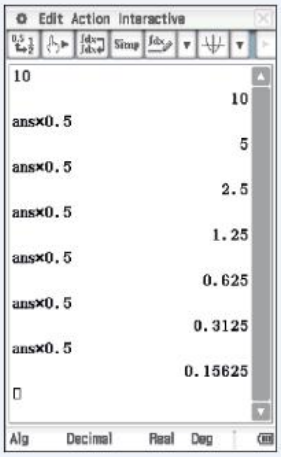
6, −4.8, 3.84, …

8, 3.2, 1.28, …

6, 24, 96, …

The first part of the first display should look like the following.




What happens when $r > 1$?

What happens when $r < -1$?

What conclusions can you make?

### IMPORTANT

A sequence that has a limit as $n \to \infty$ is said to **converge**.

A sequence that does not have a limit as $n \to \infty$ is said to **diverge**.

A geometric series with common ratio $-1 < r < 1$ converges to 0.

A geometric series with common ratio $r > 1$ or $r < -1$ diverges.

In the trivial cases: a GP with $r = 1$ converges to $t_1$; a GP with $r = -1$ diverges.

# EXERCISE 8.06 Geometric sequence formula

## Concepts and techniques

1 Example 16 **a** Write the 10th term of the sequence −5, 10, −20, …

**b** Write the 15th term of the GP 1.04, $1.04^2$, $1.04^3$, …

**c** Write the 7th term of the sequence where $t_1 = -5$, $t_n = -2t_{n-1}$.

**d** Write the 25th term of the sequence where $t_{21} = 1024$, $t_n = 2t_{n-1}$.

e Find the 50th term of the sequence 1, 11, 121, … (leave in index form).
f Find the 20th term of 3, 6, 12, ….
g Find the 8th term of the geometric sequence 2.1, 4.2, 8.4, …
h Find the 16th term of the GP 4, 2, 1, $\frac{1}{2}$, … (leave in index form).
i Write an expression for the $n$th term of the sequence 6, 18, 54, …
j What is the 9th term of the sequence −3, 6, −12, …?

2 Example 17 a Which term of the geometric sequence 4, 12, 36, ... is equal to 78 732?
b Which term of the geometric sequence 4, 20, 100, ... is equal to 12 500?
c Which term of the GP starting with 6, 36, 216, ... is equal to 7776?
d Which term of the sequence 8, 4, 2, ... is $\frac{1}{128}$?
e Which term of the sequence 5, 15, 45, … is the first that is greater than 1 000 000?

3 a Is 1200 a term of the sequence that starts 2, 16, 128, ... ?
b Is 224 a term of the sequence 7, 14, 28, …?
c Is 546.75 a term of the sequence where $t_1 = 0.25$, $t_n = -3\, t_{n-1}$?

4 Example 18 a The third term in a geometric sequence is 18 and the 7th term is 1458. Find the first term and the common ratio.
b The 4th term of a geometric sequence is −648 and the 5th term is 3888. Find the eighth term.
c The first and fifth terms of a geometric sequence are 9 and 2304.
i Find the two possible values of the common ratio.
ii Find the seventh term and explain why there is only one possible value.
d Find the value of $n$ in the first term of the sequence 5000, 1000, 200, ... that is less than 1.

5 State whether the following geometric sequences converge or diverge as $n \to \infty$.
a 1000, 500, 250, 125, …
b 5, 10, 20, 40, …
c −8, 4, −2, 1, …
d 0.004, 0.04, 0.4, 4, …

## Reasoning and communication

6 The value of an art collection appreciates at the average rate of 8% per year. The collection was purchased for $20 000. Find its value 12 years after its purchase.

7 Garry invested $1000 at 8% compounding annually. How much will the investment be worth after 25 years?

8 In its first month of life, the weight of a baby mouse increases geometrically. On the first day, a particular mouse weighed 50 g. On the fourth day, it weighed 66.55 g. How much would you expect this mouse to weigh on the 20th day?

9 Compound interest is another example of where geometric sequences can be used. My bank account attracts interest compounded monthly. After 2 months, the balance of the account is $3535.09. After 6 months it is $3606.32.
i How much can I expect to have in the account after one year?
ii To the nearest percent, what is the annual interest rate?

10 A block of ice is melting in such a way that at any time its volume is a certain percentage of what it was one minute ago. Three minutes after the block started to melt, its volume was 32 000 $cm^3$. After six minutes, its volume was 16 384 $cm^3$. What was the volume of the block initially? How long until it is all gone?

# 8.07 SUM OF A GEOMETRIC SEQUENCE

When the creator of the game of chess showed his invention to his country's ruler, he was so impressed that he granted the clever inventor the right to claim any prize he desired. Being a smart man he decided upon the following as his prize: that for the first square of the chessboard, he would receive one grain of rice, two for the second one, four on the third one, and so forth, doubling the amount each time. The ruler unknowingly accepted his request and was not sure that it was a sufficient prize for such a creation. The ruler ordered the man be paid for his prize and after an extended period of time the ruler's accountant indicated that there were not enough assets in the kingdom to meet the desired prize!



In fact, it is an amount many times more than the global production of rice.

To be able to count the number of grains of rice it is necessary to know how to sum a geometric series.

As is the case with an AP, a finite **geometric series** is the sum of the terms of a finite geometric sequence, so the series $t_1 + t_1r + t_1r^2 + t_1r^3 + \ldots + t_1r^{n-1}$ is the sum of the terms of the GP: $t_1, t_1r, t_1r^2, t_1r^3, \ldots, t_1r^{n-1}$

We first demonstrate the method with a numerical example.

Let $S_6 = 3 + 6 + 12 + 24 + 48 + 96$

Adding the first and last terms does not give the same answer as adding the second and second last terms, etc. However, if you multiply each term by the common ratio ($r = 2$), most of the terms reappear.

$2S_6 = 6 + 12 + 24 + 48 + 96 + 192$

Now subtract these two equations:

$2S_6 - S_6 = (6 + 12 + 24 + 48 + 96 + 192) - (3 + 6 + 12 + 24 + 48 + 96)$

$S_6 = 192 - 3$ (all but two terms cancel)

$= 189$

The general case works the same way.

$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n$

$= t_1 + t_1 r + t_1 r^2 + \ldots + t_1 r^{n-3} + t_1 r^{n-2} + t_1 r^{n-1}$ [1]

$rS_n = t_1 r + t_1 r^2 + t_1 r^3 + \ldots + t_1 r^{n-2} + t_1 r^{n-1} + t_1 r^n$ [2]

[2] – [1] gives

$rS_n - S_n = t_1 r^n - t_1$

$S_n(r-1) = t_1(r^n - 1)$

$S_n = \dfrac{t_1\left(r^n - 1\right)}{r-1}$ for $r \neq 1$

If the last term, $t_n$, is known then the sum can be found as follows.

$S_n = \dfrac{t_1\left(r^n - 1\right)}{r-1}$ (for $r > 1$)

$= \dfrac{t_1 r^n - t_1}{r-1}$

$= \dfrac{t_1 r^{n-1} \times r - t_1}{r-1}$

$= \dfrac{rt_n - t_1}{r-1}$

**IMPORTANT**

The sum of the first $n$ terms of a geometric series is given by:

$S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$ for $r > 1$ or $S_n = \dfrac{a\left(1 - r^n\right)}{1-r}$ for $r < 1$

When the last term ($t_n$) in a geometric sequence is known then the sum of the series is

$S_n = \dfrac{rt_n - t_1}{r-1}$ for $r > 1$ or $S_n = \dfrac{t_1 - rt_n}{1-r}$ for $r < 1$.

Of course, the sums for $r > 1$ and $r < 1$ are algebraically the same. It is just convenient to avoid negatives.

You can work out the sum for the trivial case with $r = 1$.

### Example 19

Evaluate the geometric series $1 + 3 + 9 + \ldots + 19\,683$.

#### Solution

Write the first term. $a = 1$

Write the common ratio. $r = 3$


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| | |
|---|---|
| Write the appropriate formula for $S_n$. | $S_n = \frac{rt_n - t_1}{r-1}$ |
| Substitute values. | $= \frac{3 \times 19\,683 - 1}{3-1}$ |
| Simplify. | $= \frac{59\,048}{2}$ |
| | $= 29\,524$ |
| Write the answer. | The sum of the series is 29 524. |

## Example 20

If $2 + 4 + \ldots + t_n = 2046$, how many terms are there in the series?

### Solution

| | |
|---|---|
| Write the value of the first term. | $t_1 = 2$ |
| Write the value of the common ratio. | $r = 2$ |
| Write the sum of $n$ terms. | $S_n = 2046$ |
| Write the formula. | $S_n = \frac{t_1(r^n - 1)}{r-1}$ |
| Substitute values. | $2046 = \frac{2(2^n - 1)}{2-1}$ |
| Simplify. | $2046 = 2(2^n - 1)$ |
| | $1023 = 2^n - 1$ |
| Isolate $2^n$. | $2^n = 1024$ |
| Find the value of $n$. | $n = 10$ |

**Alternative method**

| | |
|---|---|
| Write the last term formula. | $S_n = \frac{rt_n - t_1}{r-1}$ |
| Simplify. | $2046 = \frac{2 \times t_n - 2}{2-1}$ |
| | $2t_n - 2 = 2046$ |
| Solve for $t^n$. | $2t_n = 2048$ |
| | $t_n = 1024$ |
| Put in the formula. | $t_1 r^{n-1} = 1024$ |
| Substitute values. | $2 \times 2^{n-1} = 1024$ |
| Solve for $n$. | $2^n = 1024$ |
| | $n = 10$ |
| Write the answer. | There are 10 terms in the series. |

## Example 21

The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069. Find the first term.

### Solution

| | |
|---|---|
| Write the appropriate formula. | $S_n = \frac{t_1(r^n - 1)}{r - 1}$ |
| Substitute values. | $3069 = \frac{t_1(4^5 - 1)}{4 - 1}$ |
| Solve for the first term. | $t_1(4^5 - 1) = 9207$<br>$t_1 = \frac{9207}{1023}$.<br>$t_1 = 9$ |
| Write the answer. | The first term of the series is 9. |

WS Geometric series

# EXERCISE 8.07 Sum of a geometric sequence

## Concepts and techniques

1 Example 19 **a** Find the sum of the first seven terms of the geometric sequence 5, 15, 45, … .
**b** Find the sum of the geometric series $5 + 5.5 + 6.05 + \ldots$ to 13 terms.
**c** Find the sum of the geometric series $1 + 2 + 4 + \ldots$ to 15 terms.
**d** Find the sum of the geometric series $2 + 4 + 8 + \ldots$ to 8 terms.
**e** Find the sum of the geometric series $\frac{1}{5} + 1 + 5 + \cdots$ to 9 terms

2 Find $S_{10}$ for the geometric series $16 - 12 + 9 - 6\frac{3}{4} + \ldots$

3 Find $S_7$ for the geometric sequence $2\frac{2}{3}$, 4, 6, …

4 Find the ninth term of the geometric series $7 + 14 + 28 + \ldots$ and hence find $S_9$.

5 Example 20 How many terms are there in the geometric series if $-3 + 6 - 12 + \ldots = 255$?

6 How many terms of the geometric sequence 3, 4, $5\frac{1}{3}$, $7\frac{1}{9}$, … must be taken for the sum to exceed 500?

7 Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series that begins with $4 + 16 + 64 + \ldots$

8 Example 21 Calculate the sum of the first four terms of the geometric sequence where $t_3 = 20$ and $t_5 = 80$.

9 In a geometric series, the 4th and 6th terms are 56 and 224 respectively. Find two possible values for the sum of the first five terms.

10 The sum of the first 5 terms of a geometric sequence is 77 and the sum of the next 5 terms is −2464. Find the 4th term of the sequence.

11 The first term of a geometric sequence is 54 and the $4^{th}$ term is 16. Find the common ratio and the sum of the first 8 terms.


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## Reasoning and communication

12 A ball is dropped from a height of 24 m and rebounds to a height of 16 m. If it continues to rebound to two thirds of the previous height each time, find the total distance it travels after bouncing exactly 5 times.

13 Lucia earns $35 000 in her first year. Her wage increases by 5% each year. Find:
- a her wage in her 6th year
- b her total earnings (before tax) over the 6 years.

14 An amount of $50 is put into an investment account at the end of each month. If interest is paid at 12% p.a. compounding monthly, how much is in the account at the end of 10 years?

15 Lisan wants to save up $15 000 for a car in 5 years time. He invests $2000 at the start of each year in an account that pays 7.5% p.a. compound interest. How much more will Lisan have to pay at the end of 5 years to make up the $15 000?

# 8.08 INFINITE GEOMETRIC SERIES (OPTIONAL)

## INVESTIGATION Infinite sums of GPs

Find the sum of each of the following geometric series to the given numbers of terms.

$40 + 20 + 10 + \ldots$ to 5 terms, 8 terms and 50 terms

$2 + 4 + 8 + \ldots$ to 5 terms, 8 terms and 15 terms

$90 - 30 + 10 - \ldots$ to 5 terms, 8 terms and 50 terms

$10 - 30 + 90 - \ldots$ to 5 terms, 8 terms and 15 terms

What can you say about each of these series?

What determines whether or not the sums converge?



For $-1 < r < 1$, as $n$ increases, $r^n$ decreases, getting closer and closer to zero.

$$S_n = \frac{a(1-r^n)}{1-r}$$

So for $-1 < r < 1$, as $n$ becomes very large, $S_n$ gets close to $\frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

**IMPORTANT**

A geometric series converges if $-1 < r < 1$ and the **limiting sum** of a geometric series with first term $a$ and common ratio $r$, written as $S_\infty$, is given by $S_\infty = \frac{a}{1-r} = \frac{t_1}{1-r}$.

The limiting sum is also called the **sum to infinity**.

## Example 22

Which of the following series have a limiting sum?

**a** $\frac{3}{4}+\frac{15}{16}+1\frac{11}{64}+\cdots$ **b** $100+50+25+\cdots$ **c** $3+2\frac{1}{4}+1\frac{11}{16}+\cdots$

### Solution

**a** Find the common ratio.

$$r=\frac{1\frac{11}{64}}{\frac{5}{16}}=\frac{\frac{5}{16}}{\frac{3}{4}}=1\frac{1}{4}$$

Since $r>1$, this series does not have a limiting sum.

**b** Find the common ratio.

$$r=\frac{25}{50}=\frac{50}{100}=\frac{1}{2}$$

Since $r<1$, this series does have a limiting sum.

**c** Find the common ratio.

$$r=\frac{2\frac{1}{4}}{3}=\frac{1\frac{11}{16}}{2\frac{1}{4}}=\frac{3}{4}$$

Since $r<1$, this series does have a limiting sum.

## Example 23

Evaluate the following.

**a** $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ **b** $6+2+\frac{2}{3}+\cdots$ **c** $\frac{8}{9}+\frac{16}{27}+\frac{32}{81}+\cdots$

### Solution

**a** Find the common ratio.

$$r=\frac{1}{2}$$

Find the limiting sum.

$$S_\infty=\frac{a}{1-r}$$
$$=\frac{2}{1-\frac{1}{2}}$$
$$=4$$

**b** Find the common ratio.

$$r=\frac{\frac{2}{3}}{2}=\frac{2}{6}=\frac{1}{3}$$

Find the limiting sum.

$$S_\infty=\frac{a}{1-r}$$
$$=\frac{6}{1-\frac{1}{3}}=9$$

**c** Find the common ratio.

$$r=\frac{\frac{32}{81}}{\frac{16}{27}}=\frac{\frac{16}{27}}{\frac{8}{9}}=\frac{16}{24}=\frac{2}{3}$$

Find the limiting sum.

$$S_\infty=\frac{a}{1-r}$$
$$=\frac{\frac{8}{9}}{1-\frac{2}{3}}=\frac{8}{3}=2\frac{2}{3}$$


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You can write the recurring part of a decimal as a GP. Then you can change it to a fraction using the sum to infinity.

### Example 24

Write $0.2\overline{54}$ as a fraction.

#### Solution

| | |
|---|---|
| Separate the recurring part in long form. | $0.2\overline{54} = 0.2 + 0.054\ 545\ 4\ldots$ |
| Write out the recurring part as a GP. | $0.054\ 545\ 4\ldots = \frac{54}{1000} + \frac{54}{100\,000} + \frac{54}{10\,000\,000} + \cdots$ |
| State the first term and common ratio. | $a = \frac{54}{1000}$ and $r = \frac{1}{100}$ |
| Find the sum to infinity. | $S_\infty = \frac{a}{1-r} = \frac{54}{1000} \div \left(1 - \frac{1}{100}\right)$ |
| Simplify. | $= \frac{54}{1000} \times \frac{100}{99}$ |
| | $= \frac{3}{55}$ |
| Find the total. | $0.2 + 0.0\overline{54} = \frac{2}{10} + \frac{3}{55} = \frac{14}{55}$ |
| Write the answer. | $0.2\overline{54}$ is the same as $\frac{14}{55}$. |

## EXERCISE 8.08 Infinite geometric series (Optional)

### Concepts and techniques

1 Example 22 Which of the following series have a limiting sum?

a $9 + 3 + 1 + \ldots$

b $\frac{1}{4} + \frac{1}{2} + 1 + \cdots$

c $16 - 4 + 1 + \ldots$

d $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + \cdots$

e $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

f $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + \cdots$

g $-6 + 36 - 216 + \ldots$

h $-2\frac{1}{4} + 1\frac{7}{8} - 1\frac{27}{48} + \cdots$

i $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} + \cdots$

j $2 - \frac{4}{5} + \frac{8}{25} - \cdots$

2 Example 23 Find the limiting sum of the following series.

a $40 + 20 + 10 + \ldots$

b $320 + 80 + 20 + \ldots$

c $100 - 50 + 25 + \ldots$

d $6 + 3 + 1\frac{1}{2} + \cdots$

e $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + \cdots$

f $72 - 24 + 8 - \ldots$

g $-12 + 2 - \frac{1}{3} + \cdots$

h $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - \cdots$

i $12 + 9 + 6.75 + \ldots$

j $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + \cdots$

3 CAS For the series $4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots$, notice that after a while the terms are becoming closer and closer to zero and so will not add much to the sum of the whole series.

a Evaluate, correct to 4 decimal places, the sum of

i 10 terms ii 20 terms.

b Estimate its limiting sum.

4 Find the limiting sums of each of the following series.

a $1+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$

b $\frac{2}{7}+\frac{4}{49}+\frac{8}{343}+\cdots$

c $\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}+\cdots$

d $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots$

e $\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots$

f $-\frac{16}{9}-\frac{32}{27}-\frac{64}{81}-\cdots$

5 A geometric series has a limiting sum of 5 and a first term of 3. Find the common ratio.

6 The limiting sum of a geometric series is $9\frac{2}{3}$ and the common ratio is $\frac{2}{5}$. Find the first term of the series.

7 A geometric series has a limiting sum equal to one-and-a-half times the first term. Find the common ratio of the series.

8 The second term of a geometric series is 2 and its limiting sum is 9. Find the values of the first term and the common ratio.

9 A geometric series has a 3rd term of 12 and a 4th term of $-3$. Find $t_1$, $r$ and $S_\infty$.

10 Example 24 Write each recurring decimal as a fraction.

a $0.\overline{4}$ b $0.\overline{7}$ c $1.\overline{2}$ d $0.\overline{25}$

e $2.8\overline{1}$ f $0.2\overline{1}$ g $0.1\overline{32}$

## Reasoning and communication

11 The sum $x+\frac{x}{4}+\frac{x}{16}+\cdots$ has a limit of $\frac{7}{8}$. Find the value of $x$.

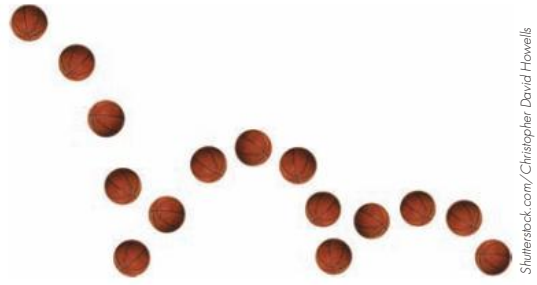
12 For the geometric series $k+k^2+k^3+\ldots$.

a for what values of $k$ does the limiting sum exist?

b find the limiting sum of the series when $k=-\frac{2}{3}$.

c evaluate $k$ if the limiting sum of the series is 3.

13 A ball is dropped from a height of 24 m and rebounds to a height of 16 m. If it continues to rebound to two-thirds of its previous height each time, find the total distance it travels.



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14 An 8 cm seedling grows by $4\frac{4}{5}$ cm in the first week, and then keeps growing by $\frac{3}{5}$ of its previous week's growth. How tall will it grow?

# CHAPTER SUMMARY
# ARITHMETIC AND GEOMETRIC SEQUENCES

8

- A **sequence** is an ordered list. It is a function whose domain is the natural numbers. Each member of a sequence is called a **term**, denoted by $t_n$ or $a_n$ where $n$ is the term number. The sum of a sequence is called a **series**. **Finite sequences** have a finite number of terms while **infinite sequences** go on indefinitely.
- An **arithmetic sequence** or **arithmetic progression** (**AP**) has a first term denoted by $t_1$ or $a$ with each term differing from the previous one by the **common difference**, $d$. The **general term** of an AP is given by $t_n = t_1 + (n-1)d$ or recursively by $t_{n+1} = t_n + d$.
- The common difference of an AP is given by: $d = t_n - t_{n-1}$ or $d = t_{n+1} - t_n$.
- For a sequence $t_1, t_2, t_3, \ldots$ the **corresponding series** is $t_1, t_1 + t_2, t_1 + t_2 + t_3, \ldots$
- The sum of $n$ terms of an **arithmetic series** with first term $t_1$ and common difference $d$ is given by $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$ or $S_n = \frac{n}{2}\left[t_1 + t_n\right]$, where $t_n$ is the last term.
- A **geometric sequence** or **geometric progression** (**GP**) has a constant called the **common ratio** that is the ratio of any two successive terms. The first term is written as $a$ or $t_1$ and the common ratio as $r$. A GP may be defined recursively by $t_{n+1} = rt_n$, where $n \geq 1$.
- The common ratio may be expressed as $r = \frac{t_n}{t_{n-1}}$.
- The **general term** of a GP is given by $t_n = ar^{n-1}$ or $t_1 r^{n-1}$
- A sequence that has a limit as $n \to \infty$ is said to converge to the limit.
- A sequence that does not have a limit as $n \to \infty$ is said to diverge.
- A geometric sequence for which $-1 < r < 1$ **converges** to 0.
- A geometric sequence for which $r < -1$ or $r > 1$ **diverges**. The size of the terms increases without bound.
- The sum of $n$ terms of a **geometric series** is given by $S_n = \frac{t_1\left(r^n - 1\right)}{r-1}$ or $S_n = \frac{t_1\left(1-r^n\right)}{1-r}$, for $r \neq 1$.
- If the last term of a geometric sequence $(t_n)$ is known, then the sum is $S_n = \frac{rt_n - t_1}{r-1}$ or $S_n = \frac{t_1 - rt_n}{1-r}$
- (optional) A geometric series converges if $-1 < r < 1$. The **limiting sum (sum to infinity)** of a series with first term $a$ and common ratio $r$ is given by $S_\infty = \frac{t_1}{1-r}$.

# 8 CHAPTER REVIEW
# ARITHMETIC AND GEOMETRIC SEQUENCES

## Multiple choice

1 Example 1 What is the fifth term of the sequence defined by $t_1 = 7$, $t_n = 2 - t_{n-1}$, where $n > 1$?
A $-7$ B $-5$ C 5 D 7 E 9

2 Example 3 Which is the recursive definition for the sequence 3, 6, 9, 12, …. ?
A $t_n = 3n$ B $a_n = a_{n-1} \times 2$ C $a_n = a_{n-1} + 3$
D $a_n = a_{n-1} - 3, a_1 = 3$ E $a_n = a_{n-1} + 3, a_1 = 3$

3 Example 5 The common difference and first term for the arithmetic sequence 4, 7, 10, 13 … are:
A 4 and $-3$ B 4 and 3 C 13 and 3 D 13 and $-3$ E 7 and 4

4 Example 13 Which of the following are geometric sequences?
A 2, 4, 8, … B 1, 2, 3, … C 9, 6, 3, … D $-2, 8, -32, \ldots$ E 5, 6, 5, …

5 Example 20 How many terms are in the sequence 16, 32, 64, … , 2048?
A 4 B 7 C 8 D 1024 E 2048

## Short answer

6 Example 2 Write down the first four terms of the sequence with general term $t_n = 2n(n + 3)$.

7 Example 4 Determine which of the following are arithmetic sequences, and for each that is, write the next 4 terms.
A $-4, 4, 12, 20, \ldots$ B $x + 16, 14 - x, 12 - 3x, 10 - 5x, \ldots$ C 5, 7, 10, 14, ...

8 Example 6 Find the 30th term of the arithmetic sequence 5, 12, 19, …

9 Example 7 Which term of the arithmetic sequence 3, 8, 12, … is equal to 123?

10 Example 8 Does 215 belong to the sequence $-20, -14, -8, \ldots$ ?

11 Example 9 Write the first four terms of the arithmetic sequence whose fourth term is 25 and 25th term is 130.

12 Example 10 An arithmetic series has a first term of 3 and a last term of 53. If there were 21 terms altogether, what is the sum of the series?

13 Example 11 If $1 + 3 + 5 + \ldots + t_n = 1225$, how many terms are there in the series?

14 Example 12 The fifth term in an arithmetic series is 40 and the sum of the first 21 terms is 2100. Find the sum of the first 10 terms.

15 Example 14 Find the common ratio of the sequence $-3, 6, -12, \ldots$ and write the next four terms.

16 Example 15 Find $x$ if 9, $x$, 36 form a geometric sequence.

17 Example 16 What is the tenth term in the geometric sequence $5, -25, 125, \ldots$?

18 Example 17 Does $-384$ belong to the geometric sequence $3, -6, 12, -24, \ldots$?


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19 Example 18 The fifth and sixth terms of a geometric sequence are 24 and 96 respectively. Find $t_{13}$.

20 Example 19 Find the sum of the geometric series 7, 21, 63, ..., 15 309.

21 Example 20 If $1 + 4 + 16 + \ldots + t_n = 5461$, how many terms are there in the series?

22 Example 21 The common ratio of a geometric series is 5 and the sum of the first 5 terms is 3124. Find the seventh term.

23 Example 22 A plant grows from ground level to a height of 1.2 metres in one year. In each subsequent year the plant grows $\frac{9}{10}$ as much as the previous year. Find the limiting height of the plant.

24 Example 24 Write $0.8\overline{1}$ as a fraction.

## Application

25 The number of members of a new social network site MathBook, doubles every month. In the first month there were 14 MathBook members.

a How many members were there in the 13th month?

b During which month would the number of MathBook members reach five million?

c The site earns 0.25 cents per member per month. How much did the members earn for MathBook in the first 20 months? Answer to the nearest dollar.

26 A city skyscraper is being built which is to have a total of 120 floors. The first floor to be built will cost $2.5 million. The cost of building each subsequent floor will be $0.5 million more than the floor immediately below.

a What will be the cost of building the 50th floor?

b What will be the cost of building all 120 floors of this city skyscraper?

27 Callum has to walk from the house to the far paddock. He takes his dog with him, but when he is 36 m from the house the dog runs from Callum to the house and back again. Each time the dog gets to Callum it does it again. The dog runs 4 times as fast as Callum is walking and does this 7 times before Callum reaches the far paddock. How far from the house is the paddock?

Qz Practice quiz

# MIXED REVISION
## CHAPTERS 7 • 8 • 9

3

### Multiple choice

1 The approximate rate of change of a quantity $x$ with respect to time is given by:

A $\frac{\Delta y}{\Delta x}$ B $\frac{\Delta x}{\Delta t}$ C $\frac{\delta x}{\delta t}$ D $\frac{\delta y}{\delta x}$ E $\frac{dx}{dt}$

2 Which of the following is NOT used with APs or GPs?

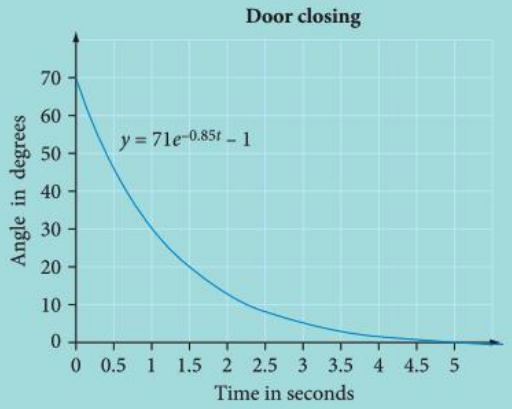
A $t_n = ar^{n-1}$ B $t_n = a(n-1)d$ C $S_n = \frac{n}{2}[a+(n-1)d]$

D $S_n = \frac{a-rl}{1-r}$ E $S_n = \frac{a(1-r^n)}{1-r}$

3 What is the value of $\lim_{x \to 2}\left(\frac{x^2-6x+5}{x-2}\right)$?

A $-3$ B $-2$ C $-1$ D 5 E 6

4 The graph below shows the angle of movement of a door with a pneumatic closer after it is released from opening through 70°. Use a tangent to find the rate of change of the angle two seconds after the door is released.




A $-125$ °/s B $-110$ °/s C $-20$ °/s D $-11$ °/s E $-1.2$ °/s

5 Calculate the 15th term of the sequence that starts $-7, -4, -1, \ldots$

A $t_{15} = 18$ B $t_{15} = 21$ C $t_{15} = 35$ D $t_{15} = 38$ E $t_{15} = 49$

6 $f(x) = 3x^4$. What is the value of $f'(-2)$?

A $-192$ B $-96$ C $-48$ D 48 E 96

7 Use $\delta = 0.02$ to find the approximate rate of change of $7x^4 - x^3 + 3x^2 - 2x - 5$ at $x = 1$

A 1.7 B 2 C 17 D 29 E 290

8 Write the rule for the $n$th term for the sequence 9, 5, 1, …

A $t_n = 9 - 4n$ B $t_n = 13 - 4n$ C $t_n = 9\left(\frac{5}{9}\right)^{n-1}$

D $t_n = 9\left(\frac{5}{9}\right)^{n}$ E $t_n = 4n + 5$

9 What is the rate of change of $y = 6x^2$ at $x = 4$?

A 12 B 24 C 48

D 96 E 192

## Short answer

1 A car travelling at 20 m/s takes 2 seconds to slow down to 15 m/s, then speeds up to 25 m/s over another 3 seconds. Find the average rate of change of the speed over the 5 seconds.

2 In a geometric sequence, $t_1 = 1000$ and $t_6 = 0.32$. Find $r$.

3 Find $n$ for $7 + 2 + -3 + -8 \ \ldots$ when $S_n = -722$.

4 Find the derivative of $f(x) = 3x^2 + 5x - 2$ from first principles.

5 Draw the graph of $f(x) = 1.5^x$ from 0 to 4 and use a tangent to find the rate of change at $x = 2$.

6 Find $n$ for the geometric sequence where, $t_1 = 3$, $r = 5$ and $t_n = 234\,375$.

7 Find the sum to 9 terms of the GP with first term 4 and common ratio 2.

8 Find the sum to 20 terms of the AP with first term 25 and common difference −3.

9 Find the equation of the tangent to $y = 0.5x^3$ at $x = 6$.

10 Find the sum to 9 terms of $4 + 12 + 36 + 108 + \ldots$.

11 Find the sum to 35 terms of $22 + 18 + 14 + 10 + \ldots$.

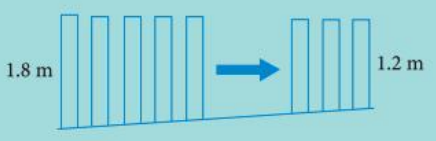
## Application

1 Find approximate rates of change of the function $f(x) = 5 - 4x^2 + 8x$ at various values of $x$ to suggest a function for the rate of change of $f(x)$.

2 a Find the approximate rates of change of the function $f(x) = (0.05)^x$ at values of $x = -2$, 0 and 2.

b By considering the ratios of the rates of change of $f(x)$ to the function values at −2, 0 and 2, suggest a function for the rate of change of $f(x)$.

3 A timber fence is to be built on sloping land, with the shortest piece of timber 1.2 m and the longest 1.8 m. There are 61 pieces of timber in the fence.




a What is the difference in height between each piece of timber?

b What length of timber is needed for the fence altogether?


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4 A sum of $2500 is invested at the beginning of each year in a superannuation fund. If compound interest is paid at 6% p.a., how much money will be available after 25 years?

5 Prove that the derivative of $e(n) = 5e^4$ is $e'(n) = 20e^3$.

6 A tram accelerates down St Kilda Road from the Federation Square stop so that its distance travelled after $t$ seconds is given by $x = 0.6t^2$ m. It maintains this distance function for the first 10 seconds of its journey.

a Find an expression for its velocity after $t$ seconds.

b How fast is it going at the end of the 10 seconds?

c How far has it travelled in that time?

d What was the acceleration 5 seconds after leaving the stop?






# 9

## TERMINOLOGY

acceleration
average rate of change
chord
derivative
differentiation
displacement
distance
first principles
gradient
instantaneous
limit
marginal
secant
tangent
velocity

Prior learning

## INTRODUCTION TO DIFFERENTIAL CALCULUS

# DERIVATIVES

9.01 Introduction to limits
9.02 The derivative
9.03 Derivatives of simple functions
9.04 The derivative of a power
9.05 Instantaneous rates of change
9.06 The slope of a curve
9.07 Applications of derivatives
Chapter summary
Chapter review

## THE CONCEPT OF THE DERIVATIVE

- examine the behaviour of the difference quotient $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$ as an informal introduction to the concept of a limit (ACMMM081)
- define the derivative $f'(x)$ as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (ACMMM082)
- use the Leibniz notation for the derivative: $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ and the correspondence $\frac{dy}{dx} = f'(x)$ where $y = f(x)$ (ACMMM083)
- interpret the derivative as the instantaneous rate of change (ACMMM084)
- interpret the derivative as the slope or gradient of a tangent line of the graph of $y = f(x)$ (ACMMM085)

## COMPUTATION OF DERIVATIVES

- estimate numerically the value of a derivative, for simple power functions (ACMMM086)
- examine examples of variable rates of change of non-linear functions (ACMMM087)
- establish the formula $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers $n$ by expanding $(x + h)^n$ or by factorising $(x + h)^n - x^n$ (ACMMM088) AC

## APPLICATIONS OF DERIVATIVES

- find the slope of a tangent and the equation of the tangent (ACMMM093)

# 9.01 INTRODUCTION TO LIMITS

The **limit** of a function $f(x)$ is the value that it gets closer and closer to as the function's independent variable $x$ gets closer and closer to a particular value, $c$, say. Getting closer and closer to is called **approaching** and means you can make the value of $f(x)$ *as close as you like* to the limiting value by making $x$ *close enough* to $c$. It is only called a limit if it is the same whether $x$ approaches $c$ from above or below. You can estimate a limit from a graph or from a table of values.



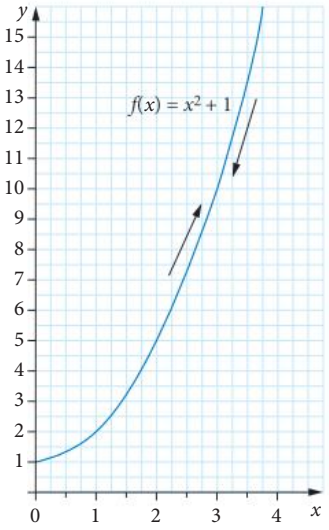
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Look at $f(x) = x^2 + 1$. From the table of values and graph below you can see that as $x$ gets close to 3 the value of the function approaches 10.

| Approaching from below | |
|---|---|
| $x$ | $f(x)$ |
| 2 | 5 |
| 2.5 | 7.25 |
| 2.9 | 9.41 |
| 2.99 | 9.9401 |
| 2.999 | 9.994 001 |
| 2.9999 | 9.999 400 01 |

| Approaching from above | |
|---|---|
| $x$ | $f(x)$ |
| 4 | 17 |
| 3.5 | 13.25 |
| 3.1 | 10.61 |
| 3.01 | 10.0601 |
| 3.001 | 10.006 001 |
| 3.0001 | 10.000 600 01 |


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As $x$ approaches 3 from the positive direction, $f(x)$ approaches 10.

As $x$ approaches 3 from the negative direction, $f(x)$ approaches 10.

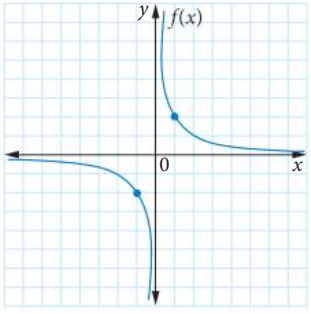
Therefore, as $x$ approaches 3 from either direction, $f(x)$ approaches 10.

You can write this as $\lim\limits_{x \to 3} (x^2 + 1) = 10$.

'Approaches' is written as an arrow, so $x \to 3$ means 'as $x$ gets close to 3', 'as $x$ tends to 3' or 'as $x$ approaches 3'.

Now look at what happens to the behaviour of $f(x) = \dfrac{1}{x}, x \neq 0$ as $x \to 0$ from each direction. If we examine the graph below, you will notice that as $x \to 0$ from the positive direction, $f(x)$ becomes large and positive (approaches infinity). Also, as $x \to 0$ from the negative direction, $f(x)$ becomes large and negative ($f(x) \to -\infty$).

There is no common value that $f(x)$ approaches as $x \to 0$, so no limit exists.




**IMPORTANT**

The **limit** of a function is the value that it approaches as the variable approaches a particular value from either direction. For a function $f(x)$ as $x$ gets close to $c$ we write the limit, if it exists, as $\lim\limits_{x \to c} f(x)$.

For the limit to exist, it must be the same whether you approach the value of $x$ from above the value or below it. In the case of $\lim\limits_{x \to 3} (x^2 + 1) = 10$, the value of the limit is the same as the value of the function at the point.

Now let's consider $f(x) = \dfrac{x^2 - 9}{x - 3}$.

If you substitute $x = 3$ to work this out you get $\frac{0}{0}$.

A denominator of zero means $f(x) = \frac{x^2 - 9}{x - 3}$ is undefined at $x = 3$.

However, you can work out a limit as $x$ gets close to 3.

## INVESTIGATION The limit of $f(x) = \frac{x^2 - 9}{x - 3}$

1 a Copy and complete the table for the function $f(x) = \frac{x^2 - 9}{x - 3}$

| $x$ | 2 | 2.5 | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x) = \frac{x^2 - 9}{x - 3}$ | | | | | | | | | | | |

b Examine the values in the table that are to the left of 3, beginning with $x = 2$. What is the value of $y$ approaching as $x$ gets closer to, or approaches, 3 from the left?

c Beginning with $x = 4$, what value is $y$ approaching as $x$ approaches 3 from the right?

d Compare the values you found in parts **b** and **c**. What do you notice?

2 a Graph $f(x) = \frac{x^2 - 9}{x - 3}$ on your CAS calculator.

b Use the trace function to move along the curve towards $x = 3$ from the left. What value does $y$ approach as $x$ approaches 3?

c Trace along the curve towards $x = 3$ from the right. What value does $y$ approach as $x$ approaches 3?

d How does the graph support your results in part **1**?

3 Does the limit of $f(x) = \frac{x^2 - 9}{x - 3}$ exist at $x = 3$?

### Example 1

Use a table of values to find $\lim_{x \to 0}\left(\frac{x^2 - 9x}{x}\right)$.

#### Solution

Write the expression as a function.

$f(x) = \frac{x^2 - 9x}{x}$

$f(x)$ is not defined at $x = 0$, so create a table of values as $x \to 0$ from both sides.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1 | −8 | −1 | −10 |
| 0.5 | −8.5 | −0.5 | −9.5 |
| 0.1 | −8.9 | −0.1 | −9.1 |
| 0.01 | −8.99 | −0.01 | −9.01 |
| 0.001 | −8.999 | −0.001 | −9.001 |

Write the result.

$\lim_{x \to 0}\left(\frac{x^2 - 9x}{x}\right) = -9$

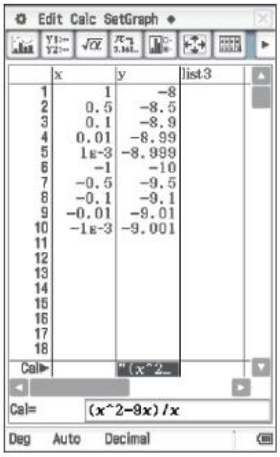
### TI-Nspire CAS

Press [on] and choose **Lists & Spreadsheet** ( ).
Enter values for $x$ in cells A1 to A5 and C1 to C5.
Select cell B1 and enter the formula '=(a1^2-9a1)/a1', then press enter.
Press [ctrl] [C] to copy the cell.
Press [shift] and ▼ to copy cells B2 to B5.
Then press [ctrl] [V] to paste.
Repeat the last three steps for cells D1 to D5.




### ClassPad

Use the **Statistics** application.
Clear the lists if necessary by tapping **Edit** at the top of the screen, tapping **Clear All** and **OK**.
Ensure that the calculator is set to **Decimal**, tapping **Standard** if necessary.
Tap list1 at the top and press [x] [EXE] to rename the column $x$. Similarly, replace list2 by $y$.
Enter values in the $x$ cells 1 to 10.
Notice that 0.001 is shown as 1E-3.
Now tap the empty cell in the $y$ column opposite Cal▶ . In the box to the right of **Cal=** enter [(] [x] [^] 2 [−] 9 [x] [)] [÷] [x]. Press [EXE] to perform the calculations.




Write the answer.

It appears that the limit is −9.

For *most* functions, if the function is defined at a value of $x$, then there is a limit and it is equal to $f(x)$.

## Example 2

If $f(x)=3(x-1)^2$, find $\lim_{x\to 2} 3(x-1)^2$

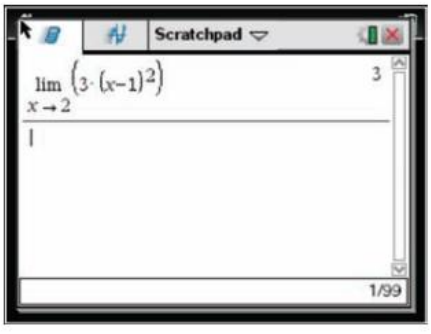
### Solution

$f(3)$ is defined, and it is clear that the values on either side will be close to the value at $x = 3$.

$\lim_{x\to 2} 3(x-1)^2 = 3\times(2-1)^2 = 3$

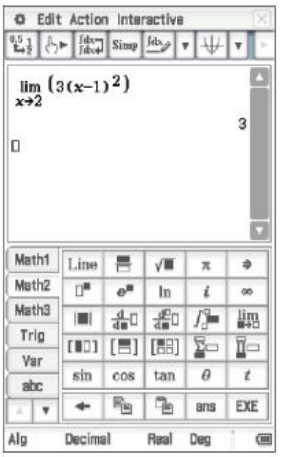
### TI-Nspire CAS

Press [on] and choose **Calculator** ( ).
Press [menu], >4: Calculus and >4: Limit.
Enter the values and function into the template as shown and press [enter].
There is an extra box to the top right of the $x \to 2$ position in the template. Leave it blank.



### ClassPad

Use the **Main** application. Press Keyboard for the soft keyboards.
Select the Math2 keyboard and tap the limit template on the middle right.
Enter the values and function into the limit template, tapping for the power.
Tap EXE to calculate the limit.




## Example 3

Find $\lim_{x \to -2}\left(\frac{2x^2+3x-2}{x+2}\right)$

### Solution

The expression is not defined at $x = -2$, so try factorisation.

$$\frac{2x^2+3x-2}{x+2} = \frac{(2x-1)(x+2)}{x+2}$$
$$= 2x-1$$

Rewrite the limit.

$$\lim_{x \to -2}\left(\frac{2x^2+3x-2}{x+2}\right) = \lim_{x \to -2}(2x-1) \text{ if } x \neq 2$$

Find the limit as $x \to 2$.

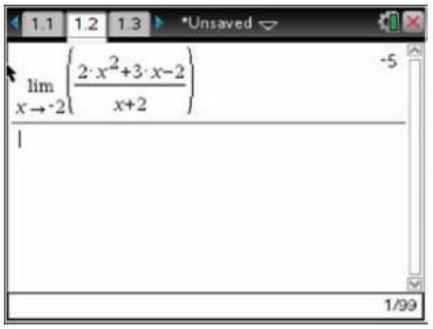
$$= -5$$

Write the result.

$$\lim_{x \to -2}\left(\frac{2x^2+3x-2}{x+2}\right) = -5$$

### TI-Nspire CAS

Press on and choose Calculator ( ).
Press menu, >4: Calculus and >4: Limit.
Enter the values and function into the limit template as shown. Press and choose $\frac{\square}{\square}$ to do the fraction. Press enter to evaluate.

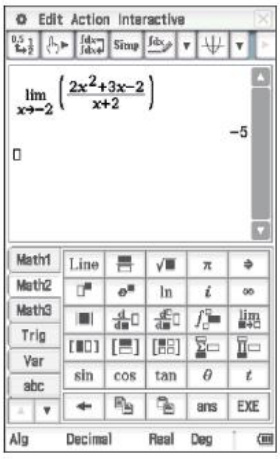




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### ClassPad

Tap the Main $\sqrt{a}$ application.
Press Keyboard for the soft keyboards.
Select the Math2 keyboard and tap the limit template $\lim_{\square}$.
Enter the values and function into the limit template as shown. Tap $\frac{\square}{\square}$ for the fraction and EXE to evaluate the expression.




In Examples **1** and **3**, *you do not substitute* the value of $x$, you only look at what happens as it gets close to the limit.

WS
Limits


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## EXERCISE 9.01 Introduction to limits

### Concepts and techniques

**1** Example 1 Use a table of values to find the following, if they exist.

**a** $\lim_{x\to 3}(2x+1)$ **b** $\lim_{x\to 2}(4-3x)$ **c** $\lim_{x\to 5}(x^2-4)$

**d** $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$ **e** $\lim_{x\to 2}\left(\frac{x^2-4}{x+2}\right)$ **f** $\lim_{x\to 3}\left(\frac{5}{x-1}\right)$

2 Example 2, 3 Evaluate the following limits, if they exist.

a $\lim_{x\to 4}(4x)$ b $\lim_{x\to -1}\left(9-x^2\right)$ c $\lim_{x\to -2}(x-3)(x+4)$

d $\lim_{x\to 3}\left(\frac{x^2-5}{x+2}\right)$ e $\lim_{x\to 0}\left(\frac{x^2+5x}{x}\right)$ f $\lim_{x\to 1}\left[\frac{(x+5)(x+2)}{x+3}\right]$

g $\lim_{x\to -2}\left(\frac{x-1}{x^2+x-2}\right)$ h $\lim_{x\to 3}\left(\frac{x^2-5x+6}{x-3}\right)$

3 CAS Find each of the following, if possible.

a $\lim_{x\to 2}\left(x^2-8x\right)$ b $\lim_{x\to 0}\left(x^2-2x-5\right)$ c $\lim_{x\to 4}2x(x+3)$

d $\lim_{x\to -3}\left[\frac{(x+3)(x+2)}{x+3}\right]$ e $\lim_{x\to -2}\left(\frac{x^3+8}{x+2}\right)$ f $\lim_{x\to 1}\left(\frac{x-1}{x^2+x-2}\right)$

g $\lim_{x\to 4}\left(\frac{x-1}{x^2-2x-3}\right)$ h $\lim_{x\to -5}\left(\frac{x-5}{2x^2-9x-5}\right)$

## Reasoning and communication

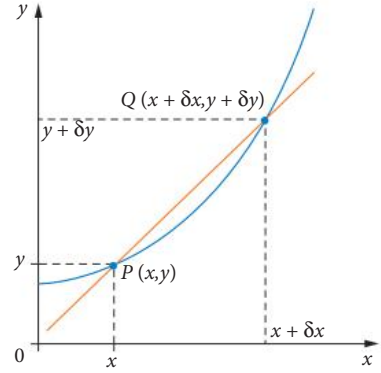
4 Given that $\lim_{x\to a}(3x+5)=2$, find the value(s) of $a$.

5 Given that $\lim_{x\to -b}\left(x^2-3x+2\right)=0$, find the value(s) of $b$.

# 9.02 THE DERIVATIVE

A non-linear curve does not have a constant gradient but you can find an approximation to the gradient by using a **secant**. A **secant** is a straight line passing through two points on the graph of a function. The line segment between the two points is called a **chord**. The closer these two points are, the closer the approximate gradient is to the exact gradient. You know that the formula $m=\frac{y_2-y_1}{x_2-x_1}$ is used to find the gradient of a straight line, given two points are $(x_1,y_1)$ and $(x_2,y_2)$. You can also write the gradient of this slope as $m=\frac{\text{change in } y}{\text{change in } x}$ or $m=\frac{\delta y}{\delta x}$.

Consider the graph of $y=f(x)$ shown below. $P$ is any point $(x, y)$ and $Q$ is another point on the same curve. The horizontal distance between $P$ and $Q$ is $\delta x$ and the vertical distance is $\delta y$. Therefore the point $Q$ is $(x+\delta x, y+\delta y)$.



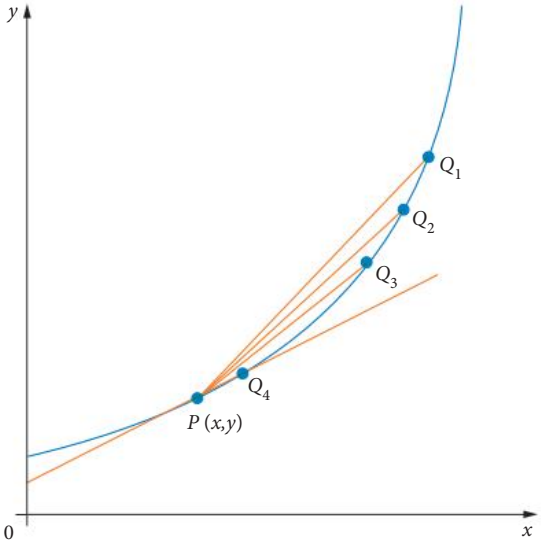


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Now you can find the gradient of the chord $PQ$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{y + \delta y - y}{x + \delta x - x}$$
$$= \frac{\delta y}{\delta x}$$

You can see from the graph below that if you bring the point $Q$ closer and closer to $P$, the distance between $P$ and $Q$ becomes smaller and smaller. Eventually the secant becomes a tangent at the point $P$. The gradient of the chord $PQ$ approximates the gradient of the tangent at $P$. The shorter the chord, the better the approximation. So the gradient of the tangent at $P$ can be written as $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

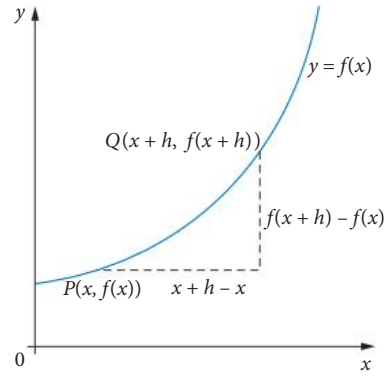



Writing $y = f(x)$, the coordinates of $P$ and $Q$ can be written as $(x, f(x))$ and $(x + \delta x, f(x + \delta x))$ respectively.

Therefore, the gradient of the secant $PQ$ can be rewritten as:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x + \delta x) - f(x)}{x + \delta x - x}$$
$$= \frac{f(x + \delta x) - f(x)}{\delta x}$$

Similarly, the gradient of the tangent at $P$ is written as $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$, provided the limit exists. It is normal practice to replace the $\delta x$ with a letter $h$ for limits of this kind to get $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. The diagram for this version of the gradient of the secant and tangent is shown below.




**IMPORTANT**

The **gradient of a function** at $P$ is the limiting value of the gradient of the secant $PQ$ as $Q$ approaches $P$. As $Q$ approaches $P$, $h$ becomes arbitrarily small as it approaches 0.

Gradient of secant $= \frac{f(x+h)-f(x)}{h}$

Gradient of tangent $= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

You have already practiced finding limits of this kind in Chapter 7 using tables, factorisation, examination of the function and CAS calculators. You also found gradients of secants in Chapter 7 (see pages 363–369).

## Example 4

Find the gradient of $f(x)=5x^2$ at the point (3, 45).

### Solution

Let $P$ be the point on the curve where $x = 3$.

$P\,(3, f(3))$ and $Q\,(3 + h, f(3 + h))$

The secant is a straight line so use $m=\frac{y_2-y_1}{x_2-x_1}$ to find its gradient.

Gradient of secant $PQ = \frac{f(3+h)-f(3)}{(3+h)-3}$

$= \frac{5(3+h)^2-5(3)^2}{h}$


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Simplify.

$$= \frac{5(9+6h+h^2)-45}{h}$$
$$= \frac{45+30h+5h^2-45}{h}$$
$$= \frac{30h+5h^2}{h}$$
$$= \frac{h(30+5h)}{h}$$
$$= 30+5h, \text{ where } h \neq 0$$

Find the limit as $h \to 0$.

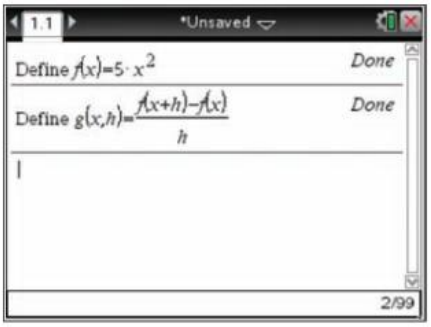
$$\text{Gradient of tangent} = \lim_{h \to 0}(30+5h)$$
$$= 30$$

Write the result.

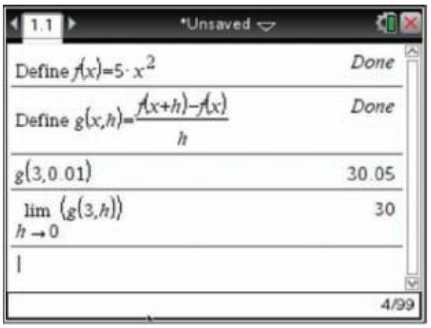
The gradient of the tangent at $x = 3$ is 30.

**TI-Nspire CAS**

Define the function $f(x) = 5x^2$ and the gradient of the secant, $g(x, h) = \frac{f(x+h)-f(x)}{h}$




You can try small values of $h$ to find the limit. You can also find the value of the gradient of the tangent at $x = 3$ using the limit function to find $\lim_{h \to 0} g(3, h)$.



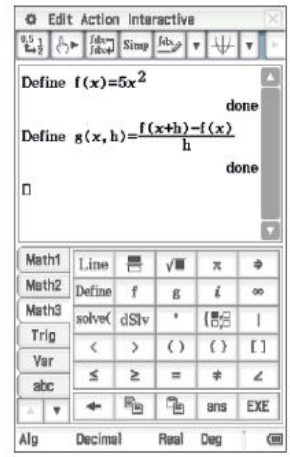

Write the answer.

The gradient of the tangent at $x = 3$ is 30.

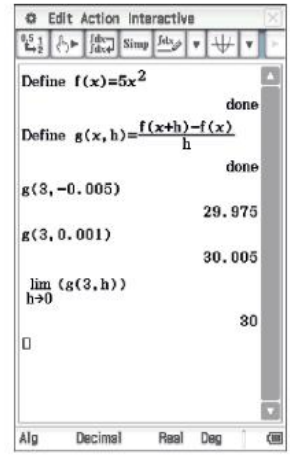


### ClassPad

Define the function $f(x) = 5x^2$ and the gradient of the secant, $g(x, h) = \dfrac{f(x+h) - f(x)}{h}$.




You can try small values of $h$ to find the limit. You can also find the value of the gradient of the tangent at $x = 3$ using the limit function to find $\lim\limits_{h \to 0} g(3, h)$.




Write the answer.

The gradient of the tangent at $x = 3$ is 30.

The gradient of a function is so important in mathematics that we give it a special name, the derivative.

You can use the method in Example **4** to find the value of a derivative.

**IMPORTANT**

The **derivative** of a function at $x$ is defined as $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided the limit exists.


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## Example 5

Find the derivative of the function $f(x)=4-3x-5x^2$ at $x=1$.

### Solution

Write the definition.
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Substitute $x=1$.
$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

Now put in $f(x)=4-3x-5x^2$.
$$=\lim_{h\to 0}\frac{4-3(1+h)-5(1+h)^2-(4-3\times 1-5\times 1^2)}{h}$$

Simplify.
$$=\lim_{h\to 0}\frac{4-3-3h-5(1+2h+h^2)-(-4)}{h}$$

Remove brackets.
$$=\lim_{h\to 0}\frac{4-3-3h-5-10h-5h^2+4}{h}$$

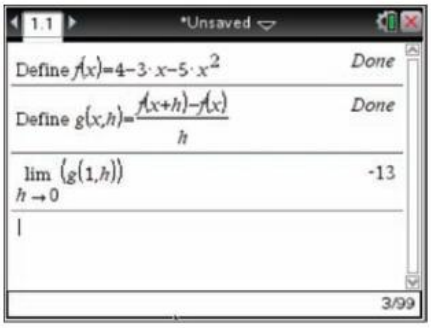
Simplify the numerator.
$$=\lim_{h\to 0}\frac{-13h-5h^2}{h}$$

Factorise and cancel.
$$=\lim_{h\to 0}\frac{h(-13-5h)}{h}$$
$$=\lim_{h\to 0}(-13-5h)$$

Find the limit.
$$=-13$$

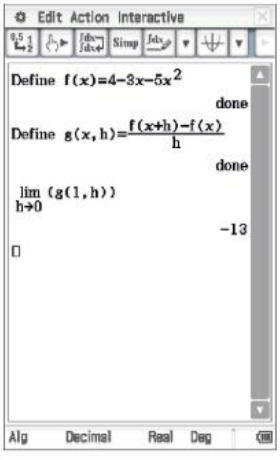
#### TI-Nspire CAS

Define the function $f(x)=4-3x-5x^2$, the gradient of the secant, $g(x,h)=\dfrac{f(x+h)-f(x)}{h}$, and find $\lim_{h\to 0} g(1,h)$.




#### Class Pad

Define the function $f(x)=4-3x-5x^2$, the gradient of the secant, $g(x,h)=\dfrac{f(x+h)-f(x)}{h}$, and find $\lim_{h\to 0} g(1,h)$.




Write the answer.

The derivative at $x=1$, $f'(1)=-13$

## EXERCISE 9.02 The derivative

### Concepts and techniques

1 Example 4 Find the gradients of the following.

a $f(x)=x^2$ at the point (2, 4) using limits and factorisation.

b $f(x)=4x^3$ at the point (1, 4) using limits and factorisation.

c CAS $f(x)=\frac{1}{x^2}$ at the point $\left(-3, \frac{1}{9}\right)$.

d CAS $f(x)=x^4+3x^3-2x+5$ at the point (0, 5).

2 For the function $f(x)=x^3+x$, find the gradient at $x = 2$.

3 For the function $f(x)=5-7x^2$, find the gradient at $x = -3$.

4 a CAS Find the gradient of the secant between the point (1, 2) and the point where $x = 1.01$, on the curve $y=x^4+1$.

b Find the gradient of the secant between the point (1, 2) and the point where $x=0.999$, on the curve $y=x^4+1$. Hint: use a negative value of $h$.

c Find the derivative of $y=x^4+1$ at the point (1, 2).

5 Example 5 Given $f(x)=7+9x-3x^2$, use limits and factorisation to find $f'(4)$.

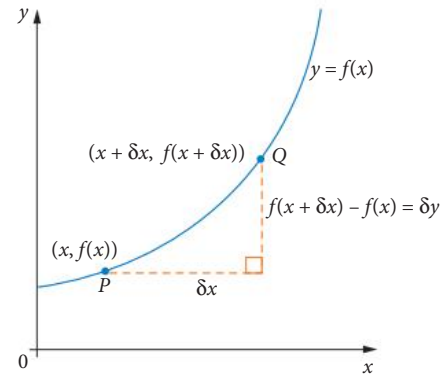
6 CAS Given $f(x)=9-x^{-1}$, $f'(-2)$.

7 For $f(x)=10x^2+3x+1$, find $f'(2)$.

8 Find the derivative of the curve $f(x)=\frac{1}{x}$ at $x = 3$, without using your calculator.

# 9.03 DERIVATIVES OF SIMPLE FUNCTIONS

The process of finding the derivative function or the derivative at a particular point by evaluating the limit is called **differentiation from first principles**. It gives the instantaneous rate of change of $y$ with respect to $x$.




Because we often make $f(x)=y$ when graphing functions, it is also common to write the derivative in terms of $x$ and $y$. Using the diagram above and the definition of the derivative stated previously, we can see that:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$y'=\lim_{h\to 0}\frac{\delta y}{\delta x}$$

(Note: $\delta x$ represents a small change in $x$ and $\delta y$ is the consequent change in $y$.)


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**IMPORTANT**

The derivative of a function written in the form $y = f(x)$ is often written as $\frac{dy}{dx}$.

Although $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$, $\frac{dy}{dx}$ does not mean a division. It is just another way of writing $f'(x)$.

## Example 6

**a** Find the gradient of a secant of $f(x) = 2 - x^2$ between any value of $x$ and the value $x + h$.

**b** Hence find $f'(x)$.

### Solution

**a** Write down the function. $f(x) = 2 - x^2$

Find $f(x+h)$ by replacing $x$ with $x + h$. $f(x+h) = 2 - (x+h)^2$

Expand the brackets. $= 2 - \left(x^2 + 2xh + h^2\right)$

Simplify. $= 2 - x^2 - 2xh - h^2$

Write the secant gradient formula. Gradient of secant $= \frac{f(x+h) - f(x)}{h}$

Substitute $f(x+h)$ and $f(x)$. $= \frac{\left(2 - x^2 - 2xh - h^2\right) - \left(2 - x^2\right)}{h}$

Simplify. $= \frac{-2xh - h^2}{h}$

Factorise. $= \frac{h(-2x - h)}{h}$

Cancel $h$. $= -2x - h$, provided $h \neq 0$

**b** Write down the formula for the derivative. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Substitute the gradient of the secant. $= \lim_{h \to 0} (-2x - h)$

Find the limit. $= -2x$

Of course, you can use your CAS calculator to find a derivative using limits in the same way as you did in Section 9.02.

## Example 7

**a** For $f(x)=x^2-2x$, find $f'(x)$ by using first principles.
**b** Hence find the gradient at $x = 2$.

### Solution

**a** Write down the function. $f(x)=x^2-2x$

Find $f(x+h)$ by replacing $x$ with $x + h$. $f(x+h)=(x+h)^2-2(x+h)$

Expand brackets. $=x^2+2xh+h^2-2x-2h$

Write the derivative formula. $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

Substitute $f(x + h)$ and $f(x)$. $=\lim_{h\to 0}\dfrac{\left(x^2+2xh+h^2-2x-2h\right)-\left(x^2-2x\right)}{h}$

Simplify and factorise. $=\lim_{h\to 0}\dfrac{h(2x+h-2)}{h}$

Cancel. $=\lim_{h\to 0}(2x+h-2)$

Find the limit. $=2x-2$

**b** Substitute $x = 2$ into $f'(x)$. $f'(2)=2(2)-2=2$

## Example 8

WS
Finding derivatives from first principles

Show that $\dfrac{d\left(4x^3\right)}{dx}=12x^2$.

### Solution

Write down the function. $f(x)=4x^3$

Find $f(x+h)$ by replacing $x$ with $x + h$. $f(x+h)=4(x+h)^3$

Expand the brackets. $=4\left(x^3+3x^2h+3xh^2+h^3\right)$

Simplify. $=4x^3+12x^2h+12xh^2+4h^3$


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| | |
|---|---|
| Write the gradient formula. | $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ |
| Substitute $f(x+h)$ and $f(x)$. | $=\lim\limits_{h\to 0}\dfrac{\left(4x^3+12x^2h+12xh^2+4h^3\right)-\left(4x^3\right)}{h}$ |
| Simplify. | $=\lim\limits_{h\to 0}\dfrac{12x^2h+12xh^2+4h^3}{h}$ |
| Factorise. | $=\lim\limits_{h\to 0}\dfrac{h\left(12x^2+12xh+4h^2\right)}{h}$ |
| Cancel. | $=\lim\limits_{h\to 0}\left(12x^2+12xh+4h^2\right)$ |
| Find the limit. | $f'(x)=12x^2$ |

# EXERCISE 9.03 Derivatives of simple functions

## Concepts and techniques

**1** Example 6 For each of the following, find the gradient of a secant at any point.

**a** $f(x)=3x-1$ **b** $f(x)=5x^2-2x+4$ **c** $f(x)=x^3-1$

**2** **a** Find the gradient of the secant of $f(x)=x^2-x-6$.

**b** Hence find $f'(x)$.

**3** **a** Find the gradient of the secant of $f(x)=x^2+4x-12$.

**b** Hence find $f'(x)$.

**4** Example 7 For each of the following, find the derivative at any point $x$ from first principles.

**a** $f(x)=4-3x$ **b** $f(x)=3x^2+2x-1$

**c** $f(x)=7x-x^2$ **d** $f(x)=3x^2-2x^3$

**5** **a** For $f(x)=x^2-2x$, find $f'(x)$ by first principles.

**b** Hence, find the gradient at $x=2$.

**6** **a** For $f(x)=3x^2-x+5$, find $f'(x)$ by first principles.

**b** Hence, find the gradient at $x=-3$.

## Reasoning and communication

**7** Example 8 Use first principles to show that $\dfrac{d\left(8cx^2\right)}{dx}=16cx$ where $c$ is a constant.

**8** Show that $\dfrac{d\left(5p\right)}{dp}=5$ from first principles.

**9** Use first principles to show that $\dfrac{d\left(k\right)}{dk}=0$ for any constant $k$.

**10** Use first principles to show that $\dfrac{d\left(11m^3\right)}{dm}=33m^2$.



# 9.04 THE DERIVATIVE OF A POWER

From the work in the previous section, you may have seen a pattern emerge for the derivatives of simple powers. You will need to recall the binomial theorem from Chapter 2 for this work.

The process of finding the derivative is called **differentiating**.

## Example 9

Differentiate $f(x) = x^4$ from first principles.

### Solution

Work out $f(x + h)$.

$$f(x + h) = (x + h)^4$$

Use the binomial theorem.

$$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

Write the derivative formula.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute the values.

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

Simplify.

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

Factorise the numerator.

$$= \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$$

Since $h \neq 0$, you can cancel.

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3)$$

Evaluate the limit, using the fact that as $h \to 0$, $6x^2h + 4xh^2 + h^3 \to 0$.

$$= 4x^3$$

Write the answer.

The derivative of $f(x) = x^4$ is $f'(x) = 4x^3$.

From the binomial theorem, you can work out the following expansions.

$(x + h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + \cdots$

$(x + h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + \cdots$

$(x + h)^7 = x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + \cdots$

and so on.

The expansions of $(x + h)^n$ always begin with $x^n + nx^{n-1}h + Ax^{n-2}h^2 + Bx^{n-3}h^3 + \cdots$,

where $A = \binom{n}{2} = \frac{n(n-1)}{1 \times 2}$ and $B = \binom{n}{3} = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}$.


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You can use this to find the derivative of $x^n$ for any positive integer power of $x$.

| | |
|---|---|
| Write the function. | $f(x) = x^n$ |
| Work out $f(x + h)$. | $f(x + h) = (x + h)^n$ |
| Use the binomial theorem. | $= x^n + nx^{n-1}h + Ax^{n-2}h^2 + Bx^{n-3}h^3 + \cdots$ |
| Write the derivative formula. | $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ |
| Substitute the values. | $= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + Ax^{n-2}h^2 + Bx^{n-3}h^3 + \cdots) - x^n}{h}$ |
| Simplify. | $= \lim_{h \to 0} \frac{nx^{n-1}h + Ax^{n-2}h^2 + Bx^{n-3}h^3 + \cdots}{h}$ |
| Factorise the numerator. | $= \lim_{h \to 0} \frac{h(nx^{n-1} + Ax^{n-2}h + Bx^{n-3}h^2 + \cdots)}{h}$ |
| Since $h \neq 0$, you can cancel. | $= \lim_{h \to 0} (nx^{n-1} + Ax^{n-2}h + Bx^{n-3}h^2 + \cdots)$ |
| | $= \lim_{h \to 0} [nx^{n-1} + h(Ax^{n-2} + Bx^{n-3}h + \cdots)]$ |
| Evaluate the limit, using the fact that as $h \to 0$, $h(Ax^{n-2} + Bx^{n-3}h + \cdots) \to 0$. | $= nx^{n-1}$ |

**IMPORTANT**

The derivative of $f(x) = x^n$ is given by $f'(x) = nx^{n-1}$

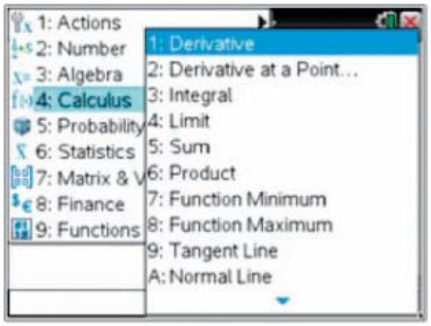
## Example 10

Differentiate $f(x) = x^5$.

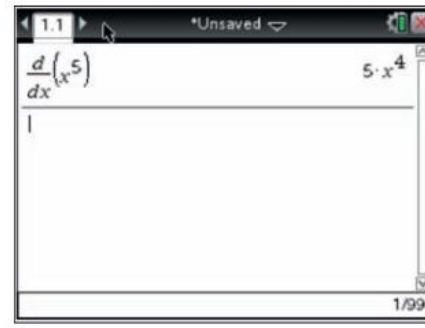
| | |
|---|---|
| Write the function. | $f(x) = x^5$ |
| Use the rule for the derivative of $x^n$ for $n = 5$. | $f'(x) = nx^{n-1}$ |
| | $= 5x^{5-1}$ |
| Simplify. | $f'(x) = 5x^4$ |

**TI-Nspire CAS**

You can also use your CAS calculator.
Press menu, press 4: Calculus and 1: Derivative.

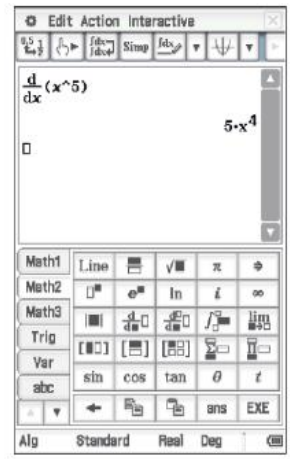


Fill in the boxes and press enter.




### ClassPad

Tap Main $\sqrt{\alpha}$ for the **Main** application. Press Keyboard and select Math2.
Tap the derivative template $\frac{d}{dx}$.
Enter the values and function into the template as shown.
Tap EXE or press EXE to evaluate.
(Note: if $x$ has been assigned values, it is not a variable and the calculation will not give the expected result.)




What happens if the power has a coefficient, like $7x^3$?

## Example 11

Find the derivative of $y = 7x^3$ from first principles.

Write the function. $f(x) = 7x^3$

Work out $f(x + h)$. $f(x + h) = (x + h)^3$

Use the binomial theorem. $= 7(x^3 + 3x^2h + 3xh^2 + h^3)$

Simplify. $= 7x^3 + 21x^2h + 21xh^2 + 7h^3$

Write the derivative formula. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Substitute the values. $= \lim_{h \to 0} \frac{7x^3 + 21x^2h + 21xh^2 + 7h^3 - 7x^3}{h}$

Simplify. $= \lim_{h \to 0} \frac{21x^2h + 21xh^2 + 7h^3}{h}$

Factorise the numerator. $= \lim_{h \to 0} \frac{7h(3x^2 + 3xh + h)}{h}$


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| | |
|---|---|
| Since $h \neq 0$, you can cancel. | $= \lim_{h \to 0} 7(3x^2 + 3xh + h)$ |
| Evaluate the limit, using the fact that as $h \to 0$, $3xh + h \to 0$. | $= 21x^2$ |
| Write the answer, with the same notation as the question. | The derivative of $y = 7x^3$ is $\frac{dy}{dx} = 21x^2$. |

## EXERCISE 9.04 The derivative of a power

### Concepts and techniques

**1** Example 9 Find the derivatives of each of the following from first principles.

a $f(x) = x^8$ b $f(x) = x^6$ c $f(x) = x^9$

d $f(x) = x^7$ e $f(x) = x^3$

**2** Example 10 Differentiate each of the following.

a $y = x^{20}$ b $f(x) = x^{12}$ c $f(x) = x^{25}$

d $y = x^{17}$ e $y = x^{22}$

**3** CAS Differentiate each of the following.

a $f(x) = x^7$ b $y = x^{18}$ c $f(x) = 4x^8$

d $y = 6x^{11}$ e $f(x) = 9x^{16}$

**4** Example 11 Find the derivatives of each of the following from first principles.

a $y = 6x^4$ b $f(x) = 9x^3$ c $y = 11x^5$

d $f(x) = 6x^5$ e $y = 2x^6$

### Reasoning and communication

**5** Prove that the derivative of $t^n$ is $nt^{n-1}$.

**6** Prove that the derivative of $p^n$ is $np^{n-1}$.

**7** Prove that the derivative of $kx^n$ is $knx^{n-1}$.

**8** Prove that the derivative of $at^n$ is $ant^{n-1}$.



# 9.05 INSTANTANEOUS RATES OF CHANGE

In the last questions of Exercise 9.04 you proved the following.

**IMPORTANT**

The derivative of $y = kx^n$ is given by $\frac{dy}{dx} = knx^{n-1}$.

In Chapter 7 you looked at the instantaneous rate of change of a function. In that chapter, you used the slope of a tangent drawn on a graph and numerical methods of approximation to find the rate of change.

The instantaneous rate of change of the function $y = f(x)$ is the limit of the average rate of change, given by $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.



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Writing $x_1 = x$, $x_2 = x + h$, $y_1 = f(x)$ and $y_2 = f(x + h)$ gives $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$.

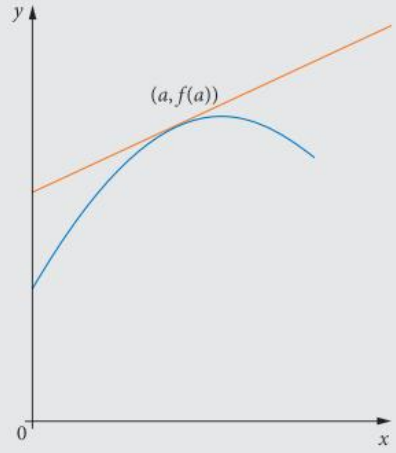
But the limit of this as $h \to 0$ is the derivative.

**IMPORTANT**

The **(instantaneous) rate of change** of a function at any point is the gradient of the tangent at that point, given by the derivative.

The gradient of the tangent is called **gradient of the graph**. At the point $a$, the rate of change of the function is the slope of the tangent at the point $(a, f(a))$, which is the derivative.

The derivative of a function at the point where $x = a$ is written as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.





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## Example 12

Find the instantaneous rate of change of the following at the value indicated.

**a** $f(x)=x^3$ at $(-2, -8)$

**b** $y=5x^4$ at $x=2$

**c** $f(x)=-6x$ at $x=21$

### Solution

**a** Write the function. $f(x)=x^3$

Find the derivative. $f'(x)=3x^2$

Substitute $x=-2$. $f'(-2)=3\times(-2)^2$

Simplify. $=12$

**b** Write the function. $y=5x^4$

Find the derivative. $\frac{dy}{dx}=20x^3$

Substitute $x=2$. $\left.\frac{dy}{dx}\right|_{x=2}=20\times 2^3$

Simplify. $=160$

**c** Write the function. $f(x)=-6x=-6x^1$

Find the derivative. $f'(x)=-6\times 1x^{1-1}=-6x^0=-6\times 1=-6$

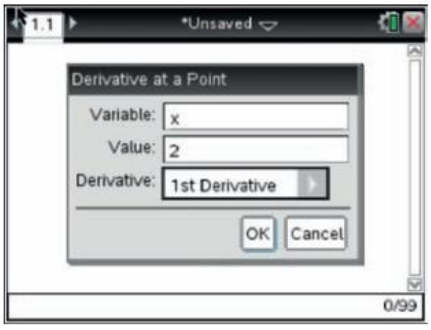
Substitute $x=21$. $f'(21)=-6$

#### TI-Nspire CAS

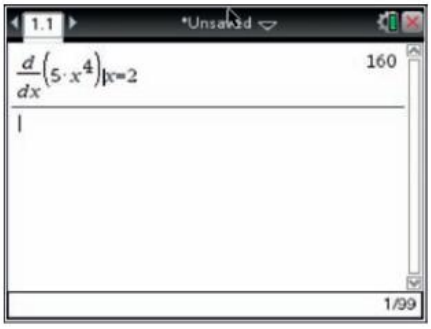
Use a calculator page.

Press menu, >4: Calculus and >2: Derivative at a point.

Enter the variable $x$ and the value 2 as shown and press enter.

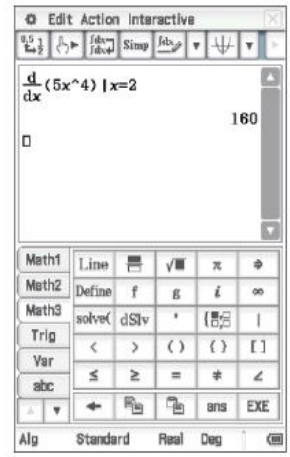



Enter the function into the template and press enter.



### CASIO ClassPad

Use the **Main** application [√α]. Press [Keyboard] and select [Math2].
Tap the derivative template [d/d□].
Enter the variable and function into the template as shown.
Tap [Math3] then [|]. Enter [x] [=] 2.
Tap [EXE] or press [EXE].




Notice in Example **12c** that the derivative is a constant. It is the derivative of a linear function. It is the gradient of the function. A constant would have a derivative of 0 as the gradient of $y = c$ is zero.

You already know that if $y$ is a function of $x$, the derivative of $y$ with respect to $x$ is written as $\frac{dy}{dx}$.

You can use other symbols for functions of course, meaning that you would use other symbols for the derivatives. For example, if $P$ is a function of $t$, you would write the function as $P(t)$ and its derivative as $P'(t)$ or $\frac{dP}{dt}$.

An object's position at any time is defined by $x$. Since its position depends on time, we often write it as $x(t)$, a function of time.

When something is moving in a straight line, you call its change of position from the starting position its **displacement**. This may be a positive or negative number depending on whether it is moving forwards or backwards (right or left).

You should also remember that the rate of change of displacement with respect to time is called **velocity** and the rate of change of velocity with respect to time is called **acceleration**.

**IMPORTANT**

**Distance** is how far an object or particle has travelled, regardless of direction.

**Displacement**, $\Delta x$, is the change in position, $x(t)$, of the object or particle.

**Velocity**, $v(t)$, is the rate of change of position with respect to time, so $v(t) = \frac{dx}{dt}$.

**Acceleration**, $a(t)$, is the rate of change of velocity with respect to time, so $a(t) = \frac{dv}{dt}$.


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## Example 13

The position of an object falling from the top of a cliff is given by $x = -5t^2$ m, where $t$ is in seconds.

Find the following.

**a** Its position after 2 seconds.

**b** Its velocity after 0.5 seconds.

**c** Its acceleration after 1 second.



Getty Images/Oliver Furrer

### Solution

**a** Substitute $t = 2$. $x(2) = -5 \times 2^2 = -20$

Write the answer. The object has fallen 20 m, so is 20 m below the top of the cliff.

**b** Find the derivative. $\frac{dx}{dt} = -10t$

Substitute $t = 0.5$. $\left.\frac{dx}{dt}\right|_{t=0.5} = -10 \times 0.5 = -5$

Write the answer. The object is going downwards at 5 m/s.

**c** Write the function. $v = -10t$

Find the derivative. $\frac{dv}{dt} = -10$

Write the answer. The object is accelerating downwards at 10 m/s$^2$.

Notice in Example **13** that the negative values of displacement and velocity indicate that the object is travelling downwards. Also, it should not be surprising that objects near the Earth accelerate downwards under gravity at 10 m/s$^2$.

## Example 14

The position of an electric train in the first minute as it moves out of the station is given by $x = \frac{1}{6}t^2$, where $t$ is in seconds and $x$ is in metres.

After the first minute, the train moves at constant speed until it approaches the next stop. Find each of the following.

**a** Its speed after 15 seconds.
**b** Its acceleration after 15 seconds.
**c** Its velocity in both m/s and km/h at the end of the first minute.
**d** Draw graphs of its velocity and acceleration for the first minute.



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### Solution

Write the distance function.

$x = \frac{1}{6}t^2$

**a** Find the derivative.

$v = \frac{dx}{dt} = \frac{1}{3}t$

Substitute $t = 15$

$\left.\frac{dx}{dt}\right|_{t=15} = \frac{1}{3} \times 15 = 5$

Write the answer.

After 15 seconds, it is moving at 5 m/s.

**b** Find the derivative of its velocity.

$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{3}t\right) = \frac{1}{3}$

Write the answer.

Its acceleration after 15 seconds is $\frac{1}{3}$ m/s$^2$.

**c** Substitute $t = 60$ in the velocity.

$$\begin{aligned}\frac{1}{3}t &= \frac{1}{3} \times 60 \\ &= 20 \text{ m/s} \\ &= \frac{20}{1000} \text{ km/s} \\ &= \frac{20 \times 60}{1000} \text{ km/min} \\ &= \frac{20 \times 60 \times 60}{1000} \text{ km/h} \\ &= 72 \text{ km/h}\end{aligned}$$

Write the answer.

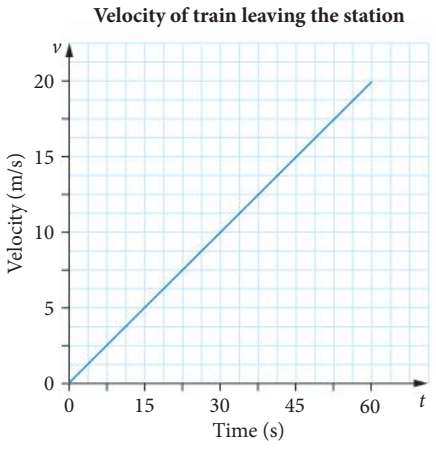
The train is travelling at 20 m/s = 72 km/h after the first minute.

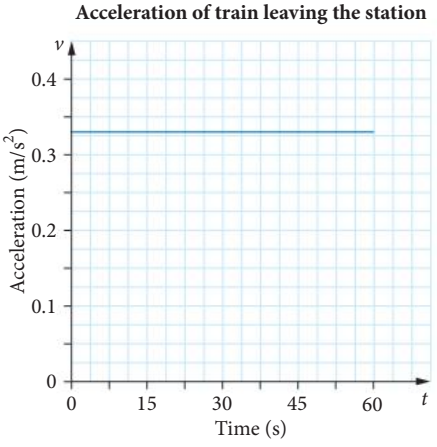
**d** Describe the graphs.

The velocity function is $v = \frac{1}{3}t$, which is a straight line through the origin, and the acceleration function is $a = \frac{1}{3}$, which is a flat line.


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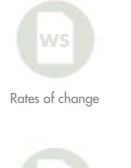
Draw the graphs.




# EXERCISE 9.05 Instantaneous rates of change

## Concepts and techniques









1 Example 12 Find the instantaneous rates of change of the following at the values indicated.
   a $f(x)=x^4$ at $x=3$ and $x=-1$
   b $y=2x^3$ at $x=-4$ and $x=5$
   c $f(x)=-3x^3$ at $x=2$ and $x=-3$
   d $y=6x^2$ at $x=5$ and $x=-5$
   e $f(x)=2x^2$ at $x=4$ and $x=-4$

2 Examples 13, 14 Find the instantaneous velocities of the following position functions at the times indicated.
   a $x=-3.1t^4$ at $t=-2$ and $t=-3$
   b $x(t)=4.2t^3$ at $t=4$ and $t=5$
   c $x=-t^5$ at $t=2$ and $t=3$
   d $x(t)=0.25t^6$ at $t=-2$ and $t=2$
   e $x=0.3t^4$ at $t=1$ and $t=-10$

3 CAS Find the accelerations of the objects with the following velocities at the times indicated.

a $v = -4.5t^6$ at $t = 5$ and $t = -5$

b $v = 2.4t^5$ at $t = 3$ and $t = -4$

c $v = -t^4$ at $t = -3$ and $t = 5$

d $v = 0.25t^3$ at $t = 4$ and $t = -5$

e $v = 0.3t^5$ at $t = 3$ and $t = 4$

## Reasoning and communication

4 A rocket blasting off is moving so that its height above the launching pad is given by $h = 24t^2$ m at $t$ seconds after the launch. Find each of the following.

a Its height after 10 seconds.

b Its velocity after 15 seconds.

c Its acceleration just before the first stage is dropped at 35 seconds.

d Its speed when the first stage is dropped.

e Its height when the first stage is dropped.



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5 A large rock is dropped into some deep water off the coast. Its depth in metres $t$ seconds after it breaks the surface is given by $d = 3.8t^2$. Find each of the following.

a Its velocity after 2 seconds.

b Its acceleration after 2 seconds.

c Its velocity after 3 seconds.

d Its acceleration after 3 seconds.

e The time it takes to reach the bottom at a depth of 120 m.

6 A rocket car is designed to break the land-speed record and will be tested on some flat salt lakes. It has two different burn rates. For the high burn rate the distance travelled is given by $d = 9.5t^2$ m, but it will only have enough fuel for 30 seconds. For the lower rate the distance covered is $d = 6t^2$, and the fuel will then last for 50 seconds.

a Find an expression for the velocity at the 30 second burn rate.

b What is the highest speed the rocket car could reach at this burn rate?

c What is the acceleration at this burn rate?

d An acceleration of 45 m/s$^2$ is written as $4.5g$ because it is 4.5 times the acceleration of gravity (10 m/s$^2$). What is the acceleration in 'g's at the 30 second burn rate?

e Find an expression for the velocity at the 50 second burn rate.

f What is the acceleration at the lower burn rate?

g What is the highest speed that could be reached at the lower burn rate?

h Draw a graph on the same axes of the velocities at the different burn rates.

i Draw a graph on the same axes of the accelerations at the different burn rates.

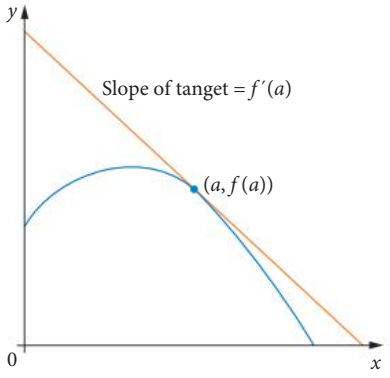


AAP Images/AP/Douglas C. Pizac

# 9.06 THE SLOPE OF A CURVE

The value of the derivative of a function at a given point is the slope of the tangent at that point. You can use this to find the equation of a tangent to a curve at a particular point. The sketch on the right shows how the tangent and the curve are related. Remember that the tangent touches the curve in only one place.

The slope of the tangent to a curve $f(x)$ when $x = a$ is $f'(a)$. Both the tangent and the curve pass through the point $(a, f(a))$.




## Example 15

Find the equation of the tangent to $f(x) = 0.5x^3$ at $x = 2$.

### Solution

| | |
|---|---|
| Substitute $x = 2$ to find the point. | $f(2) = 0.5 \times 2^3 = 4$ |
| State the point. | The tangent passes through (2, 4). |
| Find the derivative. | $f'(x) = 1.5x^2$ |
| Substitute $x = 2$ to find the gradient. | The tangent has gradient $m = f'(2) = 6$ |
| Write the equation of a straight line in point slope form. | $y - y_1 = m(x - x_1)$ |
| Substitute values. | $y - 4 = 6(x - 2)$ |
| Simplify. | $y - 4 = 6x - 12$<br>$y = 6x - 8$ |
| Express in standard form. | The tangent to $f(x) = 0.5x^3$ at $x = 2$ is $6x - y - 8 = 0$. |

## Example 16

Find the point(s) on the graph $y = x^3$ where the slope of the graph is 12.

### Solution

| | |
|---|---|
| Find the slope at $x$, the derivative. | $\frac{dy}{dx} = 3x^2$ |
| Substitute $\frac{dy}{dx} = 12$. | $12 = 3x^2$ |
| Solve the equation. | $x^2 = 4$<br>$x = 2$ or $x = -2$ |

| | |
|---|---|
| Substitute into the equation to find the points. | At $x = 2$, $y = 2^3 = 8$<br>At $x = -2$, $y = (-2)^3 = -8$ |
| State the points. | The graph of $y = x^3$ has a slope of 12 at the points (2, 8) and (–2, –8). |

## EXERCISE 9.06 The slope of a curve

WS Slopes of curves

### Concepts and techniques

1 Example 15 Find the equations of the tangents to these curves at the points stated.

a $f(x) = 4x^2$ at $x = 1$ and $x = -2$
b $y = x^3$ at $x = -1$ and $x = 2$
c $f(x) = 0.1x^5$ at $x = 2$ and $x = -4$
d $y = 3x^2$ at $x = -3$ and $x = 2$
e $f(x) = 0.25x^4$ at $x = -1$ and $x = 2$

2 Example 16 Find the point(s) on the curve $y = 3x^2$ at which the tangent has a gradient of 18.

3 Find the point(s) on the curve $y = 2x^3$ at which the tangent has a gradient of 24.

4 Find the point(s) on the curve $y = x^4$ at which the tangent has a gradient of –32.

5 Find the point(s) on the curve $f(x) = \frac{5}{6}x^3$ at which the tangent has a gradient of 40.

### Reasoning and communication

6 Find the point(s) on the graph of $y = x^3$ where it has the same slope as the slope of $y = x^2$ at the point (3, 9).

7 Find the value of $c$ such that $y = cx^2$ has a slope of 8 at $x = 5$.

8 Find any points on the graph of $y = x^3$ that have the same gradient as $y = 12x + 5$.

# 9.07 APPLICATIONS OF DERIVATIVES

You can use derivatives to solve other kinds of problems involving rates of change.

### Example 17

The volume of air in a balloon (in litres) after $t$ seconds of pumping is given by $V = 0.1t^2$. What is the rate of change of the volume after:

a 5 seconds?
b 15 seconds?



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| | | |
|---|---|---|
| | Write down the function. | $V(t) = 0.1t^2$ |
| | Differentiate. | $V'(t) = 0.2t$ |
| a | Substitute $t = 5$ into the derivative to find the instantaneous rate of change of the volume. | $V'(5) = 0.2 \times 5 = 1$ |
| | Write the answer. | The rate of change of volume after 5 seconds is 1 L/s. |
| b | Substitute $t = 15$ into the derivative. | $V'(15) = 0.2 \times 15 = 3$ |
| | Write the answer. | The rate of change of volume after 15 seconds is 3 L/s. |

Miners, farmers and manufacturers dig up, grow and make goods for sale. The costs for producing goods are not independent of the quantity produced. The first goods produced generally cost more per item than later ones because of start-up costs. Even before anything is produced, setting up a mine, ploughing fields and setting up a production line all cost a lot of money. As more is produced, the cost per item falls.



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In contrast, there comes a point where it becomes more expensive to produce more. Trucks and miners start to get in each other's way, farmers have to start using less productive land and more fertiliser, and machines in factories break down more as they are worked harder.

In Economics, the **marginal** cost, return and profit means the cost, return and profit on the last item (or tonne) produced. The cost, return or profit for 1 item is called the **unit** cost, return or profit. Economists use the value of the derivative of the cost, return or profit function to calculate marginal rates.

If the cost function for making $x$ tents was $C(x) = 0.15x^2$ for more than $x = 100$ tents, then he marginal cost would be $C'(x) = 0.3x$, so the cost of making the 102nd tent would be \$30.60.

If the profit function was $P(x) = 350x - 25\,000 - x^2$ for $x$ tents then no profit would be made for 250 tents, and the marginal profit for the 250th tent would actually \$−150, a loss of \$150.

## Example 18

The cost in dollars to make $n$ picnic tables is given by $C(n)=\frac{n^2}{11}$ for quantities over 500 tables. The tables are sold for $120 each.

a Find the marginal cost of quantities of over 500 tables.
b Find the cost of producing the 550th table.
c Find the cost of producing the 600th table.
d Find the cost of producing the 700th table.
e At what quantity is it no longer profitable to produce more tables?

### Solution

| | |
|---|---|
| a The marginal cost is the derivative. | Marginal cost $= C'(n)$ |
| Differentiate $C(n)$. | $= \frac{2n}{11}$ |
| b Substitute $n = 550$ | Cost of 550th $= \frac{2\times550}{11} = 100$ |
| Write the answer. | It costs $100 to produce the 550th table. |
| c Substitute $n = 600$ | Cost of 600th $= \frac{2\times600}{11} \approx 109.09$ |
| Write the answer. | It costs about $109 to produce the 600th table. |
| d Substitute $n = 700$ | Cost of 700th $= \frac{2\times700}{11} = 127.27$ |
| Write the answer. | It costs about $127 to produce the 700th table. |
| e When the cost to produce one more table is the same as the price, no profit is made. | $\frac{2n}{11} = 120$ |
| Solve the equation. | $n = 660$ |
| State the meaning. | The 660th table produced will make zero profit. |
| Write the answer. | Making more than 659 tables will not be profitable. |

# EXERCISE 9.07 Applications of derivatives

## Reasoning and communication



Alamy/Yogesh More

1 Example 17 An ant-lion makes a hole in the sand in the shape of an inverted cone and sits in the bottom of the hole. The slope of the sides is 45°. When an ant falls into the trap, the ant-lion gets its reward but has to redig the hole.
   a Write an expression for the volume of the hole in terms of its depth $h$.
   b What is the rate of change of volume at depth $h$?
   c If the hole is already 1.5 cm deep, approximately how much more sand must the ant-lion move to make it 1.7 cm deep?

2 Example 18 The cost of producing more than 4999 bottles in a factory is given by $C(n) = 0.0007n^2$, where $C$ is in cents and $n$ is the number of bottles produced in a day.
   a Find an expression for the marginal cost of producing bottles.
   b What is the marginal cost of producing the 5000th bottle in a day?
   c What is the marginal cost of producing the 6000th bottle in a day?
   d What is the marginal cost of producing the 7000th bottle in a day?
   e The bottles are sold to drink bottlers for 9 cents each. What is the largest number for which it is profitable to make more in a day?

3 After starting up, the profit made by a factory for producing and selling $n$ items increases according to the function $P(n) = 0.002n^3$, where $P$ is in dollars and $100 < n < 700$.
   a Find an expression for the marginal profit.
   b What is the profit made for the 120th item sold?
   c What is the profit made for the 200th item sold?
   d What is the profit made for the 500th item sold?
   e What is the profit made for the 503rd item sold?

4 The cost function for the production of ovens is given by $C(n) = 0.000\,35n^3$, where the cost $C(n)$ of producing $n$ ovens in a week is in dollars. The ovens are sold for $620 each.
   a What is the cost of producing 500 ovens?
   b What is the average cost per oven if 500 ovens are produced?
   c Find an expression for the average cost of producing $n$ ovens.
   d Find the largest number of ovens that can be produced that will still make an overall profit.
   e Find an expression for the marginal cost of producing $n$ ovens
   f What is the largest number of ovens for which it is profitable to make more?
   g Explain the difference between the answers in **d** and **f**.

5 A water tower has water pumped up to it and the town water supply is drawn from the tower. As the level of water in the tower increases, it costs more to pump water to the higher level. This means that high pressure in the town water costs more than low pressure. The cost with respect to pressure is given by $C(P) = 0.0004P^3$, where $C(P)$ is the cost per day for pressures above $P = 100$ Pascals. The pressure is normally maintained at 240 Pascals. The minimum safe pressure for fire equipment operation is considered to be 200 Pascals, provided that there is only one fire at a time in the town.
   a What is the daily cost of a pressure of 120 Pascals?
   b What is the daily cost of a pressure of 180 Pascals?
   c What is the daily cost of a pressure of 240 Pascals?
   d What is the daily saving if the pressure is reduced to the minimum fire safety level?
   e The town has 3000 houses. What is the cost per household for maintenance of the higher water pressure, compared to the minimum fire safety level?
   f Find an expression for the marginal cost of pressure.
   g At a pressure of 200 Pascals, what is the cost per household for every increase of 1 Pascal in pressure?
   h At a pressure of 240 Pascals, what is the cost per household for every increase of 1 Pascal in pressure?

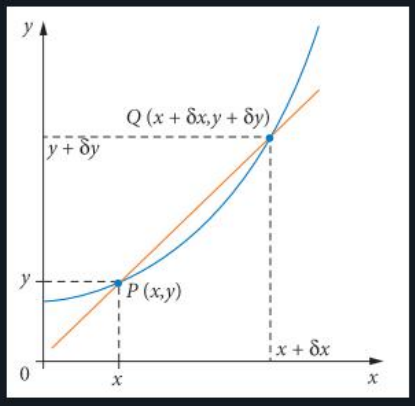


# 9 CHAPTER SUMMARY
# DERIVATIVES

- The **limit** of a function $f(x)$ at $a$ is the value that it approaches as $x$ gets closer and closer to $a$. If it exists, it is written as $\lim_{x \to a} f(x)$.
- A **secant** of a function is a straight line passing through two points on the graph. The line segment between the two points is called a **chord**.
- The **gradient of a function** at $P$ is the limiting value of the gradient of the secant at $PQ$ as $Q$ approaches $P$. As $Q$ approaches $P$, $h$ becomes arbitrarily small as it approaches 0.

$$\text{Gradient of secant} = \frac{f(x+h)-f(x)}{h}, h \neq 0$$

$$\text{Gradient of tangent} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$




- The **derivative** of a function is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided that the limit exists. The process of finding the gradient function of a function by evaluating the limit is called **differentiation from first principles**.
- The derivative of a function in the form $y = f(x)$ is often written as $\frac{dy}{dx}$.
- The derivative of $f(x) = x^n$ is given by $f'(x) = nx^{n-1}$
- The derivative of $f(x) = kx^n$ is given by $f'(x) = knx^{n-1}$, where k is a constant
- The **(instantaneous) rate of change** of a function at any point is the gradient of the tangent at that point. For a function $y = f(x)$ the gradient of the tangent to the graph is called the **gradient of the graph**.
- The derivative of a function at the point where $x = a$ is written as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.
- **Displacement** is the change of position of an object from some starting point.
- **Velocity** is the rate of change of displacement with respect to time.
- **Acceleration** is the rate of change of velocity with respect to time.


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# CHAPTER REVIEW
# DERIVATIVES
# 9

## Multiple choice

1 Examples 1–3 Find $\lim_{x \to 3}\left(\frac{3x^2 - 27}{x - 3}\right)$

A $-3$ B $0$ C $6$ D $18$ E $27$

2 Examples 4–5 What is the gradient of $f(x) = 5x^2$ at $x = -3$?

A $-90$ B $-45$ C $-30$ D $30$ E $45$

3 Example 6 What is the gradient of a secant to $y = x^2 + 3x - 4$?

A $\frac{2xh + 3h + h^2}{h}$ B $\frac{5xh + h^2}{h}$ C $\frac{5xh + h}{h}$ D $\frac{2xh + h^2}{h}$ E $\frac{2x + 3h + h^2}{h}$

4 Example 11 What is the derivative of $y = 5x^3$?

A $10x$ B $15x^2$ C $10x^2$ D $10x^3$ E $5x^2$

5 Example 12 What is the rate of change of $f(x) = 6x^2$ at $x = -5$?

A $12$ B $150$ C $-60$ D $6x^2$ E $-150$

## Short answer

6 Examples 1–3 Find $\lim_{x \to 4}\left(\frac{2x^2 - 3x - 20}{x - 4}\right)$.

7 Examples 1–3 CAS Find $\lim_{x \to -3}\left(\frac{3x^2 + 5x - 12}{x^2 + x - 6}\right)$.

8 Example 7 Find the derivative of $f(x) = 2x^3 - 3x + 2$ from first principles.

9 Example 9 Find the derivative of $p(q) = q^5$ from first principles.

10 Example 10 Find the derivatives of each of the following.

a $y = x^{14}$ b $f(x) = 5x^7$

11 Example 12 Find the rate of change of $f(x) = -5x^4$ at $x = 2$.

12 Example 15 Find the equation of the tangent to $y = 0.2x^3$ at $x = -4$

13 Example 16 Find point(s) on the graph of $f(x) = \frac{1}{15}x^5$ that have a gradient of 27.

## Application

**14** A student riding to school starts from rest at the top of a steep hill. He pedals down the hill so that his displacement is given by $x = 1.5t^2$ m $t$ seconds after starting. It is a distance of 180 m to the bottom of the hill.

- **a** Find an expression for his speed at time $t$.
- **b** How long does it take him to reach the bottom?
- **c** What is his acceleration?
- **d** What is his speed when he reaches the bottom?



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Practice quiz

**15** The cost of producing and distributing executive desks per quarter is given by the expression $C(n) = \frac{n^3}{280}$, for desk quantities greater than 200.
The desks sell for $1299.

- **a** What is the cost of producing 300 desks per quarter?
- **b** What is the cost of producing 400 desks per quarter?
- **c** What is the marginal cost of making the desks?
- **d** At what number of desks is it no longer profitable to make more?

**16** Prove that $\frac{d(3x^5)}{dx} = 15x^4$.


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# 10

## TERMINOLOGY

antiderivative
antidifferentiation
average rate of change
decreasing
derivative
differentiation
gradient function
increasing
instantaneous rate of change
integration
integral
linear product
linear combination
marginal cost
primitive
stationary
tangent

Prior learning

## INTRODUCTION TO DIFFERENTIAL CALCULUS

# PROPERTIES OF DERIVATIVES

10.01 The gradient function
10.02 The derivative of a sum
10.03 Linear products and combinations
10.04 The derivative of a polynomial
10.05 Tangents
10.06 Applications of derivatives
10.07 Antiderivatives
Chapter summary
Chapter review

## PROPERTIES OF DERIVATIVES

- understand the concept of the derivative as a function (ACMMM089)
- recognise and use linearity properties of the derivative (ACMMM090)
- calculate derivatives of polynomials and other linear combinations of power functions (ACMMM091)

## APPLICATIONS OF DERIVATIVES

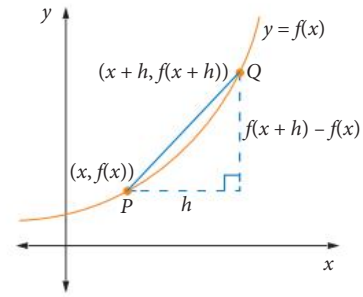
- find instantaneous rates of change (ACMMM092)
- find the slope of a tangent and the equation of the tangent (ACMMM093)

## ANTIDERIVATIVES

- calculate antiderivatives of polynomial functions and apply to solving simple problems involving motion in a straight line (ACMMM097) AC

# 10.01 THE GRADIENT FUNCTION

The diagram shows the secant (or chord) $PQ$. The gradient of the secant passing through $P$ and $Q$ gets closer and closer to the gradient of the curve at $P$ as the distance between $P$ and $Q$ gets smaller. As the distance between $P$ and $Q$ approaches zero, the gradient of the secant passing through $P$ and $Q$ approaches the gradient of the curve at $P$.




Gradient of secant $PQ = m_{PQ} = \dfrac{f(x+h)-f(x)}{h}$

We can use limit notation to write 'as the distance between $P$ and $Q$ approaches zero' as

$$\lim_{Q\to P} m_{PQ} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \text{gradient of curve at } P$$

However, from the work done previously, we know that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

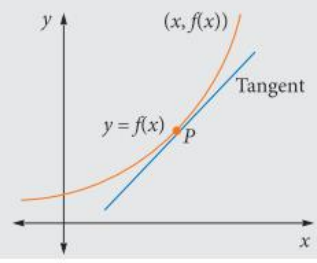
So for any point on the curve $y = f(x)$, $f'(x)$ = gradient of the curve at that point.

The gradient is also known as the **gradient function** as it is a new function calculated using the derivative of the original function.

**IMPORTANT**

### Derivative and gradient function

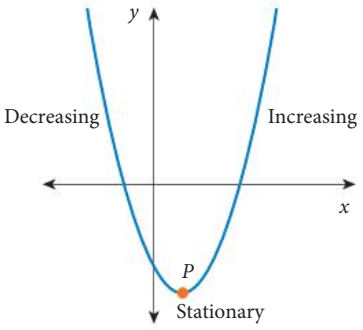
For a function $f(x)$, the gradient of the function (**gradient function**) at any point $P(x, f(x))$ is $f'(x)$.



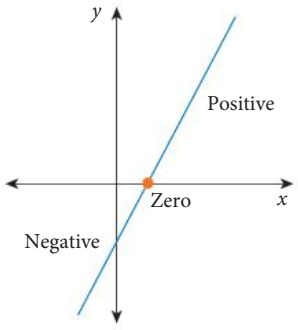


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The gradient of a curve can be used to determine how a function is changing. Consider the graph of the quadratic function shown on the right.

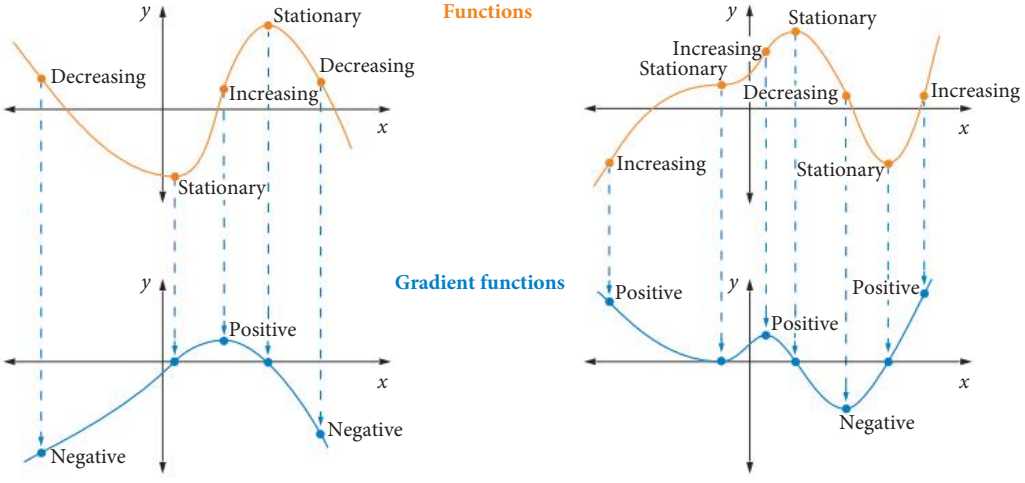



The gradient function is negative to the left of $P$, zero at $P$ and positive to the right of $P$. The graph of the gradient function is shown here.




The function is **decreasing** on the left of $P$ and **increasing** on the right of $P$. At $P$, the function is **stationary** as it is neither increasing nor decreasing.

The diagram below shows some points where functions are increasing, decreasing or stationary, with the corresponding gradient functions shown underneath.





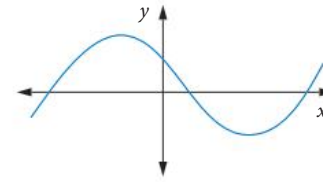

**IMPORTANT**

**Function and gradient function**

- If a function is **increasing**, the gradient will be **positive**. The gradient function is above the $x$-axis.
- If the function is **decreasing**, the gradient will be **negative**. The gradient function is below the $x$-axis.
- We say that the function is **stationary** at points where the gradient is zero. The gradient function intersects or touches the $x$-axis.

## Example 1

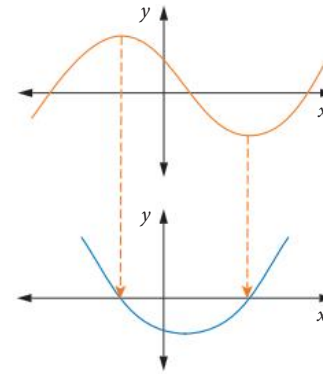
Sketch a possible gradient function for the function shown on the right.




### Solution

The function is increasing at first, so the gradient function is above the $x$-axis (positive).
There is a stationary point just before the $y$-axis, so the gradient function has an $x$-intercept.
Between the top and bottom of the curve, the function is decreasing, so the gradient function is below the $x$-axis (negative).
The gradient function again cuts the $x$-axis at the $x$ value where the second stationary point occurs.
After the second stationary point, the function is increasing, so the gradient function is above the $x$-axis (positive).

Draw the gradient function so that the points match the relevant points on the original function.



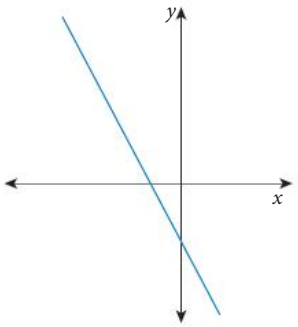


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It is also possible to construct a possible function from a gradient function.

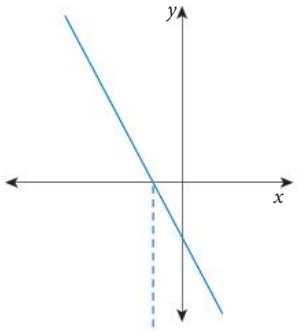
## Example 2

Sketch a possible function for the gradient function shown on the right.

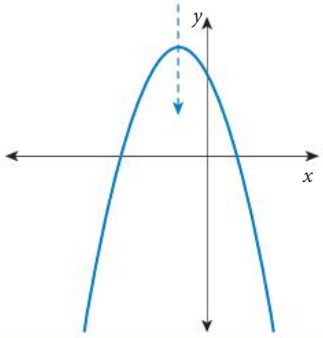



### Solution

The gradient function is at first positive (above the $x$-axis) so the function is increasing.
The gradient function has an $x$-intercept just before the $y$-axis so there is a stationary point.
After the stationary point, the gradient function is negative (below the $x$-axis) so the function is decreasing.




Draw the diagram so that the important points match with the gradient function.




## INVESTIGATION Functions and gradient functions

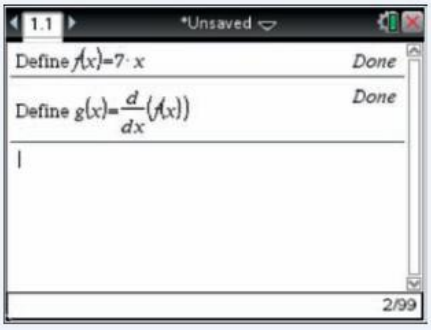
In this investigation you will use a CAS calculator to examine some functions and their gradient functions.

First we will see how to enter and graph the function $y = 7x$ and its gradient function.

### TI-Nspire CAS

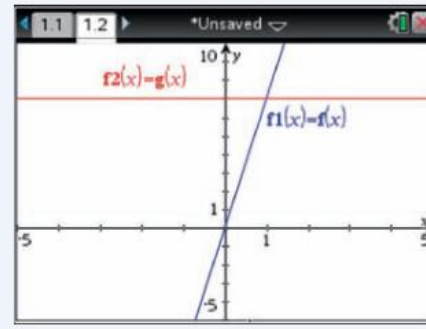
Define $f(x) = 7x$.

Define $g(x) = \frac{d}{dx}(f(x))$ using 4: Calculus and 1: Derivative from the menu.

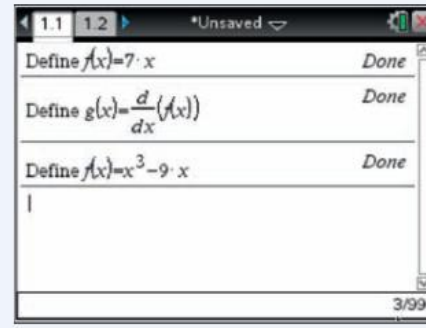




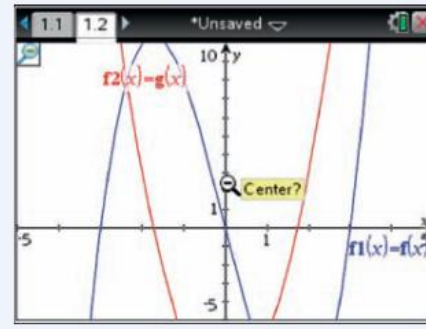

Now add a graph screen using ctrl doc for +page.
Set $f1(x) = f(x)$.
Use menu, 3: Graph Enty/Edit and
1: Function to add $f2(x) = g(x)$.
Use menu, 4 Window/Zoom and
1: Window settings to set $-5 \le x \le 5$ and $-5 \le y \le 10$ with both Xscale and YScale at 1 and enter for OK.




Use ctrl ◀ for ⟲ to go back to the Calculator screen (1.1) and redefine $f(x) = x^3 - 9x$ (similar to the graphs in Example **1**).

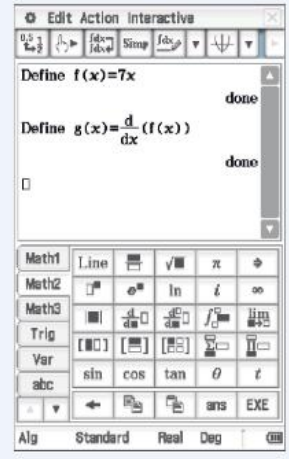



Now use ctrl ▶ for ⟲ to go back to the Graph screen (1.2).
Change the Window settings or use Zoom – Out to focus on the area of interest. Move the centre to where you want to Zoom from and compare the graphs to those in Example **1**.




### ClassPad

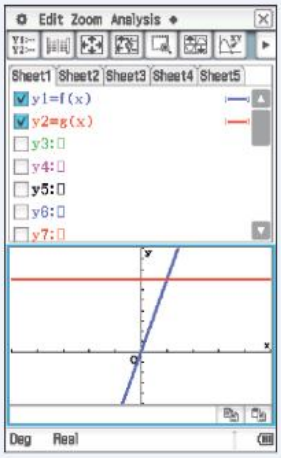
Start with the Main $\sqrt{\alpha}$ menu and press Keyboard.
Define $f(x) = 7x$, using the calculator key $x$ for $x$.
Define $g(x) = \frac{d}{dx}(f(x))$ using the derivative template $\frac{d}{dx}$ found under the Math2 tab.



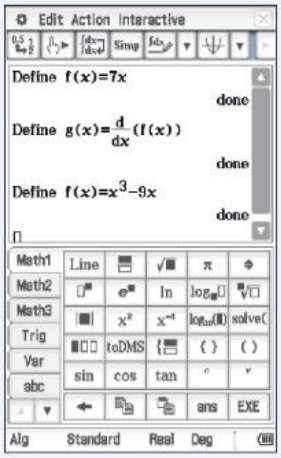


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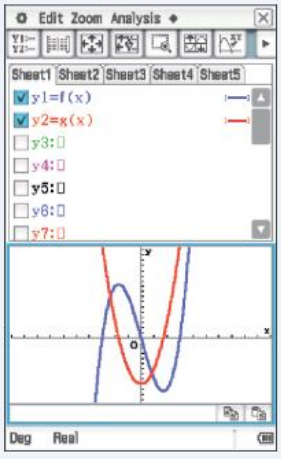
Go to the **Graph&Table** menu.
Enter $y1 = f(x)$ and $y2 = g(x)$.
Tap and use **View Window** to set $-5 \le x \le 5$ and $-5 \le y \le 10$.
Tap to draw the function and its gradient function on the same set of axes.




Go back to the main menu and redefine $f(x) = x^3 - 9x$ (similar to Example **1**).




Now go back to the **Graph&Table** menu and tap to draw the new function and its gradient function on the same set of axes. Use **Zoom Out** (from the **Zoom** menu up the top) or **View Window** to reset the view to the size shown. Compare the graphs to Example **1**.




- Describe the graph of the function and its gradient function.
- Follow the same procedure with the function $y = 0.5x^2$.
- Describe the graphs of the function $y = 0.5x^2$ and its gradient function.
- Experiment with other functions and their corresponding gradient functions, such as $f(x) = x^3 - x^2$.



You can use the gradient function to find out if a function is increasing, decreasing or stationary at a particular point on the function.

### Example 3

Consider the function $y = 2x^2$. State whether the function is increasing, decreasing or stationary at (–3, 18).

#### Solution

| | |
|---|---|
| Calculate the gradient of $y$. | $y' = 4x$ |
| Substitute $x = -3$. | $= 4 \times (-3)$<br>$= -12$ |
| The gradient is negative at (–3, 18). | The function is decreasing at (–3, 18). |

## INVESTIGATION Tangents on a curve

Tangent tracer

It is possible to look at the tangents on simple curves to see how the gradient of a function changes. The spreadsheet 'Tangent tracer' can be loaded from the website to investigate the way in which the tangent and its slope change.

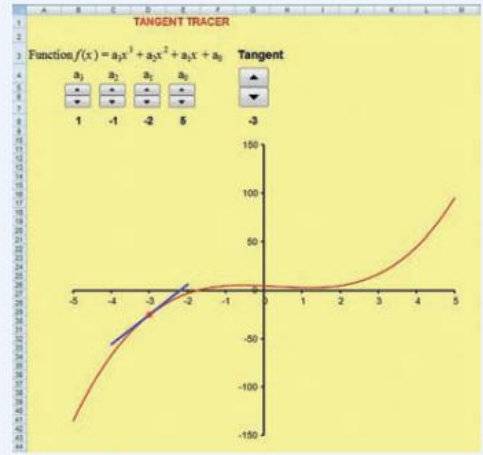
1 Load the spreadsheet 'Tangent tracer' from the website.

2 Change the function to $f(x) = x^2 - 4x + 1$ by making $a_3 = 0$, $a_2 = 1$, $a_1 = -4$ and $a_0 = 1$.

3 Move the tangent along the curve.

- Is the curve increasing or decreasing before $x = 2$?
- What happens to the slope at $x = 2$?
- Is the curve rising or falling after $x = 2$?

4 Change the function and observe how the tangent and slope of the curve change. Note that the vertical scale changes to accommodate the curve.





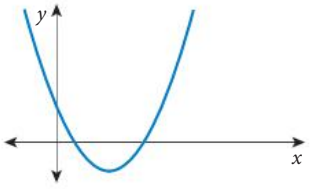
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# EXERCISE 10.01 The gradient function

## Concepts and techniques

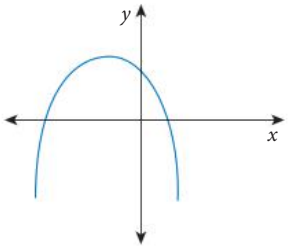
WS Gradient functions

1 **Example 1** The graph of a function is shown on the right.

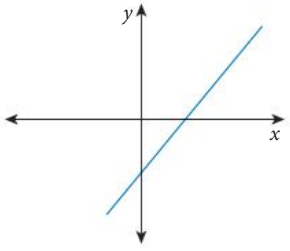



Which one of the following is most likely to be the graph of the gradient function?

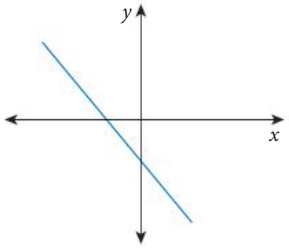
A



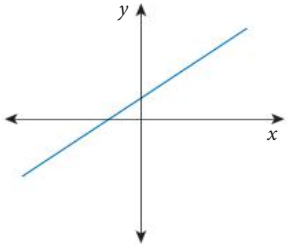

B



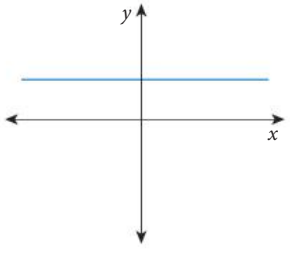

C



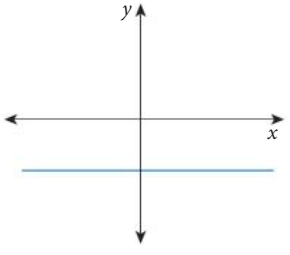

D




E

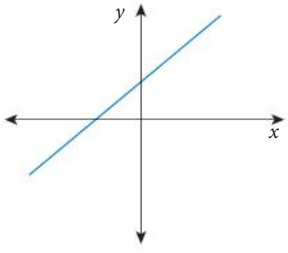



2 **Example 2** The graph of a gradient function is shown on the right.

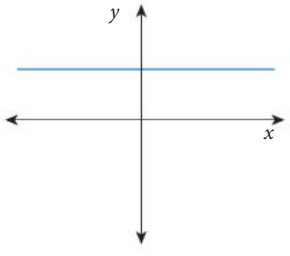



Which one of the following is most likely to be the graph of the function?

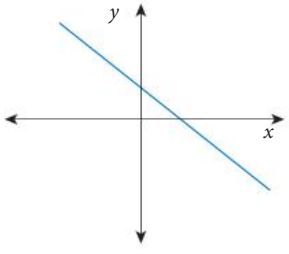
A




B

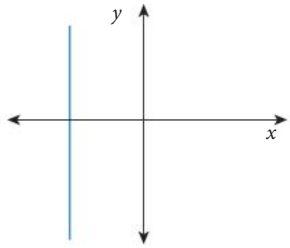



C

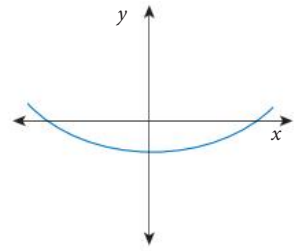





D




E




3 If a function, $y = f(x)$, is increasing, which of the following best describes the function?

A The values of $y$ increase as the values of $x$ decrease.

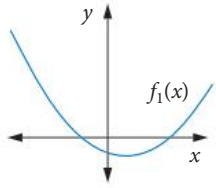
B The values of $y$ decrease as the values of $x$ increase.

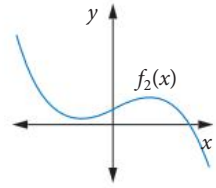
C The gradient is positive.

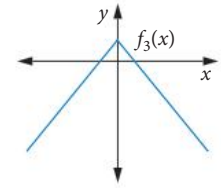
D The graph is a straight line.

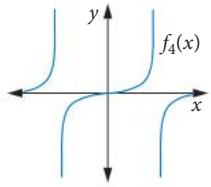
E The gradient is zero.

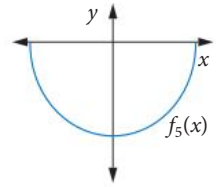
4 The graphs of 10 functions $f_1(x)$ to $f_{10}(x)$ and gradient functions $g_1(x)$ to $g_{10}(x)$ are shown. Match each graph to the graph of its derivative.

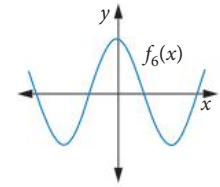


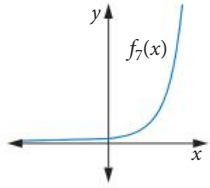



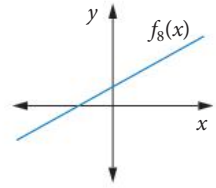



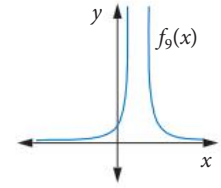



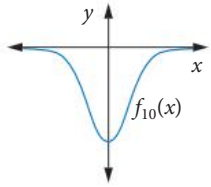



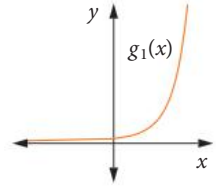



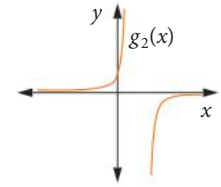



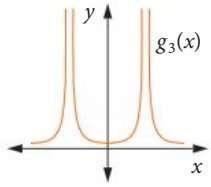



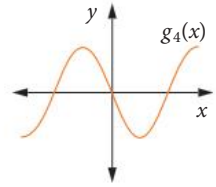



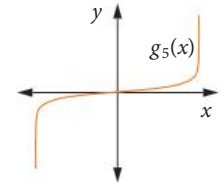



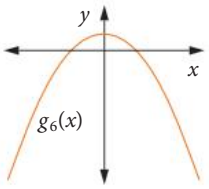



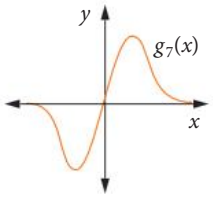



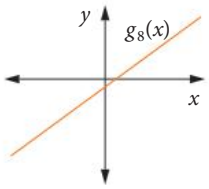



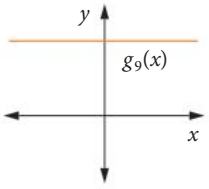


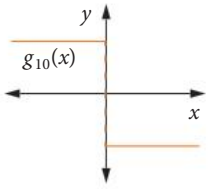
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**5** Example 3 Consider the function $f(x) = 4x^2$. Show that the function is:

**a** increasing when $x = 8$ **b** decreasing when $x = -4$

**c** stationary at $x = 0$

**6** Consider the function $f(t) = t^3$. Show that the function is:

**a** increasing when $t = -1$ **b** increasing when $t = 1$

**c** increasing at $t = 3$.

**7** If $y = x^2$, state whether $y$ is increasing, decreasing or stationary at:

**a** $x = -4$ **b** $x = -3$ **c** $x = -2$ **d** $x = -1$

**e** $x = 0$ **f** $x = 1$ **g** $x = 2$ **h** $x = 3$

**i** $x = 4$ **j** $x = 5$

**8** If $r'(t) = -21t^2 + 24t + 4$, state whether $r(t)$ is increasing, decreasing or stationary at:

**a** $t = -3$ **b** $t = -2$ **c** $t = -1$ **d** $t = 0$

**e** $t = 1$ **f** $t = 2$ **g** $t = 3$ **h** $t = 4$

**i** $t = 5$ **j** $t = 6$

**9** If $f'(m) = 3m^2 - 12$, state whether $f(m)$ is increasing, decreasing or stationary at:

**a** $m = -4$ **b** $m = -2$ **c** $m = -1$ **d** $m = 0$

**e** $m = 1$ **f** $m = 2$ **g** $m = 4$ **h** $m = 6$

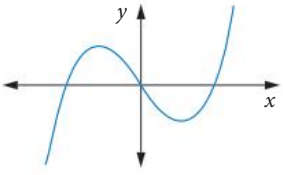
**10** If $y' = 4x^3 - 28x - 24$, state whether $y$ is increasing, decreasing or stationary at:

**a** $x = -4$ **b** $x = -2$ **c** $x = -1$ **d** $x = 0$

**e** $x = 1$ **f** $x = 2$ **g** $x = 3$ **h** $x = 4$

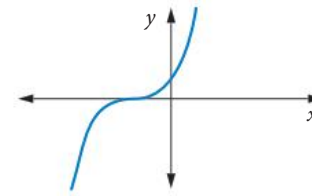
## Reasoning and communication

**11** The diagram on the right represents a function. Sketch a possible corresponding gradient function.

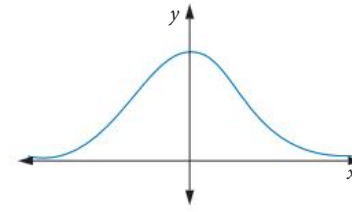




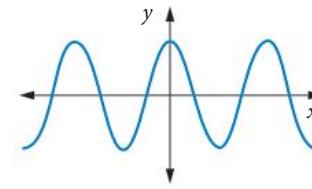

12 The diagram on the right represents a function. Sketch a possible corresponding gradient function.



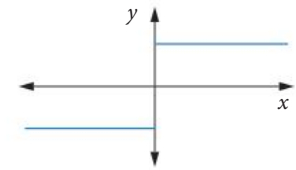

13 The diagram on the right represents a function. Sketch a possible corresponding gradient function.



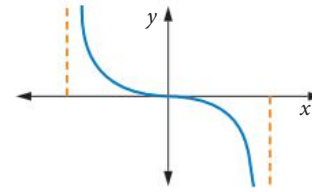

14 The diagram on the right represents a gradient function. Sketch a possible corresponding function.




15 The diagram on the right represents a gradient function. Sketch a possible corresponding function.




16 The diagram on the right represents a gradient function. Sketch a possible corresponding function.




# 10.02 THE DERIVATIVE OF A SUM

We will now look at the sums and differences of some basic functions.

## Example 4

Differentiate $f(x) = x + x^3$ from first principles.

### Solution

Write down the function. $f(x) = x + x^3$

Find $f(x + h)$. $f(x + h) = (x + h) + (x + h)^3$

Expand using $(a + b)^n$. $= x + h + x^3 + 3x^2h + 3xh^2 + h^3$


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Find $f(x+h) - f(x)$.

$$\begin{aligned} f(x+h) - f(x) &= x + h + x^3 + 3x^2h + 3xh^2 + h^3 - (x + x^3) \\ &= x + h + x^3 + 3x^2h + 3xh^2 + h^3 - x - x^3 \\ &= h + 3x^2h + 3xh^2 + h^3 \end{aligned}$$

Factorise.

$$= h(1 + 3x^2 + 3xh + h^2)$$

Write the formula for the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute $f(x+h) - f(x)$ into the formula.

$$= \lim_{h \to 0} \frac{h(1 + 3x^2 + 3xh + h^2)}{h}$$

Cancel the common factor.

$$= \lim_{h \to 0} (1 + 3x^2 + 3xh + h^2)$$

Write the limit.

$$= 1 + 3x^2$$

## Example 5

Differentiate $f(x) = x^4 - x^2$ from first principles.

### Solution

Write down the function.

$$f(x) = x^4 - x^2$$

Find $f(x+h)$.

$$f(x+h) = (x+h)^4 - (x+h)^2$$

Expand using $(a+b)^n$.

$$\begin{aligned} &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - (x^2 + 2xh + h^2) \\ &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^2 - 2xh - h^2 \end{aligned}$$

Find $f(x+h) - f(x)$.

$$\begin{aligned} f(x+h) - f(x) &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^2 - 2xh - h^2 - (x^4 - x^2) \\ &= 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2xh - h^2 \end{aligned}$$

Factorise.

$$= h(4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h)$$

Write the formula for the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute into the formula.

$$= \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h)}{h}$$

Cancel the common factor.

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h)$$

Write the limit.

$$= 4x^3 - 2x$$



The previous examples show that expressions with more than one term can be differentiated by adding or subtracting the derivatives of the terms. This can be proved for the sums and differences of *any* functions as shown below.

$$\begin{aligned}\frac{d}{dx}[f(x)-g(x)] &= \lim_{h\to 0}\frac{[f(x+h)-g(x+h)]-[f(x)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-f(x)+g(x)}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)-g'(x)\end{aligned}$$

The rule for addition follows the same method.

**IMPORTANT**

The **derivative of a sum** of functions is the sum of the derivatives.

The **derivative of a difference** of functions is the difference of the derivatives.

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

You have previously established the rule $\frac{d}{dx}x^n = nx^{n-1}$.

These rules are easily proved from the definition of a derivative.

### Example 6

Differentiate $f(x) = x^5 - x^3$.

#### Solution

Use the rule for differentiating differences.

$$\frac{d}{dx}\left(x^5 - x^3\right) = \frac{d}{dx}x^5 - \frac{d}{dx}x^3$$

Find the derivatives.

$$= 5x^4 - 3x^2$$


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## EXERCISE 10.02 The derivative of a sum

### Concepts and techniques



Derivative of a sum of terms

**1** Examples 4, 5 Find the derivative of each of the following using first principles.

**a** $x^2 - x$ **b** $x + x^2$ **c** $x^3 - x^2$

**d** $x^2 + x^4$ **e** $x^4 + x$ **f** $x^3 - x^4$

**2** Calculate the derivative of each of the following using first principles.

**a** $x + x^2 + 5$ **b** $x^2 - x + 8$ **c** $x^4 - x + x^2$

**d** $x^3 + x^2 - 3$ **e** $x^3 + x^5 - x$ **f** $x^5 - x^2 + x$

**3** Example 6 Find the derivative of each of the following using the rule for the derivatives of sums and differences.

**a** $x^4 - x^3$ **b** $x^6 + x^3$ **c** $x^7 + x^5$

**d** $x^8 - x^3$ **e** $x^9 + x$ **f** $x^7 - x^4$

**4** Calculate the derivative of each of the following.

**a** $m^6 + m^2 + m$ **b** $r^6 - r - r^3$ **c** $x^7 + x^4 + x^9$

**d** $p^5 - p^6 + p^8$ **e** $w^6 + w^7 - w^3$ **f** $n^2 - n^4 + n^6$

**5** Find the derivatives of the following with respect to the given variables.

**a** $d^5 + d^4 + d^3 + d^2$ **b** $x^8 - x^6 + x^4 - x^2$

**c** $t + t^5 - t^7 + t^8$ **d** $h^5 - h^7 + h^9 - h^{11}$

**e** $z^2 + z^5 - z^3 + z^6 - z^4$ **f** $u^3 - u^2 + u - u^5 + u^{10}$

### Reasoning and communication

**6** Prove the rule for the derivative of a sum from first principles.

# 10.03 LINEAR PRODUCTS AND COMBINATIONS

You have seen in Chapter 9 that the derivative of $x^n$ is $nx^{n-1}$. You also saw that the derivative of $kx^n$ is given by $knx^{n-1}$. The term $kx^n$ is called a **linear product**. The rule for a linear product can be extended to a linear product of *any* function.

$$\begin{aligned}\frac{d}{dx}[kf(x)] &= \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}\\ &= \lim_{h\to 0}\frac{k[f(x+h)-f(x)]}{h}\\ &= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= kf'(x)\end{aligned}$$



**IMPORTANT**

**Linear product rule**

$\frac{d}{dx}\left[kf(x)\right] = kf'(x)$, where $k$ is a constant, so

$\frac{d}{dx}\left(kx^n\right) = knx^{n-1}$

This is easy to show from the definition.

The rule for the derivative of $kx^n$ just combines the rules for the derivatives of $kf(x)$ and $x^n$.

## Example 7

Calculate the following.

a $\frac{d}{dx}(3x^5)$ b $\frac{d}{dt}(7t^4)$

### Solution

a Use $\frac{d}{dx}\left(kx^n\right) = knx^{n-1}$ $\frac{d}{dx}(3x^5) = 3 \times 5x^{5-1}$

Evaluate. $= 15x^4$

b Now use $\frac{d}{dx}\left(kx^n\right) = knx^{n-1}$ $\frac{d}{dt}\left(7t^4\right) = 7 \times 4t^{4-1}$

Evaluate. $= 28t^3$

A **linear combination** is a combination of sums or differences of linear products. A linear combination of $a$ and $b$ could be:

$3a + 7b$, $11b - 5a$, $\sqrt{3}a - \frac{2}{3}b$, etc.

It is also possible to have linear combinations of functions. The rule for differentiating linear products is useful when differentiating linear combinations of functions.

**IMPORTANT**

**Derivative of a linear combination of two functions**

$\frac{d}{dx}\left[af(x) + bg(x)\right] = af'(x) + bg'(x)$, where $a$ and $b$ are constants.

The rule for a linear combination of two functions is just a combination of the rules for the sums of functions and linear products.

### Example 8

Calculate the derivative of $5x^3 - 2x^2$.

#### Solution

| | |
|---|---|
| Write the problem. | $\frac{d}{dx}\left(5x^3 - 2x^2\right)$ |
| Use the linear combination rule with $a = 5$, $f(x) = x^3$, $b = -2$, and $g(x) = x^2$. | $= 5\frac{d}{dx}\left(x^3\right) + (-2)\frac{d}{dx}\left(x^2\right)$ |
| Work out the derivatives. | $= 5 \times 3x^2 - 2 \times 2x$ |
| Simplify. | $= 15x^2 - 4x$ |

# EXERCISE 10.03 Linear products and combinations

## Concepts and techniques

WS Derivatives of linear products

**1** Use first principles to find the derivatives of the following with respect to $x$.

**a** $3x^2 - 5x$ **b** $7x^3 + x^2$ **c** $2x^4 - 3x^3$
**d** $5x^3 + 2x$ **e** $2x^5 - 3x^4$

**2** Example 7 Find the derivatives of the following with respect to the given variables.

**a** $8m^2$ **b** $-4x^6$ **c** $9n^3$ **d** $12g^2$ **e** $-6m^7$
**f** $6p^4$ **g** $11z^5$ **h** $-10r^7$ **i** $-7t^{10}$ **j** $14x^6$

**3** Example 8 Find the derivatives of the following with respect to the given variables.

**a** $14u^8 + 11u^2$ **b** $15u^8 - 14u^3$ **c** $3u^2 + 9u^9$
**d** $2z^6 + 10z^3$ **e** $8a^8 + 9a^9$ **f** $10v^3 - 15v$
**g** $6w^7 + 11w^8$ **h** $12b^2 - 3b^7$ **i** $12c^4 + 6c^2$
**j** $6r^7 + 7r^3$

## Reasoning and communication

**4** If $g(x) = 7x^3 - 4x^2$, calculate:

**a** $4g'(x)$ **b** $-7g'(x)$ **c** $2\sqrt{3}g'(x)$ **d** $\frac{2}{3}g'(x)$ **e** $3.8g'(x)$

**5** If $y = -2x^4 + 5x^2$, calculate:

**a** $3y'$ **b** $-5y'$ **c** $5\sqrt{2}y'$ **d** $\frac{5}{2}y'$ **e** $4.5y'$

**6** If $f'(x) = 3x^3 - 4x^2$ and $g'(x) = -6x^2 + 5x$, calculate the derivative of each of the following linear combinations of functions.

**a** $3f(x) - 7g(x)$ **b** $4f(x) + 5g(x)$ **c** $-3f(x) - 2g(x)$

**7** If $y' = x^4 + 3x^3$ and $z' = 2x^3 - 4x^2$, calculate the derivative of each of the following linear combinations.

**a** $2y - 5z$ **b** $-6y + 3z$ **c** $-5y - 2z$



8 If $f'(x) = 2x^2 - 5x$, $g'(x) = -3x^3 + 4x^2$, $h'(x) = -x^4 + 3x^3$ and $m'(x) = -2x^3 + 6x^2$ calculate the derivative of each of the following.

a $2f(x) - 3g(x) + 2h(x)$ b $-3f(x) + 2h(x) - 5m(x)$

c $4g(x) - 2m(x) - 6f(x)$ d $-4m(x) + 3h(x) + 5f(x) - 7g(x)$

9 If $f(x) = 2x^3 + 5x^2 - 11$ and $g(x) = -3x^4 + 4x^3 - 9$, calculate the derivative of each of the following.

a $3f(x) + 6g(x)$ b $2f(x) - 5g(x)$ c $-3f(x) - 4g(x)$

# 10.04 THE DERIVATIVE OF A POLYNOMIAL

You have previously seen that a polynomial function is defined as:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$

where $n = 0, 1, 2, 3\ldots$

Examples of polynomial expressions include:

$3x^4 - 5x^2 - 8$, $\sqrt{3}x^2 - 12x$ and $\frac{x^5}{4} - 3x^3 + \frac{5x}{7}$

but do not include expressions such as:

$3xy + 2x^2$, $3x^2 - \frac{2}{x} + 5$ and $4x^2 - 3\sqrt{x} + 7$.

A polynomial expression can be differentiated using the rule for the differentiation of a linear combination of functions.

## Example 9

For the function $f(x) = 5x^3 - 4x^2 + 7x + 2$, find:

a the derivative $f'(x)$ b $f'(2)$ c the gradient of $f(x)$ at $-1$.

### Solution

a Write down the function. $f(x) = 5x^3 - 4x^2 + 7x + 2$

Differentiate each term. $\frac{d}{dx}(5x^3) = 15x^2$

$\frac{d}{dx}(-4x^2) = 8x$

$\frac{d}{dx}(7x) = 7$

$\frac{d}{dx}(2) = 0$

Find $f'(x)$ by adding derivatives of terms. $f'(x) = 15x^2 - 8x + 7$

b Write the derivative. $f'(x) = 15x^2 - 8x + 7$

Substitute $x = 2$. $f'(2) = 15 \times 2^2 - 8 \times 2 + 7$

Evaluate. $= 51$

c The gradient of $f(x)$ at $-1$ is $f'(-1)$. $f'(-1) = 15 \times (-1)^2 - 8 \times (-1) + 7$

Evaluate. $= 30$


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If a polynomial is in a factorised form, it must be expanded before it can be differentiated.

## Example 10

Calculate the derivative of the polynomial expression $y = (2x^2 - 3x)(4x - 5)$.

### Solution

Expand the RHS of the equation.

$$\begin{aligned} y &= (2x^2 - 3x)(4x - 5) \\ &= 8x^3 - 10x^2 - 12x^2 + 15x \\ &= 8x^3 - 22x^2 + 15x \end{aligned}$$

Differentiate.

$$y' = 24x^2 - 44x + 15$$

## Example 11

If $y_1 = 5x^3 - 3x^2 + 1$ and $y_2 = 4x^2 + 2$, calculate:

**a** $\frac{d}{dx}(y_1 \times y_2)$ **b** $\frac{d}{dx}(y_1) \times \frac{d}{dx}(y_2)$

**c** Compare the answers to Part **a** and Part **b**.

### Solution

**a** Find the product $y_1 \times y_2$.

$$\begin{aligned} y_1 \times y_2 &= (5x^3 - 3x^2 + 1) \times (4x^2 + 2) \\ &= 20x^5 + 10x^3 - 12x^4 - 6x^2 + 4x^2 + 2 \end{aligned}$$

Simplify.

$$= 20x^5 - 12x^4 + 10x^3 - 2x^2 + 2$$

Differentiate.

$$\begin{aligned} \frac{d}{dx}(y_1 \times y_2) &= \frac{d}{dx}(20x^5 - 12x^4 + 10x^3 - 2x^2 + 2) \\ &= 100x^4 - 48x^3 + 30x^2 - 4x \end{aligned}$$

**b** Calculate the derivative of $y_1$.

$$\begin{aligned} \frac{d}{dx}(y_1) &= \frac{d}{dx}(5x^3 - 3x^2 + 1) \\ &= 15x^2 - 6x \end{aligned}$$

Calculate the derivative of $y_2$.

$$\begin{aligned} \frac{d}{dx}(y_2) &= \frac{d}{dx}(4x^2 + 2) \\ &= 8x \end{aligned}$$

Calculate the product of the derivatives.

$$\begin{aligned} \frac{d}{dx}(y_1) \times \frac{d}{dx}(y_2) &= (15x^2 - 6x)(8x) \\ &= 120x^3 - 48x^2 \end{aligned}$$

**c** Are the answers equal?

$$100x^4 - 48x^3 + 30x^2 - 4x \neq 120x^3 - 48x^2$$

State a conclusion.

$$\frac{d}{dx}(y_1 \times y_2) \neq \frac{d}{dx}(y_1) \times \frac{d}{dx}(y_2)$$



Derivatives of polynomials

Example **11** clearly demonstrates that the derivative of a product of two functions is not equal to the product of their derivatives. So, for two functions $f(x)$ and $g(x)$,

$$\frac{d}{dx}\left[f(x)\times g(x)\right] \neq f'(x)\times g'(x)$$

You will examine the derivatives of products and quotients of functions in Year 12.

# EXERCISE 10.04 The derivative of a polynomial

## Concepts and techniques

**1** Which of the following is not a polynomial function?

**A** $3x^2 - 10$ **B** $5x^3 - \frac{3x}{10} + 6x^2$ **C** $5\sqrt{3}x^4 - \frac{4}{5}x + 12$

**D** $4x^2 - 6x^{-2} + 9$ **E** $\frac{6x^4 - 2x^3 + 7}{15}$

**2** If $y = 3x^2 + 7$ and $z = 9 - 4x$, which of the following is the derivative of $y + z$?

**A** $6x - 4$ **B** $2x$ **C** $4 + 6x$

**D** $2x + 16$ **E** $-24x$

**3** If $f(x) = 3x - 5$ and $g(x) = 2x + 4$, which of the following is the derivative of $f(x) \times g(x)$?

**A** $6x + 1$ **B** $-6$ **C** $12x - 2$

**D** $6$ **E** $2(6x + 1)$

**4** Example 9 Differentiate the following polynomial functions.

**a** $x^3 + x^2 + x$ **b** $6x^7 + 10x^9 - 5x^{12} + 8x^6$

**c** $14 - 2x^6 + 9x^8 + x^{11}$ **d** $3x^8 - 5x^{12} + 31$

**e** $6x^{12} + 11x^{13} - 24x^5$ **f** $12x^3 + 7x^4 - 11x^6 + 38$

**g** $4x^8 - 15x + 28x^2 + 14x^3$ **h** $10 - 6x + 7x^2 + 14x^3 - 9x^4$

**i** $15x^8 - 7x^5 + 4x^7 + 12x - 21$ **j** $16x^2 - 9x^4 + 7x^3 + 18x - 32$

**5** Use differentiation rules to find the derivatives of the following.

**a** $6x^4 - 5x^3 + 2x^2 + 8x - 9$ **b** $9x^3 + 10x^4 - 3x^5 + 18$

**c** $x^3 + x^2 + x + 1$ **d** $4x^9 - 8x^{10} + 20$

**e** $15x^{12} + 12x^{15} - 36x^5$ **f** $5 - x^7 + 7x^9 + x^{10}$

**g** $5x^8 + 12x^9 - 6x^{10} + x^{12}$ **h** $11 - 3x + x^2 + 8x^3 - x^4$

**i** $5x^4 - 17x + 28 + 16x^3$ **j** $25x^7 - 8x^6 + 14x^5 + 2x - 2$

**6** Calculate each of the following derivatives.

**a** $\frac{d}{dx}\left(5x^5 - 3x^2 + 10\right)$ **b** $\frac{d}{du}\left(8u^3 - 9u^4 + 16\right)$

**c** $\frac{d}{dm}\left(7m^8 + 6m^5 - 17m + 18\right)$ **d** $\frac{d}{dy}\left(19y + 7y^5 - 11y^4 + 2y^7\right)$

**e** $\frac{d}{dz}\left(13z^2 - 6z^3 + 3z - 15\right)$ **f** $\frac{d}{dt}\left(3t^6 + 8t^5 - 15t - 17\right)$


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**7** Calculate the following.

**a** $\frac{d}{dx}\left(3x^4 - 6x^3 + 18\right)$ **b** $\frac{d}{du}\left(3u^4 - 6u^2 + 18\right)$

**c** $\frac{d}{dm}\left(5m^6 + 8m^5 - 24m + 8\right)$ **d** $\frac{d}{dy}\left(9y + 6y^5 - 4y^3 + 2y^7\right)$

**e** $\frac{d}{dz}\left(3z^2 - 9z^3 + 38z - 5\right)$ **f** $\frac{d}{dt}\left(5t^7 + 9t^4 - 16t\right)$

**g** $\frac{d}{dr}\left(21 - 13r - 6r^2 + 9r^3\right)$ **h** $\frac{d}{da}\left(-6a^4 + 2a^2 - 10\right)$

**i** $\frac{d}{dp}\left(8p - 3p^4 - 6p^7 - 5p^9\right)$ **j** $\frac{d}{dt}\left(8 + 4t^2 - 2t^3 + 11t^4\right)$

**8** Example 10 Calculate the derivatives of each of the following.

**a** $(v + 7)(v - 6)$ **b** $(6 - 5r)(2r + 9)$ **c** $(8 - 6h)(-4 - 3h)$

**d** $(5t + 8)(2t - 3)$ **e** $(3p + 10)(8p + 1)$ **f** $(-3 - z)(4z + 15)$

**9** Calculate the derivatives of each of the following.

**a** $(a - 6)^2$ **b** $(2x + 5)^2$ **c** $(3y - 2)^2$ **d** $(5q + 7)^2$

**e** $(3 - 5a)^2$ **f** $(5g + 3)^2$ **g** $(-9 - 3v)^2$ **h** $(-7m + 4)^2$

**10** Calculate the derivatives of each of the following.

**a** $(z^2 - z - 1)(9 + 3z)$ **b** $(3p + 4)(p^2 - 2p - 4)$

**c** $(3v^2 + v - 9)(8v + 6)$ **d** $(3t^2 - 2t + 5)(-7t + 8)$

**e** $(3b^3 - 6b^2 + 2)(-4b + 3)$ **f** $(8u^3 - 6u^2 + 2u)(4u^2 + 9u - 6)$

**11** Example 11 Use the functions $u(x) = 7x - 4$ and $z(x) = 5x^2 - 8x + 3$ to demonstrate that

$$\frac{d}{dx}[u(x) \times z(x)] \neq u'(x) \times z'(x)$$

## Reasoning and communication

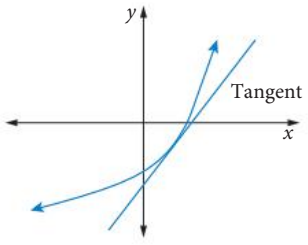
**12** If $y_1 = 3x^2 - 7x + 4$, $y_2 = x^3 - 5x^2$, $y_3 = 2x^4 + 3x^2 - 5x$ and $y_4 = 4x^3 - 5x^2 + 1$, find the derivatives of:

**a** $y_1 + y_2$ **b** $y_1 - y_2$ **c** $y_3 + y_4$ **d** $y_4 - y_1$ **e** $y_3 + y_2$

**f** $y_2 + y_3 - y_4$ **g** $y_1 - y_4 - y_3$ **h** $y_3 - y_2 + y_1$ **i** $y_1 + y_4 - y_3$ **j** $y_2 - y_4 - y_1$

# 10.05 TANGENTS

The value of the derivative of a function at a given point is the slope of the tangent at that point on the graph. Since the tangent is a straight line, we can find its equation using the slope and the point.






## Example 12

The equation of a parabola is $y = 5x - 3x^2 + 8$.

a What is the gradient at $x = 2$?

b What is the equation of the tangent at $x = 2$?

### Solution

| | | |
|---|---|---|
| a | Find the derivative. | $f'(x) = 5 - 6x$ |
| | Find its value at $x = 2$. | $f'(2) = 5 - 12 = -7$ |
| | Write the answer. | The gradient of the tangent at $x = 2$ is $m = -7$. |
| b | Find $y$ at $x = 2$. | $y = 5 \times 2 - 3 \times 2^2 + 8 = 6$ |
| | Write the values of $m$ and $(x_1, y_1)$. | $m = -7$ and $(x_1, y_1) = (2, 6)$ |
| | Write the gradient–point formula. | $y - y_1 = m(x - x_1)$ |
| | Substitute and simplify. | $y - 6 = -7(x - 2)$ |
| | Expand the brackets. | $= -7x + 14$ |
| | Write in standard form. | $7x + y - 20 = 0$ |
| | Write the answer. | The equation of the tangent is $7x + y - 20 = 0$. |

## Example 13

Find values of $x$ for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18.

### Solution

| | |
|---|---|
| Find the derivative. | $y' = 6x^2 - 12x$ |
| The derivative is the gradient of the tangent. | $6x^2 - 12x = 18$ |
| Rearrange and factorise. | $6x^2 - 12x - 18 = 0$<br>$6(x - 3)(x + 1) = 0$ |
| Solve for $x$. | $x = 3$ or $-1$. |

The result of Example **13** means that there are two points on the curve $y = 2x^3 - 6x^2 + 1$ where the tangent has a gradient of 18.

You can use calculus and coordinate geometry to find additional features of the tangent to a curve.


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## Example 14

The tangent $AB$ touches the curve $f(x) = 12 + 4x - x^2$ at point $A$ and cuts the $x$-axis at point $B$. $A$ has an $x$-coordinate of 3. Find the length $AB$ and the acute angle that it makes with the $x$-axis.

### Solution

**Point $A$ and slope of tangent**

Find $A$. $f(3) = 15$, so $A$ is the point $(3, 15)$.

Find the slope of the tangent. $f'(x) = 4 - 2x$, so $f'(3) = -2 = m$

**Equation of tangent**

Write values for the formula. $m = -2$ and $(x_1, y_1) = (3, 15)$

State the point slope form of a straight line. $y - y_1 = m(x - x_1)$

Substitute in the values. $y - 15 = -2(x - 3)$

Simplify. $y - 15 = -2x + 6$

Express in standard form. $2x + y - 21 = 0$

**Point $B$**

$B$ is on the $x$-axis, so $y = 0$. $2x + 0 - 21 = 0$

$x = 10.5$

Find $B$. $B$ is the point $(10.5, 0)$.

**Length and angle of $AB$**

Write the formula for length. $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Substitute and calculate. $= \sqrt{(10.5 - 3)^2 + (0 - 15)^2}$

$\approx 16.77$

Use $m = \tan\theta$ to find the angle. $\tan^{-1}(-2) \approx -63.4°$ or $116.6°$

Write the answers. $AB$ is about 16.77 units and the tangent makes an acute angle of about 63.4° with the $x$-axis.

# EXERCISE 10.05 Tangents

## Concepts and techniques






1 What is the gradient of the tangent for the function $f(x) = 3x^2 - 8x - 5$ at $(-1, 6)$?

A $-14$ B $-2$ C $6$ D $28$ E $40$

2 What angle does the tangent to $y = \frac{1}{3}x^2 - 3x + 4$ at the point on the curve where $x = 1$ make with the positive direction of the $x$-axis?

A $-70°$ B $-20°$ C $110°$ D $20°$ E $140°$

3 Example 12 Find the equation of each of the following.
   a The tangent to $y = 5x^2 - 2x - 4$ at $x = 1$
   b The tangent to $y = x - x^3 - 5$ at $x = 2$
   c The tangent to $y = x^2 - x^3 + 3x + 2$ at $x = -1$
   d The tangent to $y = x^2 - 2x - 3$ at $x = 4$.

4 Example 13 For the function $y = x^3 - 27x - 5$, find the values of $x$ for which $\frac{dy}{dx} = 0$.

5 Find the coordinates of the point at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3.

6 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of $-6$ at a point $P$ on the curve. Find the coordinates of $P$.

7 The tangent at point $Q$ on the curve $y = 4x^2 + 1$ is parallel to the $x$-axis. Find the coordinates of $Q$.

8 Find the coordinates of the point $H$ on the curve $y = 5x^3 - 3x$ that is parallel to the line $7x - y + 3 = 0$.

9 Example 14 $f(x) = 2x^2 + x - 4$
   a Find the acute angle that the tangent at $x = -2$ makes with the $x$-axis.
   b Find the acute angle that the tangent at $x = 0$ makes with the $x$-axis.

10 $y = x^2 - 2x^3 + 5x - 1$
   a Find the acute angle that the tangent at $x = -2$ makes with the $x$-axis.
   b Find the acute angle that the tangent at $x = 1$ makes with the $x$-axis.

11 $p = 3x^2 - 5x + 2$
   a Find the acute angle that the tangent at $x = 1$ makes with the $x$-axis.
   b Find the length of the tangent at $x = 5$ cut off by the axes.
   c Find the length of the tangent at $x = -3$ cut off by the axes.
   d Find the acute angle that the tangent at $x = -3$ makes with the $x$-axis.

12 $P = t^2 + 4t - 4t^3 - 4$
   a Find the gradient of the tangent at $t = 0$.
   b Find the acute angle that the tangent in part **a** makes with the $t$-axis.
   c Find the equation of the tangent at $t = -2$.
   d Find the acute angle that the tangent in part **c** makes with the $t$-axis.
   e Find a general expression for the gradient of the tangent.
   f Find the point(s) where the gradient of the tangent is zero.
   g Find the equation of the tangent at $t = 3$.
   h Find the gradient of the tangent at $t = 1$.

## Reasoning and communication

13 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point $P$. Find the equation of the tangent at $P$.

# 10.06 APPLICATIONS OF DERIVATIVES

Consider the function shown in the diagram on the right. You have previously seen that the **average rate of change** of a function $f(x)$ is given by

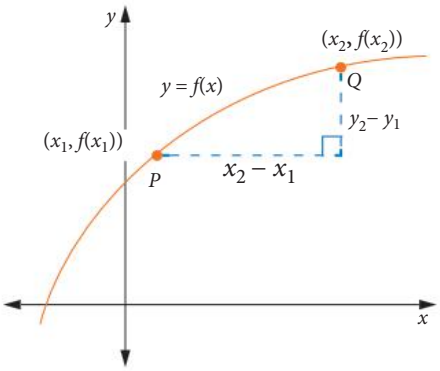
$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This may also be written as

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{instantaneous rate of change}$$

However, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y' = f'(x)$




So, for any function $f(x)$, the instantaneous rate of change and the derivative of the function are the same thing.

**IMPORTANT**

The derivative of a function is the (**instantaneous**) **rate of change** of the function.

## Example 15

Find the rate of change of $f(x) = -2x^3 - 10x + 8$ at $x = 3$.

### Solution

| | |
|---|---|
| Write down the function. | $f(x) = -2x^3 - 10x + 8$ |
| Differentiate. | $f'(x) = -6x^2 - 10$ |
| Find $f'(3)$. | $f'(3) = -6 \times 3^2 - 10$ |
| Evaluate. | $= -64$ |
| State the result. | The rate of change of $f(x)$ at $x = 3$ is $-64$. |

Instantaneous rates of change also have practical applications.



## Example 16

The number of bacteria in a culture increases according to the function

$B = 2t^4 - t^2 + 2000$ where $t$ is the time in hours. Find:

**a** the initial number of bacteria

**b** the rate at which the number of bacteria will be increasing after 5 hours.

### Solution

**a** Write the function. $B = 2t^4 - t^2 + 2000$

Find $B$ when $t = 0$. $= 2 \times 0^4 - 0^2 + 2000$

$= 2000$

State the result. There are initially 2000 bacteria.

**b** Calculate the derivative. $\frac{dB}{dt} = 8t^3 - 2t$

At 5 hours, $t = 5$. $= 8 \times 5^3 - 2 \times 5$

$= 990$

State the result. The rate of increase of bacteria after 5 hours will be 990 bacteria per hour.

## Example 17

The volume of liquid in an underground tank being gravity-filled from a tanker is given by

$V = 1200 + 1020t - 17t^2$

where the volume $V$ is in litres and the time $t$ is in minutes.

**a** What is the filling rate after 5 minutes?

**b** What is the flow rate in the pipe after 10 minutes?

### Solution

Both the filling rate and the flow rate are given by the instantaneous rate of change of the volume, so we use the derivative.

Calculate the derivative. $\frac{dV}{dt} = 1020 - 34t$

**a** Find value of the derivative at $t = 5$. $\left.\frac{dV}{dt}\right|_{t=5} = 1020 - 34 \times 5$

Evaluate. $= 850$

State the result. The filling rate after 5 minutes is 850 L/min.

**b** Find value of the derivative at $t = 10$. $\left.\frac{dV}{dt}\right|_{t=10} = 1020 - 34 \times 10$

Evaluate. $= 680$

State the result. The flow rate after 10 minutes is 680 L/min.

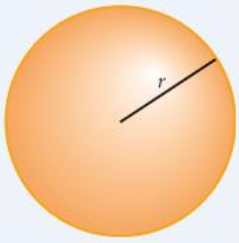
WS Instantaneous rates of change



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## INVESTIGATION Errors and derivatives

The volume of a sphere is given by $V = \frac{4\pi r^3}{3}$.




1 Calculate the volume of a sphere of radius 8 m.

2 Calculate the volume of a sphere of radius 8.1 m.

3 What effect does an error of 0.1 m in the radius have on the volume of a sphere (from these calculations)?

4 Calculate the effect on the volume of an error of 1 m in the radius (0.5 m either side of 8 m).

5 Calculate the value of $\left.\frac{dV}{dr}\right|_{r=8}$. What unit does this value have?


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6 Compare your answers to questions **4** and **5**.

7 What would you expect the approximate error to be if there was a 0.4 m error in the radius (at 8 m again)?

8 Repeat questions **4** to **6** for $r = 50$ m (error of 1 m).

9 Use the derivative to calculate the approximate error in volume if a 1 m error is made in the radius and $r \approx 700$ m.

10 What approximate error in the volume would occur if a mistake of 0.2 m were made in the radius at a radius of 700 m?

# EXERCISE 10.06 Applications of derivatives

## Concepts and techniques

1 Example 15 Find a formula for the rate of change for each of the following functions.

a $h = 30t - 4t^2$ b $D = 5t^3 + 2t^2 + 1$ c $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ e $V = r^n + 4$ f $S = 2\pi r + \frac{50}{r^2}$

g $S = 800r + \frac{400}{r}$ h $V = \frac{4}{3}\pi r^3$

2 Examples 16, 17 The volume of water flowing through a pipe is given by $V = t^2 - 3t$ L. Find the rate at which the water is flowing when $t = 5$ s.

3 Find the rate of change of $y = 7x^3 + 2x^2 - 12x + 5$ at:

a $x = 1$ b $x = -2$ c $x = 3$ d $x = 0$ e $x = 5$

4 Find the rate of change of $E = 2x^2 + 4x^3 - 18x$ at:

a $x = -2$ b $x = 3$ c $x = -1$ d $x = 0$ e $x = -14$



**5** $s = t^2 - 2t^3 + 5t - 6$

**a** Find the average rate of change from $t = 2$ to $t = 8$.

**b** Find the instantaneous rate of change at:

i $t = 2$ ii $t = 8$ iii $t = 5$

**c** Compare the average rate of change and the instantaneous rates of change.

## Reasoning and communication

**6** The approximate height of a pebble thrown into a well is given by $h = 2 + 9t - 5t^2$

where height $h$ is in metres and time $t$ is in seconds.

**a** Draw a graph showing the height from 0 s to 3 s in 0.2 s intervals.

**b** Calculate the velocity $\frac{dh}{dt}$ of the pebble.

**c** Draw a graph of the velocity of the pebble from 0 s to 3 s in 0.2 s intervals.

**d** Calculate the average velocity from 0 s to 3 s.

**e** Compare the average velocity to the instantaneous velocities at 0 s and 3 s.

**7** Refer back to the investigation 'Errors and derivatives' which precedes this exercise.

**a** Use derivatives to find the approximate error made in the volume of a cylindrical water tank of radius 4 m and height 3 m if the radius has an error of 0.09 m.

**b** What approximate error would be made in the volume of the water tank above if an error of 0.09 m were made in the height instead of the radius?

**8** An iceberg approximates the shape of a rectangular prism and is about 800 m long, 400 m wide and 120 m thick. As it moves into warmer water it melts evenly by 1 m per day along each measurement.

**a** Find the volume of the iceberg after $t$ days.

**b** Find the rate of change of the volume after $t$ days.

**c** How much ice melts on the 13th day?

**d** How much melts on the 15th day?



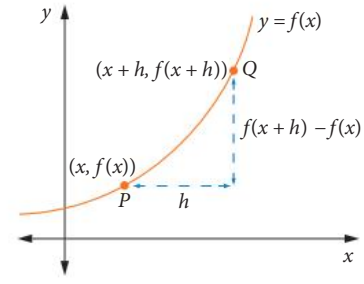
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# 10.07 ANTIDERIVATIVES

We have previously defined the derivative of the function $y = f(x)$ as

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.





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The derivative of a function is also known as the gradient function. The derivative gives the rate of change of the original function. For a polynomial function, we can use the derivative to calculate the gradient of the polynomial at any point on the curve. This is also the instantaneous rate of change at that point. Work through the following investigation.

## INVESTIGATION Differentiating forwards and backwards

We use the process called differentiation to derive the gradient function or derivative, $f'(x)$, from the original function, $f(x)$. The reverse of differentiation is known as **antidifferentiation** or **integration**. If we know the derivative, we can differentiate in reverse or work backwards from it to find a possible original function from the gradient function.

1. Copy the table below into your workbook.
2. Complete the table by working down the list in order.
3. As you go down the list, look for patterns that emerge.
4. Describe in words how the original function can be found from the derivative.

| $f(x)$ | $f'(x)$ |
|---|---|
| $5x$ | |
| | 7 |
| $x^2$ | |
| | $2x$ |
| | $x$ |
| | $10x$ |
| | $7x$ |

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^3$ | |
| | $6x^2$ |
| | $x^2$ |
| | $12x^2$ |
| | $5x^2$ |
| $x^4$ | $4x^3$ |
| | $16x^3$ |

| $f(x)$ | $f'(x)$ |
|---|---|
| | $25x^3$ |
| | $20x^4$ |
| | $17x^4$ |
| | $24x^5$ |
| | $3x^5$ |

What is the pattern for finding a function whose derivative is a power?

The pattern can be 'Add 1 to the power of $x$ and divide by the new power'. We can write this in symbols as follows.

**IMPORTANT**

An **antiderivative** of a function is another function whose derivative is the given function.

If $f'(x) = ax^n$, then $f(x) = \frac{ax^{n+1}}{n+1}$ (where $n \geq 0$), so $\frac{ax^{n+1}}{n+1}$ is a possible antiderivative of $ax^n$.

You can check this by differentiating $\frac{ax^{n+1}}{n+1}$



## Example 18

Find an antiderivative of $f(x) = 3x^2 - 10x + 4$.

### Solution

Write down the function.

$f(x) = 3x^2 - 10x + 4$

Let $g(x)$ be an antiderivative of $f(x)$ and apply the pattern to each term.

$$g(x) = \frac{3x^{2+1}}{2+1} - \frac{10x^{1+1}}{1+1} + \frac{4x^{0+1}}{0+1}$$
$$= \frac{3x^3}{3} - \frac{10x^2}{2} + \frac{4x}{1}$$

Simplify.

$g(x) = x^3 - 5x^2 + 4x$

In Example **18**, you could find many other antiderivatives of $f(x)$, such as $x^3 - 5x^2 + 4x + 1$. $x^3 - 5x^2 + 4x + 5$ and $x^3 - 5x^2 + 4x - 7$ are also antiderivatives of $f(x) = 3x^2 - 10x + 4$. The derivative of a constant is zero, so you can make as many antiderivatives as you like by adding constants to one antiderivative. The antiderivative of a function is usually written with a constant term '$+ c$'. Any particular antiderivative can then be obtained by substituting the appropriate value for the constant. An antiderivative is conventionally shown using the capital letter of the function name.

**IMPORTANT**

For a function $f(x)$, if $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

The **antiderivative** is also called a **primitive** or **indefinite integral**.

All other antiderivatives of $f(x)$ are of the form $F(x) + c$, where $c$ is a constant.

It is very important to keep in mind that an antiderivative is a function. However, all the antiderivative functions of a particular function differ only by constants.

It may be necessary to express a function in polynomial form before you can find the antiderivative.

## Example 19

Find $y$ if $\frac{dy}{dx} = (3x - 5)^2$.

### Solution

Write down the information.

$\frac{dy}{dx} = (3x - 5)^2$

Expand the RHS.

$= 9x^2 - 30x + 25$

Find the antiderivative.

$$y = \frac{9x^3}{3} - \frac{30x^2}{2} + 25x + c$$

Simplify.

$= 3x^3 - 15x^2 + 25x + c$


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More information is required if the value of the constant is to be calculated.

### Example 20

Find the integral of $f(x) = 3x - 5$, given that $F(3) = 6.5$.

WS Antiderivatives

#### Solution

| | |
|---|---|
| Write down the function. | $f(x) = 3x - 5$ |
| Find the antiderivative from the rules. | $F(x) = \frac{3x^2}{2} - 5x + c$ |
| Write the given information. | $F(3) = 6.5$ |
| Substitute $x = 3$. | $6.5 = \frac{3 \times 3^2}{2} - 5 \times 3 + c$ |
| Simplify. | $6.5 = 13.5 - 15 + c$ |
| Solve for $c$. | $c = 8$ |
| Write the result. | $F(x) = \frac{3x^2}{2} - 5x + 8$ |

## EXERCISE 10.07 Antiderivatives

### Concepts and techniques

**1** Examples 18 Find any two possible antiderivatives for each of the following functions.

**a** $f(x) = 6x^2 + 4x + 1$
**b** $m(t) = 8t - 7$
**c** $g(z) = 4z^3 + 3z - 5z^4$
**d** $h(x) = 4$
**e** $p(y) = y^2 - y - 1$
**f** $S(v) = 3v^3 - v^4 + 4v^2 - 2$
**g** $v(t) = 9.8t - 0.3t^2 - 6$
**h** $j(m) = 4.5(m + 0.3)^2 + 0.8$
**i** $p(d) = 0.1d^3 + 7d - 0.38$
**j** $h(k) = 5k^2 - 2k^3 + 2k - 5$

**2** Find the antiderivative of each of the following functions.

**a** $q(h) = 4h^3 - 6h^2 - 2h - 3$
**b** $f(x) = x^3 - 10x^4 - 6x^2 + 12$
**c** $y(u) = 2u - 5u^3 - 4u^2 + 8$
**d** $t(m) = 0.8m^2 - 6.3m^3 - 0.7m + 4$
**e** $r(t) = 0.07t + t^2$
**f** $v(x) = 4.9x - 5x^2$

**3** Example 19 Find the antiderivative of each of the following functions.

**a** $f(x) = (3x + 4)^2$
**b** $m(p) = 2p(p - 7)$
**c** $g(n) = (n - 4)^3$
**d** $v(y) = y(y - 5)(4y + 3)$

**4** Find $f(x)$ if:

**a** $f'(x) = x^2 - 5$
**b** $f'(x) = 3x + 2$
**c** $f'(x) = 7 - x^3 + x^2$
**d** $f'(x) = x^4$
**e** $f'(x) = (x + 4)^2$
**f** $f'(x) = x^4 - x^3 + x^2 - x + 1$



**5** Find $y$ in each of the following.

**a** $\frac{dy}{dx} = 4x^3 + 6x^2 - 6x + 7$

**b** $\frac{dy}{dx} = 12x^3 - 12x^2 + 12x - 12$

**c** $\frac{dy}{dt} = 3t + 5$

**d** $\frac{dy}{dz} = z - z^3 + 8$

**e** $\frac{dy}{dx} = 0.5x - 0.04x^3 + 0.7x^3 - 0.52$

**f** $\frac{dy}{dt} = 0.7t - 0.4t^2 + 0.6$

**6** Example 20 Find the integral of each of the following functions using the given conditions.

**a** $g(v) = 10v - 3$, where $G(2) = 16$

**b** $f(x) = 9x^2 - 8x + 4$, where $F(-1) = -16$

**c** $p(y) = 4y^3 - 24y^2$, where $P(-2) = 86$

**d** $h(x) = 3x(x - 2)$, where $H(1) = 3$

## Reasoning and communication

**7** $\frac{dy}{dx} = 4x - 5$ and (3, 2) is a point on the graph of $y$. Find the function $y$ and its value at $x = 5$. Justify your reasoning.

**8** When $t = 1$, the value of $x$ is $-2$. Find $x$ at $t = 4$, given that $\frac{dx}{dt} = 0.6t^2 - 2t + 4$. Show logical steps in your working.

**9** If $f'(x) = 6x - 12x^2$ and $f(-1) = 5$, find $f(1)$. Give clear reasons for your working.

**10** If $g'(v) = v - v^3 - 1$ and $g(v)$ has a zero at $v = 3$, find $g(4)$ explaining your steps.

**11** A train slows down to take a tight curve. When it enters the straight, it accelerates at 0.4 m/s$^2$. 10 seconds after entering the straight, it has travelled a distance of 100 m down the straight.

**a** Find an expression for the train's velocity after entering the straight.

**b** Find an expression for its position on the straight.

**c** Find the speed at which it left the curve.

**12** A future space exploration team on a 5-year mission to explore the galaxy lands on a lonely planet circling a far distant star. Scotty, the star-ship's engineer, throws a spanner from the top of the star-ship down to Mr Sulu 33 m directly below on the ground with an initial speed of 2 m/s. Mr Sulu wisely decided not to catch the spanner, but did notice it took 3 seconds to reach the ground.

**a** Assuming that the acceleration of gravity on the planet is $p$ m/s$^2$, write an expression for the velocity of the spanner $t$ seconds after Scotty threw it.

**b** Write an expression for its position after $t$ seconds, taking the ground level as zero.

**c** Find $p$, the acceleration due to gravity on the planet.

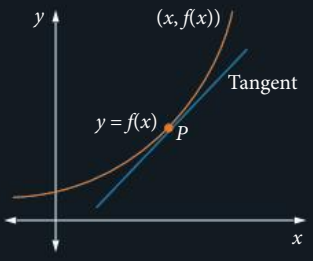
**d** Compare the size of the planet to that of Earth.


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# CHAPTER SUMMARY 10

# PROPERTIES OF DERIVATIVES

- The **derivative** of the function $f(x)$ is written as $f'(x)$, $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$.




- The derivative of the function $f(x)$ is the gradient of the function at the point $x$, so $f'(x)$ is called the **gradient function**.
- The sign of the derivative shows whether a function is **increasing** ($f'(x) > 0$), **decreasing** ($f'(x) < 0$) or **stationary** ($f'(x) = 0$).
- The rules for finding derivatives where $a$, $b$ and $k$ are constants are:

If $y = f(x) = k$, then $\frac{dy}{dx} = f'(x) = 0$

If $y = kf(x)$, then $\frac{dy}{dx} = kf'(x) = k\frac{df}{dx}$

If $y = f(x) = kx^n$, then $\frac{dy}{dx} = f'(x) = knx^{n-1}$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$$

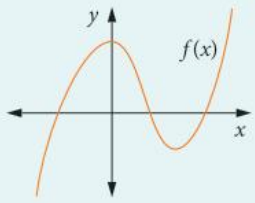
- The derivative of a polynomial is found using the rules for the derivative of $x^n$ and linear combinations.
- The derivative of a function is the **instantaneous rate of change** of the function.
- The reverse of the process of differentiation is called **antidifferentiation** or **integration**.
- An **antiderivative** is also called a **primitive** or **indefinite integral**.
- The antiderivative of $ax^n$ is $\frac{ax^{n+1}}{n+1} + c$ (where $n \geq 0$)
- If $F'(x) = f(x)$, then all antiderivatives are of the form $F(x) + c$.

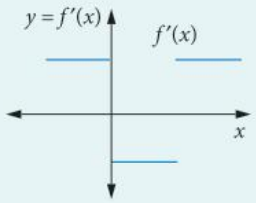
# 10 CHAPTER REVIEW

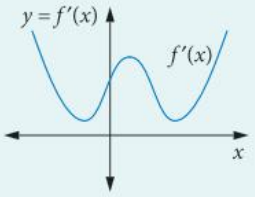
# PROPERTIES OF DERIVATIVES

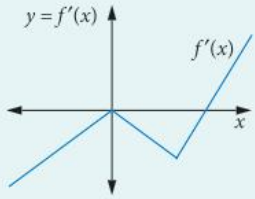
## Multiple choice

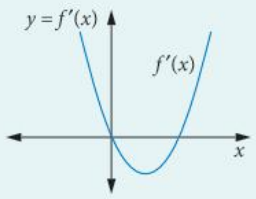
1 Example 1 Which of the possible derivative graphs $f'(x)$ is most likely to be the gradient function of $f(x)$?

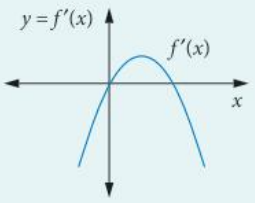



A 


B 


C 


D 

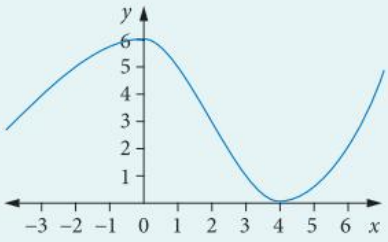

E 


2 Example 3 $P = 3r^3 - 8r + 4$. At $r = 2$, $P$ is:

A increasing B stationary C negative
D decreasing E 0

3 Example 3 For the function shown on the right, the derivative is positive for:

A $0 < x < 6$ B $x > 4$ only
C $x < 0$ and $x > 4$ D $0 < x < 4$
E $x < 0$ only





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4 Example 9 If $f(x) = 3x^2 - x + 2$, then $f'(4) =$

A 5 B 10 C 11 D 23 E 46

5 Example 9 For $f(x) = x^3 - 6x^2 + 7$, $f'(x) =$

A $3x^2 - 12x + 7$ B $3x^3 - 12x + 7$ C $x^2 - 6x$

D $6x^2 - 12x + 7$ E $3x^2 - 12x$

6 Example 15 The rate of change of $f(x) = 4x^2 - 5x + 7$ at $x = 2$ is:

A 3 B 7 C 10 D 11 E 18

7 Example 18 The function $f(x) = 12x^2 - 10x + 2$ has an antiderivative of:

A $12x^3 - 10x^2 + 2x$ B $4x^2 - 5x + 2$ C $4x^3 - 5x^2 + 2x + 7$

D $6x^3 - 5x^2 + 2x + 7$ E $12x^3 - 10x^2 + 2x + 7$

8 Examples 18, 19 The antiderivative of $(5x - 2)^2$ is:

A $5x^3 - 10x^2 + 4x + c$ B $\frac{5}{3}x^3 - 10x^2 + 4x + c$ C $\frac{25}{3}x^3 + 10x^2 - 4x + c$

D $\frac{25}{3}x^3 - 10x^2 + 4x + c$ E $\frac{15}{3}x^2 + 10x^2 + 4x + c$

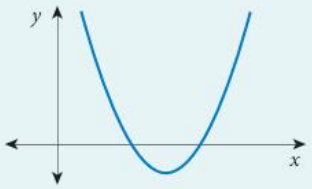
9 Example 20 Find the integral of $f(x) = 4x^3 - 12x$, given that $F(2) = 2$.

A $12x^2 - 12$ B $12x^2 - 12x + 2$ C $x^4 - 6x^2 + 10$

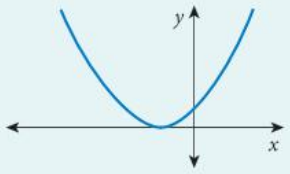
D $x^4 - 6x^2 + 2$ E $x^4 - 6x^2 - 6$

## Short answer

10 Example 1 The graph of a function is shown on the right. Use it to draw a graph of the corresponding gradient function.




11 Example 2 The graph of a gradient function is shown on the right. Use it to draw the graph of a corresponding function.




12 Example 3 State whether the function $f(x) = x^3 - 4x^2 - 4x + 16$ is increasing, decreasing or stationary at:

a $x = -3$ b $x = -2$ c $x = 0$ d $x = 2$

e $x = 3$ f $x = 4$ g $x = 5$

13 Examples 4, 5 Find the derivatives of the following from first principles.

a $x^4 + x$ b $x^3 - x^2$

14 Examples 7, 8 Find the derivatives of the following.

a $5x^4$ b $2x^5 + 3x^2$

15 Examples 7–9 Calculate the following.

a $\frac{d}{dx}(5x^4 - 3x + 8)$

b $\frac{d}{dx}(3x^2 - 4x + 1)\Big|_{x=4}$

c $\frac{d}{dp}(8p - 3p^5 + p^8)\Big|_{p=-2}$

d $\frac{d}{dm}(20m)\Big|_{m=10}$

e $\frac{d}{dt}(8)\Big|_{t=4}$

16 Example 9 From the following, find the function that is steepest at $x = 4$.

A $f(x) = 3x^2 - x + 4$

B $g(x) = 5x - 6x^2 - 10$

C $h(x) = x^3$

D $i(x) = x^2 - 2x^3$

17 Example 9 Find the derivatives of the following functions.

a $p(x) = 5x^6 - 3x^2 + x + 10$

b $g(x) = 4 - u - 3u^2 + 6u^3$

c $q(x) = q^7 + 4q^3 - q + 1$

d $m(x) = 15y^6 - y^4 - 28y$

e $n(x) = 12 + w - 3w^2 + w^3$

18 Example 10 Find the derivative of the function $y = (3x^3 - 7)^2$.

19 Example 10 Find the derivative of the function $y = (2x + 5)^3$.

20 Example 12 $P = 5v^2 - 3v + 9$. Find the gradient of the tangent at:

a $v = -1$ b $v = 3$ c $v = 0$ d $v = -3$ e $v = 10$

21 Example 12 $y = 3x^4 - 4x^3 + x - 3$. Find the equation of the tangent at:

a $x = 1$ b $x = -1$ c $x = 0$ d $x = 2$ e $x = -2$

22 Example 12 $y = x^3 - 4x^2 + 3x + 8$. Find the gradient of the tangent at:

a $x = 2$ b $x = -3$ c $x = 1$ d $x = 0$ e $x = -5$

23 Example 13 Find the values of $x$ for which the tangent to the curve $y = \frac{x^3}{3} + \frac{x^2}{2} - 12x + 4$ is equal to 0.

24 Example 15 Find the rate of change of $f(x) = 6x^4 - 5x^3 + 8x$ at:

a $x = 1$ b $x = -2$ c $x = -4$ d $x = 5$ e $x = 0$

25 Example 15 Find the instantaneous rate of change of $y = 3t^2 + 5t + 7$ at:

a $t = 3$ b $t = -5$ c $t = -1$ d $t = 0$ e $t = 8$

26 Example 18 Find two antiderivatives of:

a $f(x) = 4x^2 - 2x + 5$

b $h(x) = 3 - 4x^5$

27 Example 19 Find $y$ if:

a $\frac{dx}{dy} = 7x - 3x^2$

b $y' = 5x^4 + 2x$



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## Application

**28** Find where the gradient function of $f(x) = x^2 + 7x - 3$ is higher than the gradient function of $g(x) = 3x^2 - 5x - 8$.

**29** $y = x^2 + 2x^3 + 5x - 1$.

- **a** Find the equation of the tangent at $x = 3$.
- **b** Find the equation of the tangent at $x = -1$.
- **c** Find the acute angle that the tangent at $x = -1$ makes with the $x$-axis.

**30** $p = 3x^2 - 5x + 2$.

- **a** Find the equation of the tangent at $x = 2$.
- **b** Find where the tangent at $x = 2$ cuts the $x$-axis.
- **c** Find where the tangent at $x = 2$ cuts the $p$-axis.
- **d** Find the length of the tangent at $x = 2$ cut off by the axes.
- **e** Find the length of the tangent at $x = 1$ cut off by the axes.

**31** A circular disc expands as it is heated. The area, in $cm^2$, of the disc increases according to the formula:

$$A = 4t^2 + t$$

where time is in minutes. Find the rate of increase in the area after 5 minutes.

**32** The electrical resistance of a metal wire is given by

$$R \approx R_0(1 + 0.004T + 0.000\,001T^2)$$

where $R_0$ is the resistance at 0°C and $T$ is the temperature in degrees Celsius. A particular platinum wire has a resistance of 12 Ω (ohms) at 0°C.

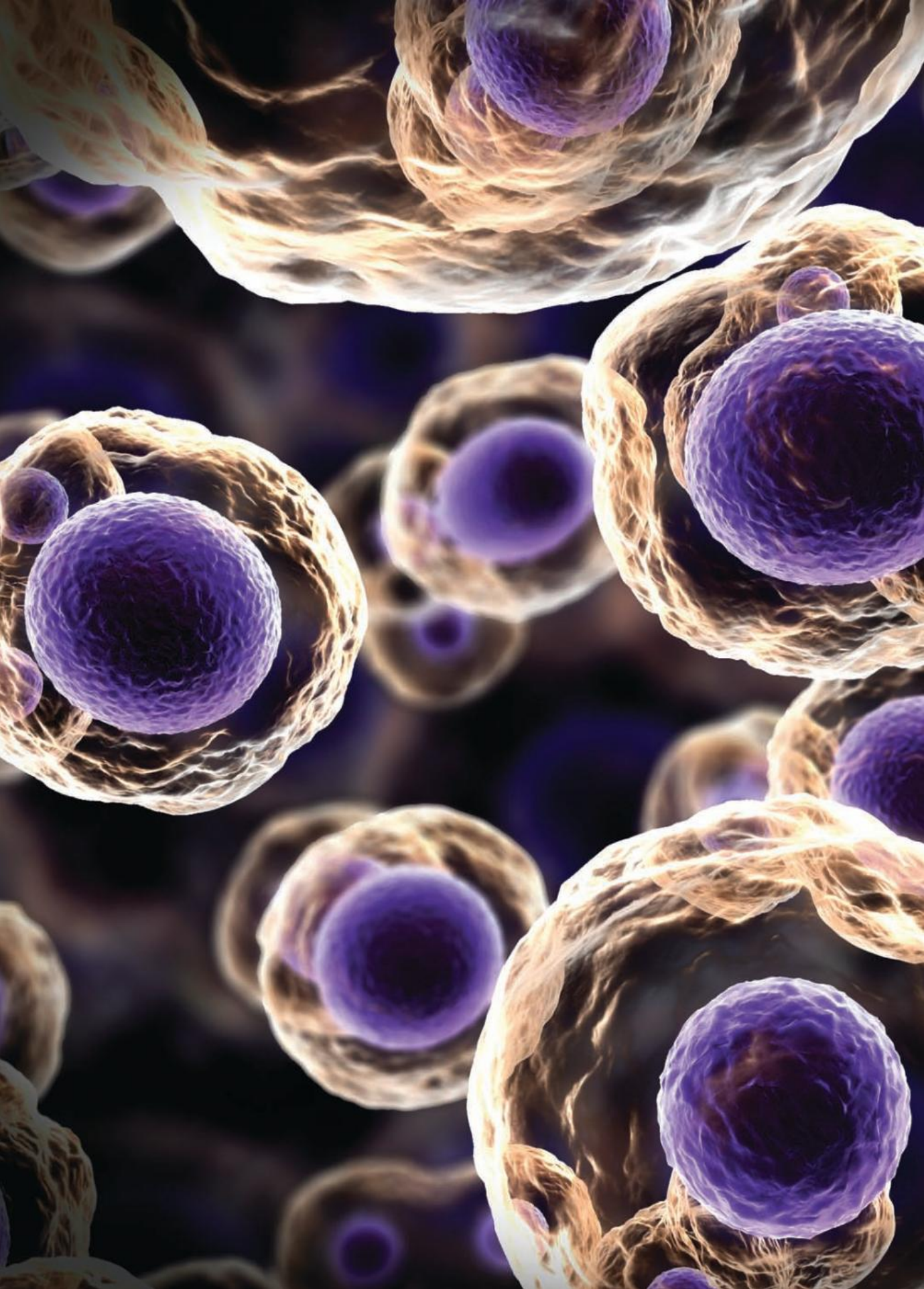
- **a** What is its resistance at 400°C?
- **b** What is the rate of change of resistance with temperature at 400°C?
- **c** What is the resistance at 500°C?
- **d** What is the average rate of change of resistance from 400°C to 500°C?
- **e** What is the resistance at −120°C?
- **f** What is the rate of change of resistance at −120°C?
- **g** Suggest a use for such a wire. Note that platinum has a melting point of 1770°C.

**33** Use derivatives to find the approximate error in volume if a 2% error in radius occurred when cylinders of a four-cylinder car engine were bored. The correct ratio of radius to length was 1 : 1 and the engine capacity was supposed to be 2000 cc. What would be the capacity if the cylinders were over-bored?

**34** $m'(x) = 3x - 2x^2 + 4$ and $m(x)$ has a zero at $x = 5$. Find the $y$-intercept of $m(x)$, explaining your steps.



Practice quiz





# 11

## TERMINOLOGY

base
decay function
exponent
exponential equation
exponential function
growth factor
growth function
half-life
horizontal asymptote
index
indicial equation
initial value
power
radical
radiocarbon dating
scientific notation
significant figures
transformation

Prior learning

## EXPONENTIAL FUNCTIONS

# EXPONENTIAL FUNCTIONS


11.01 Index laws
11.02 Radicals and fractional indices
11.03 Scientific notation and accuracy
11.04 Exponential functions
11.05 Exponential graphs
11.06 Transformations of exponential graphs
11.07 Exponential equations
11.08 Exponential models
Chapter summary
Chapter review

## INDICES AND THE INDEX LAWS

- review of indices (including fractional indices) and the index laws (ACMMM061)
- use radicals and convert to and from fractional indices (ACMMM062)
- understand and use scientific notation and significant figures (ACMMM063)

## EXPONENTIAL FUNCTIONS

- establish and use the algebraic properties of exponential functions (ACMMM064)
- recognise the qualitative features of the graph of $y = a^x$ ($a > 0$) including asymptotes, and of its translations ($y = a^x + b$ and $y = a^{x+c}$) (ACMMM065)
- identify contexts suitable for modelling by exponential functions and use them to solve practical problems (ACMMM066)
- solve equations involving exponential functions using technology, and algebraically in simple cases (ACMMM067) AC

# 11.01 INDEX LAWS

You have previously seen that the term $a^n$ is called a **power** where:

$$a^n = a \times a \times a \times \ldots \times a \text{ (to } n \text{ factors)}$$

For the power, $a^n$, $n$ is called the **index** or **exponent** and $a$ is called the **base**.

There are some general laws that simplify calculations with indices.

Each of the index laws can be proved from first principles using the definition:

$a^n = a \times a \times a \times \ldots$ (to $n$ factors)

**IMPORTANT**

**Index laws**

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$ (for $m > n$)

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

### Example 1

Prove that $(ab)^n = a^n b^n$.

#### Solution

Expand $(ab)^n$.

$(ab)^n = ab \times ab \times ab \times \ldots$ (to $n$ factors)

Regroup.

$= \underbrace{a \times a \times a \times \ldots}_{n \text{ factors}} \times \underbrace{b \times b \times b \times \ldots}_{n \text{ factors}}$

Use the definition of an index.

$= a^n \times b^n$

$= a^n b^n$


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## Example 2

Simplify:

**a** $m^9 \times m^7 \div m^2$ **b** $(2y^4)^3$

### Solution

**a** Write the expression. $m^9 \times m^7 \div m^2$

Apply the first and second index laws. $= m^{9+7-2}$

Evaluate the power. $= m^{14}$

**b** Write the expression. $(2y^4)^3$

Apply the third index law. $= 2^3 \times y^{4\times 3}$

Evaluate. $= 8y^{12}$

You can extend the definition of powers to cover the index of zero and negative indices.

**IMPORTANT**

$a^0 = 1 \ (a \neq 0)$

$a^{-n} = \frac{1}{a^n} \ (a \neq 0)$

## Example 3

Simplify and express with positive indices:

**a** $\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5}$ **b** $\frac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m}$



Review of index laws

### Solution

**a** Write the expression. $\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5}$

Simplify the numerical part and apply the index laws. $= \frac{12}{6} \times h^{-6+2-(-3)-5}$

Evaluate and simplify. $= 2h^{-6}$

Write with a positive exponent. $= \frac{2}{h^6}$

**b** Write the expression. $\frac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m}$

Express the bases as products of prime factors. $= \frac{2^{m-n} \times 3^{m-n} \times (2\times 3)^n}{(2^2 \times 3)^m}$

Apply the index laws. $= \frac{2^{m-n} \times 3^{m-n} \times 2^n \times 3^n}{2^{2m} \times 3^m}$

Apply the index laws. $= 2^{m-n+n-2m} \times 3^{m-n+n-m}$

Evaluate the exponents. $= 2^{-m} \times 3^0$

Apply the definition of powers. $= \frac{1}{2^m} \times 1$

Evaluate. $= \frac{1}{2^m}$



# EXERCISE 11.01 Index laws

## Concepts and techniques

**1** Example 2 Simplify the following.

a $a^6 \times a^9 \times a^2$ b $y^3 \times y^{-8} \times y^5$ c $a^{-1} \times a^{-3}$

d $w^{\frac{1}{2}} \times w^{\frac{1}{2}}$ e $x^6 \div x$ f $p^3 \div p^{-7}$

g $\dfrac{y^{11}}{y^3}$ h $(x^7)^3$ i $(2x^5)^2$

j $(3y^{-2})^4$ k $a^3 \times a^5 \div a^7$ l $\left(\dfrac{x^2}{y^9}\right)^5$

m $\dfrac{w^6 \times w^7}{w^3}$ n $\dfrac{p^2 \times (p^3)^4}{p^9}$ o $\dfrac{x^6 \div x^7}{x^2}$

p $\dfrac{a^2 \times (b^2)^6}{a^4 \times b^9}$ q $\dfrac{(x^2)^{-3} \times (y^3)^2}{x^{-1} \times y^4}$

**2** Simplify the following.

a $x^5 \times x^9$ b $a^{-1} \times a^{-6}$ c $\dfrac{m^7}{m^3}$

d $k^{23} \times k^6 \div k^9$ e $a^{-5} \times a^4 \times a^{-7}$ f $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

g $\dfrac{m^5 \times n^4}{m^4 \times n^2}$ h $\dfrac{p^{\frac{1}{2}} \times p^{\frac{1}{2}}}{p^2}$ i $(3x^{11})^2$

j $\dfrac{(x^4)^6}{x^3}$

**3** Express with positive indices.

a $a^{-2}$ b $4^{-3}$ c $g^{-7}$ d $5^{-4}$

e $y^{-6}$ f $(cd^2)^{-4}$ g $4h^{-4}$ h $(4h)^{-4}$

i $2k^{-1}$ j $7p^{-4}$ k $8q^{-6}$ l $(4v^{-2})^{-1}$

**4** Express without denominators.

a $\dfrac{1}{y^4}$ b $\dfrac{1}{4^3}$ c $\dfrac{1}{h^9}$ d $\dfrac{t^4}{k^6}$

e $\dfrac{y^2}{p^6}$ f $\dfrac{4j^8}{i^6}$ g $\dfrac{3y^{-2}}{m^5}$ h $\dfrac{2m^{-1}}{z^4}$

i $\dfrac{1}{x^{-8}}$ j $\dfrac{2x^4}{y^{-4}}$ k $\dfrac{3k^{-3}}{b^{-1}}$ l $\dfrac{20r^{-9}}{5w^{-6}}$


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**5** Example 3 Simplify the following.

a $(pq^3)^5$

b $\left(\frac{a}{b}\right)^8$

c $\left(\frac{4a}{b^4}\right)^3$

d $7(a^3b)^2$

e $\frac{(2m^7)^3}{m^4}$

f $\frac{xy^3 \times (xy^2)^4}{xy}$

g $\frac{(2k^8)^4}{(6k^3)^3}$

h $(2y^5)^7 \times \frac{y^{12}}{8}$

i $\left(\frac{a^6 \times a^4}{a^{11}}\right)^{-3}$

j $\left(\frac{5xy^9}{x^8 \times y^3}\right)^3$

**6** Simplify and express with positive indices.

a $r^{-6} \times r^4$

b $4y^{-5} \times 3y^{-4}$

c $\frac{12p^6 \times 6p^{-2}}{8p^{-1} \times 3p}$

d $\frac{4j^5 \times 5j^{-4}}{10j^7}$

e $\frac{2m^3 \times m^{-10}}{12m^6}$

f $\frac{5k^{-3}}{2k^4} \div \frac{10k^5}{k^{-2}}$

g $3g^{-3}h^4 \times 5g^4h^{-6}$

h $5k^3m^{-6} \times (k^{-3}m^{-4})^{-5}$

i $\frac{8v^{-4}w^2 \times 4v^{10}w^{-8}}{2vw^{-9} \times v^{-6}w^3}$

j $\frac{12p^{-4}q^{-3} \times (2pq^4)^2}{(3p^{-1}q^4)^{-3} \times (18p^4)^2}$

k $\left(\frac{e^2f^4}{e^{-5}f}\right)^3 \times \frac{4e^{-3}f^4}{3e^6f^6}$

l $\frac{c^7d^8}{d^{-5}} \div \frac{(8c^{-2}d^3)^2}{4^{-2}c^{-2}d^3}$

**7** Evaluate $a^3b^2$ when $a = 2$ and $b = \frac{3}{4}$.

**8** If $x = \frac{2}{3}$ and $y = \frac{1}{9}$, find the value of $\frac{x^3y^2}{xy^5}$.

**9** If $a = \frac{1}{2}$, $b = \frac{1}{3}$ and $c = \frac{1}{4}$, find the value of $\frac{a^2b^3}{c^4}$.

**10** Simplify $\frac{a^{11}b^8}{a^8b^7}$ and hence find the value of $\frac{a^{11}b^8}{a^8b^7}$ as a fraction when $a = \frac{2}{5}$ and $b = \frac{5}{8}$.

**11** Find the value of $\frac{p^5q^8r^4}{p^4q^6r^2}$ as a fraction when $p = \frac{7}{8}$, $q = \frac{2}{3}$ and $r = \frac{3}{4}$.

## Reasoning and communication

**12** Example 1 Prove that $a^m \times a^n = a^{m+n}$.

**13** Prove that $a^m \div a^n = a^{m-n}$ (for $m > n$).

**14** Prove that $(a^m)^n = a^{mn}$



# 11.02 RADICALS AND FRACTIONAL INDICES

It is possible for indices to be fractional.

Suppose that $a^{\frac{1}{q}}$ exists. Then you can do the following.

$$\left(a^{\frac{1}{q}}\right)^q = a^{\left(\frac{1}{q}\times q\right)} = a^1 = a$$

So, $a^{\frac{1}{q}}$ to the power $q$ would be $a$. In other words, for the laws to be consistent, $a^{\frac{1}{q}}$ would have to be the $q$th root of $a$. This means that:

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

and $a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$ or $(a^{\frac{1}{q}})^p$ so $a^{\frac{p}{q}} = \sqrt[q]{(a)^p} = \left(\sqrt[q]{a}\right)^p$

Now the square root of a negative number is not real, and neither is any even root of a negative number. For example, there are no real numbers for $\sqrt{-4}$ or $\sqrt[4]{-81}$.

This means that the procedure for $a^{\frac{p}{q}}$ will not always work in cases where $a$ is negative. As well as that, the concept of a $q$th root only makes sense if $q$ is positive. For this reason it is assumed that $q$ is positive for fractional indices. If $q$ is negative, then the fractional power has to be rewritten so that the $q$th root part is positive.

The square root sign, $\sqrt{\ }$, is also called the **radical** sign and numbers such as $\sqrt{7}$ and $\sqrt[3]{12}$ are known as **radicals**.

The definition of powers can be extended to rational indices.

**IMPORTANT**

**Index laws for rational indices**

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

$$a^{-\frac{1}{q}} = \frac{1}{\sqrt[q]{a}}$$

$$a^{\frac{p}{q}} = \sqrt[q]{(a)^p} = \left(\sqrt[q]{a}\right)^p$$

where $p$ is an integer, $q$ is a positive integer and $a$ is positive.

## Example 4

Evaluate

**a** $49^{\frac{1}{2}}$ **b** $27^{\frac{2}{3}}$

### Solution

**a** Apply the index law for rational indices. $49^{\frac{1}{2}} = \sqrt{49}$

Evaluate. $= 7$


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**b** Express the base as a product of prime numbers. $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}$

Apply the relevant index law. $= 3^{3 \times \frac{2}{3}}$

Evaluate. $= 3^2$

$= 9$

## Example 5

Write $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}}$ in index form.

### Solution

Apply the index law for rational indices. $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}} = \dfrac{1}{(4x^2-1)^{\frac{2}{3}}}$

Apply the relevant index law. $= (4x^2-1)^{-\frac{2}{3}}$

## Example 6

Write the following using index notation.

**a** $x^3 \times \sqrt{x}$ **b** $\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$

### Solution

**a** Apply the law for rational indices. $x^3 \times \sqrt{x} = x^3 \times x^{\frac{1}{2}}$

Apply index laws. $= x^{3+\frac{1}{2}}$

Evaluate exponents. $= x^{\frac{7}{2}}$

**b** Write the expression. $\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$

Express bases as products of prime factors. $= \sqrt[6]{2^3a^4} \times \sqrt{2^3} \times a^{-\frac{8}{3}}$

Apply the law for rational indices. $= (2^3a^4)^{\frac{1}{6}} \times (2^3)^{\frac{1}{2}} \times a^{-\frac{8}{3}}$

Apply index laws. $= 2^{\frac{3}{6}}a^{\frac{4}{6}} \times 2^{\frac{3}{2}} \times a^{-\frac{8}{3}}$

Simplify exponents and apply index laws. $= 2^{\frac{1}{2}+\frac{3}{2}} \times a^{\left(\frac{2}{3}+-\frac{8}{3}\right)}$

Evaluate exponents and simplify. $= 2^2a^{-2}$

Evaluate and write with a positive exponent. $= \dfrac{4}{a^2}$



# EXERCISE 11.02 Radicals and fractional indices

WS
Fractional indices and radicals

## Concepts and techniques

**1** Example 4 Evaluate the following.

a $81^{\frac{1}{2}}$ b $27^{\frac{1}{3}}$ c $16^{\frac{1}{2}}$ d $8^{\frac{1}{3}}$ e $49^{\frac{1}{2}}$

f $1000^{\frac{1}{3}}$ g $16^{\frac{1}{4}}$ h $64^{\frac{1}{2}}$ i $64^{\frac{1}{3}}$ j $1^{\frac{1}{7}}$

k $81^{\frac{1}{4}}$ l $32^{\frac{1}{5}}$ m $0^{\frac{1}{8}}$ n $125^{\frac{1}{3}}$ o $343^{\frac{1}{3}}$

p $128^{\frac{1}{7}}$ q $256^{\frac{1}{4}}$ r $9^{\frac{3}{2}}$ s $8^{-\frac{1}{3}}$ t $64^{\frac{2}{3}}$

**2** Express as roots (using radical signs)

a $6^{\frac{1}{2}}$ b $z^{\frac{1}{4}}$ c $216^{\frac{1}{3}}$ d $y^{\frac{3}{5}}$

e $49^{\frac{3}{2}}$ f $v^{\frac{7}{3}}$ g $8^{2\frac{1}{3}}$ h $16^{1\frac{1}{4}}$

i $m^{2\frac{1}{2}}$ j $u^{-\frac{1}{4}}$ k $\left(\frac{1}{9}\right)^{-\frac{1}{2}}$ l $(p^{-3})^{\frac{1}{5}}$

**3** Express the following as powers.

a $\sqrt{k}$ b $\sqrt[4]{3}$ c $\sqrt{5}$ d $\sqrt[5]{4}$

e $\sqrt[3]{w}$ f $\sqrt{c^3}$ g $(\sqrt[3]{64})^4$ h $\sqrt[4]{e^3}$

i $\sqrt[9]{28^4}$ j $\sqrt[5]{q^6}$ k $\sqrt[5]{8}$ l $(\sqrt[6]{16})^7$

**4** Example 5 Write in index form.

a $\sqrt{t}$ b $\sqrt[5]{y}$ c $\sqrt{x^3}$ d $\sqrt[3]{9-x}$

e $\sqrt{4s+1}$ f $\dfrac{1}{\sqrt{2t+3}}$ g $\dfrac{1}{\sqrt{(5x-y)^3}}$ h $\sqrt{(3x+1)^5}$

i $\dfrac{1}{\sqrt[3]{(x-2)^2}}$ j $\dfrac{1}{2\sqrt{y+7}}$ k $\dfrac{5}{\sqrt[3]{x+4}}$ l $\dfrac{3}{5\sqrt[4]{(x^2+2)^3}}$

**5** Example 6 Simplify and express with positive indices.

a $a^4 \times \sqrt{a}$ b $b^2 \times \sqrt{b^3}$

c $3n^{-3} \times 2\sqrt{n^5}$ d $6g^5 \times 2\sqrt[3]{g^4}$

e $4\sqrt[3]{w^2} \times 3\sqrt{p^5 w^6}$ f $5\sqrt[3]{m^3 n^{-2}} \times 2\sqrt{m^5 n^6}$

g $7\sqrt[4]{s^2 u^6} \times \sqrt{9s^{-8} u^3}$ h $8\sqrt[3]{p^{12} q^{-8}} \times 3\sqrt[4]{p^{-10} q^9}$

**6** Expand and simplify and write in index form.

a $\left(\sqrt{x}+x\right)^2$ b $\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{a}-\sqrt[3]{b}\right)$ c $\left(p+\dfrac{1}{\sqrt{p}}\right)^2$

d $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ e $\dfrac{\sqrt{x}(x^2-3x+1)}{x^3}$

**7** Write without fractional or negative indices.

a $(a-2b)^{-\frac{1}{3}}$ b $(y-3)^{-\frac{2}{3}}$ c $4(6a+1)^{-\frac{4}{7}}$

d $\dfrac{(x+y)^{-\frac{4}{5}}}{3}$ e $\dfrac{6(3x+8)^{-\frac{2}{9}}}{7}$


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# 11.03 SCIENTIFIC NOTATION AND ACCURACY

Very large or very small numbers are usually written in **scientific notation** to make them easier to read. What could be done to make the figures in the following facts easier to read?

- The Bay of Fundy, Canada, has the largest tidal changes in the world.

About 100 000 000 000 tonnes of water are moved with each tide change.

- The dinosaurs dwelt on Earth for 185 000 000 years until they died out 65 000 000 years ago.
- The width of one plant cell is about 0.000 06 m.
- The total storage capacity of dams in Australia is about 83 853 000 000 000 litres and households in Australia use about 2 108 000 000 000 litres of water annually.

A number in scientific notation is written as a number between 1 and 10 multiplied by a power of 10.

### Example 7

Write 320 000 000 in scientific notation.

#### Solution

The number must be between 1 and 10. $320\,000\,000 = \square \times 10^{\square}$

Insert a decimal point after 3. $= 3.2 \times 10^{\square}$

The number of places after the decimal point tells you the power of 10 by which 3.2 must be multiplied. $= 3.2 \times 10^{8}$

### Example 8

Write $7.1 \times 10^{-5}$ as a decimal number.

#### Solution

Rewrite with a positive power. $7.1 \times 10^{-5} = 7.1 \div 10^{5}$

The power of 10 tells you how many places to move the decimal point.
The division sign tells you the direction to move the decimal point.
Move the decimal point 5 places to the left. $= 0.000\,071$



The concept of **significant figures** is related to rounding off. When you look at very large (or very small) numbers, some of the smaller digits are not significant.

For example, in a football crowd of 49 976, the 6 people are not really significant in terms of a crowd of about 50 000! Even the 76 people are not significant.

To round off to a certain number of significant figures, you count from the first non-zero digit.

In any number, non-zero digits are always significant. Zeros are not significant, except between two non-zero digits or at the end of a decimal number.

Scientific notation uses the significant figures in a number, so:

| | |
|---|---|
| $12\,000 = 1.20 \times 10^4$ | correct to 3 significant figures |
| $0.000\,043\,50 = 4.350 \times 10^{-5}$ | correct to 4 significant figures |
| $0.0207 = 2.07 \times 10^{-2}$ | correct to 3 significant figures |

When rounding off to significant figures, use the usual rules for rounding off.

## Example 9

In each case below, round off to the number of significant figures shown in brackets.

**a** 4 592 170 (3 significant figures)

**b** 0.248 391 (2 significant figures)

**c** 1.396 794 (3 significant figures)

### Solution

**a**

| | |
|---|---|
| Write the number. | 4 592 170 |
| 9 is the third significant figure. | 4 59**2** 170 |
| The number after 9 is 2. | 4 59**2** 170 |
| $2 < 5$, so 9 is not rounded up. | 4 592 170 ≈ 4 590 000 to |
| Make the remaining digits zeros. | 3 significant figures. |

**b**

| | |
|---|---|
| Write the number. | 0.248 391 |
| 4 is the second significant figure. | 0.2**4**8 391 |
| The number after 4 is 8. | 0.2**48** 391 |
| $8 \geq 5$, so 4 is rounded up. | 0.2**50** 000 |
| Because this is a decimal number, numbers after | 0.248 391 ≈ 0.25 to |
| the last significant figure are not shown. | 2 significant figures |

**c**

| | |
|---|---|
| Write the number. | 1.396 794 |
| 9 is the third significant figure. | 1.3**9**6 794 |
| The number after 9 is 6. | 1.3**96** 794 |
| $6 \geq 5$, so 9 is rounded up and 1.39 becomes 1.40. | 1.**400** 000 |
| The first 0 results from rounding and is the third significant | 1.**400** |
| figure so is included. Numbers after the last significant | 1.396 794 ≈ 1.40 to |
| figure are not shown. | 3 significant figures |


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# EXERCISE 11.03 Scientific notation and accuracy

## Concepts and techniques

WS
Scientific notation and significant figures

**1** Example 7 Write the following in scientific notation.

| | | |
|---|---|---|
| **a** 3 800 | **b** 1 230 000 | **c** 61 900 |
| **d** 12 000 000 | **e** 8 670 000 000 | **f** 416 000 |
| **g** 900 | **h** 13 760 | **i** 20 000 000 |
| **j** 80 000 | | |

**2** Write in scientific notation.

| | | |
|---|---|---|
| **a** 0.057 | **b** 0.000 055 | **c** 0.004 |
| **d** 0.000 62 | **e** 0.000 002 | **f** 0.000 000 08 |
| **g** 0.000 007 6 | **h** 0.23 | **i** 0.008 5 |
| **j** 0.000 000 000 07 | | |

**3** Example 8 Write as decimal numbers.

| | | |
|---|---|---|
| **a** $3.6 \times 10^4$ | **b** $2.78 \times 10^7$ | **c** $9.25 \times 10^3$ |
| **d** $6.33 \times 10^6$ | **e** $4 \times 10^5$ | **f** $7.23 \times 10^{-2}$ |
| **g** $9.7 \times 10^{-5}$ | **h** $3.8 \times 10^{-8}$ | **i** $7 \times 10^{-6}$ |
| **j** $5 \times 10^{-4}$ | | |

**4** Example 9 Round these numbers to 2 significant figures.

| | | |
|---|---|---|
| **a** 235 980 | **b** 9 234 605 | **c** 10 742 |
| **d** 0.364 258 | **e** 1.293 542 | **f** 8.973 498 011 |
| **g** 15.694 | **h** 322.78 | **i** 2904.686 |
| **j** 9.0741 | | |

**5** Evaluate correct to 3 significant figures.

**a** $14.6 \times 0.453$ **b** $4.8 \div 7$ **c** $4.47 + 2.59 \times 1.46$ **d** $\dfrac{1}{3.47 - 2.7}$

**6** Evaluate $4.5 \times 10^4 \times 2.9 \times 10^5$, giving your answer in scientific notation.

**7** Calculate $\dfrac{8.73 \times 10^{-3}}{1.34 \times 10^7}$ and write your answer using scientific notation, correct to 3 significant figures.



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# 11.04 EXPONENTIAL FUNCTIONS

An epidemic occurs when new cases of a disease appear at a much greater rate than normally expected. If an epidemic spreads through human populations across a large area, it is known as a pandemic. Famous examples of pandemics include the bubonic plague (black death) in Medieval Europe, the Spanish influenza (flu) pandemic in 1918–19 and the sudden acute respiratory syndrome (SARS) epidemic in 2003.

One of the reasons that a bacterial disease like cholera spreads so quickly through untreated water supplies is that the bacterium multiplies rapidly. In a research laboratory, a culture containing 20 μg of bacteria was observed over a period of time. It was initially found to increase by 30% every 10 minutes. Growth at this rate for an hour is shown in the following table.

| Number of 10-minutes time periods | Growth ratio | Mass of bacteria (μg) |
|---|---|---|
| 0 | | 20 |
| 1 | 1.3 times | $20 \times 1.3$ |
| 2 | 1.3 times | $20 \times 1.3 \times 1.3 = 20 \times (1.3)^2$ |
| 3 | 1.3 times | $20 \times (1.3)^2 \times 1.3 = 20 \times (1.3)^3$ |
| 4 | 1.3 times | $20 \times (1.3)^3 \times 1.3 = 20 \times (1.3)^4$ |
| 5 | 1.3 times | $20 \times (1.3)^4 \times 1.3 = 20 \times (1.3)^5$ |
| 6 | 1.3 times | $20 \times (1.3)^5 \times 1.3 = 20 \times (1.3)^6$ |

So after an hour the bacteria had grown from 20 μg to $20 \times (1.3)^6$ μg or about 97 μg, which is a growth of almost 500%.

From the table, it can be seen that the mass in μg of the bacterium can be modelled as

$$M(t) = 20 \times (1.3)^t$$

where $t$ is the number of 10-minute periods after the commencement of the observations.

The growth of this bacteria is an example of a function where the variable is an exponent. This is known as an **exponential function**.

**IMPORTANT**

The general form of a simple **exponential function**, is:

$f(x) = Aa^x$

where $A$ and $a$ are constants, $a > 0$ and $a \neq 1$.

For $A$, $a > 0$, the domain is the real numbers and the range is the positive real numbers.


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It is important not to confuse exponential functions ($y = a^x$) with power functions ($y = x^n$). Functions like $y = 6 \times 5^{4x+3}$ and $f(x) = 7^{x^2}$ are also exponential functions, but not simple ones.

## Example 10

An exponential function is given by $f(x) = 12 \times 3^x$. Calculate the value of:

**a** $f(0)$ **b** $f(2)$ **c** $f(-2)$

### Solution

**a** Write down the function. $f(x) = 12 \times 3^x$

Substitute for $x$. $= 12 \times 3^0$

Evaluate. $= 12$

**b** Substitute for $x$. $f(2) = 12 \times 3^2$

Evaluate. $= 108$

**c** Substitute for $x$. $f(-2) = 12 \times 3^{-2}$

Evaluate. $= 12 \times \frac{1}{3^2}$

Simplify. $= \frac{12}{9}$

$= \frac{4}{3}$

You need to use the index laws when you are working with exponential functions.

## Example 11

Simplify $3^{2x-5} \times 3^{5-3x}$.



Exponential functions

### Solution

Write the expression. $3^{2x-5} \times 3^{6-3x}$

Apply the first property of exponential functions. $= 3^{2x-5+6-3x}$

Evaluate the index. $= 3^{1-x}$



# EXERCISE 11.04 Exponential functions

## Concepts and techniques

**1** Which of the following is not an exponential function?

**A** $y = 2.7^h + 5$ **B** $h(x) = 5 \times 2^{3x}$ **C** $y = \left(\frac{1}{2}\right)^{x+1} + 3$

**D** $f(x) = 5x - 2x^7$ **E** $y = 2^{-x^2}$

**2** Evaluate the following.

**a** $16^{\frac{3}{4}}$ **b** $9^0$ **c** $8^{\frac{1}{3}}$ **d** $36^0$

**e** $36^{\frac{1}{2}}$ **f** $27^{\frac{2}{3}}$ **g** $17^0$ **h** $81^{\frac{1}{2}}$

**i** $49^{\frac{3}{2}}$ **j** $36^{\frac{6}{4}}$ **k** $64^{\frac{2}{3}}$ **l** $32^{\frac{6}{5}}$

**m** $8^{-\frac{2}{3}}$ **n** $\left(\frac{3}{4}\right)^{-2}$ **o** $\left(\frac{1}{16}\right)^{-4}$ **p** $18^0$

**q** $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$ **r** $216^{-\frac{2}{3}}$ **s** $25^{1\frac{1}{2}}$ **t** $\left(6\frac{1}{4}\right)^{-\frac{1}{2}}$

**u** $\left(2\frac{1}{4}\right)^{\frac{3}{2}}$ **v** $4^{\frac{1}{3}} \times \sqrt[3]{16}$ **w** $(0.01)^{-\frac{3}{2}}$ **x** $1000^{-\frac{4}{3}}$

**3** Example 10 $f(x) = 3 \times 4^x$. Find the value of:

**a** $f(3)$ **b** $f(8)$ **c** $f(1.5)$ **d** $f(4)$ **e** $f(-2)$

**4** $h(t) = 1.85 \times (2.3)^t$. Find the value of:

**a** $h(4)$ **b** $h(2)$ **c** $h(10)$ **d** $h(4.7)$ **e** $h(-1)$

**5** $m(x) = 12.9 \times (0.85)^x$. Find the value of:

**a** $m(3)$ **b** $m(7)$ **c** $m(2.5)$ **d** $m(10)$ **e** $m(-3)$

**6** Example 11 Simplify each of the following, expressing your answers in index form.

**a** $2^{2x+5} \times 2^{3x-4}$ **b** $5^{4n-7} \times 5^{5n+3}$ **c** $7^{6m-8} \times 7^{m-2}$

**d** $3^{7y-2} \div 3^{2y+5}$ **e** $11^{3p+4} \times 11^{2p+8}$ **f** $13^{6q-3} \div 13^{4q-7}$

**7** Simplify each of the following, expressing your answers in index form.

**a** $2^{4v-3} \times 4^{5v+2}$ **b** $3^{7m+6} \times 9^{4m+5}$ **c** $3^{3x+10} \div 15^{4x-6}$

**d** $8^{4n-5} \times 16^{6n-7}$ **e** $12^{8r+3} \div 24^{7r-2}$ **f** $21^{4z-8} \times 35^{2z+11}$

**8** Simplify each of the following, expressing your answers in index form.

**a** $\frac{2^{n+2} \times 2^{n-3}}{2^{2n-5}}$ **b** $\frac{5^{2k} \times 5^{k-1}}{5^{2k-2}}$ **c** $\frac{3^{j+1} \times 2^{j+3}}{6^j}$

**d** $\frac{6^{2n-m} \times 9^{n+m}}{36^{n-m}}$ **e** $\frac{15^{4i-5} \times 21^{2i+1}}{35^{2i-2} \times 45^{2i-3}}$ **f** $\frac{8^{2f-3} \times 16^{4-f}}{4^{f+2}}$

## Reasoning and communication

**9** Use first principles to prove $(ab)^x = a^x b^x$ where $a, b > 0$ and $x$ is a real number.

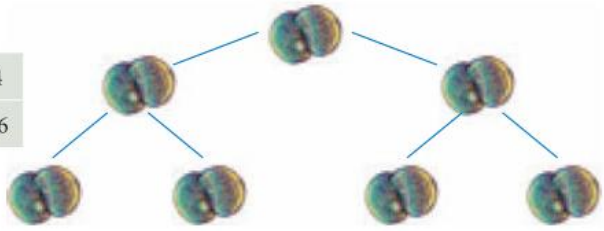
**10** Use first principles to prove $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ where $a, b > 0$ and $x$ is a real number.


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# 11.05 EXPONENTIAL GRAPHS

Most plants and animals begin as a single cell. This cell divides into two, the subsequent cells divide, and so on. The process continues in this way until (in some organisms) the cells begin to specialise, when some will divide faster or slower than others. The division of cells and the number of cells can be shown as follows.

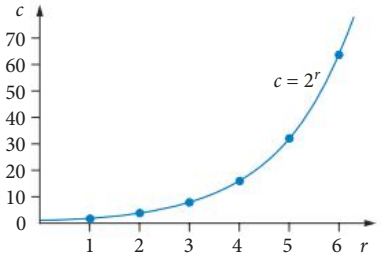
| Reproductions, $r$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cells, $c$ | 1 | 2 | 4 | 8 | 16 |



The rule for the number of cells is $f(r) = c = 2^r$. Calculating further values of the function gives:

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $c = 2^r$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 |

The graph of the function can then be sketched, as shown on the right. The graph has a $c$-intercept of 1 and the value of $c$ is always positive.




## Example 12

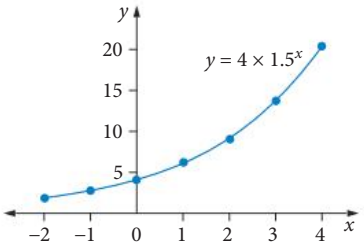
Plot the graph of $y = 4 \times 1.5^x$ for $-2 \le x \le 4$ and comment on the graph's features.

### Solution

Draw up a table of values using some values from the specified domain.

| $x$ | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 1.78 | 2.67 | 4 | 6 | 9 | 13.5 | 20.25 |

Plot the points and join with a smooth curve.




State the features of the graph.

The graph has a $y$-intercept of 4. The graph is always positive and is increasing as $x$ increases. The slope of the graph is also increasing as $x$ increases. As $x$ becomes larger in the negative direction, the graph gets closer and closer to 0.



A CAS calculator can also be used to draw the graph of an exponential function.

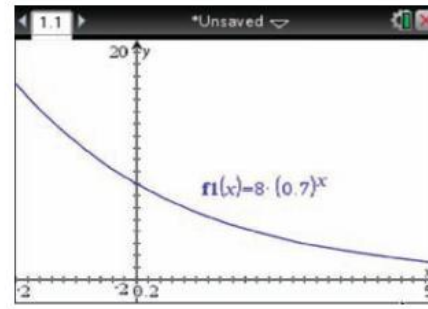
## Example 13

CAS Draw the graph of $y = 8 \times 0.7^x$ for $-2 \leq x \leq 5$ and comment on the features of the graph.

### Solution

#### TI-Nspire CAS

Use a graph page.
Put in the function $f1(x) = 8 \times 0.7^x$.
Press menu, 4: Window/Zoom and 1: Window settings to make $-2 \leq x \leq 5$ and $-2 \leq y \leq 20$.
The display should look like that shown.

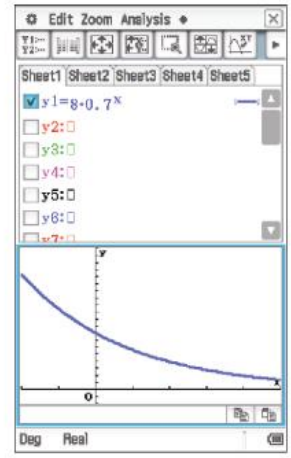



#### ClassPad

Use the **Graph&Table** application.
Put in y1=8 × 0.7 ^ x.

Tap to make $-2 \leq x \leq 5$ and $-2 \leq y \leq 20$.
Tap **OK**.

Tap to graph the function.




State the features of the graph.

The graph is decreasing as $x$ increases, but the slope is increasing (becoming less negative). The graph is positive for all values of $x$ and has a $y$-intercept of 8. As $x \to \infty$, $y \to 0$.

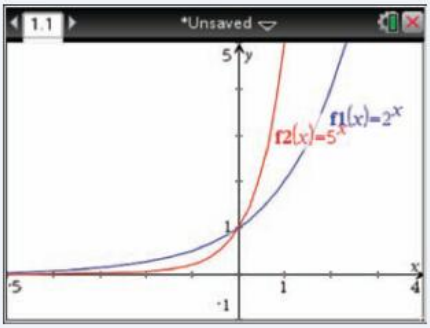

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## INVESTIGATION Exponential graphs

In the first part of this investigation you will examine the differences between the graphs of exponential functions of the form $y = a^x$ as the value of $a$ changes for $a > 1$.

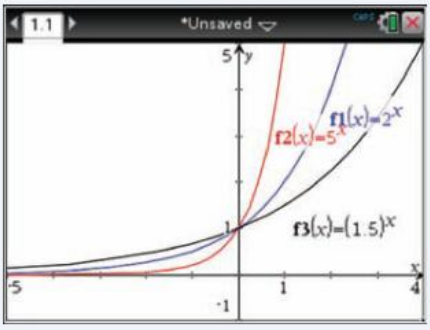
### TI-Nspire CAS

Use a graph screen. Put in the function $f1(x) = 2^x$.
Use menu, 3: Graph Entry/edit and 1: function to put in $f2(x) = 5^x$.
Change the Window settings to $-5 \le x \le 4$ and $-1 \le x \le 5$.




- Comment on the differences between the graphs.
- List the features of the graphs that remain the same or are similar.

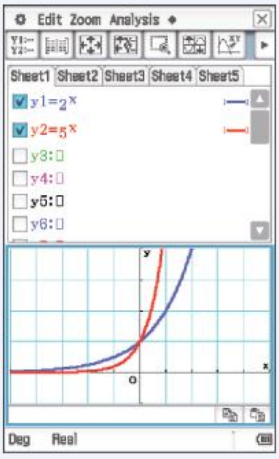
Now put in the function $f3(x) = 1.5^x$ and comment again.




### ClassPad

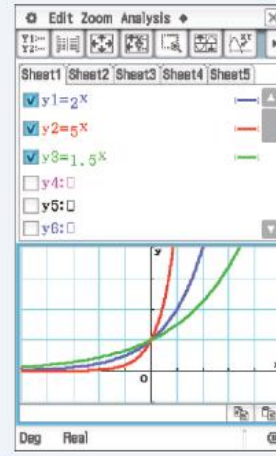
Use the **Graph&Table** menu.
Set y1=2 ^ x and Set y2=5 ^ x

Tap and change the **View Window** settings to $-5 \le x \le 5$ and $-1 \le y \le 4$. Tap **OK**.
Tap to graph the functions.






Now put in y3 = 1.5 [^] [x] and draw the graph.

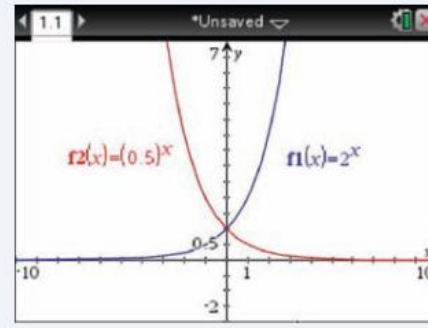



- Comment on the differences between the graphs.
- List the features of the graphs that remain the same or are similar.
- Write a statement that describes how the graph of $y = a^x$ changes as the value of $a$ changes.
- State the features that remain similar as $a$ changes.

In the next part of this investigation you will examine the differences between the graphs of exponential functions of the form $y = a^x$ as the value of $a$ changes for $0 < a < 1$.

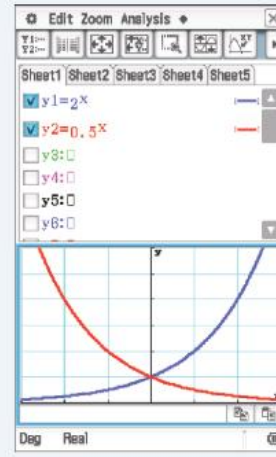
### TI-Nspire CAS

Use a graph page. Put in $f1(x) = 2^x$ and $f2(x) = (0.5)^x$. Change the Window settings so that the domain of $x$ is between −10 and 10 and the range of $y$ is between −2 and 7.




### ClassPad

Use the **Graph&Table** application.
Set y1=2 [^] [x] and y2=0.5 [^] [x].
Use [⊞] to open the **View Window** so that $x$ has a domain from −3 to 3 and $y$ has a range from 0 to 6.
Tap **OK**.
Tap [⊞] to graph the functions.





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On the same set of axes, draw graphs of the functions $y = 2^x$ and $y = 2^{-x}$.

- Comment on the differences and similarities between the graphs.

Now consider the function $y = a^{kx}$ where $k > 0$.

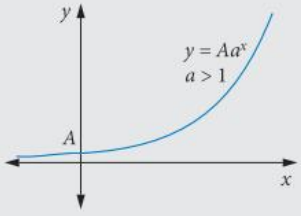
Use the procedures described in this investigation to examine the graphs of functions of this form as the value of $k$ changes.

- Write a statement that describes how the graph of $y = a^{kx}$ changes as the value of $k$ changes.
- State the features that remain similar as $k$ changes.

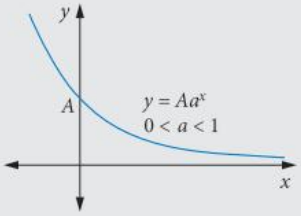
Exponential functions of the form $y = Aa^x$ have two basic shapes.

**IMPORTANT**

**Graphs of exponential functions of the form $y = Aa^x$**




Increasing exponential function




Decreasing exponential function

The graphs of $y = Aa^x$ (where $A$, $a > 0$) have a $y$-intercept at $A$ and are always positive.

For an increasing exponential function, $y = Aa^x$, when $x \to -\infty$, $y \to 0$.

For a decreasing exponential function, $y = Aa^x$, when $x \to \infty$, $y \to 0$.

This means that the $x$-axis is a **horizontal asymptote** for the function.

From the previous investigation, you can also see that the value of $A$ influences the general shape of the graph of $y = Aa^x$.

- When $a > 1$, larger values of $A$ result in functions that increase more rapidly than for smaller values of $A$.
- When $0 < a < 1$, larger values of $A$ result in functions that decrease more rapidly than for smaller values of $A$.



## EXERCISE 11.05 Exponential graphs

### Concepts and techniques

1 Which of the following is an increasing exponential function?

**A** $y = 3 \times 0.9^x$ **B** $y = 7 \times 0.8^x$ **C** $y = 2 \times 3.4^x$
**D** $y = 5 \times 0.1^x$ **E** $y = 11 \times 0.7^x$

2 Which of the following is true for the graph of $y = 2.5 \times 1.8^x$?

**A** The graph has a $y$-intercept at $(0, 1.8)$ .
**B** As $x \to \infty$, $y \to 0$.
**C** The graph is decreasing.
**D** As $x \to -\infty$, $y \to 0$.
**E** The graph has an $x$-intercept at $(2.5, 0)$.

3 Which of the following graphs of exponential functions rises the most steeply?

**A** $y = 4 \times 1.8^x$ **B** $y = 4 \times 2.1^x$ **C** $y = 4 \times 3.2^x$
**D** $y = 4 \times 4.5^x$ **E** $y = 4 \times 6.3^x$

4 Example 12 Plot a graph for each function for the stated domain. Comment on the features of the graph.

**a** $y = 2^x$ for $-3 \le x \le 4$
**b** $y = 5 \times 2^x$ for $-4 \le x \le 3$
**c** $y = 4 \times 0.6^x$ for $-2 \le x \le 4$
**d** $y = 10 \times 1.8^x$ for $-3 \le x \le 3$
**e** $y = 40 \times 0.2^x$ for $-2 \le x \le 4$
**f** $y = 10^x$ for $-2 \le x \le 3$

5 Example 13 CAS Draw the graph of each function for the stated domain. Comment on the features of the graph.

a $y = 8.5 \times 2.5^x$ for $-3 \le x \le 3$
**b** $y = 20 \times 0.9^x$ for $-5 \le x \le 5$
**c** $y = 16 \times 1.2^x$ for $-5 \le x \le 5$
**d** $y = 16 \times 2^x$ for $-5 \le x \le 5$
**e** $y = 16 \times 0.1^x$ for $-3 \le x \le 3$
**f** $y = 8 \times 10^x$ for $-3 \le x \le 3$

### Reasoning and communication

6 Draw the graph of the function $y = 3 \times 1.7^x$ and use it to solve the equation: $3 \times 1.7^x = 2$

# 11.06 TRANSFORMATIONS OF EXPONENTIAL GRAPHS

In the previous section of this chapter you looked at graphs of functions of the form $y = Aa^x$, where $A, a > 0$.

We will now look at the graphs of other exponential functions.


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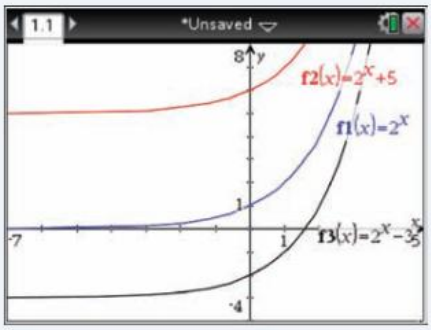
## INVESTIGATION Translations of exponential graphs

In the first part of this investigation you will examine the graphs of functions of the form

$y = Aa^x + c$.

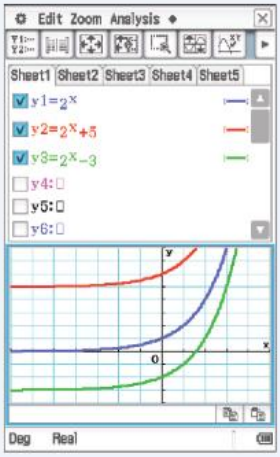
### TI-Nspire CAS

Use a graph page. Put in the functions $f1(x) = 2^x$, $f2(x) = 2^x + 5$ and $f3(x) = 2^x - 3$.
Change the Window Settings so that the domain of $x$ is between $-7$ and 5 and the range of $y$ is between $-4$ and 8.




### ClassPad

Use the **Graph&Table** application.
Put in the functions $y1=2^x$, $y2 = 2^x + 5$ and $y3 = 2^x - 3$.
Tap and change the **View Window** so that x has a domain from $-7$ to 5 and y has a range from $-4$ to 8.
Tap **OK**.
Tap to graph the functions.

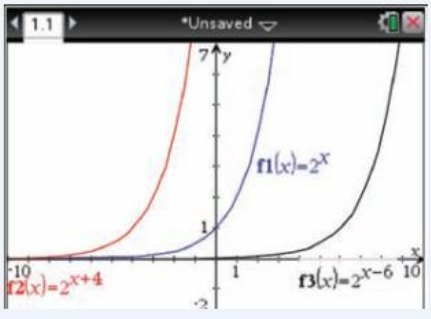



- Describe the similarities and differences between the graphs.
- What transformations would change the graph of $y = 2^x$ into the other graphs?
- Graph other exponential functions of the form $y = Aa^x + c$.
- Explain the significance of the value of $c$ when comparing the graphs of $y = Aa^x$ and $y = Aa^x + c$.

In the next part of this investigation you will examine the graphs of functions of the form $y = Aa^{x+b}$.

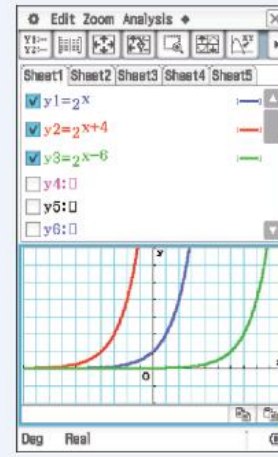
### TI-Nspire CAS

Use a graph page. Put in the functions $f1(x) = 2^x$, $f2(x) = 2^{x+4}$, $f3(x) = 2^{x-6}$. Change the Window settings so that the domain of $x$ is between $-10$ and 10 and the range of $y$ is between $-2$ and 7.



### ClassPad

Use the **Graph&Table** application.
Put in $y1=2^x$, $y2 = y1=2^{x+4}$ and $y3 = y1=2^{x-6}$.
Tap and change the **View Window** so that $x$ has a domain from $-10$ to 10 and $y$ has a range from $-2$ to 7. Tap **OK**.
Tap to graph the functions.




- Write a statement that describes the similarities and differences between the graphs.
- What transformations would change the graph of $y = 2^x$ into the other graph of $2^{x+4}$?
- Graph other exponential functions of the form $y = Aa^{x+b}$.
- Explain the significance of the value of $B$ when comparing the graphs of $y = Aa^x$ and $y = Aa^{x+b}$.

The function $y = Aa^{x+b} + c$ is the general exponential function. Based on the work of this investigation, you can see that the graphs of $y = Aa^{x+b} + c$ and $y = Aa^x$ can be compared using the values of $A$, $b$ and $c$.

**IMPORTANT**

#### Translations of exponential functions

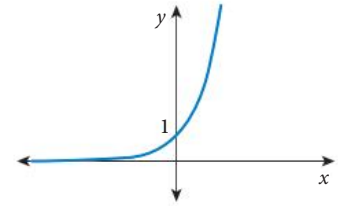
When comparing graphs of the exponential functions $y = Aa^x$ and $y = Aa^{x+b} + c$ where $a$ is the base ($a > 0$) and $A$, $b$ and $c$ are real constants ($A > 0$):

- the value of $b$ represents a translation of the graph of $y = Aa^x$, $b$ units to the left when $b > 0$ or $b$ units to the right when $b < 0$
- the value of $c$ represents a translation of the graph of $y = Aa^x$, $c$ units up when $c > 0$ or $c$ units down when $c < 0$.
- the horizontal asymptote of the exponential function $y = Aa^{x+b} + c$ is $y = c$.

## Example 14

The graph of $y = a^x$ is shown to the right.

Use this graph to draw a sketch of the function $y = a^{x-3}$.





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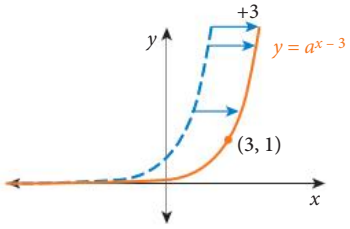
## Solution

Rewrite the function without a subtraction sign. $y = a^{x-3} = a^{x+(-3)}$

Examine the value of $b$. $b < 0$

State the significance of $b < 0$. $b < 0$ means the graph of $y = a^x$ is translated 3 units to the right to form the graph of $y = a^{x-3}$.

Sketch the graph.




The $y$-intercept is an important piece of information that is required to sketch or identify an xponential function.

When sketching graphs of exponential functions of the form $y = Aa^{x+b} + c$, remember that the function is increasing if $a > 1$ and decreasing if $0 < a < 1$.

## Example 15

Sketch the graph of:

**a** $y = 3(2^x) + 1$ **b** $y = \frac{1}{2} \times 10^{x-1}$

## Solution

**a** Compare the function with the general exponential function.

$y = Aa^{x+b} + c$
$= 3(2^x) + 1$
$a = 2 > 1, A = 3, b = 0, c = 1$

Consider the basic shape. Since $a > 1$, the graph is increasing.

Consider the horizontal translation. $b = 0$ so there is no horizontal translation.

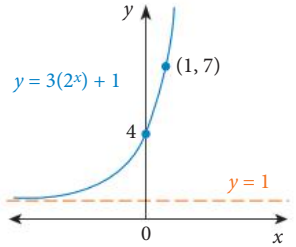
Consider the vertical translation. $c = 1$ so the graph is translated up 1.

State the asymptote. The horizontal asymptote is at $y = 1$.

Find the $y$-intercept. The $y$-intercept is at $3 \times 2^0 + 1 = 4$

Find an important point for scaling. At $x = 1$, $y = 3 \times 2^1 + 1 = 7$

Sketch the graph.



**b** Compare the function with the general exponential function.

$y = Aa^{x+b} + c$

$y = \frac{1}{2} \times 10^{x-1}$

$a = 10 > 1, A = \frac{1}{2}, b = -1, c = 0$

Consider the basic shape.

Since $a > 1$, the graph is increasing.

Consider the horizontal translation.

$b = -1$ so there is a translation 1 to the right.

Consider the vertical translation.

$c = 0$ so there is no vertical translation.

State the asymptote.

The horizontal asymptote is at $y = 0$

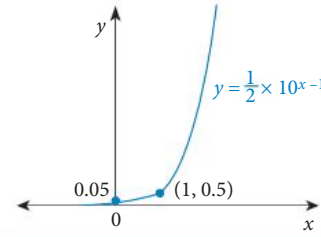
Find the $y$-intercept.

The $y$-intercept is at $\frac{1}{2} \times 10^{-1} = 0.05$

Find an important point for scaling.

At $x = 1$, $y = \frac{1}{2} \times 10^{0} = 0.5$

Sketch the graph.




Exponential graphs of the form $y = Aa^{x+b} + c$ can be sketched by **transforming** the graph of $y = a^x$.

**IMPORTANT**

### Sketching exponentials using transformations

For the general exponential function $f(x) = Aa^{x+b} + c$:

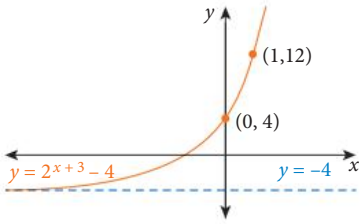
- the graph of $y = a^x$ has horizontal asymptote $y = 0$, $y$-intercept $(0, 1)$ and passes though the points $(-1, \frac{1}{a})$ and $(1, a)$.
- For $a > 1$ it is increasing and for $0 < a < 1$ it is decreasing
- the constant $A$ dilates the graph by a factor of $A$ vertically from the $x$-axis – that is, the graph is stretched vertically when $A > 1$ and compressed vertically when $0 < A < 1$
- the constant $b$ translates the graph horizontally left ($b > 0$) or right ($b < 0$)
- the constant $c$ translates the graph vertically up ($c > 0$) or down ($c < 0$)
- the horizontal asymptote becomes $y = c$ and the $y$-intercept translates to $(-b, A + c)$


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## Example 16

Sketch the graph of $y = 2^{x+3} - 4$ by transforming the graph of $y = 2^x$.

### Solution

| | |
|---|---|
| Compare the function with the general exponential function. | $y = Aa^{x+b} + c$<br>$y = 2^{x+3} - 4$<br>$a = 2, A = 1, b = 3, c = -4$ |
| Consider the basic shape. | Since $a > 1$, the graph is increasing. |
| Consider the horizontal translation. | $b = 3$ so there is a translation 3 to the left. |
| Consider the vertical translation. | $c = -4$ so the graph is translated down 4. |
| State the asymptote. | The horizontal asymptote is at $y = -4$. |
| Find the $y$-intercept. | The $y$-intercept is at $2^3 - 4 = 4$ |
| Find important point(s) for scaling. | At $x = 1$, $y = 2^4 - 4 = 12$ |
| Sketch the graph. |  |



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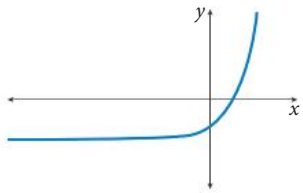
# EXERCISE 11.06 Transformations of exponential graphs

WS
Translating exponential graphs

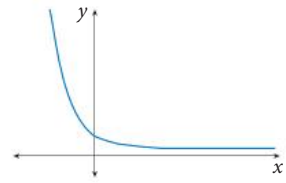
## Concepts and techniques

Use the graphs below to answer questions **1**–**5**.

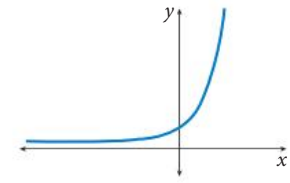
**A**



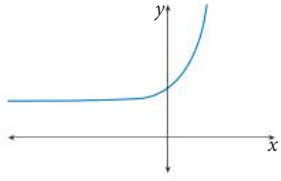

**B**



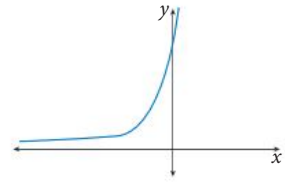

**C**




**D**




**E**




1 Which of the graphs **A**–**E** above is most likely to represent $y = 2^x + 3$?

2 Which of the graphs **A**–**E** above is most likely to represent $y = 2^x$?

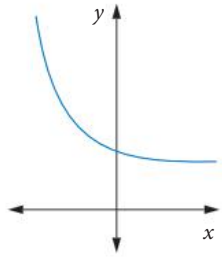
3 Which of the graphs **A**–**E** above is most likely to represent $y = 2^{x+3}$?

4 Which of the graphs **A**–**E** above is most likely to represent $y = 2^x - 3$?

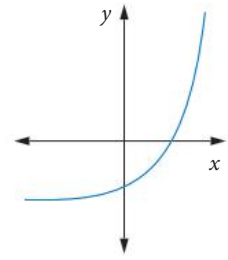
5 Which of the graphs **A**–**E** above is most likely to represent $y = \left(\frac{1}{2}\right)^x$?

6 Which of the following best represents the graph of the function $y = a^x - p$ where $a > 1$ and $p > 0$?

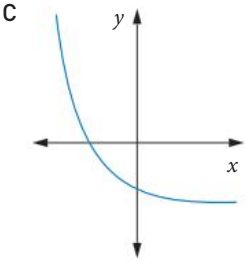
**A**



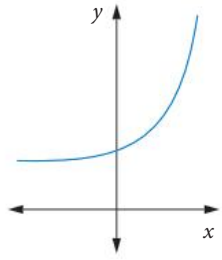

**B**



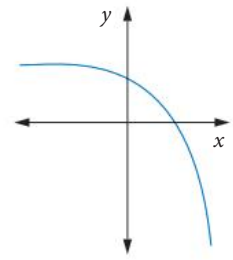

**C**




**D**



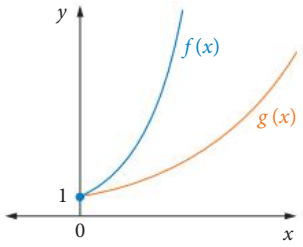

**E**





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**7** The diagram on the right shows the graphs of two functions for $x \geq 0$. Which of the following best describes the functions?




**A** $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$

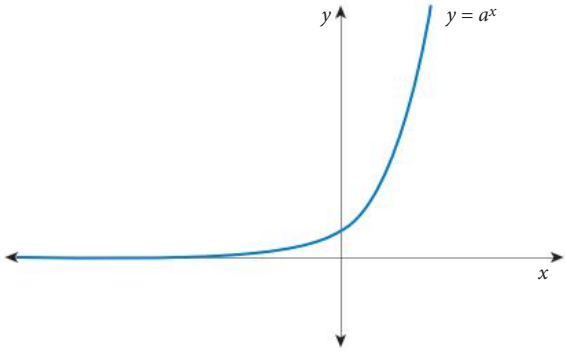
**B** $f(x) = a^x$ and $g(x) = b^x$ where $0 < a < b$

**C** $f(x) = a^x$ and $g(x) = b^x$ where $a = b$

**D** $f(x) = a^x$ and $g(x) = b^x$ where $0 < a < b < 1$

**E** $f(x) = a^x$ and $g(x) = b^x$ where $a > b > 1$

**8** Example 14 The graph of $y = a^x$ is shown to the right.
Use this graph to draw a sketch of the following functions.




**a** $y = a^{x-5}$ **b** $y = a^x - 6$ **c** $y = a^{x+2}$ **d** $y = a^x + 4$

**9** For each of the following exponential functions, calculate the $y$-intercept and the equation of the horizontal asymptote.

**a** $y = 10^x$ **b** $y = 2^{x+1}$ **c** $y = 3 \times 0.8^{x+2} + 1$

**d** $y = 2^{x-1} - 5$ **e** $y = 6 \times 4^x$ **f** $y = 7 \times 1.2^{x+3} + 3$

## Reasoning and communication

**10** Example 15 Sketch the graph of each of the following by first determining the key features of the graph.

**a** $y = 4^x - 3$ **b** $y = 2^{x-1} - 1$ **c** $y = 3(2^x) + 2$

**d** $y = \left(\frac{1}{2}\right)^x - 1$ **e** $y = 5^x - 1$ **f** $y = 2^{x+1} + 3$

**11** Example 16 Sketch the graph of each of the following by transforming the graph of $y = 3^x$.

**a** $y = 3^x + 5$ **b** $y = 3^{x+4}$ **c** $y = 3^x - 6$

**d** $y = 3^{x-5}$ **e** $y = 3^{x+2} - 4$ **f** $y = 3^{x-5} + 2$



# 11.07 EXPONENTIAL EQUATIONS

You will have encountered many methods for solving equations by this stage, but none of them would have shown you how to solve an equation when the variable is in the exponent.

You can solve some exponential equations by:

- making the base on both sides the same
- equating the indices
- solving the resulting equation.

**IMPORTANT**

An **exponential equation** is one where the variable is part of an exponent or power in the equation. These are also known as **indicial equations**.

## Example 17

Solve $3^{2x-5} = 27$.

### Solution

| | |
|---|---|
| Write down the equation. | $3^{2x-5} = 27$ |
| Express the RHS with a base of 3. | $3^{2x-5} = 3^3$ |
| Indices must be equal. | $2x - 5 = 3$ |
| Solve the resulting equation. | $2x = 8$ |
| | $x = 4$ |

You can solve some exponential equations by making simpler exponential equations using an appropriate substitution. In some cases, false solutions must be discarded.

## Example 18

Solve $16^x - 5 \times 4^x - 24 = 0$.

### Solution

| | |
|---|---|
| Write down the equation. | $16^x - 5 \times 4^x - 24 = 0$ |
| Express both powers with the same base. | $4^{2x} - 5 \times 4^x - 24 = 0$ |
| Substitute $y$ for $4^x$. | $y^2 - 5y - 24 = 0$ |
| Factorise the resulting quadratic. | $(y - 8)(y + 3) = 0$ |
| Solve for $y$. | $y = 8$ or $y = -3$ |
| Now substitute $4^x$ back for $y$. | $4^x = 8$ or $4^x = -3$ |
| You know that $4^x = -3$ has no real solution. | $4^x = 8$ |


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| | |
|---|---|
| Write both sides with base 2. | $2^{2x} = 2^3$ |
| Indices must be equal. | $2x = 3$ |
| Solve for $x$. | $x = 1.5$ |

You cannot always algebraically solve exponential equations. One way to overcome this problem is to use the inbuilt equation-solving function of a CAS calculator. It will find approximate solutions using numerical methods from applied mathematics.

## Example 19

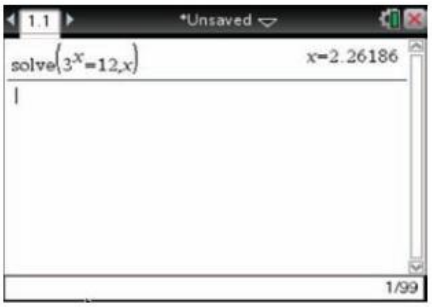
CAS Solve the equation $3^x = 12$.

WS Solving exponential equations

### Solution

#### TI-Nspire CAS

Use a calculator page
Press menu, 3:Algebra and 1:Solve.
Put in the equation $3^x = 12$ and complete the solve instruction with ,$x$ and press enter.




#### ClassPad

Use the Main $\sqrt{\alpha}$ application. Set the calculator to **Decimal** as the ClassPad II can also find an expression for the answer.
Press Keyboard and tap **Action**, **Advanced** and **solve**, type solve or find it in the Catalog. Use $\wedge$ or $\square^{\blacksquare}$ from the Math1 tab for the powers. Put in the equation and press , $x$ and enclose the expression in brackets. Press EXE.




Write down the solution. $x \approx 2.262$

# EXERCISE 11.07 Exponential equations

## Concepts and techniques

**1** Example 17 Solve the following equations algebraically.

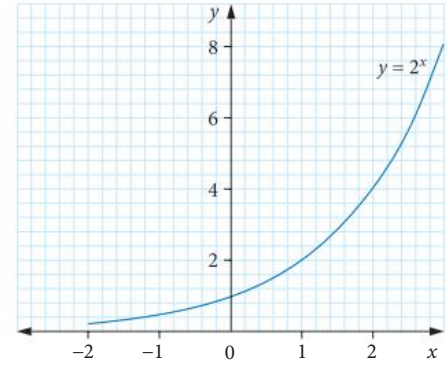
**a** $2^{x+4} = 128$
**b** $3^{x-2} - 81 = 0$
**c** $4^{2x-1} - 32 = 0$
**d** $5 \times 3^{2x-5} = 135$
**e** $7 \times 2^{4x+3} = 224$
**f** $13 \times 5^{2x+3} - 1625 = 0$

**2** Example 18 Solve the following equations algebraically.

**a** $6^x - 216^{3x-5} = 0$
**b** $\dfrac{2^x - 4}{3} = 4$
**c** $4 \times 3^x - 1 = 35$
**d** $3 \times 4^{2x-1} + 1 = 13$
**e** $6^{x-1} - 16 = 20$
**f** $9^x - 90 \times 3^x + 729 = 0$
**g** $4^x + 32 = 12 \times 2^x$
**h** $16^x + 1024 = 5 \times 4^{x+2}$
**i** $5^{2x+1} + 25 = 5^{x+3} + 5^x$
**j** $8^x + 7 \times 2^{x+1} = 7 \times 4^x + 8$
**k** $3^{3x} - 12 \times 9^x + 63 \times 3^x = 4 \times 3^3$

**3** The graph of $y = 2^x$ is shown. Use the graph to solve:

**a** $3 \times 2^x - 15 = 0$
**b** $2^{x+1} - 3.5 = 0$
**c** $5 \times 2^{x-1} - 30 = 0$
**d** Comment on the accuracy of your solutions.




**4** Example 19 CAS Solve the following.

**a** $3^x = 20$
**b** $3^x = 10$
**c** $3^{x-2} - 15 = 0$
**d** $2 \times 3^{x+2} = 50$
**e** $3^x - 0.5 = 0$
**f** $4 \times 3^{2x-1} - 3 = 0$

**5** CAS Solve the following.

**a** $10^x = 8$
**b** $10^x - 80 = 0$
**c** $5 \times 10^x - 600 = 0$
**d** $4 \times 10^{x+2} - 1800 = 0$

**6** CAS Solve the following.

**a** $\left(\frac{1}{2}\right)^x = 0.4$
**b** $\left(\frac{1}{2}\right)^x - 0.8 = 0$
**c** $4 - 3 \times \left(\frac{1}{2}\right)^x = 0$
**d** $3 \times 0.5^{2x-3} - 4 = 5$

**7** CAS Solve the following.

**a** $6^x = 2$
**b** $6^x - 72 = 0$
**c** $6 \times 6^{2x} + 3 = 75$
**d** $100 - 6^{3x-2} = 0$

**8** CAS Solve $50^{2x-3} - 28 = 0$.

**9** CAS Solve $28^{2x-3} = 50$.

**10** CAS Solve $5^x - 4^x - 3 = 0$.

**11** CAS Solve $10^{3x} - 150 = 2^{4x}$.

**12** CAS Solve $2 \times 6^{3x} - 7 \times 6^{2x} - 80 = 0$.


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# 11.08 EXPONENTIAL MODELS

In Section 11.04 of this chapter you looked at an example about a culture containing 20 μg of a bacterium that increased by 30% every 10 minutes.

The mass in μg of the bacterium can be modelled as $M(t) = 20 \times (1.3)^t$, where $t$ is the number of 10-minute periods after the commencement of the observations. As long as $t$ is a positive integer, you could model this as a geometric progression with $t_1 = 20$ and $r = 1.3$. An exponential function is better because you can have negative and decimal values of $t$.

The growth of this bacteria is an example of a **growth function** that increases by a *constant factor*. The general rule for such functions is $f(x) = Aa^x$. At $x = 0$, $f(0) = A$, so $A$ is the **initial value** (starting value) and $a$ is the **growth factor**.

## Example 20

A bacterial colony has about 2000 bacteria. The number of bacteria triples every day. Assuming that growth continues in this way:

a how many bacteria will be in the colony after 60 hours?

b how many bacteria will be in the colony after 5 days?

### Solution

| | |
|---|---|
| Write the growth function for the number of bacteria. | $N(x) = ab^x$ |
| Write down the given information. | Initial value $a \approx 2000$<br>Growth factor $b = 3$ |
| Write the function for the number of bacteria after $d$ days. | $N(d) \approx 2000 \times 3^d$ |
| **a** Convert 60 hours to days. | 60 hours = 2.5 days |
| Substitute $d = 2.5$. | $N(2.5) \approx 2000 \times 3^{2.5}$ |
| Evaluate. | $\approx 31\,200$ |
| State the result. | There will be about 31 200 bacteria after 60 hours. |
| **b** Substitute $d = 5$. | $N(5) \approx 2000 \times 3^5$ |
| Evaluate. | $\approx 486\,000$ |
| State the result. | There will be about 486 000 bacteria after 5 days. |

Exponential models can also be used when the values are decreasing by constant factors. For these models, the growth factor is less than 1, so we call the function a **decay function** instead of a growth function.



**IMPORTANT**

**Exponential growth and decay**

The general form of a growth or decay function is

$f(x) = Aa^x$

where $A = f(0)$ is the **initial value** or starting point and $a$ is the **growth factor**.

For a **growth function**, $a > 1$.

For a **decay function**, $0 < a < 1$.

## Example 21

The amount of air left in a leaking spaceship is decreasing by 30% every minute and the pressure decreases proportionally.
When the leak first occurred, the pressure was 105 kPa. What will the pressure be after 6 minutes?

### Solution

| | |
|---|---|
| The pressure decreases by 30% every minute.<br>The new pressure is 70% of the previous value.<br>The initial value is 105 kPa. | Growth factor $a = 0.7$<br>$A = 105$ |
| Write the growth function for pressure after $m$ minutes. | $P(m) = 105 \times 0.7^m$ |
| Let $m = 6$. | $P(6) = 105 \times 0.7^6$ |
| Evaluate. | $\approx 12.4$ kPa |
| State the result. | After 6 minutes, the pressure will be about 12.4 kPa. Find that leak! |

## Example 22

The difference between the temperature of a cup of coffee and its surroundings decreases by 4% every minute. A fresh cup has a temperature of 70°C, and the room temperature is 20°C. A cup of coffee is considered too cold to drink if it is below 40°C. Model the temperature of the cup of coffee and find how long it takes to become 'undrinkable'.


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## Solution

The *difference* in temperature between the coffee and the room decreases by 4% each minute. This is an exponential decay function. Let the coffee temperature after $m$ minutes be $C(m)$. Assume that the room temperature stays at 20°C. Let the difference between the coffee and room temperatures after $m$ minutes be $D(m)$.

Write the relationship between the functions. $D(m) = C(m) - 20$

Rearrange. $C(m) = D(m) + 20$

Substitute an exponential expression for $D(m)$. $C(m) = Aa^m + 20$

For $D(m)$, $A = 70 - 20 = 50$ and $a = 0.96$. $C(m) = 50 \times (0.96)^m + 20$

To find the time taken to become undrinkable, try some different values of $m$.

$C(5) = 50 \times (0.96)^5 + 20 \approx 60.8$

$C(10) \approx 53.2$ $C(20) \approx 42.1$

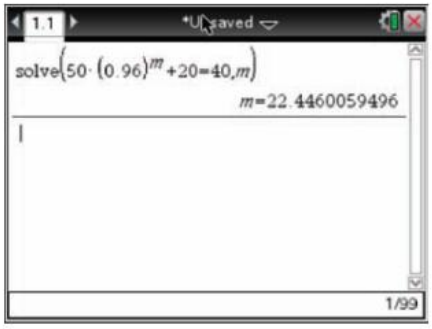
Keep trying different values of $m$ until $C(m)$ is close to 40.

$C(25) \approx 38.0$ $C(22) \approx 40.4$

$C(23) \approx 39.6$ $C(22.5) \approx 39.96$

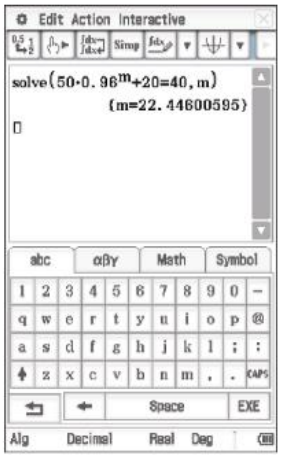
### TI-Nspire CAS

Use solve() to find a more exact answer.




### ClassPad

Use **solve()** to find a more exact answer.




State the result.

The temperature of the coffee is modelled as $C(m) = 50 \times (0.96)^m + 20$ and it takes about 22.4 minutes for it to become undrinkable.



A growth or decay function can sometimes be modelled as an exponential one from a table of values. In order to model as an exponential function, the growth or decay factor must be *constant*.

Look at the values below.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $m$ | 5 | 10 | 20 | 40 | 80 |

You can see that every time $n$ increase by 1, $m$ doubles. Clearly $m = 5 \times 2^n$. You should check that this is true for *the whole* table.

WS
Using exponential models

## INVESTIGATION Radiocarbon dating

**Radiocarbon dating** is a dating method that uses the naturally occurring isotope carbon-14 ($^{14}_{6}C$) to determine the age of material containing carbon. The carbon-14 isotope is produced in the upper atmosphere as a result of cosmic rays hitting nitrogen atoms. The proportion of carbon-14 in carbon dioxide stays the same because the radioactive decay is balanced by production of more carbon-14 in the upper atmosphere by cosmic rays.

Carbon-14 is absorbed by plants during photosynthesis, so it is present in the same proportion as in the atmosphere in all living organisms. Of course, when something dies, its stops getting more carbon-14, so the carbon-14 present just decays.

For a quantity subject to exponential decay the time required for the quantity to decay to half of its initial value is known as its **half-life**. The half-life of carbon-14 is 5730 years.

In the late 1940s an American scientist, Williard Libby, developed a technique for 'dating' archaeological material by measuring the amount of carbon-14 present compared with that found in things living now. The **becquerel (Bq)** is the SI unit of radioactivity. It is defined as the activity of a quantity of radioactive material in which one nucleus decays per second.


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Consider the case of an old bone that produced 10 Bq per gram of bone carbon-14 in 2010, compared with the bones of recently dead animals that produce 20 Bq per gram of bone carbon. Because the half-life of carbon-14 is 5730 years, the animal from which the old bone came would have died in about 3720 BCE.


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1 The table below shows the radiation produced by an organic object at different times from its death. Use the table on the right to construct an exponential model of radioactive decay.

| Age (years) | Radiation (Bq) |
|---|---|
| 0 | 20 |
| 5730 | $20 \times \frac{1}{2} = 20 \times 2^{-1}$ |
| 11 460 | $20 \times \left(\frac{1}{2}\right)^2 = 20 \times 2^{-2}$ |

2 Use your model to estimate the ages of bones with radiation readings of:

- 9.7 Bq per gram of bone carbon
- 2.3 Bq per gram of bone carbon

3 Can radiocarbon dating be used to date any fossils?

4 Is there a limit to the age of fossils that can be dated?

5 The method relies on the production of carbon-14 in the atmosphere being constant. Is this true?

6 What is the 'Suess effect'?

# EXERCISE 11.08 Exponential models

## Concepts and techniques

1 An exponential function is given by $f(x) = 45 \times (1.15)^x$.
a What is the initial amount? b What is the growth factor?

2 An exponential function is given by $f(t) = 1100 \times (0.82)^t$.
a What is the initial amount? b What is the growth factor?

## Reasoning and communication

3 Example 20 An algal bloom in a polluted lake covers an area of 3 $m^2$. It grows in size by 5% every day.
a Set up an exponential model for this situation..
b Find the area covered by the algae after:
i 3 days ii 10 days iii 20 days
c Find when the algae will cover 500 $m^2$.



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4 An investment of $3000 at compound interest grows by 8% every year.
   a Set up an exponential model for the value of the investment.
   b Find the value of the investment after:
      i 3 years  ii 10 years  iii 25 years
   c Find the time taken for the investment to double in value.

5 Example 21 The value of a car depreciates by 15% each year. The car was originally worth $24 900.
   a Set up an exponential model of the value of the car.
   b Find the value of the car after:
      i 3 years  ii 5 years  iii 10 years
   c Find the time taken for its value to reduce to $1000.



AAP Images/Ford

6 Example 22 CAS An oven has a theoretical maximum temperature of 500°C. The temperature of the room is 20°C. Because of radiation losses, when the oven is switched on its temperature $T$ increases each minute by 10% of the difference between its temperature and the theoretical maximum; that is, $(500 - T)$ decreases by 10% each minute.
   a Set up a model of the oven temperature $m$ minutes after it is switched on.
   b Find the temperature of the oven after:
      i 5 minutes  ii 10 minutes  iii 25 minutes
   c Find the time taken for the oven to reach a temperature of 400°C.

7 For the following table of values:

| $m$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p$ | 3 | 6 | 12 | 24 | 48 |

   a find the rule for $p$ as a function of $m$
   b find the value of $p$ when $m = 8$.

8 For the following table of values:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $q$ | 24 | 43.2 | 77.76 | 139.968 | 251.9424 |

   a find the rule for $q$ as a function of $x$
   b find the value of $q$ when $x = 2.7$.

9 The variable $y$ is known to be an exponential function of $x$. When $x = 2$, $y = 98$; when $x = 3$, $y = 137.2$; and when $x = 4$, $y = 192.08$. Find $y$ as a function of $x$ and hence find the value of $y$ when $x = 5.5$.


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# CHAPTER SUMMARY
# EXPONENTIAL FUNCTIONS

11

- $a^n$ is known as a **power**.
- $a^n = a \times a \times \ldots \times a$ (to $n$ factors) where $n$ is a positive integer and $a$ is a real number.
- $n$ is called the **index** or **exponent** and $a$ is called the **base**.
- The **index laws** allow us to multiply and divide by powers. The index laws and extensions are:

  $a^m \times a^n = a^{m+n} \quad a^m \div a^n = a^{m-n}$

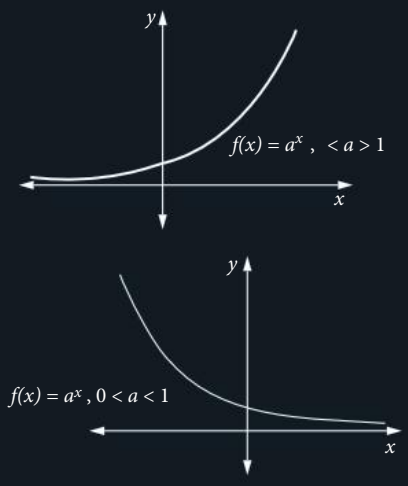
  $(a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$

  $a^0 = 1 \ (a \neq 0) \quad a^{-n} = \dfrac{1}{a^n} \ (a \neq 0)$

  $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$ where $p$ is an integer, $q$ is a positive integer and $a$ is positive.
- The square root sign, $\sqrt{\ }$, is also called the **radical** sign.
- A number written in **scientific notation** is written as a number between 1 and 10 multiplied by a power of 10.
- Numbers can be expressed to a desired number of **significant figures** according to the precision required. Digits that have meaning in terms of a number's precision are called significant figures.
- A function of the form $f(x) = ab^x$ is called an **exponential function**. For $a, b > 0$ the domain is the real numbers and the range is the positive real numbers.
- The general form of a simple exponential function is

  $f(x) = Aa^{x+b} + c$

  where $a$ is the base and $A$, $b$ and $c$ are real constants.
- The **graph of an exponential function** is always positive. For a growth function the slope increases with increasing values of $x$, while for a decay function the slope decreases.




- When sketching exponential functions in general form, key features are used:
  - the general shape;
  - the **asymptote** (which is affected by the value of the constant $c$)
  - the $y$-intercept (when $x = 0$);
  - one other **significant point**.
- Exponential graphs in the form $f(x) = Aa^{x+b} + c$ can also be sketched by **transforming** the graph of $y = a^x$
  - $A$ dilates the graph vertically by a factor of $A$ (stretched for $A > 1$, compressed for $0 < A < 1$).
  - $b$ translates the graph horizontally (left for $b > 0$, right for $b < 0$).
  - $c$ translates the graph vertically (up for $c > 0$, down for $c < 0$).

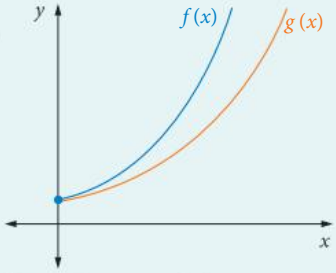
- An **exponential equation** (**indicial equation**) is one where the variable is in an exponent.
- **Exponential** (or **indicial**) **equations** may be solved algebraically by using the fact that if $a^m = a^n$, then $m = n$.
- Exponential equations may also be solved using a CAS calculator.
- A function that grows or decays by a **constant factor** can be modelled as an exponential function of the form $f(x) = Aa^x$, where $A = f(0)$ is the **initial value** or starting point, $a$ is the **growth factor** or **base**, and $x$ is the **exponent** or **index**. For a **growth function** $a > 1$, and for a **decay function** $a < 1$.


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# CHAPTER REVIEW
# EXPONENTIAL FUNCTIONS
# 11

## Multiple choice

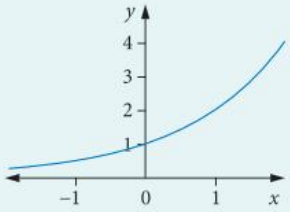
1 Examples 14,15 Which pair of functions best describes the graphs for $x \geq 0$ shown here?




A $f(x) = a^x, g(x) = b^x$ where $a > b > 1$
B $f(x) = a^x, g(x) = b^x$ where $a < b < 1$
C $f(x) = a^x, g(x) = b^x$ where $0 < a < b$
D $f(x) = a^x, g(x) = b^x$ where $a < b < 1$
E $f(x) = a^x, g(x) = b^x$ where $a < b < 0$

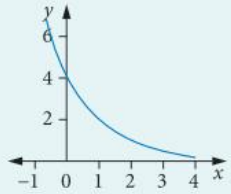
2 Example 10 Consider $f(x) = 2^x$, $g(x) = 5^x$ and $h(x) = 10^x$. Which of the following is false?
A The $y$-intercepts are all 1.
B $g(x) > f(x)$ for all values of $x > 0$.
C The slope of $h(x)$ is greater than that of the other two graphs.
D $h(x) > g(x)$ for all values of $x > 0$.
E The graph of $f(x)$ is always below the graphs of the other two functions.

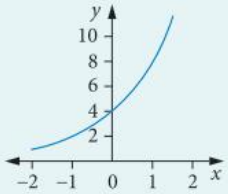
3 Examples 14,15 The graph at the right can be described by both:

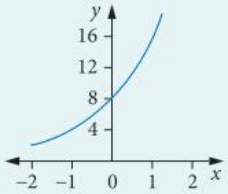


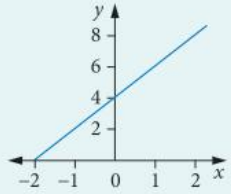

A $f(x) = a^x, g(x) = b^x$ for $a > 1, 0 < b < 1$
B $f(x) = a^{-x}, g(x) = b^x$ for $a > 1, 0 < b < 1$
C $f(x) = a^x, g(x) = b^{-x}$ for $a > 1, 0 < b < 1$
D $f(x) = a^{-x}, g(x) = b^{-x}$ for $a > 1, b > 1$
E $f(x) = a^x, g(x) = b^x$ for $a < 1, b > 1$

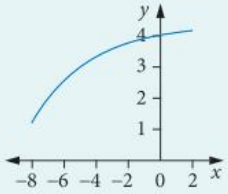
4 Example 12 The graph of $y = 4 \times 2^x$ is most like:

A



B



C



D



E


5 Example 10 The domain and range of $y = 10^x$ are respectively:
A $x \geq 1$ and positive real numbers
B real numbers and positive real numbers
C positive real numbers and positive real numbers
D positive real numbers and real numbers
E $x \geq 1$ and $y \geq 10$

6 Example 17 $25 = 3125^{2x}$ has a solution of:
A $x = 0.2$ B $x = 1$ C $x = 3$ D $x = 5$ E $x = 11.18$

7 Example 19 If $x = 2$ is a solution of $2 \times 2^{kx} - 23 = 3$, then $k$ is closest to:
A 0.14 B 1.85 C 7.4 D 7.5 E 13

8 Example 18 The solution of $2^{2x+2} - 48 \times 2^x - 256 = 0$ is:
A $x = -4$ B $x = -2$ C $x = 2$ D $x = 4$ E $x = 8$

9 Example 19 The solution of $3^{2x} - 18 = 0$ is closest to:
A $x = 1.32$ B $x = 2$ C $x = 2.63$ D $x = 3$ E $x = 6$

10 Example 10 The table on the right shows some points of an exponential function $f(x)$.
The initial value $a$ and growth factor $b$ are:
A $a = 6, b = 3$ B $a = 6, b = 54$ C $a = 3, b = 2$
D $a = 2, b = 81$ E $a = 2, b = 3$

| $x$ | 1 | 3 | 2 | 0 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | 6 | 54 | 18 | 2 | 162 |

11 Example 12 Which of the following tables does not show data from an exponential rule?

A

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |

B

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 4 |
| 2 | 16 |
| 3 | 64 |

C

| $x$ | $y$ |
|---|---|
| 0 | 16 |
| 1 | 8 |
| 2 | 4 |
| 3 | 2 |

D

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 4 |
| 2 | 25 |
| 3 | 82 |

E

| $x$ | $y$ |
|---|---|
| 0 | 2 |
| 1 | 1 |
| 2 | 0.5 |
| 3 | 0.25 |

12 Example 20 Which of the following shows the rule for an exponential function with a growth factor of 1.2 and initial value 5.8?
A $y = 5.8 + 1.2^x$ B $y = 5.8^{1.2x}$ C $y = 5.8 \times 1.2^x$
D $y = 1.2 \times 5.8^x$ E $y = 5.8 \times 2^x$

13 Example 21 An investment grows by 6% every year. If \$6000 was invested, after 10 years it will be worth about:
A \$6360 B \$9600 C \$10 137 D \$10 745 E \$11 390

## Short answer

14 Example 2 Simplify the following.
a $x^3 \times x^8$ b $20k^{12} \div 4k^3$ c $(3a^4b^2)^3$
d $\dfrac{6t^4 \times 9t^7}{12t^5}$ e $\dfrac{14^5 \times 9^2}{21^4 \times 49}$


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15 Examples 3, 4 Simplify the following and where possible express with positive indices.

a $\dfrac{24p^3q \times 3p^5q^{-3}}{27p^{-6}q^4}$ b $\dfrac{9^4 \times 12^{-2} \times 8^2}{18^3}$ c $81^{-\frac{3}{4}}$ d $5a^0$

16 Example 5 Write in index form.

a $\sqrt[3]{x^4}$ b $\sqrt[7]{(5x+3)^9}$ c $\dfrac{1}{\sqrt[4]{m^3}}$ d $\dfrac{1}{\sqrt[5]{(2p^2-1)^3}}$

17 Example 6 Write the following using index notation.

a $x^5 \times \sqrt{x^3}$ b $\sqrt[4]{9^3 \times b^7} \times \sqrt{27} \times b^{\frac{3}{2}}$

18 Example 7 Write in scientific notation.

a 0.000 013 b 123 000 000 000

19 Example 8 Write as decimal numbers.

a $1.627 \times 10^6$ b $7.002 \times 10^{-5}$

20 Example 9 Evaluate $\dfrac{-3.4 \times 10^{-3} + 1.7 \times 10^{-2}}{(6.9 \times 10^5)^3}$ and express your answer in scientific notation correct to 3 significant figures.

21 Example 10 An exponential function is given by $f(x) = 4.5 \times 7^x$. Calculate:

a $f(3)$ b $f(-2)$

22 Example 11 Simplify:

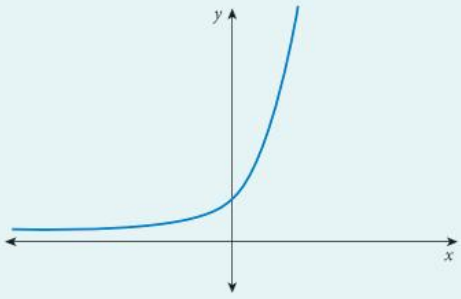
a $3^{5x-6} \times 3^{8-3x}$ b $4^{3n+5} \div 8^{2n-7}$

23 Examples 12, 13 Draw a graph of each function for the stated domain and comment on the graph's features.

a $y = 3 \times 2^x$ for $-2 \leq x \leq 2$ b $y = 4 \times 0.6^x$ for $-3 \leq x \leq 3$

c $y = 0.6 \times 1.2^x$ for $-4 \leq x \leq 4$ d $y = 1.5 \times 2.4^x$ for $-2 \leq x \leq 2$

24 Example 14 The graph of $y = a^x$ is shown on the right. Use it to draw the graph of $y = a^{x-3} + 2$.




25 Example 15 For the function $f(x) = Aa^{x+b} + c$, what effect will changing the value of $c$ have on the graph of the function?

26 Example 15 Calculate the coordinates of the $y$-intercept of the graph of the function $y = 4 \times 3^{2x+1} - 2$.

27 Example 15 Sketch the graph of $y = 3 \times 2^{x-1} - 5$ by first determining the key features of the graph.

28 Example 16 Draw the graph of $y = 2^{x+2} + 3$ by transforming the graph of $y = 2^x$.

29 Example 17 Solve the following.

a $25^{x-3} = 125$ b $9^{2x-4} - 81 = 0$ c $8^{2x-1} = \frac{1}{4}$ d $36^{5-3x} - 216^x = 0$

30 Example 18 Solve the following.

a $5^{2x} - 24 \times 5^x = 25$ b $3^{2x} + 243 = 84 \times 3^x$

c $2^{2x+2} - 8 \times 2^x - 32 = 0$ d $2^{6x-2} + 16 = 10 \times 2^{3x-1}$

31 Example 19 CAS Find the solution to the equation $2^{1.2x} + 2 = 5$, correct to 2 decimal places.

32 Example 19 CAS Find approximate solutions for the following.

a $4 \times 2^x - 12 = 0$ b $2^{x+1} - 4.5 = 0$

## Analysis

33 Prove that $\left(\frac{u}{z}\right)^n = \frac{u^n}{z^n}$.

34 How many whole minutes does it take 10 000 viruses to grow to 1 million if their population in the cells of an infected person doubles every 15 minutes?

35 If $f(x) = a \times 2^{kx}$, find the values of $a$ and $k$, such that the graph of $f(x)$ passes through the points (2, 10) and (4, 20).

36 An investment grows by 6.4% each year. After 4 years, the investment value is $15 320.

a Develop an exponential model of the investment.
b Find the initial value of the investment.
c Find the value after 7 years.
d Find the time it will take for the value of the investment to reach $30 000.

37 A car loses value by a factor of 12% each year. When new, the car cost $32 000. It is now 4 years old.

a Set up an exponential model of the value of the car.
b Find its present value.
c Find its value after a further 4 years.
d Find the time taken for the car to be worth less than $5000.

38 CAS The half-life of a radioactive isotope is the time taken for half the substance to decay. For a particular isotope, the half-life is 5 days. Originally, there is 100 g of the isotope.

a Find an exponential function for the amount of the isotope present after $5n$ days (i.e. $n$ half-lives).
b Hence find an exponential function $A(d)$ for the amount of isotope present after $d$ days.
c What is the initial value and the growth factor of the exponential function $A(d)$?
d 300 g of another isotope has a half-life of 9 days. Write an exponential function for the amount of this isotope present after $d$ days.
e Write a general expression for the amount of an isotope present after $d$ days if it begins with $G$ grams and has a half-life of $h$ days.


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**39** CAS The number of assaults in a particular suburb has increased by 12% every year since 1980 from an initial level of 12 assaults per 1000.

- **a** Find an exponential model for the number of assaults each year since 1980.
- **b** How many assaults would be predicted for 2015?
- **c** When will the assault level have increased 100-fold?

**40** CAS In 1980, computers could perform 400 000 operations per second. By 1990, they could perform 20 000 000. Assuming that the speed $V$ increases according to the exponential rule

$$V = A \times g^t$$

where $t$ is the number of years since 1980,

- **a** find an approximate value for $g$ correct to 4 decimal places
- **b** find the speed of computers in 2015.

**41** At the beginning of an experiment there are 50 bacteria in a culture. After 1 hour there were 1600 bacteria in the culture. What is the doubling period for this particular bacterium?

**42** The table on the right shows the values of a function that follows an exponential rule.

| $x$ | 1 | 0 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 40.8 | 48 | 34.68 | 29.478 |

- **a** Is the function a growth or decay function?
- **b** Find the growth factor.
- **c** Write the rule for the function.
- **d** Find the value of the function when $x = 7$.



Practice quiz

# MIXED REVISION
# CHAPTERS 10 • 11 • 12
# 4

## Multiple choice

1 Which of the following is true for the function $y = x^2 - x + 6$?

A The function is increasing when $x < \frac{1}{2}$.

B The function is increasing when $x < -2$ or $x > 3$.

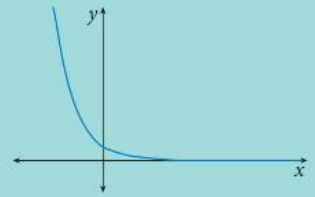
C The function is increasing when $x > -\frac{1}{2}$.

D The function is decreasing when $x < \frac{1}{2}$.

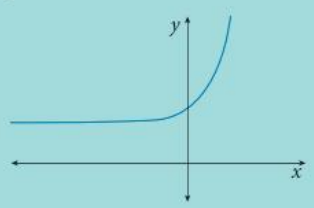
E The function is decreasing when $x > -2$ or $x < 3$.

The following graphs relate to questions **2**, **5** and **8**.

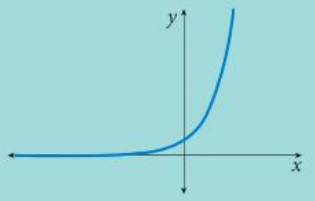
A



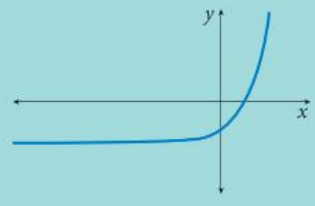

B



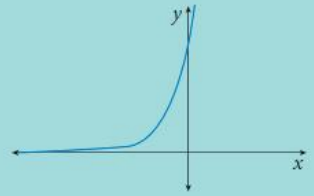

C




D

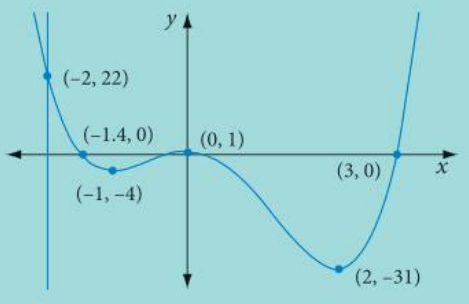



E




2 Which of the graphs **A–E** above would best represent $y = 4^x - 3$?

The following diagram relates to questions **3**, **6** and **9**. The diagram below shows the graph of a function, $f(x)$.



3 A local minimum for this function can be found at:
A (0, 1) B (−1, −4) C (−2, 22) D (3, 0) E (−1.4, 0)

4 Which of the following is not a polynomial function?
A $15 - 3x^2$ B $3\sqrt{x} - 2x^2$ C $12 - 4x + 6x^2$
D $5x^2 - \frac{3x}{7} + 4$ E $\frac{2x^3 - 3x - 17}{21}$

5 Which of the graphs **A–E** above would best represent $y = 0.4^x$?

6 A global maximum for this function can be found at:
A (2, −31) B (−1, −4) C (−2, 22) D (3, 0) E (−1.4, 0)

7 What is the gradient of the tangent to the function $y = 4x^2 + 5x + 6$ at (−2, 12)
A −21 B −12 C −11 D −5 E 21

8 Which of the graphs **A–E** above would best represent $y = 4^{x+3}$?

9 A global minimum for this function can be found at:
A (2, −31) B (−1, −4) C (−2, 22) D (3, 0) E (−1.4, 0)

## Short answer

1 If $f'(x) = 4x^3 - 7x^2 + 3$, state whether $f(x)$ is increasing, decreasing or stationary at $x = -3$.

2 Plot a graph of $y = 6^x$ for $0 \leq x \leq 3$ and use it to solve:
a $3 \times 6^x = 600$ b $6^x \times 6^{2x-1} - 150 = 0$

3 The volume in litres of a rectangular container that is leaking over time, $t$ minutes, is given by $V = -t^2 + 4t + 100$. Find:
a the initial volume.
b the volume after 10 minutes.
c the rate of change in volume after 10 minutes.
d how long it will take, to 1 decimal place, until the container is empty.

4 Find the coordinates of the point at which the curve $y = x^3 + 1$ has a tangent with a gradient of 6.

5 Use a graph drawn with a graphics calculator to solve $24^{x+1} = 298$.

6 Find the stationary points of $f(x) = 2x^3 + 15x^2 + 36x - 8$.

## Application

1 The volume of water in a pipe is given by $V = 3t^2 - 2t$, where $V$ is the volume in litres and $t$ is the time in minutes. Find the rate at which the water is flowing after 5 minutes.

2 If $f'(x) = 4x - 9x^2$ and $f(-2) = 25$, find $f(x)$.

3 Sketch the graph of each of $y = 2^{x-3} + 4$ by transforming the graph of $y = 2^x$.


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4 A bacterial colony increases in size by 8% every hour from an initial size of 300 organisms.
   a Write an exponential equation that models the number of organisms.
   b Find the number of bacteria after 8 hours.
   c Find the number of bacteria after 2 days.
   d Find the time taken for the colony to reach a size of 20 000 bacteria.
   e Find the time taken to reach 200 000 bacteria.

5 For the curve $y = 2x^3 - 9x^2 + 12x - 7$
   a find any stationary points on the curve and determine their nature
   b sketch the curve in the domain $-3 \leq x \leq 3$ .

6 The surface area of a cylinder is given by the formula $S = 2\pi r(r + h)$. A particular cylinder is to have a surface area of 160 $cm^2$.
   a Show that the volume is given by $V = 80r - \pi r^3$.
   b Find the value of $r$, correct to 2 decimal places, that gives the maximum volume.
   c Find the maximum volume, correct to 1 decimal place.


50
30
70
50
30
10



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# 12

## TERMINOLOGY

acceleration
average rate of change
displacement
dominant term
flow rate
global maximum
global minimum
gradient function
greatest value
instantaneous rate of change
least value
local maximum
local minimum
optimisation
point of horizontal inflection
stationary point
velocity

Prior learning

CALCULUS 2

# APPLICATIONS OF DERIVATIVES


12.01 Instantaneous rates of change
12.02 Straight-line motion
12.03 Using antiderivatives
12.04 Stationary points
12.05 Curve sketching
12.06 Extremes on an interval
12.07 Optimisation problems
Chapter summary
Chapter review

## APPLICATIONS OF DERIVATIVES

- find instantaneous rates of change (ACMMM092)
- construct and interpret position-time graphs, with velocity as the slope of the tangent (ACMMM094)
- sketch curves associated with simple polynomials; find stationary points, and local and global maxima and minima, and examine behaviour as $x \to \infty$ and $x \to -\infty$ (ACMMM095)
- solve optimisation problems arising in a variety of contexts involving simple polynomials on finite interval domains (ACMMM096)

## ANTIDERIVATIVES

- calculate antiderivatives of polynomial functions and apply to solving simple problems involving motion in a straight line (ACMMM097) AC

# 12.01 INSTANTANEOUS RATES OF CHANGE

The change in one quantity compared to the change in another is the **rate of change**. The rate of change can be worked out as an average or as an instantaneous rate of change at a particular time.

**IMPORTANT**

### Rates of change

The **average rate of change** of a function $f(x)$ from $x_1$ to $x_2$ is given by

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{\Delta y}{\Delta x}$$

The **instantaneous rate of change** of a function $f(x)$ is given by the derivative so at $x_1$ the instantaneous rate of change is $f'(x_1)$.

The average rate of change is the slope of the chord joining two points, while the instantaneous rate of change is the slope of the tangent at one point.

For physical quantities, we refer to the rate of change *with respect to* (**w.r.t.**) the independent variable. If the variable is not stated, it is assumed to be time.

**Flow rate** measures the speed at which a fluid travels through a pipe and is the rate of change of volume.


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## Example 1

The volume of water flowing through a pipe is given by $V = t^2 + 4t + 6$, where $V$ is in litres and $t$ is in minutes. Find the flow rate 5 minutes after the water begins flowing through the pipe.

### Solution

| | |
|---|---|
| Find the derivative of $V$. | $V' = 2t + 4$ |
| Let $t = 5$. | $= 2 \times 5 + 4$<br>$= 14$ |
| State the result. | The flow rate is 14 L/min 5 minutes after water starts flowing through the pipe. |

The derivative of a function is the slope of the tangent to the graph, which we also call the gradient of the function at that point. The derivative makes a new function called the **gradient function** of the original function. For example, the gradient function of $f(x) = 4x^3 - 2x + 5$ is the new function $f'(x) = 12x^2 - 2$. It is clear that the gradient function of a cubic function must be a quadratic.

Similarly, the gradient function of a quadratic must be a straight line.

For any polynomial, the degree of the gradient function must be 1 less than the degree of the original function.

You can sketch a gradient function without knowing the exact expression of the original function by examining the shape of the original function.

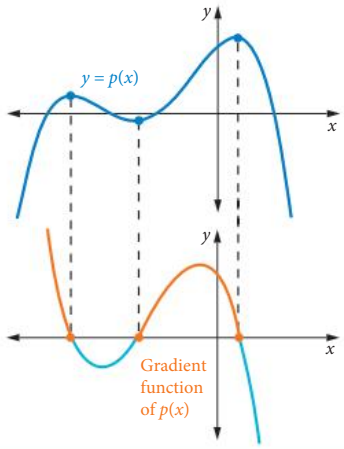
Where a function has a **stationary point**, the gradient is zero, so the gradient function must have a zero and cross or touch the $x$-axis. Where the function is *increasing*, the gradient is positive, so the gradient function will be above the $x$-axis. Where the function is *decreasing*, the gradient is negative, so the gradient function will be below the $x$-axis.

## Example 2

The function $p(x)$ is shown. Sketch the graph of the gradient function.

### Solution

Put zeros where the function is stationary.
The function is increasing before the first stationary value, so make the gradient function positive.
Between the first two stationary values the function is decreasing, so make the gradient function negative.
Continue in this way to get the entire gradient function.



# EXERCISE 12.01 Instantaneous rates of change

WS Gradient functions

## Concepts and techniques

**1** Example 1 If $y = 3x^2 - 7x + 5$, what is the instantaneous rate of change of $y$ with respect to $x$ when $x = 2$?

**A** 2 **B** 3 **C** 5 **D** 19 **E** 31

**2** The gradient of the tangent to $y = 4x^5 - 8x + 9$ at $x = -1$ is:

**A** $-28$ **B** $-12$ **C** 8 **D** 12 **E** 13

**3** If $f(x) = -\frac{6}{x^2} - 2x + 1$, the gradient function is:

**A** $-2\left(\frac{3}{x^3} - 1\right)$ **B** $-12x - 2$ **C** $-12x^3 - 2$ **D** $\frac{12}{x^3} - 2$ **E** Not defined

**4** $y = x^4 + 3x - 5x^3 + 7$

**a** Find the average rate of change from:

i $x = -2$ to $x = 2$ ii $x = 2$ to $x = 4$ iii $x = -2$ to $x = 4$.

**b** Find the instantaneous rate of change at:

i $x = -2$ ii $x = 2$ iii $x = 4$.

**5** $f(x) = x^3 - 6x^2 + 8x + 5$

**a** Find the average rate of change from $x = 2$ to $x = 4$.

**b** Find the instantaneous rate of change at $x = 2$.

**c** Find the instantaneous rate of change at $x = 4$.

**d** Find the average of the instantaneous rates of change at $x = 2$ and $x = 4$ and comment on the answer compared to part **a**.

**6** $g(t) = 7t^2 - 2t^3 + 3t + 3$

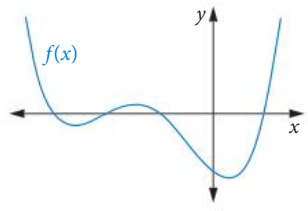
**a** Find the average rate of change from $t = 1$ to $t = 3$.

**b** Find the instantaneous rate of change at $t = 1$.

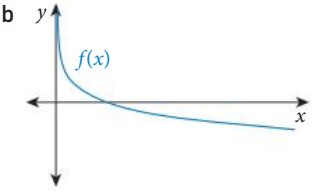
**c** Find the instantaneous rate of change at $t = 3$.

**7** Example 2 Sketch the gradient function for each of the following functions and comment on its domain and range.

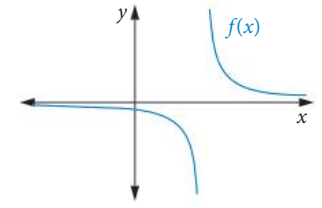
**a**



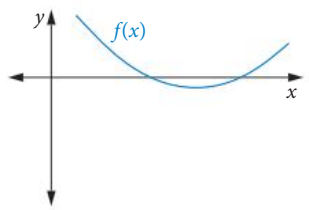

**b**



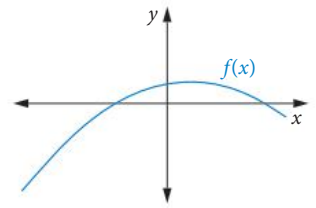

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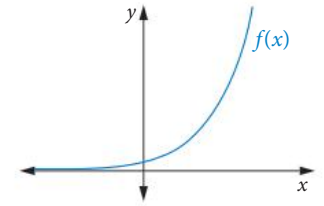

**d**




**e**



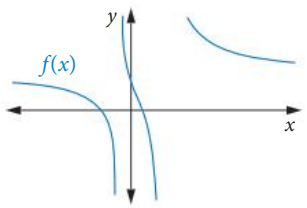

**f**



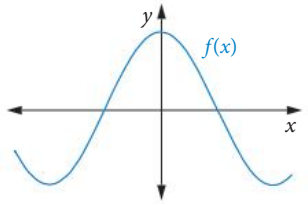


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g




h




8 Consider the function $y = x^3 - x^2 - 5x - 3$. Find the equation of the tangent to the curve at:

a $x = 3$ b $x = -2$

## Reasoning and communication

9 A tank which is initially full, can be partially drained for cleaning by opening an outlet valve. The volume of water in the tank when the valve is opened is given by:

$$V(t) = 30 + 5(40 - t)^2 \text{ for } 0 \leq t \leq 40$$

where $V$ is in litres and $t$ is in minutes.

a How much water is initially in the tank?

b What is the minimum amount of water in the tank?

c What is the average flow rate out of the tank from $t = 5$ to 10 minutes?

d What is the instantaneous flow rate at $t = 10$ min?

10 A ball is thrown upward from a platform 27 m above the ground at a velocity of 21m/s. Its height in metres above the ground at $t$ seconds is given by:

$$s(t) = -5t^2 + 21t + 27$$

a What does the rate of change of height with respect to time represent?

b What is the average velocity from $t = 2$ to $t = 4$ seconds?

c What does $s'(t)$ represent?

d Find the instantaneous velocity at $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 4$ seconds.

e What is the instantaneous velocity when the ball reaches its maximum height?

f When does the ball reach its maximum height and how high does it go?

11 The revenue in dollars generated from the sale of $x$ folding chairs is given by

$$R(x) = 20x - \frac{x^2}{500}$$

a Find the average rate of change of revenue from $x = 500$ to $x = 1000$.

b Find the average rate of change of revenue from $x = 5000$ to $x = 6000$.
$R'(x)$ is an estimate of the marginal revenue from the sale of the $(x+1)$th chair.

c Use $R'(1000)$ to estimate the marginal revenue from the sale of the 1001st chair.

d Determine the actual revenue generated by the sale of the 1001st chair.

12 Consider the function $y = x^3 - 4x^2 + 7x - 3$. Use the gradient function to show that the curve has no stationary points.

# 12.02 STRAIGHT-LINE MOTION

In this section, you will examine the motion of a particle moving along a straight line. The term 'particle' means a point. It can be the centre or any other point of an object, such as the front of a racing car. A particle is generally considered to have the mass of the object to which it refers. The word particle describes any moving object. You can find the particle's position, velocity or acceleration at any instant of time.

**Position** (usually shown as $x$) is relative to a fixed point called the origin. For straight-line motion, the position is a single coordinate whose sign indicates the direction. **Displacement** is a change of position and is usually shown by $s$. Unlike distance, position and displacement can be positive or negative, according to direction. It is common for the right to be considered as positive and the left to be considered negative.

When the particle is at the origin, its position is zero. That is, $x = 0$. The particle is often considered to begin at the origin, in which case its displacement and position are the same.

**Velocity** ($v$) is a measure of the rate of change of displacement with respect to time, so $v = \frac{ds}{dt}$.

Velocity (unlike speed) can be positive or negative, according to which direction the object is travelling in. It is common for velocity upwards to be considered as positive and downwards to be considered negative. The rate of change of displacement is also the rate of change of position.

When the object is not moving, we say that it is at rest. That is, $v = 0$.

**Acceleration** ($a$) is the measure of the rate of change of velocity with respect to time, so $a = \frac{dv}{dt}$.

Like displacement and velocity, acceleration can also be positive or negative, according to its direction.

If the acceleration is in the same direction as the velocity, the particle is speeding up (accelerating). If the acceleration is in the opposite direction from the velocity, the particle is slowing down (decelerating).

If the object is travelling at a constant velocity, there is no acceleration. That is, $a = 0$.

In most cases, time ($t$) starts from zero. Unless otherwise stated, you should assume that motion starts from $t = 0$.

**IMPORTANT**

**Motion in a straight line**

Position $= x$

Displacement = change of position $= s$

$$\text{Velocity} = v = \frac{ds}{dt} = \frac{dx}{dt}$$

$$\text{Acceleration} = a = \frac{dv}{dt}$$


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## Example 3

The displacement $s$ cm of a particle over time $t$ seconds is given by $s = 4t - t^2$.

a Find any times when the particle is at rest.

b How far does the particle travel in the first 3 seconds?

### Solution

a At rest means that the velocity is 0. Find an expression for velocity and let it equal 0.

$$\frac{ds}{dt} = 4 - 2t$$
$$0 = 4 - 2t$$

Solve. $t = 2$

State the result. The particle is at rest after 2 seconds.

b What does 'how far' mean? The distance travelled is required.

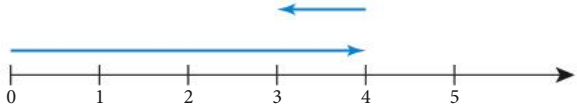
Check the displacement at $t = 0$. $s(0) = 4 \times 0 - 0^2 = 0$

What happens at $t = 2$ s? The particle changes direction at 2 s.

Find the displacement at 2 s. $s = 4 \times 2 - 2^2 = 4$ cm

Find the displacement after 3 s. $s = 4 \times 3 - 3^2 = 3$ cm

Sketch a diagram to represent the movement of the particle.




Calculate the total distance travelled. Distance travelled = 4 + 1 = 5 cm

State the result. The particle travels a total of 5 cm in the first 3 seconds.

Displacement and position are sometimes interchanged, so you should always check which is really meant in a problem.

## Example 4

The height of a particle projected into the air is given by $h = -5t^2 + 10t + 15$ m, where $t$ is in seconds.

a Find the initial height and velocity of the particle.

b Show that the particle has constant acceleration.

c Find when the particle will be at the origin.

d Find the particle's maximum height.

e Sketch the graph showing the particle's motion.

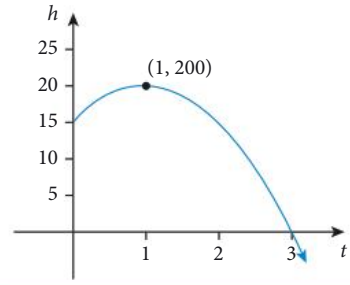
### Solution

a Find the height at 0 s. $h(0) = -5 \times 0^2 + 10 \times 0 + 15 = 15$ m

Find the equation for velocity. $v = \frac{dh}{dt} = -10t + 10$

Find the velocity at 0 s. $v(0) = -10 \times 0 + 10 = 10$ m/s

State the answer. The initial height and velocity are 15 m and 10 m/s

| | | |
|---|---|---|
| **b** | Find the equation for acceleration. | $a = \frac{dv}{dt} = -10$ |
| | State the answer. | The acceleration is a constant 10 m/s$^2$ downwards. |
| **c** | Substitute $h = 0$ in the equation. | $0 = -5t^2 + 10t + 15$ |
| | Simplify the equation. | $t^2 - 2t + 3 = 0$ |
| | Factorise. | $(t + 1)(t - 3) = 0$ |
| | Solve for $t$. | $t = -1$ or $t = 3$ |
| | State the answer. | The time starts from 0, so it is at the origin after 3 s. |
| **d** | When does the greatest height occur? | The greatest height is at $v = 0$. |
| | Substitute $v = 0$ in the equation. | $0 = -10t + 10$ |
| | Solve. | $t = 1$ |
| | Find the greatest height. | $h(1) = -5 \times 1^2 + 10 \times 0 + 15 = 20$ |
| | Write the answer. | The greatest height is 20 m. |
| **e** | Use the information to draw a graph of the particle's displacement. |  |



## Example 5

The displacement of an object from a fixed point is given by

$$x = 2t^3 - 5t + 4$$

where $t$ is in seconds and $x$ is in metres. Find the velocity and acceleration of the object after 3 s.

### Solution

| | |
|---|---|
| Differentiate to find the velocity. | $v = \frac{dx}{dt} = 6t^2 - 5$ |
| Differentiate again to find the acceleration. | $a = \frac{dv}{dt} = 12t$ |
| Substitute the value of $t$. | $v(3) = 6 \times 3^2 - 5 = 49$<br>$a(3) = 12 \times 3 = 36$ |
| Write the answer. | The velocity and acceleration after 3 s are 49 m/s and 36 m/s$^2$ respectively. |


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# EXERCISE 12.02 Straight-line motion

## Concepts and techniques

1 Example 3 The position of a particle is given by $x = t^3 - 9t$ cm, where $t$ is time in seconds.
- **a** Find the velocity of the particle after 3 s.
- **b** Find the acceleration after 2 s.
- **c** Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.
- **d** After what time will the acceleration will be 30 cm/s?

2 A particle is moving such that its position is given by $s = 2t^2 - 8t + 3$, where $s$ is in metres and $t$ is in seconds.
- **a** Find the initial velocity.
- **b** Show that acceleration is constant and find its value.
- **c** Find the position after 5 s.
- **d** Find when the particle will be at rest.
- **e** What will the particle's position be at that time?
- **f** Sketch the graph of the position against time.

3 Example 4 A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where $t$ is in seconds.
- **a** Find the initial height.
- **b** Find the initial velocity.
- **c** Find the height after 1 s.
- **d** What is the maximum height of the projectile?
- **e** Sketch the graph of the height against time.

4 The position of an object from a particular point is given by

$$x(t) = t^3 - 5t^2 + 6t + 10$$

where $x(t)$ is in metres and $t$ is in seconds.
- **a** Find the average velocity from 2 s to 4 s.
- **b** Find the velocity at 2 s.
- **c** Find the velocity at 4 s.
- **d** Find the average of the velocities at 2 s and 4 s and comment on the answer compared to part **a**.
- **e** Find the acceleration at 2 s.
- **f** Find the acceleration at 5 s.

5 The displacement of an object from a particular point is given by

$$d(t) = 3t^2 - 2t^3 + 8t + 12$$

where $d(t)$ is in metres and $t$ is in seconds.
- **a** Find the average velocity from 1 s to 3 s.
- **b** Find the velocity at 1 s.
- **c** Find the velocity at 3 s.
- **d** Find the average acceleration from 1 s to 3 s.
- **e** Find the acceleration at 1 s.
- **f** Find the acceleration at 3 s.

6 Example 6 An object is travelling along a straight line over time $t$ seconds, with its position (in metres) given by the formula $x = t^3 + 6t^2 - 2t + 1$.

a Find the equations of its velocity and acceleration.

b What will its position be after 5 s?

c What will its velocity be after 5 s?

d Find its acceleration after 5 s.

7 The displacement of a particle is given by $s = t^3 - 4t^2 + 3t$, where $s$ is in metres and $t$ is in seconds.

a Find the initial velocity.

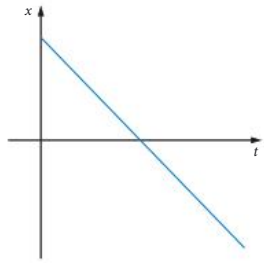
b Find the times when the particle will be at the origin.

c Find the acceleration after 3 s.

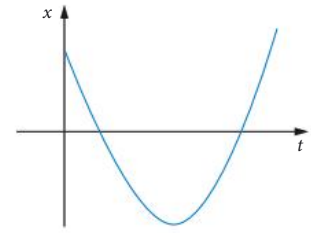
WS
Straight-line motion 1

8 The following graphs show the displacement of an object. Use the fact that $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$ to sketch the graphs for velocity and acceleration in each case.

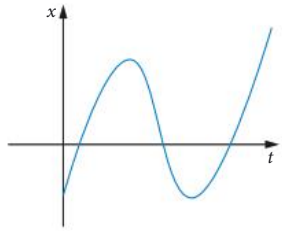
a



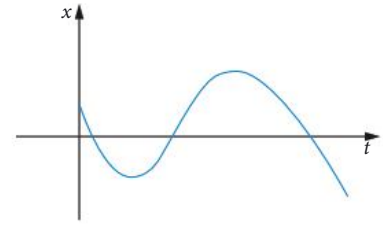

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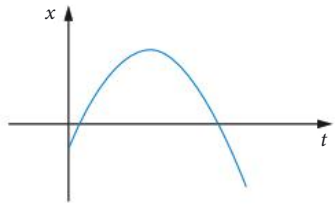

c




d




e





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## Reasoning and communication

**9** The displacement of a particle, in metres, over time $t$ seconds is $s = ut + \frac{1}{2}gt^2$ where $u = 5$ and $g = -10$.

- **a** Find the equation of the velocity of the particle.
- **b** Find the velocity after 10 s.
- **c** Show that the acceleration is equal to $g$.

**10** The height of a projectile is given by $h = 7 + 6t - t^2$, where height is in metres and time is in seconds.

- **a** Find the initial height.
- **b** Find the maximum height reached.
- **c** When will the projectile reach the ground?
- **d** Sketch the graph showing the height of the projectile over time $t$.
- **e** How far will the projectile travel in the first 4 s?

**11** The distance travelled by a train travelling between two consecutive stations is given by

$$d = 0.86t^2 - 0.0082t^3$$

where $d$ is in metres and $t$ is in seconds.

- **a** Find an expression for the velocity of the train.
- **b** Find the time taken to travel between the stations.
- **c** How far is it between the stations?
- **d** Use your graphics calculator to find the maximum speed of the train.
- **e** Find the maximum acceleration of the train.

**12** The groundsman at a high school is recovering tennis balls from the top of the gym, which is 5.1 m from the level ground around it. He throws one ball from the edge of the roof towards the oval. The vertical ($v$) and horizontal ($h$) displacements of the ball are given by

$$v(t) = 5.1 + 7t - 5t^2 \text{ and } h(t) = 9t$$

where $v(t)$ and $h(t)$ are in metres and $t$ is in seconds.

- **a** Work out the vertical displacement after 1 s.
- **b** Work out the vertical velocity after 1 s.
- **c** Find when the vertical velocity is zero. What is the significance of this value?
- **d** Find the greatest height reached.
- **e** Find when the vertical displacement is zero. What is the significance of this value?
- **f** Find how far the ball is from the gym when it hits the ground.

# 12.03 USING ANTIDERIVATIVES

The derivative of the position or displacement of a particle gives its velocity, and the derivative of the velocity gives the acceleration. What are the opposites of these statements?

## Example 6

The velocity of a particle is given by $v = 3t^2 + 2t + 1$. The particle starts 2 cm to the left of the origin. Find its position after 5 s.

### Solution

| | |
|---|---|
| Find an equation for the position. | Let $x$ = the antiderivative of $v$<br>$x = \frac{3t^{2+1}}{2+1} + \frac{2t^{1+1}}{1+1} + t + c$<br>$= t^3 + t^2 + t + c$ |
| Use the given information. | When $t = 0$, $x = -2$<br>$-2 = 0^3 + 0^2 + 0 + c$<br>$c = -2$ |
| Write the equation for position. | $x = t^3 + t^2 + t - 2$ |
| Substitute $t = 5$. | $= 5^3 + 5^2 + 5 - 2$<br>$= 125 + 25 + 5 - 2$<br>$= 153$ |
| State the result. | After 5 seconds the particle will be 153 cm to the right of the origin. |

## Example 7

The acceleration of a particle is given by $a = 12t - 10$.

If the particle is initially at rest 8 m to the right of the origin, find its exact position after 3 s.

### Solution

| | |
|---|---|
| The velocity is the antiderivative of the acceleration. | $v = \frac{12t^{1+1}}{1+1} - 10t + c$<br>$v = 6t^2 - 10t + c$ |
| The particle is initially at rest. | When $t = 0$, $v = 0$ |
| Substitute in the known values. | $0 = 6 \times 0^2 - 10 \times 0 + c$ |
| Evaluate. | $c = 0$ |
| Write the equation for velocity. | $v = 6t^2 - 10t$ |


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| | |
|---|---|
| The position is the antiderivative of the velocity. | $x = \frac{6t^{2+1}}{2+1} - \frac{10t^{1+1}}{1+1} + C$ |
| | $x = 2t^3 - 5t^2 + C$ |
| The particle is initially 8 m to the right of the origin. | When $t = 0$, $x = 8$ |
| Substitute the known values. | $8 = 2 \times 0^3 - 5 \times 0^2 + C$ |
| Evaluate. | $C = 8$ |
| Write the equation for displacement. | $x = 2t^3 - 5t^2 + 8$ |
| Find $x$ when $t = 3$. | $x = 2 \times 3^3 - 5 \times 3^2 + 8 = 17$ |
| State the result. | The position of the particle after 3 s is 17 m to the right of the origin. |

In Example 7, two different constants of integration were found. Instead of using $c$ and $C$ to distinguish them, you could use different letters like $c$ and $k$.

## Example 8

A truck travelling at 5 m/s accelerates for 4 s according to the equation $a(t) = 10 - 2t$.

What is its new speed?

### Solution

| | |
|---|---|
| Acceleration is the rate of change of velocity, so $v(t)$ must be the antiderivative of $a(t)$. | $a(t) = \frac{dv}{dt}$ |
| Write down the acceleration function. | $a(t) = 10 - 2t$ |
| Find the antiderivative. | $v(t) = 10t - t^2 + c$ |
| Initial velocity is 5 m/s. | $v(0) = 5$, so $c = 5$ |
| Write the velocity function. | $v(t) = 10t - t^2 + 5$ |
| Calculate the velocity after 4 s. | $v(4) = 10 \times 4 - 4^2 + 5$ |
| Evaluate. | $= 40 - 16 + 5 = 29$ |
| Write the result. | The new speed of the truck is 29 m/s. |

# EXERCISE 12.03 Using antiderivatives

WS
Straight-line motion 2

## Concepts and techniques

1 Example 6 The velocity of a particle is given by $v = 3t^2 - 5$ cm/s. If the particle is initially at the origin, find its position after 3 s.

2 A particle has a velocity given by $\frac{dx}{dt} = 2t - 3$ m/s. After 3 s the particle is at the origin. Find its position after 7 s.

3 The velocity of a particle is given by $v = 4 - 3t$ cm/s. If the particle is 5 cm from the origin after 2 s, find the position after 7 s.

4 The velocity of a particle is given by $v = 8t^3 - 3t^2$ cm/s. If the particle is initially at the origin, find
   a its acceleration after 5 s
   b its displacement after 3 s
   c when it will be at the origin again.

5 Example 7 A particle has a constant acceleration of 12 m/s. If the particle has a velocity of 2 m/s and is 3 m from the origin after 5 s, find its position after 10 s.

6 The acceleration (in cm/s) of an object is given by $6t + 4$. The particle is initially at rest at the origin. Find
   a its velocity after 5 s
   b its displacement after 5 s.

7 Example 8 A particle has an initial velocity of 10 cm/s. It accelerates for 7 s with an acceleration defined by:

$$a = 3t - 4 \text{ m/s}^2$$

What is the final velocity of the particle?

## Reasoning and communication

8 A projectile is accelerating at a constant rate of $-9.8$ m/s$^2$. If it is initially 2 m above the ground and has a velocity of 4 m/s, find the equation of the projectile's height above the ground.

9 A particle is accelerating according to the equation $a = (3t + 1)^2$ m/s$^2$. If the particle is initially at rest 2 m to the left of the origin, find its displacement after 4 s.

10 The velocity (in cm/s) of an object is given by $x = t^2(t^3 + 1)^2$ and the object is initially 2 cm to the right of the origin.
   a Find its acceleration after 1 second.
   b Find the position of the object after 2 seconds.


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11 Candice threw a stone upwards from a tree house at a speed of 5 m/s. Gravity accelerated the stone downwards with an acceleration of at 10 m/s$^2$. The tree house is 6 m off the ground.
   a Find an expression for the velocity.
   b At what time did the stone reach its greatest height (when $v = 0$)?
   c Find an expression for the height.
   d What was the greatest height reached?
   e Write an equation to find the time when the stone returned to the height of the tree house, and find its velocity on its return.

12 In a test conducted by a motoring organisation, a car starting from rest accelerated constantly until it reached a velocity of 100 km/h in 12 seconds. What was the acceleration of the car in m/s$^2$?

13 Barry threw a stone from the beach vertically up towards the top of a cliff. After he threw the stone, it accelerated downwards at about 10 m/s$^2$ under gravity. After a few attempts where the stone fell back, he just managed to reach the top of the cliff, which is 8 m above the beach. How fast did he throw the last stone? What is this in km/h?

# 12.04 STATIONARY POINTS

You have previously seen that the gradient function (derivative) can be used to identify the behaviour of a function. If the derivative is positive, a function is increasing, if the gradient is negative, the function is decreasing and if the gradient is zero, the function is stationary.

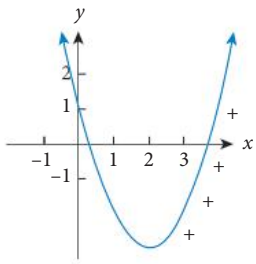
### Example 9

Find all $x$ values for which the curve $f(x) = x^2 - 4x + 1$ is increasing.

#### Solution

| | |
|---|---|
| Find the gradient function. | $f'(x) = 2x - 4$ |
| $f'(x) > 0$ for increasing curve | Let $2x - 4 > 0$<br>$x > 2$ |
| State the result. | The curve is increasing for $x > 2$. |

The result of Example 9 can be shown using a sketch of the curve, with + signs to indicate where the function is increasing.

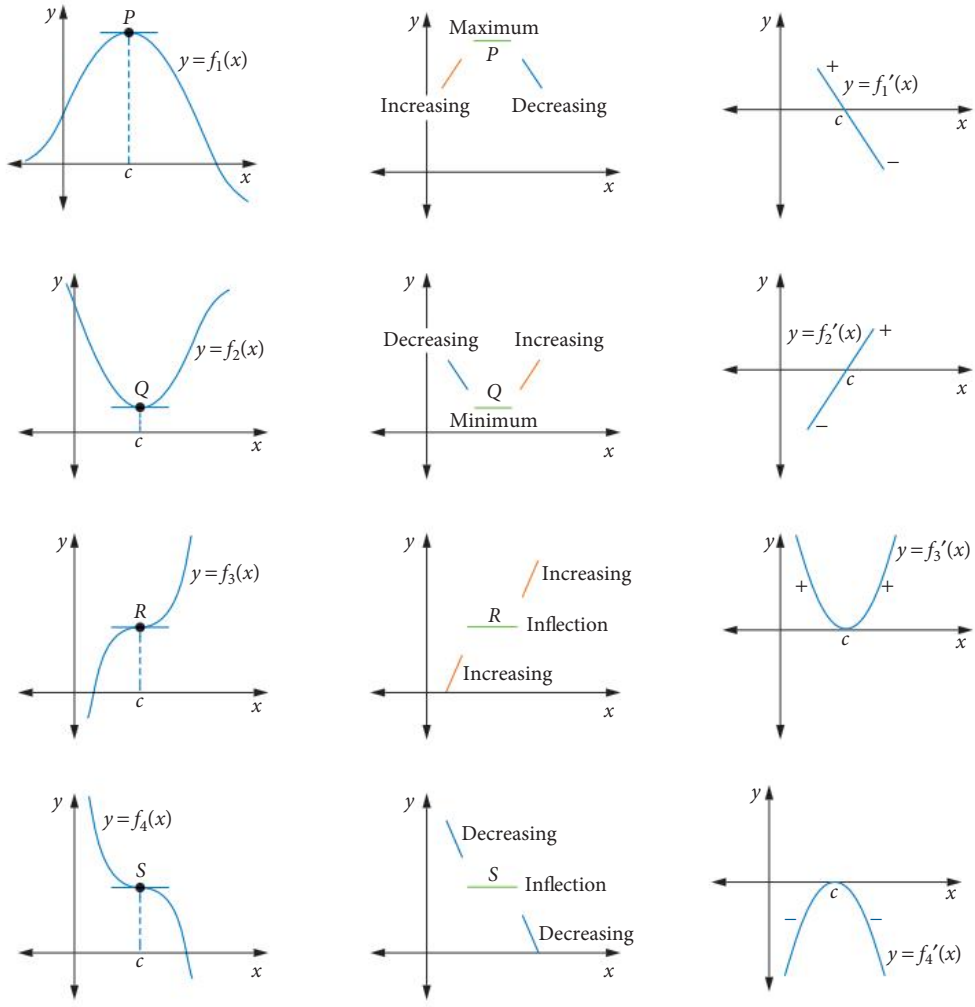



In addition to finding the stationary points, you can use the derivative to identify them as a **local maximum**, **local minimum** or point of horizontal **inflection**. The term *local* indicates that the function has a maximum or minimum value for $x$ values *near the stationary point* and not necessarily for all values of $x$ in the domain of the function. You can find local maxima, minima and points of inflection by examining the *sign of the derivative* just before and just after the stationary point.

There are four cases to consider:

| | $f'(x)$ before the stationary point | $f'(x)$ at the stationary point | $f'(x)$ after the stationary point | Type of point | Change in $f'(x)$ as $x$ increases |
|---|---|---|---|---|---|
| $f(x)$ | + | 0 | − | Local maximum | Decreasing through zero |
| $f(x)$ | − | 0 | + | Local minimum | Increasing through zero |
| $f(x)$ | + | 0 | + | Inflection on a rising curve | Decreasing to zero, then increasing again |
| $f(x)$ | − | 0 | − | Inflection on a falling curve | Increasing to zero, then decreasing again |

These are illustrated below.





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## Example 10

Identify the stationary points of $f(x) = x^3 + x^2 - 5x - 4$.

### Solution

| | |
|---|---|
| Write down the function. | $f(x) = x^3 + x^2 - 5x - 4$ |
| Differentiate. | $f'(x) = 3x^2 + 2x - 5$ |
| Let $f'(x) = 0$. | $3x^2 + 2x - 5 = 0$ |
| Factorise the LHS. | $(3x + 5)(x - 1) = 0$ |
| Solve. | $3x + 5 = 0$ or $x - 1 = 0$<br>$x = -\frac{5}{3}$ or $x = 1$ |
| Find $f\left(-\frac{5}{3}\right)$. | $f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2 - 5\left(-\frac{5}{3}\right) - 4$<br>$= 2\frac{13}{27}$ |
| Find $f(1)$. | $f(1) = (1)^3 + (1)^2 - 5 \times 1 - 4$<br>$= -7$ |
| State the result. | The stationary points are at $\left(-1\frac{2}{3}, 2\frac{13}{27}\right)$ and $(1, -7)$. |

Make a small table of values of $f'(x)$ with convenient values around and at the stationary points.

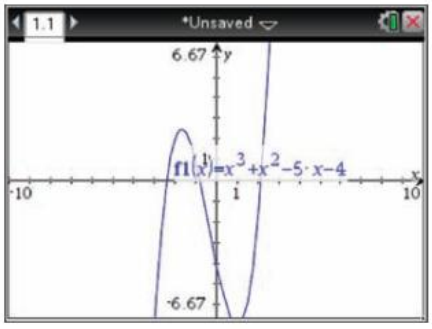
**Values on either side of stationary points**

| $x$ | $x = -2$ | $-1\frac{2}{3}$ | $x = 0$ | 1 | $x = 2$ |
|---|---|---|---|---|---|
| $f'(x)$ | 3 | 0 | $-5$ | 0 | 11 |
| $f(x)$ | Increasing | Stationary | Decreasing | Stationary | Increasing |
| shape |  |  |  |  |  |

You can also use your CAS calculator to identify the type of point.

#### TI-Nspire CAS

Use the Graphs screen and enter the function as $f1(x) = x^3 + x^2 - 5x - 4$.
Use menu 4:Windows/Zoom if necessary to see the graph clearly. The first stationary point is a maximum and the second is a minimum.

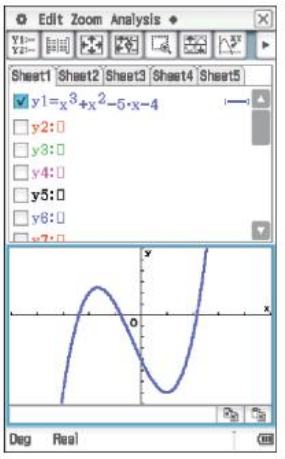


**ClassPad**

Use the **Graph&Table** application to graph the function.

Tap and change the settings in the **View Window** if necessary to see the stationary points of the graph.

The first stationary point is a maximum and the second is a minimum.




State the result.

There is a maximum at $\left(-1\frac{2}{3}, 2\frac{13}{27}\right)$ and a minimum at $(1, -7)$.



Alamy/Oleksiy Maksymenko Photography


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## Example 11

Identify any stationary points of $f(x) = x^3 + 6x^2 + 12x + 9$.

### Solution

| | |
|---|---|
| Write down the function. | $f(x) = x^3 + 6x^2 + 12x + 9$ |
| Differentiate. | $f'(x) = 3x^2 + 12x + 12$ |
| Let $f'(x) = 0$. | $3x^2 + 12x + 12 = 0$ |
| Factorise the LHS. | $3(x^2 + 4x + 4) = 0$ |
| The brackets contain a perfect square. | $3(x + 2)^2 = 0$ |
| Solve. | $x = -2$ |
| Find $f(-2)$. | $f(-2) = (-2)^3 + 6(-2)^2 + 12 \times -2 + 9$<br>$= 1$ |
| State the result. | There is only one stationary point at $(-2, 1)$. |

Now you need to investigate the gradient on either side of the stationary point.

| $x$-interval/value | $-\infty \to -2$ | $-2$ | $-2 \to \infty$ |
|---|---|---|---|
| $f'(x)$ | $f'(-3) = 3 > 0$ | $f'(-2) = 0$ | $f'(0) = 12 > 0$ |
| $f(x)$ | Increasing | Stationary | Increasing |
| shape |  |  |  |

State the result. $(-2, 1)$ is a point of horizontal inflection on a rising curve.

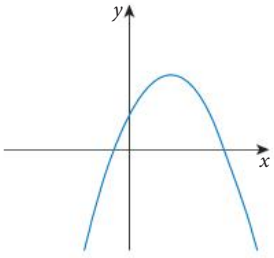
# EXERCISE 12.04 Stationary points

## Concepts and techniques

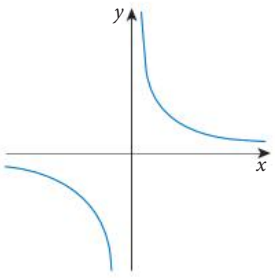
Stationary points

1 Identify the parts of each curve where the gradient of the tangent is positive, negative or zero. Label each curve with +, – or 0 to indicate the sign of the gradient.

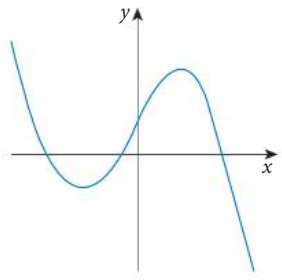
a



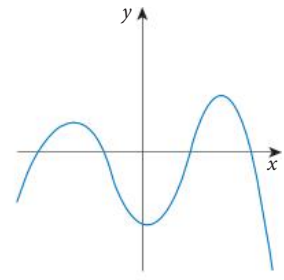

b



c




d




**2** Example 9 Find all values of $x$ for which the curve $y = 2x^2 - x$ is decreasing.

**3** Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

**4** Find values of $x$ for which the curve $y = x^2 - 3x - 4$ is

**a** decreasing **b** increasing **c** stationary.

**5** Show that the function $f(x) = -2x - 7$ is always decreasing.

**6** Consider the function $f(x) = 4x^2 + 3x - 1$. Show that the function is

**a** increasing when $x = 8$ **b** decreasing when $x = -4$

**c** stationary at $x = -0.375$.

**7** Consider the function $f(t) = t^3 - 3t^2 + t - 5$. Show that the function is

**a** increasing when $t = -1$ **b** decreasing when $t = 1$

**c** increasing at $t = 3$.

**8** $y = x^2 - 4x$. State whether $y$ is increasing, decreasing or stationary at

**a** $x = -4$ **b** $x = 2$ **c** $x = 3$ **d** $x = 4$

**9** If $r(t) = -7t^3 + 12t^2 + 4t - 16$, state whether $r(t)$ is increasing, decreasing or stationary at

**a** $t = -3$ **b** $t = 0$ **c** $t = 1$ **d** $t = 4$

**10** For $w = m^3 - 12m + 9$, state whether the function is increasing, decreasing or stationary at

**a** $m = -4$ **b** $m = -2$ **c** $m = -1$ **d** $m = 0$

**11** For $y = u^4 - 14u^2 - 24u + 24$, state whether the function is increasing, decreasing or stationary at

**a** $u = -4$ **b** $u = -2$ **c** $u = -1$ **d** $u = 0$

**12** Example 10 Identify any stationary points of the following functions.

**a** $f(x) = 2x^2 - 8x + 3$ **b** $f(x) = 6 + 4x - x^2$

**c** $f(x) = x^3 - x^2 - 8x + 10$ **d** $f(x) = -2x^3 - 3x^2 + 12x$

**e** $f(x) = x^3 + 3x - 5$ **f** $f(x) = x^3 + 3x^2 - 9x$

**13** Example 11 Identify any stationary points of the following functions.

**a** $f(x) = x^3 - 3x^2 + 3x - 5$ **b** $y = 36 + 9x^2 - 27x - x^3$

**14** Identify all stationary points on the following curves.

**a** $y = x^2 - 2x - 3$ **b** $f(x) = 9 - x^2$

**c** $y = 2x^3 - 9x^2 + 12x - 4$ **d** $y = x^4 - 2x^2 + 1$

**e** $f(x) = 3x^4 - 8x^3 + 6x^2 + 3$ **f** $y = 15x^4 - 6x^5 - 10x^3 - 10$


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## Reasoning and communication

**15** Find any stationary points on the curve $y = (x - 2)^4$.

**16** The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Find $p$.

**17** Find $a$ and $b$ if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

**18** Identify any stationary points of the following functions.

**a** $g(x) = -x^2 + 4x - 3$
**b** $f(x) = -x^2 - 2x + 4$
**c** $y = x^3 - 12x$
**d** $g(x) = 3x - x^3$
**e** $f(x) = 2x^3 - 12x^2 + 7$
**f** $h(r) = 4\pi r^2 - \pi r$
**g** $y = x^3 - x^2 - 8x + 2$
**h** $y = x^3 - 6x + 1$
**i** $y = x^4 - 4x + 5$
**j** $g(x) = x^2(8 - x)$

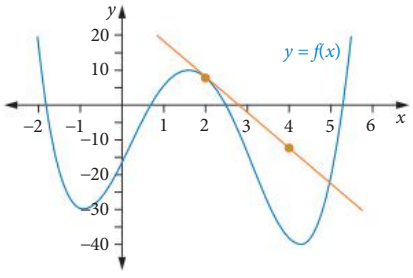
# 12.05 CURVE SKETCHING

Curve sketching is an important skill in mathematics. It helps you to understand what a function is likely to be like.

### Example 12

The graph of $f(x)$ is shown. Use the graph to estimate:

**a** the coordinates of the stationary points
**b** the $y$ value at $x = 3$
**c** the intervals where the function is increasing
**d** the rate of change at $x = 2$.




#### Solution

**a** Read directly from the graph.

The stationary points are at about (−0.9, −30), (1.6, 10) and (4.3, −40).

**b** Read from the graph.

The $y$ value at $x = 3$ is about −13.

**c** Find the upward-sloping sections.

The function is increasing for about $-0.9 < x < 1.6$ and $x > 4.3$.

**d** Draw a tangent (in pencil) or use a clear ruler. This is shown as an orange line in the diagram above. Choose convenient points on the tangent line to calculate the gradients.

(2, 8) and (4, −13) lie on the tangent.

Rate of change = gradient of tangent

$$\approx \frac{-13-8}{4-2}$$

$$= -10.5$$

The rate of change at $x = 2$ is about −10.5.

The material in this section covers curve sketching with polynomial functions. There are key features of polynomial functions that provide important aids for sketching graphs.

These include the nature of stationary points, the $y$-intercept, zeros ($x$-intercepts) and behaviour as $x \to \pm\infty$.

**IMPORTANT**

**Procedure and headings for sketching functions**

1 **Zeros:** If possible, determine the $x$-intercepts—the values of $x$ for which $f(x) = 0$.

2 **Sign:** Determine the sign of $f(x)$ either side of zeros (for functions with them).

3 **Derivative (gradient function):** Use the derivative to:

- identify stationary points
- determine the intervals in which the function is increasing or decreasing.

4 ***y*-intercept:** Find the value of $f(0)$.

5 Behaviour as $x \to \pm\infty$: a **dominant term** is of higher power than any other term. As $x \to \pm\infty$, the magnitude of the dominant term becomes much larger than the magnitudes of the other terms, so the function behaves like the dominant term.

Then use your results to sketch the graph.

## Example 13

Sketch $f(x) = (x + 1)^2(x - 3)$.

### Solution

**Zeros**

State zeros, including type.
Let $f(x) = 0$ and solve for $x$.

There is a double zero at $x = -1$ and a single zero at $x = 3$.

**Sign of function**

Use a table to show the sign of $f(x)$ before, between and after the zeros.

| $x$ | $-2$ | $-1$ | $0$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $f(x)$ | $-5$ | $0$ | $-3$ | $0$ | $25$ |
| Sign | $-$ | | $-$ | | $+$ |

Write the effect in words.

$f(x)$ is below the $x$-axis before $x = -1$, and between $x = -1$ and $x = 3$, and above after $x = 3$.

**Derivative**

Expand $(x + 1)^2$.

Multiply out the brackets.
Simplify

$$\begin{aligned} f(x) &= (x + 1)^2(x - 3) \\ &= (x^2 + 2x + 1)(x - 3) \\ &= x^3 + 2x^2 + x - 3x^2 - 6x - 3 \\ &= x^3 - x^2 - 5x - 3 \end{aligned}$$

Now find the derivative.

$$f'(x) = 3x^2 - 2x - 5$$


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### Stationary points

Find the zeros of $f'(x)$.
Factorise.
Slove

$3x^2 - 2x - 5 = 0$
$(3x - 5)(x + 1) = 0$
$x = -1$ or $x = \frac{5}{3}$

Write the conclusion.

There are stationary points at $x = -1$ and $x = \frac{5}{3}$.

Use a table to show the sign of $f'(x)$ before, between and after the stationary points.

| $x$ | $-2$ | $-1$ | $0$ | $\frac{5}{3}$ | $2$ |
|---|---|---|---|---|---|
| $f'(x)$ | 11 | 0 | $-5$ | 0 | 3 |
| **Sign** | + | 0 | − | 0 | + |

Use diagrams to show the slope.



Find the values at the points.

$f(-1) = 0$ and $f\left(\frac{5}{3}\right) = -9\frac{13}{27}$
Write the stationary points. There is a local maximum at $(-1, 0)$ and a local minimum at $\left(1\frac{2}{3}, -9\frac{13}{27}\right)$.

### Increasing and decreasing intervals

Use the information from the table of the signs of $f'(x)$.

$f(x)$ increases before $x = -1$, decreases between $x = -1$ and $x = \frac{5}{3}$, and increases after $x = \frac{5}{3}$.

### *y*-intercept

Find the value of $f(0)$.

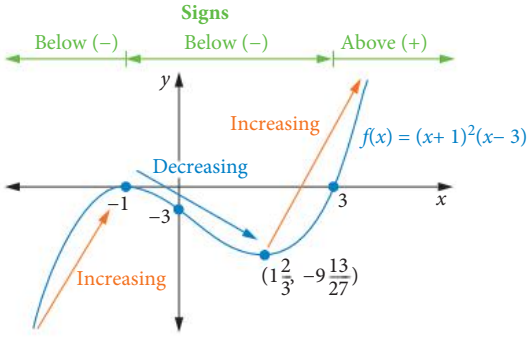
$f(0) = -3$

### Behaviour as $x \to \pm\infty$

Polynomials have a dominant term.

$f(x) = x^3 - x^2 - 5x - 3$ has dominant term $x^3$.
As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$

Place all the known points on the graph.
Use the signs to guide where the graph is above and below the $x$-axis.
Join up with smooth curves to finish the diagram.



## Example 14

Sketch $y = 2x^2 + 3x - x^3 - 5$.

### Solution

**Zeros**

Can you find the zeros?

The expression does not factorise easily, so the zeros and sign regions cannot be found easily.

**Derivative**

Find the derivative.

$$\frac{dy}{dx} = 4x + 3 - 3x^2$$

**Stationary points**

Find the zeros of the derivative.

$$4x + 3 - 3x^2 = 0$$

Rearrange with $x^2$ positive.

$$3x^2 - 4x - 3 = 0$$

Cannot factorise, so write coefficients.

$$a = 3,\ b = -4,\ c = -3$$

Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute values.

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-3)}}{2 \times 3}$$

Simplify.

$$= \frac{4 \pm \sqrt{52}}{6}$$

Keep exact values on your calculator.

$$\approx -0.54 \text{ or } 1.87$$

Write conclusion.

Stationary points are at $x \approx -0.54$ and $x \approx 1.87$.

Use a table to show the sign of $\frac{dy}{dx}$ before, between and after the stationary points.

| $x$ | $-1$ | $-0.54$ | $0$ | $1.87$ | $2$ |
|---|---|---|---|---|---|
| $\frac{dy}{dx}$ | $-4$ | $0$ | $3$ | $0$ | $1$ |
| Sign | $-$ | $0$ | $+$ | $0$ | $-$ |

Use diagrams of the slope.



Use the exact values in your calculator.

$f(-0.54) \approx -5.88$ and $f(1.87) \approx 1.06$

Write the stationary points.

There is a minimum at about $(-0.54, -5.88)$ and a maximum at about $(1.87, 1.06)$.

**Increasing and decreasing intervals**

Use the information from the table of the signs of $f'(x)$.

$f(x)$ is decreasing before $x \approx -0.54$, increasing between $x = -0.54$ and $x \approx 1.87$, and decreasing after $x \approx 1.87$.


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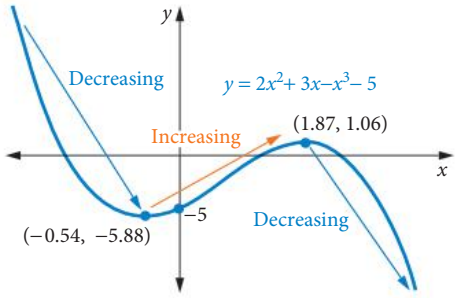
**$y$-intercept**

Find $y$ when $x = 0$. $y = -5$

**Behaviour as $x \to \pm\infty$**

Polynomials have a dominant term. $y = 2x^2 + 3x - x^3 - 5$ has dominant term $-x^3$.
As $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$

Place all the known points on the graph. Join up with smooth curves to finish the sketch.



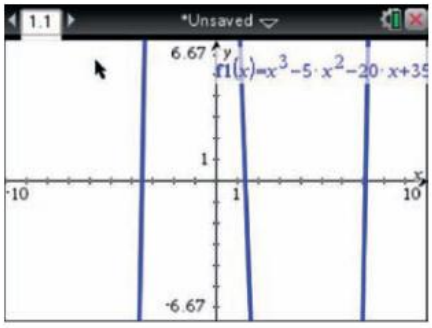

# TECHNOLOGY

## Curve sketching with a CAS calculator

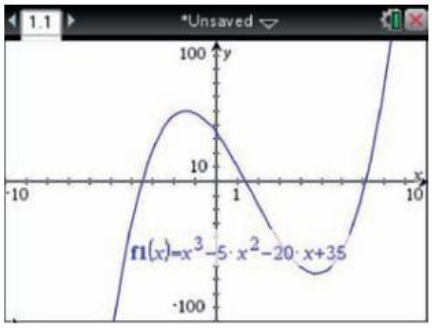
**a** CAS Use a CAS calculator to sketch the graph of $y = x^3 - 5x^2 - 20x + 35$ from $x = -10$ to $x = 10$.
**b** Identify local maxima and minima.
**c** Find the rate of change at $x = 2$.

### TI-Nspire CAS

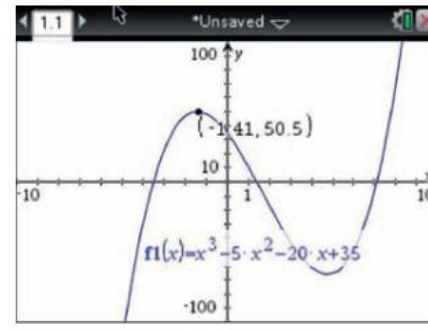
**a** Use a graph page.
Put in the function
$y = x^3 - 5x^2 - 20x + 35$.




Use Window Settings to make the domain, say, $-10 \le x \le 10$ and the range $-100 \le y \le 100$ to see the graph properly.



b Press menu and select 6: Analyze Graph, then select 3: Maximum. Click the hand symbol either side of the maximum to set the lower and upper bounds.




State the result.

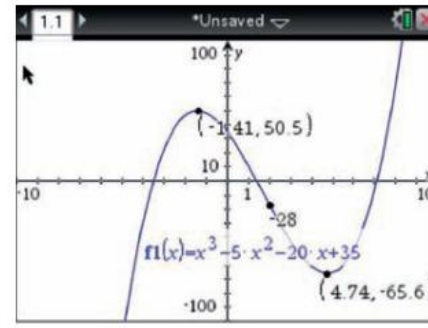
There is a local maximum at about (–1.41, 50.5).

Repeat the previously described procedure and select 2: Minimum to identify the local minimum.

State the result.

There is a local minimum at about (4.74, –65.6).

c Press menu and select 6:Analyze Graph, then select 6:dy/dx. Click the hand symbol on 2 on the $x$-axis.

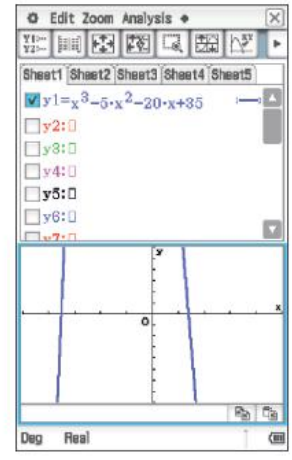



State the result.

The rate of change of the function at $x = 2$ is –28.

### ClassPad

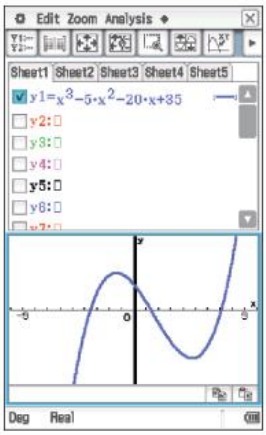
a Use the **Graph&Table** application. Put in the function $y = x^3 - 5x^2 - 20x + 35$ and tap .



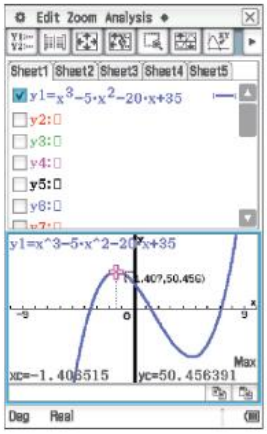


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Use **View Window** to make the domain, say, $-10 \leq x \leq 10$ and the range $-100 \leq y \leq 100$ to see the graph properly.




**b** Find the maximum as follows.
Tap **Analysis**, then **G-Solve**, then **Max**.
The local maximum is marked and the point displayed.




State the result.

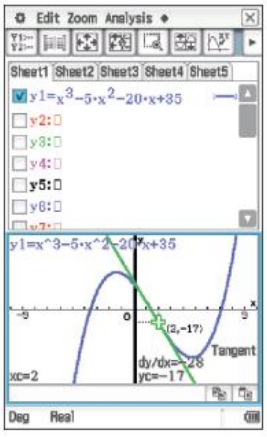
There is a local maximum at about $(-1.41, 50.5)$.

Repeat for the minimum, using **Min** instead of **Max**. You may have to tap the graph first.

State the result.

There is a local minimum at about $(4.74, -65.6)$.

**c** To find the derivative, in the top menu bar, tap **Analysis**, then **Sketch** and **Tangent**. Pressing 2 causes a pop-up box to appear with the x-value set to 2. Tap **OK**.
The value of dy/dx at the point $(2, -17)$ is displayed. (If not, tap then **Graph Format** and tap the box by **Derivative/Slope** and **Set**.)
This value is the value of the derivative of the function at the point and is slope of the tangent to the graph at this point.




State the result.

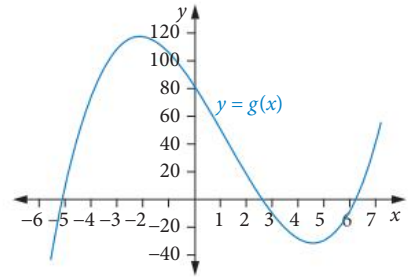
The rate of change of the function at $x = 2$ is $-28$.

# EXERCISE 12.05 Curve sketching

WS
Sketching curves

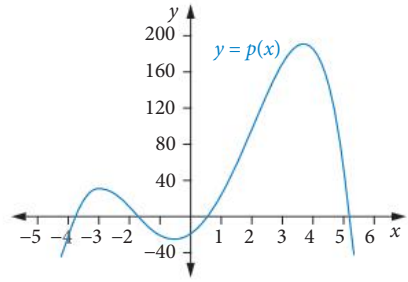
## Concepts and techniques

1 Example 12 The graph of $g(x)$ is shown. Use the graph to estimate:




- a the coordinates of the stationary points
- b the $y$ value at $x = -4$
- c the $y$ value at $x = 1$
- d the intervals where the function is decreasing
- e the intervals where the function is increasing
- f the rate of change at $x = -4$
- g the rate of change at $x = 1$.

2 The graph of $p(x)$ is shown. Use the graph to estimate:




- a the coordinates of the stationary points
- b the $y$ value at $x = 1$
- c the $y$ value at $x = 2$
- d the intervals where the rate of change of the function is positive
- e the intervals where the rate of change of the function is negative
- f the rate of change at $x = 2$
- g the rate of change at $x = -4$.

3 a CAS Draw the graph of $y = 10x^2 - 2x^3 + 14x - 33$ from $x = -10$ to $x = 10$, and hence estimate the coordinates of the stationary points.
- b Estimate where the rate of change is positive.
- c Estimate the rate of change at $x = -4$.
- d Estimate the rate of change at $x = 3$.

4 a CAS Draw the graph of $y = 2x^4 - 17x^3 + x^2 + 137x - 79$ from $x = -10$ to $x = 10$, and hence estimate the coordinates of the stationary points.
- b Estimate where the rate of change is negative.
- c Estimate the rate of change at $x = -1$.
- d Estimate the rate of change at $x = 2$.

5 Example 13 Sketch the graphs of the following functions.

a $y = 4x^2 - 9x + 5$ b $g(x) = 10x - x^2 - 21$ c $f(x) = (x - 4)^2(x + 5)$

d $y = (2x - 1)(x - 2)^2$ e $y = (2x - 1)^2(x + 4)$ f $f(x) = (x - 6)^2(x + 3)^2$

## Reasoning and communication

6 Example 14 Find stationary points of $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

7 Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$ and sketch the curve.

8 Find any stationary points on the curve $y = x^3 - 6x^2 + 3$ and sketch the curve.

9 Find any stationary points on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

10 Sketch the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.


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**11** Find the stationary points on the curve $y = (x - 4)(x + 2)^2$ and hence sketch the curve.

**12** Find all stationary points on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

**13** Sketch the graphs of the following functions.

**a** $h(x) = 2x^3 - 6x$
**b** $y = 2x^3 - 12x^2$
**c** $g(x) = x^3 - 12x$
**d** $f(x) = x^3 - 2x^2 + x - 2$
**e** $h(x) = 3x^3 - x^2 - 7x - 3$
**f** $y = (x + 5)(x + 2)(x - 3)$

**14** CAS Sketch the graph of each function and identify local maxima and minima.

**a** $q(x) = x^3 - x^2 - x - 1$
**b** $f(x) = 3x - x^3 - 18$
**c** $f(x) = x^3 + 3x^2 - 24x$
**d** $f(x) = 3x^5 - 5x^3 + 5$
**e** $f(x) = 2x^6 - 6x^4$
**f** $f(x) = 7 + 2x^2 - x^4$

# 12.06 EXTREMES ON AN INTERVAL

You can use stationary points to find the **maximum** or **minimum** value of a function over a restricted domain or, in some cases, over the whole domain of the function. A maximum over the whole domain is called a **global maximum** and similarly for a **global minimum**. In cases where the domain is restricted, you must look at the boundary conditions. For continuous curves like polynomials it is not necessary to look at a full graph; stationary points, the $y$-intercept and boundary conditions are normally sufficient.

The maximum of a function over a given domain is also known as the **greatest value** of the function while the minimum of a function over a given domain is known as the **least value** of the function.

## Example 15

Find the greatest and least values of the function $f(x) = x^2 - 4x + 3$ in the domain $-4 \leq x \leq 3$.

### Solution

**Stationary points**

Find the derivative.
$f'(x) = 2x - 4$

Find any stationary points.
$2x - 4 = 0$
$x = 2$
$f(2) = 2^2 - 4 \times 2 + 3 = -1$

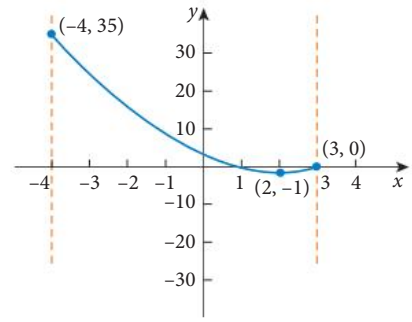
Identify the stationary point.
When $x < 2, f'(x) < 0$
When $x > 2, f'(x) > 0$
The function has a local minimum at $(2, -1)$.

**Boundaries**

Examine boundary values.
$f(-4) = (-4)^2 - 4 \times (-4) + 3 = 35$
$f(3) = 3^2 - 4 \times 3 + 3 = 0$
Boundary values are $(-4, 35)$ and $(3, 0)$.

**Sketch**

Make a rough sketch.




Use the sketch to determine the maximum and minimum values.

The greatest value of $f(x)$ in the domain $-4 \leq x \leq 3$ is 35 and the least value is $-1$.

## Example 16

Find the greatest and least values of the function $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 4$ in the interval $-3 \leq x \leq 3$.

### Solution

**Stationary points**

Find the derivative.

$$f'(x) = 3x^2 + 3x - 6$$

Find the zeros of $f'(x)$.

$$3x^2 + 3x - 6 = 0$$
$$3(x^2 + x - 2) = 0$$
$$3(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

There are stationary points at $x = -2$ and $x = 1$.

Find the signs of $f'(x)$ just before and after the stationary points.

| $x$ | −3 | −2 | 0 | 1 | 3 |
|---|---|---|---|---|---|
| $f'(x)$ | 12 | 0 | −6 | 0 | 30 |
| Sign | + | 0 | − | 0 | + |

Draw diagrams of the slopes.



Find the values at the points.

$f(-2) = 14$ and $f(1) = \frac{1}{2}$

Write in words.

There is a local maximum at $(-2, 14)$ and a local minimum at $(1, \frac{1}{2})$.

***y*-intercept**

Find the value of $f(0)$.

$f(0) = 4$


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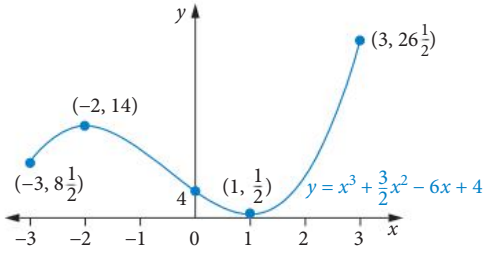
**Boundaries**

Find the values at the ends of the interval.

$f(-3) = 8\frac{1}{2}$ and $f(3) = 26\frac{1}{2}$

**Sketch**

Use the information to make a sketch of the required domain.

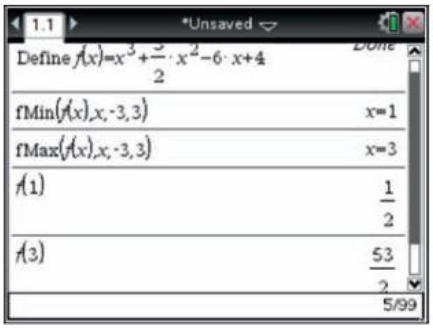



Write the answer.

The greatest value is $26\frac{1}{2}$ and the least value is $\frac{1}{2}$.

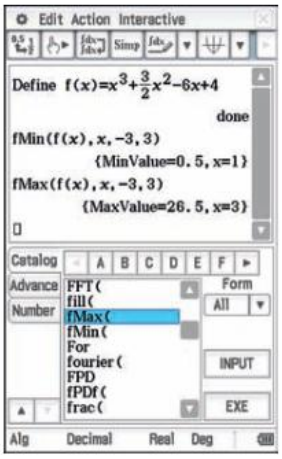
### TI-Nspire CAS

Use a calculator page.
Define the function.
Type in fMin($f(x)$, $x$, 3, 3) or get fMin by pressing menu, 4: Calculus and 7: Function Minimum. You can also use the catalogue.
You must enter the expression, the variable and the boundaries.
fMax works the same way.
Put the values of $x$ into the function.




### ClassPad

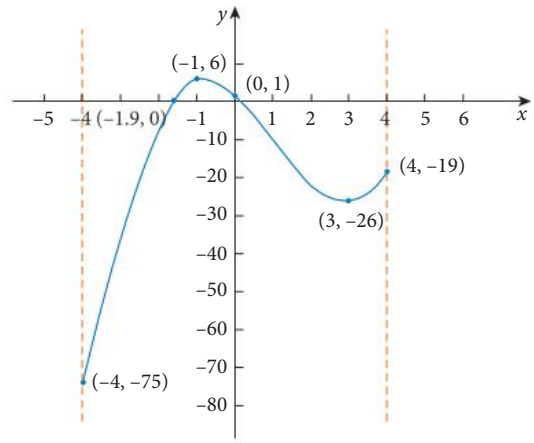
Use the Main $\sqrt{\alpha}$ application.
Define the function.
Type in fMin (f($x$), $x$, −3, 3) or get **fMin** by tapping **Action**, **Calculation**, **fMin/fMax**, then **fMin**.
You must enter the expression, the variable and the boundaries. You can also find **fMin** as one of the functions obtainable from the Catalog tab.
**fMax** works in the same way.



# EXERCISE 12.06 Extremes on an interval

## Concepts and techniques

The diagram below shows the graph of a function, $f(x)$, over the domain $-4 \leq x \leq 4$. Use the graph to answer questions **1** to **4**.




**1** A zero for this function can be found at:

**A** $(-4, -75)$ **B** $(-1, 6)$
**C** $(4, -19)$ **D** $(0, 1)$
**E** $(-1.9, 0)$

**2** A local minimum for this function can be found at:

**A** $(0, 1)$ **B** $(4, -19)$
**C** $(3, -26)$ **D** $(-1.9, 0)$
**E** $(-4, -75)$

**3** The greatest value for this function can be found at:

**A** $(4, -19)$ **B** $(3, -26)$ **C** $(-4, -75)$ **D** $(-1, 6)$ **E** $(0, 1)$

**4** The least value for this function can be found at:

**A** $(-4, -75)$ **B** $(3, -26)$ **C** $(-1, 6)$ **D** $(4, -19)$ **E** $(0, 1)$

**5** Example 15 Sketch $y = x^2 + x - 2$ for $-2 \leq x \leq 2$ and find the maximum value of $y$.

**6** Sketch $f(x) = 9 - x^2$ over the domain $-4 \leq x \leq 2$. Hence find the maximum and minimum values of the curve over this domain.

**7** Find the greatest and least values of the function $f(x) = x^2 - 2x + 3$ for $-2 \leq x \leq 2$.

**8** Example 16 Find the greatest and least values of the function $f(x) = x^3 - 5x^2 + 6x + 4$ for $-3 \leq x \leq 1$.

**9** Find the greatest and least values of the function $f(x) = 8x - x^3 + 1$ for $-3 \leq x \leq 4$.

**10** Find the greatest value of the curve $y = x^2 - 4x + 4$ for $-3 \leq x \leq 3$.

**11** Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \leq x \leq 4$ and find the greatest and least values of the function over this domain.

WS
Greatest and least values

**12** Sketch $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \leq x \leq 3$, showing any stationary points. Find the greatest and least values of the function for $-3 \leq x \leq 3$.

**13** Find the greatest value of $y$ for the curve $y = x^5 - 3$ for $-2 \leq x \leq 1$.

**14** Find the local maximum, local minimum, greatest and least values of the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ for $-2 \leq x \leq 2$.

**15** CAS Find the greatest and least values of $y = x(x + 2)(x + 3)$ for $-2 \leq x \leq 2$.

**16** CAS Find the greatest and least values of the function $f(x) = x^4 + x^3 - 16x^2 - 4x + 48$ for $-3 \leq x \leq 3$.

**17** Sketch $y = x^3 + 2$ over the domain $-3 \leq x \leq 3$ and find its greatest and least values in that domain.


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## Reasoning and communication

**18** Find the point on the graph of the function $f(x) = 1 - 2x + 3x^2 - x^3$ where the gradient of the tangent has the greatest value.

**19** Find the point on the graph of the function $f(x) = x^3 - 6x^2 + 1$ where the gradient of the tangent has the least value.

**20** Find the points on the graph of the function $f(x) = 1 - 2x + 3x^2 - x^3$ between $x = 0$ and $x = 3$ where the gradient is greatest and least.

# 12.07 OPTIMISATION PROBLEMS

Often a problem involves finding maximum or minimum values that will provide an *optimal* outcome. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise harvests. The process used to find optimum outcomes is called **optimisation**.

There is no single method for optimisation but calculus can be used for many of these problems.

**IMPORTANT**

**Solving optimisation problems on an interval**

1. Read the problem carefully—several times if needed. Think about the given facts, together with the unknown quantities that are to be found/optimised.
2. If possible, make a diagram and label it appropriately, introducing variables for the unknown quantities. Even when a diagram is not possible, identify variables and represent them with an appropriate symbol.
3. List all known facts and relationships, showing relationships by equations if possible.
4. Determine the quantity to be maximised or minimised and express it as a function of the other variables. Use the other equations to express all variables in terms of a single variable. Express the quantity as a function of the same variable.
5. Find the interval over which the quantity is to be optimised by examining restrictions on the domain, whether explicit or implied.
6. Find the maximum or minimum and express your answer in terms of the question.
7. State any assumptions that you have made and examine your solution for possible extension to other cases.

## Example 17

The expense per year (in units of ten thousand dollars) of running a certain business is given by:

$$E = x^2 - 6x + 12$$

where $x$ is the number (in hundreds) of items manufactured.

**a** Find the expense of running the business if no items are manufactured.

**b** Find the number of items needed to minimise the expense of the business and the minimum expense of running the business.

### Solution

**a** When no items are manufactured, $x = 0$.

Let $x = 0$: $E = 0^2 - 6 \times 0 + 12 = 12$

State the result.

The expense of running the business when no items are manufactured is $12 \times \$10\,000$, or $\$120\,000$ per year.

**b** We need to identify the least value for the function.
Differentiate $E$.

$$\frac{dE}{dx} = 2x - 6$$

Identify any stationary points.

Let $\frac{dE}{dx} = 0$

$2x - 6 = 0$

$x = 3$

Identify the stationary point.

When $x < 3$, $\frac{dE}{dx} < 0$

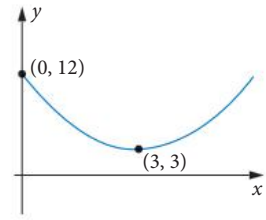
When $x > 3$, $\frac{dE}{dx} > 0$

Let $x = 3$: $E = 3^2 - 6 \times 3 + 12 = 3$

There is a local minimum at (3, 3).

Investigate the boundary values for this situation.
Draw a rough sketch of the curve.

The lower boundary is $x = 0$ because you can't manufacture fewer than 0 items.
We know when $x = 0$, $E = 12$.
The upper boundary is not stated.




Examine the graph.

$E = 3$ is the least value

State the result. Remember that $x$ is in units of 100 and $E$ is in units of $10 000.

300 items manufactured each year will give the minimum expenses and the minimum expense per year is $30 000.


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## Example 18

Trinh and Soi set out from two towns. They travel on straight roads that cross at right angles, and they each walk towards towns beyond the intersection. Trinh is initially 15 km from the intersection and walks at 3 km/h. Soi is initially 10 km from the intersection and walks at 4 km/h.

**a** Show that their distance apart after $t$ hours is given by $D^2 = 25t^2 - 170t + 325$.

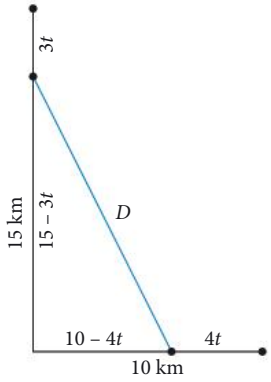
**b** Hence find how long it takes them to reach their least distance apart and the least distance.

### Solution

**a** Calculate distance travelled by each after $t$ hours.

After $t$ hours:
Trinh has walked $3t$ km. She is now $15 - 3t$ km from the intersection.
Soi has walked $4t$ km. He is now $10 - 4t$ km from the intersection.

Draw a diagram to represent their positions after $t$ hours.




Use Pythagoras' theorem.

$$\begin{aligned} D^2 &= (15 - 3t)^2 + (10 - 4t)^2 \\ &= 225 - 90t + 9t^2 + 100 - 80t + 16t^2 \\ &= 25t^2 - 170t + 325 \end{aligned}$$

**b** We need to identify the least value for the function.
Differentiate $D^2$ and find any stationary points.

$$\frac{dD^2}{dt} = 50t - 170$$

Let $\frac{dD^2}{dt} = 0$.

$$\begin{aligned} 50t - 170 &= 0 \\ 50t &= 170 \\ t &= 3.4 \end{aligned}$$

Identify the stationary point.

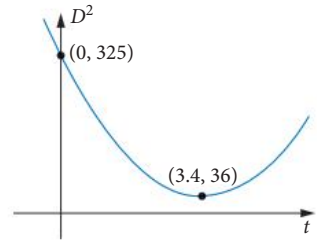
When $t < 3.4$, $\frac{dD^2}{dt} < 0$

When $t > 3.4$, $\frac{dD^2}{dt} > 0$

Let $t = 3.4$:

$$\begin{aligned} D^2 &= 25 \times 3.4^2 - 170 \times 3.4 + 325 \\ &= 36 \\ D &= \pm 6 \end{aligned}$$

So $D = 6$ as a negative distance does not make sense.
There is a local minimum at (3.4, 6).

| | |
|---|---|
| Investigate the boundary values for this situation.<br>Make a rough sketch of the curve.<br>Examine the graph. | The lower boundary is $t = 0$.<br>We know when $t = 0$, $D^2 = 325$.<br>The upper boundary is not stated.<br>  |
| Time is measured in hours and distance is measured in km. | The least value is at (3.4, 36).<br>$t = 3.4$ hours = 3 hours 24 minutes<br>$D^2 = 36$, so distance ($D$) = 6 km |
| State the result. | Trinh and Soi are at their least distance apart after 3 hours and 24 minutes and the least distance between them is 6 km. |

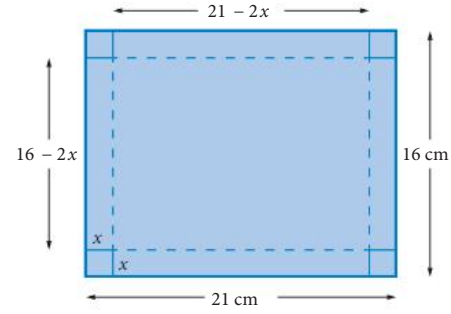
## Example 19

An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 cm wide and 21 cm long by cutting a square from each corner and then bending up the sides. Find the size of the corner square that will produce a box that has the largest possible volume.




### Solution

Make a sketch of the cardboard and mark in the corners that are to be cut out.
Let the corner cutout be of length $x$.



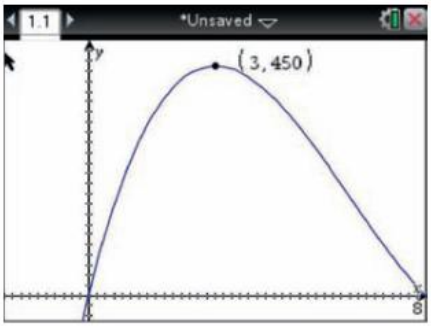


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| | |
|---|---|
| Write an equation for the volume of the box. | $V$ = length × width × height |
| Express it as a function of $x$. | $V(x) = x(16 - 2x)(21 - 2x)$ |
| Expand the brackets. | $= x(336 - 74x + 4x^2)$ |
| Expand again. | $= 336x - 74x^2 + 4x^3$ |
| Write restrictions on the domain of $x$ so there is a box. | $x > 0$ and $2x < 16$, i.e. $x < 8$ |
| State the aim. | We want the greatest value of $V(x)$ over $0 < x < 8$. |
| Differentiate. | $V'(x) = 336 - 148x + 12x^2$ |
| Factorise and rearrange. | $= 4(3x^2 - 37x + 84)$ |
| Factorise the quadratic. | $= 4(3x - 28)(x - 3)$ |
| Let $V'(x) = 0$. | $0 = 4(3x - 28)(x - 3)$ |
| Solve. | $x = 3$ or $x = \frac{28}{3} = 9\frac{1}{3}$ |
| Find the values at the boundaries (ends of the interval) and the stationary point in the interval. | $V(0) = 0$, $V(3) = 450$ and $V(8) = 0$ |

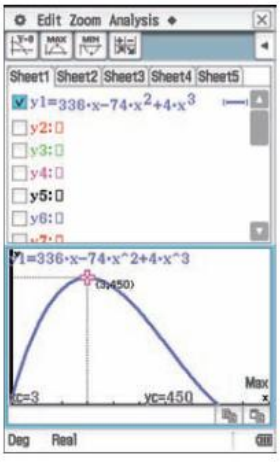
**TI-Nspire CAS**

Check the graph of $V$ on your CAS calculator.




**ClassPad**

Check the graph of $V$ on your CAS calculator. Having drawn the graph, tap ▶ on the top toolbar, then MAX to show the maximum of the graph.




| | |
|---|---|
| Write the result. | The corner square needs to be 3 cm long to produce the largest volume of 450 cm$^3$. |

# EXERCISE 12.07 Optimisation problems

WS
Application of optimisation

## Concepts and techniques

**1** Example 17 The cost per hour of a bike ride is given by the formula

$$C = x^2 - 15x + 70$$

where $x$ is the distance travelled in km. Find the distance that gives the least cost.

**2** The height, in metres, of a projectile fired into the air is given by the equation $h = 10t - 5t^2$, where $t$ is the time in seconds. The greatest height reached by the projectile is calculated. What is the interval for $t$ over which the height needs to be optimised?

**A** $t > 0$ **B** $t < 0$ and $t > 2$ **C** $t \leq 2$
**D** $0 \leq t \leq 2$ **E** $t \geq 0$

**3** The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where $t$ is the time in seconds. Find when the ball will reach its greatest height, and what the greatest height will be.

**4** The perimeter of a rectangle is 60 m and its length is $x$ m. Show that the area of the rectangle is given by the equation $A = 30x - x^2$. Hence find the greatest area of the rectangle.

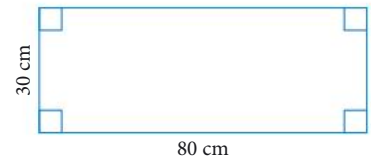
**5** Find two numbers whose sum is 28 and whose product is a maximum.

**6** The difference of two numbers is 5. Find these numbers if their product is to be a minimum.

## Reasoning and communication

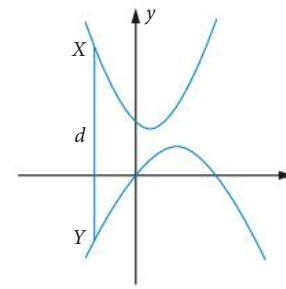
**7** Example 18 Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 $kmh^{-1}$, while the other starts at 120 km away and travels towards the intersection at 60 $kmh^{-1}$. Show that their distance apart after $t$ hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$ and hence find their minimum distance apart.

**8** Example 19 A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of size $x$ cm from each corner. The edges are turned up to make an open box.




- **a** Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$.
- **b** Find the value of $x$ that gives the box its greatest volume.
- **c** Find the maximum volume of the box.

**9** $X$ is a point on the curve $y = x^2 - 2x + 5$. Point $Y$ lies directly below $X$ and is on the curve $y = 4x - x^2$.

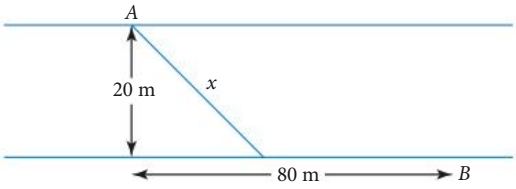



- **a** Show that the distance $d$ between $X$ and $Y$ is given by $d = 2x^2 - 6x + 5$.
- **b** Find the least distance between $X$ and $Y$.


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10 In an experiment it is found that the temperature (°C) of an object is approximated by the function $f(t) = 40 - 24t + 9t^2 - t^3$ where $t$ is the time in minutes after the start of the experiment, given that $1\frac{1}{2} \leq t \leq 6$. Find the greatest and least temperatures of the object during the given time interval.

11 Grant is at point $A$ on one side of a 20 m wide river and needs to get to point $B$ on the other side 80 m along the bank as shown. Grant swims to any point on the other bank and then runs along the side of the river to point $B$. If he can swim at 7 km/h and run at 11 km/h, find the distance he swims ($x$) to minimise the time taken to reach point $B$. Answer to the nearest metre.

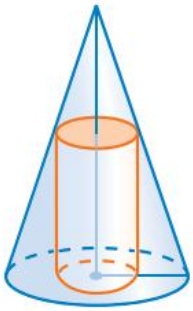



12 A person standing on the top of a vertical cliff 40 m above sea level throws a stone out to sea. The equation of the path taken by the stone is approximately $y = 40 + 5x - x^2$ where $y$ is the height of the stone above sea level and $x$ is the horizontal distance from the vertical cliff face (both in metres). What is the greatest height above sea level reached by the stone in the first 5 s?

13 Find the greatest volume of a rectangular box with a square base if the sum of the height and the side of the base must not exceed 24 cm.

14 A rectangular field shares one side with an existing paddock and so this requires no fence. There is only 1000 m of fencing material to fence the remaining sides. Find the maximum possible area of the field.

15 Find the maximum volume of a right circular cylinder that can be inscribed in a cone of altitude 12 cm and base radius 4 cm, given that the axes of the cylinder and cone coincide.



# 12 CHAPTER SUMMARY
# APPLICATIONS OF DERIVATIVES

- The **average rate of change** of a function from $x_1$ to $x_2$ is given by $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$
- The **instantaneous rate of change** of a function is given by the derivative. It is called the **gradient function** and it gives the slope of the tangent at a point.
- **Speed**, **velocity**, **acceleration** and **flow rate** are all rates of change.
- For motion in a straight line:
  - **displacement** is a change of position.
  - the derivative of **position** ($x$) or displacement ($s$) with respect to time ($t$) is **velocity** ($v$) and the derivative of velocity is **acceleration** ($a$).

    $\dfrac{ds}{dt}=v$ and $\dfrac{dv}{dt}=a$
  - the antiderivative of velocity is position or displacement and the antiderivative of acceleration is velocity.
- The sign of the derivative shows whether a function is **increasing**, **decreasing** or **stationary**.
- At **stationary points**, the derivative is zero. The type of stationary point is determined by the change of **sign of the derivative** near the stationary point.
  - A **local minimum** occurs when the derivative changes from negative to 0 to positive as $x$ increases.
  - A **local maximum** occurs when the derivative changes from positive to 0 to negative as $x$ increases.
  - A **point of inflection** occurs when the derivative has the same sign each side of a stationary point.

| | $f'(x)$ before the stationary point | $f'(x)$ at the stationary point | $f'(x)$ after the stationary point | Type of point | Change in $f'(x)$ as $x$ increases |
|---|---|---|---|---|---|
| $f(x)$ | + | 0 | − | Local maximum | Decreasing through zero |
| $f(x)$ | − | 0 | + | Local minimum | Increasing through zero |
| $f(x)$ | + | 0 | + | Inflection on a rising curve | Decreasing to zero, then increasing again |
| $f(x)$ | − | 0 | − | Inflection on a falling curve | Increasing to zero, then decreasing again |

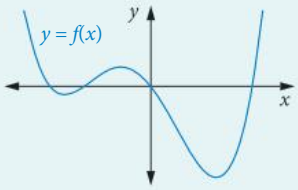

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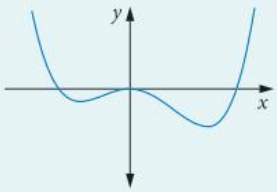
- A **global maximum** or **global minimum** value of a function is a maximum or minimum over the whole domain.
- To sketch a polynomial function, investigate the **zeros**, **sign**, **stationary points**, intervals where the function is increasing and decreasing, ***y*-intercept** and behaviour as the magnitude of $x$ becomes large.
- The **dominant term** is of higher power than any other term. As $x \to \pm\infty$, the magnitude of the dominant term becomes much larger than the magnitudes of the other terms, so the function behaves like the dominant term.
- Values of a function, its derivatives and stationary points can be estimated by reading from the graph of the function or by using numerical methods (on a calculator).
- To find the **greatest** and **least** values of a function in an interval, investigate the stationary points, $y$-intercept and values of the function at the boundaries of the interval.
- When solving optimisation problems on an interval, use the following steps.
  1. Read the problem carefully.
  2. Draw a diagram and introduce variables for the unknown quantities.
  3. List all known facts and relationships, showing relationships by equations.
  4. Determine the quantity to be maximised or minimised and express it as a function of one variable.
  5. Find the interval over which the quantity is to be optimised.
  6. Find the maximum or minimum and express the answer in terms of the question.
  7. State any assumptions and examine the solution for possible extension to other cases.

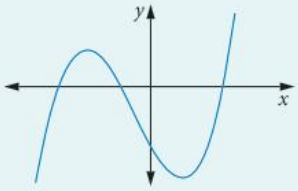
# 12 CHAPTER REVIEW
# APPLICATIONS OF DERIVATIVES

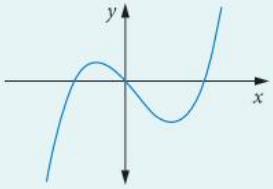
## Multiple choice

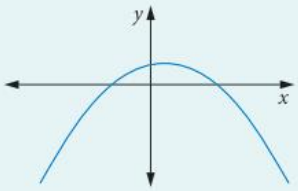
1 Example 2 The function $f(x)$ is shown on the right. Which of the graphs below best shows the gradient function?

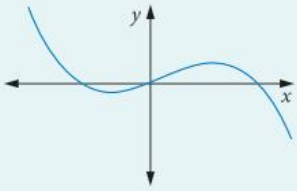



A



B



C


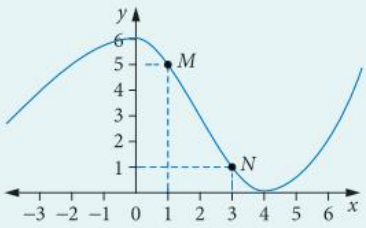

D



E



2 Example 1 $g(x) = x^2 + 4x + 6$. The average rate of change from $x = 2$ to $x = 5$ is:

A 8 B 11 C 14 D 32.25 E 33

3 Example 5 For the function shown at the right, the average rate of change between $M$ and $N$ is:




A $-2$ B $-\frac{3}{5}$ C $-0.5$

D $\frac{5}{3}$ E 2

4 Example 9 When the function $f(x)$ has a maximum:

A $f'(x) = 0$ and increases B $f'(x) > 0$ and increases C $f'(x) < 0$ and increases

D $f'(x) = 0$ and decreases E $f'(x) > 0$ and decreases.


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5 Example 10 The values of a function and its derivative for selected points are given in the table. All values for which the derivative is zero are shown. The function has no discontinuities.

| $t$ | −2 | −1 | 1 | 3 | 4 |
|---|---|---|---|---|---|
| $h(t)$ | | 4 | | 2 | |
| $h'(t)$ | 2 | 0 | −3 | 0 | −1 |

The function has:

A a minimum at (4, −1)
B a maximum at (−1, 0)
C a point of inflection at (2, 3)
D a maximum at (−1, 4)
E a point of inflection at (2, 0).

6 Example 11 The function $f(x)$ has a derivative that can be factorised to $f'(x) = (x + 2)(1 - x)$. $f(x)$ has:

A a minimum at $x = 2$ and a maximum at $x = -1$
B a maximum at $x = 2$ and a minimum at $x = -1$
C a minimum at $x = -2$ and a maximum at $x = 1$
D a maximum at $x = -2$ and a minimum at $x = -1$
E points of inflection at $x = -2$ and $x = -1$.

7 Example 11 The function $f(x) = 2x^3 + 3x^2 - 36x + 7$ has a:

A minimum at (−3, 88)
B maximum at (−3, 88)
C minimum at (3, −88)
D maximum at (3, 88)
E maximum at (2, 37).

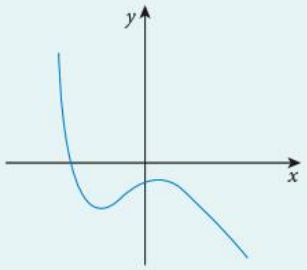
8 Example 11 A cubic function has:

A at most one stationary point
B at most two stationary points
C any number of stationary points
D at least two stationary points
E three stationary points.

## Short answer

9 Example 1 The volume of water flowing through a pipe is given by $V = t^2 + 3t$. Find the flow rate when $t = 5$.

10 Example 2 The diagram below shows the graph of a function $y = f(x)$.
Copy this diagram into your writing book.
On the same set of axes, sketch the gradient function of the function.




11 Example 3 The position in cm after time $t$ s of a particle moving in a straight line is given by $x = 2 - t - t^2$.

a Find the initial position.
b Find when the particle will be at the origin.
c Find the position after 2 s.
d How far will the particle move in the first 2 seconds?
e Find its velocity after 3 s.

12 Example 4 The position of a particle is given by $x = -5t^2 + 3t + 9$ cm, where $t$ is in seconds.
- a Find the initial velocity of the particle.
- b Show that the particle has constant acceleration.
- c Find when the particle will be at the origin.
- d Find the particle's maximum distance from the origin.
- e Sketch the graph showing the particle's motion.

13 Example 5 A marble dropped into a barrel of water falls such that its height after $t$ s is given by $h(t) = 1.5 - 0.1t^2$
Find the height, velocity and acceleration of the marble after 3 s.

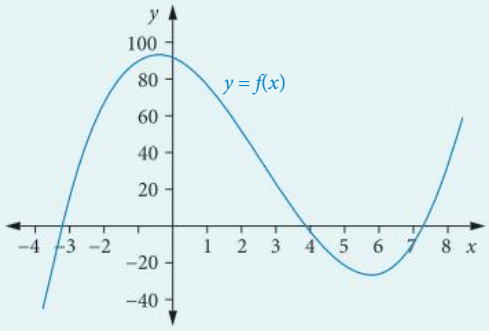
14 Example 6 The velocity of a particle is given by $v = 8t^3 - 3t^2$ cm/s. If the particle is initially at the origin, find:
- a its acceleration after 5 s
- b its displacement after 3 s
- c when it will be at the origin again.

15 Example 7 The acceleration of an object is given by $6t + 4$ cm/s$^2$. The particle is initially at rest at the origin. Find:
- a its velocity after 5 s
- b its position after 5 s.

16 Example 8 A particle is accelerating according to the equation $a = (3t + 1)^2$ m/s$^2$. If the particle is initially at rest 2 m to the left of the origin, find its position after 4 s.

17 Example 9 Find the values of $x$ for which the function $y = x^2 - 3x - 4$ is:
- a decreasing
- b increasing
- c stationary

18 Example 12 The graph of $f(x)$ is shown. Use the graph to estimate:
- a the coordinates of stationary points
- b the $y$ value at $x = 1$
- c the rate of change at $x = 2$.
- d The intervals where the function is increasing and decreasing.




19 Example 12 CAS
- a Draw the graph of $y = 25x - 3x^2 + 36 - 2x^3$ from $x = -10$ to $x = 10$, and hence estimate the coordinates of the stationary points.
- b Estimate where the rate of change is negative.

20 Example 13 Sketch the graphs of the following functions.
- a $f(x) = x^2 - 5x - 36$
- b $f(x) = (x - 2)^2 (x - 3)$

21 Example 15 Find the greatest and least values of the function $f(x) = 20x - 2x^2$ over the domain $3 < x < 8$.

22 Example 16 Find the greatest and least values of the function $f(x) = 9x^2 - x^3 - 24x + 20$ for $1 \leq x \leq 6$.


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**23** The cost to a business of manufacturing $x$ products a week is given by $C = x^2 - 300x + 9000$.
Find the number of products that will give the least cost each week.

## Application

**24** Sketch the graph of $h(x) = x^3 - x^2 - 5x + 9$.

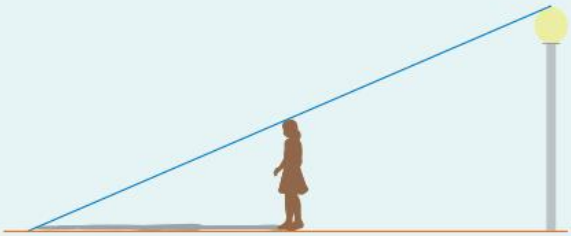
**25** Sketch the graph of $y = 36x - 3x^2 - 2x^3 + 5$.

**26** Sketch $f(x) = 2x^2 + 5x - x^3 - 12$.

**27** Find the greatest and least values of the function

$y = 3x^4 + 8x^3 + 6x^2$ for $-3 \leq x \leq 3$.

**28** A woman of height 1.8 m is walking away from a street light at a speed of 1.2 m/s. The street light is 4.2 m above the road. We are interested in the rate of change of the length of the woman's shadow.

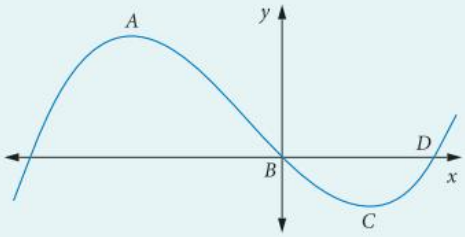


**a** Find the relationship between the length $y$ of the shadow and the woman's distance $x$ from the street light.

**b** State the relationship between the rate of increase of the length of the shadow and the woman's walking speed.

**c** Find the rate of increase of the length of the shadow when she is:
**i** 3 m from the light **ii** 6 m from the light.

**29** A flume ride track follows the curve $y = \dfrac{x^3 + 20x^2 - 1500x}{720}$ as shown in the diagram on the right.
The highest point is at $A$ and the lowest point is at $C$. The 'log' enters a tunnel at $B$ and leaves the tunnel at $D$.




**a** Find the greatest height.

**b** Find the lowest point.

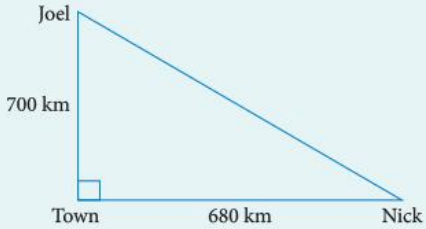
**c** Find the drop from the highest point to the lowest.

**d** Find the average gradient of the track from $A$ to $C$.

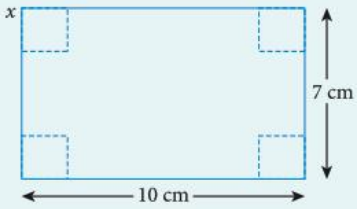
**e** Find the steepest point on the track between $A$ and $C$.

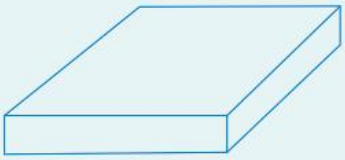
**f** What is the gradient at the steepest point?

**30** Joel is 700 km north of a town, travelling towards it at an average speed of 75 km/h. Nick is 680 km east of the town, travelling towards it at 80 km/h.
Find the least distance between Joel and Nick.




**31** A 10 cm by 7 cm rectangular piece of cardboard has equal square corners with side $x$ cm cut out and folded up to make an open box.



Calculate the maximum volume of the box.

Practice quiz


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# ANSWERS

## 1.01

**1** **a** $-5b-6$ **b** $-3m+1$
**c** $8h-19$ **d** $a^2-2a+4$
**e** $2ab-2a^2b+b$ **f** $5t-6$

**2** **a** $a^2+7a+10$ **b** $2y^2+7y-15$
**c** $x^2+7x+12$ **d** $2x^2+x-6$
**e** $x^2-25$ **f** $xy+7x-4y-28$

**3** **a** $x^3-2x^2+3x-6$
**b** $x^2-2xy+11x-18y+18$
**c** $2ab+2b^2-7b-6a+3$
**d** $x^3+8$ **e** $a^3-27$

**4** **a** $y^2+16y+64$ **b** $4b^2+20b+25$
**c** $9-6x+x^2$ **d** $9y^2-6y+1$
**e** $9a^2-6ab+b^2$ **f** $16d^2+40de+25e^2$
**g** $t^2-16$ **h** $x^2-9$
**i** $4a^2-9$ **j** $x^2-25y^2$

**5** **a** $x^4-4$ **b** $x^4+10x^2+25$
**c** $9a^2b^2-16c^2$ **d** $x^2+4+\frac{4}{x^2}$
**e** $a^2-\frac{1}{a^2}$ **f** $12a$
**g** $32-z^2$ **h** $x^2$

**6** **a** $6(4-3y)$ **b** $2y(y+2)$
**c** $ab(b+1)$ **d** $3mn(n^2+3)$
**e** $2xz(4x-z)$ **f** $q^2(3q^3-2)$

**7** **a** $(m+5)(x+7)$ **b** $(y-1)(2-y)$
**c** $(x-3)(x+2)$ **d** $(5a-3)(b+2)$
**e** $(x-4)(x^3-5)$ **f** $(x-2y)(3-x)$

**8** **a** $a+2$ **b** $2t-1$ **c** $\frac{4y+1}{3}$
**d** $\frac{4}{2d-1}$ **e** $\frac{x}{5x-2}$ **f** $2b$
**g** $\frac{13x-14}{20}$ **h** $\frac{6a+23b}{12}$

**9** **a** $\frac{5}{x}$ **b** $\frac{-x+2}{x(x-1)}$
**c** $\frac{2x}{x+2}$ **d** $\frac{2(x-1)}{(x+1)(x-3)}$
**e** $\frac{5(m+3n)}{2m}$ **f** $\frac{b^2(x+2y)}{10(2b-1)}$
**g** $\frac{3p^2+5pq-2q^2}{pq(p+q)(p-q)}$ **h** $3$

## 1.02

**1** **a** $(y+4)(y+3)$ **b** $(m+1)^2$
**c** $(z+3)(z-2)$ **d** $(x+1)(x-6)$
**e** $(v-3)(v-5)$ **f** $(x+10)(x-1)$
**g** $(y-7)(y-3)$ **h** $(y+12)(y-3)$
**i** $(x-8)(x+3)$ **j** $(y+4)(y-9)$
**k** $(n-6)(n-4)$ **l** $(q+10)(q+2)$

**2** **a** $(y-1)^2$ **b** $(x+3)^2$
**c** $(t-2)^2$ **d** $(x-6)^2$
**e** $(4b-1)^2$ **f** $(a+2)(a-2)$
**g** $(x+3)(x-3)$ **h** $(y+1)(y-1)$
**i** $(x+5)(x-5)$ **j** $(1+2z)(1-2z)$
**k** $(x+2y)(x-2y)$
**l** $(x^2+1)(x^2-1)=(x^2+1)(x+1)(x-1)$

**3** **a** $(2a+1)(a+5)$ **b** $(5y+2)(y+1)$
**c** $(3x+7)(x+1)$ **d** $(3x+2)(x+2)$
**e** $(2b-3)(b-1)$ **f** $(2x+3)(x+4)$
**g** $(2y+1)(y-6)$ **h** $(4t-1)(2t-3)$
**i** $(3x+4)(2x-3)$ **j** $(6y-1)(y+8)$
**k** $(4n-3)(n-2)$ **l** $(3q+2)(4q+5)$

**4** **a** $(4r-1)(2r+6)=2(4r-1)(r+3)$
**b** $(2p-3)(3p+2)$ **c** $(3b-4)(4b-9)$
**d** $(7y+1)^2$ **e** $(4k-3)^2$
**f** $\left(5k-\frac{2}{k}\right)^2$ **g** $(2x+7)(2x-7)$
**h** $(6x+y)(6x-y)$ **i** $(2a+9b)(2a-9b)$
**j** $(a+b-3)(a-b+1)$ **k** $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$
**l** $(3x^3+2y)(3x^3-2y)$

**5** **a** $2(x+3)(x-3)$ **b** $3(p+3)(p-4)$
**c** $2ab(a+2b)(2a-1)$ **d** $5(a-1)^2$
**e** $-(2x-3)(x-4)$ **f** $3z(z+5)(z+4)$
**g** $ab(3+2ab)(3-2ab)$ **h** $x(x+1)(x-1)$
**i** $2(3x-2)(x+2)$ **j** $(m-5)(3+n)$
**k** $-7(2x+1)$ **l** $(y+5)(y+4)(y-4)$

**6** **a** $x(x+2)(x-5)$ **b** $(x+3)(x-3)^2$
**c** $y(2xy+1)(2xy-1)$ **d** $3(3x-2)(2x+5)$
**e** $3(x-1)^2$
**f** $(x+2)(x+5)(x-5)$
**g** $z(z+3)^2$
**h** $(x+1)(x-1)(2x+3)(2x-3)$
**i** $2(x+2)(x-2)(x+y)(x^2-xy+y^2)$
**j** $4a(a+3)(a-3)$
**k** $4k(k+5)^2$
**l** $3(x+1)(x-1)(x+3)$

## 1.03

**1** **a** $z=-5.6$ **b** $x=12$ **c** $n=-16$
**d** $k=6$ **e** $x=5$ **f** $x=20$
**g** $x=4$ **h** $y=3$ **i** $x=3$
**j** $t=-4$ **k** $a=1.6$ **l** $t=39$
**m** $x=4.41$

**2** **a** $b=3\frac{1}{3}$ **b** $y=4\frac{4}{9}$ **c** $k=5\frac{5}{8}$ **d** $t=0.6$
**e** $y=-1.2$ **f** $w=13$ **g** $x=14$
**h** $x=-0.4$ **i** $t=8.2$ **j** $q=22$
**k** $b=0.8$ **l** $x=3$ **m** $t=-1\frac{2}{3}$


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**3** **a** $x = 35$ **b** $x = 1\frac{9}{35}$

**c** $x = 36$ **d** $x = -3$

**e** $x = 69$ **f** $t = 30$

**g** $x = -1$ **h** $p = 3$

**i** $x = -9.5$ **j** $x = -3$

**k** $a = -0.375$ **l** $y = 1$

**4** **a** $x_2 = 2x - x_1$ **b** $A = \frac{3V}{h}$

**c** $a = \frac{2S - dn(n-1)}{2n}$ **d** $s = \frac{v^2 - u^2}{2a}$

**e** $b = \sqrt{c^2 - a^2}$

**5** **a** $t = 8.5$ **b** $b = 8$

**c** $y = 4$ **d** $x = 6.44$

**e** $y_1 = 3\frac{2}{3}$ **f** $r = 0.072$

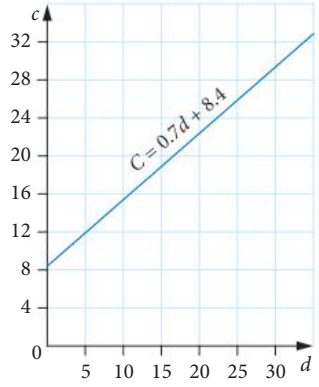
**g** $t = 2.14$ **h** $r = 2.12$

**i** $x = 1.19$ **j** $r = 3.3$

## 1.04

**1** **a** and **h** are linear relations.

**2**

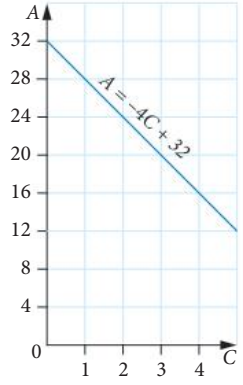



**3** **a** $C$

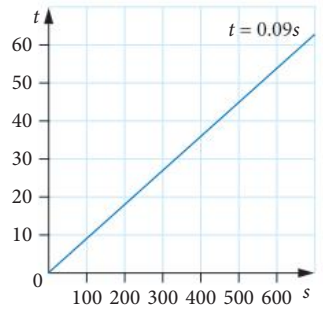
**b**

| Number of police cars, $C$. | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Number of accidents per month, $A$. | 32 | 28 | 24 | 20 | 16 |

**c**

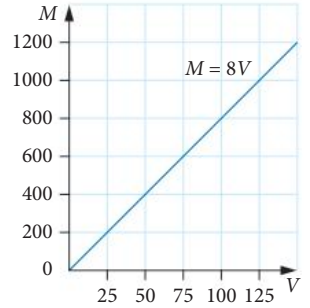



**4**



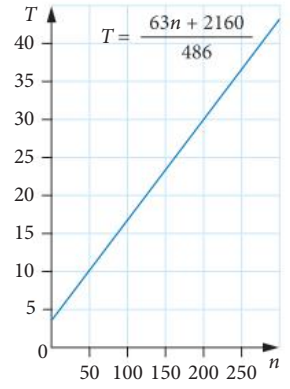

Yes, the graph is a straight line.

**5**




The dependent variable is $M$.

**6** **a**



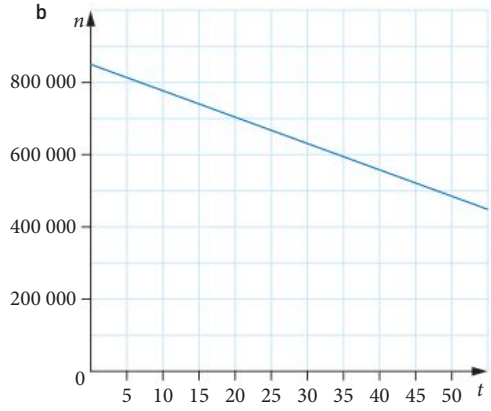

**b** $T$ is the dependent variable, $n$ is the independent variable.

**c** The chirp rate increases by 15.4 chirps/minute.

**7** **a**

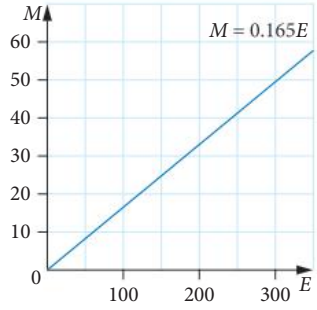
| Time in years, $t$. | 0 | 5 | 10 | 20 | 50 |
|---|---|---|---|---|---|
| Number of trees, $n$. | 850 000 | 814 000 | 778 000 | 706 000 | 490 000 |

**b**




**c** The graph shows that it is a linear relationship. It is not an example of direct proportion because it does not go through 0.

**8 a**




**b** Yes, it is a linear relationship because the graph is a straight line.

**c** Yes, it is an example of direct proprotion, as $E$ increases, $M$ increase at the same rate and the graph passes through the origin.

**d** 33 kg

## 1.05

**1 a** $x = -5, y = -1$ **b** $x = -1, y = -1$

**c** $x = -1\frac{1}{2}, y = -\frac{1}{2}$ **d** $x = 11, y = -15$

**e** $x = 2, y = -1$ **f** $x = 5, y = 2$

**g** $x = 3, y = 2$ **h** $x = 1, y = -2$

**i** $x = 3, y = -3$ **j** $x = -2, y = 1$

**2 a** $x = 3, y = 2\frac{1}{2}$ **b** $x = -2, y = 3$

**c** $x = 2, y = 5$ **d** $x = 2\frac{1}{2}, y = 5$

**e** $x = 4, y = 1$ **f** $x = 5, y = -1$

## 1.06

**1 a** $a = 2, b = 3$ **b** $x = 4, y = 1$

**c** $k = -2, j = 3$ **d** $f = 3, g = -4$

**e** $p = 5, q = -1$ **f** $f = -3, g = 6$

**2** $m = 3, n = 1$

**3 a** $m = 4, k = 3$ **b** $k = 5, j = 4$

**c** $p = -2, q = -5$ **d** $a = -6, b = 2$

**e** $x = -1, y = -3$ **f** $v = -2, w = 6$

**4 a** $b = 1, u = 3$ **b** $a = \frac{1}{2}, r = 3$

**c** $e = \frac{1}{4}, m = 1\frac{2}{3}$ **d** $y = -4, t = -6$

**e** $a = 5, n = 6$ **f** $i = -8, s = 6$

**g** $b = 1\frac{2}{7}, y = -1\frac{5}{7}$ **h** $u = 10, p = 3$

**5 a** $e = 7, f = 4$ **b** $x = 4, g = 3$

**c** $k = -6\frac{1}{4}, m = -17\frac{1}{6}$ **d** $a = 1, b = 2$

**6 a** $h = 2, i = -3$ **b** $c = -1, d = -5$

**c** $t = -4, u = 4$ **d** $f = 3, g = 7$

**e** $a = -\frac{8}{17}, b = -1\frac{9}{17}$

**7** 6 adults and 15 children

**8** In 16 years Petra will be 48 and Philippa 24.

**9** 15 kg of copper and 10 kg of zinc

**10** Boat 12 km/h, river 6 km/h

## 1.07

**1 a** 0, 12 **b** −2, 5 **c** 0, 8

**d** $-1\frac{1}{2}, 0$ **e** −7, −2 **f** 3, 5

**2 a** $-3, 1\frac{1}{3}$ **b** 1, 11 **c** −4, −1

**d** −1, 8 **e** −5, −2 **f** −1, 3

**3 a** −1, 5 **b** −2, 3 **c** −3, 4

**d** 3 **e** $-2, -\frac{1}{2}$ **f** $-3, -\frac{1}{3}$

**4 a** $-2, \frac{2}{5}$ **b** $3\frac{1}{2}, 6$ **c** −8, 8

**d** $3\frac{1}{2}$ **e** $-1\frac{1}{2}, -\frac{6}{7}$ **f** $-2, \frac{8}{9}$

**5 a** $2, 3\frac{1}{2}$ **b** $-2, -\frac{5}{7}$ **c** −7, 7

**d** 4 **e** $-5, -\frac{3}{4}$ **f** $\frac{3}{8}, \frac{1}{2}$

**6 a** 1, 7 **b** −3, −6 **c** $-2, \frac{4}{9}$

**d** −3, 2 **e** $1\frac{1}{8}, 3$ **f** $-2\frac{1}{3}, 6$

**7 a** −5, 1 **b** −7, 5 **c** 1, 8

**d** −11, 12 **e** $-1\frac{1}{3}, \frac{2}{3}$ **f** $-2 \pm \sqrt{10} \approx -5.16, 1.16$

## 1.08

**1 a** −9, −5 **b** −17, −3

**c** −14, 21 **d** $-2 \pm \sqrt{11}$

**e** $-3 \pm \sqrt{30}$ **f** −7, 20

**2 a** $-2\frac{1}{3}, 2$ **b** −1.68, 10.68 **c** $-\frac{1}{6}, 1$

**d** −1.79, 2.79 **e** −0.28, 1.95

**f** −2.64, 1.14

**3 a** −7.36, 1.36 **b** $-1\frac{1}{3}, \frac{1}{2}$

**c** −2.12, 6.12 **d** 3.413, 2.733

**e** 2.54, 9.46 **f** $-2, -1\frac{1}{2}$

**g** −28.79, −0.21 **h** 0.11, 1.76


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**4** **a** $\Delta = 20$; 2 irrational solutions
**b** $\Delta = -47$; no real solutions
**c** $\Delta = -12$; no real solutions
**d** $\Delta = 49$; 2 rational solutions
**e** $\Delta = 9$; 2 rational solutions
**f** $\Delta = -16$; no real solutions
**g** $\Delta = 0$; 1, rational solution
**h** $\Delta = 64$; 2 rational solutions
**i** $\Delta = 17$; 2 irrational solutions
**j** $\Delta = 0$; 1 rational solution

**5** **a** $x = \frac{-1 \pm \sqrt{17}}{2}$ **b** $x = \frac{5 \pm \sqrt{13}}{6}$
**c** $q = \frac{4 \pm \sqrt{28}}{2} = 2 \pm \sqrt{7}$
**d** $h = \frac{-12 \pm \sqrt{128}}{8} = \frac{-3 \pm 2\sqrt{2}}{2}$
**e** $s = \frac{8 \pm \sqrt{40}}{6} = \frac{4 \pm \sqrt{10}}{3}$
**f** $x = \frac{-11 \pm \sqrt{133}}{2}$
**g** $d = \frac{-5 \pm \sqrt{73}}{12}$ **h** $x = \frac{2 \pm \sqrt{32}}{2} = 1 \pm 2\sqrt{2}$
**i** $t = \frac{1 \pm \sqrt{5}}{2}$ **j** $x = \frac{7 \pm \sqrt{41}}{4}$

**6** **a** $y = -0.354, -5.65$ **b** $x = 1, 1.5$
**c** $b = 3.54, -2.54$ **d** $x = 1, -0.5$
**e** $x = -0.553, 0.678$ **f** $n = 0.243, -8.24$
**g** $m = -2, -5$ **h** $x = 0, 7$
**i** $x = 1, -6$ **j** $y = 2.62, 0.382$

## 1.09

**1** **a** $v = 9.8t$ **b** $p = 3.2q$ **c** $E = 3.6v^2$
**d** $d = \frac{300}{a}$ **e** $C = \frac{4000}{V}$

**2** **a** 810 km **b** 73 km/h
**3** **a** 90 mm **b** 24.5 g
**4** **a** 25.6 $m^2$ **b** 9.5 m
**5** **a** 6 min 40 s **b** 300 Megabytes
**6** 562.5 kg
**7** 10 125 m
**8** \$578.75
**9** 3 hours
**10** 90 L
**11** 16 days
**12** 14 h 24 min

## 1.10

**1** a, b, e and g
**2** **a** $7, 5x^7$ **b** $4, 2x^4$
**c** $1, 3x$ **d** $11, x^{11}$, monic
**e** $3, 3x^3$ **f** 0, 3
**g** $4, 2x^4$
**3** **a** 5 **b** 4 **c** −3 **d** 0
**4** **a** $a = 0$ **b** $b = 10$ **c** $c = -6$
**d** $a = -1$ **e** $a = 4$

**5** **a** $a^3 + 7a^2 + 10a$ **b** $m^3 + m^2 - 20m$
**c** $-3y^3 + 18y^2 - 24y$ **d** $-5n^3 + 40n^2 - 80n$
**e** $x^4 - x^2$ **f** $a^3 - a^2 - 6a$
**g** $-2x^3 + 5x^2 + 39x - 108$ **h** $-6x^3 - 9x^2 + 42x + 45$
**i** $2x^3 - 5x^2 - 24x + 63$ **j** $x^3 + x^2 - 64x - 64$
**k** $-4x^3 + 27x + 27$ **l** $x^3 + 7x^2 - 36x - 252$

**6** **a** $x^3 + 6x^2 + 12x + 8$
**b** $-x^3 + 15x^2 - 75x + 125$
**c** $8x^3 + 36x^2 + 54x + 27$
**d** $-27x^3 + 27x^2 - 9x + 1$
**e** $64x^3 - 240x^2y + 300xy^2 - 125y^3$

**7** **a** $x^4 - 32x^2 + 256$
**b** $2x^4 - x^3 - 28x^2 - 36x$
**c** $8x^4 + 60x^3 + 150x^2 + 125x$
**d** $-54x^4 - 18x^3 + 6x^2 + 2x$
**e** $-70x^4 + 119x^3 - 42x^2$

**8** **a** −19 **b** −10 **c** $x^3 - x^2 - 15x - 19$

**9** **a** $f(1) = 4, f(-3) = 0$
**b** $h(0) = -2, h(2) = 2, h(-4) = 14$
**c** $f(5) = -25, f(-1) = -1, f(3) = -9, f(-2) = -4$
**d** 14
**e** −35 **f** $x = 9$
**g** $x = \pm 5$
**h** $f(p) = 2p - 9, f(x + h) = 2x + 2h - 9$
**i** $g(x - 1) = x^2 + 2$ **j** $t = -1; t = 2, -4$

## 1.11

**1** **a** $3x^2 + 2x + 5 = (x + 4)(3x - 10) + 45$
**b** $x^2 - 7x + 4 = (x - 1)(x - 6) - 2$
**c** $x^3 + x^2 + 2x - 1 = (x - 3)(x^2 + 4x + 14) + 41$
**d** $4x^2 + 2x - 3 = (2x + 3)(2x - 2) + 3$
**e** $x^3 + x^2 - x - 3 = (x - 2)(x^2 + 3x + 5) + 7$
**f** $x^4 - x^3 - 2x^2 + x - 3 = (x + 4)(x^3 - 5x^2 + 18x - 71) + 281$
**g** $4x^3 - 2x^2 + 6x - 1 = (2x + 1)(2x^2 - 2x + 4) - 5$
**h** $6x^2 - 3x + 1 = (3x - 2)\left(2x + \frac{1}{3}\right) + 1\frac{2}{3}$
**i** $x^2 + 5x - 2 = (x + 1)(x + 4) - 6$
**j** $x^4 - 2x^2 + 5x + 4 = (x - 3)(x^3 + 3x^2 + 7x + 26) + 82$

**2** **a** 41 **b** −3 **c** −43 **d** 9424
**e** 0 **f** 37 **g** 47 **h** 2321
**i** 31 174 **j** −3

**3** **a** 70, 4, 38 **b** 2, 5, 58
**c** −14, −4, 6 **d** −44, 28, −14
**e** 257, −1, −103

**4** **a** $k = 8$ **b** $k = \frac{2}{7}$ **c** $k = 15\,299$
**d** $k = 6\frac{8}{9}$ **e** $k = \pm 2$

**5** **a** 0 **b** Yes
**c** $x^3 - 4x^2 + x + 6 = (x - 2)(x^2 - 2x - 3)$
**d** $f(x) = (x - 2)(x - 3)(x + 1)$

**6** **a** $P(-3) = 81 - 81 - 81 + 81 = 0$; $\therefore x + 3$ is a factor.
**b** $P(x) = x(x + 3)^2(x - 3)$

**7** **a** $(x + 3)(x^2 - 3x + 9)$
**b** $(1 - x)(1 + x + x^2)$
**c** $8(5 + t)(25 - 5t + t^2)$

**d** $\left(\frac{x}{2}-3\right)\left(\frac{x^2}{4}+\frac{3x}{2}+9\right)$

**e** $(x+y+3)(y^2-3y-xy+9+6x+x^2)$

**f** $(x+2)(x-4)$

**g** $x(x+2)(x-1)$ **h** $(x-1)(x+4)(x-2)$

**i** $(x+5)(x-3)(x+2)$ **j** $(x-3)(x-1)(x-7)$

**k** $(x+2)(x-9)(x-5)$ **l** $(x-3)(x-2)^2$

**m** $x(x+4)(x+1)^2$ **n** $(x-1)(x+2)^2$

**o** $(x+1)(x-3)(x+2)$

**8 a** $f(x)=(x+5)(x-2)(x+3)(x+4)$

**b** $P(x)=(x+1)(x-4)(x+3)^2$

**c** $P(x)=(x-4)(x+6)(x-5)$

**9** $a=-1\frac{7}{12}$, $b=-1\frac{17}{48}$

**10** $a=-6$

**11 a** $P(3)=140\neq 0$; $\therefore$ $x-3$ is not a factor of $P(x)$.

**b** $k=-39$

**12** $a=-2$, $b=-1$

**13 a** $a=3$, $b=11$

**b** $f(x)=(x+1)(3x^3+8x^2+7x+2)$

**c** $g(-1)=0$

**d** $f(x)=(3x+2)(x+1)^3$

**14 a** $P(x)=(x-1)(x+3)(x-2)$

**b** $-3, 1, 2$ **c** Yes

**15 a** $P(u)=(u-2)(u-1)^2$ **b** $x=2, 3$

**16 a** $f(p)=(p-1)(p+2)(p-3)$

**b** $x=0, -1\frac{1}{2}, 1$

**17 a** $a=1, b=3, c=4, d=-2$

**b** $a=1, b=-1, c=8, d=-12$

**c** $a=2, b=0, c=-1, d=6$

**d** $a=1, b=1, c=11, d=-12$

**e** $a=3, b=0, c=-1, d=8$

**18** $P(x)=x^3-x^2-12x$

**19** $a=1, b=-3, c=-6$

**20** $P(x)=2x^4-4x^3-10x^2+12x$

**21** Proof Suppose $P(x)$ has 4 zeros, $a_1$, $a_2$, $a_3$ and $a_4$. Then $(x-a_1)(x-a_2)(x-a_3)(x-a_4)$ is a factor of $P(x)$. So $P(x)=(x-a_1)(x-a_2)(x-a_3)(x-a_4)\,Q(x)$ $\therefore$ $P(x)$ has at least degree 4. But $P(x)$ only has degree 3. So it cannot have 4 zeros.

## INVESTIGATION: GENERAL FORMULA FOR SOLVING CUBIC EQUATIONS

There are formulas for the roots of the general cubic equation $ax^3+bx^2+cx+d=0$, but they are very complicated. Approximate methods give 'good enough' answers much more quickly.

### 1.12

**1 a** $-1$ **b** $0.5$ **c** $-4$ **d** $-5, 0, 1$

**e** $-3, 0, 2.5$ **f** $-4, -2, 3$ **g** $-3, -1, 3$ **h** $1, 3, 4$

**i** $-4, 1, 3$ **j** $-3, 2, 3$ **k** $-2, 2$ **l** $1, 2$

**2 a** $0, -2, 3, 9$ **b** $-2, -1, 0$

**c** $-1, 0, 3$ **d** $-2, 0, 2, 3$

**e** $-2, -0.5, 0, 0.5$

**3 a** $-\frac{2}{3}, \frac{1}{7}, 1\frac{1}{2}$ **b** $-\frac{2}{9}, \frac{1}{4}, \frac{1}{3}$

**c** $-3.5, 3$ **d** $-1.5, 3$

**e** $-7, -6, 0, 6$

## CHAPTER 1 REVIEW

**1** D

**2** E

**3** B

**4** C

**5** C

**6** D

**7** E

**8** D

**9** B

**10** B

**11** $4a+17$

**12** $6a^2-4ab+7a+2b-5$

**13 a** $9p^2+6p+1$ **b** $16d^2-24d+9$

**c** $4x^2-121$

**14 a** $(m+5)(n-3)$

**b** $(2t+5)(3t-8)$ **c** $4xy(y-2)$

**15 a** $\frac{3x-7}{20}$ **b** $\frac{3a-11}{10}$

**c** $\frac{4}{xy^2}$ **d** $\frac{8n}{3(m-3)}$

**16 a** $(m+4)(m+5)$ **b** $(p-6)^2$

**c** $(t+7)(t-7)$

**17 a** $(2x+9)(x+4)$ **b** $(3x-2)(4x+1)$

**c** $(3a-2)^2$

**18 a** $10$ **b** $-7$

**19** $-116$

**20** $-4$

**21 a** $a=\frac{2A-hb}{h}$ **b** $36$ **c** $12$

**22** $(-1, 4)$

**23** $m=2, n=-3$

**24** $x=-5\frac{3}{4}$, $y=-6\frac{3}{5}$

**25** $x=-2, y=5$

**26 a** $-4.5, 5$ **b** $-8, 2$ **c** $-0.75, 4$ **d** $-5, 5$

**27** $\frac{-9\pm\sqrt{21}}{6}$

**28** Two irrational roots.

**29** $\frac{-2\pm\sqrt{14}}{2}$

**30 a** $3$ **b** $-3$ **c** $-3, 0, 1$ **d** $x^3$

**31** $-2x^3+5x^2+48x-135$

**32 a** $2$ **b** $x=3\frac{2}{3}$ **c** $1\frac{1}{3}$

**33** $3x^5-7x^3+8x^2-5=(x-2)(3x^4+6x^3+5x^2+18x+36)+67$

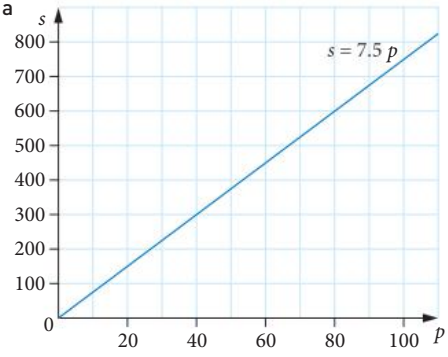
**34** $-39$

**35** $P(x)=(x-3)(x+5)^2$

**36** $-7, -1, 2$


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**37** **a**




**b** Yes. It forms a straight line.
**c** Yes. The graph is a straight line through (0, 0).
**d** 600 kg

**38** Washing powder 280 kg, sugar 160 kg
**39** 6750 L
**40** 1200 people
**41** $k = 7.4$

## 2.01

**1** **a** Yes
**b** No, self-governing can mean various things.
**c** Yes
**d** Not as it stands, as it is not clear if only positive integer powers are meant.
**e** Yes, $x = 3^n \times 7^m$ where $n, m \in N$

**2** Yes, even if the number of roots cannot be determined, as the condition is clear.

**3** Yes, because 'exactly divisible' implies that only integers are to be considered.

**4** **a** {2, 3, 5, 7, 11, 13} **b** {3}
**c** {2, 4, 6, 8, 10, …}
**d** {} **e** {−5, −4, −3, −2, −1, 0, 1, 2, 3}

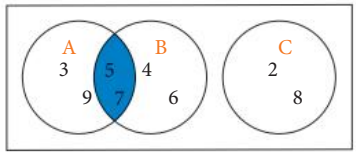
**5** **a** $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ **b** $\left\{x : x = \frac{1}{n}, n \in N\right\}$
**c** {1} **d** $\left\{\frac{1}{4}, \frac{1}{5}\right\}$
**e** All rational numbers except $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ and $\frac{1}{7}$
**f** All rational numbers except $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ and $\frac{1}{7}$
**g** $\left\{x : x = \frac{1}{n}, n \in N\right\}$ **h** $\left\{\frac{1}{4}, \frac{1}{5}\right\}$ **i** $\left\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$

**6** {x, y, z}, {x, y}, {x, z}, {y, z}, {x}, {y}, {z}, {}

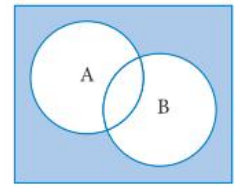
**7** **a** True **b** False **c** True

**8** **a** 0 **b** 8 **c** 0
**d** 1 **e** 2

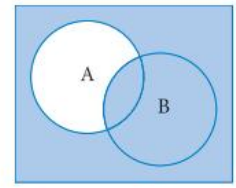
**9**



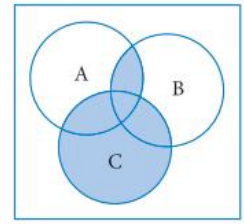

**10** **a**



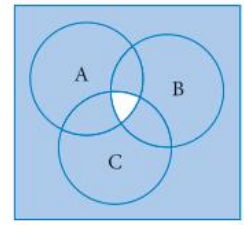

**b**



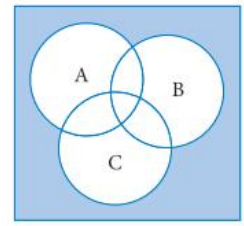

**c**



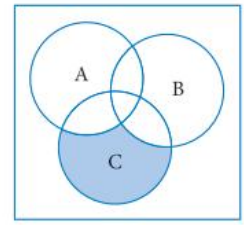

**d**



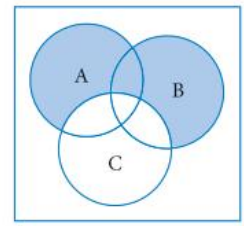

**e**



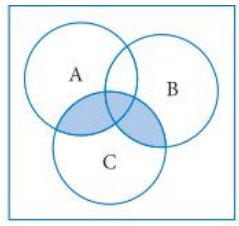

**f**



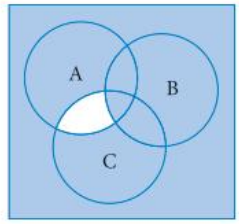

**g**



h




i




## INVESTIGATION: PASCAL'S TRIANGLE

1 The natural numbers: 1, 2, 3, 4, 5, …
2 The triangular numbers: 1, 3, 6, 10, 15, …
3 $16 = 2^4$
4 $32 = 2^5$
5 There are many patterns, you might even find some not already on the internet.

## 2.02

1 $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
2 $(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$
3 **a** 56 **b** 36 **c** 330 **d** 21 **e** 28
4 **a** 66 **b** 560 **c** 462 **d** 1716 **e** 3003
5 $264a^{10}$
6 $3003a^{10}b^5$
7 $81\,081b^{10}$
8 $-842\,688v^5w^{17}$
9 $(2x + 3y)^7 = 128x^7 + 1344x^6y + 6048x^5y^2 + 15\,120x^4y^3 + 22\,680x^3y^4 + 20\,412x^2y^5 + 10\,206xy^6 + 2187y^7$
10 $(3x - 2y)^9 = 19\,683x^9 - 118\,098x^8y + 314\,928x^7y^2 - 489\,888x^6y^3 + 489\,888x^5y^4 - 326\,592x^4y^5 + 145\,152x^3y^6 - 41\,472x^2y^7 + 6912xy^8 - 512y^9$

## 2.03

1 **a** 84 **b** 21 **c** 70 **d** 1140 **e** 105
2 **a** 36 **b** 66 **c** 455 **d** 45 **e** 56
3 **a** 495 **b** 2024 **c** 136 **d** 330 **e** 2002
4 55
5 120
6 210
7 15
8 120
9 126
10 ${}^{40}C_6 = 3\,838\,380$ and ${}^{45}C_7 = 45\,379\,620$ so 6 from 40 has a smaller number.

## 2.04

1 {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, HTTH, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
2 {1, 2, 3, 4, 5, 6}
3 {B, B, B, B, B, B, B, B, B, B, G, G, G, G, G, G, G, G, G, G, G, G, G, G}
4

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| H | H & 1 | H & 2 | H & 3 | H & 4 | H & 5 | H & 6 |
| T | T & 1 | T & 2 | T & 3 | T & 4 | T & 5 | T & 6 |

5 6
6 11
7 11
8 **a** {R1R2, R1R3, R1R4, R2R3, R2R4, R3R4}
**b** {R1R2, R1R3, R1R4, R2R3, R2R4, R3R4, W1W2, W1W3, W2W3, B1B2, R1W1, R1W2, R1W3, R2W1, R2W2, R2W3, R3W1, R3W2, R3W3, R4W1, R4W2, R4W3, W1B1, W1B2, W2B1, W2B2, W3B1, W3B2}
**c** {W1B1, W1B2, W2B1, W2B2, W3B1, W3B2}
9 {HHHHT, HHHTH, HHTHH, HTHHH, THHHH, HHHTT, HHTHT, HHTTH, THHHT, THHTH, TTHHH, HHTTT, THHTT, TTHHT, TTTHH}
10 $n(A) = |A|$, $n(B) = |B|$, $n(A \text{ or } B) = |A \cup B|$ and $n(A \text{ and } B) = |A \cap B|$.
For finite sets $A$ and $B$, $A = (A \cap \overline{B}) \cup (A \cap B)$, and since $(A \cap \overline{B})$ and $A \cap B$ are disjoint, $|A| = |A \cap \overline{B}| + |A \cap B|$. Similarly $|B| = |\overline{A} \cap B| + |A \cap B|$ and

$$\begin{aligned}|A \cup B| &= |A \cap \overline{B}| + |A \cap B| + |\overline{A} \cap B| \\ &= |A \cap \overline{B}| + |A \cap B| + |\overline{A} \cap B| + |A \cap B| \\ &\quad - |A \cap B| \\ &= |A| + |B| - |A \cap B|\end{aligned}$$

so $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$ QED
11 $\overline{A \text{ and } B} = \overline{A \cap B}$ and $\overline{A} \text{ or } \overline{B} = \overline{A} \cup \overline{B}$
Now $A \cap B = \varnothing$ because $A$ and $B$ are disjoint, and $\overline{\varnothing} = U = S$
so $\overline{A \cap B} = S$.
Since $A \cap B = \varnothing$, $A \subseteq \overline{B}$
Hence $A \cup \overline{A} \subseteq \overline{B} \cup \overline{A}$
But $A \cup \overline{A} = S$ and $\overline{B} \cup \overline{A} \subseteq S$ so $\overline{B} \cup \overline{A} = \overline{A} \cup \overline{B} = S$
QED
12 From question **10**,
$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$
But if $A$ and $B$ are mutually exclusive
then $A$ and $B = \varnothing$.
So $n(A \text{ or } B) = n(A) + n(B) - n(\varnothing)$
$= n(A) + n(B) - 0$
$= n(A) + n(B)$ QED


9780170250269

## 2.05

1 $S = \{\text{blue, red, green, yellow, black, white, purple, orange, grey}\}$, $P(A) = \frac{n(A)}{9}, \frac{1}{9}$

2 Numbering the cards from 1 to 20, $S = \{n \in N: 1 \leq n \leq 20\}$, $P(A) = \frac{n(A)}{20}, \frac{1}{20}$

3 $S = \{\text{cards from a normal pack}\}$, $P(A) = \frac{n(A)}{52}, \frac{1}{52}$

4 $S = \{1, 2, 3, 5, 8\}$, $P(\text{score}) = \frac{\text{score area}}{\text{total area}}$ so $P(1) = \frac{5}{18}$, $P(2) = \frac{1}{9}$, $P(3) = \frac{1}{3}$, $P(5) = \frac{1}{4}$, $P(8) = \frac{1}{36}$

5 Number the centre area as 1 and the areas between the circles from 2 to 10 outwards. $P(\text{score}) = \frac{\text{score area}}{\text{total area}}$. The probabilities from the inner circle outwards are 0.01, 0.03, 0.05, 0.07, 0.09, 0.11, 0.13, 0.15, 0.17 and 0.19.

6 To score, the centre of the coin must be within a quarter diameter of the centre of two lines. $S = \{1, 0\}$, $P(1) = \frac{4D}{14D} = \frac{2}{7}$, $P(0) = \frac{5}{7}$

## INVESTIGATION: TRAFFIC LIGHTS

The simulation mirrors the most common settings in Australia.
If you put together simulations for many students you will get close to 44% green, 9% amber and 47% red.

## 2.06

1 $\frac{1}{20\,000} = 0.000\,05$

2 a $\frac{4}{7}$ b $\frac{3}{7}$

3 $\frac{1}{12}$

4 99.8%

5 0.73

6 8

7 34

8 $\frac{1}{3}$

9 a $\frac{1}{6}$ b $\frac{1}{3}$ c $\frac{5}{6}$

10 $\frac{1}{12}$

11 a $\frac{7}{16}$ b $\frac{11}{16}$

12 $\frac{2}{5}$

13 8

14 735

## 2.07

1 $\frac{1}{45}$

2 $\frac{1}{220}$

3 $\frac{1}{28}$

4 $\frac{33}{66\,640} \approx 0.000\,4952$, $\frac{33}{16\,660} \approx 0.001\,988$

5 $\frac{2}{19} \approx 0.1053$

6 a $\frac{1}{11}$ b $\frac{2}{11}$ c $\frac{9}{11}$ d $\frac{9}{22}$

7 $\frac{6}{1771} \approx 0.003\,388$

8 $\frac{7}{17} \approx 0.4118$

9 $\frac{4}{33}$

10 $\frac{10}{21}$

## INVESTIGATION: RELATIVE FREQUENCY AND PROBABILITY

The numbers in each suit will get close to 25% each for the super average, but there will be wide variations for each person, with some sets of 12 cards having none of a suit or in rare cases all of one suit.

## 2.08

1 a 0.075 b 0.025 c 0.275

2 a 0.2037 b 0.1697 c 0.2310

3 a 0.43 b 0.072 c 40 fires
d 22 fires e 9 fires

4 a 200 trials
b 0.12, 0.1, 0.075, 0.08, 0.065, 0.105, 0.085, 0.09, 0.11, 0.11, 0.06
c 40 wins d 4 or 5 wins e 16 wins

5 a 0.458 b No, as the time is set differently

6 0.32

7 a 0.0135 b 0.0016 c 0.0025
d 0.000 02 e Yes

8 About 23%, but other states have different climates so the probability could be different.

## 2.09

1 13%

2 0.525

3 0.6

4 a 0.3 b 0.6 c 0.55 d 0.7

5 a 0.2 b 0.5 c 0.6 d 0.6

6 a $\frac{5}{26}$ b $\frac{9}{26}$ c $\frac{12}{13}$

7 a $\frac{3}{14}$ b $\frac{13}{28}$ c $\frac{9}{28}$

**8** 35%

**9** 59%

**10** $\frac{2}{5}$

## CHAPTER 2 REVIEW

**1** D

**2** A

**3** C

**4** A

**5** D

**6** D

**7** A

**8** **a** {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18}

**b** {6, 12, 18}

**9** {△△, △□, △×, △+, □△ □□, □×, □+, ×△, ×□, ××, ×+, +△, +□, +×, ++}

**10** **a** {1-6, 2-5, 3-4, 4-3, 5-2, 6-1}

**b** {2-1, 2-2, 2-3, 2-4, 2-5, 2-6, 2-7, 2-8, 1-2, 3-2, 4-2, 5-2, 6-2, 7-2, 8-2}

**c** {2-5, 5-2} **d** {1-6, 3-4, 4-3, 6-1}

**11** About 7.

**12** About 0.54, about 9.

**13** Men: 0.65%, women: 0.16%. It doesn't *necessarily* mean it is more dangerous for men. Women in general may not have been exposed to the cause as often.

**14** 11

**15** $S = \{33, 34, 35, 36, 37, 43, 44, 45, 46, 47, 53, 54, 55, 56, 57, 63, 64, 65, 66, 67, 73, 74, 75, 76, 77\}$;

$P(6) = P(14) = \frac{1}{25}$, $P(7) = P(13) = \frac{2}{25}$,

$P(8) = P(12) = \frac{3}{25}$, $P(9) = P(11) = \frac{4}{25}$, $P(10) = \frac{1}{5}$

**16** $S = \{1, 2, 3, 4, 5, 6\}$, $P(1) = \frac{2}{5}$, $P(2) = \frac{1}{5}$, $P(3) = \frac{2}{15}$,

$P(4) = \frac{2}{15}$, $P(5) = \frac{1}{10}$, $P(6) = \frac{1}{30}$

**17** $\frac{94}{367} \approx 0.256$

## 3.01

**1** **a** $x = 6.3$ **b** $y = 5.6$ **c** $b = 3.9$

**d** $x = 5.6$ **e** $m = 2.9$ **f** $x = 13.5$

**g** $y = 10.0$ **h** $p = 3.3$ **i** $x = 5.1$

**j** $t = 28.3$ **k** $x = 3.3$ cm **l** $x = 2.9$ cm

**m** $x = 20.7$ mm **n** $x = 20.5$ mm **o** $y = 4.4$ m

**p** $k = 20.6$ cm **q** $h = 17.3$ m **r** $d = 1.2$ m

**s** $x = 17.4$ cm **t** $b = 163.2$ m

**2** **a** $x = 39°48'$ **b** $\alpha = 35°06'$ **c** $\theta = 37°59'$

**d** $\alpha = 50°37'$ **e** $\alpha = 38°54'$ **f** $\beta = 50°42'$

**g** $x = 44°50'$ **h** $\theta = 30°51'$ **i** $\alpha = 29°43'$

**j** $\theta = 45°37'$ **k** $\alpha = 57°43'$ **l** $\theta = 43°22'$

**m** $\theta = 37°38'$ **n** $\theta = 64°37'$ **o** $\beta = 66°16'$

**p** $\alpha = 29°56'$ **q** $\theta = 54°37'$ **r** $\alpha = 35°58'$

**s** $\theta = 59°2'$ **t** $\gamma = 56°59'$

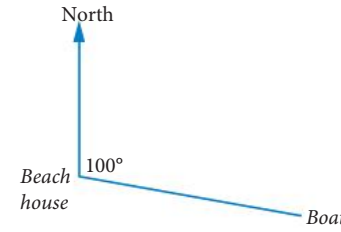
**3** **a** 10.4 m **b** 21 m **c** 126.9 m

**d** 83.1 m **e** 9.2 m

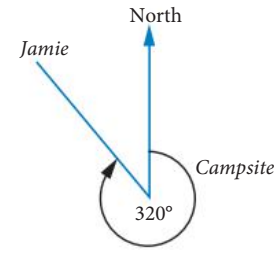
**4** **a** 72°48′ **b** 12 m **c** 171 m

**d** 35 m **e** 198 m

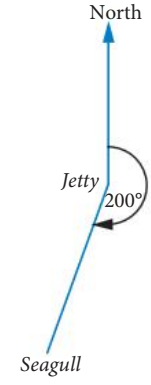
**5** **a**



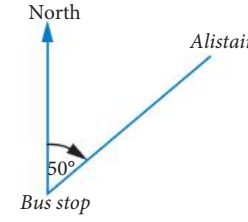

**b**



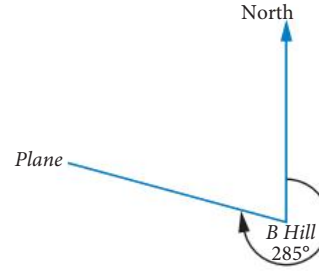

**c**




**d**



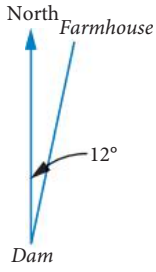

**e**



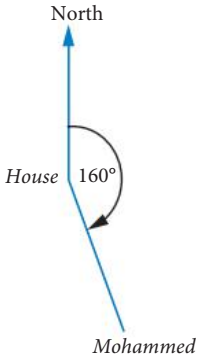



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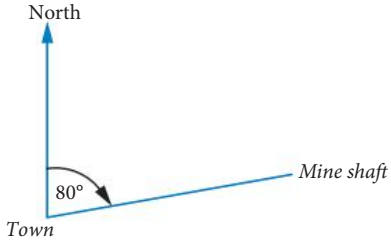
f



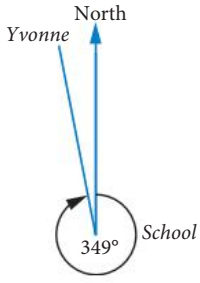

g



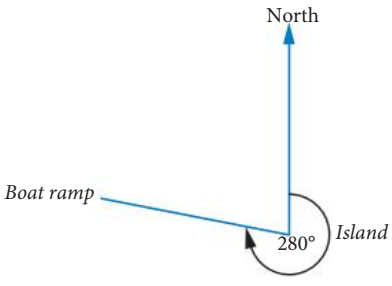

h




i




j




**6** **a** 248° **b** 145° **c** 080°
**d** 337° **e** 180°

**7** **a** 080°
**b** 210°
**c** 160°
**d** **i** 1056.5 km **ii** 2265.8 km **iii** 245°
**e** 1.8 km
**f** 035°
**g** 9.8 km
**h** 1931.9 km
**i** 149°
**j** 4.8 km

**8** 2.6 m

**9** 217°

**10** **a** 1.2 km **b** 7.2 km

## INVESTIGATION: EXACT RATIOS

The opposite for one angle is the adjacent for the other one, so the sine of an angle must be the same as the cosine of its complement. Similarly the tangent of an angle must be the reciprocal of the tangent of its complement.

### 3.02

**1** **a** $\sin(180° - \theta) = \sin(\theta)$
**b** $\cos(180° + x) = -\cos(x)$
**c** $\cos(360° - \theta) = \cos(\theta)$
**d** $\tan(360° - \phi) = -\tan(\phi)$
**e** $\sin(360° - \beta) = -\sin(\beta)$
**f** $\tan(180° - t) = -\tan(t)$
**g** $\cos(180° - x) = -\cos(x)$
**h** $\sin(180° + \alpha) = -\sin(\alpha)$
**i** $\tan(180° + \theta) = \tan(\theta)$

**2** **a** $\frac{\sqrt{3}}{2}$ **b** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ **c** $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
**d** 1 **e** $-\frac{1}{2}$ **f** $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
**g** $-\sqrt{3}$ **h** $-\frac{1}{2}$ **i** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
**j** $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

**3** **a** $\frac{1}{2}$ **b** $-\sqrt{3}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $-1$
**e** $\frac{\sqrt{3}}{2}$ **f** 1 **g** $\frac{\sqrt{3}}{2}$

**4** **a** 4th **b** 2nd **c** 3rd
**d** 1st **e** 2nd

**5** **a** $\sin(x)$ **b** $\cos(\theta)$ **c** $-\tan(\theta)$
**d** $-\cos(x)$ **e** $-\sin(t)$

**6** **a** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{3}}{2}$
**c** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ **d** $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
**e** $\frac{1}{2}$ **f** $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ **g** $\sqrt{3}$

**7** **a** $1+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2}$

**b** $-\frac{1}{2}+\frac{\sqrt{3}}{2}=\frac{\sqrt{3}-1}{2}$

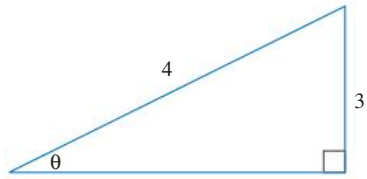
**c** $-\sqrt{3}$

**d** 3 **e** 1

**8** **a** $2\cos(\theta)$ **b** $-3\cos(x)$

**c** $-\tan(\theta)$ **d** $\sin(\theta)$ **e** $-\tan(x)$

**9** **a**




**b** $\sqrt{7}$ units

**c** **i** $\tan(\theta)=-\frac{3}{\sqrt{7}}=-\frac{3\sqrt{7}}{7}$

**ii** $\cos(\theta)=-\frac{\sqrt{7}}{4}$

**10** **a** $\sin(\theta)=-\frac{5}{\sqrt{89}}$ or $-\frac{5\sqrt{89}}{89}$

**b** $\cos(\theta)=-\frac{8}{\sqrt{89}}$ or $-\frac{8\sqrt{89}}{89}$

## 3.03

**1** **a** 7.5 $cm^2$ **b** 32.3 $units^2$

**c** 9.9 $mm^2$ **d** 30.2 $units^2$

**e** 6.3 $cm^2$

**2** $\frac{15\sqrt{3}}{2}m^2$

**3** 7.5 $cm^2$

**4** 15.5 $cm^2$

**5** 34.8 $cm^2$

**6** **a** 7.8 cm **b** 180.8 $cm^2$

**7** 1.16 $m^2$

**8** 73.1 $cm^3$

**9** 21.5 cm

**10** 2.9 $m^3$

**11** **a** 917.6 $mm^3$ **b** $27.53

## 3.04

**1** **a** $a = 13.7, b = 13.4$ **b** $p = 13.4, q = 10.9$

**c** $h = 17.2, k = 13.9$ **d** $c = 34.3, d = 21.1$

**e** $m = 11.8, n = 6.6$ **f** $x = 38.5, y = 51.0$

**2** **a** $d = 8.0$ **b** $y = 9.4$ cm

**c** $a = 10.0$ **d** $b = 10.7$ m

**e** $x = 8.9$

**3** **a** 40.6° **b** 31.7° **c** 54.1°

**d** 38.1° **e** 27.1° **f** 17.5°

**4** **a** $X = 77°, Y = 38°, x = 17.4$ m

**b** $R = 91.9°, r = 84.7$ cm, $p = 49.0$ cm

**c** $E = 27°, D = 42°, e = 6.1$ km

**d** $A = 37.2°, a = 178.4$ cm, $b = 244.3$ cm

**e** $Z = 57.4°, y = 11.1$ m, $z = 11.5$ m

**f** $L = 65.6°, m = 1.9$ km, $k = 2.7$ km

**5** **a** 14.1 cm **b** 15.6 cm

**6** **a** 54.7 mm **b** 35.1 mm

**7** 31.4 cm

**8** 111.4°

**9** 6.69 km

**10** **a** Her height is 2992 m above the hill, so her altitude is really about 3397 m.

**b** The altimeter is reading high.

## 3.05

**1** **a** 5.4 cm **b** 7 m **c** 88.2 cm

**d** 55.1 m **e** 21.4 m **f** 43.8 cm

**2** **a** $m = 5.8$ **b** $b = 10.4$ m **c** $h = 7.4$ cm

**d** $n = 16.4$ **e** $y = 9.3$

**3** **a** $m = 26$ **b** $z = 283$ **c** $f = 64.6$

**d** $g = 43.7$ **e** $s = 157$ **f** $q = 952.9$

**4** **a** 22.3° **b** 53.1° **c** 95.7°

**d** 78.0° **e** 109.5° **f** 134.4°

**5** **a** $\theta = 54° 19'$ **b** $\theta = 60° 27'$ **c** $x = 57° 42'$

**d** $\beta = 131° 31'$ **e** $\theta = 73° 49'$

**6** **a** $73° 47'$ **b** 16.8 mm

**7** **a** 6.2 cm **b** 12.7 cm

**8** 27 m

**9** 26.4° and 36.3°

**10** 170°

## 3.06

**1** **a** 0.47 **b** 1.38 **c** 3.93

**d** 0.26 **e** 2.59

**2** **a** $26°34'$ **b** −45° or 135°

**c** $68°53'$ **d** $-11°19'$ or $168°41'$

**e** 58°

**3** **a** 57.5 m **b** 10.8 m **c** 322 m **d** 9.2 cm

**4** 85 m

**5** Around 20 m.

**6** **a** 401 m **b** 233.27°

**7** 15.3°, 410.4 m

**8** 54 m

**9** 166 km

**10** 77.1 m

## 3.07

**1** **a** 36° **b** 120° **c** 225° **d** 210°

**e** 540° **f** 140° **g** 240° **h** 420°

**i** 20° **j** 50° **k** 62.5° **l** 44.0°

**m** 66.5° **n** 56.7° **o** 18.3° **p** 183.3°

**q** 154.7° **r** 246.9° **s** 320.9° **t** 6.3°

**2** **a** $\frac{3\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{5\pi}{6}$ **d** $\frac{4\pi}{3}$

**e** $\frac{5\pi}{3}$ **f** $\frac{7\pi}{20}$ **g** $\frac{\pi}{12}$ **h** $\frac{5\pi}{2}$

**i** $\frac{5\pi}{4}$ **j** $\frac{2\pi}{3}$


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**3** **a** 0.98 **b** 1.19 **c** 2.22 **d** 5.04
**e** 5.45 **f** 0.32 **g** 0.61 **h** 1.78
**i** 1.54 **j** 0.88

**4** **a** 0.34 **b** 0.07 **c** 0.06
**d** 0.83 **e** −1.14

**5** **a** −1 **b** $-\frac{1}{2}$ **c** $-\sqrt{3}$
**d** $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ **e** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **f** $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
**g** $\sqrt{3}$ **h** $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ **i** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
**j** $-\frac{\sqrt{3}}{2}$

**6** **a** 0 **b** 1 **c** 2.449
**d** 1.333 **e** 0.172 **f** −0.5

**7** **a** 3 sin (*t*) **b** cos (*x*) **c** 0
**d** cos (θ) **e** −4 sin (*p*)

**8** $\text{LHS} = \sin\left(\pi + \frac{\pi}{4}\right)$
$= -\sin\left(\frac{\pi}{4}\right)$ (third quadrant)
$= -\frac{\sqrt{2}}{2}$

$\text{RHS} = \sin(\pi)\cos\left(\frac{\pi}{4}\right) + \cos(\pi)\sin\left(\frac{\pi}{4}\right)$
$= 0 \times \frac{\sqrt{2}}{2} + (-1) \times \frac{\sqrt{2}}{2}$
$= -\frac{\sqrt{2}}{2}$

Since LHS = RHS,
$\sin\left(\pi + \frac{\pi}{4}\right) = \sin(\pi)\cos\left(\frac{\pi}{4}\right) + \cos(\pi)\sin\left(\frac{\pi}{4}\right)$

**9** $\text{LHS} = \cos\left(\frac{2\pi}{3}\right)$
$= -\cos\left(\frac{\pi}{3}\right)$ (second quadrant)
$= -\frac{1}{2}$

$\text{RHS} = \cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$
$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$
$= \frac{1}{4} - \frac{3}{4}$
$= -\frac{1}{2}$

Since LHS = RHS, $\cos\left(\frac{2\pi}{3}\right) = \cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$

**10** $\text{LHS} = \tan\left(\pi + \frac{\pi}{3}\right)$
$= \tan\left(\frac{\pi}{3}\right)$ (3rd quadrant)
$= \sqrt{3}$

$$\text{RHS} = \frac{\tan(\pi) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{3}\right)}$$
$$= \frac{0 + \sqrt{3}}{1 - 0 \times \sqrt{3}}$$
$$= \sqrt{3}$$

Since LHS = RHS, $\tan\left(\pi + \frac{\pi}{3}\right) = \frac{\tan(\pi) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan(\pi)\tan\left(\frac{\pi}{3}\right)}$

## INVESTIGATION: AREA OF A SEGMENT

**1** $\frac{1}{2}r^2\sin(\theta)$

**2** $\frac{1}{2}r^2\theta$

**3** $\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin(\theta) = \frac{1}{2}r^2[\theta - \sin(\theta)]$

## 3.08

**1** **a** $4\pi$ cm **b** $\pi$ m **c** $\frac{25\pi}{3}$ cm **d** $\frac{\pi}{2}$ cm
**e** $\frac{7\pi}{4}$ mm **f** 0.65 m **g** 3.92 cm
**h** 6.91 mm **i** 2.39 cm **j** 3.03 m

**2** 7.5 m

**3** $\frac{2\pi}{21}$

**4** **a** $8\pi$ cm$^2$ **b** $\frac{3\pi}{2}$ m$^2$ **c** $\frac{125\pi}{3}$ cm$^2$
**d** $\frac{3\pi}{4}$ cm$^2$ **e** $\frac{49\pi}{8}$ mm$^2$ **f** 0.48 m$^2$
**g** 6.29 cm$^2$ **h** 24.88 mm$^2$
**i** 7.05 cm$^2$ **j** 3.18 m$^2$

**5** **a** 4.44 **b** 6 m

**6** **a** 25 mm **b** 1.83
**c** 13.78 mm **d** 272.4 mm$^2$
**e** 75 cm$^2$ **f** 11.97 cm$^2$
**g** **i** $\frac{7\pi}{6}$ cm **ii** $\frac{49\pi}{12}$ cm$^2$

**7** $\theta = \frac{\pi}{15}$, $r = 3$ cm

**8** **a** $\frac{3\pi}{7}$ cm **b** $\frac{9\pi}{14}$ cm$^2$ **c** 0.07 cm$^2$

**9** **a** 5 cm$^2$ **b** 0.3% **c** 15.6 cm

**10** **a** $10\pi$ cm **b** $24\pi$ cm$^2$

## CHAPTER 3 REVIEW

**1** C

**2** D

**3** B

**4** D

**5** **a** 9.4 cm **b** 9.0 m **c** θ = 49°

**6** 9.9 m

**7** 218°

8 a $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$ b $-\frac{\sqrt{3}}{2}$ c $-\frac{\sqrt{3}}{2}$

d $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ e $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

9 1289 $m^2$

10 210 $mm^2$

11 498.8 $cm^3$

12 $x = 5$ cm

13 $\theta = 77° 6'$

14 61.2 m

15 $x = 10.2$ mm

16 $\theta = 105.5°$

17 26°

18 0.21

19 a 7.9 cm b 7.0 cm

20 7 m

21 a i $\frac{\pi}{12}$ ii $\frac{4\pi}{9}$ iii $\frac{17\pi}{18}$

iv $\frac{7\pi}{4}$ v $\frac{3\pi}{2}$

b i 0.99 ii 1.88 iii 1.27

iv 1.04 v 1.79

c i 26° ii 54° iii 42°

iv 218° v 281°

22 a 0.38 b 0.48 c −0.03

23 a $\frac{\sqrt{3}}{2}$ b $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

c $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ d 1 e $\frac{\sqrt{3}}{2}$

24 a $\frac{10\pi}{9}$ cm b $\frac{25\pi}{27}$ $cm^2$

25 11 cm

26 $\frac{2\pi}{5}$ $cm^2$

27 a Area of sector $= \frac{1}{2}r^2\theta$

Area of triangle $= \frac{1}{2}r^2 \sin(\theta)$

Area of segment = area of sector − area of triangle

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin(\theta)$

$= \frac{1}{2}r^2[\theta - \sin(\theta)]$

b $\frac{4\left(2\pi - 3\sqrt{3}\right)}{3}$ $cm^2$

28 31 m

29 About 17 m (16.87 m)

30 About 56 m (55.94 m)

## MIXED REVISION 1

**Multiple choice**

1 C

2 B

3 C

4 B

5 D

6 A

7 D

8 C

9 E

**Short answer**

1 $3x^3 - 5x^2 + 4x - 7 = (x + 3)(3x^2 - 14x + 46) - 145$

2 $S = \{1 - 1, 1 - 3, 1 - 5, 1 - 8, 3 - 1, 3 - 3, 3 - 5, 3 - 8, 5 - 1, 5 - 3, 5 - 5, 5 - 8, 8 - 1, 8 - 3, 8 - 5, 8 - 8\}$, $P(-7) = \frac{1}{16}$, $P(-5) = \frac{1}{16}$, $P(-4) = \frac{1}{16}$, $P(-3) = \frac{1}{16}$, $P(-2) = \frac{1}{8}$, $P(0) = \frac{1}{4}$, $P(2) = \frac{1}{8}$, $P(3) = \frac{1}{16}$, $P(4) = \frac{1}{16}$, $P(5) = \frac{1}{16}$, $P(7) = \frac{1}{16}$

3 a $\frac{7\pi}{36}$ b 144° c 0.51

d 60.7° e $\frac{41\pi}{18}$

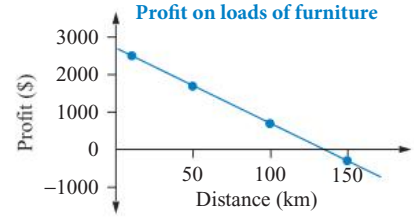
4 124 lettuces

5 40

6 4.1 cm

**Application**

1




a i $1500 ii $2100 iii $300 iv $900

b 135 km

2 The number of leaves on the plant has increased nine times.

3 $P(A \text{ or } (B \text{ and } C)) = P(A) + P(B \text{ and } C) - P(A \text{ and } (B \text{ and } C))$

$= P(A) + P(B \text{ and } C) - P(A \text{ and } B \text{ and } C)$

$= P(A) + P(B \text{ and } C) - P((A \text{ and } B) \text{ and } C)$

$= P(A) + P(B \text{ and } C) - P(\varnothing \text{ and } C)$

$= P(A) + P(B \text{ and } C) - P(\varnothing)$

$= P(A) + P(B \text{ and } C)$, since $P(\varnothing) = 0$ QED

4 $S = \{15, 20, 25, 30, 35\}$, $P(15) = \frac{11\pi - 6\sqrt{3}}{54\pi} \approx 0.142$,

$P(20) = \frac{11\pi - 6\sqrt{3}}{54\pi} \approx 0.142$, $P(25) = \frac{8\pi - 6\sqrt{3}}{27\pi} \approx 0.174$,

$P(30) = \frac{4\pi + 6\sqrt{3}}{27\pi} \approx 0.271$,

$P(35) = \frac{4\pi + 6\sqrt{3}}{27\pi} \approx 0.271$

5 a $\theta = 56°$ b $\alpha = 30°$

6 a 21.8° b 3.2 m

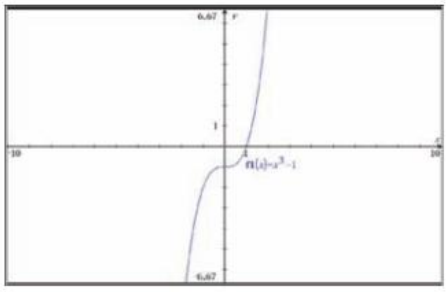
c 3.6 $m^2$ d 3.45 $m^2$


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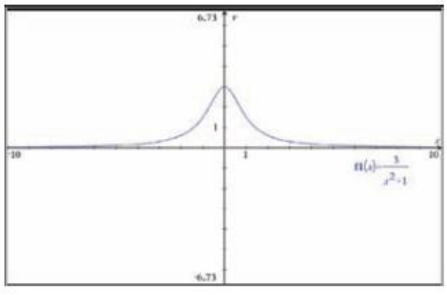
## 4.01

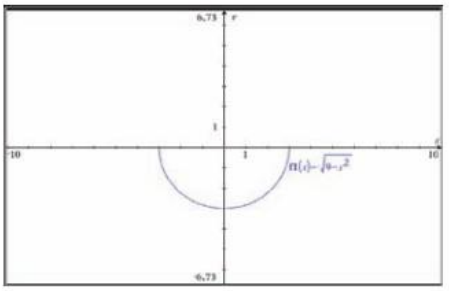
1 a Domain: {3, 4, 5, 6, 7}, range: {5, 6, 7, 10}
b Domain: {*A*, *B*, *C*, *D*, *E*}, range: {2, 3, 5, 7}
c Domain: {*x*: all real *x*}, range: {*y*: $y \geq 0$ }
d Domain: {*x*: all real *x*}, range: {*y*: $y \leq 3$ }
e Domain: {*x*: $-3 \leq x \leq 3$}, range: {*y*: $-2 \leq y \leq 4$ }

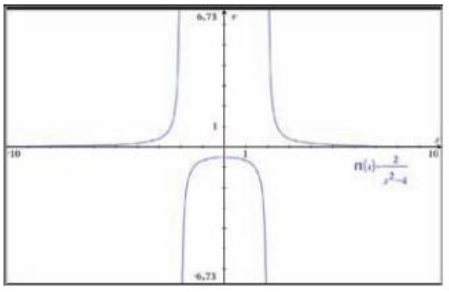
2 a Domain: {*x*: all real *x*}, range: {*y*: all real *y*}
b Domain: {*x*: $0 \leq x \leq 3$}, range: {*y*: $3 \leq y \leq 9$}
c Domain: {*x*: all real $x \neq 2$}, range: {*y*: all real $y \neq 0$} or $R \backslash 0$
d Domain: {*x*: $x \geq 0$}, range: {*y*: $y \geq 0$ }
e Domain: {1, 2, 3, 4, 5}, range: {2, 4, 8, 16, 32}
f Domain: {−3, −2, −1, 0, 1, 2, 3}, range: {0, 1, 4, 9, 16}
g Domain: {*x*: $1 \leq x \leq 4$}, range: $\{y: \frac{1}{4} \leq y \leq 1\}$
h Domain: {0, 1, 2, 3, 4}, range: {0, 14, 78, 252}
i Domain: {*x*: $x \geq 1$}, range: {*y*: $y \geq 0$}
j Domain: {*x*: $x \geq -5$}, range: {*y*: $y \leq 0$}

3 a 
Domain: {*x*: all real *x*}, range: {*y*: all real *y*}

b 
Domain: {*x*: all real *x*}, range: {*y*: $y = 3$}

c 
Domain: {*x*: all real *x*}, range: {*y*: $0 < y \leq 3$}

d 
Domain: {*x*: $-3 \leq x \leq 3$}, range: {*y*: $-3 \leq y \leq 0$}

e 
Domain: {*x*: all real $x \neq \pm 2$},
range: $\left\{y : y > 0, y \leq -\frac{1}{2}\right\}$

4 a Yes b No c No d Yes
e Yes f Yes g No h Yes
i Yes j No k Yes l No
m Yes n No o Yes

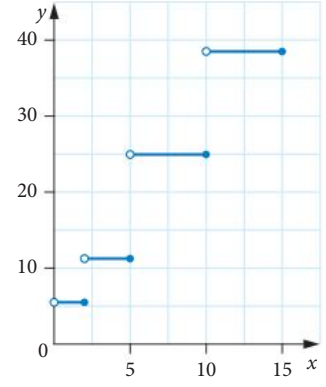
5 a No b Yes c No
d Yes e No

6 a Continuous b Discrete
c Continuous d Continuous
e Discrete f Continuous
g Continuous h Discrete
i Continuous j Discrete

7 a Continuous
b Discontinuous at $x = 10, 20, 30$
c Continuous d Discontinuous at $x = 0$
e Discontinuous for $-2 < x \leq 1$

8 a Discrete b Continuous
c Continuous d Discrete
e Continuous f Discrete

9 c Yes d Discrete

10 a 

**b** Yes **c** Discontinuous
**d** **i** \$24.95 **ii** \$11.25 **iii** \$24.95
**e** Domain $\{x: 0 < x \leq 15\}$, range {\$5.50, \$11.25, \$24.95, \$38.50}

## INVESTIGATION: TRANSLATIONS, DILATIONS AND REFLECTIONS

1 For $k$ positive, the graph moves up and for $k$ negative it moves down.
2 For $k$ positive, the graph moves left and for $k$ negative it moves right.
3 For $k$ bigger than 1 the graph stretches vertically (gets thinner) and for $0 < k < 1$ it compresses vertically (gets fatter). If $k$ is negative it reflects in the $x$-axis.
4 For $k$ bigger than 1 the graph compresses horizontally (gets thinner) and for $0 < k < 1$ it stretches horizontally (gets fatter). If $k$ is negative it reflects in the $y$-axis.

### 4.02

1 **a** $f(1) = 4, f(-3) = 0$
**b** $h(0) = -2, h(2) = 2, h(-4) = 14$
**c** $f(5) = -25, f(-1) = -1, f(3) = -9, f(-2) = -4$
**d** 14 **e** $-35$ **f** $x = 9$
**g** $x = \pm 5$ **h** $x = -3$
**i** $f(p) = 2p - 9, f(x + h) = 2x + 2h - 9$
2 $g(x - 1) = x^2 + 2$
3 **a** $f(k) = (k - 1)(k^2 + k + 1)$
**b** $t = -1; t = 2, -4$ **c** 0
4 **a** $F$ **b** $H$ **c** $G$ **d** $B$
**e** $A$ **f** $D$ **g** $C$ **h** $E$
5 **a** $f(x^2) = x^6 + 1$
**b** $[f(x)]^2 = x^6 + 2x^3 + 1$
**c** $f(-x) = -x^3 + 1$
6 **a** $g(-x) = g(x)$ since all powers of x are even.
$\therefore$ even function
**b** $f(-x) = -x = -f(x)$
$\therefore$ odd function
**c** $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$
$\therefore$ even function
**d** $f(-x) = 4(-x) - (-x)^3 = -4x + x^3$
$= -(4x - x^3) = -f(x)$
$\therefore$ odd function
**e** $f(-x) = (-x)^4 + (-x)^2$
$= x^4 + x^2 = f(x)$
$\therefore$ even function
$f(x) - f(-x) = 0$
7 **a** $f(5) = 125, f(1) = 1, f(-1) = -1$
**b** $f(2) - f(-2) + f(-1) = 0 - 4 + 1 = -3$
**c** 10 **d** 7 **e** $-22$
8 **a** Even values i.e. $n = 2, 4, 6, \ldots$
**b** Odd values i.e. $n = 1, 3, 5, \ldots$
9 **a** No value of $n$
**b** Yes, when $n$ is odd $(1, 3, 5, \ldots)$

## INVESTIGATION: ANGLES AND GRADIENTS

1 Angles less than 90°
2 Angles more than 90°, because they are 'going downhill'
3 0, and the angle is also 0° because they are parallel
4 90°, no, it is undefined because the run is 0

### 4.03

1 **a** **i** 5 units **ii** $\left(4, 1\frac{1}{2}\right)$ **iii** $-\frac{3}{4}$
**b** **i** 13 units **ii** $\left(-2\frac{1}{2}, -1\right)$ **iii** $\frac{12}{5}$
**c** **i** 4.24 units **ii** $\left(\frac{1}{2}, 3\frac{1}{2}\right)$ **iii** 1
**d** **i** 4.47 units **ii** (3, 0) **iii** $-2$
**e** **i** 6.32 units **ii** $(-5, 2)$ **iii** $-\frac{1}{3}$
2 **a** **i** 0.84 **ii** $\sqrt{3}$ **iii** 1
**iv** $-1.23$ **v** $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
**b** **i** 27° **ii** 53° **iii** 165°
**iv** 30° **v** 110°
3 $a = 3$
4 $AB = AC = \sqrt{34}, BC = \sqrt{128}$ so $\triangle ABC$ is isosceles.
5 $(-8, 13)$
6 **a** $-\frac{4}{3}$ **b** $\frac{2}{5}$ **c** 0 **d** $-\frac{3}{7}$
**e** 6 **f** Not defined **g** 4
**h** $-46$ **i** $-2$ **j** $-1$ **k** $-\frac{1}{4}$
**l** 3 **m** $-\frac{1}{5}$ **n** $-\frac{1}{5}$ **o** 1
**p** $-\frac{2}{3}$
7 Right-angled triangle
8 Parallelogram
9 **a** 6.24 m **b** No **c** 667 mm
10 45°; proof.

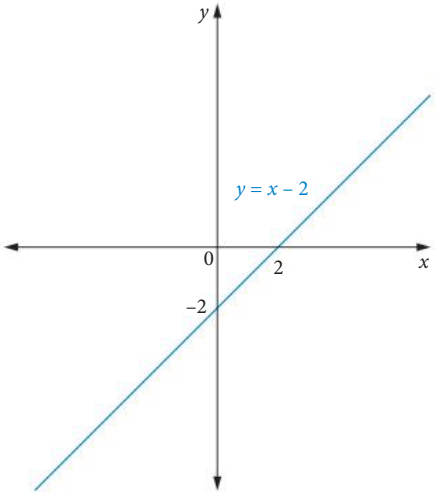
## INVESTIGATION: GRADIENT-INTERCEPT FORM OF A STRAIGHT LINE

1 For $k$ positive the line gets steeper, for $k$ negative it goes down instead of up and for $k$ a fraction (i.e. $0 < k < 1$) the slope decreases.
2 The graph goes up or down to go through $k$ on the $y$-axis.

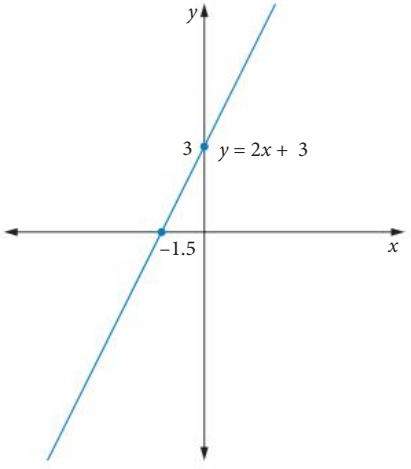

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## 4.04

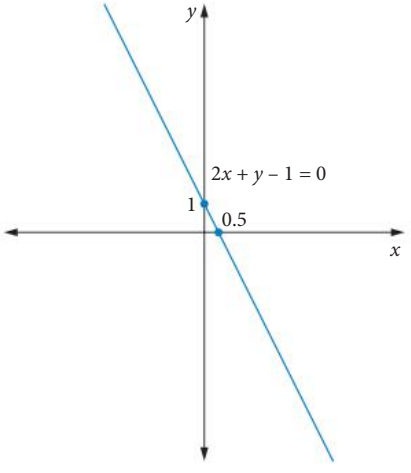
**1** **a** $x$-intercept 2, $y$-intercept $-2$



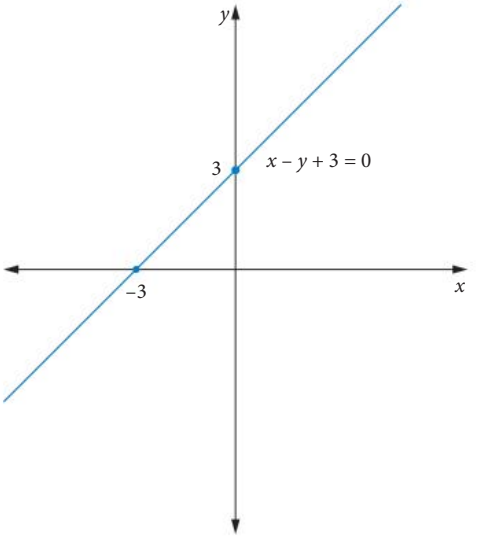

**b** $x$-intercept $-1\frac{1}{2}$, $y$-intercept 3



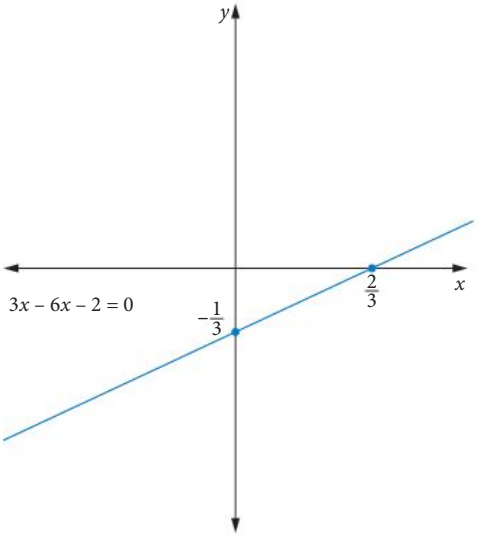

**c** $x$-intercept $\frac{1}{2}$, $y$-intercept 1




**d** $x$-intercept $-3$, $y$-intercept 3




**e** $x$-intercept $\frac{2}{3}$, $y$-intercept $-\frac{1}{3}$




**2** **a** $y = 4x - 1$ **b** $y = -3x + 4$ **c** $y = 5x$
**d** $y = 4x + 20$ **e** $y = -3x + 3$
**f** $y = \frac{4x}{3} - 4$ **g** $y = x - 1$

**3** **a** $x - y + 1 = 0$ **b** $x - 3y + 16 = 0$
**c** $x + y - 5 = 0$ **d** $x + 2y + 5 = 0$
**e** $x - 2y + 4 = 0$ **f** $x + 3y - 1 = 0$
**g** $3x + 4y + 13 = 0$

**4** $2x + y - 3 = 0$

**5** **a** $y = 3$ **b** $x = -1$

**6** $y = 7$

**7** $x = 4$

**8** **a** $a = 4$ **b** $4x + 3y - 13 = 0$
**c** $3x - 4y + 9 = 0$

9 a $-\frac{a}{b}$

b $l_2$ has gradient $\frac{b}{a}$

$m_1 m_2 = -\frac{a}{b} \times \frac{b}{a} = -1$ so the lines are perpendicular

c $bx - ay - bp = 0$

10 Given points $(x_1, y_1)$ and $(x_2, y_2)$ on the line, the gradient can be written as $m = \frac{y_2 - y_1}{x_2 - x_1}$.

For any point $(x, y)$ and $(x_1, y_1)$, the gradient can also be written as $m = \frac{y - y_1}{x - x_1}$.

Thus $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, which can be re-arranged to give $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

11 Given points $(a, 0)$ and $(0, b)$ on the line:

Gradient, $m = -\frac{b}{a}$

$y$-intercept, $c = b$

So equation is $y = -\frac{b}{a}x + b$

Thus $\frac{y}{b} = -\frac{b}{a}\frac{x}{a} + 1$

So $\frac{x}{a} + \frac{y}{b} = 1$ QED

## INVESTIGATION: SHAPE OF THE QUADRATIC FUNCTION

1 a For $k > 1$ it gets thinner and for $0 < k < 1$ it gets fatter. For $k$ negative is turns upside down.

b For $k$ positive it moves up and for $k$ negative it moves down.

c For $k$ positive it moves left and for $k$ negative it moves right.

d For $k > 1$ it gets thinner and for $0 < k < 1$ it gets fatter.

e For $k$ positive or negative the graph goes through the $x$-axis at 0. It also goes through at $-k$.

2 The graph goes through the $x$-axis at $a$ and $b$

3 The graph gets thinner or fatter or turns upside down like **1a** as $a$ changes, and the turning point moves along the $x$-axis to $b$.

## 4.05

1 a $x$-intercept 3, $y$-intercept 9

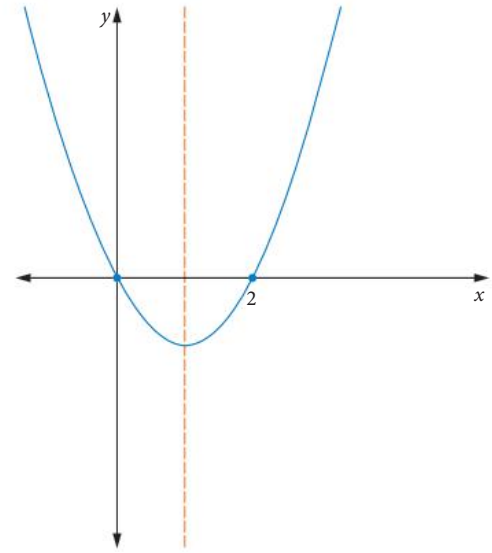
b $x$-intercepts $\pm 2$, $y$-intercept 4

c $x$-intercepts $-1$, 2, $y$-intercept 2

d $x$-intercepts $-5$, 0, $y$-intercept 0

e $x$-intercepts $1\frac{1}{3}$, 3, $y$-intercept 12

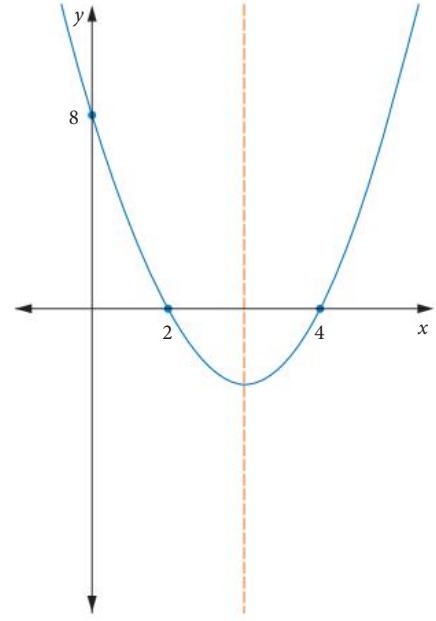
2 a i




ii Curve is increasing when $x > 1$, decreasing when $x < 1$.

iii Gradient is positive for $x > 1$, negative for $x < 1$, zero for $x = 1$.

b i

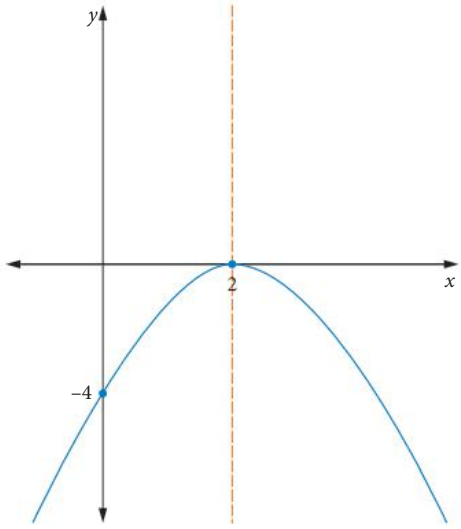



ii Curve increases when $x > 3$, decreases when $x < 3$

iii Gradient is positive for $x > 3$, negative for $x < 3$, zero for $x = 3$


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**c** i




ii Curve increases when $x < 2$, decreases when $x > 2$.

iii Gradient is positive for $x < 2$, negative for $x > 2$, zero for $x = 2$.

**3** **a** i $x = 2$ ii (2, 1) minimum

**b** i $x = -1$ ii (−1, 5) maximum

**c** i $x = 1$ ii (1, 2) minimum

**d** i $x = -3$ ii (−3, −2) maximum

**e** i $x = 5$ ii (5, −4) maximum

**f** i $x = -1$ ii (−1, −2) minimum

**g** i $x = 2$ ii (2, 3) minimum

**h** i $x = 1$ ii (1, 9) minimum

**i** i $x = -3$ ii (−3, 1) maximum

**j** i $x = -2$ ii (−2, 60) maximum

**4** **a** $x = -3$; (−3, −12)

**b** $x = -4$; (−4, 17)

**c** $x = 1\frac{1}{4}; \left(1\frac{1}{4}, 3\frac{1}{8}\right)$

**d** $x = -1\frac{1}{4}; \left(-1\frac{1}{4}, -13\frac{1}{4}\right)$

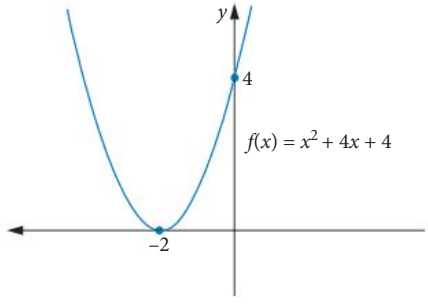
**e** $x = -3$; (−3, −23)

**5** **a** i $x = -1$ ii −3 iii (−1, −3)

**b** i $x = 1$ ii 1 iii (1, 1)

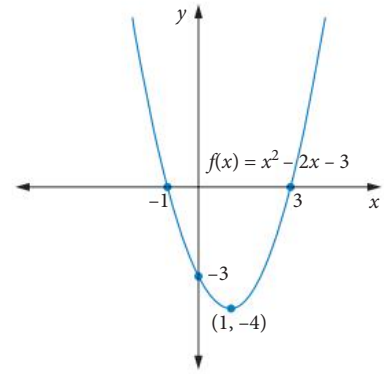
**6** **a** i −2 ii Minimum 0

iii



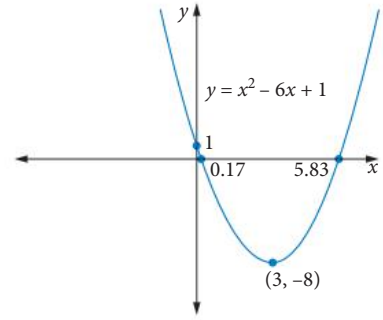

**b** i −1, 3 ii Minimum −4

iii



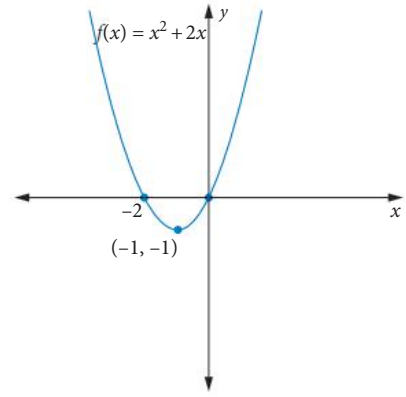

**c** i 5.83, 0.17 ii Minimum −8

iii



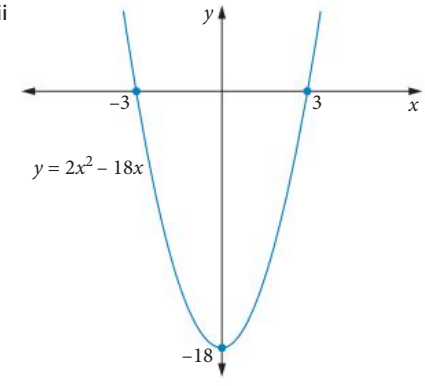

**d** i −2, 0 ii Minimum −1

iii



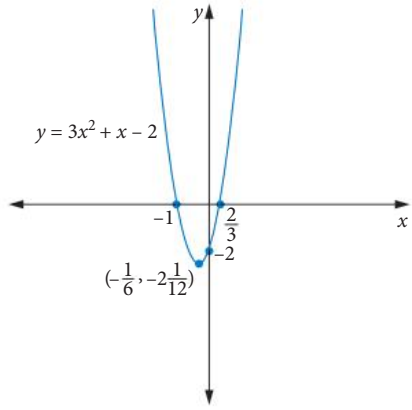

**e** i ±3 ii Minimum −18

iii



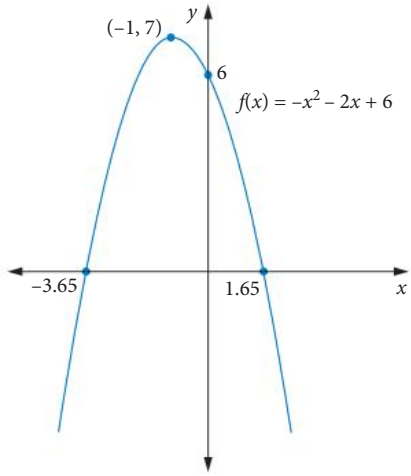
**f** i $-1, \frac{2}{3}$ ii Minimum $-2\frac{1}{12}$

iii



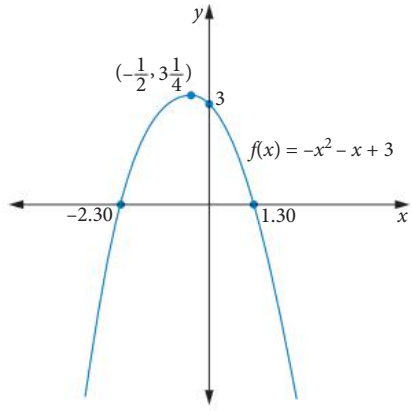

**g** i 1.65, –3.65 ii Maximum 7

iii




**h** i 1.3, –2.3 ii Maximum $3\frac{1}{4}$

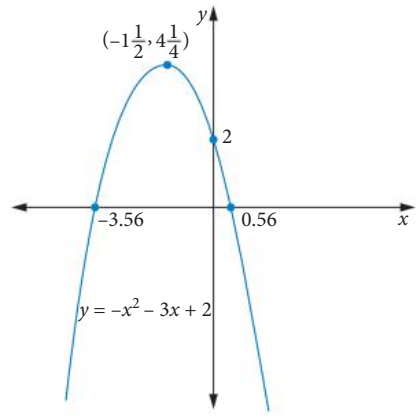
iii




**i** i 0.56, –3.56

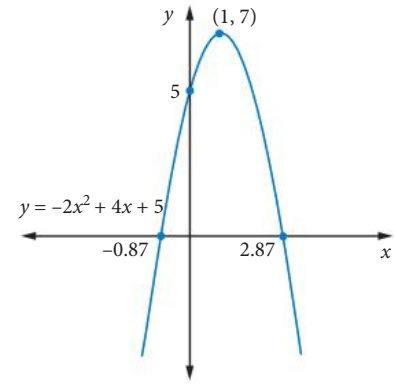
ii Maximum $4\frac{1}{4}$

iii

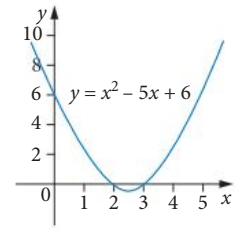



**j** i 2.87, –0.87 ii Maximum 7

iii




**7** a




**b** $x < 2, x > 3$ **c** $2 \le x \le 3$

**8** Teacher to check proof. Area is greatest when $x = 20$ m.

**9** **a** \$6400 **b** 21 **c** \$3600

**10** Max (2.5, 33.25); $h$-intercept 2; $t$-intercepts –0.08, 5.08. It takes 2.5 s to reach maximum height and 5.08 s to reach the ground.


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## INVESTIGATION: SOME USES OF THE PARABOLA IN SOCIETY

1 Car headlight reflectors, telescope mirrors, satellite receiver dishes
2 The catenary is the shape that a chain assumes when suspended at each end. It has a complicated equation involving exponentials.

## 4.06

1 a $y = x^2 - x - 5$
b $y = x^2 - 3x$
c $y = 2x^2 - 3x + 7$
d $y = x^2 + 4x - 9$
e $y = -x^2 - 2x + 1$

2 a $y = -x^2 + 4x - 6$ b $y = -2x^2 - 4x + 16$
c $y = x^2 + 4x + 1$ d $y = -5x^2 + 20x - 15$
e $y = 3x^2 + 6x + 6$

3 a 17; unequal real irrational zeros
b $-39$; no real zeros
c 1; unequal real rational zeros
d 0; equal real rational zeros
e 33; unequal real irrational zeros
f $-16$; no real zeros
g 49; unequal real rational zeros
h $-116$; no real zeros
i 1; unequal real rational zeros
j 48; unequal real irrational zeros

4 a $p < 1$ b $p = 1$ c $p > 1$

5 Solving simultaneous equations gives $x^2 + 8x - 1 = 0$; $\Delta = 68 > 0$ so 2 intersection points.

6 Solving simultaneous equations gives $x^2 + x + 4 = 0$; $\Delta = -15 < 0$ so no intersection points.

7 $p = 3\frac{1}{4}$

8 a $\Delta = p^2 - 4k^2$
For equal roots (1 solution) $\Delta = 0$:
$p^2 - 4k^2 = 0$
$p^2 = 4k^2$
$p = \pm 2k$
b For $x$-intercepts $y = 0$
When $p = 2k$:
$kx^2 + 2kx + k = 0$
$k(x^2 + 2x + 1) = 0$
$k(x + 1)^2 = 0$
$x + 1 = 0$
$x = -1$
So $x$-intercept is $-1$
When $p = -2k$:
$kx^2 - 2kx + k = 0$
$k(x^2 - 2x + 1) = 0$
$k(x - 1)^2 = 0$
$x - 1 = 0$
$x = 1$
So $x$-intercept is 1

9 a $h = -t^2 + 4t + 5$
b Maximum height is 9 m.
c It reaches the ground after 5 s.

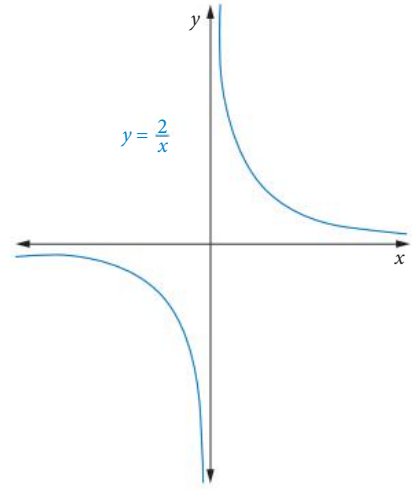
10 a $y = \frac{5x^2}{24}$ or $x^2 = 4.8y$ b About 3.1 m

## INVESTIGATION: EXPLORING THE HYPERBOLA

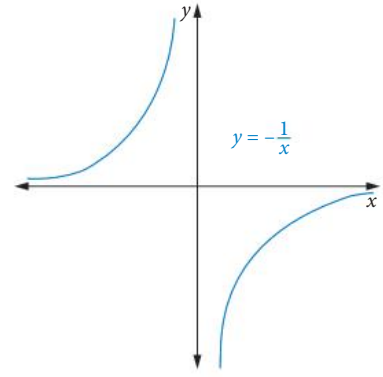
1 As $k$ increases, the graph moves away from the origin. For $k$ negative it reflects across the axis.
2 For $k$ positive it moves up and for $k$ negative it moves down.
3 For $k$ positive it moves left and for $k$ negative it moves right.

## 4.07

1 a i {all real $x$: $x \neq 0$}, {all real $y$: $y \neq 0$}
ii no $y$-intercept
iii

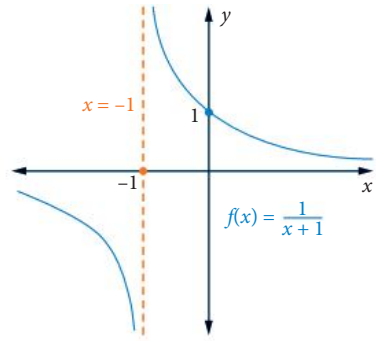



b i {all real $x$: $x \neq 0$}, {all real $y$: $y \neq 0$}
ii no $y$-intercept
iii




c i {all real $x$: $x \neq -1$}, {all real $y$: $y \neq 0$}
ii 1

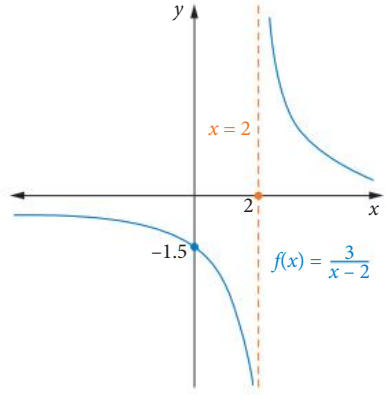
iii




**d** i {all real $x$: $x \neq 2$}, {all real $y$: $y \neq 0$}

ii $-1\frac{1}{2}$

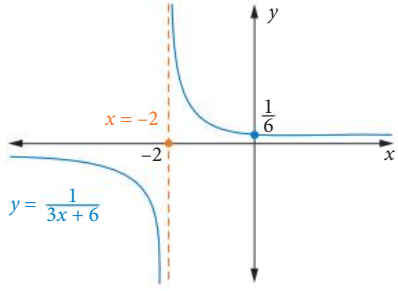
iii




**e** i {all real $x$: $x \neq -2$}, {all real $y$: $y \neq 0$}

ii $\frac{1}{6}$

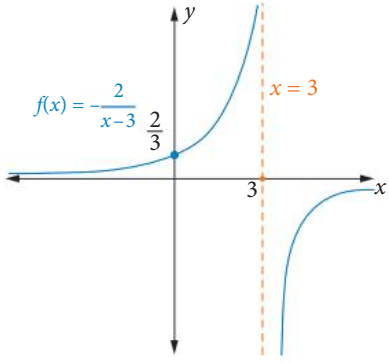
iii




**f** i {all real $x$: $x \neq 3$}, {all real $y$: $y \neq 0$}

ii $\frac{2}{3}$

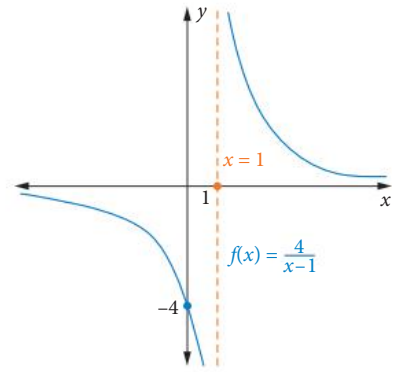
iii




**g** i {all real $x$: $x \neq 1$}, {all real $y$: $y \neq 0$}

ii $-4$

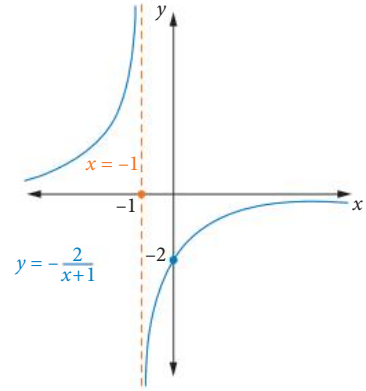
iii




**h** i {all real $x$: $x \neq -1$}, {all real $y$: $y \neq 0$}

ii $-2$

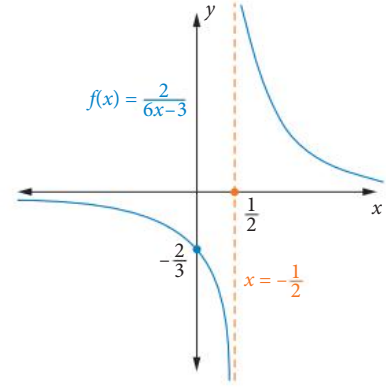
iii




**i** i {all real $x$: $x \neq \frac{1}{2}$}, {all real $y$: $y \neq 0$}

ii $-\frac{2}{3}$

iii

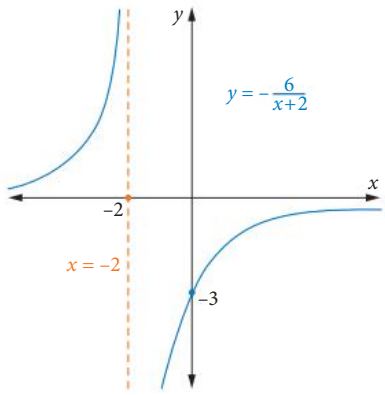



**j** i {all real $x$: $x \neq -2$}, {all real $y$: $y \neq 0$}

ii $-3$


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iii




2 $f(-x) = \frac{2}{-x}$

$= -\frac{2}{x}$

$= -f(x)$

$\therefore$ odd function

3 a $\frac{1}{9} \le y \le 1$ b $\frac{1}{3} \le y \le 1$

c $-2\frac{1}{2} \le y \le -\frac{1}{2}$ d $\frac{3}{7} \le y \le 3$

e $-2 \le y \le -\frac{1}{8}$

4 a Translate the graph of $y = x^3$ up 4 units.

b Translate the graph of $y = x^3$ down 4 units.

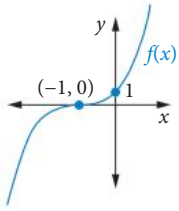
c Stretch the graph of $y = x^3$ vertically by a factor of 2 to look thinner.

d Compress the graph of $y = x^3$ vertically by a factor of 2 to look fatter.

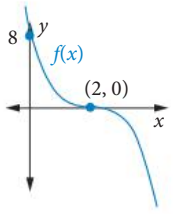
e Translate the graph of $y = x^3$ left 4 units.

f Translate the graph of $y = x^3$ right 5 units.

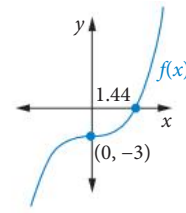
5 a



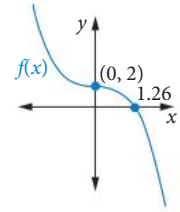

b



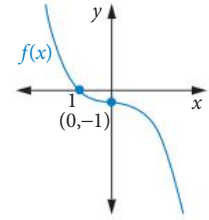

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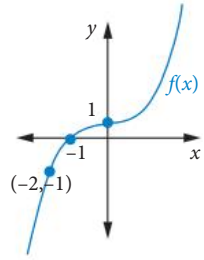

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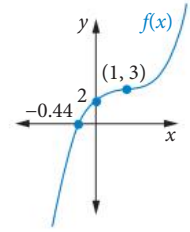

e



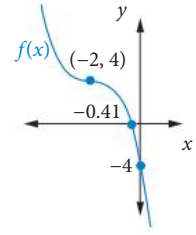

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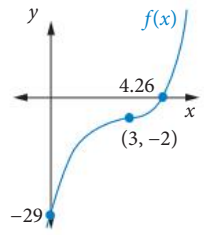

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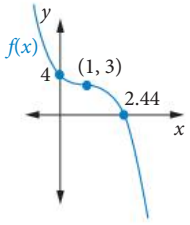

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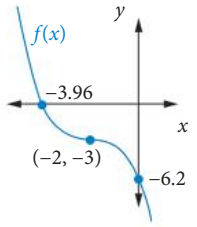
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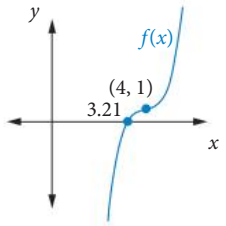

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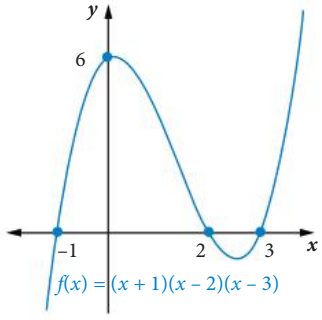

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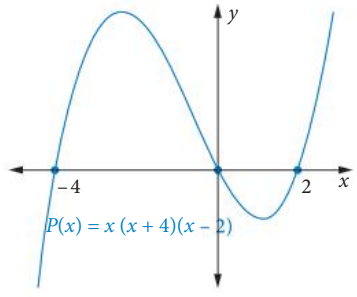

l



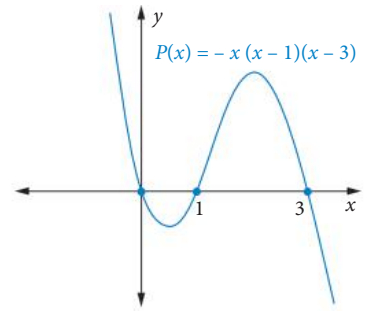

6 a



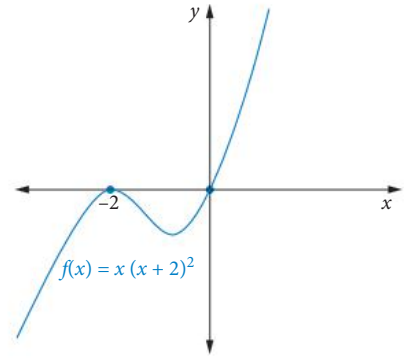

b



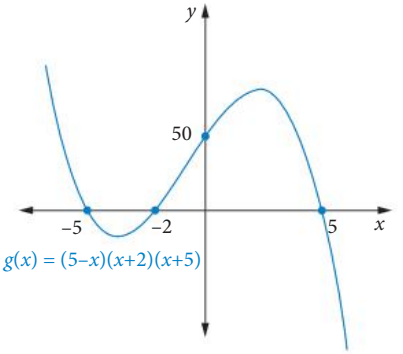

c




d

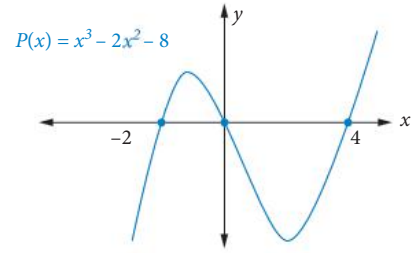



e



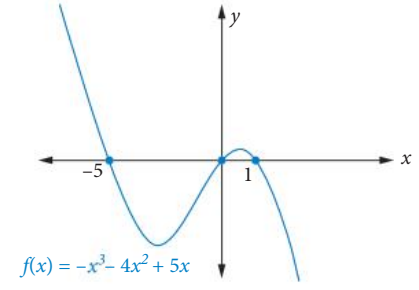

7 a i $P(x) = x(x - 4)(x + 2)$

ii




b i $f(x) = -x(x - 1)(x + 5)$

ii

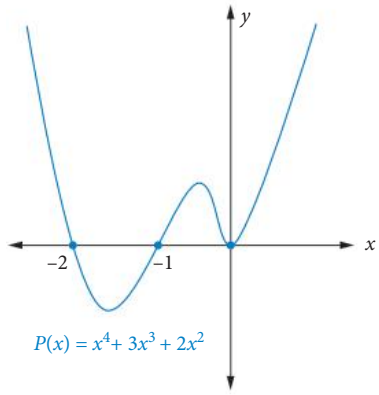




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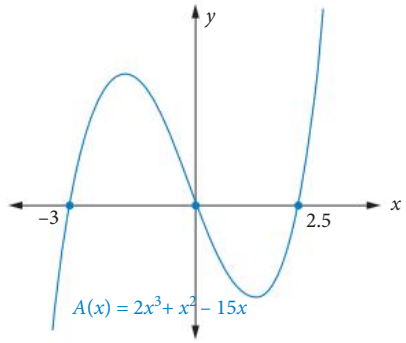
**c** i $P(x) = x^2(x + 1)(x + 2)$

ii



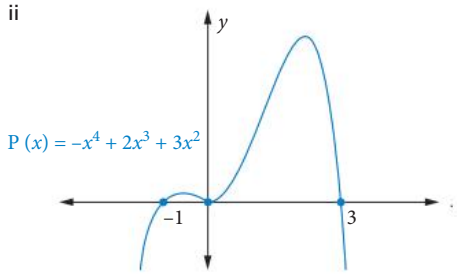

**d** i $A(x) = x(2x - 5)(x + 3)$

ii

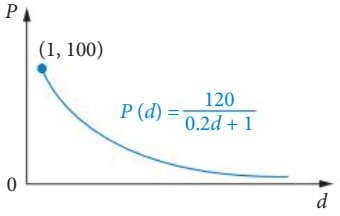



**e** i $P(x) = -x^2(x - 3)(x + 1)$

ii




**8** a




**b** 60% **c** 40% **d** 7 days

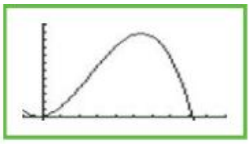
**e** 115 days

**f** $P(d) = \frac{120}{0.2d + 1.4}$, $P(d) = \frac{90}{0.2d + 1}$

or $P(d) = \frac{120}{0.6d + 1}$

**g** 113, 85 or 38 days, depending on the model used.

**9** a



**b** –1.8366, 0, 40.8366

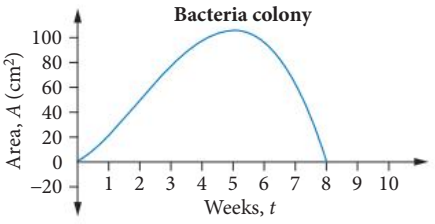
**c** 36.5, 91, 103.5 g/L

**d** 107.73 g/L. More nutrient, more algae

**e** The mass cannot be negative, so the domain must be $0 \le d \le 40.84$ or smaller.

**10** a 0.5 $cm^2$

**b**




**c** After 8 weeks **d** 85.3% **e** 93.3%

## INVESTIGATION: SEMICIRCLES

Yes, one above the axis and the other below the axis.

$y = \pm\sqrt{r^2 - x^2}$

Draw $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$ for $-r \le x \le r$.

## INVESTIGATION: EXPLORING THE PARABOLA WITH A HORIZONTAL AXIS

1. For $k > 1$, the graph compresses horizontally (gets fatter) and for $0 < k < 1$ it stretches horizontally (gets fatter). It doesn't make any difference if $k$ is negative.
2. For $k$ positive it moves left and for $k$ negative it moves right.
3. For $k$ positive it moves down and for $k$ negative it moves up.

## 4.08

**1** a Radius 10, centre (0, 0)

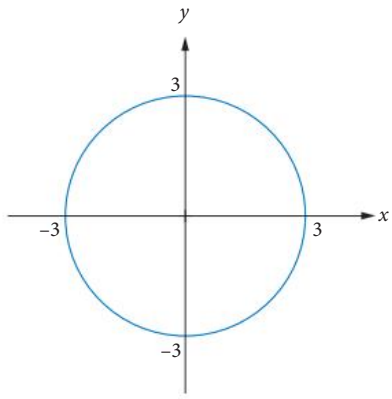
**b** Radius $\sqrt{5}$, centre (0, 0)

**c** Radius 4, centre (4, 5)

**d** Radius 7, centre (5, –6)

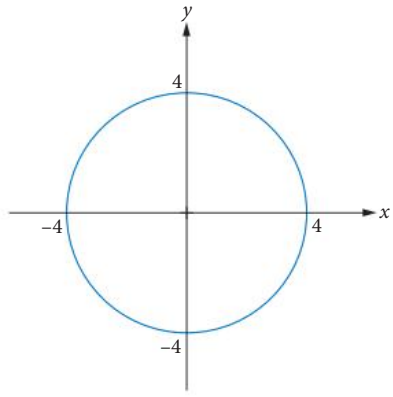
**e** Radius 9, centre (0, 3)

**2 a i**



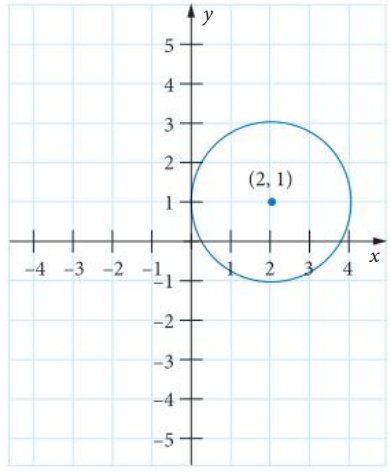

**ii** $\{x: -3 \leq x \leq 3\}, \{y: -3 \leq y \leq 3\}$

**b i**



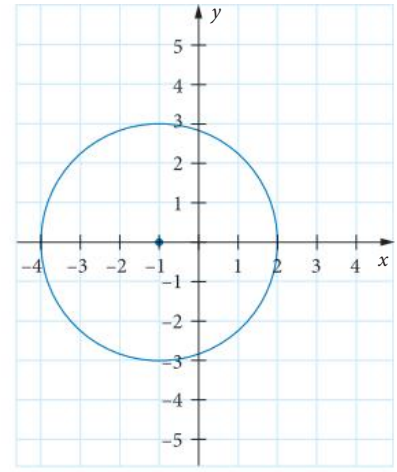

**ii** $\{x: -4 \leq x \leq 4\}, \{y: -4 \leq y \leq 4\}$

**c i**



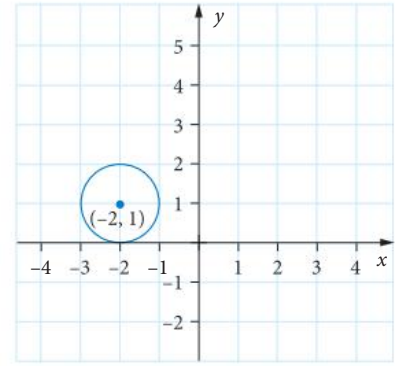

**ii** $\{x: 0 \leq x \leq 4\}, \{y: -1 \leq y \leq 3\}$

**d i**




**ii** $\{x: -4 \leq x \leq 2\}, \{y: -3 \leq y \leq 3\}$

**e i**



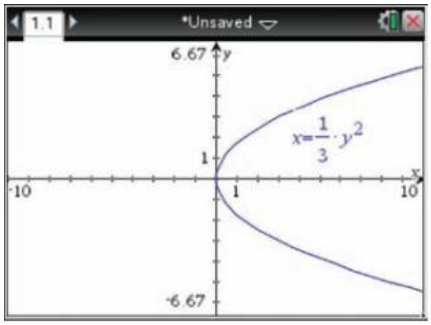

**ii** $\{x: -3 \leq x \leq -1\}, \{y: 0 \leq y \leq 2\}$

**3 a** Radius 3, centre (2, 1)
**b** Radius 5, centre (−4, 2)
**c** Radius 1, centre (0, 1)
**d** Radius 6, centre (5, −3)
**e** Radius 1, centre (−1, 1)
**f** Radius 6, centre (6, 0)
**g** Radius 5, centre (−3, 4)
**h** Radius 8, centre (−10, 2)
**i** Radius 5, centre (7, −1)
**j** Radius $\sqrt{10}$, centre (−1, −2)

**4 a** $x^2 + y^2 = 16$
**b** $x^2 - 6x + y^2 - 4y - 12 = 0$
**c** $x^2 + 2x + y^2 - 10y + 17 = 0$
**d** $x^2 - 4x + y^2 - 6y - 23 = 0$
**e** $x^2 + 8x + y^2 - 4y - 5 = 0$
**f** $x^2 + y^2 + 4y + 3 = 0$
**g** $x^2 - 8x + y^2 - 4y - 29 = 0$
**h** $x^2 + 6x + y^2 + 8y - 56 = 0$
**i** $x^2 + 4x + y^2 - 1 = 0$
**j** $x^2 + 8x + y^2 + 14y + 62 = 0$


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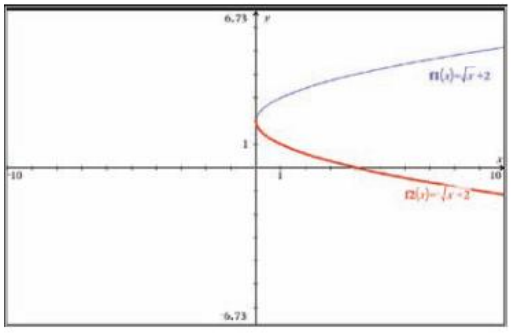
5



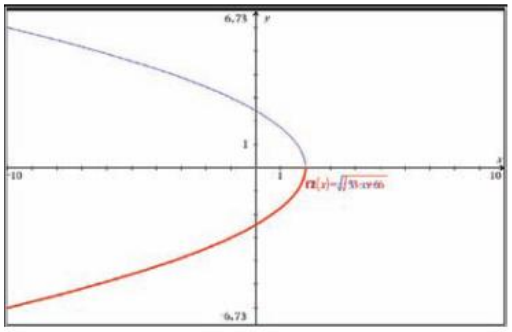

Domain is $\{x: x \geq 0\}$
Range is $\{y$: all real $y\}$

6 a $y=\sqrt{5x}$ and $y=-\sqrt{5x}$
b $y=\sqrt{x}-1$ and $y=-\sqrt{x}-1$
c $y=\sqrt{x+1}+3$ and $y=-\sqrt{x+1}+3$

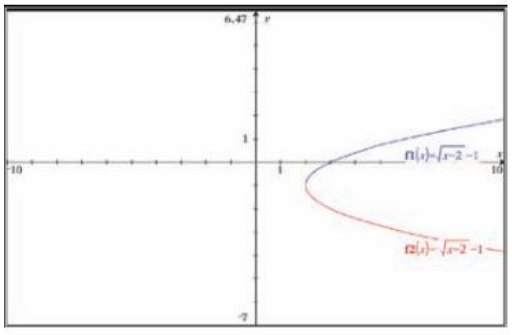
7 a



b



c



8 The circles are $(x+1)^2+(y-2)^2=25$ and $(x+1)^2+(y-2)^2=9$, which both have centre $(-1, 2)$ so are concentric.

9 a $x^2+2x+y^2-6y-15=0$
b $(2, 7), (-1, -2)$
c $Z=(-4, -1)$ or $Z=(-1, 8)$
d $m_{zx} \times m_{yx} = -\frac{1}{3} \times 3$
$= -1$
$\therefore \angle ZXY = 90°$

10 $y^2-4y-4x+8=0$

## CHAPTER 4 REVIEW

1 D
2 B
3 A
4 E
5 B
6 C

7 a Domain is $\{-3, 0, 1, 2, 4\}$ and range is $\{-7, 0, 1, 2, 5\}$
b Domain is $\{x: x \geq 2\}$ and range is $\{y: y \geq 0\}$
c Domain is $\{x$: all real $x \neq -5\}$ and range is $\{y$: all real $y \neq 0\}$

8 a Function b Not a function
c Not a function d Function
e Function

9 a Discontinuous b Continuous
c Continuous d Discontinuous
e Continuous

10 a $-3$ b $-11$ c $27x^3-3$
d $(x-1)^3-3=x^3-3x^2+3x-4$ e $x=2$

11 a Translated up 2 units.
b Dilated by a factor of 2 from the $x$-axis.
c Translated 2 units to the right.

12 a $f(-x)=\frac{1}{(-x)^4+3}=\frac{1}{x^4+3}=f(x)$ so even
b $f(-x)=(-x)^5-(-x)=-x^5+x=-(x^5-x)=-f(x)$ so odd
c $f(-x)=(-x)^2-2(-x)+3=x^2+2x+3 \neq f(x)$ or $-f(x)$
so neither odd nor even

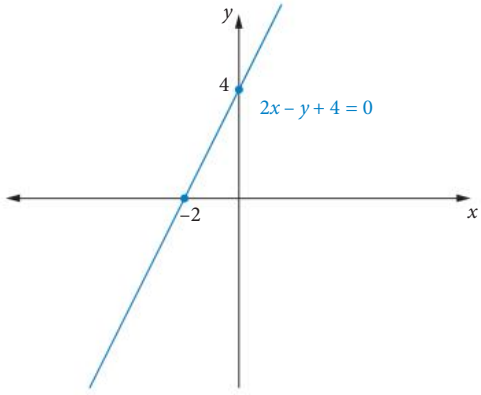
13 a $\sqrt{50} \approx 7.07$ units b $\left(\frac{1}{2}, 1\frac{1}{2}\right)$ c $\frac{1}{7}$

14 $y=2$

15 a $\frac{1}{2}$ b 4 c $\frac{5}{3}$
d 1 e $\frac{1}{3}$ f 1

**16** **a** $x$-intercept $-2$, $y$-intercept 4

**b**



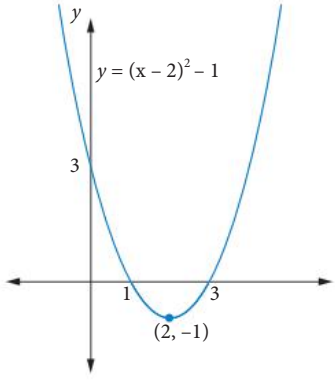

**17** **a** $2x + y - 6 = 0$ **b** $2x - y - 7 = 0$

**c** $x - 4y - 4 = 0$ **d** $x + y - 4 = 0$

**e** $3x - y - 5 = 0$ **f** $x - 2y + 2 = 0$

**g** $y = -5$ **h** $x = -2$

**18** **a**




**b** Domain is $\{x: \text{all real } x\}$ and range is $\{y: \text{all real } y \geq -1\}$

**19** **a** $f(x) = (x + 3)^2 - 11$

**b** Axis of symmetry is $x = -3$, turning point $(-3, -11)$

**20** **a** Axis of symmetry is $x = -1$, turning point $(-1, -5)$

**b** Minimum value is $-5$

**21** **a** $y = -x^2 - x + 6$ **b** $y = x^2 + 1$

**22** **a** 1; 2 real rational zeros **b** $-79$; no real zeros

**c** 0; 1 real zero (2 equal zeros)

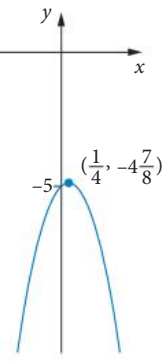
**d** 37; 2 real (irrational) zeros

**23** **a** Axis of symmetry $x = \frac{1}{4}$, turning point $\left(\frac{1}{4}, -4\frac{7}{8}\right)$

**b** Maximum. Since this is below the $x$-axis, the curve is always below the $x$-axis and there are no $x$-intercepts.

**c** $-39$; no real zeros so function has no $x$-intercepts.

**d**

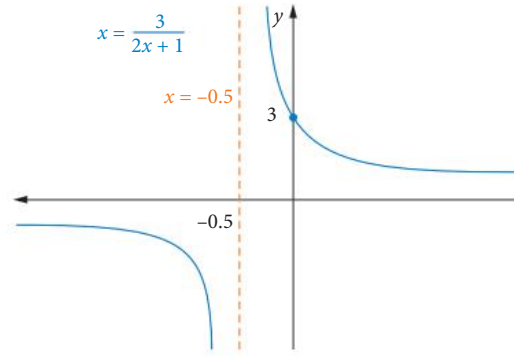



**24** $p \leq 3\frac{1}{4}$

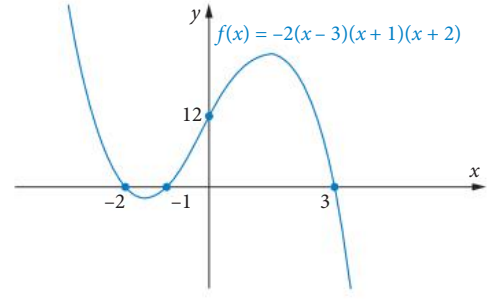
**25** **a** Solving simultaneous equations gives $3x^2 - 7x + 2 = 0$; $\Delta = 25$ so 2 real zeros. So there are 2 points of intersection.

**b** $(2, 3)$ and $\left(\frac{1}{3}, 2\frac{4}{9}\right)$

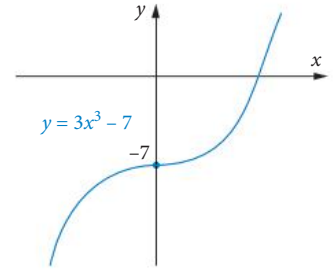
**26**




**27** **a**



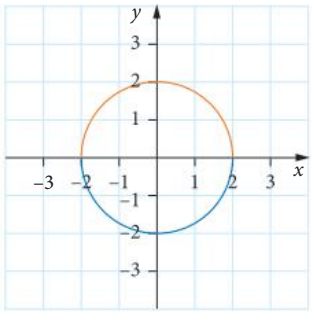

**b**





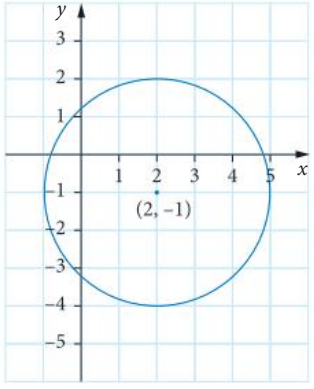
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**28** a




Domain is $\{x: -2 \le x \le 2\}$ and range is $\{y: -2 \le y \le 2\}$

b



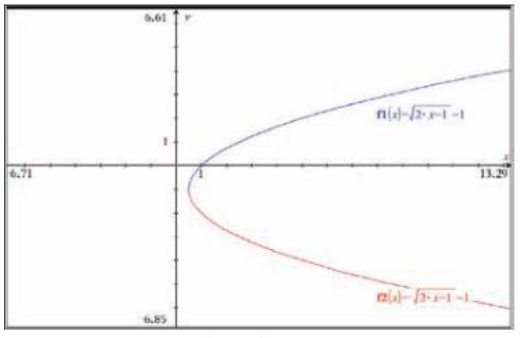

Domain is $\{x: -1 \le x \le 5\}$ and range is $\{y: -4 \le y \le 2\}$

**29** Radius 4, centre (−4, 1)

**30** a $x^2 + y^2 = 49$

b $x^2 + 10x + y^2 - 6y + 30 = 0$

**31** a




b $y = -1$ c $\left(\frac{1}{2}, -1\right)$

d Domain is $\left\{x : x \ge \frac{1}{2}\right\}$ and range is $\{y: \text{all real } y\}$

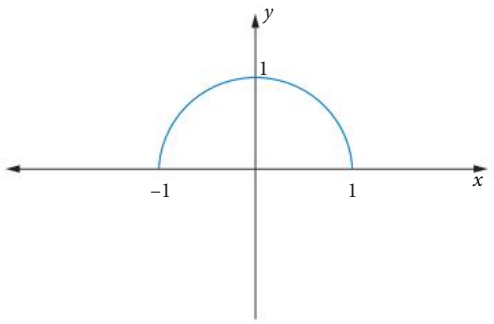
e No

**32** a $x$ b $\{x: x \ne 2\}$ and $\{y: y \ne 2\}$

c The straight line $y = x$ with a discontinuity at (2, 2).

**33** a 2 rational zeros b $k = 2$

**34** a




b Domain is $\{x: -1 \le x \le 1\}$ and range is $\{y: 0 \le y \le 1\}$

**35** 21 cm

## 5.01

**1** a $\frac{2}{7}$ b $\frac{5}{14}$ c $\frac{1}{5}$ d $\frac{2}{5}$ e $\frac{1}{3}$

**2** a $\frac{3}{8}$ b $\frac{6}{11}$ c $\frac{2}{5}$ d $\frac{3}{4}$ e $\frac{1}{2}$

**3** $\frac{15}{91}$

**4** $\frac{19}{99}$

**5** a $\frac{5}{39}$ b $\frac{14}{39}$ c $\frac{20}{39}$

**6** a $\frac{1}{161\,700}$ b $\frac{97}{53\,900}$ c $\frac{7372}{8085}$

**7** $\frac{1}{19}$ **8** $\frac{1}{8}$ **9** $\frac{1}{36}$

**10** a $\frac{1}{7776}$ b $\frac{3125}{7776}$ c $\frac{4651}{7776}$

**11** $\frac{999}{1000} = 99.9\%$

**12** $\frac{7}{64}$

## 5.02

**1** a $\frac{2}{5}$ b $\frac{4}{11}$ c $\frac{1}{5}$ d $\frac{1}{6}$ e $\frac{1}{3}$

**2** a $\frac{5}{6}$ b $\frac{9}{10}$ c $\frac{13}{50}$ d $\frac{13}{50}$ e $\frac{9}{50}$ f $\frac{1}{10}$ g $\frac{1}{10}$

**3** a $\frac{15}{19} \approx 0.79$ b $\frac{15}{77} \approx 0.19$ c $\frac{15}{146} \approx 0.10$ d $\frac{4}{7} \approx 0.57$ e $\frac{30}{77} \approx 0.39$

f Because they are not complementary.

4 a $\frac{13}{27} \approx 48\%$ b $\frac{13}{47} \approx 48\%$

c $\frac{3}{20} = 15\%$ d $\frac{16}{57} \approx 28\%$

e Because there are different numbers of Office, Workshop and On-site employees.

5 a

| | Assaults at work since 15 (thousands) | |
|---|---|---|
| | On men | On women |
| Assaults by men | 11.2% | 72.8% |
| Assaults by women | 12.2% | 3.7% |

b 84.0% c 13.3% d 76.5%

e 52.2% f 4.9%

6 a

| Worker type | Years of education | | | | |
|---|---|---|---|---|---|
| | 11–12 | 13 | 14–15 | 16–17 | 18–19 |
| Blue-collar | 33.3% | 30% | 36.7% | 0% | 0% |
| Management | 0% | 0% | 25% | 35% | 40% |
| White-collar | 20% | 22% | 38% | 20% | 0% |

b

| Worker type | Years of education | | | | |
|---|---|---|---|---|---|
| | 11–12 | 13 | 14–15 | 16–17 | 18–19 |
| Blue-collar | 50% | 45% | 31.4% | 0% | 0% |
| Management | 0% | 0% | 14.3% | 41.2% | 100% |
| White-collar | 50% | 55% | 54.3% | 58.8% | 0% |

c 31.4% d 40% e 20%

## 5.03

1 a $\frac{1}{2}$ b $\frac{2}{7}$ c $\frac{5}{6}$

2 a $\frac{3}{4}$ b $\frac{1}{5}$ c $\frac{10}{19}$

3 a $\frac{1}{2}$ b $\frac{17}{50}$ c $\frac{1}{5}$

4 a 35 b $\frac{14}{25}$ c $\frac{9}{17}$ d $\frac{6}{11}$

5 a $\frac{9}{17}$ b $\frac{2}{3}$ c $\frac{5}{6}$

6 a $\frac{7}{15}$ b $\frac{7}{11}$ c $\frac{2}{7}$

d $\frac{3}{5}$ e $\frac{1}{2}$

## 5.04

1 a $\frac{4}{17}$ b $\frac{1}{17}$ c $\frac{25}{51}$

2 a $\frac{2}{11}$ b $\frac{5}{11}$ c $\frac{4}{11}$ d $\frac{5}{11}$

e $\frac{3}{11}$

3 $\frac{1}{17}$

4 $\frac{33}{66\,640} \approx 0.000\,4951$

5 a $\frac{1}{22}$ b $\frac{1}{55}$ c $\frac{1}{220}$

6 a $\frac{1}{203}$ b $\frac{1}{4060}$ c $\frac{1}{116}$

d $\frac{1}{174}$ e $\frac{1}{58}$

7 0.000 000 03 or 0.000 003%

8 a $\frac{1}{3}$ b $\frac{1}{60}$ c $\frac{1}{150}$ d $\frac{5}{7}$

## INVESTIGATION: MISUSE OF PROBABILITY AND STATISTICS

Sally Clark was released because evidence had been suppressed, rather than just because of the misuse of probability. However, the publicity did have the convictions of 2 other women overturned as well. She had developed problems as a result of the prosecution and died young.

The gambler's fallacy is that after a run of heads, the probability of tails must be higher. It is not true because the events are independent.

## 5.05

1 a $\frac{1}{36}$ b $\frac{1}{216}$ c $\frac{1}{1296}$

2 a $\frac{1}{169}$ b $\frac{1}{64}$ c $\frac{1}{28\,561}$

d $\frac{729}{2197}$ e $\frac{243}{371\,293}$

3 $\frac{1}{1024}$

4 a $\frac{343}{5832}$ b $\frac{35}{816}$

5 $\frac{243}{3125} = 0.077\,76$

6 a $\frac{1}{625}$ b 0.002 43 c $\frac{1}{64}$

7 a $\frac{1}{27}$ b $\frac{2}{91}$ c $\frac{1}{9}$

d $\frac{2}{21}$ e $\frac{1}{81}$ f $\frac{1}{273}$

8 a $\frac{15}{56}$ b $\frac{3}{14}$ c $\frac{29}{56}$

d $\frac{13}{56}$ e $\frac{43}{56}$

9 a 9.95% b 0.45% c 0.05%

d Most positive results (21 out of 22) will be wrong.

e Even though the doctor's expertise would increase the proportion sent who actually have the disease, the results should be regarded with caution (probably do a second test).

10 About 0.007%.

11 a 2.25% b 0.0076% c About $2.9 \times 10^{-12}$


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## 5.06

1 a $\frac{2}{15}$ b $\frac{1}{15}$ c $\frac{1}{3}$ d $\frac{7}{15}$

2 a $\frac{28}{37}$ b $\frac{26}{37}$

3 $\frac{1}{3}$

4 0.52

5 a The same, $\frac{1}{36}$ b Less, $\frac{11}{450}$ to $\frac{1}{36}$

c More, $\frac{29}{300}$ to $\frac{1}{12}$ d Less, $\frac{11}{150}$ to $\frac{1}{12}$

6 $\frac{32}{663}$

7 13.52%

8 0.5785

## INVESTIGATION: DEATHS FROM CANCER

Being offered screening may change the way that a woman looks at her breasts so that cancer is detected sooner. The issue of cost of screening can be considered in terms of public and private good. Having a large number of people suffer a disease means a loss to society as well as the individual. It is a matter of degree. For example, smoking-related diseases cost society much more than the benefits in taxes.

## 5.07

1 a About 0.122% b About 0.193%

c No, more prevalent in men.

d About 7.67% e About 9.15%

f No, more prevalent in men.

g About 11.3% h About 26.6%

i No much more prevalent in women.

2 a About 30.9% b About 34.0%

c No, girls were more likely to work.

d About 31.6% e About 36.2%

f No, girls were more likely to work.

g There was a slight increase, but a big increase in girls and a decrease in boys.

3 a 19.82% b 0.72% c 20.54%

d About 3.5%

e The testimony was unlikely to be right because a positive result was about 27.5 times more likely to be from an adult than a baby.

4 a About 20.5% b About 16.7%

c About 37.3% d About 55.1%

e About 26.7%

f People who are obese or overweight are about 2.1 times more likely to have heart disease.

5 a About 38% b About 54%

c About 19%

d When you are exposed to the virus, poor sleep (less than 7 hours) makes you about 2.8 times more likely to catch a cold than having good sleep (more than 8 hours).

## CHAPTER 5 REVIEW

1 D

2 A

3 E

4 C

5 C

6 a $\frac{7}{92}$ b $\frac{191}{276} \approx 0.692$

7 a 0.49 b $\frac{2}{9} \approx 0.222$ c $\frac{23}{81} \approx 0.284$

d $\frac{1}{2}$

8 a 40 b $\frac{7}{15}$ c $\frac{1}{5}$

9 a $\frac{6}{25}$ b $\frac{7}{26}$ c $\frac{483}{10\,208} \approx 0.0473$

10 a $\frac{1}{7} \approx 14\%$ b $\frac{1}{2} = 50\%$

11 a 1 in 4225 or about 0.024%.

b No, because twins do run in families and the occurrence is also influenced by IVF.

12 $\frac{148}{225} \approx 0.658$

13 $\frac{16}{25} = 0.64$

14 0.44

15 a $\frac{16}{33} \approx 0.485$ b $\frac{4}{5} = 0.8$ c No

d $\frac{2}{9}$ to $\frac{4}{5}$, so taller students are 3.6 times as likely to have big feet than shorter students.

e $\frac{7}{9}$ to $\frac{1}{5}$, so shorter students are 3.9 times as likely to have small feet than taller students.

## INVESTIGATION: TRIGONOMETRIC RATIOS OF SPECIAL ANGLES

For sine and cosine, the values at 0, $\frac{\pi}{2}$, etc. are always 0, 1 or –1, because $x$ and $y$ are 0, 1 or –1 and $r = 1$. For tangent, the value is 0 for multiples of $\pi$ and undefined for odd multiples of $\frac{\pi}{2}$, as $x = 0$ on the $y$-axis.

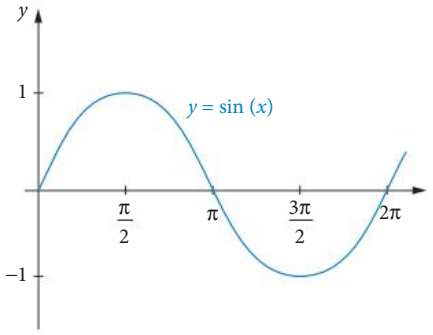
The values repeat after $2\pi$. You need other values to draw the graphs.

## 6.01

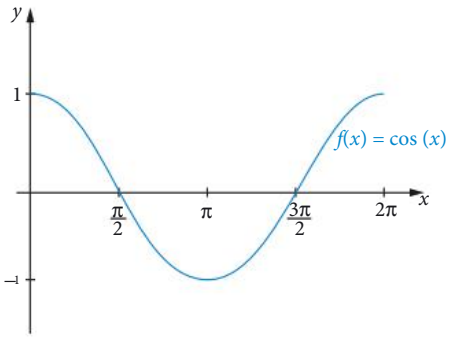
1 a 0 b 1 c –1 d 0

e 0 f 0 g 0 h 1

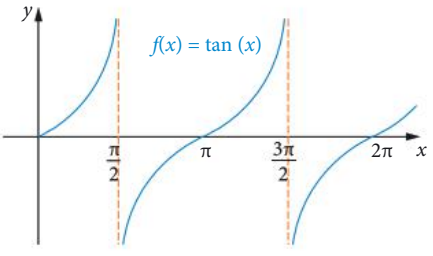
**2 a**



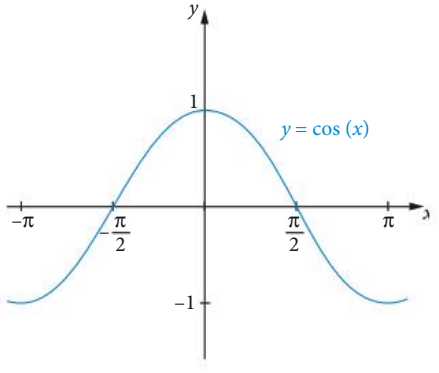

**b**



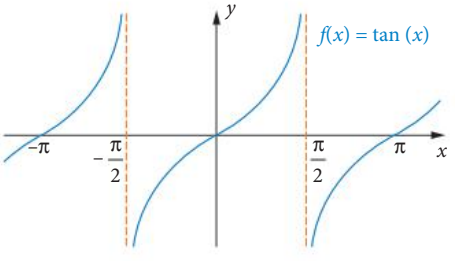

**c**



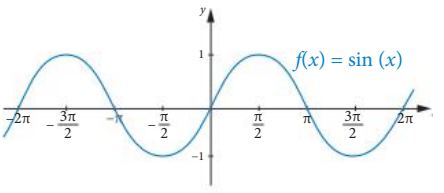

**3 a**



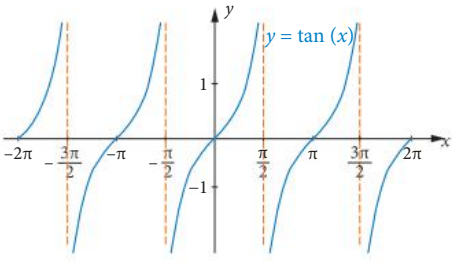

**b**



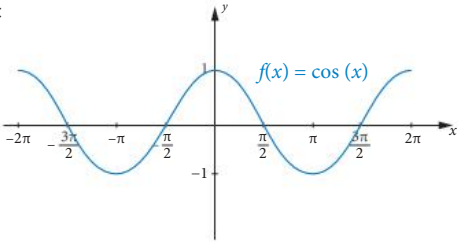

**4 a**



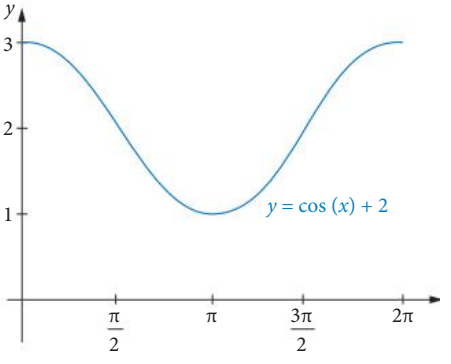

**b**



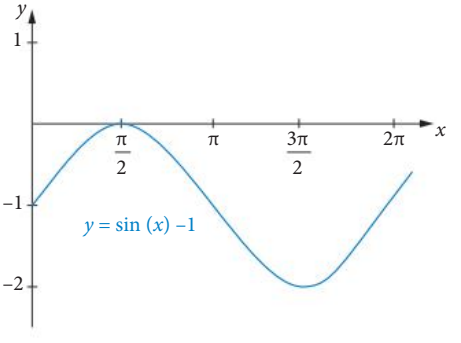

**c**




**5 a**



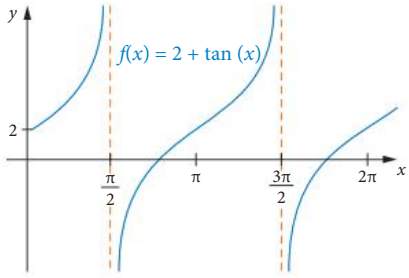

**b**



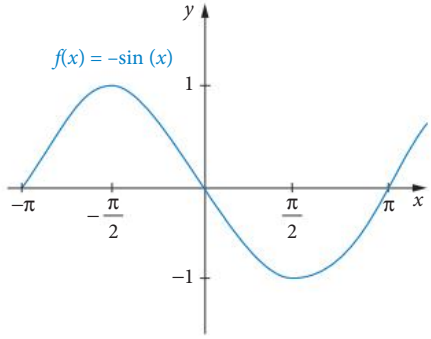


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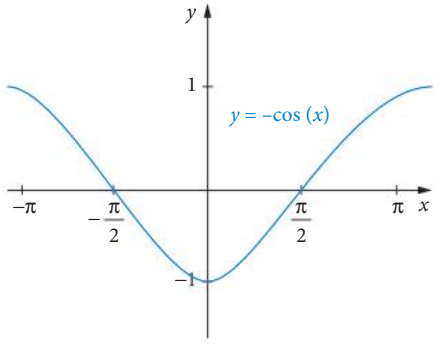
c



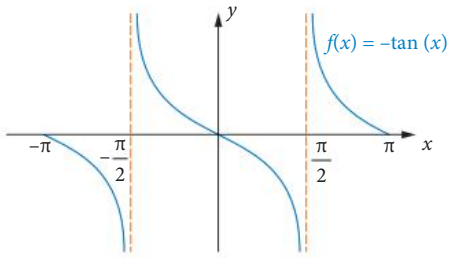

6 a



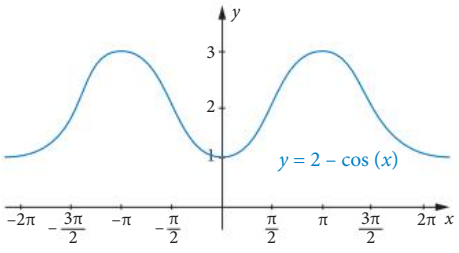

b



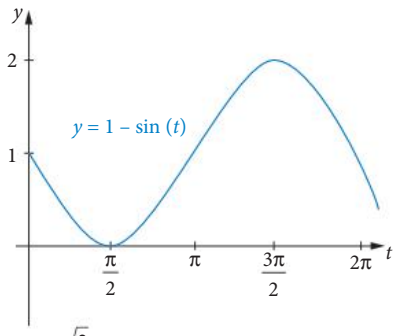

c




7




8 a



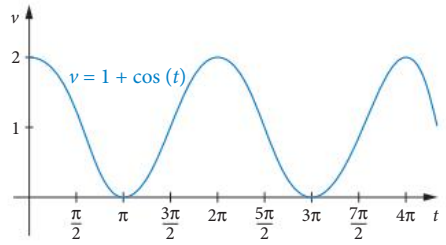

b $y = 1 - \frac{\sqrt{3}}{2}$

c i $t = 0, \pi, 2\pi$ ii $y = 1.96$

9 a i 1 ii $1\frac{1}{2}$ iii $1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}}$

iv 0 v 1

b



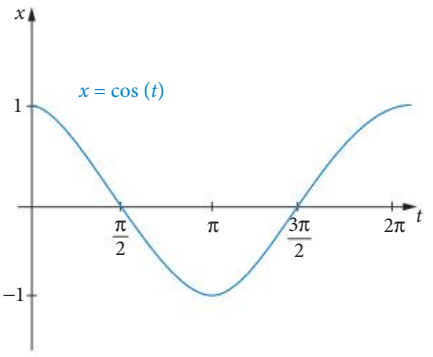

c $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ s

10 a 0.54 mm

b i $\frac{\sqrt{2}}{2}$ mm ii $\frac{\sqrt{3}}{2}$ mm

c 1 mm

d




e i $\frac{\pi}{2}, \frac{3\pi}{2}$ s ii 0, π, 2π s

f 0, π, 2π, 3π, … or $n\pi$ s where $n = 0, 1, 2, \ldots$

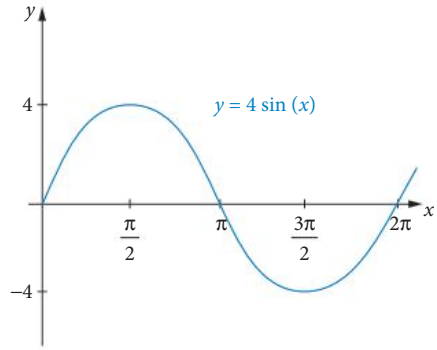
## INVESTIGATION: $y = a \sin(x)$ AND $y = a \cos(x)$

The value of $a$ stretches or compresses the graph vertically, and turns it upside down if $a$ is negative. $a$ has the same effect on tangent, although it is more difficult to see.

## 6.02

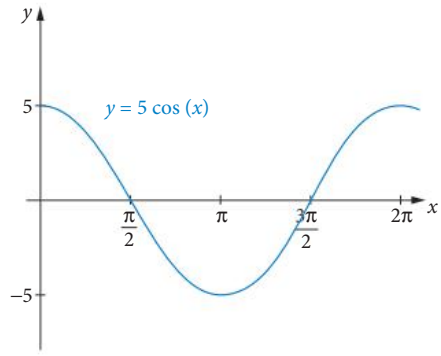
**1** **a** 5 **b** 9 **c** 4 **d** 1 **e** 2
**f** 6 **g** 8 **h** 11 **i** 4 **j** 1

**2** **a**



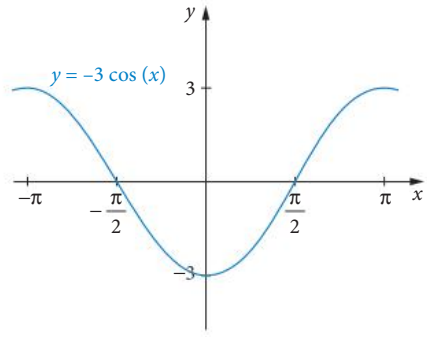

Amplitude = 4

**b**



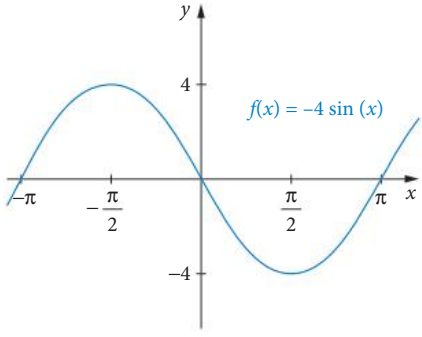

Amplitude = 5

**3** **a**



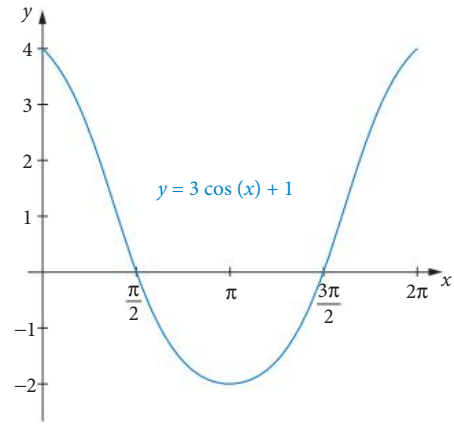

Amplitude = 3

**b**



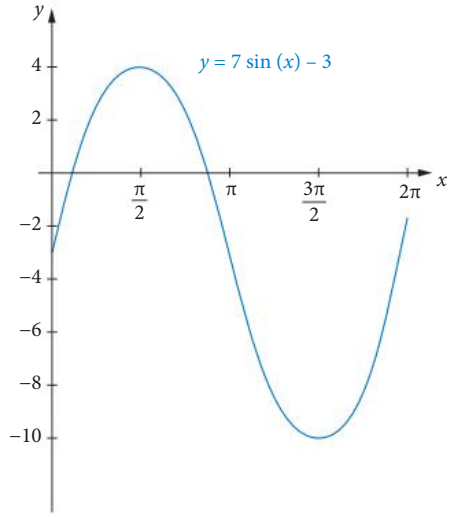

Amplitude = 4

**4** **a**




Amplitude = 3

**b**



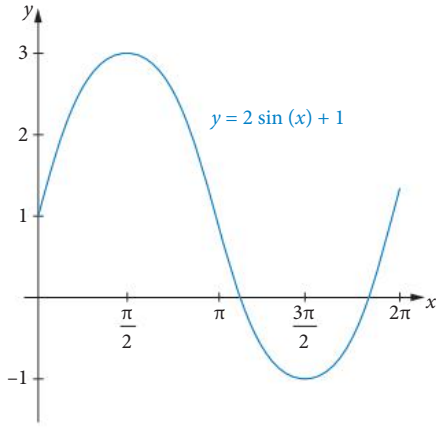

Amplitude = 7

**5** Amplitude 6, period 300

**6** Amplitude 1, period 2

**7** **a** $a$ **b** $a = 2, b = 1, y = 2 \sin(x) + 1$

**c**




**8** **a** 4° **b** 24 hours **c** $a = 4, b = 24$

**9** 5, 2π

**10** **a** $y = 5 \cos(x) + 10$ **b** $y = -2 \sin(x) + 1$


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## INVESTIGATION: $y = \sin(bx)$, $y = \cos(bx)$ AND $y = \tan(bx)$

The value of $a$ compresses or stretches the graph horizontally and if $b$ is negative reflects it in the $y$-axis. The period changes to $\frac{2\pi}{b}$ for sin and cos and to $\frac{\pi}{2}$ for tan.

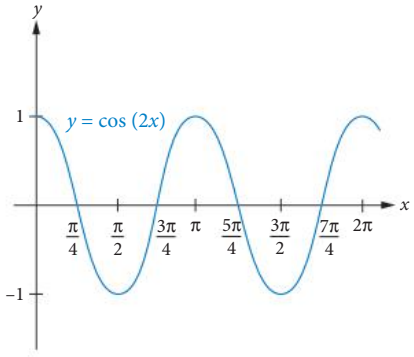
### 6.03

**1** **a** $\frac{\pi}{3}$ **b** $\frac{2\pi}{9}$ **c** $\pi$ **d** $\frac{\pi}{8}$

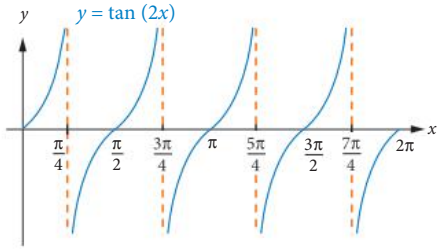
**e** $\pi$ **f** $\frac{2\pi}{7}$ **g** $3\pi$ **h** $10\pi$

**i** $8\pi$ **j** $2$ **k** $8$ **l** $\frac{1}{3}$

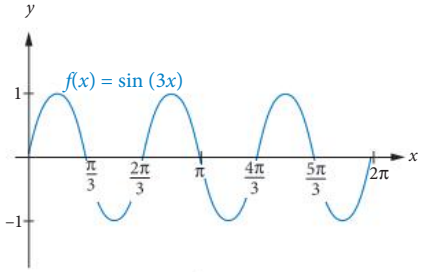
**2** **a**



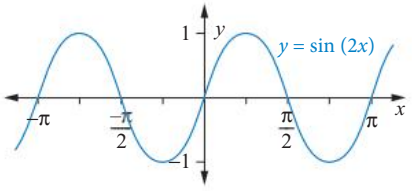

**b**



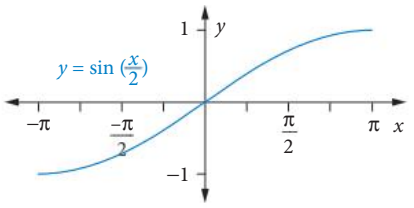

**c**



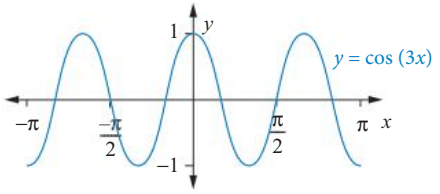

**3** **a**



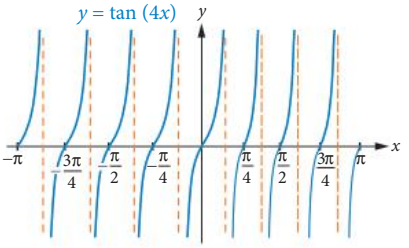

**b**



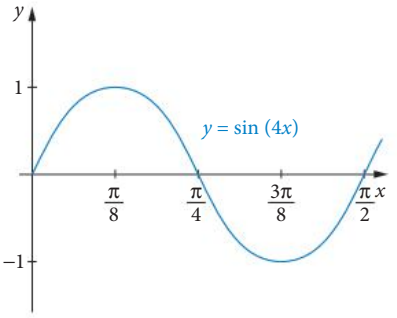

**c**



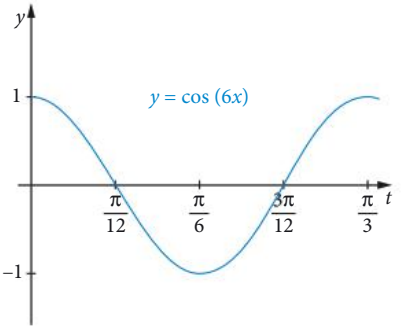

**d**



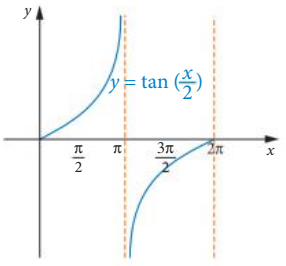

**4** **a**




**b**




**c**



**5** **a** Period $\frac{2\pi}{5}$, amplitude 4

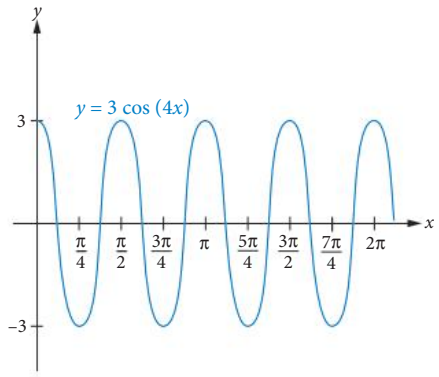
**b** Period $\frac{2\pi}{3}$, amplitude 2

**c** Period $4\pi$, amplitude 8

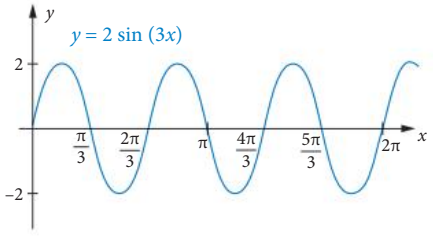
**d** Period $\frac{\pi}{2}$, no amplitude

**e** Period $\frac{2\pi}{11}$, amplitude 1

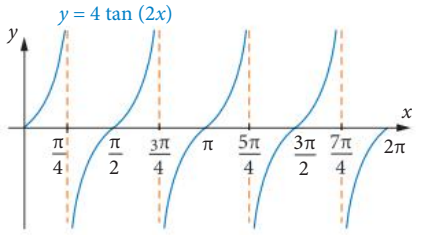
**6** **a**



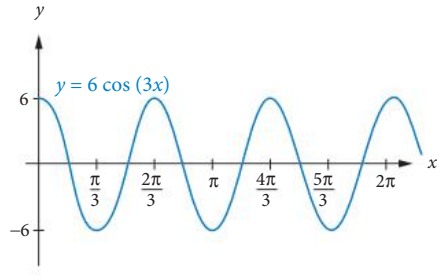

**b**



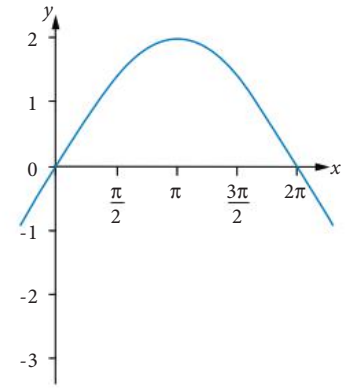

**c**



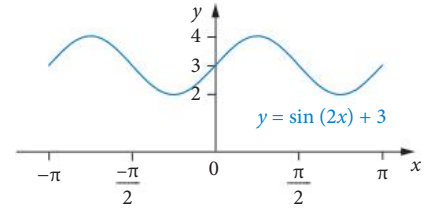

**d**



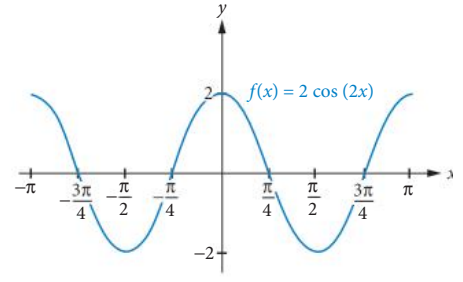

**e**



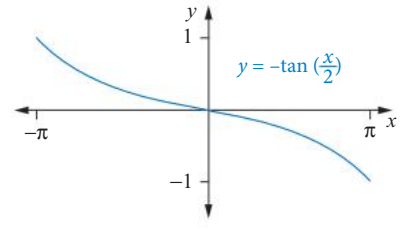

**7** **a**



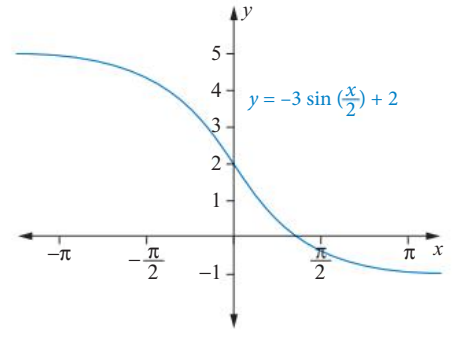

**b**




**c**



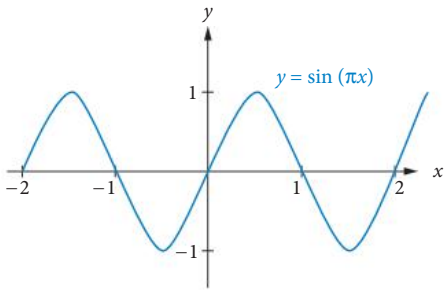

**d**



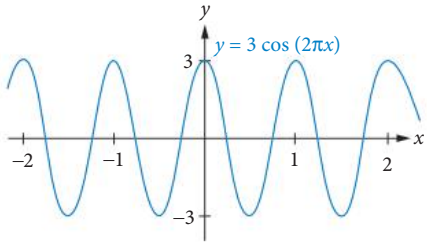


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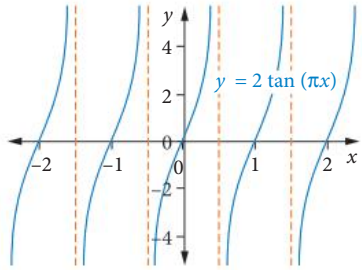
8 a



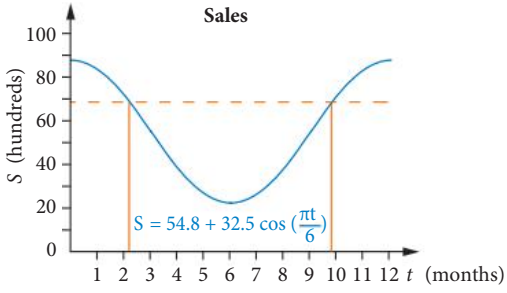

b




c

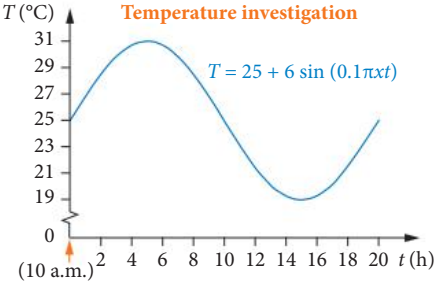



9 a




b Jan, Feb, Oct, Nov and Dec

10 a




b 31°C, 19°C C 3 p.m., 1 a.m.

d i 11:05 a.m., 6:55 p.m., ...

ii 11:08 p.m., 2:52 a.m., ...

e Because the period was not 24 hours

## INVESTIGATION: $y = \sin(x + c)$, $y = \cos(x + c)$ AND $y = \tan(x + c)$

For $c$ positive, the graph moves left and for $c$ negative it moves right.

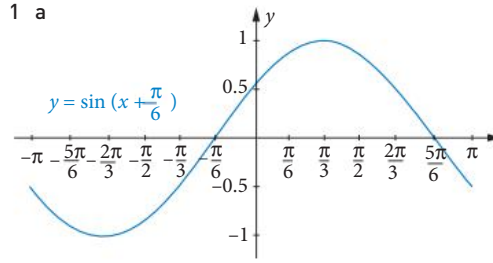
$\sin(x)$ and $\cos(x + \frac{\pi}{2})$ are exactly the same.

## INVESTIGATION: PHASE CHANGE

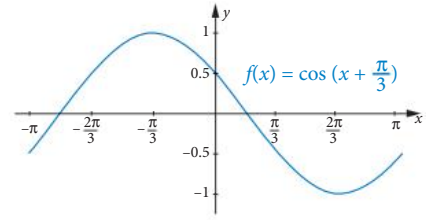
The phase change of $y = \sin(2x + \frac{\pi}{2})$ is not $\frac{\pi}{2}$ because the phase change is the movement from $f(x)$ to $f(x + c)$, which would be $\sin(2(x + \frac{\pi}{4}))$, so the phase change is $\frac{\pi}{4}$. For any trigonometric function with a coefficient for $x$, the same thing applies. The phase change of $\cos(5x + 3)$ would be 0.6, not 3.

## 6.04

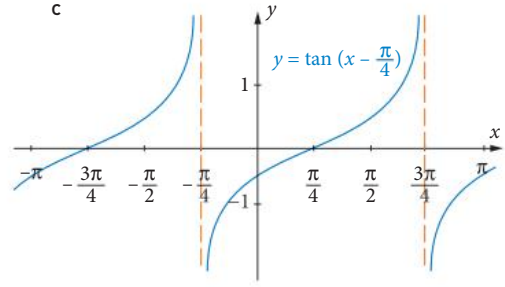
1 a



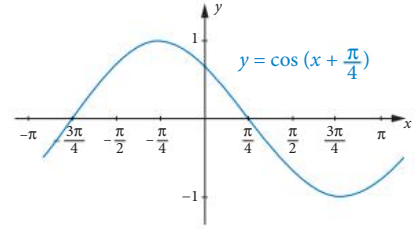

b



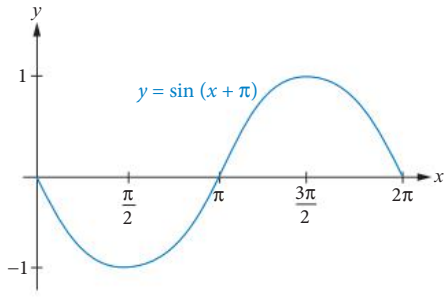

c



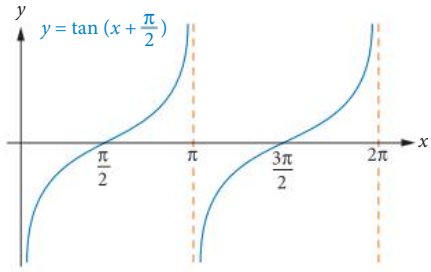

d



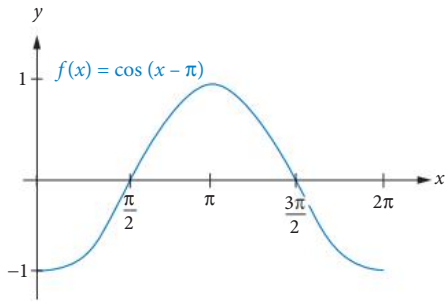
**2 a**



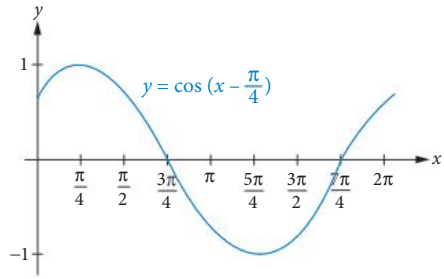

**b**



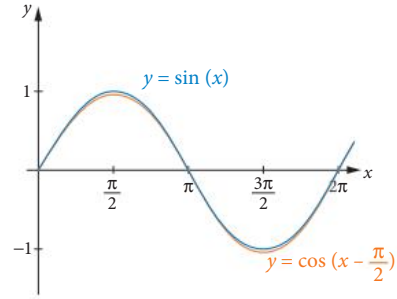

**c**




**d**



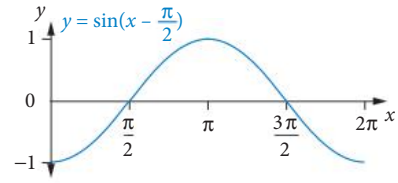

**3 a**



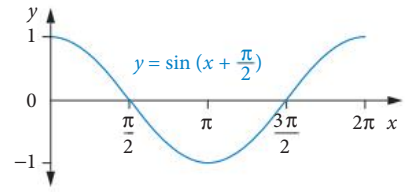

**b** Since the two graphs are identical,

$\sin x = \cos\left(x - \frac{\pi}{2}\right)$

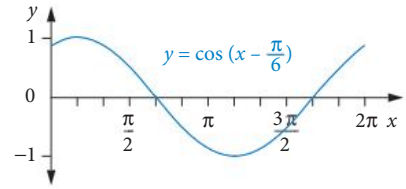
**4 a**



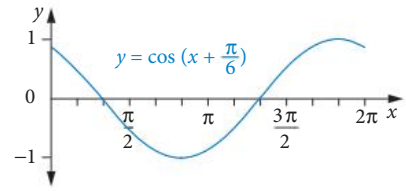

**b**



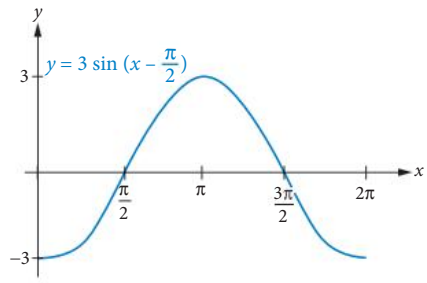

**c**



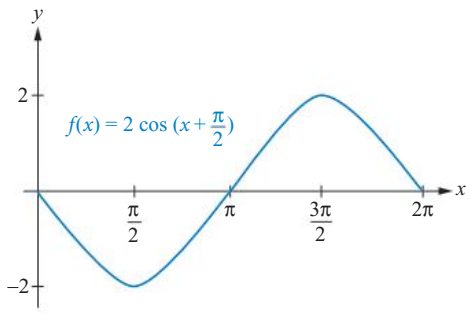

**d**




**5 a**



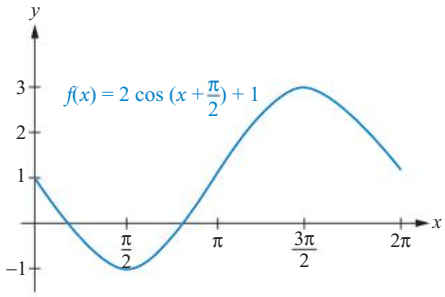

**b**



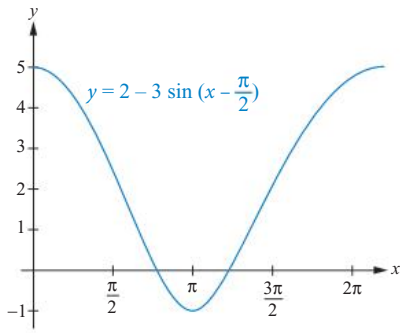


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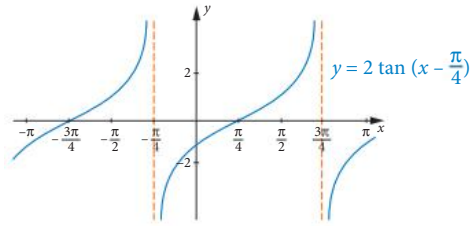
c



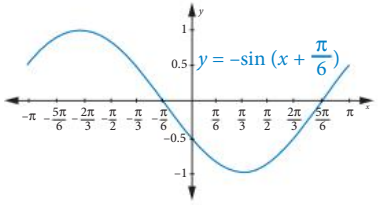

d



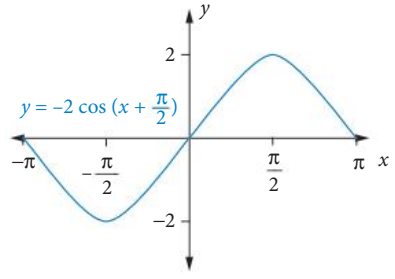

6 a



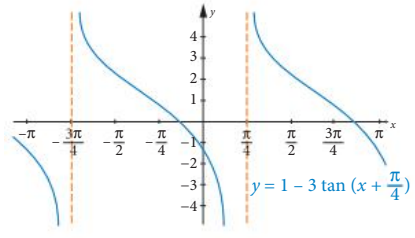

b



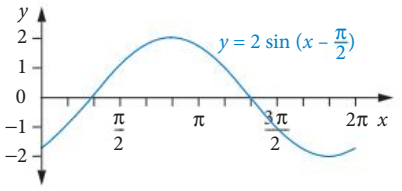

c



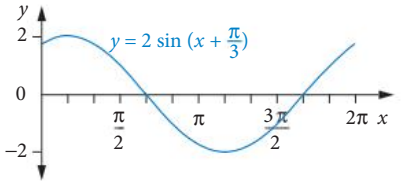

d



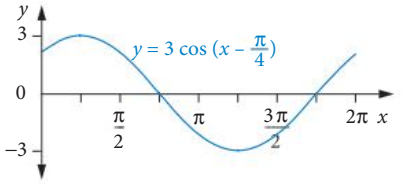

7 a



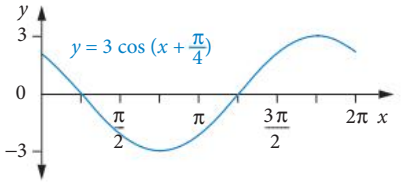

b



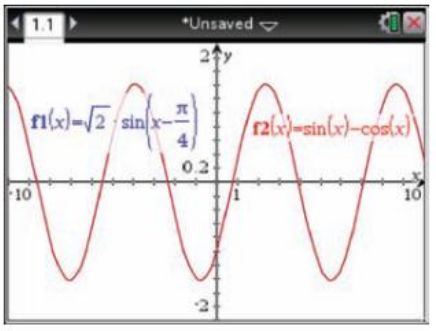

c




d



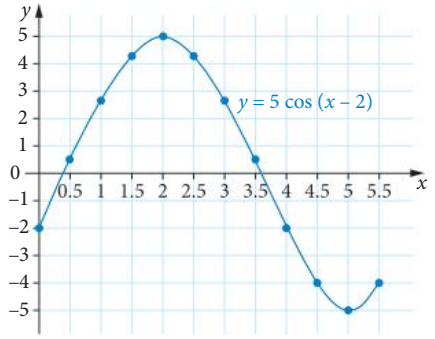

8 a and b




c $\sqrt{2}$

9 a amplitude 2 b $y = 2\sin\left(x - \frac{\pi}{3}\right)$

10 a




b $y = 5\cos(x - 2)$

## 6.05

**1** **a** $\sin(3\alpha)\cos(\beta) + \cos(3\alpha)\sin(\beta)$

**b** $\cos(5m)\cos(3n) + \sin(5m)\sin(3n)$

**c** $\dfrac{\tan(\theta)-\tan(3\psi)}{1-\tan(\theta)\tan(3\psi)}$

**d** $\cos(x)\cos(7y) - \sin(x)\sin(7y)$

**e** $\dfrac{\tan(34^\circ)+\tan(y)}{1+\tan(34^\circ)\tan(y)}$

**f** $\sin(2\theta)\cos(3\delta) - \cos(2\theta)\sin(3\delta)$

**2** **a** $\sin(a+b)$ **b** $\tan(65^\circ)$ **c** $\cos(55^\circ)$

**d** $\sin(2x+3y)$ **e** $\tan(2\theta)$ **f** $\sin(32^\circ)$

**g** $2\sin(a)\cos(b)$ **h** $2\cos(x)\sin(y)$

**i** $-2\sin(x)\sin(y)$ **j** $2\cos(m)\cos(n)$

**3** $\tan(2x)$

**4** **a** $\dfrac{\sqrt{3}+1}{2\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ **b** $\dfrac{1+\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

**c** $\dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \sqrt{3}+2$ **d** $\dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

**e** $\dfrac{\sqrt{3}}{3}$ **f** $\dfrac{\sqrt{3}}{2}$

**g** $1$ **h** $\dfrac{\sqrt{3}}{2}$

**i** $\dfrac{\sqrt{2}}{2}$ **j** $-\dfrac{\sqrt{2}}{2}$

**5** **a** $\dfrac{6+\sqrt{35}}{12}$ **b** $\dfrac{3\sqrt{5}+2\sqrt{7}}{12}$

**c** $\dfrac{6+\sqrt{35}}{3\sqrt{5}-2\sqrt{7}} = \dfrac{32\sqrt{5}+27\sqrt{7}}{17}$

**6** **a** $-\dfrac{4}{5}$ **b** $-\dfrac{12}{13}$ **c** $-\dfrac{33}{65}$ **d** $-\dfrac{12}{5}$ **e** $3\dfrac{15}{16}$

**7** **a** $2\sin(x)\cos(x)$ **b** $\dfrac{2\tan(x)}{1-\tan^2(x)}$

**8** $\cos\left[\left(\dfrac{\pi}{2}-x\right)-y\right]$

$= \cos\left(\dfrac{\pi}{2}-x\right)\cos(y) + \sin\left(\dfrac{\pi}{2}-x\right)\sin(y)$

$\cos\left[\dfrac{\pi}{2}-(x+y)\right] = \sin(x)\cos(y) + \cos(x)\sin(y)$

So $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

**9** **a** $2\sin(x)\cos(y)$ **b** $-2\sin(x)\sin(y)$

**c** $2\cos(x)\sin(y)$

**d** $\cos(x)\cos(y) - \sin(x)\sin(y) + \sin(x)\cos(y) - \cos(x)\sin(y)$

**10** $3\sin(x) - 4\sin^3(x)$

## 6.06

**1** **a** $\theta = 20^\circ 29', 159^\circ 31'$ **b** $\theta = 120^\circ, 240^\circ$

**c** $\theta = 135^\circ, 315^\circ$ **d** $\theta = 60^\circ, 120^\circ$

**e** $\theta = 150^\circ, 330^\circ$ **f** $\theta = 43^\circ 57', 316^\circ 03'$

**2** **a** $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **b** $\dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ **c** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

**d** $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$ **e** $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ **f** $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

**3** **a** $x = 0, 4.5$ **b** $x = 1.52$ **c** $x = 0, 0.95$

**d** $x = 0.79, 3.93$ **e** $x = 0$

**4** **a** $\theta = 45^\circ, 225^\circ$ **b** $\theta = 30^\circ, 150^\circ$

**c** $\theta = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ **d** $\theta = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**e** $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **f** $120^\circ, 240^\circ$

**5** **a** $x = 0, 2\pi$ **b** $x = 0, \pi, 2\pi$ **c** $x = \dfrac{3\pi}{2}$

**d** $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ **e** $x = \dfrac{\pi}{2}$

**6** **a** $\theta = \dfrac{\pi}{3}, -\dfrac{2\pi}{3}$ **b** $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

**c** $\theta = \pm\dfrac{\pi}{6}, \pm\dfrac{5\pi}{6}$ **d** $\theta = -\dfrac{\pi}{2}$

**e** $-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

**7** **a** $x = \pm 45^\circ, \pm 135^\circ$ **b** $x = 0, \pm 180^\circ, 30^\circ, 150^\circ$

**c** $x = 22.5^\circ, 112.5^\circ, -67.5^\circ, -157.5^\circ$

**d** $x = \pm 75^\circ, \pm 105^\circ$

**8** $x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

**9** $x = 0.47, -1.51$

**10** **a** $x = 90^\circ, 270^\circ, 60^\circ, 300^\circ$

**b** $\theta = \dfrac{\pi}{4}, -\dfrac{3\pi}{4}, 1.11, -2.03$

## 6.07

**1** **a** Period 12 months, amplitude 1.5

**b** 5.30 p.m.

**2** **a** 0.8 s **b** 75/min

**c** Reciprocally **d** 13 min 20 s

**3** **a** 0.6 ms **b** 1.5 mN

**c** 0.04 ms, 0.64 ms, 1.24 ms, 1.84 ms, 2.44 ms

**d** About 16 667 times

**4** **a** $y = 7\sin(2\pi x)$ **b** $y = 3\cos(2\pi x)$

**c** $y = 10\sin\left(\dfrac{2\pi x}{3}\right)$ **d** $y = 5\cos\left(\dfrac{\pi x}{4}\right)$

**5** **a** $y = 2 + 3\sin(\pi x)$ **b** $y = 5 + 2\cos\left(\dfrac{3\pi x}{2}\right)$

**6** **a** 10 cm

**b** **i** 9.8 cm **ii** 5.7 cm **iii** 9.1 cm

**c** **i** $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$ s

**ii** $t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \ldots$ s

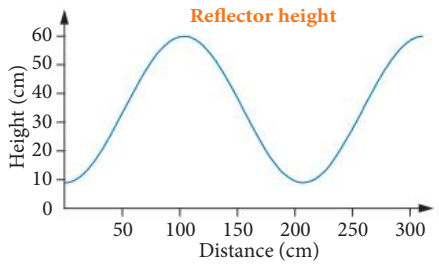
**iii** $t = 0, \pi, 2\pi, \ldots$ s

**d** 1.1 s


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**7** **a** 8 cm **b** 58 cm **c** $66\pi = 207.3$ cm

**d**



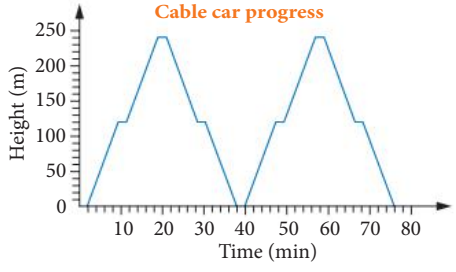

**e** About 52 cm **f** About 60 cm

**8** **a** 5 m **b** 5 m **c** 9 a.m. **d** 3 a.m., 3 p.m.

**9** **a** 4 times (if evenly spaced on cable)

**b** 38 minutes

**c**




**d** Two cars do 19 trips; and 2 do 18 trips each.

**e** 740 people **f** About 7 minutes

**10** $d = 4+0.9\sin\left(\frac{6\pi t}{37}\right)$. The boat with a draught of 3.5 m can use the pier for 8.5 hours around high tide and the one with a draught of 4.5 m for 3.35 hours.

## CHAPTER 6 REVIEW

**1** B

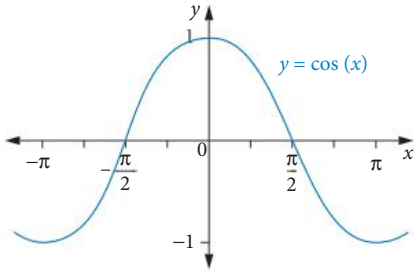
**2** D

**3** D

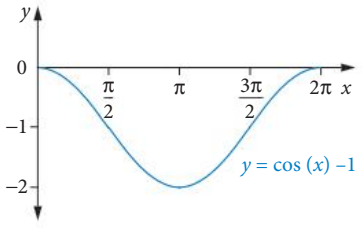
**4** A

**5** **a** $-1$ **b** 1 **c** 0 **d** 0 **e** 0

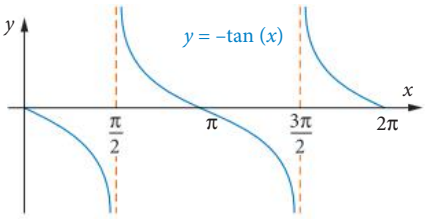
**6**




**7** **a**

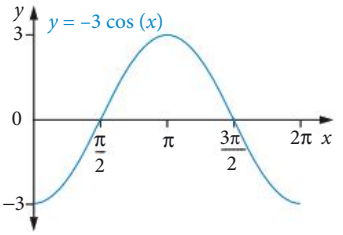



**b**




**8** **a** 7 **b** 4

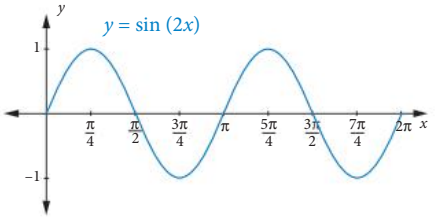
**9**



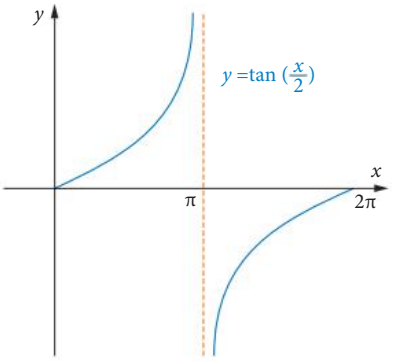

**10** Amplitude 4, period 20.

**11** **a** $\frac{\pi}{4}$ **b** $10\pi$ **c** $\frac{2}{3}$

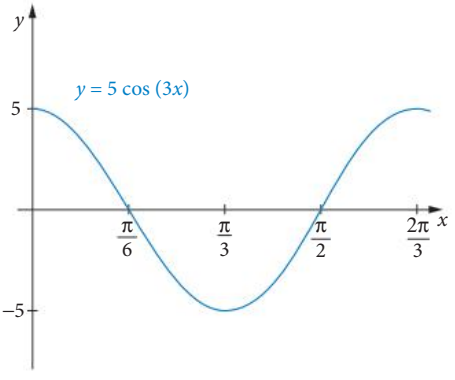
**12** **a**



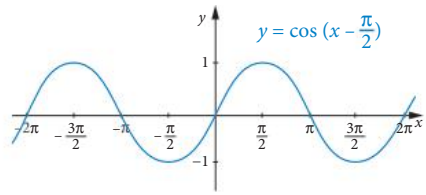

**b**




**13** Period $\frac{2\pi}{3}$, amplitude 5

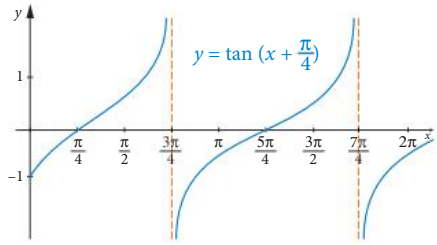


**14** a

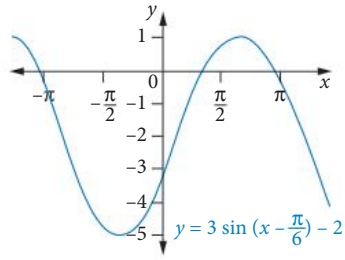



b $y = \sin(x)$

**15**




**16**

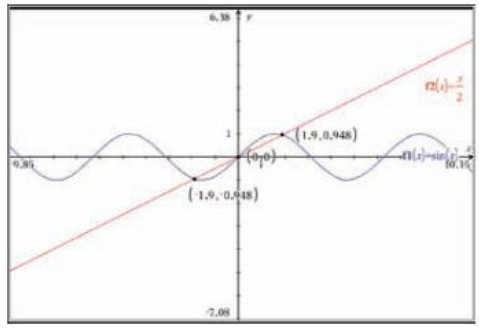



**17** a $\sin(2a + 3b)$ b $\tan(2\theta - \beta)$
c $\cos(3x + 4y)$

**18** a $\frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$ b $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

**19** $\sin(2x) = \sin(x + x)$
$= \sin(x)\cos(x) + \cos(x)\sin(x)$
$= 2\sin(x)\cos(x)$

**20**




$x = 0, \pm 1.9$

**21** a $x = \frac{\pi}{4}, \frac{7\pi}{4}$ b $x = \frac{2\pi}{3}, \frac{5\pi}{3}$ c 3.521, 5.904

**22** $\alpha = 45°, 135°, 225°, 315°$

**23** a About 1300 b i 1600 ii 1020
c Amplitude 300, period 10 years.

**24** $y = 3\sin\left(\frac{\pi x}{4}\right)$

**25** About 0.352 s, 0.648 s and 1.352 s.

**26** a 15°C, January
b 10.5°C, April and October.
c August to December

**27** a 12 h
b 12 a.m. – 3 a.m., 9 a.m. – 3 p.m., 9 p.m. – 12 a.m.
c 4 a.m. – 8 a.m., 4 p.m. – 8 p.m.

**28** $d = 15 + 10\cos\left(\frac{\pi t}{30}\right)$; the tip is 10 cm above the shelf 48 times a day (20 past and 20 to the hour).

## MIXED REVISION 2

**Multiple choice**

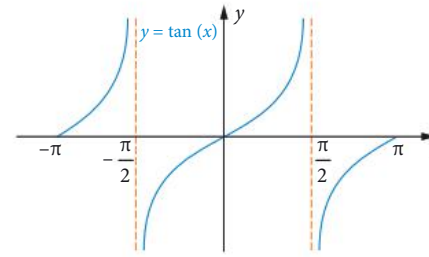
**1** D
**2** D
**3** D
**4** D
**5** E
**6** B
**7** A
**8** D
**9** C

**Short answer**

**1** $x - 3y - 2 = 0$

**2** $\frac{31}{45}$

**3**




**4** Domain is $\{x: \text{all real } x \neq 1\frac{2}{3}\}$, range is $\{y: \text{all real } y \neq 0\}$

**5** 0.0512%

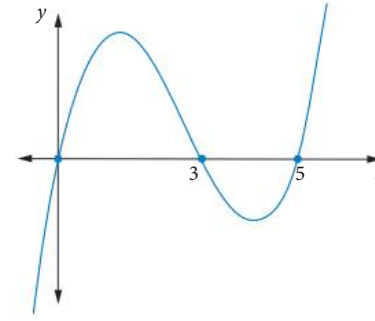
**6** $x = 60°, 120°, 240°, 300°$

**Application**

**1** a 0.051 b Yes (close to 1 : 20)

**2** a 6 cm b $d = t(t - 3)(t - 5)$

c



d 0, 3 and 5 s e −5 cm/s

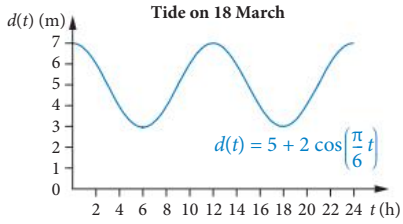

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**3** **a** 2 **b** $\frac{2}{5}$ **c** $\frac{1}{4}$

**4** **a** 11.1% **b** 41.2% **c** 18.8% **d** 25.8%

**e** The probability of high achievement among instrumental music students is about 1.6 times that of student not learning an instrument.

**5** **a**




**b** 1:30 a.m.

**c** 5:30 a.m., 9:30 a.m., 5:30 p.m. 9:30 p.m.

**d** 6 hours

**6** **a** 14 **b** Midnight **c** 24 hours

## INVESTIGATION: FORENSIC SCIENCE

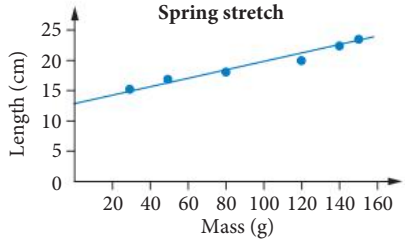
The lines of best fit on the graphs will be straight lines, but they are scattered, so estimations of height and weight based on bone measurements should be stated as ranges, rather than just single numbers.

### 7.01

**1** A

**2** B

**3** **a**

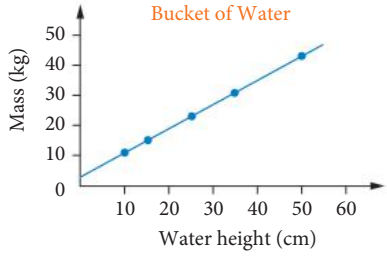



**b** The graph is a straight line, within experimental error.

**c** About 0.07 cm/g

**d** About 13 cm, the unstretched length.

**4** **a**




**b** Straight line, not through origin.

**c** 0.8 kg/cm **d** The mass of the bucket.

**5** Probably 90 km/h rounded to the nearest 10 min (91.09 km/h)

**6** 4 m/s$^2$

**7** 3.5 km/h

**8** 75 km/h

**9** 32 km/h.

**10** **a** About 88 000 km **b** About 3700 km/h

**c** Because of the motion of the Earth around the Sun, it has to orbit about 30° further than a complete orbit to appear at the same place in the sky.

**11** **a** A **b** 10 km/h

**c** 15 minutes **d** About 6.7 km/h

**e** About 8.6 km/h

**12** **a** 266 407 km **b** 11 100 km/h, 3083 m/s

**13** **a** 13.3 km/h **b** The first leg

**c** The second leg **d** 8 km/h

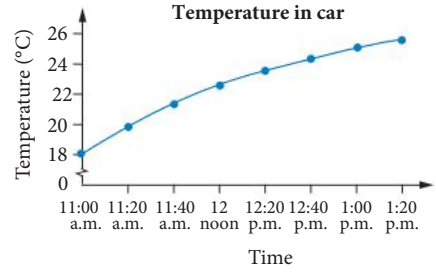
**14** **a** 400 km/h **b** 169 km/h **c** 4 h

**d** Average speed = 300 km/h, average velocity = 0

**15** **a** About 4.8 cm/day **b** about 2.3 cm/day

**c** About 1 cm/day and the rate of growth gradually decreases.

**16** **a**




**b** The rate of change of temperature gradually decreases.

**c** 3.2°/h

**d** It will probably be lowering the temperature about 1°C/h, so it won't be much use.

**e** No

**17** **a** 0.5 m/s **b** 1.5 m/s **c** 2.5 m/s **d** 1.5 m/s

**18** Let the distance each way be $x$. Then the total time for the trip $= \frac{x}{v_1} + \frac{x}{v_2} = \frac{x(v_2 + v_1)}{v_1 v_2}$. Total distance $= 2x$. Average speed $= 2x \div \frac{x(v_2 + v_1)}{v_1 v_2} = \frac{2x}{1} \times \frac{v_1 v_2}{x(v_2 + v_1)} = \frac{2v_1 v_2}{v_1 + v_2}$

### 7.02

**1** 11 m/s$^2$, −8 m/s$^2$, −20 m/s$^2$

**2** 10.5 mL/day, 4 mL/day

**3** 110°C/min, 0°C/min, −100°C/min

**4** 15 m/s, 5 m/s, −4 m/s, −14 m/s

**5** 12 km/h/s, 40 km/h/s

### 7.03

**1** **a** 6 **b** 7 **c** 11 **d** 15 **e** 13

**2** **a** 9 **b** 6 **c** 8 **d** 50 **e** 29

**3** **a** 12 **b** 15 **c** 14 **d** −5 **e** 3

**4** **a** 26 **b** 47 **c** 41 **d** −4 **e** −25

**5** **a** 10 **b** 42 **c** 26 **d** 135 **e** 143

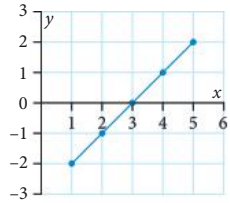
**6** **a** 3 **b** −7

**7** **a** 7 **b** 25

**8** **a** About −0.333 **b** About −0.0357

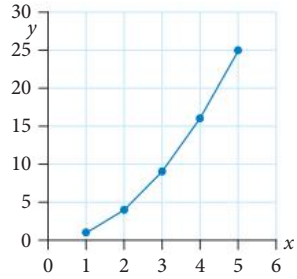
**9** **a** About 0.414 **b** About 0.293

**10** (1, −2), (2, −1), (3, 0), (4, 1), (5, 2)



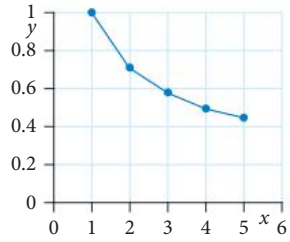

It is the straight line $y = x - 3$.

**11** (1, 1), (2, 4, (3, 9), (4, 16), (5, 25)




It is the parabola $y = x^2$.

**12** $(1, 1), \left(2, \frac{\sqrt{2}}{2}\right), \left(3, \frac{\sqrt{3}}{3}\right), \left(4, \frac{1}{2}\right), \left(5, \frac{\sqrt{5}}{5}\right)$




It is the function $y = \frac{1}{\sqrt{x}}$.

## 7.04

**1** **a** 6 **b** 2 **c** 0 **d** −2 **e** −4

**2** **a** −10 **b** −2 **c** 2 **d** 4 **e** 6

**3** **a** −4 **b** $-2\frac{1}{4}$ **c** −1 **d** $-\frac{1}{4}$ **e** $-\frac{1}{9}$

**4** **a** −4 **b** −1 **c** 2 **d** 3 **e** 5

**5** **a** 7 **b** −4 **c** −9 **d** −8 **e** −1

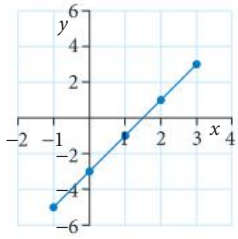
**6** **a** 3 **b** −5

**7** **a** 8 **b** 24

**8** **a** −1 **b** −0.04

**9** **a** 0.5 **b** 0.25

**10** (−1, −5), (0, −3), (1, −1), (2, 1), (3, 3)




It is the straight line $y = 2x - 3$.

# INVESTIGATION: VARIATION OF CALCULATION OF RATES OF CHANGE

In general, the rates of change for secants around the point are more accurate for the same value of $\Delta x$, but as $\Delta x$ is reduced they all become more accurate.

## 7.05

Note: answers have been rounded.

**1** **a** 6.1 **b** −3.9 **c** 3.1
**d** 1.9 **e** 14.1

**2** **a** 29 666 **b** 1.899 **c** 11.151
**d** 5933 **e** 0.0759

**3** **a** 0.167 **b** 0.174 **c** 0.180
**d** 0.181 **e** 0.182

**4** **a** −266 **b** −66.35 **c** −113.5
**d** −14.91 **e** −0.001

**5** **a** −0.062 81 **b** −0.0402 **c** −0.0157
**d** −0.9804 **e** −1.0204

**6** **a** −0.0774 **b** −0.0690 **c** −0.0637
**d** −0.0631 **e** −0.0626

**7** **a** −3.0404 **b** −0.002 506 **c** 0.7498
**d** −10.43 **e** 0.9375

**8** **a** 0.3125 **b** 0.2500 **c** 2.0004
**d** 3.9075 **e** 2041

**9** **a** 2.8285 **b** 2.8284 **c** 2.8284
**d** 2.8284 **e** 2.8284

**10** **a** 2.500 78 **b** 12.9908 **c** 20.0004
**d** 0.633 692 **e** 0.225 358

**11** **a** −22.1851 **b** 1.3864 **c** 0
**d** 181 810 **e** −181 810

**12** **a** 0.176 091 **b** 0.1743 **c** 0.173 741
**d** 0.173 724 **e** 0.173 718

**13** **a** 7.25 **b** 6.53 **c** 6.88
**d** The "around" value is between the others.

**14** **a** 2.996 **b** 2.957 **c** 2.977
**d** The "around" value is between the others and they are close in value.

**15** **a** −0.245 **b** −0.255 **c** −0.250
**d** The "around" value is between the others and they are close in value.

**16** **a** undefined **b** undefined **c** 100 **d** 40 000
**e** The function is undefined at $x = 4$, so the rate of change at $x = 4$ makes no sense.

**17** **a** 4.016 **b** 100.016 **c** 10 000
**d** The function is undefined at $x = 2$, so the rate of change at $x = 2$ makes no sense.

**18** ±3

## 7.06

**1** $g(x) = 6x - 4$

**2** $g(x) = 2x + 6$

**3** $g(x) = 10x + 7$

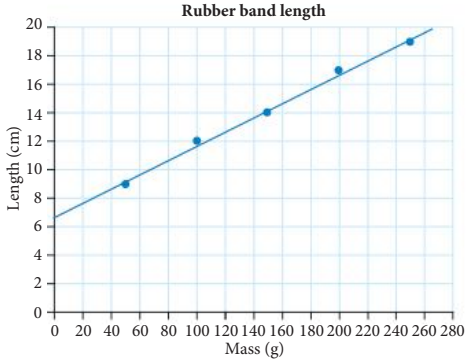
**4** $g(x) = 12x^2 + 16x + 9$


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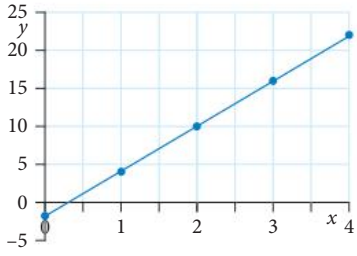
5 $g(x) = 9x^2 - 10x - 7$
6 $g(x) = 28x^3 - 3x^2 + 6x - 2$
7 $g(x) = 2x$
8 $g(x) = 3x^2$
9 $g(x) = 4x^3$
10 $g(x) = 5x^4$

## CHAPTER 7 REVIEW

1 B
2 B
3 C
4 E
5 C
6 a




b 6.7 cm – the length of the unstretched rubber band.
7 a 1.5 ML/h b 1 a.m.
8 5.26, 4.90, 4.76, 4.85 and 4.94 m/s
9 About −26 cm/s, −5 cm/s and 9 cm/s.
10 a 3 b 18
11 −12
12 About 50
13 11
14 About 62.41
15 About 0.1667
16 About −0.3466
17 a 40 km/h, 50 km/h, 65.625 km/h
b 52.5 km/h c 0
18 −2, 4, 10, 16, 22




It is a straight line with equation $y = 6x - 2$
19 a About −0.2513 b About −0.2111
c About −0.2028
d It is appears as if it is getting closer to about −0.2, but lower values of $\delta x$ suggest −0.2019.
20 $g(x) = 8x - 5$

## INVESTIGATION: HAILSTONE SEQUENCES

You can use a simple Excel spreadsheet or CAS calculator spreadsheet to investigate the sequences. Put the starting number in cell A1.
In A2, put the formula =IF(MOD(A1,2)=0,A1/2,3*A1+1). Copy the formula down the column. Try starting with 36, 27 or 71.
At present no-one knows if there could be a sequence that doesn't get to 1.

### 8.01

1 a 1, 2, 3, 4, 5 b 10, 17, 31, 59, 115
c −3, 5, 2, 7, 9 d −10, 5, 25, 15, −35
2 a 3, 9, 19, 33 b 7, 5, $4\frac{1}{3}$, 4
c 4, $9\frac{1}{2}$, $27\frac{1}{3}$, $81\frac{1}{4}$ d $3^1, 3^3, 3^5, 3^7$
e −1, 2, −3, 4 f $2^1, 2^3, 2^{10}, 2^{41}$
3 a 20 b −16 c 27 d 55
4 a

| Square number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Number of rice grains | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |

b 23
c 2048 pieces of rice will NOT fit in a square on a regular chess board.
5 a

| Number of folds | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Number of creases | 1 | 3 | 7 | 15 | 31 | 63 | 127 |

b $2^{10} - 1 = 1023$

### 8.02

1 a yes, $d = 3$ b no
c yes, $d = 3$ d no
e yes, $d = \frac{1}{3}$ f no
2 a 5, 8, 11, 14, … b −13, −7, −1, 5, …
c $3, 1\frac{3}{4}, \frac{1}{2}, -\frac{3}{4}, \cdots$
d $m, m + 2n, m + 4n, m + 6n$, ...
e 4, 9, 14, 19, … f 1, 4, 7, 10, …
3 a −4 b 7 c 1.6
d $2 + 2\sqrt{3}$ e $2x - 2$ f $2 - \pi$
4 a $r = 1$ b $m = -2$ c $t = 3$
d $m = 2\sqrt{3}$ e $n = 3\sqrt{2}$
5 There are 26 rows in the pile.
6 a 210 bulbs in 34th row
b $n > 12$, therefore 13th row has more than 100 bulbs.
7 a 5.2 km b 6.5 km
c yes, $d = 0.1$ km d 29 December
8 a There are 25 cans in the 11th row.
b There are 16 rows altogether.

## 8.03

**1** **a** 3, 8, 13, 18. **b** 6, 3, 0, –3.
**c** –5, –7, –9, –11. **d** $x, x+y, x+2y, x+3y$.
**e** $-15, -15+3k, -15+6k, -15+9k$.
**2** **a** 37 **b** –41 **c** –2.3
**d** 32 **e** –75
**3** **a** 43 **b** $a + 117d$ **c** 26
**d** –76 **e** 228
**4** **a** $t_n = 2n - 5$ **b** $t_n = 20 - n$
**c** $t_n = n + 4$ **d** $t_n = 4n - 24$
**e** $t_n = 5n + x - 5$
**5** **a** $n = 20$ **b** $n = 32$ **c** $n = 61$
**d** $n = 71$ **e** $n = 33$
**6** **a** $n = 16$ **b** $n = 33$ **c** $n = 88$
**d** $n = 13$ **e** $n = 30$
**7** **a** $t_n = 5 + 2n$ **b** $t_n = 3 + 4n$
**c** $t_n = 2 - 5n$ **d** $t_n = 51 - 7n$
**e** $t_n = 2.5n + 0.5$ **f** $t_n = 3n - 7$
**g** $t_n = 46 - 6n$ **h** $t_n = 14 - 4n$
**i** $t_n = 20 - 3n$ **j** $t_n = 9\frac{1}{2} - 1\frac{1}{2}n$
**8** **a** No, $n = 81\frac{2}{3}$ **b** $t_{20} = 65$
**c** **i** $n = 14$ **ii** $u_{14} = -4$
**9** 5 m, 3 m
**10** 14 L/h, 8 L

## 8.04

**1** **a** 28 875 **b** 3276 **c** –1419 **d** 6426
**e** 6604 **f** 598 **g** –2700 **h** 11 704
**i** –290 **j** 2751
**2** **a** 375 **b** 555 **c** 480
**d** –870 **e** 465
**3** **a** 21 terms **b** 11 terms **c** $n = 2$ or 11
**d** 8 or 13 terms **e** 47
**4** **a** $t_1 = 14, d = 4$ **b** $t_1 = -3, d = 5$
**c** $S_{40} = 3420$ **d** $S_{20} = 1010$
**5** Proof
**6** $S_{100} = 5050$, $S_{6s} = 816$, Numbers not divisible by 6 total is 4234.
**7** **a** 11 seconds **b** 605 m
**8** 203 kg, 5 kg
**9** **a** 96 rows **b** 6960 logs
**10** $2452

## 8.05

**1** **a** geometric, $r = 2$ **b** geometric, $r = 3$
**c** geometric, $r = -2$ **d** not geometric
**e** not geometric **f** geometric, $r = \sqrt{2}$
**g** not geometric
**2** **a** $r = 3, 27, 81$ **b** $r = -2, -24, 48$
**c** $r = 4, 32, 128$ **d** $r = -1, -13, 13$
**e** $r = \frac{1}{4}, 1, \frac{1}{4}$ **f** $r = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$
**3** **a** $x = \pm 10$ **b** $x = \pm 2$ **c** $x = 1\frac{1}{4}$ **d** $x = 9$
**e** $x = \frac{1}{5}$ **f** $x = 3$ **g** $x = -10\frac{1}{8}$ **h** $x = \pm 8$
**4** 23 972
**5** $6848.48
**6** $6821.15
**7** **a** About 318. **b** About 28 years.
**8** **a** 77.4% **b** ~ 13.5 years
**c** ~ 31.4 years
**9** **a** About 23.9 metres. **b** After about 5 years.
**10** $6805.83

## INVESTIGATION: TERMS OF A GP AS $N \to \infty$

When $r > 1$, the sequence keeps getting bigger: it diverges.
When $r < 1$ it keeps getting smaller (closer to 0): it converges.
When $r$ is negative it also alternates in sign.

## 8.06

**1** **a** 2560 **b** 1.8009… **c** –320
**d** 16 384 **e** $t_{50} = 11^{49}$ **f** 1 572 864
**g** 268.8 **h** $t_{16} = 2^{-13}$
**i** $t_n = 6 \times 3^{n-1}$ or $t_n = 2 \times 3^n$
**j** $t_9 = -3 \times 2^8 = -768$
**2** **a** $t_{10}$ **b** $t_6$ **c** $t_5$
**d** $t_{11}$ **e** $t_{13} = 2\,657\,205$
**3** **a** No; $t_4 = 1024$, $t_5 = 8192$
**b** Yes; $t_6 = 224$
**c** No; $t_8 = 546.75$
**4** **a** $a = 2, r = \pm 3$ **b** $t_8 = -839\,808$
**c** $r = \pm 4$, $t_7 = 36\,864$, even power makes ± positive
**d** $n = 7$; $t_6 = 1.6$, $t_7 = 0.32$
**5** **a** converges **b** diverges
**c** converges **d** diverges
**6** $50 363.40
**7** $6848.48
**8** 305.8 g
**9** 3715.87; 6%
**10** 62 500 $cm^3$. There is less than 1 $mm^3$ left after 81 minutes.

## 8.07

**1** **a** 5465 **b** 122.61 **c** 32 767
**d** 510 **e** 97 656.2
**2** 8.628
**3** 85.792
**4** 3577
**5** $n = 8$
**6** $n = 15$
**7** $n = 10$
**8** 75 or –25


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9 217 or −77
10 −56
11 $155\frac{55}{81}$
12 $101\frac{1}{27}$ metres
13 a $46 669.85 b $238 066.95
14 $11 501.93
15 $2511.86

## INVESTIGATION: INFINITE SUMS OF GPS

$40 + 20 + 10 + \ldots$: $S_5 = 77.5$, $S_8 = 79.6875$, $S_{50} \approx 80$
$2 + 4 + 8 + \ldots$: $S_5 = 62$, $S_8 = 510$, $S_{50} \approx 2.251 \times 10^{15}$
$90 - 30 + 10 - \ldots$: $S_5 = 6.777\ldots$, $S_8 = 6.748\ldots$, $S_{50} \approx 6.75$
$10 - 30 + 90 - \ldots$: $S_5 = 610$, $S_8 = -16\,400$, $S_{50} \approx -1.795 \times 10^{24}$
The ones with $-1 < r < 1$ converge and the others diverge.

## 8.08

1 a $r = \frac{1}{3}$, yes b $r = 2$, no
c $r = -\frac{1}{4}$, yes d $r = \frac{7}{6}$, no
e $r = \frac{2}{3}$, yes f $r = \frac{1}{5}$, yes
g $r = -6$, no h $r = -\frac{5}{6}$, yes
i $r = \frac{3}{2}$, no j $r = -\frac{2}{5}$, yes
2 a $S_\infty = 80$ b $S_\infty = 426\frac{2}{3}$
c $S_\infty = 66\frac{2}{3}$ d $S_\infty = 12$
e $S_\infty = \frac{7}{10}$ f $S_\infty = 54$
g $S_\infty = -10\frac{2}{7}$ h $S_\infty = \frac{9}{20}$
i $S_\infty = 48$ j $S_\infty = -\frac{16}{39}$
3 a i 7.9922 ii 8.0000
b 8
4 a $\frac{5}{4}$ b $\frac{2}{5}$ c $\frac{1}{48}$
d 3 e 5 f $-\frac{16}{3}$
5 $r = \frac{2}{5}$
6 $t_1 = 5\frac{4}{5}$
7 $r = \frac{1}{3}$
8 $r = \frac{1}{3}$, $t_1 = 6$ or $r = \frac{2}{3}$, $t_1 = 3$
9 $t_1 = 192$, $r = -\frac{1}{4}$, $S_\infty = 153\frac{3}{5}$
10 a $\frac{4}{9}$ b $\frac{7}{9}$ c $1\frac{2}{9}$ d $\frac{25}{99}$
e $2\frac{81}{99}$ f $\frac{19}{90}$ g $\frac{131}{990}$
11 $x = \frac{21}{32}$
12 a $-1 < k < 1$ b $S_\infty = -\frac{2}{5}$ c $k = \frac{3}{4}$
13 120 metres
14 20 centimetres

## CHAPTER 8 REVIEW

1 D
2 E
3 B
4 A & D
5 C
6 8, 20, 36, 56
7 a 28, 36, 44, 52
b $8 - 7x$, $6 - 9x$, $4 - 11x$, $2 - 13x$
c Not an AP
8 208
9 $t_{25}$
10 No
11 10, 15, 20, 25
12 588
13 35
14 450
15 −2: 24, −48, 96, −192
16 ± 18
17 $-5^{10}$
18 Yes
19 1 572 864
20 22 960
21 7
22 62 500
23 12 metres
24 $\frac{73}{90}$
25 a 57 344 b 20th month
c $3 670 013
26 a $27 million b $3870 million
27 About 1286 m

## INVESTIGATION: THE LIMIT OF $f(x) = \frac{x^2 - 9}{x - 3}$

1 a

| $x$ | 2 | 2.5 | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x) = \frac{x^2 - 9}{x - 3}$ | 5 | 5.5 | 5.9 | 5.99 | 5.999 | Un-defined | 6.001 | 6.01 | 6.1 | 6.5 | 7 |

b 3
c 3
d They are the same

2 The graph supports the same conclusion.
3 Yes, the limit is 3.

## 9.01

1 a 7 b −2 c 21
d 4 e 0 f 2.5
2 a 16 b 8 c −10
d $\frac{4}{5}$ e 5 f $\frac{9}{2}$
g no limit exists h 1
3 a −12 b −5 c 56 d −1
e 12 f $\frac{1}{3}$ g $\frac{3}{5}$ h $-\frac{1}{9}$
4 −1
5 2, 1

## 9.02

1 a 4 b 12 c $\frac{2}{27}$ d −2
2 13
3 42
4 a 4.06 b 3.99 c 4
5 −15
6 $\frac{1}{4}$
7 43
8 $-\frac{1}{9}$

## 9.03

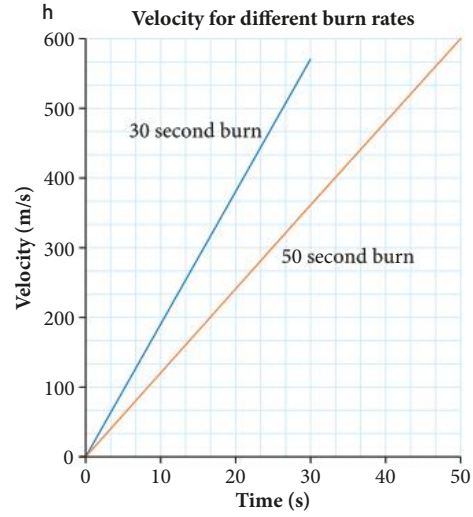
1 a 3 b $10x + 5h - 2$
c $3x^2 + 3hx + h^2$
2 a $2x + h - 1$ b $f'(x) = 2x - 1$
3 a $2x + h + 4$ b $f'(x) = 2x + 4$
4 a $f'(x) = -3$ b $f'(x) = 6x + 2$
c $f'(x) = 7 - 2x$ d $f'(x) = 6x - 6x^2$
5 a $f'(x) = 2x - 2$ b $f'(2) = 2$
6 a $f'(x) = 6x - 1$ b $f'(-3) = -19$
7–10 Proofs

## 9.04

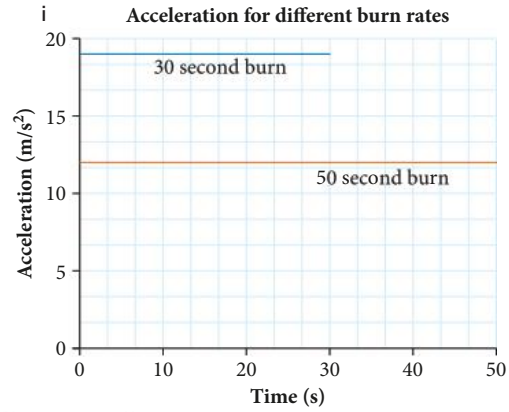
1 a $8x^7$ b $6x^5$ c $9x^8$
d $7x^6$ e $3x^2$
2 a $20x^{19}$ b $12x^{11}$ c $25x^{24}$
d $17x^{16}$ e $22x^{21}$
3 a $7x^6$ b $18x^{17}$ c $32x^7$
d $66x^{10}$ e $144x^{15}$
4 a $24x^3$ b $27x^2$ c $55x^4$
d $30x^4$ e $12x^5$
5–8 Proofs

## 9.05

1 a 108 and −4 b 96 and 150
c −36 and −81 d 60 and −60
e 16 and −16
2 a 99.2 and 334.8 b 201.6 and 315
c −80 and −405 d −48 and 48
e 1.2 and −1200
3 a −84 375 and 84 375 b 972 and 3072
c 108 and −500 d 12 and 18.75
e 121.5 and 384
4 a 2400 m b 720 m/s c 48 m/s$^2$
d 1680 m/s e 29.4 km
5 a 15.2 m/s b 7.6 m/s$^2$
c 22.8 m/s d 7.6 m/s$^2$
e About 5.6 seconds.
6 a $v = 19t$ b 570 m/s c 19 m/s$^2$ d $1.9g$
e $v = 12t$ f 12 m/s$^2$ g 600 m/s
h




i




## 9.06

1 a $8x - y - 4 = 0$ and $16x + y + 16 = 0$
b $3x - y + 2 = 0$ and $12x - y - 16 = 0$
c $40x - 5y - 64 = 0$ and $640x - 5y + 2048 = 0$
d $18x + y + 27 = 0$ and $12x - y - 12 = 0$
e $4x + 4y + 3 = 0$ and $8x - y - 12 = 0$
2 (3, 27)
3 (2, 16) and (−2, −16)
4 (−2, 16)
5 $\left(4, 53\frac{1}{3}\right)$ and $\left(-4, -53\frac{1}{3}\right)$


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6 $\left(\sqrt{2},\ 2\sqrt{2}\right)$ and $\left(-\sqrt{2},-2\sqrt{2}\right)$
7 $c = 0.8$
8 (2, 8) and (–2, –8)

## 9.07

1 a $v=\frac{1}{3}\pi h^3$ b $\frac{dv}{dh}=\pi h^2$
c At 1.5 cm, $\frac{dV}{dh} \approx 7$ cm$^3$/cm, so about 1.4 cm$^3$.
2 a Marginal cost = $0.0014n$
b 7 cents c 8.4 cents d 9.8 cents
e 6427, the 6428th will cost more than 9 cents.
3 a Marginal profit = $0.006n^2$ b $86.40
c $240 d $1500 e $1518.05
4 a $43 750 b $87.50
c Average cost = $0.000\ 35n^2$ d 1330
e Marginal cost = $0.001\ 05n^2$ f 767
g If they make 1331 ovens they will make a loss, and if they make 769 ovens they will make a loss on the 769th oven, and their profit will start going down.
5 a $691.20 b $2332.80
c $5529.60 d $2329.60
e About $0.78 per day.
f Marginal pressure cost = $0.0012P^2$
g About 1.6 cents per day.
h About 2.3 cents

# CHAPTER 9 REVIEW

1 D
2 C
3 A
4 B
5 C
6 13
7 2.6
8 $f'(x) = 6x^2 - 3$
9 $p'(q) = 5q^4$
10 a $\frac{dy}{dx}=14x^{13}$ b $f'(x) = 35x^6$
11 –160
12 $48x - 5y + 128 = 0$
13 $\left(3,\ 16\frac{1}{5}\right)$ and $\left(-3,-16\frac{1}{5}\right)$
14 a $v = 3t$ b About 11 seconds.
c 3 m/s$^2$ d About 33 m/s.
15 a About $96 429. b About $228 571.
c Marginal cost = $\frac{3n^2}{280}$ d 348 desks
16 Proof

# MIXED REVISION 3

**Multiple choice**
1 C
2 C
3 C
4 D
5 C
6 B
7 D
8 B
9 C

**Short answer**
1 1 m/s$^2$
2 $\frac{1}{5}$
3 19
4 $6x^2 + 5$
5 0.91
6 8
7 2044
8 –70
9 $54x - y - 216 = 0$
10 39 364
11 –1610

**Application**
1 $8 - 8x$
2 a –1200, –3, –0.0075 b $y = -3 \times (0.05)^x$
3 a 1 cm b 91.5 m
4 $ 145 390.96
5 Proof
6 a $v = 1.2t$ b 12 m/s
c 60 m d 1.2 m/s$^2$

# INVESTIGATION: FUNCTIONS AND GRADIENT FUNCTIONS

The graph of $y = 7x$ is a straight line with slope 7.
The graph of the gradient function is the horizontal line $y = 7$.
The graph of $y = 0.5x^2$ is a parabola.
The graph of the gradient function is straight line $y = x$.
The graph of $y = x^3 - x^2$ is a cubic with zeros at –1, 0 and 1.
The graph of its derivative is a parabola with zeros at 0 and 1.5.

# INVESTIGATION: TANGENTS ON A CURVE

The curve is decreasing before $x = 2$, stationary at $x = 2$ and increasing after $x = 2$.
Similar things happen with other curves.

## 10.01

1 B
2 C
3 D
4 $f_1 \leftrightarrow g_8, f_2 \leftrightarrow g_6, f_3 \leftrightarrow g_{10}, f_4 \leftrightarrow g_3, f_5 \leftrightarrow g_5, f_6 \leftrightarrow g_4, f_7 \leftrightarrow g_1, f_8 \leftrightarrow g_9, f_9 \leftrightarrow g_2, f_{10} \leftrightarrow g_7$
5 a $f'(8) > 0$ b $f'(-4) < 0$ c $f'(0) = 0$

6 a $f'(-1) > 0$ b $f'(1) > 0$ c $f'(3) > 0$

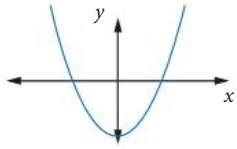
7 a–d Decreasing e Stationary
f–j Increasing

8 a–c Decreasing d–e Increasing
f–j Decreasing

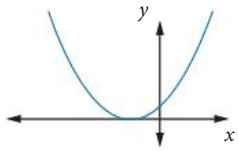
9 a Increasing b Stationary
c Decreasing d Decreasing
e Decreasing f Stationary
g Increasing h Increasing

10 a Decreasing b Stationary
c Stationary d Decreasing
e Decreasing f Decreasing
g Stationary h Increasing

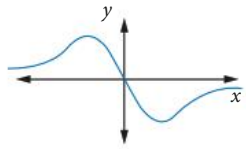
11



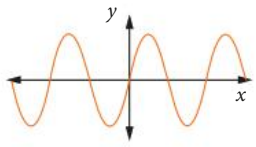

12



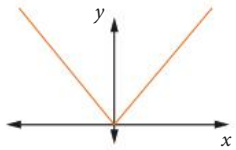

13



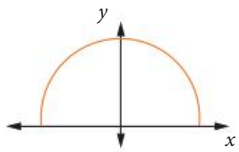

14




15




16




## 10.02

1 a $2x - 1$ b $1 + 2x$ c $3x^2 - 2x$
d $2x + 4x^3$ e $4x^3 + 1$ f $3x^2 - 4x^3$

2 a $1 + 2x$ b $2x - 1$
c $4x^3 - 1 + 2x$ d $3x^2 + 2x$
e $3x^2 + 5x^4 - 1$ f $5x^4 - 2x + 1$

3 a $4x^3 - 3x^2$ b $6x^5 + 3x^2$
c $7x^6 + 5x^4$ d $8x^7 - 3x^2$
e $9x^8 + 1$ f $7x^6 - 4x^3$

4 a $6m^5 + 2m + 1$ b $6r^5 - 1 - 3r^2$
c $7x^6 + 4x^3 + 9x^8$ d $5p^4 - 6p^5 + 8p^7$
e $6w^5 + 7w^6 - 3w^2$ f $2n - 4n^3 + 6n^5$

5 a $5d^4 + 4d^3 + 3d^2 + 2d$ b $8x^7 - 6x^5 + 4x^3 - 2x$
c $1 + 5t^4 - 7t^6 + 8t^7$
d $5h^4 - 7h^6 + 9h^8 - 11h^{10}$
e $2z + 5z^4 - 3z^2 + 6z^5 - 4z^3$
f $3u^2 - 2u + 1 - 5u^4 + 10u^9$

6 Proof.

## 10.03

1 a $6x - 5$ b $21x^2 + 2x$ c $8x^3 - 9x^2$
d $15x^2 + 2$ e $10x^4 - 12x^3$

2 a $16m$ b $-24x^5$ c $27n^2$ d $24g$
e $-42m^6$ f $24p^3$ g $55z^4$
h $-70r^6$ i $-70t^9$ j $84x^5$

3 a $112u^7 + 22u$ b $120u^7 - 42u^2$
c $6u + 81u^8$ d $12z^5 + 30z^2$
e $64a^7 + 81a^8$ f $30v^2 - 15$
g $42w^6 + 88w^7$ h $24b - 21b^6$
i $48c^3 + 12c$ j $42r^6 + 21r^2$

4 a $84x^2 - 32x$ b $-147x^2 + 56x$
c $42\sqrt{3}x^2 - 16\sqrt{3}x$ d $14x^2 - \frac{16x}{3}$
e $79.8x^2 - 30.4x$

5 a $-24x^3 + 30x$ b $40x^3 - 50x$
c $-40\sqrt{2}x^3 + 50\sqrt{2}x$ d $-20x^3 + 25x$
e $-36x^3 + 45x$

6 a $9x^3 + 30x^2 - 35x$ b $12x^3 - 46x^2 + 25x$
c $-9x^3 + 24x^2 - 10x$

7 a $2x^4 - 4x^3 + 20x^2$
b $-6x^4 - 12x^3 - 12x^2$
c $-5x^4 - 19x^3 + 8x^2$

8 a $-2x^4 + 15x^3 - 8x^2 - 10x$
b $-2x^4 + 16x^3 - 36x^2 + 15x$
c $-8x^3 - 8x^2 + 30x$
d $-3x^4 + 38x^3 - 42x^2 - 25x$

9 a $-72x^3 + 90x^2 + 30x$
b $60x^3 - 48x^2 + 20x$
c $48x^3 - 66x^2 - 30x$


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## 10.04

1 D
2 A
3 E
4 a $3x^2 + 2x + 1$
b $42x^6 + 90x^8 - 60x^{11} + 48x^5$
c $-12x^5 + 72x^7 + 11x^{10}$
d $24x^7 - 60x^{11}$
e $72x^{11} + 143x^{12} - 120x^4$
f $36x^2 + 28x^3 - 66x^5$
g $32x^7 - 15 + 56x + 42x^2$
h $-6 + 14x + 42x^2 - 36x^3$
i $120x^7 - 35x^4 + 28x^6 + 12$
j $32x - 36x^3 + 21x^2 + 18$
5 a $24x^3 - 15x^2 + 4x + 8$ b $27x^2 + 40x^3 - 15x^4$
c $3x^2 + 2x + 1$ d $36x^8 - 80x^9$
e $180x^{11} + 180x^{14} - 180x^4$
f $-7x^6 + 63x^8 + 10x^9$
g $40x^7 + 108x^8 - 60x^9 + 12x^{11}$
h $-3 + 2x + 24x^2 - 4x^3$
i $20x^3 - 17 + 48x^2$
j $175x^6 - 48x^5 + 70x^4 + 2$
6 a $25x^4 - 6x$ b $24u^2 - 36u^3$
c $56m^7 + 30m^4 - 17$
d $19 + 35y^4 - 44y^3 + 14y^6$
e $26z - 18z^2 + 3$
f $18t^5 + 40t^4 - 15$
7 a $12x^3 - 18x^2$ b $12u^3 - 12u$
c $30m^5 + 40m^4 - 24$
d $9 + 30y^4 - 12y^2 + 14y^6$
e $6z - 27z^2 + 38$ f $35t^6 + 36t^3 - 16$
g $27r^2 - 13 - 12r$ h $4a - 24a^3$
i $8 - 12p^3 - 42p^6 - 45p^8$ j $8t - 6t^2 + 44t^3$
8 a $2v + 1$ b $-20r - 33$ c $36h$
d $20t + 1$ e $48p + 83$ f $-8z - 27$
9 a $2a - 12$ b $8x + 20$ c $18y - 12$
d $50q + 70$ e $50a - 30$ f $50g + 30$
g $18v + 54$ h $98m - 56$
10 a $9z^2 + 12z - 12$ b $9p^2 - 4p - 20$
c $72v^2 + 52v - 66$ d $-63t^2 + 76t - 51$
e $-48b^3 + 99b^2 - 36b - 8$
f $160u^4 + 192u^3 - 282u^2 + 108u - 12$
11 Demonstration using process in Example 12.
12 a $3x^2 - 4x - 7$ b $-3x^2 + 16x - 7$
c $8x^3 + 12x^2 - 4x - 5$ d $12x^2 - 16x + 7$
e $8x^3 + 3x^2 - 4x - 5$ f $8x^3 - 9x^2 + 6x - 5$
g $-8x^3 - 12x^2 + 10x - 2$ h $8x^3 - 3x^2 + 22x - 12$
i $-8x^3 + 12x^2 - 10x - 2$ j $-9x^2 - 6x + 7$

## 10.05

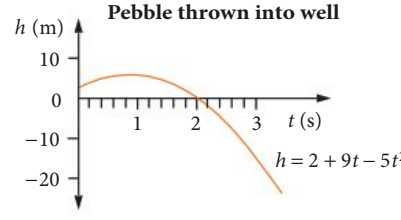
1 A
2 D
3 a $8x - y - 9 = 0$ b $11x + y - 11 = 0$
c $2x + y + 1 = 0$ d $6x - y - 19 = 0$
4 $x = \pm 3$
5 (1, 2) and (–1, 0)
6 $P = (-5, -7)$
7 $Q = (0, 1)$
8 $H = (1, 2)$
9 a 81.9° b 45°
10 a (–)87.5° b 45°
11 a 45° b 73.06 c 25.02 d (–)87.5°
12 a 4 b 76° c $P = -48t - 72$
d 88.8° e $2t + 4 - 12t^2$
f $\left(-\frac{1}{4}, -5\frac{1}{4}\right)$ and $\left(\frac{2}{3}, -2\frac{2}{27}\right)$
g $P = -98t + 203$ h –6
13 $4x - 2y - 19 = 0$

## INVESTIGATION: ERRORS AND DERIVATIVES

1 2144.66 $m^3$
2 2226.09 $m^3$
3 81.43 $m^3$
4 377.51 and 427.78 $m^3$, total 805.29
5 804.25 $m^2$
6 They are almost the same.
7 $0.8 \times 804.25 = 643.4$
8 Total 31 426, 31 416, almost the same
9 $6.157 \times 10^6$
10 $1.231 \times 10^6$

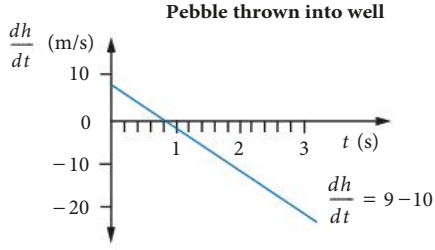
## 10.06

1 a $30 - 8t$ b $15t^2 + 4t$
c $16 - 4x$ d $15t^4 - 4t^3 + 2$
e $nr^{n-1}$ f $2\pi - \frac{100}{r^3}$
g $800 - \frac{400}{r^2}$ h $4\pi r^2$
2 7 L/s
3 a 13 b 64 c 189
d –12 e 533
4 a 22 b 102 c –10
d –18 e 2278
5 a –153
b i –15 ii –363 iii –135
c The average rate is not the average of the instantaneous rates.
6 a




b $9 - 10t$

c




d −6 m/s

e In this case, the average speed is the average of the instantaneous speeds.

7 a 6.786 $m^3$ b 4.524 $m^3$

8 a $V(t) = 38\,400\,000 - 464\,000t + 1320t^2 - t^3$

b $V'(t) = -464\,000 + 2640t - 3t^2$

c 431 469 $m^3$ d 426 351 $m^3$

## INVESTIGATION: DIFFERENTIATING FORWARDS AND BACKWARDS

1–3

| $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ |
|---|---|---|---|---|---|
| $5x$ | 5 | $x^3$ | $3x^2$ | $6\frac{1}{4}x^4$ | $25x^3$ |
| $7x$ | 7 | $2x^3$ | $6x^2$ | $4x^5$ | $20x^4$ |
| $x^2$ | $2x$ | $\frac{1}{3}x^3$ | $x^2$ | $3\frac{2}{5}x^5$ | $17x^4$ |
| $x^2$ | $2x$ | $4x^3$ | $12x^2$ | $4x^6$ | $24x^5$ |
| $\frac{1}{2}x^2$ | $x$ | $1\frac{2}{3}x^3$ | $5x^2$ | $\frac{1}{2}x^6$ | $3x^5$ |
| $5x^2$ | $10x$ | $x^4$ | $4x^3$ | | |
| $3\frac{1}{2}x^2$ | $7x$ | $4x^4$ | $16x^3$ | | |

4 Increase the power by 1 and divide by that number as well.

## 10.07

1 Answers may differ by constants.

a $2x^3 + 2x^2 + x$, $2x^3 + 2x^2 + x + 5$

b $4t^2 - 7t + 3$, $4t^2 - 7t - 3$

c $z^4 + 1\frac{1}{2}z^2 - z^5 - 1$, $z^4 + 1\frac{1}{2}z^2 - z^5 + 8$

d $4x + 2$, $4x + 5$

e $\frac{y^3}{3} - \frac{y^2}{2} - y + 4$, $\frac{y^3}{3} - \frac{y^2}{2} - y - 4$

f $\frac{3}{4}v^4 - \frac{1}{5}v^5 + \frac{4}{3}v^3 - 2v + 1$, $\frac{3}{4}v^4 - \frac{1}{5}v^5 + \frac{4}{3}v^3 - 2v + 8$

g $4.9t^2 - 0.1t^3 - 6t + 3$, $4.9t^2 - 0.1t^3 - 6t - 3$

h $1.5m^3 + 1.35m^2 + 1.205m$, $1.5m^3 + 1.35m^2 + 1.205m + 1$

i $0.025d^4 + 3.5d^2 - 0.38d + 2$, $0.025d^4 + 3.5d^2 - 0.38d + 25$

j $\frac{5}{3}k^3 - \frac{1}{2}k^4 + k^2 - 5k - 6$, $\frac{5}{3}k^3 - \frac{1}{2}k^4 + k^2 - 5k - 5000$

2 a $h^4 - 2h^3 - h^2 - 3h + c$

b $\frac{1}{4}x^4 - 2x^5 - 2x^3 + 12x + c$

c $u^2 - \frac{5}{4}u^4 - \frac{4}{3}u^3 + 8u + c$

d $0.2\overline{6}m^3 - 1.575m^4 - 0.35m^2 + 4m + c$

e $0.035t^2 + 0.\overline{3}t^3 + c$

f $1.6\overline{3}x^3 - 24.5x^2 + 122.5x + c$

3 a $F(x) = 3x^3 + 12x + 16x + c$

b $M(p) = \frac{2p^3}{3} - 7p^2 + c$

c $G(n) = \frac{n^4}{4} - 4n^3 + 24n^2 + 64n + c$

d $V(y) = y^4 - \frac{17y^3}{3} - \frac{15y^2}{2} + c$

4 a $\frac{1}{3}x^3 - 5x + c$ b $1.5x^2 + 2x + c$

c $7x - \frac{1}{4}x^4 + \frac{1}{3}x^3 + c$ d $\frac{1}{5}x^5 + c$

e $\frac{1}{3}x^3 + 4x^2 + 16x + c$

f $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + c$

5 a $y = x^4 + 2x^3 - 3x^2 + 7x + c$

b $y = 3x^4 - 4x^3 + 6x^2 - 12x + c$

c $y = 1.5t^2 + 5t + c$

d $y = \frac{1}{2}z^2 - \frac{1}{4}z^4 + 8z + c$

e $y = 0.25x^2 - 0.01x^4 + 0.2\overline{3}x^3 - 0.52x + c$

f $y = 0.35t^2 - 0.1\overline{3}t^3 + 0.6t + c$

6 a $G(v) = 5v^2 - 3v + 2$

b $F(x) = 3x^3 - 4x^2 + 4x - 5$

c $P(y) = y^4 - 8y^3 + 6$

d $H(x) = x^3 - 3x^2 + 5$

7 $y = 2x^2 - 5x - 1$, $y(5) = 24$

8 7.6

9 −3

10 −41.25

11 a $v = 0.4t + c$, where $c$ is the speed leaving the curve.

b $x = 0.2t^2 + ct$, assuming the start of the straight is position 0.

c 8 m/s

12 a $v = pt + 2$ m/s downwards or $v = -pt - 2$ m/s upwards

b $x = 33 - \frac{1}{2}pt^2 - 2t$ m above the ground.

c 6 $m/s^2$

d Probably smaller, because its gravity is less.

## CHAPTER 10 REVIEW

1 D

2 A

3 C

4 D

5 E

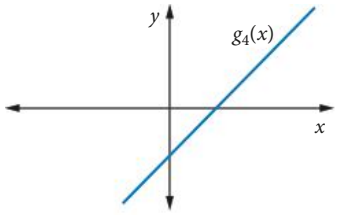
6 D

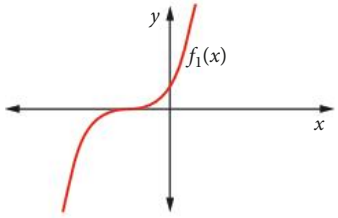
7 C


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8 D

9 C

10 


11 


12 a Increasing b Increasing
c Decreasing d Decreasing
e Decreasing f Increasing
g Increasing

13 a $4x^3 + 1$ b $3x^2 - 2x$

14 a $20x^3$ b $10x^4 + 6x$

15 a $20x^3 - 3$ b 20 c $-1256$
d 20 e 0

16 $f'(4) = 23$, $g'(4) = -43$, $h'(4) = 48$, $i'(4) = -88$, so $i(x)$ is steepest.

17 a $30x^5 - 6x + 1$ b $-1 - 6u + 18u^2$
c $7q^6 + 12q^2 - 1$ d $90y^5 - 4y^3 - 28$
e $1 - 6w + 3w^2$

18 $54x^5 - 126x^2$

19 $24x^2 + 120x + 150$

20 a $-13$ b 27 c $-3$
d $-33$ e 97

21 a $x - y - 4 = 0$
b $23x + y + 20 = 0$ c $x - y - 3 = 0$
d $49x - y - 83 = 0$ e $143x + y + 211 = 0$

22 a $-1$ b 54 c $-2$
d 3 e 118

23 $x = 3$ or $-4$

24 a 17 b $-244$
c $-1768$ d 2633 e 8

25 a 23 b $-25$ c $-1$
d 5 e 53

26 a $F_1(x) = \frac{4}{3}x^3 - x^2 + 5x + 3$, $F_2(x) = \frac{4}{3}x^3 - x^2 + 5x - 6$
b $H_1(x) = 3x - \frac{2}{3}x^6 + 3$, $H_2(x) = 3x - \frac{2}{3}x^6 - 6$

27 a $y = \frac{7}{2}x^2 - x^3 + c$ b $y = x^5 + x^2 + c$

28 $x < 3$

29 a $y = 65x - 118$ b $y = 9x + 2$ c 83.7°

30 a $y = 7x - 10$ b $x = 1\frac{3}{7}$ c $p = -10$
d 10.1 units e 1.4

31 41 $cm^2$/min

32 a 33.12 Ω b 0.0576 Ω/°C
c 39 Ω d 0.0588 Ω/°C
e 6.4128 Ω f 0.0451 Ω/°C
g A high-temperature gauge: any reasonable answer that does not expose the wire to temperatures above about 1760°C is acceptable.

33 Approximate error in volume would be 80 cc, so the capacity would be 2080 cc. Radius : height would also be incorrect.

34 Find the antiderivative, $m(x) = \frac{3x^2}{2} - \frac{2x^3}{3} + 4x + c$

Substitute in point (5, 0) to solve for $c = 25\frac{5}{6}$

## 11.01

1 a $a^{17}$ b $y^0 = 1$ c $a^{-4}$
d $w$ e $x^5$ f $p^{10}$
g $y^8$ h $x^{21}$ i $4x^{10}$
j $81y^{-8}$ k $a$ l $\frac{x^{10}}{y^{45}}$
m $w^{10}$ n $p^5$ o $x^{-3}$
p $a^{-2}b^3$ or $\frac{b^3}{a^2}$ q $x^{-5}y^2$ or $\frac{y^2}{x^5}$

2 a $x^{14}$ b $a^{-7}$ c $m^4$ d $k^{20}$
e $a^{-8}$ f $x$ g $mn^2$
h $p^{-1}$ i $9x^{22}$ j $x^{21}$

3 a $\frac{1}{a^2}$ b $\frac{1}{4^3}$ c $\frac{1}{g^7}$ d $\frac{1}{5^4}$
e $\frac{1}{y^6}$ f $\frac{1}{c^4d^8}$ g $\frac{4}{h^4}$ h $\frac{1}{256h^4}$
i $\frac{2}{k}$ j $\frac{7}{p^4}$ k $\frac{8}{q^6}$ l $\frac{v^2}{4}$

4 a $y^{-4}$ b $4^{-3}$ c $h^{-9}$
d $k^{-6}t^4$ e $p^{-6}y^2$ f $4i^{-6}j^8$
g $3m^{-5}y^{-2}$ h $2m^{-1}z^{-4}$ i $x^8$
j $2x^4y^4$ k $3bk^{-3}$ l $4r^{-9}w^6$

5 a $p^5q^{15}$ b $\frac{a^8}{b^8}$ c $\frac{64a^3}{b^{12}}$ d $7a^6b^2$
e $8m^{17}$ f $x^4y^{10}$ g $\frac{2k^{23}}{27}$
h $16y^{47}$ i $a^3$ j $125x^{-21}y^{18}$

6 a $\frac{1}{r^2}$ b $\frac{12}{y^9}$ c $3p^4$
d $\frac{2}{j^6}$ e $\frac{1}{6m^{13}}$ f $\frac{1}{4k^{14}}$
g $\frac{15g}{h^2}$ h $5k^{18}m^{14}$ i $16v^{11}$
j $\frac{4q^{17}}{p^{13}}$ k $\frac{4e^{12}f^7}{3}$ l $\frac{c^9d^{10}}{1024}$

7 $4\frac{1}{2}$

8 324

9 $\frac{64}{27}$

10 $\frac{1}{25}$

11 $\frac{7}{32}$

12–14 Proofs

## 11.02

1 a 9 b 3 c 4 d 2
e 7 f 10 g 2 h 8
i 4 j 1 k 3 l 2
m 0 n 5 o 7 p 2
q 4 r 27 s $\frac{1}{2}$ t 16

2 a $\sqrt{6}$ b $\sqrt[4]{z}$ c $\sqrt[3]{216}$
d $\sqrt[5]{y^3}$ or $\left(\sqrt[5]{y}\right)^3$ e $\sqrt{49^3}$ or $\left(\sqrt{49}\right)^3$
f $\sqrt[3]{v^7}$ or $\left(\sqrt[3]{v}\right)^7$ g $\sqrt[3]{8^7}$ or $\left(\sqrt[3]{8}\right)^7$
h $\sqrt[4]{16^5}$ or $\left(\sqrt[4]{16}\right)^5$ i $\sqrt{m^5}$ or $\left(\sqrt{m}\right)^5$
j $\sqrt[4]{u^{-1}}$ or $\left(\sqrt[4]{u}\right)^{-1}$
k $\sqrt{\left(\frac{1}{9}\right)^{-1}}$ or $\left(\sqrt{\frac{1}{9}}\right)^{-1}$ or $\sqrt{9}$
l $\sqrt[5]{u^{-3}}$ or $\left(\sqrt[5]{p}\right)^{-3}$ or $\sqrt[5]{\frac{1}{p^3}}$ or $\frac{1}{\sqrt[5]{p^3}}$ or $\frac{1}{\left(\sqrt[5]{p}\right)^3}$

3 a $k^{\frac{1}{2}}$ b $3^{\frac{1}{4}}$ c $5^{\frac{1}{2}}$
d $4^{\frac{1}{5}}$ e $w^{\frac{1}{3}}$ f $c^{\frac{3}{2}}$
g $64^{\frac{4}{3}}$ h $e^{\frac{3}{4}}$ i $28^{\frac{4}{9}}$
j $q^{\frac{6}{5}}$ k $8^{\frac{1}{5}}$ l $16^{\frac{7}{6}}$

4 a $t^{\frac{1}{2}}$ b $y^{\frac{1}{5}}$ c $x^{\frac{3}{2}}$
d $(9-x)^{\frac{1}{3}}$ e $(4s+1)^{\frac{1}{2}}$
f $(2t+3)^{-\frac{1}{2}}$ g $(5x-y)^{-\frac{3}{2}}$
h $(3x+1)^{\frac{5}{2}}$ i $(x-2)^{-\frac{2}{3}}$
j $\frac{1}{2}(y+7)^{-\frac{1}{2}}$ k $5(x+4)^{-\frac{1}{3}}$
l $\frac{3}{5}(x^2+2)^{-\frac{3}{4}}$

5 a $a^{4\frac{1}{2}}$ b $b^{3\frac{1}{2}}$ c $\frac{6}{n^{\frac{1}{2}}}$
d $12g^{6\frac{1}{3}}$ e $12w^{3\frac{2}{3}}p^{2\frac{1}{2}}$ f $10m^{3\frac{1}{2}}n^{2\frac{1}{3}}$
g $\frac{21u^3}{s^{3\frac{1}{2}}}$ h $\frac{24p^{1\frac{1}{2}}}{q^{\frac{5}{12}}}$

6 a $x+x^2+2x^{\frac{3}{2}}$ b $a^{\frac{2}{3}}-b^{\frac{2}{3}}$
c $p^2+p^{-1}+2p^{\frac{1}{2}}$ d $x+x^{-1}+2$
e $x^{-\frac{1}{2}}-3x^{-\frac{3}{2}}+x^{-\frac{5}{2}}$

7 a $\frac{1}{\sqrt[3]{a-2b}}$ b $\frac{1}{\sqrt[3]{(y-3)^2}}$ c $\frac{4}{\sqrt[7]{(6a+1)^4}}$
d $\frac{1}{3\sqrt[5]{(x+y)^4}}$ e $\frac{6}{7\sqrt[9]{(3x+8)^2}}$

## 11.03

1 a $3.8\times10^3$ b $1.23\times10^6$
c $6.19\times10^4$ d $1.2\times10^7$
e $8.67\times10^9$ f $4.16\times10^5$
g $9\times10^2$ h $1.376\times10^4$
i $2\times10^7$ j $8\times10^4$

2 a $5.7\times10^{-2}$ b $5.5\times10^{-5}$
c $4\times10^{-3}$ d $6.2\times10^{-4}$
e $2\times10^{-6}$ f $8\times10^{-8}$
g $7.6\times10^{-6}$ h $2.3\times10^{-1}$
i $8.5\times10^{-3}$ j $7\times10^{-11}$

3 a 36 000 b 27 800 000
c 9 250 d 6 330 000
e 400 000 f 0.072 3
g 0.000 097 h 0.000 000 038
i 0.000 007 j 0.000 5

4 a 240 000 b 9 200 000 c 11 000 d 0.36
e 1.3 f 9.0 g 16
h 320 i 2900 j 9.1

5 a 6.61 b 0.686 c 8.25 d 1.30

6 $1.305\times10^{10}$

7 $6.51\times10^{-10}$

## 11.04

1 D

2 a 8 b 1 c 2 d 1
e 6 f 9 g 1 h 9
i 343 j 216 k 16 l 64
m $\frac{1}{4}$ n $\frac{16}{9}$ o $2^{16} = 65\,536$
p 1 q $\frac{64}{27}$ r $\frac{1}{36}$
s 125 t $\frac{2}{5}$ u $\frac{27}{8}$
v 4 w 1000 x $\frac{1}{10000}$

3 a 192 b 196 608 c 24
d 768 e 0.1875

4 a 51.7706 b 9.7865 c 7663.9046
d 92.7454 e 0.8043

5 a 7.9222 b 4.1354 c 8.5928
d 2.5397 e 21.0055

6 a $2^{5x+1}$ b $5^{9n-4}$ c $7^{7m-10}$
d $3^{5y-7}$ e $11^{5p+12}$ f $13^{2q+4}$

7 a $2^{14v+1}$ b $3^{15m+16}$ c $3^{16-x}\times5^{6-4x}$
d $2^{36n-43}$ e $2^{12-5r}\times3^{r+5}$
f $3^{4z-8}\times5^{2z+11}\times7^{6z+3}$

8 a $2^4$ b $5^{k+1}$ c $3\times2^3$
d $2^m3^{2n+3m}$ e $3^{2i+2}\times7^3$ f $2^3$

9–10 Proofs Teacher to check.

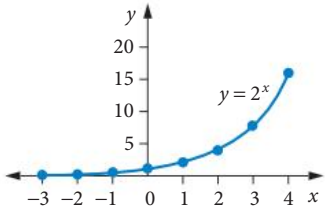

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## INVESTIGATION: EXPONENTIAL GRAPHS

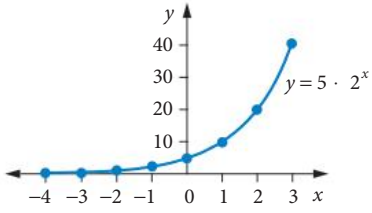
Both $f1(x) = 2^x$ and $f2(x) = 5^x$ have $y$-intercept 1.
For $x < 0$, $f1(x) = 2^x > f2(x) = 5^x$.
For $x > 0$, $f1(x) = 2^x < f2(x) = 5^x$.
They have the same shape, but f2 is steeper than f1.
The function $f3(x) = 1.5^x$ follows the same trend and it is even less steep.
As the value of $a > 1$ increases, the value below 0 decreases and the value above 0 increases. The steepness increases, but the graph still has $y$-intercept 0.
$y = 2^x$ and $y = 2^{-x}$ are reflections of each other in the $y$-axis.
$y = a^{kx}$ behaves in the same way when $k$ increases as the graph of $y = a^x$ behaves as $a$ increases.

## 11.05

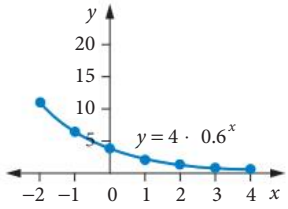
1 C
2 D
3 E
4 a The $y$-intercept is 1, the graph is increasing with increasing slope, and is always positive.



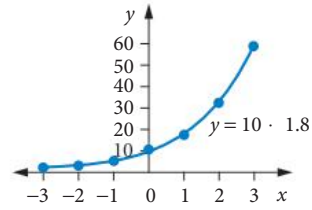

b The $y$-intercept is 5, the graph is increasing with increasing slope, and is always positive.



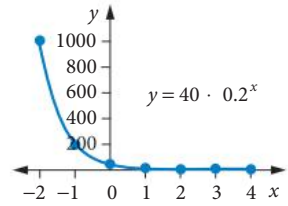

c The $y$-intercept is 4, the graph is decreasing with decreasing slope, and is always positive.



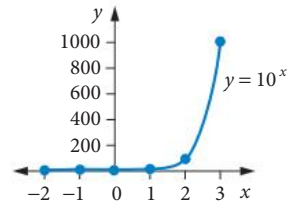

d The $y$-intercept is 10, the graph is increasing with increasing slope, and is always positive.



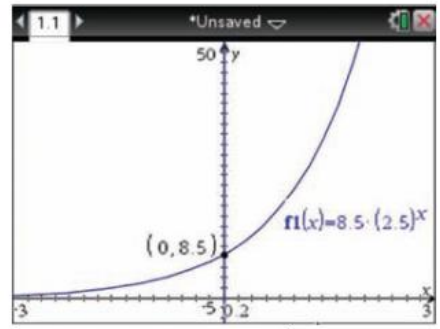

e The $y$-intercept is 40, the graph is decreasing with decreasing slope, and is always positive.

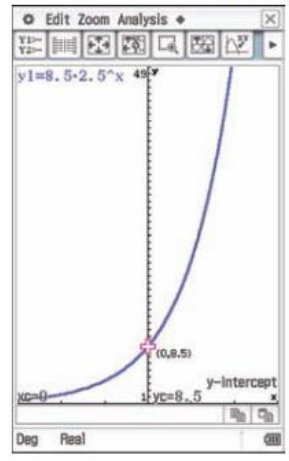



f The $y$-intercept is 1, the graph is increasing with increasing slope, and is always positive.

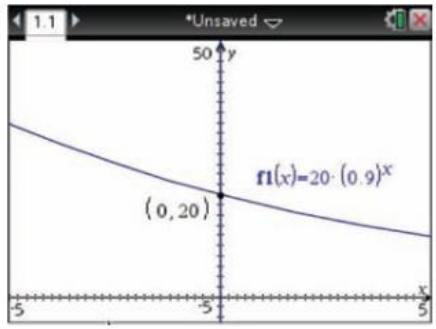


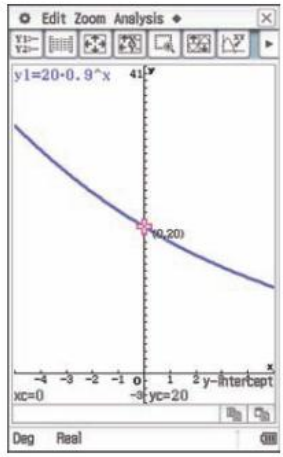

5 a The $y$-intercept is 8.5, the graph is increasing with increasing slope, and is always positive.



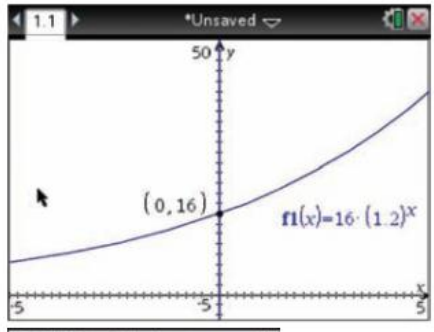



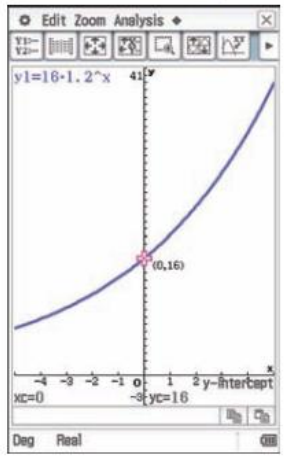
**b** The $y$-intercept is 20, the graph is decreasing with decreasing slope, and is always positive.



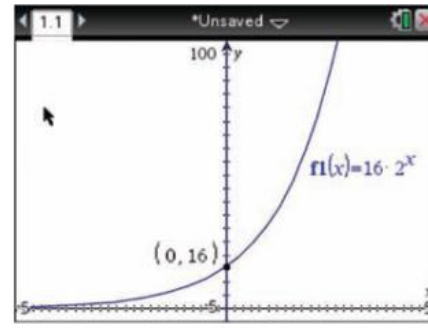



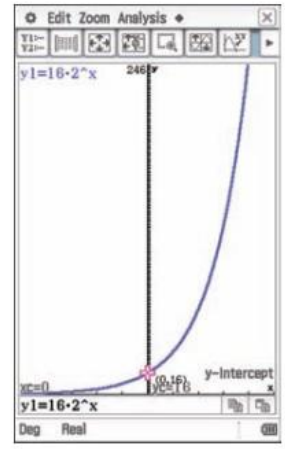

**c** The $y$-intercept is 16, the graph is increasing with increasing slope, and is always positive.



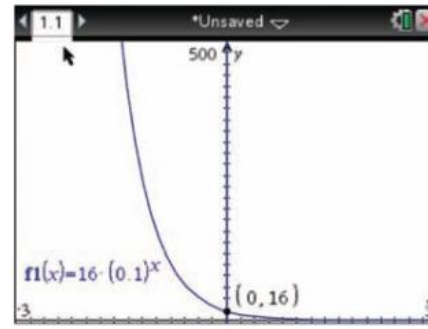



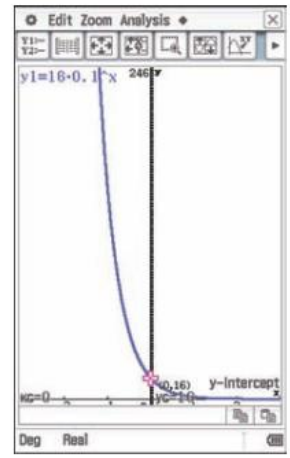

**d** The $y$-intercept is 16, the graph is increasing with increasing slope, and is always positive.




**e** The $y$-intercept is 16, the graph is decreasing with decreasing slope, and is always positive.

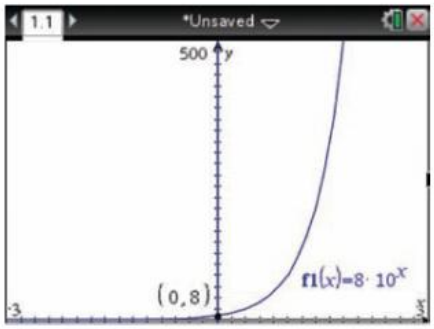


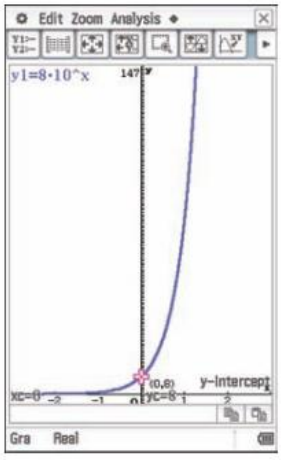





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f The $y$-intercept is 8, the graph is increasing with increasing slope, and is always positive.



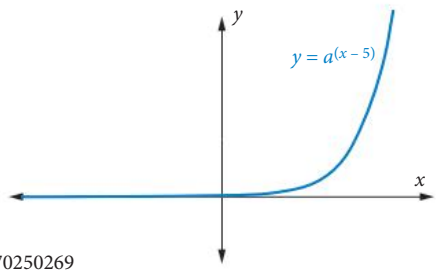

6 $x \approx -0.764$

## INVESTIGATION: TRANSLATIONS OF EXPONENTIAL GRAPHS

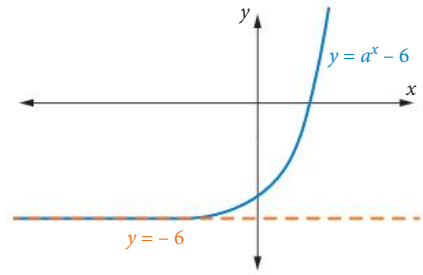
For $c$ positive the graph of $y = Aa^x + c$ moves up and for $c$ negative it moves down.
For $b$ positive the graph of $y = Aa^{x+b}$ moves left and for $b$ negative it moves right.

## 11.06

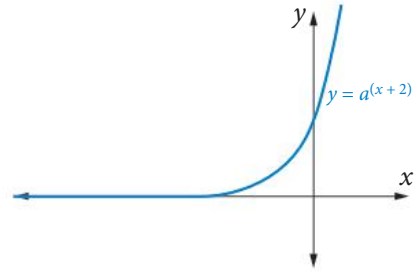
1 D
2 C
3 E
4 A
5 B
6 B
7 E
8 a



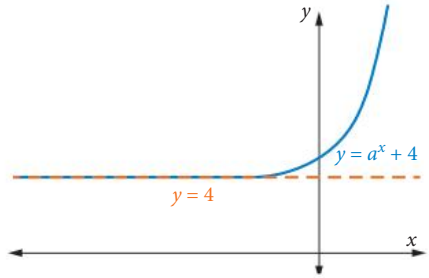

b




c

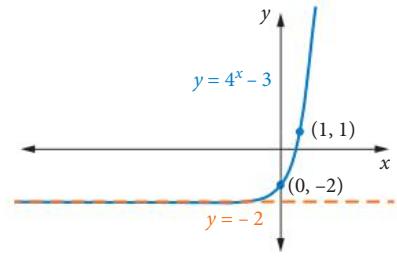



d

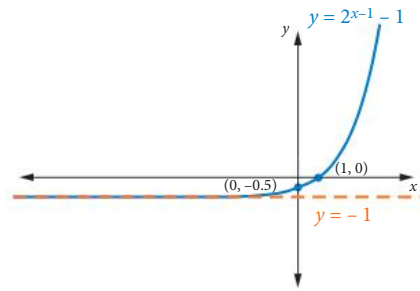



9 a (0, 1); $y = 0$ b (0, 2); $y = 0$
c (0, 2.92); $y = 1$ d (0, −4.5); $y = -5$
e (0, 6); $y = 0$ f (0, 15.1); $y = 3$

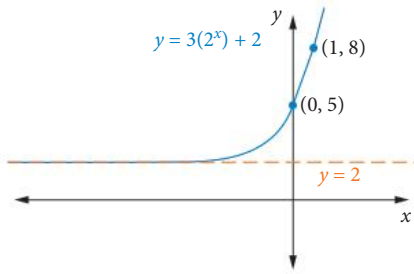
10 a



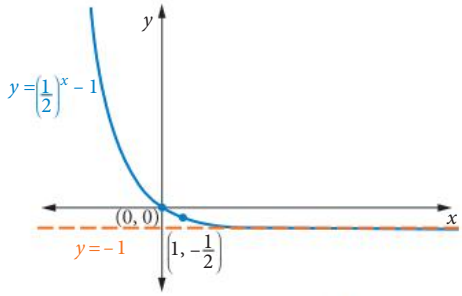

b



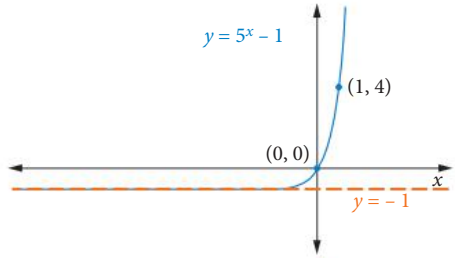
c



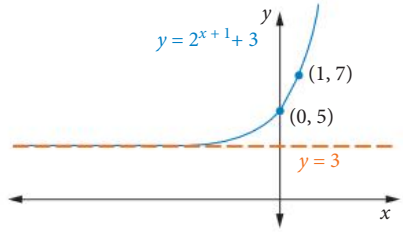

d



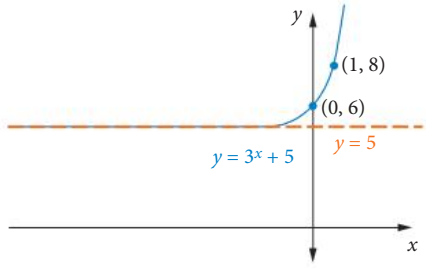

e



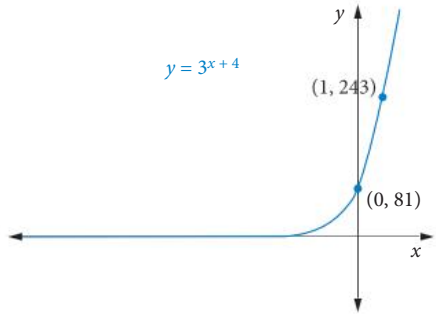

f



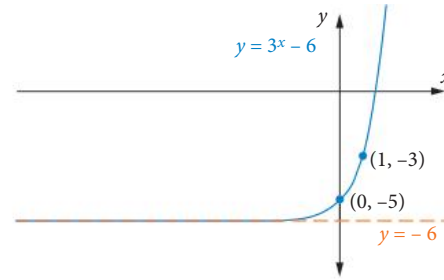

11 a



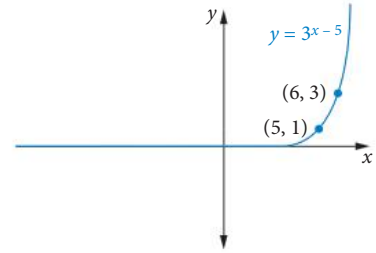

b



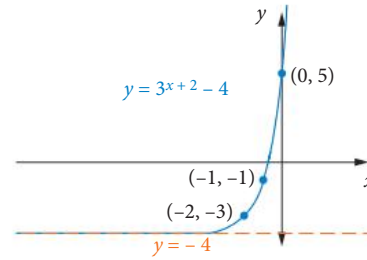

c



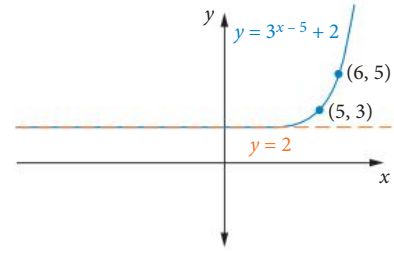

d




e




f




## 11.07

1 a $x = 3$ b $x = 6$ c $x = \frac{7}{4}$
d $x = 4$ e $x = \frac{1}{2}$ f $x = 0$

2 a $x = 1\frac{7}{8}$ b $x = 4$ c $x = 2$
d $x = 1$ e $x = 3$ f $x = 2$ or 4
g $x = 2$ or 3 h $x = 2$ or 3 i $x = -1$ or 2
j $x = 0$, 1 or 2 k $x = 1$

3 a $x \approx 2.3$ b $x \approx 0.8$ c $x \approx 3.6$
d The solutions are not accurate.

4 a $x \approx 2.73$ b $x \approx 2.10$ c $x \approx 4.46$
d $x \approx 0.93$ e $x \approx -0.63$ f $x \approx 0.37$

5 a $x \approx 0.90$ b $x \approx 1.90$
c $x \approx 2.08$ d $x \approx 0.65$

6 a $x \approx 1.32$ b $x \approx 0.32$
c $x \approx -0.42$ d $x \approx 0.71$

7 a $x \approx 0.39$ b $x \approx 2.39$
c $x \approx 0.69$ d $x \approx 1.52$


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8 $x \approx 1.926$
9 $x \approx 2.087$
10 $x \approx 1.473$
11 $x \approx 0.733$
12 $x \approx 0.905$

## INVESTIGATION: RADIOCARBON DATING

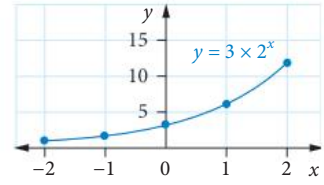
1 $R = 20 \times (0.5)^{-y/5730}$, where $y$ is the age after death.
2 5982 years and 17 879 years.
3 Only those with carbon produced by living things (i.e. things that once lived).
4 Obviously it depends how low the radiation can be reliably detected.
5 No, it is not constant.
6 The dilution of the carbon 14 in the atmosphere by burning of fossil fuels since the industrial revolution, which means that carbon dating from this period is less reliable.

## 11.08

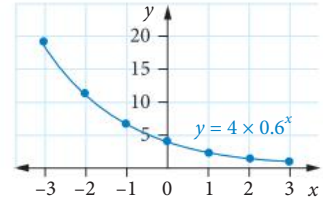
1 a 45 b 1.15
2 a 1100 b 0.82
3 a $A = 3 \times 1.05^d$
b i 3.4729 $m^2$ ii 4.8867 $m^2$ iii 7.9599 $m^2$
c 104.9 days
4 a $V = 3000 \times 1.08^y$
b i $3779.14 ii $6476.77 iii $20 545.43
c 9.01 years
5 a $V = 24\,900 \times 0.85^y$
b i $15 291.71 ii $11 048.26 iii $4902.17
c 19.8 years
6 a $T = 500 - 480 \times 0.9^m$
b i 216.6°C ii 332.6°C iii 465.5°C
c 14.9 min
7 a $p = 3 \times 2^m$ b 768
8 a $q = 24 \times 1.8^x$ b 117.3402
9 $y = 50 \times 1.4^x$, 318.1810

## CHAPTER 11 REVIEW

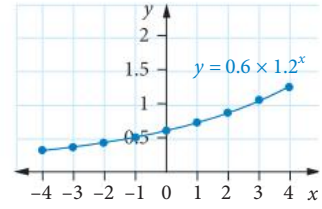
1 A
2 E
3 C
4 B
5 B
6 A
7 B
8 D
9 A
10 E
11 D
12 C
13 D
14 a $x^{11}$ b $5k^9$ c $27a^{12}b^6$
d $4\frac{1}{2}t^6$ e $4\frac{4}{7}$
15 a $\frac{8p^{14}}{3q^6}$ b $\frac{1}{2}$ c $\frac{1}{27}$ d 5
16 a $x^{\frac{4}{3}}$ b $(5x+3)^{\frac{9}{7}}$
c $m^{-\frac{3}{4}}$ d $(2p^2-1)^{-\frac{3}{5}}$
17 a $x^{\frac{13}{2}}$ b $27b^{\frac{13}{4}}$
18 a $1.3 \times 10^{-5}$ b $1.23 \times 10^{11}$
19 a 1 627 000 b 0.000 070 02
20 $4.14 \times 10^{-20}$
21 a 1543.5 b About 0.092
22 a $3^{2x+2}$ b $2^{31}$
23 a



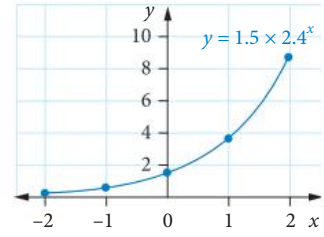

b




c

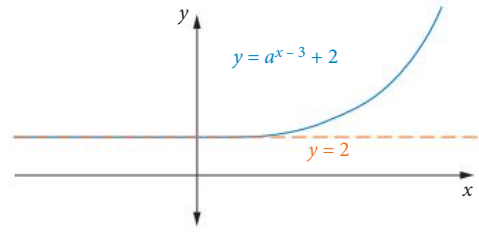



d




All values of the functions are positive. a, c and d are always increasing as $x$ increases. b is decreasing as $x$ increases. The $y$-intercepts are 3, 4, 0.6 and 1.5 respectively.

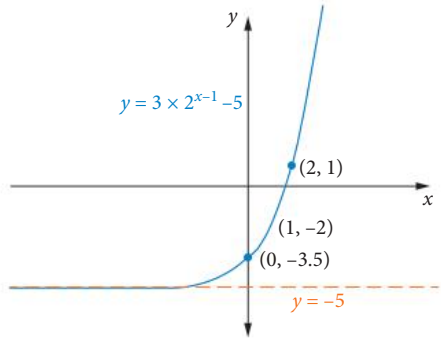
24



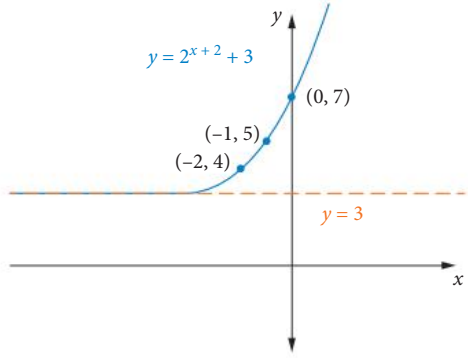
25 The constant $c$ translates the graph vertically up ($c > 0$) or down ($c < 0$).

26 (0, 10)

27




28




29 a $x = 4\frac{1}{2}$ b $x = 3$ c $x = \frac{1}{6}$ d $x = 1\frac{1}{9}$

30 a $x = 2$ b $x = 1$ or 4

c $x = 2$ d $x = \frac{2}{3}$ or $1\frac{1}{3}$

31 $x \approx 1.32$

32 a 1.58 b 1.17

33 Proof.

34 100 min

35 $a = 5, k = \frac{1}{2}$

36 a $V = P \times 1.064^y$ b \$11 953

c \$18 454 d About 15 years from the start.

37 a $V = 32\,000 \times 0.88^y$ b \$19 190

c \$11 508

d About $14\frac{1}{2}$ years from the start.

38 a $A = 100 \times 0.5^n$ b $A(d) = 100 \times 0.5^{\frac{d}{5}}$

c Initial value is 100, growth factor ≈ 0.871

d $A(d) = 300 \times 0.5^{\frac{d}{9}}$

e $A(d) = G \times 0.5^{\frac{d}{h}}$

39 a $A = 12 \times 1.12^y$ assaults per 1000

b 634 assaults c 2021

40 a 1.4788 b $3.536 \times 10^{11}$ operation per second

41 12 min

42 a A decay function b 0.85

c $f(x) = 48 \times 0.85^x$ d 15.4

## 12.01

1 C

2 D

3 D

4 a i −17 ii −17 iii −17

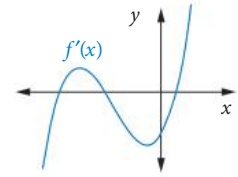
b i −89 ii −25 iii 19

5 a 0 b −4 c 8

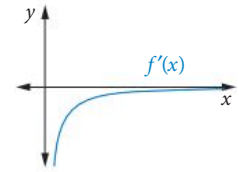
d 2, which is different from the average rate of change

6 a 5 b 11 c −9

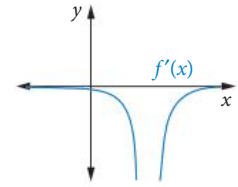
7 a



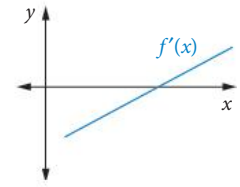

b



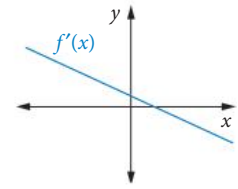

c



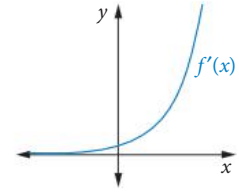

d




e



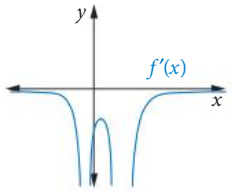

f



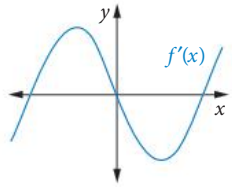


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**g**




**h**




**8** **a** $y = 16x - 48$ **b** $y = 11x + 17$

**9** **a** 8030 L **b** 30 L
**c** 325 L/min **d** 300 L/min

**10** **a** The velocity of the ball
**b** −9m/s
**c** The instantaneous velocity of the ball.
**d** 21m/s, 11 m/s, 1 m/s, −9m/s, −19m/s
**e** 0 m/s
**f** 2.1 s, 49.05 m

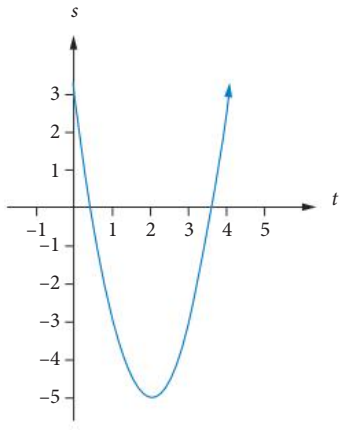
**11** **a** \$17/chair **b** −\$2/chair
**c** \$16/chair **d** \$15.998/chair

**12** $y' = 3x^2 - 8x + 7$. This means that the gradient function is a quadratic with no real roots. There are no stationary values because $y' \neq 0$.

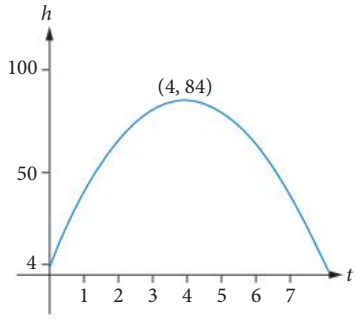
## 12.02

**1** **a** 18 cm/s **b** 12 cm/s$^2$
**c** When $t = 0$, $x = 0$; after 3 s
**d** After 5 s

**2** **a** −8 m/s
**b** $a = 4$; constant acceleration of 4 m/s$^2$
**c** 13 m **d** after 2 s **e** −5 m
**f**




**3** **a** 4 m **b** 40 m/s **c** 39 m **d** 84 m
**e**



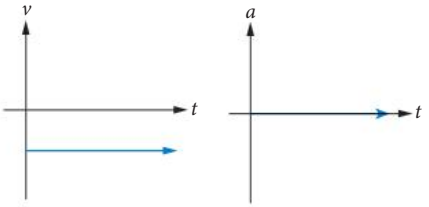

**4** **a** 4 m/s **b** −2 m/s **c** 14 m/s
**d** 6 m/s, which is different from the average velocity
**e** 2 m/s$^2$ **f** 20 m/s$^2$

**5** **a** −6 m/s **b** 8 m/s **c** −28 m/s
**d** −18 m/s$^2$ **e** −6 m/s$^2$ **f** −30 m/s$^2$

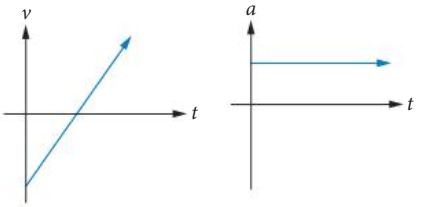
**6** **a** $v = 3t^2 + 12t - 2$; $a = 6t + 12$
**b** 266 m **c** 133 m/s **d** 42 m/s

**7** **a** 3 m/s **b** When $t = 0$ s, 1 s, 3 s
**c** 10 m/s$^2$

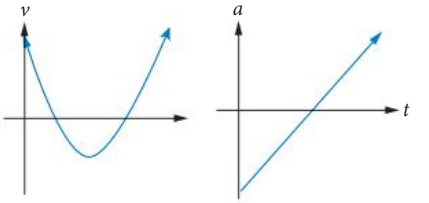
**8** **a**



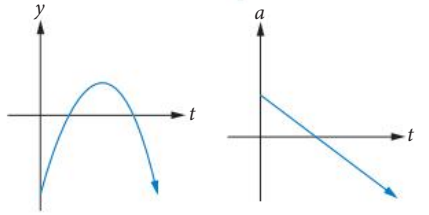

**b**



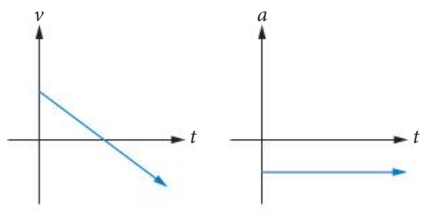

**c**




**d**

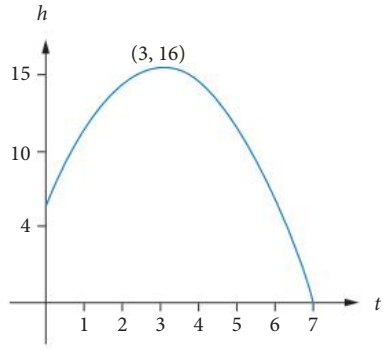



**e**



**9** **a** $v = 5 - 10t$ **b** −95 m/s
**c** $a = -10 = g$

**10** **a** 7 m **b** 16 m **c** After 7 s
**d**




**e** 10 m

**11** **a** $v = 1.72t - 0.0246t^2$ m/s
**b** 69.92 s **c** 1.401 km
**d** 30.07 m/s (108.2 km/h) **e** 1.72 m/s$^2$

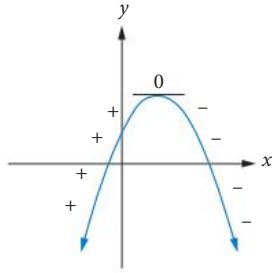
**12** **a** 7.1 m **b** −3 m/s
**c** 0.7 s, the top of the flight.
**d** 7.55 m **e** 1.929 s, it hits the ground.
**f** 17.36 m

## 12.03

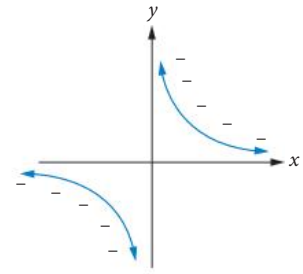
**1** 12 cm
**2** 28 m
**3** −42.5 cm
**4** **a** 570 cms$^{-2}$ **b** 135 cm **c** After 0.5 s
**5** 163 m
**6** **a** 95 cms$^{-1}$ **b** 175 cm
**7** 55.5 cm/s
**8** $h = -4.9t^2 + 4t + 2$
**9** 262 m
**10** **a** 20 cms$^{-2}$ **b** $82\frac{8}{9}$ cm
**11** **a** $v = -10t + 5$ **b** 0.5 s
**c** $h = -5t^2 + 5t + 6$ **d** 7.25 m
**e** $-5t^2 + 5t + 6 = 6$, $t = 1$, $v = -5$ m/s
**12** 2.31 m/s$^2$
**13** **a** About 12.6 m/s, 45.5 km/h

## 12.04

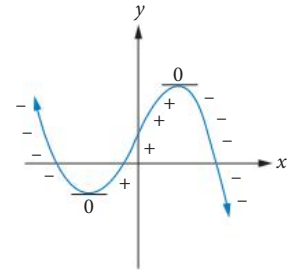
**1** **a**



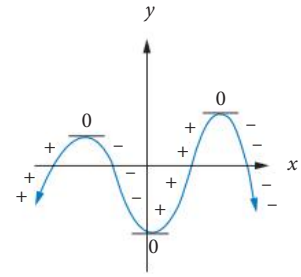

**b**




**c**




**d**




**2** $x < \frac{1}{4}$
**3** $x < 0$
**4** **a** $x < 1.5$ **b** $x > 1.5$ **c** $x = 1.5$
**5** $f'(x) = -2 < 0$ for all $x$
**6–7** Teacher to check proofs.
**8** **a** Decreasing **b** Stationary
**c, d** Increasing
**9** **a** Decreasing **b, c** Increasing
**d** Decreasing
**10** **a** Increasing **b** Stationary
**c** Decreasing **d** Decreasing
**11** **a** Decreasing **b** Stationary
**c** Stationary **d** Decreasing
**12** **a** min (2, −5) **b** max (2, 10)
**c** max $\left(-\frac{4}{3}, 16\frac{14}{27}\right)$, min (2, −2)
**d** max (1, 7), min (−2, −20)
**e** None
**f** min (1, −5), max (−3, 27)
**13** **a** Inflection on rising curve (1, −4)
**b** Inflection on falling curve (3, 9)
**14** **a** min (1, −4) **b** max (0, 9)
**c** max (1, 1) and min (2, 0)


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**d** max (0, 1), min (1, 0) and min (−1, 0)
**e** min (0, 3), inflection on rising curve (1, 4)
**f** both inflection on falling curve, (0, −10), (1, −11)

**15** (2, 0)

**16** $p = -12$

**17** $a = 1\frac{1}{2}$, $b = -6$

**18** **a** Maximum at (2, 1) **b** Maximum at (−1, 5)
**c** Maximum at (−2, 16), minimum at (2, −16)
**d** Minimum at (−1, −2), maximum at (1, 2)
**e** Maximum at (0, 7), minimum at (4, −57)
**f** Minimum at $\left(\frac{1}{8}, -\frac{\pi}{16}\right)$
**g** Maximum at $\left(-\frac{4}{3}, 8\frac{14}{27}\right)$, minimum at (2, −10)
**h** Maximum at $(-\sqrt{2}, 1 + 4\sqrt{2})$, minimum at $(\sqrt{2}, 1 - 4\sqrt{2})$
**i** Minimum at (1, 2)
**j** Minimum at (0, 0), maximum at $\left(5\frac{1}{3}, 75\frac{23}{27}\right)$

## 12.05

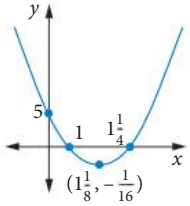
**1** **a** (−2.1, 117) and (4.6, −31)
**b** About 80. **c** About 50.
**d** $-2.1 < x < 4.6$ **e** $x < -2.1$ and $x < 4.6$
**f** 50 **g** −33

**2** **a** (−3.0, 31), (−0.5, −25) and (3.7, 190)
**b** About 20. **c** About 95.
**d** $x < -3.0$ and $-0.5 < x < 3.7$
**e** $-3.0 < x < -0.5$ and $x > 3.7$
**f** About 60.
**g** About 100.

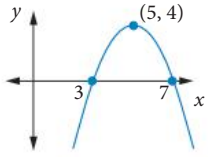
**3** **a** (−0.59, −37.37) and (3.93, 55.07)
**b** $-0.59 < x < 3.93$
**c** −162 **d** 20

**4** **a** (−1.46, −214.9), (2.01, 95) and (5.83, −304.5)
**b** $x < -1.46$ or $2.01 < x < 5.83$
**c** 76 **d** 1

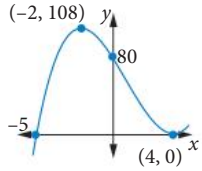
**5** **a** $y = 4x^2 - 9x + 5$



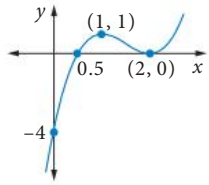

**b** $g(x) = 10x - x^2 - 21$



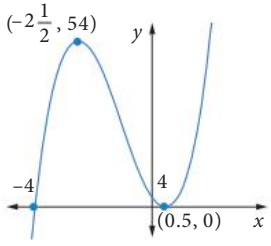

**c**



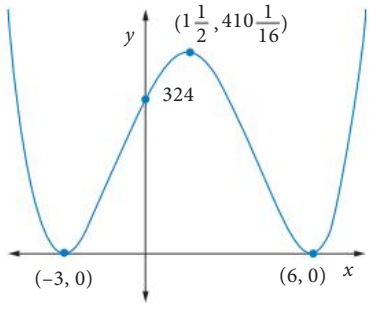

**d** $h(x) = x^2 - x - 8$



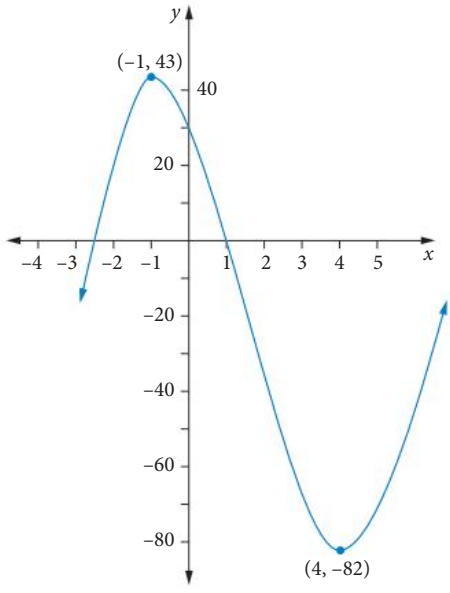

**e** $h(x) = 2x^2 + x - 4$



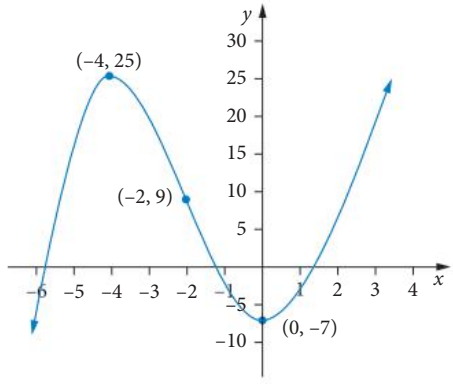

**f** $h(x) = x^2 - x - 8$



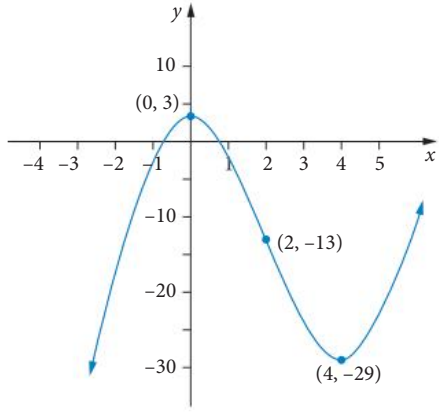

**6**



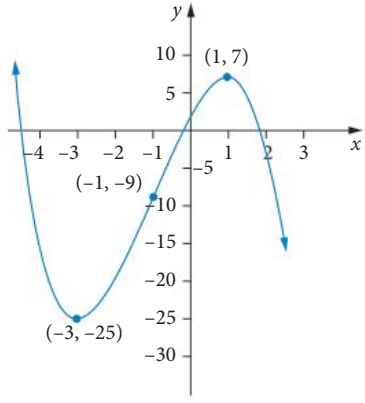
**7**



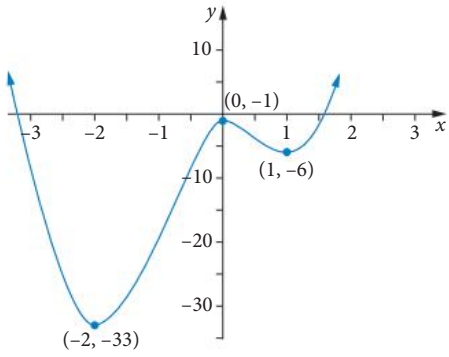

**8**



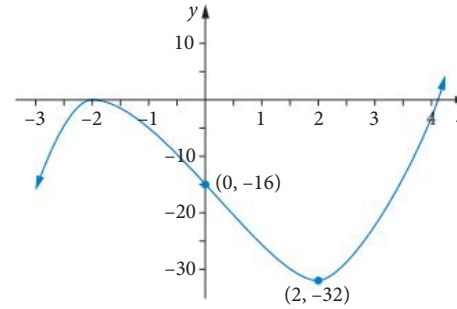

**9**



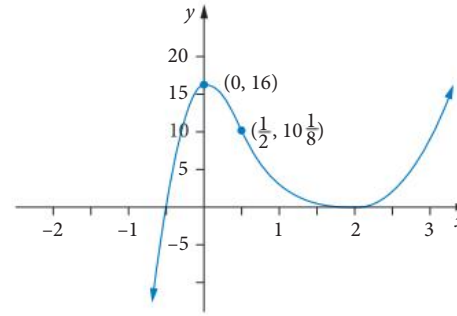

**10**



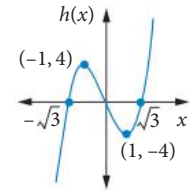

**11**



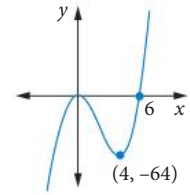

**12**



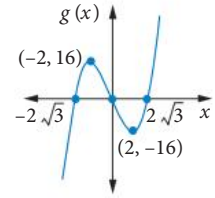

**13 a** $h(x) = 2x^3 - 6x$



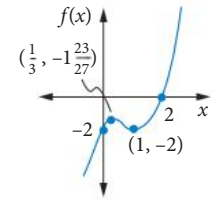

**b** $y = 2x^3 - 12x^2$




**c** $g(x) = x^3 - 12x$



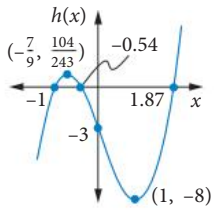

**d** $f(x) = x^3 - 2x^2 + x - 2$



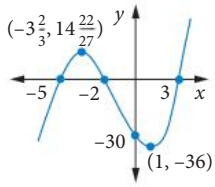


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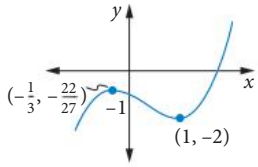
e $h(x) = 3x^3 - x^2 - 7x - 3$



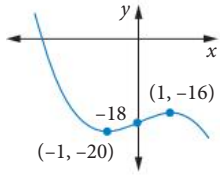

f $y = (x + 5)(x + 2)(x - 3)$



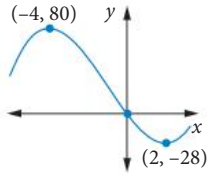

**14** a $q(x) = x^3 - x^2 - x - 1$



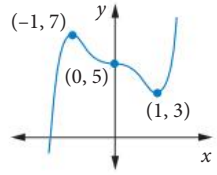

b $f(x) = 3x - x^3 - 18$



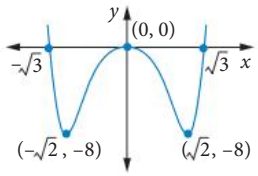

c $f(x) = x^3 + 3x^2 - 24x$



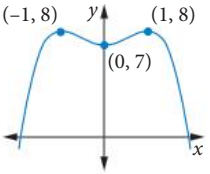

d $f(x) = 3x^5 - 5x^3 + 5$




e $f(x) = 2x^6 - 6x^4$

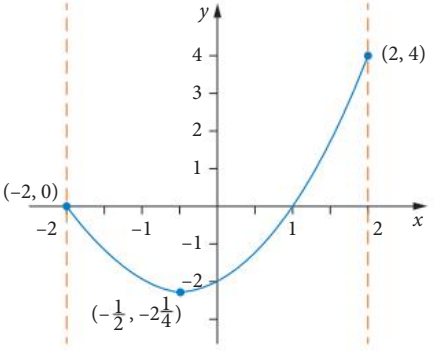



f $f(x) = 7 + 2x^2 - x^4$

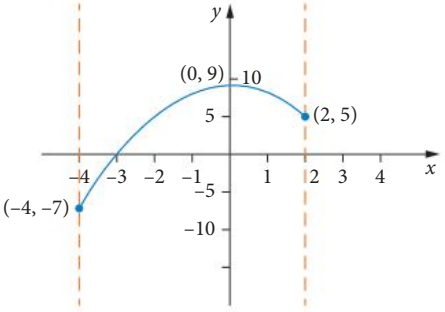



## 12.06

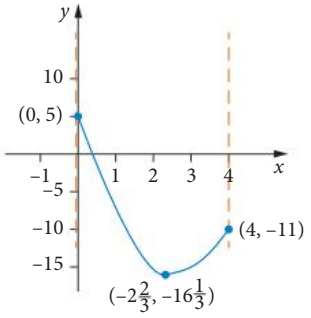
**1** E
**2** C
**3** D
**4** A
**5** Maximum value is 4.



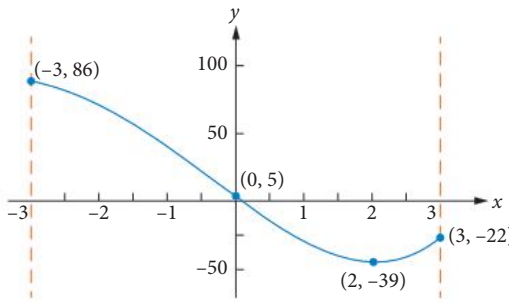

**6** Maximum value is 9, minimum value is −7.




**7** Greatest value is 11 and least value is 2.
**8** Greatest value is about 6.113 and least value is −86.
**9** Greatest value is about 9.71 and least value is −31.
**10** Greatest value is 25.
**11** Greatest value is 5, least value is $-16\frac{1}{3}$.



12 Greatest value is 86, least value is −39.



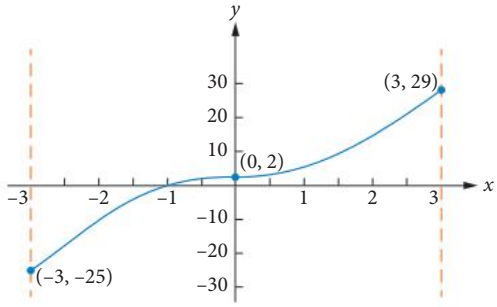

13 Greatest value is −2.

14 Greatest value 29, local maximum (0, −3), Least value −35, local minimum (1, −8)

15 −2.11 and 40

16 −30 and about 48.248

17 Least value −25, greatest value 29.




18 (1, 1)

19 (2, −15)

20 (1, 1) and (3, −5) (gradients of 1 and −11)

## 12.07

1 7.5 km

2 D

3 2 s, 16 m

4 225 $m^2$

5 14 and 14

6 −2.5 and 2.5

7 $d^2 = (200 - 80t)^2 + (120 - 60t)^2 = 10\,000t^2 - 46\,400t + 54\,400$; $d = 24$ km

8 a $V = x(30 - 2x)(80 - 2x) = x(2400 - 220x + 4x^2)$
$= 2400x - 220x^2 + 4x^3$

b $6\frac{2}{3}$ cm c 7407.4 $cm^3$

9 a $d = (x^2 - 2x + 5) - (4x - x^2)$
$= x^2 - 2x + 5 - 4x + x^2 = 2x^2 - 6x + 5$

b $\frac{1}{2}$unit

10 24°C and 4°C

11 26 m

12 46.25 m

13 2048 $cm^3$

14 125 000 $m^2$

15 89.36 $cm^3$

## CHAPTER 12 REVIEW

1 B

2 B

3 A

4 D

5 D

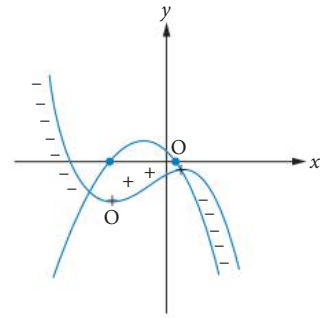
6 C

7 B

8 B

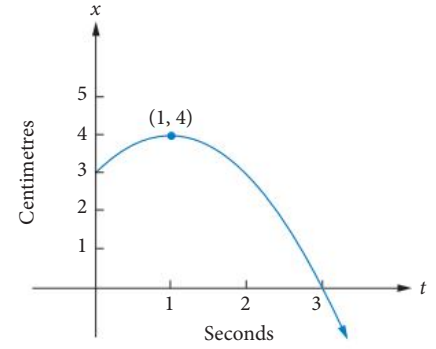
9 13

10




11 a 2 b 1 c −4
d 6 e −7

12 a 2 $cms^{-1}$ b Acceleration = $-2$ $cms^{-2}$
c After 3 s d Maximum displacement is 4 cm
e




13 0.6 m, −0.6 m/s, $-0.2$ $m/s^2$

14 a 570 $cms^{-2}$ b 135 cm c After 0.5 s

15 a 95 $cms^{-1}$ b 175 cm

16 262 m

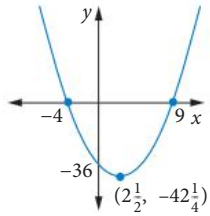
17 a $x < 1.5$ b $x > 1.5$ c $x = 1.5$

18 a About (−0.4, 93) and (5.8, −27).
b About 77. c About −30.
d Increasing for $x < -0.4$ and $x > 5.8$; decreasing for $-0.4 < x < 5.8$

19 a About (−2.60, −14.13) and (1.60, 60.13).
b About $x < -2.60$ and $x > 1.60$.

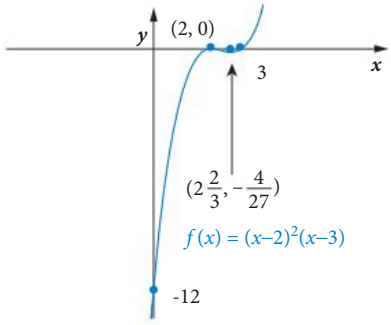

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**20** a

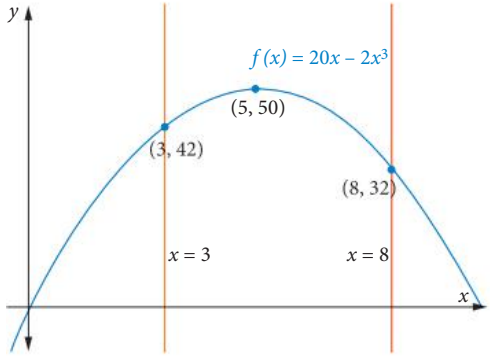



$f(x) = x^2 - 5x - 36$

b



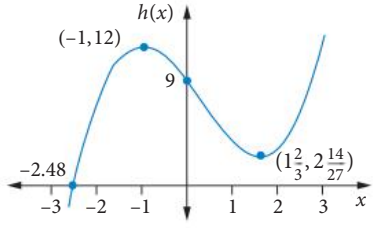

**21** Greatest value = 50, least value = 32



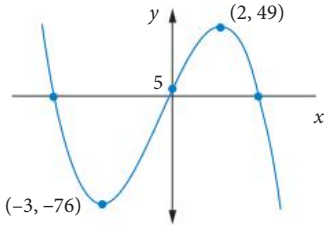

**22** 4 and −16

**23** 150 products

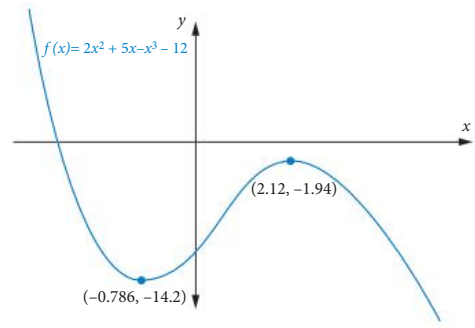
**24** $h(x) = x^3 - x^2 - 5x + 9$




**25** $y = 36x - 3x^2 - 2x^3 + 5$




**26**




**27** Least value = 0, greatest value = 513

**28** a $y = 0.75x$

b Rate of increase of shadow length = 0.75 × walking speed

c i 0.9 m/s ii 0.9 m/s

**29** a 50 m b −20.58 m

c 70.58 m d −1.51

e $\left(-6\frac{2}{3}, 14.71\right)$ f −2.27

**30** About 45.6 km ($d^2 = 952\,400 - 213\,800t + 12\,025t^2$).

**31** Maximum volume = 42.4 cm$^3$ (when $x = 1.35$ cm)

## MIXED REVISION 4

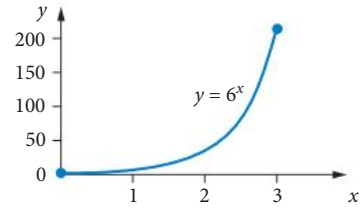
**Multiple choice**

**1** D
**2** D
**3** B
**4** B
**5** A
**6** C
**7** C
**8** E
**9** A

**Short answer**

**1** Decreasing

**2**




a $x \approx 2.96$ b $x \approx 1.27$

**3** a 100 L b 40 L

c −16 L per minute, i.e. leaking at the rate of 16 L per minute

d 12.2 minutes

**4** $\left(\sqrt{2}, 2\sqrt{2}+1\right)$ and $\left(-\sqrt{2}, -2\sqrt{2}+1\right)$

**5** $x \approx 0.793$

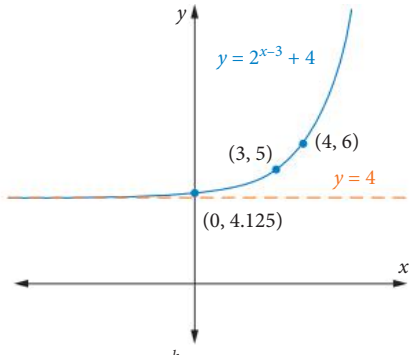
**6** Maximum (−3, −35), minimum (−2, −36)

**Application**

**1** 28 L/min

**2** $f(x) = 2x^2 - 3x^3 - 7$

**3**



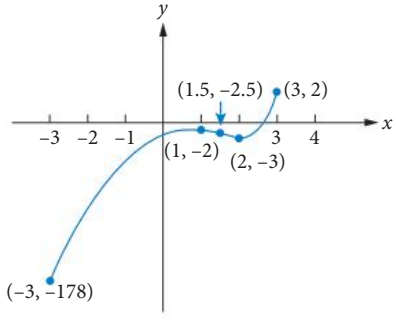

**4** **a** $N = 300 \times 1.08^h$ **b** 555 bacteria

**c** 12 063 bacteria **d** 54.57 h

**e** 84 .49 h

**5** **a** Local maximum at (1, –2); local minimum at (2, –3).

**b**




**6** **a** $h = \frac{160}{2\pi r} - r$: $V = \pi r^2 h$ so $V = 80r - \pi r^3$

**b** $r \approx 2.91$ cm

**c** $V \approx 155.4$ cm$^3$


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# GLOSSARY

## Chapter 1

**binomial product** The product of two binomial expressions such as $(3x + 2)(x + 4y)$.

**coefficient** The number multiplying a variable. In a polynomial the coefficients are $a_0, a_1, a_2 \ldots a_{n-1}, a_n$.

**constant** A fixed value such as 3 or π. A number.

**cubic** A polynomial of degree 3.

**degree** The highest power of the variable of a polynomial.

**dependent variable** A variable that is considered to be calculated from others. It is normally the vertical scale of a graph.

**direct proportion** A relationship between two variables such that they increase or decrease in the same ratio.

**discriminant** The discriminant of the quadratic equation, $ax^2 + bx + c = 0$, is $\Delta = b^2 - 4ac$.

**factor theorem** For a polynomial, $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial.

**formula** An equation with a single variable on the left-hand side that shows a fixed relationship between variables.

**highest common factor** The largest term that divides evenly into every term of an expression.

**independent variable** A variable that is considered to be used to calculate others. The variable on the horizontal scale of a graph.

**inverse proportion** Two variables related so that as one increases the other decreases in the same ratio.

**leading term** The term of a polynomial that contains the highest power.

**linear** Relating to a straight line. The power of any variable in a linear relationship is one.

**monic** A polynomial such that the coefficient of the leading term is 1.

**null factor law** If the product of two terms is zero, then at least one of the terms must equal zero. If $ab = 0$, then $a = 0$, $b = 0$ or $a = b = 0$.

**polynomial** A sum of terms containing non-negative integer powers of the variable(s). A polynomial in $x$ is of the form:

$P(x) = a_n x_n + a_{n-1} x_{n-1} + a_{n-2} x_{n-2} + \ldots + a_1 x_1 + a_0$, where $n$ is a positive whole number and $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.

**quadratic** A polynomial in one variable of degree two. It has the form $ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$.

**quadratic formula** The formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the solutions of the quadratic equation $ax^2 + bx + c = 0$.

**quartic** When you expand 4 linear factors, you get a quartic polynomial. It is a polynomial of degree 4.

**remainder theorem** When a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

**root** A value that satisfies an equation. It is a solution of the equation.

**simultaneous equations** A set of equations with more than one variable. Graphically, the solution of two simultaneous equations in two variables is the point of intersection of the graphs of the equations.

**subject** The single variable on the left-hand side of a formula.

**trinomial** An expression with three terms such as $3x^2 + 4x + 7$.

## Chapter 2

**and** $A$ and $B$ means the intersection of two events $A$ and $B$.

**atomic** Sample space elements expressed in simplest possible form.

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**binomial coefficient** Coefficients in the expansion of $(x + y)^n$, given by $\binom{n}{r} = {}^nC_r$.

**binomial expansion** Expansions of the type $(x + y)^n$.

**binomial theorem** The expansion $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \ldots + {}^nC_rx^{n-1}y^r + \ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$.

**cardinality** The size of a set. For finite sets it is the number of elements.

**certain** An event that is the whole sample space, so its probability is 1.

**combination** An un-ordered selection of some or all of the objects or symbols from a group.

**complement** The elements of the universal set not in a given set. For a sample space, the elements not in the event. The complement of $A$ is shown as $\bar{A}$ or $A'$.

**disjoint** Sets or events with no elements in common.

**element** One of the members of a set or sample space. $a$ is an element of $S$ is written as $a \in S$.

**elementary outcome** One of the elements of a sample space.

**event** A subset of a sample space. Something that could happen.

**expected frequency** The number of times an event would be expected to occur in a given number of trials.

**impossible** An event that is the null set, so its probability is 0.

**integers** The positive, negative and zero whole numbers $\boldsymbol{J} = \boldsymbol{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

**intersection** The elements that two or more sets have in common. The intersection of $A$ and $B$ is written as $A \cap B$.

**member** A member of a set is one of the elements. $a$ is a member of $S$ is written as $a \in S$.

**mutually exclusive** Sets or events with no elements in common. Their intersection is the null set.

**natural numbers** The counting numbers $\boldsymbol{N} = \{1, 2, 3, 4, \ldots\}$.

**not** Not $A$ means the complement of the event $A$, shown as $\bar{A}$ or $A'$.

**null set** The set with no elements, written as $\{\}$ or $\varnothing$.

**or** $A$ or $B$ means the union of two events $A$ and $B$.

**probability** A number $0 \leq P(A) \leq 1$ associated with the event $A$, a subset of the sample space. It measures the chance that the event will happen.

**proper subset** A subset that is smaller than the set it is a part of. $A$ is a proper subset of $B$ is written as $A \subset B$.

**random** The kind of choice where all elementary outcomes of the sample space have equal probabilities. Every possible choice has an equal chance.

**relative frequency** An estimate of probability obtained by dividing the frequency of an event by the number of observations.

**sample space** A set associated with individual occurrences in a real situation. Elements of the set are called elementary outcomes.

**set** A well-defined collection of objects, usually shown by a capital letter.

**subset** A set whose elements are all elements of another set. $A$ is a subset of $B$ is written as $A \subseteq B$.

**trial** A single observation or experiment in a probability situation.

**uniform probability distribution** A finite sample space whose elementary outcomes have equal probabilities.

**union** The elements that belong to any of two or more sets. The union of $A$ and $B$ is written as $A \cup B$.

**universal set** The fixed set containing all the elements in the sets used in a particular context.


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**Venn diagram** A diagram showing sets as circles with intersections shown as overlapping of the circles. The circles are enclosed by a rectangle representing the universal set.

**well-defined** A collection in which there is no ambiguity as to whether or not an object belongs to the collection.

## Chapter 3

**angle of depression** The angle formed by looking from the horizontal down to an object below.

**angle of elevation** The angle formed by looking from the horizontal up to an object.

**arc** A portion of the circumference of a circle.

**circular measure** The measure of angle size in radians.

**cosine rule** The lengths of the sides of a triangle are related to the cosine of one of its angles by the equation : $c^2 = a^2 + b^2 - 2ab\cos C$.

**extended angle measure** The measurement of angles beyond a full turn.

**included angle** The angle between two adjacent sides.

**radian** Angle measure defined as the arc length subtended by the angle at the centre of a unit circle with its centre at the vertex of the angle: $\pi$ radians = 180°.

**sector** An area inside a circle bounded by the centre, two radii and an arc.

**segment** An area inside a circle bounded by a chord and an arc.

**sine rule** The lengths of the sides of a triangle are related to the sines of its angles by the equation $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.

## Chapter 4

**asymptote** A line approached by a curve as $x \to a$ or $x \to \pm\infty$. It can be horizontal, vertical (for a fixed value $a$) or sloping.

**axis of symmetry** A line about which a graph is reflected equally on both sides.

**codomain** The set into which a relation or functions maps elements of the domain.

**concave** The shape of the curve of a graph in an upwards or downwards direction.

**dependent variable** A variable that is calculated from another variable according to a rule or equation.

**dilation** Narrowing or expansion of a graph.

**discontinuous** A function with a break in its graph.

**discriminant** The value $\Delta = b^2 - 4ac$ calculated from the quadratic equation $ax^2 + bx + c$ that gives information about the zeros of the quadratic function.

**domain** The set of values of the independent variable $x$ of a relation or function.

**even** A function such that $f(x) = f(-x)$ for all values of $x$.

**function** A relation such that each value in the domain has only one ordered pair.

**gradient** The slope of a line given by $m = \frac{\Delta x}{\Delta y}$.

**hyperbola** A discontinuous graph in the basic form $y = \frac{a}{x-b}$.

**independent variable** A variable that is used to calculate the value of another variable.

**intercept** The point where a graph crosses the $x$- or $y$-axis.

**linear function** A function of the form $ax + by + c = 0$ with a straight-line graph.

**midpoint** The point halfway between two points.

**odd** A function such that $f(-x) = -f(x)$ for all values of $x$.

**parabola** The shape of the graph of a quadratic function.

**point of inflection** A point where a graph changes concavity from an upwards shape to a downwards shape or vice versa.

**quadratic function** A function of the form $y = ax^2 + bx + c$.

**range** The set of values of the dependent variable $y$ of a relation or function.

**relation** A set of ordered pairs $(x, y)$ where $x \in X$ is considered as mapped to the value $y \in Y$.

**translation** Horizontal or vertical movement of a graph.

**turning point** A maximum or minimum point where a graph changes from decreasing to increasing or vice versa.

**zeros** The $x$ values of a function $f(x)$ where $f(x) = 0$.

## Chapter 5

**combination** An un-ordered selection of some or all of the objects or symbols from a group.

**complement** The elements of the universal set not in a given set. For a sample space, the elements not in the event. the complement of $A$ is shown as $\bar{A}$ or $A'$.

**conditional probability** The conditional probability of an event $A$ given event $B$, written as $P(A|B)$, is the probability that event $B$ occurs, assuming that event $A$ occurs.

**dependent events** Events where the outcome of one affects the probability of the other. Formally, $P(A|B) \neq P(A)$.

**independent events** Events where the outcome of one does not affect the probability of the other. Formally, $P(A|B) = P(A)$.

**intersection** The elements that two or more sets or events have in common. The intersection of $A$ and $B$ is written as $A \cap B$.

**mutually exclusive** Sets or events with no elements in common. Their intersection is the null set.

**tree diagram** A tree diagram of a probability situation shows successive stages of a sample space as branching.

**two-way table** A two-way table has frequencies or elements of a sample space with two characteristics listed in rows and columns.

**union** The elements that belong to any of two or more sets. The union of $A$ and $B$ is written as $A \cup B$.

**Venn diagram** A diagram showing sets as circles with intersections shown as overlapping of the circles. The circles are enclosed by a rectangle representing the universal set.

## Chapter 6

**amplitude** The vertical distance from the centre of a periodic function to its maximum or minimum.

**circular measure** The measure of angle size in radians.

**periodic** A graph or curve that repeats its shape regularly.

**phase** The horizontal translation of a trigonometric function such as $y = \sin(x + c)$ from the basic function $y = \sin(x)$. $c$ is called the phase shift.

**radian** Angle measure defined as the arc length subtended by the angle at the centre of a unit circle with its centre at the vertex of the angle: $\pi$ radians = 180°.

**sinusoidal** A function or graph that has the general form of the sine function; a sine function, cosine function or combination.

**trigonometric functions** Functions involving the trigonometric ratios sine, cosine and tangent.

**trigonometric identities** Special relationships between trigonometric ratios that are true for all values.

## Chapter 7

**acceleration** The rate of change of velocity with respect to time.

**average rate of change** The rate of change of one variable with respect to another between two points. For a function, $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

**chord** The segment of a straight line between two points on a curve.


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**dependent variable** A variable that is considered to be calculated from others. In a rate of change, it is the variable in the numerator. The variable on the vertical scale of a graph.

**displacement** The change of position of an object from its starting point to its end point, without regard for the path travelled.

**gradient of a graph** The gradient of the tangent to a graph.

**independent variable** A variable that is considered to be used to calculate others. In a rate of change, it is the variable in the denominator. The variable on the horizontal scale of a graph.

**instantaneous rate of change** The gradient of the tangent to a graph at the time for which the instantaneous rate of change is required.

**rate of change** The rate of change of $y$ with respect to $x$ is the quotient of their changes.

Rate of change $= \frac{\Delta y}{\Delta x}$.

**secant** A straight line that passes through two points on a curve.

**tangent** A straight line that touches a curve.

**velocity** The rate of change of displacement or position.

## Chapter 8

**arithmetic sequence (progression)** A sequence in which each term differs from the previous by a constant value $d$.

**common difference** The constant value $d$ which is added to any term of an arithmetic sequence to obtain the next term.

**common ratio** The constant $r$ by which any term of a geometric sequence is multiplied to get the next term.

**geometric sequence (progression)** A sequence in which any term is multiplied by a constant $r$ to get the next term.

**limiting sum (or sum to infinity)** A geometric series converges if $-1 < r < 1$. The limit of this sum is called the limiting sum or sum to infinity.

**partial sum** A sequence whose terms are the partial sums of a certain finite number of terms of a sequence.

**sequence** An ordered list; a function whose domain is the set of natural numbers.

**series** A sequence whose terms are the partial sums of another sequence.

**term** One of the values of a sequence. The term is referred to by its position, so the first term $t_1$ is the first number in the sequence.

## Chapter 9

**acceleration** The rate of change of velocity with respect to time.

**average rate of change** The rate of change of a variable with respect to another between two points. For a function, $\frac{f(x_2)-f(x_1)}{x_2-x_1}$.

**binomial theorem** The expansion of $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$.

**chord** The segment of a straight line between two points on a curve.

**derivative** For a function $f(x)$, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists.

**differentiation** The process of finding a derivative.

**displacement** The change of position of an object from some starting point, without regard for the path travelled.

**distance** The length of a path.

**first principles** A definition. Finding from first principles means to find using the definition.

**gradient** The slope of a line, given by $\frac{\Delta y}{\Delta x}$ for a straight line. For a curve it is the slope of the tangent.

**instantaneous** At a particular point or time.

**limit** The limit of a function is a value that it approaches as the variable approaches a particular value from either direction. If it exists it is written as $\lim_{x \to c} f(x)$.

**rate of change** The rate of change of $y$ with respect to $x$ is the quotient of their changes. Rate of change $= \frac{\Delta y}{\Delta x}$.

**secant** A straight line that passes through two points on a curve.

**tangent** A straight line that touches a curve at a point.

**velocity** The rate of change of displacement or position.

## Chapter 10

**antiderivative** An antiderivative of a function $f(x)$ is a function $F(x)$ whose derivative is the given function. $F'(x) = f(x)$. It is also called an indefinite integral or a primitive of $f(x)$.

**antidifferentiation** The reverse of differentiation to find an antiderivative.

**average rate of change** The average rate of change of a function $f(x)$ is given by

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

**decreasing** A function is decreasing when the gradient function or derivative is negative.

**derivative** The derivative of the function $f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

**differentiation** The process for finding a derivative of a function.

**gradient function** The gradient function is the derivative of a given function.

**increasing** A function is increasing when the gradient function (derivative) is positive.

**instantaneous rate of change** The instantaneous rate of change of a function $f(x)$, is the derivative of the function.

**integral** An indefinite integral (or just integral) is an antiderivative.

**integration** The reverse of differentiation, used to find an indefinite integral (antiderivative).

**linear combination** A linear combination of variables or functions is the sum and/or difference of scalar multiples of the variables or functions.

**linear product** A product of a function and a constant.

**primitive** A primitive is an antiderivative.

**stationary** A function is stationary when the gradient function or derivative is zero.

**tangent** A straight line that touches a curve at a point.

## Chapter 11

**base** The value $a$ in an expression of the form $a^n$.

**decay function** A function of the form $f(x) = Aa^x$ where $a$ is less than 1.

**exponent** The value $n$ in an expression of the form $a^n$.

**exponential equation** An equation where the variable is part of an exponent in the equation.

**exponential function** A function of the form $f(x) = Aa^x$ where $a > 0$ and $a \neq 1$.

**growth factor** The value $a$ in a growth or decay function of the form $f(x) = Aa^x$.

**growth function** A function of the form $f(x) = Aa^x$ where $a$ is greater than 1.

**half-life** The time required for a quantity to decay to half of its initial value.

**horizontal asymptote** An asymptote parallel to the $x$-axis of the form $y = a$.

**index** The value $n$ in an expression of the form $a^n$; an exponent.

**indicial equation** An equation where the variable is part of an exponent in the equation; an exponential equation.

**initial value** The value $A = f(0)$ in a growth or decay function of the form $f(x) = Aa^x$.


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**power** An expression of the form $a^n$ formed as the product of $n$ factors $a$: $a^n = a \times a \times a \times \ldots a \times a$ (to $n$ factors).

**radical** The square root sign, $\sqrt{\ }$ or numbers such as $\sqrt{7}$ or $\sqrt{12}$.

**radiocarbon dating** A dating method that uses the naturally occurring isotope carbon-14 ($^{14}C$) to determine the age of a material containing carbon.

**scientific notation** A number written in the form $a \times 10^n$ where $1 \leq a < 10$, such as $283 = 2.83 \times 10^2$.

**significant figures** The number of digits of a number that determine its precision; the number of digits in the non-power part of the number when written in scientific notation.

**transformation** A change to a graph or function by translation or dilation in the $x$ or $y$ direction or both to produce a new function. The transformation $y = Aa^{x+b} + c$ dilates the exponential function $y = a^x$ by the factor $A$ in the $y$-direction and translates the function $b$ to the left and $c$ up.

## Chapter 12

**acceleration** The rate of change of velocity with respect to time: $a = \frac{dv}{dt}$.

**average rate of change** The rate of change of a function $f(x)$ from $x_1$ to $x_2$ given by $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$.

**displacement** The change of position of an object from some starting point without regard for the path travelled, usually shown as $s$.

**dominant term** The term of a polynomial function that has the highest power. As $x \to \pm\infty$, the magnitude of the dominant term becomes much larger than the magnitudes of the other terms, so the function behaves like the dominant term.

**flow rate** The rate of change of volume of a fluid, especially flowing through a pipe.

**global maximum** The greatest value of a function over its domain.

**global minimum** The least value of a function over its domain.

**gradient function** The derivative of a function.

**greatest value** The largest value of a function over a given domain.

**instantaneous rate of change** The derivative of a function at a given point.

**least value** The smallest value of a function over a given domain.

**local maximum** A value of a function, $f(x)$, larger than the values at all nearby points on either side. It occurs at the point $a$ when there is a stationary point at $x = a$, $f'(x) > 0$ when $x < a$ and $f'(x) < 0$ when $x > a$.

**local minimum** A value of a function, $f(x)$, smaller than the values at all nearby points on either side. It occurs at the point $a$ when there is a stationary point at $x = a$, $f'(x) < 0$ when $x < a$ and $f'(x) > 0$ when $x > a$.

**optimisation** The process of finding the conditions to produce the best possible (optimal) outcome, often by finding a minimum or maximum.

**point of horizontal inflection** A point of a function, $f(x)$, that is a stationary point (at $x = c$), and the sign of $f'(x)$ is the same for $x < c$ and $x > c$.

**stationary point** A point where the derivative of the function is zero is called a stationary point.

**velocity** The rate of change of displacement ($s$) with respect to time: $v = \frac{ds}{dt}$.

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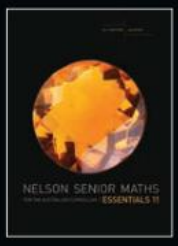
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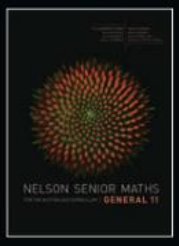
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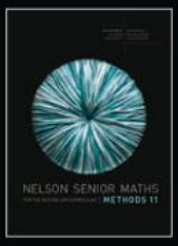



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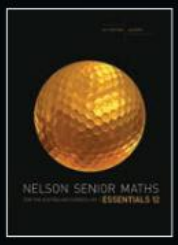



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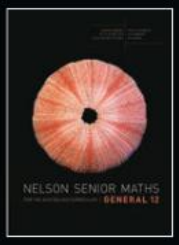
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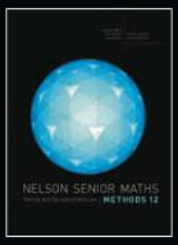



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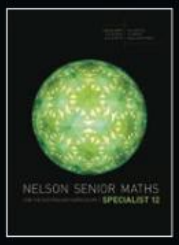

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