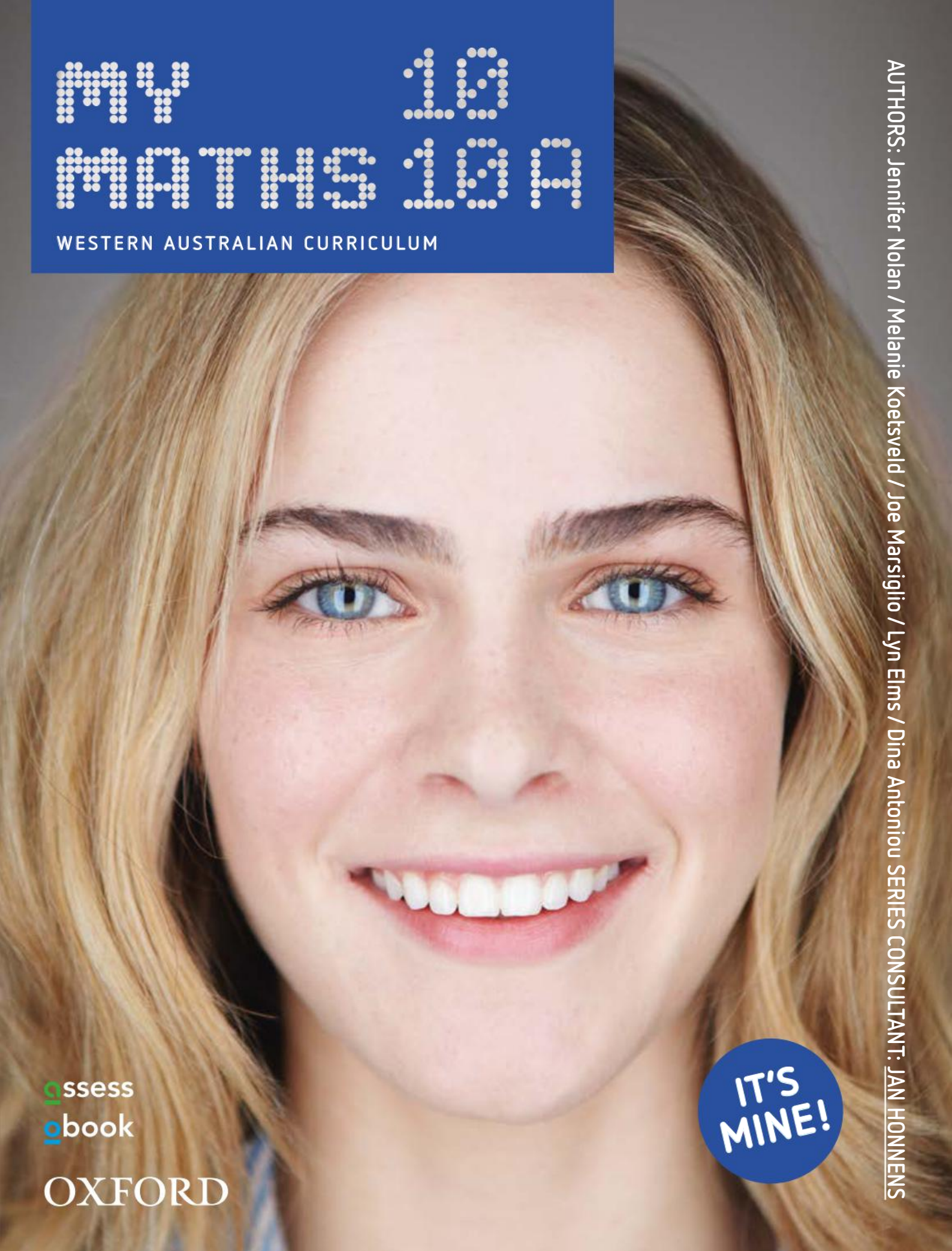
MY MATHS 10

MY MATHS 10A

WESTERN AUSTRALIAN CURRICULUM



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AUTHORS: Jennifer Nolan / Melanie Koetsveld / Joe Marsiglio / Lyn Elms / Dina Antoniou SERIES CONSULTANT: JAN HONNENS

MY MATHS

1&2

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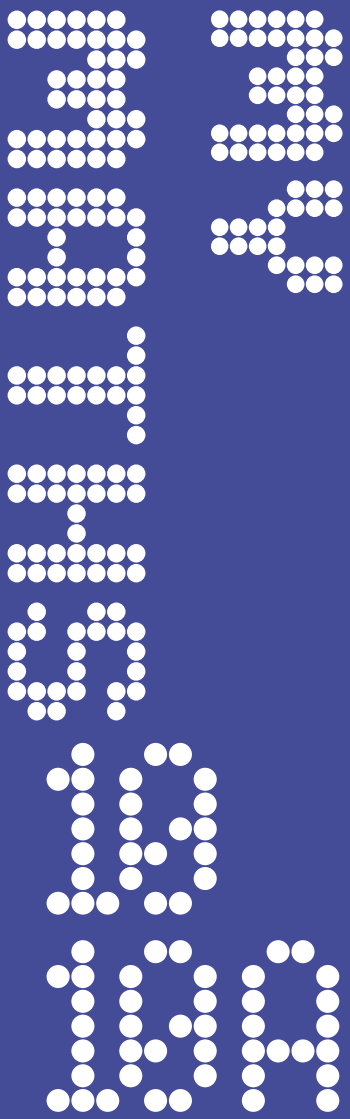
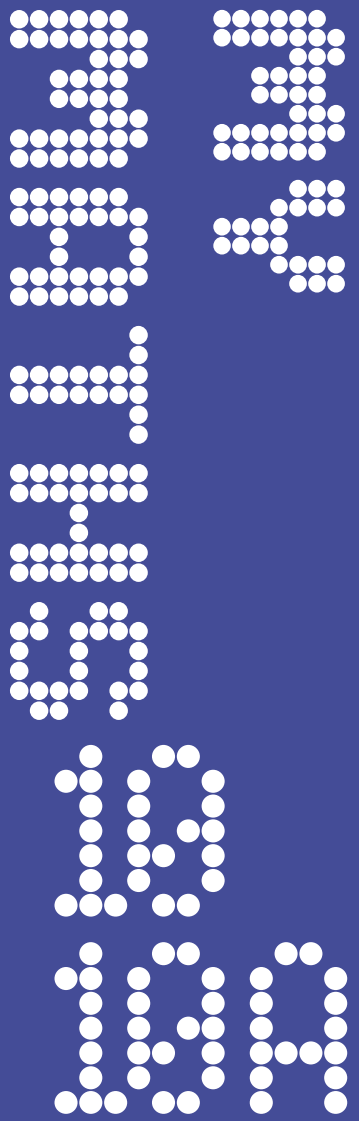
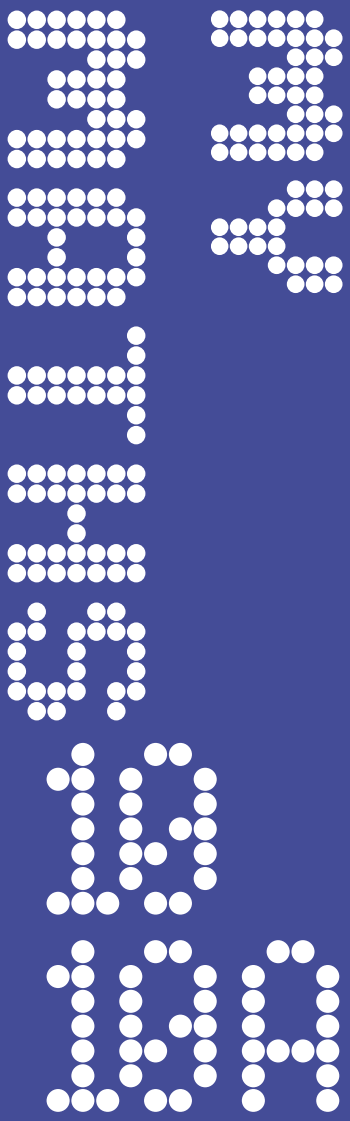




MEASUREMENT AND GEOMETRY

STATISTICS AND PROBABILITY



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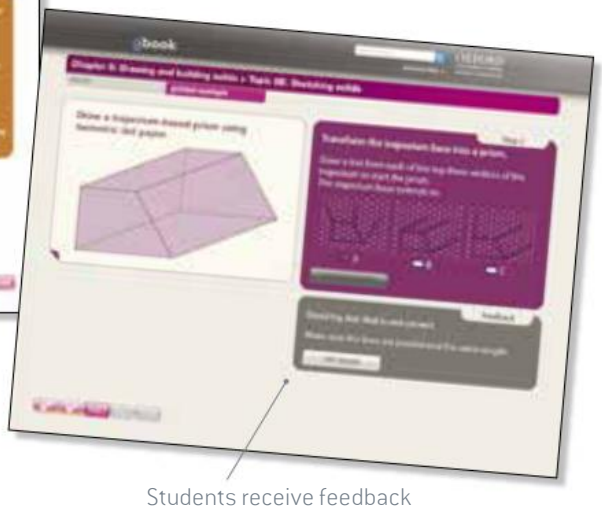

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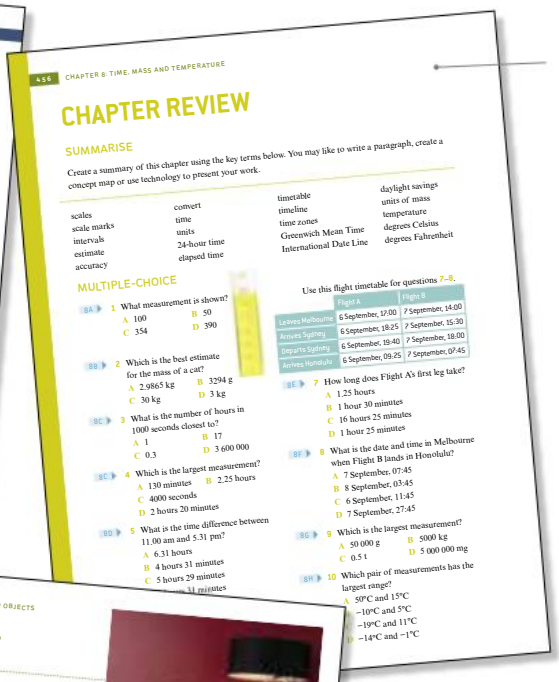

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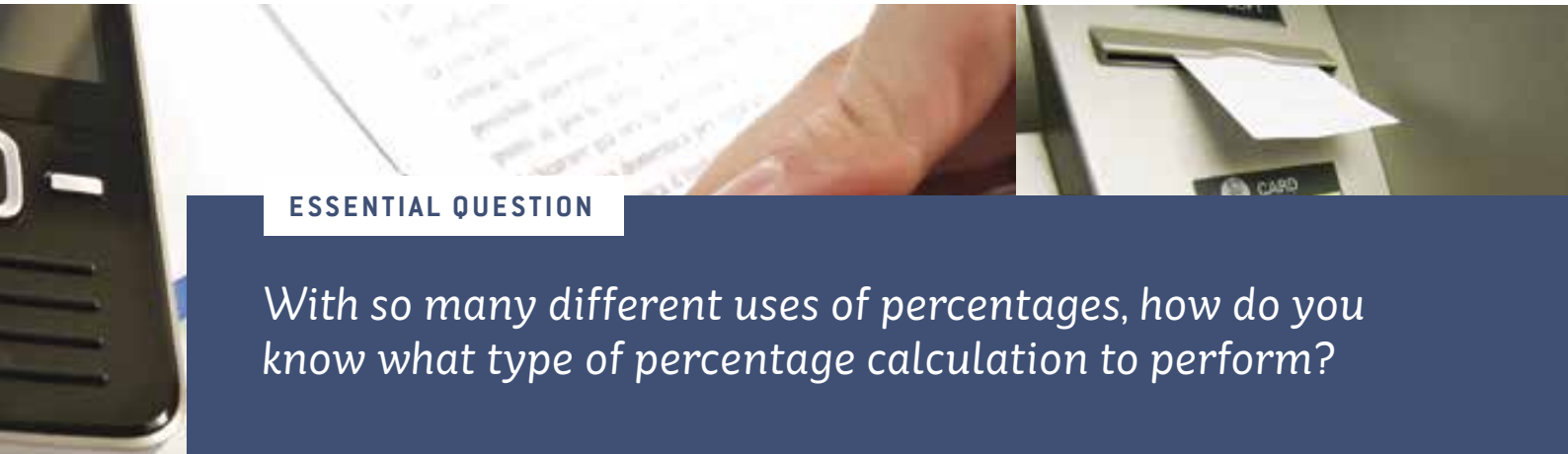
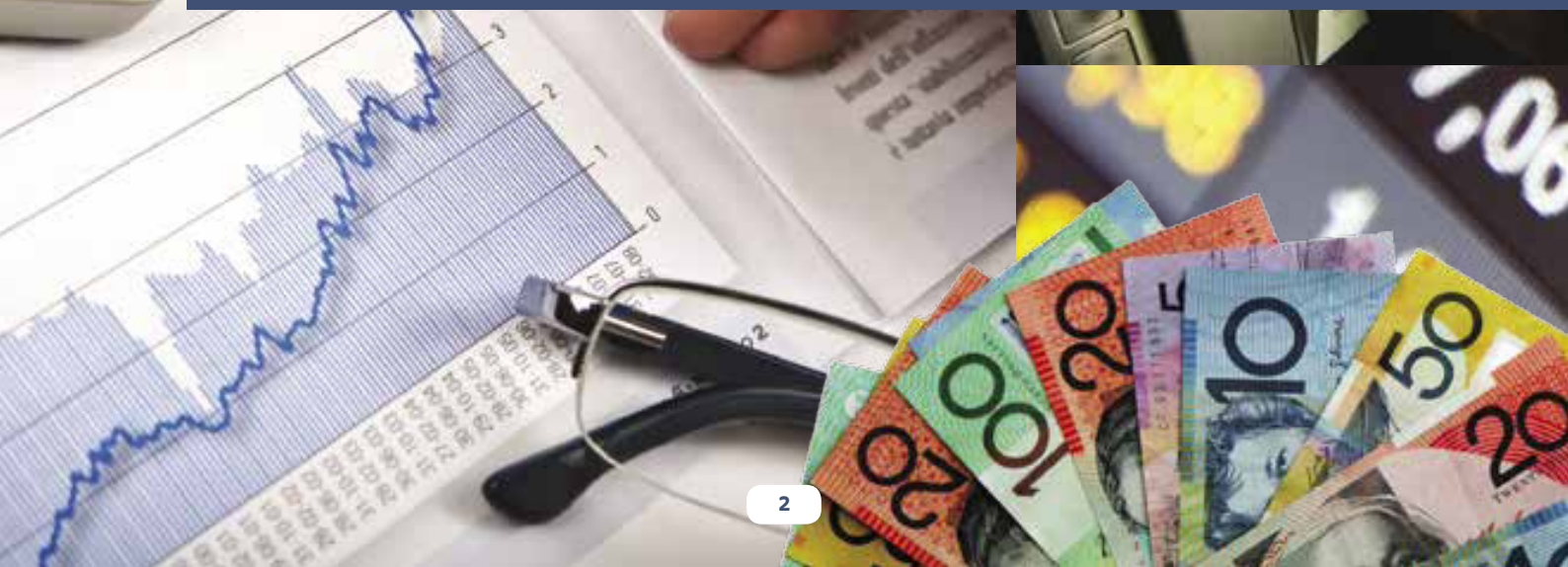
1 FINANCIAL MATHEMATICS



ESSENTIAL QUESTION

With so many different uses of percentages, how do you know what type of percentage calculation to perform?



Are you ready?

1A **1** What is $\frac{3}{4}$ as a percentage?

1A **2** The decimal 0.65 is equivalent to which percentage?

A 65% **B** 0.0065%

C 0.65% **D** 0.65

1A **3** What is the percentage $10\frac{1}{2}\%$ equivalent to as a decimal?

1A **4** Which represents 35% of \$200?

A $35 \times \$200$ **B** $\frac{35}{200} \times 100$

C $0.35 \times \$200$ **D** $\frac{65}{100} \times \$200$

1A **5** What is 22% of \$440?

1A **6** Jackson scored 21 out of a possible 24 marks on his Maths test.

a Write Jackson's marks as a fraction of the total marks in simplest form.

b What is Jackson's score as a percentage?

1B **7** A book store offers a 15% discount on all their books. What is the sale price of a book originally priced at \$25.40?

A \$3.81 **B** \$21.59

C \$10.40 **D** \$29.21

1C **8** \$2000 is invested for 15 months at an interest rate of 6.4% p.a. What are the values for the variables of the simple interest formula?

A $P = \$2000$, $R = 6.4$ and $T = 15$

B $P = \$2000$, $R = 6.4$ and $T = 1.25$

C $P = \$2000$, $R = 0.064$ and $T = 1.25$

D $P = \$2000$, $R = 0.064$ and $T = 15$

1F **9** **a** How many months are in 3 years?

b How many quarters are in 2 years?

c How many weeks are in 4 years?

d How many quarters are in 7 years?

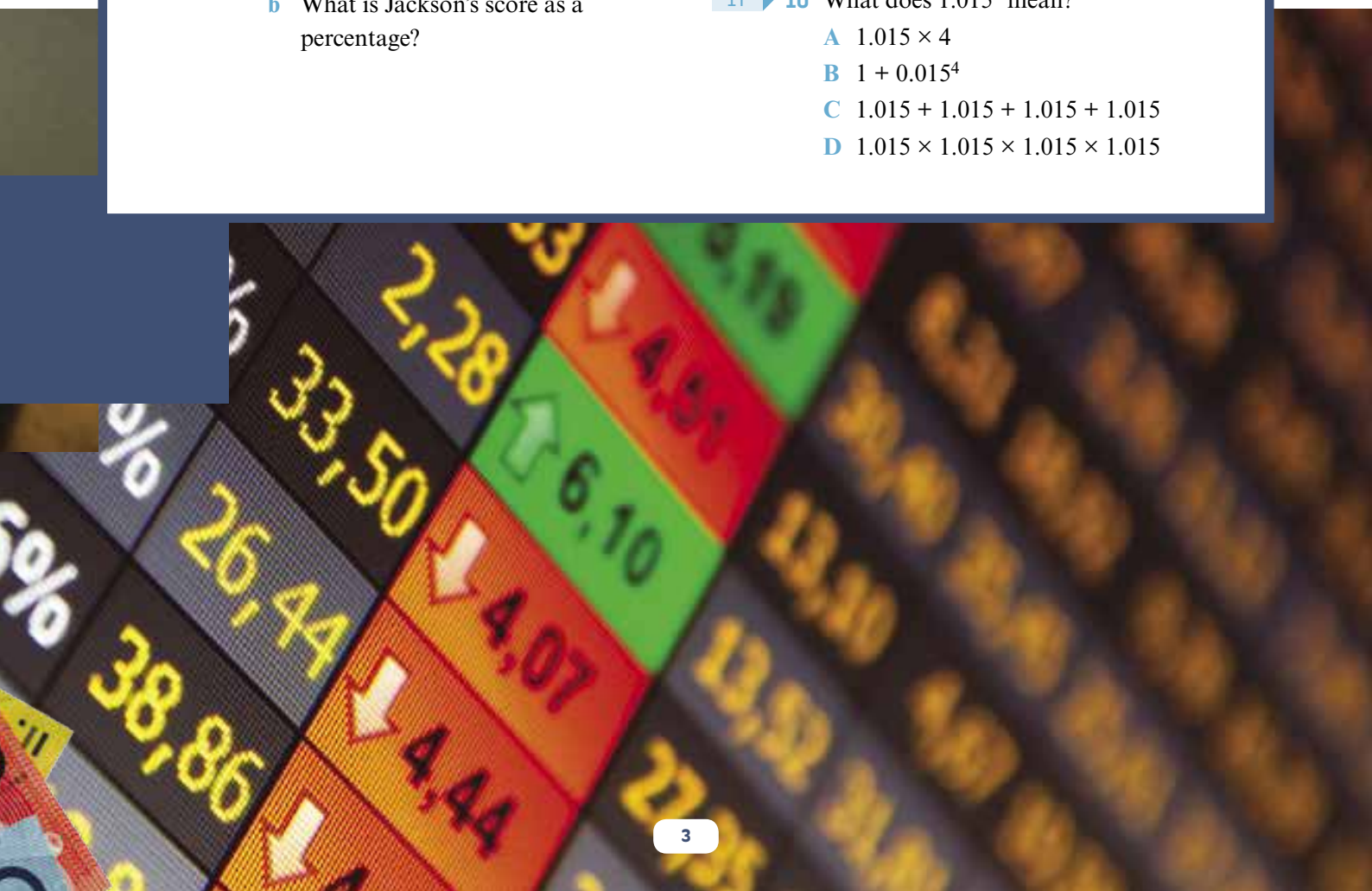
1F **10** What does 1.015^4 mean?

A 1.015×4

B $1 + 0.015^4$

C $1.015 + 1.015 + 1.015 + 1.015$

D $1.015 \times 1.015 \times 1.015 \times 1.015$



1A Review of percentages

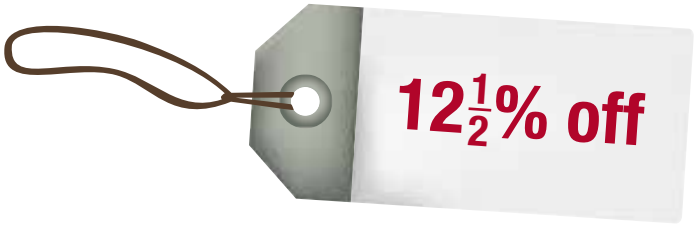
Start thinking!

Percentages, fractions and decimals are different forms that can be used to represent the same amount. Consider this table displaying the equivalent forms of three different amounts as percentages, fractions and decimals.

Percentage	Fraction	Decimal
19%	$\frac{19}{100}$	0.19
10%	$\frac{1}{10}$	0.10
80%	$\frac{4}{5}$	0.80

1 Consider each row of the table and use the values to help you explain how to convert:
- **a** a percentage to a fraction
- **b** a percentage to a decimal.

2 Reconsider each row of the table and use the values to help you explain how to convert:
- **a** a fraction to a percentage
- **b** a decimal to a percentage.



3 The value given in the price tag above advertises a **discount** amount written as a percentage.
- **a** Use your explanations from question **1** to write this percentage as a fraction and a decimal.
- **b** What was the main difference between the calculations of this in question **1** and the calculations performed in part **a**?

4 Copy and complete the missing values in this table. Remember to write the fractions in their simplest form.

Percentage	Fraction	Decimal
35%		0.35
	$\frac{2}{5}$	
		0.12
	$\frac{1}{8}$	
$15\frac{1}{2}$%		

The skills you use to convert between percentages, fractions and decimals are widely used in financial calculations.

KEY IDEAS

- Fractions, decimals and percentages are all closely related. To convert:
  - a percentage to a decimal, divide the percentage by 100
  - a percentage to a fraction, write the percentage with a **denominator** of 100 and simplify the fraction if possible
  - a decimal to a percentage, multiply the decimal by 100
  - a fraction to a percentage, multiply the fraction by 100.
- To calculate a **percentage of an amount**, write the percentage as a decimal and multiply by the amount.
- To write one quantity as a percentage of another, write the first value as a fraction of the second value and multiply by 100.

EXERCISE 1A Review of percentages

UNDERSTANDING AND FLUENCY

1 Write each percentage as a decimal.

a 15% **b** 90% **c** 112% **d** $75\frac{1}{2}\%$ **e** 18.8% **f** 0.25%

2 Write each percentage as a fraction in simplest form.

a 21% **b** 85% **c** 225% **d** 64%

e 38% **f** 80.2% **g** 110% **h** $5\frac{1}{4}\%$

i $15\frac{3}{4}\%$ **j** 0.55% **k** $8\frac{2}{5}\%$ **l** 0.004%

3 Write each fraction as a percentage.

a $\frac{3}{5}$ **b** $\frac{3}{8}$ **c** $\frac{7}{10}$ **d** $\frac{13}{20}$ **e** $\frac{11}{16}$ **f** $\frac{15}{24}$

EXAMPLE 1A-1 Calculating a percentage of an amount

Calculate $17\frac{1}{2}\%$ of \$124.95, correct to two decimal places.

THINK

1 Write the percentage as a decimal and multiply by the amount. Remember that $17\frac{1}{2}$ is the same as 17.5.

2 Round the answer to two decimal places.

3 Write your answer.

WRITE

$17\frac{1}{2}\%$ of \$124.95 $= \frac{17.5}{100} \times \$124.95$

$= 0.175 \times \$124.95$

$= \$21.866\ 25$

$\approx \$21.87$

$17\frac{1}{2}\%$ of \$124.95 is \$21.87

4 Calculate each of these.

a 12% of \$480 **b** 20% of \$3000

c 35% of \$567 **d** 150% of \$888

e 65% of \$1920 **f** 5% of \$14 000

g 40% of \$10 500 **h** 225% of \$4880

5 Calculate each of these, correct to two decimal places.

a 11% of \$145.95 **b** $42\frac{1}{2}\%$ of \$180.99

c $12\frac{1}{2}\%$ of \$95.55 **d** 45.5% of \$852

e 17.5% of \$1550.25 **f** $7\frac{1}{4}\%$ of \$39.95

g 2.75% of \$29.45 **h** $10\frac{1}{8}\%$ of \$455.95

EXAMPLE 1A-2 Writing one amount as a percentage of another

Write \$70 as a percentage of \$280.

THINK

1 Write the first quantity (\$70) as a fraction of the second (\$280).

2 Convert the fraction to a percentage by multiplying it by 100%.

3 Cancel common factors to the numerators and denominators and simplify.

WRITE

$$\frac{70}{280}$$
$$= \frac{70}{280} \times 100\%$$
$$= \frac{1}{4} \times \frac{100}{1}\%$$
$$= \frac{100}{4}\%$$
$$= 25\%$$

UNDERSTANDING AND FLUENCY

6 Write these amounts as percentages, **rounding** your answers to two decimal places where appropriate.

- **a** \$90 as a percentage of \$450
- **b** \$80 as a percentage of \$130
- **c** \$29 as a percentage of \$150
- **d** \$36.50 as a percentage of \$145
- **e** \$29.95 as a percentage of \$58
- **f** \$152.50 as a percentage of \$65
- **g** \$12.85 as a percentage of \$94.50
- **h** \$140.20 as a percentage of \$255.95
- **i** \$345.60 as a percentage of \$180.75

7 Complete the missing values in this table. Remember to write all amounts as **exact values** and fractions in their simplest form.

	Percentage	Fraction	Decimal
a		$\frac{1}{6}$	$0.1\dot{6}$
b	$46\frac{2}{3}\%$		
c		$\frac{2}{3}$	$0.\dot{6}$
d	$12\frac{1}{3}\%$		
e		$\frac{7}{6}$	
f		$\frac{11}{12}$	
g	$85\frac{1}{6}\%$		
h		$\frac{35}{9}$	

8 Shayden bought this scarf and at the end of winter sold it on eBay for $6.55.

- **a** Write Shayden's **selling price** as a fraction of the initial price in simplest form.
- **b** Write the fraction in part **a** as a percentage and a decimal, both rounded to two decimal places.

9 A store recorded the payment methods used by customers, and found that 35% of payments were in cash and the rest were electronic. Of the electronic payments, five-eighths were payments by credit card, three-tenths were electronically withdrawn from savings accounts and the rest were debit card purchases.

- **a** What fraction of the total payments were in electronic form?
- **b** Of all the electronic payments, what fraction describes the payments made with a debit card?
- **c** What percentage of electronic payments were made with a debit card?
- **d** Show that the purchases made by withdrawals from savings accounts represent 19.5% of the store's total sales.
- **e** What percentage of the store's total sales do each of the other forms of electronic payments represent?

10 Two-thirds of all customers expressed satisfaction with their regular supermarket.

- **a** What is this fraction as a decimal, correct to two decimal places?
- **b** Explain how the decimal from part **a** can be written as an exact number, rather than an approximated number.
- **c** Write the fraction as an exact decimal.
- **d** Represent the fraction as an exact percentage.
- **e** What two different forms can the exact percentage in part **d** take?

11 A bookstore offers an out-of-print novel for sale with a discount of $3.50. The novel was originally priced at $12.50.

- **a** Write the discount amount as a percentage of the **original price**.
- **b** Briefly explain how you would write the discount amount as a fraction of the original price. Both the numerator and the denominator must be whole number values and the fraction must be in simplest form.
- **c** Write the discount amount as a fraction of the original price.

12 Bruno buys raw coffee beans and roasts them to sell to customers. A 60-kg sack of raw beans has a **wholesale price** of $600. After roasting them, Bruno sells the beans for $14 per 250-g bag.

- **a** What is the wholesale price of the raw beans per 250 g?
- **b** Compare the wholesale price per 250-g bag with the selling price of the 250-g bag of roasted beans.
- **c** Write Bruno's selling price as a fraction of the wholesale price using the 250-g prices.
- **d** Convert the fraction to a percentage and briefly explain what this percentage means to Bruno's business.

13 A camping store offers a 20% discount to general customers and a 26.5% discount to VIP customers.

- **a** How much discount would a non-VIP customer receive on the tent shown in the photo?
- **b** How much discount would a VIP customer receive on the tent?
- **c** What fraction of the original price does each discount represent?

14 **a** Calculate $33\frac{1}{3}\%$ of $400. **b** Calculate $\frac{1}{3}$ of $400.

- **c** What do you notice about the answers to parts **a** and **b**? Why do you think this is so?

15 Calculations with percentages, fractions and decimals can be simplified using some basic number facts.

- **a** Calculate each of these.
  - **i** 10% of $356.00 **ii** 5% of $356.00 **iii** 1% of $356.00
- **b** Consider part **a i**. Describe a shortcut to calculating 10% of any amount without using a calculator.
- **c** How does your answer to part **a ii** compare with **a i**? Describe a shortcut to calculating 5% of any amount without using a calculator.
- **d** How does your answer to part **a iii** compare with **a i**? Describe a shortcut to calculating 1% of any amount without using a calculator.

16 Test your methods from question **15** with each of these. Check with a calculator that each answer is correct.

- **a** 10% of $438.50 **b** 5% of $438.50 **c** 1% of $438.50
- **d** 1% of $229.85 **e** 5% of $1458.99 **f** 10% of $148.35
- **g** 1% of $295.10 **h** 10% of $95.66 **i** 5% of $2458.45

17 **a** Explore how you can extend your findings from question **15**. Calculate each value.

- **i** 15% of $356.00 **ii** 20% of $356.00 **iii** $\frac{1}{2}\%$ of $356.00
- **b** Consider each percentage value in part **a**. Briefly describe a shortcut to calculate each percentage of an amount without using a calculator. (Hint: how does each percentage relate to 10%?)

PROBLEM SOLVING AND REASONING

18 Test your methods from question **17** with each of these. Check with a calculator that each answer is correct.

a 15% of $845.80	**b** 20% of $845.80	**c** $\frac{1}{2}$% of $845.80
d 15% of $336.90	**e** $\frac{1}{2}$% of $1254.00	**f** 20% of $265.00
g 20% of $4780.95	**h** $\frac{1}{2}$% of $45.50	**i** 15% of $384.40

19 Calculations involving a percentage of an amount where the original amount is not known can be solved using the **unitary method**. Consider a laptop that is discounted by 12.5%, which amounts to $56.25.

a Let x represent the original (pre-discount) price of the laptop. The discount calculation is:

12.5% of x = $56.25.

For the unitary method, find how much 1% represents (one unit). Calculate 1% of the original price. (Hint: divide the amount for 12.5% by 12.5.)

b The original price of the laptop represents the full amount, or 100%. Use your answer to part **a** to calculate 100% of the original price. (Hint: multiply the amount for 1% by 100.)

c What is the pre-discount price of the laptop?

20 You can use an alternative method to the one described in question **19** to obtain the same result.

a Reconsider the initial calculation 12.5% of x = $56.25. Write the percentage as a decimal and show that the **equation** simplifies to $0.125x = 56.25$.

b Look at the equation formed in part **a**. What operation can you perform to solve the equation for x?

c Solve the equation to determine the value for x.

d How does the value you obtained in part **c** compare with the value obtained in **19b**?

e Consider the methods used in this question and question **19**. Briefly explain why they produce the same result.

CHALLENGE

21 Calculate the pre-discount price in each situation. Where necessary, round each answer to the nearest cent.

a A football is discounted 15%, which amounts to $12.60.

b A mobile phone is discounted 10.5%, which amounts to $5.90.

c Make-up is discounted $3.20 or 8.5%.

d A furniture store offers a discount of $145.50 on a bedroom suite, which is equivalent to 17.5%.

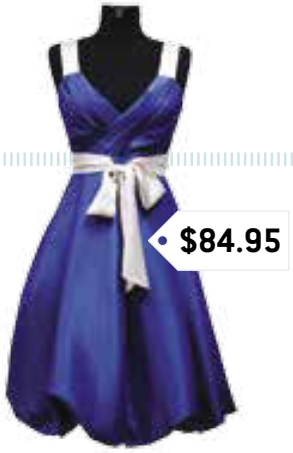
e A supermarket discounts their excess chocolates by $27\frac{1}{2}$% or $5.50.

f Sporting equipment is discounted 9.75%, which amounts to $85.30.

Reflect

How does converting percentages to decimals help to simplify a percentage of an amount calculation?

1B Financial applications of percentages



Start thinking!

Chelsea works part-time job in a clothing store. Her duties include pricing the clothes. The dress shown is currently priced at \$84.95, but will go on sale with a $22\frac{1}{2}\%$ discount.

1 The amount of the discount in dollars can be found using a percentage of an amount calculation. Determine this amount, correct to two decimal places.

2 Briefly explain how you can now determine the discounted price of the dress, and calculate the price.

3 Percentage discount is an example of a **percentage decrease**. The dress's selling price can be found by subtracting the percentage amount from 100% and finding this new percentage of the original price.

a A discount of $22\frac{1}{2}\%$ means you pay $22\frac{1}{2}\%$ less: new selling price = $(100 - 22\frac{1}{2})\%$ of original price. What percentage of the original price does this discount represent?

b Show this method produces the same selling price as question **2**.

Chelsea's duties also include marking up **retail prices** based on the wholesale prices the store pays for its goods. The store has a **mark-up** of 110% on all items.

4 A new jacket has a wholesale price of \$38. Find the actual amount of the mark-up by performing a percentage of an amount calculation (in a similar way to the discount calculation performed earlier).

5 Knowing the mark-up amount, briefly explain how you could determine the new selling price for the jacket, and calculate this selling price.

6 Percentage mark-up is an example of a **percentage increase**. The jacket's selling price can be found by adding the percentage amount to 100% and finding this new percentage of the wholesale price.

a A mark-up of 110% means you pay 110% more: selling price = (100 + 110)% of wholesale price. What percentage of the wholesale price does this mark-up represent?

b Show this method produces the same selling price as question **5**.

KEY IDEAS

- Selling price following a percentage discount can be calculated using the rule:
  selling price = (100 − percentage discount)% × original price
- Selling price following a percentage mark-up can be calculated using the rule:
  selling price = (100 + percentage mark-up)% × original price
- A **profit** occurs when the selling price is higher than the original price and a **loss** occurs when the selling price is lower than the original price.
- Percentage profit (or loss) on the original price = $\frac{\text{profit (or loss)}}{\text{original price}} \times 100\%$.

EXERCISE 1B Financial applications of percentages

1 Calculate each of these, rounding to two decimal places where appropriate.
- a 15% of $150
- b 35% of $3200
- c 8.5% of $957.65
- d $122\frac{1}{2}$% of $400

EXAMPLE 1B-1 Calculating a percentage discount

Calculate the selling price after a $17\frac{1}{2}$% discount on a necklace originally priced at $229.95.

THINK

1 Write a calculation for the selling price. A percentage discount of $17\frac{1}{2}$% means you pay $(100 - 17\frac{1}{2})$% of the original price.

2 Write the percentage as a decimal and multiply by the amount.

3 Round the amount to the nearest cent (two decimal places).

4 Write your answer.

WRITE

$$\begin{aligned}\text{selling price} &= (100 - 17\tfrac{1}{2})\% \text{ of } \$229.95\\ &= 82\tfrac{1}{2}\% \text{ of } \$229.95\\ &= \frac{82.5}{100} \times \$229.95\\ &= 0.825 \times \$229.95\\ &= \$189.708\,75\\ &\approx \$189.71\end{aligned}$$

The selling price after a $17\frac{1}{2}$% discount is $189.71.

UNDERSTANDING AND FLUENCY

2 Calculate the selling price for each of these.
- a 25% discount on $185
- b $17\frac{1}{2}$% discount on $9200
- c 31.5% discount on $650.95
- d 12.5% discount on $165.50
- e 28.5% discount on $325.99
- f $45\frac{1}{2}$% discount on $499.45

EXAMPLE 1B-2 Calculating a percentage mark-up

Calculate the selling price after a $55\frac{1}{2}\%$ mark-up is applied to sporting equipment with a wholesale price of $600.

THINK

1 Write a calculation for the selling price. A percentage mark-up of $55\frac{1}{2}\%$ means you pay $(100 + 55\frac{1}{2})\%$ of the original price.

2 Write the percentage as a decimal and multiply by the amount.

3 Write your answer.

WRITE

selling price $= (100 + 55\frac{1}{2})\%$ of $600

$= 155\frac{1}{2}\%$ of $600

$= \frac{155.5}{100} \times \$600$

$= 1.555 \times \$600$

$= \$933$

The selling price after a $55\frac{1}{2}\%$ mark-up is $933.

UNDERSTANDING AND FLUENCY

3 Calculate the selling price for each of these.

a 35% mark-up on $340

b $15\frac{1}{2}\%$ mark-up on $18 600

c 40.5% mark-up on $844.20

d 125% mark-up on $350.50

e 265% mark-up on $1420.90

f $165\frac{1}{2}\%$ mark-up on $680.95

4 For each of these, determine:

i the selling price **ii** the mark-up or discount amount.

a A smart phone is bought for $89.90 and sold later at a mark-up of 155%.

b A jewellery piece originally marked at $1495.80 is offered for sale at a discount of 12.5%.

c Sport shorts marked at $49.90 are offered for sale with a 35% discount.

d Network cabling purchased for $2.85 per metre is marked up by 480%.

5 For each situation:

i state if a profit or loss has been made and determine the amount

ii write the profit or loss amount as a percentage of the original price, correct to two decimal places where appropriate.

a Clothing is bought for $84.95 and later sold for $53.60.

b A market stallholder buys fruit for $1.70 per kilogram and sells it for $5.95 per kilogram.

c A new car is purchased for $37 935 and sold later for $17 390.

d A laptop is bought for $548 and sold for $652.

e A bookstore buys a book for $32.95 and sells it for $47.80.

f Computer games are bought at a market for $42.95 and sold later for $16.

6 The following represent the original prices and the percentage discount or percentage mark-up amounts offered on some goods. In each case, calculate:

- **i** the selling price after the discount or mark-up
- **ii** the discount or mark-up amount.

Where appropriate, round answers to the nearest cent.

- **a** \$1285; 20% discount
- **b** \$1800; $15\frac{1}{2}$% discount
- **c** \$540; 185% mark-up
- **d** \$450; 125.5% mark-up
- **e** \$346.95; 17.5% discount
- **f** \$850.40; 62.5% discount
- **g** \$3458.99; $87\frac{1}{2}$% mark-up
- **h** \$6240.55; 255% mark-up

7 A school fundraiser collected \$2915.65 for charity. Compare this with the \$2584.80 collected last year.

- **a** How much extra money was raised this year?
- **b** Write the increase as a percentage of last year's amount, correct to two decimal places.

8 Diana buys a radio for \$141.45 and sells it later for \$99.

- **a** What is the amount of the loss Diana has made?
- **b** Write the loss amount as a fraction of the price Diana originally paid.
- **c** State the percentage loss on the original price, rounded to two decimal places.

9 For each of these:

- **i** state the value of the profit or loss
- **ii** write the profit or loss amount as a fraction of the original price
- **iii** write the profit or loss as a percentage of the original price (rounded to the nearest 1%).

- **a** original price \$68, selling price \$92
- **b** original price \$30 000, selling price \$12 850
- **c** original price \$145.50, selling price \$99
- **d** original price \$37.95, selling price \$12.50
- **e** original price \$699.45, selling price \$1225.50
- **f** original price \$294.58, selling price \$346.99

10 A soccer club decided to purchase new match balls at a sale. The balls were originally priced at $84.95 each.

- **a** Based on the advertised discount, what percentage of the original price will the new selling price be?
- **b** Calculate the selling price of each soccer ball.
- **c** What is the amount saved following the discount?

If ordering more than 20 balls, the soccer club is eligible for an additional $5\frac{1}{2}\%$ discount. The club decides to order 25 balls.

- **d** How much does the club pay for the 25 balls?
- **e** Following the bulk purchase, what is each ball worth?

11 Bonnie is planning to hire a dress for her school formal. When she first enquired, the hire charge was quoted as a booking deposit of $50 plus $85 on collection of the dress. When she next contacted the store, she learned that the collection fee had increased by 12.5%.

- **a** What percentage of the original collection fee will the new collection fee be?
- **b** What amount will Bonnie pay to hire the dress, including the booking deposit?
- **c** By how much has the hire charge increased, to the nearest cent?

12 Ricardo needs to purchase a new camera for his photography course. His local camera store is offering cameras for sale at a discount of 18%. Ricardo is interested in purchasing the camera shown in the photo.

- **a** What is the selling price of the camera following the discount?
- **b** As a valued customer of the store, Ricardo receives an additional 4.5% discount. Apply this discount to the answer in part **a** to determine the amount Ricardo will pay for the camera.
- **c** Is the answer to part **b** the same as receiving a 22.5% (18% + 4.5%) discount on the original price?
- **d** What was the total discount amount Ricardo received on this sale?
- **e** Write the discount amount as a percentage of the original price.
- **f** The percentage in part **e** represents the successive discounts as a single percentage amount. How does this single amount compare with the two successive percentage discounts?

13 Calculate the original price in each situation. Where necessary, round each answer to the nearest cent.

- **a** A smart phone sells for $297.50 following a discount of 15%.
- **b** A sports skirt sells for $37.50 after a discount of 17.5%.
- **c** Jewellery sells for $229.95 after a mark-up of 245%.
- **d** An artist sells her T-shirt prints for $20.95 following a 135.5% mark-up.

PROBLEM SOLVING AND REASONING

14 Dennis plans to buy a new computer game for sale at a 15% discount. He is about to pay when he learns that a further 4.25% has been taken off from the price.

- **a** Dennis expected to pay $74.46 for the game. What is the selling price after the additional discount?
- **b** What was the original price of the game?
- **c** What single calculation can be applied to the original price to determine the selling price after two successive discounts?

15 James works in telephone sales and receives 2.75% **commission** on the total value of all his sales plus a **retainer** of $450 as his gross weekly income. In a week when James' sales total $12 452, what is his gross weekly income?

16 Laura earns a weekly retainer of $925.50 and 4.125% commission of the total value of her weekly sales. Calculate her **gross income** for a week with each total sales value.

- **a** $0
- **b** $15 000
- **c** $175
- **d** $21 587
- **e** $1648.90
- **f** $6381.45
- **g** $489
- **h** $8468.20

CHALLENGE

17 Harry applies a mark-up of 165% to the wholesale price of all items he sells in his clothing store. The goods and services tax (GST) of 10% must also be added to the marked-up price. Harry's newly arrived men's trousers have a wholesale price of $68.

- **a** What is the mark-up amount Harry needs to add to the wholesale price?
- **b** Calculate the selling price of the trousers after GST is included in the price.
- **c** Show that the calculation for the selling price is not the same as increasing the wholesale price by 175% (165% + 10%).
- **d** Write the difference between the selling price and the wholesale price as a percentage of wholesale price.
- **e** What single calculation will give the selling price of the trousers?
- **f** These prices represent the wholesale prices of different clothing items. Increase the prices by Harry's mark-up amount and add the GST to determine each selling price.
  - **i** shirts $45
  - **ii** jackets $87.50
  - **iii** belt $29.40
  - **iv** suit $189.95

Reflect

Why are percentages so important in financial calculations?

1C Understanding simple interest

Start thinking!

Last year, you were introduced to **simple interest** and its **formula**, $I = P \times R \times T$.

1 Briefly explain what **interest** means in finance.

When you borrow money from a bank, you generally repay more than you borrow. This additional repayment is known as interest.

When you invest (deposit money into a bank or other institution) rather than borrow money, interest can be paid to you on your **investment**.

2 How does interest on a **loan** differ from interest received for an investment?

3 Consider the simple interest formula, $I = P \times R \times T$.
- **a** What do the variables in this formula represent?
- **b** How must the value for R be written in the formula?
- **c** What unit must be used for T in the formula?

4 Kiara's school provides a laptop costing $1200 for each student. The laptops may be bought on a 3-year purchase plan, with interest of 6.5% p.a. for the 3 years.
- **a** What does the abbreviation p.a. represent?
- **b** Is this plan for a loan or an investment?
- **c** Identify the values for P, R and T.
- **d** Use the simple interest formula to calculate I.
- **e** What does your answer to part **d** represent?

KEY IDEAS

- Interest can be an additional charge on a loan or a bonus payment on an investment.
- Interest can be calculated using the formula $I = P \times R \times T$, where:
  - I = interest
  - P = **principal** (the amount borrowed or invested)
  - R = interest rate. The rate must be converted to a fraction or decimal. For example, 5% would be substituted as $\frac{5}{100}$ or 0.05.
  - T = **time** of the loan or investment in years.

EXERCISE 1C Understanding simple interest

UNDERSTANDING AND FLUENCY

1 Calculate the simple interest in each case.
- a $P = \$5000$, $R = 6\%$, $T = 4$ years
- b $P = \$850$, $R = 12\%$, $T = 2$ years
- c $P = \$125\,000$, $R = 8.4\%$, $T = 3$ years
- d $P = \$64\,000$, $R = 3.6\%$, $T = 5$ years
- e $P = \$9550$, $R = 10\%$, $T = 2.5$ years
- f $P = \$22\,000$, $R = 12.5\%$, $T = 3$ years

2 Convert each time to years.

a 24 months	b 48 months	c 27 months	d 45 months
e 7 months	f 15 months	g 19 months	h 1 month
i 52 weeks	j 65 weeks	k 4 weeks	l 312 weeks

EXAMPLE 1C-1 Working with the simple interest formula

For a loan of \$12 000 at an interest rate of 8.4% p.a. for 30 months, calculate:
- a the amount of simple interest charged
- b the total amount repaid at the end of the loan
- c the amount of each repayment, if equal repayments are made monthly.

THINK

a **1** Write the formula and identify the known values. Rate must be written as a fraction or a decimal and time must be written in years.

2 Substitute the values in the formula and calculate I.

b **1** The total repaid at the end of the loan is the interest added to the principal.

2 Write the answer.

c **1** The amount of each repayment is found by dividing the total amount to be paid by the number of instalments.

2 Write the answer.

WRITE

a $I = P \times R \times T$
$P = \$12\,000$
$R = 8.4\% = \frac{8.4}{100} = 0.084$
$T = 30 \text{ months} = \frac{30}{12} \text{ years} = 2.5 \text{ years}$
$I = \$12\,000 \times 0.084 \times 2.5 = \$2520$
simple interest charged = \$2520

b amount = \$12 000 + \$2520
= \$14 520

total amount repaid = \$14 520.

c amount to be paid = \$14 520
number of instalments = 30
amount of each instalment = \$14 520 ÷ 30
= \$484

amount of each repayment = \$484

3 For each loan, calculate the:
- i amount of simple interest charged
- ii total amount repaid at the end of the loan
- iii amount of each repayment if equal repayments are made monthly.

a $8000 at an interest rate of 4% p.a. over 2 years
b $15 000 at an interest rate of 10% p.a. over 4 years
c $1250 at an interest rate of 5.4% p.a. over 3 years
d $14 000 at an interest rate of 8.2% p.a. over 30 months
e $25 000 at an interest rate of 6.8% p.a. over 18 months
f $6550 at an interest rate of 4.5% p.a. over 104 weeks

4 For each investment, calculate the:
- i amount of simple interest earned
- ii value of the investment at the end of the term.

a $12 000 at an interest rate of 3% p.a. for 3 years
b $66 000 at an interest rate of 5% p.a. for 2 years
c $125 000 at an interest rate of 6% p.a. for 2.5 years
d $5500 at an interest rate of 4.2% p.a. for 12 months
e $8250 at an interest rate of 8.5% p.a. for 15 months
f $4580 at an interest rate of 3.6% p.a. for 36 months

5 Calculate the simple interest, given each of these.
a P = $8550, R = 3.6% p.a., T = 2.5 years
b P = $9500, R = 3.7% p.a., T = 1.5 years
c P = $22 000, R = 8% p.a., T = 42 months
d P = $8490, R = 8.4% p.a., T = 12 weeks
e P = $11 650, R = 9.6% p.a., T = 5.5 years
f P = $155 000, R = 12.9% p.a., T = 5 months

6 An investment of $12 000 is made at a bank that offers an annual interest rate of 6.25% p.a. The interest is calculated as simple interest and the money is invested for 3 years.
a Identify the values for P, R and T.
b How much simple interest is earned on the investment?
c What is the total value of the investment after 3 years?

7 A loan of $12 000 is taken at an interest rate of 6.25% p.a. for 3 years, charged as simple interest.

- **a** Identify the values for P, R and T.
- **b** How much simple interest is charged to the loan?
- **c** What is the total amount to be repaid at the end of the loan?

8 Consider each situation and answer in questions **6** and **7**. Use your responses to explain how you can apply and interpret simple interest calculations in loans and investments.

9 For each of the values in the table on the right, calculate:

- **i** the amount of simple interest
- **ii** the total amount at the end of the term.

	Principal	Rate (p.a.)	Time
a	$14 000	3.8%	3.5 years
b	$8 475	6.25%	12 months
c	$12 390	$9\frac{1}{2}$%	34 months
d	$115 000	$8\frac{3}{4}$%	120 days
e	$250 000	2.95%	4.75 years
f	$234 580	15.85%	17 months

PROBLEM SOLVING AND REASONING

10 An electronics store offers a purchase plan on the smart television shown, in which simple interest is charged at 12.75% p.a. for 20 months.

- **a** What is the total amount of interest charged for the term?
- **b** Using this purchase plan, what is the total cost of the television?
- **c** If the total amount to be repaid is divided into equal monthly instalments for the term of the plan, what is the value of each instalment?

The store offers another option for customers paying cash, which involves a discount of 2.5% off the advertised price.

- **d** What is the selling price for cash customers?
- **e** Calculate the difference in the television's price using the two options.

11 Jake plans to invest his savings in a term deposit at his bank. The bank offers interest of 4.8% p.a. for any period from 6 to 15 months and pays interest at maturity. Jake invests $1500 for 14 months.

- **a** Identify the values for P, R and T.
- **b** How much interest does Jake earn on his investment?
- **c** What is the total value of Jake's investment at the end of his term?

12 A bank offers an interest rate of 1.4% p.a. on its everyday accounts, with interest calculated on the daily balances. Consider the account balances shown here.

Date	Transaction	Amount	Balance
01/05	Opening balance		$985.90
09/05	Withdrawal at ATM	$60.00	$925.90
11/05	Deposit (Pay)	$875.55	$1801.45
20/05	EFTPOS Purchase	$185.95	
22/05	Gym membership	$69.50	
25/05	Deposit (Pay)	$875.55	
31/05	Interest for May		

a The account balances following three of the transactions are not shown. State the missing account balances.

b The opening balance of $985.90 applies for the first eight days of the month as each new balance applies on the date the transaction is made. How many days does each balance on this account apply for?

c For each new balance on the account, calculate the simple interest based on the number of days each balance applies. Remember that 8 days is equivalent to $\frac{8}{365}$ years.

d Add all the amounts from part **c** to calculate the total interest for the month.

e What is the account balance at the end of May, if the total interest is added to the account at the end of the last day of the month?

13 A bank is offering these interest rates to attract investment in term deposits. Interest is calculated at the end of the investment term.

	Investment amount			
Term	$5000 to <$10 000	$10 000 to <$20 000	$20 000 to <$75 000	$75 000 to <$250 000
1 to <2 months	2.5%	2.5%	2.5%	2.5%
2 to <6 months	6.25%	6.25%	6.25%	6.25%
6 to <8 months	3.25%	3.25%	3.25%	3.25%
8 to <12 months	5.35%	5.35%	5.35%	5.35%
12 to <24 months	5.4%	5.4%	5.4%	5.4%

a Answer these using the information in the table.

i What is the minimum amount that can be invested in a term deposit?

ii Do the interest rates change as the amount invested increases?

iii The bank appears to have a special rate for a period of time. What investment term receives this rate?

b What interest rate will be offered on an investment of $45 000 for 12 months?

c Calculate the amount of simple interest earned on an investment of $45 000 for 12 months.

d Compare your answer to part **c** with the option of investing the same amount for two 6-month terms. (Assume the interest rate will stay the same for the second term.)

PROBLEM SOLVING AND REASONING

e Reconsider the option of investing $45 000 for two 6-month terms as calculated in part d. If, at the end of the first 6 months, the interest was added to the investment amount for the second term, how does the amount of interest for the second term change?

f Compare all the investment options described. Which earned the greatest amount of interest?

14 Some savings accounts offered by banks include bonus interest rates if certain conditions are met. These bonus rates are seen as a reward to encourage people to save and earn more interest. Consider Maddy's account balance for November, shown here. Her account offers interest at 1.2% p.a. (calculated daily) plus a bonus 4.55% p.a. if no withdrawals are made in the month and the balance increases by at least $250 each month.

Date	Transaction	Amount	Balance
01/11	Opening balance		$8400.90
15/11	Deposit – Pay	$655.80	
19/11	Deposit – at branch	$240.00	
29/11	Deposit – Pay	$655.80	
30/11	Interest		

a Will this account receive the bonus interest rate? Provide a reason to support your answer.

b Calculate the total interest earned for the month. (Hint: first determine the account balances following each transaction).

c State the final account balance for the month.

d On the final day of the month, Maddy made an EFTPOS purchase at a clothing store for $100 and the transaction did not appear on her statement. How would the balances and interest calculations have differed if the withdrawal had appeared on the statement?

CHALLENGE

15 Kyna invests her savings in a term deposit at her bank. The bank offers her an interest rate of 4.64% p.a. for a period of 3 months and will pay the interest at maturity. Kyna invests $4250 for the 3 months.

a How much interest does Kyna earn on her investment?

b What is the value of Kyna's investment at the end of the term?

c What single calculation can be performed to determine the final value of Kyna's investment?

d Use your method from part c to determine the final value for each of these.

i an investment of $3500, interest rate 4.8% p.a., 2 years

ii an investment of $1800, interest rate 8.4% p.a., 4.5 years

iii an investment of $12 000, interest rate 6% p.a., 15 months

iv an investment of $25 000, interest rate 4.2% p.a., 4 months

Reflect

Why must interest rates be written as an annual rate and the term of an investment be in years when using the simple interest formula?

1D Working with simple interest

Start thinking!

When the value for I is known, the simple interest formula can be used to find the value for one of the other variables; that is, P, R or T.

For example, Kim is considering some different purchase options for a new music stereo system. One store offers the system shown for sale at an interest rate of 12.5% p.a. charged as simple interest. Kim is informed that the total amount of interest charged for the term of this option is $375 and that the total amount is repaid by equal monthly repayments.

1 **a** From the simple interest formula, which variable do you *not* know the value of?
b What variable does each of the given values represent?

2 Substitute the values into the simple interest formula and show that it simplifies to $375 = 150T$.

3 Solving the equation from question **2** will enable you to determine the value for T.
a Explain what method you can use to find the value for T.
b Use your method to find the value for T. What does this value mean in relation to Kim's purchase?

4 The selling price plus the interest amount represents the total Kim must pay for the stereo in this payment option. What is the size of each monthly repayment?

Kim notices another stereo system advertised for $1100. The purchase option for this stereo also has equal monthly repayments and charges $352 interest for the term of 24 months.

5 **a** From the simple interest formula, which variable do you not know the value of?
b What variable does each of the given values represent?

6 **a** Substitute the values into the simple interest formula and show that it simplifies to $352 = 2200R$.
b Solve the equation in part **a** to determine the value for R.
c The answer in part **b** represents the interest rate written as a decimal. What needs to be done to this decimal so that the value is written as a percentage amount?
d State the annual interest rate (p.a.) that is applied to this purchase option.

KEY IDEAS

- The simple interest formula is $I = P \times R \times T$ or $I = PRT$.
- Strategies for solving equations can be used to find any of these quantities when the other three are known.

EXERCISE 1D Working with simple interest

UNDERSTANDING AND FLUENCY

1 Calculate the simple interest in each case.

a $P = \$500$, $R = 12\%$, $T = 4$ years

b $P = \$3500$, $R = 9.5\%$, $T = 3$ years

c $P = \$850$, $R = 2.5\%$, $T = 1$ year

d $P = \$44\,000$, $R = 8.4\%$, $T = 2.5$ years

2 Find the value for T in each of these.

a How long will it take for an investment of $10 000 with simple interest rate 3.5% p.a. to earn $1400?

b How long does a loan of $1600 at an interest rate of 8% p.a. take to earn $320 in simple interest?

c How long will it take for an investment of $8500 with a simple interest rate of 5.6% p.a. to earn $1428?

3 Find the value for P in each of these.

a How much needs to be invested at an interest rate of 7% p.a. for 4 years to earn $4200 in simple interest?

b How much is borrowed at an interest rate of 8.8% p.a. over 2.5 years to earn $1430 in simple interest?

c How much is borrowed at an interest rate of 4.75% p.a. over 3 years to earn $2850 in simple interest?

EXAMPLE 1D-1 Using the simple interest formula to find R

A loan of $12 500 is taken over 36 months. The amount of simple interest charged on the loan is $2475. What is the annual interest rate that is applied to this loan?

THINK

1 Write the simple interest formula and identify the variables. Remember that time must be written in years.

2 Substitute the values into the formula and simplify.

3 Use the balance method to find the value for R.

4 Multiply the decimal by 100 to write as a percentage amount. Write the answer.

WRITE

$I = PRT$

$I = \$2475$

$P = \$12\,500$

$T = 36 \text{ months} = \frac{36}{12} \text{ years} = 3 \text{ years}$

$2475 = 12\,500 \times R \times 3$

$2475 = 37\,500 \times R$

$\frac{2475}{37\,500} = \frac{37\,500 \times R}{37\,500}$

$R = 0.066$

$= 0.066 \times 100\%$

$= 6.6\%$

The annual interest rate is 6.6% p.a.

4 Find the value for R in each of these.

- **a** What annual interest rate is applied to a loan of $8000 over 2 years if the simple interest charged is $704?
- **b** What annual interest rate is applied to a loan of $16 000 over 4.5 years if simple interest charged is $7560?
- **c** What annual interest rate is applied to a $25 000 investment over 30 months if simple interest earned is $2812.50?

5 Find the missing value in each of these.

- **a** $I = \$1000$, $P = \$5000$, $R = 5\%$, $T = ?$
- **b** $I = \$4680$, $P = ?$, $R = 6.5\%$, $T = 4$ years
- **c** $I = \$8160$, $P = \$64\,000$, $R = ?$, $T = 18$ months
- **d** $I = \$3864$, $P = \$9200$, $R = 8.4\%$, $T = ?$
- **e** $I = \$4320$, $P = ?$, $R = 4.8\%$, $T = 4.5$ years
- **f** $I = \$675$, $P = \$22\,500$, $R = ?$, $T = 15$ months

6 Leanne invests a sum of money into a term deposit with her bank. Her investment will earn her $2304 at maturity. The interest rate is 6.4% p.a. and Leanne invests the money for 36 months.

- **a** From the simple interest formula, which variable do you not know the value of?
- **b** What variable does each of the given values represent?
- **c** How much money does Leanne invest? Hence, state the value at the end.

7 Kevin needs to borrow the entire cost to attend an end-of-school year trip. A friend loans him the money and charges him 8.5% p.a. simple interest over an agreed term. In total, the simple interest charged totals $828.75.

- **a** From the simple interest formula, which variable do you not know the value of?
- **b** What variable does each of the given values represent?
- **c** Using the given values, how many months has the interest charge been based on?
- **d** Kevin agrees to repay his friend with equal monthly instalments for the term of the loan. What is the amount of each instalment?

UNDERSTANDING AND FLUENCY

8 Fatima plans to invest the money earned from the sale of her holiday house so it can grow in value and she can purchase some land in the country. She has $312 000 to invest following her sale and aims to have $375 000 at the end of a 2-year investment to assist her with the land purchase.

- **a** Fatima has discovered an online term deposit that offers an interest rate of 7.4% p.a. with interest paid at maturity. How much interest will Fatima earn on this investment?
- **b** Will these terms allow Fatima's investment to grow to her required value?
- **c** What is the difference between the actual growth and Fatima's required growth?
- **d** What annual rate of interest does Fatima require for her investment to grow to her required value?

9 Use the simple interest formula to determine the value for the missing amounts. For the missing T values, state the time in years.

	I	P	R (% p.a.)	T
a		$35 000	9.5	4.5 years
b	$2180.90	$9 650	5.65	
c	$15.93		2.95	3 years
d	$39 809	$110 000		5.5 years
e	$269.90		4.85	7 months
f	$137.15		21.5	31 days
g	$1890	$15 000		18 months
h	$3163.50	$8 450	29.95	

10 If the P values in the table in question **9** represent amounts borrowed, what is the total amount to be repaid in each case?

PROBLEM SOLVING AND REASONING

11 A term deposit advertised on radio offers an interest rate of 6.8% p.a. fixed for 3 months.

- **a** What do you think the expression 'fixed for 3 months' means?
- **b** How much interest is earned on an investment of $12 000 in this term deposit?
- **c** What amount of money must be invested to earn $500 in interest?

At the end of the term, the rate of interest decreases to a new amount of 6.15% p.a. fixed for 100 days.

- **d** What amount of money must be invested to earn the same amount as you calculate in part **b**?

12 Jacqui is saving money to purchase a racing bike like the one shown. She has saved $1980 and wishes to earn the remaining amount in interest on an investment. The best simple interest rate she can find is 7.2% p.a.

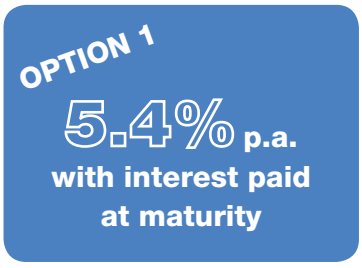
- **a** For how long must Jacqui invest the money in order to earn the extra amount she needs?
- **b** Jacqui wants to investigate ways to earn the money more quickly through a shorter investment. What interest rate does she need if she wishes to have the money in 18 months?
- **c** What will the interest rate be if Jacqui wishes to have the money in 8 months?
- **d** Consider the rates you found in parts **b** and **c**. What issues do you feel Jacqui may have in achieving these goals?

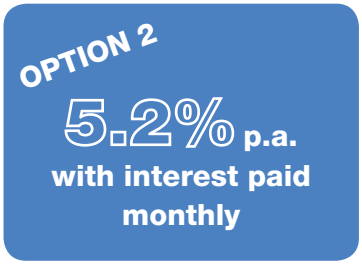
13 The statement shown is for a credit card where interest is charged from the day of purchase. To avoid extra charges, the total amount spent plus interest must be paid each month.

Date of transaction	Description	Amount
05/09	Clothing store purchase	$120.00
09/09	BPay to Electricity provider	$356.00
12/09	ABC Bookstore	$39.90
19/09	Ticketmaster	$185.00
25/09	Party hire	$150.00
30/09	Petrol	$65.85
	Interest charge for the month of September	$8.94

- **a** Determine the amounts owing throughout the month and the total number of days that interest is charged on those amounts. Remember that interest is charged from the day of purchase.
- **b** How much needs to be paid at the end of the month to avoid any extra charges?
- **c** What is the annual interest rate (% p.a.) that is charged to this credit card account? (Hint: Use the total interest shown and remember that 4 days is equivalent to $\frac{4}{365}$ years.)

14 Banks offers various interest rates depending on the length of the investment and the frequency of the interest payment. Consider these options.





- **a** Consider how much interest is earned in each option if $10 000 is invested for 1 month.
- **b** Now consider how much interest is earned in each option if $10 000 is invested for 3 months. For option 2, the monthly interest amounts are passed on to the investor, not added to the investment.

Now explore what happens to the value of investment in option 2 if the monthly interest amounts are added to the investment at the end of each month.

PROBLEM SOLVING AND REASONING

c What is the value of the investment after the first month's interest calculation? This value becomes the principal for the second month.

d Using the answer from part **c**, how much interest is earned in the second month and what is the new value of the investment?

e Using the new value from part **d**, how much interest is earned in the third month and what is the final value of the investment?

f What is the total amount of interest earned on the investment using the methods explored in parts **c**–**e**?

g Compare the amount of interest from part **f** with the total amount of interest you calculated for option 2 in part **b**. Are the amounts the same or different? Provide a brief explanation as to why this is the case.

15 In question **14**, the interest rates for the two options were different. Reconsider the calculations in parts **c**–**f** using the same interest rate as option 1, that is, 5.4% p.a.

a If the monthly interest amounts are added to the investment at the end of each month, what will the final value of the investment be after 3 months? (You may wish to use the parts of question **14** to assist you with this calculation.)

b How does the total amount of interest using this method compare to the method of interest calculation at maturity? (Compare your answer with option 1 in **14b**.)

c Consider the final value of the investment you calculated in part **a**. If planning to have your interest paid at maturity, what amount of money needs to be invested so that the interest earned is the same?

d If the total amount of interest from part **a** is to be earned on an investment of $10 000 for 3 months, with interest paid at maturity, what interest rate should be applied?

CHALLENGE

16 Consider each short-term investment. For each, calculate the total amount of interest earned if interest is calculated:

i at maturity

ii monthly, with interest added to the investment at the end of each month.

a $10 000 at 3.6% p.a. for 2 months

b $15 000 at 8.4% p.a. for 3 months

c $35 000 at 12.75% p.a. for 4 months

d $8400 at 21.5% p.a. for 3 months

17 Reconsider the amount of interest earned in each investment in question **16** with the monthly interest amounts added to the investment (part **ii**). To earn the same amount of interest, what interest rate should be applied to the principal if interest is calculated at maturity?

Reflect

Why are interest rates so important in financial calculations?

1E Understanding compound interest

Start thinking!

For investments that receive simple interest, the interest is paid at the end of the investment term. The principal and interest are combined to give the total value. Loans involving simple interest are similar. The combined principal and interest give the total amount to be repaid.

1 Consider $5000 invested at 6% p.a. for 3 years.
- **a** Calculate the amount of simple interest that is earned on this investment.
- **b** What is the value of the investment after 3 years?

2 What is the amount of interest earned on an investment of $5000 at 6% p.a. for 1 year?

3 Consider the amount of simple interest earned in each of the 3 years of the investment. Use your results from questions **1** and **2** to complete these.
- **a** Copy and complete this table.
- **b** If you add the individual I-values for each year in the table, is this sum the same as the value you calculated in question **1a**?
- **c** What do your results show you about the amount of simple interest earned in each year of an investment?

Year	P	R	I
1			
2			
3			

4 Consider the investment of $5000 at 6% p.a. for 3 years, but with interest calculated *during* the investment period at yearly intervals and added to the principal.
- **a** How much interest is earned in the first year?
- **b** Add the interest from part **a** to the principal. This new amount is the principal for the second year.
- **c** Use the new principal value to calculate how much interest is earned during the second year.
- **d** Add the interest from part **c** to the principal for the second year. This new amount is the principal for the third year.
- **e** Use the new principal value to calculate how much interest is earned during the third (final) year.
- **f** Add the interest from part **e** to the principal for the third year. This new amount is the final value of the investment.

KEY IDEAS

- Interest on the amount borrowed or amount invested is simple interest.
- Interest calculations performed at regular intervals during an investment period with interest added to the principal is known as **compound interest**.
- The regular intervals over which the interest calculations are performed are known as **compounding periods**.

EXERCISE 1E Understanding compound interest

UNDERSTANDING AND FLUENCY

1 For each investment, state the:
- **i** number of compounding periods
- **ii** time length of each compounding period.

a $5000 is invested at 4.5% p.a. for 2 years with interest calculated at the end of each year.

b $12 000 is invested at 9% p.a. for 3 years with interest calculated at the end of each year.

c $8500 is invested at 5% p.a. for 4 years with interest calculated at the end of each 6 months.

d $25 000 is invested at 8.4% p.a. for 2 years with interest calculated at the end of each month.

EXAMPLE 1E-1 Calculating compound interest

$15 000 is invested at 5% p.a. for 3 years with interest calculated at the end of each year and added to the principal amount. Calculate the total amount of interest compounded during this time.

THINK	WRITE
1 This is compound interest, as interest is calculated and added to the principal at regular intervals.	P = \$15 000 R = 5% p.a. = 0.05 Interest is compounded yearly.
2 Calculate the interest earned in the first year. Add this to the principal to determine the principal for the second year.	$I = PRT$ = \$15 000 × 0.05 × 1 = \$750 new principal = \$15 000 + \$750 = \$15 750
3 Calculate the interest earned in the second year. Add this amount to the principal to determine the principal for the third year.	$I = PRT$ = \$15 750 × 0.05 × 1 = \$787.50 new principal = \$15 750 + \$787.50 = \$16 357.50
4 Calculate the interest earned in the third year. Add this amount to the principal to determine the final value of the investment.	$I = PRT$ = \$16 357.50 × 0.05 × 1 = \$826.88 final value = \$16 357.50 + \$826.88 = \$17 364.38
5 The amount of interest compounded is found by subtracting the initial principal from the final value.	total amount of interest compounded = \$17 364.38 − \$15 000 = \$2364.38

2 In each investment, interest is calculated at the end of each year and added to the principal amount. Calculate the total amount of interest compounded during this time.

- **a** $10 000 is invested at 5% p.a. for 2 years.
- **b** $5000 is invested at 6% p.a. for 3 years.
- **c** $25 000 is invested at 8% p.a. for 3 years.
- **d** $40 000 is invested at 5% p.a. for 4 years.
- **e** $6000 is invested at 4.8% p.a. for 3 years.
- **f** $18 000 is invested at 7.2% p.a. for 4 years.

3 For each investment, calculate the amount of:

- **i** simple interest earned over the term of the investment
- **ii** compound interest earned if interest is compounded each year.

- **a** $12 000 is invested at 4% p.a. for 2 years.
- **b** $10 500 is invested at 5% p.a. for 3 years.
- **c** $45 000 is invested at 6.2% p.a. for 3 years.
- **d** $21 900 is invested at 8.5% p.a. for 4 years.

4 Mac is investigating ways to earn more interest on an investment. His bank offers him an interest rate of 5.2% p.a. with interest compounded annually. Mac has $8500 to invest for 3 years.

- **a** From the given information:
  - **i** how many compounding periods will there be in Mac's investment
  - **ii** how long is each compounding period?
- **b** How much interest will Mac earn in the first year?
- **c** Add the interest from part **b** to the principal for the first year to determine the new amount for the principal for the second year.
- **d** How much interest will Mac earn in the second year?
- **e** Add the interest from part **d** to the principal for the second year to determine the new amount for the principal for the final year.
- **f** How much interest will Mac earn in the final year of his investment?
- **g** What will be the final value of Mac's investment?
- **h** Compare the result found in part **g** with the final value that would be obtained from earning simple interest on the investment after 3 years.

5 For each of the investments shown in the table on the right, calculate the amount of interest earned if interest is compounded each year.

	Principal	Interest rate (p.a.)	Term
a	$8 400	12%	3 years
b	$12 000	5.4%	4 years
c	$25 000	3.75%	3 years
d	$4 590	9.5%	4 years
e	$150 000	6.25%	2 years
f	$35 500	15.8%	5 years

6 Pitia plans to invest his savings from a part-time job. He invests $5000 with his bank at 4.65% p.a. compounded annually for 3 years.

a Calculate the interest earned in the first year of the investment and use this to determine Pitia's principal amount for the second year of the investment.

b Calculate the interest earned in the second year of the investment and use this to determine the principal for the third year of the investment.

c Calculate the interest earned in the third year of the investment and use this to determine the final value of the investment.

7 Reconsider the information from question **6**. If Pitia had taken a loan for $5000 from his bank with interest charged at 4.65% p.a. and compounded annually for 3 years, how would this new situation compare? (Assume that there are no repayments made until the end of the loan period.)

8 Laura is investigating the best option for her savings plan. She plans to invest $14 500 for 2 years and wants to find out whether she would be better off investing it with interest calculated as simple interest or compound interest. The rates are:

Simple interest: 4.75% p.a. interest paid at maturity

Compound interest: 4.75% p.a. interest paid annually.

a For the simple interest option:

i how much interest is earned on the investment?

ii what is the final value of the investment?

b For the compound interest option:

i how much interest is earned on the investment?

ii what is the final value of the investment?

c Laura's friend tells her that the interest for both options should be the same because they both have the same interest rate. How accurate is this claim?

d Consider the interest calculations for both options in parts **a** and **b**. Compare the amount of interest for the two options for each year. Predict what will happen to the interest earned if the investment continues for a third year at the same rate.

e Which option should Laura be advised to take?

9 Reconsider the information provided in question **8**. If Laura's principal now represents a loan amount, which option would you advise Laura to take, if she makes no repayments on the loan until the end of the loan period?
Provide a brief reason.

10 To support her new tennis-coaching business, Lois borrows $80 000 from her bank. She agrees to an interest rate of 6.5% p.a. compounded annually for 4 years.

- **a** State the number of compounding periods and their length.
- **b** What is the total amount of money that Lois must repay on her loan, if she makes no repayments until the end of the loan period?
- **c** What is the total amount of interest?
- **d** Instead, Lois agrees to repay the total amount in equal monthly instalments over the 4 years. How much does she pay each month?

11 Frank plans to invest his money for the next 3 years, then use the final amount as a deposit for a house. He has $13 500 to invest and is considering these two options.

Option 1: Interest rate of 6.7% p.a. interest calculated at maturity

Option 2: Interest rate of 6.5% p.a. interest compounded annually

- **a** What will the value of Frank's investment be at the end of the term using option 1?
- **b** What will the value of Frank's investment be at the end of the term using option 2?
- **c** If the compounded rate remains the same, what rate must be applied to option 1 to make this the best option? Provide working to support your answer.

OPTION 1
Interest rate of 6.7% p.a. interest calculated at maturity

OPTION 2
Interest rate of 6.5% p.a. interest compounded annually

PROBLEM SOLVING AND REASONING

12 Banks sometimes advertise their investment rates according to the amount of money invested, length of the investment or the frequency of the interest payments. These tables display interest rates offered by a bank on term deposits of any size.

Interest paid monthly	
Term	Interest rate on amount invested (p.a.)
1 to <2 months	2.35%
2 to <3 months	2.35%
3 to <4 months	4.6%
4 to <5 months	3.05%
5 to <6 months	5.0%

Interest paid at maturity	
Term	Interest rate on amount invested (p.a.)
1 to <2 months	2.55%
2 to <3 months	2.55%
3 to <4 months	4.8%
4 to <5 months	3.25%
5 to <6 months	5.2%

a Which table appears to use compound interest as the interest calculation?

b Using the tables, what interest rate is earned on an investment for:

i 3 months with interest compounded monthly?

ii 4 months with interest paid at maturity only?

c A bank customer wishes to invest $7500 for 4 months. Calculate the amount of interest earned on the investment with interest:

i paid at maturity only

ii compounded monthly.

d Which option would you advise this customer to take? Explain your reasoning.

CHALLENGE

13 Consider an investment of $10 000 for 2 years at 4.4% p.a. compounded annually.

a What is the total value of the investment at the end of the term?

Imagine the terms of the investment were altered so the interest was compounded 6-monthly instead of annually.

b How many compounding periods are there now?

c Calculate the total value of the investment at the end of the term?
(Hint: there are four calculations, and use $\frac{1}{2}$ as the value for T.)

d Compare the final value of the investment using the two methods of calculation. Which method resulted in the greater value? Why do you think this is so?

Reflect

What is the difference between simple interest and compound interest?

1F The compound interest formula

Start thinking!

Consider the scenario described in **1E Start thinking!** on page 28, where $5000 is invested at 6% p.a. compounded annually for 3 years. You can perform compound interest calculations another way.

1 **a** Calculate the value of the investment at the end of the first year. As the interest rate is 6% p.a., the value is increased by 6%. That is, it is 106% of the value of the investment at the start of the year. (Hint: value = $5000 × 1.06.)

b Calculate the value of the investment at the end of the second year. Increase the value of the investment at the start of the second year (end of first year) by 6%.

c Calculate the value of the investment at the end of the third year. Increase the value of the investment at the start of the third year (end of second year) by 6%.

d State the final value of the investment and the amount of interest earned after three years.

2 Compare your answers in question **1** for the value of the investment at the end of each year to those obtained in 1E Start thinking! question **4** on page 28.

3 The calculations completed in question **1** follow a pattern that can be used to obtain a rule or shortcut when calculating the compounded value of an investment. Study the table at right to describe this pattern.

Year	Compounded value at end of year
1	$\$5000 \times 1.06$ $= \$5000 \times 1.06^1$
2	$(\$5000 \times 1.06^1) \times 1.06$ $= \$5000 \times 1.06^2$
3	$(\$5000 \times 1.06^2) \times 1.06$ $= \$5000 \times 1.06^3$

4 If the investment period is longer than 3 years, use the pattern to calculate the compounded value of the investment at the end of:

a year 4 **b** year 7.

KEY IDEAS

- Compound interest calculations can be performed using the formula: $A = P(1 + R)^n$, where
  A = compounded value of the investment (the final amount),
  P = principal (the initial investment amount or amount borrowed),
  R = interest rate for the compounding period, written as a decimal, and
  n = number of compounding periods.
- The compound interest is the difference between the final amount and the initial amount: compound interest = $A - P$.

EXERCISE 1F Working with the compound interest formula

1 Use the compound interest formula to find the value of A in each of these.

a $A = ?, P = \$10\,000, R = 0.05, n = 4$
b $A = ?, P = \$5000, R = 0.06, n = 3$
c $A = ?, P = \$600, R = 0.08, n = 4$
d $A = ?, P = \$44\,000, R = 0.038, n = 5$
e $A = ?, P = \$100\,000, R = 0.015, n = 8$
f $A = ?, P = \$19\,500, R = 0.0025, n = 10$

EXAMPLE 1F-1 Using the compound interest formula

For an investment of \$12 000 at an interest rate of 4.8% p.a. for 4 years with interest compounded annually, calculate:

a the value of the investment after 4 years
b the amount of compound interest earned.

THINK

a **1** Write the formula and identify the variables. Interest is compounded annually, so there are four compounding periods.

2 Substitute the values into the formula and calculate the result. Round the answer to two decimal places.

3 Write the answer.

b The amount of compound interest earned is the difference between the final amount and the initial amount.

WRITE

a $A = P(1 + R)^n$
$A = ?$
$P = \$12\,000$
$R = 4.8\%$ p.a. $= \frac{4.8}{100} = 0.048$
$n = 4$

$A = \$12\,000(1 + 0.048)^4$
$= \$12\,000 \times 1.048^4$
$= \$14\,475.260\,12$
$\approx \$14\,475.26$

The value of the investment after 4 years is \$14 475.26.

b compound interest $= A - P$
$= \$14\,475.26 - \$12\,000$
$= \$2475.26$

2 For each investment, calculate:

i the value of the investment at the end
ii the amount of compound interest earned.

a \$7500 invested at 5% p.a. for 3 years with interest compounded annually
b \$9000 invested at 4.5% p.a. for 3 years with interest compounded annually
c \$22 500 invested at 6.8% p.a. for 5 years with interest compounded annually
d \$40 000 invested at 7.3% p.a. for 4 years with interest compounded annually

3 For each investment in the table, use the compound interest formula to calculate the final value of the investment if interest is compounded annually.

4 Calculate the amount of interest earned in each investment listed in the table in question **3**.

	Principal	Interest rate (p.a.)	Term
a	$3200	6.4%	4 years
b	$4590	9.8%	5 years
c	$16 000	2.85%	2 years
d	$25 900	10.5%	3 years
e	$100 000	4.99%	5 years
f	$255 000	12.15%	6 years

5 For each investment, state the number of compounding periods, n.

a $5000 is invested at 6.7% p.a. for 4 years with interest calculated annually.

b $8450 is invested at 3.9% p.a. for 3 years with interest calculated at the end of each year.

c $25 000 is invested at 8.5% p.a. for 2 years with interest calculated at the end of each 6 months.

d $35 000 is invested at 9.6% p.a. for 4 years with interest calculated at the end of each month.

6 When using the compound interest formula, it is important that the interest rate matches the compounding period. Copy and complete each of these.

a When compounding annually, the interest rate must be per _____.

b When compounding 6-monthly, the interest rate must be per 6 months. This is achieved by dividing the annual rate (p.a.) by _____.

c When compounding quarterly, the interest rate must be per quarter. This is achieved by dividing the annual rate (p.a.) by _____.

d When compounding monthly, the interest rate must be per _____. This is achieved by dividing the annual rate (p.a.) by _____.

EXAMPLE 1F-2 Finding R and n for a different compounding period

For an investment of $3000 at 6.4% p.a. for 2 years with interest compounded quarterly:

a write the given interest rate so that it matches the compounding period

b state the value of R and n to be used in the compound interest formula.

THINK

a 1 Identify the compounding period. 'Quarterly' means one quarter of a year.

2 Write the given annual interest rate to match the compounding period of $\frac{1}{4}$ year.

b 1 State the value of R as a decimal.

2 Identify the number of compounding periods in 1 year. Hence, state the value of n which is the number of compounding periods in 2 years.

WRITE

a compounding period = $\frac{1}{4}$ year

6.4% p.a. = $\frac{1}{4} \times 6.4\%$ per quarter
= 1.6% per quarter

b $R = 1.6\% = 0.016$

number of compounding periods in 1 year = 4

$n = 2 \times 4$
$= 8$

UNDERSTANDING AND FLUENCY

7 For each investment:

- i write the given interest rate so that it matches the compounding period
- ii state the value for R and n to be used in the compound interest formula.

a $15 000 is invested at 4.8% p.a. for 3 years with interest calculated quarterly.

b $22 000 is invested at 5.9% p.a. for 4 years compounded 6-monthly.

c $3500 is invested at 8.4% p.a. for 2 years with interest calculated monthly.

d $14 500 is invested at 9.6% p.a. for 5 years with interest calculated at the end of each month.

8 Use the compound interest formula to determine the value for A given the information provided in the table.

	Principal	Interest rate (p.a.)	Term	Compounding period
a	$4800	8.15%	3 years	annually
b	$9500	8.8%	4 years	6-monthly
c	$18 000	4.84%	5 years	quarterly
d	$35 000	16.8%	4 years	monthly
e	$50 000	6.95%	3 years	6-monthly
f	$150 000	8.16%	1 year	quarterly
g	$500 000	4.98%	2 years	4-monthly
h	$750 000	10.26%	3 years	2-monthly

PROBLEM SOLVING AND REASONING

9 Maria and her family begin long-term planning for an overseas holiday. They invest their savings of $16 000 for 3 years in an account with their bank, which offers them 5.75% p.a. interest compounded annually.

a How many compounding periods will there be in Maria's investment?

b What is the length of each compounding period?

c Identify the values for P, R and n.

d Use the compound interest formula to calculate the final value of the investment.

e How much interest will be earned on this investment?

f Maria estimates that she will need $20 000 to fund the entire overseas holiday. If she invests the money for another year, will she achieve this goal?

10 Carolyn borrows $45 000 to set up a dance studio for her daughter. The loan is over 4 years with interest charged at 8.95% p.a. compounding annually. The terms of the loan require Carolyn to repay the total amount at the end of the term.

- **a** Use the compound interest formula to calculate the total amount that Carolyn must repay.
- **b** How much of the amount in part **a** represents the interest charge?

11 While investigating different investment options with his bank, Bailey decided on two possibilities to further explore. The two options are as displayed.

OPTION 1
Interest rate of
5.35% p.a.
with interest paid at maturity

OPTION 2
Interest rate of
4.95% p.a.
interest compounded annually

Bailey has $9000 to invest and starts out planning to invest the money for 3 years.

- **a** What would be the value of the investment at the end of the term for each option?
- **b** Which option would Bailey be advised to take for a 3-year investment? Provide a supporting reason with your answer.
- **c** How do the answers to parts **a** and **b** change if Bailey invests the money for 5 years?

12 Sheetal is exploring the benefits of investing with an online bank. She finds the bank will pay 8.6% p.a. interest on investments over 36 months with interest compounded quarterly. Sheetal decides to invest her savings of $8400 for 5 years.

- **a** Write the annual interest rate as a rate per quarter.
- **b** Use your answer from part **a** to state the value for R in the compound interest formula.
- **c** How many compounding periods, n, will there be in this investment?
- **d** Use the compound interest formula to determine the final value of the investment.
- **e** What is the total amount of interest Sheetal will expect to earn on this investment?

PROBLEM SOLVING AND REASONING

13 An investment of $10 000 is made for 4 years with a bank that offers an annual interest rate of 8.4%. Explore how the frequency of the compounding periods affects the final value of the investment.

- **a** Calculate the final value of the investment if the interest is calculated at the end of the term, that is, at maturity.
- **b** Calculate the final value of the investment if the interest is compounded:
  - **i** 6-monthly **ii** quarterly **iii** 2-monthly **iv** monthly
- **c** Use your results from part **b** to briefly explain what happens to the final value of an investment as the number of compounding periods increases. Will this always be the case?

14 Through a store purchase plan, Troy is informed that interest is calculated as compound interest based on the amount borrowed. In this situation, would Troy be better off with a plan involving a higher frequency of compounding periods or fewer compounding periods?

15 Erica knows she can earn 6.8% p.a. for her investments with her bank. She plans to invest some money for 2 years with interest compounded quarterly.

- **a** For this investments, state:
  - **i** the number of compounding periods, n
  - **ii** the value for R to be used in the compound interest formula.
- **b** At the end of her 2-year investment, Erica hopes to have an investment that has grown to $6500. Which variable of the compound interest formula does this value represent?
- **c** Substitute the known values into the compound interest formula and solve the equation for the unknown variable.
- **d** To the nearest dollar, how much should Erica invest at the start of her investment?

CHALLENGE

16 Consider an investment of $5000 at 5% p.a. for 5 years with interest compounded annually.

- **a** How much interest is earned in each year of the investment?
- **b** What is the value of the investment at the end of each year during this term?
- **c** Present the growth of the investment as a line graph displaying the years on the horizontal axis and the value of the investment on the vertical axis. You may wish to use digital technology to present your graph.
- **d** Repeat parts **a**–**c** with the same values, but for these compounding periods:
  - **i** quarterly **ii** 6-monthly **iii** monthly.
- **e** If the interest for this investment was calculated using simple interest, briefly explain how the value of the investment could be shown on a graph.
- **f** How will a graph displaying the investment's growth using simple interest compare with the compound interest graphs? You may wish to present all graphs on the one set of axes.

Reflect

Why is it important that the interest rate used in the compound interest formula matches the compounding period?

1G Working with compound interest

Start thinking!

Manisha received a flyer from her bank encouraging her to invest in a savings account so that her money would increase in value more quickly. Manisha agreed to invest $14 000 in an account for 2 years at an interest rate of 4.8% p.a. with interest compounded every 3 months.

1 **a** How many compounding periods will there be in each year of Manisha's investment?

b How many compounding periods will there be for the entire term of Manisha's investment?

c The answer obtained in part **b** represents the value for n in the compound interest formula. Develop a rule that can be used to determine the value for n for any compound interest situation.

2 The value for R must match the compounding period before it can be used in the compound interest formula.

a What is the annual interest rate that is applied to this investment?

b As the interest is compounded every 3 months (quarterly), what needs to be done to the annual interest rate to match the compounding period?

c Use your response from part **b** to write the interest rate as a percentage per quarter.

d Convert the percentage to a decimal value.

e The answer obtained in part **d** represents the value for R in the compound interest formula. Develop a rule that can be used to determine the value for R for any compound interest situation.

3 **a** From the compound interest formula, which variable do you not know the value of and what is the value for each of the known variables?

b Use the formula to calculate the value of the missing variable.

c What does your answer from part **b** represent in relation to Manisha's investment?

d How much compound interest is to be earned over the term of the investment?

4 How could you use the compound interest formula to find the value of P (when given A, R and n) or to find R (when given A, P and n)?

KEY IDEAS

- The compound interest formula is $A = P(1 + R)^n$.
- The amount of compound interest = $A - P$.

EXERCISE 1G Working with compound interest

UNDERSTANDING AND FLUENCY

1 Determine the number of compounding periods, n, if interest is compounded:

a annually for 3 years
b 6-monthly for 5 years
c quarterly for 4 years
d monthly for 5 years
e quarterly for 4 years and 6 months
f 6-monthly for $3\frac{1}{2}$ years.

2 Determine the value for the interest rate per compounding period, if the annual percentage rate (p.a.) is:

a 8.4% compounded quarterly
b 9.06% compounded monthly
c 7.5% compounded every 4 months
d 4.62% compounded 6-monthly.

3 For each investment, state:

i the number of compounding periods, n
ii the value for R to be used in the compound interest formula.

a $12 000 is invested at 3.8% p.a. for 4 years with interest calculated annually.
b $9500 is invested at 6.4% p.a. for 3 years with interest calculated each 6 months.
c $30 000 is invested at 12.8% p.a. for 2 years with interest calculated quarterly.
d $42 000 is invested at 8.4% p.a. for 4 years with interest calculated each month.

4 Calculate the value of A for each investment in question **3**.

EXAMPLE 1G-1 Using the compound interest formula to find P

How much must be invested at an interest rate of 6.8% p.a. in order for the value to grow to $5000 over 3 years, with interest compounded 6-monthly?

THINK	WRITE
1 Write the compound interest formula and identify the variables. You need to find the initial value (principal, P) that will grow to $5000 (final amount, A).	$A = P(1 + R)^n$ $A = \$5000$ $n = 2 \times 3 = 6$ 6.8% p.a. = 3.4% per 6 months $R = 0.034$ $P = ?$
2 Substitute the values into the formula and simplify.	$5000 = P(1 + 0.034)^6$ $= P \times 1.034^6$
3 Solve for P by dividing both sides of the equation by 1.034^6.	$\frac{5000}{1.034^6} = \frac{P \times 1.034^6}{1.034^6}$ $P = 4091.16$
4 Write the answer.	$4091.16 will grow to $5000 over 3 years at 6.8% p.a. compounded 6-monthly.

UNDERSTANDING AND FLUENCY

5 Briefly explain how you can use the compound interest formula to determine the value for P, when given the value for A, R and n.

6 Find the value for P in each of these.
- a How much must be invested at an interest rate of 10.5% p.a. in order for the value to grow to $30 000 over 4 years with interest compounded annually?
- b How much must be invested at an interest rate of 5.4% p.a. in order for the value to grow to $10 000 over 3 years with interest compounded 6-monthly?
- c How much must be invested at an interest rate of 6.4% p.a. in order for the value to grow to $5500 over 5 years with interest compounded quarterly?
- d How much must be invested at an interest rate of 7.2% p.a. in order for the value to grow to $8000 over 2 years with interest compounded every month?

7 Use the compound interest formula to determine the value for P, to the nearest dollar, given the information provided in the table.

	Final value (compounded value)	Interest rate (p.a.)	Term	Compounding period
a	$5200	6.75%	3 years	annually
b	$10 000	4.2%	6 years	monthly
c	$22 000	12.5%	4 years	6-monthly
d	$16 650	15.5%	3 years	quarterly
e	$225 000	9.95%	$4\frac{1}{2}$ years	6-monthly

8 How much interest is earned in each case in question 7?

EXAMPLE 1G-2 Using the compound interest formula to find R

For an investment of $4000 that grows to $4300 after 1 year with interest compounded 6-monthly:
- a find the value of R as a decimal, correct to three decimal places
- b state the annual interest rate, correct to one decimal place.

THINK

a 1 Write the compound interest formula and identify the variables.
2 Substitute the values into the formula.
3 Solve for R by first dividing both sides of the equation by 4000. You may like to also swap the sides of the equation.
4 Take the square root of each side (consider only the positive square root as R is positive).
5 Subtract 1 from each side to find R.

b As there are two compounding periods in 12 months, multiply the interest rate for 6 months by 2.

WRITE

a $A = P(1 + R)^n$
$A = \$4300$, $P = \$4000$, $n = 1 \times 2 = 2$, $R = ?$
$4300 = 4000(1 + R)^2$
$1.075 = (1 + R)^2$
$(1 + R)^2 = 1.075$

$\sqrt{(1 + R)^2} = \sqrt{1.075}$
$1 + R = \sqrt{1.075}$

$1 + R - 1 = \sqrt{1.075} - 1$
$R = 0.036\,822\,07$
≈ 0.037

b Interest rate for 6 months = 3.7%
Interest rate for 12 months = $3.7\% \times 2$
$= 7.4\%$

9 For each investment:

- **i** find the value of R as a decimal, correct to three decimal places
- **ii** state the annual interest rate, correct to one decimal place.

a \$1000 grows to \$1170 after 1 year with interest compounded 6-monthly

b \$2500 grows to \$2800 after 2 years with interest compounded annually

c \$3750 grows to \$4050 after 6 months with interest compounded quarterly

10 Consider an investment of \$12 000 that grows to \$15 000 over a 3-year period with interest compounded annually.

a From the compound interest formula, which variable do you not know the value of?

b Substitute the known values into the compound interest formula and show that it simplifies to $\frac{5}{4} = (1 + R)^3$.

c To solve the equation for R, you can perform inverse operations to both sides of the equation. (Note that the inverse operation to cubing a number is to take the cube root of a number.)

Copy and complete the calculation shown at right to solve the equation for R.

$$\frac{5}{4} = (1 + R)^3$$
$$1.25 = (1 + R)^3$$
$$(1 + R)^3 = 1.25$$
$$\sqrt[3]{(1 + R)^3} = \sqrt[3]{}$$
$$1 + R = \sqrt[3]{}$$
$$1 + R - 1 = \sqrt[3]{1.25} - \underline{}$$
$$R = 0.077\,217\ldots$$

d The value for R in part **c** is the rate per compounding period as a decimal. State this as an annual interest rate, correct to one decimal place.

11 For each investment:

- **i** find the value of R as a decimal, correct to three decimal places
- **ii** state the annual interest rate, correct to one decimal place.

a \$1600 grows to \$2180 after 3 years with interest compounded annually

c \$10 000 grows to \$11 000 after 18 months with interest compounded 6-monthly

b \$7500 grows to \$8200 after 2 years with interest compounded 6-monthly

12 Jade's parents invested \$5500 for 4 years into a savings account and it grew to \$8000, with interest compounding annually.

a State the values for A, P and n.

b Use the compound interest formula to solve for R.

c What annual interest rate was applied to Jade's savings account?

13 **a** Briefly explain how you can use the compound interest formula to determine the value for R when given the values for A, P and n.

b When the compounding periods are shorter than 1 year, what must be done to the R value to determine the annual rate of interest?

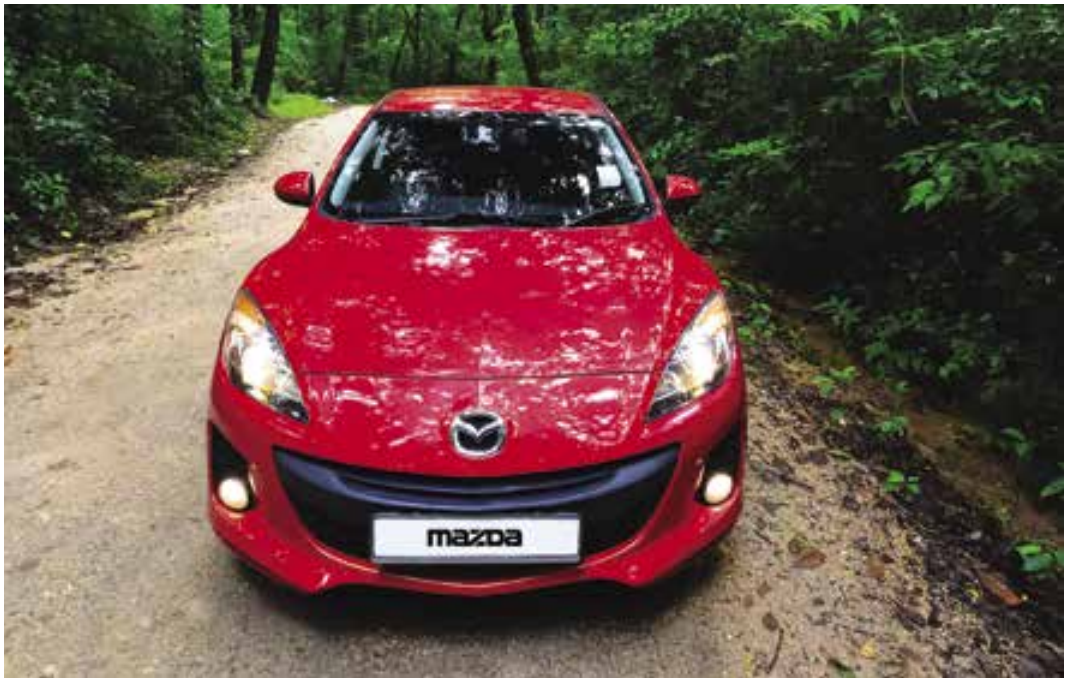
14 Use the compound interest formula to determine:

- **i** the value for R (as a decimal)
- **ii** the annual interest rate for the information given in the table.

	Principal	Final value	Term	Compounding period
a	$1000	$1500	3 years	annually
b	$8000	$12 000	4 years	annually
c	$16 000	$19 000	2 years	6-monthly
d	$3500	$4000	1 year	quarterly
e	$12 000	$16 500	3 years	6-monthly
f	$25 000	$29 500	2 years	quarterly

15 Riley aims to have $15 000 by the time he turns 18 so that he can buy a car. He opens a savings account with his bank that pays interest at a rate of 7.2% p.a. compounded monthly. Riley plans to keep the money invested on these terms for 3 years and needs to know how much he should invest.

- **a** How many compounding periods, n, will there be in Riley's investment?
- **b** What is the interest rate per compounding period? Hence, state the value of R.
- **c** From the compound interest formula, which variable do you not know the value of?
- **d** Use the compound interest formula to calculate the amount of money Riley must invest in order for it to grow to $15 000 over the 3-year term.



16 An investment of $25 000 grows to $30 000 over a period of time in a savings account that earns 5.0% p.a. with interest compounded annually.

- **a** From the compound interest formula, which variable do you not know the value of?
- **b** Substitute the known values into the compound interest formula and show that it simplifies to $1.2 = (1.05)^n$.

PROBLEM SOLVING AND REASONING

c One strategy for solving the equation for n is the guess, check and improve method. Substitute different values for n and evaluate the right side until the right side value is greater than the left side value. This value for n represents the **solution** to the equation.

Value for n	Left side	Right side	Comment
1	1.2	$1.05^1 = 1.05$	Right side value is too small
2	1.2	$1.05^2 = 1.1025$	
3	1.2	$1.05^3 =$	
4	1.2	$1.05^4 =$	

d The value for n obtained in part **c** is the number of compounding periods for the investment. Briefly explain how this relates to this investment.

17 For a period of time, Yousef watched his savings grow from \$850 to \$1100 in a savings account that earns 4.2% p.a. with interest compounded annually.

a Use the given information to state the values for A, P and R.

b Substitute the given values into the compound interest formula and simplify.

c Use the guess, check and improve method to solve the equation for n.
You may wish to use a similar table to the one used in the previous question.

d How many compounding periods did it take for Yousef's savings to grow from \$850 to \$1100?

18 For the information given in the table, state the number of compounding periods, n, for the initial principal to grow into the final value.

19 Considering each investment in question **18**, write each compounding period in years.

	Principal	Final value	Interest rate (p.a.)	Investment period
a	\$900	\$1200	5.0%	annually
b	\$12 000	\$14 000	3.5%	annually
c	\$15 000	\$18 500	4.8%	6-monthly
d	\$650	\$900	10.5%	quarterly
e	\$35 000	\$42 000	6.8%	6-monthly
f	\$50 000	\$55 590	7.2%	quarterly

CHALLENGE

20 In question **16** you were introduced to a method of guess, check and improve as a strategy to determine the value of n in the compound interest formula. Investigate other strategies that may be used to solve for n. You may wish to re-use the information in questions **16**–**18** as part of your investigations.

Write a brief report on your investigations using examples to demonstrate your findings.

(Hint: equations such as $1.2 = (1.05)^n$, where the pronumeral is in the power, are known as exponential equations.)

Reflect

What other areas in mathematics can the compound interest formula be applied to?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

- percentage
- fraction
- decimal
- percentage of an amount
- one amount as a percentage of another
- discount
- percentage decrease
- mark-up
- percentage increase
- profit
- loss
- interest
- simple interest
- principal
- interest rate
- time
- investment
- loan
- compound interest
- compounding period
- compounded value

MULTIPLE-CHOICE

1A **1** What is $22\frac{1}{2}\%$ of \$284.95, rounded to the nearest cent?

A \$63.26 **B** \$64.10
C \$64.11 **D** \$262.45

1B **2** What is the selling price following a discount of 10.5% on \$120?

A \$12.60 **B** \$107.40
C \$109.50 **D** \$132.60

1C **3** An amount of \$1500 is borrowed at 6.75% p.a. simple interest for 24 months. Which set of values should be substituted into the simple interest formula?

A $P = 1500, R = 6.75, T = 24$
B $P = 1500, R = 6.75, T = 2$
C $P = 1500, R = 0.0675, T = 2$
D $P = 1500, R = 0.0675, T = 24$

1D **4** What is the annual interest rate that is applied to a loan of \$20 000 taken over 3.5 years if the amount of simple interest charged on the loan is \$2975?

A 0.0425% **B** 0.354%
C 4.25% **D** 14.875%

1E **5** An investment of \$10 000 for 4 years earns interest at 8.4% p.a. compounded annually. What is the value of the investment after 2 years?

A \$11 750.56 **B** \$11 680
C \$12 737.61 **D** \$13 807.57

1F **6** For an investment of \$4800 for 3 years at 3.6% p.a. with interest compounded monthly, what is the number of compounding periods?

A 3 **B** 6 **C** 12 **D** 36

1G **7** Over a 4-year period, an investment grew from \$18 500 to \$27 000 with interest compounded quarterly. From the compound interest formula, which variable do you not know the value of?

A A **B** P **C** R **D** n

1G **8** The annual interest rate on an investment compounded quarterly is 8.25% p.a. What is the interest rate per compounding period?

A 8.25% **B** 2.0625%
C 0.0825% **D** 0.020 625%

SHORT ANSWER

1A **1** Write each percentage as a decimal.
a 24% **b** 35.8% **c** $16\frac{1}{4}\%$ **d** 0.272%

1A **2** Write each percentage as a fraction in simplest form.
a 86% **b** 235% **c** $15\frac{1}{5}\%$
d 8.65% **e** $13\frac{2}{3}\%$ **f** 0.0512%

1A **3** Calculate each of these.
a 15% of \$685 **b** 170% of \$9500
c $17\frac{1}{2}\%$ of \$98.50

1A **4** **a** Write \$85 as a percentage of \$680.
b Write \$39.95 as a percentage of \$72.50, rounding your answer to two decimal places.

1A **5** A CD priced at \$23.95 is sold at a sale for \$12.45. Write the selling price as an exact percentage of the original price.

1B **6** The following parts **a**–**d** list the original prices and the percentage discount or percentage mark-up on some goods. In each case, calculate:
i the selling price after the discount or mark-up
ii the discount or mark-up amount, rounding to the nearest cent.
a \$2550; $15\frac{1}{2}\%$ discount
b \$899.50; 125% mark-up
c \$18 045; 38.4% mark-up
d \$1945.90; 37.5% discount

1B **7** For each of these:
i state the value of the profit or loss
ii write the profit or loss as a percentage of the original price (rounded to the nearest 1%).
a original price \$139.85, selling price \$171.30
b original price \$27 490, selling price \$9813.95

1C **8** Calculate the simple interest if:
a P = \$2500, R = 6%, T = 3 years
b P = \$28 000, R = 4.25%, T = 5 years

1C **9** For each investment, calculate:
i the amount of simple interest earned
ii the value of the investment at the end.
a \$150 000 at an interest rate of 4.8% p.a. for $3\frac{1}{2}$ years.
b \$9450 at an interest rate of 2.95% p.a. for 5 months.

1D **10** Find the missing value in each of these. Remember to write the value for R as a percentage, and time in years.
a I = \$308, P = \$3850, R = 6.4%, T = ?
b I = \$765, P = ?, R = 4.25%, T = 12 months
c I = \$9843.75, P = 35 000, R = ?, T = 2.5 years

1E **11** In each investment, interest is calculated at the end of each year and added to the principal amount. Calculate the total amount of interest compounded during this time. Show the value of the investment at the end of each compounding period.
a \$15 000 is invested at 4.5% p.a. for 3 years.
b \$6850 is invested at 6.4% p.a. for 4 years.

1F **12** Use the compound interest formula to determine the value for A, given the information provided in the table.

	Principal	Interest rate (p.a.)	Term	Compounding period
a	\$2 250	7.5%	2 years	annually
b	\$18 750	9.2%	4 years	quarterly
c	\$150 000	6.24%	4 years	6-monthly

1G **13** Find the missing value in each of these, correct to two decimal places.
a A = ?, P = \$5650, R = 0.058, n = 3
b A = \$20 000, P = ?, R = 0.08, n = 4
c A = \$10 000, P = \$8000, R = ?, n = 6

MIXED PRACTICE

1 What is $25\frac{1}{2}\%$ written as a decimal?

2 The percentage value $12\frac{1}{5}\%$ is equivalent to which fraction in simplest form?

A $\frac{1}{5}$ **B** $12\frac{1}{5}$ **C** $\frac{61}{500}$ **D** $\frac{61}{5}$

3 Which percentage is closest in value to $\frac{9}{11}$?

A 80% **B** 81% **C** 82% **D** 83%

4 Jaymee saved $112 towards an iPod that retails for $248. Her savings as a percentage of the amount required is closest to which?

A 45% **B** 55% **C** 221% **D** $136

5 The calculation of $17\frac{3}{4}\%$ of $190.70 can be performed by completing which one of these?

A 17.75 × $190.70 **B** 17.34 × $190.70

C 0.1775 × $190.70

D 17.75 ÷ 190.70 × $100

6 A store offers computer accessories at a discount of 22.5%. A hard drive is originally priced at $145.95. Which of these shows the calculation for the sale price?

A 22.5% of $145.95 **B** 122.5% of $145.95

C 77.5% of $145.95

D $145.95 − 77.5% of $145.95

Questions **7** and **8** refer to this information.
A men's clothing store applies a mark-up of 115.25% to the wholesale price of their clothing to determine the selling price to the public.

7 A pair of jeans has a wholesale price of $85. What is the mark-up amount on these jeans to the nearest dollar?

8 Which shows the correct method to calculate the selling price of the jeans?

A $85 + 15.25% of $85

B $85 + 215.25% of $85

C 1.1525 × $85 **D** 215.25% of $85

9 Vanessa sells her jewellery for $25 per piece. This price includes her profit margin of 150%, applied to her cost price. What is the cost price to Vanessa of each jewellery piece?

10 $10 000 is invested at 6.5% p.a. simple interest for 18 months. Which set of values should be substituted into the simple interest formula?

A $P = 10\,000$, $R = 6.5$, $T = 1.5$

B $P = 10\,000$, $R = 0.065$, $T = 1.5$

C $P = 10\,000$, $R = 6.5$, $T = 18$

D $P = 10\,000$, $R = 0.065$, $T = 18$

Questions **11** and **12** refer to this information.
A loan of $15 750 is charged a simple interest rate of 9.8% p.a. for a period of 2 years.

11 How much interest is charged to the loan?

12 What is the total amount to be repaid?

13 Bryce invests $1845 at 4.5% p.a. simple interest. He hopes this investment grows to at least $2000. From the simple interest formula, which variable do you not know the value of?

A I **B** P **C** R **D** T

14 How much needs to be invested at an interest rate of 5.25% p.a. for 3 years to earn $1500 in simple interest?

Questions **15**–**18** refer to this information.
An investment of $25 000 is made with a bank that offers 4.75% p.a. interest compounded 6-monthly. The investment is made for 4 years.

15 How many compounding periods are there for the term of the investment?

16 Which represents the value used for R in the compound interest formula?

A 4.75 **B** 0.0475

C 0.023 75 **D** 0.011 875

17 Which calculation gives the final value of the investment at the end of the term?

A $A = \$25\,000(1.023\,75)^4$

B $A = \$25\,000(1.023\,75)^8$

C $A = \$25\,000(1.0475)^4$

D $A = \$25\,000(1.0475)^8$

18 How much interest is compounded over the term of the investment?

19 Missy invests $4500 with a bank who offers her interest at a rate of 6.6% p.a. with interest compounded monthly. The term of the investment is 2 years. Which statement is *not* correct?

- **A** There are 24 compounding periods for the investment.
- **B** The rate per compounding period is 0.55%.
- **C** The amount of interest earned on the investment is $594.
- **D** The value at the end of the investment is close to $5130.

Questions **20** and **21** refer to this information.
An investment of $5850 is made for 4 years with a bank that offers an interest rate of 6.45% p.a. with interest compounded annually.

20 What is the value of the investment after the second year?

21 How much interest is earned through the entire term of the loan?

22 Consider an investment of $16 000 that grows to $20 000 over a 3-year period with interest compounded annually. From the compound interest formula, which variable do you not know the value of?

A A **B** P **C** R **D** n

Questions **23** and **24** refer to this information.
Over a 5-year period, Ingrid's parents invested $9000 for her into a savings account where interest was calculated 6-monthly. Over that time, the investment grew to $12 000.

23 Which correctly displays the calculation after substituting the values into the compound interest formula?

- **A** $\$12\,000 = \$9000(1 + R)^{10}$
- **B** $\$12\,000 = \$9000(1 + R)^{5}$
- **C** $\$9000 = \$12\,000(1 + R)^{10}$
- **D** $\$12\,000 = \$9000(1 + R)^{30}$

24 What was the annual interest rate applied to the investment?

ANALYSIS

Susan made an investment of $12 000 for 4 years with her bank with interest compounded quarterly throughout the term. The interest rate earned on the investment varied at different stages.

In the first year, the interest rate was $5\frac{1}{2}$% p.a., for the next 2 years the interest rate remained fixed at 5.7% p.a., and the rate increased to 6.2% p.a. for the entire final year.

a How many compounding periods were there for the term of the investment?

b Consider the annual interest rates stated. What value for R would be used in the compound interest formula in the:
i first year **ii** second 2 years **iii** final year?

c Determine the value of the investment at the end of the first year.

d State the values for P, R and n you would use to calculate the growth in the investment from the start of the second year to the end of the third year.

e What was the value of the investment at the end of the third year?

f What was the value of the investment at the end of the term?

g How much interest did Susan earn on the investment?

h Write the interest from part **g** as a percentage of the original amount invested, correct to two decimal places.

Susan had hoped that her investment would grow to $15 500 by the end of the term.

i Keeping the compounding periods the same, what annual interest rate would Susan have had to receive to achieve her desired final value?

j If the compounding periods were increased to monthly, how does the answer to part **i** change?

CONNECT

Repayment options

Every day, people borrow money in one form or another to make large purchases. Such purchases may be from money obtained through the use of a credit card, store purchase plans or by a loan with a bank. The common feature in all these cases is that the money is expected to be repaid, generally with interest, and with payments made monthly.

In this chapter, you have performed compound interest calculations for loans where it is assumed no repayments are made until the end of the loan period. In reality, interest is calculated each month when a repayment is due. Since repayments are being made each month, the loan amount is reducing and so the amount of interest charged each month changes. This is known as reducible interest. Your task is to perform calculations for a loan of \$5000 at $12\frac{1}{2}\%$ p.a. over 2 years to determine which of two options is better for repaying the money.

- Option 1: interest is calculated as simple interest on the amount borrowed
- Option 2: interest is calculated on the amount owing at the time of calculation

Your task

You are to:

- calculate the interest for the first option if simple interest is charged on the loan
- work out the total amount to be repaid and the amount of each monthly repayment for option 1
- compare to option 2 (reducible interest loan). Calculating the amount of each monthly repayment is beyond the scope of this course but it is \$236.54. Each month, interest is added to the balance owing and a repayment is made. Complete the table at right for a full 24 months. The first 3 months have been done for you.

Month	Opening balance	Interest	Repayment	Closing balance
1	\$5000.00	\$52.08	\$236.54	\$4815.54
2	\$4815.54	\$50.16	\$236.54	\$4629.16
3	\$4629.16	\$48.22	\$236.54	\$4440.84

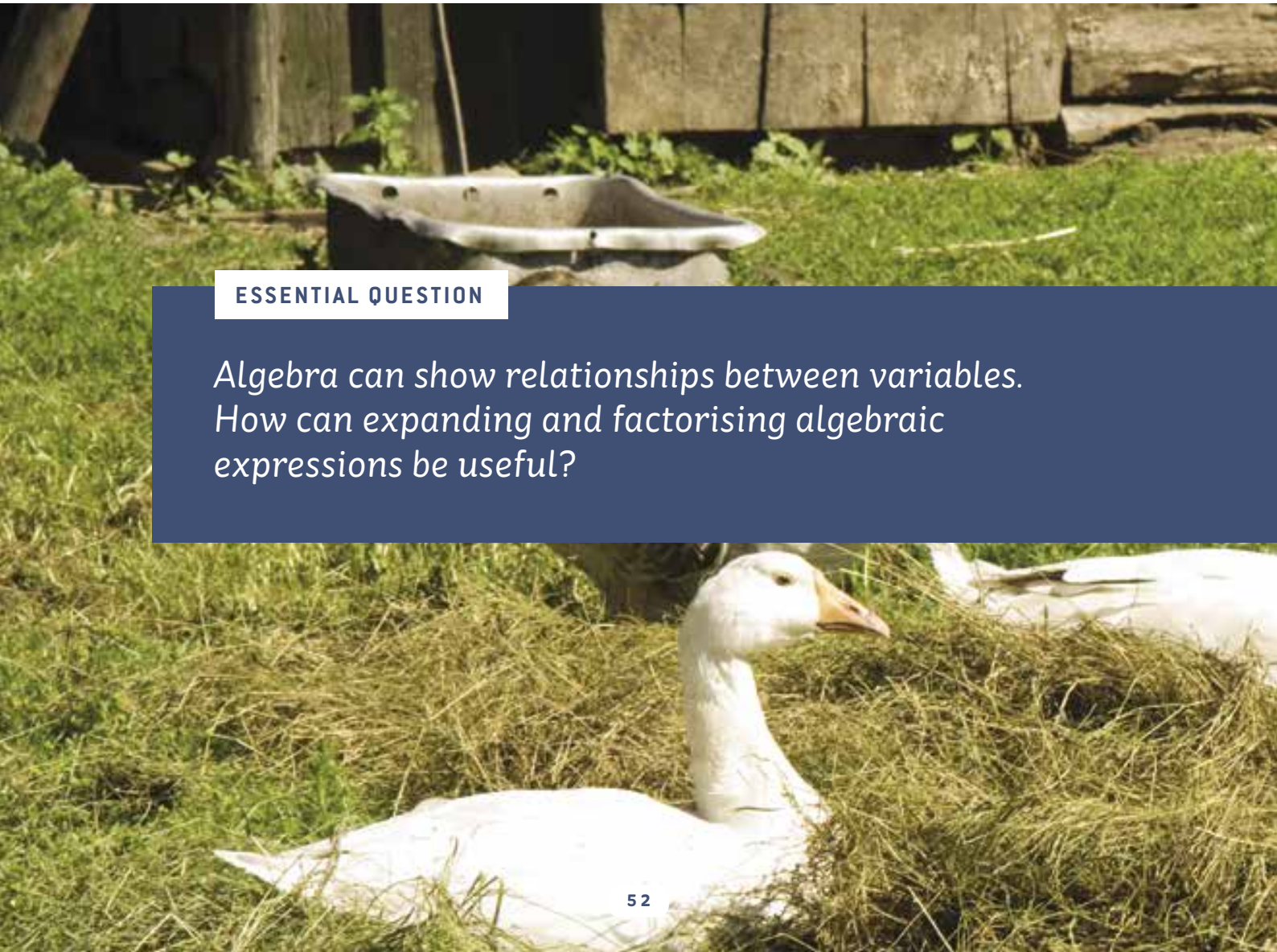
- calculate the total interest charged on the loan if reducible interest is applied
- compare the two loans by writing the interest charged on the reducible interest loan as a percentage (per annum) on the amount borrowed
- investigate the repayment options and methods of interest calculations offered by stores through their purchase plans and the methods various banks apply to their loans.

Include all necessary working to justify your answers.

You may like to present your findings as a report. Your report could include:

- an advertising brochure displaying tables comparing values
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

2 ALGEBRA



ESSENTIAL QUESTION

Algebra can show relationships between variables. How can expanding and factorising algebraic expressions be useful?

Are you ready?

2A **1** **a** Write each term in expanded form.

i $-3m$ **ii** wxy

iii $7a^2b$ **iv** $10p^3q^2r$

b Write the coefficient of each term shown in part **a**.

c Give two examples of a like term for each term shown in part **a**.

2A **2** **a** What does $8a + 4b + 7a + 3b$ simplify to?

A $22ab$ **B** $15a + 7b$

C $12a + 10b$ **D** 22

b What does $5x - 2x + 9x$ simplify to?

A $16x$ **B** 16

C $12x$ **D** 12

2A **3** Evaluate each expression after substituting $a = 5$ and $b = -2$.

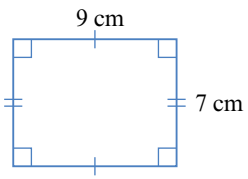
a $a + 4b$ **b** $3b^2$

c $5ab - 1$ **d** $2a^2 - 7b + 6a$

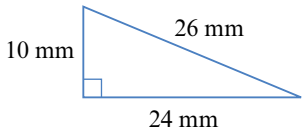
2A **4** For each shape, find

i the perimeter **ii** the area.

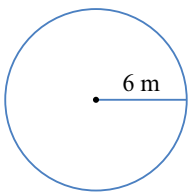
a



b



c



2B **5** Consider x^6.

a Identify:

i the base

ii the index or power.

b Write the term in expanded form.

c What does $x^6 \times x^2$ simplify to?

A x^{12} **B** x^4

C $12x$ **D** x^8

d Simplify:

i $x^6 \div x^2$ **ii** $(x^6)^2$.

2C **6** **a** What does $3(m + 4)$ expand to?

A $3m + 4$ **B** $3m + 7$

C $3m + 12$ **D** $m + 7$

b Expand $2(k - 1)$ to remove the brackets.

2D **7** Find the highest common factor (HCF) of each pair of terms.

a $6a$ and 24 **b** $8b$ and 12

c c^2 and $5c$ **d** $7d$ and $7p$

e $4xy$ and $-8x$ **f** $18m^2$ and $27m$

2D **8** Fill in the gaps to make true statements.

a $2k \times \_\_\_ = 20k$

b $5 \times \_\_\_ = 40n$

c $3x \times \_\_\_ = -15xy$

d $7p \times \_\_\_ = 28p^2$

2D **9** **a** $(\sqrt{3})^2$ simplifies to:

A $\sqrt{3}$ **B** $\sqrt{6}$ **C** 3 **D** 9

b Simplify $(\sqrt{11})^2$.

c Complete this statement using a surd: $5 = (\_\_\_)^2$.

2F **10** Complete these fraction calculations.

a $\frac{3}{8} + \frac{1}{8}$ **b** $\frac{8}{11} - \frac{3}{11}$

c $\frac{1}{2} + \frac{1}{6}$ **d** $\frac{7}{10} - \frac{2}{5}$

e $\frac{2}{5} \times \frac{15}{16}$ **f** $\frac{8}{15} \div \frac{3}{5}$

2A Working with algebraic terms

Start thinking!

State whether each of these statements is true or false. Provide a reason for deciding that a statement is false and rewrite it to make the statement true.

1 The expanded form of $7mnp$ is $7 + m + n + p$.

2 The **coefficient** of $7mnp$ is 7.

3 The coefficient of x is 0.

4 The **expression** $3k - 6m^3 + 4$ has three terms.

5 The coefficient of the second term in the expression $3k - 6m^3 + 4$ is -6.

6 The **constant term** in the expression $3k - 6m^3 + 4$ is 3.

7 $2x^2y$ and $-5x^2y$ are **like terms**.

8 a^3 and $3a^2$ are like terms.

9 $abc + abc = a^2b^2c^2$.

10 $6ab + 5a - ab$ simplifies to $10ab$.

KEY IDEAS

- Terms containing exactly the same **pronumerals** are called like terms. The order of the pronumerals can be different.
- Like terms can be added (or subtracted) by adding (or subtracting) the coefficients of the terms. Terms can be multiplied or divided whether they are like terms or unlike terms.
- When multiplying algebraic terms, multiply the coefficients together first and simplify the products of any pronumerals that are the same using index notation. For example, $k \times k = k^2$.
- When dividing algebraic terms, the division problem should be written as a fraction. Writing algebraic terms in expanded form can help identify the common factors.

EXERCISE 2A Working with algebraic terms

1 For each term:
- **i** write the term in expanded form
- **ii** identify the coefficient
- **iii** write an example of a like term.

a $3abcd$ **b** $-4mn$
c xy^2 **d** $9k^2m^4p$

EXAMPLE 2A-1 Evaluating algebraic expressions

If $a = 3$, $b = -1$ and $c = 2$, evaluate:
a a^2bc^3 **b** $2ab + ac - 5bc$

THINK

a Substitute the known values and simplify.

b Substitute the known values and simplify.

WRITE

a a^2bc^3
$= 3^2 \times -1 \times 2^3$
$= 9 \times -1 \times 8$
$= -72$

b $2ab + ac - 5bc$
$= 2 \times 3 \times -1 + 3 \times 2 - 5 \times -1 \times 2$
$= -6 + 6 + 10$
$= 10$

2 If $a = 1$, $b = 3$ and $c = -2$, evaluate:
a abc **b** $ab + bc$
c $ac - 2bc$ **d** a^4b^2
e $3ab^2c - ac$ **f** $7ac + 3bc - 4ac$
g $a^3b^2 + bc^3$ **h** $2a^2b + 3ac^2 - b^3$

EXAMPLE 2A-2 Adding and subtracting like terms

Simplify $7bd - 6bd + bd$.

THINK

Simplify like terms by adding or subtracting the coefficients. Work from left to right ($7 - 6 = 1$ and $1 + 1 = 2$).

WRITE

$7bd - 6bd + bd$
$= bd + bd$
$= 2bd$

3 Simplify each expression.

a $8x - 5x + 6x$	**b** $3ab - 4ab - 9ab$	**c** $4y^2 + y^2 + 5y^2$
d $7m^2n + 2m^2n - m^2n$	**e** $4a + 6d + 8a + 3d$	**f** $k - 3m + 5m + 7k$
g $4 + 8y - 2 - 11y$	**h** $a^2 + 6a - 5a^2 + a$	**i** $3xy + 2x^2 - xy + x^2$
j $a + ab^2 + 3a - 2ab^2$	**k** $2x^3 + 6x^2 - x^3 - 3x^2$	**l** $wxy + xy + wx + 7xy$

EXAMPLE 2A-3 Adding and subtracting algebraic terms

Simplify $4m^2n + 2m - m^2n + n^2 - 7m - 2nm^2$.

THINK

1 Rearrange the expression so that like terms are grouped together. Check that the + or − sign in front of each term has moved with that term.

2 Simplify like terms by adding or subtracting the coefficients.

WRITE

$4m^2n + 2m - m^2n + n^2 - 7m - 2nm^2$
$= 4m^2n - m^2n - 2nm^2 + 2m - 7m + n^2$

$= m^2n - 5m + n^2$

4 Simplify each expression.

a $5a + 2b - a + 3b + 7a - 5b$	**b** $6mn - 3n - n + 2n^2 + m - 7mn$
c $x + 4x^2 - 8x + 1 + 5x - x^2$	**d** $2x^2y - 5x^2y - 4xy + 3 + 7xy - 8$
e $9p^2 - 3p + 2p^3 + 6p^2 - 9 + 3p$	**f** $a^2b + 4a^2 + 8ba^2 - b^2 + 3ab^2 - 5a^2$

EXAMPLE 2A-4 Multiplying algebraic terms

Simplify: **a** $6xy \times -3wp$ **b** $p^2q \times 5kpq \times m$.

THINK

a 1 Multiply the coefficients together ($6 \times -3 = -18$). Write in expanded form (optional).

2 Simplify by leaving out the multiplication signs and write the pronumerals in alphabetical order.

b 1 Multiply the coefficients together ($1 \times 5 \times 1 = 5$). Write in expanded form.

2 Write the product of any pronumerals that are the same in index notation ($p \times p \times p = p^3$ and $q \times q = q^2$).

3 Simplify by leaving out the multiplication signs and write the pronumerals in alphabetical order.

WRITE

a $6xy \times -3wp$
$= -18 \times x \times y \times w \times p$
$= -18pwxy$

b $p^2q \times 5kpq \times m$
$= 5 \times p \times p \times q \times k \times p \times q \times m$
$= 5 \times p^3 \times q^2 \times k \times m$

$= 5kmp^3q^2$

5 Simplify each expression.

a $5ab \times 2cd$ **b** $-7xy \times 3mn$ **c** $4kp \times k$

d $6ac \times -8ad$ **e** $9x^2 \times 5wxy$ **f** $-m^2n \times -kn^3$

g $7x \times -3x \times x$ **h** $4c^2d \times abc \times 3d$ **i** $-6kp \times g^2kp \times k$

j $b^2d \times -4d^2 \times -ad$ **k** $-10mn^2 \times m^2 \times -2m^2n$ **l** $5a^2bc^3 \times a^3c \times 8ab^4$

EXAMPLE 2A-5 Dividing algebraic terms

Simplify $12ab^2c \div (8ab)$.

THINK

1 First write the division problem as an algebraic fraction.

2 Cancel the coefficients (divide 12 and 8 by the HCF of 4). Cancel any common pronumerals from numerator and denominator.

3 Simplify the numerator and the denominator.

WRITE

$12ab^2c \div (8ab)$

$= \dfrac{12ab^2c}{8ab}$

$= \dfrac{{}^{3}\cancel{12} \times {}^{1}\cancel{a} \times {}^{1}\cancel{b} \times b \times c}{{}_{2}\cancel{8} \times {}_{1}\cancel{a} \times {}_{1}\cancel{b}}$

$= \dfrac{3 \times 1 \times 1 \times b \times c}{2 \times 1 \times 1}$

$= \dfrac{3bc}{2}$

UNDERSTANDING AND FLUENCY

6 Simplify each expression.

a $abc \div b$ **b** $6mnp \div (mp)$ **c** $8wx \div (4wy)$

d $7ab \div (14bc)$ **e** $3e^2fg \div (efh)$ **f** $6ab^2c \div (15bc)$

g $-12m^2n \div (8mnp^2)$ **h** $10a^2bc \div (-5abc^2d)$ **i** $-k^2x^3y^2 \div (-2k^2xy^2)$

7 If $a = 3$, $b = -2$ and $c = 4$, evaluate each expression. (Hint: simplify each expression first.)

a $4a + 3b + 2c - 2a - c + b$ **b** $9ab + 7a - 10a + ab$

c $a^2b + ab^2 + a^2c - 2a^2b + 5a^2c$ **d** $3abc \times b^2c \times 6a$

e $12a^2b^2c \div (3bc^2)$ **f** $ac^2 \times 2ab^3 \div (8a^2bc)$

8 The surface area of this box can be calculated using the expression $2lw + 2lh + 2wh$.

Calculate the surface area of the box for each set of **dimensions**.

a $l = 25$ cm, $w = 15$ cm, $h = 10$ cm

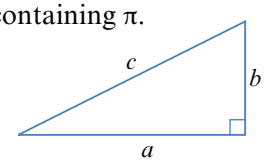
b $l = 0.5$ m, $w = 0.3$ m, $h = 0.2$ m

9 The perimeter, P, of this semicircular mat can be calculated using the formula $P = \frac{1}{2}\pi d + d$. If $d = 80$ cm, calculate the perimeter correct to one decimal place.

10 The formula for the area, A, of a circle with radius r is $A = \pi r^2$. Calculate the area when $r = 5$ cm. Leave your answer as an exact value containing π.

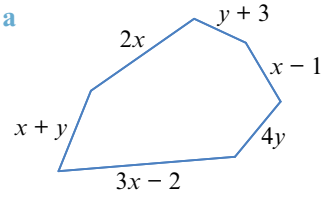
11 The formula for the volume, V, of a sphere with radius r is $V = \frac{4}{3}\pi r^3$. Calculate the volume when $r = 3$ mm. Leave your answer as an exact value containing π.

12 Using Pythagoras' Theorem, $c = \sqrt{a^2 + b^2}$. Calculate c if $a = 6$ m and $b = 8$ m.



13 Write the perimeter of each shape as an algebraic expression in simplest form.

a



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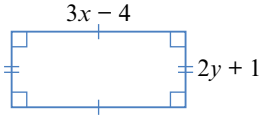

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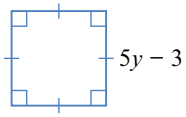
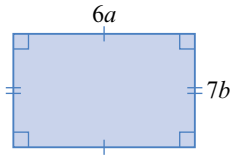


14 Calculate the perimeter of each shape in question **13** when $x = 7$ cm and $y = 2$ cm.

15 Write the area of each shaded shape as an algebraic expression in simplest form.

a



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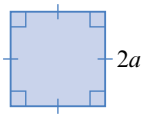

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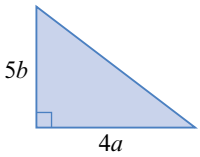

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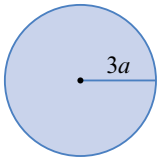

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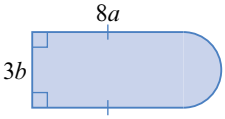

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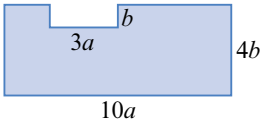


16 Calculate the area of each shape in question **15** when $a = 4$ m and $b = 3$ m.

17 The area of a rectangle is $20k^2$.

a If the length is $5k$, write an expression for the width of the rectangle.

b If the width of the rectangle is 12 m, calculate the area of the rectangle.

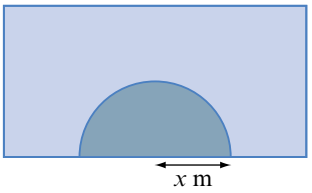
18 The area of a right-angled triangle is $15xy$.

a If the **base** length is $6x$, write an expression for the **height** of the triangle.

b If the height of the triangle is 10 cm and $x = 4$ cm, calculate the area and the base length of the triangle.

PROBLEM SOLVING AND REASONING

19 A rectangular swimming pool has a semi-circular spa of radius x metres set into it.



a Write an expression for:

i the total perimeter of the spa

ii the area of the spa.

b If the length of the pool is quadruple the radius of the spa and the width is triple the radius of the spa, write an expression using x for:

i the outside perimeter of the rectangular pool containing the spa

ii the area of the pool and spa.

c When $x = 2$, calculate:

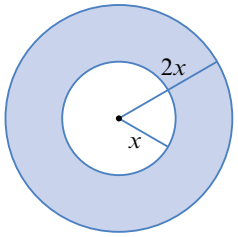
i the area of the spa

ii the area available for swimming

iii the length of edging needed around both the pool and the spa (include the curved edging needed for the spa).

CHALLENGE

20 Consider the top surface of a metal washer, which is an **annulus**.



a Write an algebraic expression for the area of the top surface of one washer.

b List three possible values for x to give an area between 400 mm^2 and 500 mm^2.

c Write an algebraic expression for the total length of the outer and inner edges of one washer.

d Use your values from part **b** to calculate the total length of the outer and inner edges of one washer, correct to one decimal place.

21 Decide whether each statement is true or false for all values of x, y and z. Provide examples to demonstrate each answer.

a $x + y = y + x$

b $x - y = y - x$

c $xy = yx$

d $x \div y = y \div x$

e $(x + y) + z = x + (y + z)$

f $(x - y) - z = x - (y - z)$

Reflect

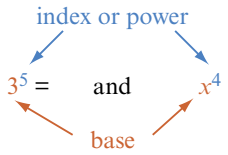
Why do like terms make it possible to add or subtract algebraic terms?

2B Review of index laws

Start thinking!

Numerical and algebraic terms can be written in **index form** (or **index notation**).

Two examples are 3^5 and x^4. They each have a **base** and an **index** or **power**.



Terms in index form can be written in expanded form.

x^4	=	$x \times x \times x \times x$
index form		expanded form

Terms in index form can be combined or simplified using the **index laws**.

1 An example to demonstrate the index law $a^m \times a^n = a^{m+n}$ is shown at right.

a Explain how this example demonstrates the law.

b Provide another example to demonstrate this law. Choose numbers for the indices and use either a number or pronumeral for the base.

$$\begin{aligned} & a^3 \times a^2 \\ &= a \times a \times a \times a \times a \\ &= a^5 \\ &= a^{3+2} \end{aligned}$$

2 Provide two examples to demonstrate each of the following index laws. You may like to discuss them first with a classmate.

a $a^m \div a^n = a^{m-n}$ **b** $(a^m)^n = a^{mn}$ **c** $(a \times b)^m = a^m \times b^m$ **d** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

3 Use the index laws in question **1** and parts **a** and **b** of question **2** to simplify each expression. Copy and complete the working shown.

a $x^5 \times x^4$
$= x^{5+\_}$
$= x^{\_}$

b $x^6 \div x^2$
$= x^{6-\_}$
$= x^{\_}$

c $(x^3)^7$
$= x^{3\times\_}$
$= x^{\_}$

KEY IDEAS

- The index laws can be used to simplify expressions that have terms written in index form.
- The index laws for multiplying or dividing terms in index form only apply when the bases are the same.
- Any base raised to the power of zero is equal to one; that is, $a^0 = 1$.
- The properties $a^{-1} = \frac{1}{a}$, $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$ can be used to write terms with positive indices only.

Index laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(a \times b)^m = a^m \times b^m$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

EXERCISE 2B Review of index laws

UNDERSTANDING AND FLUENCY

1 Use the appropriate index law to simplify each expression.

a $a^7 \div a^3$ **b** $(b^4)^5$ **c** $c^9 \times c^6$

d $d^3 \div d^2$ **e** $y^5 \times y^5$ **f** $(m^7)^2$

g $n^{15} \div n$ **h** $h \times h^6$ **i** $\frac{x^{11}}{x^5}$

j $(p^6)^3$ **k** $\frac{w^4}{w}$ **l** $a^2 \times a \times a^7$

2 The index laws also apply to terms with negative indices. Simplify each expression.

a $x^5 \times x^{-2}$ **b** $m^4 \div m^{-3}$ **c** $(k^{-6})^{-2}$

d $y^{-6} \times y^{10}$ **e** $a^{-4} \div a^{-9}$ **f** $(d^{-5})^3$

g $y^3 \times y^{-4} \times y^2$ **h** $n^{-7} \times n^{-3} \times n^{12}$ **i** $x^2 \times x^{-8} \times x$

EXAMPLE 2B-1 Writing terms with positive indices

Write each term with positive indices.

a $m^{-7}n^2$ **b** $\frac{12a^3b^{-5}}{4c^{-6}}$

THINK

a **1** Write the term as a product of two factors.

2 Write the first factor with a positive index using the property $a^{-m} = \frac{1}{a^m}$. The second factor already has a positive index. However, you may like to write it as a fraction (denominator is 1).

3 Simplify.

b **1** Cancel 12 and 4 by a common factor of 4 and simplify.

2 Write the term as a product of factors.

3 Write each base that has a negative index with a positive index. Use the properties $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$.

4 Simplify.

WRITE

a $m^{-7}n^2$

$= m^{-7} \times n^2$

$= \frac{1}{m^7} \times n^2$

$= \frac{1}{m^7} \times \frac{n^2}{1}$

$= \frac{n^2}{m^7}$

b $\frac{12a^3b^{-5}}{4c^{-6}}$

$= \frac{{}^{3}\cancel{12}a^3b^{-5}}{{}_{1}\cancel{4}c^{-6}}$

$= \frac{3a^3b^{-5}}{c^{-6}}$

$= 3 \times a^3 \times b^{-5} \times \frac{1}{c^{-6}}$

$= 3 \times a^3 \times \frac{1}{b^5} \times c^6$

$= \frac{3a^3c^6}{b^5}$

3 Write each term with positive indices.

a x^{-5}	**b** $c^{-9}d^{4}$	**c** $j^{5}k^{-3}$	**d** $x^{-1}y^{8}$
e $2a^{3}b^{-6}$	**f** $3p^{-4}q^{-5}$	**g** $w^{4}x^{-5}y^{7}$	**h** $k^{-3}m^{5}n^{-8}$
i $6d^{2}e^{-7}f$	**j** $11a^{-3}b^{-5}c^{-9}$	**k** $\frac{1}{x^{-3}}$	**l** $\frac{b^{4}}{c^{-6}}$
m $\frac{4m^{3}}{p^{-7}}$	**n** $\frac{a^{-4}}{b^{-2}}$	**o** $\frac{8x^{-2}}{2y^{-5}}$	**p** $\frac{c^{-1}d^{8}}{e^{-4}}$
q $\frac{3k^{-2}n^{-5}}{9p^{-6}}$	**r** $\frac{2^{-1}a^{3}c^{-7}}{b^{-4}d^{2}}$	**s** $\frac{w^{-2}x}{5^{-1}y^{-8}}$	**t** $\frac{6^{-1}a^{5}b^{-3}}{2m^{-7}n^{6}}$

EXAMPLE 2B-2 Using index laws to simplify expressions

Use the appropriate index laws to simplify each expression.

a $7a^{3} \times 2a^{6}$ **b** $10h^{11} \div (25h^{5})$ **c** $(k^{2})^{7} \times k^{-4}$

THINK

a 1 Multiply the coefficients.

2 Use an index law to multiply the terms with the same base. Write the base and add the indices. (3 + 6 = 9)

3 Simplify.

b 1 Write the expression as a fraction and insert the multiplication sign between the coefficient and variable.

2 Divide the coefficients. Cancel the coefficients by a common factor of 5.

3 Use an index law to divide the terms with the same base. Write the base and subtract the indices. (11 − 5 = 6)

4 Simplify.

c 1 Use an index law to simplify the first term. Write the base and multiply the indices. (2 × 7 = 14)

2 Use an index law to multiply the terms with the same base. Write the base and add the indices. (14 + (−4) = 10)

WRITE

a $7a^{3} \times 2a^{6}$
$= 14 \times a^{3} \times a^{6}$
$= 14 \times a^{9}$
$= 14a^{9}$

b $10h^{11} \div (25h^{5})$
$= \frac{10h^{11}}{25h^{5}}$
$= \frac{10 \times h^{11}}{25 \times h^{5}}$
$= \frac{2}{5} \times \frac{h^{11}}{h^{5}}$
$= \frac{2}{5} \times h^{6}$
$= \frac{2}{5}h^{6}$ or $\frac{2h^{6}}{5}$

c $(k^{2})^{7} \times k^{-4}$
$= k^{14} \times k^{-4}$
$= k^{10}$

4 Use the appropriate index laws to simplify each expression.

a $2x^{6} \times 7x^{3}$	**b** $x^{7} \times 3x^{-5}$	**c** $x^{3} \times x^{8} \times x^{4}$
d $5x^{9} \times x \times 4x^{-2}$	**e** $8x^{6} \div (2x^{3})$	**f** $30x^{9} \div (5x^{3})$
g $6x^{5} \div (14x^{4})$	**h** $12x^{10} \div (8x^{5})$	**i** $(x^{3})^{5} \times x^{2}$
j $(x^{6})^{4} \times (x^{5})^{2}$	**k** $x^{7} \times 2x^{8} \times (x^{2})^{4}$	**l** $(x^{4})^{3} \times (x^{5})^{6} \times 5x^{-7}$

UNDERSTANDING AND FLUENCY

5 Simplify each expression. Write your answer using positive indices only.

a $x^3 \times x^{-5}$ **b** $2x^{-4} \times x^{-1}$ **c** $7x^{-3} \times 3x^6 \times 2x^{-2}$

d $4x^{-8} \times 5x^3 \times x$ **e** $x^{-11} \div x^{-9}$ **f** $6x^4 \div (3x^{-5})$

g $3x^{-7} \div (12x^2)$ **h** $9x^6 \div (15x^{10})$ **i** $(x^{-3})^2 \times x^{-5}$

j $(x^{-4})^6 \times 10x^{18}$ **k** $3x^{-6} \times (x^{-1})^7 \times (x^3)^2$ **l** $(x^{-2})^7 \times (x^{-5})^{-1} \times (x^{-3})^4$

6 a What is the value of a term in index form with a power of zero?

b Write the result for each of these.

i 3^0 **ii** 100^0 **iii** m^0 **iv** $5^9 \div 5^9$

7 Use the property $a^0 = 1$ to simplify each expression.

a $3a^0$ **b** $(3a)^0$ **c** $8m^0$ **d** $(8m)^0$

e $(-5x)^0$ **f** $a^0 + b^0$ **g** $j^0 - k^0$ **h** $p^0 + p^0$

i $2c^0 + 4g^0$ **j** $w^0 + x^0 - y^0$ **k** $(m + n)^0$ **l** $(x^0)^4$

8 Use the index laws to simplify each expression.

a $x^5 \div x^5$ **b** $4x^{11} \div x^{11}$ **c** $(x^4)^2 \div x^8$

d $24(x^2)^7 \div (x^7)^2$ **e** $x^6 \times x \div x^7$ **f** $(x^5)^0 \times x^3 \times x^{-2}$

g $6x^7 \times 3(x^2)^0 \times 2(x^{-2})^2$ **h** $4(x^{-3})^4 \div (x^{-4})^3$ **i** $x^5 \times (x^4)^2 \div x^6$

j $6x^7 \times 2x^2 \div (4x^9)$ **k** $2(x^8)^2 \div (x^5)^3$ **l** $3(x^3)^4 \times 5x^{-4} \div (x^2)^4$

EXAMPLE 2B-3 Combining index laws to simplify expressions

Use the index laws to simplify each expression.

a $\dfrac{a^4 \times (a^3)^2}{a^7}$ **b** $\dfrac{2x^7 \times 6x^4}{3x^3 \times 8(x^4)^2}$

THINK

a 1 Simplify the numerator. First multiply indices to remove brackets ($3 \times 2 = 6$), then add indices ($4 + 6 = 10$).

2 Divide the numerator by the denominator. Keep the base and subtract indices ($10 - 7 = 3$).

b 1 Simplify the numerator and simplify the denominator.

2 Cancel the coefficients. Keep the base and subtract indices.

3 Use the property $a^0 = 1$ to simplify further.

WRITE

a $\dfrac{a^4 \times (a^3)^2}{a^7}$

$= \dfrac{a^4 \times a^6}{a^7}$

$= \dfrac{a^{10}}{a^7}$

$= a^3$

b $\dfrac{2x^7 \times 6x^4}{3x^3 \times 8(x^4)^2}$

$= \dfrac{12x^{11}}{24x^3 \times x^8}$

$= \dfrac{12x^{11}}{24x^{11}}$

$= \dfrac{x^0}{2}$

$= \dfrac{1}{2}$

9 Use the index laws to simplify each expression.

a $\dfrac{x^5 \times x^6}{x^3}$ **b** $\dfrac{8x^3 \times x^7}{x^5}$ **c** $\dfrac{5x^4 \times 4x^2}{10x^5}$

d $\dfrac{3x^8 \times 4x^{-5}}{12x^3}$ **e** $\dfrac{x^8 \times (x^2)^6}{x^{-4}}$ **f** $\dfrac{(x^3)^5 \times (x^{-2})^3}{(x^4)^2}$

g $\dfrac{5(x^4)^5 \times (x^{-2})^2}{25x^{16}}$ **h** $\dfrac{x^6 \times x^9}{x^4 \times x^3}$ **i** $\dfrac{(x^5)^3 \times x^7}{x^4 \times (x^3)^4}$

j $\dfrac{2x^{10} \times 8(x^2)^5}{4x^{-3} \times x^7}$ **k** $\dfrac{4(x^6)^3 \times 10(x^5)^2}{8x^4 \times (x^3)^8}$ **l** $\dfrac{(x^7)^4 \times x \times x^{11}}{x^9 \times (x^3)^5 \times x^6}$

10 Simplify each expression.

a $x^5y^9 \times x^4y^3$ **b** $2a^6b^7 \times 8a^2b$ **c** $(m^3)^5n^4 \times m^{-8}n^6$

d $\dfrac{b^3c^{11}}{c^5}$ **e** $\dfrac{w^4x^3}{wx^{-3}}$ **f** $\dfrac{a^6b^4 \times a^2b^5}{a^5b^9}$

g $\dfrac{8k^6n^5 \times 3k^8n^{11}}{6k^3n^7 \times k^5n^2}$ **h** $\dfrac{9(c^4)^5d^8 \times (c^{-3})^2}{18c^6d^7 \times cd}$ **i** $\dfrac{4x^{13}y^{18} \times 3x(y^{-2})^4}{(x^3)^4y^4 \times 12x^2(y^3)^2}$

11 Use the index law $(a \times b)^m = a^m \times b^m$ to write each expression without brackets.

a $(a \times b)^5$ **b** $(k \times p)^9$ **c** $(4 \times c)^3$

d $(7 \times y)^{13}$ **e** $(xy)^4$ **f** $(cd)^7$

g $(3w)^2$ **h** $(6g)^6$ **i** $(2p)^{-8}$

12 Use the index law $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ to write each expression without brackets.

a $\left(\dfrac{a}{b}\right)^4$ **b** $\left(\dfrac{f}{g}\right)^7$ **c** $\left(\dfrac{k}{2}\right)^6$

d $\left(\dfrac{5}{x}\right)^3$ **e** $\left(\dfrac{d}{c}\right)^{-2}$ **f** $\left(\dfrac{m}{3}\right)^{-5}$

13 Write each expression without brackets using positive indices only.

a $(x \times y)^{-4}$ **b** $(4 \times a)^{-6}$ **c** $(9 \times n)^{-1}$

d $(5p)^{-3}$ **e** $(3p)^{-2}$ **f** $(km)^{-5}$

g $\left(\dfrac{x}{4}\right)^{-3}$ **h** $\left(\dfrac{7}{d}\right)^{-1}$ **i** $(a^3b^{-2})^{-4}$

j $\left(\dfrac{k}{n}\right)^{-5}$ **k** $\left(\dfrac{d^{-4}}{c^6}\right)^{-1}$ **l** $\left(\dfrac{w^{-3}}{x^{-4}}\right)^2$

14 Use the index laws to simplify each expression. Write your answer using positive indices only.

a $(ab)^4 \times a^7b^3$ **b** $(4p)^3 \times (6p)^2$ **c** $(2y^3)^5$

d $3(m^2n)^6$ **e** $\dfrac{d^5}{h^4} \times \left(\dfrac{d}{h}\right)^7$ **f** $\left(\dfrac{5w}{y}\right)^3$

g $\left(\dfrac{a^5}{b^{-2}}\right)^6$ **h** $\left(\dfrac{a^{-4}b^2}{c^7}\right)^4$ **i** $\left(\dfrac{wx^5}{u^3}\right)^4 \times \left(\dfrac{u^5}{wx}\right)^3$

j $\left(\dfrac{c^6}{a^3n^2}\right)^2 \times \dfrac{(5a^4)^3}{c^8n^{-5}}$ **k** $\dfrac{(m^5n^4)^3 \times (m^6n)^4}{(mn^3)^5}$ **l** $\dfrac{(2x^2)^6(y^4)^2}{(4x^4y^5)^3}$

PROBLEM SOLVING AND REASONING

15 Use the index laws to decide whether each statement is true or false. Explain your reasoning. For each false statement, change the right side to make the statement true.

a $x^7 \times x \times x^7 = x^{14}$

b $(5a)^3 = 5 \times a^3$

c $-k^0 = -1$

d $a^2b^3 \times a^3b^4 = a^{12}b^{12}$

e $x^7y^4 \div y^6 = \dfrac{x^7}{y^2}$

f $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n}$

g $\dfrac{w^6 \times w^6}{w^{12}} = 0$

h $\dfrac{(b^5)^4 \times b^2}{(b^3)^7} = b$

i $\dfrac{a^8b^5}{a^4b^3} \times \dfrac{a^2b^2}{a^9b^4} = a^3$

j $\left(\dfrac{m^2}{n}\right)^3 \times \left(\dfrac{n^4}{m^5}\right)^{-1} = \dfrac{m^{11}}{n^4}$

16 Find the value of x that will make each statement true.

a $a^x \times a^3 = a^{12}$

b $b^x = 1$

c $c^x \div c^6 = c^4$

d $(d^3)^x = d^{12}$

e $e^x = \dfrac{1}{e^8}$

f $y^x = \dfrac{1}{y^{-3}}$

g $\left(\dfrac{m}{n}\right)^x = \left(\dfrac{m^7}{n^7}\right)$

h $\dfrac{w^x \times w^4}{w^7} = w^6$

i $(2a^x)^5 = 32a^{30}$

j $k^xm^7 \times k^9m^x = k^5m^3$

k $\dfrac{(n^3)^x \times n^4}{(n^2)^6} = n^7$

l $\dfrac{a^3b^6}{a^4b^x} \times \dfrac{a^7(b^2)^3}{a^5b^4} = a$

17 Write three possible sets of x and y values to make each statement true.

a $m^x \times m^y = m^{20}$

b $p^x \div p^y = p^5$

c $(a^x)^y = a^{36}$

d $(k^x)^y = 1$

CHALLENGE

18 Simplify each expression.

a $x^{2a}y^{b+3} \times x^5y^{3b-2}$

b $\dfrac{m^{3x+4}n^{x-2}}{m^xn^{x-5}}$

19 Write $\dfrac{8^5 \times 2^4}{4^7}$ in simplest index form with a base of 2.

20 Write $\dfrac{3^{4x} \times 9^{2x}}{27^x \times 3^{3x}}$ in simplest index form with a base of 3.

21 Use the appropriate index laws to simplify $\dfrac{25^{3x} \times 625^x}{125^{2x} \times 5^{3x}}$.

22 Find the value of x and y to make each statement true.

a $a^xb \times a^y(b^x)^y = a^{12}b^{33}$

b $\dfrac{a^xb^y}{a^2b^3} \times \dfrac{a^{x+y}b^y}{a^3b^7} = a^3b^{14}$

Reflect

How are index laws useful when simplifying algebraic expressions?

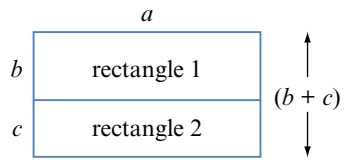
2C Expanding algebraic expressions

Start thinking!

1 A **binomial** is an expression with two terms. Write three examples of a binomial.

2 $(x + 3)(x - 5)$ is an example of a **binomial product**. Explain what the term binomial product means.

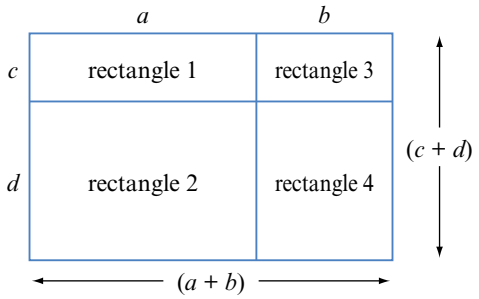
3 This area model shows a rectangle divided into two sections.



a What is the length and width of the rectangle?

b Explain how this area model can be used to demonstrate $a(b + c) = ab + ac$.

4 This area model shows a rectangle divided into four sections.



a What is the length and width of the rectangle?

b Explain how this area model can be used to demonstrate $(a + b)(c + d) = ac + ad + bc + bd$.

KEY IDEAS

- The **distributive law** can be used when expanding expressions to remove brackets.
- For one pair of brackets, the distributive law is: $a(b + c) = ab + ac$ $\quad a(b + c) = a \times b + a \times c$
- For two pairs of brackets, the distributive law is: $(a + b)(c + d) = ac + ad + bc + bd$.

$$(a + b)(c + d) = a \times c + a \times d + b \times c + b \times d$$

- $(a + b)(c + d)$ is a binomial product as each factor is an expression with two terms.
- The **difference of two squares** rule: $(a + b)(a - b) = a^2 - b^2$.
- The **expansion of a perfect square** rule: $(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$.
- The index laws can be used when multiplying algebraic terms.

EXERCISE 2C Expanding algebraic expressions

EXAMPLE 2C-1 Expanding expressions with one pair of brackets

Expand each algebraic expression to remove the brackets.

a $7(m + 5)$ **b** $3x(x - 4)$ **c** $-a(2b + 9a)$

THINK

a Use the distributive law. Multiply each term inside the brackets (m and 5) by the term in front of the brackets (7). Simplify each term.

$7(m + 5)$

b Use the distributive law. Multiply each term inside the brackets (x and -4) by the term in front of the brackets ($3x$). Simplify each term.

$3x(x - 4)$

c Use the distributive law. Multiply each term inside the brackets ($2b$ and $9a$) by the term in front of the brackets ($-a$).

$-a(2b + 9a)$

WRITE

a $7(m + 5)$
$= 7 \times m + 7 \times 5$
$= 7m + 35$

b $3x(x - 4)$
$= 3x \times x + 3x \times (-4)$
$= 3x^2 - 12x$

c $-a(2b + 9a)$
$= (-a) \times 2b + (-a) \times 9a$
$= -2ab - 9a^2$

1 Expand each algebraic expression to remove the brackets.

a $3(a + 7)$	**b** $5(b - 4)$	**c** $2(6 - c)$	**d** $4(1 + d)$
e $-6(e + 3)$	**f** $-7(f - 2)$	**g** $-3(g - 1)$	**h** $-9(5 - h)$
i $x(x + 2)$	**j** $2y(y - 4)$	**k** $-k(k + 5)$	**l** $-3p(p - 1)$
m $m(3n + k)$	**n** $4a(3b + 6c)$	**o** $-w(w - 8x)$	**p** $-2p(2 + 5p)$

EXAMPLE 2C-2 Expanding expressions with one pair of brackets using an index law

Expand each algebraic expression to remove the brackets.

a $x^2(x^3 + 6)$ **b** $5k^3m^4(3k^5 - 2m^2)$

THINK

a Multiply each term inside the brackets (x^3 and 6) by the term in front of the brackets (x^2). Simplify by using an index law if appropriate. ($x^2 \times x^3 = x^{2+3} = x^5$)

b Multiply each term inside the brackets ($3k^5$ and $-2m^2$) by the term in front of the brackets ($5k^3m^4$). Simplify each term using an index law.

WRITE

a $x^2(x^3 + 6)$
$= x^2 \times x^3 + x^2 \times 6$
$= x^5 + 6x^2$

b $5k^3m^4(3k^5 - 2m^2)$
$= 5k^3m^4 \times 3k^5 + 5k^3m^4 \times (-2m^2)$
$= 15k^8m^4 - 10k^3m^6$

UNDERSTANDING AND FLUENCY

2 Expand each algebraic expression to remove the brackets.

a $a^2(a + 4)$	**b** $b(b^5 - 2)$	**c** $c^3(c - 1)$
d $d^2(8 - d^3)$	**e** $-x(x^2 + x^5)$	**f** $-2y^4(y^3 + x^2)$
g $m^3n(m^3 + n^2)$	**h** $3xy(2xy - x^6y^4)$	**i** $4a^2b^3(a^4 + b^5)$
j $2k^4p^2(3k^6 - 7p^4)$	**k** $-8x^2y^5(x^3y - 2xy^4)$	**l** $-3a^6b^3(4ab^4 + 9a^7b^6)$

3 Expand and simplify each algebraic expression.

a $2(x + 5) + 7x$	**b** $4(y - 7) + 15$	**c** $m(n + 6) - 3m$
d $-3x(2 - x) + 5x^2 + x$	**e** $7p^3 - 8p + 5p(1 - p^2)$	**f** $x^4(x^3 - 2x) - x^5$

4 Expand and simplify each algebraic expression.

a $4(x - 2) + 2(x + 5)$	**b** $m(m + 8) + 6(m - 3)$	**c** $2a(3a + 1) - 3(2 - 4a)$
d $k^2(k^3 + 2) + 5(k^2 + 1)$	**e** $3p^4(p^2 - 2) + 2p(p^3 + 6)$	**f** $5y^3(2y^5 + 3) - 2y(7y^2 - 4)$

EXAMPLE 2C-3 Expanding binomial products

Expand each algebraic expression to remove the brackets.

a $(m + 5)(n + 2)$ **b** $(x + 7)(x - 4)$

THINK

a Use the distributive law. Multiply each term inside the second pair of brackets (n and 2) by the first term in the first pair of brackets (m) and then the second term in the first pair of brackets (5). Simplify each term.

$(m + 5)(n + 2)$

b **1** Use the distributive law. Multiply each term inside the second pair of brackets (x and -4) by the first term in the first pair of brackets (x) and then the second term in the first pair of brackets (7). Simplify each term.

$(x + 7)(x - 4)$

2 Simplify any like terms ($-4x + 7x = 3x$).

WRITE

a $(m + 5)(n + 2)$
$= m(n + 2) + 5(n + 2)$
$= mn + 2m + 5n + 10$

b $(x + 7)(x - 4)$
$= x(x - 4) + 7(x - 4)$
$= x^2 - 4x + 7x - 28$
$= x^2 + 3x - 28$

5 Expand each algebraic expression to remove the brackets.

a $(a + 4)(b + 2)$	**b** $(c + 3)(d + 6)$	**c** $(m + 7)(n + 5)$
d $(x + 6)(y + 8)$	**e** $(k + 5)(p - 1)$	**f** $(f - 2)(g + 3)$
g $(x - 9)(y - 4)$	**h** $(2m + 3)(n + 1)$	**i** $(4x + 5)(3y - 7)$
j $(3a - 2b)(5c - 4d)$	**k** $(x + 2)(x + 5)$	**l** $(y + 9)(y - 1)$
m $(m - 3)(m - 11)$	**n** $(p - 4)(p - 2)$	**o** $(5 - k)(6 - k)$
p $(4x + 7)(x + 3)$	**q** $(3a + 2)(a - 5)$	**r** $(8w - 7)(w - 2)$
s $(6p + 5)(2p + 7)$	**t** $(2x + 3)(4x - 1)$	**u** $(5y - 2)(3y - 4)$

EXAMPLE 2C-4 Expanding a binomial product using an index law

Expand $(x^5 - x^2)(x^3 + 4)$ to remove the brackets.

THINK	WRITE
1 Multiply each term inside the second pair of brackets (x^3 and 4) by the first term in the first pair of brackets (x^5) and then the second term in the first pair of brackets ($-x^2$).	$(x^5 - x^2)(x^3 + 4)$ $= x^5(x^3 + 4) - x^2(x^3 + 4)$ $= x^5 \times x^3 + x^5 \times 4 - x^2 \times x^3 - x^2 \times 4$
2 Simplify each term using an index law.	$= x^8 + 4x^5 - x^5 - 4x^2$
3 Simplify by combining the like terms.	$= x^8 + 3x^5 - 4x^2$

UNDERSTANDING AND FLUENCY

6 Expand each algebraic expression to remove the brackets.

a $(x^2 + 5)(x + 3)$ **b** $(x^3 + 4)(x^2 - 1)$ **c** $(x^5 - 7)(x^3 + 2)$
d $(x^4 - 3)(x^2 - 6)$ **e** $(x^6 + x^2)(x + 7)$ **f** $(x^7 - x^3)(x^2 + 1)$
g $(x^2 + x^3)(x^4 + x^3)$ **h** $(x^5 + x^2)(x^5 - x^2)$ **i** $(x^3 - x^2)(x^5 + x^4)$
j $(2x^2 + 5)(4x^3 - 1)$ **k** $(3x^6 - 2x)(5x^2 + x^4)$ **l** $(6x^2 - 5x^4)(3x^5 - 4x^3)$

7 a Expand each binomial product.

i $(x + 2)(x - 2)$ **ii** $(k + 7)(k - 7)$ **iii** $(m + 8)(m - 8)$
iv $(w - 6)(w + 6)$ **v** $(y - 1)(y + 1)$ **vi** $(a + b)(a - b)$

b Describe the pattern or shortcut you can see. What is special about the two binomial factors that are multiplied together?

c This pattern is known as the difference of two squares. The rule can be written as: $(a + b)(a - b) = a^2 - b^2$. Why do you think the rule is called the difference of two squares?

d Does it matter whether the product is $(a + b)(a - b)$ or $(a - b)(a + b)$? Explain.

e Use the rule to expand each binomial product.

i $(a - 3)(a + 3)$ **ii** $(x + 10)(x - 10)$ **iii** $(m + n)(m - n)$
iv $(3 + x)(3 - x)$ **v** $(1 + d)(1 - d)$ **vi** $(2x - 5)(2x + 5)$
vii $(2 - 3k)(2 + 3k)$ **viii** $(4g + h)(4g - h)$ **ix** $(5y + 2w)(5y - 2w)$

8 a Expand each binomial product.

i $(x + 3)(x + 3)$ **ii** $(y + 8)(y + 8)$ **iii** $(a + b)(a + b)$
iv $(k - 5)(k - 5)$ **v** $(p - 6)(p - 6)$ **vi** $(x - y)(x - y)$

b Describe the pattern or shortcut you can see. What is special about the two binomial factors that are multiplied together?

c This pattern is known as the expansion of a perfect square. The rule can be written as:
$(a + b)^2 = a^2 + 2ab + b^2$.
Why do you think the rule is called the expansion of a perfect square?

UNDERSTANDING AND FLUENCY

d Use the rule to expand each algebraic expression.

i $(a + 2)^2$	**ii** $(x + 5)^2$	**iii** $(p + 4)^2$
iv $(y + 10)^2$	**v** $(k + 9)^2$	**vi** $(m + n)^2$
vii $(5 + x)^2$	**viii** $(1 + 2d)^2$	**ix** $(3w + 7)^2$

e Why is $(a - b)^2 = a^2 - 2ab + b^2$ also an expansion of a perfect square?

f Use the rule to expand each algebraic expression.

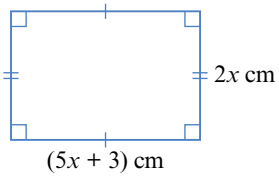
i $(a - 3)^2$	**ii** $(b - 4)^2$	**iii** $(x - 9)^2$
iv $(n - 11)^2$	**v** $(2 - x)^2$	**vi** $(g - h)^2$
vii $(4w - 1)^2$	**viii** $(5 - 2y)^2$	**ix** $(3m - 2p)^2$

9 Use the rules to expand and simplify each expression.

a $(x^2 + 2)(x^2 - 2)$	**b** $(x^2 - 5)(x^2 + 5)$	**c** $(x^2 + y^2)(x^2 - y^2)$
d $(x^2 + 4)^2$	**e** $(x^2 - 3)^2$	**f** $(x^3 - 1)^2$

PROBLEM SOLVING AND REASONING

10 **a** For the rectangle at right, write the area as a product of two factors (show brackets).



b Expand the expression to remove brackets.

c When $x = 4$, calculate the area using your answer to:

i part **a** **ii** part **b**.

d Compare your answers in part **c**. How does this show that you have expanded the area expression correctly?

e Calculate the area when:

i $x = 10$ **ii** $x = 2.5$.

11 For each shape:

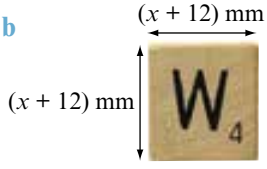
i write the area as a binomial product

ii expand the binomial product to remove brackets

iii calculate the area when $x = 3$.

a

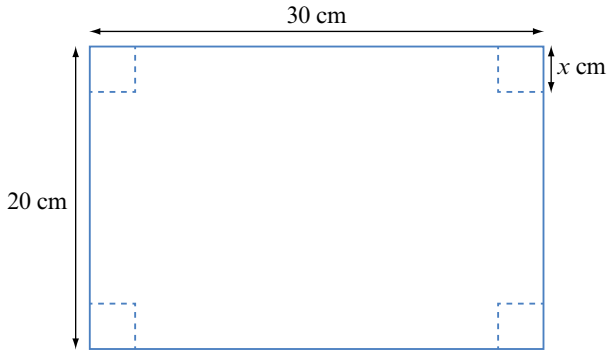
b



c

PROBLEM SOLVING AND REASONING

12 A rectangular piece of cardboard is 30 cm long and 20 cm wide. A square of side length x cm is cut at each corner so that the cardboard can be folded to form an open box.



a Write an expression containing x for:
- **i** the length
- **ii** the width
- **iii** the height of the box.

b Draw a labelled diagram of the open box.

c Write an expression for:
- **i** the volume of space contained in the box
- **ii** the inner surface area of the box.

Simplify each expression by expanding to remove any brackets.

d If $x = 3$, calculate:
- **i** the volume of space contained in the box
- **ii** the inner surface area of the box.

CHALLENGE

13 Show that $(x^2 + y^3)^2$ expands to $x^4 + 2x^2y^3 + y^6$.

14 Expand the expression $(x^2 + y^3)(x^2 - y^3)$.

15 The height of a cylindrical biscuit container is 5 cm longer than its radius.

a If the radius is x cm, write an expression for the height of the cylinder.

b Using the formula $V = \pi r^2 h$, show that the volume within this cylinder can be written as $\pi x^3 + 5\pi x^2$.

c Suggest a value for x that produces a volume between 1000 cm^3 and 2000 cm^3.

d Write an expression for the surface area inside the cylinder.

e Use your value for x from part **c** to calculate the surface area inside the cylinder.

Reflect

What type of binomial products can be expanded using a shortcut or rule?

2D Factorising algebraic expressions

Start thinking!

The expression $5(x + 3)$ is said to be in **factor form**, because it shows the product of factors; that is, $5 \times (x + 3)$. In this case, there are two factors: 5 and $(x + 3)$.

In the previous section, you used the distributive law to change an expression from factor form to **expanded form**.

You can also change an expression from expanded form to factor form by using the same law in reverse. This process is called **factorising**.

factor form ⟶ expanded form

$5(x + 3) = 5 \times x + 5 \times 3$

$= 5x + 15$

expanded form ⟶ factor form

$5x + 15 = 5(x + 3)$

1 Look at the terms $8x$ and 20. What is the **highest common factor** (**HCF**)?

2 Describe how you might use the HCF to factorise $8x + 20$ to obtain $4(2x + 5)$.

3 Show how using the HCF of x can be used to factorise $x^2 + 9x$.

4 Show how using the HCF of $(x + 2)$ can be used to factorise $y(x + 2) + 3(x + 2)$.

KEY IDEAS

- The distributive law can be used to write expressions in factor form. One of the factors needs to be the highest common factor (HCF) of each term in the expression.
- To factorise an expression, you write the HCF of each term at the front of a pair of brackets with the other factor of each term inside the brackets.
- $a \times b + a \times c = a \times (b + c)$ or $ab + ac = a(b + c)$
- The difference of two squares rule can be used to factorise expressions of the form $a^2 - b^2$. The rule is $a^2 - b^2 = (a + b)(a - b)$.
- The expansion of a perfect square rule can be used in reverse to factorise expressions of the form $a^2 + 2ab + b^2$. The perfect square rule is $a^2 + 2ab + b^2 = (a + b)^2$.
- Always check whether you can take out a common factor first.
- You can check whether you have factorised correctly by expanding the result and comparing it to the original expression.

EXERCISE 2D Factorising algebraic expressions

EXAMPLE 2D-1 Factorising simple expressions using common factors

Factorise each expression.

a $3m + 21$ b $a^2 - 4a$

THINK

a **1** Identify the HCF of $3m$ and 21. (HCF = 3.) Write each term as a product of the HCF and its other factor.

2 Use the distributive law. Write the HCF at the front of a pair of brackets and the other factor for each term inside.

b **1** Identify the HCF of a^2 and $4a$. (HCF = a.) Write each term as a product of the HCF and its other factor.

2 Use the distributive law to factorise the expression.

WRITE

a $3m + 21$
$= 3 \times m + 3 \times 7$
$= 3 \times (m + 7)$
$= 3(m + 7)$

b $a^2 - 4a$
$= a \times a + a \times (-4)$
$= a \times (a - 4)$
$= a(a - 4)$

1 Factorise each expression.

a $5a + 35$	b $30b + 6$	c $24 + 33c$	d $12d + 8$
e $8e - 18$	f $35f - 21$	g $24 + 16g$	h $10 - 40h$
i $9j + 9k$	j $16w - 28x$	k $m^2 + 2m$	l $n^2 - 6n$
m $5a + a^2$	n $9p - p^2$	o $3q^2 + 3q$	p $2r^2 - 8r$

EXAMPLE 2D-2 Factorising more complex expressions using common factors

Factorise each expression.

a $2pqr + 4pr$ b $9m^3k - 15mk^2$

THINK

a **1** Identify the HCF of $2pqr$ and $4pr$. (HCF = $2pr$) Write each term as a product of the HCF and its other factor.

2 Write the HCF at the front of a pair of brackets and the other factor for each term inside the brackets.

b **1** Identify the HCF of $9m^3k$ and $-15mk^2$. (HCF = $3mk$) Write each term as a product using the HCF.

2 Use the distributive law to factorise the expression.

WRITE

a $2pqr + 4pr$
$= 2pr \times q + 2pr \times 2$
$= 2pr \times (q + 2)$
$= 2pr(q + 2)$

b $9m^3k - 15mk^2$
$= 3mk \times 3m^2 + 3mk \times (-5k)$
$= 3mk \times (3m^2 - 5k)$
$= 3mk(3m^2 - 5k)$

UNDERSTANDING AND FLUENCY

2 Factorise each expression.

a $bcd + cd$	**b** $6ab + 6b$	**c** $15xy - 9x$
d $mpq + kmnp$	**e** $4a^2b + 3ab$	**f** $2xy - 2x^2y$
g $a^2b + 11ab^2$	**h** $m^3 + 5m^2$	**i** $8p^6 - 4p^3$
j $6x^7 + 10x^2$	**k** $m^3n + m^2n$	**l** $3p^3q + 3pq^3$
m $35x^2y + 5x^3y^2$	**n** $12a^3b^2 - 9a^2b^4$	**o** $20g^2h^2 + 28g^2h^6$
p $4a + 8b + 12c$	**q** $6xy + 3x^2 + 15wx$	**r** $2h^3 - 4h^2 + 6h$

3 Complete the following to factorise each expression by removing a negative HCF.

a $-2ab - 4a^2$

$= (-2a) \times$ ___ $+ (-2a) \times 2a$

$= -2a($___ $+ 2a)$

b $-10x^3 + 35x$

$=$ ___ $\times 2x^2 +$ ___ $\times (-7)$

$=$ ___$(2x^2 - 7)$

4 Factorise each expression by removing a negative HCF.

a $-6mn - 18$	**b** $-3ab - 3bc$	**c** $-y^2 - 10y$
d $-18x^2 - 9x$	**e** $-4k + 8k^2$	**f** $-20a^3 - 30a$
g $-12x^2y + 14xy$	**h** $-w^5 + w^7$	**i** $-4b^3 + 2b^2 - 6b^4$

5 Using the binomial factor $(x - 5)$ as the HCF, show that $y(x - 5) + 2(x - 5)$ written in factor form is $(x - 5)(y + 2)$. In the same way, show that $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2)(4a - 9)$.

6 Factorise each expression using a HCF which is a binomial factor.

a $a(x + 3) + 5(x + 3)$	**b** $m(k - 2) + 4(k - 2)$
c $y(y + 5) + 2(y + 5)$	**d** $x(x - 1) + 9(x - 1)$
e $k(k + 6) - 3(k + 6)$	**f** $p(p - 9) - 6(p - 9)$
g $7(4 - a) + a(4 - a)$	**h** $3n(2n - 5) + 4(2n - 5)$
i $x(y^2 + 2) + 8(y^2 + 2)$	**j** $a^2(a - 4) + 3(a - 4)$
k $2x(y^2 + 1) - 5(y^2 + 1)$	**l** $5x(7 - x^3) + 2(7 - x^3)$

EXAMPLE 2D-3 Factorising by grouping terms

Factorise $x^2 + 3x + 4xy + 12y$ by grouping terms.

THINK	WRITE
1 Check for a HCF of all four terms (none). Group the terms 'two and two' and identify the HCF for each pair of terms. HCF of first pair is x and HCF of second pair is $4y$.	$x^2 + 3x + 4xy + 12y$ $= (x^2 + 3x) + (4xy + 12y)$
2 Use the distributive law to factorise each pair.	$= x(x + 3) + 4y(x + 3)$
3 Write the binomial factor of $(x + 3)$ at the front of a pair of brackets and the other factor for each term inside the brackets.	$= (x + 3)(x + 4y)$

7 Factorise each expression by grouping terms.

a $ab + 4b + 3a + 12$	**b** $x^2 - 7x + 2xy - 14y$	**c** $mn + 5m - 6n - 30$
d $a^2 + 2a + 5a + 10$	**e** $x^2 - x + 4x - 4$	**f** $c^2 + 8c - 3c - 24$
g $6y - 3y^2 + 8 - 4y$	**h** $p^3 + 2p^2 + 5p + 10$	**i** $x^4 + x^2 + 3x^2 + 3$
j $2m^3 + 4m - 3m^2 - 6$	**k** $4 + 4k^2 + k^5 + k^3$	**l** $xy - 15 + 5y - 3x$

8 **a** Expand $(x + 5)(x - 5)$ using the difference of two squares rule.

b Explain how you can factorise $x^2 - 25$.

EXAMPLE 2D-4 Factorising expressions using the difference of two squares rule

Factorise each expression.

a $x^2 - 49$ **b** $16y^2 - 81$ **c** $9m^6 - 36n^2$ **d** $(k + 2)^2 - 1$

THINK	WRITE
a **1** Check for common factors (none). Write as a difference of two squares.	**a** $x^2 - 49$ $= x^2 - 7^2$
2 Factorise using the difference of two squares rule. $a^2 - b^2 = (a + b)(a - b)$ where $a = x$ and $b = 7$.	$= (x + 7)(x - 7)$
b **1** Check for common factors (none). Write as a difference of two squares.	**b** $16y^2 - 81$ $= (4y)^2 - 9^2$
2 Factorise using the difference of two squares rule.	$= (4y + 9)(4y - 9)$
c **1** Check for common factors (yes). Take out the HCF of 9.	**c** $9m^6 - 36n^2$ $= 9(m^6 - 4n^2)$
2 Write the expression in brackets as a difference of two squares.	$= 9[(m^3)^2 - (2n)^2]$
3 Factorise using the difference of two squares rule.	$= 9(m^3 + 2n)(m^3 - 2n)$
d **1** Look for common factors (none). Write as a difference of two squares.	**d** $(k + 2)^2 - 1$ $= (k + 2)^2 - 1^2$
2 Factorise using the difference of two squares rule.	$= (k + 2 + 1)(k + 2 - 1)$
3 Simplify the expression in each pair of brackets.	$= (k + 3)(k + 1)$

9 Factorise each expression.

a $x^2 - 4$	**b** $a^2 - 36$	**c** $100 - y^2$
d $64b^2 - 9$	**e** $25 - 49p^2$	**f** $4a^2 - 81b^2$
g $3m^2 - 3$	**h** $8k^2 - 18$	**i** $(xy)^2 - 16w^2$
j $m^6 - n^2$	**k** $4a^{10} - 36b^6$	**l** $(x^2y)^2 - w^4$
m $(h + 3)^2 - 25$	**n** $(c - 4)^2 - 9$	**o** $(2 - x)^2 - x^2$
p $1 - (a + b)^2$	**q** $4 - (y - 5)^2$	**r** $(x + 3)^2 - (x - 6)^2$

EXAMPLE 2D-5 Factorising expressions using the difference of two squares rule and surds

Factorise each expression.

a $x^2 - 3$ **b** $(y + 4)^2 - 11$

THINK	WRITE
a 1 Check for common factors (none). Write as a difference of two squares.	**a** $x^2 - 3$ $= x^2 - (\sqrt{3})^2$
2 Factorise using the difference of two squares rule. $a^2 - b^2 = (a + b)(a - b)$ where $a = x$ and $b = \sqrt{3}$.	$= (x + \sqrt{3})(x - \sqrt{3})$
b 1 Check for common factors (none). Write as a difference of two squares.	**b** $(y + 4)^2 - 11$ $= (y + 4)^2 - (\sqrt{11})^2$
2 Factorise using the difference of two squares rule.	$= (y + 4 + \sqrt{11})(y + 4 - \sqrt{11})$

UNDERSTANDING AND FLUENCY

10 Factorise each expression. (Hint: use a surd to write a square term.)

a $x^2 - 7$ **b** $a^2 - 13$ **c** $19 - y^2$
d $16k^2 - 5$ **e** $(p + 8)^2 - 2$ **f** $(m - 4)^2 - 6$
g $(x + 7)^2 - 3$ **h** $(y - 1)^2 - 19$ **i** $(d - 17)^2 - 17$

11 a Expand $(x + 5)^2$ using the expansion of a perfect square rule.
b Explain how you can factorise $x^2 + 10x + 25$.

12 a Expand $(y - 8)^2$ using the expansion of a perfect square rule.
b Explain how you can factorise $y^2 - 16y + 64$.

13 Factorise each expression using the perfect square rule.

a $x^2 + 12x + 36$ **b** $m^2 + 8m + 16$ **c** $p^2 + 20p + 100$
d $y^2 - 6y + 9$ **e** $x^2 - 14x + 49$ **f** $g^2 - 4g + 4$
g $1 + 2a + a^2$ **h** $25 - 10x + x^2$ **i** $81 - 18b + b^2$
j $w^2 + 2wx + x^2$ **k** $k^2 - 2km + m^2$ **l** $9x^2 + 12x + 4$

14 a Write an expression for the missing side length for each rectangular item.

i

$(x - 10)$ cm

area of welcome mat is $(x^2 - 100)$ cm^2

ii

x cm

area of stained glass panel is $(2x^2 + 3x)$ cm^2

b Write an expression for the perimeter of each item in factor form.

PROBLEM SOLVING AND REASONING

15 A rectangle has an area of $(18x - x^2)$ cm^2.

- **a** Write this in factor form.
- **b** List the expressions for the length and width of the rectangle.
- **c** Suggest values for the length and width of three different rectangles that would fit these expressions.

16 A triangle has an area of $(20x - 4x^2)$ mm^2.

- **a** List possible expressions for the height and base length of the triangle.
- **b** Suggest values for the height and base length of three different triangles that would fit these expressions.

17 Consider an odd number represented by the pronumeral n.

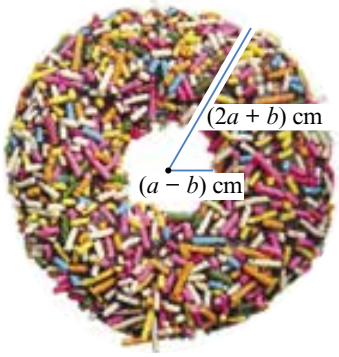
- **a** Write expressions for the next two odd numbers.
- **b** Write an expression for the sum of these three numbers.
- **c** Factorise your expression from part **b**. Explain how this relates to one of the three odd numbers.
- **d** Use the shortcut you found in part **c** to find the sum of 423, 425 and 427 without adding.
- **e** Investigate a shortcut for adding five consecutive odd numbers. Show your working.
- **f** Use the shortcut you found in part **e** to find the sum of 2671, 2673, 2675, 2677 and 2679 without adding.

CHALLENGE

18 Completely factorise each expression.

a $(a + 1)^2 - 4(a + 1)$ **b** $y(3x - 2) - 3x + 2$ **c** $(5m + 2)^2 - (2 - 3m)^2$

19 The top of a donut is in the shape of an annulus with an inner radius of $(a - b)$ cm and an outer radius of $(2a + b)$ cm. Show that the area of the annulus is $3\pi a(a + 2b)$ cm^2.



20 In question **17**, you found expressions for the sum of three and five consecutive odd numbers.

- **a** Investigate shortcuts for finding the sum of an even number of consecutive odd terms. Start with two, then four, then six consecutive odd numbers, where the first term is n.
- **b** Find the sum of this list of numbers without adding.
  73, 75, 77, 79, 81, 83, 85, 87, 89, 91

21 The difference between two numbers is 7 and the difference between the squares of the two numbers is 105.
What is the sum of the two numbers?

Reflect

What is the difference between factorising and expanding an expression?

2E Factorising quadratic trinomials of the form $x^2 + bx + c$

Start thinking!

1 If a binomial is an expression with two terms, how many terms does a **trinomial** have?

2 What is a **quadratic trinomial**? Use an example to help you explain.

3 Consider the binomial product $(x + 3)(x + 4)$.
- **a** Show how $(x + 3)(x + 4)$ expands to $x^2 + 7x + 12$.
- **b** Which terms have been multiplied together to produce: **i** x^2? **ii** 12?
- **c** Which terms have been multiplied together and the results added to produce $7x$?

4 Consider the quadratic trinomial $x^2 + 4x + 3$. In factor form, the expression is a binomial product of the form $(\_\_ + \_\_)(\_\_ + \_\_)$.
- **a** To obtain x^2, what should the first term be in each pair of brackets?
- **b** To obtain 3, what should the second term be in each pair of brackets? Show this in $(x + \_\_)(x + \_\_)$. (Hint: what two whole numbers multiply to give 3?)
- **c** Expand the binomial product you have written in part **b**. Do you obtain $4x$ as the middle term?

5 Now look at factorising $x^2 + 12x + 20$.
- **a** To obtain x^2, what should the first term be in each pair of brackets?
- **b** To obtain 20, what could the second term be in each pair of brackets? List three different factor pairs of 20. Show each factor pair in the form $(x + \_\_)(x + \_\_)$.
- **c** Expand each binomial product from part **b**. Which one produces the middle term of $12x$?
- **d** Explain why this quadratic trinomial was a little more difficult to factorise than the previous two.

6 Can you describe the pattern you are using to find the terms in the binomial product? Write a list of steps that you can follow.

KEY IDEAS

- Quadratic trinomials can be factorised to produce a binomial product.
- To factorise a quadratic trinomial of the form $x^2 + bx + c$, identify two numbers m and n that multiply to give c and add to give b. That is, $x^2 + bx + c = (x + m)(x + n)$.
- Always check whether you can take out a common factor first.
- The perfect square rule $a^2 + 2ab + b^2 = (a + b)^2$ can be used to factorise expressions of the form $x^2 + bx + c$ where $c = \left(\frac{b}{2}\right)^2$. That is, $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$.
- To use the perfect square rule, the term $\left(\frac{b}{2}\right)^2$ can be added to $x^2 + bx$ to complete the square.
- To factorise more difficult quadratic trinomials, the perfect square rule and the difference of two squares rule are used in the method called '**completing the square**'.

EXERCISE 2E Factorising quadratic trinomials of the form $x^2 + bx + c$

1 Identify which two numbers multiply to give the first number and add to give the second number.

a 7, 8 **b** 16, 10 **c** 45, 14
d 60, 16 **e** 36, 13 **f** 24, 11

2 Use your results from question **1** to factorise each quadratic trinomial.

a $x^2 + 8x + 7$ **b** $x^2 + 10x + 16$ **c** $x^2 + 14x + 45$
d $x^2 + 16x + 60$ **e** $x^2 + 13x + 36$ **f** $x^2 + 11x + 24$

EXAMPLE 2E-1 Factorising simple quadratic trinomials

Factorise each quadratic trinomial.

a $x^2 + 8x + 12$ **b** $x^2 + 13x + 30$

THINK

a **1** Show x as the first term in each set of brackets (so x^2 is obtained).

2 Identify factor pairs of 12.

3 Check which factors add to 8. (2 and 6)

4 Write the expression in factor form. (Check your result by expanding.)

b (Shorter method shown.)

1 Show x as the first term in each set of brackets.

2 Identify a factor pair of 30 that adds to 13. (3 and 10)

3 Write the expression in factor form. (Check your result by expanding.)

WRITE

a $x^2 + 8x + 12$
$= (x \quad)(x \quad)$

12: 1×12, 2×6, 3×4

$1 + 12 = 13$, $2 + 6 = 8$, $3 + 4 = 7$

$x^2 + 8x + 12$
$= (x + 2)(x + 6)$

b $x^2 + 13x + 30$

$3 \times 10 = 30$

$3 + 10 = 13$

$= (x + 3)(x + 10)$

3 Factorise each quadratic trinomial.

a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 + 11x + 10$
d $x^2 + 12x + 27$ **e** $x^2 + 7x + 10$ **f** $x^2 + 9x + 14$
g $x^2 + 6x + 9$ **h** $x^2 + 12x + 11$ **i** $x^2 + 11x + 18$
j $x^2 + 12x + 32$ **k** $x^2 + 16x + 63$ **l** $x^2 + 14x + 40$
m $x^2 + 11x + 30$ **n** $x^2 + 13x + 22$ **o** $x^2 + 15x + 36$

4 Identify which two numbers multiply to give the first number and add to give the second number.

a −8, 7 **b** 6, −7 **c** −35, −2 **d** −15, 2 **e** −8, −2 **f** 30, −11

5 Use your results from question **4** to factorise each of these quadratic trinomials.

a $x^2 + 7x - 8$ **b** $x^2 - 7x + 6$ **c** $x^2 - 2x - 35$
d $x^2 + 2x - 15$ **e** $x^2 - 2x - 8$ **f** $x^2 - 11x + 30$

EXAMPLE 2E-2 Factorising more difficult quadratic trinomials

Factorise each quadratic trinomial.

a $x^2 + 6x - 7$ **b** $x^2 - 3x - 10$ **c** $x^2 - 7x + 12$

THINK	WRITE
a **1** Show x as the first term in each set of brackets (so x^2 is obtained).	**a** $x^2 + 6x - 7$ $= (x \quad)(x \quad)$
2 Identify factor pairs of −7. One factor must be positive and the other negative.	-7: 1×-7, -1×7
3 Check which factors add to 6. (−1 and 7)	$1 + (-7) = -6$, $-1 + 7 = 6$
4 Write the expression in factor form. (Check your result by expanding.)	$x^2 + 6x - 7$ $= (x - 1)(x + 7)$
b **1** Show x as the first term in each set of brackets.	**b** $x^2 - 3x - 10$ $= (x \quad)(x \quad)$
2 Identify factor pairs of −10.	-10: 1×-10, -1×10, 2×-5, -2×5
3 Check which factors add to −3. (2 and −5)	$1 + (-10) = -9$, $-1 + 10 = 9$, $2 + (-5) = -3$, $-2 + 5 = 3$
4 Write the expression in factor form. (Check your result by expanding.)	$x^2 - 3x - 10$ $= (x + 2)(x - 5)$
c **1** Show x as the first term in each set of brackets.	**c** $x^2 - 7x + 12$ $= (x \quad)(x \quad)$
2 Identify factor pairs of 12. Only look at the negative factor pairs.	12: -1×-12, -2×-6, -3×-4
3 Check which factors add to −7. (−3 and −4)	$-1 + (-12) = -13$, $-2 + (-6) = -8$, $-3 + (-4) = -7$
4 Write the expression in factor form. (Check your result by expanding.)	$x^2 - 7x + 12$ $= (x - 3)(x - 4)$

6 Factorise each quadratic trinomial.

a $x^2 + 4x - 12$ **b** $x^2 - 8x - 20$ **c** $x^2 - 5x + 6$
d $x^2 + 6x - 27$ **e** $x^2 - 13x + 40$ **f** $x^2 - x - 20$
g $x^2 + 4x - 21$ **h** $x^2 - 14x + 33$ **i** $x^2 - 3x - 28$
j $x^2 + x - 12$ **k** $x^2 - 7x - 18$ **l** $x^2 - 15x + 56$

UNDERSTANDING AND FLUENCY

7 Factorise each quadratic trinomial by first taking out a common factor.

a $2x^2 + 18x + 16$ **b** $3x^2 + 15x + 18$ **c** $7x^2 + 14x - 56$

d $-5x^2 - 35x - 60$ **e** $-4x^2 + 28x - 40$ **f** $6 - 5x - x^2$

8 a i Expand $(x + 3)^2$ using the expansion of a perfect square rule.

ii Use your answer to part **i** to factorise $x^2 + 6x + 9$.

b i Expand $(x - 9)^2$ using the expansion of a perfect square rule.

ii Use your answer to part **i** to factorise $x^2 - 18x + 81$.

c Factorise each quadratic trinomial to produce a perfect square.

i $x^2 + 8x + 16$ **ii** $x^2 + 4x + 4$ **iii** $x^2 - 12x + 36$

iv $x^2 + 16x + 64$ **v** $x^2 - 10x + 25$ **vi** $x^2 - 20x + 100$

9 Copy and complete each expression to produce a perfect square. To find the missing term is called to 'complete the square'.

a $x^2 + 6x + \_\_ = (x + 3)^2$ **b** $x^2 + 10x + \_\_ = (x + 5)^2$

c $x^2 - 8x + \_\_ = (x - 4)^2$ **d** $x^2 - 4x + \_\_ = (x - 2)^2$

e $x^2 + 2x + \_\_ = (x + 1)^2$ **f** $x^2 - 16x + \_\_ = (x - 8)^2$

10 a Explain how you can work out what term needs to be added to an expression to complete the square. Look at your answers to question **9** to help you.

b What term should be added to $x^2 + 12x$ to complete the square?

c Explain why $\left(\frac{b}{2}\right)^2$ should be added to $x^2 + bx$ to complete the square.

EXAMPLE 2E-3 Adding a term to complete the square

Add a term to each expression to complete the square.

a $x^2 + 8x$ **b** $x^2 - 14x$

THINK

a 1 Use the coefficient of the x term to find the missing term. Halve the coefficient and then square it. That is, find $\left(\frac{b}{2}\right)^2$ where $b = 8$.

2 Write the expression obtained after completing the square.

b 1 Use the coefficient of the x term to find the missing term. Halve the coefficient and then square it. That is, find $\left(\frac{b}{2}\right)^2$ where $b = -14$.

2 Write the expression obtained after completing the square.

WRITE

a Term needed to complete the square

$= \left(\frac{8}{2}\right)^2$

$= 4^2$

$= 16$

Perfect square expression is $x^2 + 8x + 16$.

b Term needed to complete the square

$= \left(\frac{-14}{2}\right)^2$

$= (-7)^2$

$= 49$

Perfect square expression is $x^2 - 14x + 49$.

UNDERSTANDING AND FLUENCY

11 Add a term to each expression to complete the square.

a $x^2 + 14x$ **b** $x^2 - 6x$ **c** $x^2 + 18x$

d $x^2 + 20x$ **e** $x^2 - 12x$ **f** $x^2 + 22x$

12 Factorise each perfect square expression you made in question **11**.

13 **a** Think about how you would factorise $x^2 + 6x + 7$. Discuss your ideas with a classmate.

b If $x^2 + 6x + 7$ is written as $(x^2 + 6x + 9) - 2$, explain how this could be factorised using the perfect square rule and the difference of two squares rule. This method is called factorising by completing the square.

EXAMPLE 2E-4 Factorising a quadratic trinomial by completing the square

Factorise each quadratic trinomial by completing the square.

a $x^2 + 6x + 7$ **b** $x^2 - 2x - 4$

THINK	WRITE
a **1** Form a perfect square (complete the square) using the first two terms. Halve the coefficient of the second term and then square it.	**a** Term needed to complete the square for $x^2 + 6x$ $= \left(\frac{6}{2}\right)^2$ $= 3^2$ $= 9$
2 Write the expression obtained after completing the square.	Perfect square expression is $x^2 + 6x + 9$.
3 Rewrite the given quadratic trinomial using the perfect square expression. Compensate for adding 9 by subtracting 9 so that the overall expression is unchanged.	$x^2 + 6x + 7$ $= (x^2 + 6x + 9) - 9 + 7$
4 Simplify $-9 + 7$.	$= (x^2 + 6x + 9) - 2$
5 Factorise the perfect square expression.	$= (x + 3)^2 - 2$
6 Write as the difference of two squares.	$= (x + 3)^2 - (\sqrt{2})^2$
7 Factorise using the difference of two squares rule.	$= (x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$
b (Shorter method shown)	**b** $x^2 - 2x - 4$
1 Form a perfect square (complete the square) using the first two terms. $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$. Remember to compensate for adding 1 by subtracting 1.	$= (x^2 - 2x + 1) - 1 - 4$
2 Simplify $-1 - 4$.	$= (x^2 - 2x + 1) - 5$
3 Factorise the perfect square expression.	$= (x - 1)^2 - 5$
4 Write as the difference of two squares.	$= (x - 1)^2 - (\sqrt{5})^2$
5 Factorise using the difference of two squares rule.	$= (x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$

14 Factorise each quadratic trinomial by completing the square.

a $x^2 + 8x + 9$ **b** $x^2 - 10x + 14$ **c** $x^2 + 4x - 6$

d $x^2 + 2x - 5$ **e** $x^2 - 6x - 4$ **f** $x^2 - 12x + 13$

g $x^2 + 10x - 4$ **h** $x^2 - 14x + 46$ **i** $x^2 + 8x + 7$

PROBLEM SOLVING AND REASONING

15 Consider this rectangular tile, which has all lengths in centimetres. The expression for the width is $x + 5$ and the expression for the area is $x^2 + 12x + 35$.

a Factorise the expression for the area of the tile.

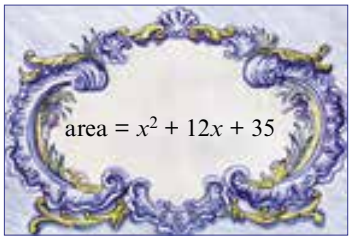
b List the two factors that multiply to give the expression for the area.

c What is the expression for the length of this tile?

d If $x = 8$, calculate the area using:

i values for length and width

ii the expression $x^2 + 12x + 35$.

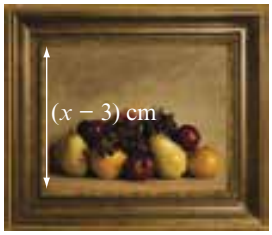


16 **a** The area of the painting within the frame is $(x^2 + 9x - 36)$ cm^2. Write an expression for the missing side length of the painting.

b Write three possible values for x.

c Find the area of the painting using your answers to part **b**.

d Find the value of x that gives an area of 646 cm^2.



17 **a** Factorise $x^2 + 6x + 5$ by completing the square.

b Look at your answer to part **a**. Is there an easier way of obtaining this? Explain.

18 Choose the most appropriate method to factorise each expression.

a $x^2 + 15x + 44$ **b** $x^2 - 2x - 15$ **c** $x^2 + 4x + 1$

d $3x^2 - 18x + 24$ **e** $5x^2 - 45$ **f** $2x^2 - 8x$

19 Explain why each expression cannot be factorised using the methods covered so far.

a $x^2 + 9$ **b** $(x - 3)^2 + 16$ **c** $(x + 1)^2 + 4$

d $x^2 - 4x + 5$ **e** $(x + 2)^2 - 4x$ **f** $x^2 + 12x + 40$

CHALLENGE

20 Factorise each expression by completing the square.

a $x^2 + 3x - 1$ **b** $x^2 - x - 7$ **c** $x^2 + 5x + 5$

d $x^2 + 7x + 7$ **e** $x^2 - 11x + 15$ **f** $x^2 - 9x - 2$

21 Factorise each expression, if possible.

a $2x^2 + 6x - 56$ **b** $x^2 - 3x + 5$ **c** $x^3 + 3x^2 - 18x$

d $12 + x - x^2$ **e** $98 - 2x^2$ **f** $4x^2 + 25$

22 What values can k have so that each quadratic trinomial can be factorised using the methods covered so far?

a $x^2 - 6x + k$ **b** $x^2 + 20x + k$

Reflect

How do you know that an expression has been factorised?

2F Working with algebraic fractions

Start thinking!

1 To simplify $\frac{3x}{3}$, you cancel a common factor in the numerator and denominator. What is the common factor?

2 Can you simplify $\frac{x+3}{3}$ by cancelling a common factor? Explain why or why not.

3 How is $\frac{x+3}{3}$ different from $\frac{3x}{3}$?

4 Can you simplify $\frac{3(x+3)}{3}$ by cancelling a common factor? If so, write the simplified fraction.

5 In each case, state whether you can simplify the algebraic fraction by cancelling a common factor.

a $\frac{8x}{4}$ **b** $\frac{x-2}{x}$ **c** $\frac{2x}{3}$ **d** $\frac{x(x+1)}{x}$

e $\frac{7}{7-x}$ **f** $\frac{18}{10x}$ **g** $\frac{x(x+5)}{4(x+5)}$ **h** $\frac{x-4}{2(x-4)}$

6 For those algebraic fractions in question **5** where you can cancel by a common factor, write the simplified fraction. Compare your results with those of a classmate. Discuss any that you don't agree on.

7 Summarise when you can and cannot simplify an algebraic fraction by cancelling a common factor.

KEY IDEAS

- To simplify an algebraic fraction, cancel any factors that are common to the numerator and denominator.
- To multiply algebraic fractions:
  - factorise where possible
  - cancel factors that are common to the numerator and denominator
  - simplify by multiplying the numerators together and the denominators together.
- To divide algebraic fractions:
  - change from a division problem to a multiplication problem by replacing the division sign with a multiplication sign and turning the fraction that follows upside down (take the reciprocal)
  - proceed as for a multiplication problem.
- To add (or subtract) algebraic fractions:
  - find the **lowest common denominator** (LCD) by finding the lowest common multiple (LCM) of the denominators
  - rewrite each fraction as an equivalent fraction with this LCD
  - add (or subtract) the new numerators and write as a single fraction
  - simplify the numerator.

EXERCISE 2F Working with algebraic fractions

EXAMPLE 2F-1 Simplifying algebraic fractions

Simplify each algebraic fraction.

a $\frac{5x}{5xy}$ b $\frac{3(x+2)}{6}$ c $\frac{(x-4)(x+1)}{(x+3)(x-4)}$

THINK

a **1** Look for common factors. It may be easier to include multiplication signs so that the factors are easily seen. Common factors are 5 and x.

2 Cancel each factor in the numerator and denominator.

3 Simplify the expression.

b **1** Look for common factors. Cancel 3 in the numerator and denominator.

2 Simplify the expression. You cannot cancel further as 2 is not a factor in the numerator.

c **1** Look for common factors. Cancel $(x-4)$ in the numerator and denominator.

2 Simplify the expression. Note that brackets are no longer needed around $x+1$ and $x+3$.

WRITE

a $\frac{5x}{5xy}$

$= \frac{5 \times x}{5 \times x \times y}$

$= \frac{{}^{1}\cancel{5} \times {}^{1}\cancel{x}}{{}_{1}\cancel{5} \times {}_{1}\cancel{x} \times y}$

$= \frac{1}{y}$

b $\frac{3(x+2)}{6}$

$= \frac{{}^{1}\cancel{3}(x+2)}{{}_{2}\cancel{6}}$

$= \frac{x+2}{2}$

c $\frac{(x-4)(x+1)}{(x+3)(x-4)}$

$= \frac{{}^{1}\cancel{(x-4)}(x+1)}{(x+3)\cancel{(x-4)}_{1}}$

$= \frac{x+1}{x+3}$

UNDERSTANDING AND FLUENCY

1 Simplify each algebraic fraction.

a $\frac{3x}{6}$ b $\frac{9x}{9xy}$ c $\frac{8x}{2x}$ d $\frac{4xy}{6y}$

e $\frac{2(x+3)}{10}$ f $\frac{8(x-5)}{4}$ g $\frac{x(x-4)}{x}$ h $\frac{3x(x+1)}{3x}$

i $\frac{6}{6(x-7)}$ j $\frac{2x(x+6)}{6x}$ k $\frac{10(x+2)}{8x}$ l $\frac{5x}{x(x-9)}$

2 Simplify each algebraic fraction.

a $\frac{(x-2)(x+5)}{(x+1)(x-2)}$ b $\frac{(x-6)(x+4)}{(x-2)(x+4)}$ c $\frac{(x-1)(x+3)}{(x+3)(x+1)}$ d $\frac{(x+7)(x+2)}{(x+7)(x-8)}$

e $\frac{(x+4)(x-5)}{(x+4)(x-5)}$ f $\frac{(x-1)(x+1)}{(x+1)(x-1)}$ g $\frac{x(x+8)}{(x+8)(x-2)}$ h $\frac{4x(x-3)}{8(x-3)}$

i $\frac{2(x-2)(x+2)}{2(x+2)(x-4)}$ j $\frac{7x(x+5)}{21x(x-5)}$ k $\frac{3(x-3)(x+9)}{6(x+2)(x-3)}$ l $\frac{10x(x+2)(x+3)}{5x(x+3)(x+2)}$

EXAMPLE 2F-2 Multiplying and dividing algebraic fractions

Simplify each expression.

a $\frac{4x}{x+6} \times \frac{x+6}{8x}$ b $\frac{(x-1)(x+2)}{(x-4)(x+3)} \div \frac{(x-1)(x+5)}{(x-7)(x+3)}$

THINK

a 1 Look for common factors. They are 4, x and $x + 6$. Cancel each factor in the numerator and denominator.

2 Simplify each fraction.

3 Multiply the numerators together and the denominators together.

b 1 Change the division problem to a multiplication problem.

2 Look for common factors. They are $(x - 1)$ and $(x + 3)$. Cancel each factor in the numerator and denominator.

3 Simplify each fraction.
Note that no further cancelling can be done.

4 Multiply the numerators together and the denominators together. Keep in factorised form.

WRITE

a $\frac{4x}{x+6} \times \frac{x+6}{8x}$

$= \frac{{}^{1}\cancel{4x}^{\cancel{1}}}{{}_{1}\cancel{x+6}} \times \frac{\cancel{x+6}^{1}}{{}_{2}\cancel{8x}_{1}}$

$= \frac{1}{1} \times \frac{1}{2}$

$= \frac{1}{2}$

b $\frac{(x-1)(x+2)}{(x-4)(x+3)} \div \frac{(x-1)(x+5)}{(x-7)(x+3)}$

$= \frac{(x-1)(x+2)}{(x-4)(x+3)} \times \frac{(x-7)(x+3)}{(x-1)(x+5)}$

$= \frac{{}^{1}\cancel{(x-1)}(x+2)}{(x-4)\cancel{(x+3)}_{1}} \times \frac{(x-7)\cancel{(x+3)}^{1}}{{}_{1}\cancel{(x-1)}(x+5)}$

$= \frac{x+2}{x-4} \times \frac{x-7}{x+5}$

$= \frac{(x+2)(x-7)}{(x-4)(x+5)}$

UNDERSTANDING AND FLUENCY

3 Simplify each expression.

a $\frac{3x}{x+2} \times \frac{x+2}{6x}$ b $\frac{x-3}{x} \times \frac{5x}{x-3}$ c $\frac{x+4}{x+1} \times \frac{x+1}{x+4}$

d $\frac{6x}{x-5} \div \frac{2x}{x-5}$ e $\frac{x+7}{4x} \div \frac{x+7}{20x}$ f $\frac{8x}{2x+1} \div \frac{8}{2x+1}$

g $\frac{4x(x-1)}{x+3} \times \frac{x+3}{2(x-1)}$ h $\frac{3(x+4)}{2(x-2)} \times \frac{4(x-2)}{3}$ i $\frac{5(x+3)}{x+1} \times \frac{x+1}{5x}$

j $\frac{2(x-6)}{x+6} \div \frac{x-6}{x+6}$ k $\frac{x+11}{3x(x+2)} \div \frac{x+11}{6(x+2)}$ l $\frac{9x(x-1)}{6(x+5)} \div \frac{3x}{2(x+5)}$

4 Simplify each expression.

a $\frac{(x-3)(x+4)}{(x-2)(x+1)} \times \frac{(x-5)(x+1)}{(x-3)(x+2)}$ b $\frac{(x+2)(x+3)}{(x-2)(x+6)} \times \frac{(x+6)(x-2)}{(x+3)(x-1)}$

c $\frac{(x+4)(x+5)}{(x-1)(x+2)} \div \frac{(x-5)(x+4)}{(x-2)(x-1)}$ d $\frac{(x-7)(x+1)}{(x+3)(x-7)} \div \frac{(x-4)(x+1)}{(x+9)(x-4)}$

e $\frac{6(x+5)(x-5)}{(x-2)(x+2)} \times \frac{(x-1)(x+2)}{3(x+5)(x-1)}$ f $\frac{(x+3)(x+8)}{4x(x+2)(x-3)} \times \frac{4x(x+2)(x-6)}{(x+3)(x+6)}$

g $\frac{(2x-1)(3x+2)}{(3x-2)(x+1)} \div \frac{(x+2)(2x-1)}{(5x+1)(3x-2)}$ h $\frac{5x(3x-5)}{(2x+5)(7x-4)} \div \frac{10(3x-5)(x+8)}{(7x-4)(2x+5)}$

EXAMPLE 2F-3 Simplifying algebraic fractions by first factorising

Factorise each expression.

a $\dfrac{x^2 - 3x - 4}{x^2 - 4x}$ **b** $\dfrac{2x^2 + 6x}{x - 2} \times \dfrac{x^2 - 5x + 6}{x^2 - 9}$ **c** $\dfrac{x^2 - x - 12}{x^2 - 16} \div \dfrac{x^2 + 2x - 3}{x^2 + 8x + 16}$

THINK

a 1 Factorise each expression in the numerator and denominator.

2 Look for common factors. Cancel $(x - 4)$ in the numerator and denominator.

3 Simplify the expression.
Note that you cannot cancel further as x is not a factor in the numerator.

b 1 Factorise each expression in the numerator and denominator of the fractions.

2 Look for common factors. They are $(x + 3)$, $(x - 2)$ and $(x - 3)$. Cancel each factor in the numerator and denominator.

3 Simplify each fraction.

4 Multiply the numerators together and the denominators together.

c 1 Change the division problem to a multiplication problem.

2 Factorise each expression in the numerator and denominator of the fractions.

3 Look for common factors. They are $(x - 4)$, $(x + 3)$ and $(x + 4)$. Cancel each factor in the numerator and denominator.

4 Simplify each fraction.

5 Multiply the numerators together and the denominators together.
Note that no further cancelling can be done.

WRITE

a $\dfrac{x^2 - 3x - 4}{x^2 - 4x}$

$= \dfrac{(x - 4)(x + 1)}{x(x - 4)}$

$= \dfrac{{}^{1}\cancel{(x - 4)}(x + 1)}{x\cancel{(x - 4)}_{1}}$

$= \dfrac{x + 1}{x}$

b $\dfrac{2x^2 + 6x}{x - 2} \times \dfrac{x^2 - 5x + 6}{x^2 - 9}$

$= \dfrac{2x(x + 3)}{x - 2} \times \dfrac{(x - 2)(x - 3)}{(x + 3)(x - 3)}$

$= \dfrac{2x\cancel{(x + 3)}^{1}}{\cancel{x - 2}_{1}} \times \dfrac{{}^{1}\cancel{(x - 2)}\cancel{(x - 3)}^{1}}{{}_{1}\cancel{(x + 3)}\cancel{(x - 3)}_{1}}$

$= \dfrac{2x}{1} \times \dfrac{1}{1}$

$= \dfrac{2x}{1}$

$= 2x$

c $\dfrac{x^2 - x - 12}{x^2 - 16} \div \dfrac{x^2 + 2x - 3}{x^2 + 8x + 16}$

$= \dfrac{x^2 - x - 12}{x^2 - 16} \times \dfrac{x^2 + 8x + 16}{x^2 + 2x - 3}$

$= \dfrac{(x - 4)(x + 3)}{(x + 4)(x - 4)} \times \dfrac{(x + 4)(x + 4)}{(x - 1)(x + 3)}$

$= \dfrac{{}^{1}\cancel{(x - 4)}\cancel{(x + 3)}^{1}}{{}_{1}\cancel{(x + 4)}\cancel{(x - 4)}_{1}} \times \dfrac{{}^{1}\cancel{(x + 4)}(x + 4)}{(x - 1)\cancel{(x + 3)}_{1}}$

$= \dfrac{1}{1} \times \dfrac{x + 4}{x - 1}$

$= \dfrac{x + 4}{x - 1}$

UNDERSTANDING AND FLUENCY

5 Simplify each expression. Remember to factorise first.

a $\frac{x^2 + 5x}{x + 5}$ **b** $\frac{x - 2}{x^2 - 2x}$ **c** $\frac{x^2 + 4x}{3x + 12}$

d $\frac{x^2 + 6x + 8}{x^2 + 10x + 24}$ **e** $\frac{x^2 + x - 6}{x^2 + 3x}$ **f** $\frac{x^2 + 5x}{2x^2 + 10x}$

g $\frac{x^2 - 36}{x^2 + 7x + 6}$ **h** $\frac{x^2 - 3x - 10}{x^2 - x - 6}$ **i** $\frac{7x^2 - 14x}{x^2 + 5x - 14}$

6 Simplify each expression.

a $\frac{x + 2}{x^2 + 4x + 3} \times \frac{x^2 + x - 6}{x^2 - 4}$ **b** $\frac{4x + 8}{x^2 + x - 2} \times \frac{x^2 - 4x + 3}{2x - 6}$

c $\frac{x^2 - 49}{x^2 + 7x} \div \frac{x^2 - 1}{x^2 - x}$ **d** $\frac{x^2 - 2x - 8}{x^2 - 3x - 4} \div \frac{x^2 + 8x + 12}{x^2 + 5x - 6}$

e $\frac{6x^2 - 24x}{3x^2 + 15x} \times \frac{x^2 + 6x + 5}{2x^2 + 2x}$ **f** $\frac{x^2 + 8x + 15}{x^2 + 9x + 8} \times \frac{x^2 + 10x + 16}{x^2 - x - 12}$

g $\frac{x^2 - 25}{x^2 - 4} \div \frac{x^2 + 2x - 35}{x^2 + 5x - 14}$ **h** $\frac{x^2 - 8x - 20}{x^2 + 10x + 21} \div \frac{x^2 - 100}{x^2 - 3x - 18}$

EXAMPLE 2F-4 Adding and subtracting algebraic fractions

Simplify each expression.

a $\frac{2x}{5} + \frac{x}{3}$ **b** $\frac{x - 5}{4} - \frac{x - 3}{6}$

THINK

a **1** Identify the lowest common denominator (LCD). The lowest common multiple (LCM) of 5 and 3 is 15. So LCD is 15. Write each fraction as an equivalent fraction with denominator of 15.

2 Write as a single fraction.

3 Simplify the numerator.

b **1** Identify the LCD. The LCM of 4 and 6 is 12. So LCD is 12. Write each fraction as an equivalent fraction with denominator of 12.

2 Write as a single fraction.

3 Expand and simplify the numerator.

WRITE

a $\frac{2x}{5} + \frac{x}{3}$

$= \frac{2x}{5} \times \frac{3}{3} + \frac{x}{3} \times \frac{5}{5}$

$= \frac{6x}{15} + \frac{5x}{15}$

$= \frac{6x + 5x}{15}$

$= \frac{11x}{15}$

b $\frac{x - 5}{4} - \frac{x - 3}{6}$

$= \frac{x - 5}{4} \times \frac{3}{3} - \frac{x - 3}{6} \times \frac{2}{2}$

$= \frac{3(x - 5)}{12} - \frac{2(x - 3)}{12}$

$= \frac{3(x - 5) - 2(x - 3)}{12}$

$= \frac{3x - 15 - 2x + 6}{12}$

$= \frac{x - 9}{12}$

UNDERSTANDING AND FLUENCY

7 Simplify each expression.

a $\frac{x}{4}+\frac{x}{5}$ **b** $\frac{x}{2}-\frac{x}{7}$ **c** $\frac{x}{6}+\frac{x}{3}$ **d** $\frac{x}{8}-\frac{x}{10}$

e $\frac{2x}{3}+\frac{x}{2}$ **f** $\frac{3x}{4}-\frac{x}{8}$ **g** $\frac{4x}{5}+\frac{3x}{7}$ **h** $\frac{x^2}{2}-\frac{x^2}{9}$

8 Simplify each expression.

a $\frac{x+1}{2}+\frac{x+5}{3}$ **b** $\frac{x+2}{5}+\frac{x-4}{6}$ **c** $\frac{x-3}{4}+\frac{x-1}{8}$ **d** $\frac{x+4}{3}-\frac{x+2}{4}$

e $\frac{x-6}{8}-\frac{x-3}{12}$ **f** $\frac{x+1}{2}-\frac{x+4}{10}$ **g** $\frac{2x+3}{3}+\frac{x-7}{5}$ **h** $\frac{x+5}{6}+\frac{2x+1}{9}$

i $\frac{4x+7}{3}+\frac{5x-3}{2}$ **j** $\frac{3x-1}{4}-\frac{2x-5}{5}$ **k** $\frac{4-x}{7}-\frac{x-4}{3}$ **l** $\frac{x^2+2}{5}-\frac{x^2+3}{8}$

PROBLEM SOLVING AND REASONING

9 Find the algebraic expression for a where $\frac{a}{2}+\frac{x+1}{4}=\frac{3x-5}{4}$.

10 Find the algebraic expression for b where $\frac{b}{5}-\frac{x-2}{2}=\frac{x+12}{10}$.

11 Simplify each expression using the most appropriate method.

a $\frac{x}{3}+\frac{2x}{5}+\frac{x}{2}$ **b** $\frac{5x}{4}+\frac{x}{8}-\frac{x}{2}$ **c** $\frac{4x}{3}-\frac{5x}{6}-\frac{3x}{8}$

d $\frac{2x^2}{7}\times\frac{3}{10x}\times\frac{5}{x}$ **e** $\frac{4}{x}\div\frac{12}{x^2}-\frac{5x}{7}$ **f** $\frac{3x}{4}\times\frac{2x^2-14x}{6x^2}+\frac{x+2}{3}$

12 Simplify each expression using the most appropriate method.

a $\frac{x^2+x-20}{x^2-6x+8}\times\frac{x^2-4}{x^2-25}\times\frac{8x^2-40x}{4x^2+8x}$ **b** $\frac{x}{x+7}\times\frac{x^2-49}{3x}\div\frac{x^2-5x-14}{x^2-4x-12}$

c $\frac{3x^2+3x}{x-3}\times\frac{x^2-7x+12}{6x^2+6x}+\frac{x+5}{3}$ **d** $\frac{x^2+9x+18}{2x+6}\div\frac{3x+18}{5x-1}+\frac{3x-5}{8}$

13 Simplify $\frac{5}{2x}+\frac{2}{5x}$ using $10x$ as the LCD.

14 Simplify each expression.

a $\frac{4}{3x}+\frac{1}{2x}$ **b** $\frac{3}{4x}-\frac{4}{7x}$ **c** $\frac{1}{x}+\frac{5}{2x}$ **d** $\frac{1}{4x}-\frac{1}{6x}$ **e** $\frac{9}{15x}+\frac{2}{5x}$ **f** $\frac{7}{6x}-\frac{3}{8x}$

15 Simplify $\frac{3}{x+2}+\frac{4}{x+5}$ using $(x+2)(x+5)$ as the LCD.

CHALLENGE

16 Simplify each expression.

a $\frac{2}{x+1}+\frac{5}{x+3}$ **b** $\frac{1}{x-4}+\frac{3}{x+6}$ **c** $\frac{4}{x+3}-\frac{2}{x+2}$

d $\frac{6}{x-5}-\frac{5}{x-1}$ **e** $\frac{3x}{x+2}+\frac{2x}{x+4}$ **f** $\frac{4}{2x+1}-\frac{5}{3x-2}$

17 Find two sets of algebraic expressions for a and b where $\frac{a}{2}+\frac{b}{3}=\frac{5x-1}{6}$.

18 Find three sets of algebraic expressions for m and n where $\frac{m}{2}-\frac{n}{4}=\frac{x}{4}$.

Reflect

How can factors be used to simplify algebraic fractions?

2G Factorising quadratic trinomials of the form $ax^2 + bx + c$

Start thinking!

So far you have factorised quadratic trinomials of the form $ax^2 + bx + c$ where $a = 1$. In this topic section, you will look at two different methods that can be used to factorise more difficult expressions. Let's first develop one of these with a simple quadratic trinomial such as $x^2 + 5x + 6$.

1 Identify a, b and c if $x^2 + 5x + 6$ is of the form $ax^2 + bx + c$.

2 To factorise $x^2 + 5x + 6$ previously, you looked for two numbers that multiply to give 6 (c) and add to 5 (b). What are the two numbers?

3 Identify the values of ac and b. What two numbers multiply to give ac and add to give b?

4 Compare the two numbers you found in questions **2** and **3**. Why are they the same in this case?

5 You can use these two numbers to split the middle term. Copy and complete the working shown at right.

$x^2 + 5x + 6$
$= x^2 + 2x + \_\_x + 6$

6 Factorise the last expression in question **5** using grouping 'two and two'.

7 Does it matter in which order the middle term bx is split? Compare factorising $x^2 + 2x + 3x + 6$ and $x^2 + 3x + 2x + 6$ using grouping.

8 This factorising method is called **splitting the middle term** and can be used for quadratic trinomials where $a \neq 1$. Repeat questions **3**, **5** and **6** to factorise $2x^2 + 11x + 12$.

9 Check your answer by expanding the result you obtained in question **8**.

KEY IDEAS

- There are a number of ways to factorise a quadratic trinomial of the form $ax^2 + bx + c$.
  - **Splitting the middle term**
    1. Identify two numbers that multiply to give ac and add to give b.
    2. Use these two numbers to split the middle term (bx) so the expression has four terms and can be factorised by grouping 'two and two'. See Example A.
  - **Cross-multiplying**
    1. Identify factor pairs of the first term (ax^2) and the last term (c).
    2. Cross-multiply to check which pair produces the middle term (bx). See Example B.

Example A

$6x^2 + 17x + 5$ $\quad ac = 30, b = 17$
$= 6x^2 + 2x + 15x + 5$ $\quad 2 \times 15 = 30$, $2 + 15 = 17$
$= 2x(3x + 1) + 5(3x + 1)$
$= (3x + 1)(2x + 5)$

Example B

$6x^2 + 17x + 5$
$= (3x + 1)(2x + 5)$

$(3x \times +1)$
$(2x \times +5)$
$15x + 2x = 17x$

EXERCISE 2G Factorising quadratic trinomials of the form $ax^2 + bx + c$

UNDERSTANDING AND FLUENCY

1 Each quadratic trinomial below is in the form $ax^2 + bx + c$. For each one:

- **i** identify ac and b
- **ii** find two numbers that multiply to give ac and add to give b
- **iii** use the two numbers to split the middle term into two terms.

a $2x^2 + 5x + 2$ **b** $3x^2 + 11x + 6$ **c** $4x^2 + 17x + 4$
d $3x^2 + 10x - 8$ **e** $6x^2 - 7x - 3$ **f** $5x^2 - 13x + 6$

2 Factorise each expression using grouping 'two and two'.

a $3x^2 + 2x + 9x + 6$ **b** $4x^2 + x + 16x + 4$ **c** $2x^2 + x + 4x + 2$
d $6x^2 - 9x + 2x - 3$ **e** $3x^2 - 2x + 12x - 8$ **f** $5x^2 - 3x - 10x + 6$

EXAMPLE 2G-1 Factorising a quadratic trinomial by splitting the middle term

Factorise $3x^2 + 10x + 8$ by first splitting the middle term.

THINK	WRITE
1 Work out ac and b where $a = 3$, $b = 10$ and $c = 8$.	$3x^2 + 10x + 8$ $ac = 24$, $b = 10$
2 Identify two numbers that multiply to give 24 and add to 10.	$4 \times 6 = 24$, $4 + 6 = 10$
3 Use the numbers 4 and 6 to split the middle term.	$3x^2 + 10x + 8$ $= 3x^2 + 4x + 6x + 8$
4 Factorise by grouping 'two and two'. Note that the same result is obtained if the middle term is split into $6x + 4x$.	$= x(3x + 4) + 2(3x + 4)$ $= (3x + 4)(x + 2)$

3 Factorise each quadratic trinomial by first splitting the middle term.

a $2x^2 + 7x + 6$ **b** $3x^2 + 5x + 2$ **c** $4x^2 + 12x + 5$
d $7x^2 + 9x + 2$ **e** $5x^2 + 14x + 8$ **f** $2x^2 + 9x + 9$
g $6x^2 + 7x + 2$ **h** $10x^2 + 17x + 3$ **i** $8x^2 + 14x + 5$
j $2x^2 + 15x + 18$ **k** $12x^2 + 7x + 1$ **l** $5x^2 + 19x + 12$

4 Factorise each quadratic trinomial by first splitting the middle term.

a $3x^2 - 8x - 3$ **b** $4x^2 + 5x - 6$ **c** $2x^2 - 9x + 10$
d $5x^2 - x - 4$ **e** $8x^2 + 10x - 3$ **f** $4x^2 + 4x - 15$
g $7x^2 - 18x + 8$ **h** $12x^2 + x - 1$ **i** $3x^2 - 11x - 20$
j $6x^2 - 19x + 14$ **k** $10x^2 + 9x - 9$ **l** $12x^2 - 23x + 5$

5 One way of expanding $(2x + 5)(x + 3)$ is to write one binomial factor above the other as shown at right.

$(2x \quad +5)$
$(x \quad +3)$

a To obtain the quadratic trinomial:

i which two terms are multiplied to give the first term? (Hint: follow the green arrow.)

ii which two terms are multiplied to give the last term? (Hint: follow the orange arrow.)

iii which terms are multiplied and the results added to give the middle term? (Hint: follow the blue arrows. Start with $2x \times +3 + \ldots$)

NOTE The middle term is worked out by cross-multiplying.

b Use your answers to part **a** to write the quadratic trinomial.

c Compare this to expanding $(2x + 5)(x + 3)$ in the following way.

$(2x + 5)(x + 3) = 2x^2 + 6x + 5x + 15$
$= 2x^2 + 11x + 15$

Can you see why it is the same?

6 Use the diagrams below to expand each pair of binomial factors by working out:

i the first term **ii** the last term **iii** the middle term.

(Hint: use cross-multiplying to work out the middle term.)

a $(2x \quad +1)$, $(x \quad +4)$ **b** $(5x \quad -2)$, $(x \quad -1)$ **c** $(3x \quad +4)$, $(2x \quad -3)$ **d** $(4x \quad -5)$, $(3x \quad +2)$

EXAMPLE 2G-2 Factorising a quadratic trinomial using cross-multiplying

Factorise $2x^2 + 13x + 15$ using cross-multiplying.

THINK

1 Identify factor pairs of the first term. Choose pairs that have x in each term.

2 Identify factor pairs of the last term. Since the middle term is positive, you only need to look at positive factors of 15.

3 Test these factor pairs. Use cross-multiplying to check which combination gives the required middle term of $13x$.

4 Write the quadratic trinomial as the product of two binomial factors. Remember that the binomial factors are written one above the other in the combinations above.

WRITE

$2x^2 + 13x + 15$

factors of $2x^2$: $2x$ and x

factors of 15: 1 and 15, 3 and 5

$(2x \quad +1)$
$(x \quad +15)$
$30x + x = 31x$ (no)

$(2x \quad +15)$
$(x \quad +1)$
$2x + 15x = 17x$ (no)

$(2x \quad +3)$
$(x \quad +5)$
$10x + 3x = 13x$ (yes)

$(2x \quad +5)$
$(x \quad +3)$
$6x + 5x = 11x$ (no)

$2x^2 + 13x + 15$
$= (2x + 3)(x + 5)$

7 Factorise each quadratic trinomial using cross-multiplying.

a $2x^2 + 7x + 3$	**b** $3x^2 + 8x + 5$	**c** $5x^2 + 12x + 7$
d $3x^2 + 17x + 10$	**e** $4x^2 + 11x + 6$	**f** $7x^2 + 30x + 8$
g $4x^2 + 12x + 5$	**h** $6x^2 + 25x + 14$	**i** $10x^2 + 13x + 4$

EXAMPLE 2G-3 Factorising more difficult quadratic trinomials using cross-multiplying

Factorise each quadratic trinomial using cross-multiplying.

a $5x^2 - 3x - 2$ **b** $6x^2 - 11x + 4$

THINK

a **1** Identify factor pairs of the first term and the last term. It is easier to always use positive factors for the first term.

2 Use cross-multiplying to check which combination gives the required middle term of $-3x$. Note: you don't need to test all the combinations. Stop when you have identified the correct one.

3 Write the quadratic trinomial in factorised form.

b **1** Identify factor pairs of the first term and the last term. Since the middle term is negative, only look at negative factors of 4.

2 Use cross-multiplying to produce $-11x$. Since there is no common factor in the trinomial, you do not need to check any combinations that have a common factor in the expression within brackets; that is, $6x$ with -4 or -2, and $2x$ with -4 or -2.

3 Write the quadratic trinomial in factorised form.

WRITE

a $5x^2 - 3x - 2$
Factors of $5x^2$: $5x$ and x
Factors of -2: -1 and $+2$, $+1$ and -2

$(5x \quad -1)$ $(5x \quad +2)$
$(x \quad +2)$ $(x \quad -1)$
$10x - x = 9x$ (no) $-5x + 2x = -3x$ (yes)

$5x^2 - 3x - 2$
$= (5x + 2)(x - 1)$

b $6x^2 - 11x + 4$
Factors of $6x^2$: $6x$ and x, $3x$ and $2x$
Factors of 4: -1 and -4, -2 and -2

$(6x \quad -1)$ $(3x \quad -4)$
$(x \quad -4)$ $(2x \quad -1)$
$-24x - x = -25x$ (no) $-3x - 8x = -11x$ (yes)

$6x^2 - 11x + 4$
$= (3x - 4)(2x - 1)$

8 Factorise each quadratic trinomial using cross-multiplying.

a $3x^2 + 7x - 6$	**b** $5x^2 - 9x - 2$	**c** $2x^2 - 11x + 12$
d $7x^2 + 4x - 3$	**e** $3x^2 - 2x - 8$	**f** $2x^2 - x - 15$
g $5x^2 - 13x + 6$	**h** $9x^2 + 8x - 1$	**i** $4x^2 - 21x - 18$

9 Factorise each quadratic trinomial using cross-multiplying.

a $6x^2 - 11x - 10$	**b** $4x^2 + 8x - 21$	**c** $10x^2 - 9x + 2$
d $8x^2 - 14x - 15$	**e** $6x^2 + x - 12$	**f** $9x^2 + 18x - 16$
g $15x^2 - x - 2$	**h** $8x^2 + 13x - 6$	**i** $6x^2 - 25x + 4$

10 Factorise each quadratic trinomial using your preferred method. Where appropriate, take out any common factors first. (Hint: the common factor can be negative.)

a $4x^2 + 10x + 6$ b $12x^2 + 27x + 6$ c $10x^2 + 38x - 8$

d $8x^2 + 22x + 15$ e $6x^2 + 3x - 30$ f $12x^2 - 8x - 4$

g $-4x^2 - 6x - 2$ h $-12x^2 - 21x + 6$ i $-6x^2 - 17x - 12$

11 Simplify each expression. Remember to factorise first.

a $\dfrac{2x^2 + 15x + 18}{x + 6}$ b $\dfrac{3x + 4}{3x^2 + 13x + 12}$ c $\dfrac{4x^2 + 18x + 14}{2x + 2}$

d $\dfrac{6x^2 - 5x + 1}{3x^2 + 14x - 5}$ e $\dfrac{5x^2 - 28x - 12}{2x^2 - 15x + 18}$ f $\dfrac{-8x^2 + 4x + 40}{2x^2 - x - 10}$

12 Simplify each expression.

a $\dfrac{5x^2 - 8x + 3}{x^2 - 1} \times \dfrac{2x^2 + x - 1}{2x^2 - 7x + 3}$ b $\dfrac{2x^2 + 13x + 21}{6x^2 - x - 2} \times \dfrac{3x^2 - 14x + 8}{x^2 - x - 12}$

c $\dfrac{4x^2 + 13x - 12}{2x^2 + 13x + 20} \div \dfrac{12x^2 - 5x - 3}{6x^2 + 17x + 5}$ d $\dfrac{4x^2 + 28x + 49}{4x^2 - 49} \div \dfrac{2x^2 - x - 28}{2x^2 + 15x + 28}$

e $\dfrac{3x^2 + 6x - 24}{3x^2 - 7x + 2} \times \dfrac{15x^2 + x - 2}{6x + 24}$ f $\dfrac{2x^3 - x^2 - 3x}{2x^2 + 9x - 18} \div \dfrac{x^3 - 9x}{x^2 + 9x + 18}$

EXAMPLE 2G-4 Factorising a quadratic trinomial by completing the square

Factorise $2x^2 + 12x - 1$ by completing the square.

THINK	WRITE
	$2x^2 + 12x - 1$
1 First check if the expression can be factorised by splitting the middle term or using cross-multiplying. (no) Since the coefficient of x^2 is 2, take out a common factor of 2.	$= 2\left(x^2 + 6x - \frac{1}{2}\right)$
2 Work with the first two terms inside the brackets to complete the square. $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Remember to compensate for adding 9 by subtracting 9.	$= 2\left[(x^2 + 6x + 9) - 9 - \frac{1}{2}\right]$
3 Simplify $-9 - \frac{1}{2}$.	$= 2\left[(x^2 + 6x + 9) - \frac{19}{2}\right]$
4 Factorise the perfect square expression.	$= 2\left[(x + 3)^2 - \frac{19}{2}\right]$
5 Write as the difference of two squares.	$= 2\left[(x + 3)^2 - \left(\sqrt{\frac{19}{2}}\right)^2\right]$
6 Factorise using the difference of two squares rule. Note that the factor of 2 remains outside the two pairs of brackets.	$= 2\left(x + 3 + \sqrt{\frac{19}{2}}\right)\left(x + 3 - \sqrt{\frac{19}{2}}\right)$

13 Factorise each quadratic trinomial by completing the square.

a $3x^2 + 18x + 21$ b $2x^2 - 20x + 40$ c $5x^2 + 10x - 30$

d $3x^2 + 12x + 3$ e $2x^2 - 12x - 3$ f $4x^2 + 8x - 1$

g $6x^2 - 24x - 5$ h $3x^2 + 24x + 1$ i $9x^2 - 18x + 2$

PROBLEM SOLVING AND REASONING

14 **a** Factorise $4x^2 + 7x - 2$.

b Show that $-4x^2 - 7x + 2$ factorises to $(1 - 4x)(x + 2)$.
(Hint: $-4x^2 - 7x + 2 = -(4x^2 + 7x - 2)$.)

c Factorise each quadratic trinomial.

i $-2x^2 + x + 3$ **ii** $-3x^2 + 16x - 5$ **iii** $-5x^2 - 3x + 14$

iv $-4x^2 + 5x + 6$ **v** $-6x^2 - x + 2$ **vi** $-25x^2 + 10x - 1$

15 An algebraic expression for the width of a 100-euro note is $x + 7$ and the expression for the area is $2x^2 + 11x - 21$.

a Factorise the expression for the area of one note.

b What is the expression for the length of this note?

c If $x = 75$ mm, calculate the area using:

i values for length and width

ii the expression $2x^2 + 11x - 21$.

d The area of an Australian $100 note is 102.7 cm^2. Compare this with the area of a 100-euro note. Which note is larger and by how much?

16 A quilt is twice as long as it is wide.

a Write an algebraic expression for the area of the quilt if the width is x cm.

Fabric is added to the edge of the quilt to form a border.

b How wide is the border if the total area of the quilt is now $(2x^2 + 27x + 81)$ cm^2? (Hint: first factorise the expression.)

c What is the length and width of the new quilt if 1566 cm^2 of fabric was used for the border?

17 Decide if each of these expressions can be factorised using one of the methods covered so far. For those that can, write the expression in factorised form.

a $2x^2 + 4x + 6$ **b** $15x^2 - 2x - 8$ **c** $3x^2 - 12x + 3$

d $4x^2 - 8x + 16$ **e** $-2x^2 + 12x - 22$ **f** $-6x^2 + 69x - 189$

CHALLENGE

18 The rectangular floor of Storage unit A has an area of $(8x^2 + 2x - 15)$ m^2. The length of the square floor of Storage unit B is the same as the smaller floor dimension of Storage unit A. Assume $x > 4$ m.

a Write expressions for the dimensions of the floor for each storage unit.

b Write an expression for the total floor area of both storage units.

c The area of the floor of Storage unit C is equivalent to the total area for Storage units A and B. In terms of x, what could be the dimensions of the rectangular floor of Storage unit C?

19 Factorise each quadratic trinomial by completing the square.

a $2x^2 + 3x - 4$ **b** $4x^2 - 5x - 2$

c $3x^2 + x - 1$ **d** $\frac{1}{2}x^2 - 5x + 7$

e $5x^2 + 5x - 3$ **f** $-6x^2 - 18x - 3$

Reflect

Which factorising method did you find easiest to use with quadratic trinomials?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

coefficient	base	difference of two squares	binomial factor
constant	index	perfect square	quadratic trinomial
like terms	index laws	expanding	completing the square
evaluating	binomial product	factorising	splitting the middle term
index form	distributive law	highest common factor	lowest common multiple

MULTIPLE-CHOICE

2A **1** What is the expanded form of $5a^2b^3$?

A $5 \times a \times b$ **B** $5 \times a^2 \times b^3$

C $5 \times a \times a \times b \times b \times b$ **D** $5a^2 \times b^3$

2A **2** What is the value of $-3(xy)^2$ when $x = -2$ and $y = 4$?

A -192 **B** 192 **C** 96 **D** 576

2B **3** What is the simplified form of $x^2 \times (x^3)^4$?

A x^9 **B** x^{24} **C** x^{20} **D** x^{14}

2B **4** What does $\frac{36x^2y}{4x^3y^{-2}}$ simplify to?

A $\frac{9}{xy^3}$ **B** $\frac{9y^3}{x}$ **C** $9x^{-1}y^{-3}$ **D** $\frac{9}{xy}$

2C **5** What is the expanded form of $6t^2w(4w^3 + 3t^4)$?

A $24t^2w^3 + 18t^6w$ **B** $24t^2w^4 + 18t^8w$

C $24t^2w^4 + 18t^6w$ **D** $24t^2w^3 + 18t^8w$

2C **6** What is the expanded form of $(x - 4)(x + 3)$?

A $x^2 + x + 12$ **B** $x^2 + x - 12$

C $x^2 - x + 12$ **D** $x^2 - x - 12$

2D **7** What is the fully factorised form of $8x^2y^3 + 12x^3y$?

A $8x^2y(y^2 + 4x)$ **B** $4x^2y(2y^2 + 3x)$

C $4xy(2xy^2 + 3x^2)$ **D** $4xy(2y^2 + 3x)$

2D **8** What is the factorised form of $3(a - 2) - b(a - 2)$?

A $(a - 2)(3 - b)$ **B** $(a - 2)(b - 3)$

C $(a - 2)(3 + b)$ **D** $(a + 2)(3 + b)$

2E **9** Which of these is a binomial product?

A $x^2(3x + 2)$ **B** $x^2 \times y^2$

C $(x + 2)(x - 4)$ **D** $4x(x - 3)$

2E **10** Which of these is a quadratic trinomial?

A $5x^2 + 4$ **B** $2x^3 + 4x^2 - 6$

C $x + 5x - 3$ **D** $4x - 2 + 3x^2$

2F **11** What is the algebraic fraction $\frac{3x(x + 4)}{6x}$ in its simplest form equivalent to?

A $x + 2$ **B** $\frac{x + 4}{2}$

C $\frac{3(x + 4)}{6}$ **D** $\frac{x(x + 4)}{2x}$

2F **12** What is $\frac{2x}{3} + \frac{x}{4}$ equal to?

A $\frac{3x}{7}$ **B** $\frac{11x}{12}$ **C** $\frac{3x}{12}$ **D** $\frac{11x}{7}$

2G 10A **13** What is the factorised form of $9x^2 + 21x + 10$?

A $(9x + 5)(x + 2)$

B $(9x + 10)(x + 1)$

C $(3x + 5)(3x + 2)$

D $(3x + 10)(3x + 1)$

SHORT ANSWER

2A **1** Simplify each expression.

a $4xy - 10xy + x^2 - 4x^2 + 2x^2y - 5xy^2$

b $ab^2 + (ab)^2 - 3a^2b^2 + 2a^2b - 4b^2a$

c $pq - 5pq + p^2 - q^2 - (3p)^2 + (2q)^2$

2A **2** Name the like terms in each list.

a $3abc^2$, $a^2b^2c^2$, $-2a^2b^2c$, $(2abc)^2$

b $5xyz^2$, $-2z^2xy$, $3yz^2x$, $4x^2y^2y^2$

c $2p^2q^2$, $-(pq)^2$, $p^2 \times q \times -q$, $(p \div q)^2$

2A **3** If $a = -2$, $b = -1$ and $c = 3$, evaluate each expression.

a $a^2 + b^2 + c^2$

b $abc - ab^2c - c^3 \div b$

c $(a + b)^2 - (c - a)^2$

2B **4** Write each term in simplified form with positive indices.

a $\dfrac{xy^{-3}}{x^{-2}y}$ b $\dfrac{4a^2b^{-1}}{12a^{-2}b^4}$ c $\dfrac{18p^3q^{-1}r^5}{15p^{-1}q^4r^{-2}}$

2B **5** Simplify each expression, writing the answers with positive indices.

a $(x^2)^3 \times x^{-2} \div x^{-4}$

b $(p^2q^{-3})^2 \div p^4q^{-3} \times p^{-1}q$

c $(2a^2)^{-3} \times (3ab^{-1})^{-2} \div 12b^5$

2C **6** Expand each expression, giving your answer in simplest form.

a $(2x^2 - 4)(3x^2 + 5)$

b $(2x + 3y)(2x - 3y)$

c $(3x^2 - 4)^2$

2C **7** Expand and simplify.

a $3(x^2 - 2) - x(x + 5)$

b $-2x(3 - x) + 5x(2 - 3x)$

c $x(x + 1) - x(x - 1) + x(x - 3)$

2D **8** Factorise each expression by grouping terms.

a $x^2 - 3x - 4x + 12$

b $x^3 + x^2 - x - 1$

c $16x + 6x^2 - 32 - 12x$

2D **9** Use the difference of two squares rule to factorise each expression. Write your answer in simplest form.

a $18x^2 - 8y^2$

b $(x - 3)^2 - x^2$

c $(x + 2)^2 - (x - 2)^2$

2D **10** Factorise each expression, including surds in your answer if necessary.

a $x^2y^2 - 13$

b $(x + 1)^2 - 3$

c $7 - (a - 1)^2$

2E **11** Factorise each expression.

a $x^2 + 8x + 12$ b $x^2 - 2x - 63$

c $-x^2 + 6x - 8$ d $-6x^2 - 6x + 36$

2E **12** Factorise each expression by first completing the square.

a $x^2 + 10x + 20$ b $x^2 - 8x - 1$

2F **13** Simplify each expression.

a $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{x^2 + x - 12}{x^2 + x - 6}$

b $\dfrac{x^2 - 5x + 6}{x^2 - 16} \div \dfrac{x^2 + x - 6}{x^2 - 3x - 4}$

2F **14** Simplify each expression.

a $\dfrac{3x}{4} + \dfrac{4x}{3}$

b $\dfrac{x - 2}{4} - \dfrac{3 - x}{5}$

c $\dfrac{4x - 1}{12} - \dfrac{3 - 2x}{8}$

d $\dfrac{4}{3x} + \dfrac{5}{4x}$

2G 10A **15** Factorise each expression.

a $3x^2 + 29x + 18$ b $8x^2 + 10x - 3$

c $2x^2 - 7x - 22$ d $12x^2 - 17x + 6$

2G 10A **16** Factorise each expression, if possible.

a $2x^2 + 12x - 4$ b $5x^2 - 20x + 30$

c $6x^2 + 21x - 90$ d $-4x^2 + 12x - 2$

MIXED PRACTICE

1 Simplify each expression, writing answers with positive indices.

a $-2x \times 4xy \times -3y^2$

b $4a^2b^2 \times -3b^3 \div 8a^3b$

c $12pq^3 \div 3p^2q \times 2q$

2 What is the factorised form of $4x^2 - 9y^2$?

A $(2x + 3y)^2$ **B** $(2x + 3y)(2x - 3y)$

C $(2x - 3y)^2$ **D** $(4x + 9y)(4x - 9y)$

3 Expand each expression to remove the brackets.

a $(5x + 2y)(5x - 2y)$

b $(x - 6)(x + 1)$

c $(3x - 2y)^2$

4 Answer true or false to each statement. If the answer is false, give the correct answer.

a $5x^{-1}$ is equivalent to $\frac{1}{5x}$.

b If a^x is equivalent to $\frac{1}{a^5}$, the value of x is 5.

c If $m^x \times m^y = m^6$, $x \times y = 6$.

5 Simplify each expression, writing answers with positive indices.

a $(a^{-2}b^4c^{-1})^{-1}$

b $\frac{6x^2(y^{-3})^2}{8x^{-1}y^{-4}}$

c $\left(\frac{3p^2q^4}{2p^{-3}q^{-2}}\right)^{-2}$

6 Simplify each expression.

a $\frac{1}{2x} + \frac{1}{3x} + \frac{1}{4x}$

b $\frac{3}{x - 1} - \frac{1}{x - 3}$

c $\frac{x}{y} + \frac{y}{x}$

7 What is $\frac{(5x)^2y^{-3}}{10x^{-2}y^{-2}}$ equivalent to?

A $\frac{5x^4}{2y}$ **B** $\frac{1}{2y}$

C $\frac{5}{2y}$ **D** $\frac{5y}{2}$

8 Factorise each expression by first completing the square. Leave any terms in surd form if necessary.

a $x^2 + 2x - 1$

b $x^2 - 6x + 4$

c $x^2 + 5x - 2$

9 If $x = -4$, $y = 0.5$ and $z = -1$, evaluate each expression.

a $xyz + (x + y + z)^2$

b $xy^2 - yz^2 - xz^2$

c $x + y \div z - xyz + xy$

10 Simplify each expression by expanding and collecting like terms.

a $(a + b)^2 - (a - b)^2$

b $x^2 - y^2 - (x - y)^2$

c $(p^2 - q^3)^2$

11 Simplify each expression.

a $\frac{3x - 6}{2x + 4} \times \frac{4x + 8}{9x - 9}$

b $\frac{x^2 - 1}{x^4 - 1}$

c $\frac{x^2 - 7x + 12}{x^2 - 9} \div \frac{x^2 - 6x + 8}{x^2 + x - 6}$

12 Completely factorise each expression.

a $(d - 1)^2 - 3(d - 1)$

b $x(y + 3) - 2y^2 - 6y$

c $x^4 - y^4$

13 Give the value of k which would make each expression a perfect square.

a $x^2 - 3x + k$

b $x^2 + 5x + k$

c $x^2 + 7x + k$

14 What is $(2x^2 - 3y^3)^2$ equivalent to?

A $4x^2 - 12x^2y^3 + 9y^3$

B $4x^4 - 6x^2y^3 + 9y^6$

C $4x^4 - 12x^2y^3 - 9y^6$

D $4x^4 - 12x^2y^3 + 9y^6$

15 If $(x-2)^2$ has a value of 9, what possible values could x take?

16 Simplify each expression.

a $\frac{2x-3}{9}+\frac{3-2x}{6}$

b $\frac{5}{x-1}-\frac{4}{2-x}$

c $\frac{4}{xy}+\frac{2}{yz}$

17 Answer true or false to each statement. For any false statements give your reasoning.

a $-3km^2n$, $2m^2kn$ and knm^2 are like terms.

b $a+a^2+a^3$ simplifies to a^6.

c If $p=-1$ and $q=-2$, the value of $q \div p^2$ is -2.

18 A triangle has an area of (x^2-9) cm^2. Suggest possible expressions for the triangle's base and height.

19 Each statement is false. Change the right side to make the statement true.

a $(m^{-3}n^{-2})^{-1}=\frac{1}{m^3n^2}$

b $pq^2r^{-1} \div q^{-2}r = p$

c $a^{b+1}=a^b+a$

20 What can x^2-2x-8 be factorised as?

A $(x-4)(x-2)$ **B** $(x+4)(x-2)$

C $(x-4)(x+2)$ **D** $(x+4)(x+2)$

21 Completely factorise each expression.

a $16x^2-48x+36$

b $k^2-\frac{1}{4}$

c $ab-3b^2-6ab+2a^2$

22 Simplify each expression, giving your answer with positive indices in the base indicated.

a $\frac{8^2 \times 4^3}{2^5}$ (base 2)

b $27 \times 3^{-4} \div 9^2$ (base 3)

c $125 \div 5^{-2} \times 625$ (base 5)

23 Find a value for a, where $\frac{a}{2}+\frac{4x-3}{6}=\frac{2x-3}{3}$.

24 Simplify each expression.

a $\frac{4}{x-1}+\frac{3}{x+1}$ **b** $\frac{1}{2-3x}-\frac{2}{3x-2}$

10A **25** Simplify each expression.

a $\frac{4x^2-11x-3}{8x^2+22x+5} \times \frac{16x^2+24x-40}{4x^2-36}$

b $\frac{6x^2+19x+10}{3x^2-10x-8} \div \frac{25-4x^2}{2x^2-13x+20}$

10A **26** What does $-2x^2+14x-13$ factorise to?

A $2\left(x+\frac{7}{2}+\frac{\sqrt{23}}{2}\right)\left(x+\frac{7}{2}-\frac{\sqrt{23}}{2}\right)$

B $-2\left(x+\frac{7}{2}+\frac{\sqrt{23}}{2}\right)\left(x+\frac{7}{2}-\frac{\sqrt{23}}{2}\right)$

C $2\left(x-\frac{7}{2}+\frac{\sqrt{23}}{2}\right)\left(x-\frac{7}{2}-\frac{\sqrt{23}}{2}\right)$

D $-2\left(x-\frac{7}{2}+\frac{\sqrt{23}}{2}\right)\left(x-\frac{7}{2}-\frac{\sqrt{23}}{2}\right)$

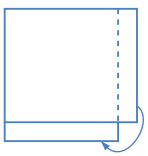
ANALYSIS

a Factorise the expression x^2+bx.

b This factorised expression can represent the area of a rectangle. What are the dimensions of the rectangle?

c This figure can represent the rectangle. Label each line segment with its pronumeral length.

d Imagine the small rectangle on the right is cut in half vertically, and one half is moved below to a horizontal position. Label the pronumeral length of each line segment.



e A shape can be added to the lower right corner, making the total shape that of a square. Describe this added shape, and give its dimensions.

f An expression for the total area of the large square with this added shape is $\left(x+\frac{b}{2}\right)^2$.

Find the sum of the individual areas in the larger square, and complete this expression:

$\left(x+\frac{b}{2}\right)^2 =$ ____ + ____ + ____ + ____

g Simplify the expression on the right side.

h Describe how this illustrates the method of completing the square.

CONNECT

Who let the geese out?

John has enough materials for 40 m of fencing to make a rectangular enclosure for his geese. He needs to work out the best dimensions for the enclosure so that the geese have the largest area possible to walk around in. He decides to use algebra to help him.

Your task

Follow these steps to complete this investigation. Show all relevant diagrams and working.

- Show that the area of the enclosure is $(20x - x^2)$ m^2 if one side of the rectangle is x m long.
- Advise John on the dimensions to use for constructing the geese enclosure so that it has the largest area possible. Clearly show your reasoning. (Hint: Use the factorised form of the expression for area. You may like to use a spreadsheet to help you.)
- Calculate the area of the enclosure using these dimensions.
- Work out how the dimensions and hence the area would change if a gate that is 1 m wide is included in the design.
- If John is offered a shed for the geese which is 3 m long and 2 m wide, investigate the best use of his fencing so that the geese still have the largest area possible to roam around in and can access the shed. (Assume the 1-m wide gate is included.)
- A feed box is in the shape of an open rectangular prism. Its length is twice the width and the sum of the height and width is 75 cm.
  - Assign a pronumeral for the width of the feed box and use this to write algebraic expressions for the length and height of the box.
  - Write an expression for the volume of feed the box can hold and the inner surface area. Show each in simplified factor form.
- Find the dimensions of the feed box so it has the largest volume possible. What is the inner surface area of this box?
- Find the dimensions of the feed box that meets the conditions above and has the smallest inner surface area. What volume does this box have?

As an extension, consider this problem.

- Three rectangular enclosures have areas of $(x^2 - 4)$ m^2, $(x^2 - 4x + 4)$ m^2 and $(-x^2 + 3x + 4)$ m^2. Which enclosure has the largest area? Start by finding expressions for the length and width of each rectangle. What values can x have? Include all necessary diagrams and working to justify your answer.

You may like to present your findings as a report. Your report could include:

- a booklet containing calculations, diagrams and plans
- a poster showing labelled diagrams and plans
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

3 REAL NUMBERS

ESSENTIAL QUESTION

How are rational and irrational numbers represented and how can different operations be applied to them?

Are you ready?

3A 10A **1** Which of these numbers is a perfect square?

A 8 **B** $\sqrt{2}$ **C** 144 **D** 128

3A 10A **2** Calculate each of these values.

a $\sqrt{81}$ **b** $\sqrt[3]{8}$ **c** $\sqrt{256}$ **d** $\sqrt[4]{81}$

3A 10A **3** Represent the recurring decimal $0.\overline{13}$ to 12 decimal places.

3A 10A **4** Represent each recurring decimal in an appropriate shorthand form.

a 0.562 626 262…

b 12.580 13 013 013…

c 45.454 545 454…

d 0.841 284 128 412…

3A 10A **5** A right-angled triangle has its smaller side lengths marked a and b and its hypotenuse length marked c. Which is the correct statement for Pythagoras' Theorem for this triangle?

A $a + b = c$ **B** $a^2 + b^2 = c^2$

C $c = a^2 + b^2$ **D** $a^2 = c^2 + b^2$

3B 10A **6** Calculate the area of each shape.

a square with side length 12 cm

b rectangle with length 15.5 m and width 5 m

c triangle with base length 25 mm and perpendicular height 14 mm

d circle with diameter 10 cm

3C 10A **7** State the prime factors in index form for each number.

a 50 **b** 72 **c** 48 **d** 100

3D 10A **8** What is the expression $5(x - 6)$ equivalent to?

A $5x - 30$ **B** $5x - 11$

C $5x + 30$ **D** $-25x$

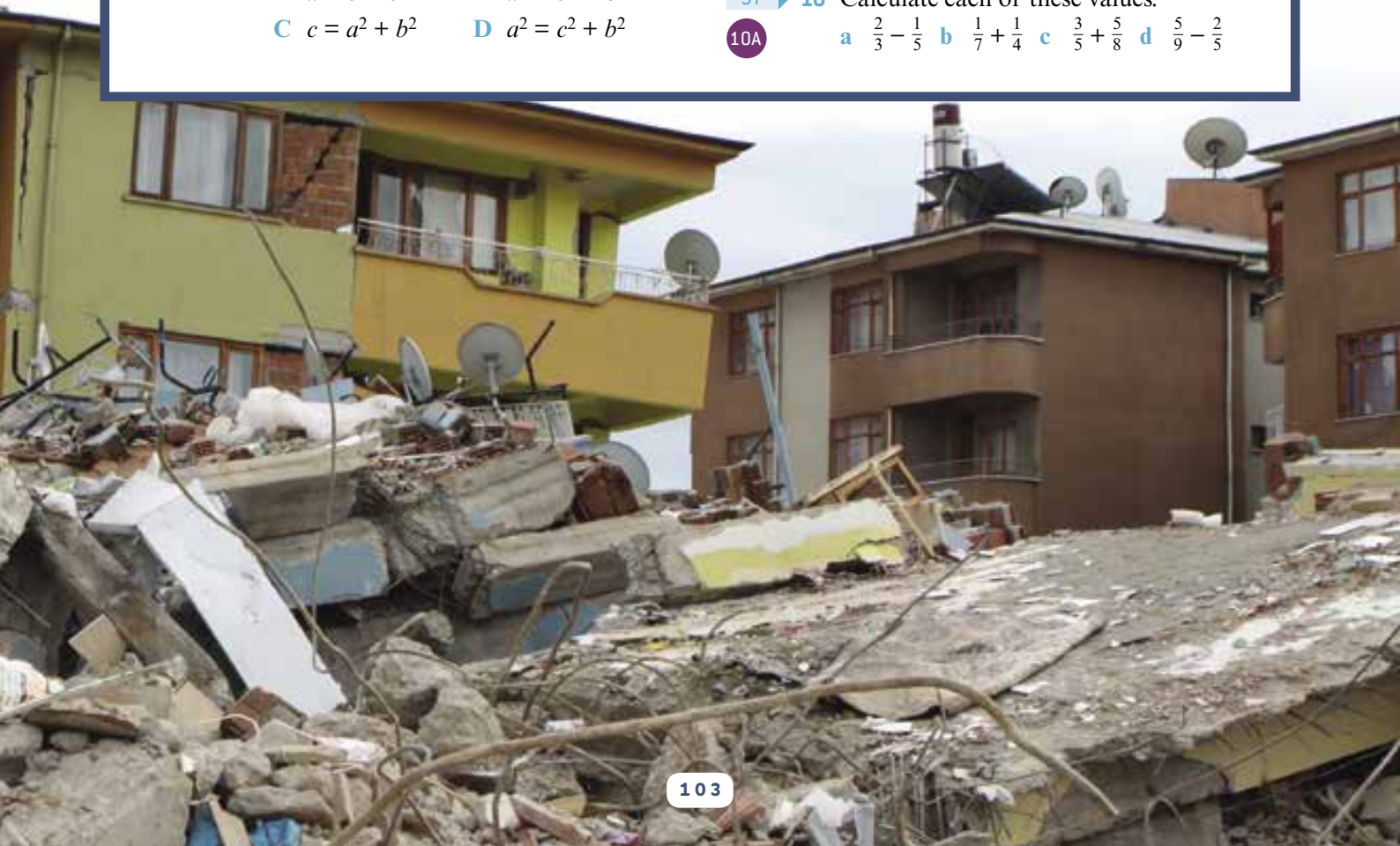
3D 10A **9** Expand each expression to remove the brackets.

a $(x + 2)(x - 7)$ **b** $(x + 6)(x + 6)$

c $(x - 4)(x + 4)$ **d** $(3x - 7)^2$

3F 10A **10** Calculate each of these values.

a $\frac{2}{3} - \frac{1}{5}$ **b** $\frac{1}{7} + \frac{1}{4}$ **c** $\frac{3}{5} + \frac{5}{8}$ **d** $\frac{5}{9} - \frac{2}{5}$



3A Understanding rational and irrational numbers

Start thinking!

The number system you have worked with up to now is called the **real number system**. Real numbers are divided into two main groups, **rational numbers** and **irrational numbers**. Rational numbers can be written exactly as fractions and irrational numbers cannot be.

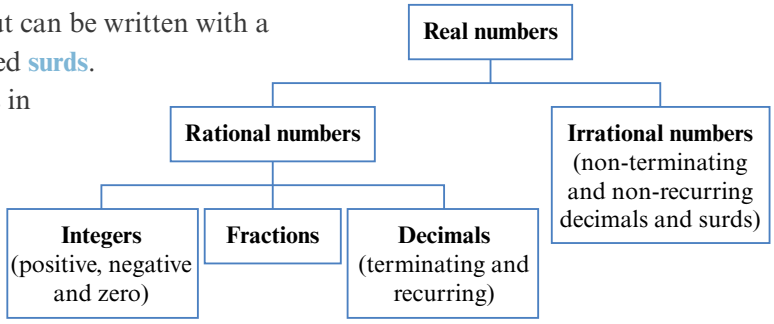
1 **a** Give some examples that show a whole number can be classified as a rational number.
b Give some examples that show how any **terminating decimal** can be classified as a rational number.
c Give some examples that show how any **non-terminating decimal** that is not a recurring decimal can be classified as an irrational number.

2 **a** Use your calculator to write a decimal approximation for the number $\sqrt{5}$. Write the number to seven decimal places.
b Repeat part **a** for each of these numbers. **i** $\sqrt{3}$ **ii** $\sqrt{6}$ **iii** $\sqrt{2}$ **iv** $\sqrt{10}$
c Calculate the value of each of these numbers. **i** $\sqrt{4}$ **ii** $\sqrt{36}$ **iii** $\sqrt{81}$ **iv** $\sqrt{100}$
d What is the difference between your answers to part **b** and your answers to part **c**?

3 **a** Briefly explain why the answers to question **2a** and **b** are all examples of irrational numbers.
b Why aren't the answers to question **2c** irrational numbers?

4 Numbers that cannot be calculated exactly but can be written with a root sign, for example, $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{5}$, … are called **surds**. Surds are irrational numbers. Which numbers in question **2** are surds and which are not?

5 The real number system can be displayed visually, as shown in this figure. Copy the diagram and provide at least two examples for each of the classifications.



KEY IDEAS

- The real number system is made up of rational numbers and irrational numbers.
- Rational numbers are numbers that can be written exactly as fractions.
- Whole numbers (integers), fractions, terminating decimals and recurring decimals are examples of rational numbers.
- Irrational numbers are numbers that cannot be written exactly as fractions.
- Surds are numbers that can only be written exactly by using a root sign.
- Surds and non-terminating decimals that are not recurring are examples of irrational numbers.

EXERCISE 3A Understanding rational and irrational numbers

1 Match each number below with a term from the list A–F.

a $\frac{3}{4}$	**A** recurring decimal
b 45. 710 326 58…	**B** integer
c -9	**C** terminating decimal
d $\sqrt{23}$	**D** fraction
e 0.154 715 471 547…	**E** surd
f 6.213 764 29	**F** non-terminating and non-recurring decimal

EXAMPLE 3A-1 Identifying rational and irrational numbers

Determine whether each number is rational or irrational.

a 0.26 **b** $-\frac{2}{3}$ **c** $-\sqrt{11}$

d $\sqrt{16}$ **e** $2.3\overline{54}$ **f** 6.584 381 257…

THINK

a 0.26 is a terminating decimal. Terminating decimals can be written exactly as a fraction ($0.26 = \frac{26}{100}$).

b Fractions are rational numbers, so $-\frac{2}{3}$ is rational.

c $-\sqrt{11}$ cannot be calculated exactly ($\approx -3.316\,624\,79\ldots$). It is a non-terminating and non-recurring decimal which cannot be written as a fraction, so $-\sqrt{11}$ is irrational.

d $\sqrt{16}$ can be calculated exactly ($\sqrt{16} = 4$). Whole numbers are rational numbers, so $\sqrt{16}$ is rational.

e $2.3\overline{54}$ represents a recurring decimal (2.354 545 454…). Recurring decimals are rational numbers, so $2.3\overline{54}$ is rational.

f 6.584 381 257… is a non-terminating number. The decimal place values do not have a pattern of repeating digits, so 6.584 381 257… is irrational.

WRITE

a 0.26 is a rational number.

b $-\frac{2}{3}$ is a rational number.

c $-\sqrt{11}$ is an irrational number.

d $\sqrt{16}$ is a rational number.

e $2.3\overline{54}$ is a rational number.

f 6.584 381 257... is an irrational number.

2 Determine whether each number is rational or irrational.

a $\frac{4}{5}$	**b** -5	**c** $\sqrt{3}$
d $\sqrt{64}$	**e** 2.548 247 001…	**f** 5.684
g $-\frac{4}{9}$	**h** $-3\sqrt{5}$	**i** $5\frac{1}{6}$
j -1.5	**k** $\sqrt{9}$	**l** $\sqrt{8}$
m $\sqrt[3]{125}$	**n** 0.001 002 142…	**o** $\sqrt[4]{32}$

EXAMPLE 3A-2 Identifying surds

Determine which of these numbers are surds.

a $\sqrt{9}$ b $\sqrt{2}$ c $2\sqrt{6}$ d $4\sqrt{25}$

THINK

a $\sqrt{9}$ can be calculated exactly ($\sqrt{9} = 3$), so $\sqrt{9}$ is not a surd.

b $\sqrt{2}$ cannot be calculated exactly ($\approx 1.414\,213\,562\ldots$), so $\sqrt{2}$ is a surd.

c $2\sqrt{6} = 2 \times \sqrt{6}$. Since $\sqrt{6}$ cannot be calculated exactly ($\approx 2.449\,489\,743\ldots$), $2\sqrt{6}$ also cannot be calculated exactly. So, $2\sqrt{6}$ is a surd.

d $4\sqrt{25} = 4 \times \sqrt{25}$. $\sqrt{25}$ can be calculated exactly ($\sqrt{25} = 5$). $4\sqrt{25} = 4 \times 5 = 20$, which is rational. So, $4\sqrt{25}$ is not a surd.

WRITE

a $\sqrt{9}$ is not a surd.

b $\sqrt{2}$ is a surd.

c $2\sqrt{6}$ is a surd.

d $4\sqrt{25}$ is not a surd.

UNDERSTANDING AND FLUENCY

3 Determine which of these numbers are surds.

a $\sqrt{21}$ b $\sqrt{12}$ c $3\sqrt{3}$ d $-\sqrt{16}$ e $2\sqrt{4}$ f $-\sqrt{6}$

g $\frac{1}{2}\sqrt{9}$ h $5\sqrt{5}$ i $\sqrt[4]{64}$ j $-\sqrt[3]{27}$ k $\sqrt[3]{3}$ l $-2\sqrt{11}$

m $\frac{\sqrt{100}}{5}$ n $\frac{\sqrt{6}}{5}$ o $\sqrt{5} - 1$ p $3 + \sqrt{3}$ q $\frac{\sqrt{6}}{\sqrt{36}}$ r $\frac{2\sqrt{2}}{4} - \sqrt{2}$

4 For each of the surds in question **3**, use your calculator to write a decimal approximation rounded to four decimal places.

5 Which irrational numbers in question **2** are surds?

6 Calculate each value without using a calculator.

a $\sqrt{121}$ b $5\sqrt{36}$ c $-4\sqrt{64}$ d $-\frac{\sqrt{16}}{4}$

e $4\sqrt{16} + \sqrt{64}$ f $\sqrt{36} + \sqrt{64}$ g $\sqrt{36 + 64}$ h $\sqrt{9 + 16}$

i $\sqrt{25 + 144}$ j $-\sqrt{9} - \frac{\sqrt{36}}{2}$ k $2\sqrt{100} - \frac{3\sqrt{144}}{4}$ l $\sqrt{9^2}$

m $4\sqrt{7^2}$ n $(\sqrt{36})^2$ o $\sqrt{256} - 4\sqrt{16}$ p $-\frac{5\sqrt{4}}{8} + \frac{\sqrt{81}}{5}$

7 Which of these are rational numbers and which are irrational?

a $-\sqrt{81}$ b $\frac{2}{3}$ c $\sqrt{0.64}$ d $\sqrt{0.036}$

e $3.358\,294\,721\ldots$ f 8.4 g $-\frac{1}{7}$ h $-3\sqrt{0.004}$

i $\sqrt{1.369}$ j $4.435\,353\,535\ldots$ k $\sqrt{20.25}$ l $\sqrt[3]{100}$

m $\sqrt[3]{0.125}$ n $5.250\,052\,255\ldots$ o $\sqrt[3]{0.27}$ p $\sqrt{75}$

UNDERSTANDING AND FLUENCY

8 Of the irrational numbers in question **7**, which are surds?

9 State whether each number is rational or irrational.

a $\sqrt{\frac{4}{16}}$ **b** $\sqrt{\frac{1}{12}}$ **c** $\frac{\sqrt{16}}{\sqrt{5}}$ **d** $\sqrt{\frac{1}{16}}$

e $\frac{2\sqrt{6}}{\sqrt{4}}$ **f** $\sqrt{\frac{1}{2}}$ **g** $\frac{\sqrt{6}}{\sqrt{25}}$ **h** $\sqrt{\frac{7}{49}}$

10 Pi (symbol π) is a number commonly used in mathematics.

a Is π rational or irrational? Provide a reason to support your answer.

b Using the figure in 3A Start thinking! (page 104) that classifies numbers within the real number system, where would π be best placed?

11 Use your calculator to find the value of each expression, correct to four decimal places.

a $\sqrt{5} + \sqrt{6}$ **b** $2\sqrt{3} + \sqrt{11}$ **c** $\sqrt{30} - \sqrt{15}$

d $4\sqrt{8} - 2\sqrt{10}$ **e** $\sqrt{7} \times \sqrt{5}$ **f** $4\sqrt{3} \times 3\sqrt{7}$

g $2\sqrt{24} \div \sqrt{8}$ **h** $(2\sqrt{7} - 4\sqrt{3})^2$ **i** $\frac{5\sqrt{6}}{\sqrt{7}}$

j $\frac{5}{4\sqrt{11}}$ **k** $\frac{4\sqrt{2}}{2 + \sqrt{3}}$ **l** $\frac{3 - \sqrt{5}}{\sqrt{3} + 5}$

PROBLEM SOLVING AND REASONING

12 Consider the two fractions $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$.

a Use your calculator to determine the approximate value for each fraction to six decimal places.

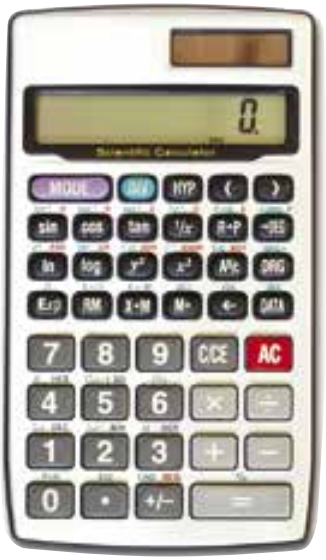
b Are the two fractions equivalent?

c Use your results to comment on the relationship between fractions written in the form $\frac{\sqrt{a}}{\sqrt{b}}$ and the form $\sqrt{\frac{a}{b}}$.

d Is $\sqrt{\frac{5}{8}}$ a rational number or an irrational number? Provide a reason to support your answer.

e Write each fraction in the form $\sqrt{\frac{a}{b}}$ and hence decide if it is a rational or irrational number.

i $\frac{\sqrt{32}}{\sqrt{2}}$ **ii** $\frac{\sqrt{15}}{\sqrt{3}}$ **iii** $\frac{\sqrt{7}}{\sqrt{28}}$



13 **a** Calculate $\sqrt{4}$ and $\sqrt{9}$.

b Between which two whole numbers will $\sqrt{5}$ lie?

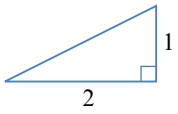
c How does a knowledge of **perfect squares** help in determining which two whole numbers a surd will lie between?

d Between which two whole number values will each of these surds lie?

i $\sqrt{11}$ **ii** $-\sqrt{20}$ **iii** $\sqrt{131}$

iv $-\sqrt{48}$ **v** $\sqrt{180}$ **vi** $\sqrt{300}$

14 As you have seen, the decimal value obtained from a calculator for a surd is an approximation, while the surd represents the exact value.

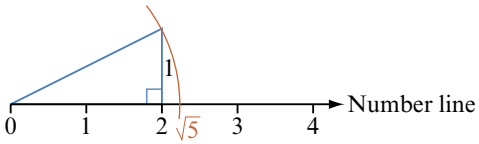


a Use **Pythagoras' Theorem** to show that the exact length of the hypotenuse in this diagram is $\sqrt{5}$.

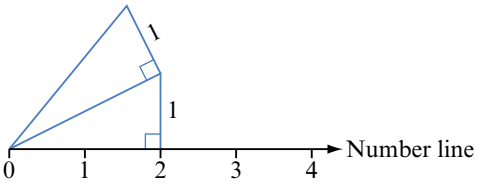
b Write a decimal approximation for the length of the hypotenuse to two decimal places.

15 Pythagoras' Theorem can be applied to determine the exact position of a surd on the number line. Using the surd $\sqrt{5}$ from the previous question, the figure shows how to construct the right-angled triangle on the number line.

a Using a pair of compasses, place the compass point at 0 and open the arms to meet the end of the hypotenuse. Draw an arc to meet the number line as shown. This point represents the exact position of $\sqrt{5}$. Accurately copy this diagram.



b Add a new triangle to the previous triangle as shown. Use Pythagoras' Theorem to show that the length of the new hypotenuse is $\sqrt{6}$.



c Repeat part **a** to determine the exact position of $\sqrt{6}$ on the number line.

16 Using the information shown in question **15**, show how you would represent these values on the number line:

a $\sqrt{2}$ and $\sqrt{3}$ **b** $\sqrt{10}$, $\sqrt{11}$ and $\sqrt{12}$ **c** $\sqrt{26}$ and $\sqrt{27}$.

17 A square has an area of 64 cm^2.

a What is the length of each side of the square?

b Calculate the length of the diagonal of the square:

i as an exact value **ii** rounded to two decimal places.

18 Repeat question **17** for a square with these areas.

a 35 cm^2 **b** 51 mm^2 **c** 95 cm^2

19 **a** Multiply the side length of each square in questions **17** and **18** by $\sqrt{2}$. Compare these results with the diagonal lengths you obtained. What do you notice?

b Use Pythagoras' Theorem to prove that the diagonal of any square is always $\sqrt{2}$ multiplied by the side length.

20 Show that each of these decimals is a rational number by writing them as a fraction in simplest form.

a 0.4 **b** 0.65 **c** 0.49 **d** 1.6 **e** 2.57 **f** 0.851

PROBLEM SOLVING AND REASONING

21 You have seen that recurring decimals are rational numbers. For example, $0.\dot{6}$ is equivalent to $\frac{2}{3}$. You can verify that $\frac{2}{3} = 0.666\,666\ldots$ by performing $2 \div 3$ on a calculator. To write a recurring decimal as a fraction (and to show it is rational), you can follow a series of steps.

a Match each Step **1–6** shown below with its corresponding Line of working **A–F** in the calculation below. Write the lines of working in the correct order.

Steps

1 Let x represent the recurring decimal.

2 Multiply the decimal by 10 to write an equation for $10x$.

3 Form a new equation from the two equations for x and $10x$. Subtract the terms on the left side of each equation and subtract the numbers on the right side of each equation.

4 Simplify the resulting equation.

5 Divide both sides of the equation by 9 to obtain x on the left side and a fraction on the right side of the equation.

6 Write the recurring decimal as a fraction in simplest form.

Lines of working

A $0.\dot{6} = \frac{2}{3}$

B $x = \frac{6}{9}$

C $10x = 6.666\,666\ldots$

D Let $x = 0.666\,666\ldots$

E $10x - x$
$= 6.666\,666\ldots - 0.666\,666\ldots$

F $9x = 6$

b Why did you multiply x by 10 in the calculation above? How did this help you obtain a whole number on the right side of the equation in step **4**?

22 Use the method shown in question **21a** to write each recurring decimal as a fraction. Check your answers with a calculator.

a $0.\dot{2}$ **b** $0.\dot{7}$ **c** $2.\dot{3}$ **d** $5.\dot{6}$

23 **a** Copy and complete the working shown at right to write $0.\overline{36}$ as a fraction.

b Explain why multiplying x by 100 was helpful in this case. (Hint: did you obtain a whole number on the right side of the equation in step 4?)

Let $x = 0.363\,636\ldots$
$100x =$ _____
$100x - x =$ _____ $- 0.363\,636\ldots$
$99x =$ _____
$x = \frac{\quad}{99}$
So $0.\overline{36} = \frac{\quad}{11}$

24 Use the method shown in question **23a** to write each recurring decimal as a fraction.

a $0.\overline{14}$ **b** $0.\dot{2}\dot{1}$ **c** $0.\overline{48}$ **d** $1.\dot{7}\dot{3}$

CHALLENGE

25 Write each recurring decimal as a fraction. Show all working.

a $0.333\,333\,333\ldots$ **b** $0.\overline{24}$
c $0.\overline{785}$ **d** $0.4\dot{1}$ **e** $0.5\dot{6}$
f $3.\overline{18}$ **g** $4.2\dot{5}$ **h** $3.5\overline{126}$

Reflect

What are some similarities and some differences between rational and irrational numbers?

3B Multiplying and dividing surds

Start thinking!

You can perform operations with surds just like with other types of numbers. In this task, you will investigate how to multiply and divide surds. You will need a calculator to perform your initial calculations. Ensure your calculator can show surds as exact values rather than decimal approximations.

1 Consider the multiplication of two surds such as $\sqrt{5} \times \sqrt{7}$.

a Use your calculator to determine the result. Your calculator should display the result as a surd.

b Repeat part **a** for each of these expressions. **i** $\sqrt{2} \times \sqrt{5}$ **ii** $\sqrt{3} \times \sqrt{7}$

2 Observe your results for each calculation in question **1**. Use your results to explain how two surds in the form $\sqrt{a} \times \sqrt{b}$ can be multiplied without using a calculator.

3 Now consider $3\sqrt{2} \times 4\sqrt{7}$.

a Enter the calculation into your calculator to determine the result.

b Repeat part **a** for each of these expressions. **i** $3\sqrt{6} \times \sqrt{5}$ **ii** $2\sqrt{7} \times 4\sqrt{11}$

4 Look at your results for each calculation in question **3**. Use your results to explain how two surds in the form $a\sqrt{b} \times c\sqrt{d}$ can be multiplied without using a calculator.

5 Consider the division of two surds such as $\sqrt{45} \div \sqrt{15}$.

a Enter the calculation into your calculator to determine the result.

b Repeat part **a** for each of these expressions. **i** $\sqrt{55} \div \sqrt{11}$ **ii** $\dfrac{\sqrt{55}}{\sqrt{5}}$

6 Look at your results for each calculation in question **5**. Use your results to explain how two surds in the form $\sqrt{a} \div \sqrt{b}$ (or $\dfrac{\sqrt{a}}{\sqrt{b}}$) can be divided without using a calculator.

7 Now consider $\dfrac{8\sqrt{15}}{4\sqrt{3}}$.

a Carefully enter the calculation into your calculator to determine the result.

b Repeat part **a** for each of these expressions. **i** $\dfrac{6\sqrt{18}}{2\sqrt{6}}$ **ii** $\dfrac{15\sqrt{21}}{5\sqrt{7}}$

8 Look at your results for each calculation in question **7**. Use your results to explain how two surds in the form $\dfrac{a\sqrt{b}}{c\sqrt{d}}$ (or $a\sqrt{b} \div c\sqrt{d}$) can be divided without using a calculator.

KEY IDEAS

- When multiplying surds, multiply the surd parts and write the product beneath the square root sign, and multiply any whole number parts separately. Write the whole number part first followed by the surd part. $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ and $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$
- Similarly, when dividing surds, divide the surd parts and divide the whole number parts. $\sqrt{a} \div \sqrt{b} = \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{a \div b}$ and $a\sqrt{b} \div (c\sqrt{d}) = \dfrac{a\sqrt{b}}{c\sqrt{d}} = (a \div c) \times \sqrt{b \div d}$

EXERCISE 3B Multiplying and dividing surds

EXAMPLE 3B-1 Multiplying two surds

Calculate each value, writing your answer in surd form.

a $\sqrt{6} \times \sqrt{5}$ b $5\sqrt{3} \times 8\sqrt{11}$

THINK

a Identify the values under the root sign and multiply them.

b 1 Multiply the surd parts and multiply the whole number parts.

2 Write the product of the whole number parts first and the product of the surd parts under the root sign.

WRITE

a $\sqrt{6} \times \sqrt{5}$
$= \sqrt{6 \times 5}$
$= \sqrt{30}$

b $5\sqrt{3} \times 8\sqrt{11}$
$= 5 \times 8 \times \sqrt{3} \times \sqrt{11}$
$= 40 \times \sqrt{3 \times 11}$
$= 40 \times \sqrt{33}$
$= 40\sqrt{33}$

UNDERSTANDING AND FLUENCY

Do not use a calculator for the questions in this Exercise.

1 Calculate each value, writing your answer in surd form.

a $\sqrt{7} \times \sqrt{5}$ b $\sqrt{3} \times \sqrt{11}$ c $\sqrt{2} \times -\sqrt{7}$ d $-\sqrt{15} \times \sqrt{7}$

e $\sqrt{13} \times \sqrt{5}$ f $-\sqrt{2} \times \sqrt{11}$ g $\sqrt{7} \times \sqrt{3}$ h $\sqrt{19} \times \sqrt{5}$

i $\sqrt{7} \times 3\sqrt{11}$ j $5\sqrt{3} \times 4\sqrt{2}$ k $-4\sqrt{2} \times 7\sqrt{11}$ l $12\sqrt{6} \times -3\sqrt{5}$

m $11\sqrt{2} \times 6\sqrt{17}$ n $8\sqrt{6} \times 8\sqrt{7}$ o $4\sqrt{3} \times -6\sqrt{11}$ p $12\sqrt{3} \times 11\sqrt{5}$

2 Calculate each value.

a $\sqrt{7} \times \sqrt{7}$ b $\sqrt{3} \times \sqrt{3}$ c $\sqrt{2} \times \sqrt{2}$ d $\sqrt{6} \times \sqrt{6}$

e $\sqrt{5} \times \sqrt{5}$ f $-\sqrt{11} \times -\sqrt{11}$ g $\sqrt{10} \times \sqrt{10}$ h $\sqrt{9} \times \sqrt{9}$

3 What did you notice about each result in question 2?

4 Calculate each value, writing your answer in surd form where appropriate.

a $3\sqrt{3} \times \sqrt{3}$ b $-\sqrt{11} \times 7\sqrt{11}$ c $4\sqrt{2} \times 5\sqrt{2}$ d $6\sqrt{5} \times 2\sqrt{5}$

e $\sqrt{5} \times 3\sqrt{2}$ f $15\sqrt{6} \times 2\sqrt{5}$ g $-4\sqrt{2} \times 8\sqrt{13}$ h $12\sqrt{3} \times \frac{1}{4}\sqrt{11}$

i $2\sqrt{12} \times 7\sqrt{3}$ j $\frac{1}{2}\sqrt{5} \times 3\sqrt{20}$ k $\frac{3}{4}\sqrt{13} \times -8\sqrt{2}$ l $\frac{3}{5}\sqrt{7} \times \frac{5}{3}\sqrt{15}$

m $-12\sqrt{6} \times -6\sqrt{7}$ n $\frac{1}{2}\sqrt{15} \times \frac{3}{4}\sqrt{13}$ o $15\sqrt{10} \times -\frac{1}{3}\sqrt{7}$ p $-10\sqrt{17} \times 1.1\sqrt{5}$

EXAMPLE 3B-2 Dividing two surds

Calculate each value, writing your answer in surd form.

a $\sqrt{42} \div \sqrt{6}$ b $\frac{36\sqrt{60}}{9\sqrt{12}}$

THINK

a Identify the values under the root sign and divide them.

b Divide the surd parts and divide the whole number parts.

WRITE

a $\sqrt{42} \div \sqrt{6}$
$= \sqrt{42 \div 6}$
$= \sqrt{7}$

b $\frac{36\sqrt{60}}{9\sqrt{12}}$
$= \frac{36 \times \sqrt{60}}{9 \times \sqrt{12}}$
$= (36 \div 9) \times \sqrt{60 \div 12}$
$= 4 \times \sqrt{5}$
$= 4\sqrt{5}$

UNDERSTANDING AND FLUENCY

5 Calculate each value, writing your answer in surd form.

a $\sqrt{34} \div \sqrt{2}$ b $\sqrt{24} \div \sqrt{8}$ c $\sqrt{15} \div \sqrt{15}$ d $\sqrt{15} \div -\sqrt{3}$

e $\frac{\sqrt{42}}{\sqrt{7}}$ f $\frac{\sqrt{56}}{\sqrt{8}}$ g $4\sqrt{18} \div \sqrt{3}$ h $22\sqrt{40} \div (11\sqrt{8})$

i $\frac{-24\sqrt{35}}{8\sqrt{7}}$ j $\frac{16\sqrt{50}}{8\sqrt{10}}$ k $-12\sqrt{60} \div 3\sqrt{6}$ l $8\sqrt{55} \div (16\sqrt{11})$

6 Calculate each value.

a $\sqrt{32} \div \sqrt{2}$ b $\sqrt{24} \div \sqrt{6}$ c $\sqrt{54} \div \sqrt{6}$ d $\sqrt{40} \div \sqrt{10}$

7 What did you notice about each result in question **6**?

8 Calculate each expression, writing your answer in surd form.

a $12\sqrt{15} \div -\sqrt{3}$ b $35\sqrt{24} \div (7\sqrt{8})$ c $-12\sqrt{48} \div (4\sqrt{24})$

d $\frac{18\sqrt{35}}{9\sqrt{7}}$ e $\frac{-32\sqrt{42}}{-8\sqrt{6}}$ f $\frac{35\sqrt{5}}{-15}$

g $\frac{-24}{-8\sqrt{3}}$ h $\sqrt{240} \div (5\sqrt{48})$ i $\frac{12\sqrt{27}}{\sqrt{432}}$

9 For the calculation $6\sqrt{3} \times 3\sqrt{2}$, show that the result is $18\sqrt{6}$.

10 Consider $\frac{24\sqrt{40}}{8\sqrt{20}}$. By dividing the whole number parts and dividing the surd parts, show that the result is $3\sqrt{2}$.

11 Consider the fraction $\frac{48}{56}$.

- **a** Briefly explain the method you would use to simplify this fraction.
- **b** Simplify the fraction.
- **c** Look at the surd fraction in question **10**.
  - **i** What is the highest common factor of the whole number parts in the numerator and the denominator?
  - **ii** What is the highest common factor of the surd parts in the numerator and the denominator?
- **d** Cancel the common factors of the whole number parts and the surd parts of the fraction and show that the result is the same as you obtained in question **10**.

12 Show that $\frac{5\sqrt{24}}{\sqrt{6}}$ is *not* the same as $\frac{\sqrt{24}}{5\sqrt{6}}$.

13 Calculate each expression, writing your answer in fractional surd form.

Remember that $6\sqrt{15} \div \sqrt{35}$ is equivalent to $\frac{6\sqrt{15}}{\sqrt{35}}$.

- **a** $6\sqrt{15} \div \sqrt{15}$
- **b** $\frac{12\sqrt{30}}{16\sqrt{6}}$
- **c** $44\sqrt{34} \div (28\sqrt{42})$
- **d** $\frac{-20\sqrt{75}}{25\sqrt{45}}$
- **e** $\frac{36\sqrt{35}}{16\sqrt{7}}$
- **f** $\frac{22\sqrt{5}}{-33\sqrt{5}}$
- **g** $\frac{8\sqrt{3} \times 4}{-\sqrt{3}}$
- **h** $\frac{1}{2}\sqrt{15} \div (4\sqrt{8})$
- **i** $9\sqrt{32} \div 18\sqrt{8}$
- **j** $\frac{4\sqrt{3} \times 6}{3\sqrt{6}}$
- **k** $\frac{9\sqrt{2} \times 5}{\sqrt{18}}$
- **l** $\frac{8\sqrt{21}}{12\sqrt{14}}$

14 Simplify each expression.

- **a** $\frac{\sqrt{54}}{\sqrt{6}}$
- **b** $\sqrt{\frac{81}{100}}$
- **c** $\sqrt{32} \times 3\sqrt{2}$
- **d** $\frac{\sqrt{3}}{\sqrt{27}}$
- **e** $5\sqrt{12} \times -2\sqrt{3}$
- **f** $\sqrt{\frac{36}{9}}$
- **g** $7\sqrt{8} \times \sqrt{18}$
- **h** $4\sqrt{6} \times \sqrt{8} \times 6\sqrt{12}$
- **i** $8\sqrt{45} \div 5\sqrt{5}$
- **j** $\frac{\sqrt{32}}{16\sqrt{8}}$
- **k** $\frac{-4\sqrt{75}}{15\sqrt{3}}$
- **l** $\frac{-5\sqrt{343}}{15\sqrt{7}}$

15 Simplify each fraction, writing your answer in fraction form where appropriate.

- **a** $\frac{3\sqrt{5} \times 4\sqrt{6}}{6\sqrt{3}}$
- **b** $\frac{5\sqrt{7} \times \sqrt{6}}{25\sqrt{14}}$
- **c** $\frac{12\sqrt{48}}{3\sqrt{2} \times 5\sqrt{6}}$
- **d** $\frac{11\sqrt{132}}{4\sqrt{11} \times -2\sqrt{6}}$
- **e** $\frac{8\sqrt{3} \times 2\sqrt{8}}{6\sqrt{6} \times \sqrt{2}}$
- **f** $\frac{4\sqrt{5} \times -3\sqrt{24}}{5\sqrt{10} \times 6\sqrt{2}}$
- **g** $\frac{\sqrt{3} \times \sqrt{15} \times \sqrt{8}}{\sqrt{6} \times 9\sqrt{12}}$
- **h** $\frac{4\sqrt{12} \times 3\sqrt{10} \times -6\sqrt{7}}{5\sqrt{8} \times 8\sqrt{15}}$
- **i** $\frac{-5\sqrt{6} \times 2\sqrt{28} \times -\sqrt{44}}{3\sqrt{21} \times -4\sqrt{18}}$

16 Squaring a number is multiplying the number by itself. Find the value of each expression.

a $(\sqrt{5})^2$ **b** $(\sqrt{8})^2$ **c** $(\sqrt{2})^2$

d $(\sqrt{7})^2$ **e** $(\sqrt{9})^2$ **f** $(\sqrt{15})^2$

g $(\sqrt{25})^2$ **h** $(\sqrt{75})^2$ **i** $(4\sqrt{5})^2$

j $(3\sqrt{6})^2$ **k** $(8\sqrt{3})^2$ **l** $(9\sqrt{11})^2$

17 Study the answer to each part of question **16**.

a Does the square of a surd in the form $\sqrt{a}$ or $a\sqrt{b}$ result in a rational or an irrational answer?

b Prove that your answer to part **a** will always be true.

18 Simplify each expression.

a $(\sqrt{x})^2$ **b** $(\sqrt{ab})^2$

c $\sqrt{6y} \times \sqrt{6y}$ **d** $\sqrt{x+y} \times \sqrt{x+y}$

19 Find the value of each expression.

a $\left(\frac{5}{\sqrt{7}}\right)^2$ **b** $\left(\frac{1}{\sqrt{5}}\right)^2$

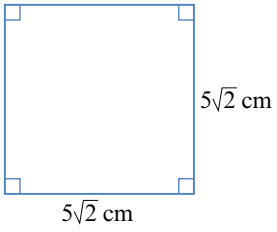
c $\left(-\frac{3}{\sqrt{6}}\right)^2$ **d** $\left(\frac{\sqrt{10}}{\sqrt{13}}\right)^2$

20 For each shape, calculate:

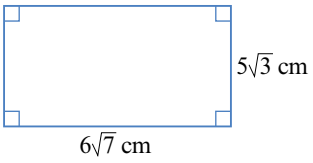
i the area, correct to two decimal places

ii the exact area.

a



b



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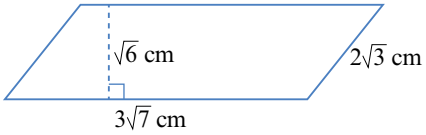

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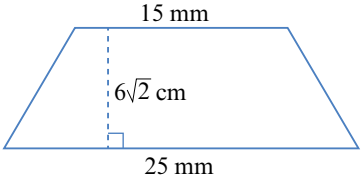

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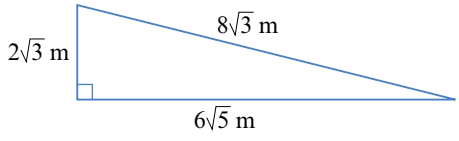

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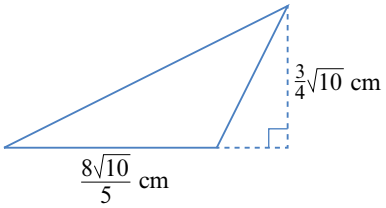


PROBLEM SOLVING AND REASONING

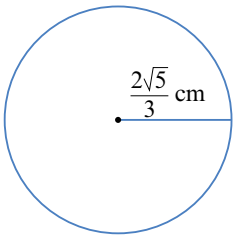
21 The glass in a ship's porthole has a diameter of $5\sqrt{6}$ cm.

- **a** What is the radius of the glass panel?
- **b** Calculate the exact area of the glass panel.
- **c** Briefly explain why the area of the glass panel is not a surd value and is not classified as a rational number.

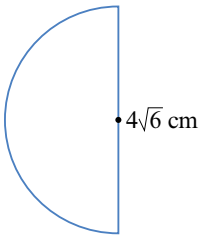
22 For each shape, calculate:

- **i** the area, correct to two decimal places
- **ii** the exact area.

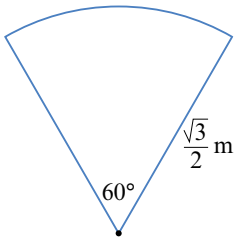
a



b



c



23 A laptop screen has an area of $182\sqrt{15}$ cm^2 and a height of $13\sqrt{3}$ cm. What is the width of the screen?

24 A circle has an area of 6π cm^2. Calculate the exact radius of the circle and the diameter.

25 A semi-circle has an area of $\frac{5\pi}{32}$ m^2.
Calculate its exact diameter.

26 A square has a diagonal length of $6\sqrt{6}$ cm. Calculate its area.

CHALLENGE

27 Reconsider the sector in question **22c**. Calculate the exact perimeter of this figure.

28 Consider a square with dimensions expressed in surd form of the type $\sqrt{a}$. Is it possible that the value representing the perimeter can be the same as the value representing the area of this square? Explain your answer.

Reflect

How are the rules for multiplying surds similar to the rules for dividing surds?

10A

3C Simplifying surds

Start thinking!

1 Consider $\sqrt{5} \times \sqrt{8}$.

a Explain how you can multiply two surds without using a calculator.

b Use your method to multiply these two surds, expressing your answer as a surd.

c Now use your calculator to multiply the two surds. How does this answer compare with your answer to part **b**?

Your answer to part **b** may have differed from the answer on your calculator. When performing operations with surds, most calculators show the answer in simplest form.

d Briefly explain how you could show your answer in part **b** is equivalent to the answer in part **c**.

2 Look at the multiplication from question **1**: $\sqrt{5} \times \sqrt{8} = \sqrt{5 \times 8} = \sqrt{40}$.

You have seen that this result can be simplified. Now investigate how this can be done.

a Copy and complete the following to write $\sqrt{40}$ as a product of two factors. Which factor pair do you think should be used to obtain the simplest form seen earlier?

$\sqrt{40} = \sqrt{\_\_ \times \_\_}$

$= \sqrt{\_\_} \times \sqrt{\_\_}$

b Repeat the method from part **a** to simplify each surd.

i $\sqrt{12}$ **ii** $\sqrt{18}$ **iii** $\sqrt{20}$ **iv** $\sqrt{32}$

c Check each answer in part **b** with your calculator.

d The method in part **a** involved writing a surd as the product of two factors. Review each of the surds simplified in parts **a** and **b**. What do you notice about one of the factors in each of the factor pairs chosen?

3 Why do you think an understanding of perfect squares is essential when working with simplifying surds?

KEY IDEAS

- In most calculations, surds need to be written in simplest form.
- To write a surd in simplest form, write the number under the root sign as a product of two factors. For the first factor use the highest possible perfect square.

EXERCISE 3C Simplifying surds

UNDERSTANDING AND FLUENCY

1 Calculate each value.

a $\sqrt{16}$ b $\sqrt{81}$ c $\sqrt{49}$ d $\sqrt{4}$

e $\sqrt{100}$ f $\sqrt{25}$ g $3\sqrt{36}$ h $4\sqrt{121}$

i $\sqrt{1}$ j $\frac{1}{2}\sqrt{16}$ k $\frac{1}{4}\sqrt{144}$ l $3\sqrt{9}$

2 Simplify each expression.

a $\sqrt{81} \times \sqrt{7}$ b $\sqrt{16} \times \sqrt{3}$ c $\sqrt{4} \times \sqrt{6}$

d $\sqrt{25} \times \sqrt{10}$ e $\sqrt{9} \times \sqrt{2}$ f $\sqrt{5} \times \sqrt{36}$

g $2 \times \sqrt{16} \times \sqrt{11}$ h $3 \times \sqrt{100} \times \sqrt{5}$ i $\frac{1}{2}\sqrt{144} \times \sqrt{10}$

3 Copy and complete each calculation to simplify the given surd.

a $\sqrt{45} = \sqrt{9} \times \sqrt{\_\_}$
$= \_\_ \times \sqrt{\_\_}$
$= \_\_\_\_\_$

b $\sqrt{20} = \sqrt{\_\_} \times \sqrt{5}$
$= \_\_ \times \sqrt{5}$
$= \_\_\_\_\_$

c $\sqrt{8} = \sqrt{\_\_} \times \sqrt{\_\_}$
$= 2 \times \sqrt{\_\_}$
$= \_\_\_\_\_$

EXAMPLE 3C-1 Simplifying a surd in $\sqrt{a}$ form

Write $\sqrt{54}$ in simplest form.

THINK

1 Write the number under the root sign as a product of two factors. For the first factor use the highest possible perfect square.

2 Calculate the exact part ($\sqrt{9} = 3$) and simplify the surd.

WRITE

$\sqrt{54} = \sqrt{9 \times 6}$
$= \sqrt{9} \times \sqrt{6}$
$= 3 \times \sqrt{6}$
$= 3\sqrt{6}$

4 Write each value in simplest form.

a $\sqrt{28}$ b $\sqrt{50}$ c $\sqrt{24}$ d $\sqrt{27}$

e $\sqrt{48}$ f $\sqrt{75}$ g $\sqrt{63}$ h $\sqrt{60}$

i $\sqrt{56}$ j $\sqrt{45}$ k $\sqrt{128}$ l $\sqrt{108}$

m $\sqrt{180}$ n $\sqrt{147}$ o $\sqrt{162}$ p $\sqrt{125}$

q $\sqrt{450}$ r $\sqrt{288}$ s $\sqrt{270}$ t $\sqrt{567}$

EXAMPLE 3C-2 Simplifying a surd in $a\sqrt{b}$ form

Write $4\sqrt{72}$ in simplest form.

THINK

1 Identify the parts of the surd. Write the number under the root sign as a product of two factors, with the first factor being the highest possible perfect square.

2 Calculate the exact part ($\sqrt{36} = 6$).

3 Multiply the whole number parts and simplify the surd.

WRITE

$4\sqrt{72} = 4 \times \sqrt{72}$
$= 4 \times \sqrt{36 \times 2}$
$= 4 \times \sqrt{36} \times \sqrt{2}$
$= 4 \times 6 \times \sqrt{2}$
$= 24 \times \sqrt{2}$
$= 24\sqrt{2}$

UNDERSTANDING AND FLUENCY

5 Write each value in simplest form.

a $5\sqrt{20}$	b $3\sqrt{8}$	c $6\sqrt{12}$	d $9\sqrt{18}$
e $4\sqrt{80}$	f $3\sqrt{90}$	g $12\sqrt{150}$	h $-3\sqrt{48}$
i $7\sqrt{96}$	j $-2\sqrt{245}$	k $8\sqrt{192}$	l $10\sqrt{252}$
m $-5\sqrt{112}$	n $7\sqrt{99}$	o $-12\sqrt{68}$	p $100\sqrt{300}$
q $\frac{1}{2}\sqrt{320}$	r $\frac{1}{2}\sqrt{216}$	s $8\sqrt{248}$	t $\frac{3}{4}\sqrt{176}$

6 Show that the surd $5\sqrt{27}$ simplifies to $15\sqrt{3}$.

7 Write each value in simplest form.

a $\sqrt{43}$	b $\sqrt{125}$	c $\sqrt{120}$	d $\sqrt{72}$
e $\sqrt{150}$	f $-\sqrt{216}$	g $\sqrt{126}$	h $\sqrt{240}$
i $-3\sqrt{18}$	j $\sqrt{245}$	k $\frac{1}{2}\sqrt{192}$	l $\frac{\sqrt{108}}{3}$
m $-2\sqrt{48}$	n $15\sqrt{80}$	o $3\sqrt{84}$	p $\frac{3}{5}\sqrt{450}$
q $-8\sqrt{800}$	r $-\frac{2}{3}\sqrt{900}$	s $-\sqrt{1000}$	t $\frac{\sqrt{288}}{4}$
u $\frac{5\sqrt{1372}}{14}$	v $\frac{-\sqrt{384}}{2}$	w $\frac{7\sqrt{1280}}{32}$	x $\frac{3}{4}\sqrt{648}$

8 Calculate each value, writing your answer in simplest surd form.

a $2\sqrt{5} \times 3\sqrt{8}$	b $4\sqrt{6} \times 2\sqrt{10}$	c $-4\sqrt{3} \times 7\sqrt{15}$	d $\sqrt{8} \times \sqrt{12}$
e $-7\sqrt{10} \times -6\sqrt{15}$	f $\frac{1}{2}\sqrt{12} \times \frac{3}{4}\sqrt{14}$	g $5\sqrt{20} \times -6\sqrt{30}$	h $5\sqrt{48} \times 4\sqrt{24}$

UNDERSTANDING AND FLUENCY

9 A surd in which the entire number is written beneath the root sign can be called an **entire surd**. For example, $\sqrt{6}$ can be classified as an entire surd, but $3\sqrt{6}$ cannot.

a Copy and complete the following to write $3\sqrt{6}$ as an entire surd.

$$\begin{aligned} 3\sqrt{6} &= \sqrt{3^2} \times \sqrt{6} \\ &= \sqrt{} \times \sqrt{6} \\ &= \sqrt{ \times 6} \\ &= \sqrt{} \end{aligned}$$

b Consider the first line of working in part **a**. Why do you think the whole number part of the surd is written as a perfect square beneath the root sign?

c How do you think the initial surd ($3\sqrt{6}$) and the answer (the entire surd) are related to each other?

d When writing a surd as an entire surd, briefly explain how you could check that your answer is correct.

e Follow the working from part **a** to write $4\sqrt{3}$ as an entire surd.

10 Write each value as an entire surd.

a $3\sqrt{5}$	**b** $2\sqrt{3}$	**c** $4\sqrt{2}$	**d** $6\sqrt{7}$
e $4\sqrt{10}$	**f** $5\sqrt{11}$	**g** $8\sqrt{6}$	**h** $5\sqrt{5}$
i $4\sqrt{15}$	**j** $-8\sqrt{2}$	**k** $12\sqrt{7}$	**l** $-5\sqrt{3}$

PROBLEM SOLVING AND REASONING

11 **a** Write $a\sqrt{b}$ as an entire surd.

b Check that the general form of the entire surd obtained in part **a** produces the same answer for each surd in question **10**.

c In your own words, explain how a surd is expressed as an entire surd.

12 Write each expression as an entire surd.

a $m\sqrt{n}$	**b** $7\sqrt{a}$	**c** $x\sqrt{5}$	**d** $10\sqrt{xy}$
e $d\sqrt{2c}$	**f** $3\sqrt{4abc}$	**g** $a\sqrt{ab^2}$	**h** $xy\sqrt{x^2yz^2}$

13 Write each expression in simplest form.

a $\sqrt{x^2y}$	**b** $\sqrt{x^2y^2}$	**c** $\sqrt{ab^2}$	**d** $\sqrt{x^2yz^2}$
e $\sqrt{x^4}$	**f** $\sqrt{x^4y}$	**g** $\sqrt{x^4y^2}$	**h** $\sqrt{a^4b^2c}$

14 An alternative method to simplify a surd involves writing the number under the root sign as a product of its prime factors.
Consider the surd $\sqrt{12}$.

a Write 12 as a product of its prime factors in index form.

b Write the product of prime factors under the root sign and show that it simplifies to $2\sqrt{3}$.

15 When using the method shown in question **14**, you will notice that not all numbers under the root sign result in perfect squares, but often they can be written differently so that they contain perfect squares. Consider $\sqrt{54}$.

- **a** Write 54 as a product of its prime factors in index form.
- **b** Use your answer to part **a** to show that $\sqrt{54}$ has an equivalent form of $\sqrt{3^2 \times 3 \times 2}$.
- **c** Simplify your answer and use your calculator to check that it is correct.

16 Use the methods from questions **14** and **15** to simplify each value.

a $\sqrt{18}$	**b** $\sqrt{28}$	**c** $\sqrt{32}$
d $\sqrt{27}$	**e** $\sqrt{24}$	**f** $\sqrt{50}$
g $3\sqrt{20}$	**h** $2\sqrt{128}$	**i** $5\sqrt{63}$
j $8\sqrt{40}$	**k** $7\sqrt{48}$	**l** $6\sqrt{45}$
m $\sqrt{180}$	**n** $5\sqrt{147}$	**o** $6\sqrt{162}$
p $\sqrt{125}$	**q** $2\sqrt{450}$	**r** $\sqrt{288}$

17 When you write the number under the root sign as a product of its prime factors in index form, what is the common feature of all surds for which the prime factors are all different?

18 Consider $3\sqrt{392} \times 5\sqrt{192}$.

- **a** Multiply the whole number parts and multiply the surd parts, then write the surd in simplest form.
- **b** Simplify each surd first and then perform the multiplication.
- **c** Do the methods in parts **a** and **b** produce the same answer?
- **d** When do you think it is best to use each method?

19 Calculate each value, writing your answer as a fraction in simplest surd form.

a $3\sqrt{48} \div \sqrt{6}$	**b** $\dfrac{2\sqrt{54}}{6\sqrt{3}}$	**c** $15\sqrt{104} \div (25\sqrt{8})$
d $\dfrac{54\sqrt{75}}{63\sqrt{15}}$	**e** $\dfrac{-21\sqrt{84}}{6\sqrt{7}}$	**f** $\dfrac{4\sqrt{300}}{3\sqrt{25}}$
g $\dfrac{8\sqrt{960}}{-\sqrt{48}}$	**h** $\sqrt{384} \div (5\sqrt{64})$	**i** $\dfrac{-\sqrt{504}}{\sqrt{54}}$

20 Calculate each value, writing your answer in simplest surd form.

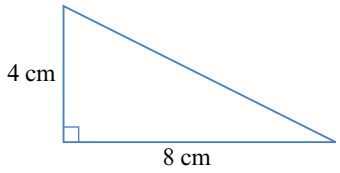
a $\dfrac{22\sqrt{48}}{8\sqrt{2} \times 4\sqrt{3}}$	**b** $\dfrac{4\sqrt{10} \times 6\sqrt{24}}{2\sqrt{5} \times 3\sqrt{2}}$
c $\dfrac{3\sqrt{8} \times 7\sqrt{12}}{12\sqrt{6} \times -8\sqrt{2}}$	**d** $\dfrac{2\sqrt{15} \times 4\sqrt{12} \times \sqrt{10}}{\sqrt{2} \times 5\sqrt{12}}$

21 Simplify $\dfrac{a\sqrt{bc^2} \times b^2\sqrt{ac}}{c\sqrt{ab^2}}$.

PROBLEM SOLVING AND REASONING

22 Use Pythagoras' Theorem to calculate the exact length of the unknown side in each triangle. Where appropriate, write the answer in simplest surd form.

a



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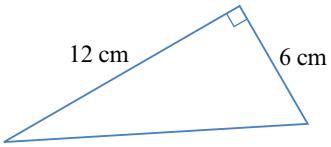

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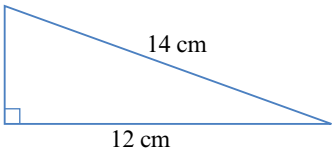

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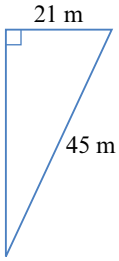
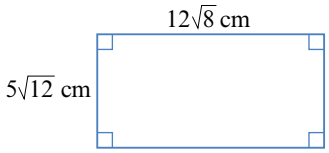


23 For each of these shapes calculate the exact area, written in simplest surd form.

a



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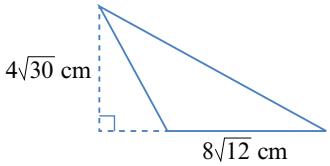

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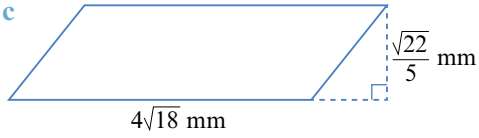
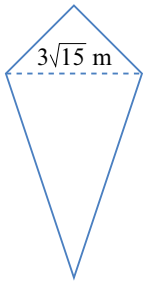


d



24 The circular face of a pocket watch has a circumference of $\frac{4\pi\sqrt{24}}{3}$ cm.

a What is the radius of the watch face, in simplest surd form?

b Calculate the exact area of the watch face.

CHALLENGE

25 Find at least three different surd values that simplify to $4\sqrt{5}$.

26 Find at least three surd values or expressions that simplify to $8\sqrt{7}$.

Reflect

Why is a knowledge of perfect squares so important when simplifying surds?

3D Adding and subtracting surds

Start thinking!

1 Consider $3x + 2y - 2x$.

- **a** How many terms are in the expression?
- **b** Using your knowledge of algebra, explain how you write expressions such as this in simplest form.
- **c** Simplify the expression.

2 Now consider $3\sqrt{5} + 2\sqrt{3} - 2\sqrt{5}$.

- **a** Use your calculator to simplify the expression.
- **b** Compare your answer to part **a** with your answer to question **1c**. What do you notice?
- **c** Compare the expression from question **1** with the surd expression in question **2**. What similarities do you notice?

3 How do you think simplifying surds involving addition and subtraction is similar to the methods for simplifying algebraic expressions involving addition and subtraction? You may wish to review your responses to questions **1** and **2** to help with your explanation.

You may recall from algebra that only like terms can be added or subtracted. Similarly, when working with surds, only **like surds** can be added or subtracted.

4 Now consider $4\sqrt{27} + 2\sqrt{8} - \sqrt{12}$.

- **a** How many surds are in this expression?
- **b** At first glance, do these surds appear to be like surds?
- **c** Write each surd in the expression in simplest form.
- **d** What do you notice about the expression after you simplified each surd term? How many of the surds are now like surds?
- **e** Simplify the expression and check your answer with the result you obtain when you enter the original expression into your calculator. Are the two results the same?

KEY IDEAS

- Like surds contain exactly the same number under the root sign, regardless of the whole number part.
- Only like surds can be added or subtracted.
- Write surds in their simplest form before adding and subtracting.

EXERCISE 3D Adding and subtracting surds

1 State whether each pair are like surds or not.

a $3\sqrt{5}$ and $\sqrt{5}$

b $4\sqrt{2}$ and $2\sqrt{4}$

c 3 and $\sqrt{3}$

d $-3\sqrt{6}$ and $6\sqrt{6}$

e $5\sqrt{5}$ and $2\sqrt{2}$

f $3\sqrt{7}$ and $3\sqrt{10}$

g $2\sqrt{2}$ and $-\sqrt{2}$

h $2\sqrt{2}$ and $\sqrt{8}$

EXAMPLE 3D-1 Adding and subtracting surds

Simplify each expression.

a $6\sqrt{5} + 3\sqrt{5} - \sqrt{5}$ b $4\sqrt{3} - 2\sqrt{2} - 6\sqrt{3} + 5\sqrt{2}$

THINK

a **1** The three surds in the expression are like surds because they contain exactly the same number under the root sign.

2 Simplify the expression by adding and subtracting the whole number parts.

b **1** Reorganise the terms by grouping the like surds together.

2 Simplify the like terms in the expression by adding and subtracting the whole number parts.

WRITE

a $6\sqrt{5} + 3\sqrt{5} - \sqrt{5}$
$= 6\sqrt{5} + 3\sqrt{5} - 1\sqrt{5}$
$= (6 + 3 - 1)\sqrt{5}$
$= 8\sqrt{5}$

b $4\sqrt{3} - 2\sqrt{2} - 6\sqrt{3} + 5\sqrt{2}$
$= 4\sqrt{3} - 6\sqrt{3} - 2\sqrt{2} + 5\sqrt{2}$
$= (4 - 6)\sqrt{3} + (-2 + 5)\sqrt{2}$
$= -2\sqrt{3} + 3\sqrt{2}$

UNDERSTANDING AND FLUENCY

2 Simplify each expression by adding (or subtracting) the like terms. (Hint: look for like surds.)

a $2\sqrt{3} + \sqrt{2} + 5\sqrt{3}$

b $4\sqrt{3} - 2\sqrt{2} + \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$

d $6\sqrt{13} + 4\sqrt{39} - \sqrt{3} + 2\sqrt{13}$

e $9\sqrt{2} - \sqrt{2} - 6\sqrt{5} + 5\sqrt{2}$

f $3\sqrt{6} + 4\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6} - 4$

3 Simplify each expression.

a $2\sqrt{3} + \sqrt{3} + 5\sqrt{3}$

b $3\sqrt{2} - 2\sqrt{2} + \sqrt{2} - 5\sqrt{2}$

c $-3\sqrt{7} - 3\sqrt{7} - \sqrt{6}$

d $9\sqrt{5} - \sqrt{3} - 4\sqrt{3} + 6\sqrt{5} - 2\sqrt{3}$

e $6\sqrt{2} - 4\sqrt{2} - 4\sqrt{2} + 2\sqrt{2}$

f $3\sqrt{10} - 4\sqrt{2} - 6\sqrt{3} + 5\sqrt{5}$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$

h $7\sqrt{21} - 6\sqrt{3} + 5\sqrt{3} - 6\sqrt{21}$

i $6\sqrt{6} + 3\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6}$

j $-3\sqrt{15} - 2\sqrt{7} - \sqrt{7} - 4\sqrt{15}$

4 Write each value in simplest form.

a $\sqrt{32}$ **b** $\sqrt{125}$ **c** $6\sqrt{54}$ **d** $-6\sqrt{18}$

e $-5\sqrt{80}$ **f** $4\sqrt{128}$ **g** $12\sqrt{243}$ **h** $5\sqrt{245}$

5 Identify the like surds in each list of surds.

a $2\sqrt{3}, \sqrt{2}, 4\sqrt{2}, 2\sqrt{5}, 3\sqrt{2}$ **b** $3\sqrt{7}, 14\sqrt{2}, 7\sqrt{3}, 2\sqrt{14}$

c $-3\sqrt{6}, 3\sqrt{12}, \sqrt{3}, \sqrt{18}, 2\sqrt{27}$ **d** $9\sqrt{32}, 4\sqrt{8}, \sqrt{128}, 5\sqrt{200}$

EXAMPLE 3D-2 Adding surds after first simplifying

Simplify $4\sqrt{45} + 5\sqrt{80}$.

THINK

1 Simplify each surd in the expression.

2 Rewrite the expression with the simplified surds. As they are like surds, add the whole number parts and simplify.

WRITE

$4\sqrt{45} = 4 \times \sqrt{45}$
$= 4 \times \sqrt{9 \times 5}$
$= 4 \times \sqrt{9} \times \sqrt{5}$
$= 4 \times 3 \times \sqrt{5}$
$= 12\sqrt{5}$

$5\sqrt{80} = 5 \times \sqrt{80}$
$= 5 \times \sqrt{16 \times 5}$
$= 5 \times \sqrt{16} \times \sqrt{5}$
$= 5 \times 4 \times \sqrt{5}$
$= 20\sqrt{5}$

$4\sqrt{45} + 5\sqrt{80}$
$= 12\sqrt{5} + 20\sqrt{5}$
$= (12 + 20)\sqrt{5}$
$= 32\sqrt{5}$

UNDERSTANDING AND FLUENCY

6 Write each term in these expressions in simplest form and then simplify the expression.

a $5\sqrt{27} + \sqrt{12} - 8\sqrt{48}$ **b** $2\sqrt{20} - 4\sqrt{80} + 2\sqrt{45}$

c $\sqrt{28} - 3\sqrt{112} + 2\sqrt{63}$ **d** $3\sqrt{24} - 7\sqrt{54} - \sqrt{96} + 4\sqrt{20}$

e $4\sqrt{108} - 5\sqrt{75} - 6\sqrt{48} + \sqrt{8}$ **f** $\sqrt{128} - 5\sqrt{27} + 2\sqrt{32} - 3\sqrt{75}$

7 Simplify each expression.

a $5\sqrt{12} + \sqrt{27} - 8\sqrt{3}$ **b** $2\sqrt{8} + 4\sqrt{32} + 3\sqrt{2}$

c $\sqrt{7} - 3\sqrt{28} + 5\sqrt{7}$ **d** $3\sqrt{45} - \sqrt{5} - 2\sqrt{80} + 2\sqrt{20}$

e $\sqrt{108} - 4\sqrt{27} - 2\sqrt{12} + 2\sqrt{75}$ **f** $3\sqrt{24} - 4\sqrt{54} + 5\sqrt{24}$

g $3\sqrt{18} - 4\sqrt{50} - \sqrt{8} + 2\sqrt{98}$ **h** $2\sqrt{75} - 6\sqrt{3} + 2\sqrt{48} - 6\sqrt{12}$

i $3\sqrt{12} - \sqrt{28} + 5\sqrt{8} - 2\sqrt{20}$ **j** $6\sqrt{54} + \sqrt{45} - 5\sqrt{96} - 2\sqrt{27}$

UNDERSTANDING AND FLUENCY

8 Simplify each expression. You may like to refer to the example shown at right.

a $5\sqrt{20} + 4\sqrt{80} + 3\sqrt{28}$

b $4\sqrt{36} - 3\sqrt{48} + 2\sqrt{12}$

c $3\sqrt{24} - 4\sqrt{125} - 2\sqrt{216} + \sqrt{45}$

d $\sqrt{63} + 7\sqrt{27} - 2\sqrt{28} + 8\sqrt{75}$

e $\sqrt{8} - 4\sqrt{108} + 5\sqrt{45} + 3\sqrt{288}$

f $-4\sqrt{300} - 5\sqrt{242} - 2\sqrt{8} + 7\sqrt{27}$

g $2\sqrt{72} + 6\sqrt{48} + 2\sqrt{128} - 6\sqrt{192}$

h $3\sqrt{75} - \sqrt{150} + 5\sqrt{80} - 5\sqrt{96} + \sqrt{180}$

$$
\begin{aligned}
&2\sqrt{12} + 5\sqrt{147} - \sqrt{48} \\
&= 2\sqrt{4 \times 3} + 5\sqrt{49 \times 3} - \sqrt{16 \times 3} \\
&= 2 \times \sqrt{4} \times \sqrt{3} + 5 \times \sqrt{49} \times \sqrt{3} - \sqrt{16} \times \sqrt{3} \\
&= 2 \times 2 \times \sqrt{3} + 5 \times 7 \times \sqrt{3} - 4 \times \sqrt{3} \\
&= 4\sqrt{3} + 35\sqrt{3} - 4\sqrt{3} \\
&= 35\sqrt{3}
\end{aligned}
$$

9 Simplify each expression.

a $\sqrt{180} + 4\sqrt{125} - 3\sqrt{245}$

b $4\sqrt{243} + 3\sqrt{300} - 2\sqrt{432}$

c $3\sqrt{338} - 12\sqrt{147} - 2\sqrt{50} + \sqrt{48}$

d $3\sqrt{216} - 4\sqrt{125} - 2\sqrt{147} + 5\sqrt{45}$

e $\sqrt{63} + 7\sqrt{567} - 2\sqrt{96} + 8\sqrt{588}$

f $\sqrt{200} - 4\sqrt{180} + 5\sqrt{864} - 3\sqrt{160}$

g $-4\sqrt{320} - 25 + \sqrt{245} - 36 + \sqrt{27}$

h $12\sqrt{72} - 6\sqrt{192} + 2\sqrt{128} - 5\sqrt{507}$

PROBLEM SOLVING AND REASONING

10 When simplifying surds, what methods can you use to ensure that each surd is written in simplest form?

11 Simplify each of these surd expressions, which contain pronumerals. You might like to refer to the example shown at right.

a $\sqrt{x} + \sqrt{16x} - \sqrt{25x}$

b $3\sqrt{4x} + 2\sqrt{x} - 5\sqrt{36x}$

c $\sqrt{9x} + \sqrt{12x} + \sqrt{16x} - 2\sqrt{48x}$

d $\sqrt{8xy} - 2\sqrt{32xy} + \frac{1}{3}\sqrt{18xy}$

e $\sqrt{150xy} + 2\sqrt{384xy} - \sqrt{6xy}$

f $\sqrt{24x^2y} + 3x\sqrt{96y} - 5\sqrt{54x^2y}$

g $\sqrt{3xy^2} - 5y\sqrt{12x} + 7\sqrt{27xy^2}$

h $3\sqrt{100x^2y} - \sqrt{147x^2y} + \frac{1}{2}\sqrt{48x^2y} + \sqrt{16x^2y}$

i $2x\sqrt{80xy} + 5\sqrt{12xy} - \sqrt{125x^2y} - 2\sqrt{432x^2y}$

$$
\begin{aligned}
&3\sqrt{27x} + 4\sqrt{x} - \sqrt{25x} + 2\sqrt{12x} \\
&= 3\sqrt{9 \times 3 \times x} + 4\sqrt{x} - \sqrt{25 \times x} + 2\sqrt{4 \times 3 \times x} \\
&= 3 \times 3\sqrt{3x} + 4\sqrt{x} - 5\sqrt{x} + 2 \times 2\sqrt{3x} \\
&= 9\sqrt{3x} + 4\sqrt{x} - 5\sqrt{x} + 4\sqrt{3x} \\
&= 13\sqrt{3x} - \sqrt{x}
\end{aligned}
$$

12 From previous work, you may recall a rule used to expand expressions that contain brackets known as the distributive law.

a Show that expanding the expression $a(b + c)$ using the distributive law produces $ab + ac$.

b Briefly explain how the result of the expansion of the expression in part **a** is obtained.

13 The distributive law can also be applied to expressions involving surds. Expand each expression. Where necessary, write all surds in simplest form.

a $6(2 + \sqrt{3})$ **b** $8(\sqrt{5} - 4)$ **c** $5(4 - 6\sqrt{2})$

d $4(3\sqrt{2} + 2\sqrt{3})$ **e** $2(5\sqrt{6} - 3\sqrt{5})$ **f** $-5\sqrt{7}(2\sqrt{6} - 3\sqrt{10} + 2)$

g $3\sqrt{6}(5\sqrt{3} + 2\sqrt{10})$ **h** $5\sqrt{7}(2\sqrt{7} - 12)$ **i** $6\sqrt{11}(2\sqrt{3} - 5\sqrt{7} + \sqrt{11})$

j $\sqrt{5}(3 - 2\sqrt{5} + 5\sqrt{15})$ **k** $-3\sqrt{3}(4\sqrt{6} - 2\sqrt{10} + 9\sqrt{15})$

14 Use a calculator or other digital technology to check your answers to the expansions in question **13**.

15 Expansion of a binomial product can be performed by repeated use of the distributive law.

a Show that the expansion of the binomial product $(a + b)(c + d)$ produces the expression $ac + ad + bc + bd$.

b Briefly explain how this result is obtained.

16 Binomial products may also contain surds. (Expressions such as $\sqrt{5} + \sqrt{3}$ are referred to as binomial surds. Can you see why?) Expand and simplify each expression.

a $(5 + 3\sqrt{3})(2 + 4\sqrt{6})$ **b** $(3\sqrt{2} + 6)(5\sqrt{5} - 8)$

c $(\sqrt{6} + 2\sqrt{3})(6\sqrt{2} + 5\sqrt{3})$ **d** $(7\sqrt{5} - 3\sqrt{2})(4\sqrt{2} + 8\sqrt{6})$

e $(4\sqrt{8} - 5\sqrt{5})(3\sqrt{3} - 9\sqrt{8})$ **f** $(4\sqrt{10} + 3\sqrt{5})(7\sqrt{2} - 6\sqrt{12})$

17 Use a calculator or other digital technology to check your answers to question **16**.

18 Expand and simplify each of these perfect squares.

a $(5 + \sqrt{6})^2$ **b** $(\sqrt{7} - 3)^2$ **c** $(4 + 5\sqrt{3})^2$ **d** $(2\sqrt{2} - 7)^2$

e $(3\sqrt{5} + 2\sqrt{3})^2$ **f** $(4\sqrt{6} - 3\sqrt{7})^2$ **g** $(7\sqrt{3} - 5\sqrt{8})^2$ **h** $(3\sqrt{10} + 6\sqrt{5})^2$

19 You may have noticed that the result of each expansion in question **18** followed a pattern.

a Show that the expansion of $(a + b)^2$ will always produce the expression $a^2 + 2ab + b^2$.

b Show that the expansion of $(a - b)^2$ will always produce the expression $a^2 - 2ab + b^2$.

c These patterns for expanding perfect squares are known as the perfect square rules. Use these rules to check your answers to question **18**.

20 Expand and simplify each expression.

a $(4 + \sqrt{6})(4 - \sqrt{6})$ **b** $(5\sqrt{5} - 3)(5\sqrt{5} + 3)$

c $(2\sqrt{3} + \sqrt{2})(2\sqrt{3} - \sqrt{2})$ **d** $(4\sqrt{5} + 7\sqrt{3})(4\sqrt{5} - 7\sqrt{3})$

e $(2\sqrt{8} - 6\sqrt{6})(2\sqrt{8} + 6\sqrt{6})$ **f** $(4\sqrt{2} + 8\sqrt{6})(4\sqrt{2} - 8\sqrt{6})$

g $(1 + 3\sqrt{5})(1 - 3\sqrt{5})$ **h** $(9\sqrt{11} - 1)\,(9\sqrt{11} + 1)$

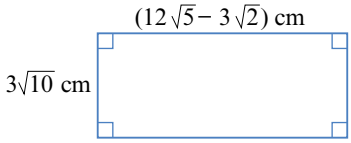
PROBLEM SOLVING AND REASONING

21 Consider each expansion in question **20**.

- **a** What do you notice about the terms inside each pair of brackets?
- **b** Is the result of the expansion a rational or an irrational number? Why do you think this is the case?
- **c** You may have noticed that the result of each expansion followed a pattern. Show that the expansion of binomial products in the form $(a + b)(a - b)$ or $(a - b)(a + b)$ will always be $a^2 - b^2$.
- **d** This pattern is known as the difference of two squares rule. Use this rule to check your answers to question **20**.

22 For each shape calculate the exact area, written in simplest surd form.

a



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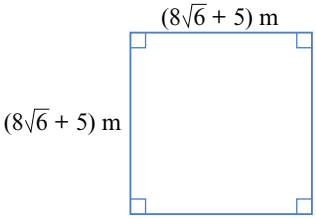

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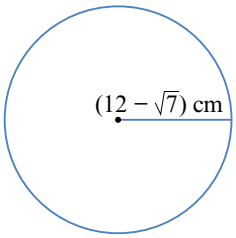

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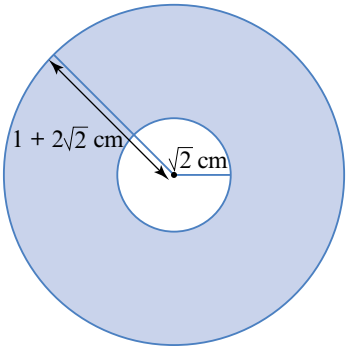
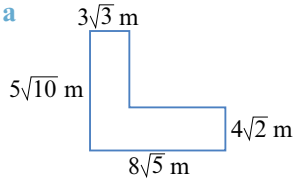


23 Use the information provided to calculate the exact area of each composite shape.

a



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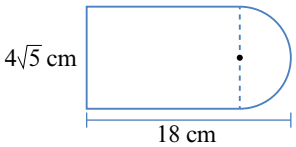

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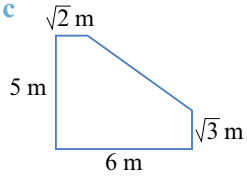


24 Calculate the exact perimeter of each shape in parts **a** and **b** of question **23**.

CHALLENGE

25 A rectangle has a perimeter of $(4\sqrt{5} + 12\sqrt{3})$ m.

- **a** What are some possible dimensions for this rectangle?
- **b** Is it possible that this rectangle could be classified as a square? Provide a reason to support your answer.

26 A rectangle has an area of 6 cm^2.

- **a** Provide at least two different sets of surd values representing possible dimensions for this rectangle.
- **b** For each set in part **a**, state the perimeter of the rectangle.

Reflect

How do the methods for simplifying expressions that have addition and subtraction of surds differ from the methods for those with multiplication and division of surds?

3E Writing surd fractions with a rational denominator

Start thinking!

1 Consider the fraction $\frac{2}{\sqrt{3}}$.

- **a** What is the numerator? Is it a rational or an irrational number?
- **b** What is the denominator? Is it a rational or an irrational number?
- **c** Is the whole fraction a rational or an irrational number? Explain.
- **d** Write the decimal approximation for the fraction, correct to four decimal places.

2 **a** Enter $\frac{2}{\sqrt{3}}$ into your calculator and press the 'enter' or '=' button. What fraction is displayed?

- **b** What is the numerator in this fraction now? Is it a rational or an irrational number?
- **c** What is the denominator? Is it a rational or an irrational number?
- **d** Write the decimal approximation for this fraction, correct to four decimal places.
- **e** How does your answer to part **d** compare with your answer to question **1d**?

The fractions in questions **1** and **2** are equivalent fractions. The fraction in question **2** is a surd fraction with a rational denominator. By convention, surd fractions are always written with a **rational denominator**.

3 **a** Calculate the result for each multiplication. Write each answer in simplest form, where appropriate.

i $\frac{1}{2} \times 1$ **ii** $\frac{3}{4} \times \frac{1}{1}$ **iii** $\frac{3}{4} \times \frac{4}{4}$ **iv** $\frac{2}{5} \times \frac{5}{5}$

- **b** What did you notice about each result in part **a**?
- **c** What is special about the second value in each multiplication?

4 **a** Calculate these results by multiplying the numerators together and multiplying the denominators together.

i $\frac{2}{\sqrt{3}} \times \frac{1}{1}$ **ii** $\frac{2}{\sqrt{3}} \times \frac{3}{3}$ **iii** $\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ **iv** $\frac{2}{\sqrt{3}} \times \frac{2}{2}$

- **b** Compare each result in part **a**. Show that each result is an equivalent fraction.
- **c** Which result in part **a** is a fraction with a rational denominator?
- **d** What operation was performed so that a rational denominator was obtained?

KEY IDEAS

- Fractions that contain surds must be written with a rational denominator.
- To write a surd fraction with a rational denominator:
  - identify the surd in the denominator
  - multiply the numerator and the denominator of the fraction by this surd
  - simplify the resulting fraction.

EXERCISE 3E Writing surd fractions with a rational denominator

1 Which fractions have an irrational denominator?

a $\frac{\sqrt{6}}{5}$ b $\frac{1}{\sqrt{5}}$ c $\frac{3\sqrt{2}}{\sqrt{4}}$ d $\frac{4\sqrt{2}}{\sqrt{7}}$ e $\frac{\sqrt{8}}{8}$ f $\frac{2\sqrt{3}}{\sqrt{3}}$

2 What would you multiply the numerator and the denominator by in each fraction to obtain a rational denominator?

a $\frac{2}{\sqrt{5}}$ b $\frac{5}{\sqrt{3}}$ c $\frac{3}{\sqrt{2}}$ d $\frac{2\sqrt{2}}{\sqrt{6}}$ e $\frac{5}{\sqrt{10}}$ f $\frac{3}{\sqrt{3}}$

EXAMPLE 3E-1 Writing simple surd fractions with a rational denominator

Write each fraction with a rational denominator.

a $\frac{2}{\sqrt{6}}$ b $\frac{7}{2\sqrt{5}}$ c $\frac{2\sqrt{3}}{\sqrt{2}}$

THINK

a **1** Identify the surd in the denominator. Multiply by a fraction that is equivalent to 1 using this surd in the numerator and the denominator.

2 Simplify the resulting fraction.

b **1** Multiply the numerator and the denominator by the surd part in the denominator.

2 Simplify the resulting fraction.

c **1** Multiply the numerator and the denominator by the surd part in the denominator.

2 Simplify the resulting fraction.

WRITE

a $\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{2\sqrt{6}}{\sqrt{36}}$

$= \frac{2\sqrt{6}}{6}$

$= \frac{\sqrt{6}}{3}$

b $\frac{7}{2\sqrt{5}} = \frac{7}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{7\sqrt{5}}{2\sqrt{25}}$

$= \frac{7\sqrt{5}}{2 \times 5}$

$= \frac{7\sqrt{5}}{10}$

c $\frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{6}}{\sqrt{4}}$

$= \frac{2\sqrt{6}}{2}$

$= \sqrt{6}$

3 Write each fraction with a rational denominator.

a $\frac{3}{\sqrt{6}}$ **b** $\frac{2}{\sqrt{7}}$ **c** $\frac{3}{\sqrt{3}}$

d $\frac{4\sqrt{5}}{\sqrt{7}}$ **e** $\frac{5}{3\sqrt{2}}$ **f** $\frac{7}{4\sqrt{5}}$

g $\frac{2\sqrt{3}}{\sqrt{5}}$ **h** $-\frac{5}{2\sqrt{5}}$ **i** $\frac{1}{7\sqrt{7}}$

j $\frac{4\sqrt{11}}{\sqrt{10}}$ **k** $\frac{8\sqrt{5}}{3\sqrt{2}}$ **l** $\frac{9\sqrt{6}}{4\sqrt{5}}$

4 Write each surd fraction in question **2** with a rational denominator.

EXAMPLE 3E-2

Writing a surd fraction with a rational denominator in simplest form

Write $\frac{\sqrt{8}}{\sqrt{5}}$ with a rational denominator. Write the numerator in simplest surd form.

THINK

1 Multiply the numerator and the denominator by the surd part in the denominator.

2 Simplify the denominator.

3 Simplify the surd in the numerator by writing the number under the root sign as a product of two factors, one of which is a perfect square.

WRITE

$$\frac{\sqrt{8}}{\sqrt{5}} = \frac{\sqrt{8}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{\sqrt{40}}{\sqrt{25}}$$
$$= \frac{\sqrt{40}}{5}$$
$$= \frac{\sqrt{4 \times 10}}{5}$$
$$= \frac{\sqrt{4} \times \sqrt{10}}{5}$$
$$= \frac{2\sqrt{10}}{5}$$

5 Write each fraction with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{\sqrt{6}}{\sqrt{3}}$ **b** $\frac{\sqrt{10}}{\sqrt{5}}$ **c** $\frac{2\sqrt{10}}{\sqrt{6}}$

d $\frac{3\sqrt{8}}{\sqrt{3}}$ **e** $\frac{\sqrt{15}}{3\sqrt{5}}$ **f** $\frac{5\sqrt{10}}{7\sqrt{2}}$

g $\frac{2\sqrt{12}}{\sqrt{5}}$ **h** $\frac{\sqrt{14}}{4\sqrt{7}}$ **i** $\frac{\sqrt{20}}{2\sqrt{2}}$

j $-\frac{7\sqrt{15}}{6\sqrt{3}}$ **k** $\frac{\sqrt{2}}{\sqrt{18}}$ **l** $\frac{6\sqrt{12}}{5\sqrt{8}}$

6 Write each fraction with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{\sqrt{6}}{\sqrt{2}}$ **b** $\frac{-\sqrt{15}}{\sqrt{6}}$ **c** $\frac{2\sqrt{8}}{3\sqrt{7}}$

d $\frac{2\sqrt{6}}{\sqrt{8}}$ **e** $\frac{6\sqrt{15}}{7\sqrt{3}}$ **f** $-\frac{3\sqrt{20}}{\sqrt{5}}$

g $\frac{5}{\sqrt{x}}$ **h** $\frac{x}{x\sqrt{y}}$ **i** $\frac{4x}{\sqrt{2y}}$

j $\frac{\sqrt{x}}{x\sqrt{y}}$ **k** $\frac{\sqrt{2y}}{2y\sqrt{x}}$ **l** $\frac{-3\sqrt{x}}{6x\sqrt{yz}}$

7 Expand each expression.

a $4(3+\sqrt{2})$ **b** $6(2-3\sqrt{3})$ **c** $5(\sqrt{3}+4)$

d $\sqrt{2}(\sqrt{5}-2)$ **e** $2\sqrt{5}(\sqrt{3}+3)$ **f** $2\sqrt{7}(3\sqrt{2}-2\sqrt{7})$

8 Consider the fraction $\frac{4}{5-\sqrt{3}}$. In this fraction, the denominator is a binomial surd.

a Multiply the denominator by $5+\sqrt{3}$.

b Is the result in part **a** a rational or an irrational number?

c Notice that you multiplied the denominator by another binomial surd. How are the surds related?

d The binomial surd $5+\sqrt{3}$ is called the **conjugate** of $5-\sqrt{3}$. What type of number will result when any binomial surd is multiplied by its conjugate?

9 State the conjugate for each expression.

a $3+\sqrt{2}$ **b** $2-\sqrt{3}$ **c** $5+\sqrt{6}$

d $4+3\sqrt{5}$ **e** $3\sqrt{5}-2$ **f** $4\sqrt{7}-1$

g $1+2\sqrt{3}$ **h** $\sqrt{2}+\sqrt{3}$ **i** $\sqrt{5}-\sqrt{3}$

j $2\sqrt{5}+5\sqrt{6}$ **k** $3\sqrt{7}-2\sqrt{5}$ **l** $5\sqrt{2}-2\sqrt{5}$

10 Multiply each binomial surd in question **9** by its conjugate. Be sure to check that each answer is a rational number.

11 Each calculation performed in question **10** follows the difference of two squares rule. Show how this rule works to produce a rational answer.

12 Look at the fraction $\frac{4}{5-\sqrt{3}}$ from question **8**. To write a fraction like this with a rational denominator, you multiply the numerator and the denominator by the conjugate of the denominator.

a Write the fraction with a rational denominator.

b Use a calculator to check that your answer to part **a** is equivalent to the original fraction $\frac{4}{5-\sqrt{3}}$.

EXAMPLE 3E-3

Writing a surd fraction with a rational denominator using the conjugate of the denominator

Write $\frac{2}{4+\sqrt{3}}$ with a rational denominator.

THINK

1 Multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of $4+\sqrt{3}$ is $4-\sqrt{3}$.

2 Simplify the fraction by expanding the expression in the numerator and then the expression in the denominator using the difference of two squares rule. Check that you obtain a rational denominator.

WRITE

$$\frac{2}{4+\sqrt{3}} = \frac{2}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}}$$
$$= \frac{2(4-\sqrt{3})}{(4+\sqrt{3})(4-\sqrt{3})}$$
$$= \frac{8-2\sqrt{3}}{16-3}$$
$$= \frac{8-2\sqrt{3}}{13}$$

UNDERSTANDING AND FLUENCY

13 Write each fraction with a rational denominator.

a $\frac{1}{2+\sqrt{3}}$ **b** $\frac{3}{3-\sqrt{2}}$ **c** $\frac{5}{6+\sqrt{5}}$

d $\frac{3}{3-4\sqrt{5}}$ **e** $\frac{\sqrt{2}}{5+3\sqrt{2}}$ **f** $\frac{2\sqrt{3}}{1+2\sqrt{3}}$

g $\frac{3}{\sqrt{5}-1}$ **h** $\frac{3\sqrt{7}}{2\sqrt{6}+5}$ **i** $\frac{-4\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

j $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}$ **k** $\frac{2\sqrt{5}}{3\sqrt{6}+2\sqrt{10}}$ **l** $\frac{6\sqrt{10}}{5\sqrt{2}-2\sqrt{5}}$

14 Expand each expression.

a $(\sqrt{2}+3)(5-2\sqrt{3})$ **b** $(4-2\sqrt{5})(\sqrt{6}+1)$

c $(2\sqrt{7}-3\sqrt{2})(\sqrt{3}+1)$ **d** $(5\sqrt{2}+4\sqrt{3})(3\sqrt{6}-\sqrt{5})$

e $(\sqrt{10}+\sqrt{7})(\sqrt{3}-\sqrt{2})$ **f** $(3\sqrt{5}-2\sqrt{3})(4\sqrt{2}+\sqrt{6})$

15 For each fraction:

i state the conjugate of the denominator

ii write the fraction with a rational denominator.

a $\frac{\sqrt{2}+1}{2\sqrt{3}+3}$ **b** $\frac{5+2\sqrt{3}}{4+\sqrt{2}}$ **c** $\frac{1+\sqrt{3}}{3-\sqrt{5}}$

d $\frac{3+\sqrt{5}}{\sqrt{6}+1}$ **e** $\frac{2\sqrt{3}-\sqrt{2}}{3\sqrt{5}+2\sqrt{1}}$ **f** $\frac{5\sqrt{2}-1}{3\sqrt{3}+2\sqrt{5}}$

g $\frac{4\sqrt{2}-3\sqrt{6}}{2\sqrt{7}-\sqrt{3}}$ **h** $\frac{4\sqrt{2}-\sqrt{6}}{\sqrt{10}-\sqrt{7}}$ **i** $\frac{7\sqrt{3}-2\sqrt{5}}{2\sqrt{5}+\sqrt{3}}$

16 Write each fraction in simplest form with a rational denominator.

a $\dfrac{\sqrt{2}}{3-\sqrt{2}}$ **b** $\dfrac{2\sqrt{5}+1}{\sqrt{7}}$ **c** $\dfrac{1-2\sqrt{3}}{3\sqrt{5}}$ **d** $\dfrac{2\sqrt{7}-3\sqrt{2}}{4\sqrt{3}-1}$

e $\dfrac{2\sqrt{5}-3\sqrt{2}}{3\sqrt{5}-2\sqrt{2}}$ **f** $\dfrac{\sqrt{7}-1}{\sqrt{7}+1}$ **g** $\dfrac{3\sqrt{2}-4}{4+3\sqrt{2}}$ **h** $\dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{5}-\sqrt{7}}$

i $\dfrac{x}{x-\sqrt{y}}$ **j** $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{y}}$ **k** $\dfrac{x+y}{x\sqrt{x}+\sqrt{y}}$ **l** $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

PROBLEM SOLVING AND REASONING

17 You can perform addition and subtraction with surd fractions in a similar way to how you would with common fractions.

Consider $\dfrac{3}{\sqrt{2}}+\dfrac{1}{2}$.

a First write each fraction with a common denominator and add the two fractions. Then write your answer with a rational denominator.

b Repeat the addition, but this time rationalise the denominator of the first fraction *before* performing the addition.

c Show that the answers to parts **a** and **b** represent the same fraction.

18 Calculate the value of each expression.

a $\dfrac{3}{\sqrt{5}}+\dfrac{1}{5}$ **b** $\dfrac{4}{\sqrt{3}}-\dfrac{1}{3}$ **c** $\dfrac{3\sqrt{2}}{2}+\dfrac{2}{\sqrt{3}}$

d $\dfrac{1}{4\sqrt{3}}+\dfrac{3}{2\sqrt{2}}$ **e** $\dfrac{5}{\sqrt{5}}-\dfrac{2}{3\sqrt{6}}$ **f** $\dfrac{2\sqrt{3}}{5\sqrt{2}}+\dfrac{\sqrt{5}}{4\sqrt{3}}$

19 Now consider $\dfrac{1}{\sqrt{5}-1}+\dfrac{3}{\sqrt{5}+1}$.

a Write each fraction as an equivalent fraction with a rational denominator.

b Add the two fractions formed in part **a**, writing your answer in simplest form.

20 Calculate the value of each expression after first expressing each fraction with a rational denominator.

a $\dfrac{2}{\sqrt{6}-2}+\dfrac{2}{\sqrt{6}+2}$ **b** $\dfrac{2}{\sqrt{5}+\sqrt{2}}-\dfrac{2}{\sqrt{5}-\sqrt{2}}$ **c** $\dfrac{2}{\sqrt{6}-1}+\dfrac{2}{\sqrt{3}+1}$

d $\dfrac{5}{\sqrt{2}+1}-\dfrac{4}{\sqrt{5}-2}$ **e** $\dfrac{2\sqrt{3}}{2\sqrt{2}+\sqrt{5}}+\dfrac{\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$ **f** $\dfrac{\sqrt{3}+1}{\sqrt{3}+4}-\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

CHALLENGE

21 Numbers written as surd fractions may represent dimensions in shapes.

Draw some basic two-dimensional shapes (for example, squares, rectangles or triangles) and use surd fractions to represent their dimensions. For each of your shapes, calculate the

a area

b perimeter

Where appropriate, write fractional answers with a rational denominator.

Reflect

When rationalising the denominator, why is it important that the numerator and the denominator are multiplied by the same value?

3F Fractional indices

Start thinking!

Previously, you looked at the index laws that you can apply to expressions with terms written in index form.

These index laws apply for all rational values, not just integer values.
That is, expressions that have fractional indices still obey the index laws.

Index laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{m \times n}$

1 Consider the expression $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$.

- **a** Which of the index laws could you use to simplify this expression? Explain.
- **b** Simplify the expression.
- **c** Check your answer with a calculator or other digital technology.

2 Now consider the expression $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$.

- **a** Which of the index laws could you use to simplify this expression? Explain.
- **b** Simplify the expression.
- **c** Check your answer with a calculator or other digital technology.

3 Now consider the expression $\left(a^{\frac{1}{2}}\right)^{\frac{1}{3}}$.

- **a** Which one of the index laws could you use to simplify this expression? Explain.
- **b** Simplify the expression.
- **c** Check your answer with a calculator or other digital technology.

4 For each of the expressions simplified in questions **1**–**3**, did your answers show that the index laws apply to the fractional indices in the same way as they apply to integer values?

KEY IDEAS

- The index laws can be applied to expressions involving terms written in index form.
- The index laws include:
  $a^m \times a^n = a^{m+n}$
  $a^m \div a^n = a^{m-n}$
  $(a^m)^n = a^{m \times n}$
- A useful index property is $a^0 = 1$.
- Expressions that contain fractional indices obey the index laws.

EXERCISE 3F Fractional indices

1 Use the appropriate index law to simplify each expression.

a $x^{\frac{1}{5}} \times x^{\frac{2}{5}}$ **b** $x^{\frac{3}{10}} \times x^{\frac{1}{5}}$ **c** $x^{\frac{3}{4}} \times x^{\frac{1}{8}}$ **d** $x^{\frac{2}{5}} \times x^{\frac{1}{6}}$

e $x^{\frac{5}{6}} \times x^{\frac{1}{6}}$ **f** $x^{\frac{2}{3}} \times x^{\frac{1}{4}}$ **g** $\dfrac{x^{\frac{4}{5}}}{x^{\frac{3}{4}}}$ **h** $\dfrac{x^{\frac{1}{2}}}{x^{\frac{3}{7}}}$

2 Use the appropriate index law to simplify each expression.

a $\left(x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(x^{\frac{3}{7}}\right)^{0}$ **c** $\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}$ **d** $\left(x^{\frac{1}{8}}\right)^{\frac{4}{9}}$

EXAMPLE 3F-1 Using the index laws to multiply and divide

Simplify each expression.

a $3x^{\frac{1}{4}} \times 4x^{\frac{2}{3}}$ **b** $\dfrac{12x^{\frac{3}{5}}}{4x^{\frac{1}{3}}}$

THINK

a **1** Identify the parts of each term. Multiply the coefficients and multiply the terms with the same bases.

2 Simplify the index terms. Write the base and add the indices together.

b **1** Identify the parts of each term. Divide the coefficients and divide the terms with the same bases. Remember that a fraction represents a division statement.

2 Simplify the index terms. Write the base and subtract the indices.

WRITE

a $3x^{\frac{1}{4}} \times 4x^{\frac{2}{3}}$

$= 3 \times 4 \times x^{\frac{1}{4}} \times x^{\frac{2}{3}}$

$= 12 \times x^{\frac{1}{4}} \times x^{\frac{2}{3}}$

$= 12 \times x^{\frac{1}{4}+\frac{2}{3}}$

$= 12 \times x^{\frac{11}{12}}$

$= 12x^{\frac{11}{12}}$

b $\dfrac{12x^{\frac{3}{5}}}{4x^{\frac{1}{3}}}$

$= 12 \times x^{\frac{3}{5}} \div \left(4 \times x^{\frac{1}{3}}\right)$

$= 12 \div 4 \times \left(x^{\frac{3}{5}} \div x^{\frac{1}{3}}\right)$

$= 3 \times \left(x^{\frac{3}{5}} \div x^{\frac{1}{3}}\right)$

$= 3 \times \left(x^{\frac{3}{5}-\frac{1}{3}}\right)$

$= 3x^{\frac{4}{15}}$

3 Simplify each expression.

a $5x^{\frac{1}{3}} \times 2x^{\frac{1}{4}}$ **b** $6x^{\frac{2}{5}} \times 5x^{\frac{3}{8}}$ **c** $9x^{\frac{3}{7}} \times -4x^{\frac{2}{9}}$

d $4x^{\frac{1}{4}} \div x^{\frac{1}{6}}$ **e** $12x^{\frac{3}{5}} \div 6x^{\frac{3}{8}}$ **f** $\dfrac{32x^{\frac{5}{6}}}{8x^{\frac{4}{5}}}$

EXAMPLE 3F-2 Raising a term to another index

Simplify $5\left(x^{\frac{2}{5}}y^2\right)^{\frac{3}{4}}$.

THINK

1 Each index term is raised to the power outside the brackets. So use $(a^m)^n = a^{m \times n}$ to simplify the expression.

2 Simplify each term by multiplying the indices.

WRITE

$$\begin{aligned} 5\left(x^{\frac{2}{5}}y^2\right)^{\frac{3}{4}} &= 5\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}(y^2)^{\frac{3}{4}} \\ &= 5 \times x^{\frac{2}{5} \times \frac{3}{4}} \times y^{2 \times \frac{3}{4}} \\ &= 5 \times x^{\frac{3}{10}} \times y^{\frac{3}{2}} \\ &= 5x^{\frac{3}{10}}y^{\frac{3}{2}} \end{aligned}$$

4 Simplify each expression.

a $\left(5x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(16x^{\frac{2}{5}}\right)^{\frac{1}{4}}$ **c** $\left(27x^{\frac{3}{8}}\right)^{\frac{2}{3}}$

d $2\left(x^{\frac{2}{5}}y^{\frac{3}{4}}\right)^{\frac{1}{2}}$ **e** $4\left(x^{\frac{1}{5}}y^{\frac{3}{7}}\right)^{\frac{2}{3}}$ **f** $5\left(x^{\frac{2}{3}}y^{\frac{4}{5}}\right)^{\frac{3}{8}}$

EXAMPLE 3F-3 Using the index laws with terms involving more than one pronumeral

Simplify $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}}$.

THINK

1 Identify the parts of each term. Multiply the coefficients and multiply the terms with the same bases.

2 Simplify the index terms with the same base.

WRITE

$$\begin{aligned} 5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}} &= 5 \times 2 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}} \\ &= 10 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}} \\ &= 10 \times x^{\frac{1}{2}+\frac{1}{5}} \times y^{\frac{1}{3}+\frac{3}{5}} \\ &= 10 \times x^{\frac{7}{10}} \times y^{\frac{14}{15}} \\ &= 10x^{\frac{7}{10}}y^{\frac{14}{15}} \end{aligned}$$

5 Simplify each expression.

a $4x^{\frac{1}{4}}y^{\frac{1}{5}} \times 7x^{\frac{1}{3}}y^{\frac{3}{4}}$ **b** $-6x^{\frac{2}{3}}y^{\frac{4}{7}} \times 12x^{\frac{5}{6}}y^{\frac{3}{7}}$ **c** $9x^{\frac{1}{6}}y^{\frac{2}{5}} \times 5x^{\frac{5}{8}}y$

d $25x^{\frac{3}{5}}y^{\frac{1}{4}} \div 5x^{\frac{1}{5}}y^{\frac{1}{12}}$ **e** $12x^{\frac{3}{8}}y^{\frac{5}{11}} \div 6x^{\frac{1}{5}}y^{\frac{1}{3}}$ **f** $\dfrac{7x^{\frac{1}{2}}y^{\frac{3}{7}}}{7x^{\frac{3}{7}}y^{\frac{2}{5}}}$

6 Use the appropriate index law to simplify each expression. Write each fractional power in simplest form.

a $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **b** $x^{\frac{1}{3}} \times x^{\frac{1}{6}}$ **c** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ **d** $x^{\frac{3}{8}} \div x^{\frac{1}{8}}$

e $3x^{\frac{1}{2}} \times 2x^{\frac{1}{3}}$ **f** $\dfrac{x^{\frac{3}{5}}}{4x^{\frac{1}{2}}}$ **g** $\left(5x^{\frac{3}{8}}\right)^2$ **h** $\frac{1}{2}x^{\frac{3}{4}} \times 3x^{\frac{1}{3}}$

7 Now explore other ways that fractional powers can be represented.

Consider the numerical expression $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.

a Which one of the index laws would you use to simplify this expression?

b Use the index law to simplify the expression and check your answer with a calculator.

c Now consider the surd expression $\sqrt{3} \times \sqrt{3}$. Multiply the two surds and write your answer in simplest form.

d Compare your answers to parts **b** and **c**. What do you notice?

e Repeat parts **b**–**d** for each of these pairs.

i $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt{2} \times \sqrt{2}$ **ii** $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$ and $\sqrt{5} \times \sqrt{5}$

iii $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$ and $\sqrt{10} \times \sqrt{10}$ **iv** $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ and $\sqrt{x} \times \sqrt{x}$

f Using your results to part **e**, explain what you notice about the relationship between the fractional power of $\frac{1}{2}$ and the square root sign.

8 Now consider the numerical expression $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$.

a Use the appropriate index law to simplify the expression and check your answer with a calculator.

b Calculate $\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}$.

c Compare your answers to parts **a** and **b**. What do you notice?

d Repeat parts **a**–**c** for each of these pairs.

i $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}}$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$

ii $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$

iii $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$ and $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10}$

iv $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ and $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$

e What are the differences between the surd expression in **8b** and that in **7c**?

f Use your results to explain what you notice about the relationship between the fractional power of $\frac{1}{3}$ and the cube root sign.

9 In questions **7** and **8** you explored some relationships to show that $x^{\frac{1}{2}}$ is equivalent to $\sqrt{x}$ and that $x^{\frac{1}{3}}$ is equivalent to $\sqrt[3]{x}$. Extend upon this to show that any fractional index of the form $\frac{1}{n}$ is equivalent to $\sqrt[n]{\ }$. That is, $x^{\frac{1}{n}}$ is equivalent to $\sqrt[n]{x}$. You may wish to provide numerical expressions as examples.

10 Notice that the fractional powers you have seen so far and the corresponding surd form have only related to fractions with a numerator of 1. Now consider the term $x^{\frac{2}{3}}$.

a Show that this term has equivalent forms of $(x^2)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$.

b Show that both expressions in part **a** are equivalent to $\sqrt[3]{x^2}$ and $(\sqrt[3]{x})^2$.

c Use a calculator to show that substituting these values for x into $x^{\frac{2}{3}}$ produces the same results as substituting into each expression in parts **a** and **b**.

i $x = 5$ **ii** $x = 2$ **iii** $x = 3$ **iv** $x = 10$ **v** $x = 12$ **vi** $x = 27$

11 Show that any fractional index of the form $\frac{m}{n}$ is equivalent to $\sqrt[n]{(\ )^m}$ or $(\sqrt[n]{\ })^m$. That is, $x^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{x^m}$ or $(\sqrt[n]{x})^m$.

12 Write your answer to each expression in question **6** in an appropriate surd form.

$x^{\frac{1}{2}} \Leftrightarrow \sqrt{x}$

$x^{\frac{1}{3}} \Leftrightarrow \sqrt[3]{x}$

$x^{\frac{1}{n}} \Leftrightarrow \sqrt[n]{x}$

$x^{\frac{m}{n}} \Leftrightarrow \sqrt[n]{x^m} \Leftrightarrow (\sqrt[n]{x})^m$

13 Calculate each value without using a calculator.

a $25^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $1^{\frac{1}{2}}$

d $144^{\frac{1}{2}}$ **e** $100^{\frac{1}{2}}$ **f** $196^{\frac{1}{2}}$

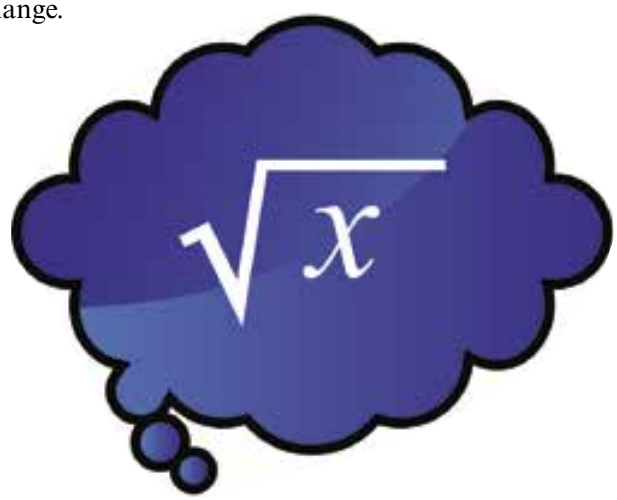
g $8^{\frac{1}{3}}$ **h** $27^{\frac{1}{3}}$ **i** $125^{\frac{1}{3}}$

j $1^{\frac{1}{3}}$ **k** $1^{\frac{1}{5}}$ **l** $32^{\frac{1}{5}}$

14 Look at each part in question **13**.

i If each fractional power was negative, briefly explain how the answers would change.

ii What would the value of each part be if the powers were negative?



15 Write each value in simplest surd form.

a $5^{\frac{1}{2}}$ **b** $8^{\frac{1}{2}}$ **c** $24^{\frac{1}{2}}$

d $108^{\frac{1}{2}}$ **e** $152^{\frac{1}{2}}$ **f** $54^{\frac{1}{3}}$

g $128^{\frac{1}{3}}$ **h** $320^{\frac{1}{2}}$ **i** $243^{\frac{1}{2}}$

j $432^{\frac{1}{2}}$ **k** $5^{\frac{3}{2}}$ **l** $6^{\frac{3}{2}}$

PROBLEM SOLVING AND REASONING

16 Write each expression with fractional indices.

a $\sqrt{x^3}$ **b** $\sqrt{6}$ **c** $\sqrt{x^3y^5}$

d $\sqrt{3xy^3}$ **e** $\sqrt{16x^{-1}y}$ **f** $\sqrt[3]{x^4y^2}$

17 Simplify each expression, giving answers with fractional indices.

a $12x^{\frac{1}{4}}y^{\frac{1}{5}} \times 9x^{\frac{1}{5}}y^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{1}{3}}y^{\frac{3}{4}}}{5x^{\frac{1}{9}}y^{\frac{2}{5}}}$ **c** $\left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$

d $\sqrt{4x^3y} \times 3\sqrt{x^5y^3}$ **e** $3\sqrt{x^4y^3} \div \sqrt{9xy^2}$ **f** $(\sqrt{x^3y^5})^{\frac{1}{3}}$

g $\sqrt{8x^3yz^5} \times 2\sqrt{3xz^3}$ **h** $\dfrac{\sqrt{2x^5y^2}}{5\sqrt{x^2yz}}$ **i** $\left(\sqrt[3]{x^6y^{\frac{1}{3}}z}\right)^2$

18 Write each expression in simplest form.

a $\sqrt{16x^2y}$ **b** $\sqrt{12x^{-3}y^7}$ **c** $\sqrt{\dfrac{72x^4y^3}{24xy}}$

d $\sqrt[3]{27xy^2}$ **e** $\sqrt[4]{16x^3y^2z}$ **f** $\sqrt{343x^{-1}y^5}$

19 Simplify each expression, writing answers with positive indices where appropriate.

a $3x^{-\frac{1}{3}}y^{\frac{2}{3}} \times 6x^{\frac{3}{4}}y^{-\frac{1}{5}}$ **b** $12x^{\frac{1}{4}}y^{\frac{1}{3}} \times 6x^{-\frac{2}{3}}y^{-\frac{1}{4}}$ **c** $\dfrac{xy^2}{x^{\frac{1}{3}}y^{\frac{5}{2}}}$

d $\dfrac{3x^{\frac{1}{3}}y^{\frac{2}{5}}}{12x^{\frac{2}{5}}y^{\frac{3}{4}}}$ **e** $\left(\dfrac{x^{-\frac{1}{3}}}{y^3}\right)^{\frac{1}{4}}$ **f** $\left(\dfrac{2x^{-\frac{2}{5}}y^{\frac{3}{4}}}{y^2}\right)^4$

g $\dfrac{5x^{-\frac{1}{2}} \times 4x^{\frac{1}{3}}}{2x^{-\frac{2}{5}}}$ **h** $\dfrac{3x^{-\frac{1}{2}}y^{\frac{1}{3}} \times 5x^{\frac{1}{6}}y^{\frac{3}{4}}}{2x^{\frac{1}{8}}y^3}$ **i** $\sqrt{\dfrac{x^{-\frac{1}{3}}y^{\frac{2}{3}}}{3x^{-\frac{1}{4}}y^{-\frac{1}{5}}}}$

CHALLENGE

20 Consider the expression $\dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$.

a Use an appropriate combination of the index laws to simplify the expression.

b Use your working and result from part **a** to assist you to solve the equation $9^x = \dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$ for x.

21 Solve each equation for x.

a $25^{x+2} = \dfrac{1}{5}$

b $27^{x-3} = 9^{x+1}$

c $8^x = \dfrac{16^{-\frac{1}{4}} \times 2^{\frac{1}{2}}}{64^{\frac{1}{6}}}$

Reflect

How are fractional powers and surds related?

3G Understanding logarithms

Start thinking!

In this section you will investigate the relationship between index form and an alternative form known as **logarithmic form** (or logarithm form).

1 Consider the equation $x = a^m$ written in index form. State the base and the index.
You are aware that the index is also referred to as the **power** or **exponent**. When written in this form, the index may also be referred to as a **logarithm** (often abbreviated to log).
Equations written in index form can also be written in an equivalent logarithmic form. Look at the original equation $x = a^m$. The equivalent logarithmic form for this is $\log_a x = m$.

This form is interpreted as 'the logarithm of x to the base a is equal to m' or 'the log of x to the base a is m'.

2 Explain how the parts of the index equation relate to the parts of the logarithmic equation.

3 Look at $2^5 = 32$, which is in index form. This can easily be written in logarithmic form.
- **a** Identify the base, logarithm and basic numeral and hence write the equation in an equivalent logarithmic form.
- **b** Repeat part **a** for $10^4 = 10\,000$.
- **c** What must you know about a number in index form for it to be written in equivalent logarithmic form?

4 Now consider 10^2.
- **a** How does this form differ from those provided in question **3**?
- **b** Explain what you need to do before you can write this as an equivalent logarithmic equation.
- **c** Write the equivalent logarithmic equation.
- **d** Follow your method from part **b** and write 5^3 in logarithmic form.

5 Just as equations written in index form can be written in logarithmic form, the reverse is also true.
Consider $\log_{10} 1000 = 3$, which is in logarithmic form.
- **a** Identify the base, logarithm and basic numeral from this equation.
- **b** Write an equivalent equation in index form.
- **c** Repeat parts **a** and **b** for each of these equations.
  i $\log_5 25 = 2$ **ii** $\log_4 64 = 3$ **iii** $\log_2 64 = 6$
- **d** What information is required from logarithmic form for an equivalent equation in index form?

KEY IDEAS

- For expressions in index form, the index may also be known as the power, exponent or logarithm.
- The equivalent logarithmic form for $x = a^m$ is $\log_a x = m$, where a is the base, m is the logarithm and x is the basic numeral.
- Writing a logarithmic equation as an equivalent equation in index form helps when calculating a logarithm.

$x = a^m$	$\log_a x = m$
index form	logarithmic form

EXERCISE 3G Understanding logarithms

1 Identify the logarithm in each part.

a $2^3 = 8$ **b** $2^2 = 4$ **c** $\log_5 25 = 2$ **d** $\log_{10} 10 = 1$

e $5^6 = 15\,625$ **f** $2^4 = 16$ **g** $\log_2 \frac{1}{2} = -1$ **h** $\log_3 \frac{1}{3} = -1$

EXAMPLE 3G-1 Converting from index form to logarithmic form

Write $3^4 = 81$ in logarithmic form.

THINK	WRITE
1 Identify the base, logarithm and basic numeral from the given form.	$3^4 = 81$ base = 3, logarithm = 4, basic numeral = 81
2 Write the equivalent equation in logarithmic form.	$\log_3 81 = 4$

UNDERSTANDING AND FLUENCY

2 Write each equation in logarithmic form.

a $3^3 = 27$ **b** $4^1 = 4$ **c** $5^4 = 625$ **d** $10^3 = 1000$

e $6^2 = 36$ **f** $9^3 = 729$ **g** $3^4 = 81$ **h** $2^7 = 128$

3 For each equation:

i state the logarithm

ii state the base

iii write in logarithmic form.

a $4^2 = 16$ **b** $5^3 = 125$ **c** $9^2 = 81$ **d** $10^2 = 100$

e $8^2 = 64$ **f** $2^6 = 64$ **g** $4^4 = 256$ **h** $x^y = z$

EXAMPLE 3G-2 Converting from logarithmic form to index form

Write $\log_2 16 = 4$ in index form.

THINK	WRITE
1 Identify the base, logarithm and basic numeral from the given form.	$\log_2 16 = 4$ base = 2, logarithm = 4, basic numeral = 16
2 Write the equivalent equation in index form.	$2^4 = 16$

4 Write each equation in index form.

a $\log_2 32 = 5$ **b** $\log_3 9 = 2$ **c** $\log_5 25 = 2$ **d** $\log_7 49 = 2$

e $\log_4 16 = 2$ **f** $\log_2 1024 = 10$ **g** $\log_6 216 = 3$ **h** $\log_{10} 100 = 2$

5 For each equation:

i state the logarithm **ii** state the base **iii** write in index form.

a $\log_3 27 = 3$ **b** $\log_4 4 = 1$ **c** $\log_6 36 = 2$ **d** $\log_2 16 = 4$

e $\log_5 625 = 4$ **f** $\log_3 243 = 5$ **g** $\log_7 343 = 3$ **h** $\log_a b = c$

6 Complete these steps to calculate $\log_3 27$.

a Let m represent the logarithm, so $\log_3 27 = m$. Write this in index form.

b Find the value of m that makes the equation a true statement.

7 Calculate each of these values.

a $\log_2 16$ **b** $\log_4 256$ **c** $\log_5 125$ **d** $\log_{10} 1000$

8 Write each value with positive indices.

a 3^{-1} **b** 5^{-1} **c** $\left(\frac{1}{2}\right)^{-1}$ **d** $(0.1)^{-1}$ **e** 2^{-2} **f** $\left(\frac{1}{3}\right)^{-3}$

g 4^{-3} **h** x^{-1} **i** x^2y^{-1} **j** $(0.2)^{-2}$ **k** 10^{-2} **l** 10^{-4}

EXAMPLE 3G-3 Converting from index form with a negative index to logarithmic form

For each part:

i find the value without using a calculator

ii write an equation in logarithmic form.

a 5^{-1} **b** $\left(\frac{1}{4}\right)^{-2}$

THINK

a 1 Write the value with a positive index and simplify.

2 Write the equation from part **i** in log form.

b 1 Write the value with a positive index and simplify.

2 Write the equation from part **i** in log form.

WRITE

a i $5^{-1} = \frac{1}{5^1}$

$5^{-1} = \frac{1}{5}$

ii $\log_5 \frac{1}{5} = -1$

b i $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2$

$\left(\frac{1}{4}\right)^{-2} = 16$

ii $\log_{\frac{1}{4}} 16 = -2$

9 For each part:

i find the value without using a calculator

ii write an equation in logarithmic form.

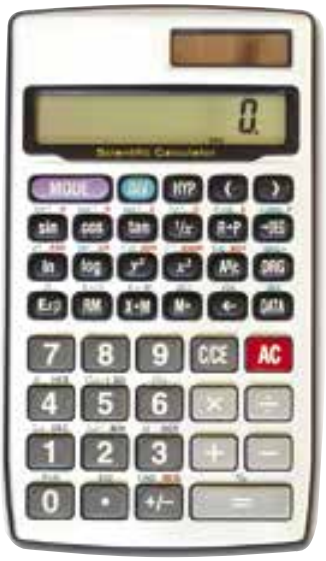
a 6^{-1} **b** 4^{-1} **c** 2^{-2} **d** 2^{-5} **e** 5^{-2} **f** 10^{-1}

g 10^{-3} **h** 3^{-3} **i** $\left(\frac{1}{2}\right)^{-1}$ **j** $\left(\frac{1}{3}\right)^{-2}$ **k** $\left(\frac{1}{5}\right)^{-3}$ **l** $(0.01)^{-1}$

10 Consider these base 10 logarithms.

i $\log_{10} 10$ **ii** $\log_{10} 100$

iii $\log_{10} 1000$ **iv** $\log_{10} 10\,000$



a Logarithms to the base 10 can be entered into a calculator by pressing the log button. Use a calculator to find the value of each logarithm.

b Another way to calculate a logarithm is to let a pronumeral represent the logarithm and write in equivalent index form.

For example, if you let x represent the logarithm, then $x = \log_{10} 100$ is equivalent to $10^x = 100$. You can now find the value for x that makes the equation a true statement. Use this method to calculate each logarithm and check that you get the same answers as in part **a**.

c The method in part **b** can be extended to include one additional step.

$x = \log_{10} 100$ is equivalent to $10^x = 100$ or $10^x = 10^2$.

Once the bases are the same, you can equate the indices. This produces $x = 2$.

Use this method for each logarithm and check that you get the same answers as in part **a**.

11 Calculate each value. You may like to refer to the example at right.

a $\log_8 64$ **b** $\log_3 27$

c $\log_9 81$ **d** $\log_2 32$

e $\log_{10} 10\,000$ **f** $\log_5 125$

g $\log_6 36$ **h** $\log_3 81$

i $\log_2 8$ **j** $\log_4 64$

k $\log_7 49$ **l** $\log_3 243$

m $\log_2 \frac{1}{8}$ **n** $\log_3 \frac{1}{27}$

o $\log_{10} 0.01$ **p** $\log_4 \frac{1}{4}$

To find $\log_3 81$, let $x = \log_3 81$.

$3^x = 81$

$3^x = 3^4$

$x = 4$

So $\log_3 81 = 4$.

12 a Write the value for each amount.

i $\log_5 5$ **ii** $\log_3 3$ **iii** $\log_{10} 10$

iv $\log_8 8$ **v** $\log_2 2$ **vi** $\log_{12} 12$

b What do you notice about each result?

c Use your answers to explain what the result of the logarithm of any number to the same base will be.

13 a Write the value for each amount.

i $\log_5 1$ **ii** $\log_3 1$ **iii** $\log_{10} 1$

iv $\log_8 1$ **v** $\log_2 1$ **vi** $\log_{12} 1$

b What do you notice about each result?

c Use your answers to explain what the result of the logarithm of 1 to any base will be.

UNDERSTANDING AND FLUENCY

14 Write the value for each expression.

a $\log_x x$ **b** $\log_x 1$

c $\log_y y$ **d** $\log_{10} 10$

e $\log_{10} 1$ **f** $\log_y 1$

$\log_a a = 1$ because $a^1 = a$
$\log_a 1 = 0$ because $a^0 = 1$

15 Consider $\log_2 (-16)$.

a Is it possible for this to be written in an equivalent index form?

b Repeat part **a** for each value.

i $\log_5 (-25)$ **ii** $\log_3 (-27)$

iii $\log_{10} (-1000)$ **iv** $\log_6 (-36)$

v $\log_4 (-64)$ **vi** $\log_8 (-8)$

c What can you conclude about taking the logarithm of any negative number to any base?

16 Now consider $\log_2 0$.

a Is it possible for this to be written in an equivalent index form?

b Repeat part **a** for each value.

i $\log_5 0$ **ii** $\log_3 0$

iii $\log_{10} 0$ **iv** $\log_6 0$

v $\log_4 0$ **vi** $\log_8 0$

c What can you conclude about taking the logarithm of zero to any base?

17 Now consider $\log_1 5$.

a Is it possible for this to be written in an equivalent index form?

b Repeat part **a** for each value.

i $\log_1 5$ **ii** $\log_1 3$

iii $\log_1 10$ **iv** $\log_1 6$

v $\log_1 4$ **vi** $\log_1 8$

c What can you conclude about taking the logarithm of any number to the base 1?

PROBLEM SOLVING AND REASONING

18 In questions **15**–**17**, you discovered that some logarithms are not defined for certain numbers and bases. In a similar way to the general summary provided with question **14**, write general summaries that cover the findings in questions **15**–**17**.

19 A knowledge of logarithms and index form is quite useful in solving logarithmic equations. Consider the equation $\log_x 64 = 2$.

a Write the equation in an equivalent index form.

b What is the value of x?

c Show that the index form in part **a** is equivalent to $x^2 = 8^2$.

d Notice that the two powers are the same in the equation shown in part **c**. If the two powers are the same, then equating the bases will produce the solution. How does this method compare with your answer to part **b**?

PROBLEM SOLVING AND REASONING

20 Use the methods from question **19** to solve each logarithmic equation for x.

a $\log_x 64 = 6$ **b** $\log_x 125 = 3$

c $\log_x 16 = 2$ **d** $\log_x 8 = 3$

e $\log_x 128 = 7$ **f** $\log_x 81 = 4$

g $\log_x \frac{1}{4} = -1$ **h** $\log_x \frac{1}{8} = -3$

21 Consider the equation $\log_4 x = 3$.

a Write the equation in an equivalent index form.

b Calculate the index term to find the value of x.

c Briefly explain how expressing a logarithmic equation in index form helps you to solve the equation.

22 Use the method from question **21** to solve each logarithmic equation for x.

a $\log_3 x = 3$ **b** $\log_5 x = 2$

c $\log_6 x = 1$ **d** $\log_{10} x = 4$

e $\log_2 x = 5$ **f** $\log_4 x = 3$

g $\log_3 x = -3$ **h** $\log_2 x = -2$

CHALLENGE

23 Investigate the change of base rule and how it can be applied to logarithms. You may wish to test this rule by repeating some of the calculations you performed in questions **7** and **11**.

24 In question **12** and **13** you encountered the general statements $\log_a a = 1$ and $\log_a 1 = 0$.

a Use your calculator to find out if these hold true when $a = 1$. That is, find the value for $\log_1 1$.

b How do you interpret the result from part **a**?

25 **a** Use the change of base rule for $\log_1 1$.

b Reconsidering your responses to question **24**, along with your answer to part **a**, provide a brief explanation as to why $\log_1 1$ is not defined.

26 Similar to π, which is an irrational number you have used with perimeter and area of circles, there is an irrational number e that is connected with logarithms. The value of e approximates to 2.718 28 when rounded to five decimal places. Logarithms to the base e ($\log_e$) are known as **natural logarithms** and can be worked out using a calculator in a similar way to logarithms to the base 10. The $\log_e$ button is usually labelled as **ln**. Identify the button and calculate each of these, correct to two decimal places.

a $\log_e 5$ **b** $\log_e 10$

c $\log_e 2$ **d** $\log_e 0.5$

e $\log_e 100$ **f** $\log_e 1$

g $\log_e 0$ **h** $\log_e 2.718\ 28$

Reflect

How is index form related to logarithmic form?

3H Working with logarithms

Start thinking!

Just as index laws apply to expressions written in index form, logarithm laws apply in a similar way to expressions in logarithmic form.

Some index laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{m \times n}$

1 Start with the index law for multiplication; that is, $a^m \times a^n = a^{m+n}$.
Let $x = a^m$ and let $y = a^n$.

- **a** Write $x = a^m$ and $y = a^n$ in equivalent logarithmic form.
- **b** Show that $x \times y$ results in the equation $xy = a^{m+n}$.
- **c** Write the equation from part **b** in an equivalent logarithmic form.
- **d** Using your answers from part **a**, show that the equation from part **c** simplifies to $\log_a (xy) = \log_a x + \log_a y$.

This logarithm law is often referred to as a **log of a product**. It shows that the logarithm of a product of two numbers (x and y) is equal to the sum of the logarithms of the two numbers with the same base.

2 Now consider the index law for division; that is, $a^m \div a^n = a^{m-n}$. Let $x = a^m$ and let $y = a^n$.

- **a** Write $x = a^m$ and $y = a^n$ in an equivalent logarithmic form.
- **b** Show that $x \div y$ results in the equation $\frac{x}{y} = a^{m-n}$.
- **c** Write the equation from part **b** in an equivalent logarithmic form.
- **d** Using your answers from part **a**, show that the equation from part **c** simplifies to $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

This logarithm law is often referred to as a **log of a quotient**. It shows that the logarithm of a quotient of two numbers (x and y) is equal to the difference of the logarithms of the two numbers with the same base.

3 Consider the index law for raising a term in index form to another index; that is, $(a^m)^n = a^{m \times n}$.

- **a** Let $x = a^m$ and raise both sides of this to the power of n. Show that this simplifies to $x^n = a^{m \times n}$.
- **b** Write the equation from part **a** in an equivalent logarithmic form.
- **c** Show that the equation from part **b** simplifies to $\log_a x^n = n \times \log_a x$.

This logarithm law is often referred to as a **log of a power**. It shows that the logarithm of a power term (x^n) is equal to the power (n) multiplied by the logarithm (of x).

KEY IDEAS

- Logarithm laws can be applied to expressions in logarithmic form.
- The log laws include:
  - the log of a product, $\log_a (xy) = \log_a x + \log_a y$
  - the log of a quotient, $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - the log of a power, $\log_a x^n = n \times \log_a x$.
- Other useful log properties include:
  - $\log_a a = 1$
  - $\log_a 1 = 0$.

EXERCISE 3H Working with logarithms

UNDERSTANDING AND FLUENCY

1 Calculate each value.

a $\log_5 5$ **b** $\log_3 3$ **c** $\log_{10} 10$ **d** $4\log_5 5$

e $3\log_3 3$ **f** $\log_3 1$ **g** $\log_2 1$ **h** $4\log_5 1$

2 Calculate each value.

a $\log_3 3^2$ **b** $\log_5 5$ **c** $\log_4 1$ **d** $\log_2 1$

e $\log_3 \sqrt{3}$ **f** $\log_4 \frac{1}{16}$ **g** $\log_4 4^{-1}$ **h** $\log_x x$

3 Use the logarithm law $\log_a (xy) = \log_a x + \log_a y$ to write each expression as a sum of two logarithms.

a $\log_{10} (3 \times 5)$ **b** $\log_2 (6 \times 3)$ **c** $\log_5 (2.5 \times 10)$ **d** $\log_2 (xy)$

4 Write each expression as a logarithm of a product.

a $\log_3 6 + \log_3 2$ **b** $\log_{10} 5 + \log_{10} 3$ **c** $\log_2 10 + \log_2 5$

d $\log_4 6 + \log_4 12$ **e** $\log_a 5 + \log_a 3$ **f** $\log_2 x + \log_2 y$

5 Use the logarithm law $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ to write each expression as a difference of two logarithms.

a $\log_{10} \left(\frac{5}{7}\right)$ **b** $\log_2 \left(\frac{5}{3}\right)$ **c** $\log_5 \left(\frac{11}{20}\right)$ **d** $\log_2 \left(\frac{a}{b}\right)$

6 Write each expression as a logarithm of a quotient.

a $\log_3 12 - \log_3 5$ **b** $\log_{10} 50 - \log_{10} 15$ **c** $\log_2 25 - \log_2 10$

d $\log_4 36 - \log_4 9$ **e** $\log_a 3 - \log_a 15$ **f** $\log_2 x - \log_2 y$

EXAMPLE 3H-1 Using the 'log of a product' and 'log of a quotient' logarithm laws

Use the appropriate logarithm law to calculate:

a $\log_4 32 + \log_4 2$ **b** $\log_5 50 - \log_5 2$.

THINK	WRITE
a 1 Each logarithm in the sum contains the same base, so use the 'log of a product' logarithm law: $\log_a (xy) = \log_a x + \log_a y$.	**a** $\log_4 32 + \log_4 2$ $= \log_4 (32 \times 2)$
2 Calculate the logarithm ($4^3 = 64$). Alternatively, $\log_4 64 = \log_4 4^3 = 3\log_4 4 = 3 \times 1 = 3$.	$= \log_4 64$ $= 3$
b 1 Each logarithm in the expression contains the same base, so use the 'log of a quotient' logarithm law. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.	**b** $\log_5 50 - \log_5 2$ $= \log_5 \left(\frac{50}{2}\right)$
2 Calculate the logarithm ($5^2 = 25$). Alternatively, $\log_5 25 = \log_5 5^2 = 2\log_5 5 = 2 \times 1 = 2$.	$= \log_5 25$ $= 2$

7 Use the appropriate logarithm law to calculate each value.

a $\log_3 9 + \log_3 9$ **b** $\log_6 108 - \log_6 3$ **c** $\log_2 8 + \log_2 4$

d $\log_{10} 500 + \log_{10} 20$ **e** $\log_4 256 - \log_4 16$ **f** $\log_3 243 - \log_3 9$

g $\log_6 9 + \log_6 4$ **h** $\log_5 175 - \log_5 7$ **i** $\log_2 32 + \log_2 16$

j $\log_{10} 25 + \log_{10} 4$ **k** $\log_3 54 - \log_3 6$ **l** $\log_4 768 - \log_4 3$

8 Use the appropriate logarithm laws to calculate each value.

a $\log_6 27 + \log_6 4 - \log_6 3$ **b** $\log_{10} 60 - \log_{10} \left(\frac{3}{5}\right)$ **c** $\log_3 \left(\frac{3}{2}\right) + \log_3 18$

9 For each expression:

i use the appropriate logarithm laws to write each amount as a single logarithm

ii use a calculator to obtain the value of the single logarithm, correct to two decimal places.

a $\log_3 5 + \log_3 7$ **b** $\log_4 18 - \log_4 3$

c $\log_6 63 - \log_6 9$ **d** $\log_2 12 + \log_2 5$

> **NOTE** On your calculator, look for the button or function that shows log to any base. Check with your teacher if you are unsure.

10 Use the logarithm law $\log_a x^n = n \times \log_a x$ to write each expression as the product of a power and a logarithm.

a $\log_3 7^2$ **b** $\log_5 2^9$ **c** $\log_4 8^3$

d $\log_9 11^4$ **e** $\log_6 5^x$ **f** $\log_2 100^k$

11 Write each expression as a logarithm of a power term.

a $4 \times \log_5 3$ **b** $8 \times \log_4 5$ **c** $2 \times \log_2 12$

d $7 \log_3 2$ **e** $y \log_6 9$ **f** $6 \log_{10} x$

EXAMPLE 3H-2 Using the 'log of a power' logarithm laws

Use the appropriate logarithm law to calculate $\log_2 (4^3)$.

THINK	WRITE
1 Use the 'log of a power' law to simplify the logarithm. $\log_a x^n = n \times \log_a x$.	$\log_2 (4^3) = 3 \times \log_2 4$
2 Calculate $\log_2 4$ (2, since $2^2 = 4$) and simplify. Alternatively, $3 \times \log_2 4 = 3 \times \log_2 2^2 = 3 \times 2 \log_2 2 = 3 \times 2 \times 1 = 6$.	$= 3 \times 2$ $= 6$

12 Use the appropriate logarithm law to calculate each value.

a $\log_3 (3^5)$ **b** $\log_4 (4^2)$ **c** $\log_6 (6^5)$

d $\log_3 (9^4)$ **e** $3\log_3 27$ **f** $4\log_4 64$

g $5\log_5 (5^2)$ **h** $\frac{1}{2}\log_4 (4^4)$ **i** $7\log_2 (2^3)$

13 Write each value in a simpler form and calculate if possible.

a $2\log_3 4$ **b** $8\log_4 2$ **c** $2\log_3 9$

d $\log_a \sqrt{a}$ **e** $-\log_3 \left(\frac{5}{3}\right)$ **f** $2\log_2 \sqrt{x}$

EXAMPLE 3H-3

Calculating logarithmic expressions using product and quotient logarithm laws

Calculate each value.

a $\log_{10} 25 + \log_{10} 5 + \log_{10} 8$ **b** $\log_5 2 + \log_5 50 - \log_5 4$ **c** $\log_2 5 - \log_2 40$

THINK

a **1** Use the 'log of a product' law to simplify the first two terms.

2 Simplify the product.

3 Use the 'log of a product' law to simplify the expression into one term.

4 Work out the value of the logarithm. ($10^3 = 1000$)

b **1** Use the 'log of a product' law to simplify the first two terms.

2 Simplify the product.

3 Use the 'log of a quotient' law to simplify the expression into one term.

4 Work out the value of the logarithm. ($5^2 = 25$)

c **1** Use the 'log of a quotient' law.

2 Simplify the quotient.

3 Work out the value of the logarithm by writing the fraction in index form with a base of 2.

WRITE

a $\log_{10} 25 + \log_{10} 5 + \log_{10} 8$
$= \log_{10} (25 \times 5) + \log_{10} 8$
$= \log_{10} 125 + \log_{10} 8$
$= \log_{10} (125 \times 8)$
$= \log_{10} 1000$
$= \log_{10} 10^3$
$= 3$

b $\log_5 2 + \log_5 50 - \log_5 4$
$= \log_5 (2 \times 50) - \log_5 4$
$= \log_5 100 - \log_5 4$
$= \log_5 \left(\frac{100}{4}\right)$
$= \log_5 25$
$= \log_5 5^2$
$= 2$

c $\log_2 5 - \log_2 40$
$= \log_2 \left(\frac{5}{40}\right)$
$= \log_2 \left(\frac{1}{8}\right)$
$= \log_2 \left(\frac{1}{2^3}\right)$
$= \log_2 2^{-3}$
$= -3$

UNDERSTANDING AND FLUENCY

14 Use the appropriate logarithm laws to calculate each value.

a $\log_6 12 + \log_6 9 + \log_6 2$ **b** $\log_2 24 + \log_2 6 - \log_2 36$

c $\log_5 40 + \log_5 10 - \log_5 16$ **d** $\log_3 15 - \log_3 45$

e $\log_4 2 - \log_4 32$ **f** $\log_{10} 20 + \log_{10} 45 + \log_{10} \left(\frac{10}{9}\right)$

15 It is important that an expression is in the correct form before applying the log of a product law or the log of a quotient law. Consider each expression.

i $3\log_9 3 + \log_9 3$ **ii** $\log_3 72 - 3\log_3 2$

a What is the first operation to be performed before each expression is simplified?

b Write each expression as a single logarithm and calculate its value.

EXAMPLE 3H-4 Calculating a logarithmic expression using logarithm laws

Calculate $\log_6 48 + \log_6 3 - 2\log_6 2$.

THINK	WRITE
1 Use the 'log of a power' law to rewrite the third term.	$\log_6 48 + \log_6 3 - 2\log_6 2$ $= \log_6 48 + \log_6 3 - \log_6 2^2$ $= \log_6 48 + \log_6 3 - \log_6 4$
2 Use the 'log of a product' law to simplify the first two terms.	$= \log_6 (48 \times 3) - \log_6 4$
3 Simplify the product.	$= \log_6 144 - \log_6 4$
4 Use the 'log of a quotient' law to simplify the expression into one term.	$= \log_6 \left(\frac{144}{4}\right)$ $= \log_6 36$
5 Work out the value of the logarithm. ($6^2 = 36$)	$= \log_6 6^2$ $= 2$

UNDERSTANDING AND FLUENCY

16 Calculate each value.

a $3\log_4 2 + \log_4 8$
b $\log_5 225 - 2\log_5 3$
c $2\log_3 9 + 2\log_3 3$
d $4\log_{10} 5 + 4\log_{10} 2$
e $\log_2 72 - 2\log_2 6$
f $5\log_2 4 - \log_2 (2^3)$
g $2\log_2 4 + \log_2 14 - \log_2 7$
h $4\log_3 2 - \log_3 32 + \log_3 54$
i $\log_5 15 + 2\log_5 5 + 3\log_5 2 - \log_5 24$
j $3\log_{10} 5 + 2\log_{10} 3 + \log_{10} \left(\frac{8}{9}\right)$

17 Solve each logarithmic equation for x. Remember to write each in an equivalent index form first.

a $\log_x 27 = 3$
b $\log_4 64 = x$
c $\log_3 x = 2$
d $\log_7 2401 = x$
e $\log_6 x = 3$
f $\log_2 x = 6$
g $\log_5 \left(\frac{1}{125}\right) = x$
h $\log_x 6 = \frac{1}{2}$
i $\log_3 x = 0$
j $\log_x \left(\frac{1}{16}\right) = 4$
k $\log_2 \sqrt{2} = x$
l $\log_x \left(\frac{1}{16}\right) = -2$

18 The approximations shown on the right can be made for these base 10 logarithms.

$\log_{10} 1.5 \approx 0.176$
$\log_{10} 2 \approx 0.301$
$\log_{10} 3 \approx 0.477$
$\log_{10} 5 \approx 0.699$
$\log_{10} 6 \approx 0.778$

a Use the logarithm laws to show that:
- **i** $\log_{10} 3 + \log_{10} 2 = \log_{10} 6$
- **ii** $\log_{10} 3 - \log_{10} 2 = \log_{10} 1.5$
- **iii** $\log_{10} 5 + \log_{10} 2 = 1$

b Use the given values along with the appropriate logarithm law to calculate each value.
- **i** $\log_{10} 15$
- **ii** $\log_{10} \left(\frac{3}{5}\right)$
- **iii** $\log_{10} \left(\frac{5}{3}\right)$
- **iv** $\log_{10} 18$
- **v** $\log_{10} 30$
- **vi** $\log_{10} \left(\frac{6}{5}\right)$

PROBLEM SOLVING AND REASONING

19 Fractions may contain logarithmic expressions in the numerator and denominator and may be simplified in a similar way as common fractions; that is, by cancelling any common factors in the numerator and denominator.

Consider $\frac{\log_{10} 3}{\log_{10} 9}$.

a Use a calculator to find the value of this fraction.

b Show that the denominator has an equivalent logarithmic expression of $2\log_{10} 3$.

c Show that the fraction simplifies to the same answer you obtained in part **a**.

20 Three students attempt to simplify the fraction $\frac{\log_{10} 8}{\log_{10} 2}$.

Hailey says the correct answer is 3, Danielle thinks the answer is 4 and Joyce is certain that it must be $\log_{10}\left(\frac{8}{2}\right)$ or $\log_{10} 4$. Which student do you think is correct and why?

21 Simplify each expression.

a $\frac{\log_{10} 16}{\log_{10} 2}$ **b** $\frac{\log_6 216}{\log_6 36}$ **c** $\frac{\log_3 8}{\log_3 2}$

d $\frac{\log_4 64}{\log_4 256}$ **e** $\frac{\log_{10} 16}{\log_{10} 64}$ **f** $\frac{\log_5 125}{\log_5 5}$

22 You can use the logarithm laws to solve further logarithmic equations.

Consider the equation $\log_5 2x + \log_5 3 = \log_5 12$. The following parts highlight two techniques that can be used to solve such equations.

a **i** Subtract $\log_5 12$ from both sides of the equation and show that the use of the logarithm laws simplifies the equation to $\log_5\left(\frac{x}{2}\right) = 0$.

ii Write the simplified equation from part **i** in index form and solve for x.

b **i** Look at the original equation. Apply the appropriate law to the left side of the equation to show that it simplifies to $\log_5 6x = \log_5 12$.

ii Notice that the same bases appear on both sides. If this is the case, then the number parts of each logarithm can be equated; that is, $6x = 12$. Show that this produces the same value for x as you found in part **a**.

23 Use either method from question **22** to solve each logarithmic equation for x.

a $\log_3 x + \log_3 5 = \log_3 15$ **b** $\log_4 2x + \log_4 8 = 3$

c $\log_5 5x - \log_5 4 = 2$ **d** $\log_5 10 - \log_5 x = \log_5 15$

e $\log_2 (2x + 1) - \log_2 x = 4$ **f** $\log_{10} 2x + \log_{10} 3 = 4 + \log_{10} 5$

CHALLENGE

24 Solve the logarithmic equation $\log_5 (x^2 - 9x + 21) = 0$ for x.

25 Show that the solution to the equation $\log_{10} 2^{2x+1} = \log_{10} 4$ is $\frac{1}{2}$.

26 If $2\log_x\left(\frac{7}{5}\right) + \log_x\left(\frac{704}{35}\right) - 3\log_x\left(\frac{4}{5}\right) = \log_x y$, find the value of y.

Reflect

How do the logarithm laws relate to the index laws?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

rational number
irrational number
whole number
integer
fraction
decimal
terminating decimal
non-terminating decimal
recurring decimal
real number system
surd
operations with surds
simplifying surds
perfect squares
entire surd
product of prime factors
like surds
distributive law
binomial surd
perfect square rules
difference of two squares rule
rational denominator
conjugate
index form
index laws
fractional indices
logarithm
exponent
logarithmic form
logarithm laws

MULTIPLE-CHOICE

3A 10A **1** Which number is irrational?

A $\sqrt{120}$ B $-\sqrt{16}$
C $3\sqrt{9}$ D $\sqrt[3]{8}$

3B 10A **2** Which surd expression is equivalent to $\sqrt{35}$?

A $\sqrt{30} + \sqrt{5}$ B $\sqrt{7} \times 5$
C $\sqrt{7} \times \sqrt{5}$ D $\sqrt{5} \times 5\sqrt{5}$

3C 10A **3** Which of these numbers is the simplest form of $4\sqrt{32}$?

A $8\sqrt{8}$ B $4\sqrt{2}$
C $8\sqrt{2}$ D $16\sqrt{2}$

3D 10A **4** Which surd expression does $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ simplify to?

A $8\sqrt{2}$ B $80\sqrt{2}$
C $16\sqrt{2}$ D $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$

3D 10A **5** After expanding, what does $2\sqrt{6}(4\sqrt{3} + 5\sqrt{10})$ simplify to?

A $8\sqrt{18} + 10\sqrt{60}$ B $24\sqrt{2} + 20\sqrt{15}$
C $24\sqrt{2} + 5\sqrt{10}$ D $72\sqrt{2} + 40\sqrt{15}$

3E 10A **6** Written with a rational denominator, what is $\frac{3\sqrt{5}}{2\sqrt{2}}$ equivalent to?

A $\frac{3\sqrt{10}}{2}$ B $\frac{3\sqrt{10}}{4}$
C $\frac{3\sqrt{7}}{4}$ D $\frac{3\sqrt{10}}{8}$

3F 10A **7** Which of these expressions is *not* equivalent to $x^{\frac{3}{4}}$?

A $\sqrt[4]{x^3}$ B $(\sqrt[4]{x})^3$
C $(\sqrt[3]{x})^4$ D $(x^3)^{\frac{1}{4}}$

3G 10A **8** What is $x^y = z$ equivalent to in logarithmic form?

A $\log_x z = y$ B $\log_x y = z$
C $\log_z x = y$ D $\log_z y = x$

3H 10A **9** What is an equivalent logarithmic expression for $\log_{10} 42$?

A $\log_{10} 20 + \log_{10} 22$
B $\log_{10} 6 + \log_{10} 7$
C $\log_{10} 6 \times \log_{10} 7$
D $\frac{\log_{10} 7}{\log_{10} 6}$

SHORT ANSWER

3A 10A **1** Determine whether each number is rational or irrational.

a $\frac{3}{7}$ **b** 0.256 371 ...

c $-\sqrt{25}$ **d** $\frac{3\sqrt{5}}{2}$

e $4\sqrt{2} + 16$ **f** $0.\overline{215}$

3A 10A **2** Determine which of these numbers are surds.

a $3\sqrt{8}$ **b** $\frac{1}{\sqrt{5}}$ **c** $6\sqrt{4}$

d $\sqrt{6} - 1$ **e** $\sqrt{\frac{9}{25}}$ **f** $\frac{\sqrt{6}}{4}$

3A 10A **3** Use a calculator to find each value, correct to three decimal places.

a $2\sqrt{6}$ **b** $\frac{\sqrt{5} - 2}{4}$

3A 10A **4** Show each number is rational.

a 0.65 **b** 0.2175

c $0.5\dot{2}$ **d** $1.\overline{23}$

3B 10A **5** Calculate each value.

a $\sqrt{11} \times \sqrt{7}$ **b** $3\sqrt{6} \times 9\sqrt{5}$

3B 10A **6** Calculate each value.

a $\sqrt{24} \div \sqrt{8}$ **b** $\frac{4\sqrt{45}}{\sqrt{15}}$

3C 10A **7** Write each expression in simplest form.

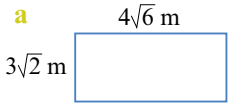
a $2\sqrt{24}$ **b** $3\sqrt{125}$ **c** $\frac{1}{2}\sqrt{108}$

3C 10A **8** Calculate each value, writing each answer in simplest form.

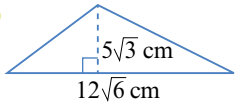
a $3\sqrt{12} \times 4\sqrt{6}$ **b** $-\sqrt{20} \times 5\sqrt{45}$

3C 10A **9** Calculate the exact area for each shape, written in simplest surd form.

a



b



3D 10A **10** Simplify each expression.

a $3\sqrt{2} - 6\sqrt{2} + \sqrt{2}$

b $3\sqrt{7} + 2\sqrt{5} - \sqrt{7} + 3\sqrt{5}$

c $2\sqrt{10} - \sqrt{2} + 4\sqrt{5} + \sqrt{2}$

d $5\sqrt{6} - 2\sqrt{6} + 5 + 3\sqrt{5} - 3$

3D 10A **11** Simplify each expression.

a $2\sqrt{24} - 2\sqrt{54} - 5\sqrt{96} + \sqrt{20}$

b $\sqrt{108} - 2\sqrt{75} + 2\sqrt{48} + 3\sqrt{8}$

c $4\sqrt{180} - 6\sqrt{125} + 3\sqrt{245}$

d $3\sqrt{200} + 4\sqrt{180} + \sqrt{160} - 3\sqrt{128}$

3D 10A **12** Expand and simplify each expression.

a $(2 + 4\sqrt{3})(3 - 2\sqrt{8})$ **b** $(2\sqrt{3} + 4\sqrt{5})^2$

3E 10A **13** Write each fraction with a rational denominator, writing the numerator in simplest surd form where appropriate.

a $\frac{5}{2\sqrt{6}}$ **b** $\frac{3\sqrt{15}}{2\sqrt{5}}$

c $\frac{4\sqrt{5}}{\sqrt{3} + 2\sqrt{6}}$ **d** $\frac{2 + \sqrt{6}}{3\sqrt{7} - 5\sqrt{3}}$

3F 10A **14** Use the appropriate index law to simplify each expression. Write each answer:

i in fractional index form

ii in an appropriate surd form.

a $\left(x^{\frac{3}{5}}\right)^{\frac{5}{2}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$ **c** $\frac{x^{\frac{2}{3}}}{9x^{\frac{1}{4}}}$

d $\sqrt{8x^3y^2} \times \sqrt{3xy}$

e $\left(9x^{\frac{1}{4}}y^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f** $\frac{\sqrt{3x^5y^2z}}{5\sqrt{8x^2yz^2}}$

3G 10A **15** Write each equation in logarithmic form.

a $3^4 = 81$ **b** $6^3 = 216$

c $2^5 = 32$ **d** $x^y = w$

3G 10A **16** Write each equation in index form.

a $\log_2 8 = 3$ **b** $\log_4 64 = 3$

c $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **d** $\log_x z = b$

3H 10A **17** Use the appropriate logarithm law to calculate each value.

a $\log_5 5^3$ **b** $\log_4 96 - \log_4 6$

c $3\log_9 3 + \log_9 27$

d $2\log_3 4 - \log_3 32 + \log_3 54$

3G 10A **18** Solve each logarithmic equation for x.

a $\log_x 32 = 5$ **b** $\log_5 x = -3$

c $\log_4 256 = x$ **d** $\log_x\left(\frac{1}{8}\right) = 3$

e $\log_2 x + \log_2 5 = \log_2 12$

f $\log_5 4x - \log_5 5 = 3$

MIXED PRACTICE

10A **1** Which of these numbers is rational?

A 5.251 681 368… B $\frac{2\sqrt{3}}{5}$

C $-\sqrt{25}$ D $2\sqrt{5}$

10A **2** When rounded to three decimal places, what is the approximation for $\frac{4+2\sqrt{7}}{3\sqrt{5}}$?

10A **3** Which number is irrational?

A $\sqrt[3]{64}$ B $2\sqrt{5}$ C $\sqrt[5]{32}$ D $2\sqrt{100}$

10A **4** Which surd represents the greatest amount?

A $2\sqrt{5}$ B $\frac{5\sqrt{3}}{2}$ C $\sqrt{10}+2$ D $\frac{15}{2\sqrt{2}}$

10A **5** Which surd expression is *not* equivalent to $\sqrt{48}$?

A $2\sqrt{12}$ B $\sqrt{8}\times\sqrt{6}$

C $4\sqrt{3}$ D $2\sqrt{6}\times2\sqrt{2}$

10A **6** What is the value of $-12\sqrt{6}\times\frac{\sqrt{7}}{3}$?

10A **7** What is the value of $\frac{4\sqrt{54}}{5\sqrt{6}}$?

10A **8** Which calculation produces the lowest value?

A $5\sqrt{15}\times2\sqrt{7}$ B $\frac{15\sqrt{250}}{2\sqrt{10}}$

C $\frac{50+2\sqrt{15}}{\sqrt{5}}$ D $\frac{8\sqrt{30}}{\sqrt{2}}$

10A **9** What is the simplest form of $5\sqrt{245}$?

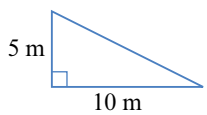
A $7\sqrt{5}$ B $12\sqrt{5}$ C $35\sqrt{5}$ D $57\sqrt{5}$

10A **10** What is $-6\sqrt{120}\times\frac{2}{3}\sqrt{8}$ in simplest form?

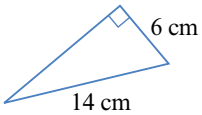
10A **11** What is $\frac{3\sqrt{972}}{8\sqrt{18}}$ in simplest form?

10A **12** Calculate the exact length of the unknown side in each right-angled triangle.

a



b



10A **13** A circle has an area of 45π cm^2. What is the exact radius of the circle?

Questions **14** and **15** refer to the expression $5\sqrt{3}-\sqrt{5}+5+3\sqrt{5}-3$.

10A **14** Which statement is *not* true?

A $3\sqrt{5}$ and $5\sqrt{3}$ are not like surds.

B The expression contains three surd terms and two integer terms.

C The like surds have a combined value of $4\sqrt{5}$ in this expression.

D There are two like surds in the expression.

10A **15** What is the simplest form of the expression?

10A **16** What does $4\sqrt{180}-3\sqrt{80}+5\sqrt{45}$ simplify to?

10A **17** Which expression is equivalent to $2\sqrt{5}(3\sqrt{2}-4\sqrt{7})$?

A $3\sqrt{5}\times3\sqrt{2}-4\sqrt{7}$

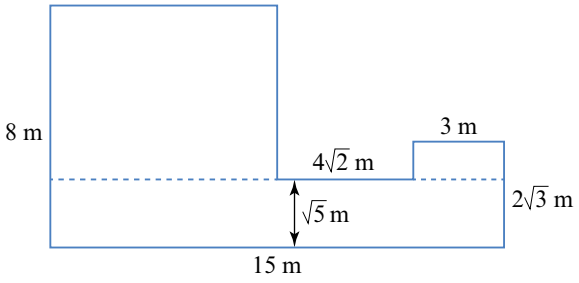
B $2\sqrt{5}\times3\sqrt{2}-2\sqrt{5}\times4\sqrt{7}$

C $2\sqrt{5}\times3\sqrt{2}+2\sqrt{5}\times4\sqrt{7}$

D $3\sqrt{2}\times2\sqrt{5}-3\sqrt{2}\times4\sqrt{7}$

10A **18** When expanded and then simplified, what is $(3-4\sqrt{5})^2$?

Questions **19** and **20** refer to this composite shape, which is divided into three small rectangles.



10A **19** What are the dimensions of the smallest rectangle?

A $2\sqrt{3}$ m × 3 m B $4\sqrt{2}$ m × 3 m

C $(2\sqrt{3}-\sqrt{5})$ m × 3 m D $2\sqrt{3}$ m × 15 m

10A **20** What is the exact area of the shape?

10A **21** When written with a rational denominator, what is $\frac{5\sqrt{6}}{3\sqrt{2}}$?

10A **22** What is the conjugate of $3 - 2\sqrt{7}$?

10A **23** Which expression shows the correct method to write $\frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}}$ with a rational denominator?

A $\frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}} \times \frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}}$ **B** $\frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}} \times \frac{2\sqrt{5} + 3}{2\sqrt{3} + 3}$

C $\frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ **D** $\frac{2\sqrt{5} - 3}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

10A **24** Which expression represents the equivalent form of $\frac{1}{4 - 3\sqrt{5}}$, written with a rational denominator?

A $\frac{1}{4}$ **B** $\frac{4 - 3\sqrt{5}}{-29}$

C $\frac{4 + 3\sqrt{5}}{61}$ **D** $\frac{4 + 3\sqrt{5}}{-29}$

10A **25** Which expression is equivalent to $\sqrt[3]{x^3y}$?

A $xy^{\frac{1}{3}}$ **B** $x\sqrt{y}$ **C** $\sqrt[3]{xy}$ **D** $x^{\frac{4}{3}}y^{\frac{1}{3}}$

10A **26** What is $96^{\frac{1}{2}}$ as a surd in simplest form?

10A **27** What is $\log_5 125 = 3$ equivalent to in index form?

A $3^5 = 125$ **B** $5^3 = 125$

C $5^{125} = 3$ **D** $3^{125} = 5$

10A **28** What is $4^6 = 4096$ equivalent to in logarithmic form?

A $\log_4 4096 = 6$ **B** $\log_4 6 = 4096$

C $\log_6 4 = 4096$ **D** $\log_6 4096 = 4$

10A **29** Calculate $\log_3 243$.

10A **30** Which expression is equivalent to $\log_{10} 35$?

A $\log_{10} 5 \times \log_{10} 7$ **B** $\log_{10} 5 + \log_{10} 7$

C $\frac{\log_{10} 7}{\log_{10} 5}$ **D** $\log_{10} 25 + \log_{10} 10$

10A **31** What is the value of x in $\log_2 x = -4$?

10A **32** What is the solution to $\log_2 3x + \log_2 5 - \log_2 6 = 3$?

A $x = \frac{6}{5}$ **B** $x = \frac{16}{5}$ **C** $x = \frac{5}{16}$ **D** $x = \frac{4}{45}$

ANALYSIS

A storage chest is a cube with a side length of $(7\sqrt{10} + 11\sqrt{7})$ cm.

a Draw a labelled diagram to represent the chest.

b Calculate the exact surface area of one of the faces of the chest.

c Is the area calculated in part **b** a rational or an irrational number?

d All external faces of the chest are to be covered with a special coating to protect it against wear and tear. What is the total area to be covered?

e If the special coating is sold for \$12.80 per m^2, how much does it cost to cover the chest?

An alternative protective paint offers the ultimate protection when three coats are applied. It is advertised that 1 L will cover an area of 5 m^2.

f Show that a 1-L can of the protective paint is enough to provide ultimate protection for the chest.

g Calculate the percentage of the paint in the 1-L can that is used.

A narrow, 80-cm long pole needs to fit inside the chest.

h Show that it is not possible for the pole to lie along the bottom of the chest.

i Is there any way that the pole can fit inside the chest? Use appropriate mathematics to show your working and to support your answer.

j What assumptions need to be considered for the calculations you have performed in this task?

CONNECT

The mathematics of earthquakes

Earthquakes occur all over the world, with some areas more prone to them than others. Many earthquakes are quite weak and result in little or no damage, while others are stronger and cause great damage and loss of life.

You may have heard expressions used to describe the 'strength' of an earthquake, such as 'a magnitude 7.0 earthquake' or 'an earthquake that measured 5.3 on the Richter scale'. This task explores how mathematics can be used to describe and compare the magnitude and intensity of earthquakes.

Class	Magnitude range	Possible effects	Estimated annual frequency worldwide
Micro	2.0–2.9	May not be felt	1 300 000
Minor	3.0–3.9	Can be felt	130 000
Light	4.0–4.9	Can be felt; may be little damage	13 000
Moderate	5.0–5.9	Slight damage to buildings and other structures	1300
Strong	6.0–6.9	Damage in highly populated areas	135
Major	7.0–7.9	Serious damage	15
Great	8.0+	Can totally destroy areas – especially around the epicentre	1

The Richter scale has numerical values that describe the strength of an earthquake. The value, or magnitude, is a measure that relates to the ground's motion (or vibrations) generated by an earthquake. Measurements are recorded using a seismometer.

One formula used to calculate the magnitude of an earthquake is:

$$\text{Magnitude} = \log_{10}\left(\frac{\text{Intensity of earthquake}}{\text{Intensity of 'standard earthquake'}}\right)$$

An earthquake's intensity also relates to ground movement and is measured as the amplitude (in millimetres) of a seismograph reading. The intensity of a 'standard earthquake' is a constant value with amplitude of 1 micron (or 10^{-3} mm).

You may like to present your findings as a report. Your report could include:

- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

Japan, 11 March 2011: 9.0 magnitude earthquake

Christchurch New Zealand, 21 February 2011: 6.3 on the Richter scale

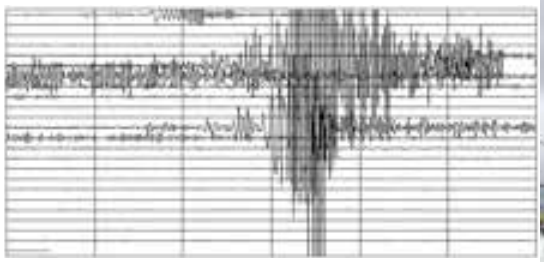
Your task

You are to:

- investigate the formula used to calculate the magnitude of an earthquake
- use the formula to calculate the magnitude of earthquakes with various intensities
- compare the intensity of earthquakes with a magnitude difference of 1 on the Richter scale
- compare the intensity of earthquakes in Japan and New Zealand in 2011
- compare the intensity of earthquakes in other parts of the world
- investigate the relationship between the increase in magnitude of an earthquake and the increase in intensity.

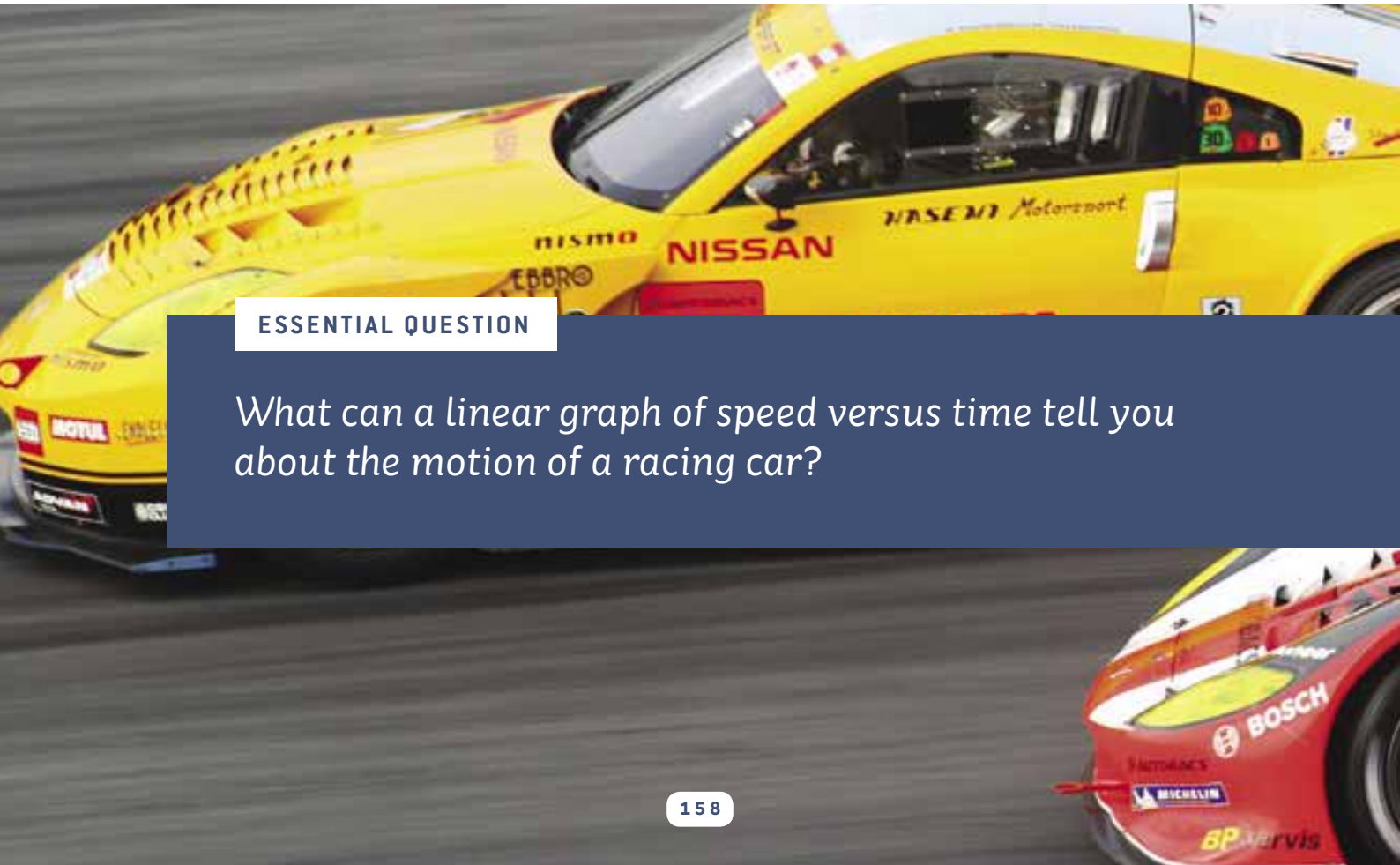
The formula you have considered for magnitude provides a basic introduction to how mathematics is applied to the study of earthquakes. More advanced technology and more complex formulas are used to provide greater accuracy in calculating earthquake magnitudes. As an extension, you may wish to explore further applications of mathematics to earthquakes.

Include examples and all necessary calculations to justify your answers and demonstrate your understanding.



A seismogram shows vibrations of the ground.

4 LINEAR RELATIONSHIPS



ESSENTIAL QUESTION

What can a linear graph of speed versus time tell you about the motion of a racing car?

4A **1** Solve each equation to find the value of x using the balance method.

a $4x = 52$ **b** $x + 11 = 3$

c $x - 7 = -2$ **d** $\frac{x}{9} = -5$

4A **2** Substitute $x = 3$ and $y = -2$ into each expression and evaluate.

a $7x - 10$ **b** $3y - x$

c $2x + 9y$ **d** $\frac{4x + y}{2}$

4A **3** Write each expression in expanded form (without brackets).

a $4(x + 7)$ **b** $-3(5x - 2)$

4A **4** Simplify each expression.

a $6x + 8 + 5x$ **b** $4x - 1 - 7x + 3$

c $2(3x - 1) - x$ **d** $1 - 2x - 5(2 - x)$

4A **5** Calculate each fraction problem.

a $\frac{7}{10} \times \frac{5}{14}$ **b** $\frac{3}{8} + \frac{1}{8}$

c $\frac{4}{5} \times \frac{2}{2} + \frac{1}{2} \times \frac{5}{5}$ **d** $\frac{2}{3} - \frac{1}{2}$

4B **6** Decide whether each statement is true or false.

a $4 > 2$ **b** $7 < 17$

c $-8 < -1$ **d** $-10 > 0$

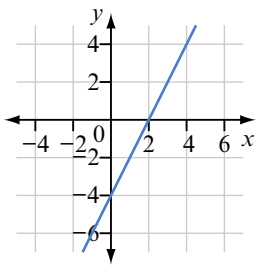
4B **7** Using the fact that x is a whole number in the range -3 to 5, show these values on a number line.

a $x = 2$ **b** $x > 2$

c $x < 4$ **d** $x \geq 0$

e $x \leq -1$ **f** $x > -2$ and $x < 2$

4C **8** Consider this linear graph.



Use the graph to identify:

a the x-intercept

b the y-intercept

c the gradient.

4C **9** **a** Identify c (y-intercept) and m (gradient) for each linear rule.

i $y = 3x - 4$ **ii** $y = -x + 2$

iii $y = 7x$ **iv** $y = -\frac{2}{3}x - 1$

b Use your answers to part **a** to identify the rise and run for each rule.

4C **10** Find the x- and y-intercepts for each rule.

a $4x + y = 4$ **b** $x + 3y = 9$

c $x - 2y = 10$ **d** $2x + y = -6$

4G **11** Multiply each equation by the integer shown in brackets to form an equivalent equation.

a $2x + y = 4$ [3]

b $-x + 3y = -1$ [−2]

c $x - 8y = 6$ [−1]

d $-4x + 5y = 0$ [4]

4A Solving linear equations

Start thinking!

1 Identify which of these are **linear equations**.

a $x + 4 = 7$ **b** $x^2 - 3 = 6$ **c** $10 - 7x = -11$ **d** $\sqrt{x} = 2$

e $2x + 8$ **f** $3x - 1 = 2x + 5$ **g** $-9x = 54$ **h** $x^3 + 4x = x - 1$

2 The equation $x + 9 = 16$ can be solved using the **balance method**.

a What operation should be performed on both sides of the equation?

b What is the solution to the equation?

3 To solve $3x - 4 = 2$ using the balance method, you need to perform two operations to obtain x by itself on one side of the equation.

a Which operation should be performed:

i first? **ii** second?

b Why is it important to perform the operations in the correct order? What would happen if you didn't?

c Copy and complete the steps shown at right to solve the equation $3x - 4 = 2$ using the balance method.

$$3x - 4 = 2$$
$$3x - 4 + 4 = 2 + \_\_$$
$$3x = \_\_$$
$$\frac{3x}{3} = \frac{\_\_}{3}$$
$$x = \_\_$$

d In part **c** you have shown five lines of working. Repeat the working for solving $3x - 4 = 2$ but this time only show three lines. (The original second and fourth lines can be done in your head.)

e How can you check that you have the correct value for the solution?

4 Repeat question **3a** and hence solve each equation.

a $\frac{x - 2}{5} = -4$ **b** $-4x + 7 = 15$

KEY IDEAS

Operation	Inverse operation
+	−
−	+
×	÷
÷	×

- To solve an equation using the balance method, perform operations on both sides of the equation to produce **equivalent equations** until you obtain the solution equation '$x = \ldots$'.
- An operation can be undone by performing the inverse operation.
- The first step in solving equations with a pronumeral term on each side is to remove the pronumeral term from one side. It can be removed by performing the inverse operation (adding or subtracting the term) on both sides so that an equivalent equation is formed.
- In some cases, it may be easier to swap the sides of an equation so that the pronumeral term with the larger coefficient is on the left side.
- A **literal equation** or formula contains two or more different pronumerals. The balance method can also be used to rearrange a literal equation or formula to make a particular pronumeral the **subject of the equation**.

EXERCISE 4A Solving linear equations

EXAMPLE 4A-1 Solving a linear equation

Solve $\frac{2x + 7}{3} = 5$.

THINK

1 First undo '÷ 3' by performing '× 3' on both sides and simplify.

2 Undo '+ 7' by performing '− 7' on both sides and simplify.

3 Undo '× 2' by performing '÷ 2' on both sides and simplify to obtain the solution.

4 Use substitution to check your solution.

WRITE

$\frac{2x + 7}{3} = 5$

$2x + 7 = 15$

$2x = 8$

$x = 4$

$\text{LS} = \frac{8 + 7}{3} = 5 = \text{RS}$

UNDERSTANDING AND FLUENCY

1 Solve each equation.

a $2x + 1 = 11$ b $7x - 4 = 3$ c $\frac{x - 3}{6} = 2$

d $\frac{x + 2}{3} = -1$ e $\frac{x}{2} + 5 = 9$ f $\frac{x}{5} - 4 = -2$

g $\frac{2x}{3} = 4$ h $\frac{5x}{2} = -10$ i $3(x - 1) = 24$

j $-2x - 5 = 9$ k $-3x + 2 = -7$ l $7 - 4x = -1$

m $4 - x = -4$ n $-2(x - 3) = 8$ o $-5(x + 1) = -5$

2 Solve each equation.

a $\frac{3x + 2}{4} = 8$ b $\frac{4x - 5}{7} = -3$ c $\frac{2x}{5} + 4 = 6$

d $\frac{9x}{4} - 1 = -19$ e $\frac{5(x + 2)}{3} = 15$ f $\frac{x - 4}{2} + 5 = 6$

g $\frac{1 - 2x}{5} = -3$ h $\frac{6 - x}{4} - 2 = 0$ i $2(7 - 3x) = -4$

3 Solve each equation. (Hint: perform operations that involve fractions and decimals.)

a $x - \frac{1}{2} = -9$ b $2x + 3.6 = 7$ c $8.5x = 34$

d $\frac{x}{0.28} + 3.2 = -5.6$ e $\frac{3 - 0.5x}{4} = 0.45$ f $\frac{2x - 5}{8} = -3$

4 Solve each equation by first expanding the expression on the left to remove brackets. Where appropriate, write the solution as a fraction.

a $2(x + 4) - 5 = -7$ b $3(x - 1) + 2 = 8$ c $4(x + 3) + 2x = 0$

d $9(x - 5) - x = 3$ e $2(3x + 4) + 7 = 13$ f $7(2x - 11) - 4x = -7$

g $5(2 - x) - 1 = -6$ h $4(1 - 3x) + 3x = 3$ i $-3(4 - x) - 2x + 5 = -1$

EXAMPLE 4A-2 Solving an equation when the variable is in the denominator

Solve $\frac{24}{x} = 4$.

THINK

1 Multiply both sides by x and simplify. You may also like to swap the sides of the equation.

2 Undo '× 4' by performing '÷ 4' on both sides and simplify to obtain the solution.

3 Use substitution to check your solution.

WRITE

$\frac{24}{x} = 4$
$24 = 4x$
$4x = 24$

$x = 6$

$\text{LS} = \frac{24}{6} = 4 = \text{RS}$

5 Solve each equation.

a $\frac{8}{x} = 4$ **b** $\frac{45}{x} = -9$ **c** $\frac{6}{x} = -1$ **d** $\frac{-12}{x} = 3$

e $\frac{-20}{x} = -5$ **f** $\frac{15}{x} = 2$ **g** $\frac{10}{3x} = 1$ **h** $\frac{-2}{5x} = -7$

i $\frac{6}{x} + 5 = 8$ **j** $\frac{11}{x} + 7 = -4$ **k** $\frac{7}{2x} - 1 = 13$ **l** $\frac{-24}{5x} - 2 = 0$

EXAMPLE 4A-3 Solving an equation with an unknown on both sides

Solve each equation. **a** $5x + 7 = 3x - 5$ **b** $3x - 4 = 11 - 2x$

THINK

a 1 Since there is a pronumeral term on each side, the first step is to remove one of them (choose $3x$ from the RS). Undo '$+ 3x$' by performing '$- 3x$' on both sides.

2 Undo '$+ 7$' by performing '$- 7$' on both sides.

3 Undo '$\times 2$' by performing '$\div 2$' on both sides to obtain the solution.

4 Use substitution to check your solution.

b 1 Remove one of the pronumeral terms (choose $-2x$ from the RS). Undo '$-2x$' by performing '$+2x$' on both sides.

2 Undo '$- 4$' by performing '$+ 4$' on both sides.

3 Undo '$\times 5$' by performing '$\div 5$' on both sides to obtain the solution.

4 Use substitution to check your solution.

WRITE

a $5x + 7 = 3x - 5$
$2x + 7 = -5$

$2x = -12$

$x = -6$

$\text{LS} = -30 + 7 = -23$
$\text{RS} = -18 - 5 = -23$

b $3x - 4 = 11 - 2x$
$5x - 4 = 11$

$5x = 15$

$x = 3$

$\text{LS} = 9 - 4 = 5$
$\text{RS} = 11 - 6 = 5$

UNDERSTANDING AND FLUENCY

6 Solve each equation.

a $5x + 3 = 2x + 9$ **b** $6x - 7 = 4x + 3$ **c** $3x + 4 = x - 2$

d $4x - 5 = 2x - 3$ **e** $3x + 8 = 2x + 1$ **f** $5x + 6 = 2 + 3x$

g $4x - 5 = 13 - 2x$ **h** $2x - 1 = 11 - x$ **i** $2x + 3 = 6 - 5x$

j $3x - 5 = 7x + 11$ **k** $2 - x = 4x - 8$ **l** $8x + 5 = 11x + 20$

7 Solve each equation by first expanding to remove brackets.

a $3(x + 2) = 2x + 11$ **b** $8x - 9 = 5(x - 3)$ **c** $7x + 3 = 6(x + 1)$

d $5(2x + 1) = 7x + 2$ **e** $4(x - 7) = 2 - x$ **f** $x - 4 = 2(3 - 2x)$

g $3(x + 6) = 2(x + 7)$ **h** $9(x - 3) = 5(x + 1)$ **i** $4(x - 1) = 2(x + 2)$

j $6(x + 5) = 4(5 - x)$ **k** $2(3x - 4) = 11(x - 3)$ **l** $7(x - 12) = -3(x - 2)$

8 Solve each equation. Start by multiplying both sides of the equation by the common denominator.

a $\frac{2x + 1}{5} = \frac{x - 7}{5}$ **b** $\frac{4x - 3}{3} = \frac{2x + 5}{3}$ **c** $\frac{7x - 2}{4} = \frac{3x - 10}{4}$

d $\frac{x + 5}{6} = \frac{11 - 2x}{6}$ **e** $\frac{9 - 2x}{11} = \frac{4 - 7x}{11}$ **f** $\frac{3(x - 4)}{2} = \frac{4 - x}{2}$

EXAMPLE 4A-4 Solving an equation with algebraic fractions on both sides

Solve $\frac{x - 3}{2} = \frac{x + 4}{9}$.

THINK	WRITE
	$\frac{x - 3}{2} = \frac{x + 4}{9}$
1 Write each fraction as an equivalent fraction with the same denominator. LCD = 18 so multiply left side by $\frac{9}{9}$ and right side by $\frac{2}{2}$.	$\frac{x - 3}{2} \times \frac{9}{9} = \frac{x + 4}{9} \times \frac{2}{2}$
	$\frac{9(x - 3)}{18} = \frac{2(x + 4)}{18}$
2 Multiply both sides by 18 to obtain an equivalent equation without denominators.	$9(x - 3) = 2(x + 4)$
3 Expand the expressions on each side of the equation.	$9x - 27 = 2x + 8$
4 Undo '$+ 2x$' by performing '$- 2x$' on both sides.	$7x - 27 = 8$
5 Undo '$- 27$' by performing '$+ 27$' on both sides.	$7x = 35$
6 Undo '$\times 7$' by performing '$\div 7$' on both sides to obtain the solution.	$x = 5$
7 Check your solution.	LS $= \frac{5 - 3}{2} = 1$
	RS $= \frac{5 + 4}{9} = 1$

UNDERSTANDING AND FLUENCY

9 Solve each equation. (Hint: some solutions may not be whole numbers.)

a $\frac{x-2}{5} = \frac{x+4}{2}$ **b** $\frac{x-4}{3} = \frac{x+5}{4}$ **c** $\frac{x+2}{2} = \frac{x+7}{3}$

d $\frac{x+3}{4} = \frac{x+6}{7}$ **e** $\frac{x-8}{3} = \frac{x+1}{6}$ **f** $\frac{3x+5}{4} = \frac{x+9}{2}$

g $\frac{2x+1}{5} = \frac{x-2}{4}$ **h** $\frac{x-7}{3} = \frac{5-x}{4}$ **i** $\frac{2x-1}{6} = \frac{3x+2}{5}$

10 Make x the subject of each formula by rearranging it using the balance method to obtain x by itself on the left side.

a $x + a = b$ **b** $k = x - p$ **c** $cx = d$

d $g + 3x = h$ **e** $y = 4x + 5$ **f** $\frac{a+x}{m} = n$

g $7x - 2w = 5$ **h** $b - \frac{x}{a} = c$ **i** $k = \frac{h}{x}$

j $6x - e = 4x + f$ **k** $2(x + y) = v$ **l** $\frac{x}{p} + q = w$

m $\frac{a}{x} - b = c$ **n** $y = \frac{kx - m}{2n}$ **o** $x + 3d = 5c - 7x$

PROBLEM SOLVING AND REASONING

11 Tom is saving to buy a tennis racquet that costs $219. He is able to save $24 per week and currently has $105. Follow the steps below to work out when he can buy the racquet.

a Represent the unknown quantity in the problem with a pronumeral.

b Use this pronumeral to write an equation to represent the problem.

c Solve the equation using the balance method.

d Write your answer to the problem.

12 For each problem, set up an equation and solve it using the balance method.

a Ruby bought four makeup kits online for a total cost of $387, which included the delivery charge of $29. What is the cost of each makeup kit?

b The perimeter of a rectangular car park is 88 m. If the length is 12 m more than the width, what are the dimensions of the car park?

c Lisa, Nicole and Tania scored a total of 29 goals in a netball match. Lisa scored five more goals than Nicole, and Tania scored three goals less than Nicole. How many goals did each girl score?

d Matt has a budget of $1800 for his birthday party. The cost of hiring a local hall is $400 and catering is $27 per person. What is the maximum number of people that can attend?

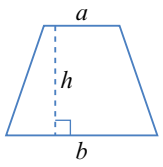
13 Shana and Paulo have the same amount of money. Shana buys six dumplings and has $3.80 left over. Paulo buys four dumplings and has $9.20 left over.

a Write an equation to represent this situation.

b Solve the equation to find the cost of one dumpling.

PROBLEM SOLVING AND REASONING

14 The formula $A = \frac{h(a+b)}{2}$ links the area of a trapezium (A) with three length measurements a, b and h, shown in the diagram.



- **a** Calculate the area of a trapezium when $a = 4$ cm, $b = 6$ cm and $h = 10$ cm.
- **b** Calculate the height h of a trapezium when $a = 15$ mm, $b = 19$ mm and $A = 119$ mm^2. (Hint: substitute the values for a, b and A into the formula and then solve the resulting equation to find h.)
- **c** Calculate a when $b = 5$ m, $h = 3$ m and $A = 18$ m^2.
- **d** Rearrange the formula to make a the subject.
- **e** Calculate a when:
  - **i** $b = 37$ cm, $h = 24$ cm and $A = 1068$ cm^2
  - **ii** $b = 2.4$ m, $h = 6.1$ m and $A = 30.5$ m^2

15 The formula $v = u + at$ can be used to calculate the final speed (v) after an amount of time (t) if you know the initial speed (u) and the acceleration (a).

- **a** What is the final speed of an object if it accelerates at 2 m/s^2 for 10 seconds from an initial speed of 14 m/s?
- **b** How long does it take an object to increase its speed from 18 m/s to 40 m/s if it is accelerating at 4 m/s^2?

CHALLENGE

16 The sum of five consecutive odd numbers is 50 more than triple the smallest number. Write an equation to solve to find the five numbers. (Hint: let the smallest number be x.)

17 Tim made cupcakes for the school fete. Twenty-five of these were sold with chocolate icing. One-third of those left were sold with lemon icing. If there were 40 cupcakes sold without icing, how many cupcakes did Tim make? Show your reasoning by writing and solving an equation.

18 Solve each equation. Use substitution to check your solutions.

- **a** $\frac{3(2-x)}{4} + 9 = 15$
- **b** $\frac{4(2x-1)}{3} - 5 = 4x - 17$
- **c** $\frac{2x-3}{5} + \frac{x+7}{3} = 1$
- **d** $\frac{x+5}{2} - \frac{x-7}{6} = 8$
- **e** $\frac{3(2x-3)}{4} + \frac{x+23}{10} = 2x$
- **f** $\frac{5x+6}{3} - \frac{1-2x}{4} = 3x + 4$

19 Investigate the 'solve' function of a calculator or other digital technology. Use this function to solve each equation in question **18** and compare your solutions.

Reflect

When solving an equation, how do you know the correct order to perform inverse operations?

4B Solving linear inequalities

Start thinking!

An inequality is a mathematical statement that uses one of the symbols $<$, $\leq$, $>$ or $\geq$.

1 Describe what each symbol means.

2 Complete these statements with the correct symbol.

a $5 \_\_ 2$ **b** $3 \_\_ 7$ **c** $-4 \_\_ 0$ **d** $-2 \_\_ -6$

3 Describe each of these inequality statements. What does it tell you about x?

a $x > 2$ **b** $x \leq 3$ **c** $x \geq 5$ **d** $x < -4$

4 Discuss with a classmate how you might show each inequality in question **3** on a number line.

5 List three values for x that make each statement true.

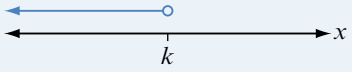
a $x > 7$ **b** $x < 10$ **c** $x \geq -3$ **d** $x \leq -5$

6 What does $2 < x < 9$ mean? List three values of x that would make this statement true.

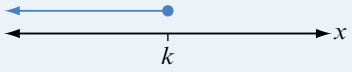
7 List the integers that make $-4 \leq x < 3$ a true statement.

KEY IDEAS

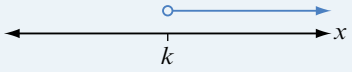
- The four inequality signs, $<$, $>$, $\leq$ and $\geq$, are used to form inequality statements.
  - $x < k$ means x is less than k



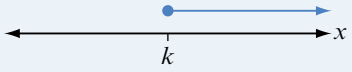
  - $x \leq k$ means x is less than or equal to k



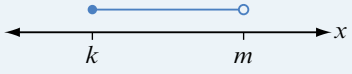
  - $x > k$ means x is greater than k



  - $x \geq k$ means x is greater than or equal to k



  - $k \leq x < m$ means $k \leq x$ and $x < m$, or x is greater than or equal to k but less than m



- On a number line:
  - a closed circle (solid dot •), indicates that the endpoint value is included ($\leq$ or $\geq$)
  - an open circle (○), indicates that the endpoint value is not included ($<$ or $>$).
- A linear inequality can be solved using the balance method. When performing an operation on both sides of an inequality, the inequality sign:
  - stays the same when you add or subtract any number
  - stays the same when you multiply or divide by a positive number
  - is reversed when you multiply or divide by a negative number.
- The inequality sign must be reversed if the sides of an inequality are swapped. For example, if $-3 < 7$ then $7 > -3$ or if $5 \geq x$ then $x \leq 5$.

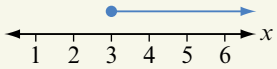
EXERCISE 4B Solving linear inequalities

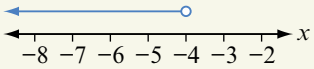
1 Consider this list of x values: -5, 2.4, -1.2, $\frac{1}{2}$, 0, $-\frac{3}{4}$, 3, 7, -10, -4.9, 8.3, $6\frac{4}{5}$.
For each inequality, choose the x values from the list that make it a true statement.

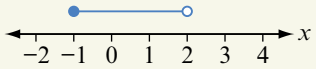
a $x > 1$ **b** $x \leq 0$ **c** $x \geq 4$ **d** $x \leq -2$
e $1 \leq x \leq 8$ **f** $0 < x < 7$ **g** $-5 \leq x < 3$ **h** $-9 < x \leq -\frac{1}{2}$

EXAMPLE 4B-1 Writing inequalities from a number line

Write the inequality that is represented on each number line.

a


b


c


THINK

a The line and arrowhead show increasing values from 3. The closed circle (solid dot) at 3 indicates that 3 is included.

b The line and arrowhead show decreasing values from −4. The open circle at −4 indicates that −4 is not included.

c The line shows the endpoints are −1 and 2. The closed circle at −1 indicates that −1 is included. The open circle at 2 indicates that 2 is not included.

WRITE

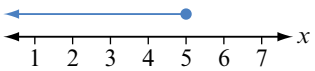
a $x \geq 3$

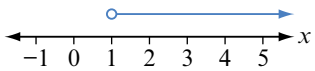
b $x < -4$

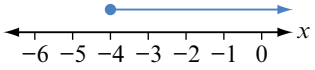
c $-1 \leq x < 2$

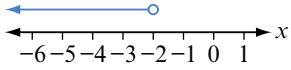
UNDERSTANDING AND FLUENCY

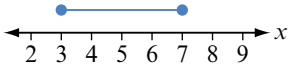
2 Write the inequality that is represented on each number line.

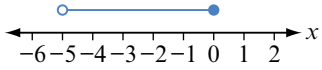
a


b


c


d


e


f


3 Represent the values of x for each inequality on a number line.

a $x > 4$ **b** $x \leq 3$ **c** $x \geq -2$ **d** $x < 0$
e $1 \leq x \leq 5$ **f** $-4 < x < 2$ **g** $0 < x \leq 6$ **h** $-2.5 \leq x < -0.5$

4 Consider the statement $2 < 6$.

a Add each of the numbers below to both sides of the statement. Is the statement still true in each case?

i 1 **ii** 3 **iii** 7 **iv** −4

b Subtract each number below from both sides of the statement. Is the statement still true in each case?

i 1 **ii** 3 **iii** 7 **iv** −4

c Multiply both sides of the statement by each number. Is the statement still true?
i 2 **ii** −2 **iii** 3 **iv** −1

d Divide both sides of the statement by each number. Is the statement still true?
i 2 **ii** −2 **iii** 3 **iv** −1

e Write a summary of what you have found.

f Write examples using $4 > -8$ to demonstrate that the inequality sign:
i stays the same when you add or subtract any number
ii stays the same when you multiply or divide by a positive number
iii is reversed when you multiply or divide by a negative number.

5 Perform the operation shown in brackets to both sides of each statement. Think about whether the inequality sign stays the same or is reversed so that the statement remains true.

a $x > 5$ [+ 3] **b** $x \leq -4$ [× −2] **c** $-x \geq -10$ [÷ −1]
d $3x < 12$ [÷ 3] **e** $5 - x \geq -4$ [− 5] **f** $\frac{x}{8} > -2$ [× 8]
g $-4x \leq -28$ [÷ −4] **h** $-\frac{x}{3} > 1$ [× 3] **i** $6x < 5$ [× −1]

EXAMPLE 4B-2 Solving inequalities (× and ÷ by positive number)

Solve each inequality.
a $4x + 7 > 3$ **b** $\frac{x-1}{3} \leq -5$

THINK

a 1 First subtract 7 from both sides to undo the '+ 7' operation. No change to the equality sign as you are subtracting.

2 Divide both sides by 4 to obtain the solution. No change to the inequality sign as you are dividing by a positive number.

3 Check your solution. Use $x = -1$ to check the endpoint value and a value greater than −1 (such as $x = 0$) to check the inequality sign.

b 1 First multiply both sides by 3 to undo the '÷ 3' operation. No change to the equality sign.

2 Add 1 to both sides to obtain the solution. No change to the inequality sign.

3 Check your solution. Use $x = -14$ to check the endpoint value and $x = -15$ to check the inequality sign.

WRITE

a $4x + 7 > 3$
$4x > -4$
$x > -1$

When $x = -1$,
LS $= -4 + 7 = 3 =$ RS
When $x = 0$,
LS $= 0 + 7 = 7 >$ RS

b $\frac{x-1}{3} \leq -5$
$x - 1 \leq -15$
$x \leq -14$

When $x = -14$,
LS $= \frac{-14-1}{3} = -5 =$ RS
When $x = -15$,
LS $= \frac{-15-1}{3} = -5\frac{1}{3} \leq$ RS

6 Solve each inequality.

a $x + 4 < 9$ **b** $x - 2 \geq 5$ **c** $3x \leq -18$

d $\frac{x}{6} > 1$ **e** $2x + 3 \leq -7$ **f** $\frac{x}{5} - 8 < 2$

g $\frac{x+2}{3} \geq -6$ **h** $7x - 5 \leq 9$ **i** $\frac{x-4}{9} > -1$

EXAMPLE 4B-3 Solving inequalities (× and ÷ by negative number)

Solve each inequality.

a $-5x - 3 < 7$ **b** $1 - \frac{x}{8} \geq -2$

THINK	WRITE
a 1 First add 3 to both sides. No change to the inequality sign.	**b** $-5x - 3 < 7$ $-5x < 10$
2 Divide both sides by -5 and reverse the inequality sign to obtain the solution.	$x > -2$
3 Check your solution. Use $x = -2$ to check the endpoint value and $x = 0$ to check the inequality sign.	When $x = -2$, LS $= 10 - 3 = 7 =$ RS When $x = 0$, LS $= 0 - 3 = -3 <$ RS
b 1 First subtract 1 from both sides. No change to the inequality sign.	**b** $1 - \frac{x}{8} \geq -2$ $-\frac{x}{8} \geq -3$
2 Multiply both sides by -8 and reverse the inequality sign to obtain the solution.	$x \leq 24$
3 Check your solution. Use $x = 24$ to check the endpoint value and $x = 16$ to check the inequality sign.	When $x = 24$, LS $= 1 - 3 = -2 =$ RS When $x = 16$, LS $= 1 - 2 = -1 \geq$ RS

UNDERSTANDING AND FLUENCY

7 Solve each inequality.

a $-5x \leq 30$ **b** $-\frac{x}{2} > 1$ **c** $-3x \geq -9$

d $-\frac{x}{6} < 4$ **e** $-2x - 3 > 11$ **f** $-9x + 1 \leq -8$

g $-\frac{x}{2} - 5 < 7$ **h** $9 - 4x > -19$ **i** $4 - \frac{x}{5} \geq 2$

8 Solve each inequality and show the solution on a number line.

a $3x + 2 \geq -13$ **b** $1 - 7x < -6$ **c** $5 - \frac{x}{6} \leq 2$

d $\frac{x-4}{2} < 5$ **e** $\frac{7-x}{3} \geq -1$ **f** $\frac{2+x}{5} > 3$

g $\frac{3x}{2} - 5 \leq 4$ **h** $\frac{8-x}{7} - 3 < -2$ **i** $\frac{1-3x}{5} > 2$

EXAMPLE 4B-4 Solving inequalities with the unknown on both sides

Solve each inequality. Use substitution to check your solutions.

a $8x + 2 < 5x - 13$ **b** $3 - 2x \geq x - 3$

THINK

a **1** Subtract $5x$ from both sides.

2 Subtract 2 from both sides.

3 Divide both sides by 3 to obtain the solution. No change to inequality sign.

4 Check your solution. Use $x = -5$ to check the endpoint value and $x = -6$ to check the inequality sign.

b **1** Add $2x$ to both sides.

2 Swap sides of equation. Reverse the inequality sign to keep the statement true.

3 Add 3 to both sides.

4 Divide both sides by 3 to obtain the solution.

5 Check your solution. Use $x = 2$ to check the endpoint value and $x = 1$ to check the inequality sign.

WRITE

a $8x + 2 < 5x - 13$
$3x + 2 < -13$
$3x > -15$
$x < -5$

When $x = -5$, LS $= -40 + 2 = -38$,
RS $= -25 - 13 = -38$, so LS = RS
When $x = -6$, LS $= -48 + 2 = -46$,
RS $= -30 - 13 = -43$, so LS < RS

b $3 - 2x \geq x - 3$
$3 \geq 3x - 3$
$3x - 3 \leq 3$
$3x \leq 6$
$x \leq 2$

When $x = 2$, LS $= 3 - 4 = -1$,
RS $= 2 - 3 = -1$, so LS = RS
When $x = 1$, LS $= 3 - 2 = 1$,
RS $= 1 - 3 = -2$, so LS ≥ RS

UNDERSTANDING AND FLUENCY

9 Solve each inequality. Use substitution to check your solutions.

a $4x - 3 > 2x + 5$ **b** $7x + 1 \leq 3x - 7$ **c** $5x - 9 \geq 9 - x$
d $2x + 4 < 5x - 8$ **e** $3 - x > 4x - 2$ **f** $x + 11 \geq 7 - 3x$
g $8 - 3x \leq 18 - 5x$ **h** $1 - 6x > -3x - 5$ **i** $2 - 9x < x - 3$

10 Solve each inequality by first expanding to remove brackets.

a $2(x - 3) > x + 5$ **b** $4x - 7 \leq 3(x + 2)$ **c** $5(x + 1) \geq -3x - 11$
d $3(2x - 5) < 7x + 4$ **e** $9(x + 5) < 10(x + 6)$ **f** $3(3x - 1) \leq -2(x - 4)$
g $4(x + 3) \geq 2(5x - 3)$ **h** $2(2 - x) > 3(1 - 2x)$ **i** $5(3 - 2x) \leq -4(2x - 7)$

11 Solve each inequality. Use substitution to check your solutions.

a $\frac{3x + 2}{7} \leq \frac{2x - 5}{7}$ **b** $\frac{2 - x}{3} < \frac{6x - 5}{3}$ **c** $\frac{1 - 4x}{2} \geq \frac{11 - 5x}{4}$
d $\frac{x + 4}{5} > \frac{x + 2}{3}$ **e** $\frac{x - 4}{7} \geq \frac{4 - x}{2}$ **f** $\frac{4x - 1}{6} < \frac{3x + 1}{4}$

12 **a** How many solutions does a linear equation have?
b How many solutions can a linear inequality have? Explain.

PROBLEM SOLVING AND REASONING

13 Write an inequality statement to represent each situation. Use a pronumeral for the unknown quantity.

- **a** A house with a reserve price of $650 000 is sold at auction. What price did the house sell for?
- **b** To go on a theme park ride, a person must be over 97 cm tall. What is the height of a person on this ride?
- **c** The speed limit along a road is 60 km/h. At what speed could a person legally travel along this road?
- **d** For the Jet Rescue ride at Sea World, your height must be in the range 125–196 cm. What is the height of a person on this ride?

14 Todd is deciding how many packs of chewing gum he should buy. Each pack costs $3 and he has $20 in his wallet.

- **a** Write an inequality to represent this situation and then solve it.
- **b** List the number of packs of gum he could possibly buy.

15 Emily and Ethan are selling watermelons at a market. They start with 20 melons and agree to share any that are left over at the end of the day.

- **a** Write an expression for the number of melons each will take home if they sell x watermelons.
- **b** They aim to take home no more than three watermelons each. Use your expression from part **a** to write an inequality for this situation and then solve it.
- **c** How many watermelons can they sell to meet their goal?

16 Jada has a pre-paid phone card with a credit of $50 to make calls from Australia to friends and family travelling overseas. For each call, she is charged a connection fee of 49c and a per minute rate depending on the country she is calling.

- **a** For how many minutes could she talk to her friend travelling in Morocco if she didn't go over her limit?
- **b** Jada's friend is now in Egypt. For how many minutes could Jada talk if she wishes to save at least $25 for another call?
- **c** Instead, Jada decides to make two calls. The first call is for 12 minutes to her brother in Japan. How long is her second call to a friend in India if she uses between $35 and $50 for both calls?

Country	Per minute rate
Egypt	$1.47
India	$1.35
Japan	$0.80
Morocco	$1.95

CHALLENGE

17 Solve each inequality.

- **a** $\frac{2(4-x)}{5} - 3 \leq 3$
- **b** $\frac{3(7x-4)}{2} - 8 < 6x + 4$
- **c** $\frac{5x-1}{3} + \frac{x-2}{4} \geq 3$
- **d** $\frac{x-4}{8} - \frac{x+3}{2} > 1$
- **e** $\frac{4(2x+1)}{5} + \frac{7-2x}{3} \geq 2x + 1$
- **f** $\frac{3x+4}{2} - \frac{x+4}{6} \leq \frac{x-5}{3}$

18 Solve each inequality.

- **a** $-3 \leq 2x - 5 \leq 7$
- **b** $15 < \frac{5(1-2x)}{3} < 25$

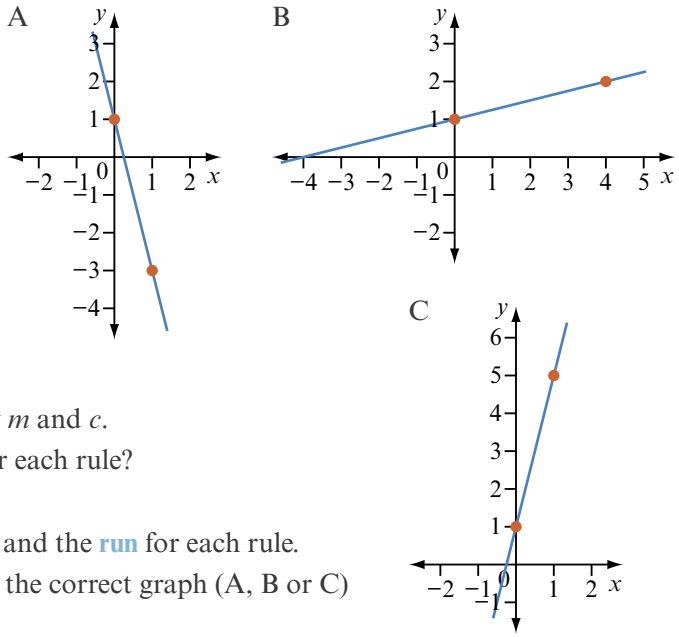
19 Investigate the 'solve' function of a calculator or other digital technology. Use this function to solve each inequality in question **17** and compare your solutions.

Reflect

How does solving linear inequalities compare to solving linear equations?

4C Sketching linear graphs

Start thinking!



1 Describe the graph of a **linear relationship**.

2 Why do you need only two points to produce a sketch graph of a linear relationship?

One way of sketching a **linear graph** is to use the **gradient–intercept method**. This provides a way of finding the two points you need to plot.

3 Consider these three linear rules.

i $y = 4x + 1$ **ii** $y = -4x + 1$ **iii** $y = \frac{1}{4}x + 1$

a Compare each rule to $y = mx + c$ and identify m and c.

b What are the coordinates of the **y-intercept** for each rule?

c What is the **gradient** for each rule?

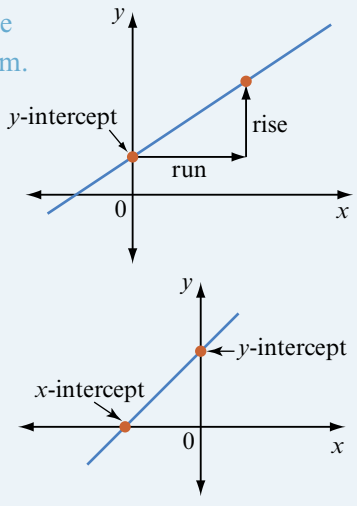
d Use your answers to part **c** to identify the **rise** and the **run** for each rule.

e Using the y-intercept and the gradient, match the correct graph (A, B or C) to each rule. Give reasons for your answers.

4 In your own words, explain how to use the gradient–intercept method to sketch a linear graph.

KEY IDEAS

- To sketch a linear graph, you require a minimum of two points. Once these two points are plotted, you can rule a straight line through them.
- In the gradient–intercept method, a point is plotted at the y-intercept and then the rise and run (identified from the gradient) is used to locate the second point.
- The general rule for a linear graph is $y = mx + c$, where m is the gradient and c is the y-intercept.
- In the ***x*- and *y*-intercept method**, one point is plotted at the ***x*-intercept** and another at the y-intercept.
- The x-intercept can be found by substituting $y = 0$ into the rule and solving for x.
- The y-intercept can be found by substituting $x = 0$ into the rule and solving for y.
- For a **line segment** between the points (x_1, y_1) and (x_2, y_2):
  - the length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - the coordinates of the **midpoint** are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



EXERCISE 4C Sketching linear graphs

EXAMPLE 4C-1 Using the gradient–intercept method to sketch a linear graph

Sketch the graph of $y = 3x - 2$ using the gradient–intercept method.

THINK

1 Compare the rule to the general formula for a linear graph, $y = mx + c$. Identify m (gradient) and c (y-intercept).

2 Write the coordinates of the y-intercept.

3 Write the gradient as a fraction and identify the rise and the run.

4 Plot a point at the y-intercept.

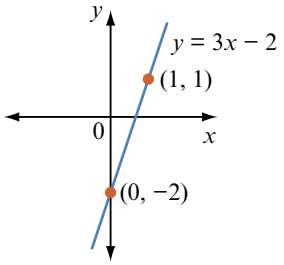
5 From this point, use the rise and the run to locate a second point (move 3 units up and 1 unit right). The second point is at (1, 1).

6 Rule a straight line through the two plotted points and label with the rule.

WRITE

$y = 3x - 2$
$m = 3, c = -2$

y-intercept at $(0, -2)$
gradient $= 3 = \frac{3}{1}$
rise = 3, run = 1



UNDERSTANDING AND FLUENCY

1 Sketch the graph of each linear rule using the gradient–intercept method.

a $y = 2x - 4$ b $y = \frac{2}{3}x + 1$ c $y = -3x + 5$
d $y = \frac{5}{4}x - 2$ e $y = -5x - 1$ f $y = -\frac{1}{2}x + 3$
g $y = 3x - 4$ h $y = x + 6$ i $y = -x - 3$

2 Rearrange each rule so it is in the form $y = mx + c$ and hence identify:
i the gradient ii the y-intercept.

a $y = 1 + 2x$ b $y = 4 - 3x$ c $y = -3 - \frac{3}{4}x$
d $y = 3(x - 2)$ e $y = -2(x - 3)$ f $y - 4 = -(5x + 6)$
g $3x + y = 2$ h $-4x + y = -5$ i $6x + 3y = 9$
j $-4x + 5y = 15$ k $3x - 2y = -4$ l $x + 3y + 6 = 0$

3 Sketch the graph of each linear rule in question 2 using the gradient–intercept method.

4 a Write $y = 3x$ in the form $y = mx + c$.
b What is the value of: i m? ii c?
c Sketch the graph of $y = 3x$ using the gradient–intercept method.

UNDERSTANDING AND FLUENCY

5 Sketch each linear rule using the gradient–intercept method.

a $y = 2x$ **b** $y = -5x$ **c** $y = -x$ **d** $y = \frac{2}{5}x$

6 **a** Write $y = 4$ in the form $y = mx + c$.

b What is the value of: **i** m? **ii** c?

c Sketch the graph of $y = 4$ using the gradient–intercept method.

7 Sketch each linear rule using the gradient–intercept method.

a $y = 6$ **b** $y = -2$ **c** $y = \frac{1}{3}$ **d** $y = -\frac{7}{2}$

8 Consider the linear graph with the rule $x = 3$. Can this rule be written in the form $y = mx + c$? Explain.

EXAMPLE 4C-2 Using the x- and y-intercept method to sketch a linear graph

Sketch the graph of $2x - 3y = 6$ using the x- and y-intercept method.

THINK

1 Write the rule.

2 Find the x-intercept by substituting $y = 0$ into the rule and solving for x.

3 Find the y-intercept by substituting $x = 0$ into the rule and solving for y.

4 Plot the points for the x- and y-intercepts on a Cartesian plane.

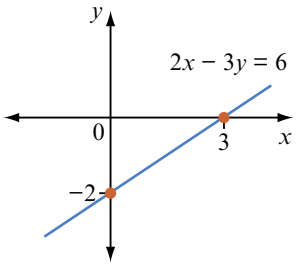
5 Rule a straight line through the points and label with the rule.

WRITE

$2x - 3y = 6$

x-intercept: when $y = 0$,
$2x - 3 \times 0 = 6$
$2x = 6$
$x = 3$
x-intercept is 3.
Coordinates of the x-intercept are (3, 0).

y-intercept: when $x = 0$,
$2 \times 0 - 3y = 6$
$-3y = 6$
$y = -2$
y-intercept is -2.
Coordinates of the y-intercept are (0, -2).



9 Sketch the graph of each linear rule using the x- and y-intercept method.

a $x + 2y = 2$ **b** $3x + y = -6$ **c** $2x + 4y = 8$

d $5x + 10y = 20$ **e** $4x + 3y = -12$ **f** $-6x + 2y = 6$

g $8x - y = -8$ **h** $-x + 4y = 2$ **i** $9x - 3y = -18$

j $-12x + 9y = 36$ **k** $7x + 4y = 21$ **l** $2x - y = 5$

10 Find the x-intercept and the y-intercept for each of these.

a $y = x + 5$ **b** $y = x - 4$ **c** $y = 3x + 6$

d $y = 2x - 8$ **e** $y = 2 - x$ **f** $y = -6x - 3$

UNDERSTANDING AND FLUENCY

11 Use your answers to question **10** to sketch the graph of each linear relationship.

12 Sketch the graph of each of these using the x- and y-intercept method.

a $2y = 3x + 6$ **b** $4y = 4x - 8$ **c** $x + 3y - 3 = 0$

d $3x + 6y + 18 = 0$ **e** $5x - 2y - 10 = 0$ **f** $2x - y - 7 = 0$

13 Consider the linear rule $y = 2x$.

a Find: **i** the x-intercept **ii** the y-intercept.

b Do you have enough information from part **a** to sketch the graph? Explain.

c What other information do you need? Discuss this with a classmate.

Another way of identifying a second point to plot is to find the matching y value for a chosen x value. Any x value can be used, but we often choose $x = 1$.

d Find the value of y when x is 1. (Hint: substitute $x = 1$ into the rule.)

e List the coordinates of the two points you can now use to sketch the graph of $y = 2x$.

f Plot the two points and rule a straight line through them to produce the sketch graph of $y = 2x$. Label your graph with its rule.

14 Use the approach from question **13** to sketch each linear graph with these rules.

a $y = 4x$ **b** $y = -3x$ **c** $y = x$ **d** $y = -\frac{1}{2}x$

EXAMPLE 4C-3 Sketching horizontal and vertical linear graphs

For each linear rule: **i** list the coordinates of any intercepts **ii** sketch the graph.

a $y = 5$ **b** $x = -2$

THINK

a 1 Write the rule.

2 Find any x- and y-intercepts. As the y-coordinate is 5 for any value of x, the graph will be a horizontal line passing through (0, 5).

3 Rule a horizontal line through (0, 5) to produce the sketch graph.

b 1 Write the rule.

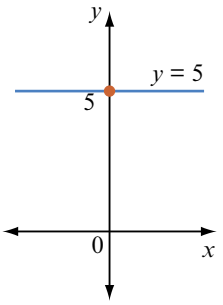
2 Find any x- and y-intercepts. As the x-coordinate is −2 for any value of y, the graph will be a vertical line passing through (−2, 0).

3 Rule a vertical line through (−2, 0) to produce the sketch graph.

WRITE

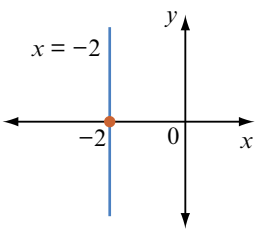
a $y = 5$

No x-intercept.
Coordinates of the y-intercept are (0, 5).



b $x = -2$

Coordinates of the x-intercept are (−2, 0).
No y-intercept.



UNDERSTANDING AND FLUENCY

15 For each linear rule:

i list the coordinates of any intercepts **ii** sketch the graph.

a $y = 8$ **b** $x = 3$ **c** $y = -5$ **d** $x = -1$

e $x = 7$ **f** $y = 0$ **g** $x = 0$ **h** $x = -4.5$

16 How many axis intercepts does the graph of each linear rule have? List them.

a $y = 2x - 4$ **b** $y = -7$ **c** $4x + 6y = 12$

d $y = 9x$ **e** $y = 5(x - 10)$ **f** $3x - 5y + 15 = 0$

g $x = 28$ **h** $x = 3y + 6$ **i** $y = 5.6$

17 Describe which method you think would be the best to use to sketch each of these.

a linear graphs with rule in general form $y = mx + c$; for example, $y = 3x + 2$.

b linear graphs with rule in general form $ax + by = d$; for example, $4x + 2y = 8$.

c linear graphs that pass through the **origin**; for example, $y = 3x$.

d linear graphs that are horizontal lines; for example, $y = 7$.

e linear graphs that are vertical lines; for example, $x = -2$.

18 Use the most appropriate method to sketch the graph of each linear relationship.

a $5x - 2y = -10$ **b** $y = 2x + 4$ **c** $x + y = 5$

d $y = 2x$ **e** $y = 5$ **f** $y = 8 - 4x$

g $x = -9$ **h** $y = 3(x - 2)$ **i** $2x + 4y - 8 = 0$

j $y = -3x$ **k** $y = -2x + 7$ **l** $y = 3 - 6x$

PROBLEM SOLVING AND REASONING

19 Sketch each linear relationship for the given x or y values.

a $y = 5x$ for $x \geq 0$ **b** $y = 2x - 4$ for $x \leq 3$

c $y = 7$ for $-2 \leq x \leq 4$ **d** $x = -3$ for $-1 \leq y \leq 1$

e $2x + 3y = 12$ for $0 \leq x \leq 6$ **f** $y = x$ for $x > 2$

g $y = 3 - x$ for $x < 3$ **h** $y = -3$ for $-1 < x < 4$

i $x - 4y = 2$ for $-2 \leq x < 6$ **j** $x = 6$ for $y < 3$

20 a Sketch these four linear relationships on the same Cartesian plane.

$x - 3y = 6$ for $0 \leq x \leq 6$ $3y = 2x + 6$ for $-3 \leq x \leq 0$

$3y = 6 - x$ for $0 \leq x \leq 6$ $2x + 3y = -6$ for $-3 \leq x \leq 0$

b What shape have you formed?

c For this shape, calculate: **i** the perimeter **ii** the area.

d Find the coordinates of the midpoint of each line segment.

e Join the midpoints to form a new quadrilateral. What is its perimeter and area?

21 The amount of simple interest (I) earned on an investment (P) at 5% p.a. for 10 years is represented by the rule $I = 0.5P$, where I and P are in dollars.

a Draw the graph of this relationship for P values from 0 to 5000.

b Use the graph to find the amount of simple interest earned for these investment amounts.

i \$1000 **ii** \$2500 **iii** \$4250

PROBLEM SOLVING AND REASONING

c Use the graph to find the amount invested to earn the following simple interest.

i \$750 **ii** \$1000 **iii** \$2250

d If the formula for simple interest is $I = PRT$, where R is the interest rate per annum written as a decimal and T is the number of years, explain what the gradient of your graph represents.

22 A racing car accelerates from a constant speed of 20 m/s. Its speed (s) after t seconds of accelerating can be calculated using the formula $s = 20 + 5t$.

a Draw the graph of this linear relationship for $0 \leq t \leq 10$; that is, t values from 0 to 10 seconds.

b What does the s-intercept represent on this graph?

c What does the gradient represent on this graph?

d From the graph, what is the speed of the racing car after:

i 3 seconds? **ii** 5 seconds? **iii** 7.5 seconds?

e Convert each speed in part **d** to kilometres per hour.

f From the graph, how long does it take for the racing car to reach:

i 30 m/s? **ii** 55 m/s? **iii** 65 m/s?

g Using the graph, how long does it take for the racing car to reach 216 km/h?

h Using the graph, what is the highest speed in kilometres per hour reached by the racing car? How long does it take to reach this speed?

CHALLENGE

23 a Sketch each of these linear relationships on the same Cartesian plane.

$2x + 4y = 8$ for $4 \leq x \leq 8$ $y = -2$ for $-2 \leq x \leq 8$

$x = -2$ for $-2 \leq y \leq 2$ $y = -x$ for $-2 \leq x \leq 0$

$y = 0$ for $0 \leq x \leq 4$.

b Calculate the perimeter of the shape formed.

c Calculate the area of the shape formed.

24 a Sketch each of these linear relationships on the same Cartesian plane.

$y = -4x + 10$ for $0 \leq x \leq 2$ $y = 4x - 10$ for $0 \leq x \leq 2$

$y = 4x + 10$ for $-2 \leq x \leq 0$ $y = -4x - 10$ for $-2 \leq x \leq 0$

$x - 4y = -10$ for $-10 \leq x \leq -2$ $x + 4y = -10$ for $-10 \leq x \leq -2$

$4y = x - 10$ for $2 \leq x \leq 10$ $x + 4y = 10$ for $2 \leq x \leq 10$

b Calculate the perimeter of the shape formed.

c Calculate the area of the shape formed.

d Join the points $(-2, -2)$ and $(2, 2)$ to form a line segment. Write a rule to represent the linear relationship formed.

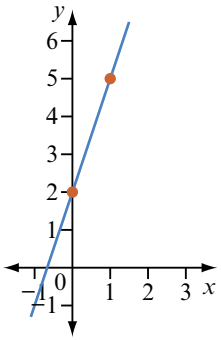
e Join the points $(-2, 2)$ and $(2, -2)$ to form a line segment. Write a rule to represent the linear relationship formed.

Reflect

What is the common goal of the two methods used to sketch linear graphs?

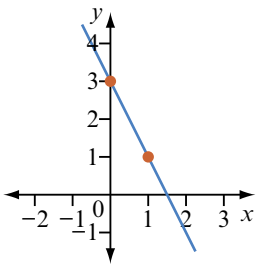
4D Finding the rule for a linear relationship

Start thinking!



1 Consider this linear graph.

- **a** What information is shown on this graph?
- **b** How might you use the information to find the rule for this linear relationship? Discuss with a classmate.
- **c** Find the gradient of this linear graph.
- **d** List the coordinates of the y-intercept.
- **e** If the rule can be represented by $y = mx + c$, list the values for m and c.
- **f** Hence, write the rule for this linear graph.



2 Repeat parts **c**–**f** of question **1** to find the rule for the linear graph shown at right.

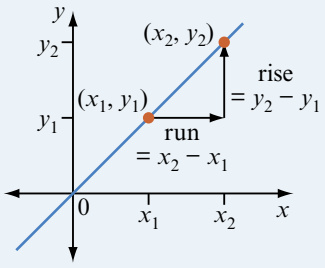
3 Explain how you can write the rule for a linear graph if you know the y-intercept and gradient.

4 Write the rule for a linear graph with:

- **a** y-intercept of -2, gradient of 3
- **b** y-intercept of 1, gradient of $-\frac{5}{3}$.

KEY IDEAS

- The gradient of a line can be found by calculating $\frac{\text{rise}}{\text{run}}$.
- The formula for the gradient (m) of a line segment between two points (x_1, y_1) and (x_2, y_2) is: $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- The rule for a linear graph with gradient m and y-intercept c is: $y = mx + c$.
- The rule for a linear graph with gradient m that passes through the point (x_1, y_1) is: $y - y_1 = m(x - x_1)$.



EXERCISE 4D Finding the rule for a linear relationship

EXAMPLE 4D-1 Finding the gradient of a line through two given points

Find the gradient of a line that passes through (1, 5) and (3, −9).

THINK

1 Write the known information.

2 Write the formula for finding gradient using two points (x_1, y_1) and (x_2, y_2).

3 Substitute the appropriate coordinates for x_1, y_1, x_2 and y_2 and simplify.

WRITE

$(x_1, y_1) = (1, 5)$ and $(x_2, y_2) = (3, -9)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-9 - 5}{3 - 1}$$

$$= \frac{-14}{2}$$

$$= -7$$

UNDERSTANDING AND FLUENCY

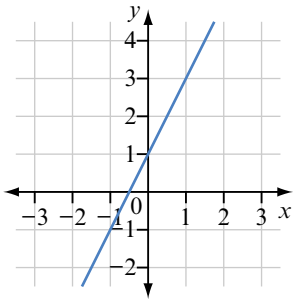
1 Find the gradient of each line that passes through the given points.

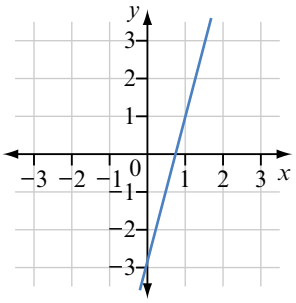
a (2, 3) and (3, 7) b (5, 4) and (10, −6) c (4, 2) and (6, 3)

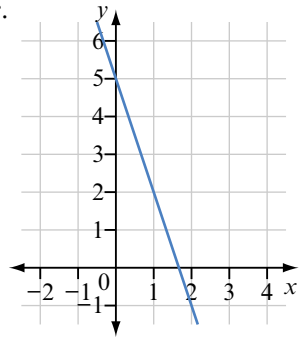
d (0, 5) and (4, 1) e (−3, −2) and (−1, 8) f (−1, −1) and (2, −10)

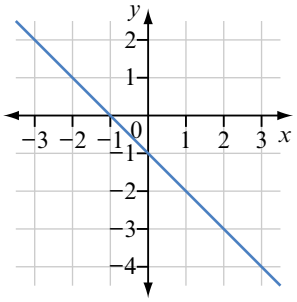
2 When finding the gradient of the line that passes through two points, does it matter which point is matched with (x_1, y_1) and which is matched with (x_2, y_2)? Use an example to help with your explanation.

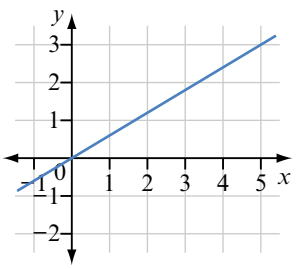
3 Write the rule for each linear graph by first finding m and c.

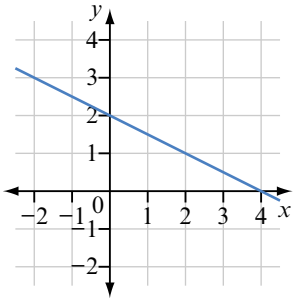
a 

b 

c 

d 

e 

f 

EXAMPLE 4D-2 Finding the rule given the y-intercept and a point

Find the rule for a line that passes through (0, 3) and (2, −1).

THINK	WRITE
1 Write the known information.	$(x_1, y_1) = (0, 3)$, $(x_2, y_2) = (2, -1)$, $c = 3$
2 Find the gradient using the two known points.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{-1 - 3}{2 - 0}$ $= -2$
3 Write the general rule for a linear graph.	$y = mx + c$
4 Substitute −2 for m and 3 for c.	$= -2x + 3$
5 Write the answer.	The rule is $y = -2x + 3$.

UNDERSTANDING AND FLUENCY

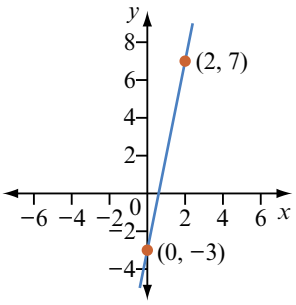
4 Find the rule for each line that passes through the given points.

a (0, 2) and (2, 10) **b** (0, −7) and (3, −4) **c** (0, 5) and (4, 7)

d (−1, −4) and (0, 6) **e** (−2, 1) and (0, −5) **f** (−8, 6) and (0, 4)

5 Find the rule for each line that passes through the points shown.

a



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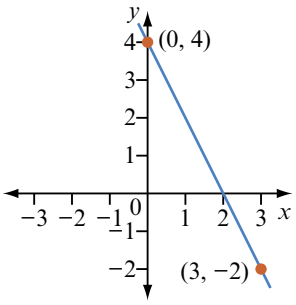
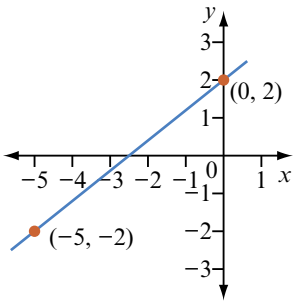


c



EXAMPLE 4D-3 Finding the rule given the gradient and a point

Find the rule for a line with a gradient of 5 that passes through (2, 3).

THINK	WRITE
1 Write the known information.	$m = 5$ and $(x, y) = (2, 3)$
2 Write the general rule for a linear graph involving m and c and substitute 5 for m.	$y = mx + c$ $y = 5x + c$
3 Substitute the x- and y-coordinates of the point and solve the equation to find c.	$3 = 5(2) + c$ for (2, 3) $3 = 10 + c$ $c = -7$
4 Write the rule using the values of m and c.	The rule is $y = 5x - 7$.

UNDERSTANDING AND FLUENCY

6 Find the rule for each line with the given information.

a gradient of 3; passes through (1, 4)
b gradient of 2; passes through (3, 9)
c gradient of −4; passes through (−1, 2)
d gradient of −1; passes through (2, 7)
e gradient of 5; passes through (−3, −6)
f gradient of −3; passes through (−4, 11)

7 Besides the general rule $y = mx + c$, there is another rule you can use.

a For a line passing through the point (x_1, y_1), the coordinates can be substituted into the rule $y = mx + c$ to produce $y_1 = mx_1 + c$. Rearrange this rule to make c the subject.
b Substitute the expression for c into $y = mx + c$.
c Rearrange your answer to part **b** and show that the rule becomes $y - y_1 = m(x - x_1)$. This new rule is useful when you know the gradient m and one point (x_1, y_1) on the line.

8 Repeat question **6** but this time use the rule $y - y_1 = m(x - x_1)$.

EXAMPLE 4D-4 Finding the rule given two points using $y - y_1 = m(x - x_1)$

Find the rule for a line that passes through (2, −3) and (4, −11), using $y - y_1 = m(x - x_1)$.

THINK	WRITE
1 Write the known information.	$(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (4, -11)$
2 Find the gradient using the two known points.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{-11 - (-3)}{4 - 2}$ $= -4$
3 Write the general rule for a linear graph involving m and (x_1, y_1).	$y - y_1 = m(x - x_1)$
4 Substitute −4 for m, 2 for x_1 and −3 for y_1 and simplify.	$y - (-3) = -4(x - 2)$ $y + 3 = -4x + 8$ $y = -4x + 5$
5 Write the answer.	The rule is $y = -4x + 5$.

9 Repeat the working in Example 4D-4 but this time use (4, −11) as (x_1, y_1). Do you obtain the same answer? Does it matter which of the two points is used in the general rule?

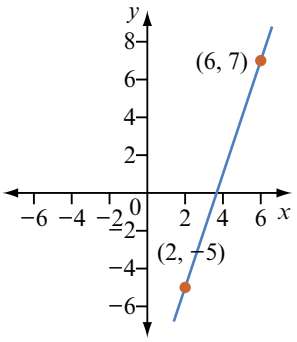
10 Find the rule for each line that passes through the given points, using $y - y_1 = m(x - x_1)$.

a (1, 3) and (3, 7)
b (5, 2) and (6, 8)
c (2, 4) and (4, −2)
d (3, −2) and (6, −8)
e (−2, −3) and (2, 1)
f (−4, −2) and (−1, −5)

UNDERSTANDING AND FLUENCY

11 Find the rule for each line that passes through the points shown, using $y - y_1 = m(x - x_1)$.

a



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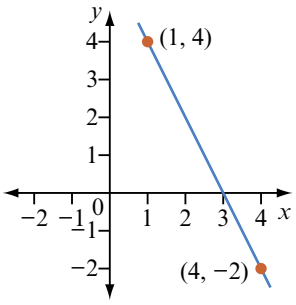

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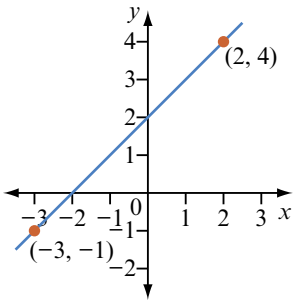


12 Use part **a** of question **10** to show that $y = mx + c$ can also be used to find the rule for a linear graph when two points are known.

13 Use the given x- and y-intercepts to write the rule for each linear graph.

a x-intercept = 2, y-intercept = 4

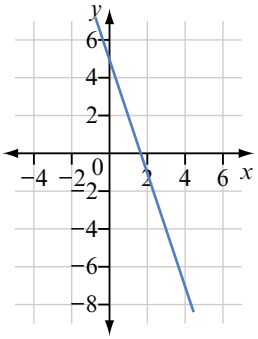
b x-intercept = −3, y-intercept = 3

c x-intercept = 3, y-intercept = −6

d x-intercept = 1, y-intercept = 5

14 Use the most appropriate method to find the rule for each line shown.

a



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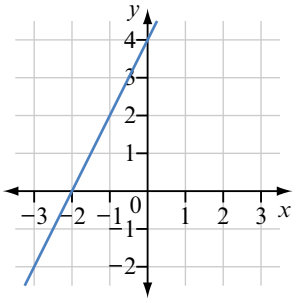

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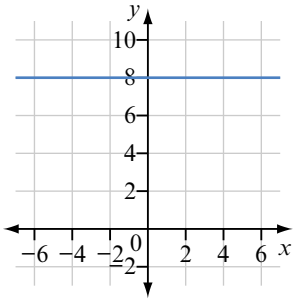

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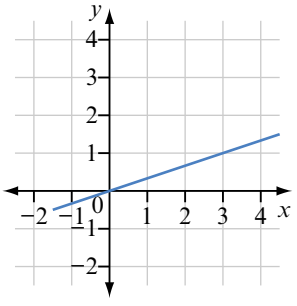

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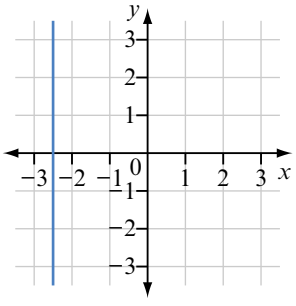

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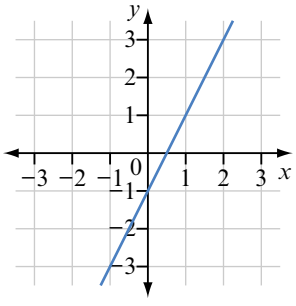


PROBLEM SOLVING AND REASONING

15 Write the rule for a linear graph to represent each of these situations.

a From an altitude of 50 m, a road rises 1 m vertically over a horizontal distance of 200 m.

b From an altitude of 6 km, a plane descends 2 km vertically over a horizontal distance of 30 km.

c A road rises 20 m vertically over a horizontal distance of 100 m.

PROBLEM SOLVING AND REASONING

16 Jasper has $80 in his bank account and decides to deposit $20 each week. If he doesn't withdraw any money:

- **a** write a rule to represent the amount of money in his account, a, after n weeks
- **b** calculate how long it takes to build his account balance to $500.

17 Inge owes $1560 on her store card and has arranged to make regular payments without interest charges until the debt is paid. A monthly direct debit of $195 from her bank account is paid to the store. If Inge doesn't buy any further items on her store card:

- **a** write a rule to represent the amount of money owed on her store card, w, after n months
- **b** use the rule to calculate the amount still owed after 3 months
- **c** calculate how long it takes to clear the debt on her store card.

18 A helicopter rises at a constant speed from a height of 20 m above the ground, to cover a vertical distance of 30 m over a horizontal distance of 100 m. For the next 100 m travelled horizontally, the helicopter maintains the same altitude. It then descends at a constant speed over a further horizontal distance of 100 m before landing.

- **a** Draw a graph to represent the height above the ground of the helicopter over the horizontal distance of 300 m.
- **b** Identify the rule for each of the three linear sections of the graph.
- **c** Use the appropriate rule to calculate the height of the helicopter for each of these horizontal distances on your graph.
  - **i** 30 m **ii** 80 m **iii** 150 m **iv** 240 m
- **d** What horizontal distance/s have been travelled when the helicopter is at a height of 40 m?

CHALLENGE

19 Find the rule for the linear graph described by each set of information. Write your answers without fractions.

- **a** x-intercept of 4, passes through (20, 8)
- **b** gradient of 0, passes through (−15, −23)
- **c** passes through (−25, 10) and (11, −14)
- **d** passes through the origin and (−17, −12)

20 Sketch the linear graph for each situation in question **15**. What values of x would be appropriate to use? Rewrite each rule with the appropriate restriction of x values written as an inequality statement.

21 Write the linear rules describing the graph in question **18** with the appropriate restriction of x values.

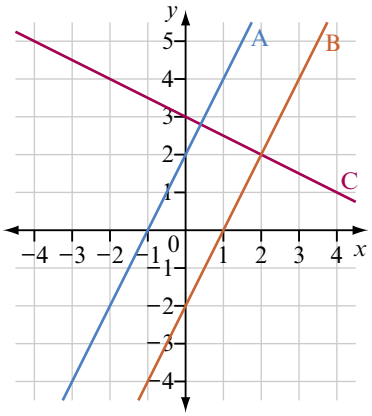
Reflect

How can you identify the rule for a linear graph?

4E Parallel and perpendicular lines

Start thinking!

Consider the lines drawn on this Cartesian plane.



1. Identify the gradient and the y-intercept for each line.
2. Write the rule for each line.
3. What do you notice about lines A and B? What name is given to lines like this?
4. Use the term *gradient* or *y-intercept* to complete this sentence: Parallel lines have the same _____.
5. What is the angle between lines B and C? Use a protractor to check. What is the name given to lines like this?
6. Calculate the product of the gradients for lines B and C.
7. Predict the product of the gradients for lines A and C. Confirm your answer by performing the multiplication.
8. Summarise what you know about the gradients of **parallel lines** and **perpendicular lines**.

KEY IDEAS

- Parallel lines never meet.
- Two lines are parallel if they have the same gradient. For example, lines with the rules $y = 3x + 2$ and $y = 3x - 7$ are parallel.
- Perpendicular lines meet at right angles.
- Two lines are perpendicular if the product of their gradients is -1. For example, lines with the rules $y = 3x + 2$ and $y = -\frac{1}{3}x - 7$ are perpendicular.
- Lines with the rules $y = m_1x + c_1$ and $y = m_2x + c_2$ or $y - y_1 = m_1(x - x_1)$ and $y - y_2 = m_2(x - x_2)$ are:
  - parallel if $m_1 = m_2$
  - perpendicular if $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.
- For a line segment between the points (x_1, y_1) and (x_2, y_2):
  - the gradient is $\frac{y_2 - y_1}{x_2 - x_1}$
  - the length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - the coordinates of the midpoint are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

EXERCISE 4E Parallel and perpendicular lines

UNDERSTANDING AND FLUENCY

1 **a** Sketch the graph of each linear relationship on the same Cartesian plane.

i $y = -3x - 4$ **iii** $y = -3x - 2$ **iv** $y = -3x$

v $y = -3x + 2$ **vi** $y = -3x + 4$ **vii** $y = -3x + \frac{11}{2}$

b What do you notice?

c How can you tell from the rules that the lines are parallel?

d Decide whether each rule describes a linear graph that is parallel to those in part **a**.

i $y = 3x + 7$ **ii** $3x + y = 5$ **iii** $y = x - 3$

2 Write the gradient of the line that is parallel to each linear graph with the given rule.

a $y = 7x - 2$ **b** $y = -x + 5$ **c** $y = 10 - \frac{3}{2}x$

d $y = -3$ **e** $4x + y = 4$ **f** $3y = 1 - 6x$

EXAMPLE 4E-1 Writing the rule for a parallel line using gradient and y-intercept

Write the rule for a line that is parallel to the graph of $y = 4x + 3$ and has a y-intercept of -5.

THINK	WRITE
1 Write the known information. Parallel lines have the same gradient.	Gradient of $y = 4x + 3$ is 4 so $m = 4$. $c = -5$
2 Write the general rule for a linear graph involving m and c.	$y = mx + c$
3 Substitute 4 for m and -5 for c.	$= 4x + (-5)$ $= 4x - 5$
4 Write the answer.	The rule is $y = 4x - 5$.

3 Write the rule for a line that is:

a parallel to graph of $y = -2x + 1$; y-intercept = 3

b parallel to graph of $y = 6x - 2$; y-intercept = -4

c parallel to graph of $y = x - 8$; y-intercept = 0

d parallel to graph of $5x + y = 10$; y-intercept = 2.

4 For the lines you drew in part **a** of question **1**:

a write the rule for a parallel line with a y-intercept of -5

b create your own rule for a line that is parallel to them.

EXAMPLE 4E-2 Writing the rule for a parallel line using gradient and a point

Write the rule for a line that is parallel to the graph of $y = 2x - 5$ and passes through (3, 1).

THINK	WRITE
1 Write the known information. Parallel lines have the same gradient.	Gradient of $y = 2x - 5$ is 2 so $m = 2$. $(x_1, y_1) = (3, -1)$
2 Write the general rule for a linear graph involving m and (x_1, y_1).	$y - y_1 = m(x - x_1)$
3 Substitute for m, x_1 and y_1 and simplify.	$y - (-1) = 2(x - 3)$ $y + 1 = 2x - 6$ $y = 2x - 7$
4 Write the answer.	The rule is $y = 2x - 7$.

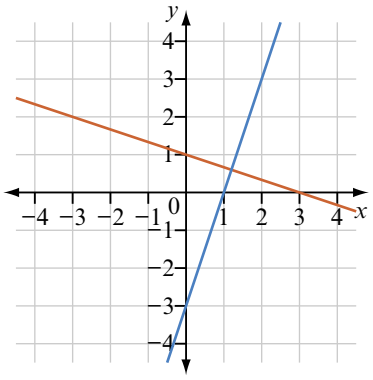
UNDERSTANDING AND FLUENCY

5 Write the rule for a line that is:

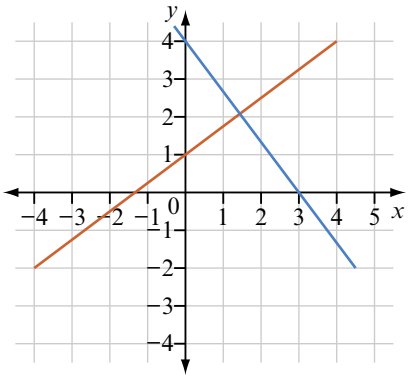
- **a** parallel to graph of $y = 4x + 3$ and passes through $(2, -5)$
- **b** parallel to graph of $y = -3x + 7$ and passes through $(-1, -3)$
- **c** parallel to graph of $y = -x - 6$ and passes through $(-4, 2)$
- **d** parallel to graph of $y = \frac{1}{2}x + 9$ and passes through $(6, 0)$.

6 Answer the following questions for the linear graphs on each Cartesian plane.

i



ii



- **a** Calculate the gradients of each pair of lines.
- **b** Find the product of the gradients for each pair of lines.
- **c** What do you notice about the product of the gradients of perpendicular lines?
- **d** Explain how, if you know the gradient of one of each pair of perpendicular lines, you could work out the other.

7 The product of the gradients of two perpendicular lines, m_1 and m_2, is -1; so you can find the gradient of one line, m_2, by taking the negative reciprocal of the other, m_1; that is, $m_2 = -\frac{1}{m_1}$. Find the value of m_2 given the m_1 values below.

a 5 **b** -8 **c** -1 **d** $\frac{1}{6}$ **e** $\frac{4}{5}$ **f** $-\frac{3}{2}$

8 Write the gradient of the line that is perpendicular to each linear graph with the given rule.

a $y = 2x - 5$ **b** $y = -3x + 1$ **c** $y = \frac{1}{7}x$

d $y = 3 - \frac{2}{5}x$ **e** $9x + y = 2$ **f** $8y = 2x - 1$

EXAMPLE 4E-3

Writing the rule for a perpendicular line using gradient and y-intercept

Write the rule for a line that is perpendicular to the graph of $y = 3x - 4$ and has a y-intercept of 2.

THINK	WRITE
1 Write the known information. For perpendicular lines, the product of their gradients is −1. The negative reciprocal of 3 is $-\frac{1}{3}$.	Gradient of $y = 3x - 4$ is 3 so $m = -\frac{1}{3}$. $c = 2$
2 Write the general rule for a linear graph involving m and c.	$y = mx + c$
3 Substitute $-\frac{1}{3}$ for m and 2 for c.	$= -\frac{1}{3}x + 2$
4 Write the answer.	The rule is $y = -\frac{1}{3}x + 2$.

UNDERSTANDING AND FLUENCY

9 Write the rule for a line that is:

a perpendicular to graph of $y = -2x - 3$; y-intercept = 4

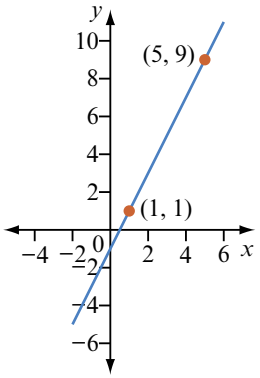
b perpendicular to graph of $y = 5x + 8$; y-intercept = −1

c perpendicular to graph of $y = x - 2$; y-intercept = 0

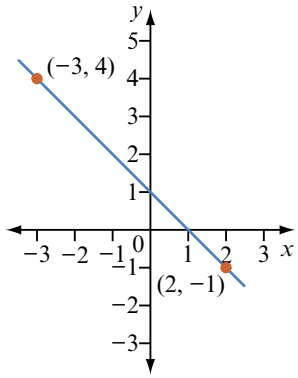
d perpendicular to graph of $7x + 2y = 14$; y-intercept = 3.

10 Write the rule for a line that is perpendicular to each line shown and has a y-intercept of 3.

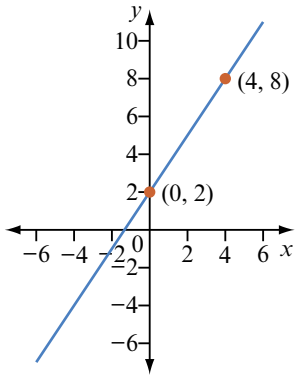
a



b



c



11 For each line shown in question **10**, write the rule for a line that has a y-intercept of −4 and is:

i parallel to the given line

ii perpendicular to the given line.

EXAMPLE 4E-4 Writing the rule for a perpendicular line using gradient and a point

Write the rule for a line that is perpendicular to the graph of $y = -4x - 7$ and passes through $(-8, 3)$.

THINK	WRITE
1 Write the known information. For perpendicular lines, the product of their gradients is -1. The negative reciprocal of -4 is $\frac{1}{4}$.	Gradient of $y = -4x - 7$ is -4 so $m = \frac{1}{4}$. $(x_1, y_1) = (-8, 3)$
2 Write the general rule for a linear graph involving m and (x_1, y_1).	$y - y_1 = m(x - x_1)$
3 Substitute for m, x_1 and y_1 and simplify.	$y - 3 = \frac{1}{4}(x + 8)$ $y - 3 = \frac{1}{4}x + 2$ $y = \frac{1}{4}x + 5$
4 Write the answer.	The rule is $y = \frac{1}{4}x + 5$.

12 Write the rule for a line that is:
- **a** perpendicular to graph of $y = 5x + 9$ and passes through $(5, -8)$
- **b** perpendicular to graph of $y = -7x - 3$ and passes through $(0, 3)$
- **c** perpendicular to graph of $y = x + 4$ and passes through $(-1, -2)$
- **d** perpendicular to graph of $y = -\frac{2}{3}x + 1$ and passes through $(-4, 4)$.

13 Decide whether each pair of lines is parallel, perpendicular or neither. Give a reason for your answer.
- **a** $y = 4x - 3$ and $4x - y = 5$
- **b** $3y - 2x = 12$ and $3x + 2y = 8$
- **c** $y = \frac{3}{8}x$ and $3y = 4 - 8x$
- **d** $2x + y = 3$ and $y = 2x + 6$
- **e** $5x + y + 4 = 0$ and $x + 5y = 7$
- **f** $2x - 7y = 11$ and $14y = 4x + 1$

14 Write two different rules for linear graphs that are parallel to $2y - 10x = 3$.

15 Write two different rules for linear graphs that are perpendicular to $x + 8y - 4 = 0$.

16 Show that the line joining $(2, -3)$ and $(4, 5)$ is parallel to the graph of $y = 4x - 7$.

17 Show that the line joining $(-11, -7)$ and $(-1, -2)$ is perpendicular to the graph of $y = -2x + 5$.

18 A line passes through two points, $(-2, 3)$ and $(7, 9)$.
- **a** Find the gradient of the line.
- **b** Write the rule for a linear graph that is:
  - **i** parallel to this line and passes through $(6, -1)$
  - **ii** perpendicular to this line and passes through $(-4, 8)$.

PROBLEM SOLVING AND REASONING

PROBLEM SOLVING AND REASONING

19 A line segment joins (−6, 7) and (0, −11). Write the rule for a linear graph that is:

- **a** parallel to this line segment and passes through the origin
- **b** perpendicular to this line segment and passes through (−6, 7).
- **c** perpendicular to this line segment and passes through its midpoint.

20 Four points are plotted on a Cartesian plane: A(−2, 1), B(5, 4), C(−6, −4) and D(8, 2).

- **a** Is line segment AB parallel to line segment CD?
- **b** Is line segment AC perpendicular to line segment BD?

21 Three points are plotted on a Cartesian plane: A(−2, 3), B(1, 12) and C(7, 10).

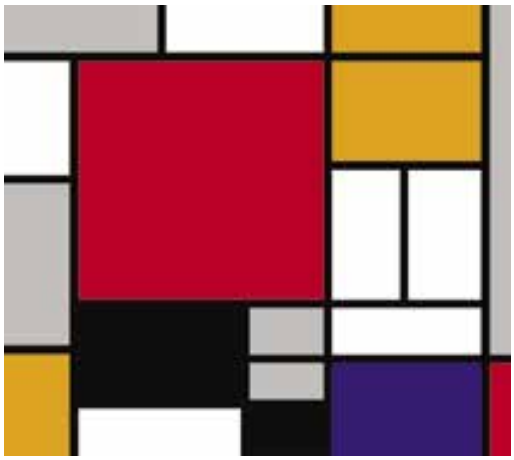
- **a** Show that line segments drawn using these points as vertices form a right-angled triangle.
- **b** Is the triangle an isosceles right-angled triangle? Show your reasoning.
- **c** Calculate the perimeter of this triangle, correct to one decimal place.
- **d** Calculate the area of this triangle, correct to one decimal place.

22 Use your knowledge of parallel lines and the formula for length of a line segment to:

- **a** show that the quadrilateral ABCD with vertices at A(1, 4), B(2, 6), C(4, 7) and D(3, 5) is a rhombus
- **b** show that the quadrilateral EFGH with vertices at E(1, 3), F(8, 6), G(4, −2) and H(−3, −5) is a parallelogram.

CHALLENGE

23 Prove that the quadrilateral KLMN with vertices at K(0, 0), L(2, 4), M(6, 2) and N(4, −2) is a square. Hence find its perimeter and area, correct to one decimal place.



24 Prove that the quadrilateral PQRS with vertices at P(−7, 5), Q(−3, 1), R(0, 4) and S(−4, 8) is a rectangle. Hence find its perimeter and area, correct to one decimal place.

25 Write the rule for the **perpendicular bisector** of the line segment joining (−2, 3) and (8, −7).

26 Prove that the quadrilateral WXYZ with vertices at W(−9, −11), X(−1, 8), Y(6, 9) and Z(7, 2) is a kite. (Hint: also consider the diagonals of the quadrilateral.)

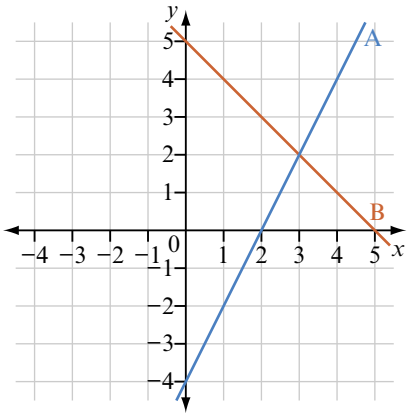
27 Without plotting points on a Cartesian plane, what type of quadrilateral is ABCD, where the vertices are A(2, 1), B(8, 4), C(9, 1) and D(−3, −5)? Show your reasoning, using your knowledge of parallel and perpendicular lines.

Reflect

What is special about the gradient of parallel lines and perpendicular lines?

4F Solving simultaneous linear equations graphically

Start thinking!



Consider the two linear graphs shown on this Cartesian plane.

1 Use lines A and B to copy and complete this table of values.

x	−1	0	1	2	3	4	5
y_A	−6	−4					
y_B	6	5					

2 From the table, which x value has a y value that is the same for both linear graphs?

3 Write the x value and the y value as a coordinate pair. This coordinate pair occurs simultaneously for both linear graphs.

4 What are the coordinates of the point of intersection of the two lines?

5 Compare your answers to questions **3** and **4**. What is special about this point?

6 Write the rule for: **a** line A **b** line B.

7 Check that substituting the coordinates of the point of intersection into each linear rule (or linear equation) makes it a true statement.

8 A linear equation such as $2x + 3 = 11$ can be solved to obtain one solution for x that makes the algebraic statement true. What is the solution?

9 Two linear equations such as $y = 2x - 4$ and $y = -x + 5$ can be solved simultaneously (at the same time) to obtain one solution for x and y that makes both statements true. What is the solution?

10 There are a number of ways of finding the solution to **simultaneous linear equations**. How can you find the solution when viewing the graphs of the linear equations?

KEY IDEAS

- Any two linear graphs will intersect or cross if the lines are not parallel.
- At the point of intersection, each line will simultaneously have the same x- and y-coordinate.
- The solution to simultaneous linear equations can be found graphically from the coordinates of the point of intersection of the linear graphs.

EXERCISE 4F Solving simultaneous linear equations graphically

EXAMPLE 4F-1 Checking solutions to simultaneous linear equations

Check if each given x and y value is the solution to the pair of simultaneous linear equations.

a $x = 2, y = 4$
$y = 3x - 2$ and $y = -2x + 8$

b $x = 5, y = -3$
$2x + y = 7$ and $y = 2x - 11$

THINK	WRITE
a **1** Substitute the given x and y values into the left side (LS) and right side (RS) of the first equation.	**a** $y = 3x - 2$ When $x = 2$ and $y = 4$, LS = 4 and RS = $3 \times 2 - 2$ $= 4$ So LS = RS
2 Substitute the given x and y values into the LS and RS of the second equation.	$y = -2x + 8$ When $x = 2$ and $y = 4$, LS = 4 and RS = $-2 \times 2 + 8$ $= 4$ So LS = RS
3 Decide whether the given x and y values is the solution to the simultaneous linear equations. Does substituting the values satisfy both equations; that is, make them true?	As $x = 2$ and $y = 4$ satisfies both equations, it is the solution.
b **1** Substitute the given x and y values into the LS and RS of the first equation.	**b** $2x + y = 7$ When $x = 5$ and $y = -3$, LS = $10 - 3$ and RS = 7 $= 7$ So LS = RS
2 Substitute the given x and y values into the LS and RS of the second equation.	$y = 2x - 11$ When $x = 5$ and $y = -3$, LS = -3 and RS = $10 - 11$ $= -1$ So LS $\neq$ RS
3 Decide whether the given x and y values is the solution to the simultaneous linear equations.	As $x = 5$ and $y = -3$ does not satisfy both equations, this is not the solution.

1 Check if each given x and y value is the solution to the pair of simultaneous linear equations.

a $x = 1, y = 5$
$y = x + 4$ and $y = -2x + 7$

b $x = -2, y = 1$
$y = x + 3$ and $y = -3x - 5$

c $x = 5, y = -2$
$y = x - 9$ and $y = -2x + 7$

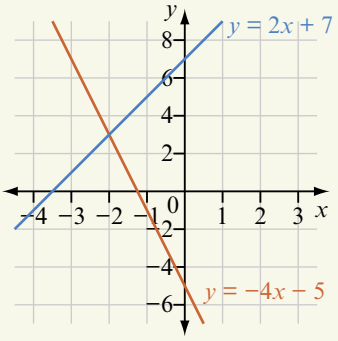
d $x = 3, y = -7$
$2x + y = -1$ and $y = -x - 4$

e $x = 4, y = 3$
$y = 2x - 5$ and $y = -x + 1$

f $x = -6, y = 0$
$y = x - 6$ and $x + y = -6$

EXAMPLE 4F-2

Identifying the solution to simultaneous linear equations from graphs



Consider the following pair of simultaneous linear equations and their graphs.

$y = 2x + 7$
$y = -4x - 5$

a Write the coordinates of the point of intersection of the two lines.
b Use the coordinates of this point to write the solution to the simultaneous linear equations.

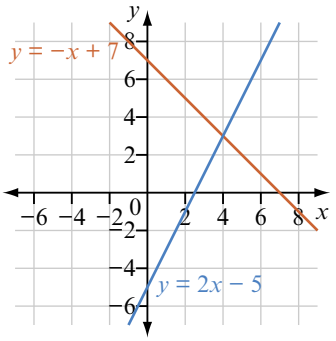
THINK	WRITE
a Identify the point of intersection of the two lines and list the coordinates.	**a** Point of intersection at $(-2, 3)$.
b Write the solution to the simultaneous linear equations.	**b** The solution to the simultaneous equations $y = 2x + 7$ and $y = -4x - 5$ is $x = -2, y = 3$.

UNDERSTANDING AND FLUENCY

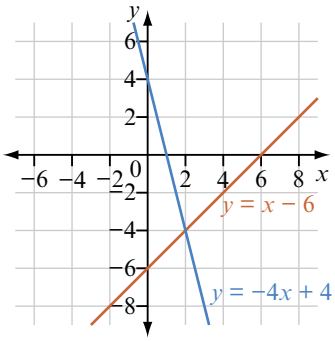
2 Consider each pair of simultaneous linear equations and their graphs.

i Write the coordinates of the point of intersection of the two lines.
ii Use the coordinates of this point to write the solution to the simultaneous linear equations.

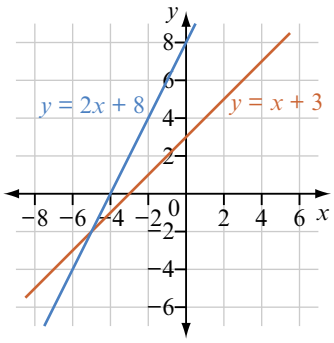
a $y = 2x - 5$
$y = -x + 7$



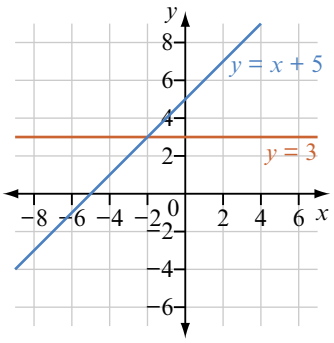
b $y = -4x + 4$
$y = x - 6$



c $y = 2x + 8$
$y = x + 3$



d $y = x + 5$
$y = 3$



3 Use substitution to check your solutions to the simultaneous linear equations in question **2**.

EXAMPLE 4F-3 Solving simultaneous linear equations graphically

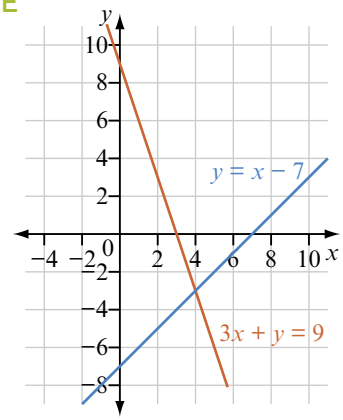
Solve the following pair of simultaneous linear equations using a graph.

$y = x - 7$
$3x + y = 9$

THINK

1 Sketch the graphs of $y = x - 7$ and $3x + y = 9$ on the same Cartesian plane, using grid or graph paper. Extend the lines so you can clearly see the point where they cross.

WRITE



2 List the coordinates of the point of intersection.

Point of intersection at (4, −3).

3 Use the coordinates to write the solution to the simultaneous linear equations.

The solution to the simultaneous equations $y = x - 7$ and $3x + y = 9$ is $x = 4$, $y = -3$.

4 Solve each pair of simultaneous linear equations using a graph.

a $y = 2x - 6$
$y = x - 1$

b $y = 3x$
$y = x + 4$

c $y = x - 2$
$y = 10 - x$

d $y = x + 7$
$y = -3x - 1$

e $y = 3x + 6$
$2x + y = 1$

f $y = -2x - 5$
$y = 2x + 7$

g $y = 2x - 4$
$y = x - 1$

h $2x + y = 6$
$y = x - 6$

i $y = 3x + 3$
$y = -x - 5$

j $2x + y = -4$
$x + y = 1$

k $x - y = 4$
$4x + y = 6$

l $3y = x + 12$
$y = x + 2$

UNDERSTANDING AND FLUENCY

5 Use substitution to check your solutions to the simultaneous linear equations in question **4**.

6 Solve each pair of simultaneous linear equations using a graph. Check your solutions.

a $y = x - 7$
$y = -4$

b $y = 2x$
$y = 3x$

c $x + y = 3$
$x = 2$

7 Solve each pair of simultaneous linear equations using a graph. Ensure that each graph is drawn accurately.

a $y = 1 - x$
$y = x$

b $2x - 2y = -1$
$2y = 4x + 7$

c $4x + 2y = -3$
$y = x - 6$

8 Explain why there is no solution to the simultaneous linear equations $y = 2x + 5$ and $2x - y = 1$.

PROBLEM SOLVING AND REASONING

9 Sophie decides to make hand-painted cards. She invests $60 in a set of paint brushes and enough paint for 40 cards. The cost of each plain white card is $2 and she plans to sell the painted cards for $6 each at the market.

a Explain how the amount of money, a, to produce n cards (where $0 \leq n \leq 40$) can be written as $a = 2n + 60$.

b Explain how the amount of money, a, received for selling n cards can be written as $a = 6n$.

c Draw graphs of the two equations on the same Cartesian plane. Label the horizontal axis as n and the vertical axis as a.

d What are the coordinates of the point of intersection? What information does this give you?

e How many cards should Sophie produce and then sell to break-even; that is, recover all her costs?

f How many cards should Sophie produce and then sell to make a profit?

10 The perimeter of this postcard is 40 cm.

a Use x and y to write a linear equation for the perimeter.

b Use x and y to write a linear equation linking the length and width of the postcard if the length is 4 cm longer than the width.

c Solve the simultaneous linear equations graphically and hence state the dimensions of the postcard.

PROBLEM SOLVING AND REASONING

11 Tia is twice as old as Lachlan. The sum of their ages is 51.

- **a** Using two variables, write two linear equations to represent this information.
- **b** Solve the pair of simultaneous equations graphically.
- **c** How old are Tia and Lachlan?

12 At the cinema, a large soft drink costs $2 more than a choc top. Cooper buys five soft drinks and seven choc tops for $70.

- **a** Write two linear equations to represent this information.
- **b** Solve the equations graphically to find the cost of each item.

13 A rectangular block of land is to be fenced along three sides with the front boundary left open. The length from front to back is 20 m longer than the width. If 124 m of fencing is used, solve two linear equations to find the dimensions of the land.

14 Chloe is planning a party and obtains prices from two catering companies. Each has a fixed price for delivery and set up, and a cost per person for the food. Chloe is unsure of how many people she will invite.

Angie's Catering: fixed price $200, $28 per person

Cool Food Club: fixed price $100, $32 per person

- **a** Write a linear equation to represent the cost of hiring Angie's Catering. Remember to define the two variables you are using.
- **b** Write a linear equation to represent the cost of hiring Cool Food Club.
- **c** Show the graph of each linear equation on the same Cartesian plane.
- **d** Which company is cheaper if catering for 18 people? Explain how you can see this from your graph.
- **e** Which company is cheaper if catering for 36 people?
- **f** How many people can be catered for so that the cost is the same using either company? What is this cost?
- **g** Write a summary advising Chloe on what her options are.

CHALLENGE

15 A linear graph has a y-intercept of -5. Another linear graph has a gradient of -3. The two linear graphs intersect at the point $(4, -6)$.

- **a** Show the two graphs on the same Cartesian plane.
- **b** Find the equation for each linear graph.
- **c** Write the solution to the simultaneous linear equations.
- **d** Use substitution to check that the solution satisfies both equations.

16 Write a pair of simultaneous linear equations that has a solution of $x = 7$, $y = -4$.

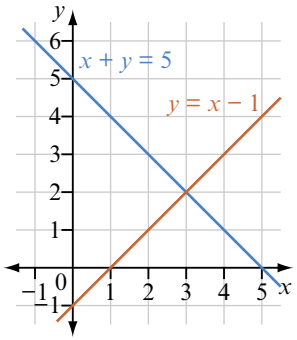
17 Explain why the pair of simultaneous linear equations $y = 4 - 3x$ and $6x + 2y = 8$ has an infinite number of solutions.

Reflect

What does the point of intersection of two linear graphs show you?

4G Solving simultaneous linear equations algebraically

Start thinking!



1 Use the diagram at right to solve the simultaneous equations $x + y = 5$ and $y = x - 1$.

Alternatively, algebra can be used to find the solution. We aim to form a new equation with only one of the variables in it.

2 **a** Substitute the expression for y from one equation (say, $y = x - 1$) into the other. That is, replace y with $x - 1$ in $x + y = 5$ and simplify.

b Solve this new equation for x and find its matching y value.

c What is the solution? This way of solving the equations is called the **substitution method**.

3 You can also use another algebraic method if the equations are in the form shown at right.

$$x + y = 5$$
$$x - y = 1$$

a What is the result of adding the left side of each equation? That is, $x + y + x - y = ?$.

b What is the result of adding the right side of each equation? That is, $5 + 1 = ?$.

c Which variable has been eliminated? Show that you can form the new equation $2x = 6$.

d Solve the new equation formed in part **c** for x and find its matching y value.

e What is the solution? This is called the **elimination method**.

f Repeat parts **a**–**d** but this time subtract the two equations. Do you get the same solution?

KEY IDEAS

- Simultaneous linear equations can be solved algebraically using substitution or elimination.
- Substitution method (used when at least one equation has one variable as the subject)
  1. choose the equation where one variable is the subject; for example, y
  2. substitute the expression for y into the other equation and simplify to form a new equation containing only one variable; in this case, x
  3. solve this new equation for x
  4. substitute the value found for x into one of the original equations and solve to find the value of y.
- Elimination method (used when both variables are on the same side of the equation)
  1. check that one of the variables will be eliminated when the equations are added or subtracted; for example, x. You may need to first multiply one or both of the equations by an integer so this can be achieved.
  2. add or subtract the two equations to form a new equation containing only one variable; in this case, y
  3. solve this new equation for y
  4. substitute the value found for y into one of the original equations and solve to find the value of x.

EXERCISE 4G Solving simultaneous linear equations algebraically

EXAMPLE 4G-1 Solving simultaneous linear equations using substitution

Solve each pair of simultaneous linear equations using the substitution method.

a $x + 3y = 2$
$y = 2x + 3$

b $y = 4x - 5$
$y = x + 4$

THINK

a **1** Write the equations with one under the other and number them for easy reference.

2 Replace y in equation ① by substituting the expression for y from equation ②.

3 Simplify the equation and then solve for x.

4 To find the matching y value, substitute $x = -1$ into either equation. In this case, choose ②.

5 Check your solution by substituting into the other equation; that is, equation ①.

6 Write the solution to the simultaneous equations.

b **1** Write the equations with one under the other and number them.

2 Replace y in equation ② by substituting the expression for y from equation ①.

3 Solve the equation for x.

4 To find the matching y value, substitute $x = 3$ into either equation. In this case, choose ②.

5 Check your solution by substituting into the other equation; that is, equation ①.

6 Write the solution to the simultaneous equations.

WRITE

a $x + 3y = 2$ ①
$y = 2x + 3$ ②

Substituting $y = 2x + 3$ into ①:
$x + 3(2x + 3) = 2$

$x + 6x + 9 = 2$
$7x + 9 = 2$
$7x = -7$
$x = -1$

Substituting $x = -1$ into ②:
$y = 2(-1) + 3 = 1$

Substituting $x = -1$, $y = 1$ into ①:
LS $= -1 + 3(1) = 2 =$ RS

The solution to the simultaneous equations $y = 2x + 3$ and $x + 3y = 2$ is $x = -1$, $y = 1$.

b $y = 4x - 5$ ①
$y = x + 4$ ②

Substituting $y = 4x - 5$ into ②:
$4x - 5 = x + 4$

$3x - 5 = 4$
$3x = 9$
$x = 3$

Substituting $x = 3$ into ②:
$y = 3 + 4 = 7$

Substituting $x = 3$, $y = 7$ into ①:
RS $= 4(3) - 5 = 7 =$ LS.

The solution to the simultaneous equations $y = 4x - 5$ and $y = x + 4$ is $x = 3$, $y = 7$.

UNDERSTANDING AND FLUENCY

1 Solve each pair of simultaneous linear equations using the substitution method.

a $x + y = 9$
$y = x + 5$

b $x + 2y = 6$
$y = x - 3$

c $2x + y = 1$
$y = x + 4$

d $y = x - 1$
$x - 3y = -9$

e $3x - y = -10$
$y = x + 2$

f $y = 6x$
$5x - y = -2$

g $6x + 5y = 21$
$y = 2x + 1$

h $y = 2 - 3x$
$4x + y = -1$

i $2x + 3y = -5$
$y = 5 - 2x$

j $x = y + 2$
$2x - 3y = 1$

k $x = 4y$
$y = 3x + 11$

l $x = 5$
$4x - 3y = 2$

2 Solve each pair of simultaneous linear equations using the substitution method.

a $y = 2x + 6$
$y = x + 7$

b $y = x - 2$
$y = 4x - 20$

c $y = 7x + 16$
$y = 2x + 1$

d $y = x - 6$
$y = 4 - x$

e $y = 5x + 9$
$y = 3 - x$

f $y = 1 - 2x$
$y = x - 11$

g $y = 3x - 5$
$y = 5 - 2x$

h $y = 4x + 14$
$y = -4x - 2$

i $y = 10 - 3x$
$y = 7 - 4x$

EXAMPLE 4G-2 Solving simultaneous linear equations using elimination

Solve this pair of simultaneous linear equations using the elimination method. $3x + 2y = 5$
$x - 2y = 7$

THINK

a 1 Write the equations with one under the other and number them.

2 Eliminate the variable y by adding the two equations. $3x + x = 4x$, $2y + (-2y) = 0$, $5 + 7 = 12$.

3 Solve for x.

4 To find the matching y value, substitute $x = 3$ into either equation. In this case, choose ①.

5 Simplify the equation and solve for y.

6 Check your solution by substituting into the other equation; that is, equation ②.

7 Write the solution to the simultaneous equations.

WRITE

a $3x + 2y = 5$ ①
$x - 2y = 7$ ②

① + ②: $4x = 12$

$x = 3$

Substituting $x = 3$ into ①:
$3(3) + 2y = 5$

$9 + 2y = 5$
$2y = -4$
$y = -2$

Substituting $x = 3$, $y = -2$ into ②:
LS $= 3 - 2(-2) = 7 =$ RS

The solution to the simultaneous equations $3x + 2y = 5$ and $x - 2y = 7$ is $x = 3$, $y = -2$.

UNDERSTANDING AND FLUENCY

3 Solve each pair of simultaneous linear equations using the elimination method.

a $2x + y = 9$
$3x - y = 1$

b $x + 2y = -1$
$-x + 5y = 8$

c $x - 2y = 11$
$5x + 2y = -5$

d $-3x + y = 10$
$3x - 4y = -4$

e $4x + 3y = 29$
$2x - 3y = 1$

f $6x - y = -8$
$-6x + 5y = 16$

4 Solve each pair of simultaneous linear equations using the elimination method. (Hint: subtract the equations to eliminate a variable.)

a $x + 4y = 17$
$x + 2y = 11$

b $3x + y = 3$
$x + y = 7$

c $4x - 3y = 33$
$4x + y = 5$

d $6x + 5y = 19$
$4x + 5y = 21$

e $x - 2y = 2$
$3x - 2y = 14$

f $-5x + y = 11$
$-5x - 2y = 23$

5 Write an equivalent equation to each of these so that it contains the term shown in brackets.

a $x + 2y = 5$ $[3x]$

b $-3x + y = 1$ $[2y]$

c $-x - 4y = 2$ $[-5x]$

d $5x + 2y = 3$ $[8y]$

e $-2x + 3y = 7$ $[-4x]$

f $4x - 2y = -3$ $[6y]$

EXAMPLE 4G-3 Solving simultaneous linear equations using elimination (multiplying one equation)

Solve this pair of simultaneous linear equations using the elimination method. $2x - 5y = -10$, $-x + 4y = 11$

THINK

1 Write the equations with one under the other and number them for easy reference.

2 Adding or subtracting the equations will not eliminate a variable. To eliminate x, the first term in equation ② needs to be $-2x$ so multiply ② by 2 to form an equivalent equation. Number this new equation as ③.

3 Eliminate the variable x by adding equations ① and ③. $2x + (-2x) = 0$, $-5y + 8y = 3y$, $-10 + 22 = 12$.

4 Solve for y.

5 To find the matching x value, substitute $y = 4$ into one of the equations. In this case, choose ①.

6 Simplify the equation and solve for x.

7 Check your solution by substituting into the other equation; that is, equation ②.

8 Write the solution to the simultaneous equations.

WRITE

$2x - 5y = -10$ ①
$-x + 4y = 11$ ②

② × 2: $-2x + 8y = 22$ ③

① + ③: $3y = 12$

$y = 4$

Substituting $y = 4$ into ①:
$2x - 5(4) = -10$

$2x - 20 = -10$
$2x = 10$
$x = 5$

Substituting $x = 5$, $y = 4$ into ②:
LS $= -5 + 4(4) = 11 =$ RS

The solution is $x = 5$, $y = 4$.

UNDERSTANDING AND FLUENCY

6 Solve each pair of simultaneous linear equations using the elimination method.

a $5x + 3y = 21$
$2x - y = 4$

b $x + 4y = -2$
$4x - 3y = 11$

c $3x + y = -5$
$7x + 2y = -14$

d $6x - 7y = 9$
$-2x + 5y = 5$

e $4x + 3y = -24$
$5x - 12y = 33$

f $2x + 5y = 26$
$10x - 3y = -38$

7 In some cases, you need to find an equivalent equation to each of the equations in the pair before adding or subtracting to eliminate a variable. Consider the pair of simultaneous linear equations shown at right.

$7x + 2y = -8$ ①
$4x + 3y = 1$ ②

a Can you eliminate a variable by adding or subtracting the equations? Explain.

b Write an equivalent equation to equation ① by multiplying each side by 3.

c Write an equivalent equation to equation ② by multiplying each side by 2.

d Which variable can you now eliminate? Should you add or subtract the new equations?

e Continue with the elimination method to solve this pair of simultaneous equations.

8 Solve each pair of simultaneous linear equations using the elimination method. (Hint: you will need to multiply both equations to form equivalent equations.)

a $4x + 3y = 16$
$3x - 2y = -5$

b $3x + 7y = -2$
$5x + 11y = -4$

c $-8x + 5y = -21$
$5x + 2y = 8$

d $3x - 10y = -32$
$-7x + 3y = 34$

e $-9x + 2y = -35$
$8x - 9y = -5$

f $-4x - 5y = 22$
$-3x + 7y = -5$

9 Compare the three methods of solving simultaneous linear equations. When is each method (graphical, substitution, elimination) the most appropriate to use?

PROBLEM SOLVING AND REASONING

10 Use the most appropriate method to solve each pair of simultaneous equations. (Hint: not all solutions are whole numbers.)

a $y = 3x - 6$
$4x + 11y = 8$

b $y = 6x - 5$
$y = 3x - 4$

c $4x - 5y = 4$
$2x + 5y = 5$

d $y = -7$
$y = 4x + 29$

e $x = 5y - 6$
$x - y = 2$

f $2y = x - 4$
$3x - 2y = 0$

g $2x - y = -4$
$4x + y = 22$

h $x + 5y = -5$
$x - 2y = 9$

i $-x - 7y = 5$
$-2x - 7y = 3$

j $-3x + 4y = 14$
$3x - y = -8$

k $x + y = 11$
$x - y = 3$

l $5x + 2y = -18$
$-5x + 2y = 22$

m $y = 2.5x + 4.3$
$y = 7.5x + 8.3$

n $4x = y - 6$
$8x + 3y = 13$

o $6x - 7y = -119$
$8x - 5y = 19$

11 The sum of two numbers is 245 and the difference is 91.

a Write two linear equations to represent this information and solve them simultaneously.

b What are the two numbers?

PROBLEM SOLVING AND REASONING

12 The perimeter around the top of a rectangular billiards table is 810 cm.

- **a** Use l and w to write a linear equation for the perimeter.
- **b** Use l and w to write a linear equation linking the length and width of the top of the table if the length is twice the width.
- **c** Solve the simultaneous linear equations and hence state the dimensions of the top of the billiards table.

13 Ella has 240 cm of ribbon to sew around the edge of a pillow case. If the length of the pillow case is to be 18 cm less than twice the width, what is the largest size she can make the pillow case? Show your reasoning using simultaneous linear equations.

14 Officials decide to trial a new method of scoring an AFL match. Geelong scored 15 goals and 6 behinds to finish with a total of 132 points while Carlton scored 13 goals and 12 points to finish with a total of 128 points. Solve a pair of simultaneous linear equations to work out the number of points allocated to a goal and to a behind.

15 At the football, Noah buys seven pies and six sausage rolls for \$63 and Lily buys five pies and five sausage rolls for \$48. Solve a pair of simultaneous equations to find the cost of each item.

CHALLENGE

16 Consider the pair of simultaneous equations shown at right.

$$7x - 6y = -14 \quad ①$$
$$5x - 3y = -1 \quad ②$$

- **a** Demonstrate that you obtain the same solution using any one of the strategies below.

  Strategy A: multiply equation ② by 2 to form equation ③ and then subtract equation ③ from ①.

  Strategy B: multiply equation ② by −2 to form equation ③ and then add equations ① and ③.

  Strategy C: multiply equation ① by 5 to form equation ③ and multiply equation ② by 7 to form equation ④. Subtract equation ④ from ③.
- **b** Are there other strategies you can use? Explain.

17 Use an algebraic method to show that the pair of simultaneous linear equations $y = 11 - 4x$ and $8x + 2y = 5$ has no solution.

18 Use an algebraic method to show that the pair of simultaneous linear equations $4x - 2y = 6$ and $y = 2x - 3$ has an infinite number of solutions.

Reflect

What is the aim of the first step in solving simultaneous linear equations algebraically?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

linear relationship	subject of equation	x- and y-intercept method	graphical solution
linear equation	linear inequality	parallel lines	algebraic solution
balance method	linear graph	perpendicular lines	substitution method
formula	gradient–intercept method	simultaneous equations	elimination method

MULTIPLE-CHOICE

4A **1** The solution to $\frac{x-4}{2} = -3$ is:
A -2 **B** 2 **C** -10 **D** 10

4A **2** Which equation has a solution of $x = -5$?
A $3x - 7 = 8$ **B** $\frac{2x+3}{13} = 1$
C $\frac{12}{x} = -2.4$ **D** $\frac{3}{2-x} = 1$

4B **3** Which statement is incorrect?
A $-7 < -4$ **B** $\frac{2}{3} < \frac{3}{4}$
C $-3.5 < -4.7$ **D** $\frac{8}{9} < \frac{9}{8}$

4B **4** If $x \geq -6.2$, which of these could *not* be a value for x?
A -4.9 **B** -6.3 **C** 6.2 **D** 0

4C **5** The y-intercept of the graph of $12x + 4y = 8$ is:
A 8 **B** -8 **C** 2 **D** -2

4C **6** The gradient of the line $12x + 4y = 8$ is:
A 12 **B** -12 **C** 3 **D** -3

4D **7** The rule for the linear graph with a gradient of $\frac{3}{4}$ and a y-intercept of $\frac{5}{8}$ is:
A $6x - 8y + 5 = 0$ **B** $8y = 6x - 5$
C $8x - 6y + 5 = 0$ **D** $6y = 8x + 5$

4D **8** The gradient of the line passing through the points (1, 8) and $(-3, -6)$ is:
A 1 **B** -1 **C** $\frac{7}{2}$ **D** $\frac{2}{7}$

4E **9** A line parallel to the line $3x + 2y = 6$ would have a gradient of:
A $\frac{2}{3}$ **B** -3 **C** $\frac{3}{2}$ **D** $-\frac{3}{2}$

4E **10** A line has a gradient of $-\frac{3}{4}$. A line perpendicular to this line would have a gradient of:
A $-\frac{4}{3}$ **B** $\frac{4}{3}$ **C** $\frac{3}{4}$ **D** $-\frac{3}{4}$

4F **11** The pair of linear equations $y = x + 3$ and $y = -2x + 6$ intersect at the point:
A $(-3, 0)$ **B** (3, 6)
C (1, 4) **D** (1, 8)

4F **12** The pair of linear equations $y = 3x + 1$ and $6x - 2y + 2 = 0$ have how many simultaneous solutions?
A none **B** one **C** two **D** many

4G **13** When $y = 3x - 4$ is substituted into the equation $x - 2y = 3$, the resulting equation is:
A $-5x + 8 = 3$ **B** $-5x - 8 = 3$
C $5x + 8 = 3$ **D** $5x - 8 = 3$

4G **14** Consider the two equations:
$2x - y + 3 = 0$ ① and $4x - y + 5 = 0$ ②
If equation ② is subtracted from equation ①, the resulting equation will be:
A $-2x - 2 = 0$ **B** $-2x - 8 = 0$
C $2x + 2 = 0$ **D** $2x + 8 = 0$

SHORT ANSWER

4A **1** Solve each equation. Leave your answer as a fraction where appropriate.

a $3 - 2(4 - 5x) = -20$

b $4(3x + 5) = 3(5x - 3)$

4A **2** Solve each equation, leaving your answer as a fraction if appropriate.

a $\frac{4 + x}{3} = \frac{2x - 5}{3}$ b $\frac{5 - 3x}{2} = \frac{2x + 5}{3}$

4A **3** Make the pronumeral in brackets the subject of each formula.

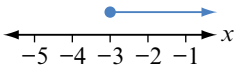
a $A = \frac{h(a + b)}{2}$ (b)

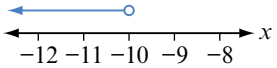
b $A = 2[(l + w)h]$ (h)

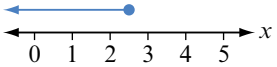
4B **4** Consider this list of x values:
$-3.9, 1\frac{5}{8}, 2.6, 8.5, -8.5, \frac{3}{4}$.
For each inequality, choose the x values from the list that make it a true statement.

a $x > -1$ b $x \leq 0.75$ c $x \geq 2.6$

4B **5** Write the inequality that is represented on each number line.

a 

b 

c 

4B **6** Solve each inequality.

a $5 - 2x < 7$ b $\frac{2 - 3x}{5} > 2$

4C **7** Sketch each linear rule using the gradient–intercept method.

a $y = -2x + 3$

b $x - 2y = 4$

c $5y - 10 = 15x$

4C **8** Sketch the graph of each, using the x- and y-intercept method.

a $2x - 4y - 12 = 0$

b $x + 3y + 9 = 0$

c $7x + 3y + 21 = 0$

4D **9** Find the rule for each line that passes through the given points.

a $(0, -4)$ and $(5, 6)$

b $(3, 6)$ and $(-2, 6)$

c $(-3, 5)$ and $(-3, -3)$

4D **10** Find the rule for each line that has:

a gradient of 2 and passes through $(1, 5)$

b gradient of -2 and passes through $(1, 5)$

4D **11** Find the rule for each line that passes through the given points. Write each answer in the form $y = mx + c$.

a $(1, 3)$ and $(2, 4)$

b $(-1, 5)$ and $(3\ -7)$

c $(-4, -4)$ and $(5, 5)$

4E **12** Write a rule for a line that is:

a parallel to graph of $y = -4x + 5$; y-intercept $= -3$

b perpendicular to graph of $y = 2x - 7$; y-intercept $= -4$

4E **13** Write a rule for a line that is:

a perpendicular to graph of $y = 3x + 2$ and passes through $(3, 2)$

b parallel to graph of $y = 3x + 2$ and passes through $(-3, -2)$

4F **14** Solve each pair of simultaneous equations using a graph.

a $y = 4x - 2$ and $y = -3x + 5$

b $2x - y + 5 = 0$ and $3x - y = 1$

4G **15** Solve each pair of simultaneous equations using substitution.

a $3x + 4y = -1$ and $y = x - 2$

b $y = 4x + 3$ and $x - 2y = 8$

c $7x - y = 4$ and $y = 4x + 2$

4G **16** Solve each pair of simultaneous equations using the elimination method.

a $2x + y = 8$ and $4x - y = 4$

b $x + 5y = 13$ and $x + 2y = 4$

c $2x - y = 7$ and $3x + 5y = 4$

MIXED PRACTICE

1 A line passes through two points (1, −4) and (−4, 1). Write the rule for a linear graph that is:

a parallel and passes through (4, 4)

b perpendicular and passes through (1, 1)

2 The solution to the inequality $2 - \frac{x}{4} < 1$ is:

A $x < 4$ **B** $x < -4$

C $x > 4$ **D** $x > -4$

3 Consider the linear equations $4x - y - 2 = 0$ and $2x + y - 4 = 0$

a Draw their graphs on a Cartesian plane.

b Write the coordinates of the intersection point.

4 Decide whether each given pair of lines is parallel, perpendicular or neither.

a $x + 2y + 10 = 0$ and $2x - y - 3 = 0$

b $x + y = 3$ and $y = x$

c $2x + 3y = 4$ and $3x + 2y = 4$

5 A rectangle is 28 m longer than it is wide. If the length is 5 times the width, find the dimensions of the rectangle.

6 If the gradient of a line is 0, this means that:

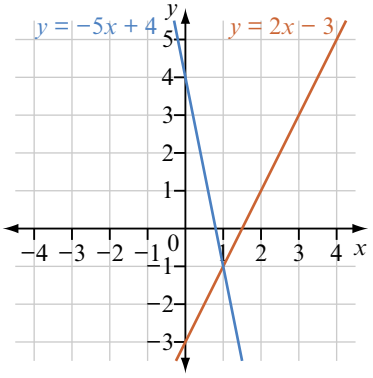
A the line slopes upwards from left to right

B the line slopes downwards from left to right

C the line is parallel to the y-axis

D the line is parallel to the x-axis

7 Use the graph to write the solution to the simultaneous linear equations $y = 2x - 3$ and $y = -5x + 4$.



8 The rise and run of the line joining (−2, −1) to (5, 9) are:

A rise 10, run 7 **B** rise 7, run 10

C rise 8, run 3 **D** rise 3, run 8

9 The average of three numbers x, y and z can be calculated using the formula:

$$V = \frac{x + y + z}{3}$$

a Write z as the subject of the formula.

b Use your formula in part **a** to calculate z when:

i $V = 12$, $x = 7$, $y = 19$

ii $V = 42$, $x = 5$, $y = 12$

10 The coordinates of the x- and y-intercepts of the rule $x - y = -2$ are (in order):

A (0, 2) and (−2, 0) **B** (−2, 0) and (0, 2)

C (0, −2) and (2, 0) **D** (2, 0) and (0, −2)

11 Give the coordinates of the intersection points of the lines:
$y = -3$, $y = 5$, $x = 1$ and $x = -4$.

12 Use an appropriate method to solve each pair of simultaneous equations.

a $y + 3x = 8$ and $y = -3x + 8$

b $3x + 2y = 1$ and $3x + 7y = 11$

c $4x - 5y = 3$ and $6x - 11y = 1$

13 If three hamburgers and two drinks cost $32, while one hamburger and four drinks cost $29, how much does each item cost?

14 Solve each inequality.

a $5 - 4x \geq 2x - 1$

b $2x + 3 < 5x + 9$

c $5(2 - 3x) \leq 3(5 - 2x)$

d $-2(x - 4) > 5(3 - 2x)$

15 Find the coordinates of the midpoint of the line joining each pair of points on a Cartesian plane.

a (5, −6) and (−7, 2)

b (−3, −1) and (9, −3)

16 Find the distance between these pairs of points on a Cartesian plane. Give your answers correct to one decimal place.

a $(-4, 3)$ and $(5, -2)$

b $(1, -4)$ and $(-7, 3)$

c $(-3, -4)$ and $(4, -3)$

17 Answer true or false to each statement. Give a reason for any false statements.

a The graph of $y = -2$ is parallel to the x-axis.

b The graph of $y = -2x$ passes through the origin.

c The equation of the x-axis is $x = 0$.

18 A line with a rule in the form $y = mx + c$ passes through the points $(3, -4)$ and $(4, -2)$. What is the value of:

a m? **b** c?

19 The value of x in the equation $\frac{2x}{7} = 8$ is:

A 56 **B** 24 **C** 48 **D** 28

20 Which equation matches this statement?
A certain number is subtracted from 8. When the result is divided by 3, the result is 4.

A $\frac{x - 8}{3} = 4$ **B** $\frac{8 - x}{3} = 4$

C $8 - \frac{x}{3} = 4$ **D** $8 - x = \frac{4}{3}$

21 Write an equation for each number problem, then find the solution.

a A number is doubled, then 3 is subtracted. The result is 17. What is the number?

b The sum of a certain number and -1 is then multiplied by 7. The result is 21. What is the number?

c Subtract 12 from treble a number. When the result is multiplied by 5, the answer is 15. What is the number?

22 Solve each inequality.

a $\frac{1 - 2x}{4} < \frac{x + 2}{3}$ **b** $\frac{x + 1}{6} \geq \frac{x - 1}{8}$

ANALYSIS

a A triangle has vertices X$(4, -4)$, Y$(-2, 2)$ and Z$(1, 5)$. Draw this triangle on a Cartesian plane.

b Write a rule for each of the line segments XY, YZ and ZX.

c Show that triangle XYZ is right-angled.

d Let the length of the side opposite vertex X be represented by x, the side opposite vertex Y by y, and the side opposite Z by z. Calculate x, y and z, correct to two decimal places.

e Use these lengths to demonstrate Pythagoras' Theorem.

f Determine the coordinates of the midpoint M of ZX. Join M to the vertex Y.

g Calculate the length of MY.

h The line MY divides the triangle XYZ into two smaller triangles. Investigate side lengths and angles to describe the shape of these triangles.

i Draw the perpendicular height of the triangle from the vertex Y to the base XZ, meeting the base at H. Write the rule for the line segment YH.

j Solve a pair of simultaneous equations to find the coordinates of H.

k Calculate the length of YH.

l Show the use of two sets of different measurements to calculate the area of triangle XYZ. Give your answers correct to one decimal place.

m Show that the length of YH $= \frac{xz}{y}$.

n Explain how a semi-circle can be drawn to enclose triangle XYZ. What does this show about the angle in a semi-circle?

CONNECT

Comparing taxi charges

Taxi fares vary across Australia. They can be different depending on where you are and what day or time of day it is.

The fare you are charged is mainly calculated using the flagfall and the distance rate. The flagfall is a set charge or minimum amount on the meter when you hire a taxi, and the distance rate is the amount per kilometre travelled. There is also a waiting time charge if the taxi is stopped, or travelling at less than a particular speed. Additional charges can be made, such as airport, tollway and booking fees.

Your task is to compare taxi charges around Australia. To simplify calculations, only the flagfall and distance rate will be considered when working out taxi fares. Information has been supplied for all the capital cities.

Brisbane

Tariff	Time and day	Flagfall	Distance rate
1	07.00–19.00 weekdays	$2.90	$2.17/km
2	19.01–06.59 weekdays All day Sat, Sun and public holidays – see also Tariff 3	$4.30	$2.17/km
3	0.00–05.00 surcharge of $2.00	$6.30	$2.17/km

Your task

You are to complete the following for each capital city. Include all necessary working to justify your answers.

- Write a rule for the linear relationship linking taxi fare and the number of kilometres travelled for each tariff.
- Use tariff 1 to calculate the taxi fare for a distance of 10 km. Which capital city is the cheapest?
- Calculate the taxi fare if you travel 20 km at these times and days: 3 pm Christmas day, 6 am Wednesday, 2 am Saturday.
- Choose another three days, times and travel distances. Compare the taxi fare in four capital cities.
- Work out which capital city allows you to travel the furthest for a fare of $50. How far can you travel?
- Use tariff 1 for Canberra and Sydney to investigate which city is cheaper over varying distances.
  - Show the graphs of the relationships on the same Cartesian plane.
  - Use an inequality distance statement to describe when Canberra is cheaper.
  - Use an inequality distance statement to describe when Sydney is cheaper.
  - For what travel distance is the cost of the taxi fare the same in Canberra and Sydney? What is the taxi fare for this distance?
- Choose two other capital cities to compare. Use an algebraic method to find the distance where the cost of the taxi fare is the same. Describe the conditions for which each city is cheaper.
- Investigate the fares for other pairs of capital cities. Write a summary of your findings.

As an extension, you may like to investigate the taxi charges in your town or in a country town near you. Compare them to one of the capital cities. Investigate, either algebraically or graphically, to find the distance where the cost of the taxi fare in your chosen town is cheaper, the same, or more expensive than in the capital city.

Sydney

Tariff	Time and day	Flagfall	Distance rate
1	06.00–21.59 weekdays	$3.60	$2.19/km
2	22.00–05.59 weekdays All day Sat, Sun and public holidays – see also Tariff 3	$3.60	$2.63/km
3	20.00–05.59 starting Fri, Sat, eve of public holiday, surcharge of $2.50	$6.10	$2.63/km

Hobart

Tariff	Time and day	Flagfall	Distance rate
1	06.00–19.59 weekdays	$3.60	$1.94/km
2	20.00–05.59 weekdays All day Sat, Sun and public holidays	$3.60	$2.32/km

Canberra

Tariff	Time and day	Flagfall	Distance rate
1	06.00–21.00 weekdays	$5.00	$2.00/km
2	21.01–05.59 weekdays All day Sat, Sun and public holidays	$5.00	$2.00/km

Adelaide

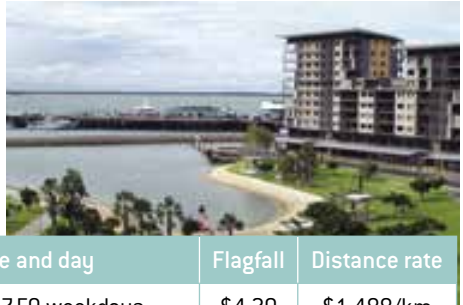
Tariff	Time and day	Flagfall	Distance rate
1	06.00–18.59 weekdays	$3.70	$1.84/km
2	19.00–05.59 weekdays All day Sat, Sun and public holidays	$4.90	$2.03/km

Melbourne

Tariff	Time and day	Flagfall	Distance rate
1	09.00–16.59 day rate	$4.20	$1.622/km
2	17.00–08.59 overnight rate	$5.20	$1.804/km
3	22.00–04.00 Fri, Sat, public holidays	$6.20	$1.986/km

Perth

Tariff	Time and day	Flagfall	Distance rate
1	06.00–17.59 weekdays	$4.10	$1.69/km
2	18.00–05.59 weekdays All day Sat, Sun and public holidays	$6.00	$1.69/km



Darwin

Tariff	Time and day	Flagfall	Distance rate
1	06.00–17.59 weekdays	$4.20	$1.488/km
2	18.00–05.59 weekdays All day Sat, Sun and public holidays	$5.00	$1.83/km

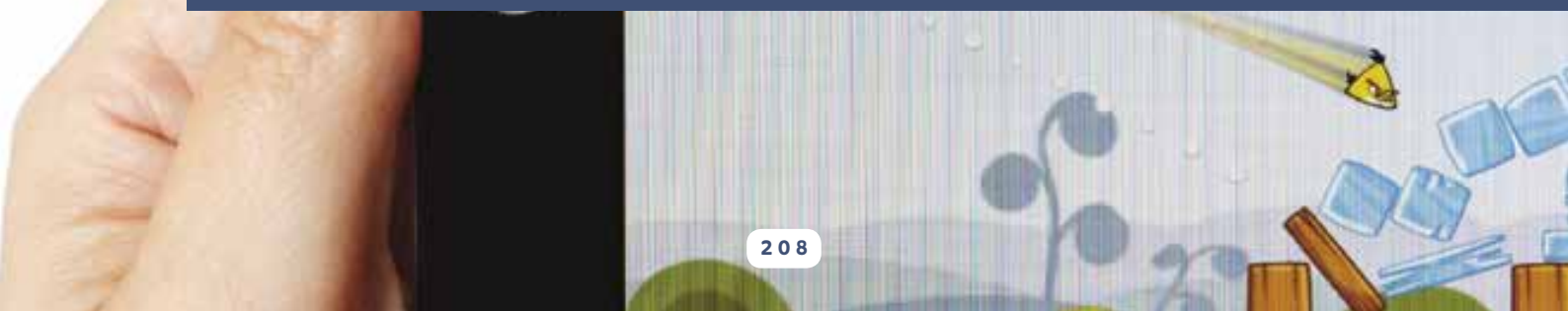
You may like to present your findings as a report. Your report could include:

- a digital taxi fare calculator
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

5 NON-LINEAR RELATIONSHIPS

ESSENTIAL QUESTION

The path of a moving object can be described by a non-linear relationship. How do games like Angry Birds and Super Mario involve parabolas?



5A **1** Factorise each expression.

a $x^2 - 3x$ **b** $x^2 - 25$

c $x^2 - 7x + 12$ **d** $-4x^2 - 18x$

5A **2** Factorise each quadratic expression by completing the square.

a $x^2 + 2x - 4$ **b** $x^2 - 3x - 1$

5A **3** Simplify each surd expression.

a $(\sqrt{3})^2$ **b** $\sqrt{24}$ **c** $7\sqrt{8}$ **d** $\frac{\sqrt{45}}{3}$

5B **4** Evaluate each expression by substituting $a = 3$, $b = -2$ and $c = 4$.

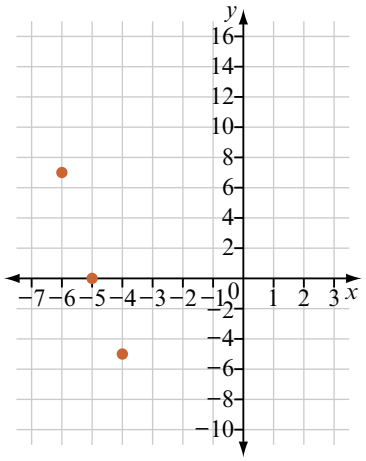
a $a^2 - b^2$ **b** $a + b + 2c$

c $b^2 - 4ac$ **b** $4^a + 3^b - 2^c$

5C **5** **a** Copy and complete this table of values using the rule for the quadratic relationship $y = x^2 + 4x - 5$. The first few y values have been calculated for you.

x	−6	−5	−4	−3	−2	−1	0	1	2
y	7	0	−5						

b Plot the values for this relationship on a Cartesian plane. The first few points have been plotted for you. Join the points with a curved line to form a parabola.



c Write the coordinates for:

i the x-intercepts

ii the y-intercept

iii the turning point.

5D **6** Consider the point A(3, 2) on a Cartesian plane. Find the coordinates of a new point after each transformation has been performed on the point A.

a translate 2 units right

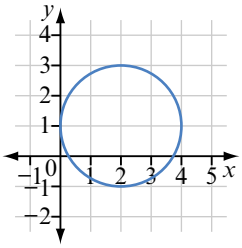
b translate 7 units down

c translate 5 units left and 1 unit up

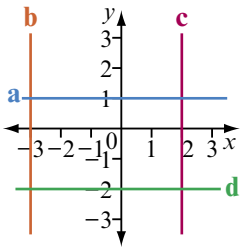
d reflect in the x-axis

e reflect in the y-axis

5E **7** Identify the coordinates of the centre and the radius of this circle.



5F **8** Write the rule for each of the lines shown.



5G 10A **9** Write each number in index form with the indicated base.

a 32 [2] **b** 729 [3]

c 25^3 [5] **d** 8^4 [2]

5G 10A **10** **a** Convert each statement from index form to logarithmic form.

i $2^6 = 64$ **ii** $10^4 = 10\,000$

b Convert each statement from logarithmic form to index form.

i $\log_3 81 = 4$ **ii** $\log_{10} 1 = 0$

5H 10A **11** **a** What is the value of $\frac{1}{x}$ when:

i $x = 0$? **ii** x approaches ∞?

b What value of x makes each expression undefined?

i $\frac{1}{x - 3}$ **ii** $\frac{1}{x + 8}$

c What does the value of each expression in part **b** approach when x approaches ∞?

5A Solving quadratic equations

Start thinking!

In Chapter 2 you became familiar with quadratic expressions and how to expand, simplify and factorise them. In this section, you will work with **quadratic equations**.

1 What is the difference between a quadratic equation and a quadratic expression?

2 Which of these are quadratic equations? Provide a reason for your decision.

i $x^2 = 1$ **ii** $x - 9 = 0$ **iii** $x^2 + 6x = 0$ **iv** $x^2 + 4x - 5$

3 One method of solving a quadratic equation involves the **Null Factor Law**.

a Calculate each of these.

i 4×0 **ii** 0×-3 **iii** 0×0 **iv** $x \times 0$ **v** $0 \times (x - 6)$ **vi** $(x + 5) \times 0$

b What do you observe with each product?

c Use your answer to part **b** to help you work out the value of x in each equation.

i $3 \times x = 0$ **ii** $x \times 8 = 0$ **iii** $-6 \times x = 0$

iv $5 \times (x - 4) = 0$ **v** $(x + 1) \times 2 = 0$ **vi** $-3 \times (x - 7) = 0$

d Copy and complete this sentence: The Null Factor Law states that if the product of two factors equals _____ then one or both of the factors must equal _____.

4 Consider the Null Factor Law with $(x - 2)(x - 7) = 0$.

a What are the two factors that form the product on the left side of the equation?

b Since the right side of the equation equals 0, what do you know about $(x - 2)$ and $(x - 7)$?

c Copy and complete the working at right to solve $(x - 2)(x - 7) = 0$ using the Null Factor Law.

$(x - 2)(x - 7) = 0$
$x - 2 = 0$ or $x - 7 =$ ___
$x = 2$ or $x =$ ___

d Check your two solutions. Explain how you did this.

5 In your own words, explain how the Null Factor Law can be used to solve a quadratic equation.

KEY IDEAS

- The general form of a quadratic equation is $ax^2 + bx + c = 0$ or $(x + m)(x + n) = 0$.
- The Null Factor Law states that if the product of two factors is 0 then one or both factors are 0. For example, if $a \times b = 0$ then $a = 0$ or $b = 0$ or both a and b are 0.
- A quadratic equation can be solved using the Null Factor Law if one side of the equation is in factor form and the other is 0. Two linear equations are produced which are easy to solve. For example, $(x + m)(x + n) = 0$ has two solutions: $x = -m$ or $x = -n$.
- To obtain the product of two factors on one side of the equation, the quadratic expression needs to be factorised.
- Solutions can be written in a shorthand way using ± (plus or minus). For example, $x = -2$ or $x = 2$ can be written as $x = \pm 2$.

EXERCISE 5A Solving quadratic equations

1 Decide whether the value given in square brackets is a solution to the equation.

a $(x - 3)(x - 2) = 0$ $[x = 2]$
b $(x + 5)(x - 1) = 0$ $[x = 5]$
c $4x(x - 2) = 0$ $[x = 0]$
d $x^2 + 2x - 8 = 0$ $[x = -2]$
e $x^2 - 4 = 0$ $[x = 4]$
f $x^2 - 6x + 9 = 0$ $[x = 3]$

EXAMPLE 5A-1 Solving quadratic equations using the Null Factor Law

Solve each equation.

a $(x + 3)(x - 5) = 0$ **b** $x(2x - 7) = 0$

THINK	WRITE
a 1 Write the equation. Check that the left side (LS) is in factor form (yes) and the right side (RS) equals 0 (yes).	**a** $(x + 3)(x - 5) = 0$
2 Apply the Null Factor Law.	$x + 3 = 0$ or $x - 5 = 0$
3 Solve each linear equation.	$x = -3$ or $x = 5$
b 1 Write the equation. Check that the LS is in factor form (yes) and the RS equals 0 (yes).	**b** $x(2x - 7) = 0$
2 Apply the Null Factor Law.	$x = 0$ or $2x - 7 = 0$
3 Solve each linear equation.	$x = 0$ or $2x - 7 = 0$ $x = 0$ or $2x = 7$ $x = 0$ or $x = 3\frac{1}{2}$

UNDERSTANDING AND FLUENCY

2 Copy and complete the steps shown to solve each quadratic equation using the Null Factor Law.

a $(x + 4)(x - 1) = 0$
$x + 4 = 0$ or $x - 1 =$ __
$x =$ __ or $x =$ __

b $3x(x - 5) = 0$
$3x =$ __ or $x - 5 =$ __
$x =$ __ or $x =$ __

c $(x + \sqrt{3})(x - \sqrt{3}) = 0$
$x + \sqrt{3} =$ __ or $x - \sqrt{3} =$ __
$x =$ __ or $x =$ __
$x = \pm\sqrt{3}$

3 Solve each equation. Show all steps of working.

a $(x + 2)(x - 4) = 0$ **b** $(x + 7)(x + 1) = 0$ **c** $(x - 3)(x - 6) = 0$
d $(x + 9)(x - 9) = 0$ **e** $x(x - 5) = 0$ **f** $x(2x - 1) = 0$
g $2x(x + 1) = 0$ **h** $7x(2x + 4) = 0$ **i** $\frac{1}{2}x(4x - 3) = 0$
j $(x + \sqrt{2})(x - \sqrt{2}) = 0$ **k** $(x - \sqrt{5})(x + \sqrt{5}) = 0$ **l** $(x + 4\sqrt{3})(x - 4\sqrt{3}) = 0$
m $(2x + 3)(x - 4) = 0$ **n** $(3x - 1)(3x + 1) = 0$ **o** $(7x + 2)(4x - 5) = 0$

EXAMPLE 5A-2 Solving quadratic equations after first factorising

Solve each equation.

a $x^2 - 4x - 12 = 0$ **b** $x^2 - 16 = 0$ **c** $5x^2 = -35x$

THINK	WRITE
a 1 Write the equation. Check that the LS is in factor form (no) and the RS equals 0 (yes).	**a** $x^2 - 4x - 12 = 0$
2 Factorise the LS of the equation (factors of −12 that add to −4).	$(x + 2)(x - 6) = 0$
3 Apply the Null Factor Law.	$x + 2 = 0$ or $x - 6 = 0$
4 Solve each linear equation.	$x = -2$ or $x = 6$
b 1 Write the equation. Check that the LS is in factor form (no) and the RS equals 0 (yes).	**b** $x^2 - 16 = 0$
2 Factorise the LS of the equation (difference of two squares).	$x^2 - 4^2 = 0$ $(x + 4)(x - 4) = 0$
3 Apply the Null Factor Law.	$x + 4 = 0$ or $x - 4 = 0$
4 Solve each linear equation.	$x = -4$ or $x = 4$
5 Alternatively, write the solution in a shorthand way using ±.	$x = \pm 4$
c 1 Write the equation.	**c** $5x^2 = -35x$
2 Rearrange the equation so the RS is zero.	$5x^2 + 35x = 0$
3 Factorise the LS of the equation (take out common factor).	$5x(x + 7) = 0$
4 Apply the Null Factor Law.	$5x = 0$ or $x + 7 = 0$
5 Solve each linear equation.	$x = 0$ or $x = -7$

UNDERSTANDING AND FLUENCY

4 Solve each equation.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$ **c** $x^2 - 9x + 8 = 0$
d $x^2 - 25 = 0$ **e** $x^2 - 81 = 0$ **f** $4 - x^2 = 0$
g $x^2 - 3x = 0$ **h** $x^2 + x = 0$ **i** $2x^2 + 8x = 0$
j $10x^2 - 4x = 0$ **k** $3x^2 = 18x$ **l** $7x^2 = -70x$

5 Use substitution to check that the solutions to each quadratic equation in question **4** are correct.

6 Consider $3(x + 6)(x - 8) = 0$.

a Why is this equivalent to $(x + 6)(x - 8) = 0$?
b Solve $(x + 6)(x - 8) = 0$ and hence write the solutions for $3(x + 6)(x - 8) = 0$.
c Use substitution to check that you have the correct solutions.

UNDERSTANDING AND FLUENCY

7 Consider $-5(x + 3)(x + 2) = 0$.

- **a** Why is this equivalent to $(x + 3)(x + 2) = 0$?
- **b** Solve $(x + 3)(x + 2) = 0$ and hence write the solutions for $-5(x + 3)(x + 2) = 0$.
- **c** Use substitution to check that you have the correct solutions.

8 Explain why you can divide both sides of $3(x - 2) = 0$ by 3 but not divide both sides of $x(x - 2) = 0$ by x.

9 Solve each quadratic equation.

- **a** $2(x + 5)(x - 6) = 0$
- **b** $7(x - 2)(x - 4) = 0$
- **c** $-5(x + 8)(x + 1) = 0$
- **d** $-4(x + 9)(x - 3) = 0$
- **e** $-11(x - 4)(x + 4) = 0$
- **f** $-x(x + 7) = 0$
- **g** $-6x(x - 2) = 0$
- **h** $-2x(3x + 5) = 0$
- **i** $-8(2x - 3)(x + 10) = 0$

10 Solve each quadratic equation. (Hint: take out a common factor first.)

- **a** $2x^2 + 6x + 4 = 0$
- **b** $3x^2 + 6x - 45 = 0$
- **c** $6x^2 - 30x + 24 = 0$
- **d** $-5x^2 - 5x + 30 = 0$
- **e** $-4x^2 + 16x + 48 = 0$
- **f** $-x^2 - x + 56 = 0$
- **g** $11x^2 - 44 = 0$
- **h** $-7x^2 + 63 = 0$
- **i** $-16x^2 + 8x = 0$

EXAMPLE 5A-3 Solving quadratic equations using surds

Solve each equation. Write the solutions as exact values.

a $x^2 - 5 = 0$ **b** $-2x^2 + 24 = 0$

THINK	WRITE
a **1** Write the equation. Check that the LS is in factor form (no) and the RS equals 0 (yes).	**a** $x^2 - 5 = 0$
2 Factorise the LS of the equation (difference of two squares).	$x^2 - (\sqrt{5})^2 = 0$ $(x + \sqrt{5})(x - \sqrt{5}) = 0$
3 Apply the Null Factor Law.	$x + \sqrt{5} = 0$ or $x - \sqrt{5} = 0$
4 Solve each linear equation.	$x = -\sqrt{5}$ or $x = \sqrt{5}$
5 Alternatively, write the solution using $\pm$.	$x = \pm\sqrt{5}$
b **1** Write the equation.	**b** $-2x^2 + 24 = 0$
2 Take out a common factor of -2.	$-2(x^2 - 12) = 0$
3 Divide both sides of the equation by -2.	$x^2 - 12 = 0$
4 Factorise the LS of the equation (difference of two squares).	$x^2 - (\sqrt{12})^2 = 0$ $(x + \sqrt{12})(x - \sqrt{12}) = 0$
5 Simplify the surd. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$.	$(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$
6 Apply the Null Factor Law.	$x + 2\sqrt{3} = 0$ or $x - 2\sqrt{3} = 0$
7 Solve each linear equation.	$x = -2\sqrt{3}$ or $x = 2\sqrt{3}$
8 Alternatively, write the solution using $\pm$.	$x = \pm 2\sqrt{3}$

11 Solve each equation. Write the solutions as exact values.

a $x^2 - 3 = 0$	**b** $x^2 - 11 = 0$	**c** $2x^2 - 14 = 0$	**d** $x^2 - 8 = 0$
e $x^2 - 27 = 0$	**f** $3x^2 - 6 = 0$	**g** $-x^2 + 5 = 0$	**h** $-4x^2 + 24 = 0$
i $-2x^2 + 64 = 0$	**j** $5x^2 = 95$	**k** $8x^2 = 160$	**l** $156 - 3x^2 = 0$

EXAMPLE 5A-4 Solving quadratic equations by completing the square

Solve $x^2 + 6x + 3 = 0$ by completing the square. Write the solutions as exact values.

THINK	WRITE
1 Write the equation.	$x^2 + 6x + 3 = 0$
2 Rewrite the LS by completing the square.	$(x^2 + 6x + 9) - 9 + 3 = 0$ $(x^2 + 6x + 9) - 6 = 0$ $(x + 3)^2 - 6 = 0$
3 Factorise the LS using the difference of two squares.	$(x + 3)^2 - (\sqrt{6})^2 = 0$ $(x + 3 + \sqrt{6})(x + 3 - \sqrt{6}) = 0$
4 Apply the Null Factor Law.	$x + 3 + \sqrt{6} = 0$ or $x + 3 - \sqrt{6} = 0$
5 Solve each linear equation.	$x = -3 - \sqrt{6}$ or $x = -3 + \sqrt{6}$
6 Alternatively, write the solution using $\pm$.	$x = -3 \pm \sqrt{6}$

UNDERSTANDING AND FLUENCY

12 Solve each equation by completing the square. Write the solutions as exact values.

a $x^2 + 2x - 6 = 0$	**b** $x^2 - 4x + 2 = 0$	**c** $x^2 + 8x + 11 = 0$
d $x^2 - 12x + 28 = 0$	**e** $x^2 + 20x + 25 = 0$	**f** $x^2 - x - 1 = 0$

13 Write the solutions to each equation in question **12** correct to two decimal places.

14 How many solutions does each quadratic equation have?

a $(x - 3)(x - 5) = 0$	**b** $(x + 2)(x + 2) = 0$	**c** $(x - 4)(x + 4) = 0$
d $(x - 7)^2 = 0$	**e** $x(x - 6) = 0$	**f** $x^2 + 9 = 0$

15 Find the solution/s to each equation. (Hint: some of these equations cannot be solved.)

a $x^2 - 9 = 0$	**b** $x^2 + 4x - 77 = 0$	**c** $x^2 - 8x + 16 = 0$
d $x^2 + 4 = 0$	**e** $x^2 + 4x = 0$	**f** $x^2 - 6x - 1 = 0$

16 Solve each quadratic equation. (Hint: first rearrange into the form $ax^2 + bx + c = 0$.)

a $x^2 + 4x = 12$	**b** $x^2 + 4 = 5x$	**c** $6x^2 = 54$
d $x^2 - 3x - 20 = 20$	**e** $x^2 - 6x = 3x$	**f** $x^2 + 1 = 4x$
g $5x - x^2 = x$	**h** $x^2 + 21 = 11x + 3$	**i** $2x^2 - 3x = x^2 + 18$
j $x(x + 4) = 21$	**k** $(x - 5)^2 = 1$	**l** $x^2 + 3x + 10 = 1 - 3x$

17 Use the Solve function of a calculator or other digital technology to solve each quadratic equation in questions **15** and **16** and compare your solutions.

PROBLEM SOLVING AND REASONING

18 Lauren dives with her outstretched arms from a platform into the diving pool. The height of her fingertips above the pool surface can be represented by the quadratic relationship $h = -2t^2 + 4t + 6$, where h is the height in metres after t seconds.

a What is the height after:

i 1 second? **ii** 2 seconds?

b What is the height of her fingertips when Lauren starts the dive?

c How long does it take for the dive? (Hint: how long does it take for her fingertips to hit the water surface?)

d Explain why there is only one time value for your answer to part **c** even though you have solved a quadratic equation that has two solutions.

19 Oliver buys rice to serve with stir fry meals over the next few weeks. The amount he has left can be represented by the quadratic relationship $y = x^2 - 10x + 25$, where y is the amount left in kilograms after x weeks.

a How much rice did Oliver buy?

b How much rice is left after 2 weeks?

c How long does it take for the rice to be used up?

d Did Oliver use the same amount of rice each week? Explain.

20 The area of a rectangular vegetable plot is 70 m^2. The length is 3 m longer than the width.

a Write a quadratic equation to represent this scenario. (Hint: let x represent the width of the vegetable plot in metres.)

b Solve the quadratic equation.

c Write the dimensions of the vegetable plot.

21 The width of a mobile phone screen is 5 cm less than its length. Write an equation to represent the scenario where the area of the screen is 66 cm^2 and then solve it to find the dimensions of the screen.

CHALLENGE

22 Solve each quadratic equation by completing the square. (Hint: first take out the coefficient of the x^2 term as a common factor.)

a $3x^2 + 18x + 21 = 0$ **b** $4x^2 + 8x - 16 = 0$ **c** $-x^2 + 4x - 1 = 0$

d $-6x^2 - 48x - 36 = 0$ **e** $2x^2 - 12x - 3 = 0$ **f** $5x^2 + 5x - 2 = 0$

23 Solve each quadratic equation. (Hint: factorise by splitting the x term or using the cross-multiply method.)

a $2x^2 + 13x + 15 = 0$ **b** $5x^2 - 22x + 8 = 0$

c $6x^2 + x - 2 = 0$ **d** $4x^2 + 25x - 21 = 0$

e $3x^2 + 2x = 1$ **f** $7x^2 = 11x + 6$

Reflect

Why do you often need to factorise a quadratic expression before solving a quadratic equation?

5B Solving quadratic equations using the quadratic formula

Start thinking!

Quadratic equations can be written in the general form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$.

1 List the values for a, b and c in the equation $2x^2 + 12x - 4 = 0$.

2 Discuss how you might solve this equation with a classmate.

3 Can you take out a common factor? If so, show the result you would get after dividing both sides of the equation by this common factor.

4 Use your result to question **3** to answer these.

a Can you factorise the left side of the equation using factors of the constant term that add to give the coefficient of the x term? If so, show the result.

b Can you factorise the left side of the equation using the completing the square method? If so, show the result.

5 The left side of the equation can be factorised so the equation becomes $(x + 3 + \sqrt{11})(x + 3 - \sqrt{11}) = 0$.

a How many solutions does the equation have?

b Write the solutions as exact values.

c Write the solutions correct to two decimal places.

Completing the square can be very laborious when solving equations like this. In Exercise 5B, you will use a formula that makes finding the solutions a lot easier.

6 CHALLENGE! Try to find the formula for yourself. Use the completing the square method to find the solutions to $ax^2 + bx + c = 0$.

KEY IDEAS

- The **quadratic formula** can be used to solve equations of the general form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- The solution is $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

  This can be written more simply as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The **discriminant** (Δ) is $b^2 - 4ac$. Evaluating this expression after substituting for a, b and c helps you identify the number of real solutions to the quadratic equation.
  - If $\Delta < 0$, there are no real solutions.
  - If $\Delta = 0$, there is one real solution.
  - If $\Delta > 0$, there are two real solutions.

NOTE In this chapter, you will be considering real solutions only.

EXERCISE 5B Solving quadratic equations using the quadratic formula

1 Compare each quadratic equation with the general form $ax^2 + bx + c = 0$ and identify a, b and c.

a $2x^2 + 7x + 3 = 0$ **b** $x^2 + 5x + 8 = 0$ **c** $3x^2 - 6x + 3 = 0$

d $5x^2 + x + 1 = 0$ **e** $-4x^2 + 12x - 9 = 0$ **f** $-x^2 + 4x + 2 = 0$

2 Use $\Delta = b^2 - 4ac$ to calculate the value of the discriminant for each equation in question **1**.

EXAMPLE 5B-1 Identifying the number of solutions to a quadratic equation

Use the discriminant to identify the number of solutions for each quadratic equation.

a $3x^2 + 7x + 2 = 0$ **b** $x^2 + 16 = 8x$ **c** $-2x^2 + 3x - 5 = 0$

THINK

a **1** Write the equation.
2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.
3 Write the formula for the discriminant.
4 Substitute values for a, b and c into the formula and simplify.
5 Identify the number of solutions.

b **1** Write the equation and rearrange so right side is 0.
2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.
3 Write the formula for the discriminant.
4 Substitute values for a, b and c into the formula and simplify.
5 Identify the number of solutions.

c **1** Write the equation.
2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.
3 Write the formula for the discriminant.
4 Substitute values for a, b and c into the formula and simplify.
5 Identify the number of solutions.

WRITE

a $3x^2 + 7x + 2 = 0$
$a = 3, b = 7, c = 2$
$\Delta = b^2 - 4ac$
$= 7^2 - 4(3)(2)$
$= 49 - 24$
$= 25$
Since $\Delta > 0$, the equation has two solutions.

b $x^2 + 16 = 8x$
$x^2 - 8x + 16 = 0$
$a = 1, b = -8, c = 16$
$\Delta = b^2 - 4ac$
$= (-8)^2 - 4(1)(16)$
$= 64 - 64$
$= 0$
Since $\Delta = 0$, the equation has one solution.

c $-2x^2 + 3x - 5 = 0$
$a = -2, b = 3, c = -5$
$\Delta = b^2 - 4ac$
$= 3^2 - 4(-2)(-5)$
$= 9 - 40$
$= -31$
Since $\Delta < 0$, the equation has no solutions.

3 Use your answers to question **2** to identify the number of solutions for each equation in question **1**.

4 Use the discriminant to identify the number of solutions for each quadratic equation.

a $x^2 + 5x + 7 = 0$ **b** $3x^2 - 8x - 4 = 0$ **c** $x^2 - 9 = 0$

d $2x^2 + 3x + 4 = 0$ **e** $12x - x^2 = 36$ **f** $6x^2 - 5 = 7x$

5 Complete the following for each quadratic equation shown below.

i Calculate the discriminant. **ii** Identify the number of solutions.

iii Solve by first factorising and then using the Null Factor Law.

iv Check whether your answer to part **iii** confirms the result to part **ii**.

a $x^2 - 2x - 15 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 - 49 = 0$

d $3x^2 - 18x = 0$ **e** $x^2 - 6x + 7 = 0$ **f** $x^2 + 2x + 3 = 0$

6 Copy and complete the lines of working below to discover the quadratic formula. You will complete the square to factorise the left side and then use the Null Factor Law to find the solutions to $ax^2 + bx + c = 0$ (where $a \neq 0$).

Step 1 Write the quadratic equation. $ax^2 + bx + c = 0$

Step 2 Take out the coefficient of x^2 as a common factor. $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$

Step 3 Divide both sides by a. $x^2 + \frac{b}{a}x + \_\_\_ = 0$

Step 4 Add a new third term to complete the square and compensate by subtracting the same term. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{\_\_\_} = 0$

Step 5 Form a perfect square using the first three terms and simplify the fourth term. $\left(x + \frac{b}{2a}\right)^2 - \frac{\_\_\_}{4a^2} + \frac{c}{a} = 0$

Step 6 Group the last two terms together. $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0$

Step 7 Combine the last two terms using a common denominator. $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\_\_\_ - 4ac}{4a^2}\right) = 0$

Step 8 Write the left side as the difference of two squares. $\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4a}{4a^2}}\right)^2 = 0$

Step 9 Simplify the second square. $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$

Step 10 Factorise the left side using the difference of two squares rule. $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \_\_\_\right)^2 = 0$

Step 11 Use the Null Factor Law. $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \_\_\_ = 0$

Step 12 Solve each linear equation. $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \_\_\_$

Step 13 Simplify each solution. $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

So the quadratic formula for finding the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 5B-2 Using the quadratic formula to solve quadratic equations

Use the quadratic formula to solve each equation. Write the solutions as:
i exact values **ii** approximate values correct to two decimal places.
a $2x^2 - 5x + 1 = 0$ **b** $9x^2 + 12x + 4 = 0$ **c** $4x^2 = 3x - 1$

THINK

a **1** Write the equation.

2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.

3 Write the quadratic formula.

4 Substitute values for a, b and c into the formula and simplify.

5 Write the solution as exact values.

6 Use a calculator to write the values correct to two decimal places.

b **1** Write the equation.

2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.

3 Write the quadratic formula.

4 Substitute values for a, b and c into the formula and simplify.

5 Write the solution as an exact value.

6 Write the value correct to two decimal places.

c **1** Write the equation and rearrange so the right side is 0.

2 Compare to $ax^2 + bx + c = 0$ and identify a, b and c.

3 Write the quadratic formula.

4 Substitute values for a, b and c into the formula and simplify.

5 Provide a reason for why there are no real solutions to this equation.

WRITE

a $2x^2 - 5x + 1 = 0$

$a = 2, b = -5, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{17}}{4}$$

i $x = \frac{5 - \sqrt{17}}{4}$ or $\frac{5 + \sqrt{17}}{4}$

ii $x \approx 0.22$ or 2.28

b $9x^2 + 12x + 4 = 0$

$a = 9, b = 12, c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-12 \pm \sqrt{12^2 - 4(9)(4)}}{2(9)}$$

$$= \frac{-12 \pm \sqrt{144 - 144}}{18}$$

$$= \frac{-12 \pm 0}{18}$$

$$= -\frac{2}{3}$$

i $x = -\frac{2}{3}$

ii $x \approx -0.67$

c $4x^2 = 3x - 1$

$4x^2 - 3x + 1 = 0$

$a = 4, b = -3, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(1)}}{2(4)}$$

$$= \frac{3 \pm \sqrt{-7}}{8}$$

Since a real value cannot be found for $\sqrt{-7}$, there are no real solutions.

7 Use the quadratic formula to solve each equation. Write the solutions as:

i exact values

ii approximate values correct to two decimal places.

a $x^2 - 7x + 5 = 0$ **b** $3x^2 - x - 1 = 0$ **c** $2x^2 + x - 4 = 0$

d $-3x^2 - x + 9 = 0$ **e** $-6x^2 - 5x + 3 = 0$ **f** $x^2 - 2x - 1 = 0$

g $x^2 + 8x + 4 = 0$ **h** $4x^2 - 12x + 1 = 0$ **i** $-x^2 + 4x + 2 = 0$

j $5x^2 + 10x - 3 = 0$ **k** $2x^2 = 11 - 3x$ **l** $x^2 + 2x = 7$

8 Consider the quadratic equation $x^2 - 5x + 6 = 0$.

a Solve the equation by factorising the left side and using the Null Factor Law.

b Solve the equation using the quadratic formula.

c Which method was quicker or easier to use?

d The solutions to any quadratic equation can be found using the quadratic formula. Why might you try using the method described in part **a** first?

9 The profit p (in dollars) made from selling n soft toys is given by the relationship $p = -0.1n^2 + 17n + 200$.

a What is the profit when 30 toys are sold?

b What is the smallest number of toys that need to be sold to make a profit of at least $850? (Hint: use the quadratic formula to solve an equation.)

10 A rectangular paddock is 20 m longer than it is wide and has an area of 5000 m^2. Solve a quadratic equation to find the dimensions of the paddock correct to one decimal place.

11 The perimeter of a framed painting is 100 cm. Within the 2 cm frame, the area of the painting that can be viewed is 300 cm^2. Find the dimensions of the framed painting, correct to the nearest millimetre.

12 Consider the result to $\sqrt{b^2 - 4ac}$ in the quadratic formula when answering each of these.

- **a** Explain why there are no solutions to a quadratic equation when the discriminant is negative.
- **b** Explain why there is only one solution when the discriminant is zero.
- **c** Explain why there are two solutions when the discriminant is positive.

CHALLENGE

13 **a** Find the value of k so $kx^2 + 6x + 3 = 0$ has only one solution.

- **b** For what values of k does $kx^2 + 6x + 3 = 0$ have:
  - **i** two solutions?
  - **ii** no solutions?

14 **a** Find the two values of k so $4x^2 + kx + 1 = 0$ has only one solution.

- **b** For what values of k does $4x^2 + kx + 1 = 0$ have two solutions? Write three examples of equations that fit this condition.
- **c** For what values of k does $4x^2 + kx + 1 = 0$ have no solutions? Write three examples of equations that fit this condition.

15 Find the value of k in each equation to give the number of solutions shown in brackets.

- **a** $x^2 + 2x + k = 0$ [one solution]
- **b** $2x^2 - 4x + \text{k} = 0$ [two solutions]
- **c** $kx^2 + 6x - 1 = 0$ [no solutions]
- **d** $kx^2 - 3x + 1 = 0$ [one solution]
- **e** $3x^2 + kx + 3 = 0$ [two solutions]
- **f** $x^2 + kx + 25 = 0$ [no solutions]

16 Write three different quadratic equations that have:

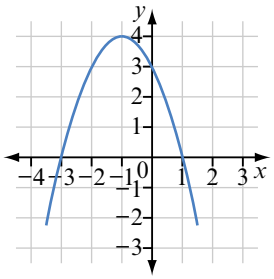
- **a** two solutions
- **b** one solution
- **c** no solutions.

Reflect

When is it an advantage to use the quadratic formula to solve a quadratic equation?

5C Sketching parabolas using intercepts

Start thinking!



The graph of a quadratic relationship is called a **parabola**.

1 Consider the graph of the parabola shown.

a Is the parabola upright or inverted?

b What is the y-intercept? **c** What are the x-intercepts?

d Does it have a **maximum** or **minimum turning point**?

e What are the coordinates of the **turning point**?

To sketch a parabola, you can use the features identified in question **1**. A **sketch graph** shows the general shape of the relationship, the x- and y-intercepts and any other distinguishing features. This is different from plotting a parabola where a table of x and y values is used.

Consider sketching the graph of $y = x^2 - 4x - 5$.

2 Look at the coefficient of the x^2 term. Will the parabola be upright or inverted?

3 **a** What is the x-coordinate at the y-intercept of any graph?

b Substitute this value for x into the rule $y = x^2 - 4x - 5$ and simplify. What is the y-intercept?

4 **a** What is the y-coordinate at the x-intercepts of any graph?

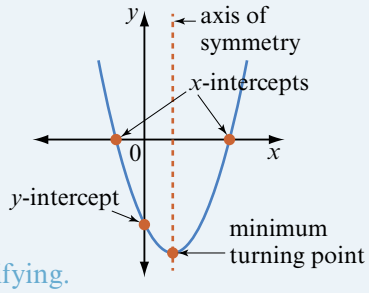
b Substitute this value for y into the rule $y = x^2 - 4x - 5$.

c Solve the equation to find the value of x. What are the x-intercepts?

5 What are the coordinates of the turning point? (Hint: as a parabola is **symmetrical**, what is the x value of the turning point?)

6 Use the information you have to sketch the graph. Label your graph with its rule.

KEY IDEAS



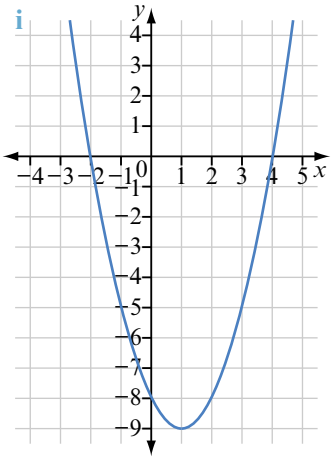
- The graph of a quadratic relationship is a parabola.
- One way of sketching a parabola is to use the x- and y-intercepts. The coordinates of the turning point and the orientation of the parabola (upright or inverted) can also be identified.
- The y-intercept is found by substituting $x = 0$ into the rule and simplifying.
- The x-intercept/s are found by substituting $y = 0$ into the rule and solving for x. A parabola can have two, one or no x-intercepts.
- The **axis of symmetry** of a parabola is a vertical line midway between the x-intercepts. Hence, the x-coordinate of the turning point is halfway between the x values at the x-intercepts. The y-coordinate of the turning point is found by substituting the x-coordinate into the rule and simplifying.

EXERCISE 5C Sketching parabolas using intercepts

UNDERSTANDING AND FLUENCY

1 Consider each parabola.

i



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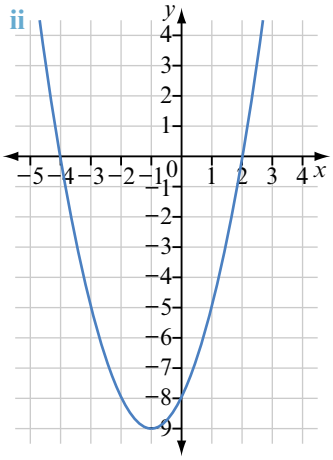

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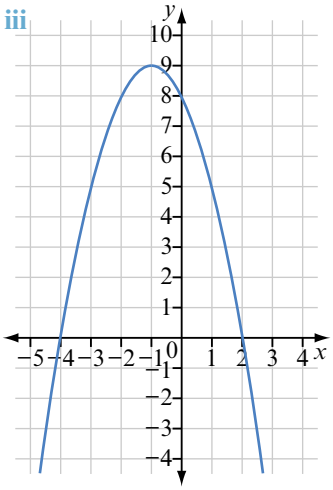


a Identify the x- and y-intercepts.

b List the coordinates of the turning points.

c Describe the nature of each turning point as maximum or minimum.

d Match each graph with its rule from the list below.

A $y = x^2 + 2x - 8$ B $y = -x^2 - 2x + 8$ C $y = x^2 - 2x - 8$

2 Explain how you can tell whether a parabola will be upright or inverted from its rule. Use the parabolas in question **1** as examples in your explanation.

EXAMPLE 5C-1 Finding the x- and y-intercepts of a quadratic relationship

Find the x- and y-intercepts for $y = x^2 - 4x$.

THINK	WRITE
1 To find the x-intercept/s, substitute $y = 0$ into the rule. (You may like to swap the sides of the equation.)	$y = x^2 - 4x$ x-intercepts: when $y = 0$, $0 = x^2 - 4x$ $x^2 - 4x = 0$
2 Factorise the quadratic expression on the left side.	$x(x - 4) = 0$
3 Solve the equation using the Null Factor Law.	$x = 0$ or $x - 4 = 0$ $x = 0$ or $x = 4$
4 List the x-intercepts.	x-intercepts are 0 and 4.
5 To find the y-intercept, substitute $x = 0$ into the rule and simplify.	y-intercept: when $x = 0$, $y = 0 - 0 = 0$
6 List the y-intercept.	y-intercept is 0.

3 For each quadratic relationship, find:
i the x-intercepts **ii** the y-intercept.

a $y = x^2 + 6x$ **b** $y = -x^2 + 10x$ **c** $y = x^2 - 6x + 5$
d $y = x^2 - 1$ **e** $y = -x^2 - 4x + 12$ **f** $y = 9 - x^2$

4 For each quadratic relationship in question **3**, state whether its graph is an upright or an inverted parabola.

EXAMPLE 5C-2 Finding coordinates of the turning point using x-intercepts

Find the coordinates of the turning point for $y = x^2 - 4x$.

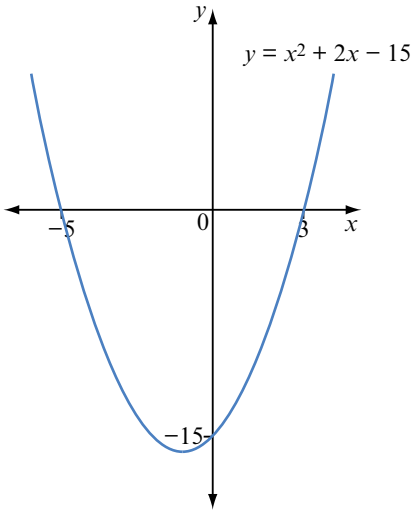
THINK	WRITE
1 Find the x-intercepts (see Example 5C-1).	$y = x^2 - 4x$ x-intercepts are 0 and 4.
2 Since a parabola is symmetrical, the x-coordinate of the turning point is halfway between the x-intercepts. Alternatively, find the average of the two x values.	Halfway between 0 and 4 is 2, or $x = \frac{0 + 4}{2} = 2$.
3 Find the y-coordinate of the turning point by substituting $x = 2$ into the rule and simplifying.	When $x = 2$, $y = 2^2 - 4(2) = -4$
4 Write the coordinates of the turning point.	Coordinates of the turning point are $(2, -4)$.

UNDERSTANDING AND FLUENCY

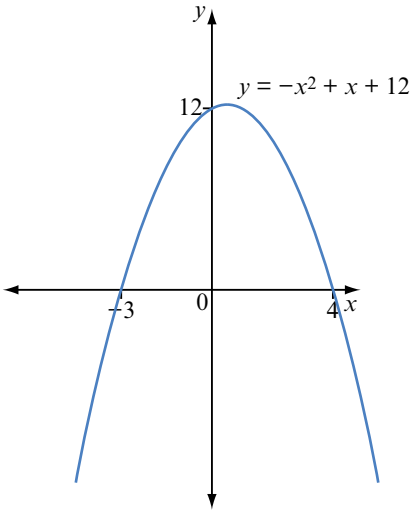
5 Find the coordinates of the turning point for each quadratic relationship in question **3**.

6 Find the coordinates of the turning point for each parabola.

a



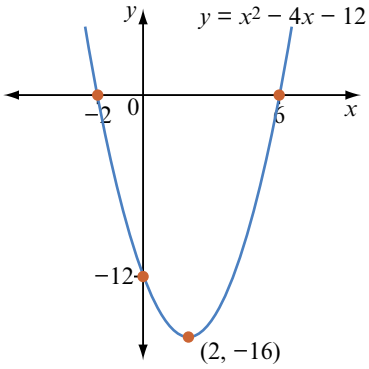
b



7 Sketch the graph of each quadratic relationship in question **3** using your answers to questions **3**–**5**.

EXAMPLE 5C-3 Sketching a parabola using *x*- and *y*-intercepts

Sketch the graph of $y = x^2 - 4x - 12$ using intercepts. Label the turning point with its coordinates.

THINK	WRITE
1 Find the *x*-intercepts by substituting $y = 0$ into the rule and solving for x. Factorise so that the Null Factor Law can be used.	$y = x^2 - 4x - 12$ *x*-intercepts: when $y = 0$, $x^2 - 4x - 12 = 0$ $(x + 2)(x - 6) = 0$ $x + 2 = 0$ or $x - 6 = 0$ $x = -2$ or $x = 6$ *x*-intercepts are -2 and 6.
2 Find the *y*-intercept by substituting $x = 0$ into the rule and simplifying.	*y*-intercept: when $x = 0$, $y = 0^2 - 4(0) - 12 = -12$ *y*-intercept is -12.
3 Find the coordinates of the turning point. The *x*-coordinate is halfway between the *x*-intercepts (or the average of the two *x* values).	At turning point, $x = \frac{-2 + 6}{2} = 2$ When $x = 2$, $y = 2^2 - 4(2) - 12 = -16$ Coordinates of turning point are $(2, -16)$.
4 Plot the points for the *x*- and *y*-intercepts and the turning point on a Cartesian plane.	
5 Draw an upright parabola through the points and label with the rule. (The parabola is upright since the coefficient of the x^2 term is positive.)	

UNDERSTANDING AND FLUENCY

8 Sketch the graph of each quadratic relationship using intercepts. Label the turning point with its coordinates.

a $y = x^2 - 6x + 8$ **b** $y = -x^2 + 10x - 16$ **c** $y = x^2 + 4x - 12$

d $y = 4 - x^2$ **e** $y = x^2 + 2x$ **f** $y = -x^2 + 10x - 25$

g $y = x^2 - 4x - 21$ **h** $y = x^2 - x - 6$ **i** $y = x^2 + 7x + 10$

9 Consider the graphs of $y = (x + 5)(x - 1)$, $y = 2(x + 5)(x - 1)$ and $y = -(x + 5)(x - 1)$.

a Find the *x*-intercepts for each graph.

b Explain why the graphs will be different even though each parabola has the same *x*-intercepts.

c Find the *y*-intercept for each parabola.

d Find the coordinates of the turning point for each parabola.

e Sketch all three parabolas on the same Cartesian plane.

EXAMPLE 5C-4

Sketching a parabola using the quadratic formula to find the *x*-intercepts

Sketch the graph of $y = x^2 + 6x + 7$ using the quadratic formula to find the x-intercepts. You do not need to label the turning point.

THINK

1 Find the x-intercepts by substituting $y = 0$ into the rule and solving for x. The quadratic expression is not easy to factorise so use the quadratic formula to find the solutions.
The surd $\sqrt{8}$ can be simplified to $2\sqrt{2}$.
$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

2 Find the y-intercept by substituting $x = 0$ into the rule and simplifying.

3 Draw an upright parabola through the x- and y-intercepts. Label the intercepts with exact values.

WRITE

$y = x^2 + 6x + 7$
x-intercepts: when $y = 0$,
$x^2 + 6x + 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 6$, $c = 7$

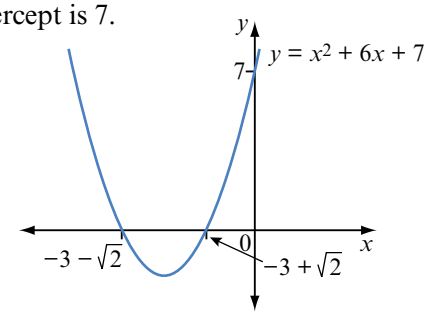
$$= \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{8}}{2}$$

$$= -3 \pm \sqrt{2}$$

x-intercepts are $-3 - \sqrt{2}$ and $-3 + \sqrt{2}$ (or approx. -4.4 and -1.6).

y-intercept: when $x = 0$,
$y = 0^2 + 6(0) + 7 = 7$
y-intercept is 7.

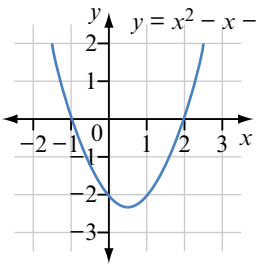


10 Sketch the graph of each quadratic relationship using the quadratic formula to find the x-intercepts. You do not need to label the turning point.

a $y = x^2 + 4x - 1$ **b** $y = x^2 - 8x + 3$ **c** $y = x^2 + 6x - 8$
d $y = x^2 - 2x - 10$ **e** $y = -x^2 + 2x + 5$ **f** $y = -x^2 - x + 4$

11 Look at the graphs of these three quadratic relationships.

i



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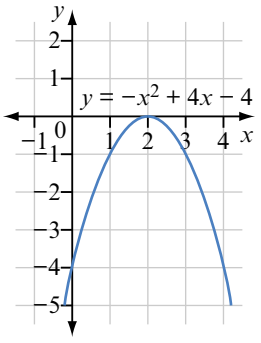

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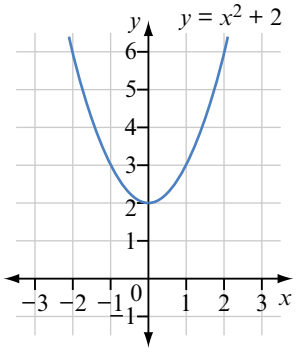


PROBLEM SOLVING AND REASONING

a How many x-intercepts does each parabola have?

b Use the graphs to solve:

i $x^2 - x - 2 = 0$ **ii** $-x^2 + 4x - 4 = 0$ **iii** $x^2 + 2 = 0$

c Show how you can use the discriminant ($\Delta = b^2 - 4ac$) to verify the number of x-intercepts for each parabola.

d Show how you can use the quadratic formula to find the x-intercepts.

12 Before sketching a parabola, you can work out the number of x-intercepts it will have.

i How many x-intercepts will each parabola with the following rules have?

ii Find the coordinates of the x-intercepts.

a $y = x^2 - 4x + 4$ **b** $y = x^2 + 2x + 3$ **c** $y = x^2 - 6x + 7$

13 Thomas throws a discus. The position of the discus can be represented by $h = -0.01(d + 4)(d - 72)$, where h is the height above the ground and d is the horizontal distance from where the discus was thrown. Both h and d are in metres.

a Sketch the graph of this relationship by finding the intercepts.

b At what height off the ground was the discus thrown?

c What horizontal distance did the discus travel before hitting the ground?

d What is the greatest height that the discus reaches?

14 A ball is kicked off the ground. Its path can be represented by $y = -0.05x^2 + 1.6x$, where x is the horizontal distance in metres and y is the vertical distance in metres.

a Sketch the graph of this relationship.

b What was the maximum height of the ball? What horizontal distance had it travelled when it reached maximum height?

c What horizontal distance did the ball travel before hitting the ground?

d If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

CHALLENGE

15 Consider the graph of $y = ax^2 + bx + c$.

a Use the quadratic formula to write expressions for the x-intercepts.

b Use the x-intercepts to show that the x-coordinate of the turning point is $-\frac{b}{2a}$.

c Write an expression for the y-coordinate of the turning point.

16 Sketch the graph of each quadratic relationship using intercepts. Label the turning point with its coordinates.

a $y = x^2 + 2x - 7$ **b** $y = -x^2 + 4x + 2$ **c** $y = x^2 - 2x - 1$

17 Write three quadratic rules that have:

a two x-intercepts

b one x-intercept

c no x-intercepts.

Reflect

How can you determine the number of x-intercepts a parabola has?

5D Sketching parabolas using transformations

Start thinking!

The **transformations** of **dilation**, **reflection** and **translation** can be performed on the graph of $y = x^2$ to produce a new parabola. You may like to use technology for this task.

For each set of quadratic rules below:

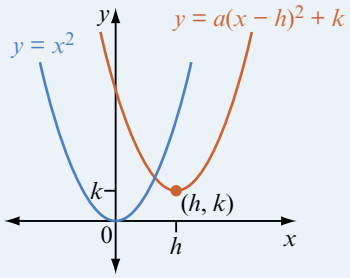
i draw the graphs on the same Cartesian plane

ii compare each parabola with the graph of $y = x^2$

iii identify the transformation/s you think have been performed. Describe the effect of different coefficients for x^2 and any constants that are added or subtracted.

1	**a** $y = x^2$	**b** $y = 2x^2$	**c** $y = 4x^2$	**d** $y = \frac{1}{2}x^2$	**e** $y = \frac{1}{4}x^2$
2	**a** $y = x^2$	**b** $y = x^2 + 1$	**c** $y = x^2 + 2$	**d** $y = x^2 - 1$	**e** $y = x^2 - 2$
3	**a** $y = x^2$	**b** $y = (x - 1)^2$	**c** $y = (x - 2)^2$	**d** $y = (x + 1)^2$	**e** $y = (x + 2)^2$
4	**a** $y = x^2$	**b** $y = -x^2$	**c** $y = -x^2 + 2$	**d** $y = -2x^2$	**e** $y = -\frac{1}{2}x^2$

KEY IDEAS



- Transformations such as dilation, reflection and translation can be performed on the graph of $y = x^2$.
  - Dilation by a factor of a produces $y = ax^2$.
  - Reflection in the x-axis produces $y = -x^2$.
  - Vertical translation of k units produces $y = x^2 + k$.
  - Horizontal translation of h units produces $y = (x - h)^2$.
- Quadratic relationships can be written in the general form $y = a(x - h)^2 + k$. This is known as the **turning point form**, as the coordinates of the turning point of the parabola can be easily identified as (h, k).
- A combination of transformations can be performed on the graph of $y = x^2$ to produce the sketch graph of $y = a(x - h)^2 + k$.

$y = a(x - h)^2 + k$

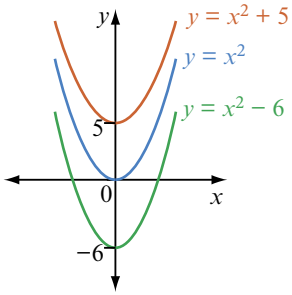
vertical translation of k units
For $k > 0$, move up.
For $k < 0$, move down.

dilation (narrower or wider)
For $a > 0$, upright parabola.
For $a < 0$, inverted parabola (reflection in x-axis).

horizontal translation of h units
For $h > 0$, move right.
For $h < 0$, move left.

EXERCISE 5D Sketching parabolas using transformations

UNDERSTANDING AND FLUENCY



1 Consider graphs with rules of the form $y = x^2 + k$, where k could be any number. Use the diagram to help you answer the following.

- **a** Describe how the value of k affects the graph of $y = x^2$.
- **b** Explain how you know whether to move the graph of $y = x^2$ up or down.
- **c** Copy and complete each sentence using the word *up* or *down*.
  When k is positive, the graph of $y = x^2$ is moved _____.
  When k is negative, the graph of $y = x^2$ is moved _____.
- **d** What are the coordinates of the turning point for $y = x^2 + k$?

2 For each rule below:

- **i** identify the value of k, if the graph has the rule $y = x^2 + k$
- **ii** describe the transformation needed to produce each parabola from the graph of $y = x^2$.

a $y = x^2 + 3$ **b** $y = x^2 - 5$ **c** $y = x^2 + 4.5$

EXAMPLE 5D-1 Sketching a parabola by performing a vertical translation

Perform a transformation on the graph of $y = x^2$ to sketch the graph of $y = x^2 + 4$. Clearly show the coordinates of the turning point.

THINK

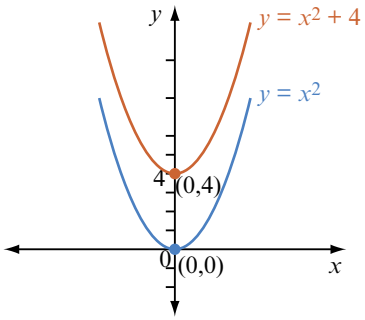
1 Identify the transformation. Translation of 4 units up. (No dilation or reflection.)

2 Sketch the graph of $y = x^2$ and locate its turning point. Translate this point 4 units up. This becomes the turning point for $y = x^2 + 4$.

3 Use the position of the turning point at (0, 4) and the orientation of the parabola (upright) to sketch the graph.

WRITE

$y = x^2 + 4$
Graph of $y = x^2$ is translated 4 units up.

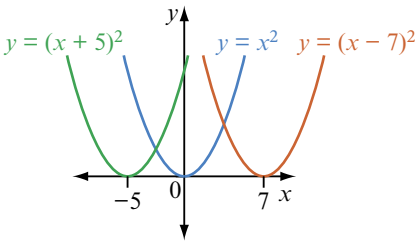


3 Perform a transformation on the graph of $y = x^2$ to sketch the graph of each quadratic rule. Clearly show the coordinates of the turning point.

a $y = x^2 + 3$ **b** $y = x^2 + 2$ **c** $y = x^2 - 5$
d $y = x^2 + 6$ **e** $y = x^2 - 4$ **f** $y = x^2 - 7.5$

UNDERSTANDING AND FLUENCY

4 Consider the graphs with rules of the form $y = (x - h)^2$, where h could be any number. Use the diagram to help you answer the following.



- **a** Describe how the value of h affects the graph of $y = x^2$.
- **b** Explain how you know whether to move the graph of $y = x^2$ left or right.
- **c** Copy and complete each sentence using the word *left* or *right*.
  When h is positive, the graph of $y = x^2$ is moved _____.
  When h is negative, the graph of $y = x^2$ is moved _____.
- **d** What are the coordinates of the turning point for $y = (x - h)^2$?

5 For each rule below:

- **i** identify the value of h, if the graph has the rule $y = (x - h)^2$
- **ii** describe the transformation needed to produce each graph from the graph of $y = x^2$.

a $y = (x - 3)^2$ **b** $y = (x + 6)^2$ **c** $y = (x - 4.5)^2$

EXAMPLE 5D-2 Sketching a parabola by performing a horizontal translation

Perform a transformation on the graph of $y = x^2$ to sketch the graph of $y = (x + 4)^2$. Clearly show the y-intercept and the coordinates of the turning point.

THINK

1 Identify the transformation. Translation of 4 units left. (No dilation or reflection.)

2 Find the y-intercept.

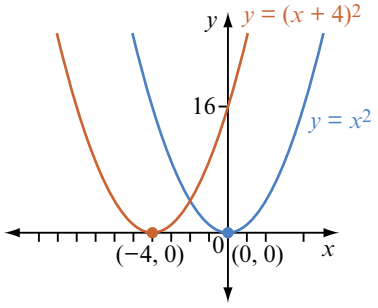
3 Sketch the graph of $y = x^2$ and locate its turning point. Translate this point 4 units left. This becomes the turning point for $y = (x + 4)^2$.

4 Use the position of the turning point at $(-4, 0)$ and the orientation of the parabola (upright) to sketch the graph. Label the y-intercept.

WRITE

$y = (x + 4)^2$
Graph of $y = x^2$ is translated 4 units left.

When $x = 0$, $y = (0 + 4)^2 = 16$
y-intercept is 16.



6 Perform a transformation on the graph of $y = x^2$ to sketch the graph of each quadratic rule. Clearly show the y-intercept and the coordinates of the turning point.

a $y = (x - 4)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 5)^2$
d $y = (x + 2)^2$ **e** $y = (x - 8)^2$ **f** $y = (x + 1.5)^2$

7 Describe how you can recognise from its rule whether a parabola will be:

a narrower or wider than the graph of $y = x^2$

b the reflection of $y = x^2$ in the x-axis.

8 For each rule below, match the description of a transformation(s) **A–F** performed on the graph of $y = x^2$ to produce its graph.

A dilated by a factor of 9

B translated 9 units down

C reflected in the x-axis and then translated 9 units right

D translated 9 units left

E reflected in the x-axis and then translated 9 units up

F reflected in the x-axis and dilated by a factor of 9

a $y = x^2 - 9$ **b** $y = (x + 9)^2$ **c** $y = 9x^2$

d $y = -9x^2$ **e** $y = -x^2 + 9$ **f** $y = -(x - 9)^2$

9 Write the rule for each parabola produced after performing the described transformation/s on the graph of $y = x^2$.

a dilation by a factor of 6

b reflection in the x-axis

c translation of 5 units right

d translation of 2 units down

e dilation by a factor of $\frac{1}{4}$ and reflection in the x-axis

f reflection in the x-axis and translation of 3 units up

10 Combining transformations gives us the turning point form of a quadratic relationship. Compare $y = (x - 2)^2 + 5$ to $y = a(x - h)^2 + k$.

a Identify a, h and k.

b What information can you identify from the values of a, h and k?

c Use (h, k) to write the coordinates of the turning point.

d Explain how (h, k) is related to the translations performed on $y = x^2$ to produce $y = (x - 2)^2 + 5$.

11 For each quadratic relationship:

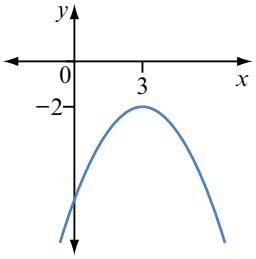
i describe the translations performed on $y = x^2$

ii write the coordinates of the turning point.

a $y = (x - 4)^2 + 3$ **b** $y = (x - 7)^2 - 2$ **c** $y = (x + 3)^2 + 1$

d $y = (x + 2)^2 - 4$ **e** $y = (x - 1)^2 + 8$ **f** $y = (x + 6)^2 - 5$

12 Look at the graph shown at right.



a Is the parabola upright or inverted?

b Identify the coordinates of the turning point.

c Which of these rules would best match the graph? Explain.

A $y = (x + 3)^2 - 2$ **B** $y = -(x + 3)^2 + 2$

C $y = -(x - 3)^2 - 2$ **D** $y = (x - 3)^2 + 2$

EXAMPLE 5D-3

Sketching a parabola by performing more than one transformation

Sketch the graph of $y = -(x + 1)^2 + 3$ by performing transformations on $y = x^2$.
Label the y-intercept and the turning point.

THINK

1 Identify the transformations: reflection then translation of 1 unit left and 3 units up. (No dilation.)

2 Identify the type and position of the turning point. General coordinates are (h, k) where $h = -1$ and $k = 3$.

3 Find the y-intercept.

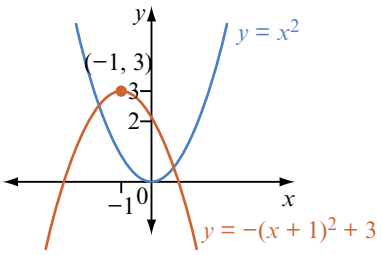
4 Use the turning point and the orientation of the parabola (inverted) to sketch the graph.
Label the y-intercept and the turning point.

WRITE

$y = -(x + 1)^2 + 3$
Graph of $y = x^2$ is reflected in x-axis, then translated 1 unit left and 3 units up.

Maximum turning point at $(-1, 3)$.

When $x = 0$, $y = -(0 + 1)^2 + 3 = 2$
y-intercept is 2.



UNDERSTANDING AND FLUENCY

13 Sketch the graph of each quadratic relationship by performing transformations on $y = x^2$. Label the y-intercept and the turning point.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 5)^2 - 3$ **c** $y = -(x - 2)^2 + 7$
d $y = -(x + 4)^2 - 6$ **e** $y = (x + 7)^2 + 4$ **f** $y = -(x - 6)^2 - 1$

14 Use a graphics calculator or other digital technology to check the graphs you sketched for question **13**.

15 For each quadratic relationship, identify:

i whether the graph will be narrower or wider than the graph of $y = x^2$
ii whether the parabola will be upright or inverted
iii the coordinates of the turning point

a $y = 2(x - 4)^2 - 8$ **b** $y = -3(x + 1)^2 + 12$ **c** $y = 5(x - 3)^2$
d $y = -4x^2 + 1$ **e** $y = \frac{1}{3}(x + 6)^2 - 3$ **f** $y = -\frac{1}{2}(x - 2)^2 - 9$

16 Write a rule for the parabola produced after performing each set of transformations on the graph of $y = x^2$.

a dilation by a factor of 2 then a translation of 5 units right
b reflection in the x-axis then a translation of 3 units down and 4 units left
c dilation by a factor of $\frac{1}{3}$ and reflection in the x-axis
d dilation by a factor of 5, reflection in the x-axis then a translation of 7 units up

PROBLEM SOLVING AND REASONING

17 A quadratic relationship has the rule $y = (x + 2)^2 + 4$. What is the smallest y value that this relationship can have? Explain.

18 A quadratic relationship has the rule $y = -(x - 6)^2 - 1$. What is the largest y value that this relationship can have? Explain.

19 Riley kicks a football to a teammate. The height of the ball can be represented by the relationship $h = -(t - 2)^2 + 5$, where h is the height in metres after t seconds.

- **a** What are the coordinates of the turning point of this relationship.
- **b** Sketch the graph of this relationship from $t = 0$ to $t = 3$.
- **c** Use the graph to find:
  - **i** the height at which the ball made contact with Riley's boot
  - **ii** the maximum height of the ball during the kick to his teammate
  - **iii** the height of the football when the teammate 'marks' the ball after 3 seconds.
- **d** If Riley's teammate had not marked the ball, how long would it have taken for the football to hit the ground? Give your answer correct to one decimal place.

20 Ella's bank account balance over the first few weeks of the year can be represented by the relationship $m = 20(t - 4)^2 + 50$, where m is the amount in dollars in her account after t weeks.

- **a** Sketch a graph of the relationship for $t = 0$ to 10.
- **b** How much did Ella have in her account at the start of the year?
- **c** What is the lowest amount she had in the account in the first 10 weeks?
- **d** After how many weeks did she have the same account balance as she did at the start of the year?
- **e** What is the account balance after 10 weeks?

21 A soccer ball is kicked from the ground up into the air. The path of the ball follows the relationship $y = -\frac{1}{40}(x - 20)^2 + 10$, where y is the height of the ball for a horizontal distance x from where the ball was kicked. Both x and y are in metres.

- **a** Sketch the graph of this relationship from when the ball was kicked to when it landed.
- **b** What was the maximum height of the soccer ball?
- **c** What horizontal distance did the ball travel before it hit the ground?

CHALLENGE

22 Write the rule for each quadratic relationship from the given information.

- **a** upright parabola, turning point at (2, 6), y-intercept of 10
- **b** upright parabola, turning point at (−4, −12), y-intercept of 20
- **c** inverted parabola, turning point at (−1, 7), y-intercept of 4
- **d** inverted parabola, turning point at (3, −5), y-intercept of −8

23 Find the x- and y-intercepts for each quadratic relationship in question **15**.

Reflect

How are transformations useful when sketching a parabola?

5E Graphs of circles

Start thinking!

Consider the relationship $x^2 + y^2 = 25$. Its graph has two x-intercepts and two y-intercepts.

1 Write the coordinates of the x-intercepts.

2 Write the coordinates of the y-intercepts.

3 Find y for each x value.

a 3 **b** -4 **c** 6 **d** -10

4 What x values are possible to satisfy the relationship? Explain.

5 Copy and complete this table of values for $x^2 + y^2 = 25$. Notice that, in most cases, there is more than one y value for each x value.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	0	± 3		$\pm\sqrt{15}$					± 4		

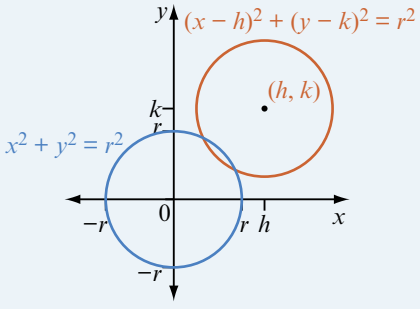
6 Plot the points from the table of values on a Cartesian plane. Use approximate values for any surds to help locate the y-coordinate. Join the points to form a circle.

7 What are the coordinates of the centre of the circle?

8 What is the radius of this circle? How does it link to the rule for the circle?

KEY IDEAS

- The relationship for a circle with centre at (0, 0) and radius r is written as $x^2 + y^2 = r^2$.
- Translations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the graph of $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) are the coordinates of the centre of the circle and r is the radius.



$$(x - h)^2 + (y - k)^2 = r^2$$

horizontal translation of h units
For $h > 0$, move right.
For $h < 0$, move left.

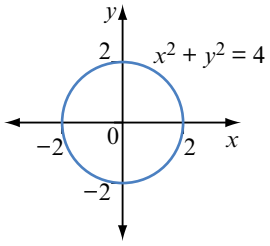
vertical translation of k units
For $k > 0$, move up.
For $k < 0$, move down.

EXERCISE 5E Graphs of circles

UNDERSTANDING AND FLUENCY

1 Look at the three circles shown.

i



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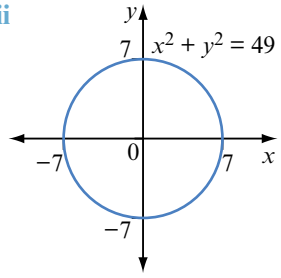

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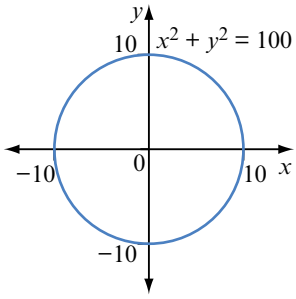


a Write the coordinates of the centre of each circle.

b Identify the radius of each circle.

c Compare the features of each circle with its rule. Can you see any patterns? Explain.

d How can you work out the radius from the rule if it is of the form $x^2 + y^2 = r^2$?

2 For each rule, identify the coordinates of the centre of the circle and the radius.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 1$ **c** $x^2 + y^2 = 64$

3 Consider the circle with the rule $x^2 + y^2 = 16$.

a What are the coordinates of the centre of this circle?

b What is the radius?

c To sketch this circle easily, you need to mark five points on the Cartesian plane. What do you think these five points will be? Discuss this with a classmate.

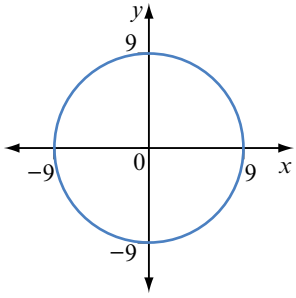
d On a Cartesian plane, mark a point for the centre of this circle. Use the radius to mark a point directly above, below, left and right of the centre. List the coordinates of the four points that sit on the circumference of the circle.

e Draw a circle through these four points. You may like to use a pair of compasses to help you. On the scale of the axes, indicate the highest and lowest values of x and y for the circle.

4 Sketch the graph of each rule in question **2**.

5 Sketch a circle on the Cartesian plane with centre at (0, 0) and radius of 6 units. Write the rule for this circle.

6 Write the rule for the circle shown at right.



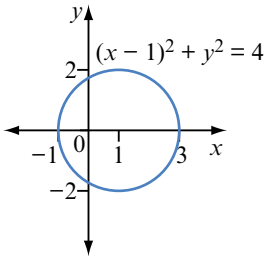
7 Write the rule for a circle with centre at (0, 0) and radius of:

a 12 units **b** 15 units

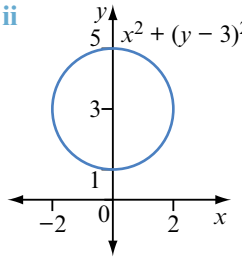
c 20 units **d** 2.5 units

8 In a similar way to when you transform parabolas, you can perform translations left or right and up or down on the basic graph of a circle with centre at (0, 0). Look at each circle.

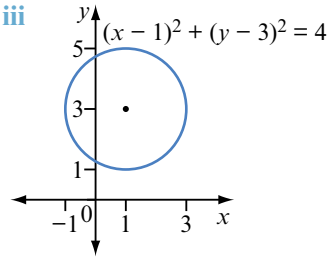
i



ii



iii



a Write the coordinates of the centre of each circle.

b Describe the translation/s that have been performed on the basic graph of $x^2 + y^2 = 4$ to produce each circle.

c Identify the radius of each circle.

d Compare the features of each circle with its rule. Can you see any patterns? Explain.

9 Consider the circle with rule $(x - 3)^2 + (y - 7)^2 = 36$. Compare this to the general rule for a circle $(x - h)^2 + (y - k)^2 = r^2$.

a Identify h, k and r.

b Use your answers to part **a** to identify the coordinates of the centre of the circle

c What is the radius of the circle?

10 Repeat question **9** for each of these rules.

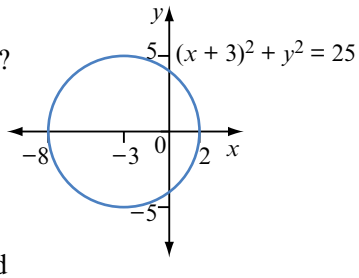
i $(x - 5)^2 + (y - 2)^2 = 9$

ii $(x - 4)^2 + (y + 3)^2 = 1$

iii $(x + 6)^2 + (y - 1)^2 = 25$

iv $(x + 2)^2 + (y + 8)^2 = 100$

11 Consider this circle drawn on the Cartesian plane.



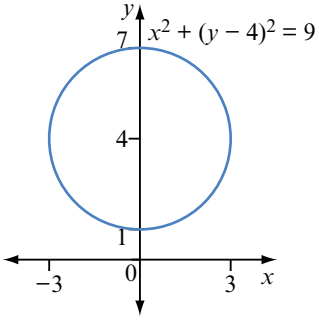
a What are the coordinates of the centre of the circle?

b What is the radius of the circle?

c Explain how the coordinates of the centre of the circle and the radius can be read from the rule if it is written as $(x + 3)^2 + (y - 0)^2 = 5^2$.

d Describe the translation/s that have been performed on the basic graph of $x^2 + y^2 = 25$ to produce the circle.

12 Consider this circle drawn on the Cartesian plane.



a What are the coordinates of the centre of the circle?

b What is the radius of the circle?

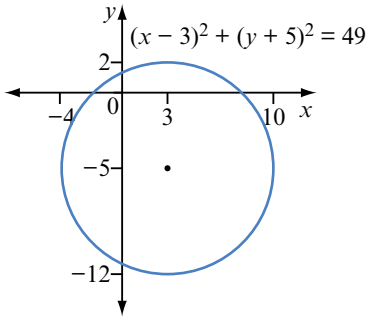
c Explain how the coordinates of the centre of the circle and the radius can be read from the rule if it is written as $(x - 0)^2 + (y - 4)^2 = 3^2$.

d Describe the translation/s that have been performed on the basic graph of $x^2 + y^2 = 9$ to produce the circle.

UNDERSTANDING AND FLUENCY

13 Consider each circle drawn on the Cartesian plane.

i



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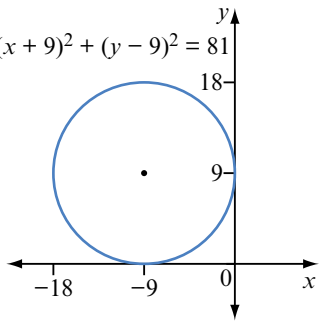


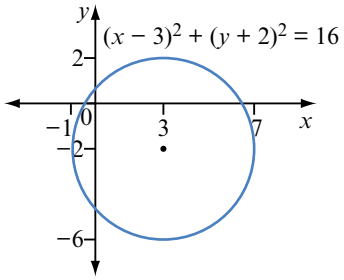
a What are the coordinates of the centre of each circle?

b What is the radius of each circle?

c Describe the translation/s that have been performed on the basic graph of $x^2 + y^2 = r^2$ to produce each circle.

EXAMPLE 5E-1 Sketching a circle by performing translations

Sketch the graph of $(x - 3)^2 + (y + 2)^2 = 16$.

THINK	WRITE
1 Identify any translations of the graph of $x^2 + y^2 = 16$. (3 units right and 2 units down) Alternatively, compare to $(x - h)^2 + (y - k)^2 = r^2$ to identify the translations. $h = 3$ (move 3 units right) and $k = -2$ (move 2 units down).	$(x - 3)^2 + (y + 2)^2 = 16$ The graph of $x^2 + y^2 = 16$ is translated 3 units right and 2 units down.
2 Identify the coordinates of the centre. Perform translations on (0, 0) or use (h, k).	centre at $(3, -2)$
3 Identify the radius. $r^2 = 16$ so $r = 4$.	radius of 4 units
4 Sketch the graph by first marking the centre of the circle at $(3, -2)$ and identifying four points that are 4 units above, below, left and right of the centre. These four points are: $(3, 2)$, $(3, -6)$, $(-1, -2)$ and $(7, -2)$.	

14 Sketch the graph of each rule.

a $(x - 3)^2 + (y - 2)^2 = 9$

b $(x - 5)^2 + (y - 7)^2 = 25$

c $(x + 1)^2 + (y - 3)^2 = 64$

d $(x - 6)^2 + (y + 6)^2 = 36$

e $(x - 2)^2 + y^2 = 16$

f $x^2 + (y + 5)^2 = 4$

g $(x + 3)^2 + (y + 5)^2 = 49$

h $(x + 4)^2 + (y + 3)^2 = 81$

15 Produce each graph in question **14** using digital technology. Compare your answers.

UNDERSTANDING AND FLUENCY

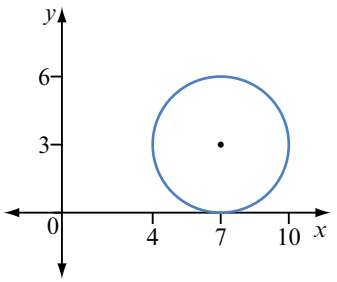
16 Write the rule for each circle using the information provided.

a circle with radius of 1 unit and centre at (4, −2)

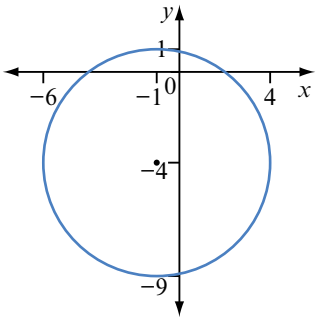
b circle with radius of 7 units and centre at (−5, −6)

17 Write the rule for each circle.

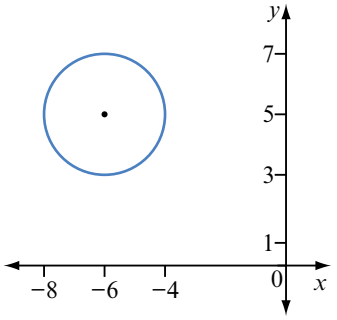
a



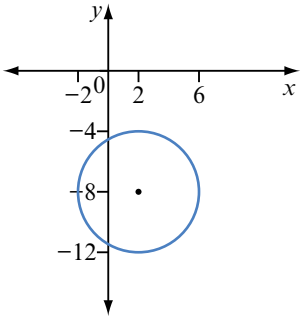
b



c



d



PROBLEM SOLVING AND REASONING

18 Consider $x^2 + y^2 = 25$.

a Draw a graph of this rule.

b Rearrange the rule to make y the subject of the equation.

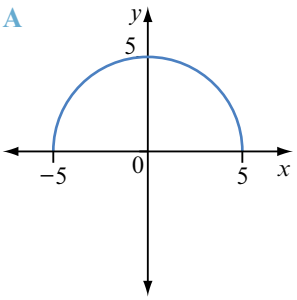
c Match each of these rules to one of the graphs shown below. Explain your reasoning.

i $y = -\sqrt{25 - x^2}$

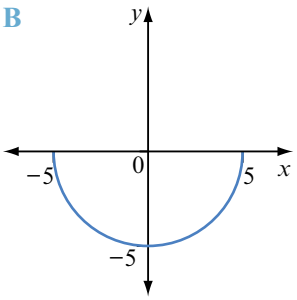
ii $y = \sqrt{25 - x^2}$

d Explain how the two half circles **A** and **B** are related to the graph of $x^2 + y^2 = 25$.

A

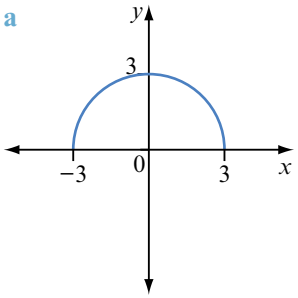


B

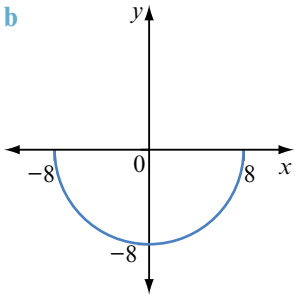


19 Write the rule for each half circle.

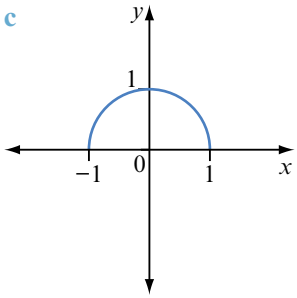
a



b



c



PROBLEM SOLVING AND REASONING

20 Sketch the graph of each half circle.

a $y = -\sqrt{49 - x^2}$ **b** $y = \sqrt{16 - x^2}$ **c** $y = -\sqrt{100 - x^2}$

21 **a** Rearrange $x^2 + y^2 = 25$ to make x the subject of the equation.

b Your answer to part **a** produces two rules. What would the graph of each rule look like?

22 An arch is modelled by the relationship $(x - 3)^2 + y^2 = 9$, where x is the horizontal distance from where the left side of the arch meets the ground and y is the corresponding height at that distance. Both x and y are measured in metres, and $x \geq 0$ and $y \geq 0$.

a Sketch the graph of this relationship.

b What is the horizontal distance along the ground from one side of the arch to the other?

c What is the maximum height of the arch?

d Describe the position on the ground below a point where the arch is at half its maximum height.

23 The outline of a pool of water is mapped on to a Cartesian plane representing the concrete floor of a warehouse using the relationship $(x - 6.2)^2 + (y - 4.8)^2 = 2.25$. All measurements are in metres. The floor of the warehouse is 15 m long (positive x-axis) and 10 m wide (positive y-axis).

a Sketch the graph of this relationship.

b What is the minimum distance from the water to a wall?

c Calculate the area of the pool of water to the nearest square metre.

d What percentage of the warehouse floor is covered in water?

CHALLENGE

24 Use a graphical method to find the points of intersection of the graphs with rules $x^2 + y^2 = 25$ and $y = x + 7$.

25 Use an algebraic method to find the points of intersection of the graphs with rules:

a $x^2 + y^2 = 9$ and $y = x - 3$

b $(x + 3)^2 + (y - 8)^2 = 4$ and $y = x + 9$.

26 Find the rule for a circle that passes through four points at $(-10, 5)$, $(-4, 11)$, $(2, 5)$ and $(-4, -1)$.

27 Can reflection be performed on the graph of $x^2 + y^2 = r^2$? Explain.

Reflect

Which feature of a circle is easiest to use when performing a translation?

5F Graphs of exponential relationships

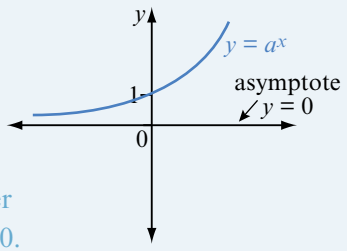
Start thinking!

An **exponential relationship** is one where x is the exponent or power of a term in index form.
Consider the rule $y = 2^x$.

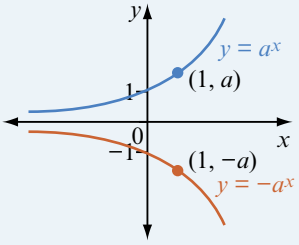
x	−3	−2	−1	0	1	2	3
y	$\frac{1}{8}$						8

1 Copy and complete this table of values for $y = 2^x$.
2 Plot the points from the table of values on a Cartesian plane. Join the points to form a curve.
3 What do you think happens to the y values as:
   a x gets bigger? **b** x gets smaller?
4 What is the y-intercept?
5 Is there an x-intercept? Explain.
6 One feature of the graph of an exponential relationship is that the curve approaches a boundary or line called an **asymptote**. What is the rule for the asymptote on the graph you have drawn? (Hint: it is the linear rule for one of the axes.)

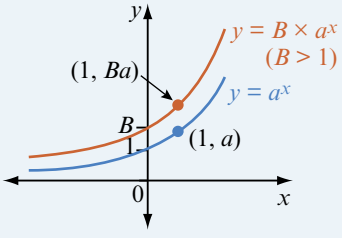
KEY IDEAS



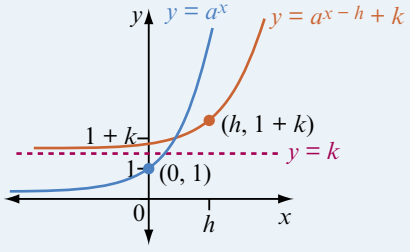
- The basic form of an exponential relationship is written as $y = a^x$, where $a > 1$. The graph has a y-intercept of 1 and a horizontal asymptote along the x-axis.
- An asymptote is a boundary or line that a curve approaches but never reaches. The asymptote for $y = a^x$ is the horizontal line with rule $y = 0$.
- A reflection in the x-axis can be performed on the graph of $y = a^x$ to produce the graph of $y = -a^x$.



- A dilation can be performed on the graph of $y = a^x$ to produce the graph of $y = B \times a^x$, where $B > 0$.



- Translations of h units right and k units up can be performed on the graph of $y = a^x$ to produce the graph of $y = a^{x-h} + k$. The horizontal asymptote at $y = 0$ is translated to $y = k$. The point $(0, 1)$ at the y-intercept is translated to $(h, 1 + k)$.



EXERCISE 5F Graphs of exponential relationships

UNDERSTANDING AND FLUENCY

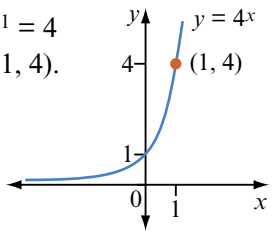
1 **a** Draw graphs of $y = 2^x$ and $y = 3^x$ for x values from -3 to 3 on the same Cartesian plane. You can plot the graphs using a table of values or use digital technology.
b Compare the two curves. How are they different?
c Which features are the same?
d Write the y-intercept for each graph.
e Write the rule for the asymptote for each graph.
f Predict how the graph of $y = 5^x$ would compare to the two curves you have drawn.

2 **a** Draw graphs of $y = 2^x$ and $y = -2^x$ for x values from -3 to 3 on the same Cartesian plane.
b Compare the two curves. How are they different?
c Which features are the same?
d Write the y-intercept for each graph.
e Write the rule for the asymptote for each graph.
f Which transformation can be performed on the graph of $y = 2^x$ to produce $y = -2^x$?

3 **a** Draw graphs of $y = 2^x$ and $y = 3 \times 2^x$ for x values from -3 to 3 on the same Cartesian plane.
b Compare the two curves. How are the graphs different?
c Which features are the same?
d Write the y-intercept for each graph.
e Write the rule for the asymptote for each graph.
f Which transformation can be performed on the graph of $y = 2^x$ to produce $y = 3 \times 2^x$?

EXAMPLE 5F-1 Sketching the graph of a simple exponential relationship

Sketch the graph of $y = 4^x$.

THINK	WRITE
1 Identify the asymptote.	Horizontal asymptote has rule $y = 0$.
2 Find the y-intercept.	When $x = 0$, $y = 4^0 = 1$ y-intercept is 1.
3 To give a sense of scale to your sketch graph, label another point on the curve with its coordinates. Find the value of y when $x = 1$.	When $x = 1$, $y = 4^1 = 4$ Point on curve is (1, 4).
4 Sketch the basic curve for an exponential relationship through the y-intercept and point, using the horizontal asymptote as a boundary.	

4 Sketch the graph of each exponential relationship.

a $y = 5^x$ **b** $y = 10^x$ **c** $y = (1.5)^x$

5 **a** Sketch the graph of $y = 3^x$. Show all key features including the coordinates of a point other than the y-intercept on the curve.

b Describe the transformation needed to produce the graph of $y = -3^x$ from the graph of $y = 3^x$.

c On the same Cartesian plane, sketch the graph of $y = -3^x$ by performing this transformation.

6 Sketch the graph of each exponential relationship by performing a reflection.

a $y = -4^x$ **b** $y = -9^x$ **c** $y = -(2.5)^x$

7 **a** Sketch the graph of $y = 3^x$.

b Describe the transformation needed to produce the graph of $y = 2 \times 3^x$ from the graph of $y = 3^x$.

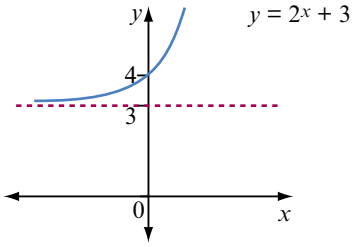
c On the same Cartesian plane, sketch the graph of $y = 2 \times 3^x$ by performing this transformation.

8 Sketch the graph of each exponential relationship by performing a dilation.

a $y = 2 \times 4^x$ **b** $y = 5 \times 2^x$ **c** $y = 0.5 \times 6^x$

9 Similar to working with parabolas and circles, you can perform a vertical translation on the basic graph of an exponential relationship. Look at each graph.

i



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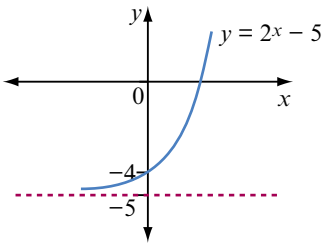


a Write the rule for the asymptote of each exponential relationship.

b Identify the y-intercept for each graph.

c Describe the translation that has been performed on the basic graph of $y = 2^x$ to produce each graph.

d Each graph above has a rule in the form $y = 2^x + k$. Copy and complete each sentence using the word *up* or *down*.

When k is positive, the graph of $y = 2^x$ is moved _____.

When k is negative, the graph of $y = 2^x$ is moved _____.

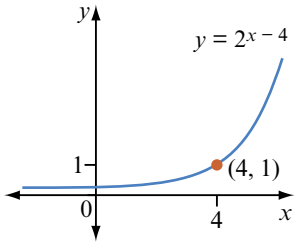
e Write the rule for the asymptote and coordinates of the y-intercept for the graph of $y = 2^x + k$.

10 Sketch the graph of each exponential relationship by performing a vertical translation.

a $y = 3^x + 2$ **b** $y = 3^x - 4$ **c** $y = 5^x + 1$

11 You can perform a horizontal translation on the basic graph of an exponential relationship. Look at each graph.

i



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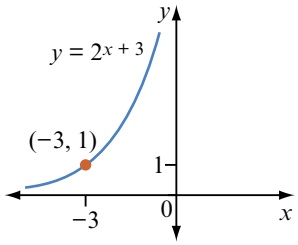


a Write the rule for the asymptote of each exponential relationship.

b Identify the y-intercept for each graph.

c The point at the y-intercept of $y = 2^x$ has been translated. Write the coordinates of this point.

d Describe the translation that has been performed on the basic graph of $y = 2^x$ to produce each graph.

e Each graph shown above has a rule in the form $y = 2^{x-h}$. Copy and complete each sentence using the word *left* or *right*.

When h is positive, the graph of $y = 2^x$ is moved _____.

When h is negative, the graph of $y = 2^x$ is moved _____.

f Write the rule for the asymptote and coordinates of the y-intercept for the graph of $y = 2^{x-h}$.

12 Sketch the graph of each exponential relationship by performing a horizontal translation.

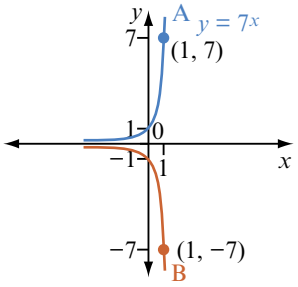
a $y = 3^{x-2}$ **b** $y = 3^{x+4}$ **c** $y = 5^{x-1}$

13 For each pair of graphs shown on a Cartesian plane below:

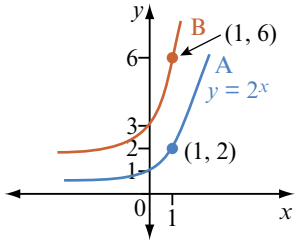
i identify the transformation performed on graph **A** to produce graph **B**

ii write the rule for graph **B**.

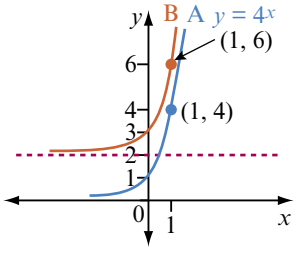
a



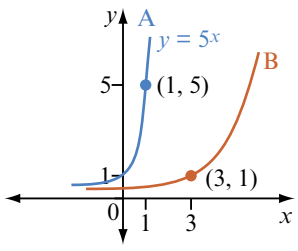
b



c



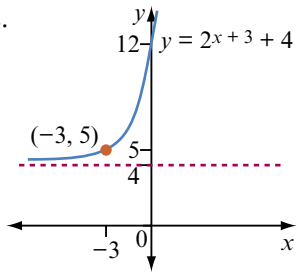
d



EXAMPLE 5F-2

Sketching the graph of an exponential relationship by performing translations

Sketch the graph of $y = 2^{x+3} + 4$. Label the y-intercept and asymptote.

THINK	WRITE
1 Identify the translations to be performed on $y = 2^x$. (3 units left and 4 units up) Alternatively, compare to $y = 2^{x-h} + k$, to identify the translations. $h = -3$ (move 3 units left) and $k = 4$ (move 4 units up).	$y = 2^{x+3} + 4$ The graph of $y = 2^x$ is translated 3 units left and 4 units up.
2 Identify the rule for the asymptote. Translate $y = 0$ or use $y = k$.	Horizontal asymptote has rule $y = 4$.
3 Find the y-intercept.	When $x = 0$, $y = 2^{0+3} + 4$ $= 8 + 4$ $= 12$ y-intercept is 12.
4 Identify another point on the curve by translating the coordinates of the y-intercept for $y = 2^x$. (0, 1) translates to $(h, 1 + k)$.	(0, 1) translates to (−3, 5)
5 Sketch the basic curve for an exponential relationship through the y-intercept and point, using the horizontal asymptote as a boundary.	

UNDERSTANDING AND FLUENCY

14 Sketch the graph of each rule.

a $y = 2^{x-1} + 3$ **b** $y = 3^{x+2} - 5$ **c** $y = 4^{x-3} + 7$

d $y = 3^{x-2} - 4$ **e** $y = 2^{x+4} + 1$ **f** $y = 5^{x+1} + 2$

15 Produce each graph in question **14** using digital technology. Compare your answers.

16 Draw the graphs of $y = 2^x$ and $y = 2^{-x}$ for x values from −3 to 3 on the same Cartesian plane and compare the two curves. You can plot the graphs using a table of values or use digital technology.

a How are they different?

b Which features are the same?

c Write the y-intercept for each graph.

d Write the rule for the asymptote for each graph.

e Describe the transformation that has been performed on the basic graph of $y = 2^x$ to produce $y = 2^{-x}$.

f What transformations are needed to produce the graph of $y = -2^{-x}$ from the graph of $y = 2^x$?

17 What is the rule obtained when the graph of $y = a^x$ is reflected in:

a the x-axis? **b** the y-axis? **c** the x-axis and the y-axis?

18 For each rule:

i describe the transformation/s needed to obtain its graph

ii sketch the graph, showing all key features including the coordinates of a point other than the y-intercept on the curve.

a $y = 3^{-x}$ **b** $y = -3^x$ **c** $y = -3^{-x}$ **d** $y = 4^{-x}$ **e** $y = -5^x$ **f** $y = -6^{-x}$

PROBLEM SOLVING AND REASONING

19 Sienna invests \$2000 at 10% p.a. interest compounded annually. The amount (in dollars) of the investment, A, after n years can be represented by $A = 2000 \times (1.1)^n$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b Describe what happens to the value of A as n increases?

c Use a table of values for $0 \leq n \leq 10$ or digital technology to produce the graph. Use the graph to find:

i the amount of the investment after 3 years

ii the number of years it takes for the investment amount to be more than \$4000.

20 The value of Gabriel's work tools depreciates at the rate of 20% p.a. compounded annually. The asset value in dollars, V, after n years can be represented by $V = 3500 \times (1.25)^{-n}$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b What was the initial value of Gabriel's tools?

c Describe what happens to the value of V as n increases?

d Use a table of values for $0 \leq n \leq 10$ or digital technology to produce the graph. Use the graph to find the number of years it takes for the asset value to be less than \$1000.

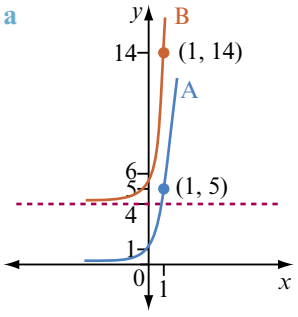
CHALLENGE

21 For each pair of graphs shown on a Cartesian plane below:

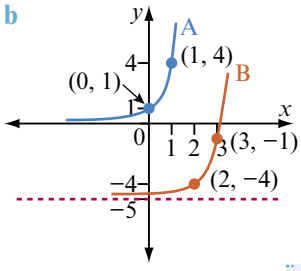
i identify the transformations performed on graph **A** to produce graph **B**

ii write the rules for graphs **A** and **B**.

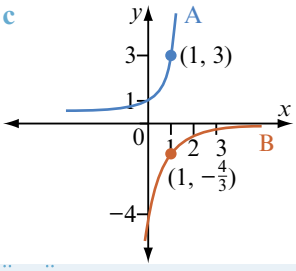
a



b



c



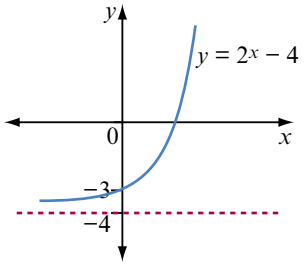
22 Describe the transformations performed on the graph of $y = 2^x$ to produce the graph of $y = -3 \times 2^{x+1} - 2$ and hence sketch its graph.

Reflect

Which two features are most useful when sketching an exponential relationship?

5G Solving exponential equations

Start thinking!



1 The sketch graph of $y = 2^x - 4$ is shown at right.

- **a** What is the rule for the horizontal asymptote?
- **b** Find the y-intercept.
- **c** Explain how you can find the x-intercept. What is the y value at this point?
- **d** Find the x-intercept by solving the **exponential equation** $2^x = 4$.

2 Consider the graph with rule $y = 2^x + 4$.

- **a** What is the rule for the horizontal asymptote?
- **b** Find the y-intercept.
- **c** Write the exponential equation you need to solve to find the x-intercept.
- **d** Can you solve this equation? Does this graph have an x-intercept? Explain.

3 Repeat question **2** for the graph with rule $y = 2^x - 5$.

KEY IDEAS

- For terms in index form, the index is also known as the power or the exponent or the logarithm.
- To solve exponential equations of the form $a^x = y$ for x, use one of these three methods.
  1. **Equate** the exponents if y can be written in index form as a term with the base a. That is, if $a^x = a^m$ then $x = m$. For example, $2^x = 8$ can be written as $2^x = 2^3$, so $x = 3$.

     In some cases, you need to rewrite both sides of the equation so you have the same base. For example, $8^x = 4$ can be written as $2^{3x} = 2^2$, so $x = \frac{2}{3}$.
  2. Write $a^x = y$ in **logarithmic form** as $x = \log_a y$ and use a calculator to find the value for $\log_a y$. For example, $2^x = 7$ can be written as $x = \log_2 7$ so $x \approx 2.8$. If your calculator does not have this function then use the third method.
  3. Take $\log_{10}$ of both sides of the equation. That is, $\log_{10} a^x = \log_{10} y$ which becomes $x\log_{10} a = \log_{10} y$ or $x = \dfrac{\log_{10} a}{\log_{10} y}$. Use the Log function on your calculator to find the value of $\dfrac{\log_{10} a}{\log_{10} y}$. For example, for $2^x = 7$, $x = \dfrac{\log_{10} 7}{\log_{10} 2} = (\log_{10} 7) \div (\log_{10} 2) \approx 2.8$.

  See Exercises 3G and 3H for more on logarithms.
- Solving an exponential equation is useful when finding the x-intercept of an exponential relationship.
- Exponential relationships can be used to model many real situations involving growth and decay; for example, for populations or investments.

EXERCISE 5G Solving exponential equations

1 Solve each equation to find x by equating the exponents (powers).

a $2^x = 2^9$ **b** $3^x = 3^6$ **c** $4^x = 4^{-3}$ **d** $8^x = 8$

e $5^{x-2} = 5^6$ **f** $2^{x+4} = 2^8$ **g** $10^{2-x} = 10^{-5}$ **h** $2^{-x-7} = 2^3$

i $9^{2x} = 9^6$ **j** $6^{3x} = 6^{15}$ **k** $7^{2x-1} = 7^{11}$ **l** $3^{4x+5} = 3^5$

EXAMPLE 5G-1 Solving exponential equations by writing with the same base

Solve each equation.

a $2^x = 32$ **b** $5^{x-4} = 125$ **c** $9^x = 27$

THINK	WRITE
a 1 Write the equation so that both sides have the same base of 2.	**a** $2^x = 32$ $2^x = 2^5$
2 Solve the equation by equating the exponents (powers). The bases are the same so the exponents are equal.	$x = 5$
b 1 Write the equation so that both sides have the same base of 5.	**b** $5^{x-4} = 125$ $5^{x-4} = 5^3$
2 Equate the exponents (powers) as the bases are the same.	$x - 4 = 3$
3 Solve the linear equation for x.	$x = 7$
c 1 Write the equation so that both sides have the same base of 3.	**c** $9^x = 27$ $(3^2)^x = 3^3$ $3^{2x} = 3^3$
2 Equate the exponents (powers) as the bases are the same.	$2x = 3$
3 Solve the linear equation for x.	$x = \frac{3}{2}$

UNDERSTANDING AND FLUENCY

2 Solve each equation.

a $2^x = 8$ **b** $3^x = 9$ **c** $4^x = 1024$ **d** $10^x = 10\,000$

e $6^{x-5} = 36$ **f** $2^{x+3} = 64$ **g** $9^{2x-1} = 729$ **h** $3^{-x+4} = 243$

i $8^x = 128$ **j** $25^x = 125$ **k** $81^x = 27$ **l** $49^{2x} = 16\,807$

3 The solution to each of these exponential equations is not a whole number. Between which two whole numbers is the value of x in each case? For example, $2^x = 10$ has a solution that is between 3 and 4, since $2^3 = 8$ and $2^4 = 16$.

a $2^x = 5$ **b** $3^x = 290$ **c** $5^x = 17$ **d** $10^x = 6850$

e $3^{2x} = 14$ **f** $7^{2x} = 3000$ **g** $4^{x+1} = 38$ **h** $6^{x-2} = 210$

UNDERSTANDING AND FLUENCY

4 Consider $2^x = 11$.
- **a** Which part of the equation is the logarithm? (Hint: is it 2, x or 11?)
- **b** Write the equation in logarithmic form.

5 Consider $5^{x-3} = 9$.
- **a** Which part of the equation is the logarithm?
- **b** Write the equation in logarithmic form.

6 Write each equation in logarithmic form. (Hint: $2^x = 16$ becomes $x = \log_2 16$.)

a $3^x = 8$	**b** $7^x = 4$	**c** $2^x = 37$	**d** $4^x = 3$
e $6^{x-2} = 20$	**f** $5^{x+8} = 17$	**g** $8^{5-x} = 9$	**h** $2^{-x-1} = 50$
i $7^{3x} = 11$	**j** $9^{5x} = 22$	**k** $3^{2x-7} = 19$	**l** $5^{6x+3} = 41$

EXAMPLE 5G-2 Solving exponential equations using log of the given base

Solve each equation. Write the value of x correct to two decimal places.

a $2^x = 5$ **b** $4 \times 3^x = 2$ **c** $7^{2x-5} = 68$

THINK	WRITE
a 1 Write the equation in logarithmic form.	**a** $2^x = 5$ $x = \log_2 5$
2 Use a calculator to obtain the value of x correct to two decimal places.	$x = 2.321\,928\,08\ldots$ $x \approx 2.32$
b 1 Simplify by dividing both sides of the equation by 4.	**b** $4 \times 3^x = 2$ $3^x = 0.5$
2 Write the equation in logarithmic form.	$x = \log_3(0.5)$
3 Use a calculator to obtain the value of x correct to two decimal places.	$x = -0.630\,929\,8\ldots$ $x \approx -0.63$
c 1 Write the equation in logarithmic form.	**c** $7^{2x-5} = 68$ $2x - 5 = \log_7 68$
2 Solve for x by adding 5 to both sides and then dividing both sides by 2.	$2x = \log_7 68 + 5$ $x = \frac{\log_7 68 + 5}{2}$
3 Use a calculator to obtain the value of x correct to two decimal places.	$x = 3.584\,199\,00\ldots$ $x \approx 3.58$

7 Solve each equation. Write the value of x correct to two decimal places.

a $2^x = 11$	**b** $5^x = 92$	**c** $6 \times 3^x = 3$	**d** $2 \times 7^x = 53$
e $8^{x-1} = 35$	**f** $3^{x+6} = 29$	**g** $2^{3-x} = 48$	**h** $9^{-x-2} = 60$
i $5^{2x} = 17$	**j** $10^{3x} = 75$	**k** $4^{2x-3} = 24$	**l** $3^{4x+1} = 87$

EXAMPLE 5G-3 Solving a simple exponential equation using log of base 10

Solve $2^x = 5$. Write the value of x correct to two decimal places.

THINK

1 Take $\log_{10}$ of both sides of the equation.

2 Use the 'log of a power' law: $\log_a x^n = n \times \log_a x$.

3 Divide both sides by $\log_{10} 2$ to make x the subject.

4 Use the Log function on a calculator to obtain the value of x correct to two decimal places.

WRITE

$2^x = 5$

$\log_{10} 2^x = \log_{10} 5$

$x \times \log_{10} 2 = \log_{10} 5$

$x = \dfrac{\log_{10} 5}{\log_{10} 2}$

$x = 2.321\ 928\ 08\ldots$

$x \approx 2.32$

8 Solve each equation. Write the value of x correct to two decimal places.

a $3^x = 7$ **b** $2^x = 9$ **c** $5^x = 26$ **d** $4^x = 98$

e $2^x = 35$ **f** $7^x = 2$ **g** $6^x = 2000$ **h** $9^x = 1.5$

EXAMPLE 5G-4 Solving more complex exponential equations using log of base 10

Solve each equation. Write the value of x correct to two decimal places.

a $4 \times 3^x = 2$ **b** $7^{2x-5} = 68$

THINK

a **1** Simplify by dividing both sides of the equation by 4.

2 Take $\log_{10}$ of both sides of the equation.

3 Use the 'log of a power' law.

4 Divide both sides by $\log_{10} 3$ to make x the subject.

5 Use a calculator to obtain the value of x correct to two decimal places.

b **1** Take $\log_{10}$ of both sides of the equation.

2 Use the 'log of a power' law.

3 Divide both sides by $\log_{10} 7$.

4 Calculate the value of the right side.

5 Solve for x by adding 5 to both sides and then dividing both sides by 2. Write the value of x correct to two decimal places.

WRITE

a $4 \times 3^x = 2$

$3^x = 0.5$

$\log_{10} 3^x = \log_{10}(0.5)$

$x \times \log_{10} 3 = \log_{10}(0.5)$

$x = \dfrac{\log_{10}(0.5)}{\log_{10} 3}$

$x = -0.630\ 929\ 8\ldots$

$x \approx -0.63$

b $7^{2x-5} = 68$

$\log_{10} 7^{2x-5} = \log_{10} 68$

$(2x - 5) \times \log_{10} 7 = \log_{10} 68$

$2x - 5 = \dfrac{\log_{10} 68}{\log_{10} 7}$

$2x - 5 = 2.168\ 398\ 01\ldots$

$2x = 7.168\ 398\ 01\ldots$

$x = 3.584\ 199\ 00\ldots$

$x \approx 3.58$

9 Solve each equation. Write the value of x correct to two decimal places.

a $5 \times 2^x = 15$ **b** $4 \times 7^x = 10$ **c** $9 \times 3^x = 5$ **d** $3 \times 4^x = 2.8$

e $2^{x-3} = 19$ **f** $8^{x+6} = 58$ **g** $(0.5)^{x-1} = 22$ **h** $11^{-x-4} = 31.6$

i $3^{2x-5} = 14$ **j** $(0.9)^{4x+3} = 25$ **k** $2 \times 4^{3x-8} = 3$ **l** $7 \times 6^{2x+4} = 11$

EXAMPLE 5G-5

Finding the *x*-intercept for the graph of an exponential relationship

Find the x-intercept for the graph of $y = 2^{x-3} - 9$, correct to two decimal places.

THINK

1. To find the x-intercept, substitute $y = 0$ into the equation.
2. Add 9 to both sides of the equation and swap sides.
3. Take $\log_{10}$ of both sides of the equation.
4. Use the 'log of a power' law.
5. Divide both sides by $\log_{10}2$.
6. Calculate the value of the right side.
7. Solve for x by adding 3 to both sides.
8. Write the x-intercept correct to two decimal places.

WRITE

$y = 2^{x-3} - 9$

When $y = 0$, $0 = 2^{x-3} - 9$

$2^{x-3} = 9$

$\log_{10}2^{x-3} = \log_{10}9$

$(x-3) \times \log_{10}2 = \log_{10}9$

$x - 3 = \dfrac{\log_{10}9}{\log_{10}2}$

$x - 3 = 3.169\,925\ldots$

$x = 6.169\,925\ldots$

x-intercept is approx. 6.17.

10 Find the x-intercept for each exponential relationship correct to two decimal places.

a $y = 2^x - 11$ **b** $y = -5^x + 4$ **c** $y = 3^{x-2} - 7$

d $y = -4^{x+2} + 3$ **e** $y = 3 \times 2^x - 5$ **f** $y = -4 \times 3^{x-5} + 8$

11 Sketch the graph of each exponential relationship in question **10** showing the key features. Clearly label the x- and y-intercepts (correct to two decimal places) and the asymptote.

PROBLEM SOLVING AND REASONING

12 A population of mice can be modelled by the relationship $n = 9^t + 1$, where n is the number of mice after t months.

a What was the initial number of mice?

b How many mice are there after 2 months?

c How many months, correct to two decimal places, does it take for there to be 1000 mice?

d Is the model showing population growth or decay? Sketch the graph of the relationship to demonstrate your answer.

PROBLEM SOLVING AND REASONING

13 After the water boils in an electric kettle, the temperature of the water T (in °C) after m minutes follows the relationship $T = 20 + 80 \times (0.8)^m$.

- **a** What was the initial temperature of the water?
- **b** What is the temperature of the water, to the nearest degree, after 3 minutes?
- **c** How many minutes, correct to one decimal place, does it take for the temperature to be half the initial temperature?
- **d** What is the lowest temperature that this kettle of water can reach?
- **e** Is the relationship showing growth or decay? Sketch the graph of the relationship to demonstrate your answer.

14 The human population can be modelled using different exponential relationships. One model uses the relationship $P = 2.6 \times (1.017)^t$, where P is the population in billions and t is the number of years after 1950.

- **a** Use this model to predict the population in:
  - **i** 2020
  - **ii** 2040.
- **b** In what year does this model predict that the population will reach 15 billion?
- **c** The population in 2010 was estimated to be 6.9 billion. How does this compare to the population predicted by this model? Give reasons why you think they may be different.

15 Repeat question **14** using the model $P = 6.07 \times (1.012)^t$, where P is the population in billions and t is the number of years after 2000. Is this a better model? Explain.

CHALLENGE

16 Investigate other population models. Check their accuracy by comparing the predictions with actual population estimates from the last few years.

17 A water lily population in a pond doubles in size each day. If left unchecked, it will smother the pond (and kill other living things in the water) in 30 days. How long have you got to save the pond if you wait until the pond is half covered?

Reflect

Why do exponential relationships provide a good model for population growth?

5H Graphs of hyperbolas

Start thinking!

The graph of a relationship involving the reciprocal of x is called a **hyperbola**.

Consider the rule $y = \frac{1}{x}$.

x	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$						-4							

1 Copy and complete this table of values for $y = \frac{1}{x}$.

2 Plot the points from the table of values on a Cartesian plane. Join the points for $x = -4$ to $-\frac{1}{4}$ as one curve and $x = \frac{1}{4}$ to 4 as a separate curve. This graph is called a **rectangular hyperbola**.

3 What do you think happens to the y values as:

a positive x values increase (towards ∞)?

b positive x values decrease (towards 0)?

c negative x values decrease (towards $-\infty$)?

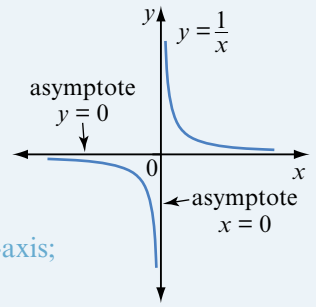
d negative x values increase (towards 0)?

4 Why isn't there a y-intercept?

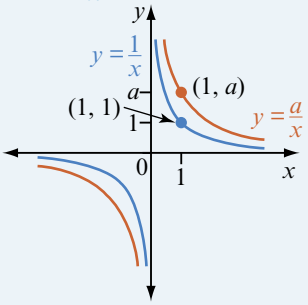
5 Is there an x-intercept? Explain.

6 A hyperbola has two asymptotes. What is the rule for each asymptote on the graph you have drawn?

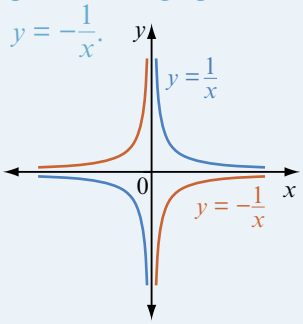
KEY IDEAS



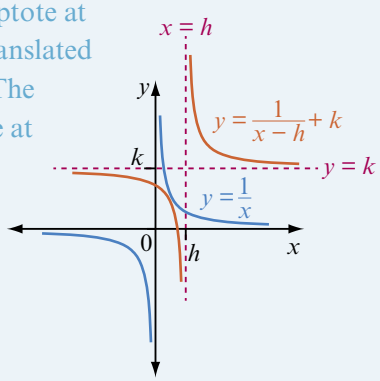
- The basic rectangular hyperbola is the graph of the rule $y = \frac{1}{x}$. The graph has two asymptotes and no x- or y-intercepts.
- An asymptote is a boundary or line that a curve approaches but never reaches. The asymptotes for $y = \frac{1}{x}$ lie along the x-axis and the y-axis; that is, have the rules $y = 0$ and $x = 0$.
- A dilation can be performed on the graph of $y = \frac{1}{x}$ to produce the graph of $y = \frac{a}{x}$, where $a > 0$.



- A reflection in the x-axis can be performed on the graph of $y = \frac{1}{x}$ to produce the graph of $y = -\frac{1}{x}$.



- Translations of h units right and k units up can be performed on the graph of $y = \frac{1}{x}$ to produce the graph of $y = \frac{1}{x - h} + k$. The asymptote at $x = 0$ is translated to $x = h$. The asymptote at $y = 0$ is translated to $y = k$.



EXERCISE 5H Graphs of hyperbolas

UNDERSTANDING AND FLUENCY

1 a Draw graphs of $y = \frac{1}{x}$ and $y = \frac{2}{x}$ for x values from -4 to 4 on the same Cartesian plane. You can plot the graphs using a table of values or using digital technology.
b Compare the two curves. How are they different?
c Which features are the same?
d Write the rules for the asymptotes for each graph.
e Which transformation can be performed on the graph of $y = \frac{1}{x}$ to produce $y = \frac{2}{x}$?
f Predict how the graph of $y = \frac{5}{x}$ would compare to the two curves you have drawn.

2 a Draw graphs of $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ for x values from -4 to 4 on the same Cartesian plane.
b Compare the two curves. How are they different?
c Which features are the same?
d Which transformation can be performed on the graph of $y = \frac{1}{x}$ to produce $y = -\frac{1}{x}$?

EXAMPLE 5H-1 Sketching the graph of a simple hyperbola

Sketch the graph of $y = \frac{3}{x}$ by performing a transformation.

THINK

1 Identify the type of graph and any transformations to the basic graph of $y = \frac{1}{x}$.

2 Identify the asymptotes and any intercepts.

3 To give a sense of scale to your sketch graph, label a point on the curve with its coordinates. Find the value of y when $x = 1$.

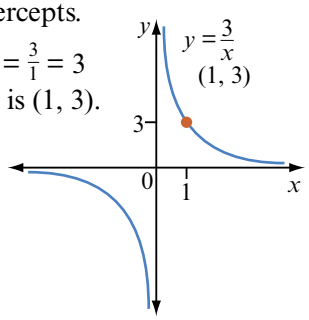
4 Sketch the basic curve for a rectangular hyperbola using the asymptotes as boundaries. Label a known point on the curve to indicate that dilation has occurred.

WRITE

The graph of $y = \frac{3}{x}$ is a rectangular hyperbola produced by performing a dilation by factor of 3 on $y = \frac{1}{x}$.

Horizontal asymptote has rule $y = 0$.
Vertical asymptote has rule $x = 0$.
No x- or y-intercepts.

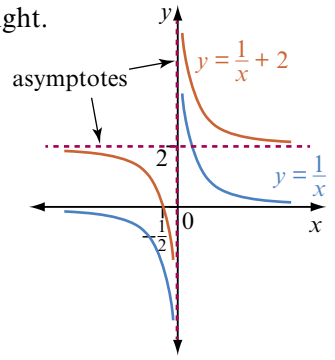
When $x = 1$, $y = \frac{3}{1} = 3$
Point on curve is (1, 3).



3 Sketch the graph of each relationship by performing one or more transformations on the graph of $y = \frac{1}{x}$.

a $y = \frac{4}{x}$ b $y = \frac{7}{x}$ c $y = -\frac{2}{x}$ d $y = -\frac{5}{x}$

4 The sketch graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x} + 2$ are shown at right.



a Write the rule of each asymptote (shown in red) for the graph of $y = \frac{1}{x} + 2$.

b Are there any intercepts? List them.

c Describe how you could use a translation to produce the graph of $y = \frac{1}{x} + 2$ from the graph of $y = \frac{1}{x}$. (Hint: consider the position of the asymptotes.)

d Describe the translation needed to produce the graph of $y = \frac{1}{x} - 3$ from the graph of $y = \frac{1}{x}$.

e Sketch the graph of $y = \frac{1}{x} - 3$.

f Each graph above has a rule in the form $y = \frac{1}{x} + k$. Copy and complete each sentence using the word *up* or *down*.

When k is positive, the graph of $y = \frac{1}{x}$ is moved _____.

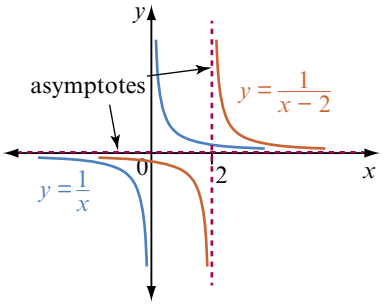
When k is negative, the graph of $y = \frac{1}{x}$ is moved _____.

g Write the rule for each asymptote and coordinates of the x-intercept for the graph of $y = \frac{1}{x} + k$.

5 Sketch the graph of each relationship by performing a vertical translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x} + 4$ **b** $y = \frac{1}{x} - 7$ **c** $y = \frac{1}{x} + 5$ **d** $y = \frac{1}{x} - 1$

6 The sketch graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x - 2}$ are shown at right.



a Write the rule of each asymptote for the graph of $y = \frac{1}{x - 2}$ (shown in red).

b Are there any intercepts? List them.

c Describe how you could use a translation to produce the graph of $y = \frac{1}{x - 2}$ from the graph of $y = \frac{1}{x}$. (Hint: consider the position of the asymptotes.)

d Describe the translation needed to produce the graph of $y = \frac{1}{x + 3}$ from the graph of $y = \frac{1}{x}$.

e Sketch the graph of $y = \frac{1}{x + 3}$.

f Each graph above has a rule in the form $y = \frac{1}{x - h}$. Copy and complete each sentence using the word *left* or *right*.

When h is positive, the graph of $y = \frac{1}{x}$ is moved _____.

When h is negative, the graph of $y = \frac{1}{x}$ is moved _____.

g Write the rule for the asymptotes and coordinates of the y-intercept for the graph of $y = \frac{1}{x-h}$.

7 Sketch the graph of each relationship by performing a horizontal translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-5}$ b $y = \frac{1}{x+4}$ c $y = \frac{1}{x-7}$ d $y = \frac{1}{x+1}$

8 Use your understanding of translations to describe how the graphs of these relationships can be produced from the graph of $y = \frac{1}{x}$.

a $y = \frac{1}{x} - 2$ b $y = \frac{1}{x-4}$ c $y = \frac{1}{x} + 9$ d $y = \frac{1}{x+6}$

e $y = \frac{1}{x-3} + 8$ f $y = \frac{1}{x+6} - 2$ g $y = \frac{1}{x-4} - 5$ h $y = \frac{1}{x+9} + 11$

EXAMPLE 5H-2 Sketching the graph of a hyperbola by performing translations

Sketch the graph of $y = \frac{1}{x+2} - 4$. Clearly show the asymptotes and intercepts.

THINK

1 Identify the translations to be performed on $y = \frac{1}{x}$. (2 units left and 4 units down)
Alternatively, compare to $y = \frac{1}{x-h} + k$, to identify the translations. $h = -2$ (move 2 units left) and $k = -4$ (move 4 units down).

2 Identify the rule for each asymptote.
Translate $x = 0$ or use $x = h$.
Translate $y = 0$ or use $y = k$.

3 Find the x-intercept.

4 Find the y-intercept.

5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts.

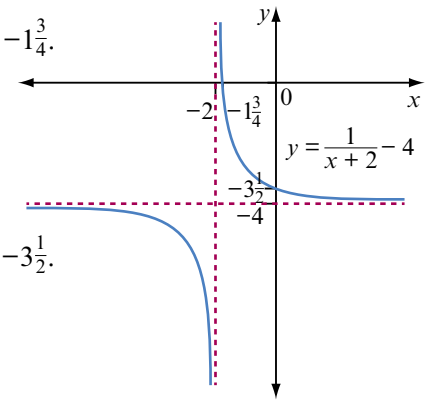
WRITE

$y = \frac{1}{x+2} - 4$
The graph of $y = \frac{1}{x}$ is translated 2 units left and 4 units down.

Vertical asymptote has rule $x = -2$.
Horizontal asymptote has rule $y = -4$.

When $y = 0$,
$0 = \frac{1}{x+2} - 4$
$\frac{1}{x+2} = 4$
$x + 2 = \frac{1}{4}$
$x = -1\frac{3}{4}$
x-intercept is $-1\frac{3}{4}$.

When $x = 0$,
$y = \frac{1}{0+2} - 4$
$= \frac{1}{2} - 4$
$= -3\frac{1}{2}$
y-intercept is $-3\frac{1}{2}$.



9 Sketch the graph of each rule. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-3} - 2$ **b** $y = \frac{1}{x+5} + 3$ **c** $y = \frac{1}{x-2} + 4$

d $y = \frac{1}{x+4} - 5$ **e** $y = \frac{1}{x-1} - 8$ **f** $y = \frac{1}{x+6} + 6$

10 Produce each graph in question **9** using digital technology. Compare your answers.

11 For each rule below:

i identify the transformation/s to be performed on the graph of $y = \frac{1}{x}$ to produce the graph of the rule

ii sketch the graph, clearly showing the asymptotes and intercepts

iii use a calculator or other digital technology to produce the graph.

a $y = -\frac{1}{x} + 3$ **b** $y = -\frac{1}{x-2}$ **c** $y = -\frac{1}{x-2} + 3$

12 Sketch the graph of each rule using transformations. Clearly show the asymptotes and intercepts.

a $y = -\frac{1}{x+4}$ **b** $y = -\frac{1}{x} - 2$ **c** $y = -\frac{1}{x-3} + 5$

d $y = -\frac{1}{x+5} - 1$ **e** $y = -\frac{1}{x-1} - 4$ **f** $y = -\frac{1}{x+2} + 2$

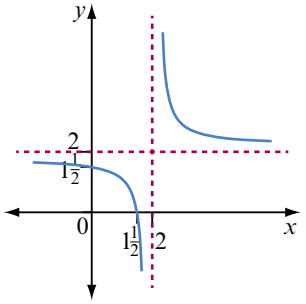
13 Check your graphs in question **12** by producing each hyperbola using digital technology.

14 For each graph shown below:

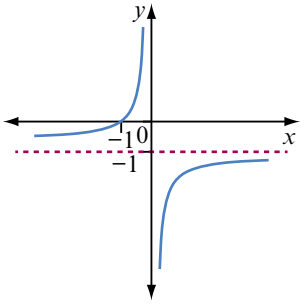
i identify the transformation/s performed on the graph of $y = \frac{1}{x}$ to produce the hyperbola

ii write the rule for the hyperbola.

a



b



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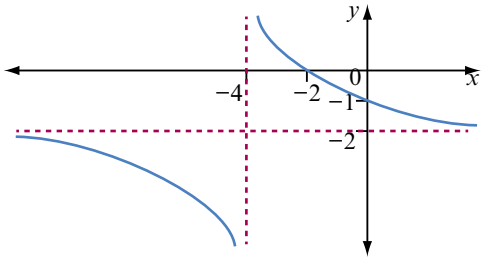
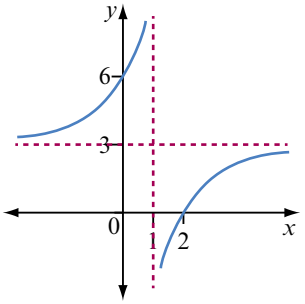


d



PROBLEM SOLVING AND REASONING

15 Why does reflecting the graph of $y = \frac{1}{x}$ in the x-axis produce the same graph as reflecting $y = \frac{1}{x}$ in the y-axis?

16 One way to describe the graph of $y = \frac{1}{x}$ is to describe the behaviour of the curve as x approaches very large or very small values. Copy and complete each statement using a term from this list: *infinity*, *zero*, *negative infinity*.

a When x approaches infinity, y approaches _____.

b When x approaches zero from the right, y approaches _____.

c When x approaches zero from the left, y approaches _____.

d When x approaches negative infinity, y approaches _____.

17 At a bank, the average waiting time of customers in a queue is related to the number of bank tellers on duty and can be modelled by the relationship $w = \frac{12}{n}$, where w is the average waiting time in minutes when there are n bank tellers on duty.

a Sketch a graph of this relationship.

b On average, how long does a customer wait when the number of bank tellers on duty is:

i 2? **ii** 3? **iii** 4?

iv 6? **v** 12? **vi** 24?

c What is the maximum waiting time for a customer? When does this occur?

d According to this model, will there be occasions when the waiting time is zero? Explain.

CHALLENGE

18 For the graphs of each pair of relationships:

i identify the number of points of intersection by considering their sketch graphs

ii use an algebraic method to find the points of intersection.

a $y = \frac{1}{x}$ and $y = x$

b $y = \frac{1}{x}$ and $y = -x$

c $y = \frac{6}{x-2}$ and $y = x + 3$

d $y = \frac{1}{x-4}$ and $y = -\frac{1}{x}$

19 Find the rule for a hyperbola of the form $y = \frac{8}{x-h}$ that passes through (6, 2).

20 Find the rule for a hyperbola of the form $y = \frac{3}{x-h} + 2$ that passes through (−2, 1).

21 Find the rule for a hyperbola of the form $y = \frac{a}{x-h}$ that passes through (−7, −1) and (1, 1).

Reflect

Why are hyperbolas made up of two separate curves?

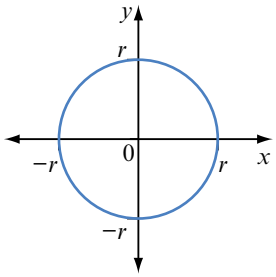
5I Sketching non-linear relationships using transformations

Start thinking!

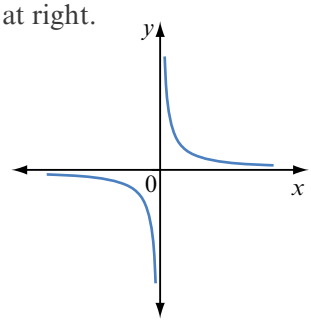
1 Match each graph with its rule from the list at right.

A $y = 2^x$
B $y = x^2$
C $x^2 + y^2 = r^2$
D $y = \frac{1}{x}$

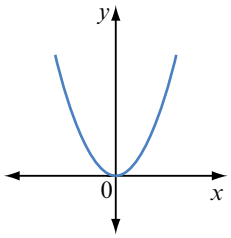
a



b



c



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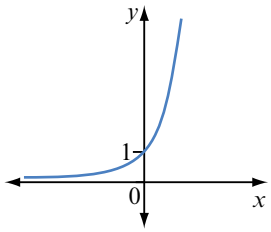


2 Make a list of key features for each graph.

3 Why are these graphs described as non-linear relationships?

KEY IDEAS

- Transformations of dilation, reflection and translation can be performed on the basic graph of many non-linear relationships to produce a graph with the general rule shown in the table below.

Type of graph	Parabola	Rectangular hyperbola	Circle	Exponential relationship
Basic rule	$y = x^2$	$y = \frac{1}{x}$	$x^2 + y^2 = r^2$	$y = a^x, a > 1$
General rule	$y = a(x - h)^2 + k$	$y = \frac{a}{x - h} + k$	$(x - h)^2 + (y - k)^2 = r^2$	$y = B \times a^{x-h} + k$, $a > 1, B > 0$
Dilation	Dilation by factor a.			Dilation of basic graph by factor B.
Reflection	Reflection in x-axis if $a < 0$.			Reflection in x-axis if $B < 0$.
Translation	Horizontal translation of h units. For $h > 0$, move right. For $h < 0$, move left. Vertical translation of k units. For $k > 0$, move up. For $k < 0$, move down.			

EXERCISE 5I Sketching non-linear relationships using transformations

1 For each rule:

- **i** describe the type of non-linear relationship
- **ii** identify the transformations to be performed on the basic graph
- **iii** list key features including intercepts
- **iv** sketch the graph using transformations, showing all key features.

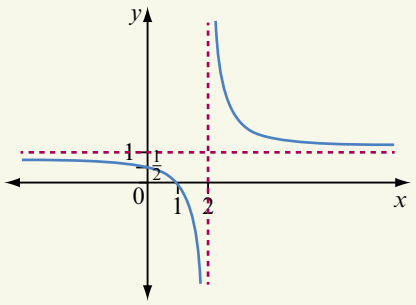
a $y = -(x + 3)^2 - 5$ **b** $y = 3^{x+2} - 1$ **c** $(x - 4)^2 + (y + 2)^2 = 9$

d $y = \frac{1}{x - 6} + 2$ **e** $y = 3(x - 2)^2 - 12$ **f** $y = -2^x + 1$

EXAMPLE 5I-1 Identifying the rule for a non-linear relationship

For the graph shown:

a describe the relationship
b identify the basic graph used to produce the graph of the relationship
c identify the transformations that have been performed on the basic graph
d write the rule for the relationship.



THINK

a Describe the trend shown in the graph. The symbol $\rightarrow$ means 'approaches', so $x \rightarrow 2+$ means 'x approaches 2 from the positive side'.

b Use the shape and any distinguishing features to identify the type of relationship.

c **1** Check if reflection has been performed.

2 Identify the translations performed.

3 Check for dilation by substituting known x and y values.

d Write the rule using parts **b** and **c**.

WRITE

a Graph is a hyperbola with two curves that approach asymptotes.
As $x \rightarrow \infty$, $y \rightarrow 1^+$ and as $x \rightarrow 2^+$, $y \rightarrow \infty$.
As $x \rightarrow -\infty$, $y \rightarrow 1^-$ and as $x \rightarrow 2^-$, $y \rightarrow -\infty$.

b The basic graph is $y = \frac{1}{x}$ with asymptotes at $y = 0$ and $x = 0$ and no intercepts.

c No reflection as curves in same quadrants as basic graph.

Asymptote at $y = 1$ so translation 1 unit up.
Asymptote at $x = 2$ so translation 2 units right.

Rule of the form $y = \frac{a}{x - 2} + 1$.

For $x = 1$, $y = 0$: $0 = \frac{a}{1 - 2} + 1$

$\frac{a}{-1} = -1$

$a = 1$

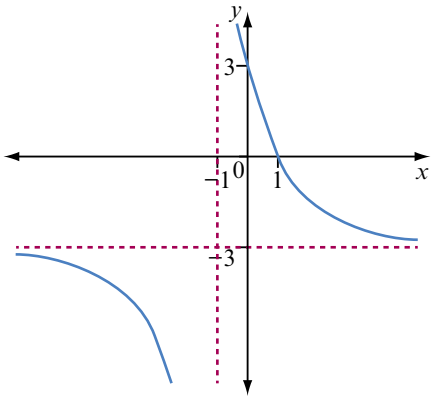
d $y = \frac{1}{x - 2} + 1$

UNDERSTANDING AND FLUENCY

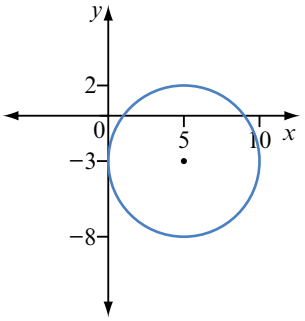
2 For each of the following graphs:

- **i** describe the relationship
- **ii** identify the basic graph used to produce the graph of the relationship
- **iii** identify the transformations that have been performed on the basic graph
- **iv** write the rule for the relationship.

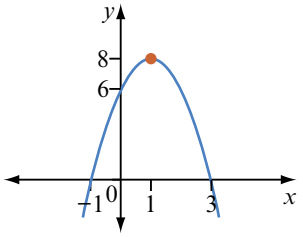
a



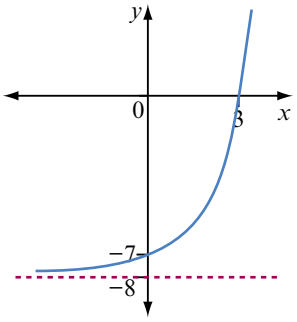
b



c



d



PROBLEM SOLVING AND REASONING

3 The area of grass watered by a sprinkler forms a circle of radius 4 m.

- **a** Write a rule for the relationship that describes this circle if the sprinkler is located at the origin of the Cartesian plane.
- **b** If the sprinkler is moved 5 m north and 7 m west of its original position, write a new Cartesian rule for the circle enclosing the area that is watered.
- **c** Does the new area overlap with the original area?

4 The amount of time it takes to move a distance of 100 m depends on the average speed of travel.

- **a** Write a rule for the relationship between time t (in seconds, s) and average speed s (in metres per second, m/s).
- **b** What type of graph would show this relationship?
- **c** Sketch a graph of this relationship.
- **d** Interpret the graph to describe how the amount of time changes when the speed is increased.

5 Explain how the graph of $y = -\frac{1}{3}(x - 4.6)^2 + 7.1$ can relate to the shape of an entrance to a railway tunnel if drawn on the Cartesian plane with x and y as distances in metres. Find the dimensions of the arch, correct to one decimal place.

6 Investigate the graphs of $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$. What do you notice? Explain.

PROBLEM SOLVING AND REASONING

7 To investigate what happens when a dilation is performed on a circle, consider a dilation of factor a performed on the graph of $x^2 + y^2 = 4$ to produce $ax^2 + ay^2 = 4$.

- **a** Why is there a dilation factor on both the x and y terms?
- **b** Describe the circle produced when $a > 1$; for example, when $a = 2$.
- **c** Describe the circle produced when $0 < a < 1$; for example, when $a = \frac{1}{2}$.

8 There are many other non-linear relationships. Consider a relationship involving the square root of x.

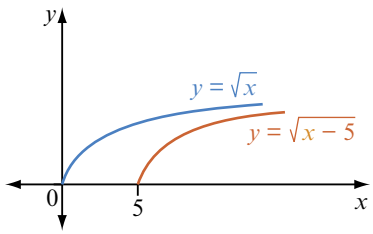
- **a** Plot the graph of $y = \sqrt{x}$ after completing this table of values. Draw a smooth line through the points.

x	0	1	4	9	16	25
y	0			3		

- **b** Describe the shape of the graph. Can you plot points for negative values of x? Explain.
- **c** An important feature of the graph of $y = \sqrt{x}$ is the point on the graph where y is a minimum. The coordinates of this point are (0, 0). Mark this point on your graph.

9 The sketch graphs of $y = \sqrt{x}$ and $y = \sqrt{x - 5}$ are shown at right.

- **a** Identify the coordinates of the point on the graph of $y = \sqrt{x - 5}$ where y is a minimum.
- **b** Describe how you could use a translation to produce the graph of $y = \sqrt{x - 5}$ from the graph of $y = \sqrt{x}$.



10 For each relationship:

- **i** use your understanding of translations to describe how the graph can be produced from the graph of $y = \sqrt{x}$
- **ii** identify the end point of the curve
- **iii** sketch the graph
- **iv** use a calculator or other digital technology to produce the same graph.

a $y = \sqrt{x} + 2$ **b** $y = \sqrt{x} - 1$ **c** $y = \sqrt{x - 1}$

d $y = \sqrt{x + 4}$ **e** $y = \sqrt{x - 2} + 3$ **f** $y = \sqrt{x + 3} - 2$

11 Sketch the graphs of $y = -\sqrt{x}$, $y = -\sqrt{x} + 3$ and $y = -\sqrt{x - 4}$ on the same Cartesian plane. Show the shape and end point only.

CHALLENGE

12 Sketch the graph of each relationship using transformations. Show all key features including intercepts (to one decimal place where necessary) and asymptotes.

- **a** $y = -3 \times 2^{x-1} + 7$
- **b** $y = -2(x + 3)^2 + 5$
- **c** $y = -\sqrt{x - 4} + 3$

Reflect

How are transformations useful in graphing a relationship if you know the basic graph?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

quadratic equations	parabola	dilation	exponential equation
Null Factor Law	sketch graph	reflection	logarithmic form
completing the square	x-intercept	translation	exponential growth
quadratic formula	y-intercept	circle	exponential decay
discriminant	turning point	exponential relationship	hyperbola
number of solutions	transformations	asymptote	non-linear relationships

MULTIPLE-CHOICE

5A **1** Which is a quadratic equation?

A $y - 2 = x^2 + 4$ **B** $x^2 + y^2 = 9$

C $y = \frac{5}{x}$ **D** $3^x = 9$

5B **2** For the quadratic equation $2x - 4x^2 = 3$, the value of the discriminant is:

A 52 **B** −8 **C** −44 **D** 44

5C **3** The coordinates of the x-intercepts of the graph of $y = x^2 - 6x + 5$ are:

A (1, 0), (5, 0) **B** (0, 1), (0, 5)

C (−3, 0), (2, 0) **D** (−1, 0), (−5, 0)

5C **4** Which statement about the graph of $y = x^2 + 2x - 3$ is *incorrect*?

A It is upright.

B The graph has two x-intercepts.

C The turning point has coordinates (−4, −1).

D The graph shows a minimum.

5D **5** Compared with the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is:

A dilated by a factor of 2

B reflected in the x-axis

C translated 2 units up

D translated 2 units right

5D **6** The coordinates of the turning point of the graph of $y = (x - 2)^2 - 5$ are:

A (−2, −5) **B** (2, −5)

C (−2, 5) **D** (2, 5)

5E **7** The coordinates of the centre of the circle $(x - 2)^2 + (y + 3)^2 = 16$ are:

A (2, −3) **B** (−2, 3)

C (−2, −3) **D** (2, 3)

5F **8** Which rule represents an exponential relationship?

A $y = 3x^2 + 4$ **B** $x^2 + y^2 = 4$

C $y = 3^x + 4$ **D** $y = \frac{1}{x}$

5G 10A **9** If $8^x = 32$, the solution to the equation is:

A $x = \frac{5}{3}$ **B** $x = \frac{3}{5}$

C $x = 4$ **D** $x = \frac{1}{4}$

5H 10A **10** Which rule represents a hyperbola?

A $y = x^2 - 4$ **B** $y = x^4$

C $y = 4^x$ **D** $y = \frac{1}{x - 4}$

5I 10A **11** The coordinates of the x-intercept of the graph of $y = \frac{1}{x + 2} + 5$ are:

A (0, −2.2) **B** (−2.2, 0)

C (−0.4, 0) **D** (−7, 0)

SHORT ANSWER

5A **1** Solve each equation. Write solutions as exact values.

a $(x - 2)^2 = 0$

b $(4x - 3)(3x + 4) = 0$

c $(\sqrt{2} - x)(\sqrt{2} + x) = 0$

d $3x^2 - 6x - 24 = 0$

e $5x^2 - 7 = 0$

5A **2** Solve each equation by completing the square. Write solutions as exact values.

a $x^2 + 6x + 4 = 0$

b $x^2 - 3x - 3 = 0$

5B **3** Use the discriminant to identify the number of real solutions for each quadratic equation.

a $x^2 - 4x + 4 = 0$ b $3x^2 = 2x - 4$

c $5x - 4x^2 = 2$ d $4 - 3x^2 = 8x$

e $1 - x - x^2 = 0$ f $6x^2 + 4x = -1$

5B **4** Use the quadratic formula to solve each equation. Write solutions as:

i exact values

ii approximate values to two decimal places.

a $x^2 + 3x = 10$ b $5x^2 = 3x + 1$

c $x^2 = 5x - 2$ d $x^2 + 5x + 2 = 0$

e $2x^2 = x + 2$ f $3x^2 + 7x + 3 = 0$

5C **5** For each quadratic relationship, find the coordinates of the:

i x-intercept(s)

ii y-intercept.

a $y = -x^2 + 25$

b $y = x^2 - 3x - 4$

c $y = 2x^2 - 8x + 8$

d $y = -x^2 - x + 6$

e $y = x^2 - 11x + 10$

5C **6** Sketch the graph of each relationship using intercepts. Label the turning point with its coordinates.

a $y = x^2 - 4x + 3$

b $y = -x^2 - 4x + 5$

5D **7** For each quadratic relationship:

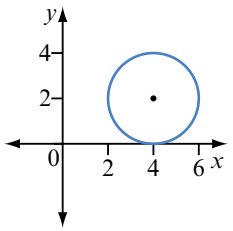
i describe the translations performed on $y = x^2$

ii write the coordinates of the turning point.

a $y = (x - 5)^2 + 2$

b $y = (x + 6)^2 - 4$

5E **8** Write the equation for this circle.



5F **9** a Sketch the graph of $y = 5^x$.

b Describe the transformations to be performed on $y = 5^x$ to produce the graph of:

i $y = 5^{x+3}$ ii $y = 5^x + 3$.

c Sketch these graphs on the same Cartesian plane.

5G 10A **10** Solve each equation.

a $4^{x+2} = 16$ b $81^{x+1} = 27$

c $5^{3x} = 15$ d $6^{2x+1} = 42$

5H 10A **11** Describe the transformations performed on the graph of $y = \frac{1}{x}$ to produce the graph of each rule.

a $y = \frac{1}{x} - 5$ b $y = -\frac{1}{x}$

c $y = \frac{4}{x}$ d $y = \frac{1}{x+5} + 6$

5I 10A **12** Identify the graph of each rule.

a $x^2 + y^2 = 2$ b $y = \frac{1}{x+4}$

c $y = 5^x$ d $y = 3x^2 - 5$

5I 10A **13** The graph of $y = \frac{1}{x}$ is translated 4 units left and 6 units down. What is the rule of the new curve?

MIXED PRACTICE

1 If $(x + 3)(x - 4) = 0$, the solution to the equation is:

A $x = 3$ or 4 **B** $x = -3$ or -4
C $x = 3$ or -4 **D** $x = -3$ or 4

2 Give the *number* of solutions to each quadratic equation.

a $x^2 = 3$ **b** $(x - 3)^2 = 0$
c $x^2 - 8x + 16 = 0$ **d** $(x+\sqrt{5})(x-\sqrt{5})=0$
e $x^2 + 5x + 1 = 0$ **f** $x^2 + x + 1 = 0$

3 The quadratic equation $ax^2 + bx + c = 0$ has two real solutions only if:

A $b^2 > 4ac$ **B** $b^2 < 4ac$
C $b^2 = 4ac$ **D** $a = 1$

4 Find the coordinates of the turning point for the graph of each quadratic relationship.

a $y = x^2 + 2x - 8$ **b** $y = -x^2 + 16x$
c $y = -x^2 + 4x + 5$ **d** $y = x^2 + 10x + 16$
e $y = x^2 + 8x + 15$ **f** $y = x^2 - 5x + 6$

5 For each quadratic relationship, identify:

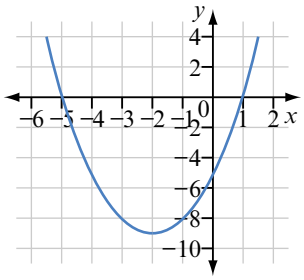
i whether the graph will be narrower or wider than the graph of $y = x^2$
ii whether the parabola will be upright or inverted
iii the coordinates of the turning point

a $y = 4(x - 3)^2 - 1$ **b** $y = \frac{1}{4}(x + 1)^2 + 2$
c $y = -2(x - 3)^2$

6 Describe the transformation needed to produce each graph from the graph of $y = 2^x$.

a $y = -2^x$ **b** $y = 3 \times 2^x$
c $y = 2^{x-3}$ **d** $y = 2^x - 3$

7 What is the rule for this parabola?



A $y = -x^2 + 4x - 5$ **B** $y = x^2 - 4x - 5$
C $y = x^2 + 4x - 5$ **D** $y = x^2 - 4x + 5$

8 For the graph $(x + 5)^2 + (y + 2)^2 = 9$, write the coordinates of the two x-intercepts, correct to one decimal place.

9 Consider the circle with the rule $(x + 3)^2 + (y - 2)^2 = 25$.

a What are the coordinates of the centre of this circle?
b What is the radius of the circle?
c Translations are performed by moving the centre of the circle 4 units to the right and 3 units down. Give the coordinates of the centre of the new circle.

10 The solution to $x^2 + 4 = 0$ is:

A -2 **B** ± 2
C 2 **D** there is no solution

11 For what values of k will $x^2 + 10x + k = 0$ have:

a one real solution
b no real solutions
c two real solutions.

12 Compared with the graph of $y = x^2$, the graph of $y = (x - 3)^2 - 3$ is:

A translated 3 units left and 3 units down
B translated 3 units right and 3 units down
C translated 3 units left and 3 units up
D translated 3 units right and 3 units up

13 The rule for the horizontal asymptote for the graph of $y = 2^x + 3$ is:

A $y = 3$ **B** $y = 2$
C $x = 3$ **D** $x = 2$

14 The coordinates of the y-intercept of the graph of the equation $y = 2^{x+3}$ are:

A $(2, 3)$ **B** $(0, 8)$
C $(8, 0)$ **D** $(0, 3)$

10A **15** Solve each equation, correct to two decimal places.

a $2^x = 25$ **b** $9^{-x} = 2.3$
c $3^x = \frac{1}{8}$ **d** $4 \times 5^{2x+1} = 30$

10A **16** Find the x-intercept for each exponential relationship to two decimal places.

a $y = 3^x - 5$ **b** $y = 2^{x+3} - 10$

c $y = 4 \times 5^x - 3$ **d** $y = -2 \times 4^{x-1} + 6$

10A **17** The coordinates of the y-intercept of the graph of $y = \frac{1}{x-5} - 4$ are:

A $(-4.2, 0)$ **B** $(5.25, 0)$

C $(0, -4.2)$ **D** $(0, 5.25)$

10A **18** The graph of the circle with the equation $(x + 3)^2 + (y - 1)^2 = 16$ has two x-intercepts and two y-intercepts. Find the values for all these intercepts, correct to one decimal place.

10A **19** The equation for the vertical asymptote of the graph of $y = \frac{1}{x-4}$ is:

A $y = 4$ **B** $y = -4$

C $x = 4$ **D** $x = -4$

10A **20** Which statement about the graph of $(x + 3)^2 + (y - 2)^2 = 4$ is true?

A There is one x-intercept.

B The circle lies fully in quadrant 2.

C There is one y-intercept.

D The circle touches one of the axes.

10A **21** **a** Describe how the graph of $y = \frac{1}{x-2}$ could be drawn from the graph of $y = \frac{1}{x}$.

b Sketch both graphs on the same Cartesian plane.

ANALYSIS

1 The parabolas you have looked at so far are vertical parabolas, as the axis of symmetry is vertical. There are also horizontal parabolas, where the axis of symmetry is horizontal. Similar to using $y = x^2$ as the basic rule for a vertical parabola, we can use $x = y^2$ as the basic rule for a horizontal parabola.

A horizontal parabola can be represented by the general rule $x - h = a(y - k)^2$ where (h, k) is the turning point of the parabola, and a is a constant.

a Consider the graph of $x + 4 = (y + 1)^2$.

i Give values for a, h and k.

ii What are the coordinates of the turning point?

iii There is one x-intercept. Calculate its coordinates.

iv There are two y-intercepts. Calculate the coordinates of both these points.

v Use the coordinates of the turning point and the intercepts to sketch the graph on a Cartesian plane.

vi Describe the transformations performed on the graph of $x = y^2$ to produce the same graph.

b Follow the procedure in part **a** to sketch the graph of $x - 4 = -(y + 1)^2$ on the same Cartesian plane as the previous one.

c Compare the two graphs in parts **a** and **b**. What is the effect of the negative value of a?

2 You are familiar with the quadratic formula in the form $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

a Consider the quadratic equation $x^2 - x - 3 = 0$. Use both formulas to show that they give the same solution, correct to one decimal place.

10A **b** Start with the first formula. Show how algebra can be used to transform the formula into the second form. (Hint: take the positive value of the square root, then rationalise the numerator.)

CONNECT

Bridges, domes and arches

What do these images have in common? They have features, such as the cross-section of a dome or the shape of bridge spans and arches, that can be modelled by a non-linear relationship. We can map the 2D shape on to a Cartesian plane with x and y representing distances in metres.

Dulles Airport, Virginia, USA
designed by Eero Saarinen

The curving roof of this airport is in the shape of a hyperbolic paraboloid. One cross-section is shaped as a parabola and another as a hyperbola.

Your task

You are to analyse the information provided with each example, using the guidelines below. Include all necessary graphs and working to justify your answers. Where appropriate, describe how transformations of the basic graph for the relationship were used.

Sydney Harbour Bridge

Use the given algebraic rules for the lower and upper arches to:

- identify the type of non-linear relationship used to model the shapes
- sketch the graphs of the relationships on the same Cartesian plane
- explain how the rule relates to the shape and features of the arch
- determine the maximum height above water level for each arch
- state the length of the bridge between the pylons
- find the difference in height between the arches where they meet the pylons and at the top of the bridge.

McDonald's Golden Arches

Use the information provided to determine two different quadratic rules that model one of the arches. (Hint: think about where to place the x- and y-axes in each case.)

The Pantheon

Determine the rule for the semi-circle that describes the shape of the vertical cross-section of the dome through its top.

Gateway Arch

Use the given information to determine a quadratic relationship to best match the shape of the arch. This arch is actually an example of a catenary curve. (You may like to research this.)

As an extension, discover a rule of your own that would best model a non-linear relationship. Some ideas you could consider are the path of a moving object, cross-sectional shape of an object, population or investment growth, and amount owing on a bank loan after repayments. Fully explain how you obtained the relationship.

Sydney Harbour Bridge, Sydney, Australia

The lower arch of the bridge can be modelled by the rule $y = -0.00183(x - 251.5)^2 + 116$ and the upper arch by $y = -0.00122(x - 251.5)^2 + 134$.

McDonald's Golden Arches
designed by Stanley Meston

The Golden Arches come in different sizes and consist of two overlapping parabolas. The outer edge of one of these arches has a maximum width of 3 m and height of 4 m.

The Pantheon, Rome, Italy

The dome has a diameter of 43.4 m.

Gateway Arch, St Louis, Missouri, USA
designed by Eero Saarinen

The arch is 192 m high with a width across the base of 192 m.

You may like to present your findings as a report. Your report could include:

- a poster showing diagrams and calculations
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

6 POLYNOMIALS

ESSENTIAL QUESTION

What are polynomials and how are they useful when modelling a relationship?

6A 10A **1** Simplify each expression.

a $4x^2 - 7x + 3 + 2x^2 - 5 - x$

b $(3x^2 - 5x) - (x^2 + 6x)$

c $(2x^3 + x^2 + 1) - (5x^3 - 4x^2 - 3x)$

d $(x^3 - 2x^2 + 3x - 4) - (x^3 + 5x^2 - 2x - 9)$

6A 10A **2** Simplify each expression.

a $3x \times x^2$ b $7x^3 \times 2x^2$

c $5x^2 \div x$ d $24x^3 \div 4x$

6A 10A **3** Expand and simplify each product.

a $x^2(3x^3 - 5)$ b $(x - 4)(x + 7)$

c $2(x + 3)^2$ d $x(3x - 2)(2x + 1)$

6A 10A **4** Evaluate $2x^3 - 3x^2 - 4x + 3$ after substituting each x value.

a $x = 1$ b $x = 3$

c $x = 0$ d $x = -2$

6B 10A **5** Consider this long division problem.

$$\begin{array}{r} 253 \\ 13\overline{)3296} \\ \underline{26} \\ 69 \\ \underline{65} \\ 46 \\ \underline{39} \\ 7 \end{array}$$

a The dividend is:

A 7 B 13

C 253 D 3296

b The divisor is:

A 7 B 13

C 253 D 3296

c The quotient is:

A 7 B 13

C 253 D 3296

d The remainder is:

A 7 B 13

C 253 D 3296

6B 10A **6** Use long division to find the quotient and remainder for:

a $456 \div 18$ b $2783 \div 23$

6D 10A **7** Solve each equation.

a $2x(x - 7) = 0$

b $x^2 - 3x - 18 = 0$

c $x^2 - 16 = 0$

d $x^2 + 4x - 1 = 0$

e $3x^2 - 6x + 3 = 0$

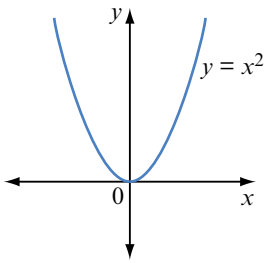
f $25 + x^2 = 0$

6E 10A **8** Find the x- and y-intercepts for each graph with the given rule.

a $y = 2x - 6$ b $y = x^2 + x - 12$

c $y = x^2 - 9$ d $y = x^2 + 4x + 4$

6F 10A **9** Describe the transformation/s to be performed on the graph of $y = x^2$ to produce the graph of:



a $y = x^2 + 4$

b $y = (x - 2)^2$

c $y = 3x^2$

d $y = -x^2$

e $y = (x + 1)^2 - 2$

f $y = -2(x - 5)^2 + 3$

6F 10A **10** Write the coordinates of the turning point for each new graph in question 9.

6A Understanding polynomials

Start thinking!

A **polynomial** is an expression with terms containing one variable only, and with that variable raised to a power that is a positive integer or zero.

1 Consider the quadratic expression $3x^2 - 5x + 7$.

a What variable is used in the expression?

b What is the power of x in:

i the first term? **ii** the second term? **iii** the third term? (Hint: what does x^0 equal?)

c Is the expression a polynomial? Explain.

d What is the highest power of x used? This is the **degree** of the polynomial.

e Which term contains the highest power of x? This is the **leading term** of the polynomial.

f What is the coefficient of the leading term? This is the leading **coefficient** of the polynomial.

g Which term is the constant term?

2 Decide if each expression is a polynomial. Give a reason for your answer.

a $x^3 + 7x^2 + 3x + 2$ **b** $8x^2 - 2x^{\frac{1}{4}}$ **c** $4x + 9x^4 - x^2$

d $\frac{6}{x} + 5x^7$ **e** $2x^5 - x^4 + 1 - \sqrt{x}$ **f** $3x^2 + 2xy - y^3$

KEY IDEAS

- A polynomial is an expression that contains only one variable, such as x. Each term contains the variable raised to a non-negative integer value (0, 1, 2, 3, …).
- A polynomial can be written as $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0x^0$ where n is the highest power and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are coefficients. $P(x)$ is read as 'P of x' meaning the polynomial P using the variable x.
- The degree of a polynomial (n) is the highest power of the variable. Polynomials are given names according to their degree. Some common polynomials are listed in the table at right.
- The leading term ($a_n x^n$) contains the highest power of the variable and is usually written first.
- To add or subtract polynomials, add or subtract any like terms.
- To expand the product of two polynomials, multiply each term in the first polynomial by each term in the second. If there are more than two polynomial factors to multiply together, expand two factors and then multiply the result by the remaining factors, one at a time.

Degree	Name	Example
0	constant	5 (or $5x^0$)
1	linear	$2x + 9$ (or $2x^1 + 9$)
2	quadratic	$-x^2 + 3x - 1$
3	cubic	$4x^3 - x^2 + 2x + 7$
4	quartic	$\frac{1}{2}x^4 + x^2 - 2$

EXERCISE 6A Understanding polynomials

1 Decide if each expression is a polynomial. For each polynomial, give its name as constant, linear, quadratic, cubic or quartic.

a $4x^3 + 2x^2 + 5x + 1$ **b** $1 - x^3y^2 + 7x$ **c** $6 - 9x$

d $3x + x^2 - 4$ **e** $2x^3 + 3\sqrt{x} - 6$ **f** 8

g $5x^3 - 2x^{\frac{1}{5}} + 7$ **h** $x^4 + x + 1$ **i** $\frac{3}{7}x^3$

EXAMPLE 6A-1 Identifying features of a polynomial

For the polynomial $2x^3 - x^2 + 7x + 4$, identify:

a the number of terms **b** the degree of the polynomial
c the constant term **d** the leading term
e the leading coefficient **f** the coefficient of the x^2 term

THINK

a Terms are separated by + and − signs.

b Look for the highest power of x.

c Look for a term without a variable (power of variable is 0).

d Look for the term that has the highest power of x.

e Write the coefficient of the leading term.

f Write the coefficient of the term containing x^2. The sign shown between it and the term before belongs to the x^2 term. ($-x^2$ means $-1x^2$.)

WRITE

a There are four terms.

b Degree is 3.

c Constant term is 4.

d Leading term is $2x^3$.

e Leading coefficient is 2.

f Coefficient of x^2 is -1.

UNDERSTANDING AND FLUENCY

2 For each polynomial below, identify:

i the number of terms **ii** the degree of the polynomial
iii the constant term **iv** the leading term
v the leading coefficient **vi** the coefficient of the x^2 term

a $2x^3 + 3x^2 + 4x + 5$ **b** $4x^5 + x^4 + 7x^3 - 2x^2 + 9x - 3$

c $-5x^4 - 2x^3 + 5x^2 + 1$ **d** $x^7 + x^6 - x^5 + x^4 + x^3 - x^2 + x$

e $9 - 3x - 6x^3 + 2x^2 - 7x^6$ **f** $3 - 11x^{10} + 5x^8$

3 Simplify each polynomial by collecting like terms.

a $4x^3 + 2x^2 - 4x - 1 + x^3 - 5x^2 + 3$ **b** $x^5 - 3x^2 + x^3 - 2x^4 - x^3 + 7x^2$

c $x^4 + 2 + 4x^2 + x^4 - 2x^2 - 5 + 6x^3$ **d** $(2x^2 - 5x + 1) + (3x^2 - 6x + 8)$

e $(5x^3 - 2x + 1) - (2x^3 - 7x + 4)$ **f** $(3x^2 + x + x^3) - (4x^4 - 3x^2 + 5x^3)$

EXAMPLE 6A-2 Evaluating a polynomial

If $P(x) = 2x^3 - x^2 + 7x + 4$, evaluate: **a** $P(3)$ **b** $P(0)$ **c** $P(-1)$

THINK

a Substitute $x = 3$ into the expression and evaluate.

b Substitute $x = 0$ into the expression and evaluate.

c Substitute $x = -1$ into the expression and evaluate.

WRITE

a $P(3) = 2(3)^3 - (3)^2 + 7(3) + 4$
$= 54 - 9 + 21 + 4$
$= 70$

b $P(0) = 2(0)^3 - (0)^2 + 7(0) + 4$
$= 0 - 0 + 0 + 4$
$= 4$

c $P(-1) = 2(-1)^3 - (-1)^2 + 7(-1) + 4$
$= -2 - 1 - 7 + 4$
$= -6$

4 If $P(x) = x^3 - 2x^2 + 5x - 8$, evaluate:
a $P(2)$ **b** $P(0)$ **c** $P(3)$ **d** $P(-3)$

5 If $P(x) = 3x^4 - 4x^3 + x^2 - 6x + 1$, evaluate:
a $P(3)$ **b** $P(1)$ **c** $P(-1)$ **d** $P(-2)$

6 If $P(x) = 2x^3 - x^2 + 3x - 6$ and $Q(x) = 3x^2 - 7x + 2$, find:
a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x)$
e $-2Q(x)$ **f** $2P(x) + Q(x)$ **g** $P(x) - 4Q(x)$ **h** $3P(x) - 2Q(x)$
i $2P(-1)$ **j** $-3Q(2)$ **k** $P(3) + Q(3)$ **l** $5P(0) - 4Q(0)$

EXAMPLE 6A-3 Expanding the product of two polynomials

Expand and simplify each product.
a $2x^3(x^2 - 3x + 4)$ **b** $(x^2 - 2)(x^3 - x + 5)$ **c** $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$

THINK

a Multiply the term outside the brackets with each term inside the pair of brackets.

b Multiply each term in the first pair of brackets with each term in the second pair of brackets. Simplify like terms.

c Multiply each term in the first pair of brackets with each term in the second pair of brackets. Simplify like terms.

WRITE

a $2x^3(x^2 - 3x + 4)$
$= 2x^5 - 6x^4 + 8x^3$

b $(x^2 - 2)(x^3 - x + 5)$
$= x^2(x^3 - x + 5) - 2(x^3 - x + 5)$
$= x^5 - x^3 + 5x^2 - 2x^3 + 2x - 10$
$= x^5 - 3x^3 + 5x^2 + 2x - 10$

c $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$
$= 2x^4(x^3 + 4x - 1) - x^2(x^3 + 4x - 1) + 3(x^3 + 4x - 1)$
$= 2x^7 + 8x^5 - 2x^4 - x^5 - 4x^3 + x^2 + 3x^3 + 12x - 3$
$= 2x^7 + 7x^5 - 2x^4 - x^3 + x^2 + 12x - 3$

7 Expand and simplify each product.

a $x^3(x^2 + 2x + 7)$ **b** $4x^2(x^2 - 5x + 2)$

c $-6x^3(x^4 - 4x^2 + 1)$ **d** $(x + 3)(x^2 - 2x + 4)$

e $(x^2 - 1)(x^2 + 3x - 7)$ **f** $(x^2 - 5)(x^3 - x + 3)$

g $(x^4 - 3x + 2)(x^3 + 4x^2 - 1)$ **h** $(2x^5 - 3x^2 + 1)(x^3 - 5x + 2)$

EXAMPLE 6A-4 Expanding the product of three polynomials

Expand and simplify $(x + 2)(x - 3)(x + 4)$.

THINK	WRITE
1 Multiply two of the factors and simplify. It is easier to choose the last two.	$(x + 2)(x - 3)(x + 4)$ $= (x + 2)(x^2 + 4x - 3x - 12)$ $= (x + 2)(x^2 + x - 12)$
2 Multiply the linear factor by the quadratic factor.	$= x(x^2 + x - 12) + 2(x^2 + x - 12)$ $= x^3 + x^2 - 12x + 2x^2 + 2x - 24$
3 Simplify like terms.	$= x^3 + 3x^2 - 10x - 24$

8 Expand and simplify each product.

a $3x(x + 2)(x + 3)$ **b** $-7x^2(x + 5)(x - 2)$ **c** $(x + 3)(x + 1)(x + 6)$

d $(x - 3)(x - 4)(x + 1)$ **e** $(3x - 2)(x - 5)(x - 2)$ **f** $(4x + 1)(2x + 7)(x - 4)$

9 Expand and simplify each expression.

a $(x^3 + 4)^2$ **b** $(x^2 + 3x)^2$ **c** $2x(x^2 - 2)^2$ **d** $(x^3 - 5x^2)^2$

e $(x + 2)^3$ **f** $(2x - 3)^3$ **g** $(x + 3)^4$ **h** $(1 - x)^4$

10 What is the maximum number of terms in a polynomial of degree 5?

11 What is the minimum number of terms in a polynomial of degree 3?

12 If $P(x)$ is a polynomial of degree n, what is:

a the maximum number of terms in $P(x)$?

b the minimum number of terms in $P(x)$?

c the degree of $2P(x)$? **d** the degree of $[P(x)]^2$?

13 Find the value of k in $P(x) = x^3 - 2x^2 + 3x + k$, if $P(2) = 2$.

14 Find the value of k in $P(x) = 3x^4 + 2x^3 - kx - 5$, if $P(-1) = 7$.

15 Find the value of a and b in $P(x) = x^3 + ax^2 + bx + 1$, if $P(3) = 31$ and $P(-2) = -19$.

16 If $P(x) = x^3 - 3x^2 + 2x + 1$, write simplified expressions for:

a $P(a)$ **b** $P(-3a)$ **c** $P(a^2)$ **d** $P(-a^2)$ **e** $P(a + 2)$ **f** $P(2a - 1)$

CHALLENGE

17 Show that $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$.

18 Simplify each expression.

a $(x^2 - 2)^3$ **b** $(x^3 + x - 1)^2$ **c** $(2x^3 + 1)^4$

Reflect

What makes an algebraic expression a polynomial?

10A

6B Division of polynomials

Start thinking!

So far you have added, subtracted and multiplied polynomials. They can also be divided, which is useful in factorising polynomials.

Before looking at long division of polynomials, first consider long division of numbers.

$$\begin{array}{r} 26 \\ 9\overline{)237} \\ \underline{18} \\ 57 \\ \underline{54} \\ 3 \end{array}$$

1 The working for using long division to find the result to 237 ÷ 9 is shown at right.

- **a** Explain how 2 is obtained in the top line.
- **b** Explain why 18 is written below 23.
- **c** What is the result of 23 − 18? Identify where this is written in the working.
- **d** How is the new number to divide into (57) formed?
- **e** How many times does 9 go into 57? Where is this written?
- **f** Where does 54 come from?
- **g** How is 3 obtained?

2 In long division, when the **dividend** is divided by the **divisor**, the **quotient** is written on the top line and the **remainder** is in the last line. Identify the dividend, divisor, quotient and remainder for 237 ÷ 9.

3 Copy and complete these statements.

- **a** 237 ÷ 9 = _____ remainder _____
- **b** 237 = 9 × _____ + 3

KEY IDEAS

- Long division is used to divide polynomials. At each stage, the leading term of the dividend is divided by the leading term of the divisor.

$$\begin{array}{rl} & x + 3 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow & x + 2\overline{)x^2 + 5x + 8} \leftarrow \text{dividend} \\ & \underline{x^2 + 2x} \\ & 3x + 8 \\ & \underline{3x + 6} \\ & 2 \leftarrow \text{remainder} \end{array}$$

- The quotient and remainder are important elements to identify. In the example, the quotient is $x + 3$ and the remainder is 2. This means that:
  $(x^2 + 5x + 8) \div (x + 2) = x + 3$ remainder 2
  or $\dfrac{x^2 + 5x + 8}{x + 2} = x + 3 + \dfrac{2}{x + 2}$.
- Alternatively, this can be written as $x^2 + 5x + 8 = (x + 2)(x + 3) + 2$; that is:
  dividend = divisor × quotient + remainder
- In general, if $P(x)$ is divided by $(x - a)$ to produce the quotient $Q(x)$ and the remainder R, then $P(x) = (x - a)Q(x) + R$.

EXERCISE 6B Division of polynomials

UNDERSTANDING AND FLUENCY

1 Copy and complete each statement.

a If $152 \div 8 = 19$, then $152 = 8 \times$ ___.

b If $9475 \div 25 = 379$, then $9475 =$ ___ $\times$ ___.

c If $321 \div 7 = 45$ remainder 6, then $351 = 7 \times 45 +$ ___.

d If $2386 \div 16 = 149$ remainder 2, then $2386 = 16 \times$ ___ + ___.

2 Copy and complete each statement.

a If $(x^2 + 5x + 6) \div (x + 2) = x + 3$,
then $x^2 + 5x + 6 = (x + 2)$(____).

b If $(x^3 + 2x^2 + 3x + 2) \div (x + 1) = x^2 + x + 2$,
then $x^3 + 2x^2 + 3x + 2 =$ (____)(____).

c If $(2x^2 - 3x - 4) \div (x - 3) = 2x + 3$ remainder 5,
then $2x^2 - 3x - 4 = (x - 3)$(____) + 5.

d If $(x^3 + 4x^2 - x + 3) \div (x - 1) = x^2 + 5x + 4$ remainder 7,
then $x^3 + 4x^2 - x + 3 =$ (____)(____) + ____.

EXAMPLE 6B-1 Dividing a quadratic polynomial by a linear expression

Use long division to find the quotient and remainder for $(2x^2 + 5x - 1) \div (x + 4)$.

THINK

1 Divide the leading term in the dividend ($2x^2$) by the leading term of the divisor (x). $2x^2 \div x = 2x$. Write the result of $2x$ above $5x$ in the quotient line. Remember to align like terms in columns.

2 Work out the remainder by first multiplying $2x$ by the divisor. $2x(x + 4) = 2x^2 + 8x$. Write the result underneath, then subtract like terms.

3 Divide the leading term of $-3x - 1$ by the leading term of the divisor. $-3x \div x = -3$. Write the result of -3 above -1 in the quotient line.

4 Work out the remainder by first multiplying -3 by the divisor. $-3(x + 4) = -3x - 12$. Write the result underneath, then subtract like terms.

5 Identify the quotient and the remainder.

WRITE

Note: each stage of working has been shown separately.

$$\begin{array}{r} 2x \\ x + 4 \overline{)\,2x^2 + 5x - 1} \end{array}$$

$$\begin{array}{rr} & 2x \\ & x + 4 \overline{)\,2x^2 + 5x - 1} \\ 2x(x + 4) & \underline{2x^2 + 8x} \\ & -3x - 1 \end{array}$$

$$\begin{array}{rr} & 2x - 3 \\ & x + 4 \overline{)\,2x^2 + 5x - 1} \\ 2x(x + 4) & \underline{2x^2 + 8x} \\ & -3x - 1 \\ -3(x + 4) & \underline{-3x - 12} \\ & 11 \end{array}$$

Quotient is $2x - 3$, remainder is 11.

3 Copy and complete the working for each long division problem.

a
$$\begin{array}{r} x + \\ x + 2 \overline{)\, x^2 + 6x + 1} \\ \underline{x^2 + } \\ + 1 \\ \underline{4x + } \\ -7 \end{array}$$

Quotient is _____, remainder is ____.

b
$$\begin{array}{r} 4x - \\ x - 1 \overline{)\, 4x^2 - 7x + 5} \\ \underline{4x^2 - } \\ -3x + \\ \underline{ + 3} \\ 2 \end{array}$$

Quotient is _____, remainder is ____.

4 Use long division to find the quotient and remainder for each division problem.

a $(x^2 + 7x + 8) \div (x + 3)$

b $(x^2 + 3x - 1) \div (x - 2)$

c $(x^2 + 2x + 3) \div (x + 4)$

d $(x^2 - 5x - 4) \div (x + 2)$

e $(2x^2 - 3x - 11) \div (x - 3)$

f $(3x^2 + x + 2) \div (x - 1)$

5 Write your answers to question **4** in the form: dividend = divisor × quotient + remainder. For example, $2x^2 + 5x - 1 = (x + 4)(2x - 3) + 11$.

6 Expand and simplify the right side of each statement found in question **5** to verify that the statement is true.

7 **a** Divide $x^2 - 2x - 24$ by $(x + 4)$ to find the quotient and remainder.

b Write the division problem in the form:
dividend = divisor × quotient + remainder.

c What does the value of the remainder tell you about the divisor?

EXAMPLE 6B-2 Dividing a polynomial by a linear expression

Use long division to find the quotient and remainder for $(x^3 + 2x^2 - 9x - 3) \div (x - 2)$.

THINK

1. Divide leading term in dividend (x^3) by leading term of divisor (x) and write the result of x^2 in the quotient line.
2. Expand $x^2(x - 2)$ to give $x^3 - 2x^2$ and write the result underneath. Subtract like terms.
3. Divide $4x^2$ by x and write the result of $4x$ in the quotient line.
4. Expand $4x(x - 2)$ to give $4x^2 - 8x$ and write the result underneath. Subtract like terms.
5. Divide $-x$ by x and write the result of -1 in the quotient line.
6. Expand $-1(x - 2)$ to give $-x + 2$ and write the result underneath. Subtract like terms.
7. Identify the quotient and the remainder.

WRITE

$$\begin{array}{rr} & x^2 + 4x - 1 \\ & x - 2 \overline{)\, x^3 + 2x^2 - 9x - 3} \\ x^2(x - 2) & \underline{x^3 - 2x^2 } \\ & 4x^2 - 9x - 3 \\ 4x(x - 2) & \underline{4x^2 - 8x } \\ & -x - 3 \\ -1(x - 2) & \underline{-x + 2} \\ & -5 \end{array}$$

Quotient is $x^2 + 4x - 1$, remainder is -5.

8 Use long division to find the quotient and remainder for each division problem.

a $(x^3 - 2x^2 - 5x + 7) \div (x - 3)$
b $(x^3 + 3x^2 + 7x - 1) \div (x + 1)$
c $(x^3 + 6x^2 - 4x - 15) \div (x - 2)$
d $(x^3 - 3x^2 - 20x + 25) \div (x + 4)$
e $(2x^3 + 3x^2 - 5x - 4) \div (x + 2)$
f $(3x^3 - 2x^2 - 7x + 2) \div (x - 1)$
g $(5x^3 - 12x^2 - 6x - 15) \div (x - 3)$
h $(4x^3 + 9x^2 - 8x - 1) \div (x + 1)$
i $(x^4 + 2x^3 - 4x^2 - 2x + 8) \div (x + 1)$
j $(x^4 - 5x^3 + 2x^2 - x - 3) \div (x - 2)$

EXAMPLE 6B-3

Dividing a polynomial that has coefficients of zero by a linear expression

Use long division to find the quotient and remainder for $(3x^4 + 2x^2 - 5) \div (x + 1)$.

THINK

1. Write the 'missing' terms in the dividend with zero as the coefficient to keep like terms in columns.
2. Divide $3x^4$ by x and write $3x^3$ in quotient line.
3. Expand $3x^3(x + 1)$ to give $3x^4 + 3x^3$. Subtract like terms.
4. Divide $-3x^3$ by x and write $-3x^2$ in quotient line.
5. Expand $-3x^2(x + 1)$ to give $-3x^3 - 3x^2$. Subtract like terms.
6. Divide $5x^2$ by x and write $5x$ in quotient line.
7. Expand $5x(x + 1)$ to give $5x^2 + 5x$. Subtract like terms.
8. Divide $-5x$ by x and write -5 in quotient line.
9. Expand $-5(x + 1)$ to give $-5x - 5$. Subtract like terms.
10. Identify the quotient and the remainder.

WRITE

$$
\begin{array}{rr}
 & 3x^3 - 3x^2 + 5x - 5 \\
x + 1 \overline{)} & 3x^4 + 0x^3 + 2x^2 + 0x - 5 \\
3x^3(x + 1) & \underline{3x^4 + 3x^3} \\
 & -3x^3 + 2x^2 + 0x - 5 \\
-3x^2(x + 1) & \underline{-3x^3 - 3x^2} \\
 & 5x^2 + 0x - 5 \\
5x(x + 1) & \underline{5x^2 + 5x} \\
 & -5x - 5 \\
-5(x + 1) & \underline{-5x - 5} \\
 & 0
\end{array}
$$

Quotient is $3x^3 - 3x^2 + 5x - 5$, remainder is 0.

9 Use long division to find the quotient and remainder for each division problem.

a $(x^3 + x + 21) \div (x + 3)$
b $(2x^3 - 3x^2 - 6) \div (x - 2)$
c $(4x^4 - 3x^2 - 5) \div (x - 1)$
d $(3x^4 - 41) \div (x + 2)$

10 a If $P(x) = x^3 - 3x^2 - 10x + k$, for what value of k does $P(x) \div (x - 2)$ give a remainder of zero.

b Use the divisor and quotient to write the linear factor and the quadratic factor of $P(x)$. Hence, show that you can write $P(x)$ as the product of three linear factors.

11 a If $P(x) = x^3 + 9x^2 + 23x + 15$, for what three values of k does $P(x) \div (x + k)$ give a remainder of zero.

b Write $P(x)$ as the product of three linear factors.

CHALLENGE

12 Use long division to find the quotient and remainder for each division problem.

a $(x^3 - 3x^2 + 2x - 4) \div (x^2 + 1)$
b $(x^4 + x^3 - 7x^2 + 3x + 5) \div (x^2 - 3)$
c $(2x^4 + 6x^2 - 1) \div (x^2 + 2)$

Reflect

Why is knowing how to divide polynomials useful?

6C Remainder and factor theorems

Start thinking!

Consider the polynomial $P(x) = x^3 + x^2 - 10x + 8$.

1 a Use long division to find the remainder when $P(x)$ is divided by $(x - 3)$.
b Evaluate $P(3)$. What do you notice?

2 a Use long division to find the remainder when $P(x)$ is divided by $(x + 2)$.
b Evaluate $P(-2)$. What do you notice?

3 a Without using long division, use the pattern you noticed in questions **1** and **2** to show how to work out the remainder when $P(x)$ is divided by $(x - 4)$.
b Use long division to verify your answer to part **a**.

4 a Evaluate $P(1)$ to find the remainder when $P(x)$ is divided by $(x - 1)$.
b Use long division to verify your answer to part **a**.

5 The pattern or shortcut you have used is called the **remainder theorem**. Explain how you can find the remainder without using long division when $P(x)$ is divided by:
a $x - 5$ **b** $x + 1$ **c** $x - 2$

6 What does the remainder in question **4** tell you about the relationship between $P(x)$ and $x - 1$? This is the basis for the **factor theorem**.

KEY IDEAS

- Remainder theorem
  - When a polynomial P(x) is divided by $(x - a)$, the remainder is $P(a)$. For example:
    when $P(x)$ is divided by $(x - 2)$, the remainder is $P(2)$
    when $P(x)$ is divided by $(x + 3)$, the remainder is $P(-3)$.
- Factor theorem
  - When a polynomial $P(x)$ is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$. For example, $P(x) = x^3 + x^2 - 10x + 8$ has a factor of $(x - 1)$, as $P(1) = 0$.
- If a linear factor of a cubic polynomial is known, long division can be used to find the quadratic factor. For example, dividing $P(x) = x^3 + x^2 - 10x + 8$ by $(x - 1)$ gives a quotient of $x^2 + 2x - 8$.
- Factorising this quotient allows us to write the polynomial as a product of its linear factors.
- In general, if $P(x)$ is divided by a factor $(x - a)$ to produce the quotient $Q(x)$, then $P(x) = (x - a)Q(x)$.

$$\begin{aligned} P(x) &= x^3 + x^2 - 10x + 8 \\ &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$

EXERCISE 6C Remainder and factor theorems

EXAMPLE 6C-1 Using the remainder theorem

Find the remainder when $x^3 + 4x^2 + x - 6$ is divided by: **a** $x - 2$ **b** $x + 3$

THINK

Name the polynomial.

a For divisor of $(x - 2)$, calculate $P(2)$ and hence write the remainder.

b For divisor of $(x + 3)$, calculate $P(-3)$ and hence write the remainder.

WRITE

Let $P(x) = x^3 + 4x^2 + x - 6$.

a $P(2) = 8 + 16 + 2 - 6 = 20$
Remainder is 20.

b $P(-3) = -27 + 36 - 3 - 6 = 0$
Remainder is 0.

1 Find the remainder when $x^3 - 5x^2 - 8x + 12$ is divided by:
a $x - 1$ **b** $x + 1$ **c** $x - 2$ **d** $x + 2$.

2 Find the remainder when $2x^3 + 3x^2 - 8x + 3$ is divided by:
a $x + 1$ **b** $x - 2$ **c** $x + 2$ **d** $x + 3$.

3 Find the remainder when $x^4 - 7x^3 + 5x^2 + 31x - 30$ is divided by:
a $x - 2$ **b** $x + 1$ **c** $x - 3$ **d** $x + 4$.

EXAMPLE 6C-2 Deciding if the divisor is a factor of the polynomial

Decide if each divisor is a factor of $x^3 + 2x^2 - x - 2$. **a** $x - 3$ **b** $x + 1$

THINK

Name the polynomial.

a For $(x - 3)$ to be a factor, the remainder should be zero. Check if $P(3) = 0$.

b For $(x + 1)$ to be a factor, the remainder should be zero. Check if $P(-1) = 0$.

WRITE

Let $P(x) = x^3 + 2x^2 - x - 2$.

a $P(3) = 27 + 18 - 3 - 2 = 40$
$(x - 3)$ is not a factor of $P(x)$.

b $P(-1) = -1 + 2 + 1 - 2 = 0$
$(x + 1)$ is a factor of $P(x)$.

4 Explain how you know if the divisor is a factor of the polynomial.

5 Decide if each divisor is a factor of $x^3 + 3x^2 - 6x - 8$.
a $x - 1$ **b** $x + 1$ **c** $x - 3$ **d** $x + 3$
e $x - 4$ **f** $x + 4$ **g** $x + 2$ **h** $x - 2$

6 Decide if each divisor is a factor of $x^3 - 4x^2 - 7x + 10$.
a $x - 2$ **b** $x - 3$ **c** $x - 1$ **d** $x + 3$
e $x + 1$ **f** $x + 2$ **g** $x - 5$ **h** $x - 4$

EXAMPLE 6C-3 Using the factor theorem to find a linear factor of a polynomial

Use the factor theorem to find a linear factor of the polynomial $P(x) = x^3 + 5x^2 - 2x - 24$.

THINK

1 Write the polynomial.

2 Look for a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, etc. That is, try x values that are factors of the constant term in $P(x)$.

3 Use the factor theorem to identify a linear factor of $P(x)$.

WRITE

$P(x) = x^3 + 5x^2 - 2x - 24$

$P(1) = 1 + 5 - 2 - 24 = -20 \neq 0$
$P(-1) = -1 + 5 + 2 - 24 = -18 \neq 0$
$P(2) = 8 + 20 - 4 - 24 = 0$

So $(x - 2)$ is a factor of $P(x)$.

7 Use the factor theorem to find a linear factor of each polynomial.

a $P(x) = x^3 + 8x^2 + 9x - 18$
b $P(x) = x^3 - x^2 - 14x + 24$
c $P(x) = x^3 - 4x^2 - 9x + 36$
d $P(x) = x^3 - 19x - 30$
e $P(x) = 3x^3 - 10x^2 + x + 6$
f $P(x) = 2x^3 - 5x^2 - 14x + 8$
g $P(x) = 8x^3 - 26x^2 + 17x + 6$
h $P(x) = 4x^3 + 4x^2 - 21x + 9$

8 Fully factorise each expression.

a $(x - 6)(x^2 + 7x + 12)$
b $(x + 4)(x^2 - 9x + 14)$
c $(x + 3)(x^2 - x - 30)$
d $(x - 2)(x^2 - 16)$
e $(x + 1)(x^2 + 4x + 4)$
f $(x - 7)(x^2 + 5x - 24)$

EXAMPLE 6C-4 Factorising a cubic polynomial

Factorise $x^3 - 7x^2 + 7x + 15$.

THINK

1 Name the polynomial.

2 Look for a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, … (factors of 15).

3 Identify a linear factor of $P(x)$.

4 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$. You will know you have performed the division correctly as the remainder should be zero.

5 Write $P(x)$ as the product of the divisor and quotient.

6 Factorise the quadratic factor.

WRITE

Let $P(x) = x^3 - 7x^2 + 7x + 15$.

$P(1) = 1 - 7 + 7 + 15 \neq 0$
$P(-1) = -1 - 7 - 7 + 15 = 0$

So $(x + 1)$ is a factor of $P(x)$.

$$
\begin{array}{r}
x^2 + 8x + 15 \\
x + 1 \overline{)\, x^3 - 7x^2 + 7x + 15} \\
\underline{x^3 + \ \ x^2} \\
-8x^2 + 7x + 15 \\
\underline{-8x^2 - 8x} \\
15x + 15 \\
\underline{15x + 15} \\
0
\end{array}
$$

$P(x) = (x + 1)(x^2 - 8x + 15)$
$= (x + 1)(x - 3)(x - 5)$

9 Factorise each polynomial.

a $x^3 + 6x^2 + 3x - 10$ **b** $x^3 + 3x^2 - 4x - 12$

c $x^3 + 4x^2 - 19x + 14$ **d** $x^3 + 5x^2 - 4x - 20$

e $x^3 - 5x^2 - 8x + 48$ **f** $x^3 - 2x^2 - 9x + 18$

g $2x^3 + x^2 - 8x - 4$ **h** $4x^3 - 9x^2 - 19x + 30$

i $6x^3 - 23x^2 - 6x + 8$ **j** $12x^3 - 17x^2 + 2x + 3$

10 Factorise each polynomial in question **7**.

PROBLEM SOLVING AND REASONING

11 What is the maximum number of factors $P(x)$ can have if $P(x)$ is:

a a quadratic polynomial? **b** cubic polynomial?

c quartic polynomial? **d** polynomial of degree 7?

e polynomial of degree 12? **f** polynomial of degree n?

12 **a** How many linear factors does the polynomial $P(x)$ have if $P(x) = (x + 2)(x - 3)(x + 4)$?

b Without expanding, what is the constant term of $P(x)$? Explain.

c To find linear factors using the factor theorem, you look for values of x that make $P(x) = 0$. What are these x values and how do they relate to the constant term?

13 Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

a Use the factor theorem to find a linear factor of $P(x)$.

b Use the factor theorem to find another linear factor of $P(x)$.

c Use the factor theorem to find a third linear factor of $P(x)$.

d Write $P(x)$ as a product of three factors.

e Check your answer to part **d** by expanding and simplifying the product.

NOTE Remember to try factors of −6 to identify the x value that makes $P(x) = 0$.

14 **a** Use your answers to question **5** to identify three factors of the polynomial $P(x)$.

b Write $P(x)$ as a product of three factors.

c Check your answer to part **b** by expanding and simplifying the product.

15 **a** Use your answers to question **6** to identify three factors of the polynomial $P(x)$.

b Write $P(x)$ as a product of three factors.

16 Using only the factor theorem, find three factors of $x^3 + 2x^2 - 11x - 12$ and hence write the polynomial as a product of three factors.

CHALLENGE

17 Explain why the strategy used in question **16** is not suitable to factorise each of these polynomials.

a $6x^3 + 13x^2 + 4x - 3$ **b** $x^3 + 5x^2 + 10x + 8$

18 Find a polynomial of degree 3 with leading coefficient of 1 that has a remainder of 5 when divided by $x - 2$.

19 Factorise each quartic polynomial.

a $x^4 - x^3 - 7x^2 + x + 6$

b $x^4 + 8x^3 + 17x^2 - 2x - 24$

c $2x^4 + 13x^3 + 21x^2 + 2x - 8$

Reflect

How does the factor theorem help you factorise a polynomial?

6D Solving polynomial equations

Start thinking!

1 Consider the quadratic polynomial equation $x^2 + 3x - 18 = 0$.

a What is the maximum number of solutions this equation can have?

b Check which of these x values is a solution to the equation.

i $x = -1$ **ii** $x = 1$ **iii** $x = -2$ **iv** $x = 3$

v $x = -4$ **vi** $x = 5$ **vii** $x = -6$ **viii** $x = 7$

c Without substitution, how could you identify if each x value is a *possible* solution? (Hint: look at the constant term on the left side of the equation.)

2 Consider the quadratic polynomial equation $x^2 + x - 6 = 0$.

a Without substitution, identify which of these x values is a *possible* solution to the equation? Explain your decision.

i $x = -1$ **ii** $x = 5$ **iii** $x = -3$ **iv** $x = 4$

v $x = -2$ **vi** $x = 6$ **vii** $x = 2$ **viii** $x = 8$

b From the set of possible x values identified in part **a**, check which ones are a solution.

3 Repeat question **2** for the polynomial equation $x^3 + x^2 - 4x - 4 = 0$.

4 **a** Factorise the left side of the equation in question **1**.

b Explain how you can use the Null Factor Law to solve the equation.

5 Solve the equation in question **2** using the Null Factor Law.

KEY IDEAS

- Null Factor Law: if $a \times b = 0$ then $a = 0$ or $b = 0$ or both a and b are 0.
- The polynomial equation $P(x) = 0$ can be solved by applying the Null Factor Law. $P(x)$ must be in factor form.
- If $P(x) = (x - a)(x - b)(x - c)$ then $P(x) = 0$ has the solution $x = a$, $x = b$ or $x = c$.
- To factorise $P(x)$ so it is a product of linear factors:
  1 apply the factor theorem to find a linear factor
  2 divide $P(x)$ by the linear factor to obtain the quotient
  3 factorise the quotient.
- If $P(x)$ is a cubic polynomial, the quotient will be a quadratic factor that may be factorised into two linear factors.
- If $P(x)$ is a quartic polynomial, the quotient will be a cubic factor. The strategy of using the factor theorem and long division will need to be repeated with the quotient.

EXERCISE 6D Solving polynomial equations

1 Factorise the left side of each quadratic equation.

a $x^2 + 5x + 4 = 0$ b $x^2 - 9x + 18 = 0$ c $x^2 - 2x - 8 = 0$

d $x^2 + 6x + 9 = 0$ e $x^2 - 25 = 0$ f $2x^2 - 14x = 0$

2 Solve each equation in question 1 using the Null Factor Law.

EXAMPLE 6D-1 Solving polynomial equations in factor form

Solve each equation. a $(x - 1)(x + 3)(x - 2) = 0$ b $(2x + 1)(x - 5)(x - 4)(x + 4) = 0$

THINK	WRITE
a 1 As the left side (LS) is in factor form and the right side (RS) is 0, apply the Null Factor Law.	a $(x - 1)(x + 3)(x - 2) = 0$ $x - 1 = 0$ or $x + 3 = 0$ or $x - 2 = 0$
2 Solve each linear equation.	$x = 1$ or $x = -3$ or $x = 2$
3 Write the solution.	$x = -3$, 1 or 2
b 1 Apply the Null Factor Law.	b $(2x + 1)(x - 5)(x - 4)(x + 4) = 0$ $2x + 1 = 0$ or $x - 5 = 0$ or $x - 4 = 0$ or $x + 4 = 0$
2 Solve each linear equation.	$x = -\frac{1}{2}$ or $x = 5$ or $x = 4$ or $x = -4$
3 Write the solution.	$x = -4, -\frac{1}{2}, 4$ or 5

UNDERSTANDING AND FLUENCY

3 Solve each equation.

a $(x + 2)(x + 5)(x - 4) = 0$ b $(x + 1)(x - 3)(x + 4) = 0$

c $(x - 6)(x - 2)(x + 3) = 0$ d $x(x + 2)(x - 9) = 0$

e $(2x - 1)(x + 1)(x - 1) = 0$ f $(3x - 2)(2x + 7)(x + 5) = 0$

4 Solve each equation.

a $(x + 3)(x - 4)(x + 7)(x - 1) = 0$ b $(x - 2)(x - 5)(x - 3)(x + 6) = 0$

c $(x + 4)(x + 1)(x + 2)^2 = 0$ d $x^2(x + 5)(x - 6) = 0$

e $5x(3x + 1)(x + 4)(4 - x) = 0$ f $(4x - 3)(2x + 5)(x^2 + 1) = 0$

5 Fully factorise the left side of each equation. Hence, solve each equation.

a $(x + 1)(x^2 + 8x + 12) = 0$ b $(x - 3)(x^2 - x - 20) = 0$

c $(x - 2)(x^2 - 6x + 8) = 0$ d $(x + 4)(x^2 + 6x - 7) = 0$

e $(x + 3)(x^2 - 16) = 0$ f $x(x - 1)(x^2 - 1) = 0$

g $(x - 5)(x^2 - 10x + 25) = 0$ h $(x + 2)(x^2 + 6x + 9) = 0$

i $(x - 4)(x^2 - x - 6) = 0$ j $(x + 1)(x^2 + 2x + 2) = 0$

k $3(x + 3)(5x^2 + 30x) = 0$ l $(x - 2)(2x^2 + 4x - 6) = 0$

EXAMPLE 6D-2 Solving a cubic polynomial equation

Solve $x^3 - 7x^2 + 7x + 15 = 0$.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 - 7x^2 + 7x + 15$.
2 Use the factor theorem to identify a linear factor of $P(x)$.	$P(1) = 1 - 7 + 7 + 15 \neq 0$ $P(-1) = -1 - 7 - 7 + 15 = 0$ So $(x + 1)$ is a factor of $P(x)$.
3 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$.	$x^2 - 8x + 15$ $x + 1 \overline{)\, x^3 - 7x^2 + 7x + 15}$ $x^3 + x^2$ $-8x^2 + 7x + 15$ $-8x^2 - 8x$ $15x + 15$ $15x + 15$ 0
4 Write $P(x)$ as the product of the divisor and quotient.	$P(x) = (x + 1)(x^2 - 8x + 15)$
5 Factorise the quadratic factor.	$= (x + 1)(x - 3)(x - 5)$
6 Solve $P(x) = 0$ using the Null Factor Law.	For $P(x) = 0$, $(x + 1)(x - 3)(x - 5) = 0$ $x = -1$, 3 or 5

UNDERSTANDING AND FLUENCY

6 Solve each equation.

a $x^3 + 6x^2 + 5x - 12 = 0$
b $x^3 - 5x^2 - 4x + 20 = 0$
c $x^3 + x^2 - 36x - 36 = 0$
d $x^3 + 10x^2 + 21x = 0$
e $x^3 - 7x - 6 = 0$
f $x^3 + 2x^2 + 5x + 10 = 0$
g $x^3 + 3x^2 - 9x - 27 = 0$
h $x^3 + 9x^2 + 24x + 16 = 0$
i $x^3 - 3x^2 - 3x - 4 = 0$
j $x^3 - 6x^2 + 12x - 8 = 0$

7 Solve each equation by first taking out a common factor.

a $2x^3 - 2x^2 - 20x - 16 = 0$
b $3x^3 - 15x^2 - 3x + 15 = 0$
c $5x^3 + 10x^2 - 20x - 40 = 0$
d $4x^3 + 24x^2 + 44x + 24 = 0$

8 Solve each equation by first taking out a negative common factor.

a $-x^3 - 2x^2 + 9x + 18 = 0$
b $-x^3 + 4x^2 + 17x - 60 = 0$
c $-2x^3 + 8x^2 + 2x - 8 = 0$
d $-3x^3 - 3x^2 + 24x + 36 = 0$

9 Solve each equation using the quadratic formula. Write the solutions as exact values.

a $x^3 + x^2 - 3x + 1 = 0$
b $x^3 + 6x^2 + 9x + 2 = 0$
c $x^3 + x^2 - 10x - 12 = 0$
d $x^3 + 4x^2 - 27x - 20 = 0$

UNDERSTANDING AND FLUENCY

10 Follow these steps to solve $x^4 + x^3 - 7x^2 - x + 6 = 0$.

- **a** Name the polynomial on the left side of the equation as $P(x)$ and identify one of its linear factors.
- **b** Use long division to find the quotient when $P(x)$ is divided by this linear factor. Write $P(x)$ as the product of the divisor and quotient, $Q(x)$.
- **c** Identify a linear factor of $Q(x)$ and use long division to find its quadratic factor.
- **d** Factorise the quadratic factor.
- **e** Write $P(x)$ as a product of four linear factors and hence solve $P(x) = 0$ using the Null Factor Law.

11 Solve each equation.

- **a** $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$
- **b** $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$
- **c** $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$
- **d** $x^4 - x^3 - 19x^2 - 11x + 30 = 0$
- **e** $x^4 - 5x^3 + 20x - 16 = 0$
- **f** $x^4 - 13x^2 + 36 = 0$

PROBLEM SOLVING AND REASONING

12 What is the maximum number of solutions to $P(x) = 0$ if $P(x)$ is:

- **a** a linear polynomial?
- **b** a quadratic polynomial?
- **c** a cubic polynomial?
- **d** a quartic polynomial?
- **e** a polynomial of degree 6?
- **f** a polynomial of degree n?

13 **a** Using only the factor theorem, find three linear factors of $P(x)$ where $P(x) = x^3 - 2x^2 - 5x + 6$.

- **b** Hence, solve $P(x) = 0$.

14 Explain why $x^3 + x^2 - 2x - 8 = 0$ has only one solution.

15 Explain why $x^3 - 5x^2 + 3x + 9 = 0$ has only two solutions.

16 The volume of a vanilla slice is 192 cm^3. Its length is twice the height and the width is 2 cm more than the height.

- **a** Write a polynomial to represent the volume of the slice.
- **b** Solve a polynomial equation to find the dimensions of the slice.

17 The volume of a Toblerone chocolate box is 450 cm^3. The height of its triangular face is 1 cm less than the base and its perpendicular length is five times the size of the base. Find its dimensions.

CHALLENGE

18 $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ has the quadratic factor $(x^2 - x - 2)$. Factorise $P(x)$ and hence solve $P(x) = 0$.

19 Solve each equation.

- **a** $x^5 - x^4 - 17x^3 - 19x^2 + 16x + 20 = 0$
- **b** $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48 = 0$

Reflect

In what form does a polynomial equation need to be before the Null Factor Law can be used?

6E Graphs of polynomial relationships

Start thinking!

A polynomial relationship can be shown as a graph. In chapters 4 and 5, you worked with linear and quadratic relationships. Here you will look at cubic and quartic relationships.

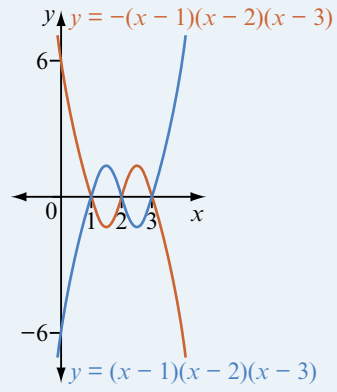
1 The basic cubic graph has the rule $y = x^3$.
- **a** Draw the graph using a table of values (for $-3 \leq x \leq 3$) or digital technology.
- **b** Identify the x- and y-intercepts.
- **c** An important feature on this graph is the **point of inflection** at (0, 0). Mark this on your graph.

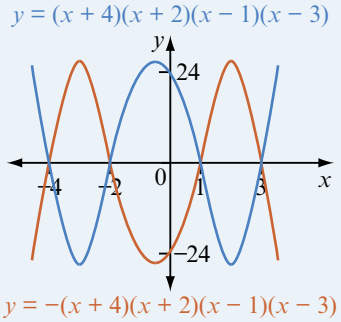
2 The basic quartic graph has the rule $y = x^4$.
- **a** Draw the graph using a table of values (for $-3 \leq x \leq 3$) or digital technology.
- **b** Identify the x- and y-intercepts.
- **c** This graph looks similar to $y = x^2$. How is it different? Draw the graph of $y = x^2$ on the same Cartesian plane to illustrate your answer.

3 Sketch the graph of each polynomial relationship. To give a sense of scale, you can show the coordinates of a point on the graph. Label the point where $x = 2$.
- **a** $y = -x^3$ **b** $y = -x^4$

KEY IDEAS

- To sketch a polynomial relationship:
  1. write the polynomial in factor form
  2. identify the x-intercepts (find x when $y = 0$)
  3. identify the y-intercept (find y when $x = 0$)
  4. draw a smooth curve through the known points
  5. if necessary, find the coordinates of another point to confirm the orientation of the graph.
- Graphs of cubic relationships
  - The graph of $y = (x - a)(x - b)(x - c)$ has x-intercepts a, b and c, and y-intercept $-abc$. The curve starts from the bottom left of the Cartesian plane.
  - The graph of $y = -(x - a)(x - b)(x - c)$ is the reflection in the x-axis of $y = (x - a)(x - b)(x - c)$.
- Graphs of quartic relationships
  - The graph of $y = (x - a)(x - b)(x - c)(x - d)$ has x-intercepts a, b, c and d, and y-intercept $abcd$. The curve starts from the top left of the Cartesian plane.
  - The graph of $y = -(x - a)(x - b)(x - c)(x - d)$ is the reflection in the x-axis of $y = (x - a)(x - b)(x - c)(x - d)$.





EXERCISE 6E Graphs of polynomial relationships

EXAMPLE 6E-1 Sketching a cubic relationship using intercepts

Sketch the graph of $y = (x + 2)(x + 4)(x - 3)$.

THINK

1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

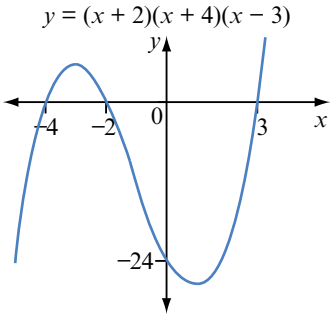
2 Substitute $x = 0$ to find the y-intercept.

3 Mark the four intercepts on a Cartesian plane and draw a smooth curve through them. Curve starts from bottom left of Cartesian plane. Label with the rule.

WRITE

$y = (x + 2)(x + 4)(x - 3)$
When $y = 0$,
$(x + 2)(x + 4)(x - 3) = 0$
$x + 2 = 0$ or $x + 4 = 0$ or $x - 3 = 0$
$x = -2, -4$ or 3
x-intercepts are -4, -2 and 3.

When $x = 0$, $y = (2)(4)(-3) = -24$
y-intercept is -24.



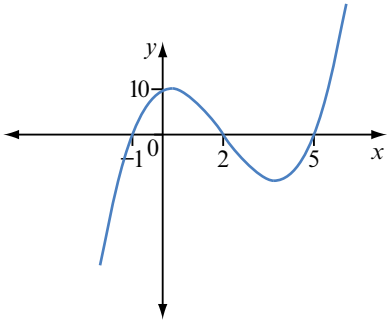
UNDERSTANDING AND FLUENCY

1 Sketch the graph of each cubic relationship.

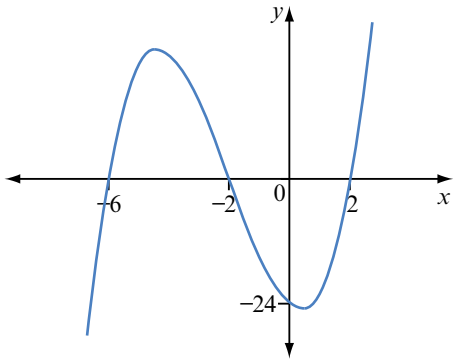
a $y = (x + 1)(x + 3)(x - 1)$
b $y = (x - 2)(x - 5)(x + 1)$
c $y = (x - 4)(x + 2)(x + 3)$
d $y = (x - 3)(x - 1)(x - 5)$
e $y = x(x + 3)(x - 3)$
f $y = 2x(x + 4)(x - 1)$

2 For each cubic relationship:
i identify the x-intercepts
ii identify the y-intercept
iii write the rule.

a



b



3 **a** Find the x- and y-intercepts for:

i $y = (x + 7)(x - 2)(x - 3)$ **ii** $y = -(x + 7)(x - 2)(x - 3)$

b How is the graph of $y = -(x + 7)(x - 2)(x - 3)$ different from the graph of $y = (x + 7)(x - 2)(x - 3)$?

4 Sketch the graph of each cubic relationship.

a $y = -(x + 5)(x - 1)(x - 3)$ **b** $y = -(x - 6)(x + 1)(x - 2)$

c $y = -(x + 2)(x + 3)(x - 5)$ **d** $y = -x(x + 7)(x - 4)$

EXAMPLE 6E-2 Sketching a quartic relationship using intercepts

Sketch the graph of $y = (x - 2)(x + 3)(x + 1)(x - 4)$.

THINK

1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

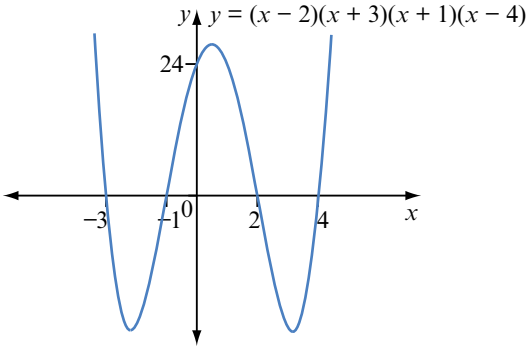
2 Substitute $x = 0$ to find the y-intercept.

3 Mark the five intercepts on a Cartesian plane and draw a smooth curve through them. Curve starts from top left of Cartesian plane. Label with the rule.

WRITE

$y = (x - 2)(x + 3)(x + 1)(x - 4)$
When $y = 0$,
$(x - 2)(x + 3)(x + 1)(x - 4) = 0$
$x = 2$ or $x = -3$ or $x = -1$ or $x = 4$
x-intercepts are -3, -1, 2 and 4.

When $x = 0$, $y = (-2)(3)(1)(-4) = 24$
y-intercept is 24.



5 Sketch the graph of each quartic relationship.

a $y = (x - 1)(x + 2)(x + 3)(x - 3)$ **b** $y = (x + 4)(x + 1)(x - 1)(x + 3)$

c $y = x(x - 2)(x - 6)(x + 5)$ **d** $y = -(x - 3)(x + 3)(x + 1)(x - 1)$

e $y = (2x - 3)(x + 2)(x + 1)(x - 4)$ **f** $y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 Sketch the graph of each polynomial relationship. (Hint: you will need to first factorise the polynomial.)

a $y = x^3 - 3x^2 - 13x + 15$ **b** $y = -x^3 - 2x^2 + 16x + 32$

c $y = x^3 - 2x^2 - 3x$ **d** $y = -2x^3 + 8x^2 - 2x - 12$

e $y = x^4 - 15x^2 - 10x + 24$ **f** $y = x^4 - 9x^3 + 6x^2 + 56x$

g $y = -x^4 + 10x^2 - 9$ **h** $y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

7 Use digital technology to check your graphs in question **6**.

PROBLEM SOLVING AND REASONING

8 Consider the relationship $y = (x + 2)(x - 3)^2$.

- **a** What type of polynomial relationship is this?
- **b** Find the x- and y-intercepts.
- **c** As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?
- **d** Sketch the graph. Use digital technology to verify your answer.
- **e** What effect does the repeated factor of $(x - 3)$ have on the graph?

9 Sketch the graph of each cubic relationship.

- **a** $y = (x + 4)(x - 1)^2$
- **b** $y = (x + 2)^2(x - 5)$
- **c** $y = -(x - 2)(x - 7)(x - 2)$
- **d** $y = -(2x + 1)(x - 4)^2$

10 Sketch the graph of each quartic relationship.

- **a** $y = (x - 3)(x + 2)(x + 1)^2$
- **b** $y = -(x - 2)^2(x - 4)(x + 2)$
- **c** $y = -x(x - 2)(x + 3)^2$
- **d** $y = x^2(x + 5)(x - 3)$
- **e** $y = (x - 1)^2(x + 2)^2$
- **f** $y = -x^2(x - 4)^2$

11 The effect of having a repeated factor in a polynomial relationship means that an x-intercept is also a turning point. Investigate the effect of having a cubed repeated factor in a quartic relationship. That is, find what happens at $x = a$ for the graph of the form $y = (x - a)^3(x - b)$. Use digital technology and try different values for a and b.

12 A water ride can be modelled by the polynomial relationship $h = -\frac{1}{5}(t^3 - 11t^2 + 39t - 45)$, where h is the height above the ground in metres and t is the time in seconds from the start of the ride.

- **a** Sketch a graph of this relationship.
- **b** At what height above the ground does a person start the ride?
- **c** How high is a person after 1 second?
- **d** The ride descends to its lowest point. How long does this take?
- **e** How long does it take for the ride to ascend and then descend again to its lowest point?

13 Amelia monitors the change in value of shares over the month of June. She finds that the dollar change in value, y, after x days can be approximated by a cubic relationship, where y increases from \$0 to \$21 after 5 days and is zero again after 12 days and 20 days.

- **a** Sketch a graph of this cubic relationship.
- **b** Find the rule for this relationship.
- **c** Use this relationship to estimate the change in value at the end of June.

CHALLENGE

14 Sketch the graph of each polynomial relationship, clearly labelling all intercepts.

- **a** $y = (x - 1)(x + 2)(x + 1)(x - 4)(x + 5)$
- **b** $y = -x(x - 6)(x + 3)(x - 2)^2$

15 Use digital technology to produce the graphs in question **14** and locate and identify the coordinates of the turning points.

Reflect

What are the key features used to sketch a polynomial relationship?

10A

6F Polynomials and transformations

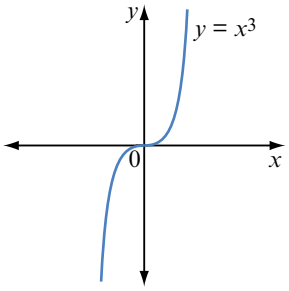
Start thinking!

Graphs of polynomial relationships can also be transformed by performing dilation, reflection, translation or a combination of these.

1 Consider the graph of $y = x^3$. For each of the transformations listed below:

i describe how the graph of $y = x^3$ will be changed
ii sketch the new graph that is produced
iii write the rule for the new graph.

a dilate by a factor of 2
b reflect in the x-axis
c translate 3 units up
d translate 2 units right
e translate 1 unit left and 4 units down



2 Repeat question **1** for the graph of $y = x^4$.

KEY IDEAS

- Transformations such as dilation, reflection and translation can be performed on the graph of $y = P(x)$.
  - Dilation by a factor of a produces $y = aP(x)$.
  - Reflection in the x-axis produces $y = -P(x)$.
  - Vertical translation of k units produces $y = P(x) + k$.
  - Horizontal translation of h units produces $y = P(x - h)$.

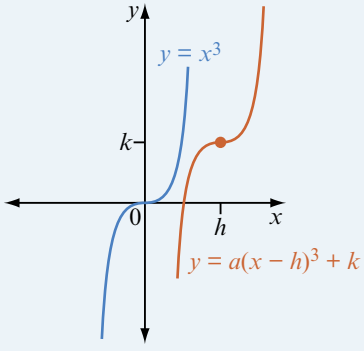
- A combination of transformations can be performed on the graph of $y = P(x)$ to produce the graph of $y = aP(x - h) + k$.

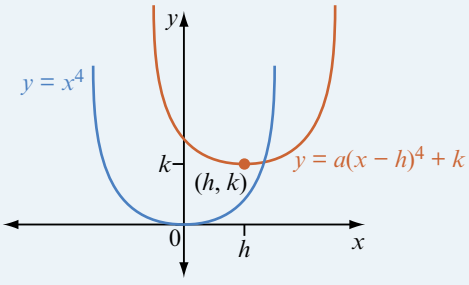
$y = aP(x - h) + k$

vertical translation of k units
For $k > 0$, move up.
For $k < 0$, move down.

dilation by factor of a
For $a < 0$, reflection in x-axis.

horizontal translation of h units
For $h > 0$, move right.
For $h < 0$, move left.



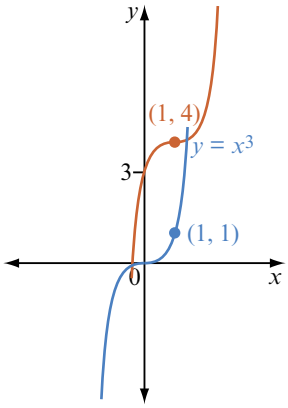


EXERCISE 6F Polynomials and transformations

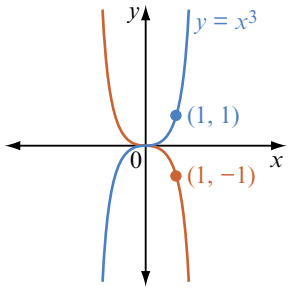
UNDERSTANDING AND FLUENCY

1 Identify the transformations performed on the graph of $y = x^3$ to produce each graph shown in orange.

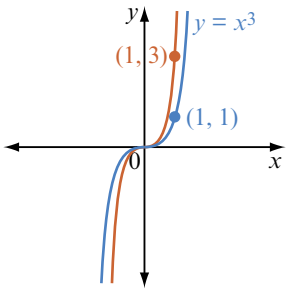
a



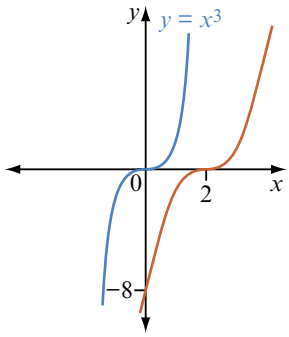
b



c



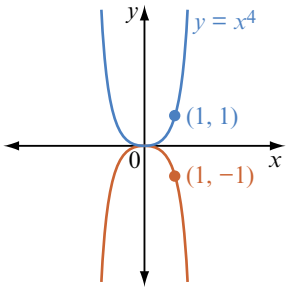
d



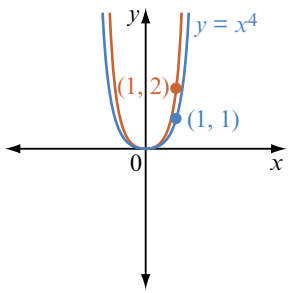
2 Write the rule for each graph in question **1**.

3 Identify the transformations performed on the graph of $y = x^4$ to produce each graph shown in orange.

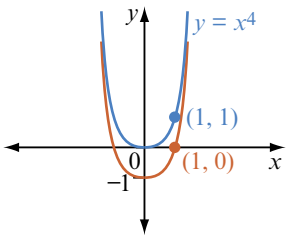
a



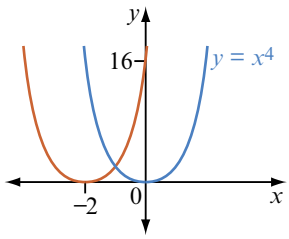
b



c



d



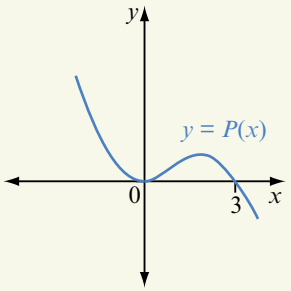
4 Write the rule for each graph in question **3**.

EXAMPLE 6F-1

Performing transformations on the graph of a polynomial relationship

Perform a transformation on the graph of $y = P(x)$ to produce the graph of:

a $y = P(x) + 2$
b $y = P(x + 2)$
c $y = -P(x)$.



THINK

a **1** Identify the transformation.

2 No dilation or reflection so the shape remains the same. Move the original graph 2 units up. (0, 0) moves to (0, 2) and (3, 0) moves to (3, 2). Draw $y = P(x)$ and $y = P(x) + 2$ on the same Cartesian plane.

b **1** Identify the transformation.

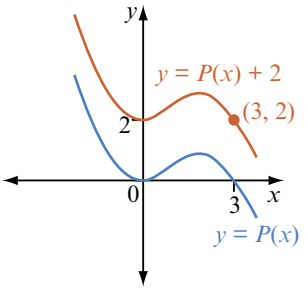
2 No dilation or reflection so the shape remains the same. Move the original graph 2 units left. (0, 0) moves to (−2, 0) and (3, 0) moves to (1, 0). Draw $y = P(x)$ and $y = P(x + 2)$ on the same Cartesian plane.

c **1** Identify the transformation.

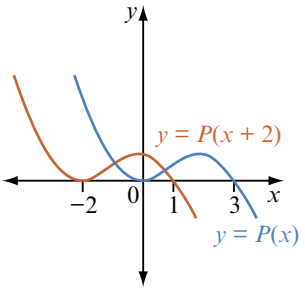
2 No dilation or translation. Draw $y = P(x)$ and $y = -P(x)$ on the same Cartesian plane.

WRITE

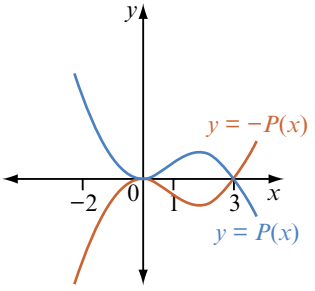
a Graph of $y = P(x)$ to be translated 2 units up.



b Graph of $y = P(x)$ to be translated 2 units left.

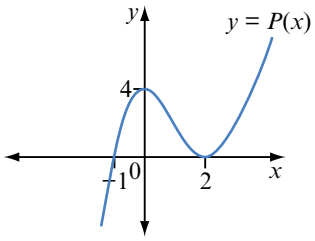


c Graph of $y = P(x)$ to be reflected in the x-axis.



UNDERSTANDING AND FLUENCY

5 Perform a transformation on the graph of $y = P(x)$ to produce the graph of:

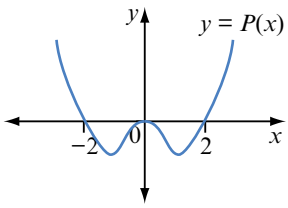


a $y = P(x) + 1$ **b** $y = P(x) - 2$

c $y = P(x - 1)$ **d** $y = P(x + 3)$

e $y = -P(x)$ **f** $y = 2P(x)$

6 Perform a transformation on the graph of $y = P(x)$ to produce the graph of:



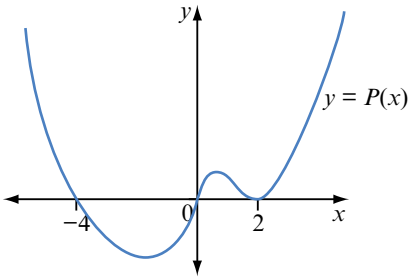
a $y = P(x) + 3$ **b** $y = P(x) - 1$

c $y = P(x - 3)$ **d** $y = P(x + 4)$

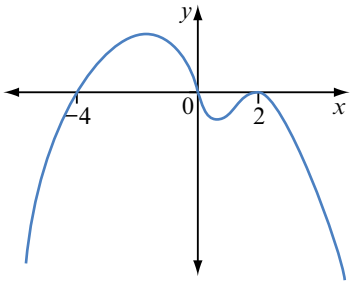
e $y = -P(x)$ **f** $y = 2P(x)$

PROBLEM SOLVING AND REASONING

7 Identify the rule for each graph in terms of $P(x)$ using the graph of $y = P(x)$ shown at right.



a



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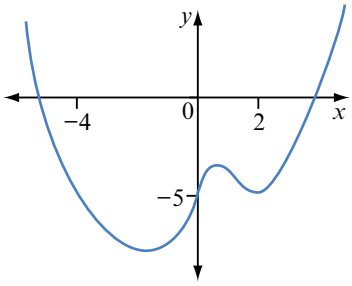
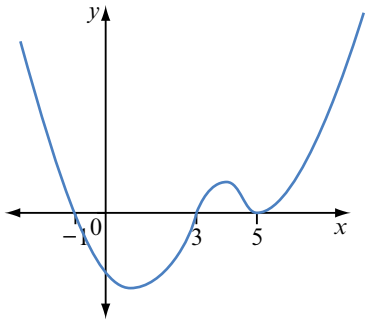


c



8 Compare the graph produced after reflecting $y = x^3$ in the x-axis with the graph produced after reflecting $y = x^3$ in the y-axis. What is the rule for each transformed graph?

CHALLENGE

9 For each cubic relationship:

i describe the transformations to be performed on $y = x^3$ to produce the graph of the relationship

ii identify the coordinates of the point of inflection

iii find the x- and y-intercepts

iv sketch the graph.

a $y = \frac{1}{2}(x - 3)^3 + 4$ **b** $y = -2(x + 1)^3 - 2$

10 Use digital technology to verify your answers to question **9**.

Reflect

How are transformations useful when sketching a polynomial relationship?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

polynomial	dividend	factor theorem	turning point
leading term	divisor	polynomial relationship	transformations
leading coefficient	quotient	cubic relationship	dilation
constant	remainder	point of inflection	reflection
degree of polynomial	remainder theorem	quartic relationship	translation

MULTIPLE-CHOICE

6A 10A **1** Which expression is a polynomial?

A $x^2 + \sqrt{x}$ **B** $x^3 - 2x$

C $\dfrac{3x}{x^2 + 1}$ **D** $x^2 + x^{-1}$

6A 10A **2** The degree of the polynomial $4x^2 - 3x^4 + x^3 - 2x$ is:

A 1 **B** 2 **C** 3 **D** 4

6A 10A **3** The coefficient of the leading term in the expression $3 - 2x^2 + 5x^3 + 7x$ is:

A 3 **B** -2 **C** 5 **D** 7

6B 10A **4** When $x^2 + 6x - 3$ is divided by $x - 2$, the remainder is:

A 13 **B** -19 **C** 5 **D** -11

6B 10A **5** Which of these is correct?

A $(5x^2 - x + 2) \div (x - 4) = 5x + 19$ remainder -74

B $(3x^2 - 2x + 1) \div (x - 1) = 3x - 5$ remainder -9

C $(4x^2 - x + 8) \div (x - 2) = 4x - 9$ remainder -10

D $(2x^2 - x + 1) \div (x - 5) = 2x + 9$ remainder 46

6C 10A **6** When $P(x) = x^3 - 2x^2$ is divided by $(x - 1)$, the remainder is:

A $P(-1)$ **B** $P(1)$

C $P(-2)$ **D** $P(2)$

6C 10A **7** Which expression is a factor of $x^3 - 4x^2 + x + 6$?

A $x - 1$ **B** $x - 2$

C $x + 3$ **D** $x + 6$

6D 10A **8** The leading term in a polynomial is raised to the power n. The maximum number of solutions it could have is:

A n **B** $n + 1$

C $n - 1$ **D** impossible to tell

6D 10A **9** How many different solutions does $(x + 1)(x - 1)(x + 2)(x + 1) = 0$ have?

A 1 **B** 2 **C** 3 **D** 4

6E 10A **10** The graph of $y = (x - 3)(x + 2)(x - 5)$ has a y-intercept of:

A 3 **B** -2 **C** 5 **D** 30

6E 10A **11** The graph of $y = (x - 1)^3$ has a point of inflection at:

A $x = 1$ **B** $x = -1$

C $x = 0$ **D** $y = 1$

6F 10A **12** The transformation performed on the graph of $y = x^3$ to produce the graph of $y = (x + 5)^3$ is a translation of:

A 5 units right **B** 5 units left

C 5 units up **D** 5 units down

SHORT ANSWER

6A 10A **1** Decide if each expression is a polynomial. For each polynomial, give its name as linear, quadratic, cubic or quartic.

a $5 - 3x^2 + 4x - 6x^3$ b $1 - 3x$

c $\frac{4x}{5}$ d $\frac{6}{7x}$

e $6x^2 - 5x + 12$ f $\frac{1}{2}x - 3x^4$

6A 10A **2** For the polynomial $3x^4 + 2x^6 - 4x^5 + x - 8x^3 - 7x^2 - 5x^8$, identify:

a the number of terms

b the degree of the polynomial

c the constant term

d the leading term

e the leading coefficient

f the coefficient of the x^2 term

6B 10A **3** Use long division to find the quotient and remainder for each problem.

a $(x^2 + 5x - 2) \div (x - 2)$

b $(3x^2 - x + 4) \div (x + 1)$

c $(x^2 - x + 8) \div (x - 4)$

d $(x^3 + 2x^2 - x + 3) \div (x^2 + 1)$

6B 10A **4** Expand and simplify the right side to verify whether each statement is true.

a $2x^2 + 5x - 1 = (2x + 3)(x + 1) - 4$

b $x^2 - 3x + 5 = (x - 1)(x - 2) + 7$

6C 10A **5** Use the factor theorem to find a linear factor of each polynomial $P(x)$.

a $P(x) = x^3 - x^2 - 5x - 3$

b $P(x) = x^3 + 4x^2 + 7x + 12$

c $P(x) = x^3 + 3x^2 - 6x + 2$

d $P(x) = x^3 + x^2 + 4$

6C 10A **6** Fully factorise each polynomial.

a $x^3 - 7x^2 + 14x - 8$

b $x^3 - 7x - 6$

6C 10A **7** Two factors of the polynomial $x^3 - 39x - 70$ are $(x - 7)$ and $(x + 5)$. What is the third factor?

6D 10A **8** Solve each equation.

a $(x - 5)(x + 2)(x - 1)(x + 1) = 0$

b $x(x + 3)(x - 3)(x - 2) = 0$

c $(x + 3)^2(x - 5)^2 = 0$

d $x^2(x + 4)(x - 4) = 0$

6D 10A **9** Fully factorise the left side of each equation, then solve.

a $x(x - 2)(x^2 - 4) = 0$

b $(x - 4)(x^2 - 8x + 16) = 0$

6E 10A **10** Sketch the graphs of:

a $y = (x - 3)(x + 1)(x - 2)$

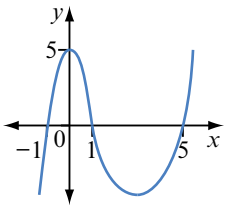
b $y = -(x + 2)^2(x - 4)$.

6E 10A **11** For the cubic relationship shown,

a identify the x-intercepts

b identify the y-intercept

c write the rule as a product of factors.



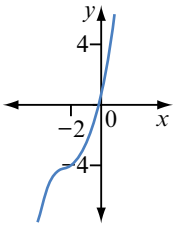
6E 10A **12** Consider $y = x^4 - 5x^2 + 4$.

a Write the relationship in terms of its factors.

b Identify the x- and y-intercepts

c Sketch the graph.

6F 10A **13** Describe the transformation/s performed on the graph of $y = x^3$ to produce the graph shown.



6F 10A **14** Consider the graph of $y = x(x - 1)^3$.

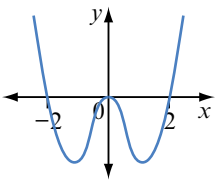
a Identify the x- and y-intercepts.

b Sketch the graph.

6F 10A **15** Write the rule for this polynomial relationship in:

a factor form

b expanded form.



MIXED PRACTICE

10A **1** If $P(x) = 4x^3 - x^2 + 2x - 1$, the value of $P(-1)$ is:

A -8 **B** 2 **C** 0 **D** 4

10A **2** If $P(x) = 2x^3 - 3x^2 + 5x - 4$, calculate values for:

a $P(2)$ **b** $P(-1)$ **c** $P(-2)$

d $P(-3)$ **e** $P(4)$ **f** $P(0)$

10A **3** The solution to $x^3 - 19x + 30 = 0$ is:

A $x = -5, -2$ or 3 **B** $x = -5$, 2 or 3

C $x = 5, -2$ or 3 **D** $x = 5$, 2 or -3

10A **4** If $P(x) = 3x^3 - 4x^2 + x$ and $Q(x) = -x^3 + 2x^2 - 4$, evaluate:

a $P(-1)$ **b** $Q(-1)$

c $P(x) + Q(x)$ **d** $Q(x) - P(x)$

e $P(0) - Q(0)$ **f** $P(1) - 4Q(1)$

10A **5** Use long division to find the quotient and remainder for each problem.

a $(x^3 - 4x^2 + 5x - 8) \div (x - 2)$

b $(x^3 - 2x^2 - 3x + 4) \div (x - 1)$

c $(x^3 + 3x^2 + 8x - 5) \div (x - 3)$

10A **6** Expand and simplify each product.

a $3x^2(4x - 8)$ **b** $(x^3 - 5x)^2$

c $(x^4 - 3x^2)(5x^2 + 2x^4)$ **d** $(x^2 - 1)^2(x + 1)$

10A **7** The graph of the polynomial $y = (x + 3)^2(x - 3)^2$ has how many solutions?

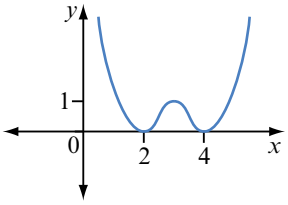
A 1 **B** 2

C 3 **D** 4

10A **8** Write the rule for this graph in:

a factor form

b expanded form.



10A **9** Decide if $(x + 2)$ is a factor of:

a $x^3 - 3x^2 - 6x + 8$

b $x^3 - 4x^2 - 3x + 18$

c $x^3 + 4x^2 - 3x - 18$

10A **10** The expansion of $(1 - x)^3$ is:

A $3x^2 - x^3 + 1 - 3x$

B $1 - 2x + 2x^2 + x^3$

C $1 - x^3$

D $3 - 3x - x^3$

10A **11** The graph of $y = -(x + 5)(x - 3)(x - k)$ has a y-intercept of 30. What is the value of k?

10A **12** The division of the polynomial $4x^3 - 2x^2 - x + 10$ by $(x - 2)$ gives a quotient and remainder of:

A $4x^2 - 4x + 7$ remainder -4

B $4x^2 + 6x - 13$ remainder -16

C $4x^2 + 6x + 11$ remainder 32

D $4x^2 + 4x - 7$ remainder 4

10A **13** One linear factor of $x^3 - 7x + 6$ is:

A $x + 3$ **B** $x + 2$

C $x + 1$ **D** $x + 4$

10A **14** The polynomial $x^3 - 2x^2 - 5x + k$ is divisible by $(x - 1)$, $(x + 2)$ and $(x - 3)$. What is the value of the constant k?

A -1 **B** 2 **C** -3 **D** 6

10A **15** One factor of the polynomial $x^3 - 43x - 42$ is $(x + 1)$. What are the other factors?

10A **16** Consider $y = (x + 1)^2(x - 3)^2$.

a What type of polynomial relationship is this?

b Find the x- and y-intercepts.

c As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?

d What effect does the repeated factors of $(x + 1)$ and $(x - 3)$ have on the graph?

10A **17** Give all the x-intercepts of the graph of the polynomial $P(x) = x(x + 1)(x - 4)(x^2 - 4)$.

10A **18** Fully factorise the polynomial $x^4 - 2x^2 + 1$.

10A **19** Find the solution to the equation $-3x^3 + 9x^2 + 30x - 72 = 0$.

ANALYSIS

The infinity symbol

You are familiar with the infinity symbol – it looks like the number 8 on its side. Its shape is like that of a cubic relationship with three x-intercepts.

a It is possible to model the infinity sign using two cubic relationships graphed for the same set of x values.

i Complete this table for $y = x^3 - x$.

x	−1	−0.6	−0.5	0	0.5	0.6	1
y							

ii Repeat part **i** for $y = -x^3 + x$.

iii On a Cartesian plane, plot the points in the tables from parts **i** and **ii**. Join the points with a smooth curve.

iv Describe the set of x values used for these two graphs.

v Do the turning points for the two relationships occur at $x = -0.5$ and 0.5? Explain.

vi Use digital technology to draw the graphs of these two relationships for the set of x values described in part **iv**. Find the coordinates of the turning points.

b It is also possible to model the infinity symbol by tracing the path a point takes as it moves from an angle of 0° with the x-axis to an angle of 360° with the x-axis.

The coordinates of all points (x, y) on the infinity symbol can be represented by the values $(\cos\theta, \frac{\sin 2\theta}{2})$, where θ is the angle the line joining the point with the origin makes with the x-axis.

i Construct a table for x and y using angles of θ every 15° from 0° to 360°. (This number of points is necessary to get an accurate representation of the graph.)

Angle (θ)	x ($\cos\theta$)	y ($\frac{\sin 2\theta}{2}$)
0°		
15°		
30°		
...		
345°		
360°		

ii Complete the table, giving your answers correct to two decimal places, if necessary. You may find some of the trigonometric values negative as the angle increases beyond 90°. You will understand the reason for this when you study the trigonometric ratios for angles greater than 90° in Chapter 8.

iii Plot the points (x, y) on a Cartesian plane. Join them with a smooth curve. (Alternatively, you could use a spreadsheet to plot the graph.)

iv Describe the shape of the graph.

v Describe the set of x and y values used for the graph.

c Compare the two models. Do you consider one to be a better model of the infinity symbol than the other? Explain.

CONNECT

Modelling a roller coaster ride

Engineers use polynomials to model roller coaster rides. Relationships can be formed for the height of the ride after a given time. For example, one relationship where h is the height in metres of a roller coaster after t seconds is

$h = -0.5t^6 + 5t^5 - 13.75t^4 - 5t^3 + 63.5t^2 - 59t + 20.$

You can use digital technology to produce its graph.

In this task, you will be looking at a few simpler relationships to model sections of roller coaster rides.

Your task

You are to complete the following three problems. Include all necessary graphs and working to justify your answers.

Problem 1

There are three rollercoaster rides at a fun park. Each can be represented by a polynomial relationship with height h in metres after t seconds.

- Ride of terror: $h = -0.1t^3 + 1.8t^2 - 9.6t + 16$
- Fear factor: $h = 0.3t^3 - 5t^2 + 21t$
- Ride of your life: $h = -2t^4 + 21t^3 - 61t^2 + 36t + 36$

Describe each ride to your friend who will be visiting the fun park the next day. Include information such as the initial height of the ride, times when the ride skims the ground or goes through an underground tunnel and the realistic duration of the ride. Use a sketch graph to help you.

Your friend is nervous of extreme heights. Use digital technology to find the maximum height of each ride.

Problem 2

A new roller coaster ride is to be designed so that, after completing a loop, it moves up from ground level, skims the ground again after a further 3 seconds and finishes on the ground after 7 seconds. It needs to have two thrilling 'up and down' sections. Determine a suitable polynomial relationship to model the conditions of this ride after completing the loop. Explain your reasoning.

Problem 3

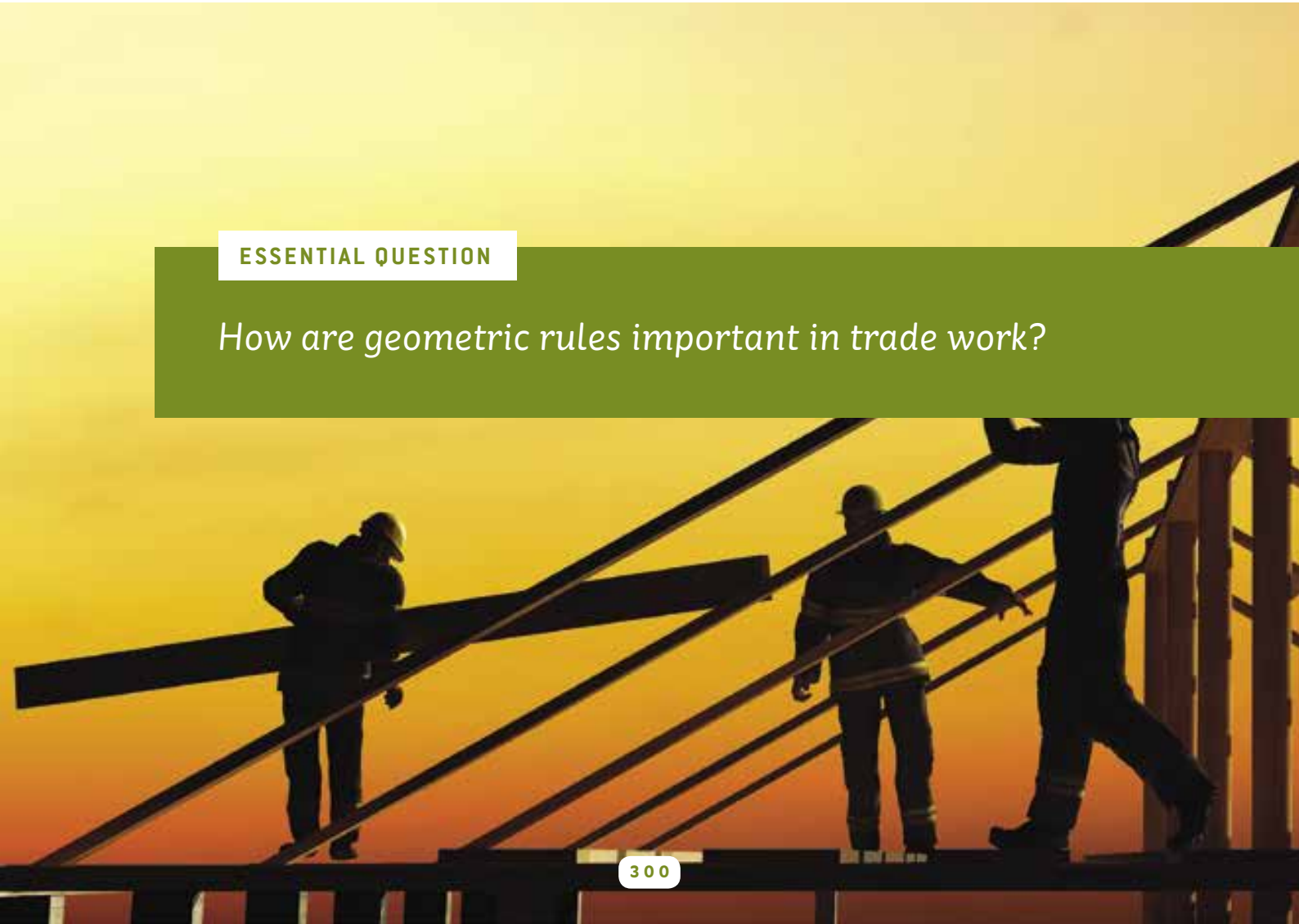
Design your own roller coaster ride. Use a polynomial of degree 4 or higher and fully explain how you modelled the relationship. Describe this ride to your friend.

As an extension, choose another scenario that could be modelled by a polynomial relationship of degree 3 or higher. Instead of using time as the independent variable, you may like to use a length or distance variable. Include all reasoning, working and diagrams.

You may like to present your findings as a report. Your report could include:

- a poster showing diagrams and calculations
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).

7 GEOMETRY



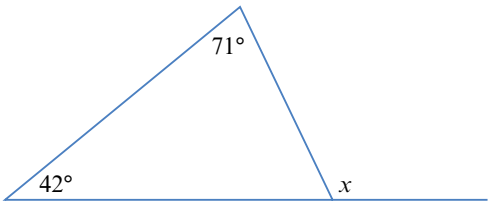
ESSENTIAL QUESTION

How are geometric rules important in trade work?

Are you ready?

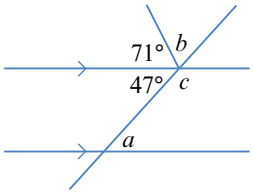
7A **1** What is the internal angle sum of a triangle?

7A **2** What is the value of x in this diagram?



A 71° **B** 29°
C 67° **D** 113°

7A **3** Consider this diagram.



a What is the value of a?
b What is the value of b?

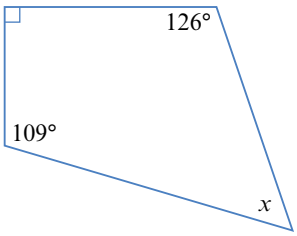
A 47° **B** 71°
C 62° **D** 118°

c What is the value of c?

7A **4** Which statement is false?

A A rhombus is a square.
B A rectangle is a parallelogram.
C A square is a rectangle.
D A rhombus is a parallelogram.

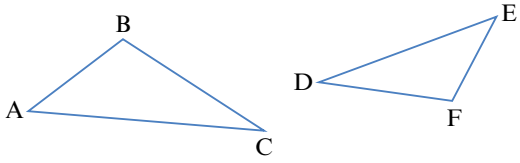
7A **5** What is the value of x in this diagram?



7B **6** Which of these is *not* a condition for congruence in triangles?

A SAS **B** AAA
C SSS **D** AAS

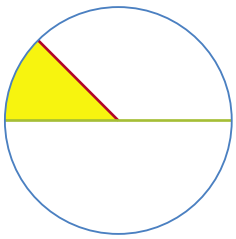
7C **7** Which option would allow you to calculate the scale factor for these similar figures?



A $\frac{AB}{DE}$ **B** $\frac{AC}{ED}$
C $\frac{BC}{DE}$ **D** $\frac{AB}{DF}$

7G 10A **8** **a** If $4x + 10 = 24$, find the value of x.
b If $5x - 2 = 3x + 2$, find the value of x.

7G 10A **9** For this diagram of a circle, name the part shown:

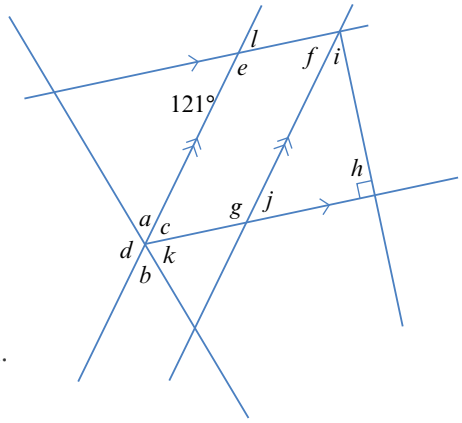


a in red
b in blue
c in green
d in yellow.

7A Geometry review

Start thinking!

Consider this diagram.



1 What is the relationship between angles:

a a and b? **b** a and d?

c f and i? **d** c and g?

e f and j? **f** c and l?

2 Make a glossary list of all angle relationships that you can see within the diagram. Be sure to include angles around a point, angles and parallel lines, angles within a triangle and a **quadrilateral**.

3 What type of angle is angle e? How is this different from angle k?

4 Make a glossary list of all the types of angles.

5 What is the angle sum of angles h, i and j? How do you know?

6 What type of triangle is this?

7 Make a list of the six different types of triangle and their attributes.

8 How is angle g related to the triangle from question **6**?

9 What is the angle sum of angles c, e, f and g? How do you know?

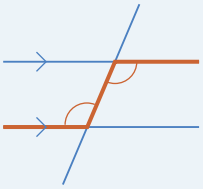
10 What type of quadrilateral is this?

11 Make a list of the six special types of quadrilateral and their attributes.

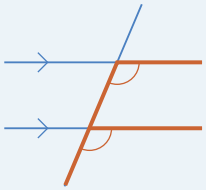
12 Explain why, if you knew the value of angles a and e, you could find the value of every other angle in the diagram, including those not labelled.

KEY IDEAS

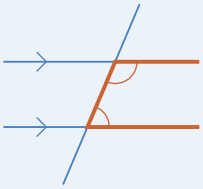
- **Complementary angles** add to 90°.
- **Supplementary angles** add to 180°.
- Angles around a point add to 360°.
- **Vertically opposite angles** are equal.
- When parallel lines are crossed by a **transversal**, a number of angles are formed, as shown at right.
- A **polygon** is a closed shape with straight sides.
- A **triangle** has an angle sum of 180°.
- A quadrilateral has an angle sum of 360°.



alternate angles
(equal)



corresponding angles
(equal)

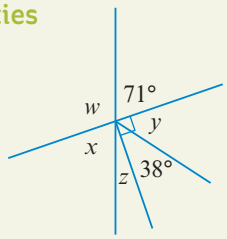


co-interior angles
(supplementary)

EXERCISE 7A Geometry review

EXAMPLE 7A-1 Finding angle size using angle properties

Find the size of the labelled unknown angles in this diagram.

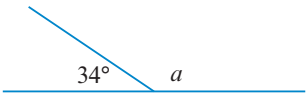


THINK	WRITE
1 Angle w is supplementary to 71°.	$w = 180° - 71° = 109°$
2 Angle x is vertically opposite 71°.	$x = 71°$
3 Angle y is complementary to 38°.	$y = 90° - 38° = 52°$
4 Angle z is supplementary to angles 71°, y and 38°.	$z = 180° - 71° - 52° - 38° = 19°$

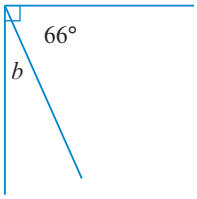
UNDERSTANDING AND FLUENCY

1 Find the size of the labelled unknown angles in these diagrams.

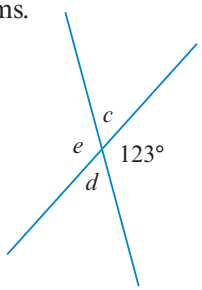
a



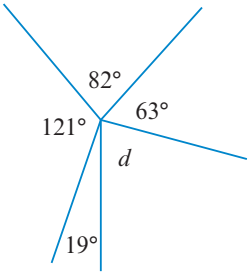
b



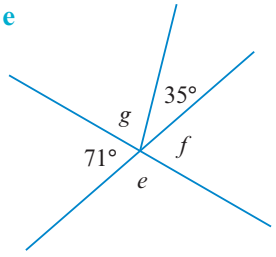
c



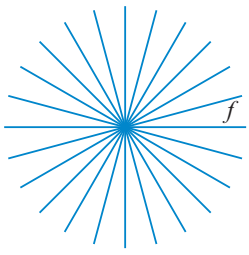
d



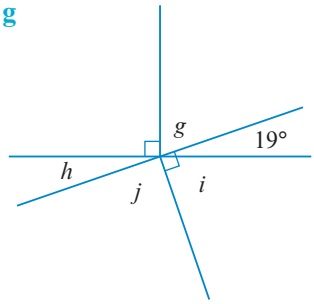
e



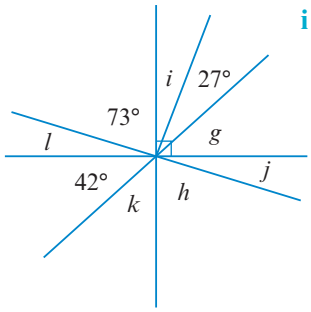
f



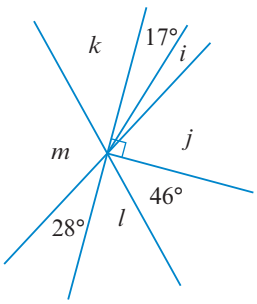
g



h



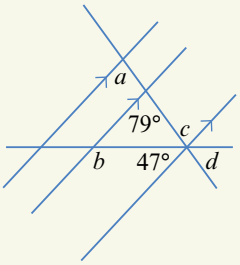
i



EXAMPLE 7A-2

Finding angle size using angle relationships with parallel lines

Find the value of the pronumerals in this diagram.

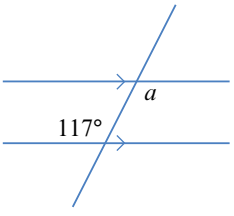


THINK	WRITE
1 Angle a is corresponding to 79°. Corresponding angles are equal.	$a = 79°$
2 Angle b is co-interior to angle 47°. Co-interior angles are supplementary.	$b = 180° - 47° = 133°$
3 Angle c is alternate to 79°. Alternate angles are equal.	$c = 79°$
4 Angle d is supplementary to angle c and the angle which is vertically opposite and hence equal to 47°.	$d = 180° - 79° - 47° = 54°$

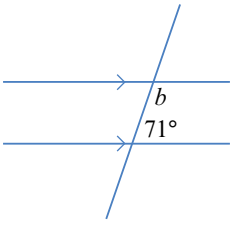
UNDERSTANDING AND FLUENCY

2 Find the value of each pronumeral in these diagrams.

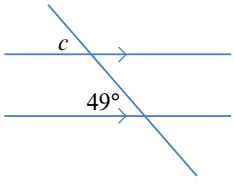
a



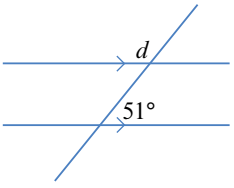
b



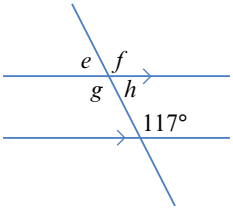
c



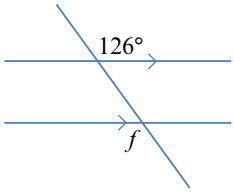
d



e

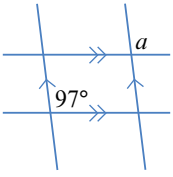


f

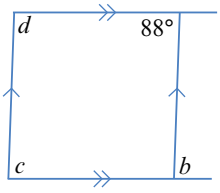


3 Find the value of each pronumeral in these diagrams.

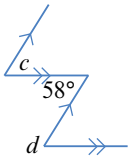
a



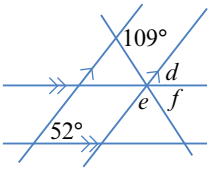
b



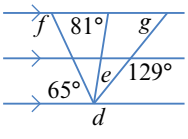
c



d



e



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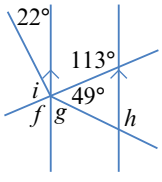
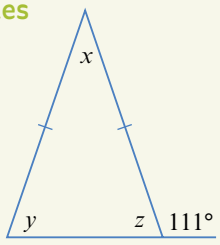


EXAMPLE 7A-3 Finding angle size using triangle properties

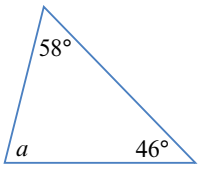
Find the value of each pronumeral in this diagram.



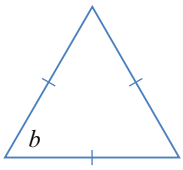
THINK	WRITE
1 Angle z is supplementary to 111°.	$z = 180° - 111° = 69°$
2 Angle y is equal to angle z because the triangle is an isosceles triangle.	$y = z = 69°$
3 Angle x is supplementary to angles y and z because there are 180° in a triangle.	$x = 180° - 69° - 69° = 42°$

4 Find the value of each pronumeral in these diagrams.

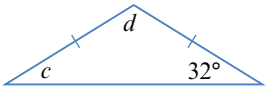
a



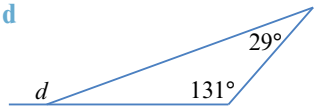
b



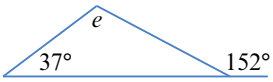
c



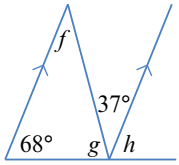
d



e

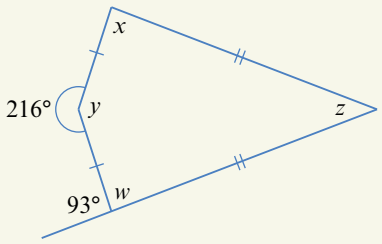


f



EXAMPLE 7A-4 Finding angle size using quadrilateral properties

Find the value of each pronumeral in this diagram.

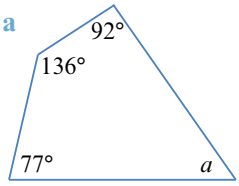


THINK	WRITE
1 Angle w is supplementary to 93°.	$w = 180° - 93° = 87°$
2 Angle x is equal to angle w because a **kite** has a pair of equal and opposite angles.	$x = w = 87°$
3 Angle y and 216° add to 360°.	$y = 360° - 216° = 144°$
4 Angles w, x, y and z add to 360° because the angle sum in a quadrilateral is 360°.	$z = 360° - 144° - 87° - 87° = 42°$

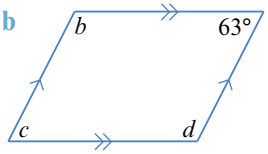
UNDERSTANDING AND FLUENCY

5 Find the value of each pronumeral in these diagrams.

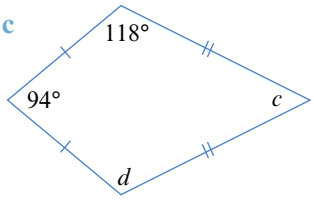
a



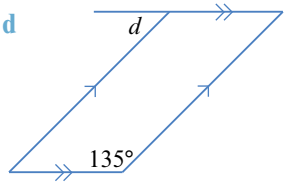
b



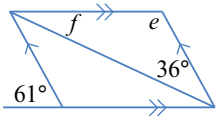
c



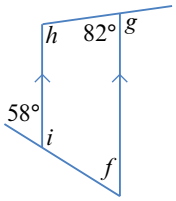
d



e

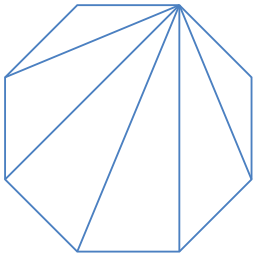


f



PROBLEM SOLVING AND REASONING

6 You can use the internal angle sum of a triangle to find the internal angle sum for any polygon. Consider this octagon.



a How many sides does it have?

b How many triangles is it split into?

c What is the difference between these two numbers?

d Use the triangles to calculate the internal angle sum of an octagon.

e Repeat parts **a**–**d** to find the internal angle sum of a:

i pentagon **ii** decagon **iii** heptagon **iv** dodecagon.

f Explain why the internal angle sum of any polygon can be found using the formula $(n - 2) \times 180°$, where n = number of sides.

7 Use the formula obtained in question **6** to find the internal angle sum of a polygon with:

a 20 sides **b** 50 sides **c** 100 sides.

8 **a** Explain why you can use the formula $x = \frac{180°(n - 2)}{n}$ to find the size of an individual internal angle in any **regular polygon**.

b Why can this formula only be used for regular polygons?

9 Find the size of an individual internal angle for a regular:

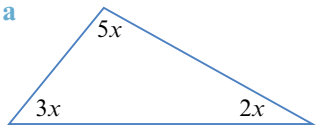
a heptagon **b** nonagon **c** hexagon.

10 Find the number of sides for a regular polygon that has each internal angle equal to:

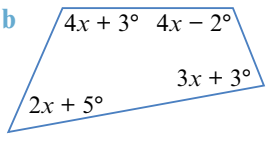
a 160° **b** 144° **c** 170° **d** 179.64°.

11 Find the value of x in each diagram.

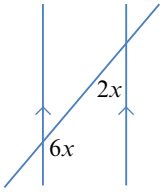
a



b

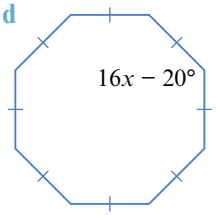


c

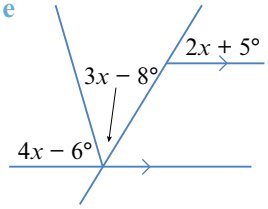


PROBLEM SOLVING AND REASONING

d



e



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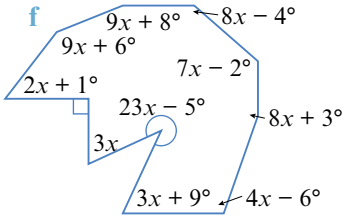
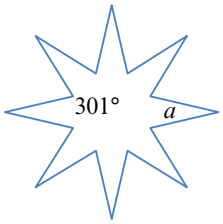


12 Decide if each statement is true or false. If false, write a correct version of the statement.

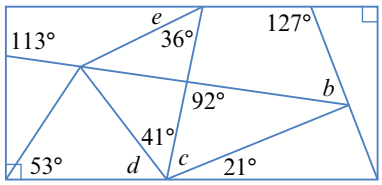
- **a** A kite has a pair of equal opposite angles.
- **b** Angles around a point add to 180°.
- **c** An **isosceles** triangles has two pairs of equal angles.
- **d** Corresponding angles in parallel lines are supplementary.
- **e** A square is a rhombus.
- **f** A parallelogram is a rectangle.
- **g** Complementary angles add to 180°.
- **h** A triangle that has one 60° angle must be **equilateral**.
- **i** A quadrilateral can have a maximum of one **concave** angle.

13 Find the value of each pronumeral, giving reasons.

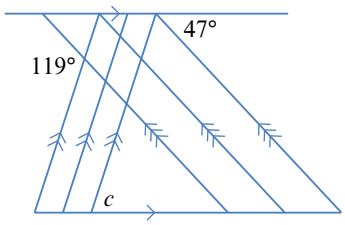
a



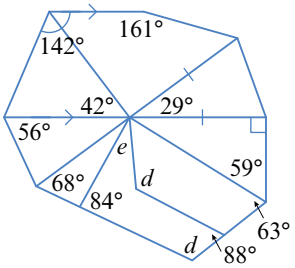
b



c



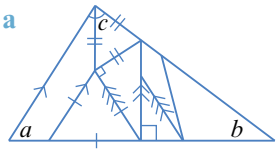
d



CHALLENGE

14 Find the value of each pronumeral, giving reasons.

a



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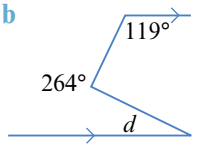
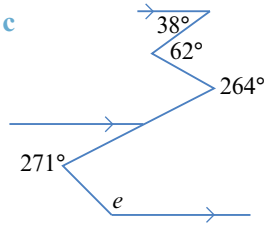


c



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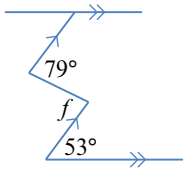
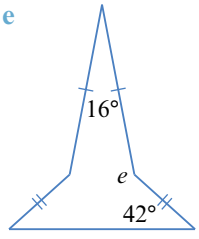
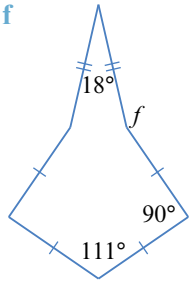


e



f

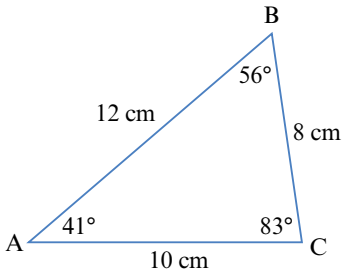


Reflect

What numbers are important when considering the relationships between angles, lines and polygons?

7B Congruence

Start thinking!



1 List the length of each side and the size of each angle in ΔABC.
For example, AB = 12 cm, ∠ABC = 56°, etc.

2 Draw ΔDEF, where DE = 8 cm, EF = 10 cm and DF = 12 cm.

For any two figures to be **congruent** (identical in shape and size), they must have all corresponding sides the same length and all corresponding angles the same size.

3 Is ΔDEF congruent to ΔABC? How do you know?

When two triangles have all their corresponding sides the same length (**SSS**), the corresponding angles are also the same size. This means that the triangles are congruent.

4 Measure the angles of ΔDEF to confirm that these triangles are congruent.

When two figures are congruent, you write this as a statement with the congruency symbol ≅. The same order of corresponding vertices is used in naming matching sides, angles and shapes.

5 List the corresponding vertices and hence write a congruency statement for the two triangles.

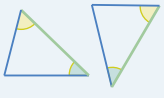
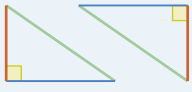
6 Draw ΔGHI, where ∠GHI = 41°, ∠HIG = 56° and ∠IGH = 83°.

7 Can you tell if ΔGHI is congruent to ΔABC? How do you know?

8 Explain why, if two triangles have all their corresponding angles the same size (**AAA**), this does not mean that they are congruent. You may like to include diagrams with your explanation.

KEY IDEAS

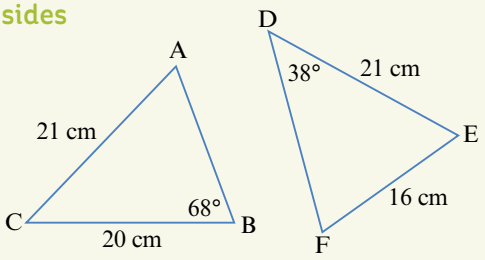
- Congruent figures are identical in shape and size but can be in any position or orientation.
- Two figures are congruent if their corresponding sides are all the same length and their corresponding angles all the same size. The symbol for congruence is ≅.
- Matching sides, angles and shapes in congruent figures are named using the same order of corresponding vertices.
- For triangles, there are four conditions for congruence (shown in the table).
- The specifications AAA and SSA do not necessarily mean congruence (more information is needed).

SSS	SAS	AAS	RHS
Three corresponding pairs of sides are equal in length. 	Two corresponding pairs of sides are equal in length and the corresponding pair of angles in between are equal. 	Two pairs of angles are equal and a corresponding pair of sides are equal in length. 	The hypotenuses and a corresponding pair of sides are equal in length in a right-angled triangle. 

EXERCISE 7B Congruence

EXAMPLE 7B-1 Matching corresponding sides in congruent triangles

Assuming this pair of triangles is congruent, match the corresponding sides.



THINK

1 Name the two sides that are 21 cm long according to the order of corresponding vertices; that is, A and E, C and D.

2 BC is 20 cm and EF is 16 cm, so if the triangles are congruent, BC must be equal to FD.

3 Write the last pair of corresponding sides.

WRITE

AC = ED

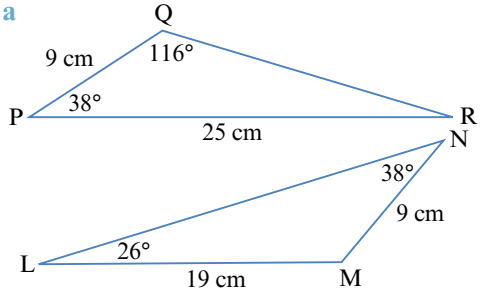
BC = FD

AB = EF

UNDERSTANDING AND FLUENCY

1 Assuming each pair of triangles is congruent, match the corresponding sides.

a



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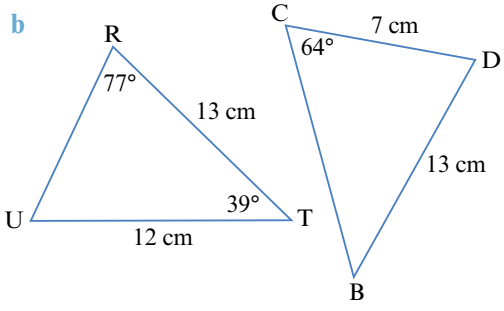

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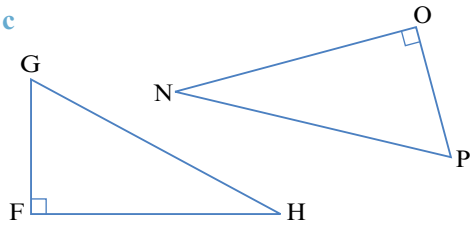

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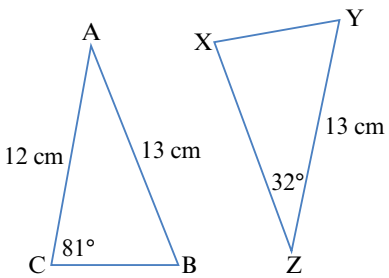

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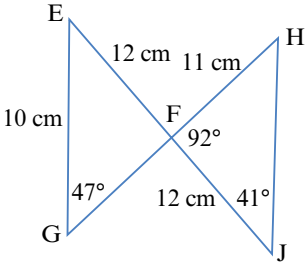

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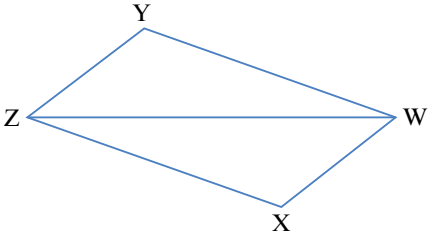
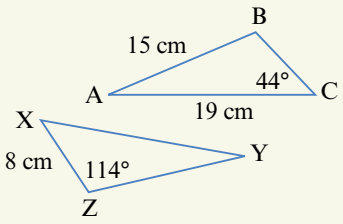


EXAMPLE 7B-2 Finding unknown side lengths and angles in congruent triangles

Find the unknown side lengths and angles in this pair of congruent triangles.



THINK

1 Match corresponding sides to identify the unknown side lengths.

2 Match corresponding angles to identify the unknown angles. Use the triangle sum of 180° if necessary.

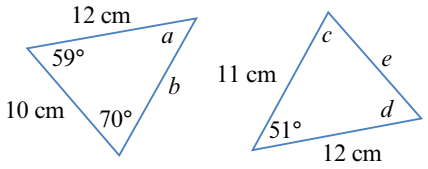
WRITE

AB = YZ = 15 cm
BC = ZX = 8 cm
AC = YX = 19 cm

∠ABC = ∠YZX = 114°
∠ACB = ∠YXZ = 44°
∠CAB = ∠XYZ = 180° − 114° − 44° = 22°

2 Find the unknown side lengths and angles in each pair of congruent triangles.

a



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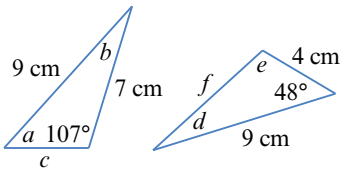

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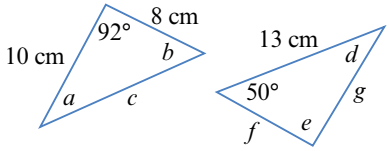

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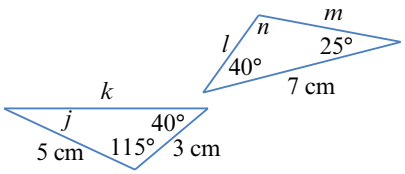
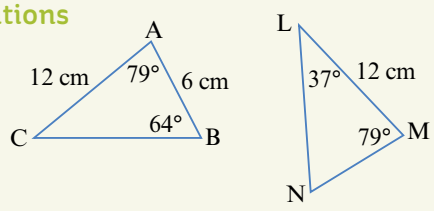


EXAMPLE 7B-3 Identifying congruence conditions

Decide which condition you would use to check if this pair of triangles is congruent.



THINK

1 Look at the triangles. ΔABC has two given side lengths and two angles and ΔLMN has two given angles and a side length.

2 Can any information be added to the triangles? Both triangles have two angles, so the third for each can be found.

3 ΔABC now has information for AAS and SAS, but ΔLMN only has information for AAS.

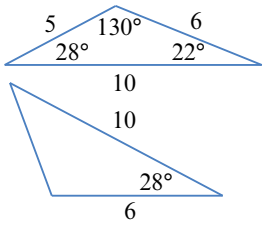
WRITE

You would use the congruence condition AAS to check if these triangles are congruent.

UNDERSTANDING AND FLUENCY

3 Decide which condition you would use to check if each pair of triangles is congruent.

a



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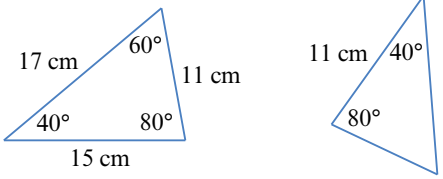

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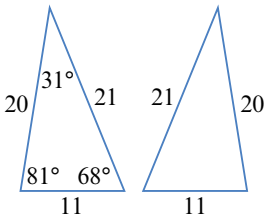

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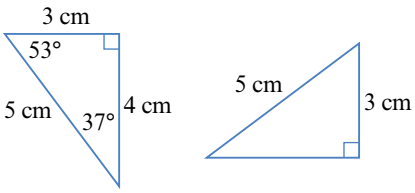

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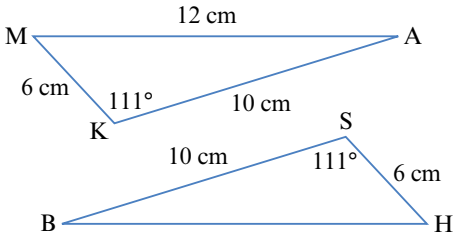

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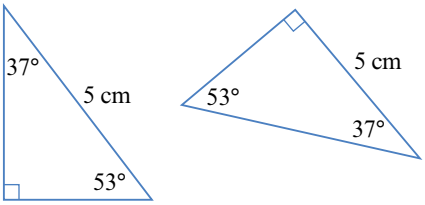
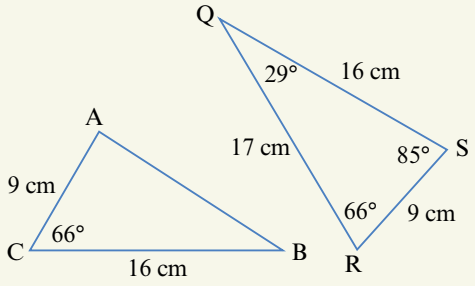


EXAMPLE 7B-4 Identifying congruence

Decide if this pair of triangles is congruent, giving a reason for your answer.



THINK

1 Two side lengths and an angle are given that are the same in both triangles (9 cm, 16 cm and 66°).

2 In ΔABC, the known angle is between the two sides, which fits the SAS condition for congruence.

3 The same sized angle in ΔQRS is not between the same two sides. A different sized angle (85°) is between the two sides, which means that these two triangles fail the SAS condition for congruence.

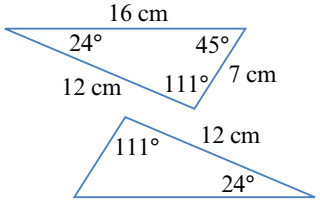
WRITE

The two triangles are not congruent as they fail the SAS condition for congruence.

UNDERSTANDING AND FLUENCY

4 Decide if each pair of triangles is congruent, giving a reason for your answer.

a



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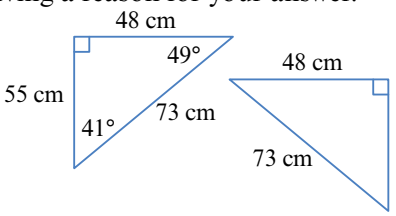

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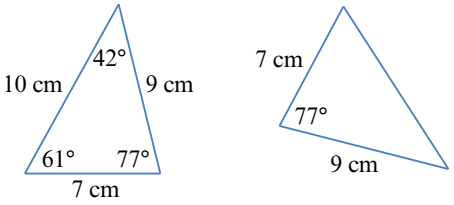

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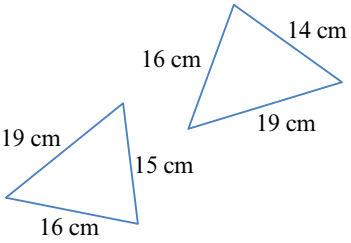

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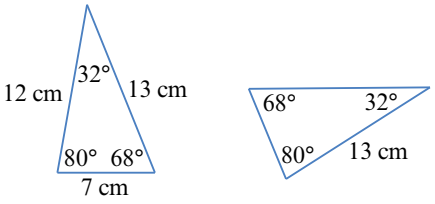

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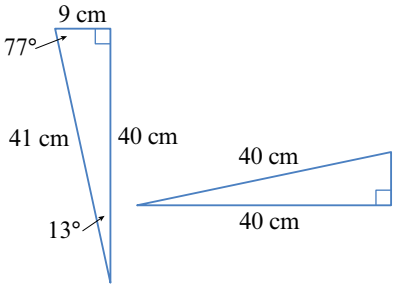
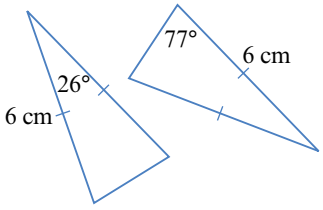


PROBLEM SOLVING AND REASONING

5 Use your understanding of triangle properties to decide whether each pair of triangles is congruent.

a



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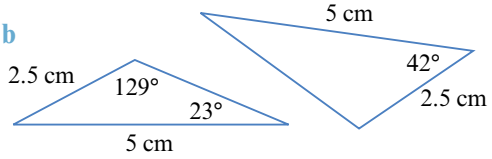

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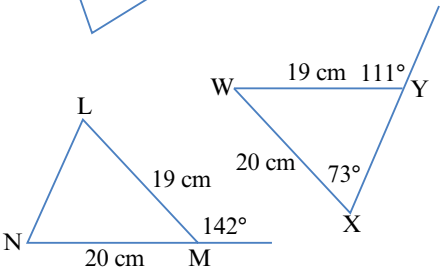

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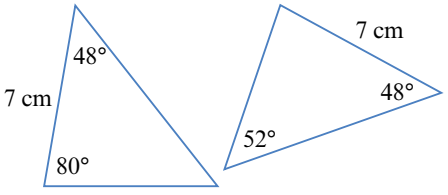


6 **a** Use a **square** and a **rhombus** to show that having all corresponding sides the same length is not enough to prove congruence for shapes other than triangles.

b Use a **parallelogram** and a rhombus to show that meeting SAS does not prove congruence for shapes other than triangles.

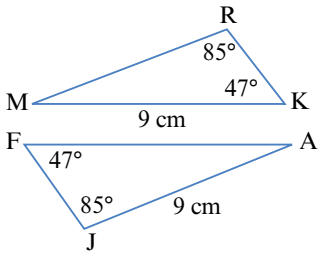
c Use a square and a **rectangle** to show that meeting AAS does not prove congruence for shapes other than triangles.

d Can you think of an alternative to SAS and AAS that could prove congruence in quadrilaterals? Investigate if this pattern continues for other polygons.

PROBLEM SOLVING AND REASONING

7 Lachlan said that each pair of triangles is congruent. Explain why he is wrong in each case.

a



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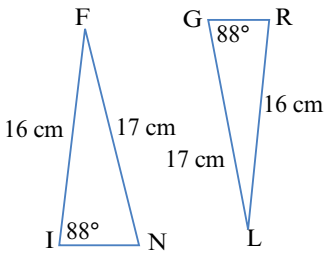

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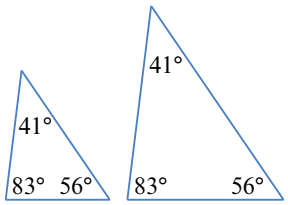

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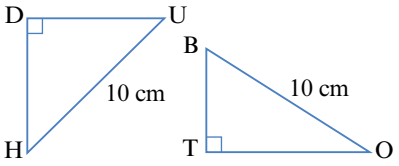


8 Explain why correspondence of side lengths and angles is important when deciding if a pair of triangles meets a congruency condition.

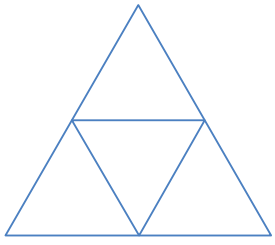
9 Decide if each statement is true or false.

- **a** If two triangles meet AAA, they are not congruent.
- **b** If two triangles each have two corresponding sides the same length and one corresponding angle of the same size, the triangles are congruent.
- **c** A pair of triangles can meet one congruence condition but fail another.
- **d** If ΔABC is congruent to ΔDEF, and ΔDEF is congruent to ΔGHI, then ΔGHI is congruent to ΔABC.
- **e** If two quadrilaterals have all sides equal in length, then they are congruent.
- **f** All equilateral triangles are congruent.

CHALLENGE

10 Use your understanding of the relevant geometrical properties to draw each description and show that the triangles within them are congruent.

- **a** A circle with centre O has a right-angled triangle drawn where the right angle is at the centre and the short sides are radii of the circle. A second right-angled triangle is drawn in the same circle under the same conditions.
- **b** An equilateral triangle is split into four smaller triangles by drawing the three **vertices** of a single triangle at the midpoint of each side, as seen in the diagram at right.



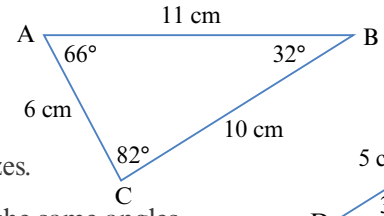
- **c** A kite is split into two triangles by a line drawn vertically down its centre.
- **d** A regular hexagon is split into six triangles by drawing 3 diagonal lines joining opposite vertices.

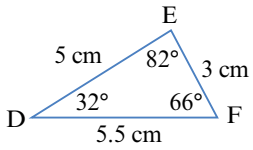
Reflect

What do you need to check to see if any pair of shapes is congruent?

7C Similarity

Start thinking!



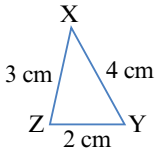


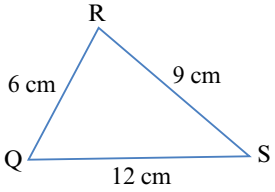
1 What is similar and what is different about the first pair of triangles?

Similar figures are exactly the same shape but can be different sizes.

2 Explain how you know two triangles are similar if they have the same angles.

3 How might you figure out if the second pair of triangles are similar?





4 How long is QS compared with YX?

It is important when checking that corresponding sides are in the same **ratio** that you check every pair of sides.

5 Check that the other two pairs of sides are in the same ratio.

This ratio is called **scale factor**, can be calculated using the formula: scale factor $= \frac{\text{image length}}{\text{original length}}$.

6 Explain why, if the **image** is bigger than the original, the scale factor will be bigger than 1, but if the image is smaller than the original, the scale factor will be a fraction less than 1.

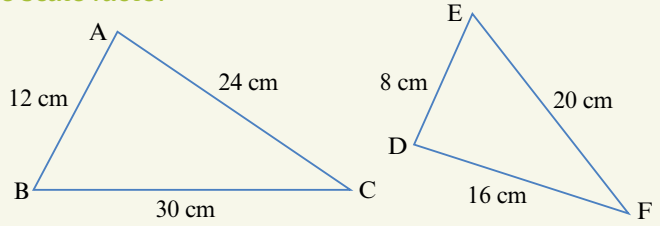
KEY IDEAS

- Similar figures are identical in shape but can be different in size.
- For two figures to be similar, all angles must be equal in size and all corresponding sides must be in the same ratio.
- If two figures are similar, an unknown side length or angle can be found if the scale factor is known.
- Scale factor is calculated using the formula: scale factor $= \frac{\text{image length}}{\text{original length}}$.
- For two triangles to be similar, they must meet one of four similarity conditions:
  - SSS: All three corresponding sides are in the same ratio.
  - SAS: Two corresponding sides are in the same ratio and the angles in between are equal.
  - AAA: All three corresponding angles are equal.
  - **RHS**: The hypotenuses are in the same ratio as another pair of sides in right-angled triangles.

EXERCISE 7C Similarity

EXAMPLE 7C-1 Finding the scale factor

Find the scale factor for this pair of similar triangles. Assume that the first triangle is the original.

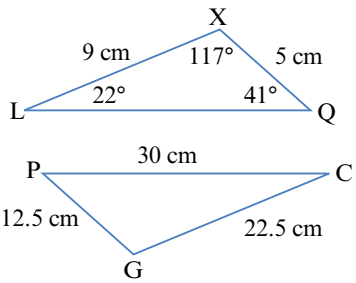


THINK	WRITE
1 To find the scale factor, compare corresponding sides. Choose the longest side of the original triangle (ΔABC) and the longest side of the image (ΔDEF).	BC = 30 cm, EF = 20 cm
2 Substitute these two side lengths into the formula for scale factor.	Scale factor $= \frac{\text{image length}}{\text{original length}}$
3 Check that the scale factor seems reasonable. (Number between 0 and 1 means a **reduction**.)	$= \frac{20\text{ cm}}{30\text{ cm}}$ $= \frac{2}{3}$
4 Match the remaining corresponding sides and check that each side is in the same ratio.	Scale factor $= \frac{16}{24} = \frac{8}{12} = \frac{2}{3}$
5 Write your answer.	Scale factor $= \frac{2}{3}$

UNDERSTANDING AND FLUENCY

1 Find the scale factor for each pair of similar triangles. Assume that the first triangle in the pair is the original.

a



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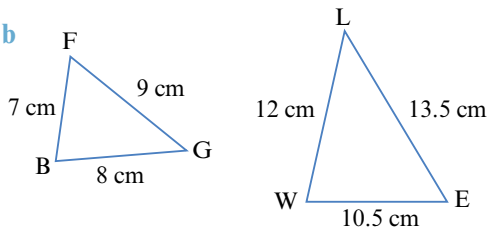
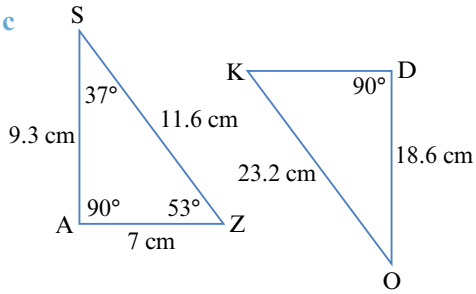


c



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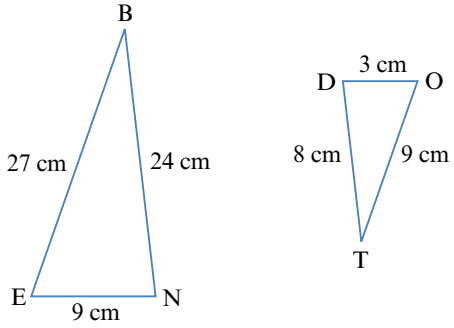
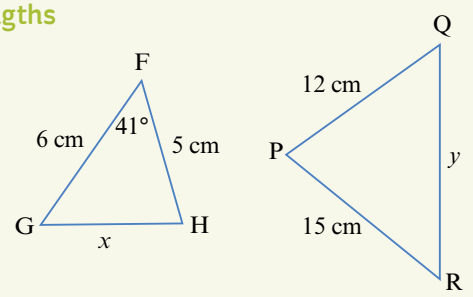


EXAMPLE 7C-2 Finding unknown side lengths in similar triangles

Find the unknown side lengths in this pair of similar triangles.



THINK

1 As the triangles are similar, all corresponding sides must be in the same ratio. Match corresponding sides.

2 Use the pair of corresponding sides that both have measurements (FH and RP) to find the scale factor from ΔFGH to ΔRQP.
Check that the scale factor seems reasonable. (Number larger than 1 means an **enlargement**.)

3 Since x is on the 'original' triangle, divide the length of the corresponding side in ΔRQP by the scale factor.

4 Since y is on the 'image' triangle, multiply the length of the corresponding side in ΔFGH by the scale factor.

5 Write your final answer.

WRITE

Corresponding pairs of sides are:
FG and RQ, GH and QP, FH and RP

$$\text{Scale factor} = \frac{\text{image length}}{\text{original length}}$$
$$= \frac{RP}{FH}$$
$$= \frac{15\text{ cm}}{5\text{ cm}}$$
$$= 3$$

$$x = \frac{QP}{\text{scale factor}}$$
$$= \frac{12\text{ cm}}{3}$$
$$= 4\text{ cm}$$

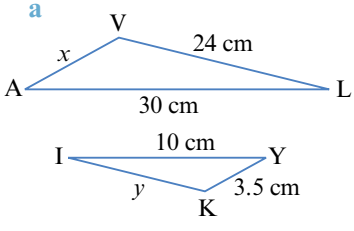
$$y = FG \times \text{scale factor}$$
$$= 6\text{ cm} \times 3$$
$$= 18\text{ cm}$$

$x = 4$ cm and $y = 18$ cm.

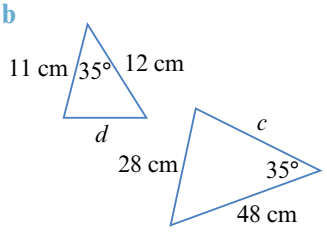
UNDERSTANDING AND FLUENCY

2 Find the unknown side lengths in each pair of similar triangles.

a



b



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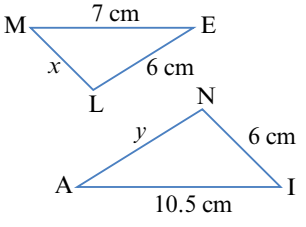

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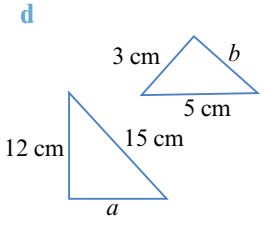
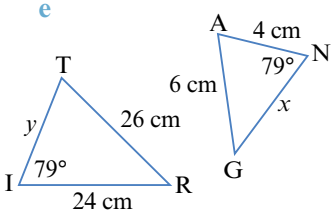


e



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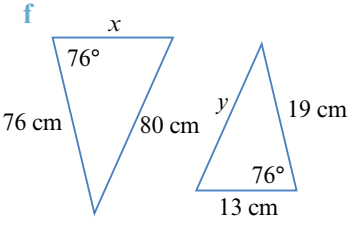
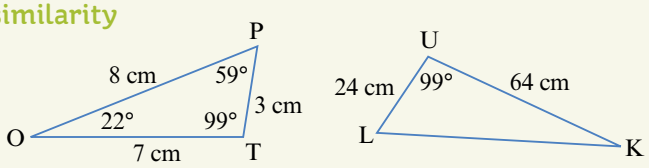


EXAMPLE 7C-3 Identifying similarity

Decide if this pair of triangles is similar, giving a reason for your answer.

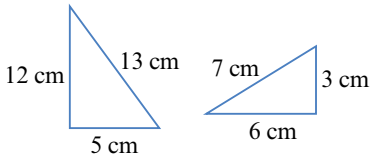


THINK	WRITE
1 ΔLUK has two sides and an angle in between, so use the similarity condition SAS to compare. Match corresponding sides and angles.	PT corresponds to LU ∠PTO corresponds to ∠LUK ∠TOP corresponds to ∠UKL ∠TPO corresponds to ∠ULK TO corresponds to UK OP corresponds to KL
2 Since ∠PTO corresponds to ∠LUK, check if the angles are equal.	∠PTO = ∠LUK = 99°
3 PT corresponds to LU. Find the scale factor from ΔPTO to ΔLUK.	$\text{Scale factor} = \frac{\text{image length}}{\text{original length}}$ $= \frac{LU}{PT}$ $= \frac{24\text{ cm}}{3\text{ cm}}$ $= 8$
4 TO corresponds to UK. Find the scale factor from ΔPTO to ΔLUK.	$\text{Scale factor} = \frac{\text{image length}}{\text{original length}}$ $= \frac{UK}{TO}$ $= \frac{64\text{ cm}}{7\text{ cm}}$ ≈ 9.14
5 Look at your results and write your final answer.	The two triangles are not similar as the two scale factors are not the same, so they fail the similarity condition SAS.

UNDERSTANDING AND FLUENCY

3 Decide if each pair of triangles is similar, giving a reason for your answer.

a



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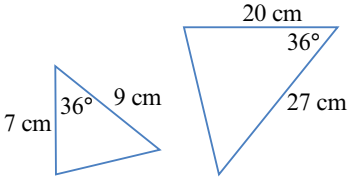

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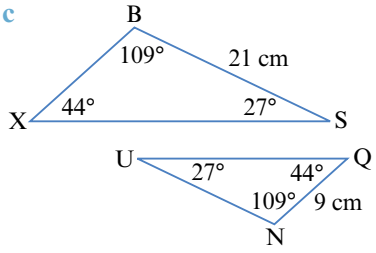

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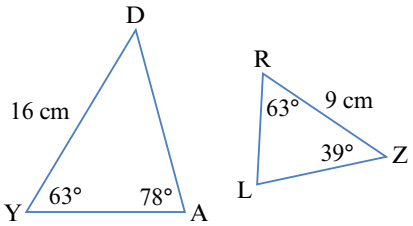

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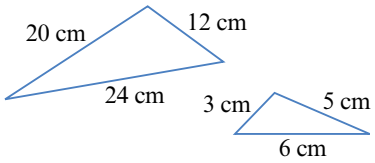

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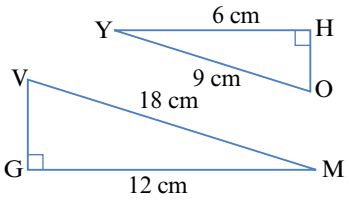
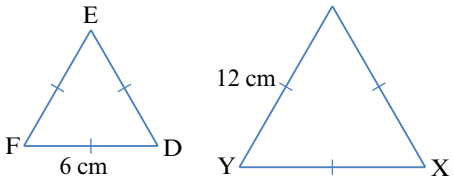
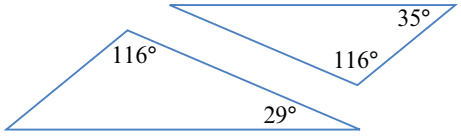


4 Use your understanding of triangle properties to decide whether each pair of triangles is similar.

a



b



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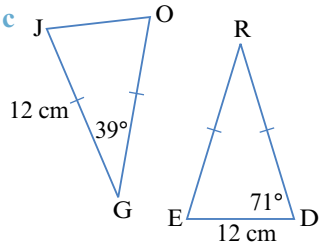

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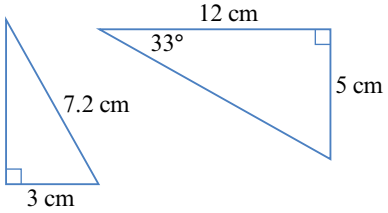


5 **a** Use a rectangle and a parallelogram to show that having all corresponding sides in the same ratio is not enough to prove similarity for shapes other than triangles.

b Use a parallelogram and a rhombus to show that having all corresponding angles the same size is not enough to prove similarity for shapes other than triangles.

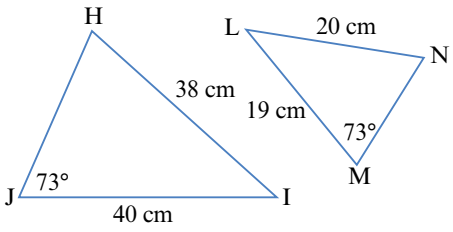
c Use a rectangle and a square to show that meeting SAS does not prove similarity for shapes other than triangles.

d Use a parallelogram and a rhombus to show that meeting AAS does not prove similarity for shapes other than triangles.

e Can you think of an alternative to SAS and AAS that could prove similarity in quadrilaterals? Investigate if this pattern continues for other polygons.

6 Jessica said that each pair of triangles is similar. Explain why she is wrong in each case.

a



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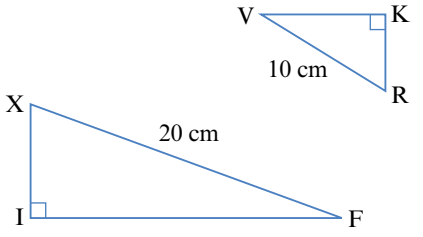
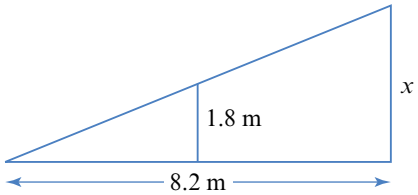


7 Explain why correspondence of sides and angle is important when deciding if a pair of triangles meets a similarity condition.

8 A 1-m ruler is placed upright next to a tree. If the ruler casts a 2.3 m shadow and the tree casts an 11.2 m shadow, how tall is the tree?

PROBLEM SOLVING AND REASONING

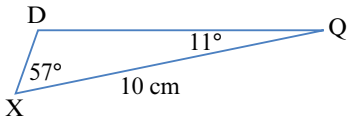
9 A ski jump is 8.2 m long. It has vertical supports at the half-way mark that are 1.8 m tall. How tall is the ski jump at its end?



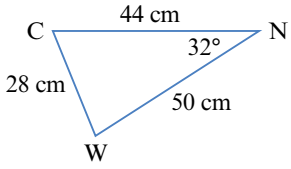
10 Use similar triangles to calculate the unknown height or length of something in your classroom. Include a diagram in your answer.

11 Select two pairs of similar triangles from these options. Provide reasons for your selection.

a



b



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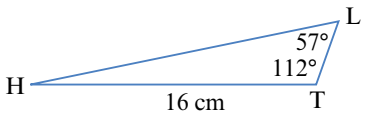

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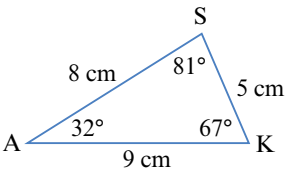

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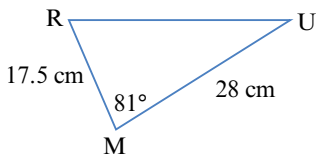

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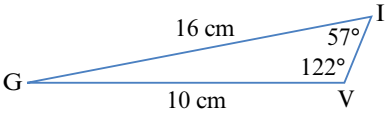


12 Decide if each statement is true or false.

- **a** If two triangles meet AAA, they are not necessarily similar.
- **b** If ΔABC is similar to ΔDEF, and ΔDEF is similar to ΔGHI, then ΔGHI is similar to ΔABC.
- **c** All isosceles triangles are similar.
- **d** All squares are similar.
- **e** If two triangles fail SAS, they are not necessarily similar.
- **f** If two quadrilaterals have all sides in the same ratio, they are not necessarily similar.

13 Dilating ΔSIJ by $\frac{4}{3}$ gives ΔPWX. Is this a reduction or enlargement? Explain.

CHALLENGE

14 A flagpole has a guide wire attached at its midpoint, tethered to the ground 2.7 m away. A metre ruler standing vertically touches the guide wire when placed 75 cm away from where it is tethered to the ground. How tall is the flagpole?

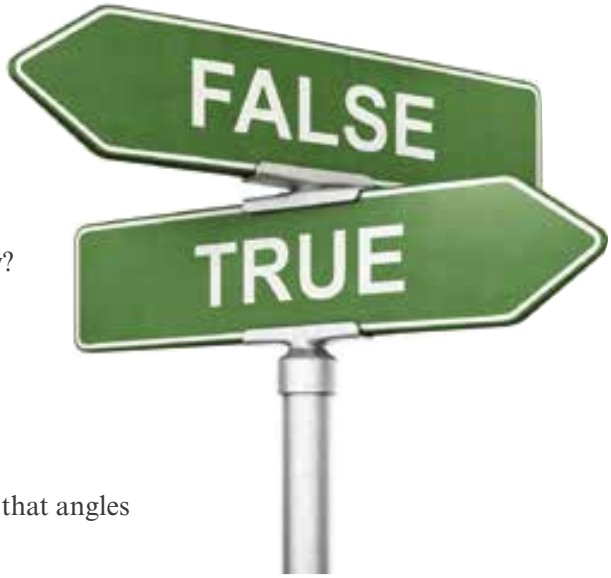
15 Explain how you can use similar triangles to find the width of a river. (Hint: you will need to use four markers on one side of the river, one of which lines up with a landmark on the opposite side.) Draw a diagram to support your answer.

Reflect

What is the difference between the congruence and similarity conditions?

7D Understanding proofs

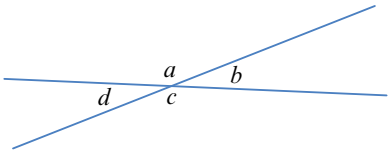
Start thinking!



1 How do you know when something is true?

It is important in mathematics not to just demonstrate that something is true or that it works, but to *prove* that it is true.

2 Which angles are equal in this diagram? How do you know?



3 Copy the diagram and cut out the angles and demonstrate that angles a and c are equal in size.

This is a practical demonstration that vertically opposite angles are equal, but it is not a proof.

A mathematical **proof** is a series of statements that show that a theory is true in all cases. Usually the proof uses some self-evident or assumed statements, known as **axioms**, in showing that something is true. One of the most famous mathematicians, Euclid (who lived around 300 BC), was able to prove much of the mathematics you use today, using only five axioms.

You can use the fact that there is 180° around a straight line as an axiom to prove that vertically opposite angles are equal.

4 What is $a + b$ equal to?

5 What is $b + c$ equal to?

6 How do your answers to questions **4** and **5** show that $a + b$ is equal to $c + b$?

7 How does this show that a is equal to c?

8 Write these steps as a mathematical proof showing that angle a is equal to angle c.

9 Why do you think proofs are important in mathematics?

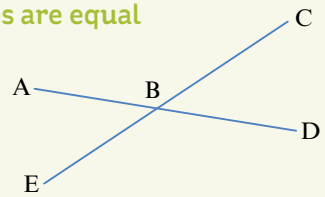
KEY IDEAS

- A mathematical proof is a series of statements that show that a theory is true in all cases.

EXERCISE 7D Understanding proofs

EXAMPLE 7D-1 Proving that vertically opposite angles are equal

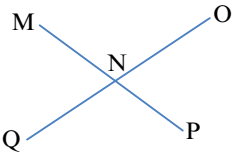
Prove that ∠ABC = ∠DBE.



THINK	WRITE
1 Use angles around a straight line to write a statement about ∠ABC and a statement about ∠DBE that share a common angle.	∠ABC + ∠CBD = 180° ∠DBE + ∠CBD = 180°
2 Since the right side of each equation is the same, the left sides must be equal.	∠ABC + ∠CBD = ∠DBE + ∠CBD
3 Simplify by subtracting ∠CBD from both sides of the equation.	∠ABC = ∠DBE

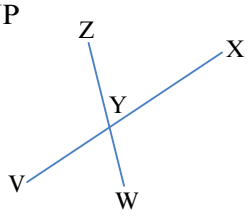
UNDERSTANDING AND FLUENCY

1 Fill in the gaps to prove that ∠MNQ = ∠ONP.

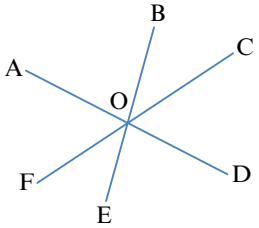


∠MNQ + ∠QNP = _____°
∠ONP + ∠_____ = 180°
∠_____ + ∠QNP = ∠_____ + ∠QNP
∠_____ = ∠_____

2 Prove that ∠XYW = ∠ZYV.

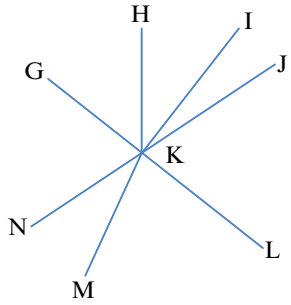


3 Show that ∠EOF = ∠BOC, starting with the opening statement:



a ∠EOF + ∠FOB = 180°
b ∠EOF + ∠AOF + ∠AOB = 180°.

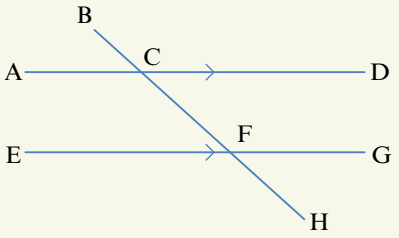
4 Show that ∠GKJ = ∠NKL.



NOTE In some books you may notice the letters 'QED' written at the end of a proof. This is an acronym for the Latin phrase *quod erat demonstrandum*, which means 'which had to be demonstrated'.

EXAMPLE 7D-2 Proving that alternate angles are equal

Prove that ∠ACF = ∠CFG,
given that ∠CFG = ∠BCD.

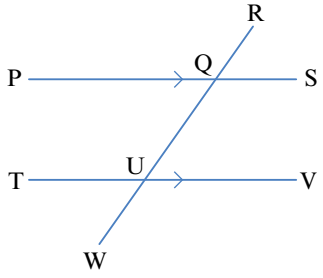


THINK	WRITE
1 Write the given information.	∠CFG = ∠BCD
2 Write statements linking ∠ACF and ∠BCD.	∠ACF + ∠DCF = 180° ∠BCD + ∠DCF = 180° ∠ACF + ∠DCF = ∠BCD + ∠DCF ∠ACF = ∠BCD
3 Use the given information to complete the proof.	but ∠BCD = ∠CFG so ∠ACF = ∠CFG

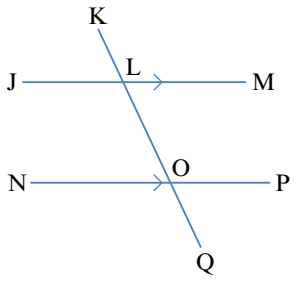
UNDERSTANDING AND FLUENCY

5 Fill in the gaps to prove that ∠TUQ = ∠SQU, given that ∠TUQ = ∠PQR.

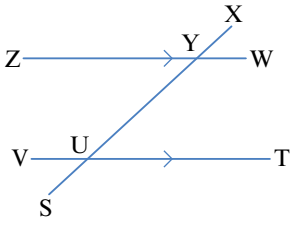
∠TUQ = ∠PQR
∠PQR + ∠PQU = _____
∠SQU + ∠_____ = 180°
∠PQR + ∠_____ = ∠SQU + ∠_____
∠PQR = ∠_____
but ∠PQR = ∠TUQ
so ∠TUQ = ∠_____



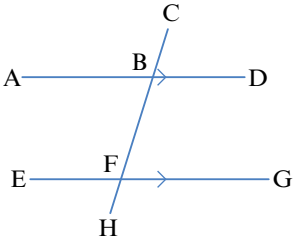
6 Given that ∠JLK = ∠LON, show that ∠MLO = ∠LON.



7 Given that ∠XYW = ∠YUT, show that ∠WYU = ∠VUY.



8 Given that ∠ABC = ∠EFB, show that ∠DBF is supplementary to ∠BFG.



9 Is the sum of two even integers always even? To investigate, consider any two even numbers; call them x and y.

- **a** How do you define an even number?
- **b** Explain why it follows that you can write x as $2a$.
- **c** Write y in the same format using the pronumeral b.
- **d** Write $x + y$ using the format from parts **b** and **c**.
- **e** Factorise your answer to part **d**.
- **f** Explain why your answer to part **e** proves that the sum of two even integers is always even.
- **g** Write the proof in full.

10 Use a similar method to that shown in question **9** to prove that the square of any even number is always even.

11 There are different types of mathematical proof – so far you have seen a type called **direct proof**. Another method of proof is called **proof by contradiction**, where for a statement to be true a contradiction would have to occur, so that the statement is shown to be false.

One such proof involves the square root of 2. To prove that it is an irrational number, you instead try to prove that it is a rational number.

- **a** What is a rational number?

A rational number can be written in the form $\frac{a}{b}$, where a and b are whole numbers, and b is not equal to zero.

- **b** So, if the square root of 2 is rational, $\sqrt{2} =$ _____.

You assume that this fraction is in simplest form; that is, a and b have no common factors.

- **c** Explain why a or b may be even, but a and b cannot both be even.
- **d** Square both sides of the equation you found in part **b**.
- **e** Explain why you can write this in the form $a^2 = 2b^2$.
- **f** How do you know that a^2 is even?
- **g** How do you know that a is an even number? (Hint: look at question **10**.)

So you know that a is an even number, which means that b cannot also be an even number.

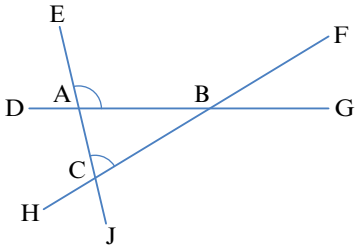
- **h** Explain why you can now write a as $2k$.
- **i** Substitute this into the equation $a^2 = 2b^2$.
- **j** Expand this and explain why it shows that b is also an even number.
- **k** Explain why this means that $\sqrt{2}$ is not rational.
- **l** Write the proof in full.

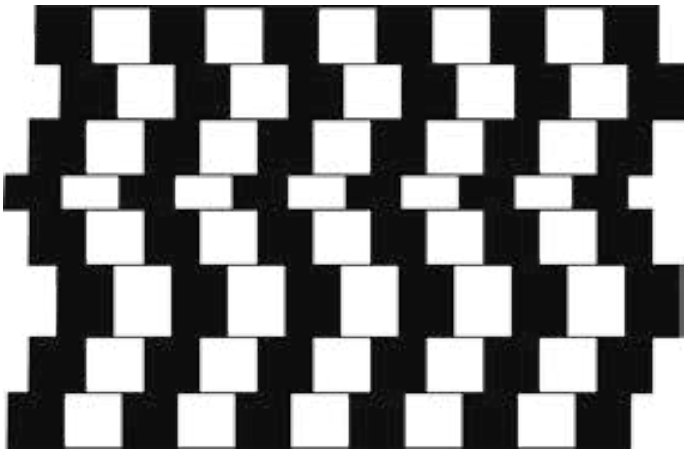
12 Follow a process similar to that in question **11** to show that $\sqrt[3]{2}$ is an irrational number.

13 Another example of proof by contradiction is used in proving that if corresponding angles are equal, the lines must be parallel. Start with the (false) assumption that if two corresponding angles are equal, then the lines are *not* parallel.

a Why does this mean that the lines would intersect?

Consider this diagram, showing intersecting lines DG and HF, cut by the transversal EJ.





b Name the two angles in the diagram that you are assuming are equal.

The other assumption that you must make is that the length AC > 0.

c Why do you need to make this assumption?

Say that $\angle EAB = \angle ACB = x$.

d Show that $\angle BAC = 180° - x$

e If $\angle ABC$ can be represented by y, write a statement adding all the angles in ΔABC.

f Show that simplifying this statement leaves you with $y = 0°$.

g How does this imply that AC does in fact equal 0?

h Explain how this is proof by contradiction that if corresponding angles are equal, then the lines must be parallel.

i Write the proof in full.

14 Another method of proof is called **proof by contraposition**. In its simplest form, you can prove that 'if a then b' by first proving 'if not b then not a'. One example of proof by contraposition is to show if p^2 is odd, then p must be odd.

a What is the contrapositive statement to 'if p^2 is odd, then p must be odd'?

b Prove this contrapositive statement. Hint: this is what you are asked to do for question **10**.

c Write a final statement to complete the proof by contraposition.

15 Use proof by contraposition to prove this statement.

If the product of a and b is odd, then both a and b must be odd.

Hint: write the contrapositive statement and prove this first.

16 Use proof by contraposition to prove that if $a^2 - 2a + 3$ is even, then a must be odd.

CHALLENGE

17 Another method of proof is **proof by mathematical induction**. This is where you initially prove the statement for a single case, for example $n = 1$, and then prove it to be true for a rule, such as $n = k + 1$, where k is an integer.

a Explain why, if something is true for $n = 1$ and $n = k + 1$, then it must be true for all integers.

An example is to prove that all integers that can be written as $2n + 1$ must be odd.

b Prove this statement for:

i $n = 1$ **ii** $n = 2$

iii $n = 99$ **iv** $n = 100$.

c Substitute $n = k + 1$ into $2n + 1$ and prove that this is also an odd number.

d Write a closing statement to finish the proof.

18 Use proof by induction to prove that $3^n - 1$, where n is a positive integer, is a multiple of 2.

19 It is important when you complete proofs that you do not make errors because, if you are not careful, you may think you have proven something that is actually false. The following example 'proves' that 2 = 1. Can you find the error?

$a = b$, where a and b are not equal to zero.

Then $a^2 = ab$

and $a^2 - b^2 = ab - b^2$

Factorising gives $(a + b)(a - b) = b(a - b)$

Dividing by $(a - b)$ gives $a + b = b$

Remember that $a = b$

So $b + b = b$

So $2b = b$

Dividing by b gives $2 = 1$

20 Follow a process similar to that in question **11** to show that $\sqrt{5}$ is an irrational number.

21 Create your own proofs using:

a direct proof

b proof by contradiction

c proof by contraposition

d proof by mathematical induction.

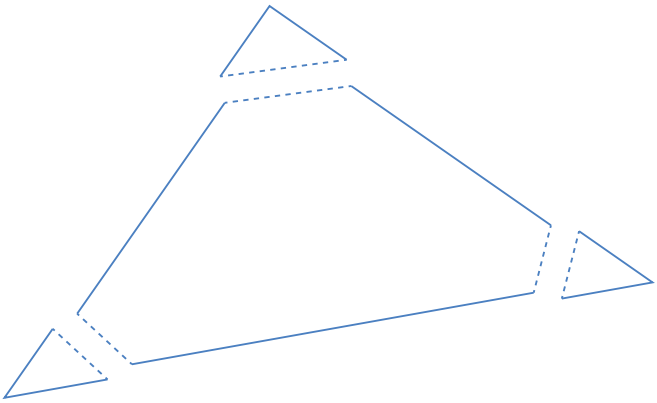
You may wish to use the Internet to help you.

Reflect

How are the different methods of proof similar and how are they different?

7E Proofs and triangles

Start thinking!

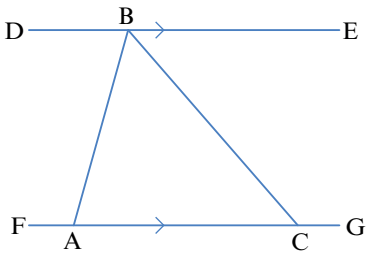


1 What is the internal angle sum of a triangle?

2 Draw a triangle and by either measuring its angles or by cutting out its angles and placing them in a straight line, show that it has an internal angle sum of 180°.

3 Why is drawing 10 different triangles and showing each has an angle sum of 180° not a proof?

You can use your knowledge of angles and parallel lines to prove that the internal angle sum of a triangle is 180°.



4 Name the internal angles of the triangle.

5 Show that ∠CBE = ∠BCA.

6 Which angle is ∠BAC equal to?

7 Write an equation for the angles around the straight line DE. (Hint: the three angles add to 180°).

8 Substitute the values you found from questions **5** and **6** into this equation.

9 Explain why this is the end of the proof.

10 Write the proof in full.

KEY IDEAS

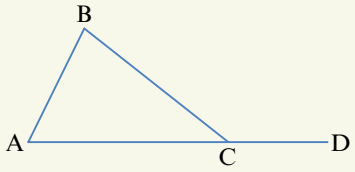
- Many properties of triangles can be proved using knowledge of angles around a straight line and in parallel lines.
- This information can also be used to prove congruence in triangles.
- The symbol for congruence is ≅.

EXERCISE 7E Proofs and triangles

EXAMPLE 7E-1

Proving the exterior angle of a triangle is the sum of the two opposite internal angles

Prove that ∠BCD = ∠ABC + ∠CAB.

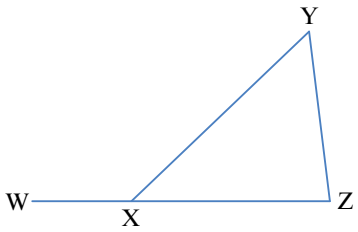


THINK	WRITE
1 Write an equation linking the **exterior angle** of the triangle and the straight line it lies on.	∠BCD + ∠BCA = 180°
2 Write an equation linking the internal angles of the triangle.	∠ABC + ∠CAB + ∠BCA = 180°
3 Equate the left sides of the two equations.	∠BCD + ∠BCA = ∠ABC + ∠CAB + ∠BCA
4 Simplify by subtracting ∠BCA from both sides.	∠BCD = ∠ABC + ∠CAB

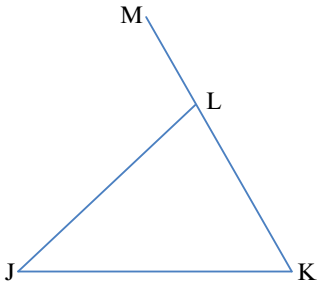
UNDERSTANDING AND FLUENCY

1 Prove that ∠WXY = ∠XZY + ∠ZYX by following these steps.

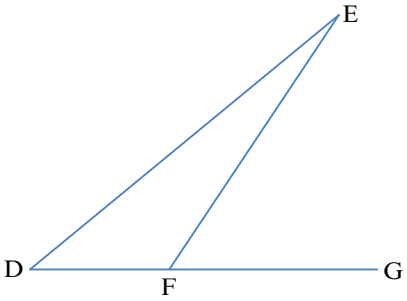
- **a** Write an equation linking the exterior angle of the triangle and the straight line it lies on.
- **b** Write an equation linking the internal angles of a triangle.
- **c** Equate the left sides of the two equations.
- **d** Simplify the resulting equation.



2 Prove that ∠JLM = ∠JKL + ∠LJK.



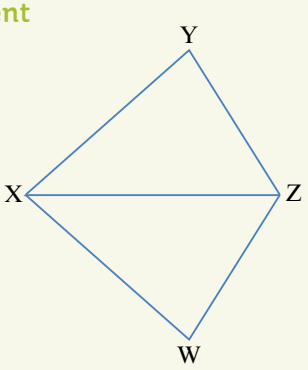
3 Prove that ∠EFG = ∠FDE + ∠DEF.



EXAMPLE 7E-2

Proving that two triangles are congruent using equal sides

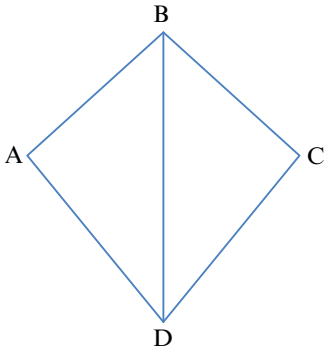
If XY = XW and ZY = ZW, prove that ΔXYZ ≅ ΔXWZ.



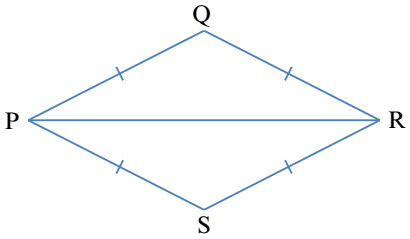
THINK	WRITE
1 If ΔXYZ and ΔXWZ are congruent, they have corresponding vertices. Match them for the two triangles.	X is common to both triangles. Z is common to both triangles. Y corresponds to W.
2 Decide if the given information is enough to satisfy a congruence condition.	Two pairs of corresponding sides are equal in length. Need another piece of information to check for congruence.
3 List a third piece of information.	XZ is common to both triangles.
4 Check for congruence and write the proof.	XY = XW (given) ZY = ZW (given) XZ is common to both triangles. ΔXYZ ≈ ΔXWZ (using SSS)

UNDERSTANDING AND FLUENCY

4 If AB = CB and AD = CD, prove that ΔABD ≅ ΔCBD.



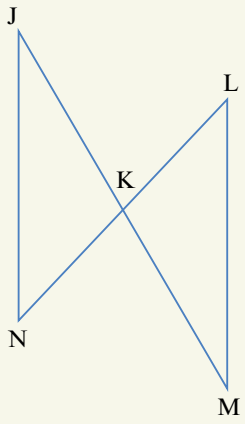
5 Prove that ΔPQR ≅ ΔPSR.



EXAMPLE 7E-3

Proving that two triangles are congruent using AAS

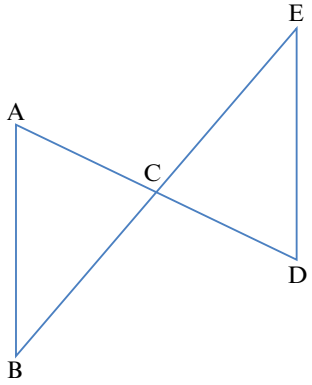
If KL = KN, ∠LKM = ∠NKJ and ∠LMK = ∠NJK, prove that ΔJKN ≅ ΔMKL using a congruence condition.



THINK	WRITE
1 If ΔJKN and ΔMKL are congruent, they have corresponding vertices. Match them for the two triangles.	J corresponds to M N corresponds to L K is common to both triangles
2 Consider the given information.	Sides KL and KN are corresponding and equal in length. Angles LKM and NKJ are corresponding and are equal. Angles LMK and NJK are corresponding and are equal.
3 Decide if you have enough information to meet a congruence condition (two pairs of equal angles and a corresponding pair of sides that are equal in length). Write the proof.	KL = KN (given) ∠LKM = ∠NKJ (given) ∠LMK = ∠NJK (given) ΔJKN ≈ ΔMKL (using AAS)

UNDERSTANDING AND FLUENCY

6 If AC = DC, ∠ACB = ∠DCE and ∠CBA = ∠CED, prove that ΔABC ≅ ΔDEC by following these steps.



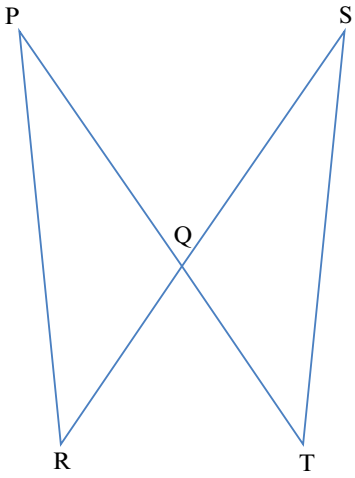
a Write the given information.

b Write a statement about the third angle of each triangle.

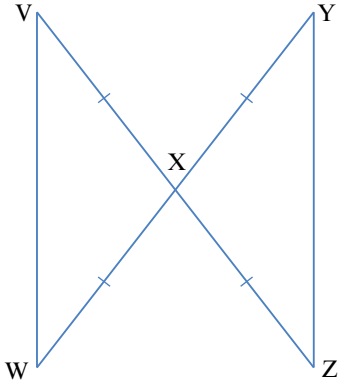
c Prove that the triangles are congruent by showing that they meet a congruence condition.

UNDERSTANDING AND FLUENCY

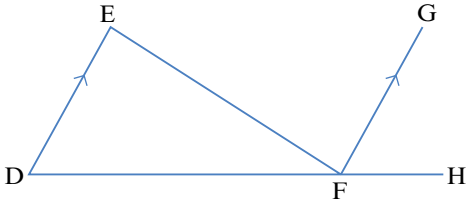
7 If QR = QT, ∠PQR = ∠SQT and ∠RPQ = ∠TSQ, prove that ΔPQR ≅ ΔSQT.



8 Prove that ΔVWX ≅ ΔYZX.

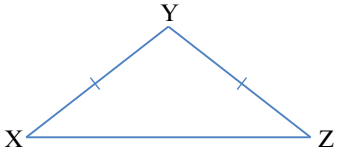


9 Use the fact that DE and FG are parallel to prove that ∠EFH = ∠DEF + ∠FDE.



PROBLEM SOLVING AND REASONING

10 Question **8** used the fact that if a triangle has two sides that are equal in length, then the angles opposite those sides are equal. You can prove this fact using congruent triangles. Consider ΔXYZ.



a Draw a line from Y to the midpoint of XZ, and name this point W.

b Name the two triangles now formed.

c Which two sides do you know are equal in these two triangles?

d Name the side common to both triangles.

e Explain why ∠XYW = ∠ZYW.

f Use a congruence condition to prove that ΔWXY ≅ ΔWZY.

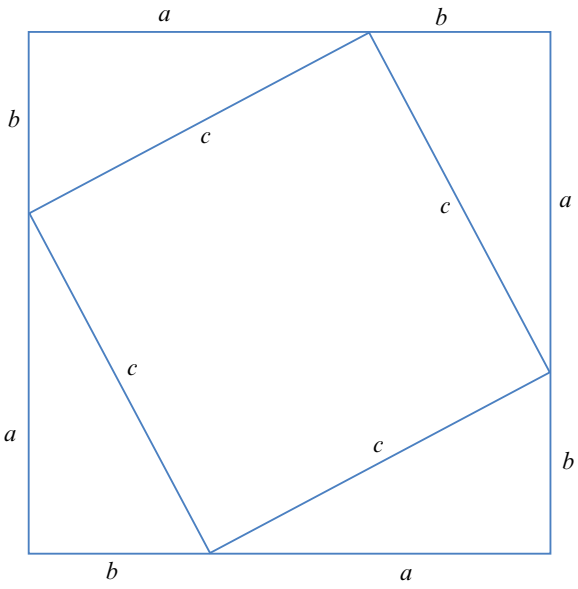
g Explain why you now know that ∠WXY = ∠WZY.

h Write the proof in full.

PROBLEM SOLVING AND REASONING

11 Use what you have proven in question **10** to prove that the angles of an equilateral triangle are 60°.

12 Pythagoras' Theorem is another theorem that is reasonably simple to prove. Consider this diagram, which shows one square drawn inside another so that four congruent right-angled triangles are formed.



a How do you know that the hypotenuse of each triangle is equal in length?

b Explain why the area of the larger square can be represented by the equation $A = (a + b)^2$.

c Write an expression for the area of the smaller square.

d Write an expression for the area of one of the triangles.

e Explain why the area of the smaller square plus the four triangles can be represented by the equation $A = c^2 + 2ab$.

f Why can you now write $(a + b)^2 = c^2 + 2ab$?

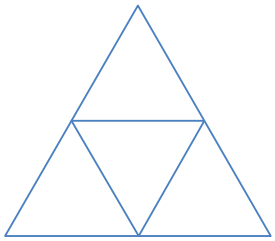
g Expand and simplify the expression to prove Pythagoras' Theorem.

h Write the proof in full.

CHALLENGE

13 Prove that the four right-angled triangles in question **12** are congruent and hence show that the corresponding angles in the triangles are equal.

14 Prove that drawing a triangle along the midpoints of an equilateral triangle splits that equilateral triangle into four congruent, equilateral triangles. (Hint: you will need to make use of your knowledge of angles around a straight line and the isosceles triangle theorem.)



15 Use the Internet to research other proofs involving triangles.

Reflect

How are angles in parallel lines useful in proofs involving triangles?

7F Proofs and quadrilaterals

Start thinking!

Different quadrilaterals have different properties, and you can use your understanding of angles in parallel lines and triangles to prove these properties.

Shape	Properties
Parallelogram	Opposite angles are equal Opposite sides are parallel and equal Diagonals bisect each other
Rhombus	All sides are equal and opposite sides are parallel Opposite angles are equal Diagonals bisect each other at right angles Diagonals bisect the interior angles
Square	All sides are equal and opposite sides are parallel All angles are right angles Diagonals are equal in length and bisect each other at right angles Diagonals bisect the interior angles
Rectangle	Opposite angles are equal Opposite sides are parallel and equal Diagonals are equal and bisect each other
Trapezium	One pair of opposite sides is parallel

1 For each quadrilateral:
- **a** draw an example
- **b** demonstrate that each property is true.

2 Use the table of properties to decide if each statement is true or false. If false, give a reason.
- **a** A square is a rhombus.
- **b** A **trapezium** is a parallelogram.
- **c** A parallelogram is a rectangle.
- **d** A rectangle is a parallelogram.
- **e** A square is a parallelogram.
- **f** A rhombus is a square.

3 How might you prove properties involving:
- **a** angles?
- **b** side lengths?
- **c** diagonals?

4 Why do you think that congruence might be important in these proofs?

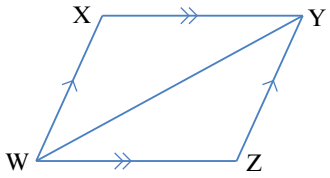
KEY IDEAS

- Quadrilaterals have many properties that can be shown using mathematical proofs.
- Understanding of angle and triangle properties is essential to these proofs.
- Bisect means to cut in half.
- The symbol for 'is parallel to' is $\parallel$.
- The symbol for 'is perpendicular to' is $\perp$.

EXERCISE 7F Proofs and quadrilaterals

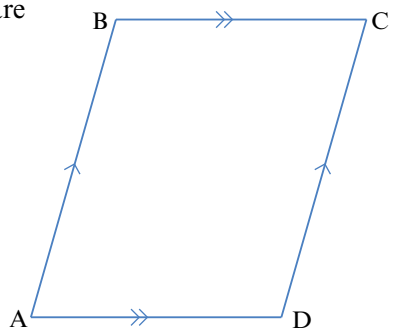
EXAMPLE 7F-1 Proving opposite sides of a parallelogram are equal

Prove that the opposite sides of a parallelogram are equal in length.

THINK	WRITE
1 Draw and label a parallelogram. Mark in a diagonal.	
2 Name the two triangles.	The two triangles are ΔWXY and ΔYZW.
3 Name the common side (the diagonal of parallelogram).	The common side is WY.
4 Use knowledge of angles and parallel lines to show angle equivalence.	∠XYW = ∠ZWY (alternate angles) ∠YWX = ∠WYZ (alternate angles)
5 Use a congruence condition to show the triangles are congruent.	ΔWXY ≅ ΔYZW (AAS)
6 Show that the opposite sides of the parallelogram are equal in length by matching corresponding sides of the congruent triangles.	WX = YZ because they are corresponding sides of congruent triangles. WZ = YX because they are corresponding sides of congruent triangles.
7 Complete the proof with a closing statement.	Therefore, the opposite sides of a parallelogram are equal in length.

UNDERSTANDING AND FLUENCY

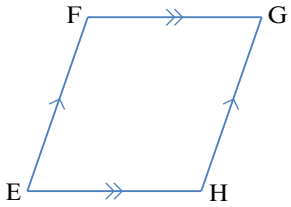
1 Prove that the opposite sides of this parallelogram are equal in length by following these steps.



- **a** Copy the parallelogram and draw a diagonal between vertices A and C.
- **b** Name the two triangles formed by the diagonal AC.
- **c** Name the side common to both triangles.
- **d** Use an understanding of angles and parallel lines to show angle equivalence.
- **e** Use a congruence condition to show the triangles are congruent.
- **f** Show that the opposite sides of the parallelogram are equal in length by matching corresponding sides of the congruent triangles.

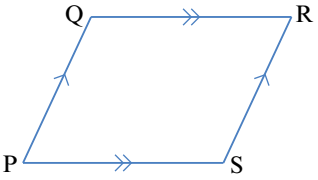
2 Draw the parallelogram LMNO and prove that its opposite sides are equal in length.

3 If FG = EF, prove that this shape is a rhombus.



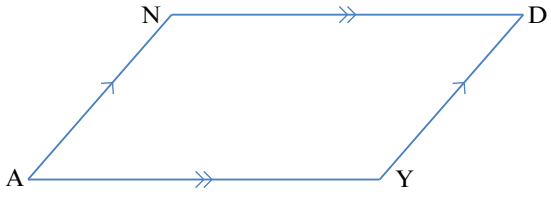
EXAMPLE 7F-2 Proving opposite angles in a parallelogram are equal

Prove that the opposite angles of a parallelogram are equal in size.

THINK	WRITE
1 Draw and label a parallelogram.	
2 Use an understanding of angles and parallel lines to write statements about co-interior angles.	∠QPS + ∠PSR = 180° (co-interior angles) ∠QPS + ∠PQR = 180° (co-interior angles)
3 Equate the left sides of the equations.	∠QPS + ∠PSR = ∠QPS + ∠PQR
4 Simplify by subtracting the same angle from both sides.	∠PSR = ∠PQR
5 Repeat steps **2**–**4** for the other pair of opposite angles.	Similarly, ∠PSR + ∠QRS = 180° (co-interior angles) ∠PSR + ∠QPS = 180° (co-interior angles) ∠PSR + ∠QRS = ∠PSR + ∠QPS ∠QRS = ∠QPS
6 Complete the proof with a closing statement.	Therefore, the opposite angles of a parallelogram are equal in size.

UNDERSTANDING AND FLUENCY

4 Prove that the opposite angles of this parallelogram are equal by following these steps.



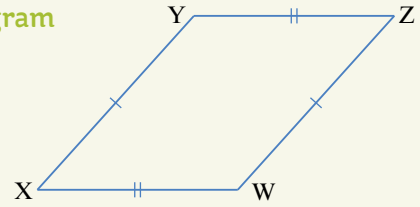
- a Use an understanding of angles and parallel lines to write statements about co-interior angles.
- b Equate the left sides of the equations and simplify the resulting equation by subtracting the same angle from both sides.
- c Repeat steps a and b to show equivalence for the other angle pair.
- d Complete the proof with a closing statement.

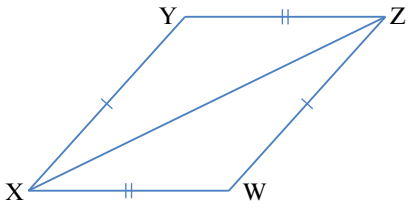
5 Draw the parallelogram MARK and prove that its opposite angles are equal.

6 Draw the rhombus ISAC and prove that its opposite angles are equal.

EXAMPLE 7F-3 Proving a shape is a parallelogram

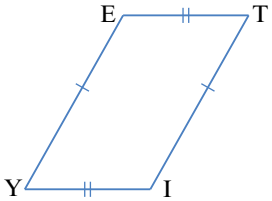
Prove that XYZW is a parallelogram.



THINK	WRITE
1 Copy the figure and include a diagonal.	
2 Show that the two triangles formed are congruent.	XY = WZ YZ = WX XZ is common side ΔXWZ ≅ ΔZYX (using SSS)
3 Match the corresponding angles in these congruent triangles.	∠WXZ = ∠YZX ∠XZW = ∠ZXY
4 If alternate angles are equal, then the lines must be parallel. Note: ‖ means 'is parallel to'.	YZ ‖ WX because the alternate angles ∠WXZ and ∠YZX are equal. XY ‖ ZW because the alternate angles ∠XZW and ∠ZXY are equal.
5 Use the definition of a parallelogram to prove that WXYZ is a parallelogram.	WXYZ is a parallelogram because opposite sides are parallel and equal in length.

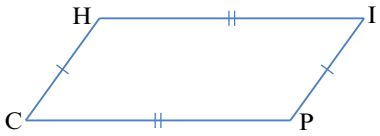
UNDERSTANDING AND FLUENCY

7 Prove that YETI is a parallelogram by following these steps.



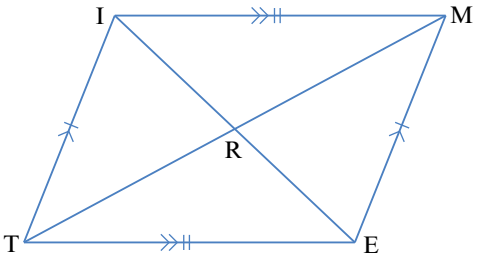
- a Copy the figure and include a diagonal.
- b Show that the two triangles formed are congruent.
- c Match the corresponding angles in these congruent triangles.
- d If alternate angles are equal, then the lines must be parallel.
- e Use the definition of a parallelogram to prove that YETI is a parallelogram.

8 Prove that CHIP is a parallelogram.



9 Prove that a quadrilateral ABCD with four equal side lengths is a rhombus.

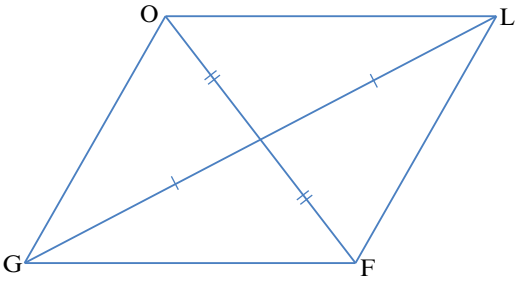
10 Consider this parallelogram.



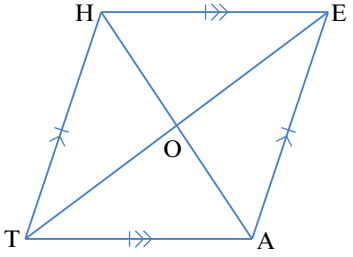
- **a** What angle is equal to ∠IMR?
- **b** What angle is equal to ∠MIR?
- **c** Show that ΔMIR ≅ ΔTER.
- **d** Copy the figure and mark corresponding sides for the congruent triangles.
- **e** Explain how this shows that the diagonals of the parallelogram TIME bisect each other.
- **f** Write the proof in full.

11 Draw a parallelogram and prove that its diagonals bisect each other.

12 Prove that this shape is a parallelogram.



13 Consider this rhombus.



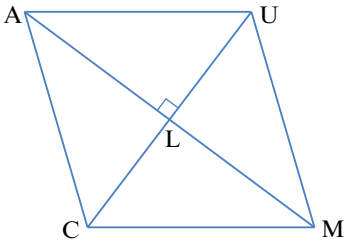
- **a** Follow the steps from question **10** to prove that the diagonals bisect each other.
- **b** Hence show that the four triangles are congruent to one another.
- **c** Redraw the rhombus with markings showing the corresponding sides.
- **d** Show that ∠HOE = ∠AOE.
- **e** Use your understanding of angles around a straight line to prove that the diagonals bisect each other at right angles.
- **f** Write the proof in full.

> **NOTE** Remember you can use the symbol ⊥ for 'is perpendicular to' or 'at right angles to'.

14 Draw a rhombus and prove that its diagonals bisect each other at right angles.

15 Prove that this is a rhombus if:

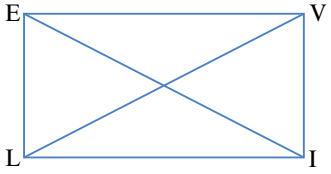
- **a** AL = ML and CL = UL
- **b** AU ∥ MC and AC ∥ MU.



16 Prove that parallelogram REMY is a rectangle if ∠REM = 90°.

PROBLEM SOLVING AND REASONING

17 Consider figure LEVI, where EV = IL, EV || IL and ∠EVI = 90°.



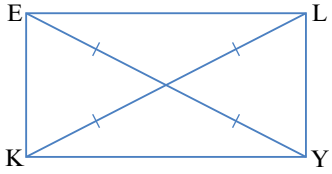
a Prove that the remaining angles are 90°.

b Prove that EL = IV and EL || IV.

c Use Pythagoras' Theorem to show that LV = EI.

d Hence prove that ΔLEV ≅ ΔELI.

18 Prove that this shape is a rectangle by following these steps.



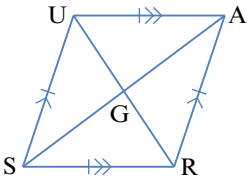
a Use the isosceles triangle theorem and what you know about equal alternate angles to first prove that it is a parallelogram.

b Use ∠LYE + ∠YEL + ∠ELK + ∠KLY to prove that ∠ELY = 90°.

c Prove that all interior angles are 90°.

19 Draw the quadrilateral DANE where the diagonals DN and AE are equal and bisect each other. Prove that it is a rectangle.

20 Consider this rhombus.



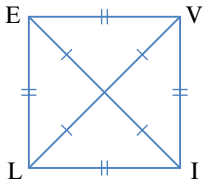
a Prove that UG = RG and AG = SG (that the diagonals bisect each other).

b Prove that ΔUAG ≅ ΔRAG.

c Hence show that ∠UAS = ∠RAS.

d Explain why this shows that the diagonals of a rhombus bisect the interior angles.

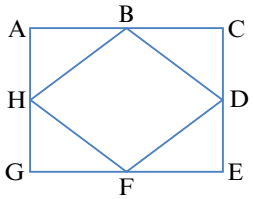
21 Prove that ∠EVL = 45°.



22 What property of rhombuses (including squares) means that the diagonals bisect the internal angles?

CHALLENGE

23 ACEG is a rectangle. If the vertices of HBDF touch the midpoints of each of the rectangle's sides, prove that HBDF is a rhombus.



Reflect

How are congruent triangles useful in proofs involving quadrilaterals?

7G Circle geometry: circles and angles

Start thinking!

There are many different theorems involving circles that allow you to solve problems. Before working with these theorems, it is important you understand the special terms used in circle geometry.

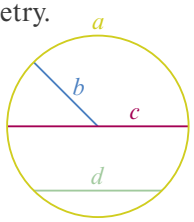


Figure 1

1 Consider each term in this list:
arc, centre, **chord**, **circumference**, **diameter**, **radius**, **sector**, **segment**

- **a** Look them up in the glossary and write a definition.
- **b** Match each term with one of the pronumerals, *a*–*h*, in figures 1 and 2.

2 Consider figure 3.

- **a** Name the two chords.
- **b** Name the minor arc formed by the two chords.
- **c** Name the angle formed by the two chords.

In circle geometry, you say that the angle is subtended by the arc, rather than saying that an angle is formed by two chords.

- **d** Copy and complete this sentence.
  The arc _____ subtends the angle _____.
- **e** Imagine that there is a chord AC. Copy and complete this sentence about it:
  The chord AC subtends the angle _____.

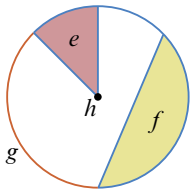


Figure 2

3 Explain what 'subtend' means.

4 Why is it important to be familiar with the terms used in circle geometry before carrying out any proofs?

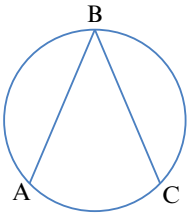


Figure 3

KEY IDEAS

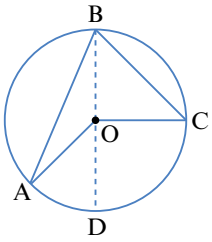
- Understanding the terms arc, centre, chord, circumference, diameter, radius, sector and segment is important in circle geometry. If necessary, refer to the glossary.
- An angle is subtended by the chord or arc that connects the endpoints of the two chords that form the angle.
- Theorem 1: the angle subtended at the centre of the circle is twice the size of the angle on the circumference subtended by the same arc.
- Theorem 2: any angle subtended by the diameter is a right angle.
- Theorem 3: all angles at the circumference of a circle that are subtended by the same arc are equal in size.

EXERCISE 7G Circle geometry: circles and angles

UNDERSTANDING AND FLUENCY

1 Draw an example of a circle with each of these.
- a a chord XY
- b an arc between points S and T
- c the major segment formed by the chord UP
- d the minor sector formed by the radii OM and ON
- e the arc that subtends ∠XYZ
- f a chord that subtends ∠DEF

2 Consider this diagram.

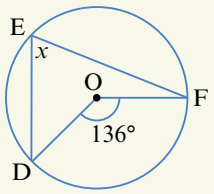


- a How do you know that AO = BO = CO?
- b How do you know that ∠ABO = ∠OAB?
- c Use the theorem for the exterior angle of a triangle to show that ∠AOD = 2∠ABO.
- d Use your answers to parts b and c to show that ∠DOC = 2∠OBC.
- e Explain how this shows that ∠AOC = 2∠ABC.

This question demonstrates Circle Theorem 1:
the angle subtended at the centre of the circle is twice the size of the angle on the circumference subtended by the same arc.

EXAMPLE 7G-1 Using Circle Theorem 1

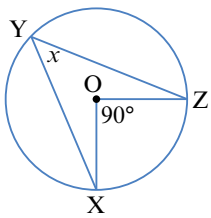
Find the value of x in this circle.



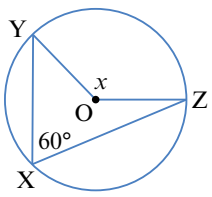
THINK	WRITE
1 Using Theorem 1 means that ∠DOF = 2∠DEF.	$136^\circ = 2x$
2 Write your answer.	$x = 68^\circ$

3 Find the value of x in each circle.

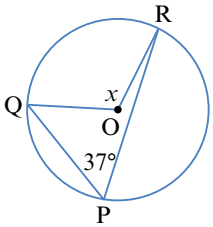
a



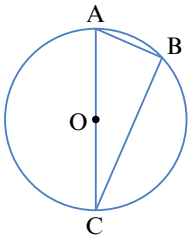
b



c



4 Consider this circle.



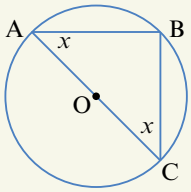
This question demonstrates Circle Theorem 2:
any angle subtended by the diameter is a right angle.

a What is the angle at the centre of the circle?

b Use Theorem 1 to show that $\angle ABC = 90^\circ$.

EXAMPLE 7G-2 Using Circle Theorem 2

Find the value of x in this circle.

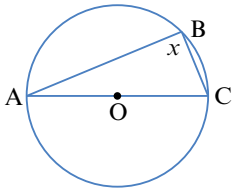


THINK	WRITE
1 Using Theorem 2, $\angle ABC$ is a right angle.	$\angle ABC = 90^\circ$
2 Use the angle sum of a triangle to write an equation to find the value of x.	$2x + 90^\circ = 180^\circ$
3 Solve the equation and write your answer.	$2x = 90^\circ$ $x = 45^\circ$

UNDERSTANDING AND FLUENCY

5 Find the value of x in each circle.

a



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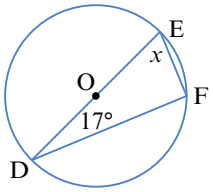

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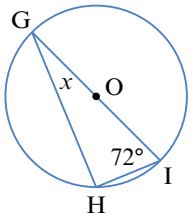
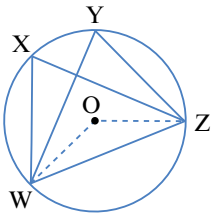


6 Consider this circle.



This question demonstrates Circle Theorem 3:
All angles at the circumference of a circle that are subtended by the same arc are equal in size.

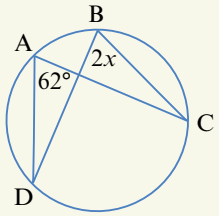
a Use Theorem 1 to explain why $\angle WOZ = 2\angle WXZ$.

b Similarly, show that $\angle WOZ = 2\angle WYZ$.

c What can you say about $\angle WYZ$ and $\angle WXZ$?

EXAMPLE 7G-3 Using Circle Theorem 3

Find the value of x in this circle.

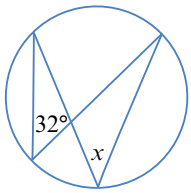


THINK	WRITE
1 Using Theorem 3 means that ∠DBC and ∠DAC are equal in size.	∠DBC = ∠DAC
2 Substitute values and solve for x.	$2x = 62°$ $x = 31°$

PROBLEM SOLVING AND REASONING

7 Find the value of each pronumeral.

a



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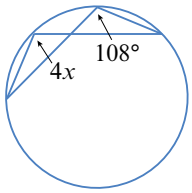

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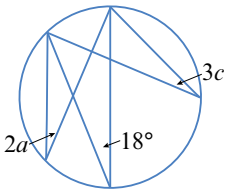
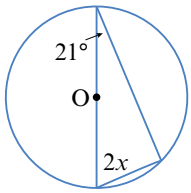


8 Find the value of each pronumeral, giving a reason.

a



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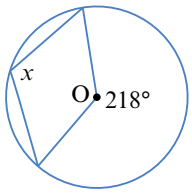

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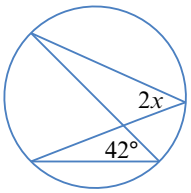

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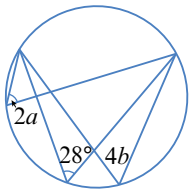

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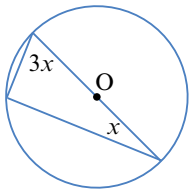

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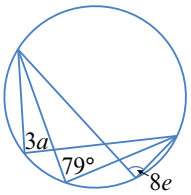

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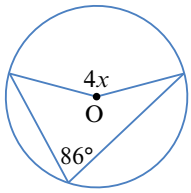

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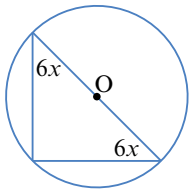

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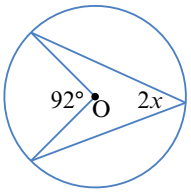
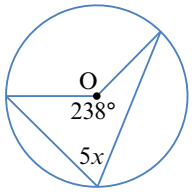


9 Use your understanding of triangle and circle properties and theorems to find the value of x.

a



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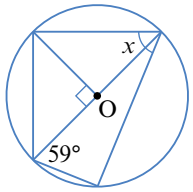

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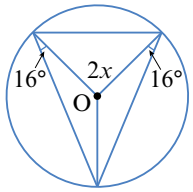

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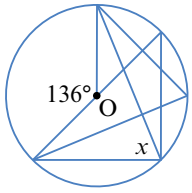

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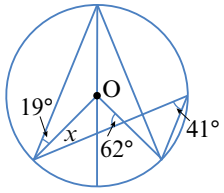

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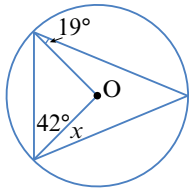
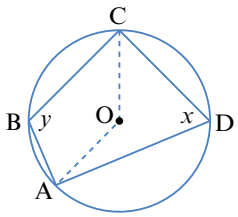


10 Consider this diagram, showing a cyclic quadrilateral (a quadrilateral that has each of its vertices on the circumference of a circle).



a Explain why the obtuse angle $\angle AOC = 2x$ and the reflex angle $\angle AOC = 2y$.

b Use angles around a point to explain why the reflex $\angle AOC = 360^\circ - 2x$.

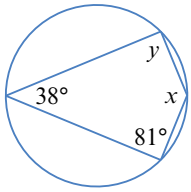
c Use your answers from parts **a** and **b** to show that $y = 180^\circ - x$.

d Copy the figure, replacing radii OC and OA with OB and OD, and use this to show that $\angle BAD = 180^\circ - \angle BCD$.

e Find the value of each pronumeral.

This question demonstrates Circle Theorem 4:
the opposite angles of a cyclic quadrilateral are supplementary.

i



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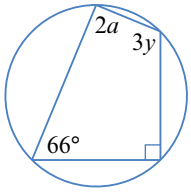
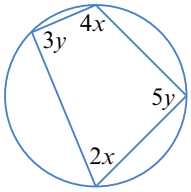
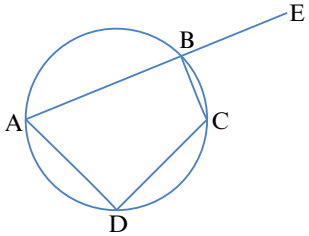


iii



11 Consider this diagram, showing another cyclic quadrilateral.



a Use Theorem 4 to explain why $\angle ADC = 180^\circ - \angle ABC$.

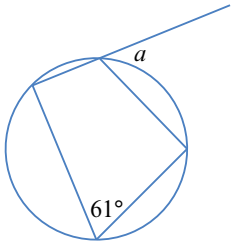
b Why does $\angle EBC = 180^\circ - \angle ABC$?

c Explain why this shows that $\angle ADC = \angle EBC$.

This question demonstrates Circle Theorem 5:
the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

d Find the value of each pronumeral.

i



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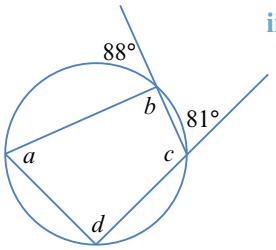

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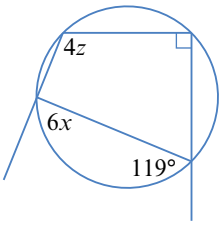
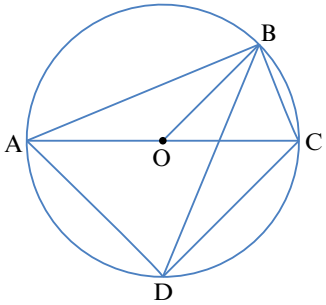


CHALLENGE

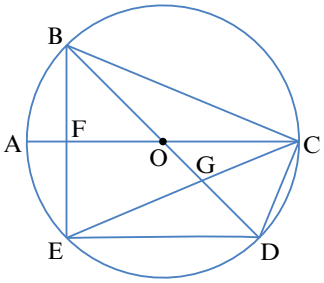
12 Write a complete proof for each of the Circle Theorems, 1 to 5.

13 Use the given information to answer these.

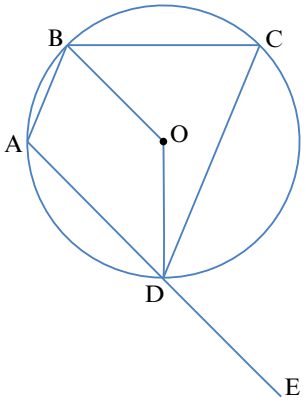
a Find ∠DBC, given that ∠ADB = 68°, OA = OB = OC and ∠BCD = 92°.



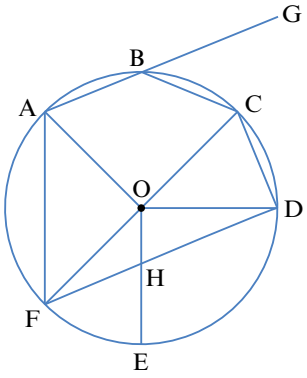
b Find ∠DEG, given that BE ⊥ AC, ∠OCG = 19° and ∠FOG = 142°.



c Find ∠ODC, given that ∠BAD = 96° and ∠CBO = 46°.



d Find ∠OAB, given that ∠GBC = 46°, ∠BCD = 138° and ∠AFD = 70°.



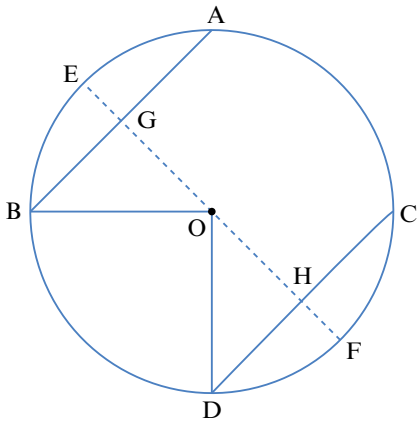
14 Use dynamic geometry software or other digital technology to demonstrate each theorem.

Reflect

How are all the proofs in this section reliant on triangle properties?

7H Circle geometry: chords

Start thinking!



Consider this diagram, ignoring the dashed line to begin with.

1 Name the two chords shown.

2 Name the two radii shown.

3 The two radii are perpendicular to one another. What does this mean?

4 Name the arc that subtends ∠BOD.

Now consider the dashed line.

5 Name the dashed line and describe it using circle terminology.

6 EF bisects AB and DC. What does this mean?

GO = OH. This means that CD and AB are **equidistant** from the centre of the circle.

7 What does equidistant mean?

8 Explain how you know CD and AB are equidistant from the centre of the circle.

9 What else can you describe from this diagram?

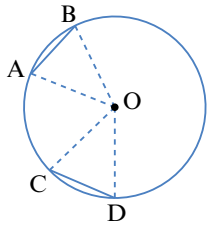
KEY IDEAS

- A chord is a straight line from one point on the circumference of a circle to another.
- Theorem 6: chords that are equal in length subtend equal angles at the centre of the circle
- Theorem 7: if a radius and a chord intersect perpendicularly, then the radius bisects the chord.
- Theorem 8: chords that are equal in length are equidistant from the circle centre.
- Theorem 9: when two chords intersect inside a circle, this divides each chord into two line segments in which the product of the lengths of the line segments for both chords is the same.

EXERCISE 7H Circle geometry: chords

UNDERSTANDING AND FLUENCY

1 Consider this diagram, where AB = CD.



This question demonstrates Circle Theorem 6:
chords that are equal in length subtend equal angles at the centre of the circle.

a Given that OA, OB, OC and OD are radii, what can you say about them?

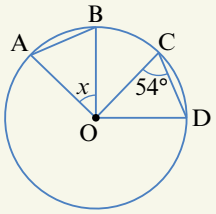
b What type of triangle does this mean ΔOAB and ΔOCD are?

c Explain why ΔOAB ≅ ΔOCD.

d Explain why ∠COD = ∠AOB.

EXAMPLE 7H-1 Using Circle Theorem 6

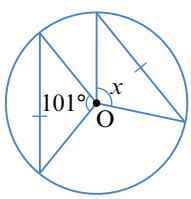
Find the value of x in this circle if AB = CD.



THINK	WRITE
1 Since OC and OD are radii, they have equal length.	OC = OD
2 This means that ΔCOD is an isosceles triangle so you can find the other two angles within the triangle.	ΔCOD is isosceles, so ∠OCD = ∠ODC = 54° ∠COD = 180° − 54° − 54° = 72°
3 Use Theorem 6 to identify the equal angles at the centre of the circle.	∠AOB = ∠COD = 72°
4 Write your answer.	x = 72°

2 Find the value of x in each circle.

a



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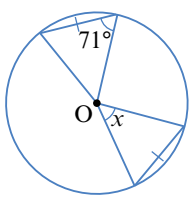

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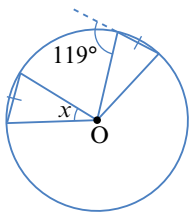


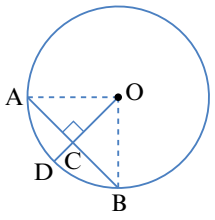
3 Consider this diagram.

a Explain why OA = OB.

b Explain why ∠OCA = ∠OCB.

c Use a triangle congruency condition to explain why ΔACO ≅ ΔBCO.

d Explain why AC = CB.

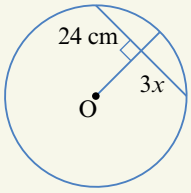


This question demonstrates Circle Theorem 7:

if a radius and a chord intersect perpendicularly, then the radius bisects the chord.

EXAMPLE 7H-2 Using Circle Theorem 7

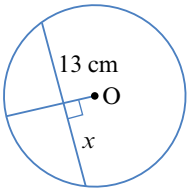
Find the value of x in this circle.



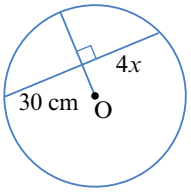
THINK	WRITE
1 The radius and chord intersect at right angles. Using Theorem 7, the length $3x$ is the same as 24 cm.	$3x = 24$ cm
2 Solve for x.	$x = 8$ cm

4 Find the value of x in each circle.

a



b



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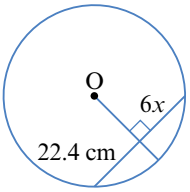


5 Consider this diagram, where AB = CD.

a Explain why OE = OB = OD = OF.

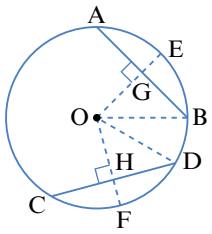
b Use Theorem 7 to explain why AG = GB and CH = HD.

c Likewise, use Theorem 7 and the fact that AB = CD to explain why GB = HD.

d Use a triangle congruency condition to explain why ΔOHD ≅ ΔOGB.

e Explain why OH = OG.

f Use your understanding of Pythagoras' Theorem to explain why the shortest distance from the centre of a circle to a chord is perpendicular to that chord.

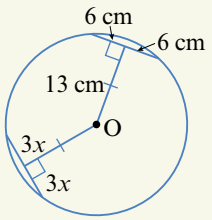


This question demonstrates Circle Theorem 8:

chords that are equal in length are equidistant from the circle centre.

EXAMPLE 7H-3 Using Circle Theorem 8

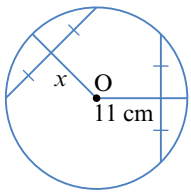
Find the value of x in this circle.



THINK	WRITE
1 The chords are equidistant from the centre circle. Using Theorem 8, the chords must be equal in length.	$3x + 3x = 6 + 6$
2 Solve for x and include the length unit.	$6x = 12$ $x = 2$ cm

6 Find the value of each pronumeral.

a



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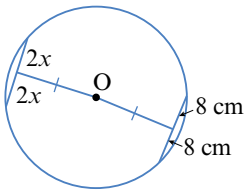

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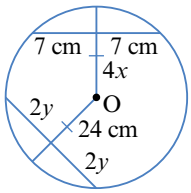
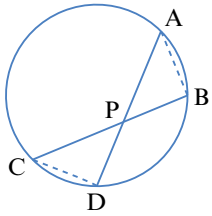


7 Consider this diagram.



a Explain why $\angle CPD = \angle APB$.

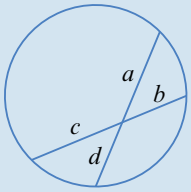
b Use Theorem 3 to explain why $\angle DCB = \angle DAB$.

c Explain why $\angle PDC = \angle PBA$.

d Explain how you can tell that ΔCPD is similar to ΔAPB.

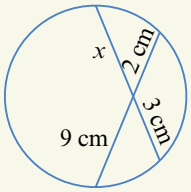
e Explain why you can state that $\frac{AP}{BP} = \frac{CP}{DP}$ and hence that $AP \times DP = CP \times BP$.

This question demonstrates Circle Theorem 9:
when two chords intersect inside a circle, this divides each chord into two line segments in which the product of the lengths of the line segments for both chords is the same. In the figure below, $a \times d = c \times b$.



EXAMPLE 7H-4 Using Circle Theorem 9

Find the value of x in this circle.



THINK

1. Write the product of the lengths of the two line segments for each chord.
2. Use Theorem 9 to write an equation.
3. Simplify the equation.
4. Solve for x and include the length unit.

WRITE

chord 1: $x \times 3$ cm
chord 2: 2 cm $\times$ 9 cm

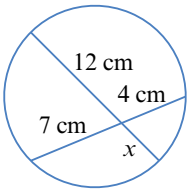
$x \times 3 = 2 \times 9$

$3x = 18$

$x = 6$ cm

8 Find the value of x in each circle, correct to one decimal place.

a



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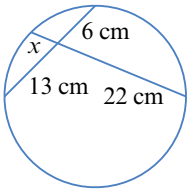

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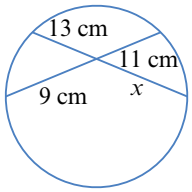
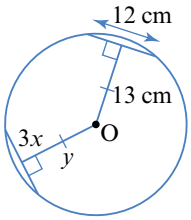


9 Find the value of each pronumeral, giving a reason.

a



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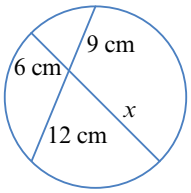

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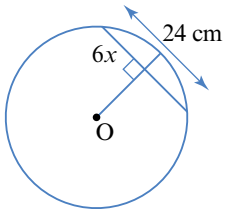

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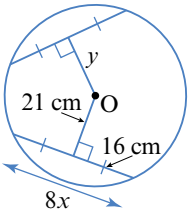

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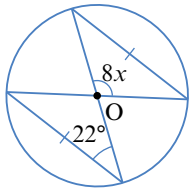

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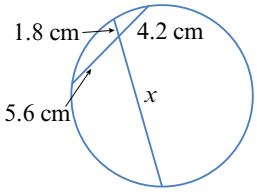

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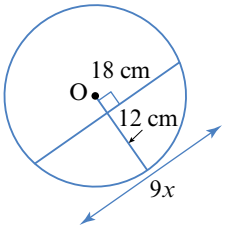

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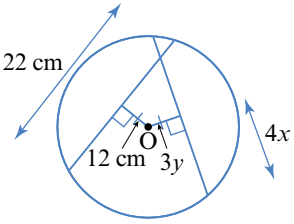

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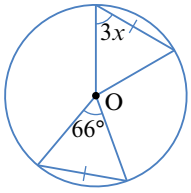
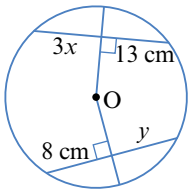


PROBLEM SOLVING AND REASONING

PROBLEM SOLVING AND REASONING

10 Use your understanding of circle properties and theorems, and any other necessary geometric properties, to find the value of each pronumeral.

a



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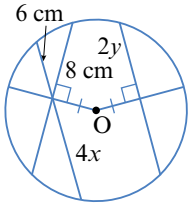

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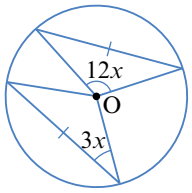

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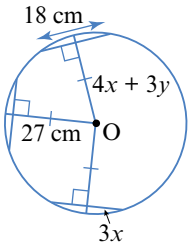

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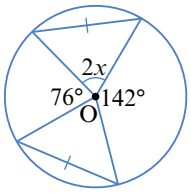

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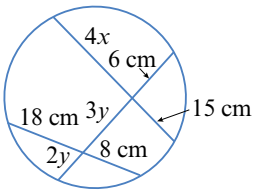
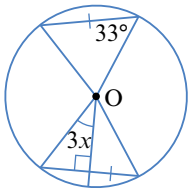


11 Find the value of each pronumeral.

a



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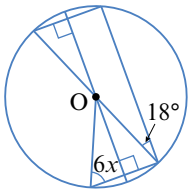

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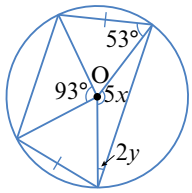

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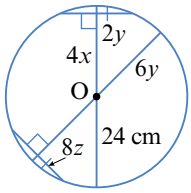

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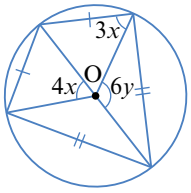

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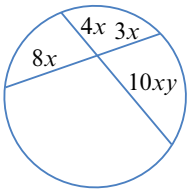


12 Draw a circle with two chords. Show that constructing perpendicular bisectors of these two chords locates the centre of the circle.

CHALLENGE

13 Write a complete proof for each theorem in this exercise.

14 Use dynamic geometry software or other digital technology to demonstrate each theorem.

15 For each theorem in this exercise, create a problem to solve that involves the use of:

a simultaneous equations

b Pythagoras' Theorem (where possible).

16 Use the Internet to investigate the alternate segment theorem. Write out a proof of what you find. Include an example of a problem and solution in your findings.

Reflect

How is the centre of the circle important in many circle theorems?

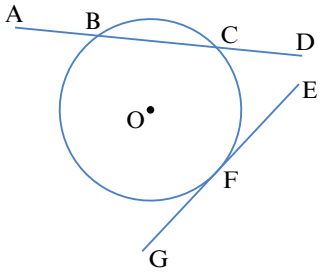
7I Circle geometry: tangents and secants

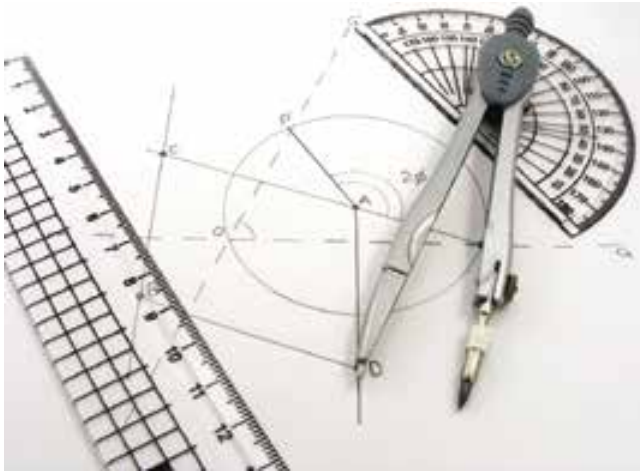
Start thinking!

Two important concepts that have not yet been covered are the **tangent** and the **secant**.

1 Use your glossary or other means to write the definitions of these terms.

2 Name the tangent and the secant on this diagram.





3 At what point does the tangent touch the circumference of the circle?

4 Name the points where the secant cuts the circle.

5 Explain the difference between a chord, a secant, a tangent and an arc.

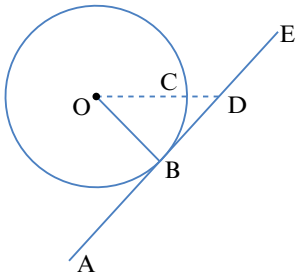
KEY IDEAS

- A secant is a line that cuts a circle twice.
- A tangent is a line that touches the circumference of a circle at one point only.
- Theorem 10: a tangent drawn at the same point to a radius will be perpendicular to that radius.
- Theorem 11: if two tangents intersect outside a circle, the distances along the tangent from the intersection to the circumference of the circle are equal.
- Theorem 12: if two secants intersect outside a circle, the products of the entire secant length by the external secant length will be the same.
- Theorem 13: if a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length will be equal to the square of the tangent length.

EXERCISE 7I Circle geometry: tangents and secants

UNDERSTANDING AND FLUENCY

1 Consider this diagram, showing the intersection of a radius and a tangent.



a Name the two radii and explain why they are equal.

b Name the tangent at point B.

You may guess that ∠OBD = 90° and, to prove this, you instead imagine that another angle, in this case ∠ODB, is 90°.

c Which side would be the hypotenuse of ΔODB?

d This would mean that OB > OD. Why does this not make sense?

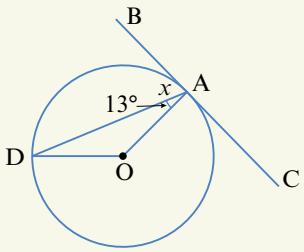
e How does this prove that ∠OBD = 90°?

This question demonstrates Circle Theorem 10:

a tangent drawn at the same point to a radius will be perpendicular to that radius.

EXAMPLE 7I-1 Using Circle Theorem 10

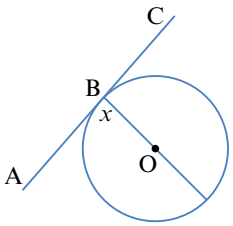
Find the value of x in this diagram.



THINK	WRITE
1 Using Theorem 10 means that the angle between the tangent BC and the radius OA is 90°.	∠OAB = 90°
2 Use your understanding of complementary angles to find x.	$x = 90° - 13°$ $= 77°$

2 Find the value of x in each circle.

a



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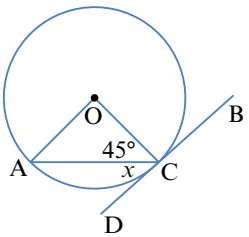
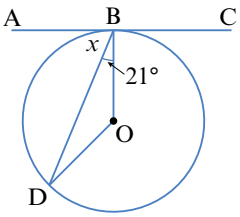
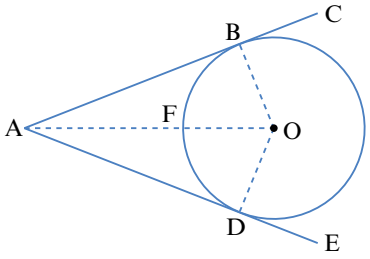


c



3 Consider this diagram, showing two tangents intersecting outside a circle.

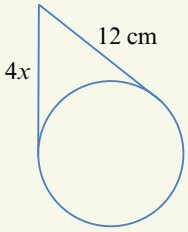


This question demonstrates Circle Theorem 11:
if two tangents intersect outside a circle, the distances from the intersection to the circumference of the circle are equal.

a How do you know that $\angle ABO = \angle ADO = 90^\circ$?
b Explain why OB = OD.
c Explain why $\Delta ABO \cong \Delta ADO$.
d Explain why this means that AB = AD.

EXAMPLE 7I-2 Using Circle Theorem 11

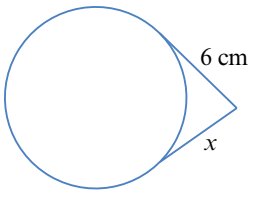
Find the value of x in this diagram.



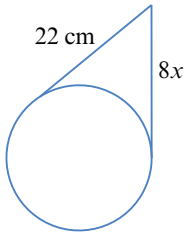
THINK	WRITE
1 Using Theorem 11 means that the distances from the intersection of the two tangents to the circumference of the circle are equal. Write this as an equation.	$4x = 12$
2 Solve for x and include the length unit.	$x = 3$ cm

4 Find the value of each pronumeral.

a



b



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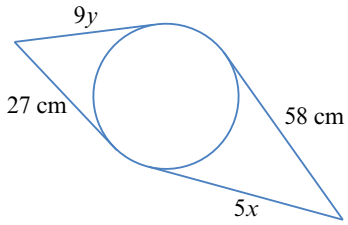
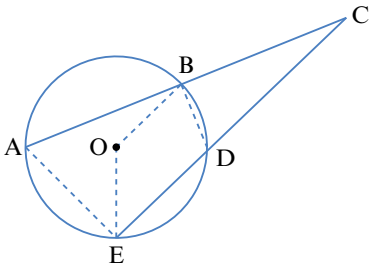


5 Consider this diagram, showing two secants intersecting outside a circle.

a Explain why $\angle BOE = 2\angle BAE$.
b Explain why the reflex angle $\angle BOE = 360^\circ - 2\angle BAE$.
c Explain how you know $\angle BDE = 180^\circ - \angle BAE$.



d Hence explain why ∠BDC = ∠BAE.

Now consider ΔDCB and ΔACE.

e What angle do they have in common?

f What pair of angles have you already shown are equal?

g Explain why these triangles must therefore be similar.

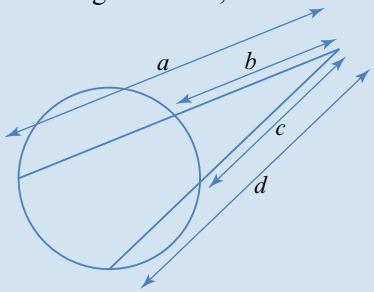
h Show that $\frac{CA}{CD} = \frac{EC}{BC}$.

i Explain why this can also be written as AC × BC = CE × CD.

This question demonstrates Circle Theorem 12:

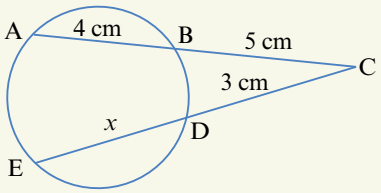
if two secants intersect outside a circle, the products of the entire secant length by the external secant length will be the same.

In the figure below, $a \times b = c \times d$.



EXAMPLE 71-3 Using Circle Theorem 12

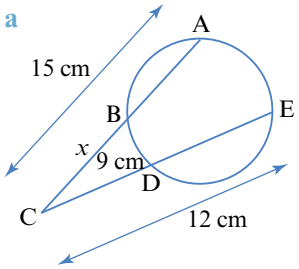
Find the value of x in this diagram.



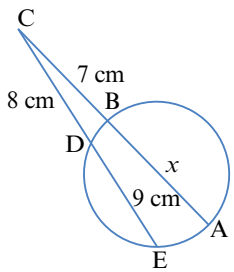
THINK	WRITE
1 Using Theorem 12, write the relationship between the secant lengths.	AC × BC = CE × CD
2 Write the length of each line segment.	AC = 4 + 5 = 9 cm BC = 5 cm CE = (3 + x) cm CD = 3 cm
3 Substitute these lengths into the relationship and simplify.	9 × 5 = (3 + x) × 3 45 = 3(3 + x)
4 Solve for x and include the length unit.	15 = 3 + x x = 12 cm

6 Find the value of x in each diagram.

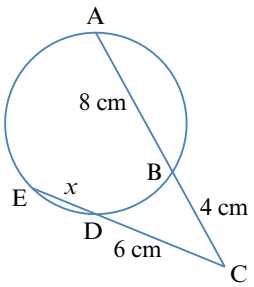
a



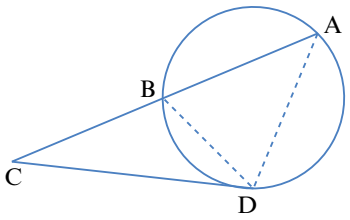
b



c



7 Consider this diagram, showing a tangent and a secant intersecting.



a Which angle do ΔBCD and ΔDAC have in common?

b Construct OD and OB, where O is the centre of the circle, and explain why ∠CAD = ∠CDB.

c Explain why ΔBCD and ΔDCA must be similar.

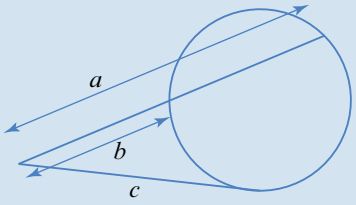
d Explain how you know that $\frac{CB}{CD} = \frac{CD}{CA}$.

e Explain why this can be written as $CA \times CB = (CD)^2$.

This question demonstrates Circle Theorem 13:

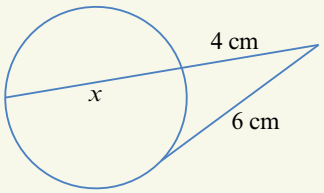
if a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length will be equal to the square of the tangent length.

In the figure below, $a \times b = c^2$.



EXAMPLE 7I-4 Using Circle Theorem 13

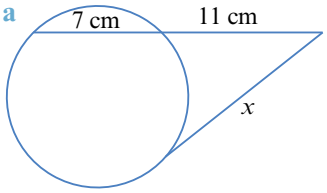
Find the value of x in this diagram.



THINK	WRITE
1 Using Theorem 13, write the relationship for an intersecting secant and tangent.	$a \times b = c^2$
2 Write the length for each line segment.	$a = (x + 4)$ cm $b = 4$ cm $c = 6$ cm
3 Substitute into the formula and simplify.	$(x + 4) \times 4 = 6^2$ $4(x + 4) = 36$
4 Solve for x and include the length unit.	$x + 4 = 9$ $x = 5$ cm

8 Find the value of x in each diagram.

a



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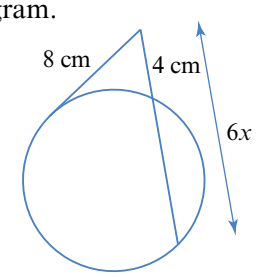

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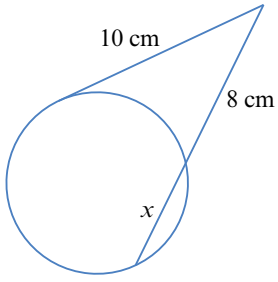
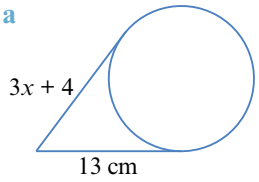


PROBLEM SOLVING AND REASONING

9 Find the value of each pronumeral, giving reasons.

a



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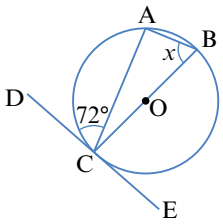

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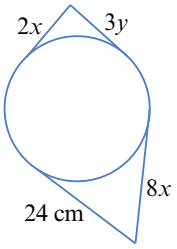

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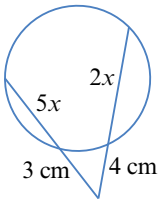

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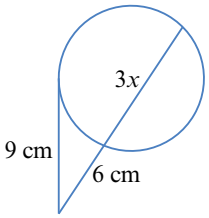

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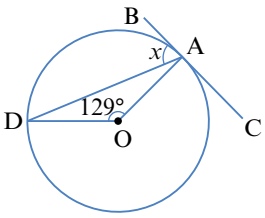
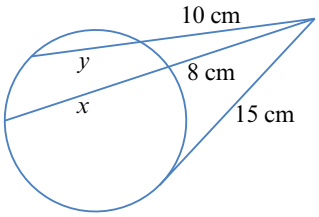


10 Use your understanding of any relevant circle theorems and geometric properties to find the value of each pronumeral, giving reasons.

a



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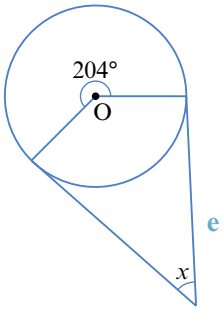

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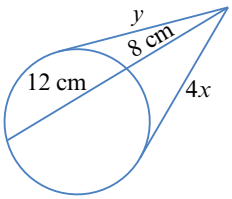


CHALLENGE

11 Write a complete proof for each theorem in this Exercise.

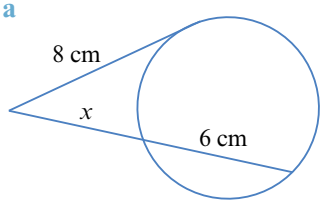
12 For each theorem in this Exercise, create a problem to solve that involves the use of:

a simultaneous equations

b Pythagoras' Theorem (where possible).

13 Find the value of each pronumeral in these diagrams.

a



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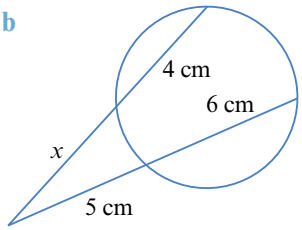

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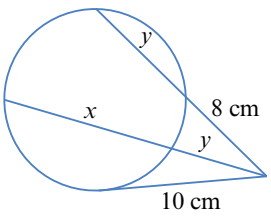


14 The diagram in question **3** shows two tangents that intersect outside a circle. Prove that a straight line drawn from the centre of the circle to the intersection point bisects the angle formed by the two tangents.

15 Use dynamic geometry software or other digital technology to demonstrate each theorem.

Reflect

What geometric properties do the circle theorems rely on?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

corresponding angles	regular polygon	trapezium	circle	diameter
alternate angles	quadrilateral	congruent figures	arc	subtended
co-interior angles	parallelogram	bisect	chord	cyclic quadrilateral
complementary angles	rectangle	similar figures	segment	equidistant
supplementary angles	square	scale factor	sector	tangent
vertically opposite angles	rhombus	axiom	circumference	secant
transversal	kite	proof	radius	line segment

MULTIPLE-CHOICE

7A **1** What angle is supplementary to 25°?

A 75° **B** 155° **C** 65° **D** 335°

7B **2** $\Delta ANP \cong \Delta GWK$, where the order of the letters represents corresponding vertices of the two triangles. Which statement is correct?

A $\angle APN = \angle KGW$

B $\angle PAN = \angle KWG$

C $\angle NPA = \angle GKW$

D $\angle NAP = \angle WGK$

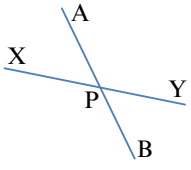
7B **3** Which of these is *not* a test for congruent triangles?

A AAA **B** SSS **C** SAS **D** AAS

7C **4** Which of these is *not* a test for similar triangles?

A AAA **B** SSS **C** SAS **D** AAS

7D **5** Straight lines AB and XY intersect at P. Given that $\angle APY = 2\angle APX$, which statement is true?



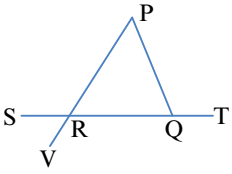
A $\angle BPX = 2\angle APY$

B $\angle XPB = 2\angle APX$

C $\angle YPB = 2\angle APX$

D $\angle APX = 2\angle BPY$

7E **6** Which statement is true?



A $\angle RPQ = \angle RQP$

B $\angle PRS = \angle PQT$

C $\angle TQP = \angle PRQ + \angle RQP$

D $\angle RPQ + \angle PQR = \angle SRP$

7F **7** Which one is *not* a parallelogram?

A square **B** kite

C rhombus **D** rectangle

7G 10A **8** The angle in the centre of a circle and the angle at the circumference subtended by the same arc are complementary angles. Which values for the two angles are possible?

A 60° and 30° **B** 45° and 45°

C 90° and 45° **D** 120° and 60°

7H 10A **9** A circle contains two chords of the same length. Which of these is *not* true?

A The arcs on which they stand are the same length.

B The chords are the same perpendicular distance from the centre.

C The chords can not intersect.

D The chords subtend equal angles at the centre of the circle.

SHORT ANSWER

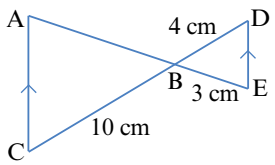
7A **1** Decide whether each statement is true or false. For any false statement, provide a reason.

- **a** The diagonals of a kite intersect at 90°.
- **b** It is not possible for a triangle to have a concave angle.
- **c** If a kite has four equal sides, it can be classified as a rhombus.
- **d** It would be possible to construct a triangle with side lengths of 3 cm, 7 cm and 4 cm.

7B **2** A rectangle ABCD has side lengths of 4 cm and 3 cm. Its diagonals intersect at the point P.

- **a** Draw a diagram to represent this rectangle.
- **b** Explain why ∠ABP is equal to ∠CDP, and ∠PAB is equal to ∠PCD.
- **c** Explain why ΔPAB ≅ ΔPCD.
- **d** If ∠BDC = 37°, find the size of ∠CBD.

7C **3** **a** Provide a reason why ΔABC is similar to ΔEBD.

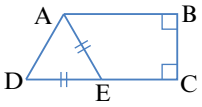


- **b** Consider ΔEBD to be the image of ΔABC. What is the scale factor?
- **c** What is the length of AB?

7D **4** ΔABC has an internal line drawn from the vertex B, meeting the opposite side AC at the point D. These are the facts known about this triangle.
∠ABD + ∠DBC = 90°
∠BAD + ∠BCD = 90°
∠BAD + ∠ADB + ∠ABD = 180°

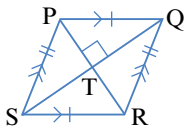
- **a** Draw a sketch of this triangle.
- **b** Use only the given information to prove that ∠ADB = ∠DBC + ∠BCD.

7E **5** Refer to this diagram and write a proof to show that:

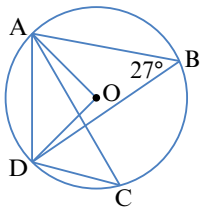


- **a** ∠AEC = 2∠ADE
- **b** 2∠ADE + ∠BAE = 180°

7F **6** Use the information in the diagram below to prove that figure PQRS is a rhombus.

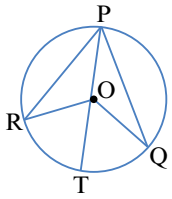


7G 10A **7** Determine the size of these angles.

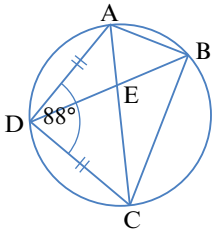


- **a** ∠AOD
- **b** ∠ACD
- **c** ∠OAD

7G 10A **8** Prove that 2∠RPO + 2∠QPO + ∠POR + ∠POQ = 360°



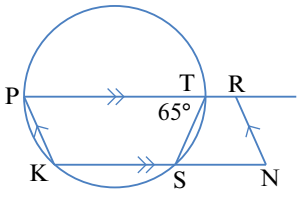
Questions **9** and **10** refer to this diagram.



7H 10A **9** Determine the size of ∠DBC.

7H 10A **10** Prove that ∠DBC = ∠ABD.

Questions **11** and **12** refer to this diagram.



7I 10A **11** Determine the size of ∠KPT.

7I 10A **12** Prove that ∠TSN = ∠RNS

MIXED PRACTICE

1 If $\angle ABC + \angle CBD = 180°$ and $\angle EBD + \angle DBC = 180°$, write a relationship between $\angle ABC$ and $\angle EBD$.

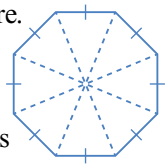
2 $\Delta ADM \cong \Delta RWZ$, where the order of the letters represents corresponding vertices of the two triangles. These details are known about ΔADM: $\angle DAM = 90°$, $\angle AMD = 67°$, AM = 5 cm, AD = 12 cm, DM = 13 cm
Use congruency to find these.

a RZ **b** $\angle WRZ$
c WZ **d** $\angle RWZ$.

3 ΔXGR is similar to ΔKFY, where the order of the letters represents corresponding vertices of the two triangles. Which of the following statements about the two triangles is true?

A $\angle GRX = \angle FKY$ **B** $\frac{XG}{KF} = \frac{RX}{YF}$
C $\angle RXG = \angle KFY$ **D** $\frac{FY}{GR} = \frac{KY}{XR}$

4 **a** Name the shape of this figure.



b Give the size of each internal angle.
There are eight smaller triangles within the figure.
c What is the size of the central angle of each of these triangles?
d Explain whether these triangles are similar.
e Are the triangles all congruent to each other? Give a reason for your answer.

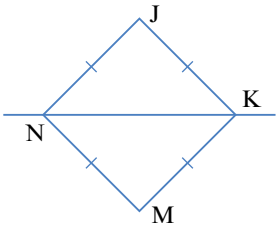
5 Explain these statements.
a Two similar figures may also be congruent.
b Two congruent figures must also be similar.

6 If $a + b = 180°$, $b + c = 180°$ and $c + d = 180°$, which of these is true?
A $a + d = 180°$ **B** $a = b$
C $b = c$ **D** $a + c = 180°$

7 A rational number n can be represented in the form $\frac{x}{y}$, where $y \neq 0$, and x and y have no common factors. Explain whether the reciprocal of n is also a rational number.

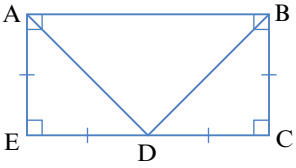
8 A regular polygon has internal angles of 150°.
a Write a formula you could use to determine the number of sides in the polygon.
b Substitute into this formula to calculate the number of sides in the polygon.
c What is the name of this polygon?

9 What would be the simplest congruency condition you could use to prove $\Delta NJK \cong \Delta NMK$?



10 Which statement is *not* correct for two congruent triangles?
A the ratio of corresponding sides is 1
B corresponding sides are equal in length
C corresponding angles are equal in size
D they are not similar

Questions **11** and **12** refer to this diagram.

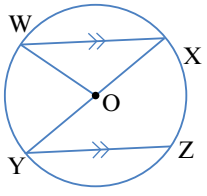


This represents the back view of a rectangular envelope with the flap folded down. The envelope is twice as long as it is wide.

11 Using only the information provided in the diagram, which congruency condition could you use to prove $\Delta ADE \cong \Delta BDC$?
A RHS **B** SSS **C** SAS **D** AAS

12 Prove that $\angle ADB$ is a right angle.

Questions **13** and **14** refer to this diagram.

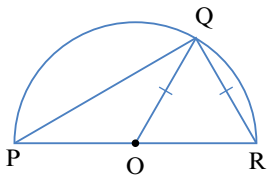


10A **13** Find a relationship between $\angle WOY$ and $\angle OYZ$.

10A **14** Which statement is incorrect?

A OW = OX
B ∠XWO = ∠WXO
C ∠WOY = ∠OWX
D WX = YZ

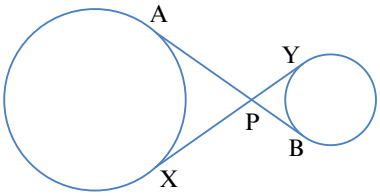
Questions **15** and **16** refer to this diagram.



10A **15** Determine the size of ∠PQO.

10A **16** State the relationship between ∠POQ and ∠OQR.

10A **17** AB and XY are tangents to the two circles. Prove that AB = XY.



10A **18** Prove that, if a cyclic quadrilateral is a parallelogram, then it must be a rectangle.

ANALYSIS

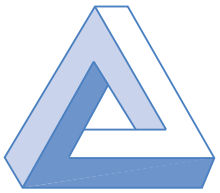
Congruent and similar figures are everywhere in everyday life. These figures can create quite pleasing, and often unusual, images.

1 Car logos

Here is a stylised image of the logo displayed on Volkswagen cars.

a How many sets of parallel lines are in the drawing of the logo?
b The angles within the V and the W are all equal in size. Explain how you know this is so.
c The logo displays vertical symmetry. Does this mean that there are congruent shapes within the letters? Explain.
d Are there any similar shapes within the letters?

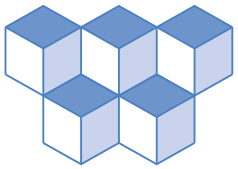
2 Art



This Penrose triangle is said to be 'impossible' because it cannot be constructed as a 3D object. Appropriate shading creates the perception of the 3D image. (Use an Internet search to see a 3D sculpture of this triangle in Perth, Western Australia. There is a trick, however!)

a Trace the figure, and cut around the edges.
b Cut out the three shaded shapes.
c Describe the congruence of these shapes, and explain how they fit together.

3 Quilting



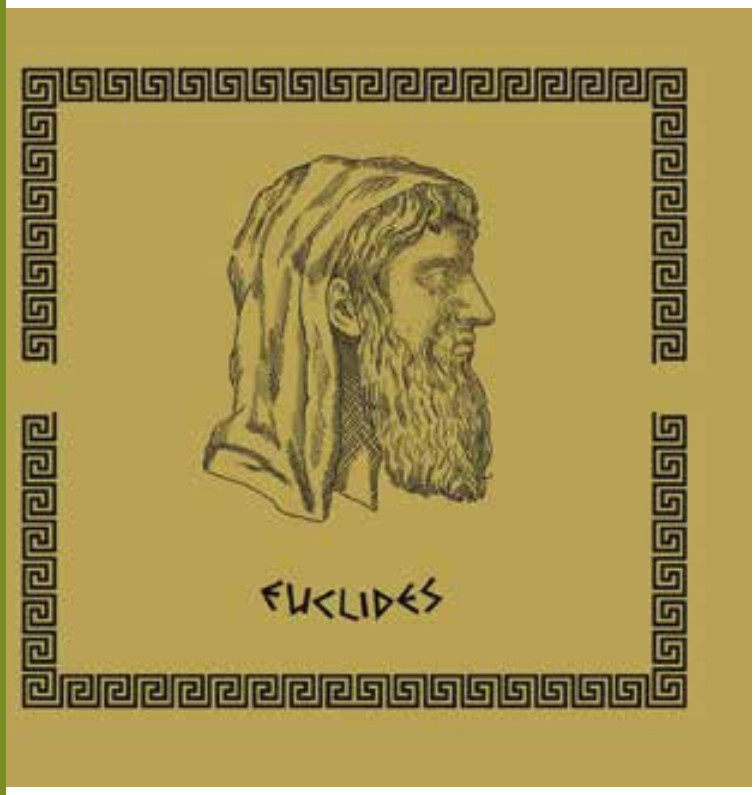
The patterns in quilting are structured around congruent figures.

a How many different congruent shapes are shown in this pattern?
b Describe the shape/s.
c Copy the pattern, and extend it in all directions.

CONNECT

Euclid's elements

Euclid was a Greek mathematician who lived around 300 BC. He is sometimes called the 'Father of Geometry'. His most famous and influential work was his series of 13 books, Elements, in which he set out definitions, axioms, theorems and proofs of the theorems.



Your task

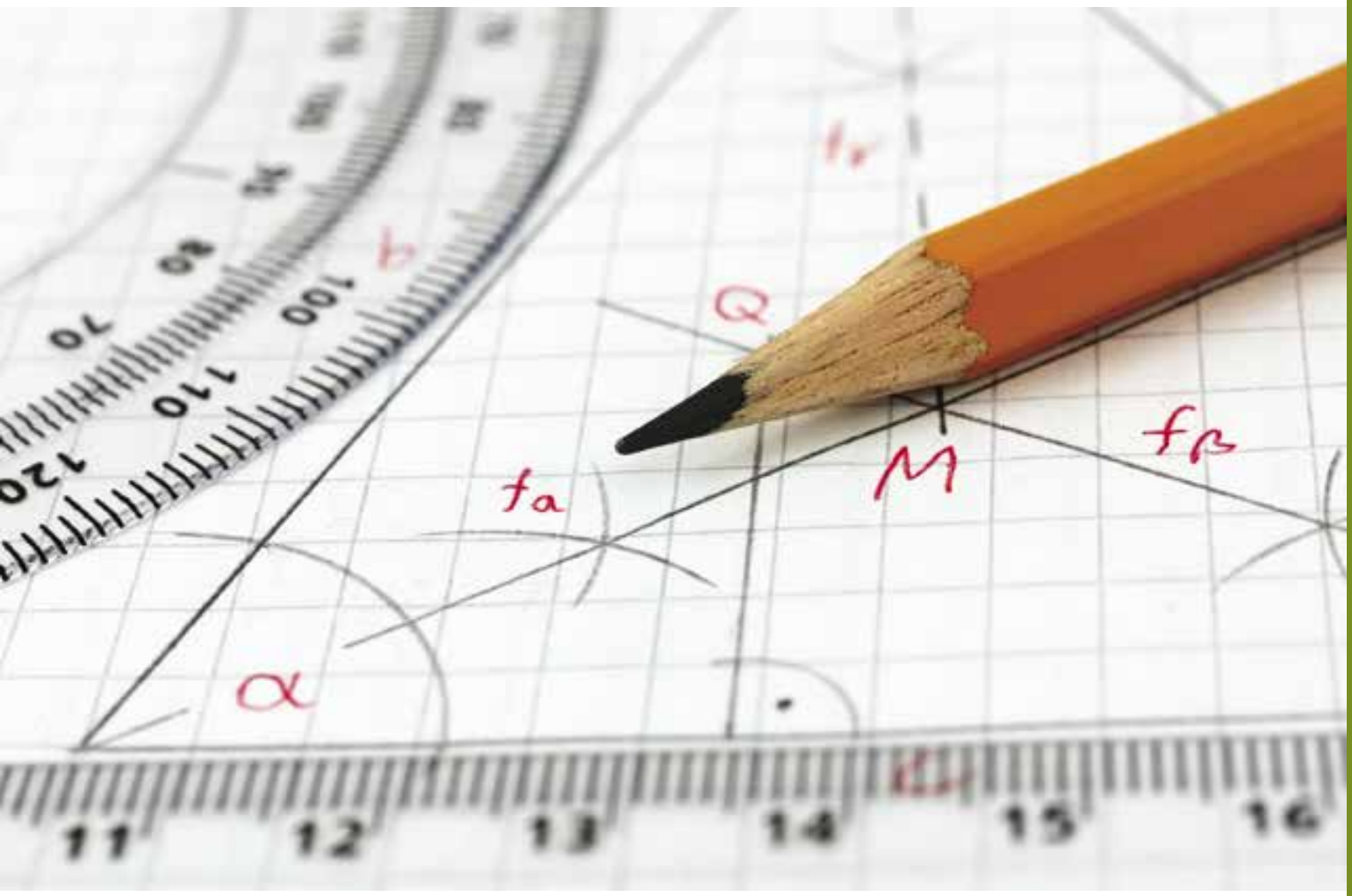
Select five different proofs from Euclid's elements and use your own examples to demonstrate them. To complete this task you will need to follow these steps.

Select five different proofs from Euclid's Elements.

For each proof:

- draw a relevant diagram complete with labels on each vertex
- use your understanding of parallel lines and angles and congruence in order to demonstrate the proof using your diagram
- set out the proof in a logical sequence; you may wish to redraw the diagram several times to help you.

Ensure that you use proper mathematical terminology.



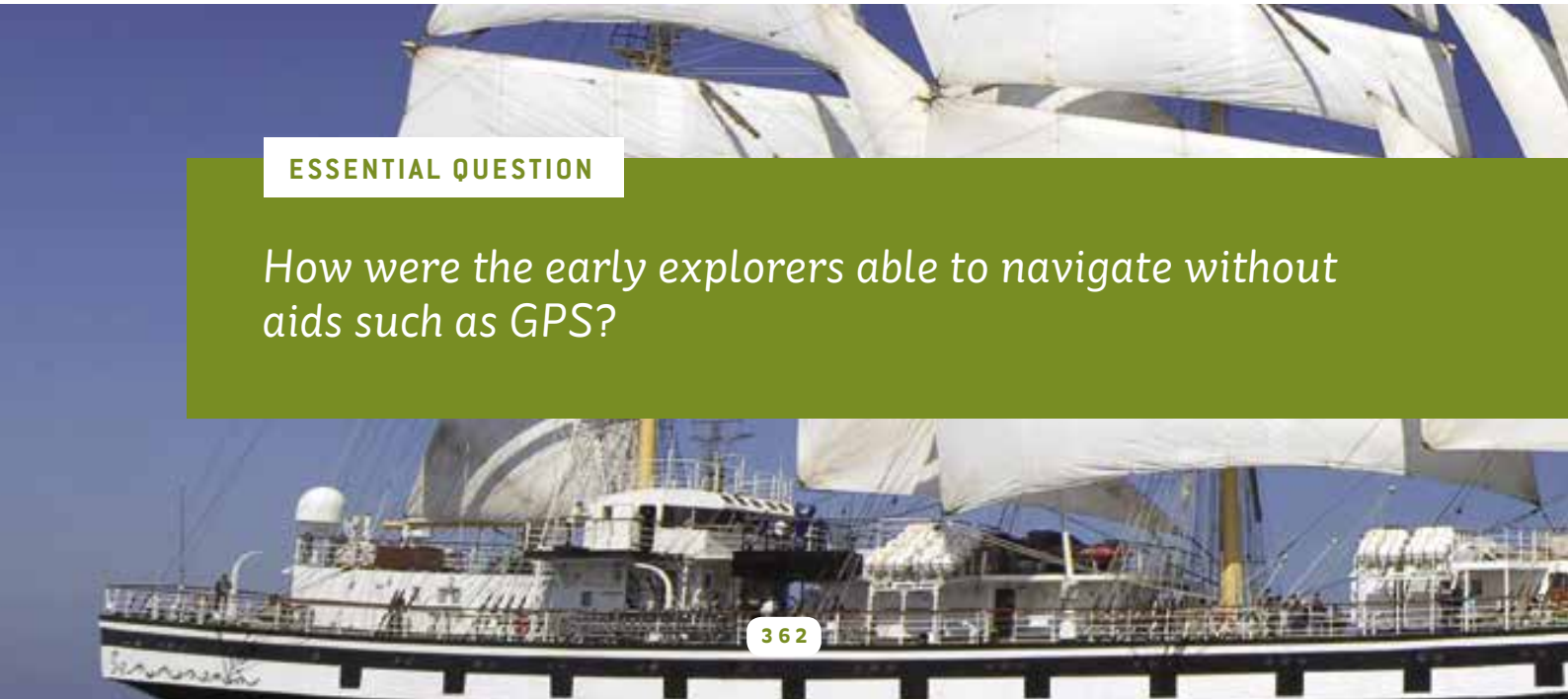
You will need:

- access to the Internet
- ruler
- protractor
- pair of compasses.

You may like to present your findings as a report. Your report could include:

- a PowerPoint presentation
- a booklet
- a poster series
- other (check with your teacher).

8 PYTHAGORAS' THEOREM AND TRIGONOMETRY



ESSENTIAL QUESTION

How were the early explorers able to navigate without aids such as GPS?

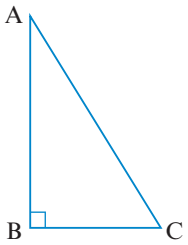
Are you ready?

8A **1** Calculate the following, correct to one decimal place.

a $\sqrt{40}$ **b** $\sqrt{14.9}$

c $\sqrt{22.04}$ **d** $\sqrt{145.3}$

Questions **2**–**6** relate to this figure.



8A **2** Which line segment is the hypotenuse?

8A **3** If AB = 9 mm and AC = 10 mm, give the length of BC, correct to the nearest millimetre.

8B **4** How can sin ∠BAC be written?

A $\frac{BC}{AB}$ **B** $\frac{BC}{AC}$ **C** $\frac{AB}{AC}$ **D** $\frac{AC}{BC}$

8B **5** What is the ratio $\frac{AB}{BC}$?

A tan ∠BAC **B** sin ∠BAC

C cos ∠ACB **D** tan ∠ACB

8B **6** Write a ratio for:

a cos ∠BAC

b cos ∠ACB

8C **7** What is the value of tan 68°, correct to two decimal places?

A 2.48 **B** 0.93 **C** 0.37 **D** 0.40

8C **8** What is the value of $\cos^{-1} 0.45$, correct to the nearest degree?

A 27° **B** 24° **C** 63° **D** 66°

8C **9** Calculate the value of each expression, correct to two decimal places.

a $3.2 \times \cos 43°$ **b** $\frac{67}{\tan 78°}$

c $\frac{4.5}{\cos 62°}$ **d** $0.7 \times \sin 85°$

8D **10** Give definitions for:

a angle of elevation

b angle of depression

c compass direction

d true bearing.

8D **11** **a** Write the true bearing that is equivalent to each compass bearing.

i NE **ii** SSE

iii ENE **iv** WNW

b Write the true bearing for each compass bearing.

i S30°E **ii** S28°W

iii S2°E **iv** N8°W

8D **12** If one of the acute angles in a right-angled triangle is 27°, what is the value of the other acute angle?

8E 10A **13** How many faces are on the surface of each of these solids?

a rectangular prism

b triangular-based pyramid

8E 10A **14** How many edges are there on each of the solids in question **13**?

8F 10A **15** **a** Draw a scalene triangle.

b Describe how you would calculate the area of the triangle.

c Label the height and base on your figure, and give a formula for its area.

8H 10A **16** Draw a diagram of the Cartesian plane.

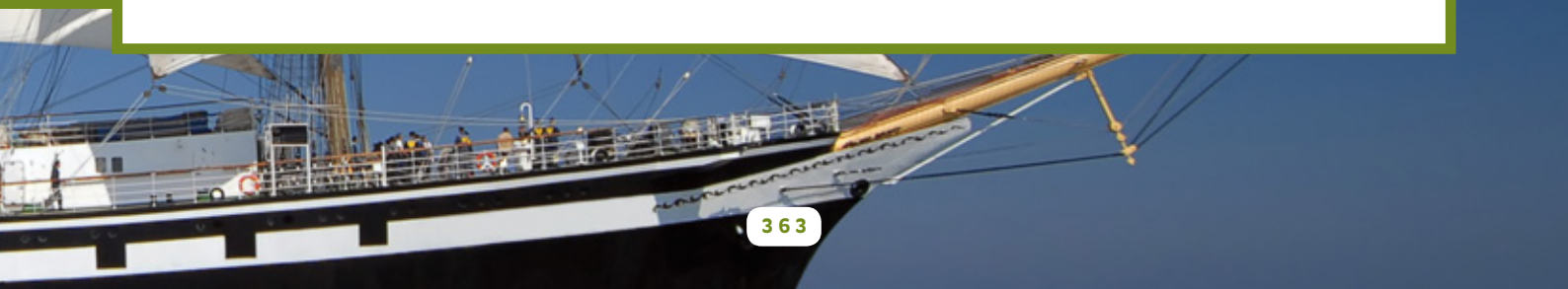
a Label the quadrants.

b Indicate the signs of x and y (positive or negative) in each of the quadrants.

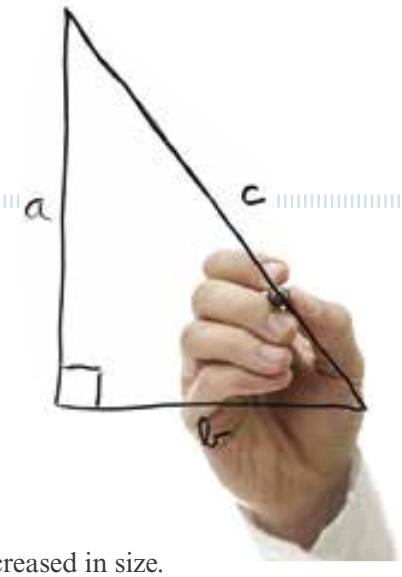
8H 10A **17** Draw a circle. Label:

a its centre **b** a radius

c an arc **d** the circumference.



8A Finding lengths using Pythagoras' Theorem



Start thinking!

Pythagoras' Theorem gives the relationship between the lengths of the sides in a right-angled triangle.

1 **a** Use the figure to state this relationship in words.
b What is the name given to side c?
c Explain why side c is always the longest side in a right-angled triangle.

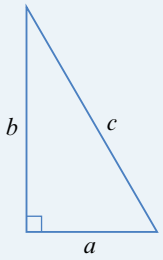
2 Imagine what happens to this triangle if the right angle is increased or decreased in size.
a First, imagine the right angle is increased to greater than 90°.
i Draw a sketch of this figure. **ii** Explain what happens to the length of side c.
iii Adjust Pythagoras' Theorem to write an algebraic relationship between the lengths of the sides.
b Next, imagine the right angle is decreased to less than 90°.
i Draw a sketch of this figure. **ii** Explain what happens to the length of side c.
iii Adjust Pythagoras' Theorem to write an algebraic relationship between the lengths of the sides.

3 Explain how you could use Pythagoras' Theorem to determine whether a triangle is, in fact, right-angled.

4 The ratio 3:4:5 is sometimes mentioned when working with Pythagoras' Theorem. Explain what this ratio refers to.

KEY IDEAS

- When two of the three lengths of a right-angled triangle are known, Pythagoras' Theorem can be used to find the third length.
- Pythagoras' Theorem states that, in any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.



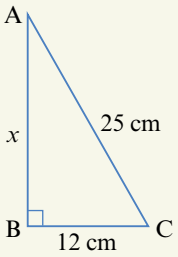
- The Theorem is stated algebraically as $c^2 = a^2 + b^2$.
- If c represents the length of the longest side of a triangle, the triangle is:
  - right-angled if $c^2 = a^2 + b^2$
  - obtuse-angled if $c^2 > a^2 + b^2$
  - acute-angled if $c^2 < a^2 + b^2$.
- **Pythagorean triads** (also called **triples**) are sets of three whole numbers which satisfy Pythagoras' Theorem, such as 3, 4 and 5.

EXERCISE 8A Finding lengths using Pythagoras' Theorem

EXAMPLE 8A-1

Using Pythagoras' Theorem to calculate an unknown side length (decimal value)

Calculate the unknown side length in this triangle, correct to one decimal place.



THINK

1. Since the triangle is right-angled, state Pythagoras' Theorem and define a, b and c.
2. Substitute the known values into the equation and simplify.
3. Solve for x by first subtracting 144 from both sides of the equation and then finding the square root.
4. Answer the question, providing the correct units.

WRITE

$c^2 = a^2 + b^2$
$a = 12$ cm, $b = x$, $c = 25$ cm

$25^2 = 12^2 + x^2$
$625 = 144 + x^2$

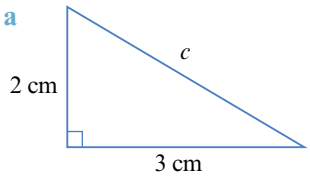
$x^2 = 625 - 144 = 481$
$x = \sqrt{481} \approx 21.9$

The length of the third side is 21.9 cm.

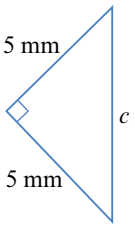
UNDERSTANDING AND FLUENCY

1 Calculate the length of the unknown side in each triangle, correct to one decimal place.

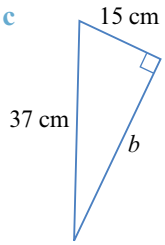
a



b



c



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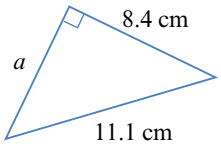

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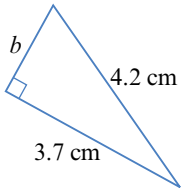

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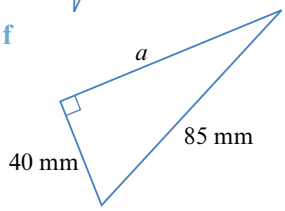

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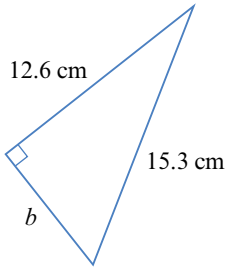

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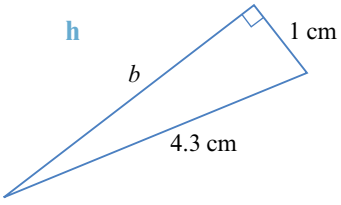
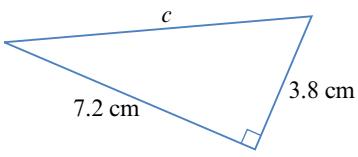


i



EXAMPLE 8A-2

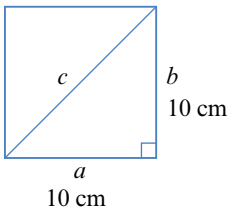
Using Pythagoras' Theorem to calculate an unknown side length (exact value)

A square piece of paper has a side length of 10 cm. What is the length of its diagonal? Give your answer as an exact value, in simplest form.

THINK

1 Draw a labelled diagram.

WRITE



2 The diagonal is the hypotenuse of the right-angled triangle. Identify a, b and c in the triangle.

$a = 10$, $b = 10$, c is unknown

3 Use Pythagoras' Theorem, leaving the answer in square root form if it is **irrational**.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 10^2 + 10^2 \\ &= 100 + 100 \\ &= 200 \\ c &= \sqrt{200} \end{aligned}$$

4 Simplify the irrational number. Look for the highest perfect square that is a factor.

$$\begin{aligned} c &= \sqrt{100 \times 2} \\ &= \sqrt{100} \times \sqrt{2} \\ &= 10 \times \sqrt{2} \\ &= 10\sqrt{2} \end{aligned}$$

5 Answer the question, providing the correct units.

The exact length of the diagonal is $10\sqrt{2}$ cm.

UNDERSTANDING AND FLUENCY

2 Calculate the length of the diagonal of a square with each given side length. Give your answer as an exact value in simplest form.

a 5 cm **b** 20 mm **c** 8 cm **d** 25 cm

3 Calculate the length of the diagonal of each square in question **2**, correct to one decimal place.

4 Determine whether each triangle with the given side lengths is right-angled, acute-angled or obtuse-angled.

a 26 cm, 24 cm, 10 cm
b 100 cm, 96 cm, 28 cm
c 3.5 cm, 8.5 cm, 9.3 cm
d 1 cm, 2.4 cm, 2.6 cm
e 7.5 cm, 4.2 cm, 8.5 cm
f 5 cm, 7 cm, 6 cm

5 An equilateral triangle has side lengths of 8 cm.

a Draw a sketch of the triangle. Draw a line from the top vertex that is perpendicular to the base. Label the lengths of all sides of the figure.

b Use Pythagoras' Theorem to calculate the height of the triangle. Give your answer as:

i an exact value

ii an approximate value, correct to one decimal place.

6 A squaring tool is used when making picture frames to ensure the corners of the frames are true right angles.

Imagine you are constructing a square picture frame with side lengths of 54 cm.

a One of the diagonals of the frame measures 78 cm. Explain how you can be sure that the frame is not truly square. Refer to acute and obtuse angles.

b The other diagonal measures 76 cm. Draw a labelled sketch of the frame, showing its true shape.

7 A sheet of writing paper is 21 cm wide and 30 cm long. The envelope used for the paper has a width of 22 cm and a diagonal of 24.6 cm.

a What is the height of the envelope?

b The paper is folded into equal-sized strips along its longer side. Show how to find the *minimum* number of folds necessary to ensure the paper will fit within the envelope.

c If the paper was folded along its shorter side, what would be the minimum number of folds needed?

UNDERSTANDING AND FLUENCY

8 A rectangular running field measures 50 m by 100 m. How much shorter is it to run directly from one corner to the diagonally opposite corner than it is to travel around the outside of the field?

9 Lisa is flying her kite high in the sky. She releases the string to its full length of 55 m, and holds the end of the string at head height. Lisa is 178 cm tall. Assume the string forms a straight line.

Lisa sees the kite flying directly over a tree which she knows is 35 m from where she is standing.

a Draw a diagram to display the scene. Remember to account for Lisa's height.

b Calculate the height the kite is flying above the ground.

10 You know that a triangle with side lengths 3 cm, 4 cm and 5 cm is right-angled. The numbers 3:4:5 are one example of a Pythagorean triad (or triple). There are many more.

a To investigate other triads, start by finding out if multiples of these numbers also satisfy Pythagoras' Theorem. Do each of these sets of numbers satisfy Pythagoras' Theorem?

i 30:40:50 **ii** 6:8:10 **iii** 12:16:20

b Use the triad to check side lengths that are not whole numbers. Show that triangles with the following side lengths are indeed right-angled triangles, using the 3:4:5 triad.

i 1.5 cm, 2 cm, 2.5 cm **ii** 0.6 cm, 0.8 cm, 1 cm **iii** 1.2 cm, 1.6 cm, 2 cm

c What do you conclude from your investigations in parts **a** and **b**?

d All the triangles in parts **a** and **b** are similar triangles. Explain why this is so.

PROBLEM SOLVING AND REASONING

11 There are different formulas which can be used to generate sets of Pythagorean triads. Consider this one. A Pythagorean triad can be written as:

$x^2 - y^2$, $2xy$ and $x^2 + y^2$, where x and y are integers, and $x > y$.

a Which expression will represent the hypotenuse? Will this always be the case?

b Choose values for x and y. Note the restriction placed on these values.

c Calculate values for:

i $x^2 - y^2$ **ii** $2xy$ **iii** $x^2 + y^2$

d Verify that the set of numbers you generate does, in fact, satisfy Pythagoras' Theorem.

e Explain the restriction $x > y$.

f Repeat the procedure to generate another set of Pythagorean triads.

g What values of x and y will generate the triad 3:4:5?

PROBLEM SOLVING AND REASONING

12 A right-angled triangle has two shorter sides and one is twice as long as the other. The hypotenuse is 25 cm long. What are the lengths of the two shorter sides?

13 Josh has an extension ladder which is 9.8 m long. He needs to clean the windows on the outside of a building. The lower part of these windows is 9.8 m from the base of the building, and they are 1.2 m tall. Unfortunately, the closest Josh can place his ladder to the base of the building is 3 m, as there are bushes preventing him from putting it any closer. Josh is 1.7 m tall.

- **a** Draw a diagram to show the situation.
- **b** Calculate the height the ladder reaches up the building.
- **c** Explain whether Josh will be able to clean the windows.

14 The flagpole which sits on Parliament House in Canberra stands 81 m high. The Australian flag flying from the top section of the pole is said to be the largest Australian flag flying in Australia: it's about the size of the side of a double-decker bus! It measures 12.8 m wide, with a diagonal of 14.3 m.

- **a** What is the height of the flag?
- **b** Calculate the perimeter of the flag.
- **c** What is the distance from the base of the flagpole to the lowest point where the flag is attached to the pole?
- **d** Assume the flag is flying flat and directly at right angles to the pole. Calculate the distance from the base of the flagpole to the:
  - **i** lower
  - **ii** higher

  unattached corner of the flag.

15 A cubby house is built in a tree on two levels, the lower level being 10 m above the ground, while the upper level is another 5 m vertically higher. The ladder from the ground to the lower level is 12 m long. Sam wants to construct another ladder to go from the ground 1 m further from the base of the tree to the top level. How long should he make this ladder?

CHALLENGE

16 A ladder rests against a vertical wall, with the top of the ladder 7 m above the base of the wall. When the bottom of the ladder is moved 1 m further away from the base of the wall, the top of the ladder rests against the base of the wall.

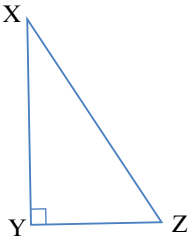
- **a** Draw a diagram to illustrate this situation.
- **b** Calculate the length of the ladder.

Reflect

Why are utility access holes and covers in roads round in shape rather than square or rectangular?

8B Finding lengths using trigonometry

Start thinking!



Trigonometry is the study of triangles and the relationship between their sides and angles.

1 Use this right-angled triangle to write ratios for:

a sin ∠YXZ **b** cos ∠YXZ **c** tan ∠YXZ

d sin ∠XZY **e** cos ∠XZY **f** tan ∠XZY

2 Consider the equilateral triangle ABC, which has a perpendicular drawn from the vertex B to the base AC.

a Write the size of: **i** ∠BAD **ii** ∠ABD.

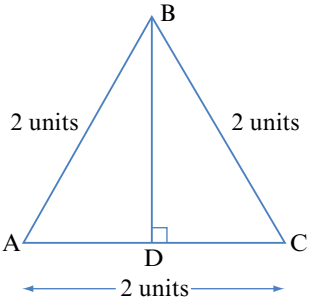
b What is the length of AD?

c Use Pythagoras' Theorem to calculate the length of BD. Leave your answer as an exact number (irrational number in surd form).

d Use your answers from parts **b** and **c** to write values for:

i sin ∠BAD **ii** cos ∠BAD **iii** tan ∠BAD

iv sin ∠ABD **v** cos ∠ABD **vi** tan ∠ABD



3 Consider the right-angled isosceles triangle XYZ. The two equal sides, XY and XZ, are 1 unit in length.

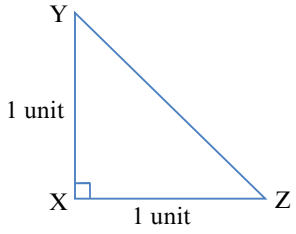
a Use Pythagoras' Theorem to calculate the length of YZ. Leave your answer as an irrational number.

b Write the size of: **i** ∠XYZ **ii** ∠XZY

c Write values (in exact form) for:

i sin ∠XYZ **ii** cos ∠XYZ **iii** tan ∠XYZ

iv sin ∠XZY **v** cos ∠XZY **vi** tan ∠XZY



4 Use your answers from questions **2** and **3** to copy and complete this table.

	30°	45°	60°
sin			
cos			
tan			

KEY IDEAS

- Trigonometry involves relationships between angles and side lengths in a triangle.
  $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$
- The exact values for the trigonometric ratios of 30°, 45° and 60° are as follows:
- Where possible, use exact values for trigonometric ratios. Otherwise, obtain them by using a calculator.

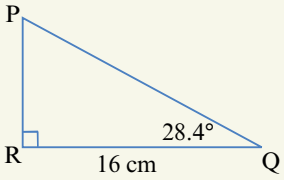
	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

EXERCISE 8B Finding lengths using trigonometry

EXAMPLE 8B-1

Using the calculator value of trigonometric ratios to calculate unknown side lengths

Calculate the lengths of the two unknown sides in triangle PQR.
Give your answers correct to one decimal place.



THINK

1 Use the tangent ratio to calculate the length of PR. Simplify to calculate its value.

2 Use the cosine ratio to calculate the length of PQ. Simplify to calculate its value.

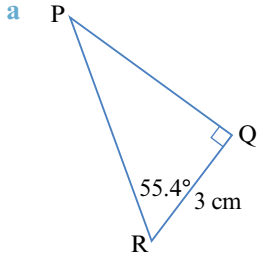
WRITE

$$\tan 28.4° = \frac{PR}{16}$$
$$\frac{PR}{16} = \tan 28.4°$$
$$16 \times \frac{PR}{16} = 16 \times \tan 28.4°$$
$$PR \approx 8.7 \text{ cm}$$
$$\cos 28.4° = \frac{16}{PQ}$$
$$PQ \times \cos 28.4° = PQ \times \frac{16}{PQ}$$
$$PQ \times \cos 28.4° = 16$$
$$PQ = \frac{16}{\cos 28.4°}$$
$$\approx 18.2 \text{ cm}$$

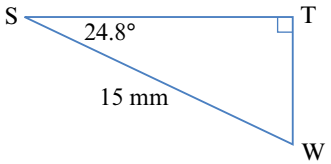
UNDERSTANDING AND FLUENCY

1 Find the length of the unknown sides in each triangle, correct to one decimal place.

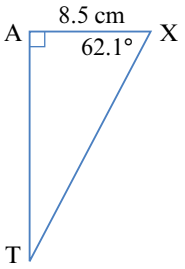
a



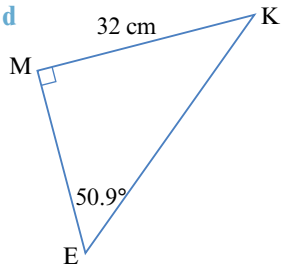
b



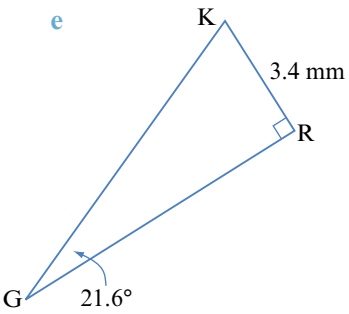
c



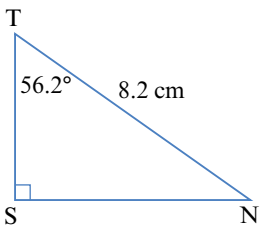
d



e



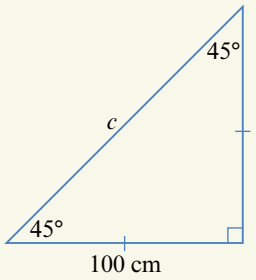
f



EXAMPLE 8B-2

Using an exact value of a trigonometric ratio to calculate an unknown side length

Use an exact value for the appropriate trigonometric ratio to calculate the length of the unknown side in this triangle. Give your answer as an exact value.



THINK

1 Form an equation using the relevant trigonometric ratio.

2 Replace sin 45° with its exact value and solve for the unknown value.

3 Write the answer as an exact value. Include the unit.

WRITE

$$\sin 45° = \frac{100}{c}$$

$$\frac{1}{\sqrt{2}} = \frac{100}{c}$$

$$c \times \frac{1}{\sqrt{2}} = \frac{100}{c} \times c$$

$$\frac{c}{\sqrt{2}} = 100$$

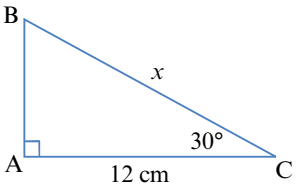
$$c = 100 \times \sqrt{2}$$

$$c = 100\sqrt{2} \text{ cm}$$

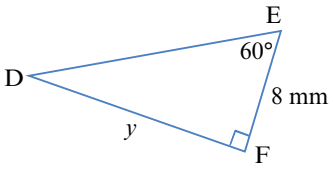
UNDERSTANDING AND FLUENCY

2 Use an exact value for the appropriate trigonometric ratio to calculate the length of the unknown side in each triangle. Give your answers as exact values.

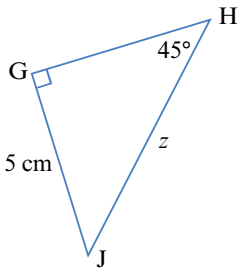
a



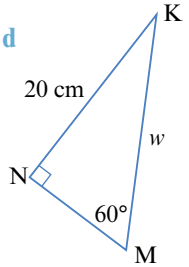
b



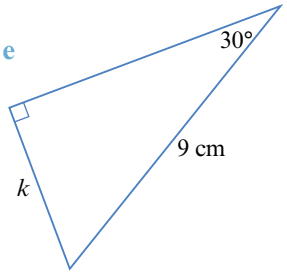
c



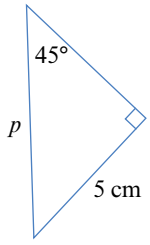
d



e



f



3 Calculate the approximate value, correct to one decimal place, for each exact length found in question **2**.

4 You may have noticed that the trigonometric values for some angles are related . Draw separate diagrams to explain each of these.

a Why does $\sin 30° = \cos 60°$ and $\cos 30° = \sin 60°$?

b Why does $\sin 45° = \cos 45°$?

c Why does $\tan 30° = \dfrac{1}{\tan 60°}$?

5 It is important when solving equations involving trigonometric ratios to take care when transposing. Each of these calculations contains an error. Explain where the error first occurs, then complete the solution correctly.

a
$\cos 45° = \dfrac{x}{12}$

$\dfrac{\sqrt{3}}{2} = \dfrac{x}{12}$

$12 \times \dfrac{\sqrt{3}}{2} = 12 \times \dfrac{x}{12}$

$6\sqrt{3} = x$

b
$\sin 30° = \dfrac{8}{y}$

$\dfrac{1}{2} = \dfrac{8}{y}$

$\dfrac{1}{2} \times 8 = y$

$4 = y$

c
$\tan 60° = \dfrac{b}{7}$

$\dfrac{1}{\sqrt{3}} = \dfrac{b}{7}$

$7 \times \sqrt{3} = b$

$7\sqrt{3} = b$

d
$\tan 20° = \dfrac{4}{z}$

$4 \times \tan 20° = \dfrac{1}{z}$

$\dfrac{1}{4 \tan 20°} = z$

e
$\cos 78° = \dfrac{m}{12}$

$\dfrac{\cos 78°}{12} = \dfrac{1}{m}$

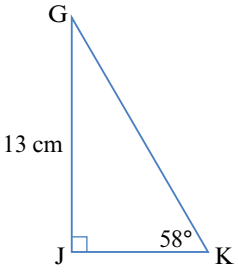
$\dfrac{12}{\cos 78°} = m$

f
$\sin 15° = \dfrac{a}{9}$

$\dfrac{1}{\sin 15°} = \dfrac{9}{a}$

$a = \dfrac{9}{\sin 15°}$

6 It is sometimes possible to simplify calculations by selecting a different angle for the trigonometric ratio.



a To calculate the length of JK you could say $\tan 58° = \dfrac{13}{\text{JK}}$. Solve this equation to find the length of JK.

b If you use the tangent of $\angle$JGK instead of $\angle$JKG, the side length JK would be in the numerator.

i What is the value of $\angle$JGK?

ii Use the tangent of this angle to calculate the length of JK.

iii Compare your answer with the one you obtained in part **a**.

c Comment on using the two different procedures for finding the length of the same line segment.

7 Since 1988, Australian currency has gradually been changed over from paper notes to polymer notes. The size has also changed.

The paper $5 note was 152 mm long, while the polymer one is 130 mm long. Interestingly, the diagonal of each of these notes forms the same angle of 26.6° with the longer side of each.

a Calculate the dimensions of the paper and the polymer $5 notes. Give your answers to the nearest millimetre.

b Compare the dimensions of the two $5 notes. What do you notice?

UNDERSTANDING AND FLUENCY

8 Kate is flying a kite on a 72-m string which makes an angle of 32° with the horizontal. Kate is 172 cm tall and is holding the end of the string at head height.

- **a** Draw a labelled diagram to display this information.
- **b** Calculate the vertical height of the kite above Kate.
- **c** How high is the kite above the ground?
- **d** What is the horizontal distance between Kate and the kite?

9 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. His dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.

- **a** Draw a diagram to display the information.
- **b** How far in front of Shun is his dog?
- **c** What is the vertical height of the end of the leash in Shun's hand above the ground?

10 Pythagoras' Theorem and the trigonometric ratios can both be used to calculate a side length in a right-angled triangle. Give an example of where:

- **a** Pythagoras' Theorem could be used, but trigonometry could not
- **b** trigonometry could be used, but Pythagoras' Theorem could not
- **c** either Pythagoras' Theorem or trigonometry could be used.

11 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick (to the nearest mm)?

PROBLEM SOLVING AND REASONING

12 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks NNE. They both walk 25 km.

- **a** Display the information on a diagram.
- **b** How far north of his starting point is Ben when he completes his walk?
- **c** How far east of his starting point is Ben at this stage?
- **d** If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk?

13 Parliament House in Canberra occupies a site of 32 hectares. The building itself occupies only 15% of the site. It is 300 m long and 300 m wide, being one of the largest buildings in the southern hemisphere.

- **a** Draw a labelled diagram to show the floor plan of the building.
- **b** Use Pythagoras' Theorem to calculate the diagonal length of the building.
- **c** Use trigonometry to calculate the diagonal length of the building. (Hint: what is the size of the angle you can use?)
- **d** Comment on your answers to parts **b** and **c**.

PROBLEM SOLVING AND REASONING

14 The Great Hall of Parliament House displays a tapestry based on a painting by the Australian artist Arthur Boyd. It measures 20 m wide and is claimed to be one of the largest tapestries in the world. The tapestry is floor to ceiling, with the angle between the diagonal and the floor being 24.2°.

- **a** Draw a diagram to display these facts.
- **b** How high is the ceiling in the Great Hall?
- **c** What is the perimeter of the tapestry?

15 A 440-g block of Toblerone chocolate is packaged in a box the shape of a triangular prism. The triangular faces of the box are equilateral in shape, with side lengths of 6 cm.

- **a** Calculate the height of the triangular face of the box using:
  - **i** Pythagoras' Theorem **ii** trigonometry.
- **b** The diagonal of the side rectangular face of the box is at 11.3° to the longer side of the face. What is the length of the chocolate box?

16 A traffic cone is 500 mm tall. (Assume that the sides meet at the top at a point.) The sides of the cone form an angle of 78.7° with the diameter of the base.

- **a** Calculate the length of the slant height of the cone.
- **b** What is the circumference of the cone at its base?

17 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide, and is hung so the bottom frame is 1.5 m from the floor. If she lowers her eyes 5.7°, she sees the bottom frame of the painting, while if she raises her eyes 16.7° she sees the top frame.

- **a** At what height are Sandy's eyes?
- **b** What are the dimensions of the painting?

CHALLENGE

18 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between the parallel sides.

19 A pilot was flying a routine route from A, directly east to B, a distance of 3500 km. He didn't know his compass was not operating correctly, and he was actually travelling 10° south of his true direction. When he didn't arrive at his destination after travelling 3500 km, the pilot realised he was off course.

- **a** How much further does he need to travel to fly directly to his destination from this point? Give your answer to the nearest kilometre.
- **b** If the pilot had become aware of the error when he was half-way into the flight, and adjusted his direction from that point, how many kilometres would he have saved in the journey?

Reflect

Why can the answers be slightly different when trigonometry is used to solve a problem rather than Pythagoras' Theorem?

8C Finding angles using trigonometry

Start thinking!

One system of units used for the measurement of an angle is **degrees-minutes-seconds** (**DMS**).

1 degree = 60 minutes

1 minute = 60 seconds

An angle of 28 degrees 43 minutes and 17 seconds is written as 28°43′17″.

1 Your calculator has a button which allows you to convert from degrees into degrees, minutes and seconds. Convert these angles to degrees, minutes and seconds. Give your answer to the nearest second.

a 43.51° **b** 78.46° **c** 62.34° **d** 1.93°

2 Explain why each of these angles is not written correctly.

a 28°63′17″ **b** 49°13′67″ **c** 33°71′17″ **d** 15°23′60″

3 Round your answers in question **1** to the nearest minute.

The inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) buttons on a calculator can be used to find the size of an angle from its trigonometric ratio.

4 Use your calculator to find the size of the angle θ for each given trigonometric ratio. Write your answers to the nearest:

i degree **ii** minute.

a $\sin\theta = 0.154$ **b** $\cos\theta = 0.926$ **c** $\tan\theta = 1.07$

d $\tan\theta = 0.864$ **e** $\sin\theta = 0.037$ **f** $\cos\theta = 0.829$

KEY IDEAS

- Angles can be written in degrees, minutes and seconds.
- There are 60 seconds in a minute and 60 minutes in a degree.
- When rounding to the nearest minute:
  - if there are less than 30 seconds, the number of minutes stays the same.
  - if there are 30 or more seconds, the number of minutes increases by 1.
- When rounding to the nearest degree:
  - if there are less than 30 minutes, the number of degrees stays the same.
  - if there are 30 or more minutes, the number of degrees increases by 1.
- A calculator can be used to obtain the size of the angle from its sine, cosine or tangent value. This is known as finding the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a value.

EXERCISE 8C Finding angles using trigonometry

1 Convert each angle to degrees and minutes. Give your answer to the nearest minute.

a 23.47°	**b** 16.28°	**c** 82.13°	**d** 7.69°
e 46.09°	**f** 3.47°	**g** 1.04°	**h** 38.99°

EXAMPLE 8C-1 Using a calculator to find angle sizes from trigonometric values

Calculate the size of θ if:

a $\tan\theta = 3.426$ **b** $\sin\theta = \frac{3}{13}$

Write your answers to the nearest:

i degree **ii** minute.

THINK

a 1 Use the $\tan^{-1}$ key to find the angle size. Leave all decimal places on the calculator.

2 Round the answer to the nearest degree and minute.

b 1 Use the $\sin^{-1}$ key to find the angle size. Leave all decimal places on the calculator.

2 Round the answer to the nearest degree and minute.

WRITE

a $\tan\theta = 3.426$

$\theta = \tan^{-1}(3.426)$

$= 73.728\ldots°$

i $\theta = 74°$

ii $\theta = 73°44'$

b $\sin\theta = \frac{3}{13}$

$\theta = \sin^{-1}\left(\frac{3}{13}\right)$

$= 13.342\ldots°$

i $\theta = 13°$

ii $\theta = 13°21'$

UNDERSTANDING AND FLUENCY

2 Use a calculator to find the size of θ for the given trigonometric ratios. Write your answers to the nearest:

i degree **ii** minute.

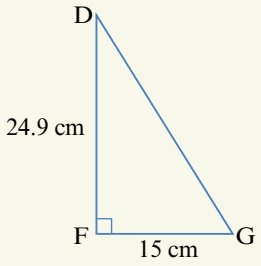
a $\sin\theta = 0.378$	**b** $\cos\theta = 0.845$	**c** $\tan\theta = 2.376$
d $\tan\theta = 0.097$	**e** $\sin\theta = 0.759$	**f** $\cos\theta = 0.238$
g $\tan\theta = 45.326$	**h** $\sin\theta = 0.019$	**i** $\cos\theta = 0.777$

3 Find the angle θ (to the nearest minute) for the given trigonometric values.

a $\cos\theta = \frac{4}{7}$	**b** $\tan\theta = \frac{15}{11}$	**c** $\sin\theta = \frac{4}{13}$
d $\tan\theta = \frac{57}{72}$	**e** $\sin\theta = \frac{29}{31}$	**f** $\cos\theta = \frac{3}{19}$
g $\tan\theta = \frac{100}{13}$	**h** $\cos\theta = \frac{44}{99}$	**i** $\tan\theta = \frac{1}{100}$

EXAMPLE 8C-2 Using trigonometric ratios to find unknown angles

Find the value of the two unknown angles in this triangle. Give your answers to the nearest minute.

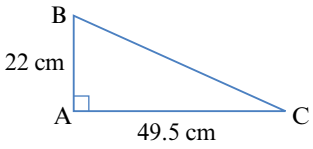


THINK	WRITE
1 The opposite and adjacent sides for ∠DGF are known. Use the tangent ratio.	$\tan \angle DGF = \frac{24.9}{15}$
2 Use the inverse tan. Leave all the decimal places showing on the calculator.	$\angle DGF = \tan^{-1}\left(\frac{24.9}{15}\right)$ $= 58.934\ldots°$
3 Convert to degrees and minutes, rounding to the nearest minute.	$\angle DGF = 58°56'$
4 Calculate the second unknown angle.	$\angle FDG = 180° - (90° + 58°56')$ $= 31°4'$

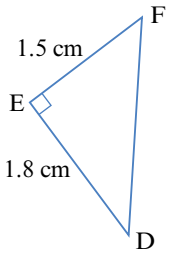
UNDERSTANDING AND FLUENCY

4 Find the value of the two unknown angles (to the nearest minute) in each triangle.

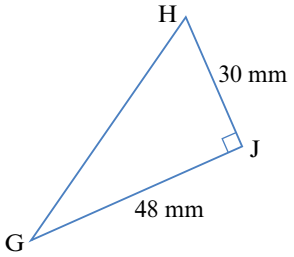
a



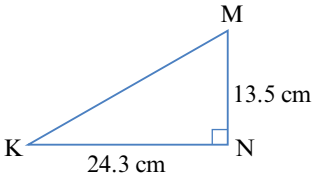
b



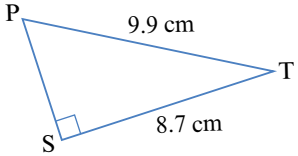
c



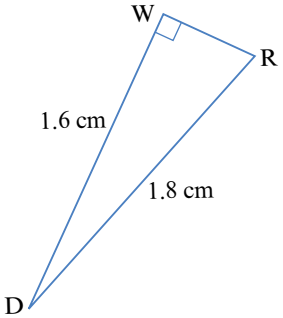
d



e



f



5 The Leaning Tower of Pisa is a freestanding bell tower, next to the cathedral in the Italian city of Pisa. The tower is 55.86 m tall on its low side and 56.7 m tall on its high side. The top of the tower is displaced 3.9 m horizontally from where it would be if it was standing vertically.

a The 'average' height of the tower above the ground represents the height of the centre of the top of the tower above the ground. Calculate this height.

b Draw a labelled sketch of the tower using this average height and its horizontal displacement. Mark the angle of 'lean' of the tower.

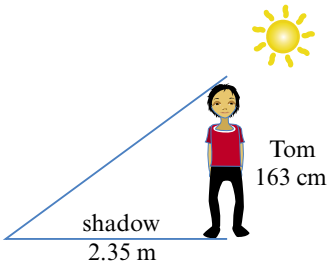
c Describe which trigonometric ratio you would use to calculate this angle.

d Calculate the angle of lean of the tower. Give your answer to the nearest:

i degree **ii** minute.

6 Tom and his brother Sam were standing in the sun measuring the length of each other's shadow. They each knew their height, so collated the measurements (see table).

	Height	Shadow length
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m



a Tom drew a diagram to represent his shadow and height. Draw a diagram to display Sam's measurements.

b Calculate the base acute angle for each triangle. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?

c **i** Explain why these two triangles must be similar in shape.

ii What is the linear scale factor in this case?

iii Draw the two triangles combined into one figure.

7 The sizes of Australian currency notes have changed over the years. Some of the new polymer notes are similar in shape to the previous paper ones, while others are a completely new shape. The table shows the dimensions of the notes.

Note description		$5	$10	$20	$50	$100
Paper note	length	152 mm	155 mm	160 mm	165 mm	172 mm
	width	76 mm	76 mm	81 mm	82 mm	82.5 mm
Polymer note	length	130 mm	137 mm	144 mm	151 mm	158 mm
	width	65 mm	65 mm	65 mm	65 mm	65 mm

a Make a list of the obvious differences between the old notes and the new notes.

b One way to check whether the new note for a particular currency is similar in shape to its old equivalent is to find the size of the angle its diagonal makes with the length of the note.

i Find this angle for each of the old notes and new notes. Give your answers in degrees, correct to one decimal place.

ii Which notes have remained similar in shape?

8 Some road signs display the gradient of a road, particularly when approaching a hill or mountain climb. The gradient can be written in different forms, commonly as a ratio, a percentage or a fraction; for example, 1 in 10, 1:10, 10% and $\frac{1}{10}$ all represent the same gradient.

Pat is a long-distance truck driver delivering goods between various locations. He generally plans his route to avoid steep hill climbs, if possible. His next delivery takes him via a mountainous stretch, where he has a choice of three possible routes.

Route 1: Over a horizontal distance of 12 km the rise is 6 km

Route 2: Over a horizontal distance of 18 km the rise is 7.5 km

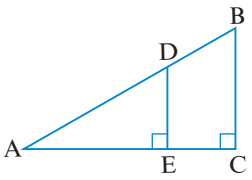
Route 3: Over a horizontal distance of 15 km the rise is 6.5 km

a Write the gradient of each route as a ratio and as a percentage.

b What is the angle of rise of each road over the distances given?

c Rank the routes in order of increasing steepness.

9 Consider this triangle. Within this figure, there are two right-angled triangles: ΔABC and ΔADE.



a Measure the relevant line segments (to the nearest millimetre), and complete the following, correct to two decimal places.

i $\sin \angle BAC = \frac{BC}{AB}$

$= \frac{\_\_ \text{ mm}}{\_\_ \text{ mm}}$

$= \_\_\_\_$

ii $\sin \angle DAE = \frac{DE}{AD}$

$= \frac{\_\_ \text{ mm}}{\_\_ \text{ mm}}$

$= \_\_\_\_$

iii Comment on your answers to parts **i** and **ii**.

iv The angle at A is the common angle in each of these calculations. Measure this angle. Use your calculator to find the sine of this angle. Comment on your answer.

b Repeat the calculations in part **a** for the cosine of angle A. Use the line segments relevant to the cosine ratio.

c Repeat the calculations in part **b** for the tangent of angle A. Use the line segments that give the tangent ratio.

d What do you conclude from your answers in parts **a**–**c**?

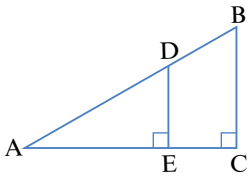
e **i** Explain why ΔABC and ΔADE are similar triangles.

ii Does similarity account for the fact that, for example, sin 30° is always 0.5, irrespective of the size of the right-angled triangle in which the 30° angle occurs? Explain your understanding.

iii Why does the ratio of a particular angle remain the same irrespective of the size of the triangle in which it occurs?

PROBLEM SOLVING AND REASONING

10 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider ΔABC in the diagram from question **9** shown on the right.



- **a** The sine of ∠BAC is the ratio of length of the opposite side to the length of the hypotenuse; that is, BC:AB. When the angle at A is 0°:
  - **i** what is the length of BC?
  - **ii** what does this mean for the value of the ratio BC:AB?
- **b** Now consider when the angle at A becomes almost 90°.
  - **i** What is the value of BC compared with AB?
  - **ii** What is the value of the ratio BC:AB?
- **c** Describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.
- **d** Repeat parts **a**–**c** for the cosine ratio. Consider the relevant line segments for the cosine ratio.
- **e** Once more, repeat parts **a**–**c** for the tangent ratio, considering the relevant line segments for the tangent ratio.

11 From your investigations in question **10**, you can determine the minimum and maximum values for the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of _____ and a maximum value of _____.

The cosine ratio has a minimum value of _____ and a maximum value of _____.

The tangent ratio has a minimum value of _____ and a maximum value of _____.

12 Check your answers to question **11** with a calculator. Provide some values to support your conclusions.

CHALLENGE

13 The A series of standard paper sizes used in Australia is based on the A0 size of paper having an area of 1 m^2. Its measurements are 841 mm by 1189 mm. The A1 size is formed from the A0 size by dividing the paper in half along its long side. This process continues for A2, A3, etc., with each size being derived from the previous one. You will be familiar with A4 size. When the measurements are halved, rounding down to the nearest millimetre generally occurs.

- **a** Draw a table to show the sizes of A0 to A5 paper.
- **b** Add a column to your table to show the size of the angle the diagonal makes with the longer side in each case. Give your values in degrees, correct to one decimal place.
- **c** It has been said that all the A series paper sizes are similar in shape. Comment on this statement.

Reflect

How can the value of the sine, cosine or tangent of an angle give an indication of the size of the angle?

8D Applications of trigonometry

Start thinking!

Trigonometry can be applied to many practical problems.

1 **a** Explain why you could not use Pythagoras' Theorem to calculate the height of this Qantas jet.

b Imagine you are standing 20 km from a point directly beneath the plane. You use a **clinometer** to measure the **angle of elevation** to the plane as 30°.

i Draw a labelled diagram to display this information.

ii Use a trigonometric ratio to calculate the height of this plane.

iii Calculate the straight-line distance between you and the plane.

c The plane approaches the airport at an altitude of 1000 m when it is 10 km horizontally from the airport. What is the **angle of depression** from the plane to the airport at that point?

2 Consider a Qantas jet on a flight to a destination 1000 km away with a **compass bearing** of N20°E.

a Draw a labelled diagram to display the distance travelled:

i north **ii** east.

b Use trigonometry to calculate these distances.

3 Sometimes a direction is given as a **true bearing**.

a What is a true bearing?

b Write the true bearings for flights to and from the destination in question **2**.

KEY IDEAS

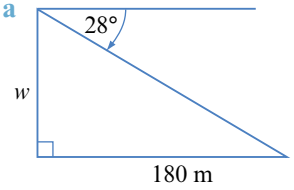
- The angle of elevation is the angle measured up from the horizontal to the line of vision.
- The angle of depression is the angle measured down from the horizontal to the line of vision.
- Compass bearings indicate direction from north or south towards east or west. For example, N30°W represents a bearing of 30° from north towards west.
- True bearings are angles measured from north in a clockwise direction. They are generally written using three digits, followed by a capital T. For example, 045°T represents 45° east of north, which is also north-east (NE).

EXERCISE 8D Applications of trigonometry

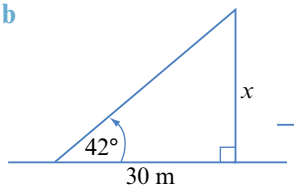
UNDERSTANDING AND FLUENCY

1 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

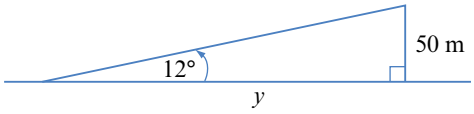
a



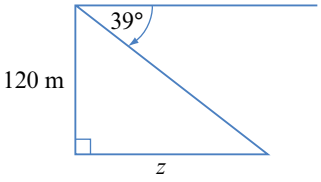
b



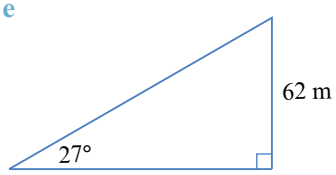
c



d



e



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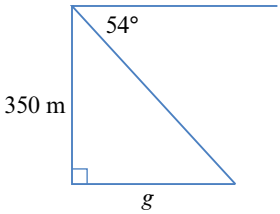


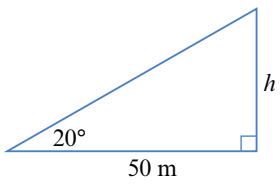
EXAMPLE 8D-1 Using trigonometry to calculate distances

The Cape Byron lighthouse is situated on a cliff at the most easterly point of the Australian mainland. From a point on the ground 50 m from the base of the lighthouse, the angle of elevation to the top of the lighthouse is 20°. Calculate the height of the lighthouse, to the nearest metre.

THINK

1 Draw a diagram to represent the situation.

WRITE



2 Use the tangent ratio.

$$\tan 20° = \frac{h}{50}$$
$$h = 50 \times \tan 20°$$
$$\approx 18.2$$

3 Answer the question, rounding to the nearest metre.

The lighthouse is 18 m high.

2 A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°. Find the height of the lighthouse, to the nearest metre.

3 The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea.

EXAMPLE 8D-2 Using bearings with trigonometry

A ship at sea has travelled 1000 km on a bearing of N40°E since it left port.

a Draw a labelled diagram to display the distance travelled:
i east **ii** north.

b Use trigonometry to calculate these distances to the nearest kilometre.

THINK

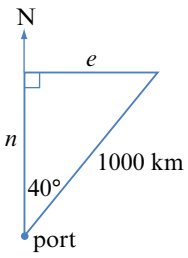
a Draw a diagram to represent the situation (e represents distance east, n represents distance north).

b 1 Use the sine ratio to calculate the distance east.

2 Use the cosine ratio to calculate the distance north.

WRITE

a



b $\sin 40° = \frac{e}{1000}$

$1000 \times \sin 40° = e$

$e \approx 642.8$

i The ship is 643 km east of the port.

$\cos 40° = \frac{n}{1000}$

$1000 \times \cos 40° = n$

$n \approx 766.0$

ii The ship is 766 km north of the port.

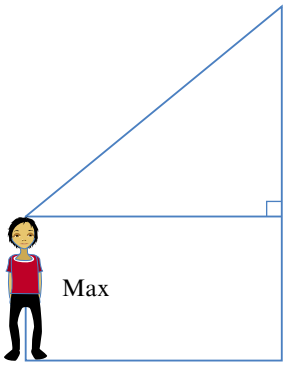
UNDERSTANDING AND FLUENCY

4 A ship at sea has travelled 250 km on a bearing of N30°E since it left port.
a Draw a labelled diagram to display the distance travelled:
i east **ii** north.
b Use trigonometry to calculate these distances to the nearest kilometre.

5 If the ship in question **4** turns around at this point and returns to its original port, write a description of the ship's path, giving the bearing and distances with respect to its turn-around position.

6 Write the true bearings for the ship's paths in questions **4** and **5**.

7 Max threw a ball into the air. He viewed the ball at its highest point to be at an angle of elevation of 40° from him, and 3 m horizontally away from him. Max's eye level is 172 cm above ground.

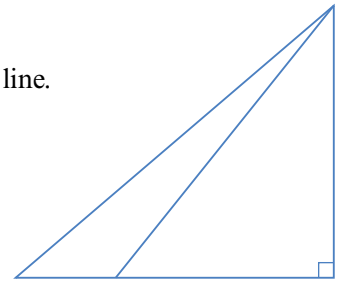


- **a** Copy and label this diagram with the known values.
- **b** What height does the ball reach above the level of Max's eyes?
- **c** What is the maximum height the ball reaches above the ground?
- **d** Why is it important to consider the height of a person's eyes in a situation like this?

8 Sedona is standing 20 m in front of a wall. From her eye level 168 cm above the ground, the angle of elevation to the top of the wall is 25°.

- **a** Draw a diagram to display the information.
- **b** Find the height of the wall.

9 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)



- **a** Copy and label this diagram with the known values.
- **b** Find the height of the tree from Ann's observations.
- **c** Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance to the tree.
- **d** What is the distance between Ann and Beth?
- **e** Suppose Beth had been standing in the same line as Ann, but on the opposite side of the tree.
  - **i** Draw a diagram to show this situation.
  - **ii** What would be your answer to part **d** in this case?

10 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

- **a** Draw a diagram to display this information.
- **b** Find the height of the tower.

11 Jake is riding his bike from home to his friend Luke's home. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's home.

- **a** Draw and label a sketch of Jake's route. Mark the north direction at each point.
- **b** Calculate the direct distance between the two homes.
- **c** Use trigonometry to find the bearing from Jake's home to Luke's. (Remember that bearings are measured from the north in a clockwise direction.)
- **d** What is the bearing from Luke's home to Jake's?
- **e** The bearing in part **d** is known as the **back-bearing** of the bearing in part **c** (and vice versa). What do you notice about the relationship between the two bearings?
- **f** Draw a scale drawing of Jake's route. From your drawing:
  - **i** find the distance between the two homes
  - **ii** use a protractor to measure the bearings in both directions.
- **g** Comment on any differences between your previous answers and those using your scale drawing.

12 Here's an activity you can do with a classmate.

- **a** Draw a scale drawing of the perimeter of your classroom. Mark your position anywhere within the room. Call this point X.
- **b** Measure the bearings from two adjacent vertices of your room plan to the point X.
- **c** Give this information to a classmate to locate your position. Note any questions your classmate asks regarding your instructions.
- **d** Describe how this information will define your position X.
- **e** Explore to see whether the position of X can still be located if bearings are made from any two vertices of your room plan.
- **f** Explain how your position could also be located by one bearing measurement and one distance measurement.

13 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled A, and walk 35 km at a bearing of 055°T to B.
- From B, walk 25 km at a bearing of 015°T to finish walk at C.

- **a** Draw a sketch of the walk. Try to make it roughly to scale as it can lead to incorrect calculations if the relative position of points is misleading.
- **b** Find how far north:
  - **i** B is from A
  - **ii** C is from B
  - **iii** C is from A.
- **c** Find how far east:
  - **i** B is from A
  - **ii** C is from B
  - **iii** C is from A.

PROBLEM SOLVING AND REASONING

d Consider the right-angled triangle in which AC is the hypotenuse. Calculate:
- **i** the distance from A to C
- **ii** the bearing from A to C
- **iii** the bearing from C to A.

e Draw a scale drawing of the bushwalkers' route.
- **i** Measure distances and bearings to obtain answers for part **d**.
- **ii** Compare your answers with those for part **d**.

14 A new hike for bushwalkers took them along a different route.
From the starting point, walk 18 km at a bearing of 300°T.
Change direction and walk 23 km at a bearing of 200°T.

a Find the direct distance from the starting point to the end point.

b What is the bearing of the starting point from the end point?

15 Jan is standing 2 m in front of a painting on a wall. The bottom frame of the painting is 1.4 m from the floor, and Jan is 165 cm tall. The painting has a height of 70 cm.

a Draw a diagram to show this information.

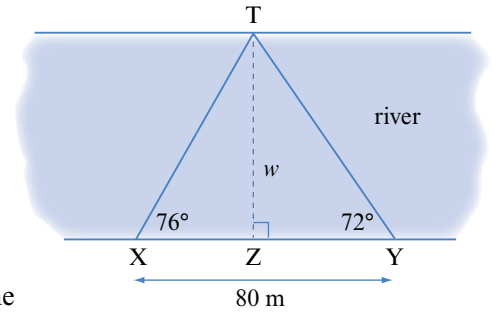
b Find the angle of elevation from Jan's eyes to the top frame of the painting.

c What is the angle of depression from Jan's eyes to the bottom frame?

d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to view from the top to the bottom frame?

CHALLENGE

16 Council surveyors have been set the task of measuring the width of the river along a straight stretch in preparation for building a bridge. They have located a tree on the opposite side of the river and taken angle measurements from two points X and Y, 80 m apart.
Calculate the width of the river. (Hint: let the distance XZ be represented by a pronumeral.)



Reflect

How are contour lines on maps related to angles of elevation?

8E Three-dimensional problems

Start thinking!

1 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.
The base of the pyramid is a square with side lengths of 230 m.
It stands 140 m tall.

- **a** Draw a labelled sketch of the pyramid.
- **b** Calculate the diagonal length of the base.
- **c** The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle the sloping edge makes with the base diagonal.
- **d** A different right-angled triangle can be used to find the angle the sloping sides make with the base.
  - **i** Draw a labelled diagram to display its dimensions.
  - **ii** Calculate the angle the sloping side makes with the base.

2 Gustave Eiffel was the engineer in the company which designed and built the Eiffel Tower in Paris. The base of the tower is 120 m square, and its height is 320 m.

- **a** Draw a sketch of the tower.
- **b** Find the length of the diagonal of the base.
- **c** The height of the tower from the midpoint of the base is 320 m.
  - **i** Draw a labelled diagram to find the distance from a corner of the base to the top of the tower.
  - **ii** What is this distance?
  - **iii** What is the angle this line makes with the base diagonal?

KEY IDEAS

- There can be many right-angled triangles in three-dimensional objects.
- Pythagoras' Theorem can be used to find an unknown side length of a right-angled triangle on the surface and within a 3D object.
- Trigonometry can be used to find an unknown angle or side length of a right-angled triangle on the surface and within a 3D object.

EXERCISE 8E Three-dimensional problems

1 Draw a sketch of each three-dimensional object. Highlight a right-angled triangle within each object.

a rectangular prism
b square-based pyramid
c triangular-based pyramid
d cylinder

EXAMPLE 8E-1 Finding lengths and angles within 3D objects

A box is in the shape of a cube with side lengths of 20 cm.

a Calculate the length of the diagonal of the base of the box.
b Find the length of the diagonal within the box.
c Find the angle this internal diagonal makes with the base diagonal

THINK

a 1 Draw a labelled diagram of the box. Mark the relevant diagonals.

2 Find the length of the base diagonal.

b Find the diagonal length within the box.

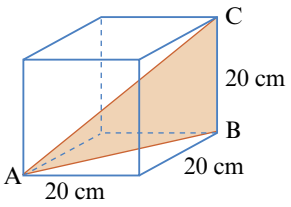
c 1 Define the right-angled triangle which contains the box's base diagonal and internal diagonal.

2 Decide on the trigonometric ratio to use. Calculate the angle.

NOTE Any of the trigonometric ratios could be used here, as the lengths of all three sides of the triangle are known.

WRITE

a



length of base diagonal
$= \sqrt{20^2 + 20^2}$
$= \sqrt{800}$
≈ 28.3 cm

b diagonal length within box
$= \sqrt{800 + 20^2}$
$= \sqrt{1200}$
≈ 34.6 cm

c The base diagonal is AB.
The internal diagonal is AC.
The angle between these two is $\angle CAB$. It is within the right-angled triangle CAB.

$$\sin \angle CAB = \frac{20}{34.6}$$

$$\angle CAB = \sin^{-1}\left(\frac{20}{34.6}\right)$$

$$\approx 35°$$

The diagonal within the box makes an angle of 35° with the base diagonal.

2 A box is in the shape of a cube with side lengths of 15 cm.

- **a** Calculate the length of the diagonal of the base of the box.
- **b** Find the length of the diagonal within the box.
- **c** Find the angle this internal diagonal makes with the base diagonal.

3 A box has a base 20 cm by 15 cm and is 10 cm tall.

- **a** Provide working to show whether a 30 cm rod will fit in the box.
- **b** If a rod was a snug fit, what angle would it make with the base diagonal of the box?

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and its height is 5 cm.

- **a** Draw a labelled diagram of the triangular prism, showing all dimensions.
- **b** Draw a labelled diagram of:
  - **i** the front view **ii** the top view (or base view).
- **c** On your diagram of the front view, draw a diagonal.
  - **i** Find the length of this diagonal.
  - **ii** What angle does this diagonal make with the longer side?
  - **iii** Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.
- **d** Calculate:
  - **i** the height of the base **ii** the area of the base.
- **e** The cheese is sliced vertically through the base height.
  - **i** Draw the shape of the face exposed. Label its dimensions.
  - **ii** Find the area exposed.
  - **iii** What is the diagonal length of the exposed face?

5 Jack was given a tent for Christmas. He wants to make a cover the same size to protect it when it rains. The front of the tent forms an equilateral triangle, with a height of 1.5 m.

- **a** Sketch the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.
- **b** Use Pythagoras' Theorem to find the value of x.
- **c** Jack has bought 3.5 m of material of the correct width to make the protective cover. Explain whether this is sufficient.

6 The rectangular perimeter of a football field is lined with trees. Ethan is standing 70 m in front of one of the trees. He notices a much taller tree 37° to his left. He measures the angle of elevation to the top of the taller tree as 13°.

- **a** Draw a sketch showing the relative position of the two trees described. Label the known values.
- **b** Find the distance along the ground from Ethan to the base of the taller tree.
- **c** Draw a sketch showing the elevation from Ethan to the tall tree. Label the known values.
- **d** Use this diagram to find the height of the tall tree.

7 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of an cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

- **a** Find the diameter of the saucepan.
- **b** A stirring spoon is 20 cm in length. Explain whether it can fall into the saucepan.

8 An ice cream cone has a height of 10 cm. The circumference of the top circle of the cone is 15.5 cm.

- **a** Find the radius of the top circle.
- **b** Draw a labelled two-dimensional sketch of the outline of the cone. Define any right angles within the shape.
- **c** Find the slant length of the cone.

9 A Christmas tree in the shape of a cone is 3 m tall with a base diameter of 2 m.

Sandy wants to put streamers from the top to the base of the tree, along the outer surface of the tree. She has a roll of streamers 30 m long she can cut them up from. How many full length streamers is she able to attach?

10 A solid cube of side length 5 cm is sliced vertically through its base diagonal.

- **a** What is the shape of its exposed face? Draw a diagram with its dimensions labelled.
- **b** Compare the area of the exposed face with that of one of the faces of the cube.

11 A hollow cube is made from a length of wire, bent to form the edges. The edges are each 5 cm long.

- **a** Draw a diagram to display this cube.
- **b** An ant starts crawling from a corner along the wire edges. What is the maximum distance the ant can crawl along the wire edges without going along the same edge twice? Explain your answer.

12 A **tetrahedron** is a three-dimensional object with four equilateral triangular faces.

- **a** Identify the tetrahedron in this photo and draw a sketch. The length of each side is 10 cm and its height from base to the upper apex is 8.2 cm. Add these measurements to your sketch.
- **b** **i** Draw a right-angled triangle displaying the height of the tetrahedron and the length of a side. Label the measurements.
  - **ii** Use your diagram to calculate the angle between the edge and a face in a tetrahedron.
- **c** Draw the shape of a face. Mark in the perpendicular height, and label all known values.
  - **i** Find this perpendicular height.
  - **ii** Draw a right-angled triangle showing this perpendicular height of a face and the height of the tetrahedron. Use your diagram to calculate the angle between two faces of a tetrahedron.

13 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower (A) is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

- **a** Draw a labelled diagram to display the information. Label the height of the tower (AT) as h and the ground distance XT as d.
- **b** In ΔAXT, use a trigonometric ratio to find an equation for h in terms of d.
- **c** In ΔAYT, use a trigonometric ratio to find an equation for h in terms of d. (Note that the length of YT is $(25 + d)$ m.)
- **d** Equate your two equations from parts **b** and **c**. Solve to find a value for d.
- **e** Use this value for d in ΔAXT to find the height of the tower.

PROBLEM SOLVING AND REASONING

14 Three markers are placed on the ground at points A, B and C. The bearing of C from A is 030°T, and the bearing of C from B is 300°T. The bearing of B from A is 075°T, while the distance from A to C is 50 m.

a Draw a diagram to show the position of the markers on the ground. Try to make your drawing as accurate as possible.

b Find the distance from:

i B to C

ii B to A.

c A flagpole is erected at marker A. The angle of elevation from C to the top of the flagpole is 35°. What is the angle of elevation from B to the top of the flagpole?

15 A rectangular prism has a base whose length is twice its width (w). The height of the prism is three times its width.

a Write the length and height of the prism in terms of its width w.

b Draw a labelled diagram to display the prism.

c Find the length of the base diagonal in terms of w.

d Calculate the length of the diagonal within the prism in terms of w.

e Find the angle the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

CHALLENGE

16 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

17 Consider a cube of side length x cm.

a Draw a diagram of the cube.

b An ant crawls from the top front left corner to the bottom back right corner along the outer surface.

i Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.

ii Find the length of this shortest path. Give your answer in terms of the side length x.

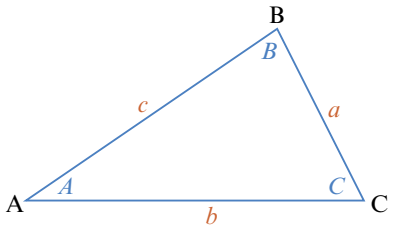
Reflect

How can you decide whether to use Pythagoras' Theorem or trigonometry when determining a length in a right-angled triangle?

8F Sine and area rules

Start thinking!

Trigonometry can also be applied in non-right-angled triangles.



1 The convention is to label the side lengths with lowercase letters of the angles opposite these sides.

a Copy this figure.

b Draw a perpendicular from vertex B to base AC, meeting the base at D. Label the length of BD as h.

c **i** Use the sine ratio for angle A to write a relationship between h and c. Make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Make h the subject of the formula.

iii Equate your two answers and show how this can be written as $\frac{a}{\sin A} = \frac{a}{\sin C}$.

Similarly, relationships can be deduced using angle B. This relationship is called the **sine rule**, and can be written as: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or alternatively as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

These ratios are used in pairs to find an unknown angle or side length.

2 The area of a triangle is calculated as $\frac{1}{2} \times$ base $\times$ perpendicular height. If the triangle is not right-angled, and the perpendicular height is not given, it can be calculated using trigonometry.

a Use the relationship you developed in question **1** part **ci** to show that the area of the triangle can be written as: area $= \frac{1}{2}bc \sin A$.

b Using relationships between the other sides and angles, the formula for area can be written as:
area $= \frac{1}{2}ab \sin C$ or area $= \frac{1}{2}ac \sin B$ or area $= \frac{1}{2}bc \sin A$
To use this area formula, what necessary information do you need to know about a triangle?

KEY IDEAS

- The sine rule can be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or alternatively as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. These ratios are used in pairs.
- The general formula for the area of a non-right-angled triangle can be written:
  area $= \frac{1}{2}ab \sin C$ or area $= \frac{1}{2}ac \sin B$ or area $= \frac{1}{2}bc \sin A$
- To find the area, two side lengths and the included angle must be known.
- If all the side lengths are known, but no angles are known, Heron's formula can be used to find the area of a triangular shape.
  area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

EXERCISE 8F Sine and area rules

EXAMPLE 8F-1 Using the sine rule

Find the unknown angle and side lengths (correct to one decimal place) for the triangle ABC, where $b = 8$ cm, $B = 75°$ and $C = 70°$.

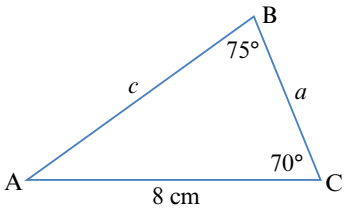
THINK

1 Draw a labelled triangle roughly to scale.

2 Two angles are given, so the third angle can be calculated.

3 Use the sine rule to calculate the lengths of the other two sides. (Use the formula with the side length in the numerator.)

WRITE



$A = 180° - (75° + 70°)$
$= 35°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 35°} = \frac{8}{\sin 75°}$$
$$a = \frac{8 \times \sin 35°}{\sin 75°}$$
$$\approx 4.8 \text{ cm}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{8}{\sin 75°} = \frac{c}{\sin 70°}$$
$$c = \frac{8 \times \sin 70°}{\sin 75°}$$
$$\approx 7.8 \text{ cm}$$

UNDERSTANDING AND FLUENCY

1 Use the sine rule to find the unknown value for each triangle ABC.

- **a** If $A = 45°$, $B = 72°$ and $a = 10$ mm, calculate b.
- **b** If $C = 65°$, $a = 12$ cm and $c = 13.4$ cm, find the size of angle A.
- **c** $B = 37°$, $c = 8$ cm and $b = 15$ cm. Calculate A.

2 Find the unknown angles and side lengths for each triangle ABC using the given information.

- **a** $a = 8.5$ cm, $B = 81°$ and $A = 37°$
- **b** $c = 37.3$ mm, $b = 33.7$ mm, $C = 85°$
- **c** $a = 22.5$ cm, $c = 28.3$ cm, $A = 49°$
- **d** $b = 5.8$ cm, $B = 26°$, $A = 67°$

EXAMPLE 8F-2 Using the area rule

Find the area of triangle ABC shown in Example 8F-1 (correct to one decimal place).

THINK

All the side lengths and angles are known, so any of the formulas can be used.

WRITE

area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 4.8 \times 8 \times \sin 70°$

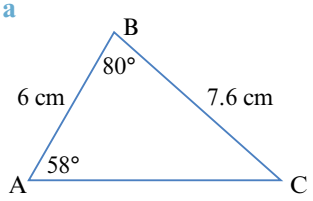
$\approx 18.0 \text{ cm}^2$

3 Find the area of each triangle ABC.

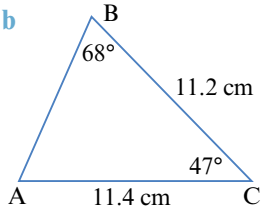
- **a** $a = 9.5$ cm, $b = 7.6$ cm, $C = 28°$
- **b** $b = 2.8$ mm, $c = 8.9$ mm, $A = 87°$
- **c** $a = 4.7$ cm, $c = 9.2$ cm, $B = 57°$
- **d** $c = 12.5$ cm, $b = 9.2$ cm, $A = 43°$

4 Find the perimeter of each triangle.

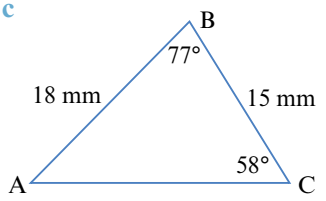
a



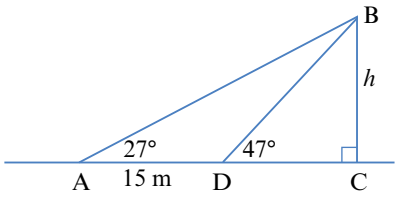
b



c

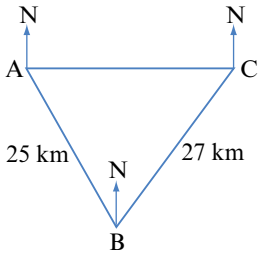


5 Chloe measures the angle of elevation to the top of a building as 27°. When she walks 15 m closer to the building, she measures the angle of elevation to be 47°.



- **a** Find the size of:
  - **i** ∠ADB
  - **ii** ∠ABD.
- **b** Use the sine rule in ΔABD to calculate the length of BD, to one decimal place.
- **c** Use the sine ratio in ΔBCD to find the height of the building.
- **d** Use the cosine ratio in ΔBCD to find how far Chloe is from the building when she took her second reading.
- **e** How far was she from the building initially?

6 A group of bushwalkers start from a point A, travels 25 km to a point B at a bearing of S30°E, then continues for 27 km to point C which lies due east of their starting point.



- **a** What is the value of ∠BAC?
- **b** Use the sine rule in ΔABC to find the size of ∠ACB to the nearest degree.
- **c** What is the size of ∠ABC?
- **d** Use the sine rule again to find the distance the bushwalkers would need to travel to return directly to their starting point. Give your answer to one decimal place.
- **e** Give the bearing:
  - **i** of C from B
  - **ii** of B from C
  - **iii** the bushwalkers would need to take to return directly to their starting point.
- **f** Calculate the area enclosed by the bushwalkers' hike.

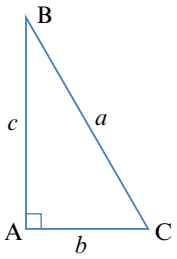
7 The banks of a creek run directly east–west. From where Kim stands, the bearing to a tree on the opposite side of the creek is 032°T. If she moves 20 m due east on the bank of the creek, the bearing to the same tree is 310°T.

- **a** Draw a labelled diagram displaying as much information as possible.
- **b** Find the distance (in metres to one decimal place) of the tree from:
  - **i** Kim's first viewing point
  - **ii** Kim's second viewing point.
- **c** Calculate the width of the creek.

8 A lighthouse stands 50 m tall on top of a cliff. From a ship at sea, the angle of elevation to the base of the lighthouse is 3°, while the angle of elevation to the top of the lighthouse is 4.2°.

- **a** Draw a labelled diagram to display this information.
- **b** Use the sine rule to find the straight-line distance from the ship to:
  - **i** the base of the lighthouse
  - **ii** top of the lighthouse.
- **c** Calculate the distance in kilometres (to one decimal place) from the ship to the base of the cliff.
- **d** Find the height of the cliff, to the nearest metre.

9 Consider this right-angled triangle ABC.



The sine rule is quoted as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

- **a** What is the value of sin A?
- **b** Use the triangle to write expressions for sin B and for sin C.
- **c** Substitute your expressions from parts **a** and **b** into the sine rule. Simplify.
- **d** Explain your answer. Does the sine rule hold for right-angled triangles?
- **e** What would have been your answer for part **c** if the triangle had been right-angled at B or C?

10 A give-way sign is the shape of an equilateral triangle, with side lengths of 40 cm. Calculate the area of the sign to the nearest square centimetre.

11 Refer to the diagram in question **9**. The area of the triangle can be calculated using the formula area = $\frac{1}{2}ab \sin C$.

- **a** Use the diagram to obtain an expression for sin C.
- **b** Substitute this expression into the area formula and simplify.
- **c** Explain the resulting formula.

12 The formula you have used to find the area of a non-right-angled triangle relies on the fact that you know the length of two sides of the triangle, and the angle between these sides. If all the side lengths are known, but no angles are known, Heron's formula can be used to find the area of a triangular shape. (The proof is beyond this course, so is simply quoted here.)

$A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$, and a, b and c are the side lengths of the triangle. Note that s is the **semi-perimeter** of the triangle.

A triangular block of land has dimensions 200 m by 300 m by 400 m.

- **a** Draw a labelled diagram of the block. State values for a, b and c.
- **b** Find the value for s.
- **c** Substitute into Heron's formula. Simplify to find the area of the land to the nearest square metre.
- **d** If the block had been 300 m by 400 m by 500 m, it would be right-angled in shape.
  - **i** Explain why this is so.
  - **ii** Find its area using Heron's formula.
  - **iii** Calculate its area using the traditional area formula.
  - **iv** Compare your answers to parts **ii** and **iii**.

PROBLEM SOLVING AND REASONING

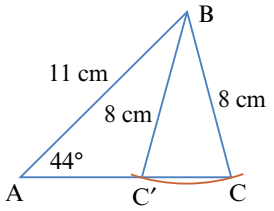
13 When using the sine rule, you have to make sure you are given enough information to define a specific triangle. If this is not the case, you may be able to draw two triangles from the given information, and therefore get two sets of answers. This is called the ambiguous case of the sine rule.

Consider a triangle ABC with $A = 44°$, $a = 8$ cm and $c = 11$ cm.

a From this information, try to draw two different triangles. (Make sure they are roughly to scale.) Why is it possible to draw more than one triangle?

The two triangles can be drawn superimposed as shown here.

One solution belongs to triangle ABC (this is the one you would find with the usual sine rule calculations), while the other solution belongs to triangle ABC′. It is necessary to understand the relationship between angle C in the first triangle and angle C′ in the second triangle.



b Consider ΔC′BC.

- **i** What type of triangle is this?
- **ii** What is the relationship between ∠BCC′ and ∠BC′C?
- **iii** ∠BC′A is supplementary to ∠BC′C. Why is this so?
- **iv** How would you find the size of ∠BC′A once you have found the size of ∠BCC′?

c Use the sine rule within ΔABC to find:

- **i** $\angle C$
- **ii** $\angle B$
- **iii** b.

d In ΔABC′, using the sine rule where necessary, find:

- **i** ∠BC′A
- **ii** ∠ABC′
- **iii** b.

e Collate your two separate results to give the dimensions of the two triangles which satisfy the given information. Confirm that these results seem to be realistic values for the triangles drawn.

CHALLENGE

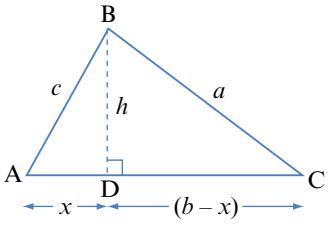
14 Question **4** part **c** is actually an ambiguous case where two distinct triangles can be drawn. Find a second set of solutions for this question.

Reflect

When would you use the area formula involving sine and when would you use Heron's formula to find the area of a triangle?

8G Cosine rule

Start thinking!



1 Consider triangle ABC, with the perpendicular from B dividing the base b into two sections of length x and $(b - x)$.

a Use ΔCDB and Pythagoras' Theorem to complete the relationship $a^2 = h^2 +$ _____.

b Use ΔADB and Pythagoras' Theorem to complete the relationship $h^2 =$ _____.

c Substitute the value of h^2 from part **b** into your equation in part **a**. Simplify the resulting equation and confirm that it can be written as $a^2 = b^2 + c^2 - 2bx$.

d Use ΔADB and the cosine ratio to write a relationship with x as the subject of the formula.

e Substitute your value for x into your equation in part **c**. Your resulting equation should be: $a^2 = b^2 + c^2 - 2bc \cos A$. This is known as the **cosine rule**.

f Similarly, expressions for b^2 and c^2 can be deduced.

g The cosine rule can be rearranged to find the size of an angle of a triangle. Confirm it can be written in the following forms.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

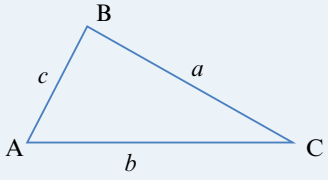
h Using the cosine rule, what information would you need to calculate:

i a side length?

ii an angle?

The cosine rule can be summarised as:
in any triangle ABC,
$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

KEY IDEAS



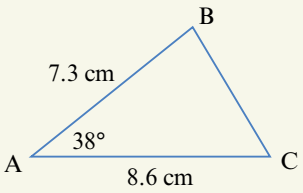
- The cosine rule can be used to find the side length or angle of a triangle.
- For calculating a side length, the cosine rule can be written as:
  $a^2 = b^2 + c^2 - 2bc \cos A$ or $b^2 = a^2 + c^2 - 2ac \cos B$ or $c^2 = a^2 + b^2 - 2ab \cos C$
  Two side lengths and the size of the included angle must be known.
- For calculating an angle, the cosine rule can be written as
  $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
  The lengths of the three sides of the triangle must be known in this case.

EXERCISE 8G Cosine rule

EXAMPLE 8G-1 Using the cosine rule to find a side length

Consider this triangle.

Find the unknown side length, correct to one decimal place.



THINK

1 Identify the pronumerals.

2 Substitute into the cosine rule, leaving all the decimal places on the calculator.

3 Take the square root, and provide the answer.

WRITE

$b = 8.6$ cm, $c = 7.3$ cm, $A = 38°$

$a^2 = b^2 + c^2 - 2bc \cos A$
$= 8.6^2 + 7.3^2 - 2 \times 8.6 \times 7.3 \times \cos 38°$
$= 28.307\ldots$

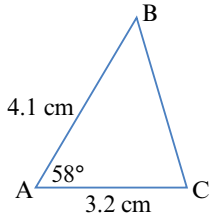
$a = \sqrt{28.307\ldots}$
≈ 5.3

The unknown side BC measures 5.3 cm.

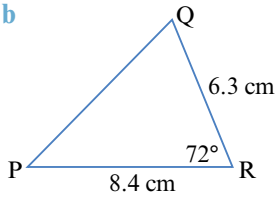
UNDERSTANDING AND FLUENCY

1 Find the unknown side length in each triangle, correct to one decimal place.

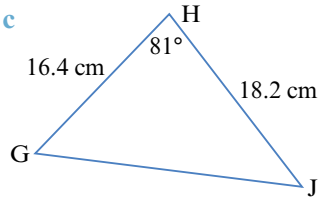
a



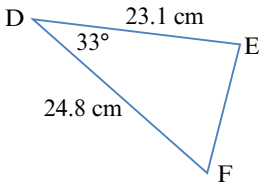
b



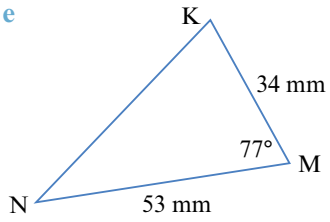
c



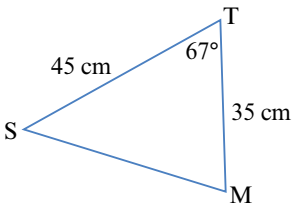
d



e



f



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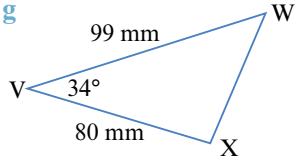
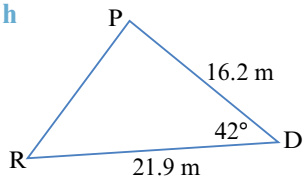
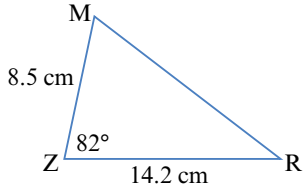


h



i



EXAMPLE 8G-2 Using the cosine rule to find an angle

Triangle ABC has side lengths of $a = 5.8$ cm, $b = 4$ cm and $c = 6.4$ cm.
Find the size of all angles in the triangle to the nearest degree.

THINK

1 Use the angle form of the cosine rule for angle A. Substitute and simplify, leaving all decimal places on the calculator.

2 Take the inverse cosine, rounding to the nearest degree.

3 Repeat steps 1 and 2 to calculate the size of angle B.

4 Calculate angle C.

WRITE

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{4^2 + 6.4^2 - 5.8^2}{2 \times 4 \times 6.4}$$
$$= 0.4554\ldots$$
$$A = \cos^{-1}(0.4554\ldots)$$
$$\approx 63°$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{5.8^2 + 6.4^2 - 4^2}{2 \times 5.8 \times 6.4}$$
$$= 0.7893\ldots$$
$$B = \cos^{-1}(0.7893\ldots)$$
$$\approx 38°$$
$$C = 180° - (63° + 38°)$$
$$= 79°$$

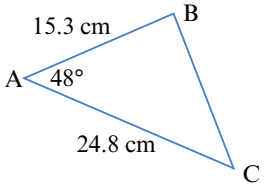
UNDERSTANDING AND FLUENCY

2 The following lengths are given for the sides in ΔABC. Find the size of all the angles.

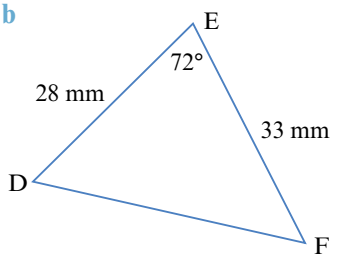
- **a** $a = 4.5$ cm, $b = 6$ cm and $c = 5$ cm
- **b** $a = 13$ mm, $b = 12$ mm and $c = 7$ mm
- **c** $a = 1.5$ m, $b = 2.3$ m and $c = 1.8$ m
- **d** $a = 16$ cm, $b = 19$ cm and $c = 12$ cm

3 Find the unknown side lengths and angles in each triangle.

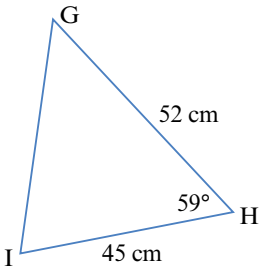
a



b



c



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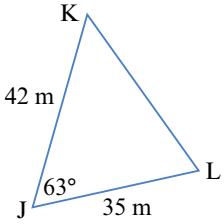

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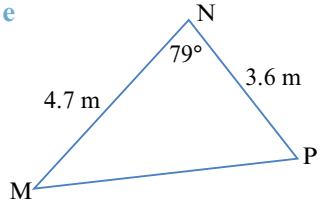

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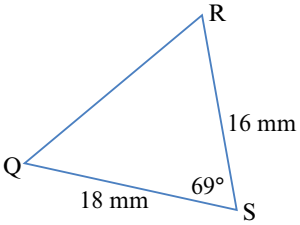
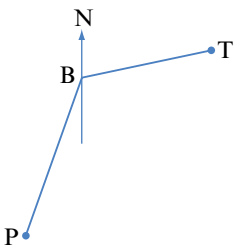


4 From a buoy at sea, Todd rows 2.5 km at a bearing of N80°E, while Pete rows 3.5 km at a bearing of S20°W.



- **a** Copy the diagram and label all known sides and angles.
- **b** What is the size of ∠TBP?
- **c** Join TP. This represents the distance between the two rowers. Use the cosine rule in ΔBTP to calculate this distance.
- **d** Use the cosine rule in ΔBTP to calculate the size of ∠BPT.
- **e** Use your answer from part **d** to find the bearing of Todd from Pete.
- **f** Calculate the size of ∠BTP.
- **g** Use your answer from part **f** to find the bearing of Pete from Todd.

5 A triangular garden bed has side lengths 5.3 m, 4.8 m and 4.5 m.

- **a** Draw a labelled sketch of the garden bed.
- **b** Identify the vertex with the largest angle. (Hint: the largest angle lies opposite the longest side.)
- **c** Use the cosine rule to find the size of this angle.
- **d** Explain how, using this angle, you can now calculate the area of the bed. What is its area?

6 A deep-sea diver is attached to two safety ropes from two boats on the surface of the ocean. The boats are 50 m apart. One rope is 110 m long, while the other is 100 m long.

- **a** Draw a labelled diagram to describe the situation.
- **b** Use the cosine rule to find the angles the two ropes make with the surface of the water.
- **c** Construct a right-angled triangle within your diagram which would enable you to find the maximum depth to which the diver could descend while attached to the two safety ropes. Explain a technique you could use to find this depth.
- **d** Calculate this maximum depth.

7 A give-way sign is in the shape of a triangle. Two adjacent sides measure 35 cm, and the angle between these sides measures 60°. Use the cosine rule to prove that the triangle is equilateral.

8 Investigate the cosine rule in right-angled triangles.

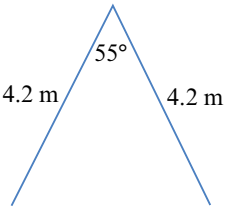
- **a** Draw a right-angled triangle ABC with the right angle at A.
- **b** Copy and complete the cosine rule: $\cos A =$ _____.
- **c** What is the value of $\cos A$ in your figure?
- **d** Substitute this value into your equation in part **b** and simplify. Describe the result.

9 Emily is practising shooting hockey goals. The hockey goal is 3 m wide. She stands 5 m directly in front of one of the posts.

- **a** Draw a diagram to display the situation.
- **b** Use Pythagoras' Theorem to calculate her distance to the other goal post.
- **c** Use the cosine rule to find the angle within which she must shoot to score a goal.
- **d** Emily moves her position so she is 5 m from each of the goal posts. Explain whether she has a greater chance of scoring a goal from this position.

10 The end support pipes of a swing form a triangular shape with the ground.

The pipes are each 4.2 m long with an angle of 55° between them.



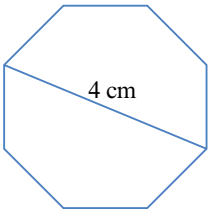
- **a** How far apart are the feet on the ground?
- **b** Find the vertical height of the top of the frame above the ground.

11 The frame for a bridge consists of a series of triangular shapes connected together as shown in the photo.

The sloping steel parts are 3.9 m long, while the horizontal steel parts are 2.8 m long.

- **a** Find the angle between the two sloping pieces of steel.
- **b** Draw a sketch of several sections of the bridge, showing the angles and lengths of the connecting pieces.
- **c** Find the height of the outside frame of the bridge if the horizontal beams are 10 cm wide.

12 The diagonal length of an octagon measures 4 cm.



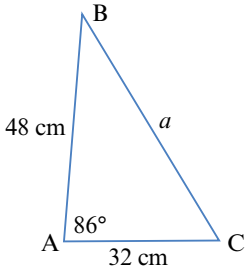
- **a** Draw a sketch of the octagon, joining all the diagonals of the figure.
- **b** Consider one of the triangles within your figure.
  - **i** What is the size of the angle at the centre?
  - **ii** Find the sizes of the other two angles of the triangle.
- **c** Calculate the length of the outer edge of the octagon.

PROBLEM SOLVING AND REASONING

13 A light aircraft is set to fly on a course N78°E for a distance of 350 km. A strong easterly wind blows the plane off course by 6°.

- **a** Draw a labelled diagram to show this situation.
- **b** If the pilot flies the planned distance on this incorrect path, how far off course will he be at the end of his journey?

14 Sometimes there is a choice as to whether the cosine rule or the sine rule can be used to solve a problem. Consider this triangle.



- **a** Explain why it is not possible to use the sine rule to calculate the value of a.
- **b** Use the cosine rule to find the length of a.
- **c** **i** Explain how you could now use the sine rule to find the sizes of angles B and C.
  - **ii** Calculate the values of these two angles.
- **d** Angles B and C from part **c** could also have been calculated using the cosine rule. Find their values using this rule.
- **e** Comment on your answers to parts **c ii** and **d**.

15 A cube has a side length of 25 cm.

- **a** Draw a labelled sketch of the cube.
- **b** Use the cosine rule to find the length of the diagonal of the base. Leave your answer in a simplified exact form.
- **c** Use the cosine rule again to find the length of the diagonal of the cube. Leave your answer in a simplified exact form.

16 The angles of a triangle are in the ratio 3:4:5.
The two sides that form the largest angle measure 3.5 cm and 4.3 cm.

- **a** Find the size of all the angles.
- **b** Calculate the length of the third side.
- **c** Draw a labelled diagram to display the measurements of all the angles and sides.

CHALLENGE

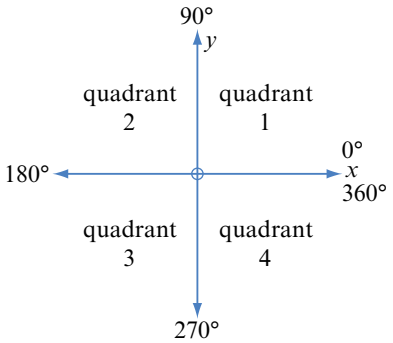
17 A right-angled isosceles triangle is inscribed in a semi-circle of radius r.

- **a** Draw a diagram to display this information.
- **b** Find the lengths of the two equal sides in terms of r using:
  - **i** Pythagoras' Theorem
  - **ii** trigonometry.
- **c** Comment on your two answers to part **b**.

Reflect

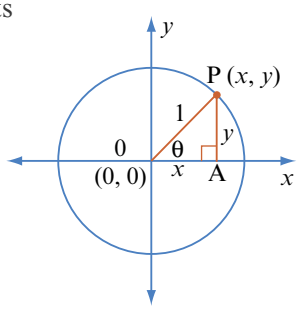
How can you tell when to use the sine rule and when to use the cosine rule in non-right-angled triangles?

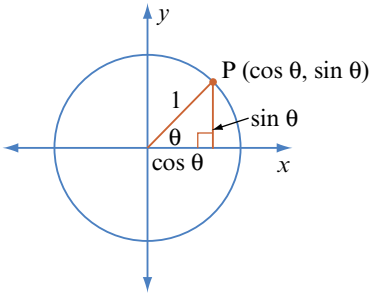
8H The unit circle and trigonometric graphs



Start thinking!

The **unit circle** is a circle drawn on the Cartesian plane with a radius of 1 unit and its centre at the origin. The quadrants on the Cartesian plane are labelled in an anticlockwise direction.





1 Imagine there is a point, P, with coordinates (x, y), on the circumference of the unit circle. As P moves around the circumference, the value of its coordinates changes. Consider its position as shown here, making an angle of θ with the positive direction of the x-axis.

a Use a trigonometric ratio to find the values of x and y in terms of θ.

b These labels can now be attached to the point P. So, P also has the coordinates (cos θ, sin θ). Which trigonometric ratio represents:

i the horizontal direction? **ii** the vertical direction?

2 The point P could be in any quadrant. The x- and y-values are positive or negative depending on the quadrant. Copy and complete this table.

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
sign of x-coordinate (cos θ)				
sign of y-coordinate (sin θ)				
sign of tan θ $\left(\frac{\sin\theta}{\cos\theta}\right)$				

An easy way to remember the sign of the trigonometric functions in the four quadrants is with a mnemonic – something like **A**ll **S**tations **T**o **C**anberra. In this case:
A: sin, cos and tan **all** positive in quadrant 1
S: **sin** positive in quadrant 2
T: **tan** positive in quadrant 3
C: **cos** positive in quadrant 4

KEY IDEAS

- The unit circle is a circle on the Cartesian plane with its centre at the origin and a radius of 1 unit.
- The x-coordinate of a point on the unit circle is represented by the cosine ratio.
- The y-coordinate of a point on the unit circle is represented by the sine ratio.
- Angles greater than 90° are generally represented by the angle made by the radius connecting the point with the x-axis.
- Sine, cosine and tangent values have different signs depending on the quadrant in which they lie. The mnemonic **A**ll **S**tations **T**o **C**anberra can be an aid to remembering these signs.
- A **periodic function** is one that repeats itself continuously in cycles.

EXERCISE 8H The unit circle and trigonometric graphs

1 Identify the quadrant in which each angle lies.

a 345° **b** 103° **c** 204°
d 265° **e** 74° **f** 139°

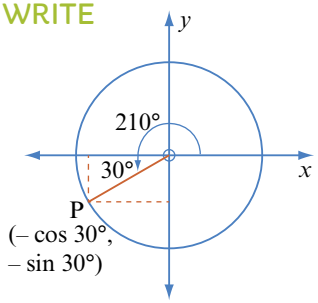
EXAMPLE 8H-1 Using the unit circle to calculate exact trigonometric values for an angle greater than 90°

Use the unit circle and exact values for the trigonometric ratios to calculate the sin, cos and tan of 210°.

THINK

1 Sketch the unit circle and mark the angle 210°. It is in quadrant 3 and is equivalent to (180° + 30°).

WRITE



2 Write values for sin 30° and cos 30° in exact form to calculate the exact value of tan 30°.

$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$

$$\tan 30° = \frac{\sin 30°}{\cos 30°}$$
$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$
$$= \frac{1}{\sqrt{3}}$$

3 Apply the correct signs to the trigonometric ratios for quadrant 3.

In quadrant 3, sin and cos are both negative, while tan is positive.

$\sin 210° = -\frac{1}{2}$

$\cos 210° = -\frac{\sqrt{3}}{2}$

$\tan 210° = \frac{1}{\sqrt{3}}$

2 Use the unit circle and exact values for the trigonometric ratios to calculate the sin, cos and tan of each angle.

a 240° **b** 300° **c** 120°
d 315° **e** 135° **f** 150°

EXAMPLE 8H-2

Using the unit circle to calculate decimal trigonometric values for an angle greater than 90°

Use the unit circle to calculate the values of sin, cos and tan of 290°.
Give your answers correct to three decimal places.

THINK

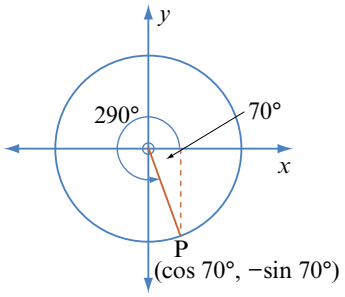
1 Sketch the unit circle and mark the angle of 290°. It lies in quadrant 4 and is equivalent to (360° − 70°).

2 Calculate values for sin 70° and cos 70°, correct to three decimal places. Use your calculator values to find the value of tan 70°.

3 Apply the correct signs to the ratios.

NOTE These answers can be checked using a calculator. There may be a slight variation due to rounding.

WRITE



In quadrant 4, sin is negative and cos is positive, so the coordinates of P are (cos 70°, −sin 70°).

$\sin 70° = 0.940$
$\cos 70° = 0.342$
$\tan 70° = \frac{0.940}{0.342}$
$= 2.747$

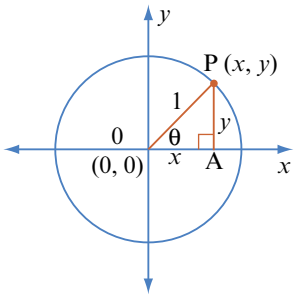
In quadrant 4, sin and tan are both negative, while cos is positive.
$\sin 290° = -0.940$
$\cos 290° = 0.342$
$\tan 290° = -2.747$

UNDERSTANDING AND FLUENCY

3 Use the unit circle to calculate the values of sin, cos and tan of each angle. Give your answers correct to three decimal places.

a 87° **b** 285° **c** 202°
d 340° **e** 170° **f** 95°

4 Trace the point P around the unit circle in an anticlockwise direction from its starting point on the positive x-axis back to its starting point.



a You will notice that the value of the angle increases from 0° to 360°. If you consider the x value of the point as it moves from 0° to 90°, you notice that its value decreases. Describe what happens to the x value from:

i 90° to 180° **ii** 180° to 270° **iii** 270° to 360°.

b This change in x value is actually the change in the cosine of the angle as it sweeps from 0° to 360°. To obtain a clearer picture of this movement, you can create a table of cosine values for angles from 0° to 360°. Copy and complete this table with the aid of your calculator. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ													

c Plot these points on grid or graph paper, with the angle (θ) along the horizontal axis and the cosine value (cos θ) along the vertical axis. Join your points with a smooth curve. Describe the shape of the curve. What is the starting value of the curve?

d Sketch what would happen if you extend the angle values to 720°. This type of curve is known as a periodic function as it repeats itself continuously in cycles.

e The curve progresses through repeating peaks and troughs. The **period** of the curve is the distance between repeating peaks or troughs.

i Note the value of cos θ at its first peak. How many degrees does the curve pass through before it reaches its next peak?

ii What is the period of the **cosine graph**?

f The distance from the horizontal axis to a peak or a trough is called the amplitude of the graph. It is half the distance between the maximum and minimum points of the graph.

i How far above the horizontal axis does the curve reach its maximum point?

ii What is the distance from the horizontal axis to the curve's minimum point? (Ignore any negative signs.)

iii What is the amplitude of the graph?

5 Consider the change in the y value as the angle sweeps from 0° to 360°. This actually represents the change in the sine value of the angle. Repeat the procedure in question **4** to graph the sine values for angles from 0° to 360° and then answer these questions.

a Describe what happens to the value of the sine of the angle from 0° to 360°.

b What is the starting sine value of the curve?

c Draw a sketch of the **sine graph**.

d Why would this be classed as a periodic function?

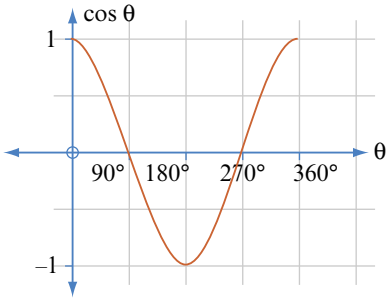
e What is the period of the graph?

f How far above/below the horizontal axis does the curve reach its maximum/minimum?

g What is the amplitude of the graph?

6 The graphs of the sine function and the cosine function are quite similar. If you were presented with a cosine curve and a sine curve, and the graphs weren't labelled, how could you tell which was which?

7 Consider this graph of the cosine function.



a Use the graph to read the cosine value for each of these angles.

i 360° **ii** 90° **iii** 270° **iv** 180°

b Use the graph to read the approximate cosine value (correct to one decimal place) for each of these angles.

i 35° **ii** 100° **iii** 350°

iv 200° **v** 280° **vi** 140°

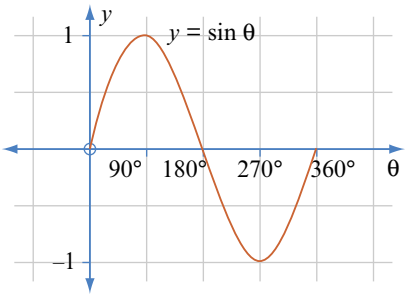
c Use the graph to find angles with these cosine values. Give your answers as approximate values, if necessary.

i 1 **ii** −1 **iii** 0

iv 0.4 **v** −0.6 **vi** 0.8

d Describe what you notice about your answers in part **c**.

8 Consider this graph of the sine function.



a Use the graph to read the sine value for each of these angles.

i 360° **ii** 90°

iii 270° **iv** 180°

b Use the graph to read the approximate sine value (correct to one decimal place) for each of these angles.

i 35° **ii** 100° **iii** 350°

iv 200° **v** 280° **vi** 140°

c Use the graph to find angles with these sine values. Give your answers as approximate values, if necessary.

i 1 **ii** −1 **iii** 0

iv 0.4 **v** −0.6 **vi** 0.8

d Describe what you notice about your answers in part **c**.

9 A grandfather clock has a pendulum which swings back and forth in a regular periodic motion. This has been one method of keeping time for hundreds of years, since the motion is so regular. The weight of the bob does not affect the period; however, the period is affected by the length of the pendulum.

a Explain why the motion of a pendulum is periodic.

b The equation used to find the period of a pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$, where T is the time (in seconds), l is the length of the pendulum (in metres) and g is the acceleration due to gravity (9.8 m/s^2). Find the period of a pendulum with lengths:

i 1 m **ii** 1.5 m **iii** 2 m

c Explain how the length of a pendulum affects its period.

PROBLEM SOLVING AND REASONING

10 Periodic functions occur in many everyday activities. Consider these sunrise and sunset times, recorded for the east coast of Australia on the first day of the month throughout a year. The times are recorded as Eastern Standard Time.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sunrise (am)	4.55	5.20	5.40	5.57	6.12	6.29	6.38	6.29	6.02	5.28	4.57	4.44
Sunset (pm)	6.46	6.42	6.20	5.46	5.17	5.01	5.04	5.19	5.34	5.47	6.05	6.28

a Plot the sunrise and sunset figures on separate graphs, placing the sunrise graph directly below the sunset graph. Write the months as the horizontal axis values and the time as the vertical axis values.

b Describe the shapes of the two curves.

c Are these two curves similar to those for sine and cosine functions? Explain any similarities or differences.

d Are the curves periodic? Explain why or why not.

e Because of the way you positioned these two graphs, the trend in daylight hours throughout the year is visually apparent. Explain how this is so.

f If you were not told that these were figures for the east coast of Australia, how could you reason that they must be figures for somewhere in the southern hemisphere?

CHALLENGE

11 The tangent ratio is also a periodic function.

a Complete a table of values for $\tan\theta$ for θ values from 0° to 360°.

b Plot these points on a Cartesian plane and join them with a smooth curve.

c **i** Describe the shape of the graph.

ii It is different from the sine curve and the cosine curve. Why is this so?

iii At $\theta = 90°$, a vertical asymptote occurs. Explain what you understand this to mean. Where does the next asymptote occur?

iv Explain why the graph is periodic. What is its period?

v What is the amplitude of the graph?

12 Repeat some of these questions using digital technology to confirm your answers.

Reflect

Music uses terms such as 'amplitude' and 'frequency'. What is the connection between these terms and trigonometric graphs?

8I Solving trigonometric equations

Start thinking!

Trigonometric functions are periodic, so trigonometric equations have infinitely many solutions.

1 Consider the graph for $y = \cos\theta$.

a How many angles have a cosine value of 0.5 in the interval: **i** 0° to 360°? **ii** 0° to 720°

b Use your calculator to find the number of answers.

A similar situation occurs when considering the graph of $y = \sin\theta$. In fact, any of the **trigonometric graphs** present the same property of multiple solutions. For this reason, in order to restrict the number of solutions, you generally solve the equation within a particular **domain** (the domain is the set of θ values).

Note that a calculator may not supply all the solutions within a quoted domain. You must look at the calculator answer, together with the domain, the period of the function and the sign of the trigonometric function in the various quadrants of the Cartesian plane. Considering all these options allows you to decide on the number of appropriate answers and their values.

2 Consider solving the equation $\sin\theta = \frac{1}{\sqrt{2}}$, $\theta \in [0°, 360°]$. (Note: [0°, 360°] means the interval of values from 0° to 360°. The initial and final values of the domain are shown within square brackets.)

a What angle in the first quadrant has a sine value of $\frac{1}{\sqrt{2}}$?

b What is the stated domain?

c What is the period of the function and how many periods occur within this domain?

d How many solutions for θ should you expect within this domain?

e Is the sine value positive or negative in this equation?

f What other quadrant/s have a sine value with this same sign?

g Remembering that the angle is always measured from the positive or negative x-axis, what is another possible solution?

h If the domain had been 0° to 720°, how many solutions would you expect? What are they?

3 Consider the equation $\cos\theta = -0.2$, $\theta \in [0°, 360°]$.

a What angle in the first quadrant has a cosine of 0.2? Give your answer correct to the nearest degree.

b The cosine value is negative in this equation. In which quadrants is cosine negative?

c What are the solutions to this equation?

KEY IDEAS

- There are multiple solutions for trigonometric equations, as they relate to periodic functions.
- Sine, cosine and tangent values have different signs depending on the quadrant in which the angle lies. The mnemonic **A**ll **S**tations **T**o **C**anberra can be an aid to remembering these signs.
- A calculator supplies only one solution. Consider the quadrant in which the angle lies when searching for extra solutions.

EXERCISE 8I Solving trigonometric equations

UNDERSTANDING AND FLUENCY

1 Indicate whether the trigonometric value is positive or negative in each case.

a sin 100° **b** cos 200° **c** tan 140° **d** cos 295° **e** tan 260
f sin 340° **g** cos 94° **h** sin 130° **i** cos 190° **j** tan 350°

2 Write each angle in the form 180° − θ.

a 140° **b** 92° **c** 170° **d** 150° **e** 100°

3 Write each angle in the form 180° + θ.

a 200° **b** 260° **c** 190° **d** 220° **e** 255°

4 Write each angle in the form 360° − θ.

a 280° **b** 315° **c** 345° **d** 299° **e** 302°

EXAMPLE 8I-1 Solving a trigonometric equation within a given domain (exact value)

Solve the equation $\sin\theta = -\frac{1}{2}$, $\theta \in [0°, 360°]$.

THINK	WRITE
1 Write the equation.	$\sin\theta = -\frac{1}{2}$
2 First find the angle in the first quadrant that has a sine of $\frac{1}{2}$.	$\sin^{-1}\left(\frac{1}{2}\right) = 30°$
3 Identify the quadrants where sine is negative.	Sine is negative in quadrants 3 and 4.
4 Write possible values for θ.	So, $\theta = 180° + 30° = 210°$ or $\theta = 360° - 30° = 330°$
5 Write the solution to the equation.	There are two solutions: $\theta = 210°$ or $330°$

5 Solve each equation for the domain [0°, 360°].

a $\cos\theta = \frac{\sqrt{3}}{2}$ **b** $\sin\theta = -\frac{1}{\sqrt{2}}$

c $\cos\theta = -\frac{1}{2}$ **d** $\sin\theta = -\frac{\sqrt{3}}{2}$

e $\cos\theta = \frac{1}{\sqrt{2}}$ **f** $\sin\theta = \frac{\sqrt{3}}{2}$

EXAMPLE 8I-2 Solving a trigonometric equation within a given domain (decimal value)

Solve the equation $\cos\theta = 0.4$, $\theta \in [0°, 360°]$.

THINK	WRITE
1 Find the angle which has a cosine of 0.4.	$\cos\theta = 0.4$ $\cos^{-1}(0.4) = 66°$
2 Identify the quadrants where cosine is positive.	Cosine is positive in quadrants 1 and 4.
3 Write possible values for θ.	So, $\theta = 66°$ or $\theta = 360° - 66° = 294°$
4 Write the solution to the equation.	There are two solutions: $\theta = 66°$ or $294°$

6 Solve each equation for the domain [0°, 360°].

a $\cos\theta = -0.4$ **b** $\sin\theta = -0.3$

c $\cos\theta = 0.9$ **d** $\sin\theta = 0.6$

e $\cos\theta = 0.6$ **f** $\sin\theta = -0.7$

7 Use digital technology to confirm your answers to questions **5** and **6**.

PROBLEM SOLVING AND REASONING

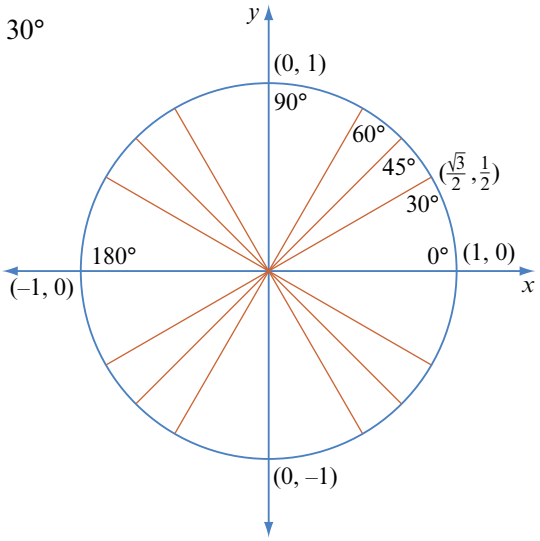
8 The exact values of sine and cosine for the angles 30°, 45° and 60°, together with their multiples, can be recorded on the unit circle.

a Draw a unit circle. Mark the coordinates of the points where the circle cuts the horizontal and vertical axes. Also, mark points on the circle representing angles of 30°, 45° and 60°, together with their multiples up to 360°.

b Label each of the points shown on the unit circle with its exact cosine and sine value as coordinates.

For example, the point representing 30° would be labelled as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Take care to use the correct signs.



c Give the exact coordinates of each of these angles on the unit circle.

i 225° **ii** 300

iii 150° **iv** 135°

v 315° **vi** 90°

9 Use your diagram from question **8** to find solutions to each equation for the domain $[0°, 360°]$. Remember to look for all the solutions.

a $\sin\theta = \frac{\sqrt{3}}{2}$ **b** $\cos\theta = \frac{\sqrt{3}}{2}$ **c** $\sin\theta = -\frac{\sqrt{3}}{2}$

d $\cos\theta = -\frac{\sqrt{3}}{2}$ **e** $\sin\theta = \frac{1}{2}$ **f** $\cos\theta = \frac{1}{\sqrt{2}}$

g $\sin\theta = -\frac{1}{\sqrt{2}}$ **h** $\cos\theta = -\frac{1}{2}$ **i** $\sin\theta = -1$

10 Consider the equation $\sin\theta + 2 = 3$, $\theta \in [0°, 360°]$.

a The equation simplifies to $\sin\theta = 1$. What angle has a sine value of 1? (Hint: refer to the unit circle.)

b Are there any other angles within the domain with a sine value of 1? If so, which one/s?

c What is the solution to the equation?

d Write a list of all the solutions for the domain $[0°, 720°]$.

11 Consider the equation $\cos\theta + 1 = 0$. This can be rearranged as $\cos\theta = -1$.

a What solution/s can be found in the domain $[0°, 360°]$?

b What are the solutions in the domain $[0°, 720°]$?

12 It is possible that a trigonometric equation has no solution. Explain why each of these has no solution within any domain.

a $3 - \sin\theta = 1$ **b** $2\cos\theta = 3$ **c** $\frac{1}{4}\sin\theta = 1$

13 Indicate whether each equation has a solution within the domain $[0°, 360°]$. For those without a solution, explain why this is so. There is no need to solve the equation.

a $\cos\theta = \frac{1}{4}$ **b** $1 - \sin\theta = 1$ **c** $2\cos\theta = 2$

d $\frac{1}{2}\sin\theta = -\frac{1}{2}$ **e** $1 + \cos\theta = -1$ **f** $4\sin\theta = -\frac{1}{4}$

g $3 - \cos\theta = \frac{1}{3}$ **h** $\frac{1}{2}\cos\theta = -\frac{1}{4}$ **i** $\frac{1}{4} + \sin\theta = -1$

14 Take care when you consider the domain of a function. Don't always assume that a trigonometric function has two or more solutions. Consider the following functions within the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

a $\sin\theta = \frac{1}{2}$, $\theta \in [0°, 90°]$ **b** $\sin\theta = -\frac{1}{2}$, $\theta \in [0°, 90°]$

c $\cos\theta = -\frac{1}{2}$, $\theta \in [0°, 180°]$ **d** $\cos\theta = 0.3$, $\theta \in [0°, 180°]$

e $\sin\theta = -0.4$, $\theta \in [0°, 270°]$ **f** $\cos\theta = 0.8$, $\theta \in [0°, 270°]$

15 Solve each equation for the domain indicated.

a $2 \sin \theta - 3 = \sin \theta - 2$, $\theta \in [0°, 180°]$

b $4 \cos \theta + 5 = 4 + 5 \cos \theta$, $\theta \in [0°, 180°]$

c $\sqrt{2} \cos \theta + 1 = 0$, $\theta \in [0°, 270°]$

d $2 \sin \theta + \sqrt{3} = 0$, $\theta \in [0°, 270°]$

16 Consider the situation where the domain does not start from 0°.

a What is the solution to $\sin \theta = 1$ within the domain 0° to 360°?

b If the domain started at the angle found in part **a** instead of starting at 0°, give the starting value of θ.

c There is only one solution from this point to 360°. How far would you have to extend the domain beyond 360° to get another solution?

d What is the smallest domain you could have if you wanted two solutions for the equation?

17 Give the smallest domain for each equation, given that two solutions are required.

a $\sin \theta = -1$ **b** $\sin \theta = 0$

c $\cos \theta = -1$ **d** $\cos \theta = 0$

18 Consult the cosine and sine values for angles on the unit circle.

a What is the smallest angle with a sine value of $\frac{1}{2}$?

b There is a larger angle with the same sine value. What is it?

c What would be the smallest domain for two solutions to the equation $\sin \theta = \frac{1}{2}$?

d What would be the smallest domain for four solutions to the equation?

19 Repeat question **18** for the equation $\cos \theta = \frac{1}{2}$.

20 Give the minimum domain for there to be two solutions for each equation.

a $\sin \theta = -1$ **b** $\cos \theta = 0$

c $\cos \theta = \frac{1}{2}$ **d** $\sin \theta = \frac{\sqrt{3}}{2}$

e $\sin \theta = \frac{1}{\sqrt{2}}$ **f** $\cos \theta = -\frac{\sqrt{3}}{2}$

21 Find the smallest domain for each equation to give:

i two solutions

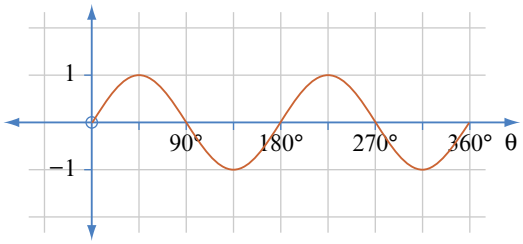
ii four solutions.

a $\sin \theta = -\frac{1}{2}$ **b** $\cos \theta = -\frac{1}{2}$

c $\sin \theta = \frac{1}{\sqrt{2}}$ **d** $\cos \theta = -\frac{1}{\sqrt{2}}$

CHALLENGE

22 Just as there are families of linear and quadratic functions, there are families of trigonometric functions.



a Explain why this graph is periodic.

b It is not the graph of $y = \sin\theta$ or $y = \cos\theta$. How can you tell?

c Does it more closely resemble a sine curve or a cosine curve? Explain.

d What is the period of the function?

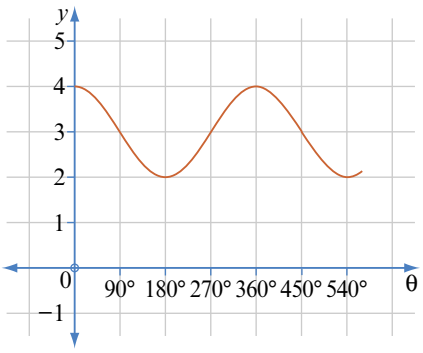
e How many solutions would you find for a particular positive value of the function (say 0.8) within the domain [0°, 360°]?

f How many solutions would you find for a particular negative value of the function (say −0.35) within the same domain?

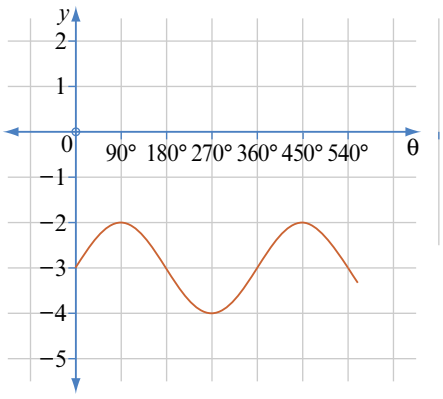
g How does this compare with the number of solutions within the same domain for $\cos\theta$ and $\sin\theta$?

23 These graphs are derived from graphs of $y = \sin\theta$ and $y = \cos\theta$. Study them carefully and indicate the trigonometric function each graph represents.

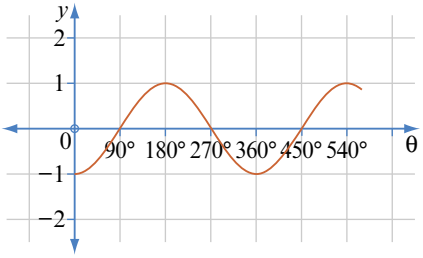
a



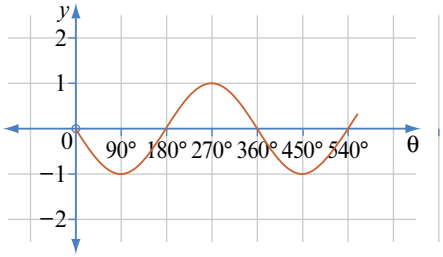
b



c



d



24 Use digital technology to check your answers to question **23**.

Reflect

How does symmetry in the unit circle help in solving trigonometric equations?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

Pythagoras' Theorem	degrees	back-bearing	peak
irrational number	minutes	sine rule	trough
Pythagorean triad	seconds	cosine rule	period of graph
trigonometry	angle of elevation	Heron's formula	amplitude
sine	angle of depression	unit circle	asymptote
cosine	compass bearings	trigonometric graphs	trigonometric equations
tangent	true bearings	periodic function	domain

MULTIPLE-CHOICE

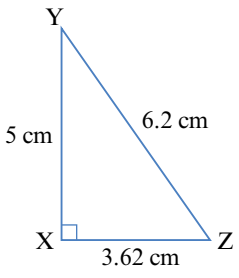
8A **1** Which of these describes a triangle with side lengths 4 cm, 5 cm and 6 cm?

A right-angled B obtuse-angled
C acute-angled D isosceles

8B **2** The ratio $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

A sine B cosine
C tangent D none of these

Questions **3** and **4** refer to this triangle.



8C **3** What is the size of ∠YZX, to the nearest degree?

A 54° B 55°
C 35° D 36°

8C **4** Which ratio represents sin ∠ZYX?

A $\frac{5}{6.2}$ B $\frac{3.62}{6.2}$ C $\frac{3.62}{5}$ D $\frac{5}{3.62}$

8D **5** What is the equivalent true bearing of N12°W?

A 348°T B 012°T
C 192°T D 168°T

8E 10A **6** Which value is closest to the diagonal length within a cube of side length 5 cm?

A 5 cm B 8 cm C 9 cm D 10 cm

8F 10A **7** What is angle A in $\frac{10}{\sin 80°} = \frac{9}{\sin A}$ closest to?

A 55° B 60° C 65° D 70°

8G 10A **8** The cosine rule is used to calculate the side length of a non-right-angled triangle ABC. If you know the value of angle B, what do you also need to know the value of?

A a and b B a and c
C c and b D angle A

8H 10A **9** A point P on the unit circle in quadrant 2 has what coordinates?

A $(-\cos\theta, \sin\theta)$ B $(\cos\theta, -\sin\theta)$
C $(-\cos\theta, -\sin\theta)$ D $(\cos\theta, \sin\theta)$

8I 10A **10** The graph of $y = \sin\theta$ has what period?

A 90° B 180° C 270° D 360°

8I 10A **11** What is the exact sine value for 30°?

A $\frac{1}{2}$ B $\frac{\sqrt{3}}{2}$ C $\frac{1}{\sqrt{3}}$ D $\frac{1}{\sqrt{2}}$

SHORT ANSWER

Calculate each angle correct to the nearest degree and each length correct to two decimal places, unless stated otherwise.

8A **1** An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:
- **a** an exact value
- **b** an approximate value, correct to one decimal place.

8B **2** A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

8C **3** A map shows a stretch of road as a horizontal distance of 18.5 km. The road rises 6.2 km from start to finish.
- **a** Find the angle rise of the road, to the nearest minute.
- **b** Find the average gradient of the road as a:
  - **i** ratio **ii** percentage.
- **c** What is the actual distance you would travel along the road?

8D **4** Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

Questions **5** and **6** refer to these three-dimensional objects.
- **a** cylinder with base circumference of 20 cm and height of 15 cm
- **b** box 10 cm long, 8 cm wide and 15 cm tall
- **c** cube of side length 25 cm

8E 10A **5** Find the length of the longest diagonal within each object.

8E 10A **6** Calculate the angle this diagonal makes with the height of the object.

8F 10A **7** Standing in a park, Anabel takes a bearing of 065°T to a tree. She walks 15 m due east and finds the bearing to the same tree to be 295°T.
- **a** Draw a labelled diagram to represent this situation.
- **b** Find the distance from Anabel, in both positions, to the tree.

8F 10A **8** In ΔABC, $a = 5$ cm, $b = 6.1$ cm and $A = 39°$.
- **a** Explain why two distinct triangles can be drawn with these measurements. Draw these two figures.
- **b** Find the dimensions of all the angles and sides of these two triangles.

8G 10A **9** A triangular garden bed in a park is 50 m by 40 m by 20 m.
- **a** Draw a sketch of the shape of the garden bed.
- **b** Find the area of the garden bed.
- **c** Find the size of the angle at each vertex of the triangular shape.
- **d** What is the shortest distance from the vertex opposite the longest side to the longest side?

8H 10A **10** Show how the unit circle can be used to find each value.

a $\sin 300°$ **b** $\cos 225°$ **c** $\sin 135°$
d $\cos 315°$ **e** $\sin 240°$ **f** $\cos 330°$

8I 10A **11** Solve each equation for the domain $[0°, 360°]$.

a $\cos\theta = -1$ **b** $\sin\theta = 0$
c $\cos\theta = \frac{\sqrt{3}}{2}$ **d** $\sin\theta = -\frac{1}{2}$
e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin\theta = -\frac{\sqrt{3}}{2}$

MIXED PRACTICE

Where appropriate, calculate each angle correct to the nearest degree and each length correct to one decimal place, unless stated otherwise.

1 A square has a side length of 3.2 cm. What is the length of its diagonal, correct to the nearest centimetre?

A 3 cm **B** 4 cm **C** 5 cm **D** 6 cm

2 What is the exact value of sin 60°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$

3 What is the angle between the diagonal and the side of a square?

4 A square has a diagonal length of $4\sqrt{2}$ cm. What is its side length?

5 From a point 5 km horizontally from the point directly beneath it, the angle of elevation to a flying plane is 9°. At what height is the plane flying?

6 What is the value of cos 60° the same as?

A cos 30° **B** sin 30° **C** sin 60° **D** cos 45°

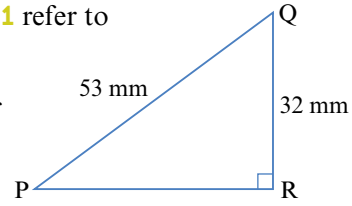
7 What true bearing is equivalent to the compass direction N52°W?

8 The hypotenuse of a right-angled triangle measures 8.4 cm. One of the other sides measures 6.3 cm. What is the length of the third side, to the nearest centmetre?

A 5 cm **B** 6 cm **C** 7 cm **D** 8 cm

9 Calculate the value of $\frac{5}{\cos 14.8°}$.

Questions **10** and **11** refer to this triangle.



10 Find the size of ∠PQR to the nearest minute.

11 Write an expression for the sine of ∠QPR.

12 What is the value of $\sin^{-1}\left(\frac{7}{9}\right)$, to the nearest minute?

A 51°3′ **B** 51°4′ **C** 51°5′ **D** 51°6′

10A **13** Write the cosine rule to find the value of an angle A in triangle ABC.

10A **14** What is the period of the graph of $y = \cos\theta$?

A 120° **B** 180° **C** 360° **D** 540°

10A **15** In which quadrant of the Cartesian plane is the sine value negative and the cosine value positive?

A quadrant 1 **B** quadrant 2
C quadrant 3 **D** quadrant 4

10A **16** Write the formula to find the length of a diagonal within a rectangular prism with dimensions x cm × y cm × z cm.

10A **17** How many solutions would the equation $\cos\theta = -0.5$ have within the domain [0°, 450]?

A 1 **B** 2 **C** 3 **D** 4

10A **18** Which angle in the first quadrant has an exact tangent value of $\sqrt{3}$?

A 30° **B** 45° **C** 60° **D** 90°

10A **19** In ΔABC, $c = 12$ cm, $A = 60°$ and $C = 50°$. Use the sine rule to calculate the value of a.

10A **20** Use Heron's formula to find the area of a triangle with side lengths 6 cm, 7 cm and 10 cm.

10A **21** Compare the graphs of $y = \cos\theta$ and $y = \sin\theta$.

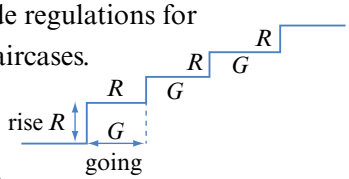
10A **22** Why is there an asymptote at 90° in the graph of $y = \tan\theta$?

10A **23** What is the amplitude of the graph of $y = \sin\theta$?

A 1 **B** 1.5 **C** 2 **D** 3

ANALYSIS

1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as shown in this diagram.



The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm. The going (G) must have a length within the range 250–355 mm.
The regulation also states that $2R + G$ must lie within the range 550–700 mm.

a Consider a staircase with the minimum rise of 115 mm and the minimum going of 250 mm.
- **i** Explain whether this staircase would fall within the building regulation guidelines.
- **ii** What is the slope of this staircase? Give your answer to the nearest degree.

b Repeat part **a** with the maximum values for the rise and going.

c If the going is its maximum value of 355, and the value of $2R + G$ falls within the required limits:
- **i** find a range of acceptable values for the rise
- **ii** calculate the range of acceptable slope values.

d Draw a set of five stairs with a going of 300 mm and a rise of 150 mm.
- **i** What is the horizontal distance of the stairs?
- **ii** Find the vertical height of the stairs.
- **iii** Calculate the direct distance between these two points.
- **iv** What is the slope of the staircase?

e What do you consider would be ideal figures for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and give its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, as a wheelchair may tip backwards or run away if it is too steep. The code specifies that an internal ramp should be no steeper than 1 to 12, while an external ramp should have a gentler slope of 1 to 20.
- **i** Draw internal and external ramps 20 m long with these slopes.
- **ii** For each one, find the going and the rise. Mark these measurements on your diagrams.

10A **2** Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel.

Tom records the height of the tag with respect to its starting position with a video. Here is his record.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	−20	−30	−20	0	20	30

a Graph the data, showing time on the x-axis.

b Suggest whether the graph models a sine curve or a cosine curve.

c How long does it take for the wheel to rotate once?

d What was the starting position of the tag on the wheel? Explain how you deduced this.

e What is the radius of the wheel?

f Calculate the circumference of the wheel.

g At what speed does Emma ride? Give your answer in metres per second.

h Draw a diagram to show the position of the tag after 10 seconds of riding at this constant speed. Explain how you arrived at this answer.

CONNECT

How high is that?

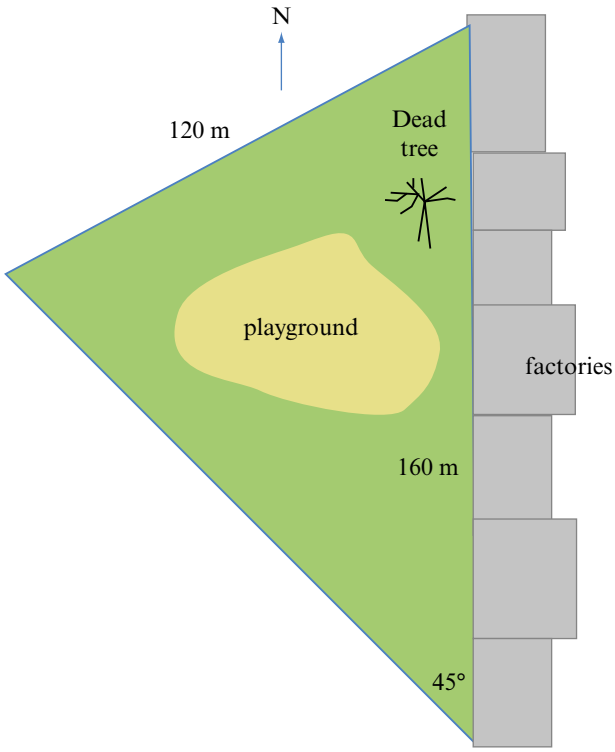
Joe is a landscape architect who works for the local council. His latest project is to design a recreation park on a disused block of land next to a block of factories. The land is triangular in shape.

A dead tree must be removed from the block before the landscaping can commence.

There's to be a playground area for the children with swings and gym equipment. The land must be level in this area.

Joe has also been asked to include a bed of sunflowers. Since these flowers follow the path of the sun in its daily journey across the sky, they must have full sun throughout the day for as much time as possible. This means that they can't be in the shade of the line of factories for any length of time.

Joe certainly has his work cut out for him! He starts by initially drawing a plan of the area, and the proposed improvements.



Your task

For your investigation of a landscape architect's work, complete these steps.

1. Research recreation sites and draw a proposal for the development of the block.
2. Explain how you catered for the constraints of:
   - **a** cutting down the dead tree
   - **b** levelling the playground
   - **c** positioning the swings and gym equipment
   - **d** positioning the sunflower garden.

Extension: The size of the land

10A 3. Use the measurements on Joe's plan to determine for the block:
   - **a** the angles of the vertices
   - **b** the lengths of all the sides
   - **c** the perimeter
   - **d** the area.

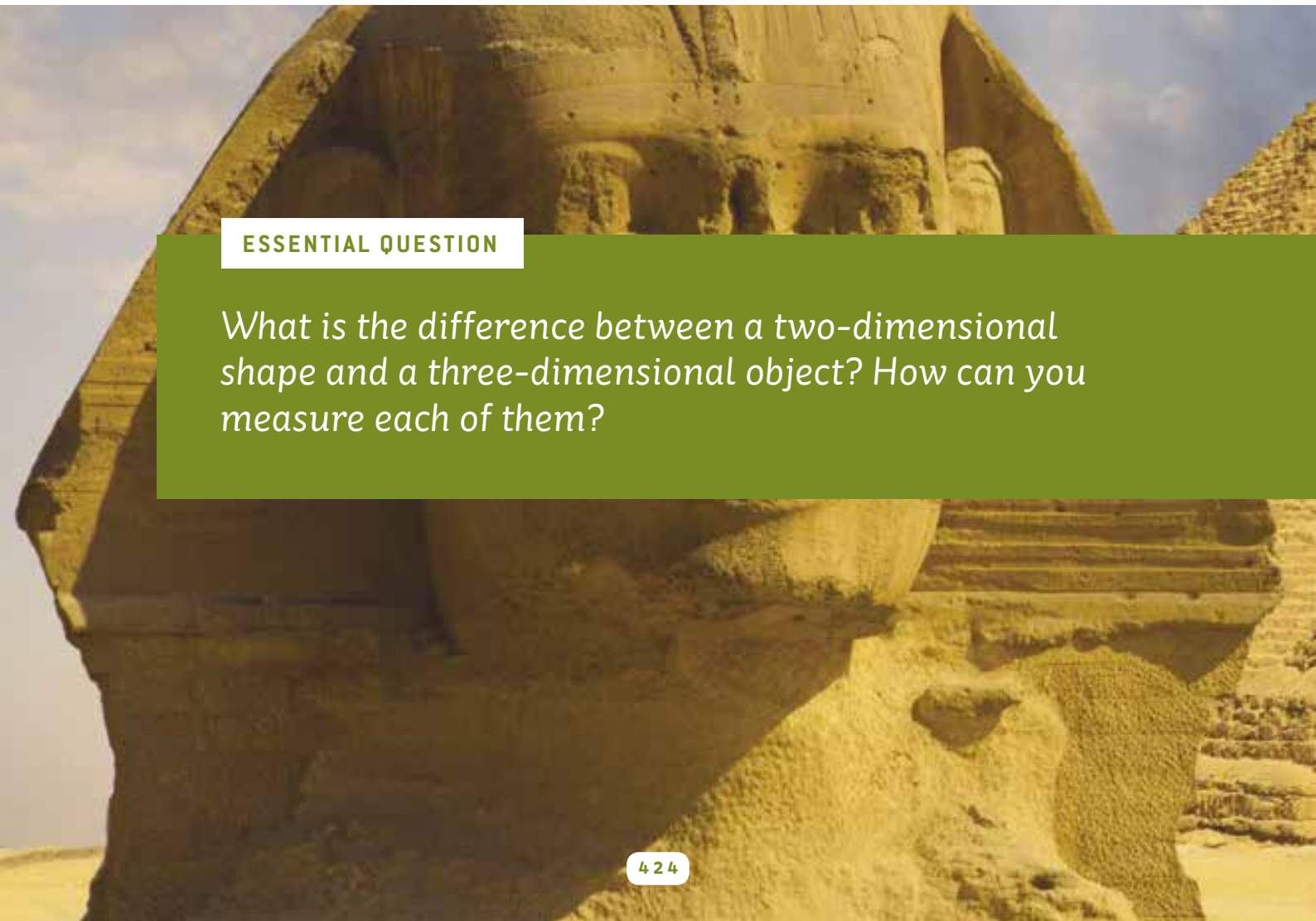
10A 4. Add this detail to your proposed plan.

Include all necessary diagrams and working to justify your answers.

You may like to present your findings as a report. Your report could include:

- an information booklet
- a PowerPoint presentation
- an advertising brochure
- other (check with your teacher).

9 MEASUREMENT



ESSENTIAL QUESTION

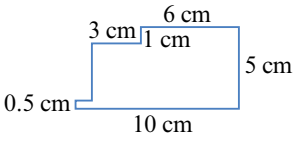
What is the difference between a two-dimensional shape and a three-dimensional object? How can you measure each of them?

9A **1** What is 14 cm written in metres?

A 0.014 m B 0.14 m
C 1.4 m D 1400 m

9A **2** What is the perimeter of this shape?

A 30 cm
B 25.5 cm
C 27 cm
D 24.5 cm

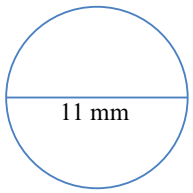


9A **3** A netball court is 15.25 m wide and 30.5 m long. What is its perimeter?

A 465.13 m B 91.5 m
C 45.75 m D 76.25 m

9A **4** What is the circumference of this circle, correct to the nearest millimetre?

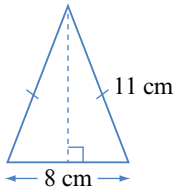
A 17 mm
B 69 mm
C 34 mm
D 35 mm



9B **5** What is the area of a parallelogram with base length 4.2 cm and height 32 mm?

A 14.8 cm^2 B 13.44 cm^2
C 7.4 cm^2 D 134.4 cm^2

Questions **6** and **7** refer to this triangle.



9B **6** Which of these represents the height, h, of the triangle?

A $h = \sqrt{11^2 - 8^2}$ B $h = \sqrt{11^2 + 4^2}$
C $h = \sqrt{11^2 - 4^2}$ D $h = \sqrt{11^2 + 8^2}$

9B **7** What is the area of the triangle?

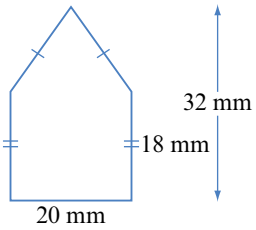
A 41 cm^2 B 30.2 cm^2
C 82 cm^2 D 44 cm^2

9B **8** What is the area of the circle in question **4**, correct to the nearest square millimetre?

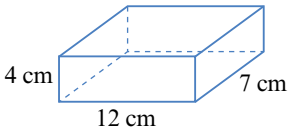
A 380 mm^2 B 95 mm^2
C 69 mm^2 D 35 mm^2

9B **9** What is the area of this composite shape?

A 500 mm^2
B 680 mm^2
C 640 mm^2
D 586 mm^2



Questions **10**–**12** refer to this prism.



9C **10** What is the total surface area of the prism?

A 23 cm^2 B 160 cm^2
C 320 cm^2 D 336 cm^2

9D **11** What is the volume of the prism?

A 23 cm^3 B 160 cm^3
C 320 cm^3 D 336 cm^3

9D **12** If the prism is filled with water to its capacity, how many millilitres would it hold?

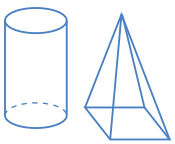
A 23 mL B 160 mL
C 320 mL D 336 mL

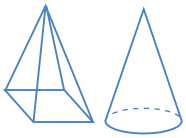
9D **13** A cylinder has a radius of 15 mm and a height of 20 mm. What is the volume of the cylinder, to the nearest cubic millimetre?

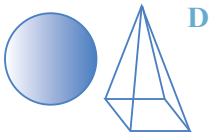
A 770 mm^3 B 14 137 mm^3
C 4500 mm^3 D 942 mm^3

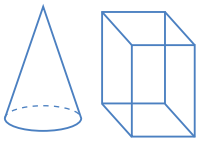
9E **14** Which figure shows a pyramid and a cone?

10A

A 

B 

C 

D 

9A Length and perimeter

Start thinking!

1 The **perimeter** of some shapes can be found using a formula.

a Create a table similar to the one shown with space for six shapes.

b Copy the shapes A to F into the first column of the table.

c Name each shape.

d Match each shape with a formula for perimeter from the following list.

$P = a + b + c$

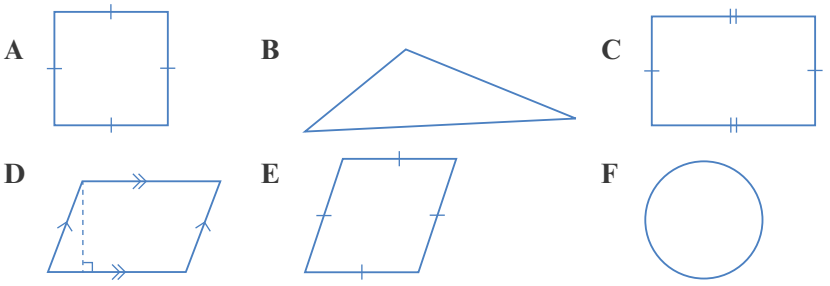
$P = 2(l + w)$

$C = \pi D$

$P = 4l$

$C = 2\pi r$

Shape	Name	Perimeter formula

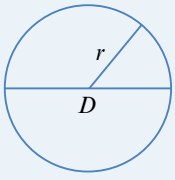


2 Which shapes share a formula for perimeter? Explain why this occurs.

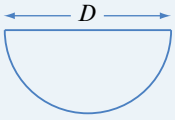
3 There are two formulas listed for the perimeter of a circle. Why does this occur? What is the relationship between the different variables in each formula?

KEY IDEAS

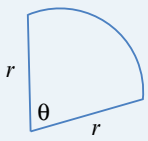
- Perimeter is the distance around the outside edge of a **two-dimensional** (2D) shape.
- To calculate the perimeter of a straight-edged shape, add the length measurements of each side. All measurements must be in the same units.
- The perimeter of a circle is called its circumference (C) and can be found using the formula $C = \pi D$ or $C = 2\pi r$, where D is the length of the diameter and r is the length of the radius.



- To calculate the perimeter of a semi-circle, find the length of the arc by halving the circumference of a circle and then add the diameter (or twice the radius) to your answer.



- The perimeter of a sector is $\frac{\theta}{360°} \times 2\pi r + 2r$, where θ is the angle in degrees between the two radii.



EXERCISE 9A Length and perimeter

UNDERSTANDING AND FLUENCY

For questions involving π, use the π button on your calculator. Where appropriate, write answers correct to one decimal place.

1 Convert each measurement to the unit shown in brackets.

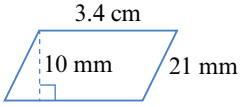
a 43 cm (mm) **b** 12.5 km (m) **c** 31 cm (m)
d 0.63 km (m) **e** 3 mm (cm) **f** 2.7 m (cm)
g 78 000 cm (km) **h** 8.9 m (mm) **i** 5.1 km (cm)
j 0.0056 km (mm) **k** 19 mm (m) **l** 997 mm (km)

2 Calculate the perimeter of each shape.

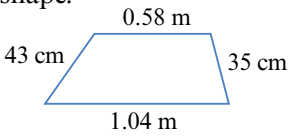
a rectangle with side lengths 16 m and 18m **b** equilateral triangle with side length 15 cm
c regular pentagon with side length 12 mm **d** square with side length 21.4 mm

3 Calculate the perimeter of each shape.

a

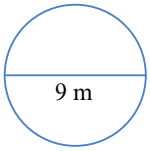


b

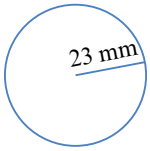


4 Calculate the circumference of each circle.

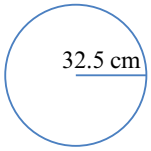
a



b



c

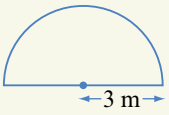


5 Calculate the circumference of a circle with:

a diameter 65.8 mm **b** radius 9.5 cm **c** diameter 15.4 cm.

EXAMPLE 9A-1 Calculating the perimeter of a semi-circle

Calculate the perimeter of this semi-circle.



THINK

1 For the curved length (arc), identify which circumference formula to use. Radius is given (3 m) so use $C = 2\pi r$. Length of arc is half of circumference, so divide $2\pi r$ by 2.

2 Substitute for r in the formula.

3 Calculate the result using the π button on a calculator.

4 Add the lengths of the two radii to the length of the arc to find the perimeter, P. Write the result correct to one decimal place and include the appropriate unit.

WRITE

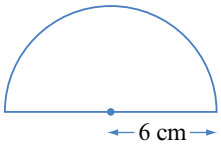
$$\text{Length of arc} = \frac{2\pi r}{2}$$
$$= \pi r$$
$$= \pi \times 3$$
$$= 9.424\,778\ldots \text{ m}$$

$P = 9.424\,778\ldots + 3 + 3$
$= 15.424\,778\ldots$
≈ 15.4 m

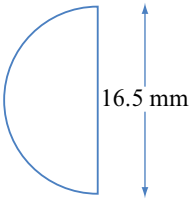
UNDERSTANDING AND FLUENCY

6 Calculate the perimeter of each semi-circle.

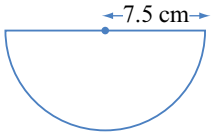
a



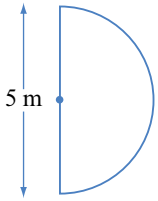
b



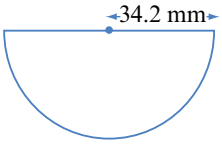
c



d



e



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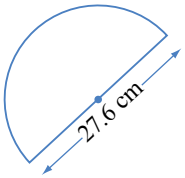
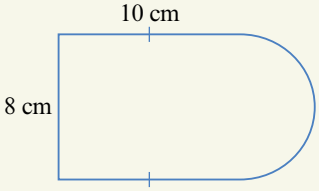


EXAMPLE 9A-2 Calculating the perimeter of a composite shape

Calculate the perimeter of this composite shape.



THINK

1 The length of the curved edge (arc) is half the circumference of a circle with diameter of 8 cm.

2 Substitute for D in the formula.

3 Calculate the result using the π button on a calculator.

4 Calculate the perimeter. Write the result correct to one decimal place and include the appropriate unit.

WRITE

$$\text{length of arc} = \frac{\pi D}{2}$$

$$= \frac{\pi \times 8}{2}$$

$$= \pi \times 4$$

$$= 12.566\,37\ldots \text{ cm}$$

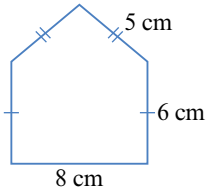
$$P = 12.566\,37\ldots + 10 + 8 + 10$$

$$= 40.566\,37\ldots$$

$$\approx 40.6 \text{ cm}$$

7 Calculate the perimeter of each shape.

a



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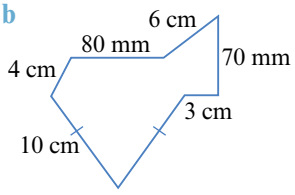
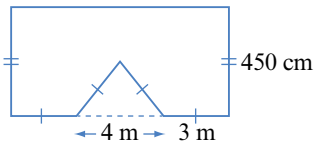


c



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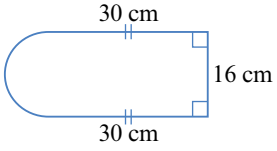

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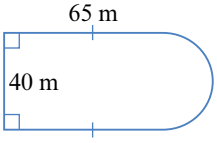

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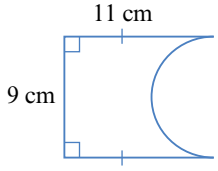
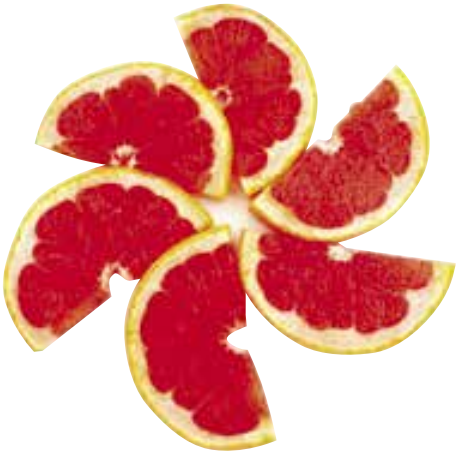


8 Each piece of grapefruit has a radius of approximately 6.25 cm.
What is the length of rind on each piece of grapefruit?



9 This pizza is divided into quarters.

a How many degrees are in the entire circular pizza?

b How many degrees are in one quarter?

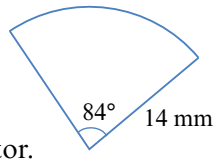
c Use your answers to parts **a** and **b** to write one quarter of the pizza as a fraction of the whole pizza. What does the fraction simplify to?

d Write a formula for finding the length of the arc of one quarter of the pizza. (One quarter of a circle is also known as a quadrant.)

e If the radius of the whole pizza is 7 cm, find the length of the arc of the quadrant.

f Use the information in part **e** to calculate the perimeter of the quadrant.

10 Consider the sector shown at right.



a What is the angle within this sector?

b Write this sector as a fraction of a full circle.

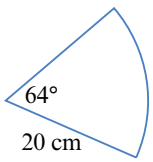
c Write an expression for finding the length of the arc for this sector. Find the length of the arc.

d Use your answer to part **c** to help you calculate the perimeter of the sector.

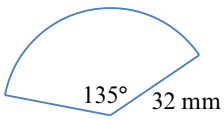
11 Using your findings from questions **6**–**8**, explain why the perimeter of any sector can be found using $\frac{\theta}{360°} \times 2\pi r + 2r$, where θ is the angle in degrees between the two radii.

12 Calculate the perimeter of each sector.

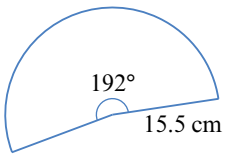
a



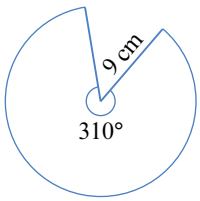
b



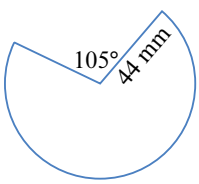
c



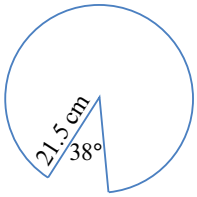
d



e



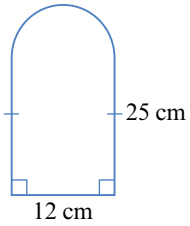
f



UNDERSTANDING AND FLUENCY

13 These composite shapes consist of a basic shape and a sector. Calculate the perimeter of each composite shape.

a



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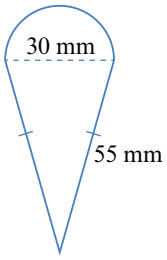

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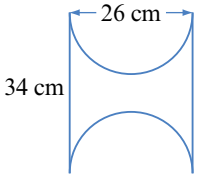

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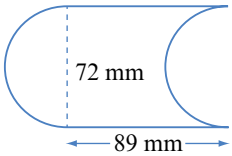

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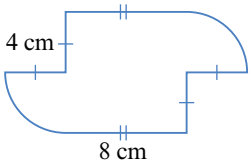

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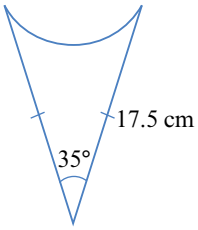


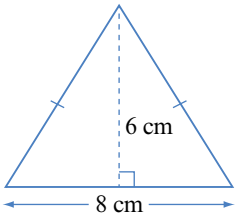
PROBLEM SOLVING AND REASONING

14 A dartboard is divided into 20 equal sectors and has a diameter of approximately 450 mm.

- **a** What is the angle contained within each sector?
- **b** Calculate the length of the outer arc for one sector. (Assume that the radii of the sectors meet in the centre of the bullseye.)
- **c** Calculate the perimeter of one sector.
- **d** Use your answer to part **b** to find the circumference of the dartboard.
- **e** Calculate the circumference of the dartboard using the diameter. How does this compare to your answer in part **d**?

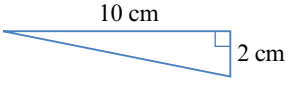
15 At times you need to use **Pythagoras' Theorem** to find the perimeter of a shape.

- **a** Draw a right-angled triangle with side lengths labelled as a, b and c. Use Pythagoras' Theorem to show the relationship between the three side lengths.
- **b** What measurement is needed before you can calculate the perimeter of this triangle?
- **c** Explain how you can use the information provided in the triangle to find the unknown side length.
- **d** Find the unknown side length and use your answer to calculate the perimeter of the triangle.



16 Calculate the perimeter of each shape. (Hint: use Pythagoras' Theorem to find any unknown measurements.)

a



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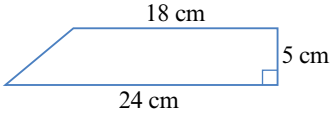

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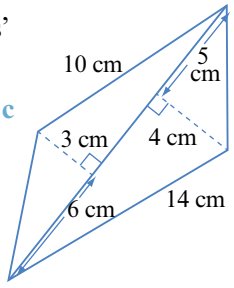
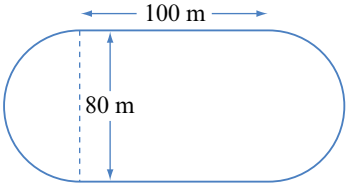


PROBLEM SOLVING AND REASONING

17 An athletics track has the dimensions shown.



a How far would an athlete travel if she ran the perimeter of the track?

A line is marked 1 m inside the boundary of the track for another athlete to run along.

b Draw a diagram showing the athletics track with the line marked in.

c Calculate the distance travelled by an athlete who follows the path of this line.

18 Denise has designed a square birthday invitation.

a If the perimeter of the invitation is 26 cm, what is its side length?

b Write a sentence explaining how to find the side length of a square when the perimeter is known.

c Use your answer to part **b** to write a formula for finding the side length of any square when its perimeter is known.

19 A photograph is rectangular, with the length being 3 cm more than the width. The perimeter of the photograph is 46 cm.

a Find the dimensions of the photograph.

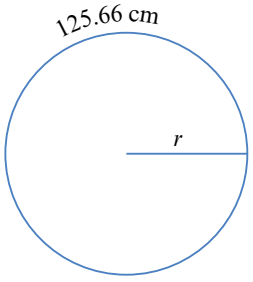
The photograph is to be mounted so that there is a 2 cm border between the photograph and the frame.

b What are the dimensions of the photo and its border?

c What is the perimeter of the shape that will be framed?

CHALLENGE

20 You can find the radius of a circle when the circumference is known.



The circumference of the circle above is 125.66 cm.

a What length do you estimate the radius of the circle to be?

b Explain how you can find the radius if the circumference is known.

c Write a formula that will give you the radius when the circumference is known. (Hint: start with $C = \ldots$ and transpose.)

d Use the formula you found in part **c** to find the radius of the circle shown, to the nearest centimetre.

21 In one revolution of Justin's bicycle wheel, he can cover a distance of 400 cm.

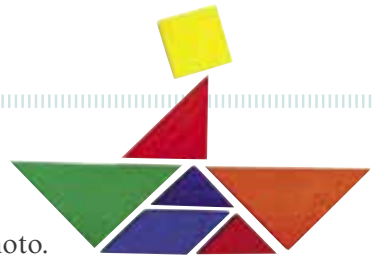
a What is the radius of each wheel on Justin's bicycle? Give your answer to the nearest centimetre.

b If Justin's house is 3.5 km from his school, how many revolutions would each of his bicycle wheels complete by the time he reaches home?

Reflect

How can finding the perimeter of simple shapes help you find the perimeter of more complex shapes?

9B Area



Start thinking!

1 Explain the difference between the perimeter and the **area** of the shapes in this photo.

2 Create a table similar to the one in 9A Start thinking! on page 310, using the headings: Shape, Name, Area formula.

a Copy the shapes A to H into your table and name them.

b Match each shape with a formula for area from the following list.

$A = \frac{1}{2}bh$

$A = \frac{1}{2}(a + b)h$

$A = lw$

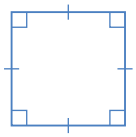
$A = \pi r^2$

$A = l^2$

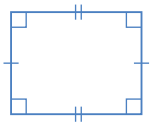
$A = bh$

$A = \frac{1}{2}xy$

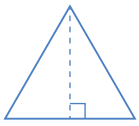
A



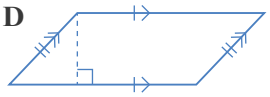
B



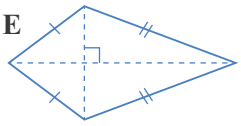
C



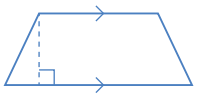
D



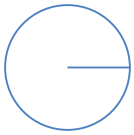
E



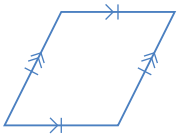
F



G

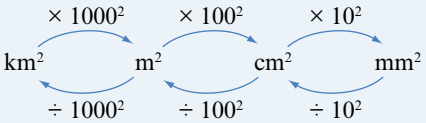


H

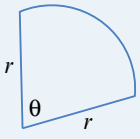


KEY IDEAS

- To calculate the area of any shape, identify the appropriate formula to use and substitute the given values.
- To find the area of a composite shape:
  1. split the shape into individual parts for which you can calculate the area using known formulas
  2. label the individual parts with any missing dimensions
  3. calculate the areas of the individual parts
  4. add or subtract the areas to find the total area.
- Common units of measurement for area are square millimetres (mm^2), square centimetres (cm^2), square metres (m^2), square kilometres (km^2) or **hectares** (ha).
- A unit of area can be converted to another unit of area by multiplying or dividing by the conversion factor.



- 1 ha = 10 000 m^2
- The area of a sector is $\frac{\theta}{360°} \times \pi r^2$, where θ is the angle in degrees between the two radii.



EXERCISE 9B Area

For questions involving π, use the π button on your calculator. Where appropriate, write answers correct to one decimal place.

EXAMPLE 9B-1 Converting units of area

Convert each area measurement to the unit shown in brackets.

a $1.2\ km^2$ (m^2) b $320\ mm^2$ (cm^2)

THINK

a Multiply by $1000^2 = 1\ 000\ 000$, as you are converting to smaller unit. ($1\ km^2 = 1000^2\ m^2$.)

b Divide by $10^2 = 100$, as you are converting to a larger unit. ($1\ cm^2 = 10^2\ mm^2$.)

WRITE

a $1.2\ km^2 = (1.2 \times 1\ 000\ 000)\ m^2$
$= 1\ 200\ 000\ m^2$

b $320\ mm^2 = (320 \div 100)\ cm^2$
$= 3.2\ cm^2$

UNDERSTANDING AND FLUENCY

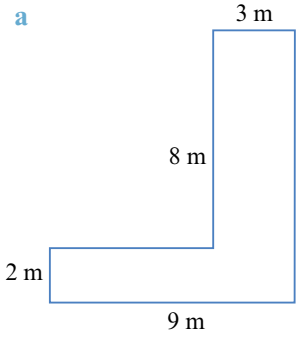
1 Convert each area measurement to the unit shown in brackets.

a $7\ m^2$ (cm^2) b $4500\ m^2$ (km^2) c $8.4\ ha$ (m^2)
d $29\ mm^2$ (cm^2) e $24\ 000\ m^2$ (ha) f $0.88\ m^2$ (mm^2)
g $34\ 200\ 000\ cm^2$ (km^2) h $10.22\ km^2$ (cm^2) i $3\ km^2$ (ha)

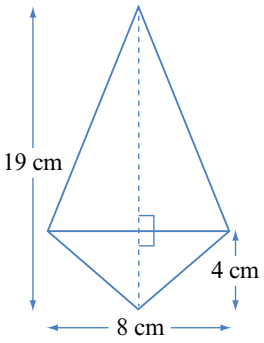
2 A composite shape is made up of two or more basic shapes. For each shape:

i identify the basic shapes ii calculate the area.

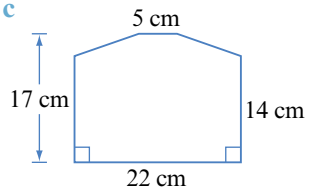
a



b



c



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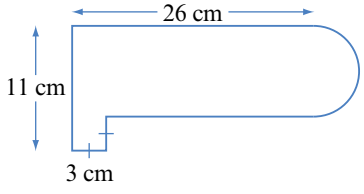
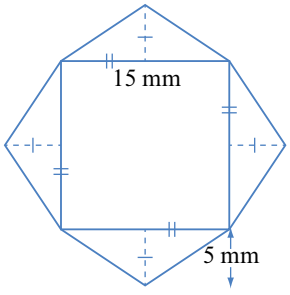


e



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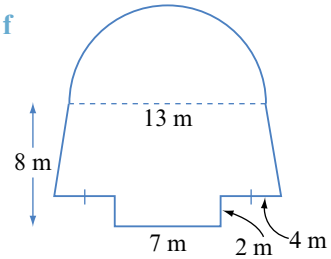
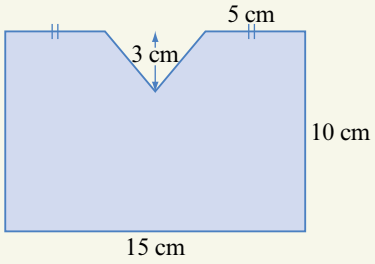


EXAMPLE 9B-2 Calculating the shaded area within a shape

Calculate the shaded area within this shape.



THINK

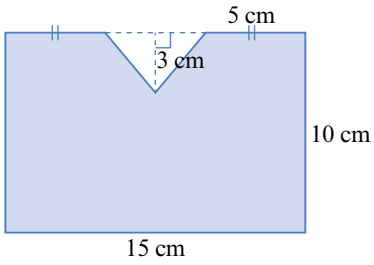
1 Redraw the shape with dashed lines to split it into individual shapes for which you can calculate the area.

2 Label any missing dimensions for each individual shape. The total length of the rectangle is 15 cm and the length either side of the triangle is 5 cm. The triangle must have a base of 5 cm ($15 - 2 \times 5 = 5$).

3 Calculate the area of each individual shape.

4 Calculate the required area by subtracting the area of the triangle from the area of the rectangle.

WRITE



rectangle: $A = lw$
$= 10 \times 15$
$= 150 \text{ cm}^2$

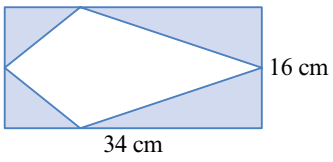
triangle: $A = \frac{1}{2}bh$
$= \frac{1}{2} \times 5 \times 3$
$= 7.5 \text{ cm}^2$

total area $= 150 \text{ cm}^2 - 7.5 \text{ cm}^2$
$= 142.5 \text{ cm}^2$

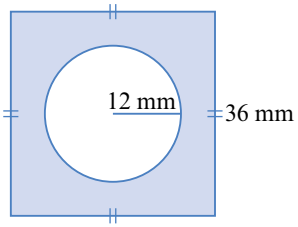
UNDERSTANDING AND FLUENCY

3 Calculate the shaded area within each shape.

a



b



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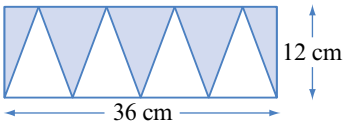

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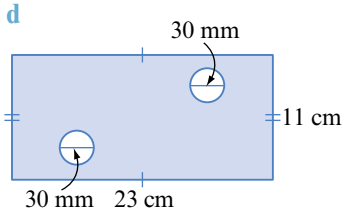
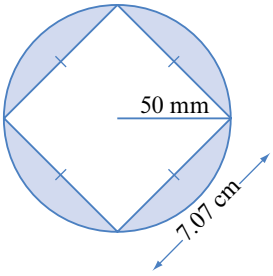


e



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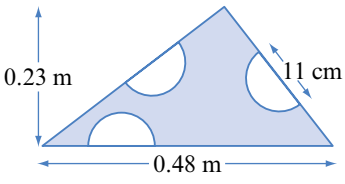
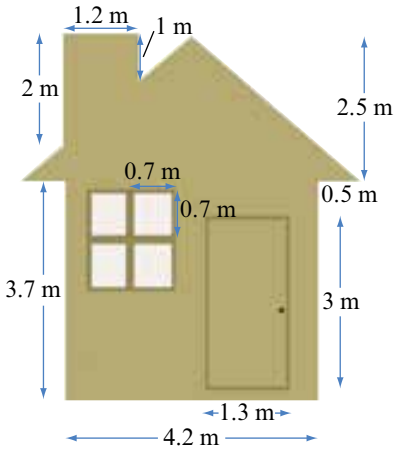


4 The façade of this house is made of a number of basic shapes.



- **a** Identify each shape.
- **b** Calculate the area of each shape you identified in part **a**.
- **c** There are four windows of equal size. Explain how you can calculate the area of all four windows if you know the area of only one window.

The owners wish to paint the façade of the house white and need to find the total area to be painted.

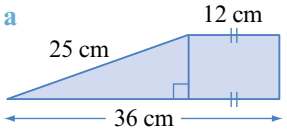
- **d** Using your knowledge of how to find the area of a composite shape, explain how to calculate the required area. (Remember the windows won't need to be painted.)
- **e** Use your answers to parts **c** and **d** to find the area of the house that will be painted white.
- **f** If 1 L of paint covers 15 m^2, how many litres of white paint do the owners need to purchase?

5 A farm covers land in the shape of a trapezium. The length of the northern boundary of the farm is 265 m and length of the southern boundary is 180 m. The perpendicular distance from the northern end to the southern end is 90 m.

- **a** Draw a diagram of the farm and find its area to the nearest square metre.
- **b** Write your answer to part **a** to the nearest hectare.
- **c** The farmer decides to purchase his neighbour's farm, which is in the shape of a rectangle but has the same area as his farm. What are possible dimensions of his neighbour's farm? Give your answer in metres.

6 Using your knowledge of Pythagoras' Theorem to find any unknown lengths, calculate the shaded area in each composite shape. (Hint: check that all measurements are written in the same units.)

a



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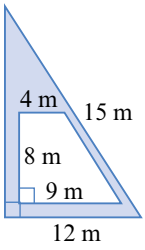
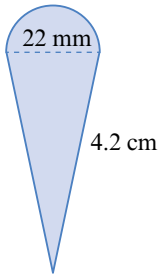
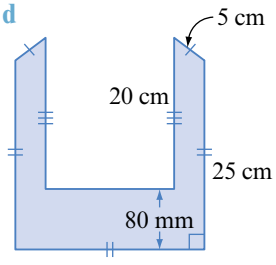


c



d



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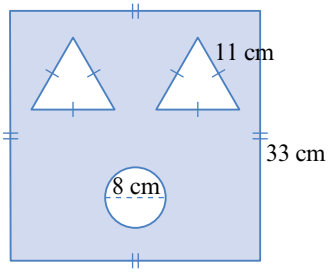

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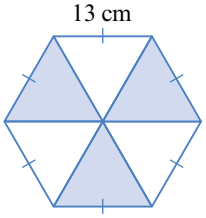
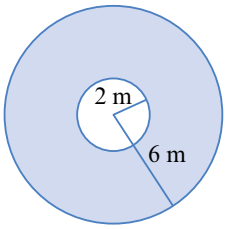


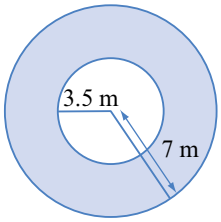
UNDERSTANDING AND FLUENCY

7 A garden bed is in the shape of an annulus.

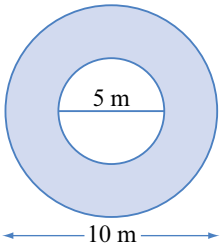


- **a** Explain how you can find the area of the garden bed.
- **b** Find the area of this garden bed.
- **c** Three more garden beds, all annuli, are to be added to the same garden. Find the area of each one.

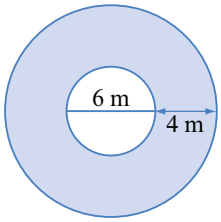
i



ii



iii

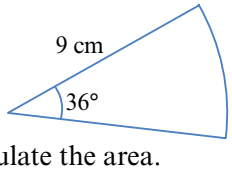


8 A log cake is cut into circular slices, with diameter 64 mm. A nut topping is then sprinkled on each slice.

> **NOTE** Knowing the formula for the length of an arc in a sector (see Exercise 9A questions 8–10) can help you find a formula for the area of a sector.

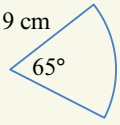
- **a** Find the area covered by topping on one slice.
- **b** Another slice is cut in half. Explain how you can use your answer to part **a** to find the area of topping on half a slice of cake.
- **c** A third slice is cut into quarters. Explain how you can use your answer to part **a** to find the area of topping on a quarter of a slice of cake.
- **d** Write a formula for finding the area of a quarter of a circle (a quadrant). Explain the relationship between this formula and the number of degrees in a quadrant.

9 **a** What is the angle within this sector?



- **b** Write the angle of the sector as a fraction of the angle in a full circle.
- **c** Write an expression for finding the area of this sector and calculate the area.

EXAMPLE 9B-3 Calculating the area of a sector



Calculate the area of this sector.

THINK

1. Write the angle of the sector as a fraction of the angle in a full circle and simplify.
2. Write an expression for finding the area of this sector.
3. Identify the value for r.
4. Calculate the area, correct to one decimal place, and include the appropriate unit.

WRITE

$$\frac{65°}{360°} = \frac{13}{72}$$

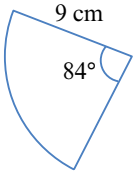
$$A = \frac{13}{72} \times \pi r^2$$

$$r = 9 \text{ cm}$$

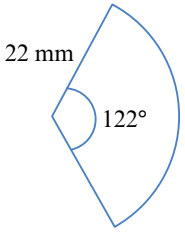
$$A = \frac{13}{72} \times \pi \times 9^2$$
$$= 45.945\,79\ldots$$
$$\approx 45.9 \text{ cm}^2$$

10 Calculate the area of each sector.

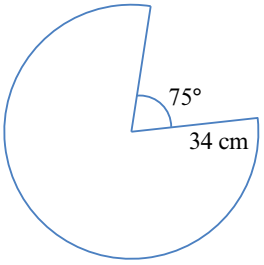
a



b



c



PROBLEM SOLVING AND REASONING

11 Use a ruler and pencil to draw each shape so that its area is 36 cm^2. Show all appropriate dimensions.

a square **b** rectangle **c** triangle **d** parallelogram

e kite **f** trapezium **g** circle **h** semi-circle

12 A rectangle has an area of 720 cm^2. One side is five times as long as the other.

a If x is the length of one side, draw a diagram of the rectangle and label the sides.

b Use this information to find the dimensions of the rectangle.

c If the dimensions are doubled, what is the area of the new rectangle?

d How does the area of the new rectangle compare to the area of the old rectangle?

e If the dimensions of the original rectangle are tripled, what is the area of the new rectangle? Compare this area to the original area.

f Explain what will happen to the area of the rectangle when the dimensions are:

i quadrupled

ii multiplied by a factor of n.

13 A clock has a 7-cm minute-hand and a 5-cm hour-hand.

a What is the difference in the area covered by each hand in a full revolution?

The time shown on the clock face is 2 o'clock.

b Calculate the angle of the sector between the minute-hand and the hour-hand.

c Find the difference in the area covered by the two hands when in this position.

d Find the difference in the area when the time is 7 o'clock.

CHALLENGE

14 A design based on the ancient yin-yang symbol is shown below.

a What is the relationship between the radius of the full circle and the radius of the semi-circles formed inside?

b If the radius of the full circle is 24 cm, find the area shaded blue (yin).

c Find the area of the white section (yang) without performing any calculations.

Reflect

How can the area of a full circle help you determine the area of parts of a circle?

9C Surface area of prisms and cylinders

Start thinking!

These objects or solids are three dimensional (3D) and can be classified as **prisms**.

1 Identify the three types of prisms in the photo.

2 A prism is identified by its base or **cross-section**.

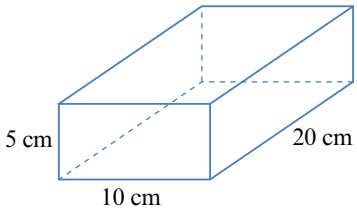
- **a** In your own words, write a definition of 'cross-section'.
- **b** What is special about the cross-section of a prism?
- **c** For each prism above, identify the polygon formed when a cross-section is made.

3 **a** Draw a diagram of each prism identified in question **1** and its matching **net**.

- **b** Explain how a net can help you find the **total surface area** (**TSA**) of any prism.

4 **a** Draw a net for this **rectangular prism** and label the dimensions.

- **b** Find the TSA of the prism.
- **c** Some of the **faces** are identical. How can this be used as a shortcut for finding the TSA of the rectangular prism?
- **d** Show that the formula for finding the TSA of a rectangular prism with length l, width w and height h is $\text{TSA} = 2(lh + lw + wh)$.

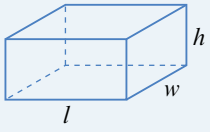


5 A **cube** is also a rectangular prism. Use the formula for the TSA of a rectangular prism to find a formula for the TSA of a cube.

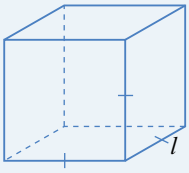
KEY IDEAS

- The total surface area (TSA) of a 3D object or solid is the total area of its outer surface.
- Prisms are 3D objects that have a uniform cross-section that is a polygon.
- Some common prisms are a rectangular prism, a cube and a **triangular prism**.
- The total surface area (TSA) of a prism is the sum of the areas of each face of the prism.
- In some cases, a formula can be used to calculate surface area.

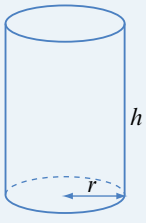
Rectangular prism of length l, width w and height h: $\text{TSA} = 2(lh + lw + wh)$



Cube of length l: $\text{TSA} = 6l^2$



Cylinder of radius r and height h: $\text{TSA} = 2\pi rh + 2\pi r^2$



EXERCISE 9C Surface area of prisms and cylinders

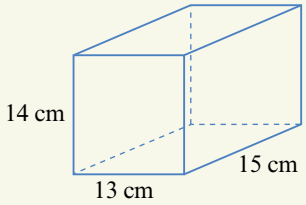
Where appropriate, write your answers to one decimal place.

1 Draw a net for each prism.

- **a** rectangular prism
- **b** triangular prism
- **c** cube

EXAMPLE 9C-1 Calculating the surface area of a rectangular prism

Calculate the surface area of this rectangular prism.

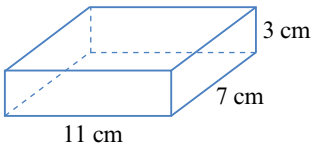


THINK	WRITE
1 Write the formula for the TSA of a rectangular prism.	TSA = $2(lh + lw + wh)$
2 Identify the values of l, w, and h.	l = 13 cm, w = 14 cm, h = 15 cm
3 Substitute these values into the formula.	TSA = $2(13 \times 14 + 13 \times 15 + 14 \times 15)$
4 Calculate the result and include the appropriate unit.	$= 2(182 + 195 + 210)$ $= 1174$ cm^2

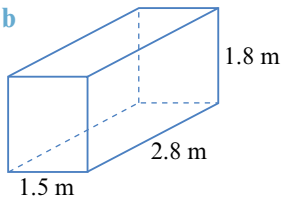
UNDERSTANDING AND FLUENCY

2 Calculate the surface area of each rectangular prism.

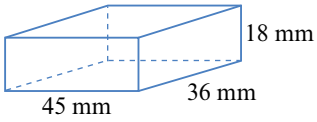
a



b

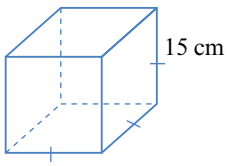


c

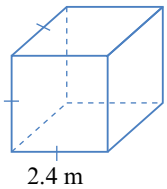


3 Calculate the surface area of each cube.

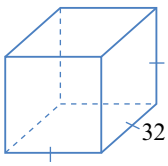
a



b

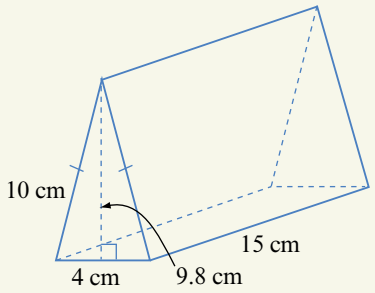


c



EXAMPLE 9C-2 Calculating the surface area of a triangular prism

Calculate the surface area of this triangular prism.



THINK

1 Identify the number and types of faces.

2 Calculate the area of each face. The triangular faces are isosceles triangles, therefore two rectangles are identical.

3 Add the areas and include the appropriate unit.

WRITE

Prism has five faces: three rectangular faces and one pair of identical triangular faces.

rectangular face 1: $A = 10 \times 15 = 150\ \text{cm}^2$

rectangular face 2: $A = 4 \times 15 = 60\ \text{cm}^2$

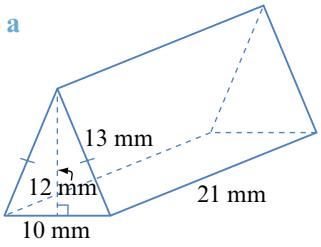
triangular face: $A = \frac{1}{2} \times 4 \times 9.8 = 19.6\ \text{cm}^2$

TSA $= 2 \times 150\ \text{cm}^2 + 60\ \text{cm}^2 + 2 \times 19.6\ \text{cm}^2$
$= 399.2\ \text{cm}^2$

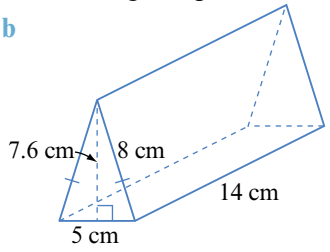
UNDERSTANDING AND FLUENCY

4 Calculate the surface area of each triangular prism.

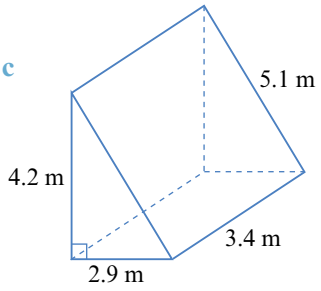
a



b

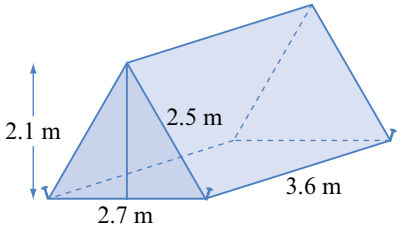


c



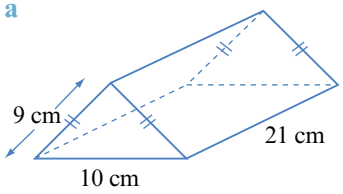
5 This money box is in the shape of a rectangular prism with length 11 cm, width 9 cm and height 18 cm. Calculate its total surface area.

6 The tent below is in the shape of a triangular prism. Find the amount of canvas required to construct it (assume it has no floor).

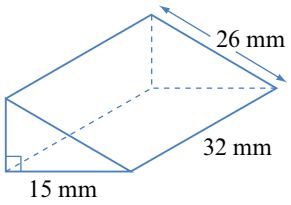


7 Calculate the surface area of each triangular prism.
(Hint: remember you can use Pythagoras' Theorem.)

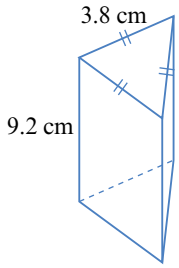
a



b



c



8 Each die in the stack is in the shape of a cube of side length 15 mm.

a Calculate the surface area of one die.

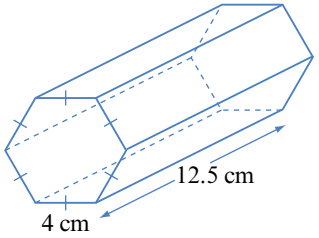
b Use your answer to part **a** to calculate the total surface area of nine separate dice.

c When stacked, the nine dice form a rectangular prism. Using the information you have been given about each die, draw the rectangular prism in your books and label its dimensions.

d Calculate the total surface area of the rectangular prism.

e Find the difference between your answers to parts **b** and **d**. Explain why these answers differ. Support your explanation with calculations.

9 If the base of this hexagonal prism has an area of 41.52 cm^2, calculate its total surface area.



10 A **cylinder** is another 3D object.

a Draw a cylinder with a radius of r cm and height of h cm.

b How does it differ from the prisms you have looked at so far?

c Draw a net for your cylinder. Label the dimensions on your net.

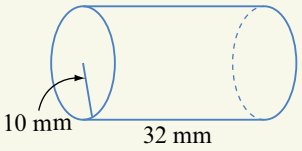
d The width of the rectangular face should be equal to the height of the cylinder. Explain how the length of the rectangle relates to the circular ends.

e Write the formula for finding the area of each shape in the net.

f Show that the formula for finding the surface area of a cylinder is TSA = $2\pi rh + 2\pi r^2$.

EXAMPLE 9C-3 Calculating the surface area of a cylinder

Calculate the surface area of this cylinder.

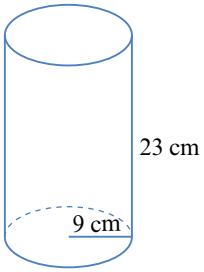


THINK	WRITE
1 Write the formula for the total surface area of a cylinder.	TSA = $2\pi rh + 2\pi r^2$
2 Identify the values of r and h.	$r = 10$ mm, $h = 32$ mm
3 Substitute these values into the formula and simplify.	TSA = $2 \times \pi \times 10 \times 32 + 2 \times \pi \times 10^2$ $= 640\pi + 200\pi$ $= 840\pi$
4 Calculate the result using the π button on your calculator. Write the answer correct to one decimal place and include the appropriate unit.	$= 2638.9378\ldots$ ≈ 2638.9 mm^2

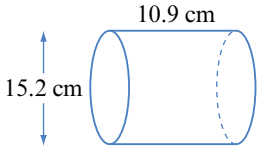
UNDERSTANDING AND FLUENCY

11 Calculate the surface area of each cylinder.

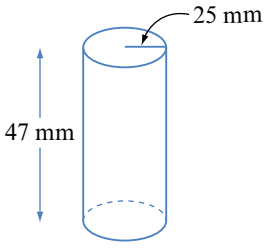
a



b



c



12 Size D batteries have a diameter of 34 mm and a height of 57 mm.

a Calculate the surface area of one battery.

b What is the total surface area of all three batteries?

13 A cube has a total surface area of 384 cm^2.

a What is the length of each side? Explain how you found your answer.

b Use your answer to part **a** to write a formula for finding the side length of a cube when the total surface area is known.

14 An open pipe has a height of 33 cm and a diameter of 55 mm.

a Calculate the surface area of the outer surface of the pipe.

b Did you need to use the entire formula for the surface area of a cylinder, TSA = $2\pi rh + 2\pi r^2$, to find your answer to part **a**? Explain.

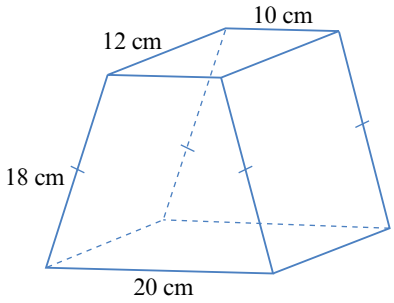
PROBLEM SOLVING AND REASONING

15 Find the surface area of the outside of an open cylinder with a radius of 4.5 cm and a height of 9.9 cm.

16 A kindergarten wishes to add two cylindrical poles to its playground. One is to be painted yellow and the other red. The vertical poles have the same diameter of 125 mm, but vary in height.

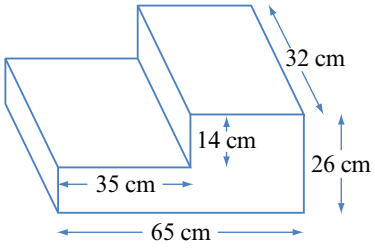
- **a** The yellow pole is 1.8 m tall. Calculate the area which needs to be painted yellow. (Assume the top end needs painting but the end in the ground does not.)
- **b** The red pole is double the height of the yellow pole. Will the kindergarten need to purchase double the amount of red paint in order to paint this pole? Why or why not?
- **c** Calculate the area to be painted red. Does your result support your answer to part **b**?

17 Calculate the surface area of this 3D solid.



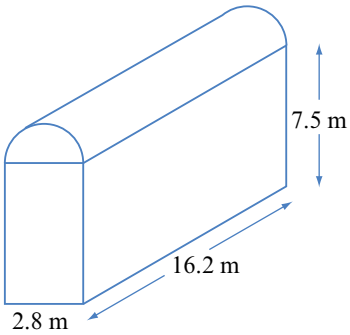
CHALLENGE

18 Consider how to calculate the total surface area of the solid step shown at right.



- **a** The front face is a composite shape. What two basic shapes can you identify?
- **b** Find the area of the face by using your knowledge of areas of composite shapes. How many of these faces are there in the solid?
- **c** The remaining faces are rectangular. Find the area of each of the remaining faces.
- **d** Add the areas from parts **b** and **c** to find the total surface area of this solid.

19 Find the total surface area of this composite object. (Hint: identify the individual prisms first.)



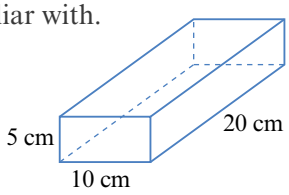
Reflect

What are three workplace situations where finding the surface area of an object could be useful?

9D Volume of prisms and cylinders

Start thinking!

The **volume** of any prism (V) can be found by multiplying the area of the base (A) by the height of the prism (H). Some prisms have a base or cross-section in a shape that you are familiar with.



1 **a** Identify the shape of the base of this prism and calculate its area.
b Use your answer to part **a** to find the volume of this prism.
c Use $V = AH$ to write a formula for finding the volume of any rectangular prism with a base of length l and width w.

2 **a** What shape is the base or cross-section of a cube?
b Use $V = AH$ to write a formula for finding the volume of any cube with a base of length l.

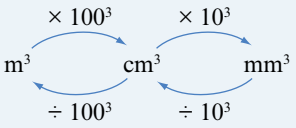
3 **a** Write the formula for the area of a triangle with base b and height h.
b Use $V = AH$ to write a formula for finding the volume of any triangular prism.

4 Can you write the formula for a trapezoidal prism? Discuss this with a classmate.

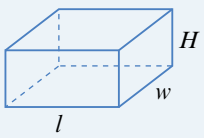
5 A cylinder is not a prism, but you can use the same approach to find the formula for its volume. Write the formula for the volume of a cylinder.

KEY IDEAS

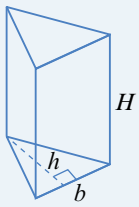
- The volume of any prism can be found using the formula $V = AH$, where A is the area of the base and H is the height of the prism. The height is perpendicular to the base.
- Note that H is used for the height, instead of h, to avoid confusion with the height of a triangular face.
- Volume is measured in cubic units, such as mm^3, cm^3 and m^3.
- A unit of volume can be converted to another unit of volume by multiplying or dividing by the conversion factor.



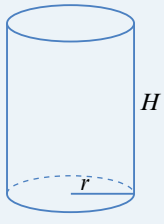
- Volume of a rectangular prism: $V = lwH$



- Volume of a triangular prism: $V = \frac{1}{2}bhH$



- Volume of a cylinder: $V = \pi r^2 H$



- **Capacity** is the amount of liquid an object can hold. You can convert units of volume to units of capacity. For example, a volume of 1 cm^3 is equivalent to a capacity of 1 mL.

EXERCISE 9D Volume of prisms and cylinders

Where appropriate, write answers correct to one decimal place.

EXAMPLE 9D-1 Converting units of volume

Convert each volume measurement to the unit shown in brackets.

a $0.89\ m^3$ (cm^3)

b $1650\ mm^3$ (cm^3)

THINK

a Multiply by $100^3 = 1\ 000\ 000$ as you are converting to smaller unit. ($1\ m^3 = 100^3\ cm^3$.)

b Divide by $10^3 = 1000$ as you are converting to a larger unit. ($1\ cm^3 = 10^3\ mm^3$.)

WRITE

a $0.89\ m^3 = (0.89 \times 1\ 000\ 000)\ cm^3$
$= 890\ 000\ cm^3$

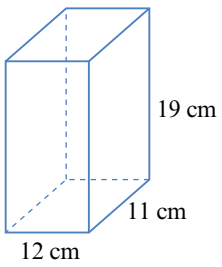
b $1650\ mm^3 = (1650 \div 1000)\ cm^3$
$= 1.65\ cm^3$

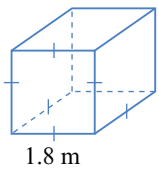
UNDERSTANDING AND FLUENCY

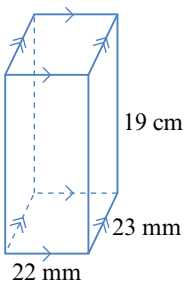
1 Convert each volume measurement to the unit shown in brackets.

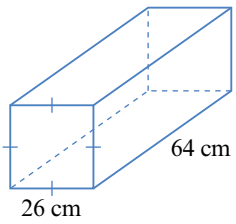
a $12.5\ m^3$ (cm^3)
b $240\ 000\ mm^3$ (cm^3)
c $34\ 200\ 000\ cm^3$ (m^3)
d $0.55\ m^3$ (mm^3)
e $67\ 200\ mm^3$ (m^3)
f $0.000\ 009\ m^3$ (mm^3)
g $7520\ mm^3$ (cm^3)
h $8.74\ m^3$ (cm^3)
i $142\ 900\ cm^3$ (m^3)
j $73\ 000\ 000\ mm^3$ (m^3)

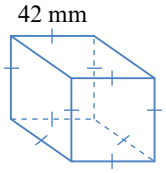
2 Calculate the volume of each rectangular prism.

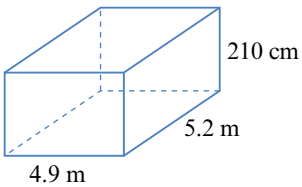
a


b


c


d


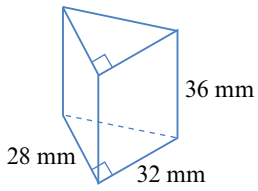
e


f


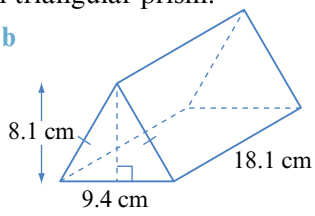
UNDERSTANDING AND FLUENCY

3 Calculate the volume of each triangular prism.

a



b



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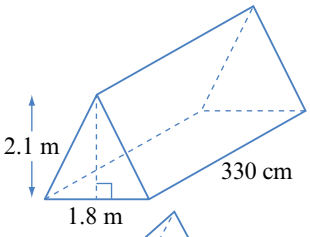

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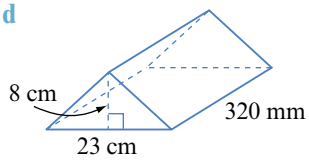

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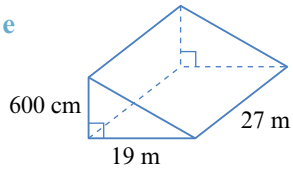

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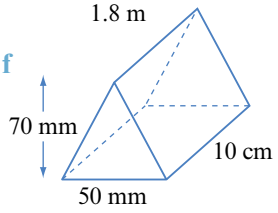
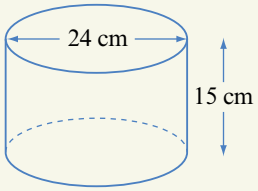


EXAMPLE 9D-2 Calculating the volume of a cylinder

Calculate the volume of this cylinder.



THINK

1 Identify the shape of the base (a circle) so you can write the formula.

2 Identify the values for r and H and substitute them into the formula.

3 Calculate the result using the π button on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

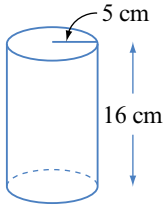
WRITE

$V = \pi r^2 H$

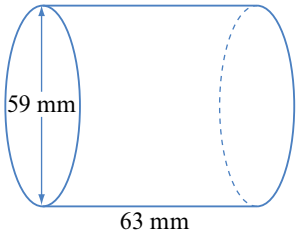
$r = D \div 2 = 24 \div 2 = 12$ cm
$H = 15$ cm
So, $V = \pi \times 12^2 \times 15$
$= 2160\pi$
$= 6785.840\,132...$
≈ 6785.8 cm^3

4 Calculate the volume of each cylinder.

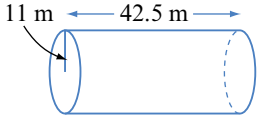
a



b



c



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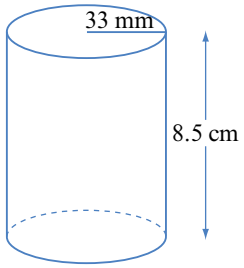

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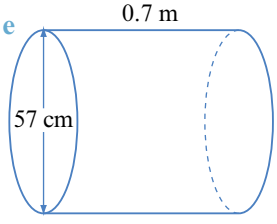

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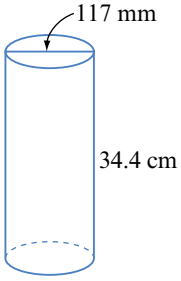


5 A suitcase measures 76 cm by 48 cm by 29 cm. What volume of luggage can this suitcase hold if it is packed to capacity?

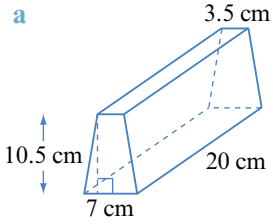
6 Paula wishes to use odd pieces of colourful fabric to construct a child's cushion. The cushion is in the shape of a cube with side length of 0.4 m.

a How much polyester fibre does she need to fill the cushion?

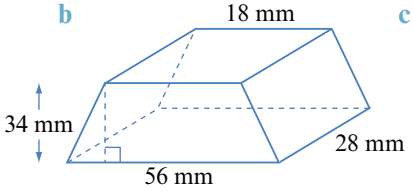
b Convert your answer to cubic centimetres.

7 Calculate the volume of each prism.

a



b



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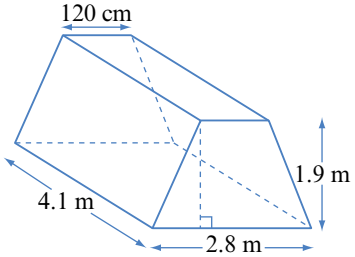


8 A round hat box has a diameter of 23 cm and a height of 12.5 cm. Calculate its volume.

9 **a** Write a sentence explaining the difference between volume and capacity.

b If $1\ m^3 = 1\ 000\ 000\ cm^3$, write $1\ m^3$ in millilitres.

c Use your answer to part **b** to write $1\ m^3$ in kilolitres.

d Use your answers above to complete these conversions.

i $1.2\ m^3 =$ _____ L

ii $15.8\ L =$ _____ cm^3

iii $1242\ kL =$ _____ m^3

iv $8.8\ cm^3 =$ _____ kL

10 A can of soft drink has a diameter of 6.4 cm and a height of 12 cm. What is the capacity of the can?

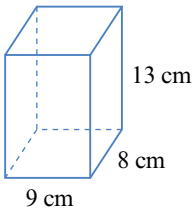
11 **a** Write the length, width and height of Prism A. Use this information to calculate the volume of Prism A.

b Write the length, width and height of Prism B. Use this information to calculate the volume of Prism B.

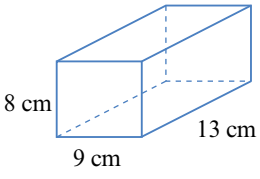
c What do you notice about your answers to parts **a** and **b**?

d Does the orientation of the prism affect its volume? Can you think of another word to better describe the height of Prism B?

Prism A

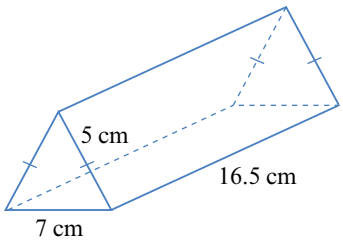


Prism B

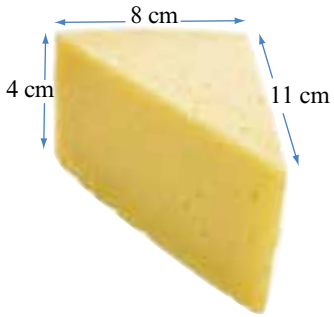


PROBLEM SOLVING AND REASONING

12 Calculate the volume of this prism. (Hint: first find the area of the base using Pythagoras' Theorem.)



13 This piece of cheese is in the shape of a triangular prism.



- **a** Calculate its volume.
- **b** If 1 cm^3 of cheese has a mass of 0.001 kg, find the mass of the cheese in kilograms.
- **c** This particular cheese costs $21 per kilogram. What is the cost of the piece of cheese shown in the photo?

14 A rectangular fish tank is 45 cm long, 20 cm wide and 50 cm tall.

- **a** Calculate the volume of the fish tank.
- **b** What is the capacity of the fish tank if it is filled to the brim? Give your answer in millilitres and litres.
- **c** The fish tank can only be filled to a line 6 cm from the top. How many litres of water will be required to fill the tank to this height?

15 The volume of a prism is 2744 cm^3.

- **a** If the prism is a cube, what is its side length?
- **b** If the prism is a rectangular prism, give two possible sets of dimensions.
- **c** If the prism is a triangular prism, give two possible sets of dimensions.

16 **a** A cylinder has a volume of 1649.34 cm^3 and a radius of 5 cm. What is the height of the cylinder to the nearest centimetre?

- **b** A cylinder has a volume of 207 cm^3 and a height of 5.7 cm. Find its radius. Give your answer to the nearest millimetre.

17 Jack installs two cylindrical water tanks. The first has a diameter of 1.5 m and a height of 2 m. The second tank has a radius of 2 m and a height of 1.5 m.

- **a** Which tank has the larger capacity?
- **b** What is the difference in the capacity of the two tanks? Give your answer to the nearest litre.

CHALLENGE

18 A cylinder has a radius of 8 cm and a height of 15 cm.

- **a** Calculate the volume of the cylinder.

For each of the situations described below:

- **i** calculate the volume
- **ii** describe how this volume compares to the volume of the original cylinder.

CHALLENGE

b The radius is halved but the height remains the same.

c The radius is doubled and the height remains the same.

d The radius remains the same but the height is halved.

e The radius remains the same but the height is doubled.

19 Rosemary wishes to make an open box from a 30 cm by 30 cm piece of blue cardboard. She decides to cut out a square of length 2 cm from each corner and fold up the sides to form her box.

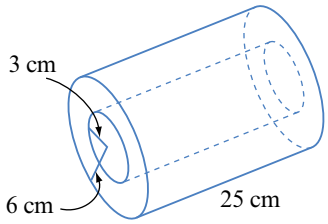
a Draw a diagram showing the piece of cardboard with the corners cut out. Label the dimensions.

b Determine the length, width and height of the box once she has folded up the sides.

c Calculate the volume of the box.

d Would the volume of the box increase or decrease if she cut out 3 cm square corners from the piece of cardboard before forming the rectangular prism? Explain your answer.

20 This plastic pipe is hollow.



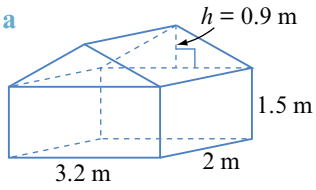
a What shape is the base of the pipe? Calculate its area. (Hint: refer to Exercise 9B question **7**.)

b Use the area of the base to calculate the volume of plastic used for this pipe.

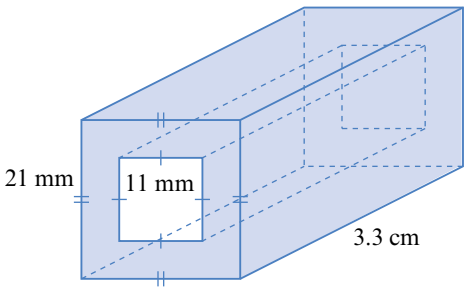
c Can you think of a different method for finding the volume? Explain your answer.

21 Calculate the volume of each **composite solid**. (Hint: identify the individual solids first.)

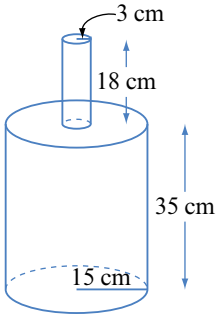
a



b



c



Reflect

What is the difference between the volume and the capacity of a prism?

9E Surface area of pyramids and cones

Start thinking!

1 Identify the solid in this photograph.

2 Write one or two sentences explaining the difference between a **pyramid** and a prism.

3 How would you find the total surface area of a pyramid? Is the method for finding the TSA of a pyramid different from finding the TSA of a prism? Explain.

4 **a** Name this pyramid object. (Hint: use the name of the base.)

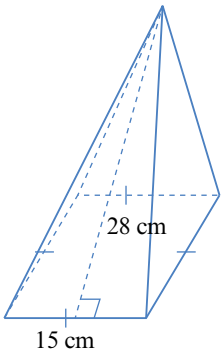
b Draw a net for this object.

c Identify the shape of each face and draw each face individually. Label the dimensions of each face.

d Using the appropriate formula, calculate the area of each face.

e Do you need to calculate the area of every face separately for this object? Explain your answer.

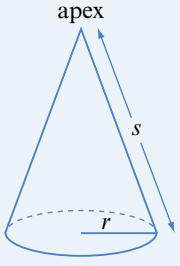
f Use your answer to part **d** to find the total surface area of this object.



KEY IDEAS

- A pyramid is named according to the shape of its base.
- The total surface area of a pyramid is the sum of the areas of each face.
- If the lengths of the base and the slant edge are known, use Pythagoras' Theorem to find the height of a triangular face.
- A **cone** has a circular base and a curved surface that meets at a point called the apex.
- Surface area of a cone with radius r and slant height s:
  $\text{TSA} = \pi r^2 + \pi rs$

NOTE In this topic, only right pyramids and right cones are considered, where the apex is directly above the centre of the base.



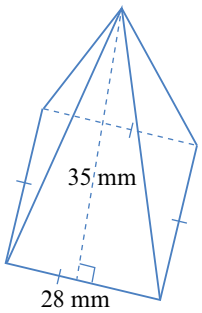
EXERCISE 9E Surface area of pyramids and cones

UNDERSTANDING AND FLUENCY

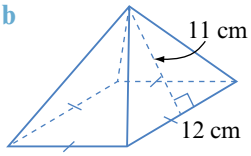
Where appropriate, write answers correct to one decimal place.

1 Calculate the surface area of each square-based pyramid.

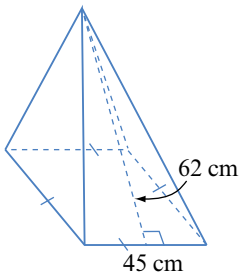
a



b

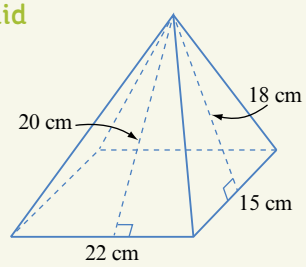


c



EXAMPLE 9E-1 Calculating the surface area of a pyramid

Calculate the surface area of this pyramid.



THINK

1 Identify the number of faces.

2 Calculate the area of each face.

3 Add the areas and include the appropriate unit.

WRITE

Pyramid has five faces: one rectangular face and two pairs of identical triangular faces.

rectangular face: $A = 22 \times 15 = 330 \text{ cm}^2$

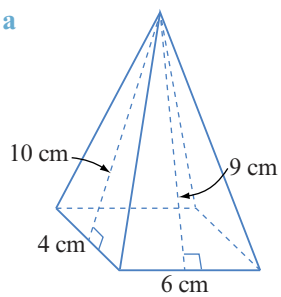
front triangular face: $A = \frac{1}{2} \times 22 \times 20 = 220 \text{ cm}^2$

side triangular face: $A = \frac{1}{2} \times 15 \times 18 = 135 \text{ cm}^2$

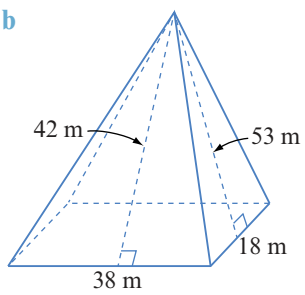
$\text{TSA} = 330 \text{ cm}^2 + 2 \times 220 \text{ cm}^2 + 2 \times 135 \text{ cm}^2$
$= 1040 \text{ cm}^2$

2 Calculate the surface area of each pyramid.

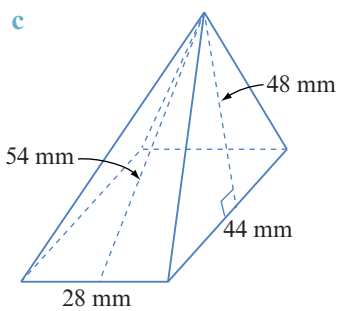
a



b



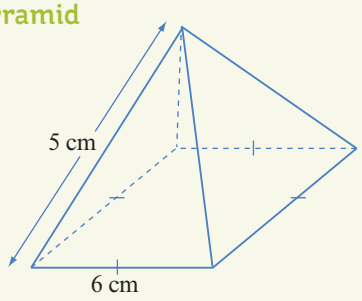
c



EXAMPLE 9E-2

Calculating the surface area of a pyramid by first using Pythagoras' Theorem

Use Pythagoras' Theorem to find the height of a triangular face and hence calculate the surface area of this pyramid.



THINK

1 Identify the number of faces.

2 Use Pythagoras' Theorem to find the height of one of the triangular faces.

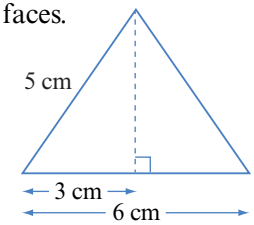
3 Calculate the area of a triangular face.

4 Calculate the area of the base.

5 Calculate the area of a triangular face.

WRITE

Pyramid has five faces: one square face and four identical triangular faces.



$5^2 = h^2 + 3^2$
$25 = h^2 + 9$
$h^2 = 16$
$h = 4$ cm

triangular face: $A = \frac{1}{2} \times 6 \times 4 = 12$ cm^2

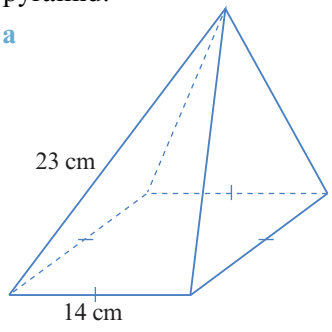
square base: $A = 6^2 = 36$ cm^2

TSA $= 36$ cm$^2 + 4 \times 12$ cm^2
$= 84$ cm^2

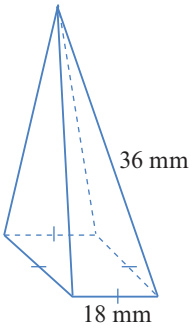
UNDERSTANDING AND FLUENCY

3 Find the height of a triangular face and hence calculate the surface area of each pyramid.

a



b



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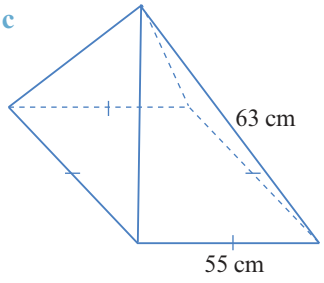
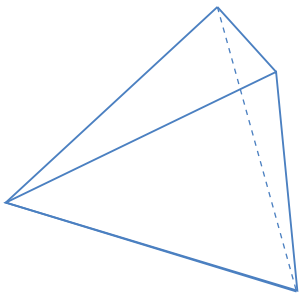


4 The main entrance to the Louvre Museum in Paris, France, has been constructed as a pyramid with a square base of length 35 m. Each identical triangular face has a height of 27 m and is made of glass segments. Find the total surface area of the glass walls of the pyramid.

5 The Red Pyramid in Dahshur, Egypt, was originally covered in white limestone. It has a slant edge of length 187 m and a square base of side length 220 m.

- **a** Draw a diagram of this pyramid and label its dimensions.
- **b** Find the surface area of the pyramid that was covered with limestone. (Hint: do not include the base in your calculations.)

6 A pyramid with a triangular base is known as a tetrahedron. If all the triangular faces are equilateral triangles then it is known as a **regular tetrahedron**.

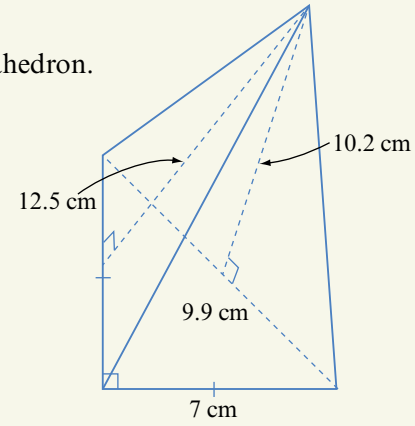


- **a** Draw a net for a regular tetrahedron.
- **b** How many faces does it have?
- **c** If the side lengths are 4 cm, find the total surface area of the tetrahedron.
- **d** Having regular triangular faces on a tetrahedron makes it easier to calculate the total surface area. Explain why this is.

NOTE Not all tetrahedra (plural for tetrahedron) are regular; that is, the faces of tetrahedra are not always equilateral triangles. You may need to work out the area of more than one triangular face to calculate the total surface area.

EXAMPLE 9E-3 Calculating the surface area of a tetrahedron

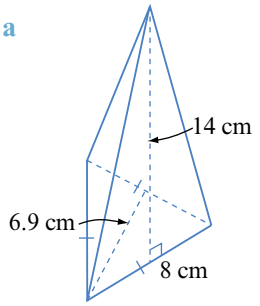
Calculate the surface area of this tetrahedron.



THINK	WRITE
1 Identify the number of faces.	Base is an isosceles right-angled triangle: $b = h = 7$ cm. 2 identical triangular faces: $b = 7$ cm, $h = 12.5$ cm. 1 triangular face: $b = 9.9$ cm, $h = 10.2$ cm
2 Calculate the area of each face.	base: $A = \frac{1}{2} \times 7 \times 7 = 24.5$ cm^2 triangular face 1: $A = \frac{1}{2} \times 7 \times 12.5 = 43.75$ cm^2 triangular face 2: $A = \frac{1}{2} \times 9.9 \times 10.2 = 50.49$ cm^2
3 Add the areas and include the appropriate unit.	TSA $= 24.5$ cm$^2 + 2 \times 43.75$ cm$^2 + 50.49$ cm^2 $= 162.49$ cm^2

7 Calculate the surface area of each tetrahedron. (Hint: Use Pythagoras' Theorem to find any missing heights.)

a



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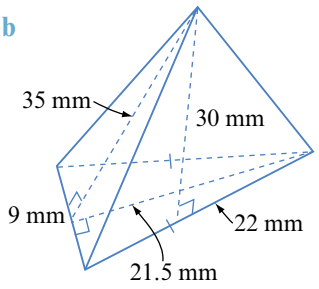

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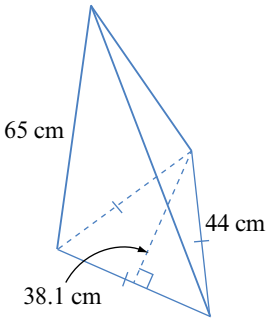

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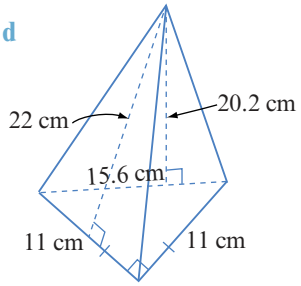

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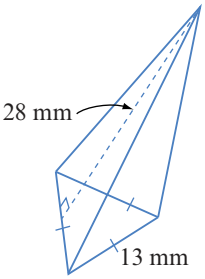

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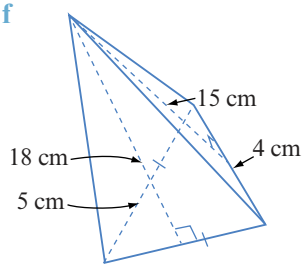


8 A cone is drawn with a radius of r and a slant height of s.

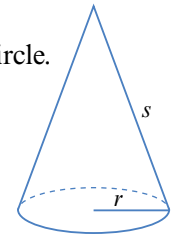
a Draw a net for the cone and label the dimensions.

b One of the shapes is a circle. Write the formula for finding the area of a circle.

c The second shape is a sector. Let the angle in the sector be θ. Mark the angle on the sector.

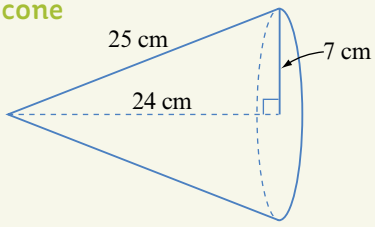
d Find the area of the sector with a radius of s and angle θ.

e Show that the formula for finding the surface area of a cone is $\pi r^2 + \pi rs$.



EXAMPLE 9E-4 Calculating the surface area of a cone

Calculate the surface area of this cone.



THINK

1 Identify the radius and the slant height.

2 Write the formula for the surface area of a cone. Substitute values for r and s and simplify.

3 Calculate the result using the π button on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$r = 7$ cm, $s = 25$ cm

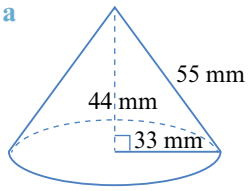
$\text{TSA} = \pi r^2 + \pi rs$
$= \pi \times 7^2 + \pi \times 7 \times 25$
$= 49\pi + 175\pi$
$= 224\pi$

$= 703.716\,75\ldots$
$\approx 703.7\ \text{cm}^2$

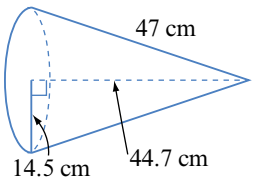
UNDERSTANDING AND FLUENCY

9 Calculate the surface area of each cone.

a



b



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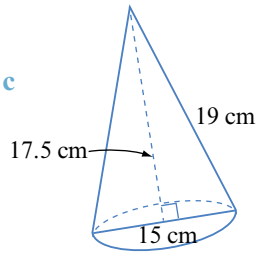

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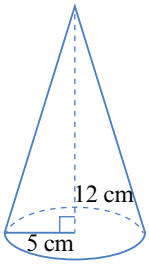

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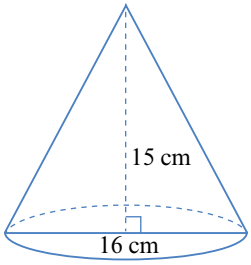

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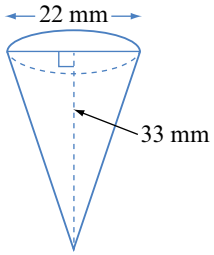


PROBLEM SOLVING AND REASONING

10 A square-based pyramid has a surface area of 561 cm^2. If the length of the base is 11 cm, find the height of the triangular faces.

11 A cone has a radius of 15.5 cm and a height of 9.5 cm. Use Pythagoras' Theorem to find the slant height of the cone and hence its total surface area.

12 A party hat is in the shape of a cone. The diameter of the base is 12 cm and the slant height is 18 cm. Find the amount of cardboard needed to make this party hat assuming there is no overlap. (Hint: Do you need to use all parts of the formula for the TSA of a cone?)

13 An eight-sided die is in the shape of an octahedron.

a How many equilateral triangles form the octahedron?

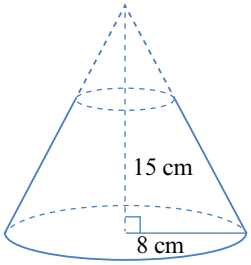
b If the sides of the triangles measure 14 mm, find the total surface area of the octahedron.

c The octahedron resembles two identical square pyramids sitting base to base. Calculate the surface area of one pyramid and double your answer.

d Are your answers to parts **b** and **c** equal? Why or why not?

CHALLENGE

14 The tip of a hollow cone is sliced and removed. The remaining object is called a **truncated cone**. If the tip was one-third of the height of the cone, find the surface area of the truncated cone. (Assume there is a base to the cone.)



Reflect

How does increasing the radius of a cone affect its total surface area?

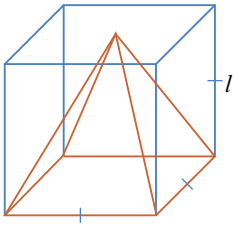
10A

9F Volume of pyramids and cones

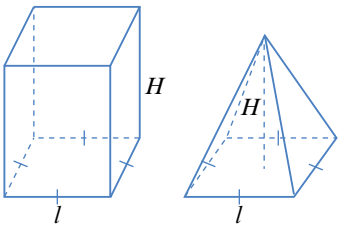
Start thinking!

1 Discuss with a classmate how the volume of a pyramid compares to the volume of a prism with the same base and height. How could you demonstrate the relationship between the two volumes?

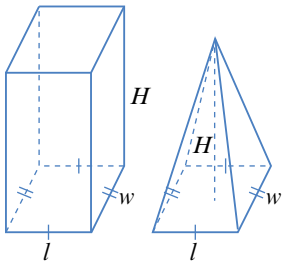
2 This cube has a square-based pyramid drawn inside. If three of these pyramids will fit exactly inside the cube and the volume of the cube is given by $V = l^3$, write a formula for finding the volume of the square-based pyramid.



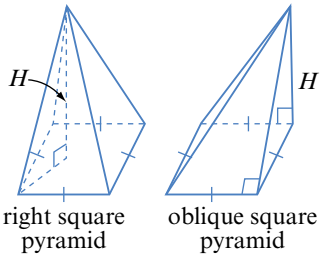
3 Not all square-based pyramids have the same height as the length of the base. If the formula for the volume of a square-based prism with height H is $V = l^2H$, write a formula for calculating the volume of a square-based pyramid with these dimensions.



4 If the volume of a rectangular prism is given by $V = lwH$, write a formula for finding the volume of a rectangular pyramid with the same dimensions.

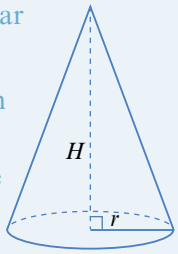


5 A right pyramid has its apex over the midpoint of its base. An **oblique pyramid** does not have its apex over the midpoint of its base. Are the volumes of a right pyramid and an oblique pyramid with the same dimensions equal? Explain.



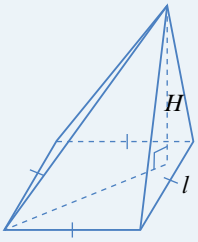
KEY IDEAS

- The volume of any pyramid can be found using the formula $V = \frac{1}{3}AH$, where A is the area of the base and H is the height of the pyramid. The height is perpendicular to the base.
- The volume of a cone can be found using $V = \frac{1}{3}AH$, where A is the area of the circular base: $V = \frac{1}{3}\pi r^2 H$

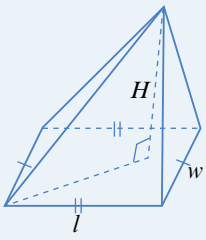


- The formula for an individual pyramid can be found by substituting for the area of its base. For example:

Volume of a square-based pyramid: $V = \frac{1}{3}l^2H$



Volume of a rectangular pyramid: $V = \frac{1}{3}lwH$



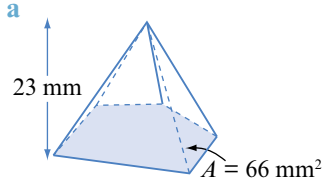
EXERCISE 9F Volume of pyramids and cones

UNDERSTANDING AND FLUENCY

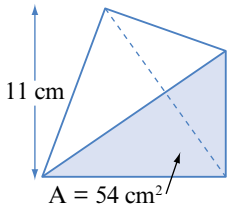
Where appropriate, write answers correct to one decimal place.

1 Find the volume of each pyramid.

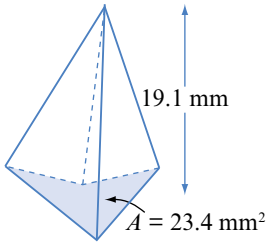
a



b

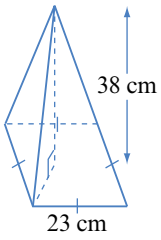


c

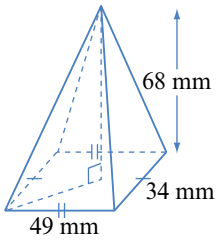


2 Find the volume of each right pyramid, using the appropriate formula.

a



b



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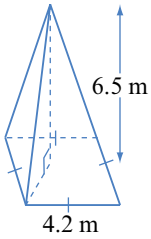

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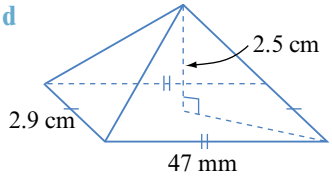

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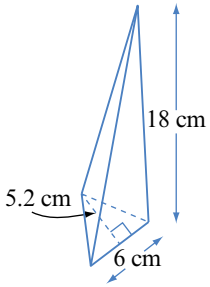

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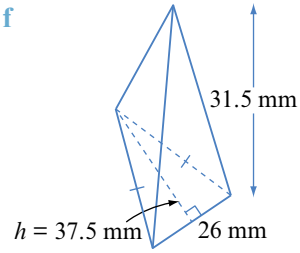
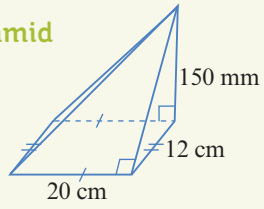


EXAMPLE 9F-1 Calculating the volume of a pyramid

Find the volume of this pyramid.



THINK

1. Identify the shape of the base of the pyramid and write the formula for its volume.
2. Identify the values for l, w and H and substitute them into the formula. Ensure that all measurements are written in the same units.
3. Calculate the result and include the appropriate unit.

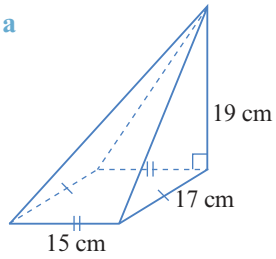
WRITE

Pyramid has a rectangular base.
$V = \frac{1}{3}lwH$

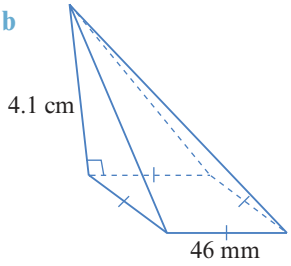
$l = 20$ cm, $w = 12$ cm
$H = 150$ mm $= 15$ cm
$V = \frac{1}{3} \times 20 \times 12 \times 15$
$= 1200$ cm^3

3 Calculate the volume of each pyramid.

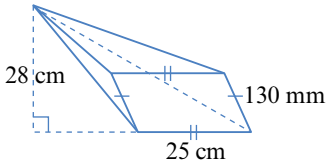
a



b



c



4 The Great Pyramid of Cheops in Giza, Egypt, is a square-based pyramid. Its present height is approximately 147 m and the length of its base is approximately 230 m. Calculate its volume.

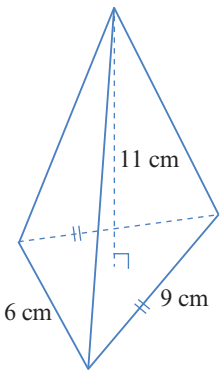
5 Convert these measurements to the units shown in brackets. (Use the conversion chart in the Key ideas on page 432 and Exercise 9D question **9**, page 447.)

a 9.8 m^3 (cm^3)
b 19 500 000 mm^3 (m^3)
c 248 mL (L)
d 32 m^3 (L)
e 250 cm^3 (mL)
f 540 L (m^3)
g 0.145 cm^3 (mm^3)
h 7 900 cm^3 (L)
i 10 200 mL (m^3)

6 **a** Use Pythagoras' Theorem to find the height of the triangular base of this pyramid. Hence, find the area of the base.

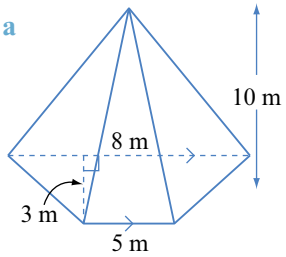
b Use your answer to part **a** to find the volume of the triangular pyramid. Give your answer in cubic millimetres and cubic centimetres.

c Can you write a formula for finding the volume of any triangular pyramid?



7 Find the volume of each pyramid by first calculating the area of its base.

a



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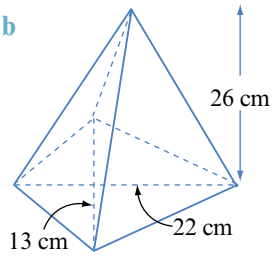

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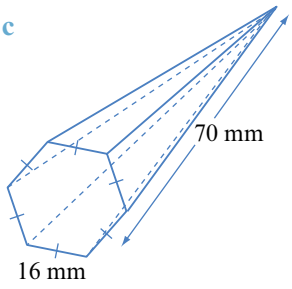
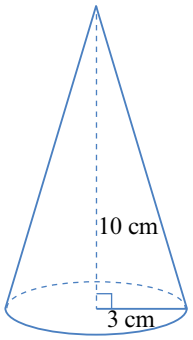


UNDERSTANDING AND FLUENCY

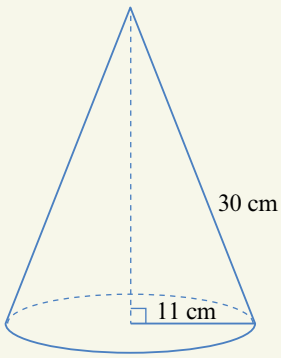
8 A cone is not a pyramid, but its volume can be worked out in a similar way.



- **a** Copy the diagram of the cone and draw a cylinder with the same base and height dimensions around it.
- **b** Write a sentence describing the figure that has been created and the relationship between the cone and the cylinder.
- **c** Find the volume of the cylinder.
- **d** How can you use your answer to part **c** to find the volume of the cone? Test your theory by finding the volume of the cone and comparing it to the volume of the cylinder.
- **e** Use your findings to complete this statement.
  The formula for finding the volume of a cone with radius r and height H is $V =$ ___ $\times \pi r^2 H$.

EXAMPLE 9F-2 Calculating the volume of a cone

- **a** Calculate the volume of this cone.
- **b** Find the capacity, in litres, of the cone.



THINK

a 1 Write the formula for the volume of a cone and identify the given values.

2 Use Pythagoras' Theorem to find the height of the cone.

3 Substitute the exact values of r and H into the formula. Write the answer correct to one decimal place and include the appropriate unit.

b 1 Convert cm^3 to L by first converting to mL. ($1\ cm^3$ is equivalent to 1 mL. 1000 mL = 1 L.)

2 Write the answer correct to one decimal place.

WRITE

a $V = \frac{1}{3}\pi r^2 H$
$r = 11$ cm, $s = 30$ cm, $H = ?$

$30^2 = H^2 + 11^2$
$900 = H^2 + 121$
$H^2 = 779$
$H = \sqrt{779}$
≈ 27.9 cm

$V = \frac{1}{3} \times \pi \times 11^2 \times \sqrt{779}$
$= 3536.5737...$
$\approx 3536.6\ cm^3$

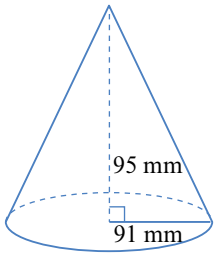
b $3536.6\ cm^3 = 3536.6$ mL
$= (3536.6 \div 1000)$ L
$= 3.5366$ L
≈ 3.5 L

9 Cones can also be either right or oblique.

i Find the volume of each cone.

ii Find the capacity, in litres, of each cone.

a



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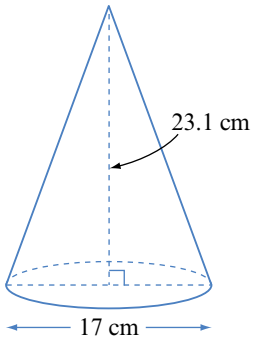

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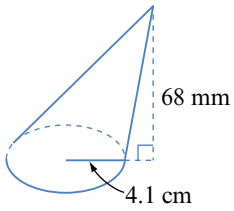

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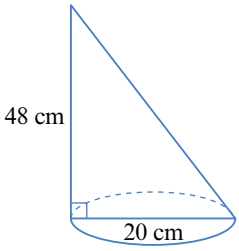

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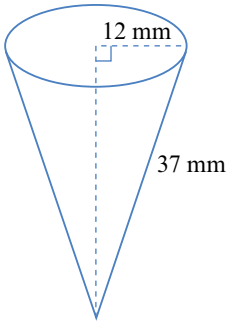

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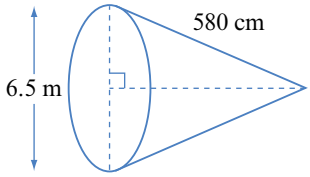
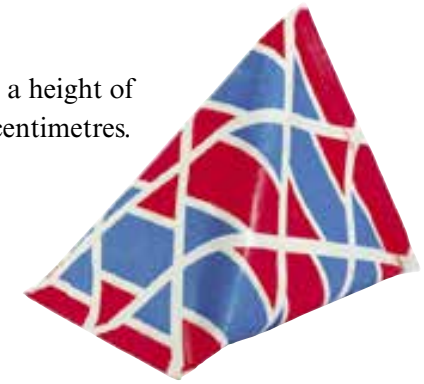


10 A traffic cone has a base with diameter of 270 mm and a height of 850 mm. Find its volume, giving your answer in cubic centimetres.



11 An orange juice carton is in the shape of a triangular pyramid. Its bottom triangular face has a base length of 8.5 cm and a height of 4 cm. If the height of the carton is 13 cm, how much orange juice can it hold? Give your answer to the nearest millilitre.

12 **a** A cone has a volume of 89.8 cm^3. If the radius of the cone is 3.5 cm, find the height of the cone.

b A second cone has the same volume, but a height of 10 cm. Find the radius of this cone.

13 A vase has the shape of an open cone sitting in a frame. The top of the vase has a radius of 12 cm and its height is 46 cm. It is filled to three-quarters of its capacity with water. Find the amount of water, in litres, that the vase contains.

14 A cone has a radius of 3 cm and a height of 5 cm.

a Calculate its volume.

b Double the radius of the cone.

i What is the new radius?

ii If the height does not change, calculate the volume of the new cone.

PROBLEM SOLVING AND REASONING

c Double the radius once again. Using the same height, find the volume.

d Compare your answers to parts a, b and c. Describe what happens to the volume of a cone if the radius is doubled and the height remains the same.

e This time double the height and leave the radius unchanged. Calculate the volume of the cone if the radius is 3 cm and the height is now 10 cm.

f Double the height of the cone again. Calculate the volume of this cone.

g Can you predict the volume of the cone if the height is doubled once again? Calculate the volume to see if your prediction is correct.

h Write a sentence describing the effect on the volume of a cone when the height is doubled but the radius remains the same.

i Why do you think the volume increases by a factor of 4 when the radius is doubled but only by a factor of 2 when the height is doubled?

CHALLENGE

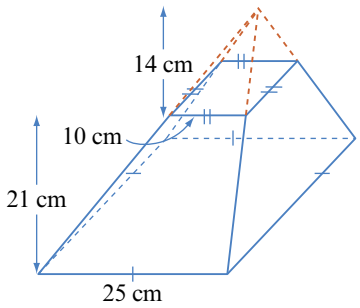
15 The top or apex of a square pyramid is cut off parallel to the base and is removed. The object remaining is called a **truncated pyramid**.

a What shape are the faces of the truncated pyramid?

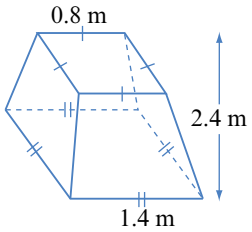
b Calculate the volume of the whole pyramid.

c Calculate the volume of the smaller pyramid that has been cut off.

d Use your answers to parts b and c to calculate the volume of the truncated pyramid.



16 The base of a statue is to be cut from a concrete slab that is formed as a square-based pyramid. If the height of the original slab was 3.1 m, what volume of concrete is needed to construct this base? Give your answer in cubic metres.



17 A drinking glass is 120 mm tall and is in the shape of truncated cone. The base of the glass has a diameter of 55 mm and the top of the glass has a diameter of 75 mm.

a If the height of the conical piece of glass used to construct the drinking glass was 192 mm, calculate the volume of the drinking glass.

b How many millilitres of iced tea can this glass hold when it is filled to its brim?

18 It is possible to write a formula for finding the volume of a truncated square-based pyramid and a truncated cone. Use your answers to questions **15**–**17** to assist you to find these formulas.

Reflect

How does knowing the volume of a prism help to find the volume of a pyramid?

9G Surface area and volume of spheres

Start thinking!

The planet Earth appears to be shaped like a **sphere**.

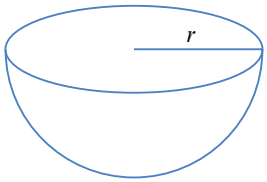
1 Write a sentence describing the features of a sphere.

2 A sphere has only one measurement which determines its size. What is this measurement?

3 With a partner, discuss what might happen to the total surface area of the sphere if the radius is:
a increased **b** decreased.

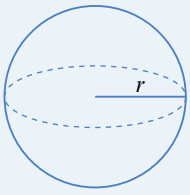
4 Will increasing or decreasing the radius of a sphere have the same effect on the volume as it does on the total surface area?

The formulas for finding the total surface area and the volume of a sphere are:
TSA $= 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$

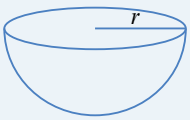


5 The shape on the right is a hemisphere.
a What fraction of a sphere is a **hemisphere**?
b Use your answer to part **a** and your formula for the TSA of a sphere to write the formula for the TSA of this open hemisphere.
c A circular disc is placed on the top of the open section of the hemisphere to form a closed hemisphere. Describe how you can adapt the formula above to find the total surface area of the closed hemisphere.
d Use your answer to part **a** to write a formula for the volume of the open hemisphere.
e Does having an open or closed hemisphere affect the volume of the hemisphere? Explain.

KEY IDEAS



- The size of a sphere is determined by one measurement: its radius (or diameter).
- The formula for the total surface area of a sphere is TSA $= 4\pi r^2$.
- The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.
- The formula for the total surface area of a hemisphere is:
  - TSA $= 2\pi r^2$ (open hemisphere)
  - TSA $= 3\pi r^2$ (closed hemisphere)
- The formula for the volume of a hemisphere is $V = \frac{2}{3}\pi r^3$.



EXERCISE 9G Surface area and volume of spheres

Where appropriate, write answers correct to one decimal place.

EXAMPLE 9G-1 Calculating the surface area of a sphere

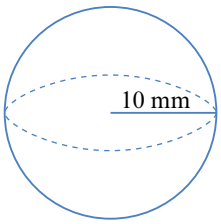
Calculate the total surface area of a sphere with a radius of 3 cm.

THINK	WRITE
1 Write the formula for total surface area of a sphere.	TSA = $4\pi r^2$
2 Identify the measurement for radius and substitute into the formula.	r = 3 cm TSA = $4 \times \pi \times 3^2$ $= 4 \times \pi \times 9$ $= 36\pi$
3 Calculate the result using the π button on your calculator.	$= 113.0973\ldots$
4 Write the answer correct to one decimal place and include the appropriate unit.	≈ 113.1 cm^2

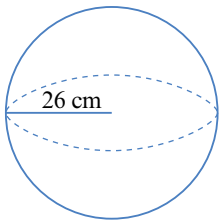
UNDERSTANDING AND FLUENCY

1 Calculate the total surface area of each sphere.

a



b



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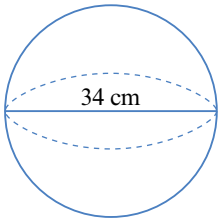

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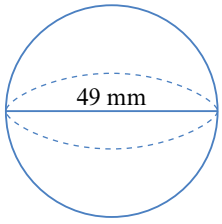

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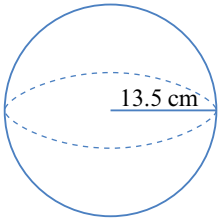

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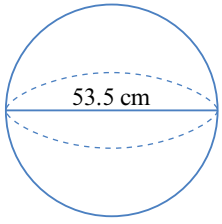
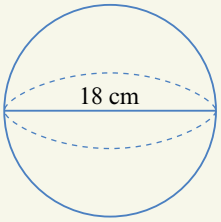


EXAMPLE 9G-2 Calculating the volume of a sphere

Calculate the volume of this sphere.



THINK

1 Write the formula for the volume of a sphere.

2 Identify the measurement for radius and substitute into the formula.

3 Calculate the result using the π button on your calculator.

4 Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$V = \frac{4}{3}\pi r^3$

$r = 18 \div 2 = 9$ cm

$V = \frac{4}{3} \times \pi \times 9^3$

$= 972\pi$

$= 3053.628\,059\ldots$

≈ 3053.6 cm^3

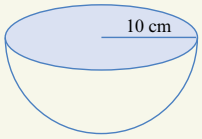
2 Calculate the volume of each sphere in question 1.

EXAMPLE 9G-3 Calculating the surface area and volume of a hemisphere

For this hemisphere, calculate:

a its total surface area

b its volume.



NOTE Closed hemispheres are drawn with a shaded face.

THINK

a 1 Determine if the hemisphere is open or closed and write the appropriate formula for the total surface area.

2 Identify the measurement for radius and substitute into the formula.

3 Calculate the result.

4 Write the answer correct to one decimal place and include the appropriate unit.

b 1 Write the formula for the volume of a hemisphere and substitute the radius into the formula.

2 Calculate the result.

3 Write the answer correct to one decimal place and include the appropriate unit.

WRITE

a TSA $= 3\pi r^2$

$r = 10$ cm

TSA $= 3 \times \pi \times 10^2$

$= 3 \times \pi \times 100$

$= 300\pi$

$= 942.477\,79\ldots$

≈ 942.5 cm^2

b $V = \frac{2}{3}\pi r^3$

$= \frac{2}{3} \times \pi \times 10^3$

$= \frac{2000\pi}{3}$

$= 2094.395\,102\ldots$

≈ 2094.4 cm^3

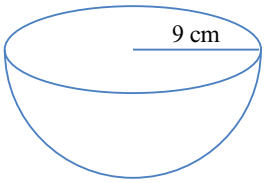
3 For each hemisphere, calculate:

i its total surface area

ii its volume.

(Hint: Remember to determine if the hemisphere is open or closed.)

a



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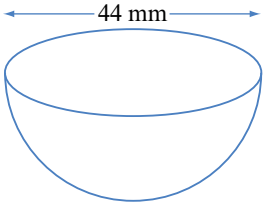

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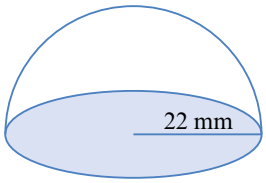

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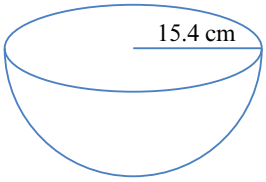

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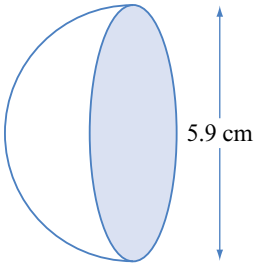

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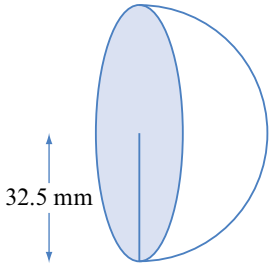
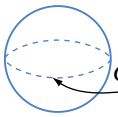


4 When inflated, a beach ball has a diameter of 76 cm.

a Calculate its total surface area.

b Find the volume of air it can hold.

5 The circumference around the middle of this golf ball is 13.5 cm.



a What is the total surface area of the golf ball?
(Hint: first calculate the radius of the golf ball.)

b What is the volume of the golf ball?

6 The total surface area of a tennis ball is approximately 564 cm^2.

a Transpose the formula for the total surface area of a sphere to make r the subject.

b Find the radius of the tennis ball.

7 The volume of a fully inflated volleyball is 0.15 m^3.

a Transpose the formula for the volume of a sphere to make r the subject.

b Find the radius of the volleyball to the nearest centimetre.

8 The approximate diameters of the planets in our solar system are listed in the table below.

Planet	Diameter (km)	Total surface area	Volume
Mercury	4 878		
Venus	12 104		
Earth	12 756		
Mars	6 787		
Jupiter	142 800		
Saturn	120 000		
Uranus	51 118		
Neptune	49 528		

a Copy and complete the table. Write the values in **scientific notation**.

b Which planet has the largest volume?

c Which planet has the smallest volume?

d Does the planet with the largest volume also have the largest surface area? Explain why this may or may not be the case.

e How many times greater is Earth's volume compared to the volume of Mercury?

f How many times greater is Jupiter's surface area compared to Earth's surface area?

g Approximately 71% of the Earth's surface is covered by water. To the nearest square kilometre, how much of the Earth's surface area is covered by water?

9 The surface area of the rind of this blood orange is 116.9 cm^2.

a What is the diameter of the orange?

b The thickness of the rind is approximately 1 cm. What amount of space does the edible part of the orange occupy?

10 A balloon is inflated to become a sphere with a radius of 9.5 cm.

a What is the volume of the balloon?

b Two more balloons are inflated to exactly the same size as the first balloon. What is the total volume of the three balloons?

When objects are filled with a liquid or gas, you can refer to their capacity rather than volume.

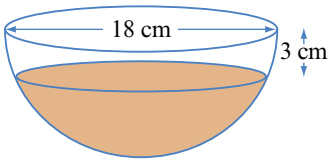
c What is the capacity of the three balloons in litres?

d If the rate that the balloons can be inflated is 2.5 L/s, how long will it take to inflate all three balloons?

e What assumptions do you need to make when answering part **d**?

PROBLEM SOLVING AND REASONING

11 A hemispherical soup bowl has a diameter of 18 cm.



- **a** Find the surface area of the inside of the bowl.
- **b** A circular cover is placed over the bowl to keep the soup warm. What is the total surface area of the inside of the bowl with its cover?
- **c** Find the capacity of the bowl if it was filled to the brim with soup.
- **d** When it is being served, however, the soup only fills to 3 cm below the edge of the bowl. Calculate the amount of soup in the bowl when it is served at the table.
- **e** What percentage of the bowl does not contain soup?

12 A ping pong ball has a diameter of 40 mm.

- **a** Find its volume.

It is also possible to find the **density** of a solid. Density is defined as the mass per unit volume. Use the formula: density $= \frac{\text{mass}}{\text{volume}}$.

- **b** If the mass of the ping pong ball is 2.7 g, calculate its density.

CHALLENGE

13 Investigate the effect that each situation has on:

- **i** the total surface area of a sphere
- **ii** the volume of a sphere.

- **a** doubling the radius
- **b** tripling the radius
- **c** halving the radius
- **d** dividing the radius by 3.

14 The balloon part of a modern hot-air balloon is called an envelope and is constructed from lightweight yet strong synthetic fabric such as Dacron.

- **a** Assuming that the envelope can be treated as if it is a perfect sphere with a radius of 6.7 m, what amount of Dacron would be required to construct the envelope?
- **b** If Dacron costs $45 per square metre, calculate the cost of purchasing enough Dacron to construct the envelope.
- **c** Calculate the amount of hot air the envelope can hold. Give your answer to the nearest cubic metre.
- **d** The amount of hot air needs to be reduced by one third.
  - **i** What is the volume of the balloon when the hot air has been reduced?
  - **ii** What is the radius of the balloon now?
- **e** Why might it be necessary to reduce (or increase) the amount of hot air in a hot-air balloon?

Reflect

How does increasing or decreasing the radius of a sphere by different scale factors affect its volume?

9H Surface area and volume of composite solids

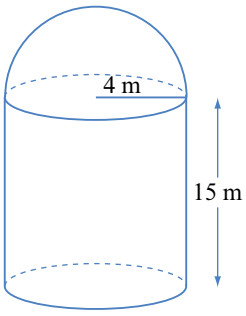
Start thinking!

Many buildings and structures are composed of more than one solid. When an object consists of more than one solid it is called a composite solid.

1 Can you recognise and name the building in the photograph? How many solids can you identify in its construction? List them.

You can find the total surface area of composite solids by using our knowledge of the formulas for the total surface area of simple solids.

2 The building shown at right is to have its entire surface painted (except for the surface sitting on the ground). Identify the two solids used to make it.



3 The lower part of the building is in the shape of a cylinder.

- **a** Write the formula for finding the total surface area of a cylinder.
- **b** Do you need to find the surface area of the entire cylinder in this case? Why or why not?
- **c** Adjust the formula for the total surface area of a cylinder and hence calculate the surface area for the lower part of the building, correct to one decimal place.

4 The upper part of the building is a hemisphere.

- **a** Explain how can you adapt the formula for finding the total surface area of a sphere to help you find the surface area of this hemisphere.
- **b** Calculate the surface area of the upper part of the building, correct to one decimal place.

5 Use your answers to questions **3** and **4** to calculate the surface area to be painted.

6 Use the approach shown in questions **3**–**5** to help you find the volume of the building.

KEY IDEAS

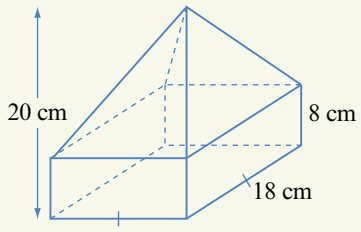
- A composite solid consists of two or more simple solids.
- To calculate the total surface area of a composite solid, identify the individual component solids and adapt their formulas to find the surface area required. Add or subtract the individual surface areas to find the total surface area of the composite solid.
- To calculate the volume of a composite solid, identify the individual component solids and calculate their volumes. Add or subtract the individual volumes to find the volume of the composite solid.

EXERCISE 9H Surface area and volume of composite solids

Where appropriate, write answers correct to one decimal place.

EXAMPLE 9H-1 Calculating the surface area of a composite solid (addition)

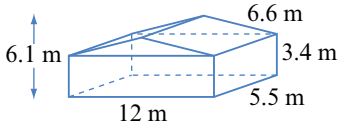
Calculate the surface area of this composite solid.



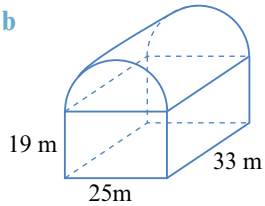
THINK	WRITE
1 Identify the two parts of the composite solid.	Composite solid made up of a rectangular prism and a square-based pyramid.
2 Write the formula for the total surface area of a rectangular prism.	rectangular prism: TSA = $2(lh + lw + wh)$
3 Adjust the formula to find the required surface area (SA_1). You do not need the top of the rectangular prism.	$SA_1 = lw + 2(lh + wh)$
4 Identify the length, width and height of the rectangular prism and substitute into the adjusted formula.	$l = 18$ cm, $w = 18$ cm, $H = 8$ cm $SA_1 = 18 \times 18 + 2(18 \times 8 + 18 \times 8)$ $= 900$ cm^2
5 Write the formula for the total surface area of a square-based pyramid.	square-based pyramid: TSA = $l^2 + 2bh$
6 Adjust the formula to find the surface area (SA_2) of the triangular faces only.	$SA_2 = 2bh$
7 Identify the base and calculate the height (using Pythagoras' Theorem) of the triangular faces and substitute into the adjusted formula.	$b = 18$ cm, $h = \sqrt{12^2 + 9^2} = 15$ cm $SA_2 = 2 \times 18 \times 15$ $= 540$ cm^2
8 Add the areas and include the appropriate unit.	TSA of composite solid $= SA_1 + SA_2$ $= 900 + 540$ $= 1440$ cm^2

1 Calculate the surface area of each composite solid.

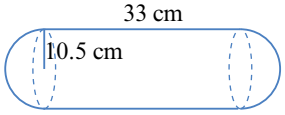
a



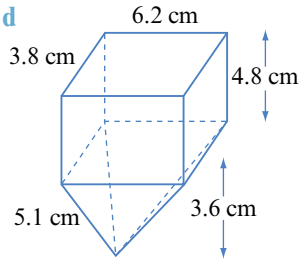
b



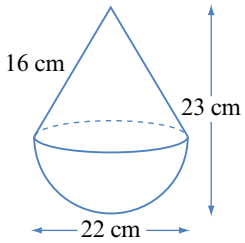
c



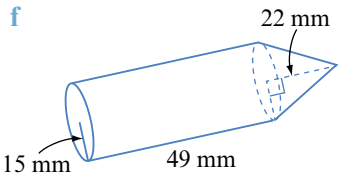
d



e



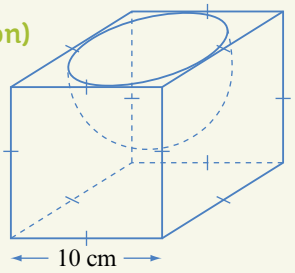
f



EXAMPLE 9H-2

Calculating the total surface area of a composite solid (subtraction and addition)

Calculate the total surface area of this composite solid.



THINK

1 Identify the two parts of the composite solid.

2 Write the formula for the TSA of a cube.

3 Adjust the formula by subtracting the area of the base of the hemisphere from the TSA of the cube (SA_1).

4 Identify the length of the cube and the radius of the hemisphere, and substitute into the adjusted formula.

5 Write the formula for the surface area of an open hemisphere (SA_2) and substitute for r.

6 Add the areas and include the appropriate unit.

WRITE

Composite solid is made up of a cube with an open hemisphere taken out.

Cube: TSA = $6l^2$

$SA_1 = 6l^2 - \pi r^2$

$l = 10$ cm, $r = 10 \div 2 = 5$ cm

$SA_1 = 6 \times 10^2 - \pi \times 5^2$

$= (600 - 25\pi)$ cm^2

open hemisphere: $SA_2 = 2\pi r^2$

$= 2 \times \pi \times 5^2$

$= 50\pi$ cm^2

TSA of composite solid = $SA_1 + SA_2$

$= 600 - 25\pi + 50\pi$

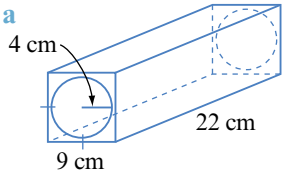
$= 600 + 25\pi$

$= 678.539\,81\ldots$

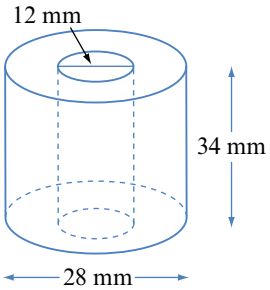
≈ 678.5 cm^2

2 Calculate the total surface area of each object.

a



b



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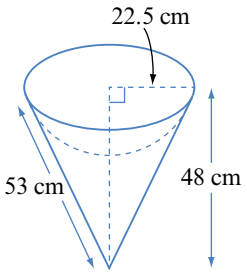
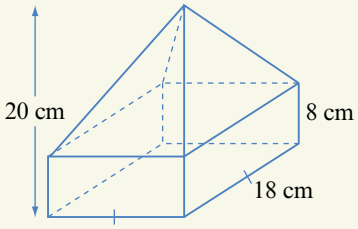


EXAMPLE 9H-3 Calculating the volume of a composite solid

Calculate the volume of this composite solid.



THINK	WRITE
1 Identify the two parts of the composite solid.	Composite solid made up of a rectangular prism and a square-based pyramid.
2 Write the formula for the volume of a rectangular prism.	rectangular prism: $V = V_1 = lwH$
3 Identify the length, width and height of the rectangular prism and calculate its volume.	$l = 18$ cm, $w = 18$ cm, $H = 8$ cm $V_1 = 18 \times 18 \times 8$ $= 2592$ cm^3
4 Write the formula for the volume of a square-based pyramid.	square-based pyramid: $V = V_2 = \frac{1}{3}l^2H$
5 Identify the length of the base of the pyramid and its height. Calculate its volume.	$l = 18$ cm, $H = 20 - 8 = 12$ cm $V_2 = \frac{1}{3} \times 18^2 \times 12$ $= 1296$ cm^3
6 Add the volumes and include the appropriate unit.	volume of composite solid $= V_1 + V_2$ $= 2592 + 1296$ $= 3888$ cm^3

UNDERSTANDING AND FLUENCY

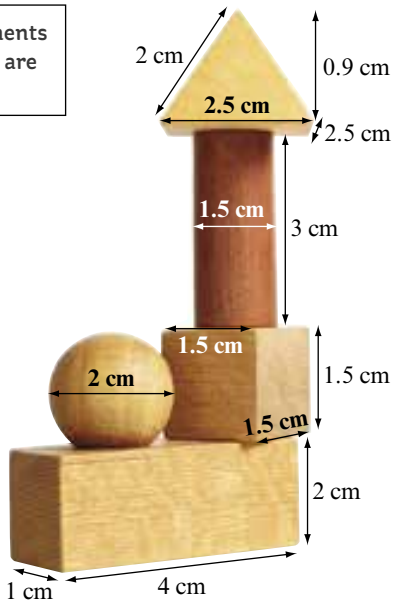
3 Calculate the volume of each composite solid in question **1**.

4 Calculate the volume of each object in question **2**.

NOTE Measurements on curved shapes are diameters.

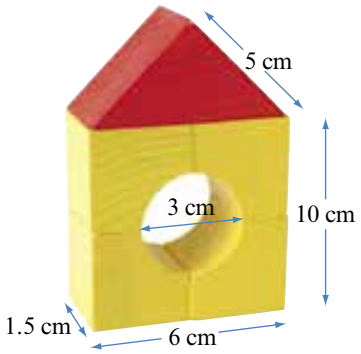
5 Each of the wooden blocks in the photo is to be separately painted.

- **a** Identify each solid and write the formula for its total surface area.
- **b** Calculate the total surface area of each solid and hence the total surface area which needs to be painted.
- **c** Calculate the volume of each solid and hence the total volume of wood used in the blocks.



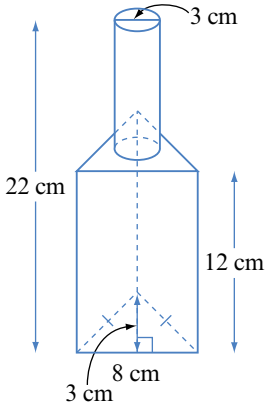
6 There are three solids that can be identified in this toy house, which has been constructed from different shaped blocks.

- **a** Identify them.
- **b** Explain how you can calculate the total surface area of the toy house.
- **c** Find the total surface area of the toy house.
- **d** Find the volume of wood used to make this toy house.



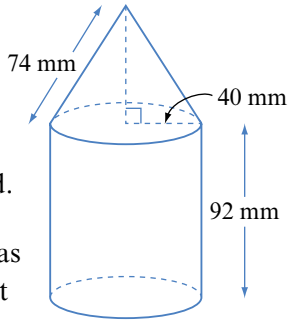
7 Rebecca buys an ice-cream cone. As well as the ice cream on top of the cone, the cone is completely filled with ice cream. Calculate the total amount of ice cream Rebecca has. Give your answer to the nearest millilitre. (Hint: assume the ice-cream forms a perfect hemisphere.)

8 A triangular prism has a cylinder sitting on it (shown far right). Calculate its total surface area. (Hint: which part of the cylinder and triangular prism are not exposed?)



9 **a** Calculate the total surface area of this solid.

- **b** Which dimension is needed to also calculate the volume of this solid?
- **c** Use Pythagoras' Theorem to find the unknown length and hence calculate the volume of the solid.



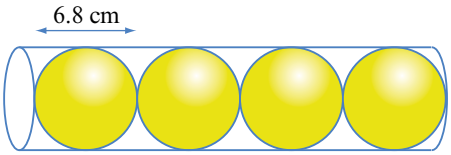
10 The coconut has a radius of 15 cm. The basketball has a radius of 12 cm. What volume of the coconut is not occupied by the basketball?

11 Each brick in this image is equal in size and measures 23 cm by 11.5 cm by 7.5 cm. The diameter of each hole is 2.5 cm. Calculate the volume of clay required to make the five bricks.

12 This pyramid is 8 cm high and has a base length of 9 cm and width of 5 cm. A coin of diameter 3 cm and thickness 0.2 cm sits in the centre of the pyramid. What percentage of space does the coin occupy within the pyramid?

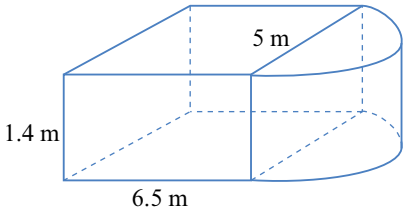
PROBLEM SOLVING AND REASONING

13 Tennis balls are sold in canisters. Each canister holds four tennis balls.



- **a** If each tennis ball has a diameter of 6.8 cm, calculate the volume of one tennis ball.
- **b** What is the volume of all four tennis balls?
- **c** Determine the dimensions of the canister and find its volume. (Assume that the tennis balls fit perfectly into the canister.)
- **d** What amount of space is not taken up by tennis balls in the canister?
- **e** If the tennis balls were to be sold in a container with the same dimensions but in the shape of a rectangular prism, would there be more or less unused space?

14 A swimming pool is the shape of a rectangular prism. A spa is built onto one end of the pool, as shown in the figure.



- **a** What amount of water is required if the pool and spa are filled to capacity? Give your answer to the nearest litre.
- **b** To make the pool and spa functional the water levels must be below the top edge. If the pool is filled to 70% of its depth and the spa is filled to 60% of its depth, what amount of water is required, to the nearest litre?

CHALLENGE

15 Seven cylinders with the same height but different diameters are stacked together as shown. The top cylinder has a diameter of 1 cm and the diameter of each cylinder increases by 1 cm. The bottom cylinder has a diameter of 7 cm. If each cylinder has the same height, 1.5 cm, calculate the total surface area when the cylinders are sitting one on top of the other.

16 These nuts have the same dimensions but different shapes. One is circular and the other is in the shape of a regular hexagon. They both have a depth of 6.2 mm.

- **a** What amount of steel is required to manufacture each nut?
- **b** Which nut requires the greater amount of steel?

Reflect

How can you use the volume of individual solids to determine the volume of a composite solid?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

perimeter	area	three dimensional (3D)	volume	right pyramid
circumference	circle	prism	capacity	oblique pyramid
diameter	rectangle	cross-section	composite solid	truncated pyramid
radius	square	net	pyramid	apex
semi-circle	parallelogram	total surface area	tetrahedron	sphere
sector	kite	rectangular prism	cone	hemisphere
quadrant	trapezium	cube	truncated cone	density
Pythagoras' Theorem	annulus	triangular prism	slant edge	
composite shapes	two dimensional (2D)	cylinder	arc	

MULTIPLE-CHOICE

9A **1** What is the circumference of a circle with a radius of 27.5 mm closest to?

A 90 mm B 170 mm
C 180 mm D 2400 mm

9B **2** What is 100 hectares written in square metres?

A 0.01 m^2 B 1 m^2
C 10 000 m^2 D 1 000 000 m^2

9B **3** A quadrant has a radius of 10 cm. What is its area closest to?

A 70 cm^2 B 80 cm^2
C 310 cm^2 D 320 cm^2

9C **4** What is the TSA of a cube with side lengths of 8 cm?

A 384 cm^2 B 48 cm^2
C 512 cm^2 D 288 cm^2

9D **5** A cylinder has a base diameter of 6 cm and a height of 20 cm. What is its volume closest to?

A 188 cm^3 B 377 cm^3
C 565 cm^3 D 2262 cm^3

9E 10A **6** How many triangular faces does a tetrahedron have?

A 3 B 4 C 5 D 6

9F 10A **7** A square-based pyramid has the same base length and height as a cube with side lengths of 6 cm. What is the volume of the pyramid?

A 216 cm^3 B 72 cm^3
C 108 cm^3 D 144 cm^3

9F 10A **8** A cone with a diameter of 2.5 cm and a height of 10 cm has a capacity closest to which of the following?

A 15 mL B 50 mL
C 65 mL D 200 mL

9G 10A **9** A solid sphere with a diameter of 25 cm is sliced in half. What is its total exposed surface area now closest to?

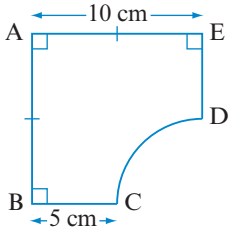
A 500 cm^2 B 1500 cm^2
C 4000 cm^2 D 6000 cm^2

9G 10A **10** A solid wooden cylinder of radius 10 cm and height 10 cm has a cylinder of radius 2 cm drilled from the centre. What is the volume of timber remaining (in cubic centimetres)?

A 640π B 1000π
C 960π D 40π

SHORT ANSWER

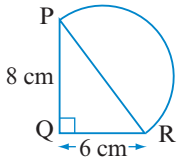
9A **1** Consider this shape.



a Determine the length of the arc CD.

c What is the perimeter of the figure?

9A **2** Find the perimeter of this shape, made of a triangle and a semi-circle.



9A **3** A flat metal washer is in the form of an annulus. The radius of the inner ring of the washer is 1 cm, while the outer ring has a radius of 1.5 cm. How much longer is the outer circumference than the inner circumference?

9B **4** Determine the area of the shape shown in:

a question 1 **b** question 2.

9B **5** Calculate the area of the upper surface of the metal washer in question 3.

9C **6** The base of a triangular prism is a right-angled triangle with a base length of 25 mm and a height of 18 mm. The height of the prism is 30 mm. What is the TSA of the prism?

9C **7** A set of wooden cubes was made for a playground. The smallest cube had a side length of 40 cm, the next sized cube had side lengths twice as long, while the side lengths on the largest cube were twice as long as those on the middle cube.

a Calculate the total surface area of each cube.

b If 1 L of paint covers an area of 15 m^2, and the paint is only available in 5-L cans, how many cans of paint would be required to paint the three cubes?

9C **8** An open can of baked beans has a cylindrical shape with a base diameter of 7 cm and a height of 10 cm. A label is placed around the circular surface of the can. Find the total outer surface area of the open can.

9D **9** Calculate the volume of each cube in question 7.

9D **10** A water hose has an internal diameter of 14 mm.

a What is the capacity, in litres, of a 1-m length of hose?

b What length of pipe would contain 1 L?

9E 10A **11** A square-based pyramid has a base with a side length of 5 cm. The height of each triangular face is 20 cm. Calculate its total surface area.

9E 10A **12** The diameter of a closed cone is 10 cm and the height is 16 cm. Calculate the total surface area of the cone.

9F 10A **13** A square-based pyramid has a base length of 30 cm. The triangular faces are equilateral in shape. Find the volume of the pyramid.

9G 10A **14** A sphere fits neatly inside a cylinder with a base radius of 8 cm. Find the volume of the cylinder and the sphere in terms of π, and compare them.

9G 10A **15** Compare the surface area of the sphere and cylinder in question 14.

9H 10A **16** **a** Calculate the total surface area of:

i a solid hemisphere with a diameter of 40 cm

ii a cube with side length of 40 cm.

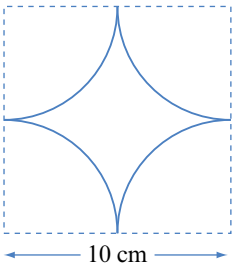
b The hemisphere is placed on top of the cube. What is the total surface area of this object?

MIXED PRACTICE

1 A closed cylinder has a base diameter of 15 cm and a height of 15 cm. What is the total surface area of the cylinder closest to?

A 700 cm^2 **B** 750 cm^2
C 1000 cm^2 **D** 2800 cm^2

2 This 10-cm square piece of paper has equal-sized quadrants removed from its four vertices.
What is the area of the paper remaining? Give your answer correct to the nearest square centimetre.



3 A sector has a radius of 12 cm and the angle between the two radii is 37°. Find the perimeter of the sector, correct to the nearest centimetre.

4 Find the area of the sector in question **3**, correct to the nearest square centimetre.

5 1.5 m^3 is equivalent to how many litres?

A 1.5 **B** 15 **C** 150 **D** 1500

6 A rectangular prism has dimensions 4 cm by 5 cm by 6 cm. Calculate its total surface area.

7 A semi-circle has a radius of 8.5 cm. What is its perimeter closest to?

A 44 cm **B** 27 cm **C** 70 cm **D** 43 cm

8 Find the area of the semi-circle in question **7**, to the nearest square centimetre.

9 A fish tank is 45 cm long, 20 cm wide and 50 cm tall. Water is added until it comes up to three-quarters the height of the tank. How many litres of water are in the tank?

10 How many faces are there on the surface of a triangular prism?

A 3 **B** 4 **C** 5 **D** 6

11 Describe what happens to the area of a two-dimensional shape when its dimensions are multiplied by a factor of n.

12 A cylinder has a diameter of 18 cm and a height of 12 cm. Calculate the volume of the cylinder to the nearest cubic centimetre.

13 A cube can hold 8 L of water when filled to the top. What is the side length of the cube?

A 2 m **B** 0.2 m
C 2 cm **D** 0.2 cm

14 The perimeter of a sector with a radius of 2 cm is 5.6 cm. What is the angle between the two radii of the sector closest to?

A 40° **B** 45° **C** 90° **D** 180°

10A **15** A square-based pyramid has a total surface area of 224 cm^2. If the length of the base is 8 cm, what is the height of each triangular face?

A 56 cm **B** 14 cm **C** 10.5 cm **D** 10 cm

10A **16** What is the volume of a cone compared with the volume of a pyramid with the same base area and height?

A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** the same
D unknown without dimensions

10A **17** What is the formula for the volume of a hemisphere?

A $4\pi r^2$ **B** $3\pi r^2$ **C** $\frac{2}{3}\pi r^3$ **D** $\frac{4}{3}\pi r^3$

10A **18** Convert 5000 cm^3 to cubic metres.

10A **19** Write the formula for the total surface area of a solid hemisphere.

10A **20** A cone has a radius of 22 mm and a height of 45 mm. What is its slant height, to the nearest millimetre?

A 50 mm **B** 39 mm
C 23 mm **D** 67 mm

10A **21** What is the volume of the cone in question **20**?

10A **22** To convert from units of cubic millimetres to cubic centimetres what would you divide by?

A 10 **B** 100 **C** 1000 **D** 10 000

10A **23** If the radius of a sphere is doubled, what happens to its volume?

10A **24** What is the difference in shape between a rectangular prism and a rectangular pyramid?

10A **25** Draw the net of a closed cone with a base diameter of 20 cm and a height of 15 cm. Label the diagram with as many measurements as you can.

ANALYSIS

Ethan bought this set of wooden blocks for his young cousins.

The set contains a sphere, a square pyramid, a triangular prism, a cube, a rectangular prism, a cone and a cylinder.

a Name the solids in order from left to right.

b The cube has side lengths of 3 cm. Calculate its:

i total surface area **ii** volume.

c The rectangular prism has a base the same size as the cube, and is twice the height of the cube.

i Compare the surface area of the rectangular prism with that of the cube.

ii Compare the volume of the rectangular prism with that of the cube.

d The triangular prism is actually the rectangular prism cut vertically along its base diagonal.

i Draw a labelled diagram showing the shape of its base.

ii Calculate its total surface area.

iii Is the TSA of the triangular prism half that of the rectangular prism? Explain.

iv Compare the volume of the triangular prism and the rectangular prism.

e The base of the cylinder has a diameter the same length as the side of the cube, and is the same height as the rectangular prism. Calculate its:

i total surface area **ii** volume.

f Place the solids discussed in parts **b**–**e** in increasing order of:

i surface area **ii** volume.

10A **g** The square base of the pyramid is the same shape as that of the base of the cube. Its height is the same as that of the rectangular prism. Calculate its:

i total surface area **ii** volume.

10A **h** The cone has the same base diameter and height as the cylinder. Calculate its:

i total surface area **ii** volume.

10A **i** The sphere has the same diameter as the cone. Calculate its:

i total surface area **ii** volume.

10A **j** Using your answers for parts **g**, **h** and **i**, create a comparison summary from the lists in part **f**.

CONNECT

Setting up a small farm

Imagine setting up a small farm where you will grow two types of grain. What do you need to consider when planning the farm?

Here is some information to work with:

The land has an area of 50 hectares and is rectangular, with one side bordering a river.

The farm is to be divided into three sections: one section to grow wheat, one to grow canola and the other to contain your home, shed, water tanks and silo.

You have ordered a wheat silo which is in the shape of a cylinder with a conical tip as shown in the photo at right.

The base has a diameter of 4.36 m. the cylinder has a height of 6.58 m and the cone has a height of 1.3 m.

Your task

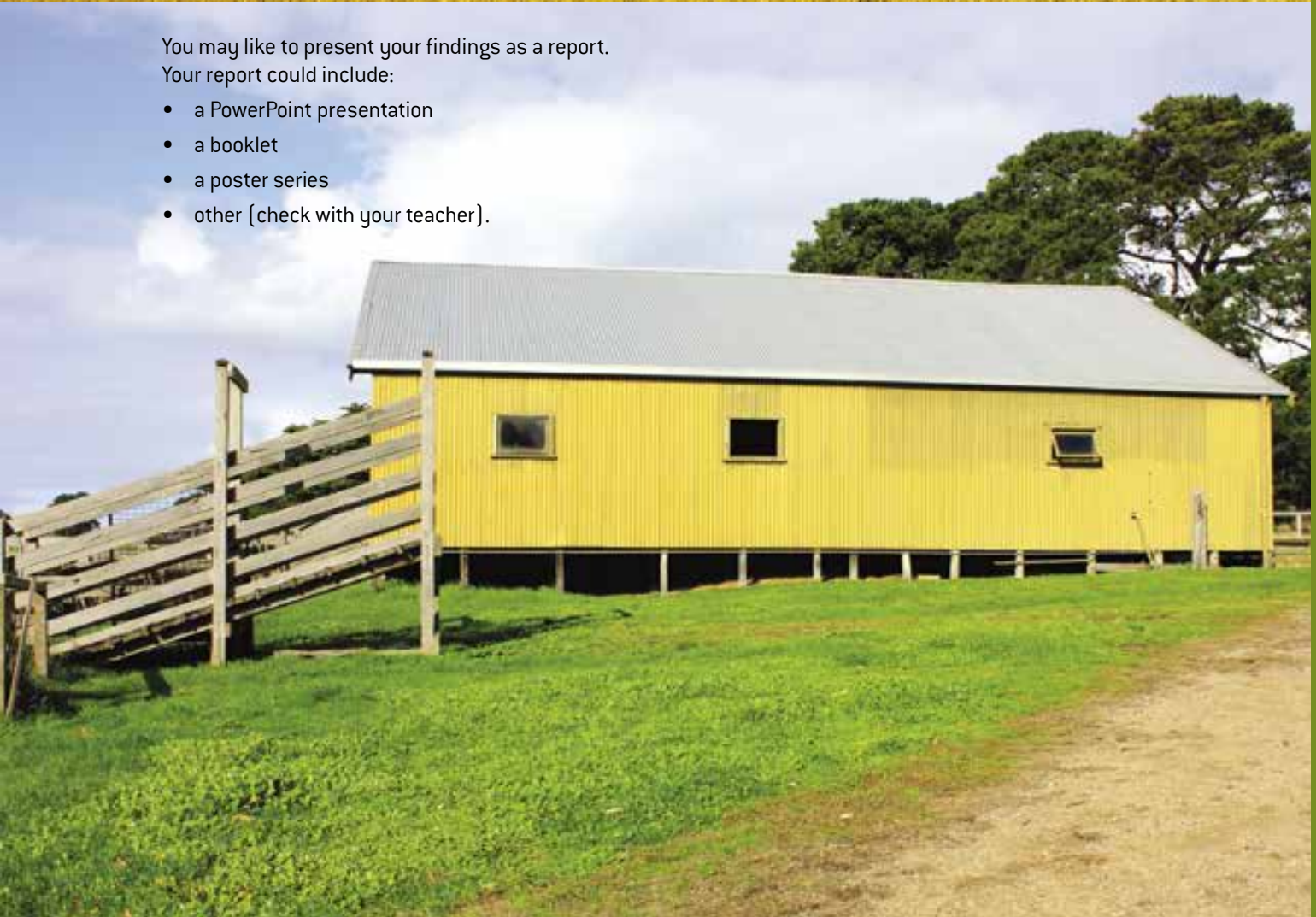
You are to:

- work out a possible set of dimensions for your land
- calculate the amount and estimated cost of fencing needed to border your land
- decide on the best size and location for the wheat field and the canola field which still allows enough space for the home paddock
- work out the area of usable land for growing wheat and canola if there must be a 3-metre border inside the perimeter of each field for movement of equipment
- design a machinery shed in the shape of a rectangular prism
- estimate the amount of paint needed and the cost of painting the shed
- compare the capacity of different water tanks (prisms, cylinders, cones and pyramids) that have a length or diameter of 3.6 m and a height of 3.9 m
- calculate the volume of wheat held by the silo if you can only fill it to 70% capacity
- draw a scale plan of the aerial view of your farm, showing the three sections and other main features
- explore and investigate other ways that measurement is used on farms.

Include all necessary working to justify your answers.

You may like to present your findings as a report.
Your report could include:

- a PowerPoint presentation
- a booklet
- a poster series
- other (check with your teacher).



10 STATISTICS

ESSENTIAL QUESTION

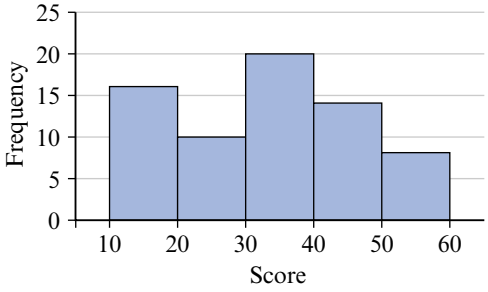
How do the media use statistics to report on facts and issues?

Questions 1 and 2 refer to this data set.
3, 7, 4, 9, 5, 4, 16, 2, 5, 5, 8, 6

10A 1 What is the median of the data set?

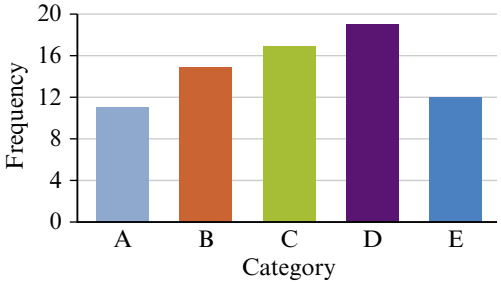
10A 2 What is the mean of the data set?

10A 3 What is the most common class interval in this histogram?



A 10–<20 B 30–<40
C 20 D 35

10A 4 How many people were surveyed to produce this column graph?

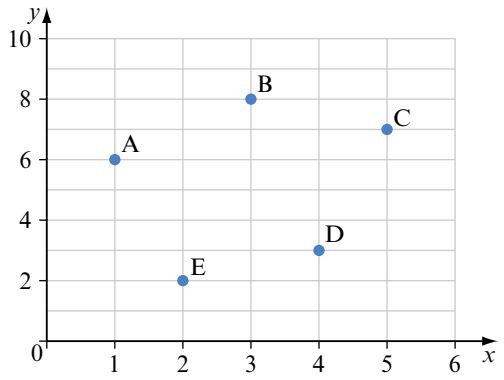


10B 5 What is the range of the data set shown in question 1?

10B 6 What is the range of this data set?

x	f
15	4
16	3
17	2
18	5
21	1

10E 7 What are the coordinates of the point C in this scatterplot?



10H 8 Which of these is *not* an example of numerical data?

A length of a running track
B coming first, second or third in a race
C number of pets in a home
D weight of chocolate in a single bar

10H 9 What is surveying every fifth person at a supermarket?

A stratified sampling
B random sampling
C location sampling
D systematic sampling

10H 10 Which of these surveys is *not* a census?

A surveying everybody in your class to find the year level's opinion of a new school rule
B surveying everybody in your family to find out what to cook for dinner
C surveying your classmates on who should be class captain
D surveying everybody in your school to find out the school's opinion on a community issue

10A Measures of centre

Start thinking!

A data set can be described by its centre. The three common measures of centre are the **mean** (μ), **median** and **mode**. Consider this data which Ava and Lily collected on the ages of people at a skate park.

16	14	18	21	21	16	16	15	15	14
8	22	45	22	18	14	14	18	21	17

1 Find the mean, median and mode of this data set and hence state what you think is the average age of the people at the skate park.

It is rare that you are given data in a raw list. Usually it is presented in a table.

2 Create a **frequency table** to represent the data that Ava and Lily collected.

It is easy to read the mode straight from the table. In order to find the mean and median, you need to add two more columns to the frequency table. Consider this incomplete frequency table.

Score (x)	Frequency (f)	$x \times f$	cf
8	1	$8 \times 1 = 8$	1
14	4		5
15	2		

3 How is the score × frequency column ($x \times f$) calculated?

4 Explain why finding the total of this column is the same as adding up all the scores in the list.

5 Copy and complete the first three columns of the frequency table.

6 Add a total row and use the two totals (in the second and third columns) to find the mean. How does this compare to your answer to question **1**?

7 Complete the **cumulative frequency** column (cf) of the table. How are these numbers found? (Hint: 'cumulative' means increasing by repeated additions.)

8 Knowing that, for n pieces of data, the median is the $\frac{n+1}{2}$th number, find the median using the cf column.

You can also calculate statistics using tables with **class intervals**.

KEY IDEAS

- To find the mean from a table of data, add a score × frequency column to the table.
- To find the median from a table of data, add a cumulative frequency column to the data.
- To find the mean or median from a table of grouped data, use the midpoint of the class interval as the value of the score.
- The mean is represented by the Greek letter mu, μ.
- A piece of data that is very different from the rest of a data set is called an **outlier**.

EXERCISE 10A Measures of centre

EXAMPLE 10A-1 Finding measures of centre for data in a list

Find the mean, median, and mode for this data set. 11, 15, 9, 16, 14, 9, 7, 13, 13, 8

THINK

1 To find the mean, add all the numbers and divide the total by how many numbers there are.

2 To find the median, rearrange the numbers in order and find the middle number. If the set has an even number of data, find the average of the two middle numbers.

3 To find the mode, look at the ordered number list and state the most common number(s).

WRITE

$$\frac{11 + 15 + 9 + 16 + 14 + 9 + 7 + 13 + 13 + 8}{10}$$

The mean is 11.5.

7, 8, 9, 9, 11, 13, 13, 14, 15, 16

The median is $\frac{11 + 13}{2} = 12$

The modes are 9 and 13.

1 Find the mean, median and mode for each data set.

a 3, 8, 4, 9, 2, 5, 5, 3, 8, 9, 7

b 5, 8, 19, 56, 11, 13, 2, 7, 7, 9, 13, 16, 6

c 9, 3, 5, 18, 15, 25, 23, 19, 13, 17, 6, 18, 17, 5

d 110, 234, 76, 134, 129, 117, 89, 96, 135, 107

e 6.9, 7.5, 2.2, 8.5, 4.6, 4.9, 8.7, 4.2, 5.1, 6.0, 1.3, 4.5, 6.9, 7.5, 3.3, 8.8, 4.9

f 11, 19, 18, 16, 23, 27, 15, 81, 12, 16, 14, 13, 18, 21, 25, 26, 27, 22, 19, 12, 13

EXAMPLE 10A-2 Finding the mean for data in a table

Find the mean for this data.

Score (x)	Frequency (f)
8	13
9	9
10	3

THINK

1 To prepare for calculations add a 'score × frequency' column ($x \times f$) and a 'total' row.

2 To find the mean, divide the $x \times f$ total by the f total.

WRITE

x	f	$x \times f$
8	13	104
9	9	81
10	3	30
Total	25	215

$\mu = 215 \div 25 = 8.6$

UNDERSTANDING AND FLUENCY

2 Find the mean for each data set.

a

Score	Frequency
1	6
2	7
3	10

b

Score	Frequency
10	9
20	3
30	7
40	11

c

Score	Frequency
14	12
15	16
16	18
17	10

d

Score	Frequency
1	4
2	9
3	14
4	11
5	7

e

Score	Frequency
8	6
9	11
10	19
11	23
12	20

f

Score	Frequency
0	8
2	19
4	12
6	9
8	2

EXAMPLE 10A-3

Finding the mean for data in a table with class intervals

Find the mean for this data.

Class interval	Frequency (f)
0–<10	8
10–<20	5
20–<30	7

THINK

1. To calculate statistics from a table with class intervals, first find the midpoint of each interval. Each class interval spans 10 scores, and half of this is 5.
2. To prepare for calculations, add a midpoint column (x), a 'score × frequency' column ($x \times f$) and a total row.
3. To find the mean, divide the $x \times f$ total by the f total.

WRITE

Class interval	f	x	$x \times f$
0–<10	8	5	40
10–<20	5	15	75
20–<30	7	25	175
Total	20		290

mean = 290 ÷ 20
= 14.5

3 Find the mean for each data set.

a

Class interval	Frequency
0–<10	3
10–<20	8
20–<30	7
30–<40	2

b

Class interval	Frequency
0–<5	13
5–<10	12
10–<15	15

c

Class interval	Frequency
20–<24	6
24–<28	7
28–<32	9
32–<36	18

d

Class interval	Frequency
0–<20	8
20–<40	11
40–<60	5
60–<80	16
80–<100	10

e

Class interval	Frequency
160–<162	4
162–<164	9
164–<166	11
166–<168	12
168–<170	8

f

Class interval	Frequency
0–<5	16
5–<10	22
10–<15	20
15–<20	19
20–<25	11

EXAMPLE 10A-4 Finding the median for data in a table

Find the median for the data shown in Example 10A-2.

THINK

1 To prepare for calculations, add a cumulative frequency column (cf).

2 To find the median, locate the $\frac{n+1}{2}$th score, where $n = 25$.

3 The 13th score will be in the row containing scores of 8.

WRITE

x	f	cf
8	13	13
9	9	22
10	3	25

$\text{median} = \frac{25+1}{2}\text{th score}$
$= 13\text{th score}$

median = 8

4 Find the median for the data shown in each table in question **2**.

EXAMPLE 10A-5 Finding the median for data in a table with class intervals

Find the median for the data shown in Example 10A-3.

THINK

1 First find the midpoint of each interval. Each class interval spans 10 scores, and half of this is 5.

2 Add a midpoint column (x) and a cumulative frequency column (cf).

3 To find the median class, find the $\frac{n+1}{2}$th score, where $n = 20$.

4 The 11.5th score will be in the row containing scores 10–<20. Use the midpoint 15.

WRITE

Class interval	f	x	cf
0–<10	8	5	8
10–<20	5	15	13
20–<30	7	25	20

$\text{median} = \frac{20+1}{2}\text{th score}$
$= 11.5\text{th score, or halfway between the 11th and 12th scores}$

median = 15

5 Find the median for the data shown in each table in question **3**.

6 Find the mode for the data shown in each table in question **2**.

> **NOTE** Remember that the mode is the row with the most scores.

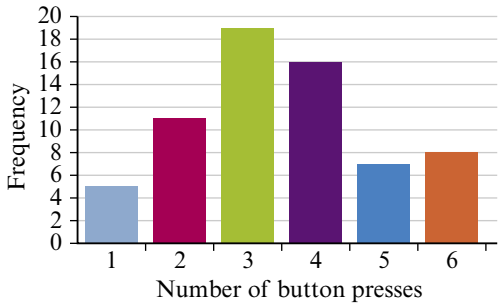
7 Find the modal class for the data shown in each table in question **3**.

8 Jordan collected this data on the number of pets in the family home. What would you say is the average number of pets in a home?

1	2	2	1	6	2	1	3	4	1
18	3	4	4	1	1	5	4	2	2

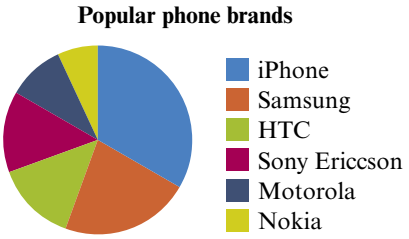
> **NOTE** Don't forget to consider whether an answer has to be a whole number.

9 This **column graph** shows the number of times a person presses a button for a green pedestrian light to cross the road safely.



- **a** Create a frequency table that displays the data represented in the graph.
- **b** Using your frequency table, find the mean, median and mode.
- **c** What would you say is the average number of button presses?

10 This **pie graph** shows the popularity of some mobile phone brands.



- **a** What is the most common mobile phone brand? What statistic is this?
- **b** Why is this the only measure of centre you can find? Give two reasons.

11 Consider the data in this table on the ages of people in a netball team.

Score	Frequency
11	1
15	3
16	6
17	8

- **a** Find the mean, median and mode.
- **b** What would you say is the average age of the netball team?

12 This data gives the number of cars in a car park over a 48-hour period.

12	18	31	46	48	50	67	58	77	82
98	97	98	76	72	47	47	31	22	17
14	13	13	19	24	28	34	51	59	64
66	67	87	96	94	82	77	59	48	42
42	33	27	19	18	13	13	12		

- **a** Calculate the mean, median and mode.
- **b** Create a frequency table with class intervals of 5 and calculate the mean, median and modal class.
- **c** Create a frequency table with class intervals of 10 and recalculate the mean, median and modal class.

PROBLEM SOLVING AND REASONING

d Create a frequency table with class intervals of 20 and recalculate the mean, median and modal class.

e Compare your answers from parts **a**, **b**, **c** and **d**. What do you notice about the values of each measure of centre as the size of class intervals is increased?

13 Consider this data set, on the ages of people in a classroom.

15 16 15 16 14 17 15 16 16 14 13 16 17
16 15 14 15 15 16 13 15 16 15 16 68

a Calculate the mean, median and mode.

b Why is there a difference between the median and mode, and the mean?

The unusual piece of data that affects the mean but not the median or mode is an outlier.

c Explain why, when there is an outlier in a data set, the median is the best measure of centre.

14 For each of these data sets:

i calculate the mean, median and mode

ii decide whether the mean or the median is the better measure of centre

iii describe the centre of the data set.

a 4, 7, 2, 5, 7, 8, 4, 2, 67, 3, 6, 4, 7, 4, 4, 9, 8, 8, 3, 5

b 23, 74, 36, 54, 19, 22, 47, 88, 33, 42, 26, 77, 55, 49, 29, 61, 18, 44, 75, 31

c 189, 172, 191, 176, 188, 184, 183, 172, 129, 178, 184, 183, 187, 186, 179, 186, 184, 181, 180, 176

15 By creating a frequency table or otherwise, find the mean, median and modal class of this **histogram** and hence describe the centre of the data.

CHALLENGE

16 The prices for various items were collected from three different milk bars. Use your understanding of measures of centre to compare the prices and decide at which milk bar you would prefer to shop.

Item	Gray St Shop	Doey's	The Corner
Chocolate bar	$2.20	$2.50	$2.30
2 L milk	$2.80	$1.90	$2.50
Bottle of juice	$1.90	$3.00	$2.00
Salad roll	$4.50	$5.00	$6.00
Ice cream	$2.50	$3.00	$3.50
Tin of corn	$1.50	$2.90	$2.50
Fried dim sim	$0.90	$0.80	$0.70

Reflect

When might it be appropriate to use each measure of centre?

10B Measures of spread

Start thinking!

Another way to describe data is to describe how it is spread. You should already be familiar with the most basic measure of **spread**, the **range**.
Consider this data Finn collected on the number of people in a household.
2, 5, 2, 7, 3, 11, 4, 6, 5, 5, 4, 5, 3, 4, 8

1 Order the data set and hence find the range.

2 Explain how you found the range.

Another measure of spread is the **interquartile range**, or IQR. The IQR measures the middle 50% of the data and relates to what is known as the **five-number summary**.

3 Using your ordered data, circle and label the minimum, maximum and median of this data set.

These three statistical measures make up three numbers of the five-number summary.

4 In what positions do you think the remaining two numbers might be?

The first of the remaining two numbers is called the first or **lower quartile** and is often represented by Q_1. It is the median of the first or lower half of the data. Note: It is important that the each 'half' of the data does not contain the median, as it is the median that divides the set into the two halves.

5 How many numbers in the first half of the data?

6 What is the lower quartile for this data set? Remember that this is the median of the first seven numbers. Circle and label this on your data set.

KEY IDEAS

- A common measure of spread used to describe a data set is the range. It is the difference between the maximum and minimum scores.
- Interquartile range (IQR) is another measure of spread used to describe a data set. It is the middle 50% of a data set.
- To calculate the interquartile range, follow these steps:
  1. Arrange the scores in order and locate the median (Q_2). The median splits the data set into two sets.
     - If there is an even number of scores, the median is the average of the two middle numbers and the data set is divided evenly.
     - If there is an odd number of scores, the median is the middle number and the data set is divided into two even sets that do not include the median.
  2. Find the lower quartile (Q_1), the median of the first or lower half of the data.
  3. Find the upper quartile (Q_3), the median of the second or upper half of the data.
  4. Subtract Q_1 from Q_3 to give the IQR.

EXERCISE 10B Measures of spread

EXAMPLE 10B-1 Finding the range

Find the range for this data set.

Class interval	Frequency (f)
0–<10	8
10–<20	5
20–<30	7

THINK

The range is the difference between the upper limit of the highest interval and the lower limit of the smallest interval.

WRITE

range = 30 − 0
= 30

UNDERSTANDING AND FLUENCY

1 Find the range for each data set.

a 5, 9, 6, 12, 16, 3, 8, 19, 11, 8, 12

b 48, 3, 7, 3, 8, 4, 12, 19, 6, 6, 8, 19, 13

c 110, 167, 189, 102, 144, 117, 166

d 54, 46, 78, 11, 19, 22, 26, 57, 67

e

Score	Frequency
18	8
19	4
20	6
21	9
22	11

f

Class interval	Frequency
25–<30	4
30–<35	8
35–<40	5
40–<45	8
45–<50	6

g

Class interval	Frequency
16–<20	1
20–<24	2
24–<28	3
28–<32	4
32–<36	5

EXAMPLE 10B-2 Finding the five-number summary

Find the five-number summary for this data set: 2, 6, 4, 8, 6, 2, 5, 3, 6

THINK

1. Arrange the scores in order, and find the median by either circling the middle number or finding the average of the middle two numbers.
2. Divide the set into two equal sets, remembering not to include the median.
3. Find Q_1 and Q_3 by finding the median of the two halves of the set.
4. Find the minimum (Q_0) and maximum (Q_4) scores of the set.
5. Write your answer.

WRITE

2 2 3 4 (5) 6 6 6 8

2 2 3 4 (5) 6 6 6 8

2 2 3 4 (5) 6 6 6 8

$Q_1 = 2.5$ $Q_3 = 6$

$Q_0 = 2$, $Q_4 = 8$

The five-number summary for this data set is:
$Q_0 = 2$, $Q_1 = 2.5$, $Q_2 = 5$, $Q_3 = 6$, $Q_4 = 8$.

UNDERSTANDING AND FLUENCY

2 Write each data set in order and circle the scores that make up the five-number summary.

- **a** 6, 9, 3, 6, 3, 8, 3, 3, 7, 4, 5
- **b** 11, 19, 61, 13, 45, 62, 78, 7, 9, 12, 35, 46, 42, 18, 13
- **c** 6, 4, 9, 1, 6, 3, 4, 8, 7, 6, 4, 3, 2, 4, 8, 6, 9, 4, 5, 2, 3, 7, 7
- **d** 8, 16, 7, 19, 32, 8, 9, 7, 16, 21, 18, 4, 6, 19, 4, 2, 19, 27, 11

3 Find the five-number summary for each data set.

- **a** 4, 8, 9, 12, 7, 6, 6, 6, 7, 4, 3, 2, 8, 9, 11
- **b** 46, 28, 16, 23, 8, 4, 46, 49, 55, 13
- **c** 72, 81, 16, 22, 42, 55, 67, 73, 19, 24, 16, 77, 37
- **d** 724, 169, 843, 120, 564, 786, 341, 186
- **e** 42, 18, 32, 18, 64, 39, 42, 18, 64, 21, 27, 34
- **f** 8, 19, 24, 26, 17, 13, 8, 4, 13, 9, 14, 21, 17, 32, 8, 4, 3, 19

EXAMPLE 10B-3 Finding the interquartile range

Find the interquartile range for this data set.
6, 32, 7, 2, 8, 9, 9, 4, 6, 2, 6, 21, 19, 13, 5

THINK	WRITE
1 Arrange the scores in order.	2, 2, 4, 5, 6, 6, 6, 7, 8, 9, 9, 13, 19, 21, 32
2 Find the five-number summary by following the steps in Example 10B-2.	$Q_0 = 2$, $Q_1 = 5$, $Q_2 = 7$, $Q_3 = 13$, $Q_4 = 32$
3 Locate and subtract Q_1 from Q_3 to obtain the IQR.	$\text{IQR} = Q_3 - Q_1$ $= 13 - 5$ $= 8$

4 Given these five-number summaries, find the interquartile range.

- **a** $Q_0 = 2$, $Q_1 = 6$, $Q_2 = 10$, $Q_3 = 16$, $Q_4 = 19$
- **b** $Q_0 = 17$, $Q_1 = 22$, $Q_2 = 25$, $Q_3 = 31$, $Q_4 = 33$
- **c** $Q_0 = 7$, $Q_1 = 9$, $Q_2 = 13$, $Q_3 = 19$, $Q_4 = 24$
- **d** $Q_0 = 34$, $Q_1 = 40$, $Q_2 = 52$, $Q_3 = 59$, $Q_4 = 66$

5 Find the interquartile range for each data set.

- **a** 46, 48, 32, 12, 62, 27, 19, 34, 86, 16, 16
- **b** 8, 1, 6, 4, 23, 5, 7, 16, 3, 3, 14, 13, 7, 5
- **c** 123, 189, 146, 179, 98, 165, 108, 142, 152
- **d** 19, 62, 53, 14, 8, 43, 62, 9, 46, 13, 19, 20
- **e** 5, 9, 11, 16, 13, 7, 13, 18, 21, 34, 12, 8, 7
- **f** 125, 465, 132, 99, 164, 222, 378

6 This **stem-and-leaf plot** shows the number of cupcakes a local bakery sold per day over the course of a month. Find the range and the interquartile range and hence describe the spread of the set.

Key: 4 | 2 = 42

Stem	Leaf
4	1 2 2 6 7 8
5	1 6 6 7 9
6	1 2 3 3 6 6 8 9
7	2 4 8 9 9
8	0 1 5 5 7
9	1 2

7 Consider these two data sets.

Set A: 9, 31, 46, 11, 39, 24, 3, 43, 25, 10, 19, 4, 36, 14, 16

Set B: 16, 26, 29, 18, 14, 23, 14, 57, 26, 21, 16, 18, 14, 19, 19

a Find the mean and median for each data set. What do you notice?

b Find the range for each data set. What do you notice?

c Find the interquartile range for each data set.

d Explain why the interquartile range must be used to describe the difference between these two sets.

8 Consider this data on the number of times a year a sample of Year 10 students dye their hair.

Number of times hair is dyed	0	1	2	3	4	5	6
Frequency	10	2	3	7	11	9	10

a Find the range.

b Find the five-number summary.

c Find the interquartile range.

9 Explain what it means if a data set has:

a a small interquartile range

b a small interquartile range and a large range

c a large interquartile range

d an interquartile range that is very similar to its range.

10 For each data set:

i place the scores in order

ii predict whether it would have a small, moderate or large interquartile range

iii calculate the interquartile range

iv compare your answers from parts **ii** and **iii**. Did your predictions get more accurate as you gained more practice?

a 15, 19, 23, 27, 15, 14, 18, 67, 16, 29, 31, 28, 27, 26, 19

b 8, 9, 6, 2, 1, 11, 18, 2, 6, 4, 5, 6, 7, 7, 7, 11, 19, 1, 3, 4

c 123, 189, 156, 148, 191, 117, 108, 168, 144, 151, 167, 177, 129, 107, 149

d 66, 78, 64, 81, 80, 79, 77, 72, 75, 769, 75, 74, 68, 65, 63, 73, 75, 79, 73, 72, 72, 67, 81, 77, 77

11 Consider this data set, showing the income of Year 10 students (in dollars) per week.

10 120 50 60 45 85 250 80 100 65 75 20
35 45 50 80 25 10 20 50 100 45 80

a Calculate the range and the interquartile range.

b Why is the range so much larger than the interquartile range?

c If the data in a set is spread evenly, you expect the range to be roughly twice the size of the interquartile range. Explain why.

d The presence of an outlier can affect the range. Explain why it does not affect the interquartile range.

e Explain why, when there is an outlier in a data set, the interquartile range is the best measure of spread.

12 For each data set:

i calculate the range and interquartile range

ii decide which of these is the better measure of spread

iii describe the spread of the data set.

a 11, 8, 19, 9, 11, 15, 7, 2, 6, 17, 13, 13, 12, 16, 8, 7, 16, 5, 4, 19

b 32, 68, 43, 31, 77, 44, 67, 42, 37, 55, 59, 79, 81, 83, 11, 56, 57, 61, 82, 63

c 123, 118, 131, 141, 142, 136, 128, 127, 119, 129, 138, 127, 191, 132, 134

13 For the data shown in this table, calculate:

a the mean **b** the median

c the range **d** the interquartile range.

Ages of people at a soccer match	Frequency
1–10	7
11–20	20
21–30	4
31–40	16
41–50	18
51–60	6
61–70	4
71–80	2

14 When finding the five-number summary, sometimes an individual point (for example, lower quartile) is one of the original scores, sometimes it is the average of two scores. If you know the number of scores in a set, can you predict when all points are one of the original scores? Investigate by following these steps.

a Write out the series of numbers 1–5 and find the five-number summary. How many points of the five-number summary are from the original scores?

b Write out the series of numbers 1–6 and find the five-number summary. How many points of the five-number summary are from the original scores?

c Write out the series of numbers 1–7 and find the five-number summary. How many points of the five-number summary are from the original scores?

d Continue the pattern shown in parts **a**–**c** until you have worked up to the series of numbers 1–15.

e Classify each of these series as having all points as an original score, the lower and upper quartiles as original scores, the median as an original score, or none as an original score.

PROBLEM SOLVING AND REASONING

f What pattern do you notice?

g Can you write a formula that will help you predict which points of the five-number summary will be an original score? Hint: one method is to assign each possibility (quartiles, all, none, median) a number from 0 to 3 and use a formula that gives a remainder.

15 Consider this **back-to-back stem-and-leaf plot** showing the heights of students in two classes.

Key: 1|6 = 16

Leaf Class A	Stem	Leaf Class B
	14	9
9	15	6 8
9 8 7 7 7 6 5 4 3 3 2 1 1	16	0 1 3 4 5 5 7 9
9 9 8 7 6 5 5 4 3 2 2 1	17	2 3 4 4 5 5 5 7 9
6 5 4 1	18	1 4 8
	19	7

a For each class, find:

i the mean **ii** the median

iii the range **iv** the interquartile range.

b Compare the heights of the two classes. Which class would you say is taller? Which class would you say is more diverse?

CHALLENGE

16 Consider this table showing the ages of people at the opening of a music festival.

Ages of people at a music festival	Frequency
11–20	28
21–30	36
31–40	19
41–50	11
51–60	4
61–70	1
71–80	1

a Add a column and find the midpoint of each class interval.

b Add another column and find the cumulative frequency of the data.

Recall that to find the median of a large data set, you find the $\frac{n+1}{2}$th score.

c Find the median of the data set.

d How many numbers are in each half of the set?

> **NOTE** Remember that the lower quartile is simply the median of the first half.

e Use the formula median = $\frac{n+1}{2}$th score to find the lower quartile.

f Find the upper quartile.

g Find the interquartile range.

h Why is using the midpoint of the class interval important in finding the interquartile range?

Reflect

Why is the interquartile range usually a better measure of spread than the range?

10C Standard deviation

Start thinking!

Another measure of spread is called the **standard deviation**. It measures the average spread from the mean.

1 What do you think a small standard deviation means? Do you think it means that most values are close to the mean or spread far from the mean?

Consider these two data sets.

Data set A:	1	1	9	1	9	9
Data set B:	2	3	1	7	2	15

2 Calculate the mean for each data set. What do you notice?

3 Which data set is more spread out from the mean? That is, which data set would you predict to have the biggest standard deviation?

4 For data set A:

a find the difference between the mean and each score

b add these numbers together

c divide by the number of scores.

5 What is the average 'difference' for data set A?

6 Repeat question **4** for data set B. What do you find?

You know that data set B is more spread out than data set A, but the average 'difference' for both sets is the same. You can use your understanding of squares to highlight the differences between these sets.

7 Find the square of:

a 3 **b** 9.

8 What is the effect of squaring a small number versus squaring a larger number? (Hint: how much bigger do the numbers get?)

You can use this property of squaring to amplify the differences in a spread-out data set.

KEY IDEAS

- Standard deviation (σ) is a measure of how much the scores in a set differ (or deviate) from the mean.
- A small standard deviation means that the scores in a set are close to the mean.
- A large standard deviation means that the scores in a set are spread out from the mean.
- Standard deviation is best found using a calculator.

EXERCISE 10C Standard deviation

For all questions in this section, use the standard deviation represented by σ in your calculator.

EXAMPLE 10C-1 Finding the standard deviation for data in a list

Find the standard deviation for this data set.
4, 8, 12, 19, 5, 22, 17, 8, 9, 11, 13, 13, 18, 42, 23, 35, 7, 13, 18, 19, 17, 21, 29

THINK	WRITE
1 Enter the data into your calculator in the appropriate mode. Check with your teacher how to do this.	
2 Calculate the standard deviation using the appropriate function. Round your answer to two decimal places.	$\sigma \approx 9.15$

1 Find the standard deviation for each data set.

a 6, 3, 8, 65, 4, 8, 11, 15, 3, 9, 5, 7, 7, 10

b 12, 17, 15, 13, 29, 21, 22, 22, 11, 19, 13, 13, 17, 24

c 140, 156, 120, 99, 187, 147, 159, 166, 122, 171, 136, 129

d 15, 19, 13, 12, 12, 11, 18, 14, 13, 14, 16, 15, 14, 16

e 16, 18, 21, 19, 35, 17, 16, 34, 29, 18, 34, 42, 49, 37, 18

f 46, 251, 32, 101, 189, 167, 278, 29, 18, 352, 267, 111, 197

EXAMPLE 10C-2 Finding the standard deviation for data in a table

Find the standard deviation for this data set.

Score	Frequency
14	6
15	7
16	5
17	6

THINK	WRITE
1 Enter the data into your calculator in the appropriate mode. Check with your teacher how to do this.	
2 Calculate the standard deviation using the appropriate function. Check with your teacher how to do this. Round your answer to two decimal places.	$\sigma \approx 1.12$

UNDERSTANDING AND FLUENCY

2 Find the standard deviation for each data set.

a

Score	Frequency
1	3
2	5
3	9
4	3
20	1

b

Score	Frequency
10	6
20	8
30	9
40	16
50	14

c

Score	Frequency
19	6
20	4
21	9
22	10
23	5

d

Score	Frequency
13	16
14	25
15	13
16	18
17	9

e

Score	Frequency
5	9
10	11
15	15
20	13
25	12

f

Score	Frequency
100	9
200	12
300	15
400	16
500	19

PROBLEM SOLVING AND REASONING

3 Consider this data set, showing the ages of players in two cricket teams.

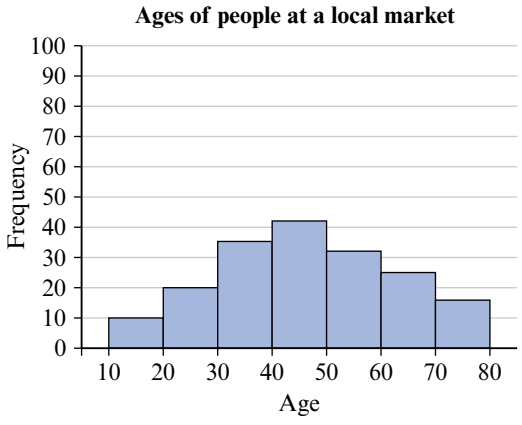
16 16 15 17 15 16 14 15 16
16 17 16 15 17 16 15 16 16
17 16 17 15 16 16 17 16

a Enter the data into a frequency table.

b Find the standard deviation for the data set.

c Write a sentence about the spread of ages of students in the cricket teams.

4 These two histograms show data collected on the ages of people attending a local market on two different days. The data sets have the same mean and number of scores, and have been shown on the same scale for ease of comparison. Which data set has a small standard deviation and which has a large standard deviation? Explain how you know.



5 For each data set:

- **i** predict whether its standard deviation will be large or small
- **ii** calculate the standard deviation
- **iii** write a sentence comparing your prediction to your calculation.

a 11 19 8 21 16 13 17 15 9 22 15
13 13 17 19 12 12 13 16 17 16 17

b 5 21 9 7 6 46 19 88 22 102 46 78 19
66 45 42 13 33 91 16 8 9 46 79 51

c 125 122 131 126 121 120 120 129 127 127 131 130 128 128
129 124 125 124 128 127 126 127 129 131 130 120 128 127
127 125

6 Consider this data, collected on the price (in dollars) of a particular brand of tablet from various stores in a city.

499 519 499 475 469 479 509 549 529 525
475 499 499 199 489 479 499 475 515 499

a Find the standard deviation for this data set.

You may notice that one of the prices is distinctly different from the rest.

b Provide a possible reason for this.

c Recalculate the standard deviation for the data set excluding the outlier.

d What do you notice about your answers to parts **a** and **c**?

e What measure of spread might you use instead of standard deviation when there is an outlier in the data set?

7 This frequency table shows the number of people at a ski resort over one ski season.

Class interval	Frequency
0–100	3
101–200	5
201–300	8
301–400	11
401–500	19
501–600	32
601–700	22

a How would you calculate the mean for this data?

b Add a column and find the midpoint of each class interval.

c Use this midpoint column and the frequency column to find the standard deviation.

d Explain why you use the midpoint of the class interval to find both the mean and the standard deviation.

8 This list shows the number of people at a shoe sale each hour over a 24-hour period.

46 26 35 49 58 13
27 48 46 39 42 48
46 51 35 29 47 41
42 42 49 45 46 51

a Find the mean.

b Find the standard deviation.

c Comment on the centre and spread of the data set.

PROBLEM SOLVING AND REASONING

9 Consider this data, showing the number of text messages sent by Year 10 students per day.

1	5	11	17	21	9	8	25	26
19	5	2	8	9	13	34	16	
29	31	26	19	18	15	20	10	

a Find the mean and standard deviation for the data.

b Place the data into a table with class intervals of 10.

c By adding appropriate columns to the table, recalculate the mean and standard deviation for the data.

d How does your answer to part **c** compare to your answer to part **a**?

e Write a sentence describing the centre and spread of the set.

10 Data was collected on the number of minutes passengers were kept waiting for their trains on two different train lines over the course of a day.

Train line A: 4, 2, 3, 5, 6, 7, 4, 4, 6, 5, 6, 7, 9, 4, 5, 6, 6, 7, 6, 7, 9, 4, 6, 6, 1

Train line B: 2, 9, 15, 3, 4, 3, 11, 1, 1, 6, 2, 3, 4, 2, 3, 19, 2, 3, 2, 2, 2, 3, 1, 4

a Find the mean for each train line.

b Find the standard deviation for each train line.

c Write a sentence comparing the centre and spread of the two data sets.

d Which train line would you say runs late more significantly?

11 This back-to-back stem-and-leaf plot shows the age of people auditioning for a talent show in two different cities. Find the mean and standard deviation for both sets and write a sentence comparing the centre and spread of ages of people auditioning in the two cities.

Key: 1|6 = 16

Leaf City A	Stem	Leaf City B
9 8 7 7 7 6	1	5 6 7 7 8 8 8 8 9
9 8 8 7 5 4 2 1	2	0 0 1 1 2 2 2 3 4 5 6 6 7 9
9 8 7 6 4 3 2 2 2 0	3	1 5 6 7 9
6 5 4 2 1 1	4	2 6
3 2 1	5	1
5	6	
1	7	

CHALLENGE

12 You may have noticed on your calculator that there are two different standard deviations you can calculate. The one that you have been using (σ) assumes the data is for an entire population. The other (s) is used when a sample of the population provides the data. The formula for the sample standard deviation is designed to minimise bias in the sample by dividing by $(n - 1)$ rather than dividing by n.

For each data set:

i decide whether it is a sample or a population

ii calculate the appropriate standard deviation

CHALLENGE

a Data is collected on the ages of students attending a school formal by surveying everybody who is there at 6.00 pm.

16	17	18	16	17	16	15	14	16	17	18	16	17	17
16	17	16	17	15	16	17	16	17	16	16	16	15	17
18	18	17	16	17	16	17	16	17	16	16	16	17	17

b Data is collected on the number of hours spent using social networks per week in a Year 10 class of 20.

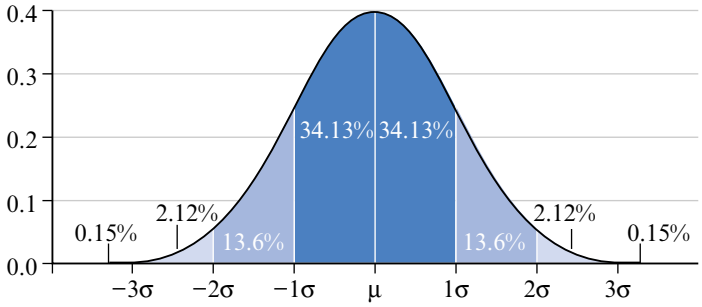
5	10	7	1	19	17	16	7	10	11
8	9	15	13	14	6	21	18	14	13

c Data is collected on the weights of newborn babies in South Australia.

3.2	4.1	2.9	3.4	3.5	3.4	3.1	3.8	4.0	3.3	3.6	2.9	3.1
3.4	3.5	4.2	2.6	3.0	3.6	3.4	4.8	2.1	3.5	3.3	3.6	

13 Standard deviation plays a useful role in making predictions about **populations** that are evenly distributed. A population that has an even, symmetric distribution is said to be normally distributed (also known as forming a bell curve). A normal distribution will have nearly all values (99.7%) within three standard deviations (also known as three sigma) of the mean.

Consider this graph.



a Show that 99.7% of values lie within three sigma of the mean.

b What percentage of values lie within:

i one sigma of the mean?

ii two sigma of the mean?

A typical example is IQ (intelligence quotient). The average (μ) IQ is said to be 100, and it is thought to have a standard deviation of 15.

c Redraw the graph so that the scale on the x-axis uses these values (that is, $\mu = 100$, $1\sigma = 115$, etc.).

d What is the IQ range of 99.7% of the population?

e What IQ would you need to have in order to be in the top 0.15%?

f What IQ would you need to have in order to be in the bottom 2.27%?

g If somebody had an IQ of 135, what top percentage would you consider them to be in?

h If somebody had an IQ of 80, what bottom percentage would you consider them to be in?

i A group of highly intelligent people had their IQs tested and the distribution was found to be normal. If 95.46% of this group had an IQ between 120 and 152, find the mean and standard deviation of this group.

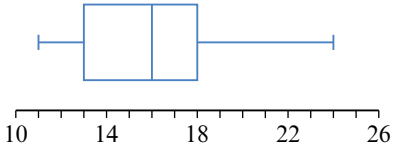
Reflect

Why is standard deviation a very useful measure of spread?

10D Box plots

Start thinking!

Box plots (or box-and-whisker plots) are a way to visually represent a data set using the five-number summary as key points.



1 What are the five numbers in the five-number summary?

Consider this box plot, showing the ages of people at a skate park on a Wednesday afternoon.

2 Identify the minimum and maximum points on the box plots and write down their values.

3 Which vertical line on the box plot do you think represents the median? Explain.

4 Write down the median.

5 Which quartile does the end of the box represent?

6 Which quartile does the start of the box represent?

7 Write down the values of the lower and upper quartiles.

8 Copy out the box plot and label it with the five-number summary. Use labels of Q_0, Q_1, Q_2, etc.

9 How does the box plot split the data into the four quartiles?

10 Explain how the box section of the box plots represents the interquartile range.

How do you draw a box plot? Consider this data set, showing the ages of people at a skate park on a Saturday morning.

16	14	22	26	28	15	19	17	17	22	16
19	18	18	27	19	31	34	24	22	18	

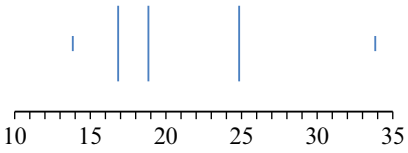
11 Find the five-number summary for this data set.

12 Draw an even scale that covers the minimum and maximum numbers for the data set.

13 Using the figure at right as a guide, draw above your scale:

a short vertical lines at the minimum and maximum numbers of the data set

b longer vertical lines at the median, and the lower and upper quartiles.

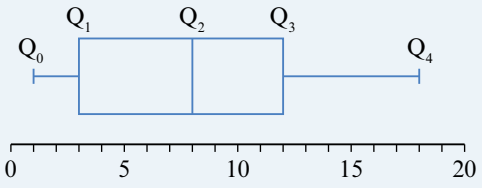


14 Use a ruler to complete your box plot.

15 Explain how a box plot easily displays both the centre and spread of a data set.

KEY IDEAS

- A box plot can be used to visually represent the five-number summary and easily display the centre and spread of a data set.
- A box plot should always be accompanied by a clear and even scale.

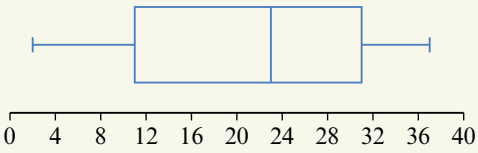


EXERCISE 10D Box plots

EXAMPLE 10D-1 Understanding a box plot

Use this box plot to state the:

a five-number summary
b interquartile range.



THINK

a Look at the labelled box plot in the Key ideas to identify the position of each number in the five-number summary. Write the values.

b The interquartile range is the difference between the third and first quartiles.

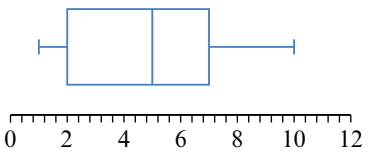
WRITE

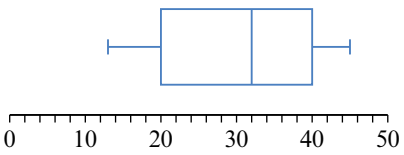
a $Q_0 = 2$, $Q_1 = 11$, $Q_2 = 23$, $Q_3 = 31$, $Q_4 = 37$

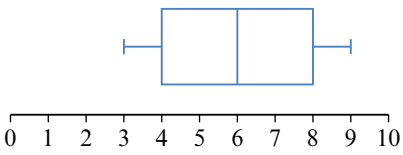
b $IQR = Q_3 - Q_1$
$= 31 - 11$
$= 20$

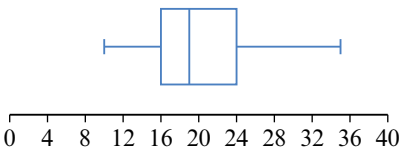
UNDERSTANDING AND FLUENCY

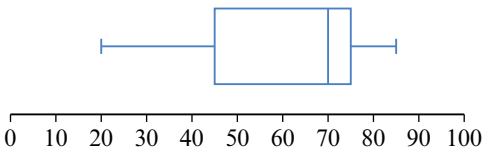
1 For each box plot, state the:
i five-number summary
ii interquartile range.

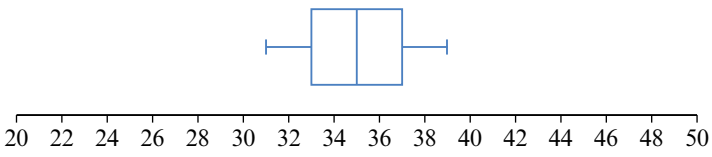
a


b


c


d


e


f


EXAMPLE 10D-2 Drawing a box plot

Draw a box plot to represent this data on the number of books that people own.

45	22	19	4	14	39	152	108	39	19	66	81
122	42	38	22	18	35	144	33	27	13	31	

THINK

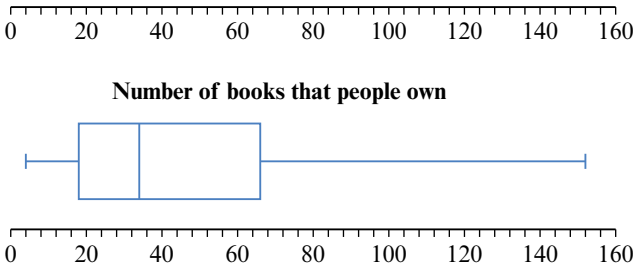
1 Find the five-number summary.

2 Draw an even scale that covers the minimum and maximum numbers.

3 Mark in the position of the five numbers from the five-number summary, and complete your box plot using a ruler. Give a title if appropriate.

WRITE

$Q_0 = 4$, $Q_1 = 19$, $Q_2 = 35$, $Q_3 = 66$, $Q_4 = 152$



UNDERSTANDING AND FLUENCY

2 Consider this data set: 2, 6, 3, 8, 9, 10, 14, 19, 4, 8, 22, 11, 13, 17, 4, 19, 7, 7, 10.

- **a** Arrange the data in order.
- **b** Find the five-number summary.
- **c** Draw an even scale that covers the minimum and maximum numbers.
- **d** Above the scale, place a mark in the position of each number of the five-number summary.
- **e** Use a ruler to create a box plot linking these data points.

3 Draw a box plot to represent each data set.

- **a** 3, 6, 2, 8, 4, 8, 10, 15, 6, 8, 11
- **b** 11, 15, 14, 19, 20, 26, 17, 13, 6, 22
- **c** 4, 7, 1, 8, 3, 7, 6, 6, 9, 3, 5, 1, 6, 9, 7, 5, 3
- **d** 12, 36, 45, 92, 42, 85, 49, 66, 51, 19, 27, 78
- **e** 16, 28, 9, 16, 13, 27, 18, 22, 33, 24, 9, 18, 8
- **f** 5, 7, 9, 19, 6, 9, 8, 5, 5, 4, 3, 9, 9, 1, 6, 5, 4
- **g** 11, 18, 49, 23, 24, 26, 37, 31
- **h** 8, 11, 19, 21, 6, 7, 12, 18, 19, 31, 4, 6, 24, 7, 9

4 Data was collected on the number of puppies in a litter.

1	6	4	2	5	4	8	6	4	5	8
3	2	4	6	5	8	7	5	6	5	6
12	4	6	5	7	6	5	6	5		

- **a** Find the five-number summary.
- **b** Draw a box plot to represent the data.

5 Use the data in this table to draw a box plot and comment on its distribution in terms of its centre and spread.

Number of cinema visits per month	Frequency
0	4
1	11
2	16
3	9
4	7
5	3
6	1

6 Box plots are also useful when describing the shape of a data set. Consider these box plots.

box plot A

box plot B

a How would you describe the shape of box plot A?

b How would you describe the shape of box plot B?

Box plot B is obviously **skewed**. When the data is skewed to the left like in box plot B, the distribution is called positively skewed.

c Draw an example of a box plot that is negatively skewed.

d Explain why the smaller the quartile section (for example, the first two sections of box plot B), the more clustered together those scores are.

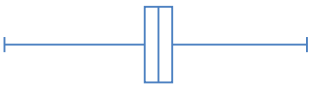
> **NOTE** Remember that each quartile still represents 25% of the data.

e Match each of these box plots (**i–iv**) with the correct graph (**A–D**).

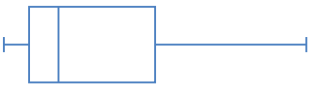
i

ii

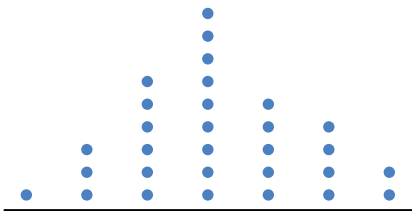
iii



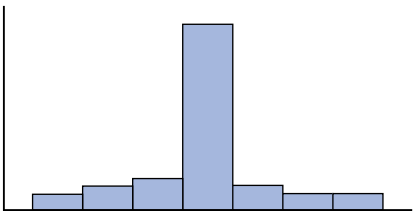
iv



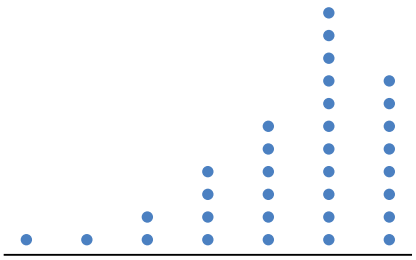
A



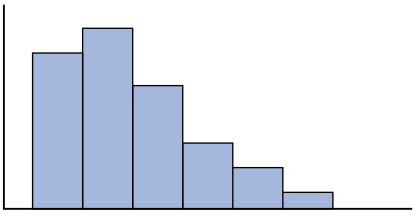
B



C



D



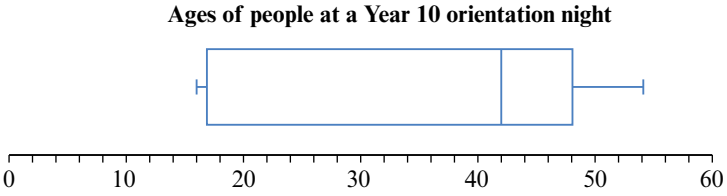
7 **a** Describe what this box plot tells you about the ages of people attending the event, by describing its:

i shape

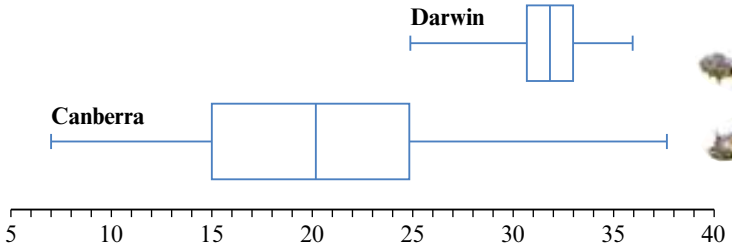
ii spread

iii centre.

b Can you suggest a reason for the unusual shape of the distribution?



8 When you want to compare two distributions, you can use parallel box plots. These are two or more box plots plotted onto the same scale. Consider these parallel box plots, showing temperature for a particular year in Darwin and Canberra.



a What is the centre and spread of the temperature in Darwin?

b What is the centre and spread of the temperature in Canberra?

c Write a paragraph comparing the two distributions. Be sure to mention the shape of the distributions. Which would you say is the hotter city?

9 Data was collected on the ages of customers at a laser tag centre on weekdays (set A) and weekends (set B).

Set A: 19, 18, 21, 22, 23, 18, 24, 25, 22, 21, 20, 19, 18, 18, 19, 20, 19, 20, 24, 22, 21, 23, 27, 19, 21

Set B: 21, 26, 23, 19, 18, 17, 15, 6, 10, 43, 36, 26, 23, 49, 7, 11, 14, 13, 51, 38, 32, 27, 18, 12, 19, 33

a Draw a set of parallel box plots to represent the data.

b Describe each set in terms of its shape, centre and spread.

c Compare the two sets. What differences do you notice?

10 A vet collected the following data on the weights (in kilograms) of dogs.

5.1 11.2 35.5 17.9 21.1 84.4 42.2 51.0 9.2 8.6 7.5 18.5 19.2 21.8 22.6
24.8 13.7 13.1 16.9 29.8 58.3 42.7 2.1 3.8 9.6 10.8 46.0 1.9 35.0 39.4

a Draw a box plot to represent the data.

b Draw a histogram to represent the data.

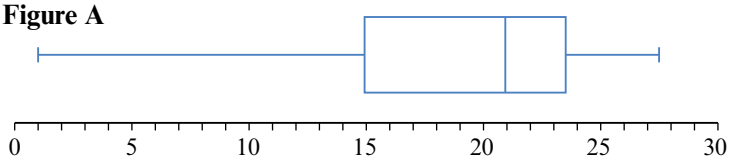
c Describe the distribution of the data.

d Which display do you think represents the data best? Explain.

CHALLENGE

11 When a box plot is heavily skewed with a long whisker at one end (such as figure A), this can often be caused by a single unusual score.

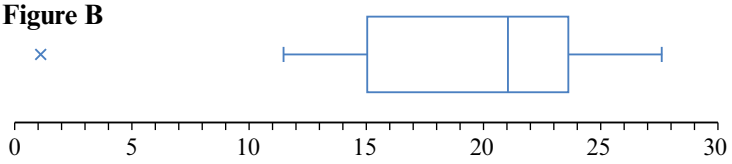
Figure A



a What name is given to an unusual score?

When one or more outliers are present, it can be more helpful to represent them on the box plot with a cross, finishing the whisker at the next smallest (or largest) score, as shown in figure B.

Figure B



Usually it is reasonably obvious when a score is an outlier, but sometimes it can be a little harder to tell. Consider this data collected on the finishing times (in seconds) of participants in a 200 m race.

32, 36, 39, 46, 29, 39, 40, 40, 42, 42,
92, 28, 36, 38, 41, 34, 37, 41, 34, 36

b Find the five-number summary and the interquartile range.

c Draw the box section of the box plot.

You can use the interquartile range to help determine if a score is an outlier.

d Find $1.5 \times$ IQR.

e From the lower quartile (35), subtract $1.5 \times$ IQR.

f Do any scores fall below this value? Draw in the lower whisker.

g From the upper quartile (41), add $1.5 \times$ IQR.

h Do any scores fall above this value? Mark this score with an X and then extend the whisker from the upper quartile to the next largest score.

i Explain why you extend the whisker to the next largest score (46) and not 50.

j Explain why you can use $Q_1 - 1.5 \times \text{IQR} < x < Q_3 + 1.5 \times \text{IQR}$ to determine if a score, x, is an outlier.

12 Draw a box plot for each of these data sets. Be sure to indicate the presence of any outliers.

a 12, 18, 22, 19, 8, 11, 13, 16, 24, 46, 17, 16,
9, 26, 25, 18, 17, 10, 13, 19, 19, 22, 23

b 59, 66, 78, 35, 46, 91, 7, 55, 43, 42, 61, 66,
72, 77, 84, 66, 52, 39, 99, 81

c 32, 40, 53, 51, 42, 44, 42, 10, 39, 36, 35, 42,
51, 48, 43, 5, 51, 34, 36, 49, 34, 42, 41, 45, 46

Reflect

How do box plots show the centre and spread of a data set?

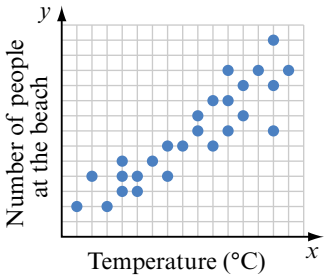
10E Scatterplots and bivariate data

Start thinking!

The data that you have looked at so far consists of only one variable, such as the ages of people, the number of people, etc. This is called **univariate data**. What about when you look at the relationship between two variables, such as a person's height and weight? This is **bivariate data**.

1 Give another two examples of bivariate data.

Bivariate data is commonly graphed using a **scatterplot**. It is a graph formed by plotting individual points. Consider this scatterplot.



2 What does it show?

When describing a bivariate relationship, you usually describe it in terms of its strength and its direction, and you use the word **correlation** rather than relationship. Scatterplots can be said to show a strong, moderate or weak correlation, or no correlation at all.

3 How would you describe the strength of the correlation in this scatterplot? Explain.

4 Would you describe this as a positive or negative correlation? Explain.

You can combine your answers to questions **3** and **4** to describe the scatterplot in context (that is, describing the actual situation shown) in a single sentence. This sentence is always written describing how the variable on the y-axis acts as the variable on the x-axis increases. For example, a scatterplot showing a strong, negative correlation between number of hours spent on the Internet on the x-axis and test results on the y-axis would be described as: test results decrease strongly as the number of hours on the Internet increases.

5 For the given scatterplot, which variable is on the x-axis and which variable is on the y-axis?

6 Use your answers to questions **3**–**5** to describe the scatterplot in the context of the bivariate relationship that it shows.

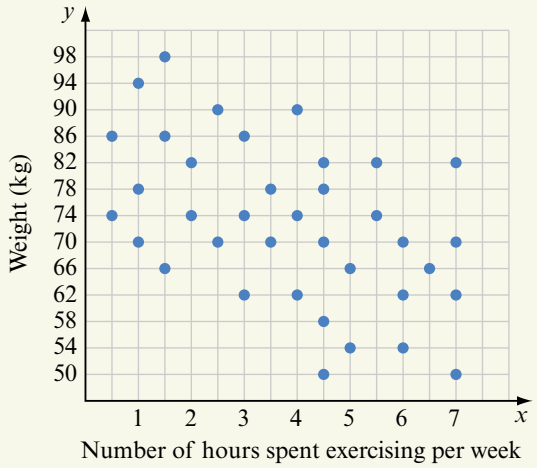
KEY IDEAS

- Bivariate data is data that has two variables.
- Scatterplots can be used to display bivariate data.
- Scatterplots can be described in terms of their strength (no correlation, weak, moderate or strong) and their direction (negative or positive).
- Bivariate data can be described by stating how one variable (on the y-axis) behaves as the other variable (on the x-axis) increases.
- In bivariate data, a **dependent variable** is one that is affected by another variable; an **independent variable** is one that is not.

EXERCISE 10E Scatterplots and bivariate data

EXAMPLE 10E-1 Describing the trend for bivariate data

Describe the trend for this scatterplot in the context of the data it represents.



THINK

1 Consider the strength and direction of the plot.

2 Describe the trend shown in context of the data.

WRITE

The scatterplot shows a weak, negative trend.

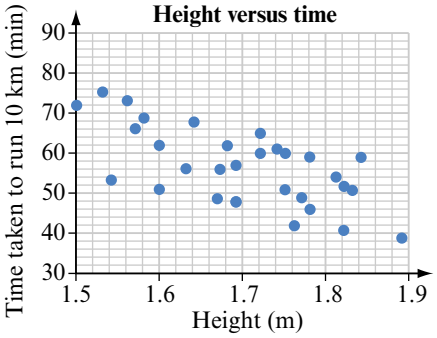
Weight decreases weakly as the number of hours spent exercising increases.

UNDERSTANDING AND FLUENCY

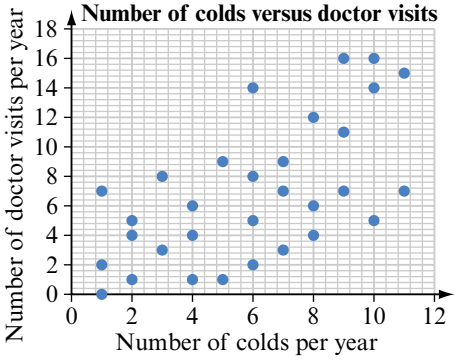
1 Describe the trend shown in each scatterplot

i in terms of strength and direction ii in the context of the data it represents.

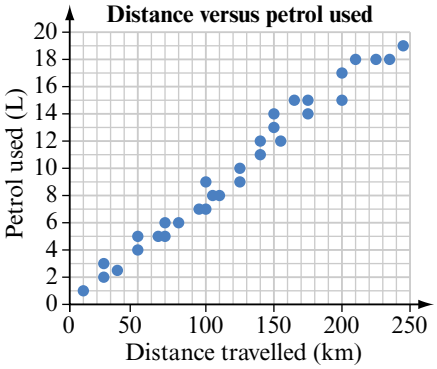
a



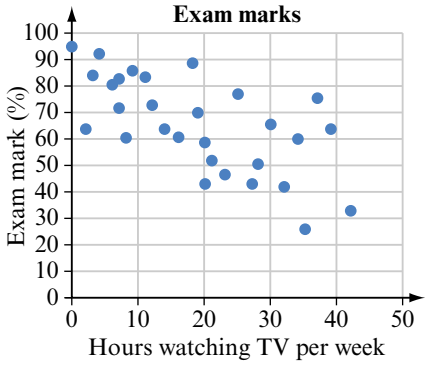
b



c



d



EXAMPLE 10E-2 Drawing a scatterplot

Draw a scatterplot for this data set.

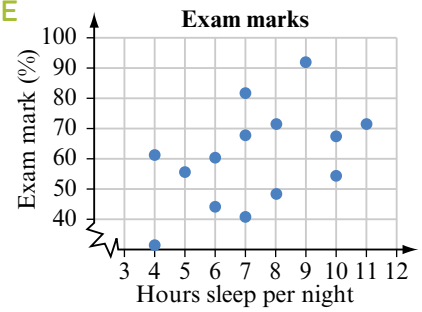
Hours sleep per night	8	6	7	9	10	11	8	7	6	5	4	8	7	6	10
Exam marks	72	61	82	92	68	72	49	68	61	56	32	62	41	45	55

THINK

1 Consider the maximum and minimum values for both axes and use them to draw and label accurately scaled axes. Be sure to indicate if either axis does not begin at zero.

2 Plot each pair of variables from the table.

WRITE



UNDERSTANDING AND FLUENCY

2 Draw a scatterplot for each data set.

a

Height (cm)	150	181	166	168	175	178	172	155	161	168	179	184	188	159	167
Shoe size	4	9	9	7	9	8	7	4	5	7	7	10	12	7	6

b

Temperature (°C)	33	15	16	25	27	36	41	11	8	32	20	25	31	18	28
Number of pies sold	5	25	33	16	19	2	3	42	35	12	21	11	10	51	15

c

Number of customers	45	56	16	32	78	98	26	66	45	76	86	42	59	31	67
Daily profit ($00s)	12	16	8	19	21	48	7	32	22	29	36	13	18	16	12

d

Age (years)	11	19	23	78	56	43	28	16	82	16	72	45	63	33	18
Seconds a person can hold their breath	59	48	62	7	35	35	32	66	11	39	21	17	9	46	19

3 Describe the trend shown in each scatterplot you drew for question 2 in context of the bivariate data.

PROBLEM SOLVING AND REASONING

4 Kieran and Marie want to investigate the relationship between the number of different types of animal in a zoo and the number of visitors that the zoo receives per day.

a Predict the correlation you expect to see between the two variables in terms of strength and direction.

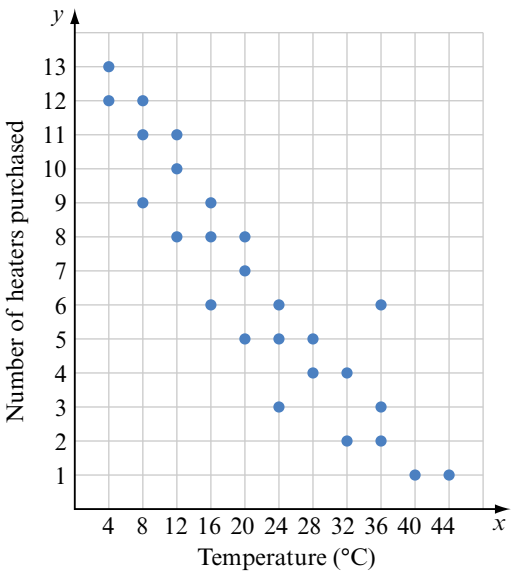
They collected this data:

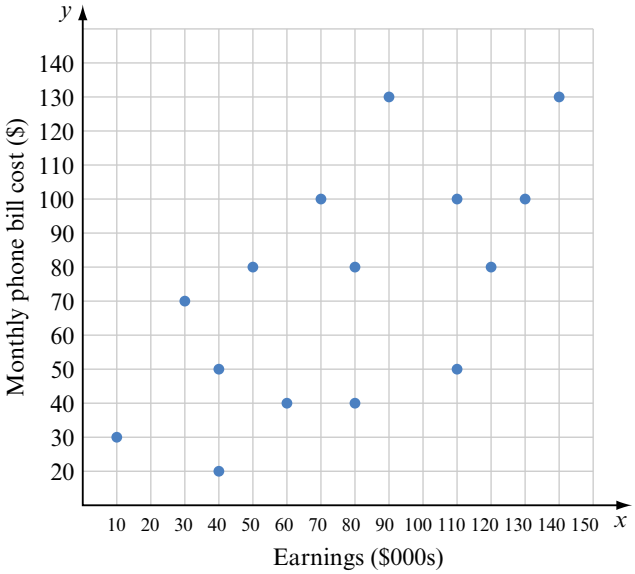
Number of different types of zoo animal	9	26	11	37	19	42	15	14	25	32	47	34	31	37	22	18
Number of visitors per day	42	261	150	541	247	387	226	291	317	401	761	396	309	311	177	151

b Draw a scatterplot to represent this data.

c Describe the trend shown by the scatterplot. Compare to your answer to part a.

5 Consider these two scatterplots





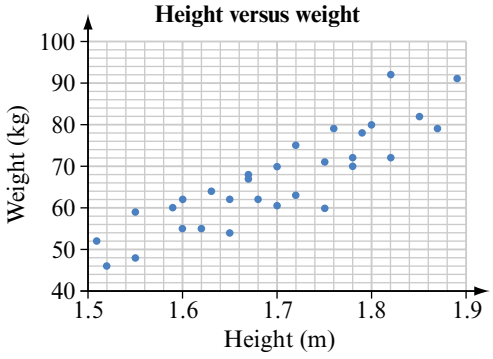
a Describe the strength and direction of each relationship in the context of what it shows.

b If a day was 26°C, how many heaters do you predict would be bought?

c If somebody had an income of $100 000, what would you predict their monthly phone bill to be?

d How confident are you of your predictions? Explain why you would not be as confident predicting a value using scatterplot B as you would be predicting a value using scatterplot A.

6 Consider this scatterplot, showing the relationship between height and weight of a Year 10 class.



a Describe the trend shown.

b What is the weight of the person who has a height of:

i 165 cm? **ii** 175 cm? **iii** 180 cm?

What if you wanted to predict the weight of a person who was 185 cm tall? If there is no actual data point, you can draw a **line of best fit** and use it to make a prediction. When drawing a line of best fit 'by eye', it is important that the data points are balanced above and below it.

c Why does it not matter if none of the original data points sit on the line of best fit?

To use a line of best fit to make a prediction, you can read straight from the graph.

d Use your chosen line of best fit to predict the weight of a person who is 185 cm tall.

It is usually best to find the equation of the straight line and use it to substitute in values. This is most easily done if some of the original data points fall exactly on the straight line, but can also be done by choosing two random points on the line, spread as far apart as possible.

e Why is more accurate to choose two points on a straight line that are as far apart as possible?

The line of best fit for this scatterplot passes through the points (1.65, 62) and (1.85, 82).

f Use the formula $y - y_1 = m(x - x_1)$ and the two data points that the line passes through to find the equation of the line of best fit. See Example 4D-4 on page 181 if you need help.

g Use the equation you found in part **f** to predict the weight of a person who is 185 cm tall. How does this compare to your answer to part **d**?

h How does a line of best fit help when there are multiple y values for a single x value?

i Use your answers to parts **b** and **c** to predict the weight of a person who is 170 cm tall.

7 Consider this data.

Number of apps	21	6	17	21	8	49	2	31	22	13	4	6	42	36	25	9	30	17	21	14
Space available on device (GB)	13	10	8	9	18	2	14	1	5	21	22	18	9	2	10	11	13	15	3	10

a Draw a scatterplot to represent the data.

b Describe the relationship between the two variables.

c Draw in a line of best fit and find its equation.

d If somebody had 30 apps, how much space would you predict is available on their device?

e Explain why you can't have much confidence in your prediction.

f Can you provide a reason why the relationship is weak?

Predicting a value within the data set, as in part **d**, is called interpolation. Sometimes you may want to predict a value outside the data set. This is called extrapolation.

g Identify whether these predictions are interpolations or extrapolations.

- **i** There will be 5 GB available on a device with 40 apps.
- **ii** There will be 2 GB available on a device with 60 apps.
- **iii** There will be 15 GB available on a device with 20 apps.
- **iv** There will be 1 GB available on a device with 50 apps.

h Imagine that somebody had 200 apps. If the graph continued its current downward trend, what prediction would you make about available space?

i Comment on your answer to part **h**. How confident can you be about making a prediction outside this data set?

8 When examining the relationship between two variables in real life, you think about how one variable changes depending on how the other changes. Height and weight are a good example of this.

a Would you say that the height of a person depends on how much they weigh? Or is it that the weight of a person depends on how tall they are?

PROBLEM SOLVING AND REASONING

Another way to look at this is to think about what effect changes in one variable would have.

b Would you say that as a person gets taller, this generally makes them heavier? Or as a person gets heavier, this generally makes them taller?

When a variable depends on another variable, it is called the dependent variable, for obvious reasons. The other variable is the independent variable.

c Of height and weight, which is:

- **i** the dependent variable?
- **ii** the independent variable?

Time is a well-known independent variable.

d Why do you think time is always the independent variable?

e Which axis (x or y) is time usually graphed on?

f Use your answer to part **e** to state which axis (x or y) the independent variable should be graphed on.

g When you are presented with a table of values, which variable is always in the first row, x or y?

h Use your answer to part **g** to state which row of a table (top or bottom) the independent variable should be in.

i Use your answers to parts **f** and **h** to explain how, if you are not certain which variable is dependent and which is independent, you can determine it by presentation of the data (in a table or a graph).

9 For each of these pairs of variables, determine:

- **i** if there might be a relationship between them
- **ii** if there is a correlation, which is the independent variable and which is the dependent variable
- **iii** the strength and direction of the correlation.

a time (years) and height of a tree (metres)

b value of a car (dollars) and age (years)

c weight of a person (kilograms) and number of languages they speak

d shoe size and height of a person (metres)

CHALLENGE

10 A strong positive correlation was found between the number of hours studied and percentage achieved on a test.

a Does this mean that if you study for many hours you will get high marks? Explain.

To help prevent assumptions like the one made in part **a**, there is a saying: 'correlation does not mean causation'.

b What does this mean?

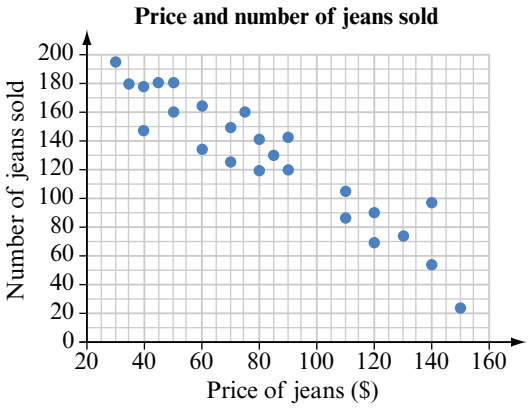
c Explain why the saying is useful to remember, particularly when dealing with strong correlations.

Reflect

How are scatterplots useful in understanding and describing bivariate data?

10F Interpreting bivariate data

Start thinking!



Consider this scatterplot, showing the relationship between the price of jeans and the number that are sold per day.

1 Describe the trend shown in the scatterplot.

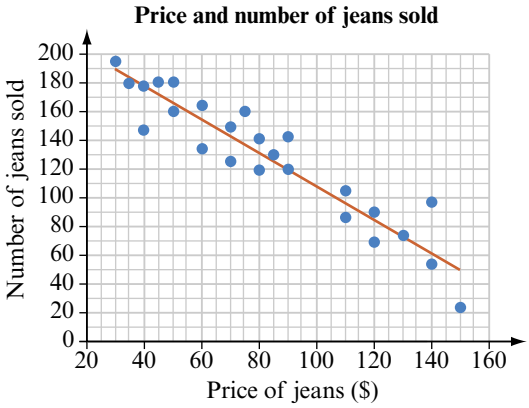
2 Copy the scatterplot and draw in a line of best fit.

3 Use the formula $y - y_1 = m(x - x_1)$ to find the equation of the line of best fit.

4 Use your equation to find the number of jeans sold if they cost $100.

Drawing in a line of best fit by eye is not the most accurate way to model the data: even if your line passes through two of the original points, another person's may not and therefore answers may differ. It is best to use digital technology (such as an appropriate calculator) to find the line of best fit. This technique is called **linear regression**.

Now consider this version of the same scatterplot.



5 Copy the scatterplot and the line of best fit.

6 Draw a vertical line between each data point and the line of best fit. This is called an error line.

7 Measure the lengths of each of these error lines, making each error line above the line of best fit positive (+) and each error line below the line of best fit negative (−).

8 Add together these positive and negative numbers. What do you find?

9 Explain why the most accurate line of best fit will have an error line sum as close to zero as possible.

KEY IDEAS

- A line of best fit can be added to a scatterplot to help predict the value of one variable when the other is known.
- Adding a line of best fit to data with a linear relationship is known as linear regression.
- The most common method for linear regression is least-squares regression.
- The equation of the line of best fit can be found using the formula $y - y_1 = m(x - x_1)$ or using technology.
- The known variable can then be substituted into the equation to predict the value of the unknown variable.

EXERCISE 10F Interpreting bivariate relationships

EXAMPLE 10F-1 Using linear regression

Use digital technology to graph this data and find the equation of the line of best fit (correct to one decimal place).

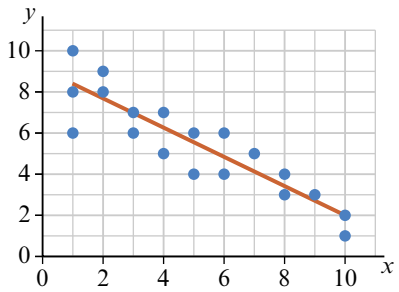
x	1	5	8	3	10	2	7	4	1	9	10	3	5	2	1	8	4
y	10	4	3	6	1	8	5	7	8	3	2	7	6	9	6	4	5

THINK

1 Enter the data into your calculator in the appropriate section and view the graph.

2 Find the linear regression function on your calculator and use it to add the line of best fit to the graph and view its equation. Check with your teacher if you are unsure how to do this. Write the equation correct to one decimal place.

WRITE



$y = -0.7048x + 8.9978$

$y = -0.7x + 9.0$ (correct to one decimal place)

UNDERSTANDING AND FLUENCY

1 For each of these data sets, use digital technology to:

i construct a scatterplot ii draw a line of best fit

iii find the equation of the line of best fit (correct to one decimal place).

a

x	3	2	7	5	6	3	1	4	10	9	7	4	8	10	1	5	7	3	8	6
y	5	4	7	6	8	2	2	3	10	9	9	8	6	8	5	4	4	6	10	5

b

x	4	7	1	2	10	5	8	6	2	9	4	3	10	7	1	9	8	5
y	5	4	9	9	2	6	4	5	7	4	8	8	1	6	8	1	5	7

c

x	6	8	3	9	11	14	5	1	15	13	8	5	11	12	7	4
y	5	10	5	6	11	9	3	2	11	13	8	7	8	10	3	1

d

x	2	8	4	6	7	7	5	1	4	5	3	9	4	8	6	2
y	4	8	9	5	4	6	7	2	8	9	4	10	5	6	9	4

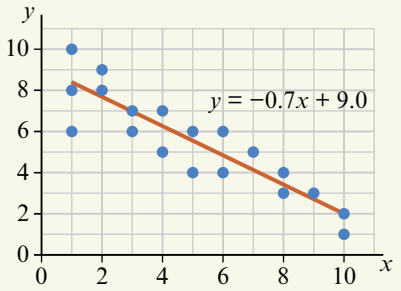
e

x	4	9	13	15	7	6	18	11	13	7	9	10	18	12	16	17
y	42	29	15	6	38	36	1	13	18	40	20	12	4	16	9	7

EXAMPLE 10F-2

Using linear regression to make predictions

Use the equation for the line of best fit for Example 10F-1 to predict the value of y when $x = 3$.



THINK

1 Write the equation for the line of best fit.

2 Substitute the value of $x = 3$ into the equation manually or in your calculator to find the predicted value of y.

WRITE

$y = -0.7x + 9.0$

$= -0.7 \times 3 + 9.0$

$= 6.9$

The predicted value of y is 6.9.

UNDERSTANDING AND FLUENCY

2 Use each equation to find the value of y for the given value of x, shown in brackets.

a $y = 21.2x + 6.9$ $(x = 5)$

b $y = -1.87x + 4.44$ $(x = 3.2)$

c $y = 19.56x - 67.22$ $(x = 19.8)$

d $y = -11.52x - 7.30$ $(x = 21)$

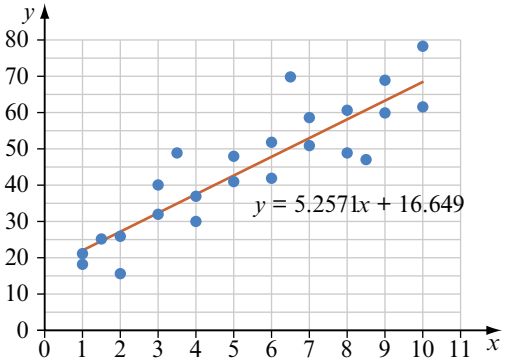
3 Use the line of best fit shown on this scatterplot to predict the value of:

a x when $y = 36$

b x when $y = 73$

c y when $x = 4$

d y when $x = 9$



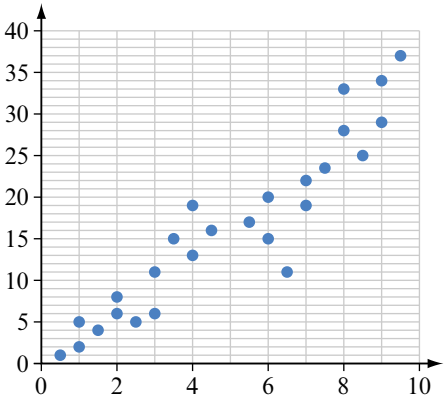
4 Use each line of best fit you found in question **1** to predict the value of y when $x = 10$.

5 For each scatterplot:

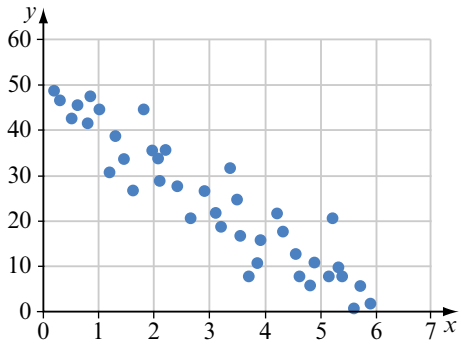
i draw in a line of best fit and find its equation (correct to one decimal place).

ii use the equation to predict the value of y when $x = 5$.

a



b



6 Andrew wanted to investigate the relationship between the number of books read in a year and the score in an English exam.

a What relationship do you predict exists?

b State which is the independent variable and which is the dependent variable.

Andrew collected this data.

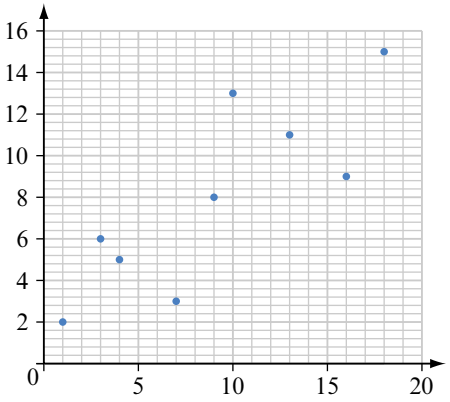
Number of books read	11	6	21	8	2	7	9	15	16	1	0	4	1	0	12	29
English exam mark	70	60	89	70	47	67	65	86	75	45	55	52	62	30	77	95

c Create a scatterplot to represent the data.

d Use a calculator or other means to find a line of best fit.

e Andrew wanted to know how many books he should read in a year to aim for a mark of 70% in the English exam. What would you tell him?

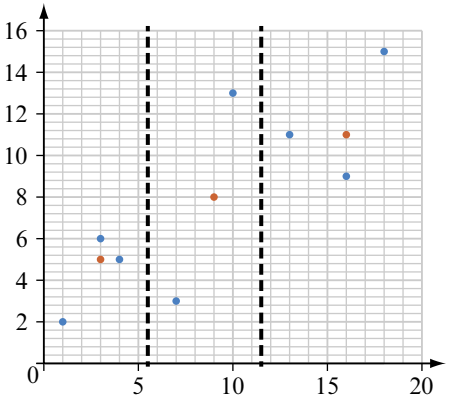
7 Another common method to find a line of best fit is called **median–median regression** or **three-median regression**.



a Copy this scatterplot and divide it vertically into three sections so that each section has the same number of points.

b If there was one extra point, which section would you include it in? What about if there were two extra points; which sections would you include them in? Explain.

For this method you need to find the median of each section. This is obtained by finding each median x-coordinate and median y-coordinate and marking these points on the graph.



c Show that the points marked in on this scatterplot are indeed the medians of each section.

d Draw in the line of best fit by placing your ruler on the lower median and the upper median, then moving it $\frac{1}{3}$ of the way toward the middle median.

e Find the equation of this line of best fit.

f Use your calculator to find the equation of the line of best fit using three-median regression. How does this equation compare to what you calculated in part **e**?

g Challenge! Explain how the gradient of the line can be found using the formula $m = \frac{y_U - y_L}{x_U - x_L}$, and the y-intercept can be found using the formula $c = \frac{1}{3}[(y_L + y_M + y_U) - m(x_L + x_M + x_U)]$.

h Explain why the three-median regression line is best used when a scatterplot has one or more outliers.

8 For each data set:

- **a** draw a scatterplot
- **b** use least-squares regression to draw a line of best fit and write down its equation
- **c** use three-median regression to draw a line of best fit and write down its equation
- **d** use both equations to predict the value of y when $x = 5$
- **e** compare the answers you obtained in part **d**. Which prediction do you trust most? Explain.

i

x	11	6	14	8	13	7	16	3	19	8	18	3	6	4	7	10	12	1
y	32	16	49	27	40	29	58	24	62	36	64	12	23	17	24	37	42	6

ii

x	3	6	4	1	2	8	9	7	4	8	7	19	9	4	3	6	2	1
y	5	11	7	2	4	10	12	12	10	15	10	16	15	5	9	9	6	5

iii

x	2	9	7	8	6	4	4	6	7	3	8	2	1	6	4	9	8	3
y	19	2	5	7	9	11	12	22	2	15	4	17	23	7	15	1	3	18

9 For each data set:

- **a** draw a scatterplot
- **b** choose three-median or least-squares regression to draw a line of best fit and write its equation
- **c** use the chosen equation to predict the value of y when $x = 7$

i

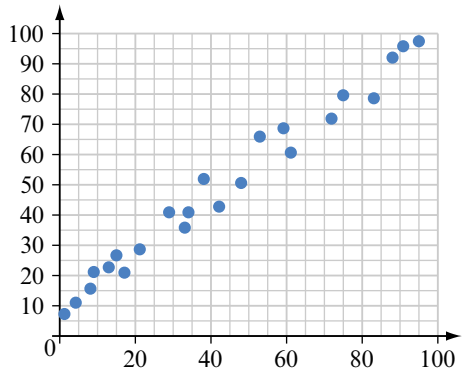
x	2	3	4	8	6	4	7	9	2	9	6	5	3	7	8	5	1	2
y	10	13	16	22	19	14	25	33	13	26	8	20	15	20	26	22	9	8

ii

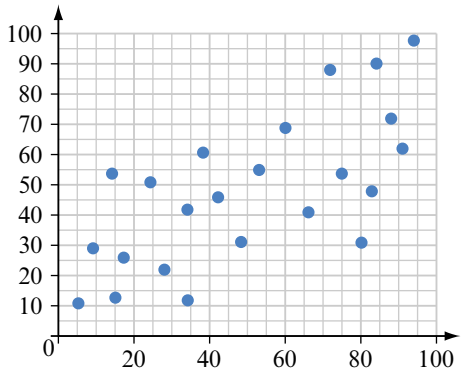
x	6	8	5	8	2	6	4	1	8	3	5	4	6	5	8	9	4	3
y	33	19	30	12	45	23	35	49	9	47	26	41	25	24	15	10	30	42

10 Consider these two scatterplots.

Scatterplot A



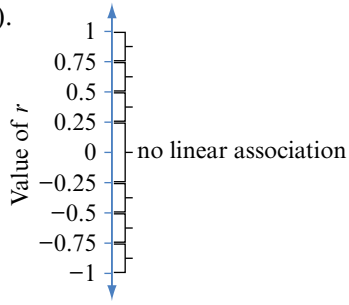
Scatterplot B



- **a** Describe the difference between the two scatterplots.
- **b** Draw a line of best fit for each graph and use it to predict the value of y when $x = 8$.
- **c** In which prediction are you more confident? Explain.

PROBLEM SOLVING AND REASONING

The strength of a bivariate relationship (its correlation) can be described using the **correlation coefficient** (represented by the pronumeral r). Scatterplot A has $r = 0.9$ and scatterplot B has $r = 0.6$.

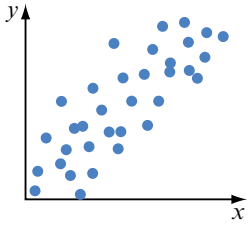


d Explain how you think the value of the correlation coefficient might work.

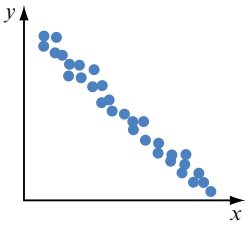
e Copy and complete this figure using the words *strong*, *moderate*, *weak* and either *positive linear association* or *negative linear association*.

f Estimate the correlation coefficient for these graphs.

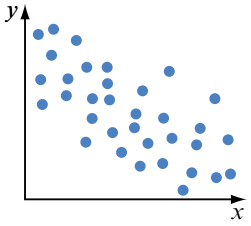
i



ii



iii

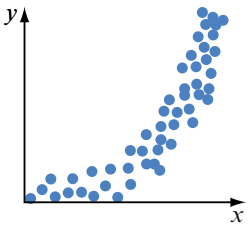


It is best to use digital technologies to calculate the correlation coefficient. Check with your teacher how to do this.

g Estimate and then use your calculator to calculate the correlation coefficient for both scatterplot A and scatterplot B. How do your estimates compare?

CHALLENGE

11 Consider this scatterplot.



a Explain why linear regression is not useful in this case.

Lines of best fit are not limited to linear graphs. They can be produced for a range of other shapes, such as quadratic, exponential, cubic, trigonometric and logarithmic graphs.

b Explain why logarithmic regression must be used in order to draw a line of best fit for this scatterplot.

c For each data set below:

i draw a scatterplot

ii identify the graph as fitting one of the shapes mentioned above.

iii use digital technology to produce a line of best fit. Check with your teacher how to access each regression technique.

c

x	4	6	3	8	9	5	7	6	4	3	2	1	2	1	6	8	7	5
y	1	3	0	5	24	1	13	7	3	3	8	12	4	8	5	19	9	2

d

x	7	9	6	3	5	4	4	7	8	9	10	5	3	4	6	7	9	8
y	15	70	10	0	6	5	7	16	30	65	130	9	2	10	9	19	75	37

e

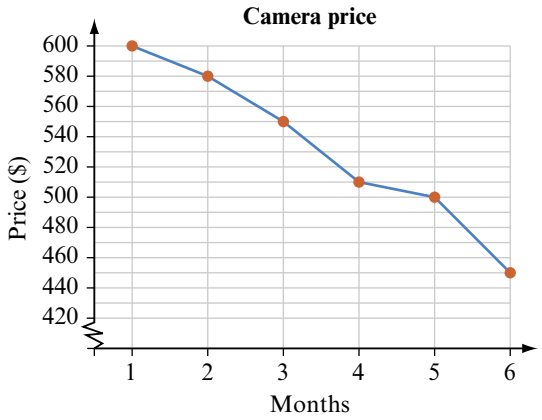
x	5	7	9	1	2	4	3	4	2	8	6	5	4	5	3	6	7	1
y	2	9	2	8	2	8	4	9	1	4	3	3	10	4	3	4	8	9

Reflect

When and how is a line of best fit useful?

10G Time series

Start thinking!



Consider this plot.

1 What is it showing?

2 What is the independent variable? (Hint: it is the variable on the x-axis.)

3 What is the dependent variable?

4 What do you notice about the intervals at which the data is taken?

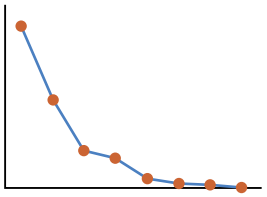
A **time series** is a sequence of data taken at regular intervals.

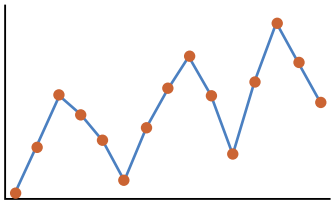
5 Give three examples of regular time intervals.

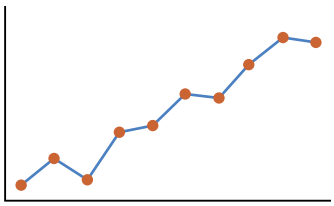
6 Give three examples of things that can be recorded over regular time intervals.

7 Explain why a line graph should be used to display this bivariate data rather than a scatterplot.

A time series can be described in the same way as a scatterplot. Consider these plots.







8 Classify each plot as either showing a linear (straight line), non-linear or seasonal (cyclic) trend. Explain your reasoning.

9 Explain why a time series does not need to be a perfect line in order to be classified as linear.

10 Look carefully at the plot you described as showing a seasonal trend. It has clear cycles. Would you say the cycles are moving upward, downward, or not moving (are stationary)?

11 Describe the other two plots as having an upward or downward trend.

Just like in scatterplots, you can describe time series plots in context: as one variable increases, the other increases or decreases.

12 What is the variable that you would always mention first as increasing when describing a time series plot?

13 Describe the trend shown in the first time series plot.

KEY IDEAS

- Time series data is bivariate data where the independent variable is time.
- Time series data is usually shown as a **line graph**.
- Trends shown in a time series plot can be described as linear, non-linear or seasonal, and upward or downward.

EXERCISE 10G Time series

EXAMPLE 10G-1 Plotting time series data

Plot this time series data, showing the number of customers in a store over an 8-week period.

Week	1	2	3	4	5	6	7	8
Number of customers	229	298	442	512	731	822	841	945

THINK

Plot the data on an appropriate set of axes, joining successive points with a straight line. Ensure your graph has an even scale on the axes and a title.

WRITE

UNDERSTANDING AND FLUENCY

1 Plot each data set.

a

Hours after midday	1	2	3	4	5	6	7	8	9	10
Temperature	34	36	37	35	34	31	27	22	19	17

b

Years of experience	1	2	3	4	5	6	7	8	9	10	11	12
Salary ('000s $)	42	44	46	49	51	54	56	59	62	65	68	72

c

Days after rain	0	5	10	15	20	25	30	35
Water storage (L)	456	421	362	321	298	254	221	182

d

Weeks experience	0	1	2	3	4	5	6	7
Time to complete task (mins)	98	71	62	55	50	46	43	42

e

Month	1	2	3	4	5	6	7	8	9	10	11	12
Sales ($00)	26	32	29	31	35	32	31	34	37	39	36	40

f

Week	1	2	3	4	5	6	7	8	9
Hours worked	36	42	38	35	38	41	48	31	38

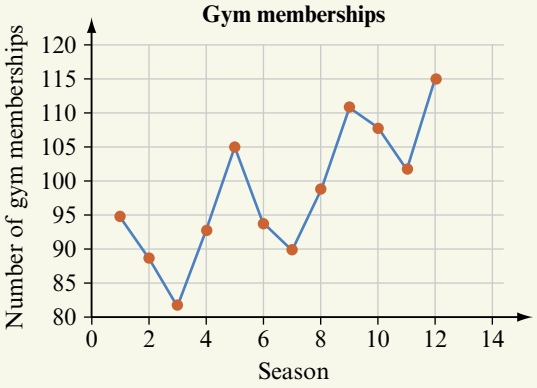
EXAMPLE 10G-2

Describing the trend for a time series

Describe the trend shown in this time series plot as:

a linear, non–linear or seasonal

b upward, downward or stationary.



THINK

a Look at the graph. Is it linear (roughly a straight line), non-linear (curved) or seasonal (showing regularly spaced peaks and troughs)?

b Look at the graph. Is it moving upward, downward, or staying roughly the same?

WRITE

a The graph is seasonal.

b The graph is moving upward.

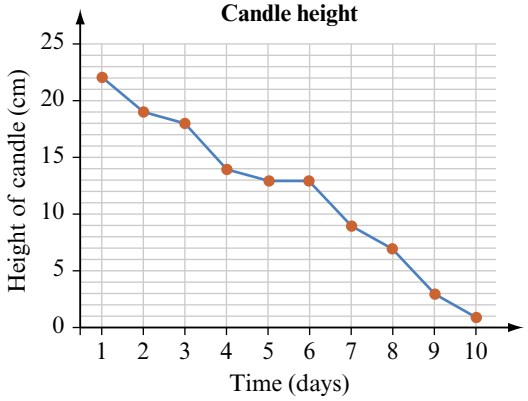
UNDERSTANDING AND FLUENCY

2 For each plot, describe the trend as:

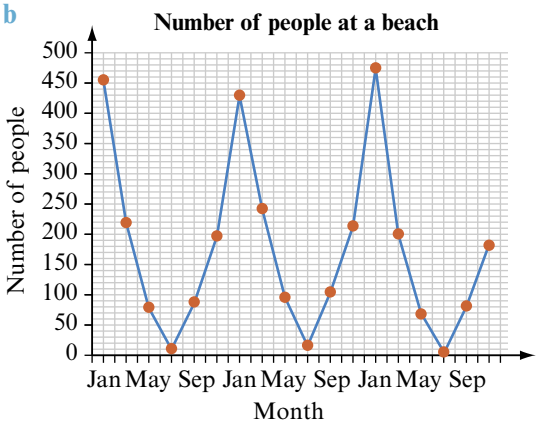
i linear, non-linear or seasonal

ii upward, downward or stationary.

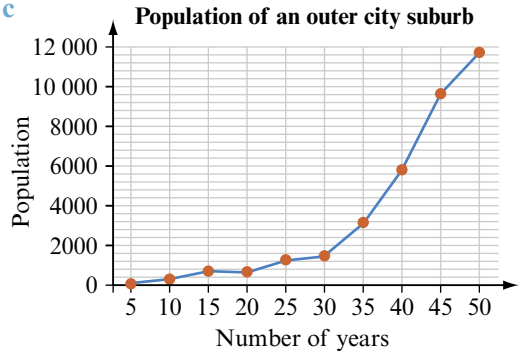
a



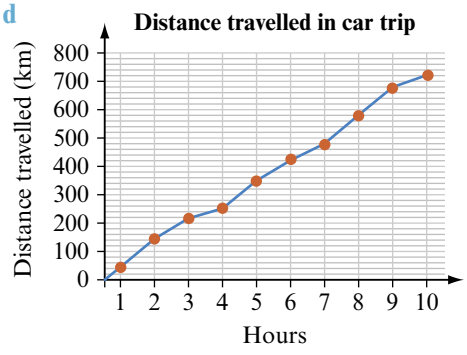
b



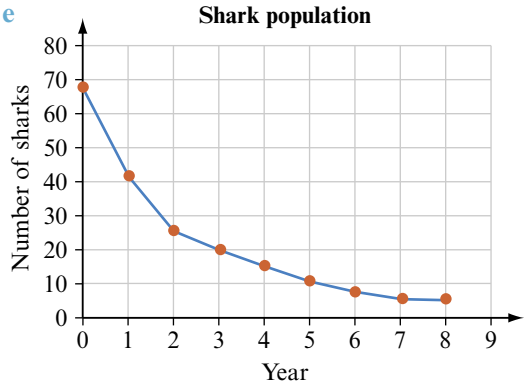
c



d



e



f

3 Describe any trend shown in the time series plots from question **1**.

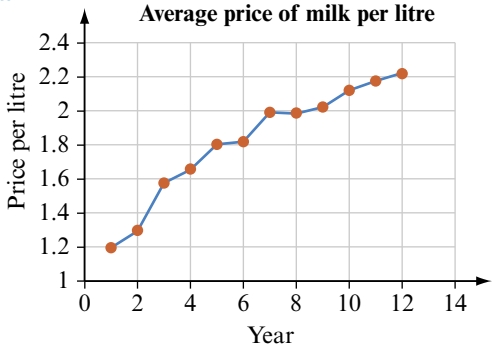
EXAMPLE 10G-3 Describing time series data

Describe the time series data from Example 10G-1 in context.

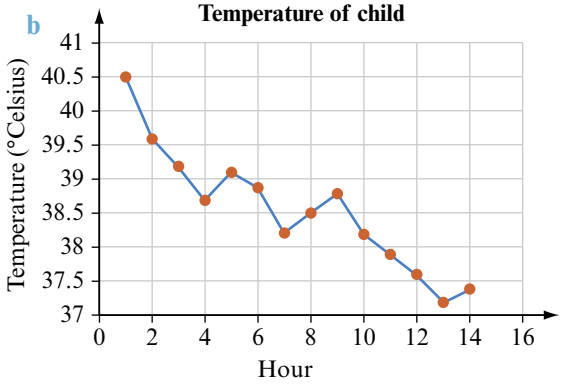
THINK	WRITE
1 Look at the graph. Decide if it shows a linear, non-linear or seasonal trend, and if this trend is upward or downward.	
2 Describe the graph in context.	As time increases, the number of customers also increases in a linear trend.

4 Describe each time series in context.

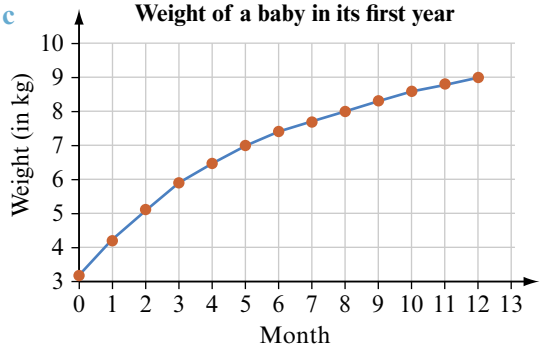
a



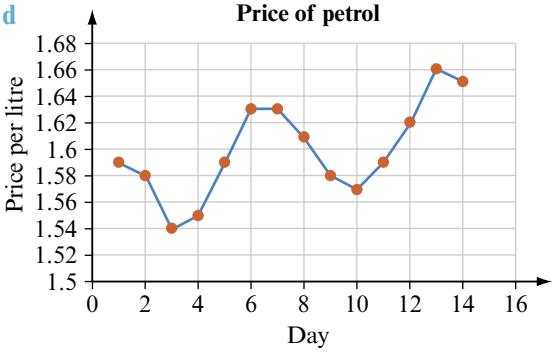
b



c



d



5 Describe each of the time series plots from question **2** in context.

6 Describe each of the time series plots from question **1** in context.

7 Consider this data set, showing the balance of a bank account over a 10-week period, to the nearest dollar.

Week	1	2	3	4	5	6	7	8	9	10
Balance ($)	787	542	312	116	816	678	457	298	998	749

a Plot the time series data and describe the trend shown.

b Over the time period, what was:

i the minimum balance? **ii** the maximum balance?

c Suggest a reason for the cycles shown in the graph.

8 Consider this data set, showing the growth of a tree over a number of years.

Time (years)	1	2	3	4	5	6	7	8
Tree height (m)	0.4	1.2	2.3	3.5	4.5	5.6	6.8	7.9

a Plot the time series data.

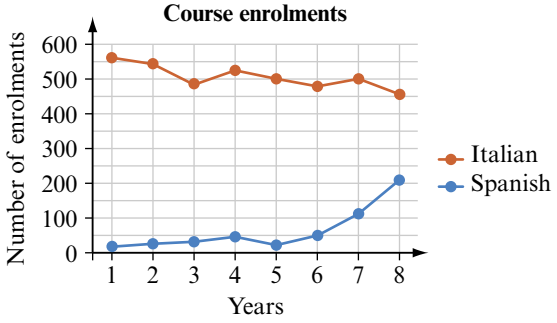
b Describe the trend shown in the plot.

c How tall would you predict the tree to be:

i at 10 years old? **ii** at 20 years old?

d Explain why you would be more confident in your first prediction than in your second.

9 Consider this graph.



a What is it showing?

b Describe the trend shown in each plot.

c At what point would you predict that Spanish has more enrolments?

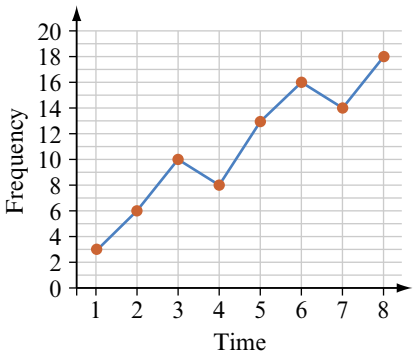
10 Even though these graphs are line graphs, a line of best fit can still be added to the plots in order to make predictions. For each of these plots:

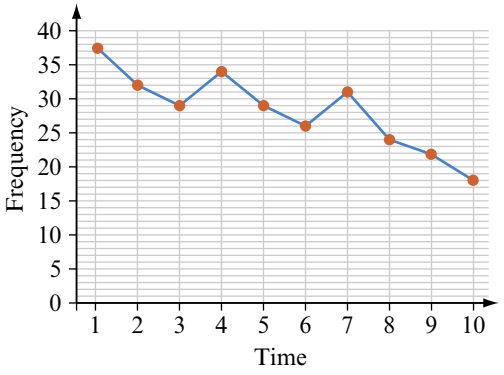
a copy the graph into your book

b add a line of best fit by eye or by using technology

c use the line of best fit to predict the value of y when:

i $x = 5$ **ii** $x = 15$.

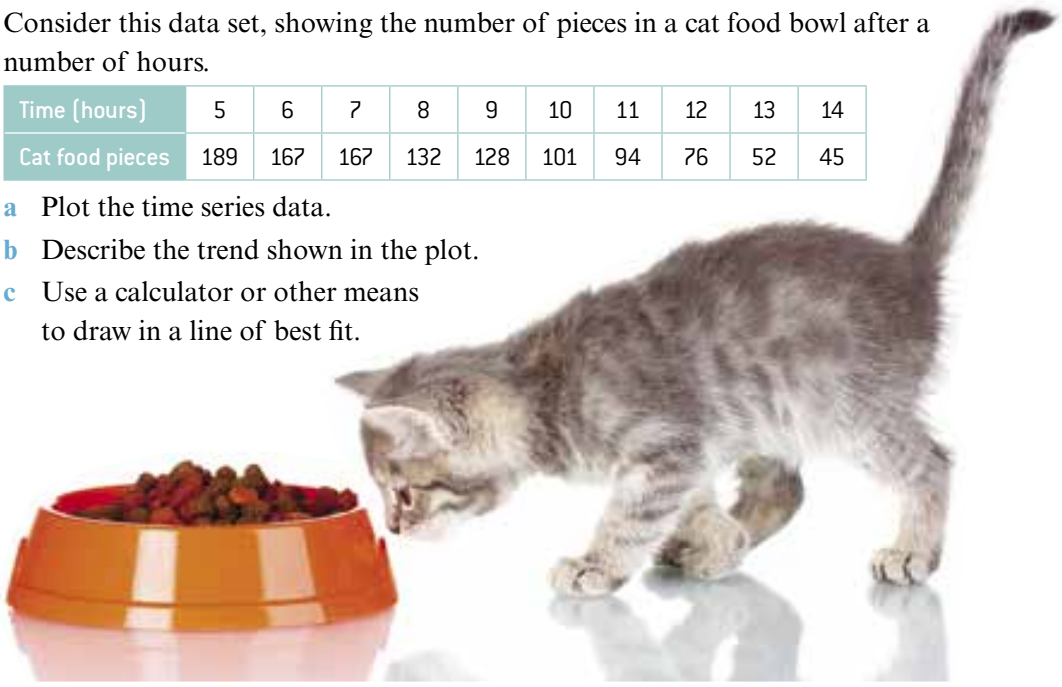




PROBLEM SOLVING AND REASONING

11 Consider this data set, showing the number of pieces in a cat food bowl after a number of hours.

Time (hours)	5	6	7	8	9	10	11	12	13	14
Cat food pieces	189	167	167	132	128	101	94	76	52	45



a Plot the time series data.

b Describe the trend shown in the plot.

c Use a calculator or other means to draw in a line of best fit.

d Use your line of best fit to predict:

i the number of cat food pieces originally in the bowl

ii the number of hours until the cat finishes all the food in the bowl.

12 Choose something to investigate for which you can create a time series plot (for example, number of people in a canteen, heart rate, etc.).

a Collect a sequence of at least 10 measurements and present them in a table.

b Plot the data you have collected onto an appropriate set of axes.

c Describe the trend shown in your time series plot.

d Add a line of best fit by eye or by using technology

e Use your line of best fit to make a prediction of your choice.

f Collect the actual measurement of your prediction (if this is not possible, you may wish to redo your answer to part **e**).

g Write a brief sentence comparing your prediction and the actual measurement.

CHALLENGE

13 Consider this data set, showing the number of hours spent studying for a Maths exam and the mark for the exam.

Time spent studying (hours)	0	1	2	3	4	5	6	7	8	2	6
Maths exam mark (%)	35	42	56	83	71	76	88	92	70	67	72

a Explain why a scatterplot must be used to plot this data.

b Explain why, although time is the independent variable, this is not a time series plot.

c Plot the data and describe the trend shown.

d Explain why, even if the last two data points were not included in the set, a scatterplot must still be used to plot the data.

Reflect

How are times series plots similar to and different from scatterplots?

10H Analysing reported statistics

Start thinking!

The media often report statistics to support claims; for example, '9 in 10 nutritionists recommend that kids eat Weetbix'. But how do you know if you can trust the statistics used?
There are two main components to consider when deciding if reported statistics are to be trusted: how the information is gathered and how this information is interpreted.

1 Brainstorm ways in which the Weetbix claim might be skewed.

2 With a classmate, brainstorm the answers to these questions. You may wish to use the glossary to help you.

- **a** What is the difference between **primary** and **secondary data**?
- **b** What is the difference between a **census** and a **sample**?
- **c** What are the advantages and disadvantages of the three main types of sampling (**stratified**, **systematic** and **random**)?
- **d** What does it mean if data or a survey is **biased**?
- **e** How could a sample size of 20 be considered a good sample in one scenario but not in another?

Imagine that you wanted to investigate the most popular music genre in your community.

3 Do you have a particular outcome you would prefer? If so, what is it?

4 What would be a good sample size for this investigation? Explain.

5
- **a** Write a question for this investigation that will give you fair results.
- **b** Write a question for this investigation that will give you the results that you want.
- **c** Explain the difference between your two questions.

6
- **a** Provide an example of how to collect the data in order to get fair results that represent the whole community.
- **b** Provide an example of how to collect the data that will provide biased results.
- **c** Explain why taking a large, random sample from your school would lead to biased results.

KEY IDEAS

- When evaluating a report or claim, you need to consider how the information was collected and also how the information has been interpreted.
- Data needs to taken from an unbiased sample that is as large as possible using fair techniques.
- All data collected needs to be represented fairly, so as not to skew the outcome. Presenting only some of the information, or choosing inaccurate statistics to describe the data, is misleading.

EXERCISE 10H Analysing reported statistics

EXAMPLE 10H-1 Evaluating surveys

Decide if each survey could provide fair or biased results, giving a reason for your answer.

a Asking every 20th person in the local phone book if they enjoy running to determine how many people in the community exercise.

b Surveying 1000 random households in Sydney in order to find the average number of people living in a home in NSW.

THINK	WRITE
a **1** Consider the surveying technique. Is the sample large enough and gathered in an unbiased way so that it should theoretically represent the population (the local community)?	a The sample is randomly selected from the community and is of sufficient size. However, this survey is likely to provide biased results, as the question does not ask how many people exercise, just how many enjoy running.
2 Consider the question asked. Does it answer what the person is trying to discover (how many people exercise?)	
b Consider the surveying technique. Is the sample large enough and gathered in an unbiased way so that it should theoretically represent the population (NSW)?	b This survey is likely to provide biased results because, although it is a reasonable size, it only samples from one city in the whole of NSW, and is therefore unlikely to be representative of the population.

UNDERSTANDING AND FLUENCY

1 Decide if each survey would provide fair or biased results, giving a reason for your answer.

a Asking 500 random people at an AFL match which is their favourite sport in order to determine the country's favourite sport.

b Asking everybody at your school their opinion on a school issue.

c Asking every 10th person on the electoral roll if they like dogs to determine the most popular pet.

d Asking a stratified sample (for gender) of 20 from around the country who they think should be prime minister at the next election.

e Asking 10 000 people randomly selected in the country their opinion on an issue in your community.

f Asking everybody you see at a local shopping mall their opinion on a global issue.

EXAMPLE 10H-2 Explaining how a statement may misrepresent data

Explain how the statement 'Cricket deemed world's most popular sport' may misrepresent the data that 67% of people surveyed at cricket matches in over 15 countries gave cricket as their favourite sport.

THINK	WRITE
1 Look at the sample chosen. Does it reflect the population (the world)?	This statement misrepresents the data by using a biased sample, people at cricket matches, who are much more likely to give cricket as their favourite sport than people not attending a cricket match.
2 Look at the question or result given. Is the conclusion drawn fair?	

2 Explain how each headline may misrepresent the data.

- **a** 'Soft rock the most popular music genre' (70% of people surveyed said they enjoyed listening to soft rock)
- **b** 'New Apple tablet tops popularity stakes' (48 people out of 50 surveyed in an Apple store said it was their favourite)
- **c** 'Women now earn more than men' (from 10 people randomly surveyed on a street).
- **d** 'New anti-aging cream is unbeatable' (100% of the company's shareholders surveyed agree).

EXAMPLE 10H-3 Analysing surveys and results

Caleb wants to know the average age of people at a skate park. He surveyed every 10th person he saw there over one week, surveying a total of 20 people. Summary statistics gave a mode of 17, a median of 18, a mean of 22 and a range of 80. Caleb concluded the average age of people at the skate park was 22.

a Decide if the data collection method is fair **b** Decide if the interpretation of the data is fair.
c If appropriate, provide a suggestion to improve the survey.

THINK	WRITE
a Caleb surveyed every 10th person over one week for a total of 20 people. Is this fair?	**a** The method appears fair, though the sample could be larger.
b Caleb concluded that the average age was 22. Is this fair?	**b** The interpretation could be improved. The sample size is small and the mean is distinctly larger than the median, implying the presence of an outlier, which skews results.
c What could Caleb do to improve this survey?	**c** Caleb could either take a larger sample or use the median as a measure of centre.

UNDERSTANDING AND FLUENCY

3 For each scenario:

- **i** decide if the data collection method is fair
- **ii** decide if the interpretation of the data is fair
- **iii** if appropriate, provide a suggestion to improve the survey.

a Tess wanted to know which is the most popular beach in NSW. She asked every 10th person on the electoral roll in her town 'What is your favourite beach in NSW?'. Seventy-five per cent of 200 people said that Bondi was their favourite beach, so Tess concluded that Bondi was the most popular beach in NSW.

b Stuart wanted to know the average height of basketball players in his town. He surveyed 10 players at random. Summary statistics gave a mode of 1.82 m, a median of 1.80 m, a mean of 1.79 m and a range of 15 cm. Stuart concluded the average height of basketball players in his town was 1.80 m.

c Isaac wanted to know the most popular computer brands. He asked a stratified sample of 1000 people 'Do you like Macs?' and found that 60% of people said yes. Isaac concluded that Mac computers were the most popular brand of computer.

d Chelsea wanted to know the most popular brand of soft drink in town. She asked all the local supermarkets which brand they sold the most of and found that Coke made up 37% of the sales, Pepsi 29%, Schweppes 17% and all other brands made up the remainder. She concluded that no one brand was the most popular.

e Sienna wanted to know the average number of hours people spent sitting down every day. She asked over 100 people she saw at random walking through a mall how many hours they sat down per day, and came up with an average of 8.5 hours.

f Kane wanted to know the most popular party foods to serve at an upcoming school dance. He surveyed every second person in his neighbourhood about their favourite party foods, and decided to serve party pies, chocolates and sandwiches.

PROBLEM SOLVING AND REASONING

4 A drug company claims that its new pain relief drug blocks pain for up to 18 hours. This is based on a sample of 15 people recording how many hours they were pain-free:

2	18	3	2	2
1	2.5	2	1.5	2.5
2	1.5	2	1.5	1.5

a Why are the company making this claim? Explain why it is misleading.

The company changes tactics and decides to claim that it blocks pain for an average of 3 hours.

b Calculate the three measures of centre for this data

c Comment on the drug company's claim. Is this claim now fair? Explain.

5 What questions should you ask yourself when being presented with a claim that uses statistics?

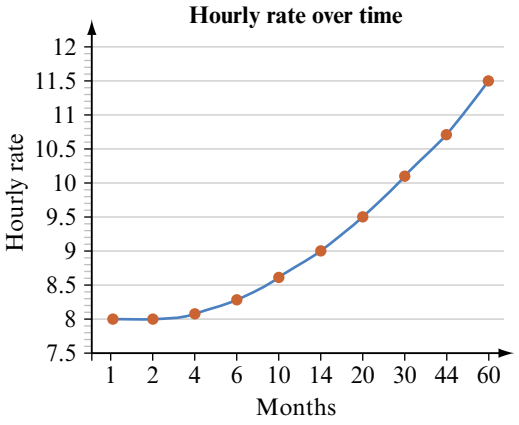
6 There is a saying that '87% of all statistics are made up'. Explain the humour in this.

7 Fatima's employer made this graph.

a What is this graph supposed to represent?

b Explain how the graph misrepresents the data.

c Redraw the graph as best you can so that it is fair.



8 Each of these media statements is followed by their 'fine print' statements in brackets. Explain why the claims are not as 'amazing' when you read the fine print.

a an advertisement for a phone's battery life claims it 'Lasts up to 7 days' (based on standby power consumption)

b An advertisement for toilet paper claims it is 'Voted Australia's favourite!' (a total of 42 people surveyed)

c An advertisement for a moisturising lotion states '67% of women notice a difference after just one week!' (compared to using regular soap)

9 Use the statistics shown in this table to comment on the number of refugees hosted by Australia in relation to the population as a whole, compared with other countries.

Country	Number of refugees hosted in 2012	Estimated total population in 2012
Australia	30 083	22 000 000
Germany	589 737	82 000 000
Jordan	302 707	6 000 000
Pakistan	1 638 456	174 000 000

*Refers to the number of people currently residing in the country of asylum. For most industrialised countries, this refers to the number of people recognised as refugees after arriving as asylum seekers over the past 10 years. Excludes resettled refugees. Source: https://www.refugeecouncil.org.au/r/stat-int.php

NOTE Find the number of refugees per million of the country's population.

10 Consider the table and text opposite, which were published online.

a Using the table; where does Australia rank in terms of:

i absolute numbers of road deaths? **ii** road deaths per 100 000 people?

b Why would it be misleading to say that Australia ranks 11th out of 30 countries?

c Why is it not useful to know that Australia ranked 7th out of 15 countries?

d Calculate the median for the first column. How do you know that there is data missing from the table?

e How does the missing data make it difficult to determine where Australia sits?

f The article does not provide any evidence to support its claims that Australia is 'falling behind the rest of the developed nations'. What could you do to research this?

Australia's road toll figures show Australia falling behind the rest of the developed nations.

In 2005, Australia ranked seventh best out of 15 nations (OECD nations) for road deaths per 100 million vehicle kilometres travelled (0.8 deaths). That might sound good, but in 2004 Australia ranked fourth and in 2003 it was third.

If we compare Australia to a greater pool of nations, we are ranked 11th. Between 2004 and 2005 there was a **seven per cent fall in the developed nations' median road death rate** per 100,000 population, but **Australian's rate rose two per cent**.

	Road deaths per 100,000 population	Population (millions)	Road deaths
Australia	**8.0**	**20.3**	**1,627**
Austria	9.3	8.2	768
Belgium	10.4	10.4	1,089
Canada	9.1	32.3	2,925
Czech Republic	12.6	10.2	1,286
Denmark	6.1	5.4	331
Finland	7.2	5.2	379
France	8.8	60.6	5,318
Germany	6.5	82.5	5,361
Great Britain	5.5	58.5	3,201
Greece	15.0	11.1	1,658
Hungary	12.7	10.1	1,278
Iceland	6.3	0.3	19
Ireland	9.5	4.2	396
Italy	–	–	–
Japan	6.2	127.8	7,931
Luxembourg	–	–	–
Netherlands	4.6	16.3	750
New Zealand	9.9	4.1	405
Norway	4.9	4.6	224
Poland	14.3	38.2	5,444
Portugal	11.8	10.6	1,247
Slovakia	–	–	–
Slovenia	12.9	2.0	258
South Korea	13.2	48.3	6,376
Spain	10.2	43.5	4,442
Sweden	4.9	9.0	440
Switzerland	5.5	7.4	409
Turkey	–	–	–
USA	14.7	296.4	43,443
OECD median	9.5		

Source: http://www.caradvice.com.au/2852/road-toll-statistics-australia-getting-worse/, viewed 13 March 2014

11 Find a media article that uses statistics (including at least one visual display) to support its claims, and analyse these claims. Write a paragraph commenting on how trustworthy you think the article is.

12 Write an informative guide to teach somebody how and what to look for when reading about statistics and the claims that they are supposedly supporting.

Reflect

What can you look for in a media report to determine if the statistics are to be trusted?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

median
median class
mean
mode
modal class
frequency table
cumulative frequency
class interval
midpoint
outlier
histogram
range
interquartile range
five-number summary
lower quartile
upper quartile
stem-and-leaf plot
back-to-back stem-and-leaf plot
box plot
whisker
positively skewed
negatively skewed
univariate data
bivariate data
scatterplot
correlation
line of best fit
interpolation
extrapolation
dependent variable
independent variable
time series
cyclic trend
upward trend
downward trend
primary data
secondary data
census
sample
biased data
standard deviation
normal distribution
bell curve
sigma
linear regression
least-squares regression
median–median regression
three-median regression
correlation coefficient

MULTIPLE-CHOICE

Questions 1–3 refer to this frequency distribution table.

Score (x)	Frequency (f)	$x \times f$	cf
10	3		
12	8		
14	4		

10A 1 What is the $x \times f$ value for the score 12?
A 30 B 12 C 8 D 96

10B 2 What is the range of scores?
A 3 B 4 C 5 D 11

10C 10A 3 What is the standard deviation?
A 12.1 B 2.0 C 1.4 D 4.0

10D 4 In a box plot, what does the 'box' represent?
A mean B interquartile range
C range D median

10D 5 The length from the lower quartile to the upper quartile in a box plot represents what percentage of the scores of the data set?
A 25% B 50% C 75% D 100%

10E 6 The line of best fit for a scatterplot has the equation $y = 1.7x + 1.3$. When x has a value of 1.5, what is the predicted value for y?
A 1.25 B 2.55 C 3.85 D 4.5

10F 10A 7 The relationship between two variables is strong and negative. What is its correlation coefficient closest to?
A 0.9 B −0.9 C 0.5 D −0.5

10H 8 How can data collected from the Internet be described?
A primary data B secondary data
C collected data D distributed data

SHORT ANSWER

10A **1** Consider this frequency distribution table.

Class interval	f
0–<10	3
10–<20	5
20–<30	12

a Calculate the mean.

b Give the modal class.

c Find the median class.

10B **2** For the data in this stem-and-leaf plot, which shows the number of ice creams sold per day, find:

Key 4|0 = 40

Stem	Leaf
4	0 1 3 5 7 9
5	2 5 5 7 8
6	2 3 3 5 7 7 9 9
7	3 5 6 7 8
8	2 4 6 7 8
9	3 6

a the range

b the median

c the interquartile range

d the five-number summary.

10C 10A **3** Below are results for five tests.

Lily: 72, 77, 67, 69, 75

Cassy: 47, 92, 87, 77, 52

a For each student, calculate:

i the mean

ii the standard deviation.

b Which of these two students performs more consistently?

10D **4** For the data in question **2**, draw a box plot and comment on its shape.

Questions **5** and **6** refer to this table. It shows the number of children enrolled at a day-care centre over the 12 years since its opening.

Years since opening	1	2	3	4	5	6	7	8	9	10	11	12
Number of children	25	27	26	33	32	35	37	39	40	39	40	45

10E **5** a Create a scatterplot for the data.

b Draw in the line of best fit by eye and use it to predict the number of children the day-care centre could expect to have after 20 years.

c Explain the problems in making such predictions.

10F 10A **6** a Enter the data into your calculator and view your graph.

b Find the equation of the line of best fit using linear regression or least-squares regression.

c Use your equation to predict the expected number of students after 20 years. Comment.

10G **7** This table shows the mean monthly temperature (°C) for a town in the years 2000 and 2015.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2000	28.8	29.0	27.4	26.9	23.1	21.5	19.8	20.5	24.5	24.5	27.3	27.6
2015	28.3	29.2	27.8	25.3	22.4	20.3	20.8	21.7	23.6	24.6	27.5	28.4

a Plot the two sets of data as time series on the same graph.

b Describe the trends of each.

c Compare the two graphs to describe the differences between them.

10H **8** Two brands of yoghurt display nutrition facts per 100-g serving on their packaging.

	Brand X 'light'	Brand Y '98% fat free'
Energy	425 kJ	397 kJ
Protein	5.8 g	5.0 g
Total fat	4.8 g	1.9 g
Carbohydrate	9.0	14.2 g
Sodium	80 mg	63 mg
Calcium	206 mg	177 mg

a Compare the information supplied.

b What would you consider the term 'light' to mean? Would you consider Brand X lives up to its classification?

The producers of Brand X defend the claim that their product is 'light' by producing a bar graph of the fat content of their product and their rival's.

c Draw a graph which they could use to support the claim. Explain.

MIXED PRACTICE

Questions 1–3 refer to this frequency distribution table.

x	f	$x \times f$	cf
3	6		
4	10		
5	8		

1 What is the number of scores in this data set?

A 12 **B** 18 **C** 24 **D** 40

2 What is the mean of the data closest to?

A 3 **B** 4 **C** 5 **D** 6

3 What is the median of the data?

A 3 **B** 4 **C** 5 **D** 10

Questions 4–8 refer to these scores:
8, 3, 9, 4, 6, 5, 7, 5, 4, 3, 6, 7, 4, 9, 8.

4 Write the scores in ascending order.

5 What is the median score?

6 Find the value of the interquartile range.

7 Write the five-number summary for the data set.

8 Draw a box plot from your five-number summary in question 7.

9 Which of these statements concerning a box plot is incorrect?

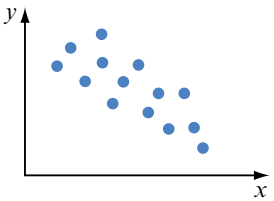
A The range of the data is represented from the lower point of the box plot to the upper point.

B It is not possible to calculate the mean of data displayed as a box plot.

C The median of the data can be found.

D It is possible to find the mode of the data.

Questions 10–12 refer to this scatterplot. It shows the relationship between two variables, x and y.



10 How could you describe the relationship between the two variables?

A strong **B** weak

C moderate **D** linear

11 Which of these terms also describes the relationship?

A positive **B** negative

C horizontal **D** vertical

12 What does the scatterplot show happens to the value of x when the value of y increases?

A stays the same **B** increases

C decreases

D can't be found from the graph

13 In a time series graph, what is the independent variable?

14 Describe the type of trend you would expect to find in a time series graph showing soup sales over the course of a year.

15 What is the difference between a scatterplot and a time series graph?

16 It is claimed that 99% of those surveyed recommended Nu-White toothpaste as the best toothpaste on the market. In a sample of 200 people, how many would this represent?

17 What techniques can you use to misrepresent data in a graph?

18 A survey was conducted by selecting the first person in the top left column of every page of the local telephone book. What type of sampling would this be?

10A **19** Use your calculator to find the standard deviation for the scores in the frequency distribution table shown for questions 1–3.

10A **20** Use your calculator to find the standard deviation for the scores shown for questions 4–8.

10A **21** What does the standard deviation in a set of data indicate?

A how large the mean is

B how the scores are spread out from the mean

C how many scores are in a data set

D where the median lies in the set

10A **22** Which statement is incorrect? The correlation coefficient between two variables can be:

A 1 **B** 0

C −1 **D** any number at all

Questions **23** and **24** refer to this table, which shows the relationship between two variables x and y.

x	4	5	7	10	8	6	9	8	4	9	8	7	5	9	9
y	8	11	14	20	17	12	23	31	11	24	6	18	13	18	24

10A **23** Use your calculator to determine the equation of the line of best fit using the least squares regression method.

10A **24** Calculate the value of the correlation coefficient.

ANALYSIS

Usain Bolt of Jamaica successfully defended his title and set a new race record of 9.63 seconds in the 100 m sprint in the London 2012 Olympic Games. A record of the winning times for this race has been kept since the early 1900s. The Olympic Games usually take place every four years, but were not held during the years of World War I or II, so there are some gaps in the record.

Consider this record of winning times for the 100 m sprint since 1952.

Year	1952	1956	1960	1964	1968	1972	1976	1980
Time (s)	10.79	10.62	10.32	10.06	9.95	10.14	10.06	10.25

Year	1984	1988	1992	1996	2000	2004	2008	2012
Time (s)	9.99	9.92	9.96	9.84	9.87	9.85	9.69	9.63

a From the raw data, describe the winning times over the period shown.

b Calculate the:

- **i** mean, median and mode
- **ii** range and interquartile range
- **iii** five-number summary.

c **i** Find the standard deviation of the set.

ii Comment on its value.

d **i** Draw a box plot to represent the data.

ii Comment on the shape of the box plot.

iii Explain how you think the shape of the box plot will change as future times are added to the data.

e Draw a scatterplot of the data.

f Draw the line of best fit, and choose two points to calculate its equation.

10A **g** Choose three-median or least squares regression to draw a line of best fit and write its equation.

10A **h** Compare your equation in part **g** with that in part **f**.

10A **i** Use your calculator to calculate the correlation coefficient for the data. Interpret your answer.

j In the 1960s, the speculation was 'When will the 10-second barrier be broken?' Today, talks centre around 'When will the 9.5-second barrier will be broken?' Use an equation you have generated to make a prediction when this might occur. Remember that the Olympic Games are only held every four years.

CONNECT

Investigating and analysing data

Every day, statistics are used to make claims and in reports. They can be great summaries of what can often be an overwhelming amount of data. However, it is important that the information is collected and interpreted in such a way that the outcome is not biased.

You are to investigate the use of statistics for one of these topics:

- age distribution and life expectancy of Aboriginal and Torres Strait Islander people compared with that of the Australian population as a whole
- biodiversity changes in Australia that have occurred since European occupation
- temperature distributions across the past 200 years over Australia compared with the world as a whole
- the growth of Australia's trade with other countries in the Asia region
- a topic of your choice (check with your teacher).

Your task

You will need to:

- decide on a topic and identify exactly what it is that you will be researching (If you choose your own topic, ensure that you can meet all other criteria before you begin.)
- research to find the appropriate data
- find or calculate any appropriate measures of centre and spread
- produce any appropriate graphs.

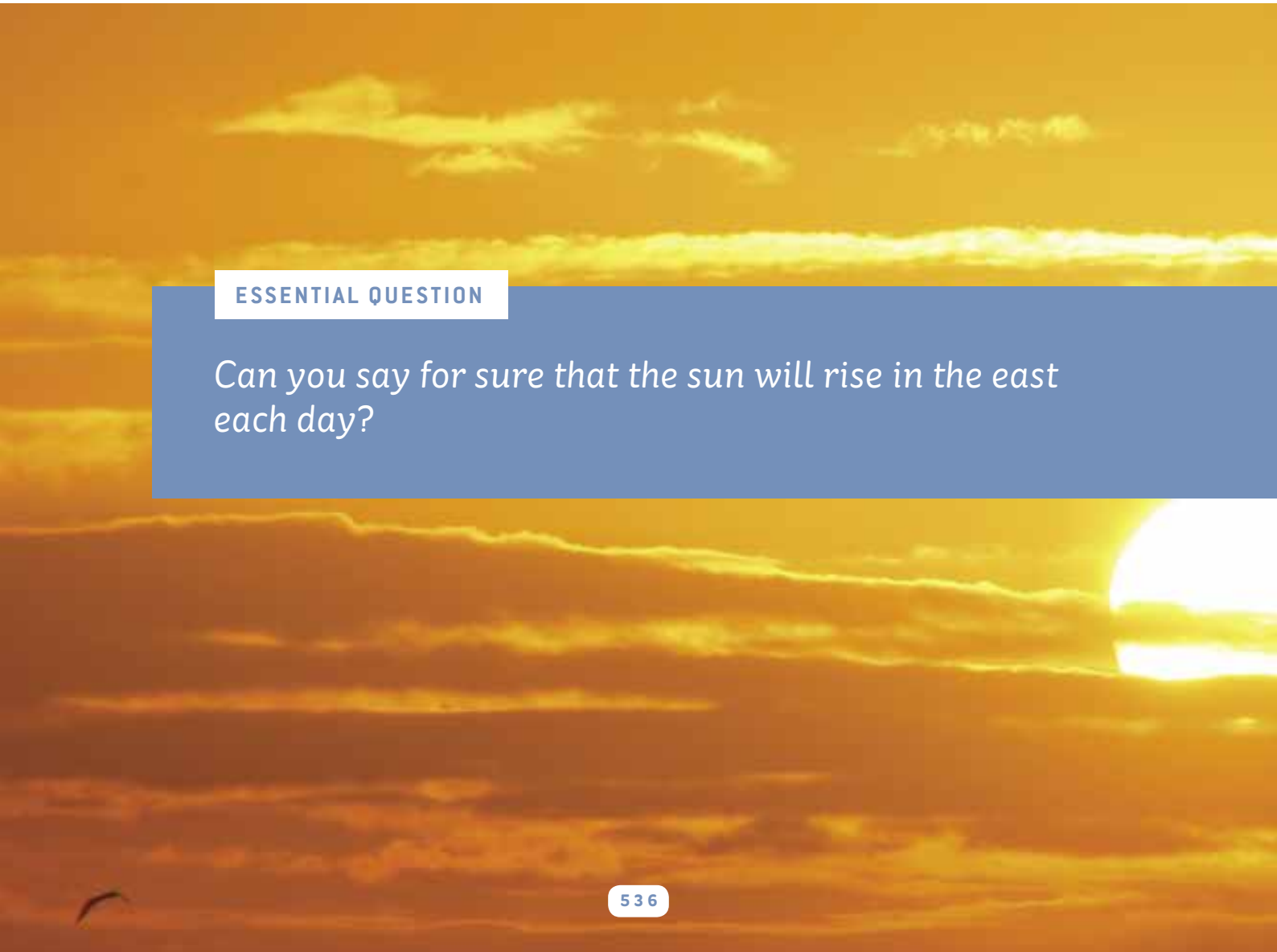
You will need:

- access to the Internet
- a calculator.

You may like to present your findings as a report. Your report could include:

- a digital presentation
- a pamphlet
- a poster
- other (check with your teacher).

11 PROBABILITY



ESSENTIAL QUESTION

Can you say for sure that the sun will rise in the east each day?

Are you ready?

11A **1** How could you describe the occurrence of an event with a probability of 1?

A impossible **B** certain
C somewhat likely **D** very unlikely

11A **2** How many outcomes are possible when a fair coin is flipped?

A 1 **B** 2 **C** 3 **D** 4

11A **3** **a** How many outcomes are possible when one fair coin is flipped twice?

A 1 **B** 2 **C** 3 **D** 4

b List all the outcomes.

11A **4** Consider a standard deck of playing cards.

a How many black cards are in the deck?

A 4 **B** 13 **C** 26 **D** 52

b How many aces are in the deck?

A 1 **B** 2 **C** 4 **D** 13

c How many black aces are in the deck?

A 1 **B** 2 **C** 4 **D** 13

11C **5** A bag of jellybeans contains three green, six yellow, four black and five red ones.

a How many jellybeans are in the bag?

b Jack ate one black and one red jellybean. How many jellybeans remain?

c Tom then ate all the yellow jellybeans. How many jellybeans remain now?

d How many red jellybeans are left?

11E **6** This table shows the number of black, blue and red pencils and biros in Sam's pencil case.

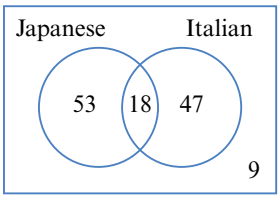
	Black	Blue	Red	Total
Pencils	6		4	
Biros		8		20
Total	13	16	9	

a How many pencils does Sam have?

b How many red biros does Sam have?

c What is the total number of pencils and biros in Sam's pencil case?

11F **7** Year 10 students at Westland High School are offered the option to study Japanese and Italian. They may elect to study no foreign language, one language, or both. The students' choices are shown in this Venn diagram.



a How many students study Italian?

b How many students do not study Italian?

c How many students study both Italian and Japanese?

d How many students study no foreign language?

e What is the total number of students in Year 10 at the school?

11A Review of theoretical probability

Start thinking!

The **probability** that something will occur is an indication of how likely an **event** or **outcome** is to happen. The **experimental probability** is often different from the **theoretical probability**. The **probability scale** ranges from 0 (an impossible event) to 1 (a certain event). Probability can be written as a decimal, a fraction or a percentage.

Imagine you had a set of the letters of the alphabet in a bag.

1 How many letters are in the alphabet?

2 How many of these letters are: **a** consonants? **b** vowels?

Probability is calculated using the formula $\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$.

3 What is the probability of choosing: **a** a consonant? **b** a vowel?
Write your answers as fractions in both cases.

4 Explain why the probability of choosing a consonant is greater than the probability of choosing a vowel.

5 Write each answer from question **3** as both a decimal and a percentage.

6 The events 'choosing a consonant' and 'choosing a vowel' are **complementary events**. Explain why this is so.

7 What is the value of Pr(consonant) + Pr(vowel)? Explain your answer.

Imagine the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 were added to the bag containing the letters.

8 Explain why your answer to question **2** would be the same, but for question **3** it would be different.

9 Are 'choosing a consonant' and 'choosing a vowel' still complementary events? Explain.

10 Find the probability of choosing a: **a** consonant **b** vowel **c** digit.

11 Show that the sum of the probabilities of your answers in question **10** is 1.

KEY IDEAS

- The probability of an event occurring can be written as a value in the range 0 (impossible) to 1 (certain). This value can be written as a fraction, a decimal or a percentage.
  $\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$
- The sum of the values of the probabilities of complementary events is 1.
- The **sample space** of an **experiment** is a list of all the different outcomes possible and is written within curly brackets.
- Different outcomes of an experiment may not be **equally likely** to occur and hence the sample space may not reflect the probability of different outcomes.

EXERCISE 11A Review of theoretical probability

UNDERSTANDING AND FLUENCY

1 Consider the letters in the word MATHEMATICS.
- **a** How many letters are in the word?
- **b** Write the sample space for the word.
- **c** How many consonants are in the word? Explain whether these consonants are all different.
- **d** How many vowels are in the word? Explain whether these vowels are all different.

2 A single letter is selected from the word MATHEMATICS. Calculate these probabilities.
- **a** Pr(selecting a consonant)
- **b** Pr(selecting a vowel)
- **c** Pr(not selecting a vowel)

3 Explain the relationship between your answers in question **2**.

4 List the sample space for each experiment.
- **a** rolling an eight-sided die
- **b** randomly selecting a letter from the word AUSTRALIA
- **c** drawing a card from a deck and recording its colour
- **d** spinning a spinner with two red, three white and four blue sections (all sections being equal in size)

5 State whether the outcomes for the experiments in question **4** are equally likely to occur or not.

EXAMPLE 11A-1 Calculating the probability of an event with alternatives

Find the theoretical probability of rolling a fair six-sided die and obtaining a multiple of 2 or 3.

THINK	WRITE
1 Write the formula.	$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$
2 Find the total number of possible outcomes.	A die shows the numbers 1 to 6. total number of outcomes = 6
3 Find the number of **favourable outcomes**.	multiples of 2: 2, 4 and 6 multiples of 3: 3 and 6 number of different favourable outcomes = 4
4 Substitute into the formula and write the fraction in simplest form.	$\Pr(\text{multiple of 2 or 3}) = \frac{4}{6}$ $= \frac{2}{3}$

6 Find the theoretical probability of:

- **a** rolling a fair six-sided die and obtaining a factor of 6 or a multiple of 2
- **b** selecting an ace from a standard deck of playing cards
- **c** spinning red or white on the spinner in question **4d**
- **d** randomly selecting a consonant from the word CALCULATOR.

7 A magician was invited to a party to perform his magic tricks. He produced these origami cranes from his magic hat.

- **a** How many cranes did he produce?
- **b** If you closed your eyes and picked a crane, what is the probability that it would be:
  - **i** red or dark blue?
  - **ii** *not* red or dark blue?
- **c** What do your two answers in part **b** add to?
- **d** State the complementary event to:
  - **i** picking a red crane
  - **ii** not picking a green crane
  - **iii** picking a green or a red crane.
- **e** For each event listed in part **d**, find the probability of the event and its complementary event.
- **f** Find the sum of the probability of each event and its complementary event from part **d**. What do you find?
- **g** What can you say about the sum of the probability of complementary events?

8 In mathematical notation, if an event is defined as A, its complementary event is defined as A′.

- **a** Show that $\Pr(A') = 1 - \Pr(A)$.
- **b** If A represents the event of choosing a card of one particular suit from a deck of cards, determine:
  - **i** $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A) + \Pr(A')$.

UNDERSTANDING AND FLUENCY

9 State whether each of these probability statements is true or false. If the statement is false, give the correct answer.

- **a** in rolling a fair four-sided die, Pr(not 4) = $\frac{3}{4}$
- **b** in choosing one card from a deck of playing cards, Pr(black or jack) = $\frac{15}{26}$
- **c** in choosing a letter from the alphabet, Pr(letter occurs before M in the alphabet) = $\frac{1}{2}$
- **d** in choosing a letter from the word PROBABILITY, Pr(vowel) = $\frac{3}{11}$
- **e** in choosing a colour from the rainbow, Pr(not red) = $\frac{6}{7}$

10 Consider the coloured blades on these two toy windmills.

a

b

For each of the windmills:

- **i** write the sample space of the blade colours
- **ii** determine Pr(bottom blade will be green) after spinning the windmill
- **iii** find the theoretical probability that the bottom blade will not be red.

PROBLEM SOLVING AND REASONING

11 Match the events that have the same probability.

- **a** drawing a red card from a deck of playing cards
- **b** obtaining a number greater than 4 with one roll of a six-sided die
- **c** not choosing a diamond from a deck of playing cards
- **d** choosing a weekday that begins with the letter T
- **e** selecting one particular quadrant from the Cartesian plane

- **A** choosing a note less than \$50 from a set of notes containing one each of \$5, \$10, \$20 and \$50
- **B** choosing a prime number from the set of numbers 1 to 10
- **C** obtaining a head with one flip of a coin
- **D** not choosing a boy from a group of nine boys and three girls
- **E** choosing a multiple of 3 from the digits 1 to 6

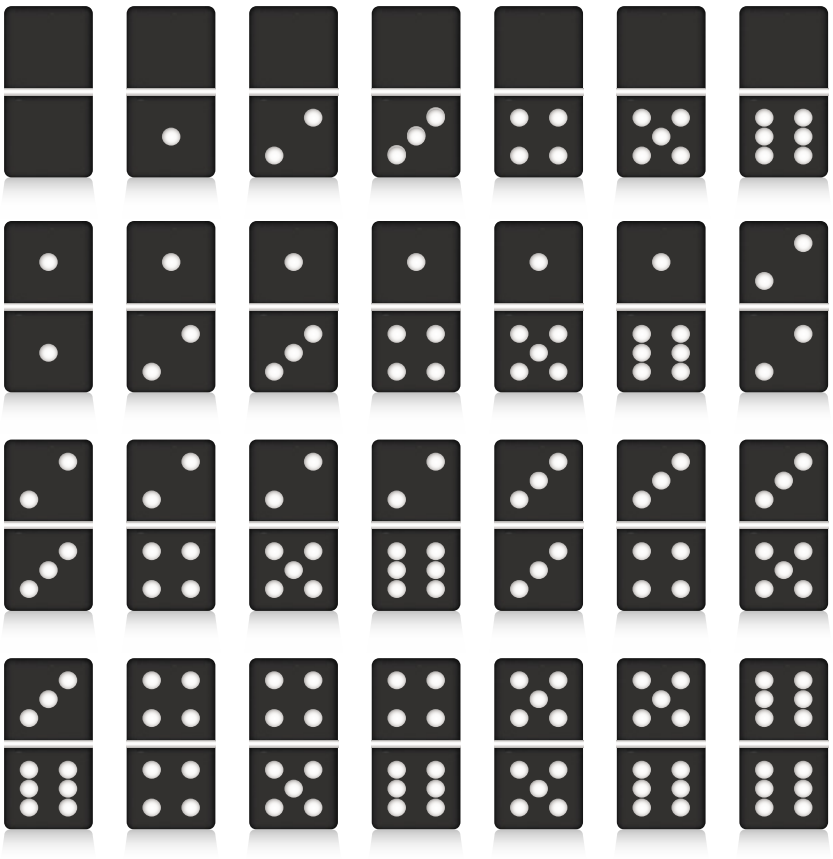
12 Consider selecting a single card from a standard deck of playing cards. What is the theoretical probability of:

- **a** selecting a red card or a king from the deck?
- **b** selecting a red card, but not a king?
- **c** not selecting a red card, but selecting a king?

13 Consider the situation where all the hearts are first removed from a standard deck of playing cards, and repeat question **12**.

14 In a group of students there are 15 boys. The probability of choosing a girl from the group is $\frac{2}{5}$. How many girls are in the group?

15 A set of dominoes contains tiles, each tile being divided in two and showing dots (similar to dice) to represent numbers. A blank on a tile represents 0.



a How many tiles are in the complete set?

b Explain the order of the tiles in the photo.

c There are the same number of half-tiles for each of the numbers 0 to 6. What is this number?

d One tile is chosen at random. What is the probability that it shows these combinations?

- **i** double 6
- **ii** the same number twice
- **iii** two different numbers
- **iv** one even number and one odd number
- **v** at least one prime number
- **vi** two numbers, one of which is a multiple of the other.

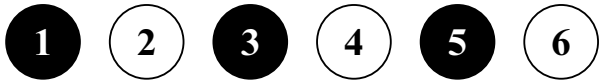
CHALLENGE

16 A bag contains four different coloured Christmas decorations: red, green, gold and silver. There are four silver decorations and eight gold decorations. One decoration is drawn from the bag. The probability it is silver or gold is 0.6. How many red decorations are in the bag, if there are three green decorations?

17 Consider this target.

- **a** Discuss whether the chance of landing on a red section is the same as that of landing on a white section.
- **b** The diameter of the innermost red circle (the bullseye) is 2 cm, and the widths of each of the subsequent rings is 2 cm. Draw a diagram showing the radius of each of the rings.
- **c** Calculate the area of each of the rings. Draw a labelled diagram showing each of these areas.
- **d** Assuming the arrow is equally likely to hit any point within the target, calculate the probability of the arrow landing:
  - **i** in the bullseye
  - **ii** in the outermost red ring.

18 Six disks are lined up in a row as shown. There are three black disks and three white disks, with the colours alternating.



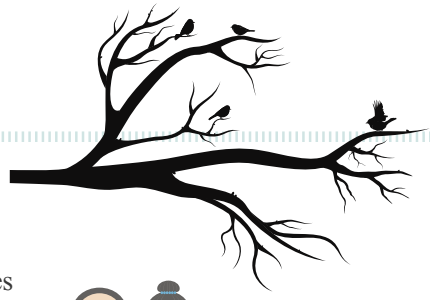
The disks are moved within the row by taking adjacent pairs of disks and jumping over other disks to occupy free spaces. (There must be free spaces within the row to allow this move.) The free space at either end of the row can be used at any time. The aim is to ultimately (in as few moves as possible) get all disks of one colour together on the left, and all disks of the other colour together on the right. The numbers showing on the disks do not have to be in numerical order.

- **a** Show how this can be done in as few moves as possible.
- **b** What is the minimum number of possible moves?

Reflect

If you calculated the probability of an event to be $\frac{13}{12}$, why would you suspect you had made an error?

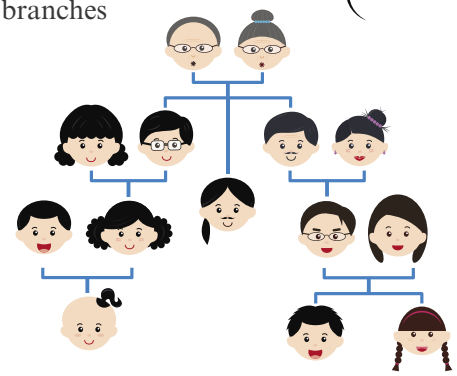
11B Tree diagrams



Start thinking!

A **tree diagram** is a diagram consisting of line segments connected like the branches and twigs of a tree.

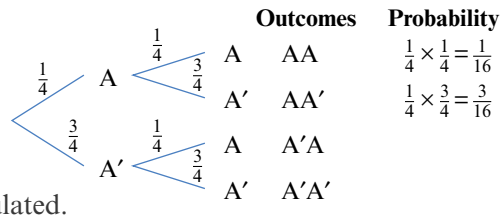
Typically, tree diagrams are drawn horizontally from left to right. However, they can be drawn vertically, as in a family tree.



1 Label the people in the figure by describing the relationship of each to the first generation couple.

Tree diagrams display the relationship between events in multi-step experiments. Following the branch from its beginning to the tips of the twigs displays all the possible outcomes.

2 An eight-sided die is rolled twice. Let A represent the outcome of rolling a number greater than 6.



An **element** of a set is another term for a member of the set.

a List the elements in the set of:
i A **ii** A′.

b Show how the values $\Pr(A) = \frac{1}{4}$ and $\Pr(A') = \frac{3}{4}$ are calculated.

c Discuss the outcomes of the experiment. Are they all different?

d Complete the calculations for the probability of all the outcomes.

e Determine the probability that the die lands on a number greater than 6:
i both times **ii** at least once.

3 **a** Extend the tree diagram to show the outcomes when the die is rolled three times. Show probability calculations for all the outcomes.

b Use your diagram to calculate probabilities for the following events.
i the die does not land on a number greater than 6 in any of the rolls
ii the die lands on a number greater than 6 at least twice

4 If three dice had been used, would any of the answers to the previous questions be different? Explain.

KEY IDEAS

- Tree diagrams display the outcomes of multi-step experiments.
- The possible outcomes for each step of the experiment are represented by the branches.
- The branches may not all have the same probability.
- The final outcomes are listed at the ends of the branches.
- The probability of the final outcomes can be calculated by multiplying the individual probabilities along the branches.

EXERCISE 11B Tree diagrams

UNDERSTANDING AND FLUENCY

1 Two fair six-sided dice are rolled. A record is made of each time a 6 is rolled on either of the dice.

- **a** If A represents the event {rolling a 6}, what does A′ represent?
- **b** Draw a tree diagram showing the results of the experiment. Mark the probabilities on each of the branches.
- **c** How many outcomes are possible?
- **d** Explain whether all the outcomes are different.
- **e** Use your tree diagram to find the probability of rolling:
  - **i** a double 6
  - **ii** at least one 6.

2 The experiment in question **1** is repeated. This time, all the numbers rolled on the two dice are recorded.

- **a** Explain how this experiment differs from that in question **1**.
- **b** How many outcomes are possible in this case?
- **c** Draw a tree diagram to represent the experiment, and use it to determine the probability that the two numbers rolled are:
  - **i** the same
  - **ii** both prime numbers.

3 A 10-sided die is rolled, and a successful event is regarded as being when it lands on a number greater than 6.

- **a** List the elements in a successful event.
- **b** What is the probability that the event will be successful?
- **c** What is the probability the event will not be successful?
- **d** If you rolled the die 100 times, how many successful outcomes would you expect to get?

4 A coin is flipped three times. Calculate the probability the outcome shows:

- **a** one head
- **b** at least one head
- **c** no heads
- **d** no more than one tail.

EXAMPLE 11B-1

Calculating the probability of a multi-step event using a tree diagram

A pile of money contains four silver coins and six gold coins. One coin is randomly chosen, its colour noted, then it is replaced in the pile. This is repeated twice more. Use a tree diagram to calculate the probability of selecting at least two gold coins.

THINK

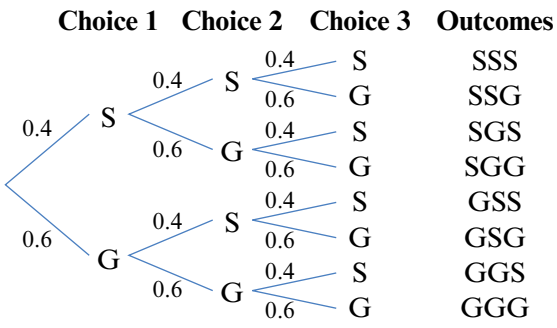
1 Calculate the probabilities of choosing a silver coin (S) and a gold coin (G) each time.

WRITE

$$\Pr(\text{silver}) = \Pr(S) = \frac{4}{10} = 0.4$$

$$\Pr(\text{gold}) = \Pr(G) = \frac{6}{10} = 0.6$$

2 Draw a tree diagram. Mark the individual probabilities on the branches, and show all the outcomes.



3 Select those outcomes which contain at least two gold coins.

Those outcomes with two or more gold coins are: SGG, GSG, GGS and GGG.

4 Calculate the probabilities of each of these outcomes.

$$\Pr(SGG) = 0.4 \times 0.6 \times 0.6 = 0.144$$
$$\Pr(GSG) = 0.6 \times 0.4 \times 0.6 = 0.144$$
$$\Pr(GGS) = 0.6 \times 0.6 \times 0.4 = 0.144$$
$$\Pr(GGG) = 0.6 \times 0.6 \times 0.6 = 0.216$$

5 Add all the selected probabilities.

$$\Pr(\text{at least two gold coins}) = 0.144 + 0.144 + 0.144 + 0.216 = 0.648$$

5 John removes the loose coins from his pockets each night and throws them into a tin. At the moment, there are 75 silver coins and 25 gold coins in the tin. He draws out a coin and notes its colour. After replacing it in the tin, he repeats this process another two times.

a Determine the probability of drawing out:
i a silver coin **ii** a gold coin.

b Draw a tree diagram to display the experiment. Mark probability values on each of the branches.

c Use your tree diagram to calculate the probability of drawing out:
i all silver coins **ii** no silver coins **iii** at least one of each colour.

6 Three eight-sided dice are thrown and the number rolled for each is noted. A record is made of whether this number is even or odd.

Use a tree diagram to find the probability of rolling:

a no odd numbers

b one or two odd numbers

c more odd numbers than even numbers.

7 A multiple-choice question consists of two parts. Each part could be true or false.

What is the probability that the answers are:

a both true? **b** both false? **c** one true, one false?

8 In a room there are two lights, each with its own on/off switch.

a Draw a tree diagram to show all the possible combinations of on/off the two lights could be.

b If it is equally likely that each light is on or off, what is the probability that the lights will:
i both be on?
ii both be off?
iii not be in the same mode?

9 'Unders and overs' is a game that is played by rolling two six-sided dice. The numbers rolled are added to give a total. Players bet on the total being under 7, equal to 7 or over 7.

a Draw a tree diagram to show all the outcomes when rolling two dice.

b List the outcomes where the total is:
i under 7 **ii** equal to 7 **iii** greater than 7.

c Calculate your probability of winning if you bet on:
i under 7 **ii** equal to 7 **iii** greater than 7.

d Describe what your strategy would be if you played this game.

UNDERSTANDING AND FLUENCY

10 Two-up is said to be the 'Digger's gambling game', because it was played by Australian troops during the World Wars. Two coins are placed tails up on a flat board then tossed into the air.

When the coins land, the rules are:
two heads up, you win
one head and one tail, you toss again
two tails, you lose.

a Draw a tree diagram to display the outcomes of the game.

b Use your tree diagram to calculate:

i Pr(winning)

ii Pr(tossing again)

iii Pr(losing).

11 There are 25 players in the school tennis singles championship. Draw a tree diagram to determine the minimum number of matches which must be played before a winner can be declared.

PROBLEM SOLVING AND REASONING

12 There are three sets of traffic lights in succession on a road. They each show red, amber or green for equal amounts of time. Assume they are not synchronised.

a If you approach the first light, and it shows red, does that influence the colour you will encounter at the following two lights? Explain.

b What is the chance you will encounter green lights for all three and get a clear run through?

c What is the chance you will encounter a green light, and one of the two lights after that is also green?

13 When rolling two unbiased dice of the same kind, the probability of rolling a double depends on the number of sides on the dice.

a Consider rolling a pair of six-sided dice.

i How many outcomes are possible when the two dice are rolled?

ii How many of these outcomes are doubles? List these.

iii What is the probability of rolling a double?

b Consider rolling a pair of eight-sided dice. Repeat the questions in part **a**.

c Write a general statement for the probability of rolling a double using a pair of dice with n sides.

PROBLEM SOLVING AND REASONING

14 Snakes and ladders is a board game played on a board with numbered grid squares by two or more players. Ladders connect certain pairs of squares together, and snakes also connect other pairs of squares. The grid size varies from board to board, but is most commonly 10 by 10. The exact arrangement of the snakes and the ladders also varies. The aim of the game is to progress from square 1 to square 100 by rolling a die and moving the number of squares rolled. You must move up a ladder or down a snake if you land on the squares that show the base of a ladder or the head of a snake.

Imagine a grid where the first ladder base is on square 7.

a What is the minimum number of die rolls which would take you to square 7?

b What is the maximum number of die rolls which would take you to square 7?

c Consider the minimum number of rolls required.

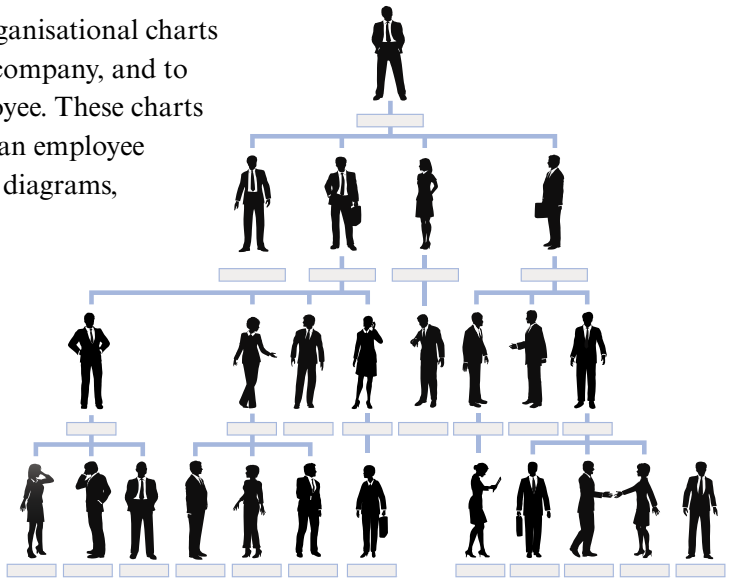
i List the die rolls which would take you to square 7.

ii How many different options are there?

d Consider using a pair of dice instead of a single die. Repeat parts **a**–**c**.

15 Companies often produce organisational charts to show the structure of the company, and to indicate the level of an employee. These charts also show the team to which an employee belongs. These are really tree diagrams, and are shown in a vertical orientation.

Consider this organisational chart.



Write a detailed description of the structure of the company. You may find it helpful to give particular employees a name.

CHALLENGE

16 A family tree for humans shows that there are two parents for each child. The family tree for a bee is quite different. A male bee has only one parent, a mother. A female bee has two parents, a mother and a father. Draw a tree diagram which traces the ancestry of a male bee back four generations.

Reflect

Why must a tree diagram be used instead of a two-way table to display the outcomes of experiments with more than two steps?

11C Experiments with and without replacement

Start thinking!

Consider the digits 1, 2, 3, 4 and 5.

1 You are asked to write a two-digit number using these digits. The digits may be used more than once.
- **a** Draw a tree diagram and write all possible outcomes.
- **b** How many outcomes are there?
- **c** How many of these outcomes have two digits which are the same?
- **d** Calculate the probability that the number has:
  - **i** two digits the same **ii** two different digits
  - **iii** the first digit smaller than the second.

2 Now consider forming the two-digit number with no repeated digits. This tree diagram displays that situation.

First digit	Second digit	Outcomes
1	2	12
	3	13
	4	
	5	
2	1	
	3	
	4	
	5	
3	1	
	2	
	4	
	5	
4	1	
	2	
	3	
	5	
5	1	
	2	
	3	
	4	

- **a** Complete the column of outcomes.
- **b** How many outcomes are possible?
- **c** Compare the number of outcomes in this situation with your answer to question **1b**. Explain any difference.
- **d** How many outcomes have two digits the same now?
- **e** Calculate the probability that the number has:
  - **i** two digits the same **ii** two different digits
  - **iii** the first digit smaller than the second.

3 Question **1** is an example of performing an experiment with replacement, while question **2** represents an experiment without replacement.
- **a** Explain the difference between the two situations.
- **b** Explain why the probability of a particular event is different for the two situations.

KEY IDEAS

- Experiments with replacement involve selecting an item, recording the results, then replacing the item before performing another trial.
- Experiments without replacement involve not replacing the item from the first selection before a second selection is made. The number of items from which to choose the second item decreases by 1.
- When outcomes are not equally likely, a tree diagram with probabilities written on the branches is useful for calculating the final probability of each outcome.

EXERCISE 11C Experiments with and without replacement

1 From a set of alphabet letters, two letters are drawn:
- a with replacement
- b without replacement

What is the probability that the letters are both vowels or both consonants?

EXAMPLE 11C-1 Calculating probability for experiments with replacement

A barrel contains coloured balls: two red, one blue and three green. One ball is withdrawn from the barrel, its colour noted, then it is replaced. A second ball is chosen and its colour noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two balls are the same colour.

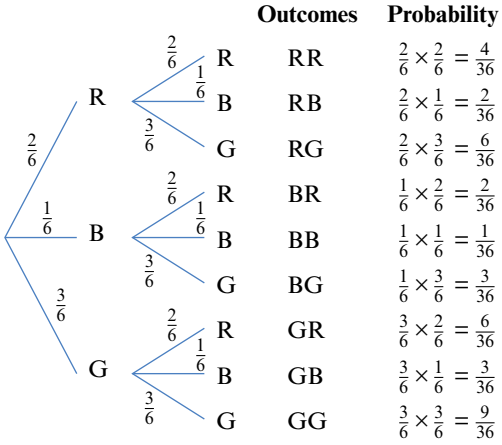
THINK

a Draw a tree diagram writing all outcomes together with their probabilities. Check that the sum of all probabilities is 1.

b Locate the outcomes where the two colours are the same. Add to find the total of these probabilities.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{2}{6} = \frac{4}{36}$
RB	$\frac{2}{6} \times \frac{1}{6} = \frac{2}{36}$
RG	$\frac{2}{6} \times \frac{3}{6} = \frac{6}{36}$
BR	$\frac{1}{6} \times \frac{2}{6} = \frac{2}{36}$
BB	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
BG	$\frac{1}{6} \times \frac{3}{6} = \frac{3}{36}$
GR	$\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$
GB	$\frac{3}{6} \times \frac{1}{6} = \frac{3}{36}$
GG	$\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

b The outcomes which have two balls the same colour are RR, BB and GG.

$\Pr(RR) = \frac{4}{36}$, $\Pr(BB) = \frac{1}{36}$, $\Pr(GG) = \frac{9}{36}$

$\Pr(\text{same colour}) = \frac{4}{36} + \frac{1}{36} + \frac{9}{36}$

$= \frac{14}{36}$

$= \frac{7}{18}$

2 Consider the experiment described in Example 11C-1. Find the probability that:
- a only one ball is blue
- b only one ball is red
- c only one ball is green.

UNDERSTANDING AND FLUENCY

3 Consider a standard deck of playing cards. One card is chosen, and its suit noted. It is replaced, then a second card is chosen.

- **a** Draw a tree diagram to represent the experiment.
- **b** What is the number of possible outcomes?
- **c** If the outcome {spade, heart}, for example, is considered to be the same as {heart, spade}, how many different outcomes are possible?

4 Repeat question **3**, using a deck of playing cards with all the hearts removed.

EXAMPLE 11C-2 Calculating probability for experiments without replacement

Repeat the experiment in Example 11C-1, but this time assume the first ball is not replaced before the second ball is chosen.

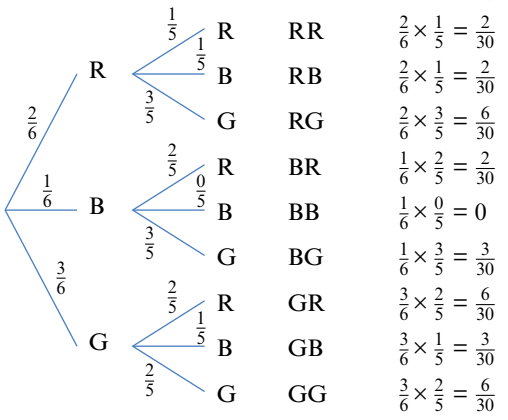
THINK

a Draw the tree diagram. The number of balls from which to choose the second ball is decreased by 1.

b Locate the outcomes where the two colours are the same. Add these to find the total of the probabilities. Note that the outcome BB is not possible.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$
RB	$\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$
RG	$\frac{2}{6}\times\frac{3}{5}=\frac{6}{30}$
BR	$\frac{1}{6}\times\frac{2}{5}=\frac{2}{30}$
BB	$\frac{1}{6}\times\frac{0}{5}=0$
BG	$\frac{1}{6}\times\frac{3}{5}=\frac{3}{30}$
GR	$\frac{3}{6}\times\frac{2}{5}=\frac{6}{30}$
GB	$\frac{3}{6}\times\frac{1}{5}=\frac{3}{30}$
GG	$\frac{3}{6}\times\frac{2}{5}=\frac{6}{30}$

b The outcomes which have two balls the same colour are RR and GG.

$\Pr(RR)=\frac{2}{30}$, $\Pr(GG)=\frac{6}{30}$

$\Pr(\text{same colour})=\frac{2}{30}+\frac{6}{30}$

$=\frac{8}{30}$

$=\frac{4}{15}$

5 Consider the experiment described in Example 11C-2. Find the probability that:

- **a** only one ball is blue
- **b** only one ball is red
- **c** only one ball is green.

6 Consider again the number problem described in Start thinking! (page 422) and imagine using the five digits 0 to 4 instead of 1 to 5. This introduces the problem that the number generated cannot begin with 0.

- **a** Draw a tree diagram to show the possible outcomes resulting from forming a two-digit number, where the digits can be repeated. How many numbers are possible?
- **b** Repeat part **a** for the situation where the digits are not repeated.
- **c** Compare the situations in parts **a** and **b**.

7 It's not always necessary to draw tree diagrams to determine the number of outcomes of an experiment. There is a shortcut.

- **a** First, consider forming a two-digit number from the digits 1 to 5 with replacement. There are five digits to choose from when writing the first digit. You can use all the digits again for the second digit, so there are five from which to choose again. This means that there should be 5×5 possible outcomes. Does this agree with your previous answer?
- **b** Secondly, consider forming a two-digit number from the digits 1 to 5 *without* replacement. There are five digits to choose from when writing the first digit. Since you are not able to repeat any of the digits, there are only four from which to choose the second digit. This means that there should be 5×4 possible outcomes. Does this agree with your previous answer?
- **c** Use this method to determine the number of possible outcomes in forming a three-digit number from the digits 1 to 5:
  - **i** with replacement
  - **ii** without replacement.
- **d** Repeat part **c** using the digits 0 to 4.

8 **a** Imagine that the number plates for cars contain six digits, and there can be repeat digits. A number plate can even start with the digit 0.

- **i** Write a list of the digits which could be used.
- **ii** How many different digits can be used in each place of the six-digit number?
- **iii** How many different number plates could be produced using this system?

b A new system was introduced whereby each number plate contained three digits followed by three letters.

- **i** How many number plates would be possible using this system?
- **ii** Explain why the system with a mixture of digits and letters is preferred to the system with six digits.

9 A fair coin is flipped five times.

- **a** How many outcomes are possible?
- **b** Write a list of all the outcomes which contain at least three heads.

10 In a family there are three children, one girl and two boys. Assume that it is equally likely a boy or a girl results from a birth. What is the probability that:

a the girl is the eldest child? **b** the girl is the middle child?

c both boys are older than the girl?

11 The letters of the word PROBABILITY are written on separate pieces of paper and placed in a box. One letter is drawn from the box and it is recorded whether it is a consonant or a vowel. A second letter is drawn without replacing the first letter. Determine the probability that:

a the letters are both vowels

b one letter is a vowel and the other is a consonant.

12 A restaurant offers a menu of six entrees, six main meals and six desserts.

a How many different combinations of courses does the restaurant offer?

b During the evening, the kitchen runs out of one of the entrees and two of the desserts. What is the decrease in the number of options on offer?

13 Sam has a small bag of Smarties. It contains 10 red ones, 8 blue, 6 orange, 4 yellow and 12 green.

a Sam prefers red Smarties. What is the probability that, if he chooses one Smartie at random, it is red?

b Sam eats all the red ones. He then chooses one at random, eats it, then chooses another one and eats it. What is the probability that the two Smarties he ate were both the same colour?

14 Jack's drawer has nine socks: four identical school socks and five identical sports socks. What is the smallest number of socks Jack can take from his drawer, without looking, so he can be certain of having a pair of school socks and a pair of sports socks?

15 Two dice are thrown and the product of the two numbers rolled is recorded.

a What is the probability that this number is:

i odd? **ii** prime? **iii** a square number?

b What is:

i the most likely outcome? **ii** the least likely outcome?

16 A tetrahedral die has four triangular faces. When the die is tossed, the number recorded is the face on which the die lands. Two of these dice are tossed, and the number rolled on each die is noted. The product of these two numbers is then recorded.

PROBLEM SOLVING AND REASONING

a How many outcomes are possible?
b How many *different* outcomes are possible?
c What is the probability that the product is:
  i prime? ii not prime?
  iii a multiple of 4? iv odd?

17 Using dice of different shapes raises the issue of whether a game is fair. Consider a game where Ann rolls two four-sided dice and Beth rolls two six-sided dice.
a If they have to roll a double to start the game, discuss whether this is fair.
b After starting, they both have to roll two odd numbers to progress. Explain whether this rule is fair.

18 A coin is biased so that the chance of it showing a tail when flipped is 0.7.
a Draw a tree diagram to show the outcomes and their probabilities for the coin being flipped three times.
b What is:
  i the most favourable outcome?
  ii the least favourable outcome?
c What is the probability of flipping:
  i only one head?
  ii only one tail?
  iii more heads than tails?

CHALLENGE

19 Scrabble is a board game using tiles containing alphabet letters to make words on a grid rather like a crossword puzzle.

There are 98 tiles, each with a letter of the alphabet on its face, and two blank tiles which can be used as any alphabet letter. You pick up seven tiles to start the game. Say you hope to write the word SCHOOLS with your seven tiles. The game contains four Ss, two Cs, two Hs, eight Os and four Ls.
a What is the probability you will pick up the letter S first?
b Assuming that your first letter is an S, what is the probability your next letter is also an S?
c Continue in this manner to determine the probability you will be successful in picking up all the letters to make the word SCHOOLS.

Reflect

Is there any difference between the sample space of an experiment with replacement and the sample space of the same experiment without replacement?

11D Independent and dependent events

Start thinking!

On a desk sits a container of 10 black pencils and another container of 20 coloured pencils. The coloured pencils include 2 red, 4 blue, 3 green and 11 yellow.

1 One pencil is selected from each container.

- **a** What is the probability of selecting a black pencil from the first container?
- **b** What is the probability of selecting a red pencil from the second container?
- **c** Explain whether the probability of selecting the black pencil affects the probability of selecting the red pencil.
- **d** Explain why these two events should be regarded as **independent events**, not **dependent events**.

2 Imagine if all the pencils are put into one container.

- **a** **i** If one pencil is selected, what is the probability it is a black one?
  - **ii** If this pencil is replaced, and another pencil is selected, what is the probability that this pencil is a red one?
  - **iii** Does the selection of the second pencil depend on what happened in the first selection?
  - **iv** Would you classify these two events as independent or dependent?
- **b** Now consider the situation where the first pencil is *not* replaced before the second one is selected.
  - **i** You have already calculated the probability that the first pencil selected is black (question **2a i**). What is the probability that the second pencil selected is red?
  - **ii** Compare this answer with your answer to question **2a ii**.
  - **iii** In this case, does the selection of the second pencil depend on what happened in the first selection?
  - **iv** Would you classify these two events as independent or dependent? Explain any difference between this case and the situation in question **2a iv**.

KEY IDEAS

- Independent events are events where the outcome of one event has no effect on the outcome of the other.
- If A and B are independent events, $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$.
- Dependent events are those where the outcome of one event does have an effect on the outcome of the other.
- If A and B are dependent events and A occurs first, $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B, \text{ once A has occurred})$
- When the order of two events is not specified, take care to look for all options.

EXERCISE 11D Independent and dependent events

1 Classify each pair of events as independent or dependent.
- a rolling two six-sided dice
- b rolling a six-sided die twice
- c rolling a four-sided die and a six-sided die
- d selecting two children from a class of 25 students
- e selecting a captain for Class 10A and a captain for Class 10B
- f selecting an aeroplane seat for you and a friend

EXAMPLE 11D-1 Calculating the probability of independent events

A fair coin is flipped twice.
- a Explain why these two events are independent.
- b Calculate the probability of the outcome being:
  - i a head then a tail
  - ii a head and a tail.

THINK

a The outcome of the second flip does not depend on what happens in the first flip.

b i Find Pr(head) and Pr(tail) then multiply to obtain the combined outcome.

ii When order is not specified, look for all the options.

WRITE

a Every time a coin is flipped, it has an equal chance of turning up a head or a tail. What happens in the first flip has no effect on what happens in the second flip, so these two events are independent.

b i $\Pr(\text{head}) = \frac{1}{2}$, $\Pr(\text{tail}) = \frac{1}{2}$

$\Pr(\text{head, tail}) = \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

ii $\Pr(\text{head and tail})$

$= \Pr(\text{head, tail}) + \Pr(\text{tail, head})$

$= \frac{1}{4} + \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{2}{4}$

$= \frac{1}{2}$

2 A fair coin is flipped three times.
- a Explain why these events are independent.
- b Determine the probability of each outcome.
  - i {head, head, tail}
  - ii two heads and a tail
  - iii at least two tails
  - iv no more than two tails

EXAMPLE 11D-2 Calculating the probability of dependent events

Dana has in her pocket five silver coins and four gold coins. She selects one coin, then another, without replacing the first coin.

a Explain why these two events are dependent.
b Determine the probability that Dana selected one silver and one gold coin.

THINK

a The outcome of the second selection depends on what happens in the first selection.

b When order is not specified, look for all the options.

WRITE

a Dana removes one coin in her first selection. This coin is not replaced, so the number of coins for the second selection is reduced by 1. This means that the two events are dependent.

b Pr(one silver and one gold)
$= \Pr(\text{silver, gold}) + \Pr(\text{gold, silver})$
$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8}$
$= \frac{20}{72} + \frac{20}{72}$
$= \frac{40}{72}$
$= \frac{5}{9}$

UNDERSTANDING AND FLUENCY

3 Sally has 10 coins in her purse: seven silver and three gold. She takes out three coins, one after the other, without replacing them.
   a Explain why these events are dependent.
   b Determine the probability of each outcome.
      i a gold coin is the first one drawn out
      ii {gold, gold, gold}
      iii gold, then two silver coins
      iv a gold coin and two silver coins

4 Two six-sided dice are rolled. Determine the probability of each event.
   a {6, not 6}
   b {even, odd}
   c {prime, not prime}
   d at least one even number

5 Consider a deck of playing cards with all the picture cards removed. Two cards are withdrawn from the pack, one after the other, without replacing the first one. What is the probability that these two cards are:
   a the same value?
   b the same suit?
   c from different suits?

6 Consider the pencil container introduced at the beginning of this section. It contains 10 black pencils and 20 coloured pencils: 10 black, 2 red, 4 blue, 3 green and 11 yellow.

Two pencils are taken out one after the other. The chance that the first one chosen is black, then the next one is red is calculated as:

$$\Pr(\text{black and red}) = \Pr(\text{black}) \times \Pr(\text{red, once black has occurred}) = \frac{10}{30} \times \frac{2}{29} = \frac{2}{87}$$

a Use this rule to determine the probability that the two pencils chosen are:

- **i** black and (not red)
- **ii** black and (red or blue)
- **iii** black and (red, blue or green)
- **iv** both black
- **v** both red
- **vi** red and blue.

b Investigate to see whether the events can be undertaken in the reverse order. Determine the probability that the two pencils are:

- **i** red and black
- **ii** (not red) and black
- **iii** blue and red.

c Comment on your answers to part **b**.

7 Scratchie tickets come in a variety of forms. You scratch boxes to reveal information underneath, and you may win a prize according to the rules of the particular scratchie.

Consider the type of ticket shown in the second image. Here, you scratch the box marked 'YOUR SCORE' to reveal your number. You then scratch the box marked 'THEIR SCORE' to reveal their number. If these two numbers match, you win a prize.

a Your chance of winning obviously depends on the range of numbers used. Explain why.

b Consider the range of numbers to be 1 to 20. What would be your chance of winning:

- **i** the prize at the top of the card?
- **ii** the prize below that?
- **iii** both prizes?

c Repeat part **b** if the range of numbers is 1 to 100.

d Compare your answers for parts **b** and **c**. What do you conclude?

e Explain whether these events are independent or dependent.

UNDERSTANDING AND FLUENCY

8 Try making up your own scratchie ticket. Consider the following rules.

There are three boxes.

Each box contains a number in the range 1 to 10.
To win a prize of $20, two of the three numbers must match.
If all three numbers match, the prize is $100.

a Draw an example of a scratchie of this type. Write instructions on the ticket. Fill in a few sample tickets with numbers to get an understanding of the variety of options.

b Look at the option where all three boxes match.

- **i** What is the probability of winning $100? Remember that there is more than one option here.
- **ii** If you scratch the first two boxes and they match, what is the probability that the third box will also match?
- **iii** After scratching the first box, what is the probability that the next two boxes will match this number?

c Consider the option where two boxes match.

- **i** If there are two boxes with a matching number 1, give some examples of how this could occur.
- **ii** In how many ways can this occur?
- **iii** What is the probability of this situation occurring?
- **iv** What is your chance of winning $20?

d Summarise your views of the chance you will win a prize when you purchase a scratchie of this type.

PROBLEM SOLVING AND REASONING

9 You are setting the table for you and your friend to have dinner. You need a knife, fork and spoon for each of you. The cutlery drawer in the kitchen contains 15 knives, 10 forks and 15 spoons.

Without looking in the drawer, you choose three pieces of cutlery.

a What is the chance that your first choice was a knife?

b If your first choice was, in fact, a knife, what is the probability that you next chose a fork, then a spoon?

c Explain whether this probability would change if you removed these three items together.

d What is the probability of choosing a knife, a fork and a spoon?

PROBLEM SOLVING AND REASONING

e Assume that you have chosen a knife, fork and spoon for yourself. What is the probability that you will now choose these three items for your friend on your next selection?

10 Chris loves shooting basketball points. He knows from experience that he can be successful in 7 throws out of 10.

a What is the probability he is successful each time he shoots?

b Chris takes 3 shots. What is the probability he is successful in:

i all 3?

ii only 2?

iii only 1?

11 A four-sided die and a six-sided die are rolled.

a What is the probability that a double is rolled?

b What is the probability that both dice show odd numbers?

c The product of the numbers on the two dice is recorded.

i What is the smallest possible number?

ii What is the largest possible number?

iii What is the probability that this number is prime?

CHALLENGE

12 Zac and Will are playing a card game. To keep track of the games, they start with the same number of tokens, and the loser gives the winner one token at the end of each game. After a number of games, Will has three more tokens than when he began and Zac has won three games. Find the number of games Zac and Will have played.

13 Meg, Nancy and Peta are seated in a room with their eyes closed. A hat is placed on each of their heads. They are told that the hat on their head was taken from a box containing two red hats and three blue hats.
They are told to open their eyes and try to determine the colour of the hat on their head.
Meg opens her eyes, looks at the other two hats and says 'I see one red hat and one blue hat, so I don't know the colour of my hat'.
Nancy opens her eyes and says 'I see two blue hats, so I think the colour of my hat is red'.
Peta, opening her eyes, sees a blue hat and a red hat and says 'I think my hat is blue'.

a Explain why Meg can't determine the colour of her hat.

b Explain why Nancy thought her hat was red.

c Explain why Peta thought her hat was blue.

d What coloured hat does each person wear?

Reflect

If event A is dependent on event B, does this mean that event B is dependent on event A?

11E Conditional probability with two-way tables and tree diagrams

Start thinking!

Two-way tables can be used to display outcomes for a two-step experiment.

You are interested in the number of hearts and aces in a deck of playing cards. This can be displayed in a two-way table, which shows hearts, its **complement**, aces and its complement.

	Ace (A)	Not ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

1 Explain what each number represents.
a 1 **b** 12 **c** 3 **d** 36

2 The intersection of the column 'Ace' and the row 'Heart' can be represented by the outcome {A and H} or {H and A}. Represent each of these outcomes:
a those aces which are not hearts
b those cards which are not aces and not hearts.

Consider drawing a single card from the deck.

3 Use the table to find the probability of drawing:
a a heart from the group of aces
b an ace from the group of hearts.

This is an example of **conditional probability**. In other words, a 'condition' has been placed on the set from which to choose. If you were to choose from the set of hearts, you would be selecting an item from the total of the first row of the table.

4 Which row or column total would you use if you were choosing a particular card from the set of:
a aces? **b** not hearts? **c** not aces?

5 Find the percentage chance of drawing:
a a heart from the deck **b** a card which is not a heart from the group of aces

KEY IDEAS

- Conditional probability is when a condition is placed on the set from which a choice is to be made.
- Sometimes conditional probability is written in different words. You may see terms like 'if … then…', 'given' or 'knowing that'.
- A two-way table can be used to find conditional probability.
- The words 'of' and 'from' indicate the set from which the choice is made.
- This set will be represented in a particular row or column of the two-way table.
- A tree diagram can also be used to find conditional probability.

EXERCISE 11E Conditional probability with two-way tables and tree diagrams

EXAMPLE 11E-1 Calculating conditional probability using a two-way table

This table shows the results of a survey taken at a library to find the users of Facebook and Twitter.

	Male	Female	Total
Facebook users	20	30	50
Twitter users	5	10	15
Not used either	10	5	15
Total	35	45	80

Use the two-way table to find the percentage of:
a females who use Facebook
b Facebook users who are female.

THINK

a 1 Look at the 'Female' column to identify the total number of females.

2 Locate the number of Facebook users in the 'Female' column.

3 Write as a percentage.

b Repeat the steps in part a. Look at the 'Facebook users' row.

WRITE

a total number of females = 45

number of females using Facebook = 30

percentage of females who use Facebook
$= \frac{30}{45} \times 100\%$
$= 66\frac{2}{3}\%$ (or approx. 66.7%)

b total number of Facebook users = 50
number of Facebook users who are female = 30
percentage of Facebook users who are female
$= \frac{30}{50} \times 100\%$
$= 60\%$

UNDERSTANDING AND FLUENCY

1 Consider the two-way table in Example 11E-1.
a How many people undertook the survey?
b How many males do not use either Facebook or Twitter?
c How many Twitter users are there?
d How many female Twitter users are there?
e How many Facebook users are not male?

2 Consider the two-way table in Example 11E-1. Find the percentage of:
a females who use Twitter
b Twitter users who are female
c non-users who are male
d males who are non-users.

3 Consider the two-way table showing aces and hearts on page 562 to answer this question. Determine the percentage of:

a hearts which are not aces

b non-hearts which are not aces.

4 This two-way table shows the mode of transport to school for a group of students.

	Primary school	High school	Total
Catch public transport	15	27	42
Driven	22	18	40
Walk/ride	4	14	18
Total	41	59	100

Find how many:

a students are in the group

b primary school students are driven to school

c high school students walk or ride to school

d of those who catch public transport are high school students

e of those who are in primary school walk or ride to school.

5 Using the table from question **4**, calculate the percentage of:

a primary students who catch public transport to school

b those driven to school who are high school students

c those who walk or ride in the whole group

d high school students in the group

e high school students who are driven to school.

6 Say you were given a two-way table like the one from question **4** with some of the entries missing.

Explain how you could complete all the entries in the table, and complete the table.

	Primary school	High school	Total
Catch public transport	22		40
Driven	37	11	
Walk/ride			
Total		38	102

EXAMPLE 11E-2 Calculating conditional probability using a tree diagram

A sandwich tray contains 30 sandwiches, 10 of which are ham and 20, salad. Jack takes two sandwiches at random, one after the other.

a Draw a tree diagram to display the possible outcomes of his two choices.
b If Jack's first sandwich is ham, what is the probability that his second one will also be ham?
c Knowing that Jack's first choice is a salad sandwich, what is the probability that his second one will be ham?
d What is the probability of Jack's second sandwich being salad, given that his first sandwich is ham?

THINK

a Draw a tree diagram to display all the outcomes. Write the probabilities on all the arms.

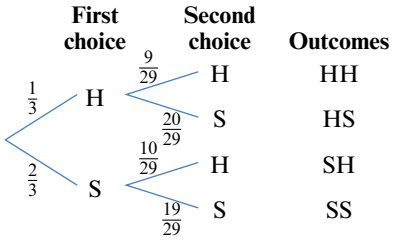
b Follow the branch where the first arm is H and the second arm is H. Take the probability of the second branch.

c Follow the branch S then H. Take the probability of the second branch.

d Follow the branch H then S. Take the probability of the second branch.

WRITE

a H represents ham and S represents salad.



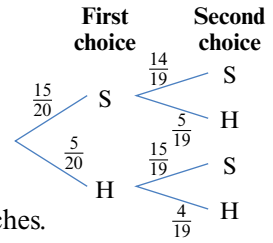
b $\Pr(H, H) = \frac{9}{29}$

c $\Pr(S, H) = \frac{10}{29}$

d $\Pr(H, S) = \frac{20}{29}$

UNDERSTANDING AND FLUENCY

7 A box of 20 chocolates contains 15 with soft centres and 5 with hard centres.
Two chocolates are taken at random, one after the other.
This tree diagram represents the outcomes of the two choices.



a Explain the probabilities of $\frac{15}{20}$ and $\frac{5}{20}$ on the first two branches.
b Explain the four probabilities on the branches for the second choice.
c If the first chocolate is hard, what is the probability the second will also be hard?
d Knowing that the first chocolate is a soft one, what is the probability the second will be hard?
e What is the probability of the second chocolate being soft, given the first is hard?

8 A computer generates random questions for students to practise. From collected data, the chance that a student gets the first question correct is 0.8. If the student gets this question correct, the computer generates a harder question, with the probability of the student getting this one correct being reduced by 0.1. If the student gets the first question wrong, the computer gives the student a question of the same level of difficulty as the first question.

a Draw a tree diagram to display the outcomes of two questions.

b Find the probability that the student gets the second question:

i correct, given they got the first question correct

ii correct, given they got the first question wrong

iii wrong, given they got the first question correct.

9 A small firm has 50 employees, 15 being female. There are 34 full-time employees, 30 of whom are males. There is a group of part-time employees.

a Draw a two-way table to record this data. Complete all the entries in the table.

b How many employees are:

i full-time females? **ii** part-time? **iii** male?

c Determine the percentage of:

i males who are part-time **ii** part-time employees who are male

iii full-time female employees **iv** female employees who are full-time.

d Explain the difference in your answers in part **c** for:

i parts **i** and **ii** **ii** parts **iii** and **iv**.

10 This two-way table shows the status of employment in a suburb.

	Male	Female	Total
Unemployed	140	130	270
Employed	2700	2400	5100
Total	2840	2530	5370

a Calculate the percentage of:

i those unemployed who are female

ii those males who are employed

iii those females who are employed

iv females in the suburb.

b A resident proudly claimed that only 5% of the unemployed in the area were male. Discuss whether this statement is correct.

11 This two-way table shows number and types of paper clips in a pile on a desk.

	Small	Large	Total
Silver	27	35	
Coloured	18	10	
Total			

a Complete the totals for the rows and columns.

b Find whether each of these statements is true, showing the calculations you used. (Percentages are rounded to the nearest whole number.) For those statements which are not true, show working to provide a correction.

i There is an equal number of small and large paper clips.

ii There are fewer coloured paper clips than non-coloured ones.

iii The percentage of small paper clips which are silver is 44%.

iv The percentage of coloured paper clips which are large is 22%.

v The whole pile contains 69% silver paper clips.

PROBLEM SOLVING AND REASONING

12 A jar contains black and white balls. Two balls are drawn without replacement. The probability of selecting a black ball on the first draw is 0.47, and the probability of selecting a second black ball, given that the first ball is black, is 0.22. What is the probability of selecting a white ball on the second draw, given that the first ball is a black one?

13 The number of girls and boys in Years 7 to 10 in a small country school is shown in this two-way table.

	Year 7	Year 8	Year 9	Year 10	Total
Girls	37	32	38	40	
Boys	45	47	40	45	
Total					

a Complete the totals for the rows and columns.

b Calculate the percentage of:

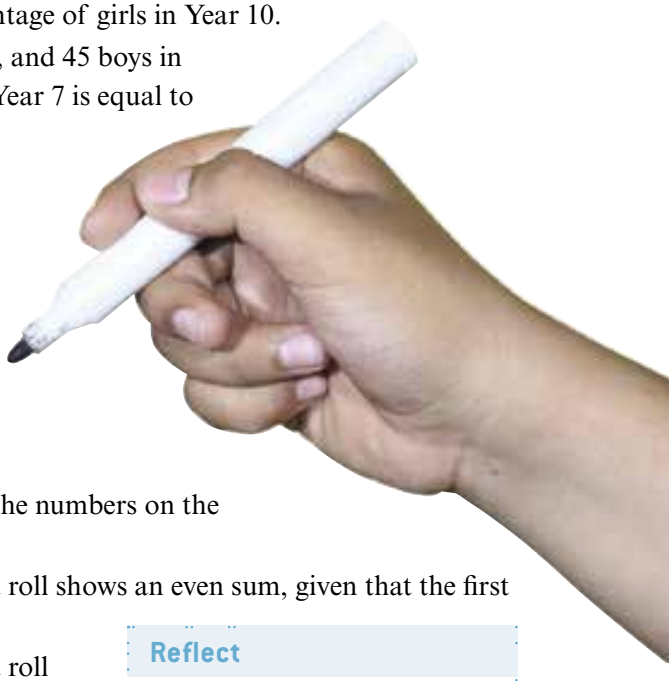
- **i** girls in Year 7
- **ii** Year 9s who are girls
- **iii** boys in Year 8
- **iv** girls who are above Year 8
- **v** boys who are in Years 7 or 8.

c Discuss whether each statement is true. Provide calculations to write a meaningful correct statement, if necessary.

- **i** Because there are 40 boys in Year 9, and 40 girls in Year 10, the percentage of boys in Year 9 is equal to the percentage of girls in Year 10.
- **ii** Because there are 45 boys in Year 7, and 45 boys in Year 10, the percentage of boys in Year 7 is equal to the percentage of boys in Year 10.

CHALLENGE

14 In a room there are 60 children, each of whom is either left-handed or right-handed. There are 30 right-handed boys, and 11 left-handed children. There are 25 girls amongst the 60 children. Complete a two-way table to find how many left-handed girls are in the room.



15 Two dice are rolled twice and the sum of the numbers on the two faces is recorded.

a What is the probability that the second roll shows an even sum, given that the first roll shows a sum greater than 7?

b What is the probability that the second roll shows a sum greater than 7, given that the first roll shows an even sum?

Reflect

How do you recognise when a question involves conditional probability?

11F Conditional probability and Venn diagrams

Start thinking!

A **Venn diagram** shows a collection of sets and their relationships.

The probability that A occurs, given that B has already occurred, can be written mathematically as Pr(A|B), where the '|' symbol stands for 'given'. Note that Pr(B|A) is different, and represents the probability that B occurs, given that A has already occurred.

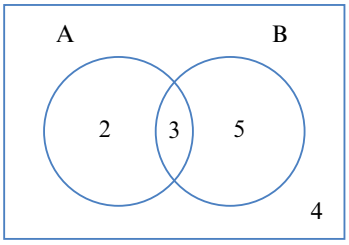


Figure 1

Consider these three Venn diagrams.

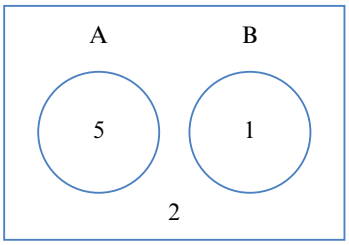


Figure 2

1 Consider figure 1.
- **i** How many elements are in set B?
- **ii** How many of the elements which are in set B are also in set A?
- **iii** Determine Pr(A|B).
- **iv** By a similar reasoning, determine Pr(B|A).

2 Consider figure 2.
- **a** Repeat question **1i–iv**.
- **b** Comment on the difference between your answers here and those for question **1**. Explain what could account for these differences.

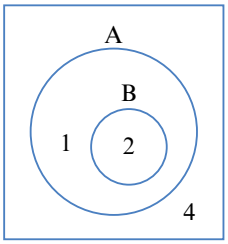


Figure 3

3 Consider figure 3.
- **a** Repeat question **1i–iv**.
- **b** Comment on anything you found different here. What could account for this difference?

KEY IDEAS

- A Venn diagram is used to display the relationship between different sets of data. Numbers indicating how many individuals belong are placed within each section of the diagram. These numbers can be used to calculate conditional probability.
- Conditional probability is shown as follows: Pr(B|A) represents the probability B occurs, given that A has already occurred.
- Conditional probability can also be found using these formulas.

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) \text{ and } \Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \text{ or } \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

NOTE When working with conditional probability, remember to determine the number of elements in the conditional (or given) set first.

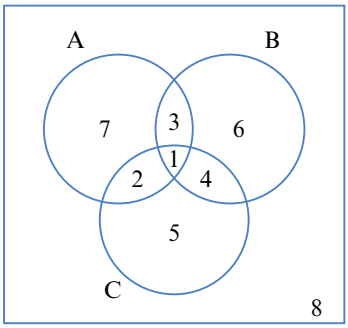
EXERCISE 11F Conditional probability and Venn diagrams

UNDERSTANDING AND FLUENCY

The correct terminology must be used when referring to sets.

- The **universal set** contains all the elements under consideration, and is represented by ε.
- The **intersection** of two sets, A and B, is the area of overlap and is denoted by $A \cap B$.
- The **union** of the elements from both sets A and B is represented by $A \cup B$.
- The complement of set A is written as A'.
- The number of elements in set A is written as n(A).
- A **subset** is a smaller set within another set. For example if A is a subset of B, this can be written $A \subset B$. If A is not a subset of B this can be written $A \not\subset B$.

1 Consider this Venn diagram with three overlapping sets.



a How many elements are in set:

i A? **ii** B?

iii C? **iv** A'?

v $(A \text{ and } B)'$?

b Calculate each probability, explaining how you determined the numerator and the denominator.

i Pr(A|B) **ii** Pr(B|A)

iii Pr(C|B) **iv** $Pr(A|C')$

v $Pr((A \text{ and } B)|C')$ **vi** $Pr(C|(A \text{ and } B)')$

2 A survey of 50 people was conducted to collect data relating to the types of movies they liked. The movie categories were comedy, drama and animation. The results are as follows.

26 like comedy,
24 like drama,
33 like animation,
13 like comedy and drama,
17 like drama and animation,
14 like comedy and animation,
while 8 like all three types of movies.

Draw a Venn diagram to display this data.

3 Use your Venn diagram from question **2** to find the probability that a person chosen:

a likes comedy

b likes comedy, given that they don't like drama

c does not like comedy, given that they like drama

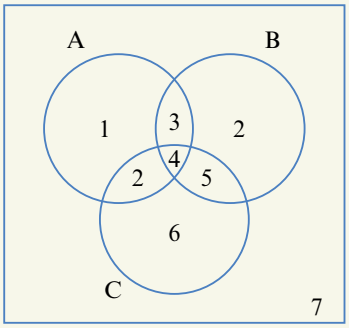
d likes comedy or drama, given that they don't like animation

e likes any of these movie forms

f does not like animation, given that they like comedy.

EXAMPLE 11F-1 Calculating conditional probability using a Venn diagram

This Venn diagram shows the distribution of a group of students who study Art (A), Business Studies (B) and Computing (C).



Use the diagram to determine the probability that a student studies:

a all three of these subjects, given that they study two of the subjects

b Art and Computing, given that they do not study Business Studies.

THINK

a 1 Find the number of students in the 'given' set. This is the total in all the overlapping sections.

2 Find the number of students within this group who do all three subjects.

3 Write these two numbers as a probability fraction, simplifying if possible.

b 1 Find the number who do not study Business Studies. This is all those outside the circle B.

2 Find the number of students within this group who do Art and Computing.

3 Write as a probability fraction, simplifying if possible.

WRITE

a number of students in 'given' set
= number studying two or more subjects
= 3 + 4 + 5 + 2
= 14

number studying all three subjects = 4

$\Pr(\text{all three subjects}|\text{two subjects}) = \frac{4}{14}$
$= \frac{2}{7}$

b number not studying Business Studies
= 1 + 2 + 6 + 7
= 16

number in this group doing Art and Computing
= 2

$\Pr(\text{Art and Computing}|\text{not Business Studies}) = \frac{2}{16}$
$= \frac{1}{8}$

UNDERSTANDING AND FLUENCY

4 Use the Venn diagram in Example 11F-1 to determine the probability that a student studies:

a only two subjects

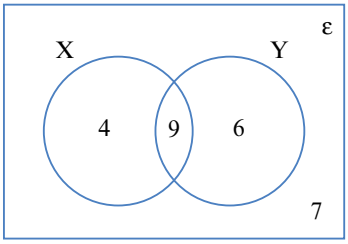
b only two subjects, given that they study Art or Business Studies

c only two subjects, given that they do not study Art or Business Studies

d three subjects, given that they do not study Art

e Art and Business Studies, given that they do not study Computing.

5 Consider this Venn diagram.



a Knowing that n(X) means the number of elements in set X, find:

i $n(X)$ **ii** $n(X \cup Y)$

iii $n(X')$ **iv** $n(X \cap Y)$

v $n(\varepsilon)$ **vi** $n(X' \cap Y)$

b Calculate each probability.

i $\Pr(X)$ **ii** $\Pr(X')$

iii $\Pr(X \cap Y)$ **iv** $\Pr(X \cup Y)'$

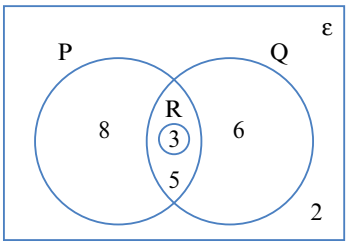
v $\Pr(X' \cap Y')$ **vi** $\Pr(X \cap Y')$

vii $\Pr(X|Y)$ **viii** $\Pr(X \cap Y|X)$

ix $\Pr(X \cup Y|X')$ **x** $\Pr(X \cup Y')$

xi $\Pr(X'|Y')$ **xii** $\Pr(Y \cup Y')$

6 Consider this Venn diagram.



a Decide whether each statement is true and explain your reasoning.

i $R \subset P$ **ii** $R \subset Q$

iii $R \not\subset (P \cap Q)$ **iv** $R \not\subset (P \cup Q)$

v $R \not\subset \varepsilon$ **vi** $R \subset Q \subset \varepsilon$

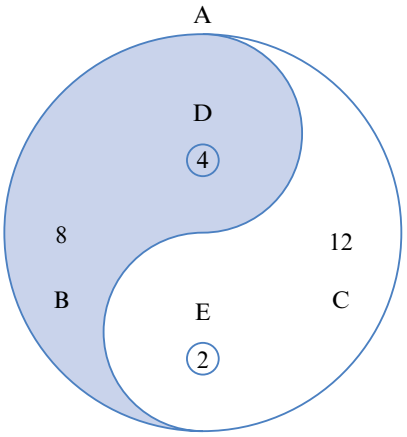
b Use the diagram to calculate each probability.

i $\Pr(P \cap R)$ **ii** $\Pr(P \cap Q')$

iii $\Pr(P' \cap Q')$ **iv** $\Pr(P \cap Q \cap R)$

v $\Pr(P \cap R|P)$ **vi** $\Pr(Q \cap R|P)$

7 The sets in Venn diagrams do not have to be in circles. Consider this yin yang symbol.
The universal set is the set A, with sets B, C, D and E being subsets of set A. The number of elements in each subset is indicated.



a Find the number of elements in each region.

i $n(A)$ **ii** $n(C)$

iii $n(E)$ **iv** $n(B \cup E)$

v $n(C \cup E')$ **vi** $n(D \cup E)$

vii $n(D \cap E)$ **viii** $n(C \cap D)$

b Calculate each probability.

i $\Pr(D|B)$ **ii** $\Pr(B|D)$

iii $\Pr(D \cup E)$ **iv** $\Pr(B \cap C)$

v $\Pr(B|A)$ **vi** $\Pr(D \cup E|C')$

vii $\Pr(D|E)$ **viii** $\Pr(C|A')$

UNDERSTANDING AND FLUENCY

8 One hundred people took part in a survey to determine the readership of three monthly magazines (A, B and C). The results were as follows.

59 read A, 68 read B, 59 read C, 42 read A and B, 45 read B and C, 39 read A and C, while 27 read all three magazines.

a Draw a Venn diagram to show the results of the survey.

b Find the number of people who:

i read magazines A or B

ii read magazine C but not magazine B

iii read two of the magazines

iv read two magazines at most

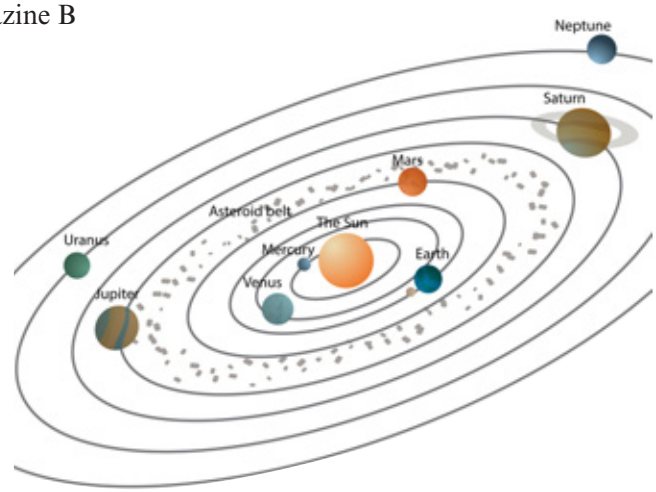
v do not read magazine A

vi read only one magazine

vii do not read B or C

viii do not read three magazines.

9 Describe the orbits of the planets in our solar system as shown here in terms of a Venn diagram.



PROBLEM SOLVING AND REASONING

10 A Venn diagram that consists of two sets can be transformed into a two-way table. Consider the Venn diagram in question **5**.

a Set up a table with headings X, X′, Y and Y′. Add a row and a column for the totals. Fill in the information you know. Your table could look something like the one at right. Complete the table.

	X	X′	Total
Y	9		15
Y′			
Total	13		26

b Give the number of elements in each set.

i n(X′) **ii** n(Y′)

iii $n(X \cap Y)$

c Use the table to calculate each probability.

i Pr(X|Y) **ii** Pr(Y|X)

iii Pr(X′|Y′) **iv** Pr(Y′|X′)

11 A two-way table can be transformed into a Venn diagram.

	Ace (A)	Not ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

a Explain how you would approach this.

b This two-way table was discussed on page 562. Transform it into a Venn diagram.

PROBLEM SOLVING AND REASONING

12 The formula for conditional probability, previously written as

Pr(A and B) = Pr(A) × Pr(B, once A has occurred)

can now be written as: $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$.

Show that it can also be written as

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \text{ or } \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

This formula is often useful when a question is not accompanied by a diagram.

13 In a survey of a group of students, the probability that a student selected at random likes cherries is $\frac{35}{59}$. The probability that a student likes grapes, given that he/she also likes cherries, is $\frac{11}{35}$.

a Draw a Venn diagram to show this information. (Hint: you may not be able to fill in all the sets. Complete as much as you can.)

b Use your Venn diagram to calculate the probability that a student selected at random likes both cherries and grapes.

c Use one of the formulas in question **12** to find the answer to part **b**.

d Comment on the two methods you used.

14 A survey was conducted to determine whether a group of people preferred milk or dark chocolate. It was found that the probability that a person likes milk chocolate is $\frac{23}{41}$. The probability that a person who likes milk chocolate also likes dark chocolate is $\frac{5}{23}$.

a Draw a Venn diagram to show this information. (Complete as much as you can.)

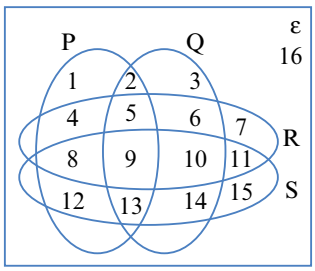
b Use your Venn diagram to find the probability that a person selected at random likes both milk and dark chocolate.

c Use a formula to find the answer to part **b**.

d Comment on the two methods you used.

CHALLENGE

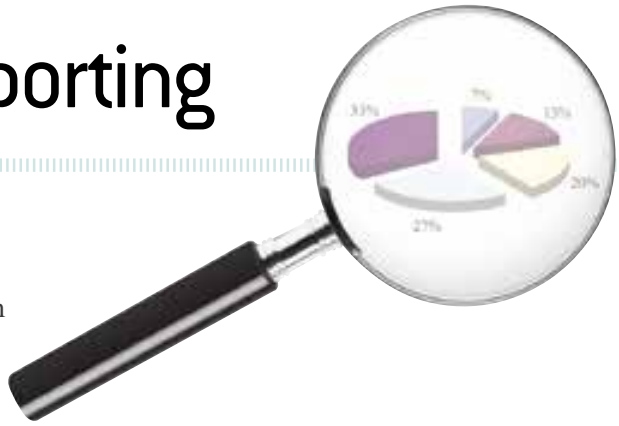
15 This Venn diagram shows the relationship between four sets P, Q, R and S. Determine the sum of the elements within the region $(P \cap Q) \cup (R \cap S')$.



Reflect

The Audi car symbol is in the shape of a Venn diagram. What is the meaning of the symbol?

11G Sampling and reporting



Start thinking!

Rumour has it that the school library is going to close at 4 pm each weekday, because very few students are making use of its extended opening hours. Will this greatly affect many students? You decide it's necessary to find out.

The first question to answer is: Who will you choose to participate in the survey?

A census is where every member of a population is surveyed. Because conducting a census is not always practical, a survey can instead be conducted on a sample of the population. The individuals in the sample should be chosen randomly, so that the sample displays the same characteristics as the population.

1 Your plan is to investigate the after-school use of your school library by Years 7 to 10 students. You decide it's not possible to ask every single student, so you need to conduct a survey.

- **a** An adequate sample size is $\sqrt{\text{population}}$. How many students would you need to survey?
- **b** Describe how you would select this sample of students.
- **c** You are made aware that more students in the lower year levels are using the school library than in the upper year levels. How would you allow for this when choosing your sample?
- **d** You have various options for conducting the survey, including interviewing and questionnaires. How would you conduct the survey?
- **e** How would you ensure that bias was not introduced into the sample or the survey?
- **f** Explain how it would be possible for you to introduce bias into your sample or survey.

2 Imagine you have collected all your data.

- **a** Describe how you would collate your data.
- **b** How would you present your results?
- **c** How would you ensure that no bias was introduced into the results you report?
- **d** Imagine you find that none of the Year 10 students in your sample used the library, and all the Year 7 students did. Would you consider this sample to be truly representative of the whole Year 10 and Year 7 population? What would you do?

KEY IDEAS

- When a survey is conducted on a sample of a population, the sample must truly represent the characteristics of the population.
- Stratified sampling is a method that involves division of a population into small groups known as strata.
- Systematic sampling occurs when every kth element of the population is selected.
- Be aware that bias can easily occur when collecting data by sampling.
- Take care that any displays of data in reports do not give a biased impression.
- Be prepared to question the claims of reports.

EXERCISE 11G Sampling and reporting

EXAMPLE 11G-1 Reviewing a published report of a survey

A new tunnel for which users pay a toll has just been opened to allow cars to travel from one side of the river to the other. The alternative route is via a bridge, which has no toll. Shortly after the tunnel opened, a newspaper reported: 'The tunnel is now so popular that the probability of a car using the tunnel rather than the bridge is 85%'. Public perception is that this is not the case. A group of residents decided to do a car count over an extended period of time and found 967 cars from a total of 2472 went over the bridge. Comment on the newspaper report.

THINK	WRITE
1 Calculate the probability from collected data of a car using the tunnel.	Using the data collected: Pr(car uses bridge) $= \frac{967}{2472} \times 100\%$ $= 39\%$ So, Pr(car uses tunnel) = 61%
2 Compare the collected data with the reported data, and comment on the result.	Collected data shows that 61% of cars use the tunnel, while the newspaper reports this figure to be 85%. It seems that the newspaper report is exaggerated.

UNDERSTANDING AND FLUENCY

1 This table shows the distribution of students in Years 7 to 10 in a school.

Year 7		Year 8		Year 9		Year 10	
Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
72	83	65	70	45	60	45	58

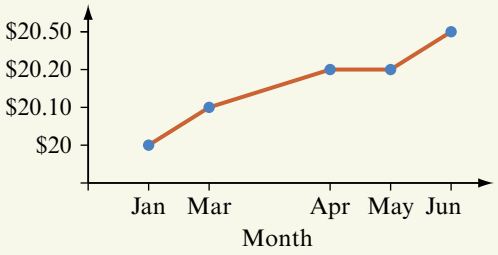
- **a** What is the population of this group?
- **b** What size sample would be suitable for a survey?
- **c** How many students would you choose from each year level for the survey?
- **d** Explain why it would not be fair to choose an equal number of boys and girls from each year level.
- **e** What would be the composition of boys and girls in each year level in your sample?
- **f** Explain how you would select these students.
- **g** Do you consider that there is any bias in the sample you have selected this way? Explain why or why not.

2 Critically analyse each report.

- **a** A survey conducted at a gym revealed that 90% of its clients felt they were getting enough exercise each week. It was later quoted that 'Australians are getting fitter, as 90% of them say they exercise regularly each week'.
- **b** There were four teenage boys left on the bus at the end of the line when a reporter asked if they had part-time jobs. Two of them said they did. It was later reported that '50% of teenagers today have part-time jobs'.
- **c** At the hardware store, a survey showed that four out of five males believe they do their fair share of house work. A report later appeared in a newspaper with the heading: 'Females lucky: 80% of males do fair share of house work'.
- **d** The girls in the cheer squad at a school swimming carnival, held indoors, were asked whether they were wearing sunscreen to prevent them getting sunburnt. They all said that they had no sunscreen on. It was later reported that 'School children ignore warnings about wearing sunscreen to prevent skin cancer'.

EXAMPLE 11G-2 Understanding misleading features in graphs

This graph displays the hourly rate for a keyboard operator employed in a large group. The period covers the first 6 months of the year.



- **a** This graph could be considered to be misleading or inadequate. Give reasons why.
- **b** The manager claims that the hourly rate of a keyboard operator has increased dramatically over the last 6 months. The graph appears to indicate this.
  - **i** Is this true?
  - **ii** How was this effect achieved?

THINK	WRITE
a What are the typical ways to make a graph favour a particular point of view?	**a** This graph is misleading for the following reasons. There is no title. There is no label on the y-axis. The scale on the x-axis is not uniform. The month of February is missing. Since the scale on the y-axis does not start at zero, a break should be marked on the y-axis at the beginning. The scale on the y-axis is not uniform.
b i Look at the values on the y-axis.	**b i** The hourly rate has only increased from \$20 to \$20.50 over the 6-month period, an increase of 50c per hour. This is not a 'dramatic' increase.
ii Consider the scale on the y-axis.	**ii** Since the scale on the y-axis does not start zero, the range of \$20 to \$20.50 can be stretched out. This gives the impression that the increase is greater than it really is.

3 **a** Redraw the graph from Example 11G-2 so that it is not misleading.

b Explain the corrections you had to make in order to achieve this.

4 Take your graph from question **3**.

a Redraw it to create a misleading graph where the hourly rate of the keyboard operators appears to have not changed very much over the 6 months.

b Explain how you achieved this.

5 Some advertisements claim that eating a good breakfast leads to success in school. Consider, however, an alternative argument:

'Students who don't eat breakfast are more likely to be absent from school or experience lethargy.'

Do these two statements mean the same thing? Why use one statement rather than the other?

6 Critically analyse each media report. Suggest what research would need to have been done to make each claim valid.

a In a study of women, having coffee with lunch seemed to reduce the diabetes risk more than having coffee at other times of day.

b Studies suggest that physically active people have a reduced risk of Alzheimer's disease in later life.

c Studies suggest that happiness wards off heart disease.

7 We are often unaware of the companies which sponsor such 'studies'. Consider this headline.

'A hot-dog a day increases risk of cancer by 21%'

Whether this is true or not is open to debate. However, what credibility does the study hold when it was later revealed to have been conducted by an animal-rights group that wants people to be vegan? Suggest how this type of reporting of information from a biased source could be monitored.

8 Celebrities often promote products on television and in magazines. Do these celebrities really believe in the product, or are they simply paid to advertise it? Are they given these products free for their own use?

What are your thoughts when you see advertisements such as these? Does it prompt you to purchase the product? Would you be more likely to purchase the product than you would if it was advertised by a non-celebrity?

UNDERSTANDING AND FLUENCY

9 Online surveys are now very common. If surveys conducted this way are to have any credibility, there are many issues to consider. For example, how can you ensure they draw on a random sample which is representative of the target audience? How can you ensure that the person completing the survey is who they say they are? How do you cope with the non-response rate?
Write some guidelines you feel would be important to follow when choosing a sample for an online survey.

10 After answering an online survey, you frequently receive unwanted advertising emails from companies you have never heard of: 'Great holiday deals on Hayman Island', 'Take advantage of this special meal', etc. Some groups make money by selling email addresses to interested companies. Is this simply part of the digital age, or can you somehow monitor this behaviour? What are your thoughts?

11 A television channel invites its viewers to vote on a topic they have just discussed. Later in the program, the results of the votes are revealed, such as: '98% of viewers agree that …' Discuss whether the reported figures in this type of survey are valid.

PROBLEM SOLVING AND REASONING

12 The CEO of a company has drawn up this table showing the composition of her company.

	Total employees		Full-time		Part-time	
	Male	Female	Male	Female	Male	Female
Managers	18	4	14	4	4	0
Non-managers	42	61	40	31	2	30

She made these claims about her employees.

a 82% of male employees are managers.

b 6% of females in the company are managers.

c 29% of employees work part-time.

d 49% of female non-managers work full-time.

Discuss whether each statement is correct. Provide mathematical evidence to support your answer. If a particular claim is incorrect, supply the correct figure.

13 The top 10 Facebook users in 2013 came from the following countries. The table shows the number of users (in millions) and the population (in millions) of these countries.

Number	Country	Users (millions)	Population (millions)	Per cent users
1	USA	160	318	
2	Brazil	68	201	
3	India	63	1241	
4	Indonesia	48	250	
5	Mexico	41	118	
6	Turkey	33	77	
7	UK	32	64	
8	Philippines	30	99	
9	France	25	66	
10	Germany	25	81	

a Complete the last column of the table (correct to one decimal place).

b Which of these countries has the:

i highest percentage of users?

ii lowest percentage of users?

c China has the largest population of any country (1.4 billion), yet it does not make the top 10 list of users. Suggest some reasons for this.

d Using this data, draw two different graphs to support these two statements:

- **i** people in the US and UK use Facebook far more than those in other countries
- **ii** People of India don't use Facebook as much as those in other countries.

e Explain why you chose to draw your graphs the way you did in part **d**.

14 A telephone survey was conducted on a sample of 1020 respondents from a variety of locations across Australia. The sample was chosen to reflect the actual Australian population.

Age (years)	Male	Female	Total
14–19	58	60	118
20–29	66	79	145
30–39	69	90	159
40–49	83	90	173
50–64	110	120	230
65+	96	99	195
Total	482	538	1020

a Comment on the age categories used.

b Comment on the composition of the sample.

c How could you check whether the figures in this sample truly represent the actual Australian population?

d It was claimed that 46% of 14–19-year-olds access social network sites every day. Does this mean that, if I asked 100 people in this age group, 46 of them would tell me that they accessed a social network site every day? Explain.

e It was also claimed that Facebook dominates social media, with 97% of the social network users. What does this mean?

15 Conduct a survey amongst the members in your class. Report on the results of your survey, justifying any figures with mathematical evidence.

Reflect

Why is it wise to always question the figures used in reports of surveys?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

probability
experimental probability
theoretical probability
probability scale
event
outcome
complementary events
sample space
tree diagram
multi-step experiments
element of set
experiments with replacement
experiments without replacement
independent event
dependent event
two-way table
conditional probability
Venn diagram
universal set
intersection of sets
union of sets
number of elements in set
subset
census
population
survey
sample
random sample

MULTIPLE-CHOICE

11A **1** Consider a set of cards numbered 1 to 10. If one card is chosen, what is the probability it is a prime number?
A $\frac{1}{5}$ B $\frac{2}{5}$ C $\frac{1}{2}$ D $\frac{3}{10}$

11B **2** A coin is flipped three times. How many outcomes would appear on a tree diagram of this experiment?
A 2 B 4 C 6 D 8

11C **3** One card is drawn from a deck of playing cards and its suit is noted. It is replaced, then a second card is drawn. What is the probability two diamonds are drawn?
A $\frac{1}{13}$ B $\frac{1}{16}$ C $\frac{1}{17}$ D $\frac{3}{52}$

11C **4** Repeat the experiment in question **3**, but this time assume the first card is not replaced before the second card is drawn. What is the probability that two diamonds are drawn?
A $\frac{1}{13}$ B $\frac{1}{16}$ C $\frac{1}{17}$ D $\frac{3}{52}$

11D **5** Two six-sided dice are rolled, and the sum of the two numbers is recorded. What is the probability that this sum is 6?
A $\frac{1}{36}$ B $\frac{1}{18}$ C $\frac{5}{36}$ D $\frac{1}{6}$

Questions **6** and **7** refer to this table, which shows the number of students at a school.

Year level	Males	Females	Total
Year 10	66		128
Year 11	63		
Year 12			105
Total	191		353

11E **6** How many females are in Year 11?
A 43 B 57 C 62 D 120

11E **7** How many males are in Year 12?
A 43 B 57 C 62 D 120

Questions **8** and **9** use this information. Two sets A and B overlap. Set A has 12 elements, set B has 10 elements. The number of elements in both set A and set B is 2.

11F **8** What is Pr(A|B)?
A $\frac{1}{4}$ B $\frac{1}{5}$ C $\frac{1}{6}$ D $\frac{2}{3}$

11F **9** What is Pr(B|A)?
A $\frac{1}{4}$ B $\frac{1}{5}$ C $\frac{1}{6}$ D $\frac{2}{3}$

11G 10A **10** A sample of 20 people is being taken from a group of 280 males and 120 females. How many females should be chosen?
A 6 B 10 C 14 D 60

SHORT ANSWER

11A **1** A box contains six red, four yellow, three green and two blue marbles. One marble is chosen from the box. Calculate the probability that the marble chosen is:

- **a** yellow
- **b** not yellow
- **c** red or yellow
- **d** not red or blue.

11B **2** A bag contains the letters of the alphabet. One letter is randomly chosen, then replaced. This is repeated another two times. Draw a tree diagram to find the probability of drawing:

- **a** no vowels
- **b** at least one vowel
- **c** three consonants
- **d** no more than two consonants.

11C **3** A three-digit number is made using the digits 0 to 9. The digits can be repeated.

- **a** How many numbers are possible?
- **b** Describe any restrictions on the formation of the number.
- **c** What is the probability the number will be 111?
- **d** What is the probability that all the digits will be the same in the number?

11C **4** The experiment in question **3** is repeated, this time not allowing any repetition of the digits.

- **a** How many numbers are possible in this case?
- **b** Describe any restrictions on the formation of the number.
- **c** What is the probability the number will be 107?
- **d** How many numbers are possible with a value greater than 500?

11D **5** Two biros are taken from a container of four blue, five black and two red biros. Calculate the probability that the:

- **a** second biro is red, given the first one is blue
- **b** second biro is red, given the first one is black
- **c** two biros are the same colour
- **d** two biros are a different colour.

11E **6** This table shows the number of iPhone users in a group of people.

	Male	Female
iPhone	20	35
Other type of phone	15	12

Find the percentage of:

- **a** males who do not use an iPhone
- **b** iPhone users who are male
- **c** non-iPhone users who are female
- **d** females who use an iPhone.

11F **7** Sets A, B and C overlap, and can be described as follows:
$n(A) = 14$, $n(B) = 13$, $n(C) = 15$,
$n(A \cap B) = 6$, $n(B \cap C) = 5$,
$n(A \cap C) = 7$ and $n(A \cap B \cap C) = 4$.

- **a** Draw a Venn diagram to display this information.
- **b** Find:
  - **i** $Pr(A|B)$
  - **ii** $Pr(B|A)$
  - **iii** $Pr(A'|B)$
  - **iv** $Pr(A|B')$
  - **v** $Pr(A \cap B|B)$
  - **vi** $Pr(A \cap B|A)$

11G 10A **8** A survey is to be conducted on a group of 480 males and 220 females.

- **a** What size sample would be appropriate?
- **b** How many males and how many females should be chosen?
- **c** Describe a suitable method which could be used to select the sample.

MIXED PRACTICE

1 What is the chance that a head will result on one flip of a coin?

A certain **B** impossible
C fifty-fifty **D** very unlikely

2 A die is rolled and the number of 6s is recorded. How many favourable outcomes are there?

A 1 **B** 2 **C** 3 **D** 4

3 When choosing a letter of the alphabet, what is the probability it is a letter before 'S'?

4 When rolling two eight-sided dice, what is the number of possible outcomes?

5 Write the sample space for flipping a coin three times.

6 Two six-sided dice are rolled. What is the probability one die will show a 1?

A $\frac{5}{18}$ **B** $\frac{5}{36}$ **C** $\frac{11}{36}$ **D** $\frac{1}{3}$

7 What is the probability a single letter selected from the word AUSTRALIA is a vowel?

8 The intersection of sets A and B is written as:

A $A \cup B$ **B** $A \cap B$ **C** $A \subset B$ **D** $A \in B$

9 A date is chosen from the month of June. How many elements are in the complement to the set {the date is not a multiple of 2 or 3}?

10 A menu displays three entrees, five main meals and four desserts. How many different combinations does the restaurant provide?

11 A single card is chosen from a deck of playing cards. What is the probability it is a black card or a king?

A $\frac{1}{2}$ **B** $\frac{1}{13}$ **C** $\frac{7}{13}$ **D** $\frac{15}{26}$

12 Consider two multiple-choice questions, one with four options and the second with five options. What is the probability the answer is A for both the questions?

A $\frac{1}{4}$ **B** $\frac{1}{5}$ **C** $\frac{1}{20}$ **D** $\frac{1}{10}$

13 For two sets, X and Y, write the complement of $X \cup Y$.

14 At a school fete, 100 tickets were sold for the raffle. What is the chance you will win first, second and third prizes if you purchase three tickets?

15 Which of these equations is correct for conditional probability?

A $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$

B $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(B)}$

C $\Pr(B|A) = \frac{\Pr(A)}{\Pr(A \cap B)}$

D $\Pr(B|A) = \frac{\Pr(B)}{\Pr(A \cap B)}$

16 What is an investigation conducted on the whole population?

A survey **B** census
C sample **D** random sample

17 What is the probability of drawing one jack after another, without replacement, from a deck of playing cards?

A $\frac{1}{221}$ **B** $\frac{1}{169}$ **C** $\frac{3}{169}$ **D** $\frac{3}{676}$

18 An online survey was sent to 20% of a population. The return rate of the survey was 12%. If this amounted to 60 returned surveys, what was the size of the population?

A 620 **B** 14 400
C 2500 **D** not able to tell.

19 If $\Pr(A \cap B) = \frac{3}{25}$, $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{13}{25}$, what is the value of $\Pr(B|A)$?

20 Consider three sets, P, Q and R. If $Q \subset P$ and $R \not\subset P$, write a relationship between Q and R.

21 What is the probability of rolling a total less than 7 in two rolls of a six-sided die?

22 In your drawer you have six identical black socks and four identical white socks. Without looking, you take out two socks. What is the probability they will make a pair?

Questions 23–25 refer to the following information.
A circular spinner has three equal sectors coloured red, blue and green. It is spun three times.

23 What is the probability it will give the same colour each time?

24 What is the probability it will give a different colour each time?

25 What is the probability it will give two colours the same?

ANALYSIS

1 The game of Mastermind can be played with pen and paper using the following rules.
Player 1 chooses a three- or four-digit number with no repeat digits, using the digits 0 to 9. Player 2 has to try to guess this number. For each attempt that player 2 makes, player 1 indicates how many digits are correct and in the correct position (have correct place value) by drawing a black dot for each, and also how many digits are correct but in the wrong position by drawing a hollow dot for each.
The game continues until player 2 guesses the correct number.
Assume you are player 2. Player 1 chooses a three-digit number.
   a Explain why you are not able to draw a two-way table to display all the outcomes.
   b How many numbers are possible?
   c Describe any restrictions on the numbers.
   d You start by writing the digit 1 as the first digit in the number. What is the probability it is correct?
   e After playing for a while, you learn that the first digit is 3. Knowing that this is the case, what is the probability you will choose:
      i the second digit correctly?
      ii the next two digits correctly?

f A little further into the game, you learn that the second digit is 5. If you have no information about the last digit, what is the probability you will get it correct in your next turn?

g The game may sound quite simple; however, it is not as easy as it first appears, and requires logical thinking to determine the correct number. Say that player 1 chose the number 354. At some stage in the game you have 543 as a suggestion. Although you have all the digits correct, they are all in the incorrect position. What is the maximum number of trials it would take for you to get the digits in the correct order?

h Play a game with a friend. Record your experiences. Suggest a technique which you found helpful.

2 A Venn diagram can be used to find the highest common factor (HCF) and the lowest common multiple (LCM) of a pair of numbers.
   a Take the numbers 156 and 520.
      i Write each number as the product of its prime factors.
      ii Draw a Venn diagram with two overlapping circles to represent the two numbers. By arranging the prime factors of the two numbers, explain how the Venn diagram can represent the HCF and LCM of the two numbers.
      iii What is the HCF and LCM?
   b Take the numbers 168, 252 and 420.
      i Use the above technique to draw a Venn diagram displaying the HCF and LCM of the three numbers.
      ii What is the HCF and LCM?

CONNECT

How secure is your password?

When you log in to a computer, you use a password. When someone uses an ATM or EFTPOS machine, they are required to enter a PIN. However, there's always the risk of identity theft: that someone will illegally gain access to your password or PIN.

Identity theft has increased dramatically since the introduction of the Internet. Many banking and shopping sites claim to be 'secure'. But in reality, if 'hackers' want to invade a web site, they will eventually find a way. You can only hope that the technology used to maintain security for these sites advances faster than that of the hackers!

Identity theft is harder if you have a strong password. Consider your options. Web sites often give specific rules with regard to length and character use. These can be quite complex, so for this task, use these rules:

- Let the number of characters in your password be represented by n.
- Let the number of a particular type of character (the count) be represented by c.
- You can use upper- and lower-case letters. Letters are case-sensitive.
- Any digit can be used.
- Any symbol can be used.

The strength of a password is scored according to the following criteria. The strength represents the probability (as a percentage) that the password will not be cracked. Points are given for the use of some characters, and deducted for the combination of certain characters.

Password: _____

These combinations count as positive scores and increase the password's strength.

	Count (c)	Score
Number of characters		$c \times 4 =$
Lower-case letters		$2(n - c) =$
Upper-case letters		$2(n - c) =$
Digits		$c \times 4 =$
Symbols		$c \times 6 =$
Middle digits or symbols		$c \times 2 =$
Total		

These combinations count as negative scores and decrease the password's strength.

	Count (c)	Score
Letters only		$c =$
Digits only		$c =$
Successive lower-case letters		$c \times 3 =$
Successive upper-case letters		$c \times 3 =$
Successive digits		$c \times 3 =$
Total		

Note: Successive characters are those of the same type which follow on from the first character. For example, the word 'Password' has six lower-case successive letters (following on from the letter 'a').

Taking the total of the negative scores from the positive scores gives the strength of a particular password.

Your task

- Test a series of different passwords. Be systematic in introducing different characters into the password. Record what you find.
- A suggestion often given is to substitute other characters for digits or letters (zero for lower- or upper-case O, ! or 1 for i, 5 for S, and so on). Investigate the effect this has on the strength of the password. Can you, for example, make a password twice as strong by doing certain things?

Imagine you have been asked by your school Student Council representative to investigate the topic of secure passwords for the school's computers, and to present your findings in a report to students. Your report should include:

- samples of weak and strong passwords
- calculations to support these claims
- recommendations for improving the strength of passwords.

You may like to present your findings as a report. Your report could include:

- an information booklet
- a PowerPoint presentation
- an advertising brochure
- other (check with your teacher).

Answers

CHAPTER 1 FINANCIAL MATHEMATICS

1 Are you ready?

1 75% **2** A **3** 0.105 **4** C
5 $96.80
6 **a** $\frac{7}{8}$ **b** 87.5%
7 B **8** C
9 **a** 36 **b** 8 **c** 208 **d** 28
10 D

1A Review of percentages

1A Start thinking!

1 **a** Put percentage over 100.
b Divide percentage by 100.
2 **a** Multiply by 100. **b** Multiply by 100.
3 **a** $\frac{25}{200} = \frac{1}{8}$, 0.125
b Percentage didn't involve a fraction.
4 **a**

Percentage	Fraction	Decimal
35%	$\frac{7}{20}$	0.35
40%	$\frac{2}{5}$	0.4
12%	$\frac{3}{25}$	0.12
12.5%	$\frac{1}{8}$	0.125
$15\frac{1}{2}$%	$\frac{31}{200}$	0.155

Exercise 1A Review of percentages

1 **a** 0.15 **b** 0.9 **c** 1.12
d 0.755 **e** 0.188 **f** 0.0025
2 **a** $\frac{21}{100}$ **b** $\frac{17}{20}$ **c** $\frac{9}{4}$ **d** $\frac{16}{25}$
e $\frac{19}{50}$ **f** $\frac{401}{500}$ **g** $1\frac{1}{10}$ **h** $\frac{21}{400}$
i $\frac{63}{400}$ **j** $\frac{11}{2000}$ **k** $\frac{21}{250}$ **l** $\frac{1}{25\,000}$
3 **a** 60% **b** 37.5% **c** 70%
d 65% **e** 68.75% **f** 62.5%
4 **a** $57.60 **b** $600 **c** $198.45 **d** $1332
e $1248 **f** $700 **g** $4200 **h** $10 980
5 **a** $16.05 **b** $76.92 **c** $11.94 **d** $387.66
e $271.29 **f** $2.90 **g** $0.81 **h** $46.16
6 **a** 20% **b** 61.54% **c** 19.33%
d 25.17% **e** 51.64% **f** 234.62%
g 13.60% **h** 54.78% **i** 191.20%
7 **a** $16\frac{2}{3}$ **b** $\frac{7}{15}$, $0.4\dot{6}$ **c** $66\frac{2}{3}$%
d $\frac{37}{300}$, $0.12\dot{3}$ **e** $116.\dot{6}$, $1.1\dot{6}$ **f** $91.\dot{6}$, $0.91\dot{6}$
g $\frac{511}{600}$, $0.8516\dot{6}$ **h** $388.\dot{8}$, $3.\dot{8}$
8 **a** $\frac{131}{399}$ **b** 32.83%, 0.33
9 **a** $\frac{13}{20}$ **b** $\frac{3}{40}$ **c** 7.5%
d $\frac{3}{10} \times 0.65 \times 100 = 19.5\%$
e credit card: 40.6%, debit card: 4.9%
10 **a** 0.67
b By placing a dot above the recurring part of the decimal
c $0.6\dot{}$ **d** $66.\dot{6}$%
e $66\frac{2}{3}$% or $66.\dot{6}$% (recurring decimal)
11 **a** 28%
b Multiply both the numerator and denominator by 2.
c $\frac{7}{25}$
12 **a** $2.50
b The selling price is 5.6 times the wholesale price.
c $\frac{28}{5}$
d 560%. Bruno's business will make a large profit if he can sell all his roasted beans.
13 **a** $109.99 **b** $145.74 **c** $\frac{1}{5}$, $\frac{53}{200}$
14 **a** $133.33 **b** $133.33
c They are the same, because $33\frac{1}{3}\% = \frac{1}{3}$
15 **a** **i** $35.60 **ii** $17.80 **iii** $3.56
b Move the decimal place one place left.
c Find 10% and halve it.
d The decimal place is one place further left. When calculating 1%, move the decimal point two places left.
16 **a** $43.85 **b** $21.93 **c** $4.39
d $2.30 **e** $72.95 **f** $14.84
g $2.95 **h** $9.57 **i** $122.92
17 **a** **i** $53.40 **ii** $71.20 **iii** $1.78
b **i** Find 10%, then halve it and add that on.
ii Find 10% and double it
iii Find 1% and halve it.
18 **a** $126.87 **b** $169.16 **c** $4.23
d $50.54 **e** $6.27 **f** $53
g $956.19 **h** $0.23 **i** $57.66
19 **a** $4.50 **b** $450 **c** $450
20 **a** 12.5% = 0.125 **b** $56.25 ÷ 0.125
c $450 **d** The same
e 12.5% is exactly the same as 0.125
21 **a** $84 **b** $56.19 **c** $37.65
d $831.43 **e** $20 **f** $874.87

1B Financial applications of percentages

1B Start thinking!

1 $19.11
2 a Subtract discount amount from original selling price of dress.
b $65.84
3 a 77.5% b $65.84
4 $41.80
5 a Add mark-up amount to wholesale price.
b $79.80
6 a 210% b $79.80

Exercise 1B Financial applications of percentages

1 a $22.50 b $1120 c $81.40 d $490
2 a $138.75 b $7590 c $445.90
d $144.81 e $233.08 f $272.20
3 a $459 b $21 483 c $1186.10
d $788.63 e $5186.29 f $1807.92
4 a i $229.25 ii $139.35
b i $1308.83 ii $186.98
c i $32.44 ii $17.47
d i $16.50 ii $13.68
5 a i loss, $31.35 ii 36.90%
b i profit, $4.25/kg ii 250%
c i loss, $20 545 ii 54.16%
d i profit, $104 ii 18.98%
e i profit, $14.85 ii 45.07%
f i loss, $26.95 ii 62.75%
6 a i $1028 ii $257
b i $1521 ii $279
c i $1539 ii $999
d i $1014.75 ii $564.75
e i $286.23 ii $60.72
f i $318.90 ii $531.50
g i $6485.61 ii $3026.62
h i $22 153.95 ii $15 913.40
7 a $330.85 b 12.80%
8 a $42.45 b $\frac{283}{943}$ c 30.01%
9 a i $24 ii $\frac{6}{17}$ iii 35%
b i $17 150 ii $\frac{343}{600}$ iii 57%
c i $46.50 ii $\frac{31}{97}$ iii 32%
d i $25.45 ii $\frac{509}{759}$ iii 67%
e i $526.05 ii $\frac{3507}{4663}$ iii 75%
f i $52.41 ii $\frac{5241}{29\,458}$ iii 18%
10 a 82% b $69.66 c $15.29
d $1624.67 e $64.99
11 a 112.5% b $145.63 c $10.63
12 a $1023.36 b $977.31 c no
d $270.69 e 21.69%
f It is lower (as 4.5% of $1248 = $56.16 and 4.5% of $1023.36 = $46.05).
13 a $350 b $45.45 c $66.65 d $8.90
14 a $71.30 b $87.60 c 81.39% of $87.60
15 $792.43
16 a $925.50 b $1544.25 c $932.72
d $1815.96 e $933.52 f $1188.73
g $945.67 h $1274.81
17 a $112.20 b $198.22
c $\frac{275}{100} \times 68 = \$187$, $\frac{265}{100} \times 68 \times \frac{110}{100} = \$198.22$
d 191.5% e 291.5% of $68
f i $131.18 ii $255.06
iii $85.70 iv $553.70

1C Understanding simple interest

1C Start thinking!

1 An additional repayment when you borrow money, or a payment to someone who invests money.
2 Interest on a loan is extra money you have to pay. Interest on an investment is extra money paid to you.
3 a P = principal, or money invested or borrowed; R = interest rate; T = number of time periods over which money is borrowed or invested.
b as a decimal or fraction
c same as period of time over which interest in calculated
4 a per annum b loan
c $P = \$1200$, $R = 6.5\% = 0.065$, $T = 3$ years
d $234 e interest paid for loan

Exercise 1C Understanding simple interest

1 a $1200 b $204 c $31 500
d $11 520 e $2387.50 f $8250
2 a 2 b 4 c 2.25 d 3.75
e $\frac{7}{12}$ f 1.25 g $1\frac{7}{12}$ h $\frac{1}{12}$
i 1 j 1.25 k $\frac{1}{13}$ l 6
3 a i $640 ii $8640 iii $360
b i $6000 ii $21 000 iii $437.50
c i $202.50 ii $1452.50 iii $40.35
d i $2870 ii $16 870 iii $562.33
e i $2550 ii $27 550 iii $1530.56
f i $589.50 ii $7139.50 iii $297.48
4 a i $1080 ii $13 080
b i $6600 ii $72 600
c i $18 750 ii $143 750
d i $231 ii $5731
e i $876.56 ii $9126.56
f i $494.64 ii $5074.64
5 a $769.50 b $527.25 c $6160
d $164.58 e $6151.20 f $8331.25
6 a $P = \$12\,000$, $R = 0.0625$, $T = 3$
b $2250 c $14 250
7 a $P = \$12\,000$, $R = 0.0625$, $T = 3$
b $2250 c $14 250
8 Simple interest calculations are the same for loans as investments. The difference is in the interpretation: interest is *earned* on an investment and *charged* on a loan.
9 a i $1862 ii $15 862
b i $529.69 ii $9004.69

c i $3334.98 ii $15 724.98
d i $3308.22 ii $118 308.22
e i $35 031.25 ii $285 031.25
f i $52 672.98 ii $287 252.98

10 a $624.75 b $3564.75 c $178.24
d $2866.50 e $698.25

11 a $P = \$1500$, $R = 0.048$, $T = \frac{14}{12}$
b $84 c $1584

12 a $1615.50, $1546, $2421.55
b 8, 2, 9, 2, 3, 7
c 0.30, 0.07, 0.62, 0.12, 0.18, 0.65
d $1.94 e $2423.49

13 a i $5000 ii no iii 2 up to 6 months
b 5.4% c $2430
d $2812.50; $382.50 more
e it increases by $43.95
f two 6-month investments with interest added to investment at end of first 6 months

14 a Yes. No withdrawals are made, and over $250 has been deposited this month.
b $42.03 c $9994.53
d No bonus interest would have been calculated; final balance $9861.27.

15 a $49.30 b $4299.30
c $4250\left(1 + 0.0464 \times \frac{3}{12}\right)$
d i $3836 ii $2480.40 iii $12 900
iv $25 350

1D Working with simple interest

1D Start thinking!

1 a T b $P = \$1200$, $R = 0.125$
2 $375 = 1200 \times 0.125T = 150T$
3 a inverse operations
b 2.5. This is how many years she will be repaying her loan.
4 $52.50
5 a R b $I = \$352$, $T = 2$
6 a $352 = 1100 \times R \times 2 = 2200R$
b 0.16 c × 100 d 16% p.a.

Exercise 1D Working with simple interest

1 a $240 b $997.50 c $21.25 d $9240
2 a 4 years b 2.5 years c 3 years
3 a $15 000 b $6500 c $20 000
4 a 4.4% b 10.5% c 4.5%
5 a 4 b $18 000 c 8.5%
d 5 e $20 000 f 2.4%
6 a P b $I = \$2304$, $R = 0.064$, $T = 3$
c $12 000, $14 304
7 a T b $P = \$6500$, $R = 0.085$, $I = \$828.75$
c 18 months d $407.15
8 a $46 176 b no; she needs $63 000
c $16 824 d 10.1% p.a.
9 a $14 962.50 b 4 years c $180
d 6.58% e $9539.91 f $7510.84
g 8.4% h 1.25

10 a $49 962.50 b $11 830.90 c $195.93
d $149 809 e $9809.81 f $7647.99
g $16 890 h $11 613.50

11 a The interest rate won't change over that time.
b $204 c $29 411.76 d $12 107.32

12 a 2.6 years b 12.5% c 28%
d She will have trouble finding a financial institution offering such high interest rates.

13 a $120.00 for 4 days
$476.00 for 3 days
$515.90 for 7 days
$700.90 for 6 days
$850.90 for 5 days
$916.75 for 1 day
b $925.69 c 21.9% p.a.

14 a $45, $43.33 b $135, $129.99
c $10 043.33 d $43.52, $10 086.85
e $43.71, $10 130.56 f $130.56
g Different, because the interest earns interest.

15 a $10 135.61 b 61 cents more
c $10 045.19 d 5.4244% p.a.

16 a i $60 ii $60.09
b i $315 ii $317.22
c i $1487.50 ii $1511.38
d i $451.50 ii $459.64

17 a $10 015 b $15 105.71
c $35 561.88 d $8551.44

1E Understanding compound interest

1E Start thinking!

1 a $900 b $5900
2 $300
3 a Year 1: $P = \$5000$, $R = 0.06$, $I = \$300$.
Year 2: $P = \$5000$, $R = 0.06$, $I = \$300$.
Year 3: $P = \$5000$, $R = 0.06$, $I = \$300$.
b yes c same each year
4 a $300 b $5300 c $318
d $5618 e $337.08 f $5955.08

Exercise 1E Understanding compound interest

1 a i 2 ii 1 year
b i 3 ii 1 year
c i 8 ii 6 months
d i 24 ii 1 month

2 a $1025 b $955.08 c $6492.80
d $8620.25 e $906.14 f $5771.23

3 a i $960 ii $979.20
b i $1575 ii $1655.06
c i $8370 ii $8899.66
d i $7446 ii $8450.31

4 a i 3 ii 1 year
b $442 c $8942 d $464.98
e $9406.98 f $489.16 g $9896.14
h Final value with simple interest is $9826, which is $70.14 less.

5 a $3401.40 b $2809.61 c $2919.29
d $2008.86 e $19 335.94 f $38 421.56

6 a $5232.50 b $5475.81 c $5730.44

7 Pitia would have paid his bank $5730.44 over the 3 years.
8 a i $1377.50 ii $15 877.50
b i $1410.22 ii $15 910.22
c It is inaccurate.
d At end of first year, interest earned is same for both options. At end of second year, option with compound interest earned $32.72 more. Prediction: in third year, difference will be greater.
e compound interest option
9 The simple interest option, because she would pay less interest.
10 a 4, 1 year b $102 917.31
c $22 917.31 d $2144.11
11 a $16 213.50 b $16 307.32
c $I = 16\,307.32 - 13500 = \$2807.32$; $I = PRT$; $2807.32 = 13\,500 \times R \times 3$; $R = 2807.32 \div (13\,500 \times 3) = 0.0693$
So the interest rate must be greater than 0.0693 to make option 1 the best option.
12 a first table
b i 4.6% p.a. ii 3.25% p.a.
c i $81.25 ii $76.54
d Interest paid at maturity, because they will earn more interest on their investment.
13 a $10 899.36 b 4 c $10 909.47
d Compounding half-yearly increased the interest earned by $10.11 because interest was added into the investment sooner, resulting in larger amounts of interest being paid.

1F Working with the compound interest formula

1F Start thinking!

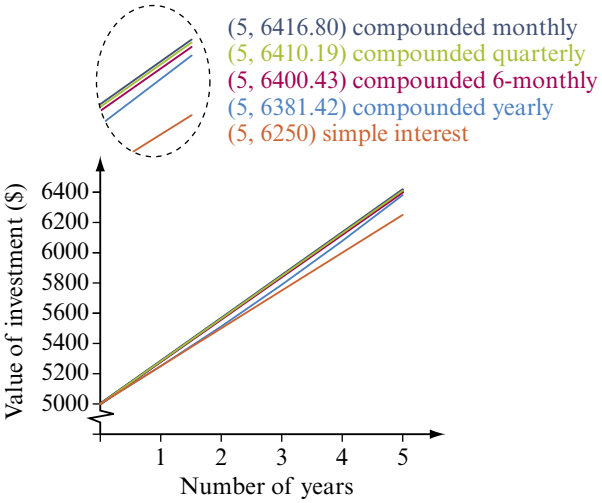
1 a $\$5000 \times 1.06 = \$5300$
b $\$5300 \times 1.06 = \$5618$
c $\$5618 \times 1.06 = \$5955.08$
d $5955.08, $955.08
2 Same values are obtained.
3 In table, previous year's value is multiplied by 1.06 to obtain new value. In general, value of investment at end of investment period, A, is: amount invested multiplied by [(1 + interest rate, written as a decimal) raised to the power of number of time periods] or $A = P(1 + R)^n$.
For this scenario, $P = \$5000$, $R = 0.06$, $n = 3$ and so
$A = \$5000(1 + 0.06)^3 = \$5000 \times 1.06^3 = \$5955.08$
4 a $\$5000 \times 1.06^4 = \$6312.38$
b $\$5000 \times 1.06^7 = \$7518.15$

Exercise 1F Working with the compound interest formula

1 a $12 155.06 b $5955.08 c $816.29
d $53 019.97 e $112 649.26 f $19 993.02
2 a i $8682.19 ii $1182.19
b i $10 270.50 ii $1270.50
c i $31 263.59 ii $8763.59
d i $53 022.34 ii $13 022.34
3 a $4101.25 b $7325.28 c $16 925.00
d $34 945.12 e $127 567.39 f $507 382.93
4 a $901.25 b $2735.28 c $925
d $9045.12 e $27 567.39 f $252 382.93
5 a 4 b 3 c 4 d 48
6 a year b 2 c 4 d month; 12
7 a i 1.2% ii $R = 0.012$, $n = 12$
b i 2.95% ii $R = 0.0295$, $n = 8$
c i 0.7% ii $R = 0.007$, $n = 24$
d i 0.8% ii $R = 0.008$, $n = 60$
8 a $6071.85 b $13 406.88 c $22 895.02
d $68 216.57 e $61373.74 f $162 619.66
g $551 913.02 h $1 017 670.16
9 a 3 b 1 year
c $P = \$16\,000$, $R = 0.0575$, $n = 3$
d $18 921.74 e $2921.74 f yes
10 a $63 404.70 b $18 404.70
11 a $10 444.50, $10 403.75
b 5.35% p.a. with interest paid at maturity
c The option with interest compounded annually becomes the better option
12 a 2.15% b $R = 0.0215$ c 20
d $12 854.25 e $4454.25
13 a $13 360
b i $13 897.66 ii $13 944.79
iii $13 960.82 iv $13 977.02
c It increases. Yes.
14 fewer
15 a i 8 ii 0.017 b A
c $6500 = P(1 + 0.017)^8$; $P = \$5679.97$
d $5680
16 a $250, $262.50, $275.63, $289.41, $303.88
b $5250, $5512.50, $5788.13, $6077.53, $6381.41
c curved line through the points: (0, 5000), (1, 5250), (2, 5512.50), (3, 5788.13), (4, 6077.53), (5, 6381.41)
d i a $254.73, $267.70, $281.34, $295.68, $310.74
b $5254.73, $5522.43, $5803.77, $6099.45, $6410.19
c curved line through the points: (0, 5000), (1, 5254.73), (2, 5522.43), (3, 5803.77), (4, 6099.45), (5, 6410.19)
ii a $253.13, $265.93, $279.41, $293.54, $308.41
b $5253.13, $5519.06, $5798.47, $6092.01, $6400.42
c curved line through the points: (0, 5000), (1, 5253.13), (2, 5519.06), (3, 5798.47), (4, 6092.01), (5, 6400.42).
iii a $255.81, $268.90, $282.65, $297.12, $312.31
b $5255.81, $5524.71, $5807.36, $6104.48, $6416.79

c curved line through the points: (0, 5000), (1, 5255.81), (2, 5524.71), (3, 5807.36), (4, 6104.48), (5, 6416.80)
e It would be a straight line with a positive gradient, while all of the compound interest graphs would be increasing curves.
f Simple interest graph would be a straight line through the points: (0, 5000), (1, 5250), (2, 5500), (3, 5750), (4, 6000), (5, 6250)

300% enlargement



1G Working with compound interest

1G Start thinking!

1 a 4 b 8
c n = number of compounding periods in a year × number of years
2 a 4.8% b divide it by 4
c 1.2% d 0.012
e interest rate ÷ (number of compounding periods per year × 100)
3 a A is unknown. P = \$14000, R = 0.012, n = 8
b \$15 401.82 c final value of investment
d \$1401.82
4 Transpose the equation to make the required variable the subject.

To find P: $P = \frac{A}{(1+R)^n}$ To find R: $R = \sqrt[n]{\frac{A}{p}} - 1$

Exercise 1G Working with compound interest

1 a 3 b 10 c 16 d 60 e 18 f 7
2 a 2.1% b 0.755% c 2.5% d 2.31%
3 a i 4 ii 0.038
b i 6 ii 0.032
c i 8 ii 0.032
d i 48 ii 0.007
4 a \$13 930.63 b \$11 476.30
c \$38 597.47 d \$58 703.48
5 $P = A \div (1+R)^n$
6 a \$20 122.05 b \$8522.70
c \$4003.95 d \$6930.08
7 a \$4274.64 b \$7775.87 c \$13 545.38
d \$10 550.71 e \$145 348.17
8 a \$925.36 b \$2224.13 c \$8454.62
d \$6099.29 e \$79 651.83
9 a i 0.082 ii 16.4%
b i 0.058 ii 5.8%
c i 0.039 ii 15.6%
10 a R
b $15000 = 12000(1+R)^3$;
$15000 \div 12000 = (1+R)^3$
c $\sqrt[3]{1.25}, \sqrt[3]{1.25}, 1$ d 7.7%
11 a i 0.109 ii 10.9%
b i 0.032 ii 6.4%
c i 0.023 ii 4.6%
12 a A = \$8000, P = \$5500, n = 4
b 0.098 c 9.8%
13 a $R = \sqrt[n]{\frac{A}{P}} - 1$ or $R = \sqrt[n]{A \div P} - 1$
b Multiply R by the number of compounding periods per year.
14 a i 0.145 ii 14.5%
b i 0.107 ii 10.7%
c i 0.044 ii 8.8%
d i 0.034 ii 13.6%
e i 0.055 ii 11%
f i 0.021 ii 8.4%
15 a 36 b 0.6%, 0.006
c P d \$12 093.83
16 a n
b $30\,000 = 25\,000(1+0.05)^n$;
$30000 \div 25000 = (1.05)^n$
c 4
d \$25 000 must be invested for 4 years at 5.0% compounded annually to grow to \$30 000
17 a A = \$1100, P = \$850, R = 0.042
b $\frac{22}{17} = (1.042)^n$ c 7
18 a 6 b 4.5 c 9 d 12.6 e 5.5 f 6
19 a 6 b 4.5 c 4.5 d 3.1 e 2.7 f 1.5
20 Check with your teacher. Some responses may include references to the Solve function on a CAS calculator or the use of logarithms.

1 Chapter review

MULTIPLE-CHOICE

1 C 2 B 3 C 4 C
5 A 6 D 7 C 8 B

SHORT ANSWER

1 a 0.24 b 0.358 c 0.1625 d 0.00272
2 a $\frac{43}{50}$ b $\frac{47}{20}$ c $\frac{19}{125}$ d $\frac{173}{2000}$ e $\frac{41}{300}$ f $\frac{8}{15\,625}$
3 a \$102.75 b \$16150 c \$17.24
4 a 12.5% b 55.10%
5 $51\frac{471}{479}\%$
6 a i \$2154.75 ii \$395.25
b i \$2023.88 ii \$1124.38
c i \$24 974.28 ii \$6929.28
d i \$1216.19 ii \$729.71

7 a i profit, $31.45 ii 22%
b i loss, $17 676.05 ii 64%
8 a $450 b $5950
9 a i $25 200 ii $175 200
b i $116.16 ii $9566.16
10 a 1.25 b $18 000 c 11.25%
11 a $15 675, $16 380.38, $17 117.49
b $7288.40, $7754.86, $8251.17, $8779.24
12 a $2600.16 b $26 978.10 c $191 793.77
13 a $6691.22 b $14 700.60 c 0.04

MIXED PRACTICE

1 0.255 2 C 3 C
4 A 5 C 6 C
7 $98 8 D 9 $10
10 B 11 $3087 12 $18 837
13 D 14 $9523.81 15 8
16 C 17 B 18 $5164.17
19 C 20 $6628.99 21 $1661.71
22 C 23 A 24 5.8%

ANALYSIS

a 16
b i 0.013 75 ii 0.014 25 iii 0.0155
c $12 673.74
d P = $12 673.74, R = 0.014 25, n = 8
e $14 192.70 f $15 093.31 g $3093.31
h 25.78% i 6.45% j 6.42%

1 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 2 ALGEBRA

2 Are you ready?

1 a i $-3 \times m$ ii $w \times x \times y$
iii $7 \times a \times a \times b$
iv $10 \times p \times p \times p \times q \times q \times r$
b i -3 ii 1 iii 7 iv 10
c i $6m$, $1.2m$ ii $3wxy$, $-2wxy$
iii $0.2a^2b$, $5a^2b$ iv $4p^3q^2r$, p^3q^2r
2 a B b C
3 a -3 b 12 c -51 d 94
4 a i 32 cm ii 63 cm^2
b i 60 mm ii 120 mm^2
c i 37.7 m ii 113.1 m^2
5 a i x ii 6
b $x \times x \times x \times x \times x \times x$ c D
d i x^4 ii x^{12}
6 a C b $2k - 2$
7 a 6 b 4 c c d 7 e $4x$ f $9m$
8 a 10 b $8n$ c $-5y$ d $4p$
9 a C b 11 c $\sqrt{5}$
10 a $\frac{4}{8} = \frac{1}{2}$ b $\frac{5}{11}$ c $\frac{2}{3}$ d $\frac{3}{10}$
e $\frac{3}{8}$ f $\frac{8}{9}$

2A Working with algebraic terms

2A Start thinking!

1 False; the coefficient and pronumerals in a term are multiplied together. $7mnp = 7 \times m \times n \times p$
2 true
3 False; if a pronumeral has no coefficient stated, it is 1. Correct answer is 1.
4 true 5 true
6 False; 3 is the coefficient of the first term, while the constant is 4.
7 true
8 False; like terms must contain the same pronumerals to the same powers.
9 False; the terms are like terms and should be added, not multiplied. $abc + abc = 2abc$
10 False; the term $5a$ is not a like term with the other two, so cannot be included in the addition. The answer should be $5ab + 5a$.

Exercise 2A Working with algebraic terms

1 a i $3 \times a \times b \times c \times d$
ii 3 iii $-4abcd$
b i $-4 \times m \times n$ ii -4 iii $2mn$
c i $x \times y \times y$ ii 1 iii $5xy^2$
d i $9 \times k \times k \times m \times m \times m \times m \times p$
ii 9 iii $-3k^2m^4p$
2 a -6 b -3 c 10 d 9
e -52 f -24 g -15 h -9
3 a $9x$ b $-10ab$ c $10y^2$
d $8m^2n$ e $12a + 9d$ f $8k + 2m$
g $2 - 3y$ h $7a - 4a^2$ i $3x^2 + 2xy$
j $4a - ab^2$ k $x^3 + 3x^2$
l $8xy + wx + wxy$
4 a $11a$ b $-4n + 2n^2 + m - mn$
c $3x^2 - 2x + 1$ d $-3x^2y + 3xy - 5$
e $2p^3 + 15p^2 - 9$ f $-a^2 - b^2 + 3ab^2 + 9a^2b$
5 a $10abcd$ b $-21mnxy$ c $4k^2p$
d $-48a^2cd$ e $45wx^3y$ f km^2n^4
g $-21x^3$ h $12abc^3d^2$ i $-6g^2k^3p^2$
j $4ab^2d^4$ k $20m^5n^3$ l $40a^6b^5c^4$
6 a ac b $6n$ c $\frac{2x}{y}$ d $\frac{a}{2c}$ e $\frac{3eg}{h}$
f $\frac{2ab}{5}$ g $-\frac{3m}{2p^2}$ h $-\frac{2a}{cd}$ i $\frac{x^2}{2}$
7 a 2 b -69 c 246 d $-20\,736$
e -18 f 4
8 a 1550 cm^2 b 0.62 m^2
9 205.7 cm 10 25π cm^2
11 36π mm^3 12 10 m
13 a $7x + 6y$ b $6x + 4y - 6$ c $20y - 12$
14 a 61 cm b 44 cm c 28 cm
15 a $42ab$ b $4a^2$ c $10ab$
d $9\pi a^2$ e $24ab + \frac{9}{8}\pi b^2$ f $37ab$
16 a 504 m^2 b 64 m^2
c 120 m^2 d 144π m^2 or 452.4 m^2
e 319.8 m^2 f 444 m^2
17 a $4k$ b 180 m^2
18 a $5y$ b area = 120 cm^2, base length = 24 cm

19 **a** **i** $\pi x + 2x$ **ii** $\frac{1}{2}\pi x^2$
b **i** $14x$ **ii** $12x^2$
c **i** 6.3 m^2 **ii** 41.7 m^2 **iii** 34.3 m

20 **a** $3\pi x^2$
b Any value larger than 6.5 mm and smaller than 7.3 mm. Some possible values are: 6.7 mm, 7 mm and 7.1 mm.
c $6\pi x$ **d** 126.3 mm, 131.9 mm, 133.8 mm

21 **a** true (addition is commutative); $3 + (-5) = -5 + 3$
b false (subtraction is not commutative); $4 - 6 \neq 6 - 4$
c true (multiplication is commutative); $2 \times 3 = 3 \times 2$
d false (division is not commutative); $3 \div 4 \neq 4 \div 3$
e true (the Associative Law holds for addition); $(2 + 4) + 5 = 2 + (4 + 5)$
f false (the Associative Law does not hold for subtraction); $(8 - 3) - 2 \neq 8 - (3 - 2)$

2B Review of index laws

2B Start thinking!

1 **a** The result using the expanded form shows the same answer as adding the indices.
b $x^2 \times x^4 = (x \times x) \times (x \times x \times x \times x) = x^6 = x^{2+4}$

2 One possible answer is given.
a $x^5 \div x^2 = \frac{x \times x \times x \times x \times x}{x \times x} = x^3 = x^{5-2}$
b $(x^2)^3 = x^2 \times x^2 \times x^2$
$= (x \times x) \times (x \times x) \times (x \times x)$
$= x^6$
$= x^{2 \times 3}$
c $(x \times y)^2 = (x \times y) \times (x \times y) = x \times x \times y \times y$
$= x^2 \times y^2$
d $\left(\frac{x}{y}\right)^2 = \frac{x}{y} \times \frac{x}{y} = \frac{x \times x}{y \times y} = \frac{x^2}{y^2}$

3 **a** $x^5 \times x^4 = x^{5+4} = x^9$
b $x^6 \div x^2 = x^{6-2} = x^4$
c $(x^3)^7 = x^{3 \times 7} = x^{21}$

Exercise 2B Review of index laws

1 **a** a^4 **b** b^{20} **c** c^{15} **d** d **e** y^{10} **f** m^{14}
g n^{14} **h** h^7 **i** x^6 **j** p^{18} **k** w^3 **l** a^{10}

2 **a** x^3 **b** m^7 **c** k^{12} **d** y^4 **e** a^5
f d^{-15} **g** y **h** n^2 **i** x^{-5}

3 **a** $\frac{1}{x^5}$ **b** $\frac{d^4}{c^9}$ **c** $\frac{j^5}{k^3}$ **d** $\frac{y^8}{x}$
e $\frac{2a^3}{b^6}$ **f** $\frac{3}{p^4q^5}$ **g** $\frac{w^4y^7}{x^5}$ **h** $\frac{m^5}{k^3n^8}$
i $\frac{6d^2f}{e^7}$ **j** $\frac{11}{a^3b^5c^9}$ **k** x^3 **l** b^4c^6
m $4m^3p^7$ **n** $\frac{b^2}{a^4}$ **o** $\frac{4y^5}{x^2}$ **p** $\frac{d^8e^4}{c}$
q $\frac{p^6}{3k^2n^5}$ **r** $\frac{a^3b^4}{2c^7d^2}$ **s** $\frac{5xy^8}{w^2}$ **t** $\frac{a^5m^7}{12b^3n^6}$

4 **a** $14x^9$ **b** $3x^2$ **c** x^{15} **d** $20x^8$
e $4x^3$ **f** $6x^6$ **g** $\frac{3x}{7}$ **h** $\frac{3x^5}{2}$
i x^{17} **j** x^{34} **k** $2x^{23}$ **l** $5x^{35}$

5 **a** $\frac{1}{x^2}$ **b** $\frac{2}{x^5}$ **c** $42x$ **d** $\frac{20}{x^4}$
e $\frac{1}{x^2}$ **f** $2x^9$ **g** $\frac{1}{4x^9}$ **h** $\frac{3}{5x^4}$
i $\frac{1}{x^{11}}$ **j** $\frac{10}{x^6}$ **k** $\frac{3}{x^7}$ **l** $\frac{1}{x^{21}}$

6 **a** 1
b **i** 1 **ii** 1 **iii** 1 **iv** 1

7 **a** 3 **b** 1 **c** 8 **d** 1 **e** 1 **f** 2
g 0 **h** 2 **i** 6 **j** 1 **k** 1 **l** 1

8 **a** 1 **b** 4 **c** 1 **d** 24 **e** 1 **f** x
g $36x^3$ **h** 4 **i** x^7 **j** 3 **k** $2x$ **l** 15

9 **a** x^8 **b** $8x^5$ **c** $2x$ **d** 1 **e** x^{24} **f** x
g $\frac{1}{5}$ **h** x^8 **i** x^6 **j** $4x^{16}$ **k** 5 **l** x^{10}

10 **a** x^9y^{12} **b** $16a^8b^8$ **c** m^7n^{10}
d b^3c^6 **e** w^3x^6 **f** a^3
g $4k^6n^7$ **h** $\frac{c^7}{2}$ **i** 1

11 **a** a^5b^5 **b** k^9p^9 **c** $64c^3$ **d** $7^{13}y^{13}$ **e** x^4y^4
f c^7d^7 **g** $9w^2$ **h** 6^6g^6 **i** $2^{-8}p^{-8}$

12 **a** $\frac{a^4}{b^4}$ **b** $\frac{f^7}{g^7}$ **c** $\frac{k^6}{64}$ **d** $\frac{125}{x^3}$ **e** $\frac{d^{-2}}{c^{-2}}$ **f** $\frac{m^{-5}}{3^{-5}}$

13 **a** $\frac{1}{x^4y^4}$ **b** $\frac{1}{4^6a^6}$ **c** $\frac{1}{9n}$ **d** $\frac{1}{5^3p^3}$
e $\frac{1}{9p^2}$ **f** $\frac{1}{k^5m^5}$ **g** $\frac{64}{x^3}$ **h** $\frac{d}{7}$
i $\frac{b^8}{a^{12}}$ **j** $\frac{n^5}{k^5}$ **k** d^4c^6 **l** $\frac{x^8}{w^6}$

14 **a** $a^{11}b^7$ **b** $2304p^5$ **c** $32y^{15}$ **d** $3m^{12}n^6$
e $\frac{d^{12}}{h^{11}}$ **f** $\frac{125w^3}{y^3}$ **g** $a^{30}b^{12}$ **h** $\frac{b^8}{a^{16}c^{28}}$
i u^3wx^{17} **j** $125a^6c^4n$ **k** $m^{34}n$ **l** $\frac{1}{y^7}$

15 **a** false; $x^7 \times x \times x^7 = x^{7+1+7} = x^{15}$
b false; $(5a)^3 = 5^3 \times a^3 = 125 \times a^3$
c true; $-k^0 = -(1) = -1$
d false; $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4} = a^5b^7$
e true; $x^7y^4 \div y^6 = x^7y^{4-6} = x^7y^{-2} = \frac{x^7}{y^2}$
f false; $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$
g true; $\frac{w^6 \times w^6}{w^{12}} = \frac{w^{6+6}}{w^{12}} = \frac{w^{12}}{w^{12}} = 1$
h true; $\frac{(b^5)^4 \times b^2}{(b^3)^7} = \frac{b^{20} \times b^2}{b^{21}} = \frac{b^{22}}{b^{21}} = b$
i false; $\frac{a^8b^5}{a^4b^3} \times \frac{a^2b^2}{a^9b^4} = \frac{a^{8+2}b^{5+2}}{a^{4+9}b^{3+4}} = \frac{a^{10}b^7}{a^{13}b^7} = \frac{1}{a^3}$
j false; $\left(\frac{m^2}{n}\right)^3 \times \left(\frac{n^4}{m^5}\right)^{-1} = \frac{m^6}{n^3} \times \frac{n^{-4}}{m^{-5}} = \frac{m^6}{n^3} \times \frac{m^5}{n^4} = \frac{m^{11}}{n^7}$

16 **a** 9 **b** 0 **c** 10 **d** 4 **e** −8 **f** 3
g 7 **h** 9 **i** 6 **j** −4 **k** 5 **l** 8

17 **a** x and y must add to 20.
One possible answer is $x = 4$, $y = 16$.
b $x - y$ must equal 5.
One possible answer is $x = 8$, $y = 3$.
c $x \times y$ must equal 36.
One possible answer is $x = 4$, $y = 9$.

d $x \times y$ must equal 0. This means that either $x = 0$ or $y = 0$ or both equal 0.
One possible answer is $x = 5$, $y = 0$.

18 **a** $x^{2a+5}y^{4b+1}$ **b** $m^{2x+4}n^3$

19 2^5 **20** 3^{2x} **21** 5^x

22 **a** $x = 4$ and $y = 8$ or $x = 8$ and $y = 4$
b $x = -2$, $y = 12$

2C Expanding algebraic expressions

2C Start thinking!

1 Some possible examples are: $3x + 4$, $y - 2x$, $x^2 + 12$.

2 $(x + 3)$ and $(x - 5)$ are both binomial expressions. They are multiplied together to form the product $(x + 3)(x - 5)$.

3 **a** a, $(b + c)$
b area of larger rectangle = $a(b + c)$
area of two smaller rectangles = $ab + ac$
These two areas are equal,
so $a(b + c) = ab + ac$.

4 **a** $(a + b)$, $(c + d)$
b area of larger rectangle = $(a + b)(c + d)$
area of four smaller rectangles
$= ac + ad + bc + bd$
These two areas are equal, so $(a + b)(c + d)$
$= ac + ad + bc + bd$.

Exercise 2C Expanding algebraic expressions

1 **a** $3a + 21$ **b** $5b - 20$ **c** $12 - 2c$
d $4 + 4d$ **e** $-6e - 18$ **f** $-7f + 14$
g $-3g + 3$ **h** $-45 + 9h$ **i** $x^2 + 2x$
j $2y^2 - 8y$ **k** $-k^2 - 5k$ **l** $-3p^2 + 3p$
m $3mn + km$ **n** $12ab + 24ac$ **o** $-w^2 + 8wx$
p $-4p - 10p^2$

2 **a** $a^3 + 4a^2$ **b** $b^6 - 2b$
c $c^4 - c^3$ **d** $8d^2 - d^5$
e $-x^3 - x^6$ **f** $-2y^7 - 2y^4x^2$
g $m^6n + m^3n^3$ **h** $6x^2y^2 - 3x^7y^5$
i $4a^6b^3 + 4a^2b^8$ **j** $6k^{10}p^2 - 14k^4p^6$
k $-8x^5y^6 + 16x^3y^9$ **l** $-12a^7b^7 - 27a^{13}b^9$

3 **a** $9x + 10$ **b** $4y - 13$ **c** $mn + 3m$
d $8x^2 - 5x$ **e** $2p^3 - 3p$ **f** $x^7 - 3x^5$

4 **a** $6x + 2$ **b** $m^2 + 14m - 18$
c $6a^2 + 14a - 6$ **d** $k^5 + 7k^2 + 5$
e $3p^6 - 4p^4 + 12p$ **f** $10y^8 + y^3 + 8y$

5 **a** $ab + 2a + 4b + 8$ **b** $cd + 6c + 3d + 18$
c $mn + 5m + 7n + 35$ **d** $xy + 8x + 6y + 48$
e $kp - k + 5p - 5$ **f** $fg + 3f - 2g - 6$
g $xy - 4x - 9y + 36$ **h** $2mn + 2m + 3n + 3$
i $12xy - 28x + 15y - 35$
j $15ac - 12ad - 10bc + 8bd$
k $x^2 + 7x + 10$ **l** $y^2 + 8y - 9$
m $m^2 - 14m + 33$ **n** $p^2 - 6p + 8$
o $30 - 11k + k^2$ **p** $4x^2 + 19x + 21$
q $3a^2 - 13a - 10$ **r** $8w^2 - 23w + 14$
s $12p^2 + 52p + 35$ **t** $8x^2 + 10x - 3$
u $15y^2 - 26y + 8$

6 **a** $x^3 + 3x^2 + 5x + 15$ **b** $x^5 - x^3 + 4x^2 - 4$
c $x^8 + 2x^5 - 7x^3 - 14$ **d** $x^6 - 6x^4 - 3x^2 + 18$
e $x^7 + 7x^6 + x^3 + 7x^2$ **f** $x^9 + x^7 - x^5 - x^3$
g $x^7 + 2x^6 + x^5$ **h** $x^{10} - x^4$
i $x^8 - x^6$ **j** $8x^5 + 20x^3 - 2x^2 - 5$
k $3x^{10} + 15x^8 - 2x^5 - 10x^3$
l $-15x^9 + 38x^7 - 24x^5$

7 **a** **i** $x^2 - 4$ **ii** $k^2 - 49$ **iii** $m^2 - 64$
iv $w^2 - 36$ **v** $y^2 - 1$ **vi** $a^2 - b^2$
b The product of the two binomial factors produces an expression which is the difference of two square numbers or pronumerals. The first term in each binomial factor is the same, and the second term in one factor is the negative of the second term in the other factor.
c In the expression $a^2 - b^2$, the terms a^2 and b^2 are both squares, and there is a minus sign between them indicating 'find the difference'.
d No, multiplication is commutative.
e **i** $a^2 - 9$ **ii** $x^2 - 100$ **iii** $m^2 - n^2$
iv $9 - x^2$ **v** $1 - d^2$ **vi** $4x^2 - 25$
vii $4 - 9k^2$ **viii** $16g^2 - h^2$ **ix** $25y^2 - 4w^2$

8 **a** **i** $x^2 + 6x + 9$ **ii** $y^2 + 16y + 64$
iii $a^2 + 2ab + b^2$ **iv** $k^2 - 10k + 25$
v $p^2 - 12p + 36$ **vi** $x^2 - 2xy + y^2$
b The two binomial factors are the same. The product of such factors produces an expression made up of the squares of the two terms in the factor, and twice the product of the two terms in the factor.
c The expression $(a + b)^2$ is the square of $(a + b)$.
d **i** $a^2 + 4a + 4$ **ii** $x^2 + 10x + 25$
iii $p^2 + 8p + 16$ **iv** $y^2 + 20y + 100$
v $k^2 + 18k + 81$ **vi** $m^2 + 2mn + n^2$
vii $25 + 10x + x^2$ **viii** $1 + 4d + 4d^2$
ix $9w^2 + 42w + 49$
e The expression $(a - b)^2$ is the square of $(a - b)$.
f **i** $a^2 - 6a + 9$ **ii** $b^2 - 8b + 16$
iii $x^2 - 18x + 81$ **iv** $n^2 - 22n + 121$
v $4 - 4x + x^2$ **vi** $g^2 - 2gh + h^2$
vii $16w^2 - 8w + 1$ **viii** $25 - 20y + 4y^2$
ix $9m^2 - 12mp + 4p^2$

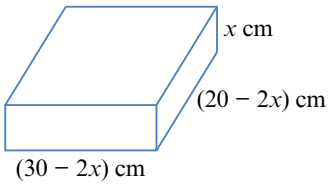
9 **a** $x^4 - 4$ **b** $x^4 - 25$
c $x^4 - y^4$ **d** $x^4 + 8x^2 + 16$
e $x^4 - 6x^2 + 9$ **f** $x^6 - 2x^3 + 1$

10 **a** $2x(5x + 3)$ **b** $10x^2 + 6x$
c **i** 184 cm^2 **ii** 184 cm^2
d The two answers are the same, indicating that the product has been expanded correctly.
e **i** 1060 cm^2 **ii** 77.5 cm^2

11 **a** **i** $(x + 8)(x + 5)$
ii $x^2 + 13x + 40$ **iii** 88 m^2
b **i** $(x + 12)(x + 12)$
ii $x^2 + 24x + 144$ **iii** 225 mm^2
c **i** $(5x - 2)(x + 4)$
ii $5x^2 + 18x - 8$ **iii** 91 cm^2

12 **a** **i** $(30 - 2x)$ cm
ii $(20 - 2x)$ cm **iii** x cm

b



c i $x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$
ii $30 \times 20 - 4x^2 = 600 - 4x^2$

d i 1008 cm^3 ii 564 cm^2

13 $(x^2 + y^3)^2 = (x^2 + y^3)(x^2 + y^3)$
$= x^4 + x^2y^3 + x^2y^3 + y^6$
$= x^4 + 2x^2y^3 + y^6$

14 $(x^2 + y^3)(x^2 - y^3) = x^4 - y^6$

15 **a** $(x + 5)$ cm

b volume $= \pi \times x^2 \times (x + 5) = \pi x^3 + 5\pi x^2$

c One possible value is 6 cm.

d $2\pi x^2 + 2\pi x(x + 5)$

e 640.9 cm^2

2D Factorising algebraic expressions

2D Start thinking!

1 4

2 $8x \div 4 = 2x$ and $20 \div 4 = 5$. Taking the highest common factor, 4, out of $8x$ and 20 leaves the expression $(2x + 5)$.

3 $x(x + 9)$ **4** $(x + 2)(y + 3)$

Exercise 2D Factorising algebraic expressions

1 **a** $5(a + 7)$ **b** $6(5b + 1)$ **c** $3(8 + 11c)$
d $4(3d + 2)$ **e** $2(4e - 9)$ **f** $7(5f - 3)$
g $8(3 + 2g)$ **h** $10(1 - 4h)$ **i** $9(j + k)$
j $4(4w - 7x)$ **k** $m(m + 2)$ **l** $n(n - 6)$
m $a(5 + a)$ **n** $p(9 - p)$ **o** $3q(q + 1)$
p $2r(r - 4)$

2 **a** $cd(b + 1)$ **b** $6b(a + 1)$
c $3x(5y - 3)$ **d** $mp(q + kn)$
e $ab(4a + 3)$ **f** $2xy(1 - x)$
g $ab(a + 11b)$ **h** $m^2(m + 5)$
i $4p^3(2p^3 - 1)$ **j** $2x^2(3x^5 + 5)$
k $m^2n(m + 1)$ **l** $3pq(p^2 + q^2)$
m $5x^2y(7 + xy)$ **n** $3a^2b^2(4a - 3b^2)$
o $4g^2h^2(5 + 7h^4)$ **p** $4(a + 2b + 3c)$
q $3x(2y + x + 5w)$ **r** $2h(h^2 - 2h + 3)$

3 **a** $b, -2a(b + 2a)$ **b** $-5x, -5x(2x^2 - 7)$

4 **a** $-6(mn + 3)$ **b** $-3b(a + c)$
c $-y(y + 10)$ **d** $-9x(2x + 1)$
e $-4k(1 - 2k)$ **f** $-10a(2a^2 + 3)$
g $-2xy(6x - 7)$ **h** $-w^5(1 - w^2)$
i $-2b^2(2b - 1 + 3b^2)$

5 $y(x - 5) + 2(x - 5)$
$= (x - 5) \times y + (x - 5) \times 2$
$= (x - 5)(y + 2)$
$4a(3 + k^2) - 9(3 + k^2)$
$= (3 + k^2) \times 4a + (3 + k^2) \times (-9)$
$= (3 + k^2)(4a - 9)$

6 **a** $(x + 3)(a + 5)$ **b** $(k - 2)(m + 4)$
c $(y + 5)(y + 2)$ **d** $(x - 1)(x + 9)$
e $(k + 6)(k - 3)$ **f** $(p - 9)(p - 6)$
g $(4 - a)(7 + a)$ **h** $(2n - 5)(3n + 4)$
i $(y^2 + 2)(x + 8)$ **j** $(a - 4)(a^2 + 3)$
k $(y^2 + 1)(2x - 5)$ **l** $(7 - x^3)(5x + 2)$

7 **a** $(a + 4)(b + 3)$ **b** $(x - 7)(x + 2y)$
c $(n + 5)(m - 6)$ **d** $(a + 2)(a + 5)$
e $(x - 1)(x + 4)$ **f** $(c + 8)(c - 3)$
g $(2 - y)(3y + 4)$ **h** $(p + 2)(p^2 + 5)$
i $(x^2 + 1)(x^2 + 3)$ **j** $(m^2 + 2)(2m - 3)$
k $(k^2 + 1)(k^3 + 4)$ **l** $(y - 3)(x + 5)$

8 **a** $x^2 - 25$

b Write each term as a square; that is, $x^2 - 25 = x^2 - 5^2$. Use the values x and 5 to write the product of the sum and difference of these two values. The factorised form is $(x + 5)(x - 5)$.

9 **a** $(x + 2)(x - 2)$ **b** $(a + 6)(a - 6)$
c $(10 + y)(10 - y)$ **d** $(8b + 3)(8b - 3)$
e $(5 + 7p)(5 - 7p)$ **f** $(2a + 9b)(2a - 9b)$
g $3(m + 1)(m - 1)$ **h** $2(2k + 3)(2k - 3)$
i $(xy + 4w)(xy - 4w)$ **j** $(m^3 + n)(m^3 - n)$
k $4(a^5 + 3b^3)(a^5 - 3b^3)$ **l** $(x^2y + w^2)(x^2y - w^2)$
m $(h + 8)(h - 2)$ **n** $(c - 1)(c - 7)$
o $4(1 - x)$ **p** $(1 + a + b)(1 - a - b)$
q $(y - 3)(7 - y)$ **r** $9(2x - 3)$

10 **a** $(x + \sqrt{7})(x - \sqrt{7})$ **b** $(a + \sqrt{13})(a - \sqrt{13})$
c $(\sqrt{19} + y)(\sqrt{19} - y)$ **d** $(4k + \sqrt{5})(4k - \sqrt{5})$
e $(p + 8 + \sqrt{2})(p + 8 - \sqrt{2})$
f $(m - 4 + \sqrt{6})(m - 4 - \sqrt{6})$
g $(x + 7 + \sqrt{3})(x + 7 - \sqrt{3})$
h $(y - 1 + \sqrt{19})(y - 1 - \sqrt{19})$
i $(d - 17 + \sqrt{17})(d - 17 - \sqrt{17})$

11 **a** $x^2 + 10x + 25$

b Take the square root of the first and last terms in the expansion (x and 5). Note the middle term is twice the product of these two values, and contains a '+' sign ($+10x$). Write the factorised form as the square of the sum of x and 5; that is, $(x + 5)^2$.

12 **a** $y^2 - 16y + 64$

b Take the square root of the first and last terms in the expansion (y and 8). Note the middle term is twice the product of these two values, and contains a '−' sign ($-16y$). Write the factorised form as the square of the difference between y and 8; that is, $(y - 8)^2$.

13 **a** $(x + 6)^2$ **b** $(m + 4)^2$ **c** $(p + 10)^2$
d $(y - 3)^2$ **e** $(x - 7)^2$ **f** $(g - 2)^2$
g $(1 + a)^2$ **h** $(5 - x)^2$ **i** $(9 - b)^2$
j $(w + x)^2$ **k** $(k - m)^2$ **l** $(3x + 2)^2$

14 **a** i $(x + 10)$ cm ii $(2x + 3)$ cm
b i $4x$ cm ii $6(x + 1)$ cm

15 **a** $x(18 - x)$ **b** x cm, $(18 - x)$ cm
c 10 cm and 8 cm, 12 cm and 6 cm, 14 cm and 4 cm

16 a Some possible expressions are:
height is $(5 - x)$ mm, base is $8x$ mm;
height is $(10 - 2x)$ mm, base is $4x$ mm;
height is $(20 - 4x)$ mm, base is $2x$ mm
b Some possible values are: height 4 mm, base 8 mm; height 3 mm, base 16 mm; height 2 mm, base 24 mm

17 a $(n + 2)$ and $(n + 4)$ b $3n + 6$
c $3(n + 2)$; this expression is 3 times the middle odd number.
d $425 \times 3 = 1275$
e $n + (n + 2) + (n + 4) + (n + 6) + (n + 8)$
$= 5n + 20$
$= 5(n + 4)$
The sum is 5 times the middle odd number.
f $2675 \times 5 = 13\,375$

18 a $(a + 1)(a - 3)$ b $(3x - 2)(y - 1)$
c $16m(m + 2)$

19 area $= \pi(2a + b)^2 - \pi(a - b)^2$
$= \pi[(2a + b) + (a - b)]\,[(2a + b) - (a - b)]$
$= \pi[3a][a + 2b]$
$= 3\pi a(a + 2b)$ cm^2

20 a sum of n odd numbers (where n is an even number)
$= n \times$ (average of two middle odd numbers)
or $n \times$ (first odd number $+ n - 1$)
b 10(average of 81 and 83) $= 10 \times 82 = 820$
or $10(73 + 10 - 1) = 820$

21 15

2E Factorising quadratic trinomials of the form $x^2 + bx + c$

2E Start thinking!

1 Three
2 An expression with three terms where the highest power of the pronumeral is 2.
3 a $(x + 3)(x + 4) = x(x + 4) + 3(x + 4)$
$= x^2 + 4x + 3x + 12$
$= x^2 + 7x + 12$
b i $x \times x$ ii 3×4
c $x \times 4 = 4x$ and $3 \times x = 3x$, $4x + 3x = 7x$
4 a x
b 1 and 3; $(x + 1)(x + 3)$
c $(x + 1)(x + 3) = x^2 + 3x + 1x + 3 = x^2 + 4x + 3$
5 a $(x + \_\_)(x + \_\_)$
b 1 and 20; $(x + 1)(x + 20)$
2 and 10; $(x + 2)(x + 10)$
4 and 5; $(x + 4)(x + 5)$
c $(x + 1)(x + 20) = x^2 + 21x + 20$
$(x + 2)(x + 10) = x^2 + 12x + 20$
$(x + 4)(x + 5) = x^2 + 9x + 20$
$(x + 2)(x + 10)$ produces the correct middle term of $12x$
d Sometimes there are more options available for the factors of the constant term.
6 Find all the pairs of factors of the constant. Identify the pair of factors which add to the same value as the coefficient of the middle term. Factorise as a binomial product of the form $(x +$ one factor$)(x +$ other factor$)$.

Exercise 2E Factorising quadratic trinomials of the form $x^2 + bx + c$

1 a 7 and 1 b 8 and 2 c 9 and 5
d 10 and 6 e 9 and 4 f 8 and 3

2 a $(x + 7)(x + 1)$ b $(x + 8)(x + 2)$
c $(x + 9)(x + 5)$ d $(x + 10)(x + 6)$
e $(x + 9)(x + 4)$ f $(x + 8)(x + 3)$

3 a $(x + 2)(x + 1)$ b $(x + 3)(x + 2)$
c $(x + 10)(x + 1)$ d $(x + 9)(x + 3)$
e $(x + 5)(x + 2)$ f $(x + 7)(x + 2)$
g $(x + 3)(x + 3)$ h $(x + 11)(x + 1)$
i $(x + 9)(x + 2)$ j $(x + 8)(x + 4)$
k $(x + 9)(x + 7)$ l $(x + 10)(x + 4)$
m $(x + 6)(x + 5)$ n $(x + 11)(x + 2)$
o $(x + 12)(x + 3)$

4 a 8 and -1 b -6 and -1 c -7 and 5
d 5 and -3 e -4 and 2 f -6 and -5

5 a $(x + 8)(x - 1)$ b $(x - 6)(x - 1)$
c $(x - 7)(x + 5)$ d $(x + 5)(x - 3)$
e $(x - 4)(x + 2)$ f $(x - 6)(x - 5)$

6 a $(x + 6)(x - 2)$ b $(x - 10)(x + 2)$
c $(x - 3)(x - 2)$ d $(x + 9)(x - 3)$
e $(x - 8)(x - 5)$ f $(x - 5)(x + 4)$
g $(x + 7)(x - 3)$ h $(x - 11)(x - 3)$
i $(x - 7)(x + 4)$ j $(x + 4)(x - 3)$
k $(x - 9)(x + 2)$ l $(x - 8)(x - 7)$

7 a $2(x + 8)(x + 1)$ b $3(x + 3)(x + 2)$
c $7(x + 4)(x - 2)$ d $-5(x + 3)(x + 4)$
e $-4(x - 5)(x - 2)$ f $-1(x + 6)(x - 1)$

8 a i $x^2 + 6x + 9$ ii $(x + 3)(x + 3)$
b i $x^2 - 18x + 81$ ii $(x - 9)(x - 9)$
c i $(x + 4)^2$ ii $(x + 2)^2$ iii $(x - 6)^2$
iv $(x + 8)^2$ v $(x - 5)^2$ vi $(x - 10)^2$

9 a 9 b 25 c 16 d 4 e 1 f 64

10 a Halve the coefficient of the x term and then square it.
b 36
c To complete the square, you halve the coefficient of the x term and then square it. Half the coefficient of bx is $\frac{b}{2}$. Squaring $\frac{b}{2}$ produces $\left(\frac{b}{2}\right)^2$.

11 a 49 b 9 c 81 d 100 e 36 f 121

12 a $(x + 7)^2$ b $(x - 3)^2$ c $(x + 9)^2$
d $(x + 10)^2$ e $(x - 6)^2$ f $(x + 11)^2$

13 b $(x + 3)^2 - (\sqrt{2})^2$

14 a $(x + 4 + \sqrt{7})(x + 4 - \sqrt{7})$
b $(x - 5 + \sqrt{11})(x - 5 - \sqrt{11})$
c $(x + 2 + \sqrt{10})(x + 2 - \sqrt{10})$
d $(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$
e $(x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$
f $(x - 6 + \sqrt{23})(x - 6 - \sqrt{23})$
g $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$

h $(x - 7 + \sqrt{3})(x - 7 - \sqrt{3})$
i $(x + 7)(x + 1)$

15 a $(x + 7)(x + 5)$ **b** $(x + 7)$ and $(x + 5)$
c $(x + 7)$
d i 195 cm^2 **ii** 195 cm^2

16 a $(x + 12)$
b x must be greater than 3. Some possible values are: 4, 5, 6.
c 16 cm^2, 34 cm^2, 54 cm^2 **d** $x = 22$

17 a $x^2 + 6x + 5 = (x + 3)^2 - 2^2$
$= (x + 3 + 2)(x + 3 - 2)$
$= (x + 5)(x + 1)$
b $5 \times 1 = 5$ and $5 + 1 = 6$, so factors are $(x + 5)$ and $(x + 1)$.

18 a $(x + 11)(x + 4)$ **b** $(x - 5)(x + 3)$
c $(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$
d $3(x - 4)(x - 2)$ **e** $5(x + 3)(x - 3)$
f $2x(x - 4)$

19 a $x^2 + 3^2$ is the sum of two squares.
b $(x - 3)^2 + 4^2$ is the sum of two squares.
c $(x + 1)^2 + 2^2$ is the sum of two squares.
d $(x - 2)^2 + 1^2$ is the sum of two squares.
e The expression simplifies to $x^2 + 2^2$, which is the sum of two squares.
f $(x + 6)^2 + 2^2$ is the sum of two squares.

20 a $\left(x + \frac{3}{2} + \frac{\sqrt{13}}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{13}}{2}\right)$
b $\left(x - \frac{1}{2} + \frac{\sqrt{29}}{2}\right)\left(x - \frac{1}{2} - \frac{\sqrt{29}}{2}\right)$
c $\left(x + \frac{5}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{5}}{2}\right)$
d $\left(x + \frac{7}{2} + \frac{\sqrt{21}}{2}\right)\left(x + \frac{7}{2} - \frac{\sqrt{21}}{2}\right)$
e $\left(x - \frac{11}{2} + \frac{\sqrt{61}}{2}\right)\left(x - \frac{11}{2} - \frac{\sqrt{61}}{2}\right)$
f $\left(x - \frac{9}{2} + \frac{\sqrt{89}}{2}\right)\left(x - \frac{9}{2} - \frac{\sqrt{89}}{2}\right)$

21 a $2(x + 7)(x - 4)$
b Not possible as it is the sum of two squares: $\left(x - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2$
c $x(x + 6)(x - 3)$
d $-1(x - 4)(x + 3)$
e $-2(x + 7)(x - 7)$
f Not possible as it is the sum of two squares: $(2x)^2 + 5^2$

22 a $k \leq 9$ **b** $k \leq 100$

2F Working with algebraic fractions

2F Start thinking!

1 Factors of 3 are 1 and 3, factors of $3x$ are 1, 3, x and $3x$. To simplify, cancel the common factor of 3. (Dividing by a common factor of 1 would not simplify the fraction.)
2 No; 3 has factors of 1 and 3 and $(x + 3)$ has factors of 1 and $(x + 3)$ so there is no common factor that can be used to simplify the fraction.
3 x and 3 are added in the numerator of the first fraction whereas x and 3 are multiplied in the second (producing a product of factors). Hence 3 can be cancelled in $3x$ but not in $(x + 3)$.
4 Yes; cancel the common factor of 3, leaving the simplified version as $(x + 3)$ or just $x + 3$. To cancel a factor, it must be multiplied by another factor (even if the other factor is 1). The product of factors in the numerator is $3 \times (x + 3)$ and the product of factors in the denominator is 3×1 so 3 is the common factor to be cancelled.
5 **a** Yes, common factor is 4.
b no **c** no
d Yes, common factor is x.
e no
f Yes, common factor is 2.
g Yes, common factor is $(x + 5)$.
h Yes, common factor is $(x - 4)$.
6 **a** $2x$ **d** $x + 1$
f $\frac{9}{5x}$ **g** $\frac{x}{4}$ **h** $\frac{1}{2}$
7 There must be a common factor in the numerator and denominator for a fraction to be simplified by cancelling.

Exercise 2F Working with algebraic fractions

1 a $\frac{x}{2}$ **b** $\frac{1}{y}$ **c** 4
d $\frac{2x}{3}$ **e** $\frac{x + 3}{5}$ **f** $2(x - 5)$
g $x - 4$ **h** $x + 1$ **i** $\frac{1}{x - 7}$
j $\frac{x + 6}{3}$ **k** $\frac{5(x + 2)}{4x}$ **l** $\frac{5}{x - 9}$

2 a $\frac{x + 5}{x + 1}$ **b** $\frac{x - 6}{x - 2}$ **c** $\frac{x - 1}{x + 1}$
d $\frac{x + 2}{x - 8}$ **e** 1 **f** 1
g $\frac{x}{x - 2}$ **h** $\frac{x}{2}$ **i** $\frac{x - 2}{x - 4}$
j $\frac{x + 5}{3(x - 5)}$ **k** $\frac{x + 9}{2(x + 2)}$ **l** 2

3 a $\frac{1}{2}$ **b** 5 **c** 1 **d** 3
e 5 **f** x **g** $2x$ **h** $2(x + 4)$
i $\frac{x + 3}{x}$ **j** 2 **k** $\frac{2}{x}$ **l** $(x - 1)$

4 a $\frac{(x + 4)(x - 5)}{(x - 2)(x + 2)}$ **b** $\frac{x + 2}{x - 1}$
c $\frac{(x + 5)(x - 2)}{(x + 2)(x - 5)}$ **d** $\frac{x + 9}{x + 3}$
e $\frac{2(x - 5)}{x - 2}$ **f** $\frac{(x + 8)(x - 6)}{(x - 3)(x + 6)}$
g $\frac{(3x + 2)(5x + 1)}{(x + 1)(x + 2)}$ **h** $\frac{x}{2(x + 8)}$

5 a x **b** $\frac{1}{x}$ **c** $\frac{x}{3}$
d $\frac{x + 2}{x + 6}$ **e** $\frac{x - 2}{x}$ **f** $\frac{1}{2}$

g $\frac{x-6}{x+1}$ h $\frac{x-5}{x-3}$ i $\frac{7x}{x+7}$

6 a $\frac{1}{x+1}$ b 2 c $\frac{x-7}{x+1}$

d $\frac{x-1}{x+1}$ e $\frac{x-4}{x}$ f $\frac{(x+5)(x+2)}{(x+1)(x-4)}$

g $\frac{x+5}{x+2}$ h $\frac{(x+2)(x-6)}{(x+7)(x+10)}$

7 a $\frac{9x}{20}$ b $\frac{5x}{14}$ c $\frac{x}{2}$

d $\frac{x}{40}$ e $\frac{7x}{6}$ f $\frac{5x}{8}$

g $\frac{43x}{35}$ h $\frac{7x^2}{18}$

8 a $\frac{5x+13}{6}$ b $\frac{11x-8}{30}$ c $\frac{3x-7}{8}$

d $\frac{x+10}{12}$ e $\frac{x-12}{24}$ f $\frac{4x+1}{10}$

g $\frac{13x-6}{15}$ h $\frac{7x+17}{18}$ i $\frac{23x+5}{6}$

j $\frac{7x+15}{20}$ k $\frac{40-10x}{21}$ l $\frac{3x^2+1}{40}$

9 $a = x - 3$

10 $b = 3x + 1$

11 a $\frac{37x}{30}$ b $\frac{7x}{8}$ c $\frac{x}{8}$

d $\frac{3}{7}$ e $-\frac{8x}{21}$ f $\frac{7x-13}{12}$

12 a 2 b $\frac{x-6}{3}$ c $\frac{5x-2}{6}$ d $\frac{29x-19}{24}$

13 $\frac{29}{10x}$

14 a $\frac{11}{6x}$ b $\frac{5}{28x}$ c $\frac{7}{2x}$

d $\frac{1}{12x}$ e $\frac{1}{x}$ f $\frac{19}{24x}$

15 $\frac{7x+23}{(x+2)(x+5)}$

16 a $\frac{7x+11}{(x+1)(x+3)}$ b $\frac{2(2x-3)}{(x-4)(x+6)}$

c $\frac{2(x+1)}{(x+3)(x+2)}$ d $\frac{x+19}{(x-5)(x-1)}$

e $\frac{x(5x+16)}{(x+2)(x+4)}$ f $\frac{2x-13}{(2x+1)(3x-2)}$

17 Some possible answers are: $a = x - 1$, $b = x + 1$; $a = 3x + 3$, $b = -2x - 5$.

18 Some possible answers are: $m = x$, $n = x$; $m = x + 1$, $n = x + 2$; $m = x - 1$, $n = x - 2$.

2G Factorising quadratic trinomials of the form $ax^2 + bx + c$

2G Start thinking!

1 $a = 1$, $b = 5$, $c = 6$

2 2, 3

3 $ac = 1 \times 6 = 6$, $b = 5$; 2, 3

4 The numbers are the same. In this case, $ac = c$ since $a = 1$.

5 $x^2 + 2x + 3x + 6$

6 $x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$

7 No, since $x^2 + 3x + 2x + 6 = x(x + 3) + 2(x + 3)$ $= (x + 3)(x + 2)$, and $(x + 3)(x + 2)$ is the same as $(x + 2)(x + 3)$.

8 $ac = 24$, $b = 11$; 3, 8
$2x^2 + 3x + 8x + 12$
$= x(2x + 3) + 4(2x + 3)$
$= (2x + 3)(x + 4)$

Exercise 2G Factorising quadratic trinomials of the form $ax^2 + bx + c$

1 a i $ac = 4$, $b = 5$ ii 1, 4 iii $x + 4x$

b i $ac = 18$, $b = 11$ ii 2, 9 iii $2x + 9x$

c i $ac = 16$, $b = 17$ ii 1, 16 iii $x + 16x$

d i $ac = -24$, $b = 10$ ii -2, 12 iii $-2x + 12x$

e i $ac = -18$, $b = -7$ ii -9, 2 iii $-9x + 2x$

f i $ac = 30$, $b = -13$ ii -3, -10 iii $-3x - 10x$

2 a $(3x + 2)(x + 3)$ b $(4x + 1)(x + 4)$

c $(2x + 1)(x + 2)$ d $(2x - 3)(3x + 1)$

e $(3x - 2)(x + 4)$ f $(5x - 3)(x - 2)$

3 a $(2x + 3)(x + 2)$ b $(3x + 2)(x + 1)$

c $(2x + 1)(2x + 5)$ d $(7x + 2)(x + 1)$

e $(5x + 4)(x + 2)$ f $(2x + 3)(x + 3)$

g $(3x + 2)(2x + 1)$ h $(5x + 1)(2x + 3)$

i $(4x + 5)(2x + 1)$ j $(2x + 3)(x + 6)$

k $(4x + 1)(3x + 1)$ l $(5x + 4)(x + 3)$

4 a $(3x + 1)(x - 3)$ b $(4x - 3)(x + 2)$

c $(2x - 5)(x - 2)$ d $(5x + 4)(x - 1)$

e $(4x - 1)(2x + 3)$ f $(2x - 3)(2x + 5)$

g $(7x - 4)(x - 2)$ h $(4x - 1)(3x + 1)$

i $(3x + 4)(x - 5)$ j $(6x - 7)(x - 2)$

k $(5x - 3)(2x + 3)$ l $(4x - 1)(3x - 5)$

5 a i $2x \times x$ ii $+5 \times +3$

iii $2x \times +3$ and $+5 \times x$

b $2x^2 + 11x + 15$ c same result

6 a i $2x^2$ ii $+4$ iii $8x + x = 9x$

b i $5x^2$ ii $+2$ iii $-5x - 2x = -7x$

c i $6x^2$ ii -12 iii $-9x + 8x = -x$

d i $12x^2$ ii -10 iii $8x - 15x = -7x$

7 a $(2x + 1)(x + 3)$ b $(3x + 5)(x + 1)$

c $(5x + 7)(x + 1)$ d $(3x + 2)(x + 5)$

e $(4x + 3)(x + 2)$ f $(7x + 2)(x + 4)$

g $(2x + 1)(2x + 5)$ h $(3x + 2)(2x + 7)$

i $(5x + 4)(2x + 1)$

8 a $(3x - 2)(x + 3)$ b $(5x + 1)(x - 2)$

c $(2x - 3)(x - 4)$ d $(7x - 3)(x + 1)$

e $(3x + 4)(x - 2)$ f $(2x + 5)(x - 3)$

g $(5x - 3)(x - 2)$ h $(9x - 1)(x + 1)$

i $(4x + 3)(x - 6)$

9 a $(3x + 2)(2x - 5)$ b $(2x - 3)(2x + 7)$

c $(5x - 2)(2x - 1)$ d $(4x + 3)(2x - 5)$

e $(3x - 4)(2x + 3)$ f $(3x - 2)(3x + 8)$

g $(5x - 2)(3x + 1)$ h $(8x - 3)(x + 2)$

i $(6x - 1)(x - 4)$

10 a $2(2x + 3)(x + 1)$ b $3(4x + 1)(x + 2)$

c $2(5x - 1)(x + 4)$ d $(4x + 5)(2x + 3)$

e $3(2x + 5)(x - 2)$ f $4(3x + 1)(x - 1)$

g $-2(2x + 1)(x + 1)$ h $-3(4x - 1)(x + 2)$

i $-(3x + 4)(2x + 3)$

11 **a** $2x + 3$ **b** $\frac{1}{x+3}$ **c** $2x + 7$

d $\frac{2x-1}{x+5}$ **e** $\frac{5x+2}{2x-3}$ **f** -4

12 **a** $\frac{5x-3}{x-3}$ **b** $\frac{2x+7}{2x+1}$

c 1 **d** $\frac{(2x+7)(x+4)}{(2x-7)(x-4)}$

e $\frac{5x+2}{2}$ **f** $\frac{x+1}{x-3}$

13 **a** $3(x+3+\sqrt{2})(x+3-\sqrt{2})$

b $2(x-5+\sqrt{5})(x-5-\sqrt{5})$

c $5(x+1+\sqrt{7})(x+1-\sqrt{7})$

d $3(x+2+\sqrt{3})(x+2-\sqrt{3})$

e $2\left(x-3+\sqrt{\frac{21}{2}}\right)\left(x-3-\sqrt{\frac{21}{2}}\right)$

f $4\left(x+1+\frac{\sqrt{5}}{2}\right)\left(x+1-\frac{\sqrt{5}}{2}\right)$

g $6\left(x-2+\sqrt{\frac{29}{6}}\right)\left(x-2-\sqrt{\frac{29}{6}}\right)$

h $3\left(x+4+\sqrt{\frac{47}{3}}\right)\left(x+4-\sqrt{\frac{47}{3}}\right)$

i $9\left(x-1+\frac{\sqrt{7}}{3}\right)\left(x-1-\frac{\sqrt{7}}{3}\right)$

14 **a** $(4x-1)(x+2)$

b $-4x^2 - 7x + 2 = -(4x^2 + 7x - 2)$
$= -(4x-1)(x+2)$
$= (-4x+1)(x+2)$
$= (1-4x)(x+2)$

c **i** $(3-2x)(x+1)$ **ii** $(1-3x)(x-5)$
iii $(7-5x)(x+2)$ **iv** $(2-x)(4x+3)$
v $(1-2x)(3x+2)$ **vi** $(1-5x)(5x-1)$

15 **a** **i** $(2x-3)(x+7)$

b $2x - 3$

c **i** $147 \times 82 = 12\,054\ \text{mm}^2 = 120.54\ \text{cm}^2$

ii $2 \times 75^2 + 11 \times 75 - 21 = 12\,054\ \text{mm}^2$
$= 120.54\ \text{cm}^2$

d 100-euro note is larger in area by $17.84\ \text{cm}^2$

16 **a** $2x \times x = 2x^2\ \text{cm}^2$

b $2x^2 + 27x + 81 = (2x+9)(x+9)$;
length $= (2x+9)$ cm, width $= (x+9)$ cm.
Border is 4.5 cm wide.

c Area of border is $(27x + 81)\ \text{cm}^2$.
For $27x + 81 = 1566$, $x = 55$ so length is 119 cm and width is 64 cm.

17 Parts a, d and e cannot be factorised as difference of two squares cannot be set up when using the completing the square method.

b $(5x-4)(3x+2)$

c $3(x-2+\sqrt{3})(x-2-\sqrt{3})$

f $-3(2x-9)(x-7)$

18 **a** A: rectangle of length $(4x-5)$ m and width $(2x+3)$ m.
B: square of length $(2x+3)$ m.

b $(12x^2 + 14x - 6)\ \text{m}^2$

c $12x^2 + 14x - 6 = 2(3x-1)(2x+3)$
Dimensions could be $(6x-2)$ m by $(2x+3)$ m or $(4x+6)$ m by $(3x-1)$ m.

19 **a** $2\left(x+\frac{3}{4}+\frac{\sqrt{41}}{4}\right)\left(x+\frac{3}{4}-\frac{\sqrt{41}}{4}\right)$

b $4\left(x-\frac{5}{8}+\frac{\sqrt{57}}{8}\right)\left(x-\frac{5}{8}-\frac{\sqrt{57}}{8}\right)$

c $3\left(x+\frac{1}{6}+\frac{\sqrt{13}}{6}\right)\left(x+\frac{1}{6}-\frac{\sqrt{13}}{6}\right)$

d $\frac{1}{2}(x-5+\sqrt{11})(x-5-\sqrt{11})$

e $5\left(x+\frac{1}{2}+\sqrt{\frac{17}{20}}\right)\left(x+\frac{1}{2}-\sqrt{\frac{17}{20}}\right)$

f $-6\left(x+\frac{3}{2}+\frac{\sqrt{7}}{2}\right)\left(x+\frac{3}{2}-\frac{\sqrt{7}}{2}\right)$

2 Chapter review

MULTIPLE-CHOICE

1 C	**2** A	**3** D	**4** B	**5** C
6 D	**7** B	**8** A	**9** C	**10** D
11 B	**12** B	**13** C		

SHORT ANSWER

1 **a** $-6xy - 3x^2 + 2x^2y - 5xy^2$

b $-3ab^2 - 2a^2b^2 + 2a^2b$

c $-4pq - 8p^2 + 3q^2$

2 **a** $a^2b^2c^2$ and $(2abc)^2$

b $5xyz^2$, $-2z^2xy$ and $3yz^2x$

c $2p^2q^2$, $-(pq)^2$ and $p^2 \times q \times -q$

3 **a** 14 **b** 39 **c** -16

4 **a** $\frac{x^3}{y^4}$ **b** $\frac{a^4}{3b^5}$ **c** $\frac{6p^4r^7}{5q^5}$

5 **a** x^8 **b** $\frac{1}{pq^2}$ **c** $\frac{1}{864a^8b^3}$

6 **a** $6x^4 - 2x^2 - 20$ **b** $4x^2 - 9y^2$

c $9x^4 - 24x^2 + 16$

7 **a** $2x^2 - 5x - 6$ **b** $4x - 13x^2$ **c** $x^2 - x$

8 **a** $(x-3)(x-4)$ **b** $(x+1)^2(x-1)$

c $2(8+3x)(x-2)$

9 **a** $2(3x+2y)(3x-2y)$

b $-3(2x-3)$ **c** $8x$

10 **a** $(xy+\sqrt{13})(xy-\sqrt{13})$

b $(x+1+\sqrt{3})(x+1-\sqrt{3})$

c $(\sqrt{7}+a-1)(\sqrt{7}-a+1)$

11 **a** $(x+2)(x+6)$ **b** $(x-9)(x+7)$

c $-1(x-4)(x-2)$ **d** $-6(x+3)(x-2)$

12 **a** $(x+5+\sqrt{5})(x+5-\sqrt{5})$

b $(x-4+\sqrt{17})(x-4-\sqrt{17})$

13 **a** $\frac{x+4}{x+3}$ **b** $\frac{(x-3)(x+1)}{(x+4)(x+3)}$

14 **a** $\frac{25x}{12}$ **b** $\frac{9x-22}{20}$ **c** $\frac{14x-11}{24}$ **d** $\frac{31}{12x}$

15 **a** $(3x+2)(x+9)$ **b** $(4x-1)(2x+3)$

c $(2x-11)(x+2)$ **d** $(4x-3)(3x-2)$

16 **a** $2(x+3+\sqrt{11})(x+3-\sqrt{11})$

b cannot be factorised

c $3(2x-5)(x+6)$

d $-4\left(x-\frac{3}{2}+\frac{\sqrt{7}}{2}\right)\left(x-\frac{3}{2}-\frac{\sqrt{7}}{2}\right)$

MIXED PRACTICE

1 a $24x^2y^3$ b $-\frac{3b^4}{2a}$ c $\frac{8q^3}{p}$

2 B

3 a $25x^2 - 4y^2$ b $x^2 - 5x - 6$
c $9x^2 - 12xy + 4y^2$

4 a False; $5x^{-1}$ is equivalent to $\frac{5}{x}$.
b False; the value of x is -5.
c false; $x + y = 6$

5 a $\frac{a^2c}{b^4}$ b $\frac{3x^3}{4y^2}$ c $\frac{4}{9p^{10}q^{12}}$

6 a $\frac{13}{12x}$ b $\frac{2(x-4)}{(x-1)(x-3)}$ c $\frac{x^2+y^2}{xy}$

7 A

8 a $(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$
b $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$
c $\left(x + \frac{5}{2} + \frac{\sqrt{33}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{33}}{2}\right)$

9 a 22.25 b 2.5 c -8.5

10 a $4ab$ b $2xy - 2y^2$
c $p^4 - 2p^2q^3 + q^6$

11 a $\frac{2(x-2)}{3(x-1)}$ b $\frac{1}{x^2+1}$ c 1

12 a $(d - 1)(d - 4)$ b $(y + 3)(x - 2y)$
c $(x^2 + y^2)(x + y)(x - y)$

13 a $\frac{9}{4}$ b $\frac{25}{4}$ c $\frac{49}{4}$

14 D

15 $x = -1$ or $x = 5$

16 a $\frac{3-2x}{18}$ b $\frac{14-9x}{(x-1)(2-x)}$
c $\frac{4z+2x}{xyz}$

17 a true
b False; these are not like terms so cannot be added.
c true

18 Some possible answers are: base $(2x + 6)$ cm, height $(x - 3)$ cm; base $(x + 3)$ cm, height $(2x - 6)$ cm.

19 a $(m^{-3}n^{-2})^{-1} = m^3n^2$ b $pq^2r^{-1} \div q^{-2}r = \frac{pq^4}{r^2}$
c $a^{b+1} = a^b \times a$

20 C

21 a $4(2x - 3)^2$ b $(k + \frac{1}{2})(k - \frac{1}{2})$
c $(a - 3b)(b + 2a)$

22 a 2^7 b $\frac{1}{3^5}$ c 5^9

23 $a = -1$

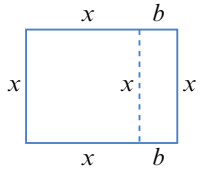
24 a $\frac{7x+1}{(x-1)(x+1)}$ b $\frac{3}{2-3x}$

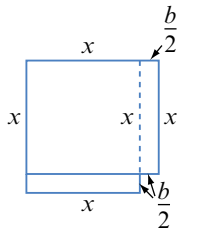
25 a $\frac{2(x-1)}{x+3}$ b -1

26 D

ANALYSIS

a $x(x + b)$
b x by $(x + b)$

c 

d 

e square with side lengths $\frac{b}{2}$

f $\left(x + \frac{b}{2}\right)^2 = x^2 + \frac{bx}{2} + \frac{bx}{2} + \frac{b^2}{4}$

g $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$

h The square is completed by halving the coefficient of the x-term and then squaring it.

2 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 3 REAL NUMBERS

3 Are you ready?

1 C

2 a 9 b 2 c 16 d 3

3 0.131 313 131 313

4 a $0.5\overline{62}$ b $12.580\,\overline{13}$ c $45.\overline{45}$
d $0.\overline{8412}$

5 B

6 a 144 cm^2 b 77.5 m^2 c 175 mm^2
d 25π cm^2

7 a 2×5^2 b $2^3 \times 3^2$ c 3×2^4 d $2^2 \times 5^2$

8 A

9 a $x^2 - 5x - 14$ b $x^2 + 12x + 36$
c $x^2 - 16$ d $9x^2 - 42x + 49$

10 a $\frac{7}{15}$ b $\frac{11}{28}$ c $1\frac{9}{40}$ d $\frac{7}{45}$

3A Understanding rational and irrational numbers

3A Start thinking!

1 a Some possible answers are:
$3 = \frac{3}{1}$, $4 = \frac{24}{6}$, $-2 = -\frac{10}{5}$
b Some possible answers are:
$0.25 = \frac{1}{4}$, $0.675 = \frac{5}{8}$, $0.37 = \frac{37}{100}$
c Some possible answers are:
0.025 673 476 134 98…,
$\sqrt{2} \approx 1.414\,213\,56\ldots$, $\sqrt[3]{5} \approx 1.709\,975\,9\ldots$

2 a 2.236 068 0
b i 1.732 050 8 ii 2.449 489 7
iii 1.414 213 6 iv 3.162 277 7
c i 2 ii 6 iii 9 iv 10

d The answers to part b are approximate whereas the answers to part c are exact whole numbers.

3 a These answers are all non-recurring and non-terminating decimals which cannot be written as a fraction.

b Each answer can be written in as a fraction.

4 All numbers in question 2a and 2b are surds. All numbers in question 2c are not surds.

5 Some possible answers are given.
Rational numbers: integers 21, −7, 0;
fractions $\frac{7}{8}$, $-\frac{2}{3}$; decimals 0.75, $0.\overline{61}$.
Irrational numbers: 0. 635 492 761 455 7…, $\sqrt{7}$, $\sqrt[3]{4}$

Exercise 3A Understanding rational and irrational numbers

1 a D b F c B d E e A f C

2 a rational b rational c irrational
d rational e irrational f rational
g rational h irrational i rational
j rational k rational l irrational
m rational n irrational o irrational

3 a, b, c, f, h, i, k, l, n, o, p, q, r

4 a 4.5826 b 3.4641 c 5.1962
f −2.4495 h 11.1803 i 2.8284
k 1.4422 l −6.6332 n 0.4899
o 1.2361 p 4.7321 q 0.4082
r −0.7071

5 $\sqrt{3}$, $-3\sqrt{5}$, $\sqrt{8}$, $\sqrt[4]{32}$

6 a 11 b 30 c −32 d −1
e 24 f 14 g 10 h 5
i 13 j −6 k 11 l 9
m 28 n 36 o 0 p $\frac{11}{20}$

7 Rational numbers: a, b, c, f, g, j, k, l, m; irrational numbers: d, e, h, i, n, o, p

8 d, h, i, o, p

9 a rational b irrational c irrational
d rational e irrational f irrational
g irrational h irrational

10 a Irrational, since it cannot be written exactly as a terminating or recurring decimal.

b irrational numbers

11 a 4.6856 b 6.7807 c 1.6042 d 4.9892
e 5.9161 f 54.9909 g 3.4641 h 2.6788
i 4.6291 j 0.3769 k 1.5157 l 0.1135

12 a $\sqrt{\frac{5}{8}} \approx 0.790\,569$, $\frac{\sqrt{5}}{\sqrt{8}} \approx 0.790\,569$ b yes

c $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

d Irrational, since it cannot be written exactly as a terminating or recurring decimal.

e i $\sqrt{\frac{32}{2}} = \sqrt{16} = 4$, rational

ii $\sqrt{\frac{15}{3}} = \sqrt{5} = 2.236\ldots$, irrational

iii $\sqrt{\frac{7}{28}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, rational

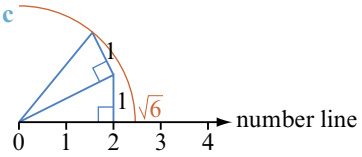
13 a 2, 3 b 2, 3

c By choosing between which two perfect squares the number lies.

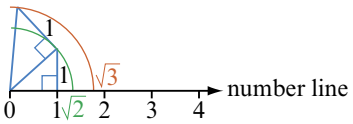
d i 3, 4 ii −4, −5 iii 11, 12
iv −6, −7 v 13, 14 vi 17, 18

14 b 2.24

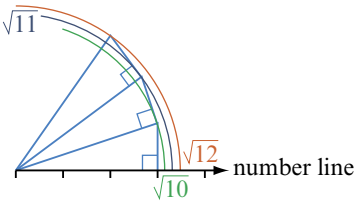
15 b, c



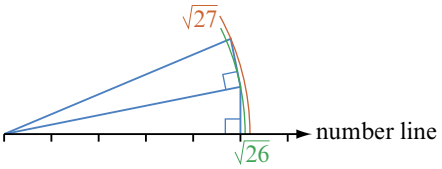
16 a



b



c



17 a 8 cm b i $\sqrt{128}$ cm ii 11.31 cm

18 a side = $\sqrt{35}$ cm, 5.92 cm;
diagonal = $\sqrt{70}$ cm, 8.37 cm

b side = $\sqrt{51}$ mm, 7.14 mm;
diagonal = $\sqrt{102}$ mm, 10.10 mm

c side = $\sqrt{95}$ cm, 9.75 cm;
diagonal = $\sqrt{190}$ cm, 13.78 cm

19 a i side = $8\sqrt{2}$ cm, diagonal = 16 cm
ii side = $\sqrt{70}$ cm, diagonal = $\sqrt{140}$ cm
iii side = $\sqrt{102}$ cm, diagonal = $\sqrt{204}$ cm
iv side = $\sqrt{190}$ cm, diagonal = $\sqrt{380}$ cm.
The length of the diagonal doubles.

20 a $\frac{2}{5}$ b $\frac{13}{20}$ c $\frac{49}{100}$ d $\frac{8}{5}$ e $\frac{257}{100}$ f $\frac{851}{1000}$

21 a 1D, 2C, 3E, 4F, 5B, 6 A

b To give another number with a whole number as well as the same recurring decimal part so that we could subtract and eliminate the decimal part.

22 a $\frac{2}{9}$ b $\frac{7}{9}$ c $\frac{7}{3}$ d $\frac{17}{3}$

23 a Let $x = 0.363\,636\ldots$
$100x = 36.363\,636\ldots$
$100x - x = 36.363\,636\ldots - 0.363\,636\ldots$
$99x = 36$
$x = \frac{36}{99}$
So $0.\overline{36} = \frac{4}{11}$

b Because the recurring part of the decimal contained two digits it was necessary to shift the decimal point two places to obtain the same recurring decimal again.

24 a $\frac{14}{99}$ b $\frac{7}{33}$ c $\frac{16}{33}$ d $\frac{172}{99}$

25 a $\frac{1}{3}$ b $\frac{8}{33}$ c $\frac{785}{999}$ d $\frac{41}{99}$
e $\frac{17}{30}$ f $\frac{35}{11}$ g $\frac{383}{90}$ h $\frac{3899}{1110}$

3B Multiplying and dividing surds

3B Start thinking!

1 a $\sqrt{35}$ b i $\sqrt{10}$ ii $\sqrt{21}$
2 $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
3 a $12\sqrt{14}$ b i $3\sqrt{30}$ ii $8\sqrt{77}$
4 $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$
5 a $\sqrt{3}$ b i $\sqrt{5}$ ii $\sqrt{11}$
6 $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$
7 a $2\sqrt{5}$ b i $3\sqrt{3}$ ii $3\sqrt{3}$
8 $\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}$

Exercise 3B Multiplying and dividing surds

1 a $\sqrt{35}$ b $\sqrt{33}$ c $-\sqrt{14}$ d $-\sqrt{105}$
e $\sqrt{65}$ f $-\sqrt{22}$ g $\sqrt{21}$ h $\sqrt{95}$
i $3\sqrt{77}$ j $20\sqrt{6}$ k $-28\sqrt{22}$ l $-36\sqrt{30}$
m $66\sqrt{34}$ n $64\sqrt{42}$ o $-24\sqrt{33}$ p $132\sqrt{15}$
2 a 7 b 3 c 2 d 6
e 5 f 11 g 10 h 9
3 The answer is rational.
4 a 9 b -77 c 40 d 60
e $3\sqrt{10}$ f $30\sqrt{30}$ g $-32\sqrt{26}$ h $3\sqrt{33}$
i 84 j 15 k $-6\sqrt{26}$ l $\sqrt{105}$
m $72\sqrt{42}$ n $\frac{3\sqrt{195}}{8}$ o $-5\sqrt{70}$ p $-11\sqrt{85}$
5 a $\sqrt{17}$ b $\sqrt{3}$ c 1 d $-\sqrt{5}$
e $\sqrt{6}$ f $\sqrt{7}$ g $4\sqrt{6}$ h $2\sqrt{5}$
i $-3\sqrt{5}$ j $2\sqrt{5}$ k $-4\sqrt{10}$ l $\frac{\sqrt{5}}{2}$
6 a 4 b 2 c 3 d 2
7 The answer is rational.
8 a $-12\sqrt{5}$ b $5\sqrt{3}$ c $-3\sqrt{2}$
d $2\sqrt{5}$ e $4\sqrt{7}$ f $-\frac{7\sqrt{5}}{3}$
g $\frac{3}{\sqrt{3}}$ h $\frac{\sqrt{5}}{5}$ i 3
11 a Divide both numerator and denominator by the highest common factor.
b $\frac{6}{7}$ c i 8 ii 20
12 $\frac{5\sqrt{24}}{\sqrt{6}} = 10$, $\frac{\sqrt{24}}{5\sqrt{6}} = \frac{2}{5}$
13 a 6 b $\frac{3\sqrt{5}}{4}$ c $\frac{11\sqrt{17}}{7\sqrt{21}}$
d $-\frac{4\sqrt{15}}{15}$ e $\frac{9\sqrt{5}}{4}$ f $-\frac{2}{3}$
g -32 h $\frac{\sqrt{15}}{8\sqrt{8}}$ i 1
j $\frac{8}{\sqrt{12}}$ k 15 l $\frac{2\sqrt{3}}{3\sqrt{2}}$
14 a 3 b $\frac{9}{10}$ c 24 d $\frac{1}{3}$
e -60 f 2 g 84 h 576
i $\frac{24}{5}$ j $\frac{1}{8}$ k $-\frac{4}{3}$ l $-\frac{7}{3}$
15 a $2\sqrt{10}$ b $\frac{\sqrt{3}}{5}$ c $\frac{8}{5}$
d $-\frac{11\sqrt{2}}{8}$ e $\frac{8\sqrt{2}}{3}$ f $-\frac{2\sqrt{6}}{5}$
g $\frac{\sqrt{5}}{9}$ h $-\frac{9\sqrt{7}}{5}$ i $-\frac{5\sqrt{44}}{9}$
16 a 5 b 8 c 2 d 7
e 9 f 15 g 25 h 75
i 80 j 54 k 192 l 891
17 a rational b $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$
18 a x b ab c $6y$ d $x + y$
19 a $\frac{25}{7}$ b $\frac{1}{5}$ c $\frac{3}{2}$ d $\frac{10}{13}$
20 a i 50.00 cm^2 ii 50 cm^2
b i 137.48 cm^2 ii $30\sqrt{21}$ cm^2
c i 19.44 cm^2 ii $3\sqrt{42}$ cm^2
d i 169.71 mm^2 ii $120\sqrt{2}$ mm^2
e i 23.24 m^2 ii $6\sqrt{15}$ m^2
f i 6.00 cm^2 ii 6 cm^2
21 a $\frac{5\sqrt{6}}{2}$ cm b $\frac{75\pi}{2}$ cm^2
c When a surd is squared the answer is rational and π is an irrational number, so the area is not rational as it contains π.
22 a i 6.98 cm^2 ii $\frac{20\pi}{9}$ cm^2
b i 37.70 cm^2 ii 12π cm^2
c i 0.39 m^2 ii $\frac{\pi}{8}$ m^2
23 $14\sqrt{5}$ cm
24 radius = $\sqrt{6}$ cm, diameter = $2\sqrt{6}$ cm
25 $\frac{\sqrt{5}}{2}$ m
26 108 cm^2
27 $\left(\sqrt{3} + \frac{\pi\sqrt{3}}{6}\right)$ m
28 no

3C Simplifying surds

3C Start thinking!

1 a $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ b $\sqrt{40}$
c The calculator answer has a whole part as well as a surd part.
d $\sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10} = 2 \times \sqrt{10} = 2\sqrt{10}$
2 a $\sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10}$
b i $2\sqrt{3}$ ii $3\sqrt{2}$ iii $2\sqrt{5}$ iv $4\sqrt{2}$
d One factor is a perfect square.
3 When simplifying surds, the number under the root sign is expressed as a product of two factors. The first factor needs to be the highest possible perfect square.

Exercise 3C Simplifying surds

1 **a** 4 **b** 9 **c** 7 **d** 2
e 10 **f** 5 **g** 18 **h** 44
i 1 **j** 2 **k** 3 **l** 9

2 **a** $9\sqrt{7}$ **b** $4\sqrt{3}$ **c** $2\sqrt{6}$
d $5\sqrt{10}$ **e** $3\sqrt{2}$ **f** $6\sqrt{5}$
g $8\sqrt{11}$ **h** $30\sqrt{5}$ **i** $6\sqrt{10}$

3 **a** $\sqrt{45} = \sqrt{9} \times \sqrt{5}$
$= 3 \times \sqrt{5}$
$= 3\sqrt{5}$

b $\sqrt{20} = \sqrt{4} \times \sqrt{5}$
$= 2 \times \sqrt{5}$
$= 2\sqrt{5}$

c $\sqrt{8} = \sqrt{4} \times \sqrt{2}$
$= 2 \times \sqrt{2}$
$= 2\sqrt{2}$

4 **a** $2\sqrt{7}$ **b** $5\sqrt{2}$ **c** $2\sqrt{6}$ **d** $3\sqrt{3}$
e $4\sqrt{3}$ **f** $5\sqrt{3}$ **g** $3\sqrt{7}$ **h** $2\sqrt{15}$
i $2\sqrt{14}$ **j** $3\sqrt{5}$ **k** $8\sqrt{2}$ **l** $6\sqrt{3}$
m $6\sqrt{5}$ **n** $7\sqrt{3}$ **o** $9\sqrt{2}$ **p** $5\sqrt{5}$
q $15\sqrt{2}$ **r** $12\sqrt{2}$ **s** $3\sqrt{30}$ **t** $9\sqrt{7}$

5 **a** $10\sqrt{5}$ **b** $6\sqrt{2}$ **c** $12\sqrt{3}$ **d** $27\sqrt{2}$
e $16\sqrt{5}$ **f** $9\sqrt{10}$ **g** $60\sqrt{6}$ **h** $-12\sqrt{3}$
i $28\sqrt{6}$ **j** $-14\sqrt{5}$ **k** $64\sqrt{3}$ **l** $60\sqrt{7}$
m $-20\sqrt{7}$ **n** $21\sqrt{11}$ **o** $-24\sqrt{17}$ **p** $1000\sqrt{3}$
q $4\sqrt{5}$ **r** $3\sqrt{6}$ **s** $16\sqrt{62}$ **s** $3\sqrt{11}$

7 **a** $\sqrt{43}$ **b** $5\sqrt{5}$ **c** $2\sqrt{30}$ **d** $6\sqrt{2}$
e $5\sqrt{6}$ **f** $-6\sqrt{6}$ **g** $3\sqrt{14}$ **h** $4\sqrt{15}$
i $-9\sqrt{2}$ **j** $7\sqrt{5}$ **k** $4\sqrt{3}$ **l** $2\sqrt{3}$
m $-8\sqrt{3}$ **n** $60\sqrt{5}$ **o** $6\sqrt{21}$ **p** $9\sqrt{2}$
q $-160\sqrt{2}$ **r** -20 **s** $-10\sqrt{10}$ **t** $3\sqrt{2}$
u $5\sqrt{7}$ **v** $-4\sqrt{6}$ **w** $\frac{7\sqrt{5}}{2}$ **x** $\frac{27\sqrt{2}}{2}$

8 **a** $12\sqrt{10}$ **b** $16\sqrt{15}$ **c** $-84\sqrt{5}$ **d** $4\sqrt{6}$
e $210\sqrt{6}$ **f** $\frac{3\sqrt{42}}{4}$ **g** $-300\sqrt{6}$ **h** $480\sqrt{2}$

9 **a** $3\sqrt{6} = \sqrt{3^2} \times \sqrt{6}$
$= \sqrt{9} \times \sqrt{6}$
$= \sqrt{9 \times 6}$
$= \sqrt{54}$

b To be simplified it must have first been a perfect square.
c They are equal.
d Put the entire surd into your calculator and press equal or enter to obtain the simplified surd.
e $\sqrt{48}$

10 **a** $\sqrt{45}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{252}$
e $\sqrt{160}$ **f** $\sqrt{275}$ **g** $\sqrt{384}$ **h** $\sqrt{125}$
i $\sqrt{240}$ **j** $-\sqrt{128}$ **k** $\sqrt{1008}$ **l** $-\sqrt{75}$

11 **a** $\sqrt{a^2b}$
c Square the whole part then multiply it by the surd part to give the number under the root sign in the entire surd.

12 **a** $\sqrt{m^2n}$ **b** $\sqrt{49a}$ **c** $\sqrt{5x^2}$ **d** $\sqrt{100xy}$
e $\sqrt{2cd^2}$ **f** $\sqrt{36abc}$ **g** $\sqrt{a^3b^2}$ **h** $\sqrt{x^4y^3z^2}$

13 **a** $x\sqrt{y}$ **b** xy **c** $b\sqrt{a}$ **d** $xz\sqrt{y}$
e x^2 **f** $x^2\sqrt{y}$ **g** x^2y **h** $a^2b\sqrt{c}$

14 **a** $2^2 \times 3$

15 **a** 2×3^3 **c** $3\sqrt{6}$

16 **a** $3\sqrt{2}$ **b** $2\sqrt{7}$ **c** $4\sqrt{2}$ **d** $3\sqrt{3}$
e $2\sqrt{6}$ **f** $5\sqrt{2}$ **g** $6\sqrt{5}$ **h** $16\sqrt{2}$
i $15\sqrt{7}$ **j** $16\sqrt{10}$ **k** $28\sqrt{3}$ **l** $18\sqrt{5}$
m $6\sqrt{5}$ **n** $35\sqrt{3}$ **o** $54\sqrt{2}$ **p** $5\sqrt{5}$
q $30\sqrt{2}$ **r** $12\sqrt{2}$

17 The surd cannot be simplified.

18 **a** $1680\sqrt{6}$ **b** $42\sqrt{2} \times 40\sqrt{3}$ **c** yes
d Use method in part b only when the surd part can be simplified.

19 **a** $6\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{3\sqrt{13}}{5}$
d $\frac{6\sqrt{5}}{7}$ **e** $-7\sqrt{3}$ **f** $\frac{8\sqrt{3}}{3}$
g $-16\sqrt{5}$ **h** $\frac{\sqrt{6}}{5}$ **i** $-\frac{2\sqrt{21}}{3}$

20 **a** $\frac{11\sqrt{2}}{8}$ **b** $8\sqrt{6}$ **c** $-\frac{7\sqrt{2}}{16}$ **d** $8\sqrt{3}$

21 $abc\sqrt{b}$

22 **a** $4\sqrt{5}$ cm **b** $6\sqrt{5}$ cm **c** $2\sqrt{13}$ cm **d** $12\sqrt{11}$ m

23 **a** $240\sqrt{6}$ cm^2 **b** $96\sqrt{10}$ cm^2 **c** $\frac{24\sqrt{11}}{5}$ mm^2
d $33\sqrt{30}$ m^2

24 **a** $\frac{4\sqrt{6}}{3}$ cm **b** $\frac{32}{3}\pi$ cm^2

25 Some possible answers are: $\sqrt{80}$, $2\sqrt{20}$, $\frac{1}{2}\sqrt{320}$.

26 Some possible answers are: $\sqrt{448}$, $2\sqrt{112}$, $\frac{1}{4}\sqrt{7168}$.

3D Adding and subtracting surds

3D Start thinking!

1 **a** 3
b Combine like terms by adding or subtracting.
c $x + 2y$

2 **a** $\sqrt{5} + 2\sqrt{3}$
b There are the same coefficients as in part 1c.
c The like surds have been combined in the same way as the like terms.

3 Like surds can be added or subtracted in the same manner as algebraic like terms.

4 **a** 3 **b** no **c** $12\sqrt{3} + 4\sqrt{2} - 2\sqrt{3}$
d Expression can be simplified; two.
e $10\sqrt{3} + 4\sqrt{2}$; yes

Exercise 3D Adding and subtracting surds

1 **a** yes **b** no **c** no **d** yes
e no **f** no **g** yes **h** yes

2 **a** $7\sqrt{3} + \sqrt{2}$ **b** $4\sqrt{3} - \sqrt{2}$
c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$
d $8\sqrt{13} + 4\sqrt{39} - \sqrt{3}$ **e** $13\sqrt{2} - 6\sqrt{5}$
f $4\sqrt{6} + 4\sqrt{5} + 1$

3 **a** $8\sqrt{3}$ **b** $-3\sqrt{2}$
c $-6\sqrt{7} - \sqrt{6}$ **d** $15\sqrt{5} - 7\sqrt{3}$
e 0 **f** $3\sqrt{10} - 4\sqrt{2} - 6\sqrt{3} + 5\sqrt{5}$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ h $\sqrt{21} - \sqrt{3}$
i $7\sqrt{6} + 3\sqrt{5} + 5$ j $-7\sqrt{15} - 3\sqrt{7}$

4 a $4\sqrt{2}$ b $5\sqrt{5}$ c $18\sqrt{6}$ d $-18\sqrt{2}$
e $-20\sqrt{5}$ f $32\sqrt{2}$ g $108\sqrt{3}$ h $35\sqrt{5}$

5 a $\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}$ b none
c $3\sqrt{12}, \sqrt{3}, 2\sqrt{27}$ d $9\sqrt{32}, 4\sqrt{8}, \sqrt{128}, 5\sqrt{200}$

6 a $-15\sqrt{3}$ b $-6\sqrt{5}$ c $-4\sqrt{7}$
d $-19\sqrt{6} + 8\sqrt{5}$ e $-25\sqrt{3} + 2\sqrt{2}$ f $16\sqrt{2} - 30\sqrt{3}$

7 a $5\sqrt{3}$ b $23\sqrt{2}$ c 0
d $4\sqrt{5}$ e 0 f $4\sqrt{6}$
g $\sqrt{2}$ h 0
i $6\sqrt{3} - 2\sqrt{7} + 10\sqrt{2} - 4\sqrt{5}$
j $3\sqrt{5} - 2\sqrt{6} - 6\sqrt{3}$

8 a $26\sqrt{5} + 6\sqrt{7}$ b $24 - 8\sqrt{3}$
c $-6\sqrt{6} - 17\sqrt{5}$ d $61\sqrt{3} - \sqrt{7}$
e $38\sqrt{2} - 24\sqrt{3} + 15\sqrt{5}$ f $-19\sqrt{3} - 59\sqrt{2}$
g $28\sqrt{2} - 24\sqrt{3}$ h $15\sqrt{3} + 26\sqrt{5} - 25\sqrt{6}$

9 a $5\sqrt{5}$ b $42\sqrt{3}$
c $29\sqrt{2} - 80\sqrt{3}$ d $18\sqrt{6} - 5\sqrt{5} - 14\sqrt{3}$
e $66\sqrt{7} - 8\sqrt{6} + 112\sqrt{3}$
f $10\sqrt{2} - 24\sqrt{5} + 60\sqrt{6} - 12\sqrt{10}$
g $3\sqrt{3} - 25\sqrt{5} - 61$ h $88\sqrt{2} - 113\sqrt{3}$

10 Use your calculator to simplify or test the perfect squares to see which are factors.

11 a 0 b $-22\sqrt{x}$ c $7\sqrt{x} - 6\sqrt{3x}$
d $-5\sqrt{2xy}$ e $20\sqrt{6xy}$ f $-x\sqrt{6y}$
g $12y\sqrt{3x}$ h $34x\sqrt{y} - 5x\sqrt{3y}$
i $8x\sqrt{5xy} + 10\sqrt{3xy} - 5x\sqrt{5y} - 24x\sqrt{3y}$

12 b Each term inside the brackets is multiplied by the term outside the brackets.

13 a $12 + 6\sqrt{3}$ b $8\sqrt{5} - 32$
c $20 - 30\sqrt{2}$ d $12\sqrt{2} + 8\sqrt{3}$
e $10\sqrt{6} - 6\sqrt{5}$ f $-10\sqrt{42} + 15\sqrt{70} - 10\sqrt{7}$
g $45\sqrt{2} + 12\sqrt{15}$ h $70 - 60\sqrt{7}$
i $12\sqrt{33} - 30\sqrt{77} + 66$ j $3\sqrt{5} - 10 + 25\sqrt{3}$
k $-36\sqrt{2} + 6\sqrt{30} - 81\sqrt{5}$

15 b Multiply each term in the second bracket by the expression outside the brackets; that is, $c(a + b) + d(a + b)$, then use the distributive law to expand each bracket.

16 a $10 + 20\sqrt{6} + 6\sqrt{3} + 36\sqrt{2}$
b $15\sqrt{10} - 24\sqrt{2} + 30\sqrt{5} - 48$
c $12\sqrt{3} + 15\sqrt{2} + 12\sqrt{6} + 30$
d $28\sqrt{10} + 56\sqrt{30} - 24 - 48\sqrt{3}$
e $24\sqrt{6} - 288 - 15\sqrt{15} + 90\sqrt{10}$
f $56\sqrt{5} - 48\sqrt{30} + 21\sqrt{10} - 36\sqrt{15}$

18 a $31 + 10\sqrt{6}$ b $16 - 6\sqrt{7}$ c $91 + 40\sqrt{3}$
d $57 - 28\sqrt{2}$ e $57 + 12\sqrt{15}$ f $159 - 24\sqrt{42}$
g $347 - 140\sqrt{6}$ h $270 + 180\sqrt{2}$

20 a 10 b 116 c 10 d -67
e -184 f -352 g -44 h 890

21 a Each term in the second bracket is exactly the same as the first except the sign between the terms in the second bracket is the opposite to the sign between the terms in the first bracket.
b Rational, because the surd terms add to zero each time.

22 a $180\sqrt{2} - 18\sqrt{5}$ cm^2 b $409 + 80\sqrt{6}$ m^2
c $(151 - 24\sqrt{7})\pi$ cm^2 d $(7 + 4\sqrt{2})\pi$ cm^2

23 a $(32\sqrt{10} + 15\sqrt{30} - 12\sqrt{6})$ m^2
b $(72\sqrt{5} - 40 + 10\pi)$ cm^2
c $(15 + 3\sqrt{3} + 2.5\sqrt{2} - 0.5\sqrt{6})$ m^2

24 a $(10\sqrt{10} + 16\sqrt{5})$ m b $(36 + 2\pi\sqrt{5})$ cm

25 a One possible set of dimensions could be $2\sqrt{5}$ m by $6\sqrt{3}$ m.
b Yes, the dimensions would be $\sqrt{5} + 3\sqrt{3}$m.

26 a, b One possible answer could be $2\sqrt{3}$ cm by $\sqrt{3}$ cm. (Look for two surd lengths that multiply to give $\sqrt{36}$ cm^2; that is, 6 cm^2.) The perimeter of this example would be $6\sqrt{3}$ cm.

3E Writing surd fractions with a rational denominator

3E Start thinking!

1 a 2; rational b $\sqrt{3}$; irrational
c Irrational. If any part of a number is irrational then the entire number is irrational.
d 1.1547

2 a $\frac{2\sqrt{3}}{3}$ b $2\sqrt{3}$, irrational
c 3, rational d 1.1547 e It is the same.

3 a i $\frac{1}{2}$ ii $\frac{3}{4}$ iii $\frac{3}{4}$ iv $\frac{2}{5}$
b The fraction remained unchanged.
c Each fraction is multiplied by 1.

4 a i $\frac{2}{\sqrt{3}}$ ii $\frac{6}{3\sqrt{3}}$ iii $\frac{2\sqrt{3}}{3}$ iv $\frac{4}{2\sqrt{3}}$
c part iii
d Multiplication of both numerator and denominator by the surd in the denominator.

Exercise 3E Writing surd fractions with a rational denominator

1 b, d, f

2 a $\sqrt{5}$ b $\sqrt{3}$ c $\sqrt{2}$
d $\sqrt{6}$ e $\sqrt{10}$ f $\sqrt{3}$

3 a $\frac{\sqrt{6}}{2}$ b $\frac{2\sqrt{7}}{7}$ c $\sqrt{3}$ d $\frac{4\sqrt{35}}{7}$
e $\frac{5\sqrt{2}}{6}$ f $\frac{7\sqrt{5}}{20}$ g $\frac{2\sqrt{15}}{5}$ h $-\frac{\sqrt{5}}{2}$
i $\frac{\sqrt{7}}{49}$ j $\frac{2\sqrt{110}}{5}$ k $\frac{4\sqrt{10}}{3}$ l $\frac{9\sqrt{30}}{20}$

4 a $\frac{2\sqrt{5}}{5}$ b $\frac{5\sqrt{3}}{3}$ c $\frac{3\sqrt{2}}{2}$ d $\frac{2\sqrt{3}}{3}$
e $\frac{\sqrt{10}}{2}$ f $\sqrt{3}$

5 **a** $\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{2\sqrt{15}}{3}$ **d** $2\sqrt{6}$
e $\frac{\sqrt{3}}{3}$ **f** $\frac{5\sqrt{5}}{7}$ **g** $\frac{4\sqrt{15}}{5}$ **h** $\frac{\sqrt{2}}{4}$
i $\frac{\sqrt{10}}{2}$ **j** $-\frac{7\sqrt{5}}{6}$ **k** $\frac{1}{3}$ **l** $\frac{3\sqrt{6}}{5}$

6 **a** $\sqrt{3}$ **b** $-\frac{\sqrt{10}}{2}$ **c** $\frac{4\sqrt{14}}{21}$ **d** $\sqrt{3}$
e $\frac{6\sqrt{5}}{7}$ **f** -6 **g** $\frac{5\sqrt{x}}{x}$ **h** $\frac{\sqrt{y}}{y}$
i $\frac{2x\sqrt{2y}}{y}$ **j** $\frac{\sqrt{xy}}{xy}$ **k** $\frac{\sqrt{2xy}}{2xy}$ **l** $-\frac{\sqrt{xyz}}{2xyz}$

7 **a** $12 + 4\sqrt{2}$ **b** $12 - 18\sqrt{3}$ **c** $5\sqrt{3} + 20$
d $\sqrt{10} - 2\sqrt{2}$ **e** $2\sqrt{15} + 6\sqrt{5}$ **f** $6\sqrt{14} - 28$

8 **a** 22 **b** rational
c They have the same terms but the opposite sign between the two terms.
d rational

9 **a** $3 - \sqrt{2}$ **b** $2 + \sqrt{3}$ **c** $5 - \sqrt{6}$
d $4 - 3\sqrt{5}$ **e** $3\sqrt{5} + 2$ **f** $4\sqrt{7} + 1$
g $1 - 2\sqrt{3}$ **h** $\sqrt{2} - \sqrt{3}$ **i** $\sqrt{5} + \sqrt{3}$
j $2\sqrt{5} - 5\sqrt{6}$ **k** $3\sqrt{7} + 2\sqrt{5}$ **l** $5\sqrt{2} + 2\sqrt{5}$

10 **a** 7 **b** 1 **c** 19
d -29 **e** 41 **f** 111
g -11 **h** -1 **i** 2
j -130 **k** 43 **l** 30

11 $(a + b)(a - b) = a^2 - b^2$

12 **a** $\frac{10 + 2\sqrt{3}}{11}$

13 **a** $2 - \sqrt{3}$ **b** $\frac{9 + 3\sqrt{2}}{7}$ **c** $\frac{30 - 5\sqrt{5}}{31}$
d $\frac{-9 - 12\sqrt{5}}{71}$ **e** $\frac{5\sqrt{2} - 6}{7}$ **f** $\frac{12 - 2\sqrt{3}}{11}$
g $\frac{3\sqrt{5} + 3}{4}$ **h** $-6\sqrt{42} + 15\sqrt{7}$ **i** $-4\sqrt{6} - 8$
j $\frac{\sqrt{15} - \sqrt{6}}{3}$ **k** $\frac{3\sqrt{30} - 10\sqrt{2}}{7}$ **l** $2\sqrt{5} + 2\sqrt{2}$

14 **a** $5\sqrt{2} - 2\sqrt{6} + 15 - 6\sqrt{3}$
b $4\sqrt{6} + 4 - 2\sqrt{30} - 2\sqrt{5}$
c $2\sqrt{21} + 2\sqrt{7} - 3\sqrt{6} - 3\sqrt{2}$
d $30\sqrt{3} - 5\sqrt{10} + 36\sqrt{2} - 4\sqrt{15}$
e $\sqrt{30} - 2\sqrt{5} + \sqrt{21} - \sqrt{14}$
f $12\sqrt{10} + 3\sqrt{30} - 8\sqrt{6} - 6\sqrt{2}$

15 **a** **i** $2\sqrt{3} - 3$ **ii** $\frac{2\sqrt{6} - 3\sqrt{2} + 2\sqrt{3} - 3}{3}$
b **i** $4 - \sqrt{2}$ **ii** $\frac{20 - 5\sqrt{2} + 8\sqrt{3} - 2\sqrt{6}}{14}$
c **i** $3 + \sqrt{5}$ **ii** $\frac{3 + \sqrt{5} + 3\sqrt{3} + \sqrt{15}}{4}$
d **i** $\sqrt{6} - 1$ **ii** $\frac{3\sqrt{6} - 3 + \sqrt{30} - \sqrt{5}}{5}$
e **i** $3\sqrt{5} - 2$ **ii** $\frac{6\sqrt{15} - 4\sqrt{3} - 3\sqrt{10} + 2\sqrt{2}}{41}$
f **i** $3\sqrt{3} - 2\sqrt{5}$ **ii** $\frac{15\sqrt{6} - 10\sqrt{10} - 3\sqrt{3} + 2\sqrt{5}}{7}$
g **i** $2\sqrt{7} + \sqrt{3}$ **ii** $\frac{8\sqrt{14} + 4\sqrt{6} - 6\sqrt{42} - 9\sqrt{2}}{25}$
h **i** $\sqrt{10} + \sqrt{7}$ **ii** $\frac{8\sqrt{5} + 4\sqrt{14} - 2\sqrt{15} - \sqrt{42}}{3}$
i **i** $2\sqrt{5} - \sqrt{3}$ **ii** $\frac{16\sqrt{15} - 41}{17}$

16 **a** $\frac{3\sqrt{2} + 2}{7}$ **b** $\frac{2\sqrt{35} + \sqrt{7}}{7}$ **c** $\frac{\sqrt{5} - 2\sqrt{15}}{15}$
d $\frac{8\sqrt{21} + 2\sqrt{7} - 12\sqrt{6} - 3\sqrt{2}}{47}$ **e** $\frac{18 - 5\sqrt{10}}{37}$
f $\frac{4 - \sqrt{7}}{3}$ **g** $17 - 12\sqrt{2}$ **h** -1
i $\frac{x^2 + x\sqrt{y}}{x^2 - y}$ **j** $\frac{x + \sqrt{xy}}{x - y}$
k $\frac{x^2\sqrt{x} - x\sqrt{y} + xy\sqrt{x} - y\sqrt{y}}{x^3 - y}$ **l** $\frac{x - 2\sqrt{xy} + y}{x - y}$

17 **a** $\frac{3\sqrt{2} + 1}{2}$ **b** $\frac{3\sqrt{2} + 1}{2}$

18 **a** $\frac{3\sqrt{5} + 1}{5}$ **b** $\frac{4\sqrt{3} - 1}{3}$ **c** $\frac{9\sqrt{2} + 4\sqrt{3}}{6}$
d $\frac{\sqrt{3} + 9\sqrt{2}}{12}$ **e** $\frac{9\sqrt{5} - \sqrt{6}}{9}$ **f** $\frac{12\sqrt{6} + 5\sqrt{15}}{60}$

19 **a** $\frac{\sqrt{5} + 1}{4} + \frac{3\sqrt{5} - 3}{4}$ **b** $\frac{2\sqrt{5} - 1}{2}$

20 **a** $2\sqrt{6}$ **b** $-\frac{4\sqrt{2}}{3}$
c $\frac{2\sqrt{6} + 5\sqrt{3} - 3}{5}$ **d** $5\sqrt{2} - 4\sqrt{5} - 13$
e $\frac{11\sqrt{6} - 4\sqrt{15} - 6}{6}$ **f** $\frac{16\sqrt{3} - 25}{13}$

3F Fractional indices

3F Start thinking!

1 **a** $a^m \times a^n = a^{m+n}$, when multiplying terms with the same bases, add the indices.
b $a^{\frac{5}{6}}$

2 **a** $a^m \div a^n = a^{m-n}$, when dividing terms with the same bases, subtract the indices.
b $a^{\frac{1}{6}}$

3 **a** $a(a^m)^n = a^{m \times n}$, when raising a term in index form to another power, multiply the indices.
b $a^{\frac{1}{6}}$

4 yes

Exercise 3F Fractional indices

1 **a** $x^{\frac{3}{5}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{7}{8}}$ **d** $x^{\frac{17}{30}}$
e x **f** $x^{\frac{11}{12}}$ **g** $x^{\frac{1}{20}}$ **h** $x^{\frac{1}{14}}$

2 **a** $x^{\frac{1}{5}}$ **b** 1 **c** $x^{\frac{3}{10}}$ **d** $x^{\frac{1}{18}}$

3 **a** $10x^{\frac{7}{12}}$ **b** $30x^{\frac{31}{40}}$ **c** $-36x^{\frac{41}{63}}$
d $4x^{\frac{1}{12}}$ **e** $2x^{\frac{9}{40}}$ **f** $4x^{\frac{1}{30}}$

4 **a** $5^{\frac{3}{5}}x^{\frac{1}{5}}$ **b** $2x^{\frac{1}{10}}$ **c** $9x^{\frac{1}{4}}$
d $2x^{\frac{1}{5}}y^{\frac{3}{8}}$ **e** $4x^{\frac{2}{15}}y^{\frac{2}{7}}$ **f** $5x^{\frac{1}{4}}y^{\frac{3}{10}}$

5 **a** $28x^{\frac{7}{12}}y^{\frac{19}{20}}$ **b** $-72x^{\frac{3}{2}}y$ **c** $45x^{\frac{19}{24}}y^{\frac{7}{5}}$
d $5x^{\frac{2}{5}}y^{\frac{1}{6}}$ **e** $2x^{\frac{7}{40}}y^{\frac{4}{33}}$ **f** $x^{\frac{1}{14}}y^{\frac{1}{35}}$

6 **a** $x^{\frac{1}{2}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{1}{2}}$ **d** $x^{\frac{1}{2}}$
e $6x^{\frac{5}{6}}$ **f** $\frac{1}{4}x^{\frac{1}{10}}$ **g** $25x^{\frac{3}{4}}$ **h** $\frac{3}{2}x^{\frac{13}{12}}$

7 **a** $a^m \times a^n = a^{m+n}$ **b** 3 **c** 3
d The answers are the same.
e **i** 2 **ii** 5 **iii** 10 **iv** x
f The fractional power of $\frac{1}{2}$ performs the same operation as $\sqrt{\ }$.

8 **a** 3 **b** 3
c The answers are the same.
d **i** 2 **ii** 5 **iii** 10 **iv** x
e $\sqrt[3]{\ }$ is a cube root and $\sqrt{\ }$ is a square root.
f The fractional power of $\frac{1}{3}$ performs the same operation as $\sqrt[3]{\ }$.

9 Some possible examples are: $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$, $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$.

12 **a** $\sqrt{x}$ **b** $\sqrt{x}$ **c** $\sqrt{x}$ **d** $\sqrt{x}$
e $6\sqrt[6]{x^5}$ **f** $\frac{1}{4}\sqrt[10]{x}$ **g** $25\sqrt[4]{x^3}$ **h** $\frac{3}{2}\sqrt[12]{x^{13}}$

13 **a** 5 **b** 6 **c** 1 **d** 12
e 10 **f** 14 **g** 2 **h** 3
i 5 **j** 1 **k** 1 **l** 2

14 **i** The answers would become the reciprocal of the given answers.
ii **a** $\frac{1}{5}$ **b** $\frac{1}{6}$ **c** 1 **d** $\frac{1}{12}$
e $\frac{1}{10}$ **f** $\frac{1}{14}$ **g** $\frac{1}{2}$ **h** $\frac{1}{3}$
i $\frac{1}{5}$ **j** 1 **k** 1 **l** $\frac{1}{2}$

15 **a** $\sqrt{5}$ **b** $2\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{3}$
e $2\sqrt{38}$ **f** $3\sqrt[3]{2}$ **g** $4\sqrt[3]{2}$ **h** $8\sqrt{5}$
i $9\sqrt{3}$ **j** $12\sqrt{3}$ **k** $5\sqrt{5}$ **l** $6\sqrt{6}$

16 **a** $x^{\frac{3}{2}}$ **b** $6^{\frac{1}{2}}$ **c** $x^{\frac{3}{2}}y^{\frac{5}{2}}$
d $3^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{2}}$ **e** $4x^{-\frac{1}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{4}{3}}y^{\frac{2}{3}}$

17 **a** $108x^{\frac{9}{20}}y^{\frac{13}{15}}$ **b** $\frac{1}{5}x^{\frac{2}{9}}y^{\frac{7}{20}}$ **c** $2x^{\frac{1}{5}}y^{\frac{1}{4}}$
d $6x^4y^2$ **e** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{1}{2}}y^{\frac{5}{6}}$
g $96^{\frac{1}{2}}x^2y^{\frac{1}{2}}z^4$ **h** $\left(\frac{2}{25}\right)^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}}$ **i** $x^4y^{\frac{2}{9}}z^{\frac{2}{3}}$

18 **a** $4x\sqrt{y}$ **b** $\frac{2y^3\sqrt{3y}}{x\sqrt{x}}$ **c** $xy\sqrt{3x}$
d $3\sqrt[3]{xy^2}$ **e** $2\sqrt[4]{x^3y^2z}$ **f** $\frac{7y^2\sqrt{7y}}{\sqrt{x}}$

19 **a** $18x^{\frac{5}{12}}y^{\frac{7}{15}}$ **b** $\frac{72y^{\frac{1}{12}}}{x^{\frac{5}{12}}}$ **c** $\frac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}$
d $\frac{1}{4x^{\frac{1}{15}}y^{\frac{7}{20}}}$ **e** $\frac{1}{x^{\frac{1}{12}}y^{\frac{3}{4}}}$ **f** $\frac{16}{x^{\frac{8}{5}}y^5}$
g $10x^{\frac{7}{30}}$ **h** $\frac{15}{2x^{\frac{11}{24}}y^{\frac{23}{12}}}$ **i** $\frac{y^{\frac{13}{30}}}{3^{\frac{1}{2}}x^{\frac{1}{24}}}$

20 **a** $\frac{3^{\frac{2}{3}}}{9}$ or $3^{-\frac{4}{3}}$ **b** $x = -\frac{2}{3}$

21 **a** $x = -\frac{5}{2}$ **b** $x = 11$ **c** $x = -\frac{1}{2}$

3G Understanding logarithms

3G Start thinking!

1 base = a, index = m
2 The logarithm of a number is the index to which the base is raised to equal that number.
3 **a** base = 2, index = 5, basic numeral = 32; $\log_2 32 = 5$
b $\log_{10} 10\,000 = 4$
c base, index, basic numeral
4 **a** The basic numeral is not given.
b Calculate the equivalent basic numeral.
c $\log_{10} 100 = 2$ **d** $\log_5 125 = 3$
5 **a** base = 10, logarithm = 3, basic numeral = 1000
b $10^3 = 1000$
c **i** $5^2 = 25$ **ii** $4^3 = 64$ **iii** $2^6 = 64$
d base, logarithm, basic numeral

Exercise 3G Understanding logarithms

1 **a** 3 **b** 2 **c** 2 **d** 1
e 6 **f** 4 **g** −1 **h** −1

2 **a** $\log_3 27 = 3$ **b** $\log_4 4 = 1$
c $\log_5 625 = 4$ **d** $\log_{10} 1000 = 3$
e $\log_6 36 = 2$ **f** $\log_9 729 = 3$
g $\log_3 81 = 4$ **h** $\log_2 128 = 7$

3 **a** **i** 2 **ii** 4 **iii** $\log_4 16 = 2$
b **i** 3 **ii** 5 **iii** $\log_5 125 = 3$
c **i** 2 **ii** 9 **iii** $\log_9 81 = 2$
d **i** 2 **ii** 10 **iii** $\log_{10} 100 = 2$
e **i** 2 **ii** 8 **iii** $\log_8 64 = 2$
f **i** 6 **ii** 2 **iii** $\log_2 64 = 6$
g **i** 4 **ii** 4 **iii** $\log_4 256 = 4$
h **i** y **ii** x **iii** $\log_x z = y$

4 **a** $2^5 = 32$ **b** $3^2 = 9$ **c** $5^2 = 25$
d $7^2 = 49$ **e** $4^2 = 16$ **f** $2^{10} = 1024$
g $6^3 = 216$ **h** $10^2 = 100$

5 **a** **i** 3 **ii** 3 **iii** $3^3 = 27$
b **i** 1 **ii** 4 **iii** $4^1 = 4$
c **i** 2 **ii** 6 **iii** $6^2 = 36$
d **i** 4 **ii** 2 **iii** $2^4 = 16$
e **i** 4 **ii** 5 **iii** $5^4 = 625$
f **i** 5 **ii** 3 **iii** $3^5 = 243$
g **i** 3 **ii** 7 **iii** $7^3 = 343$
h **i** c **ii** a **iii** $a^c = b$

6 **a** $3^m = 27$ **b** 3

7 **a** 4 **b** 4 **c** 3 **d** 3

8 **a** $\frac{1}{3}$ **b** $\frac{1}{5}$ **c** 2 **d** 10 **e** $\frac{1}{4}$ **f** 27
g $\frac{1}{64}$ **h** $\frac{1}{x}$ **i** $\frac{x^2}{y}$ **j** 25 **k** $\frac{1}{100}$ **l** $\frac{1}{10\,000}$

9 **a** **i** $\frac{1}{6}$ **ii** $\log_6 \frac{1}{6} = -1$
b **i** $\frac{1}{4}$ **ii** $\log_4 \frac{1}{4} = -1$
c **i** $\frac{1}{4}$ **ii** $\log_2 \frac{1}{4} = -2$
d **i** $\frac{1}{32}$ **ii** $\log_2 \frac{1}{32} = -5$

e i $\frac{1}{25}$ ii $\log_5 \frac{1}{25} = -2$
f i $\frac{1}{10}$ ii $\log_{10} \frac{1}{10} = -1$
g i $\frac{1}{1000}$ ii $\log_{10} \frac{1}{1000} = -3$
h i $\frac{1}{27}$ ii $\log_3 \frac{1}{27} = -3$
i i 2 ii $\log_{\frac{1}{2}} 2 = -1$
j i 9 ii $\log_{\frac{1}{3}} 9 = -2$
k i 125 ii $\log_{\frac{1}{5}} 125 = -3$
l i 100 ii $\log_{0.01} 100 = -1$

10 a i 1 ii 2 iii 3 iv 4
11 a 2 b 3 c 2 d 5
e 4 f 3 g 2 h 4
i 3 j 3 k 2 l 5
m −3 n −3 o −2 p −1
12 a i 1 ii 1 iii 1
iv 1 v 1 iv 1
b All equal 1.
c The logarithm of any number to the same base is 1.
13 a i 0 ii 0 iii 0 iv 0 v 0 vi 0
b All of the questions equal 0.
c The logarithm of 1 to any base is 0.
14 a 1 b 0 c 1 d 1 e 0 f 0
15 a no
b i no ii no iii no iv no v no vi no
c There is no logarithm of a negative number to any base.
16 a no
b i no ii no iii no iv no v no vi no
c There is no logarithm of zero to any base.
17 a no
b i no ii no iii no iv no v no vi no
c There is no logarithm of any number to base 1.
18 $\log_a(-x)$ does not exist, $\log_a 0$ does not exist, $\log_1 x$ does not exist.
19 a $x^2 = 64$ b 8 d It is the same.
20 a 2 b 5 c 4 d 2
e 2 f 3 g 4 h 2
21 a $4^3 = x$ b 64
c It can be easily solved by calculating the index term.
22 a 27 b 25 c 6 d 10 000
e 32 f 64 g $\frac{1}{27}$ h $\frac{1}{4}$
23 $\log_b x = \frac{\log_a x}{\log_a b}$
24 $\log_1 1$ is not defined.
25 $\log_1 1 = \frac{\log_a 1}{\log_a 1} = \frac{0}{0}$ (not defined because you cannot have 0 in the denominator of a fraction)
26 a 1.61 b 2.30 c 0.69
d −0.69 e 4.61 f 0
g undefined h 1.00

3H Working with logarithms

3H Start thinking!

1 a $\log_a x = m$, $\log_a y = n$ c $\log_a xy = m + n$
2 a $\log_a x = m$, $\log_a y = n$ c $\log_a\left(\frac{x}{y}\right) = m - n$
3 b $\log_a x^n = m \times n$

Exercise 3H Working with logarithms

1 a 1 b 1 c 1 d 4
3 f 0 g 0 h 0
2 a 2 b 1 c 0 d 0
e $\frac{1}{2}$ f −2 g −1 h 1
3 a $\log_{10} 3 + \log_{10} 5$ b $\log_2 6 + \log_2 3$
c $\log_5 2.5 + \log_5 10$ d $\log_2 x + \log_2 y$
4 a $\log_3 12$ b $\log_{10} 15$ c $\log_2 50$
d $\log_4 72$ e $\log_a 15$ f $\log_2 xy$
5 a $\log_{10} 5 - \log_{10} 7$ b $\log_2 5 - \log_2 3$
c $\log_5 11 - \log_5 20$ d $\log_2 a - \log_2 b$
6 a $\log_3 \frac{12}{5}$ b $\log_{10} \frac{50}{15}$ c $\log_2 \frac{25}{10}$
d $\log_4 \frac{36}{9}$ e $\log_a \frac{3}{15}$ f $\log_2 \frac{x}{y}$
7 a 4 b 2 c 5 d 4
e 2 f 3 g 2 h 2
i 9 j 2 k 2 l 4
8 a 2 b 2 c 3
9 i a $\log_3 35$ b $\log_4 6$ c $\log_6 7$ d $\log_2 60$
ii a 3.24 b 1.29 c 1.09 d 5.91
10 a $2 \times \log_3 7$ b $9 \times \log_5 2$ c $3 \times \log_4 8$
d $4 \times \log_9 11$ e $x \times \log_6 5$ f $k \times \log_2 100$
11 a $\log_5 3^4$ b $\log_4 5^8$ c $\log_2 12^2$
d $\log_3 2^7$ e $\log_6 9^y$ f $\log_{10} x^6$
12 a 5 b 2 c 5 d 8 e 9
f 12 g 10 h 2 i 21
13 a $\log_3 16$ b 4 c 4
d $\frac{1}{2}$ e $1 - \log_3 5$ f $\log_2 x$
14 a 3 b 2 c 2 d −1 e −2 f 3
15 a log of a power law
b i $\log_9 81 = 2$ ii $\log_3 9 = 2$
16 a 3 b 2 c 6 d 4 e 1
f 7 g 5 h 3 i 3 j 3
17 a 3 b 3 c 9 d 4
e 216 f 64 g −3 h 36
i 1 j $\frac{1}{2}$ k $\frac{1}{2}$ l 4
18 b i 1.176 ii −0.222 iii 0.222
iv 1.255 v 1.477 vi 0.079
19 a $\frac{1}{2}$
20 Hailey, since $\frac{\log_3 8}{\log_3 2} = \frac{3\log_3 2}{\log_3 2}$
21 a 4 b $\frac{3}{2}$ c 3 d $\frac{3}{4}$ e $\frac{2}{3}$ f 3
22 a ii $5^0 = \frac{x}{2}$, $x = 2$
23 a 3 b 4 c 20
d $\frac{2}{3}$ e $\frac{1}{14}$ f $8333\frac{1}{3}$
24 $x = 4$ or $x = 5$

25 $(2x + 1)\log_{10}2 = \log_{10}2^2$
$(2x + 1)\log_{10}2 = 2\log_{10}2$
$2x + 1 = 2$
$2x = 1$
$x = \frac{1}{2}$

26 $y = 77$

3 Chapter review

MULTIPLE-CHOICE

1 A 2 C 3 D 4 C 5 B
6 B 7 C 8 A 9 B

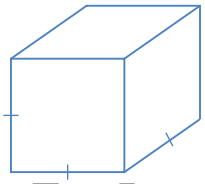
SHORT ANSWER

1 a rational b irrational c rational
d irrational e irrational f rational
2 a, b, d, f
3 a 4.899 b 0.059
4 a $\frac{13}{20}$ b $\frac{87}{400}$ c $\frac{47}{90}$ d $1\frac{23}{99}$
5 a $\sqrt{77}$ b $27\sqrt{30}$
6 a $\sqrt{3}$ b $4\sqrt{3}$
7 a $4\sqrt{6}$ b $15\sqrt{5}$ c $3\sqrt{3}$
8 a $72\sqrt{2}$ b -150
9 a $24\sqrt{3}$ m^2 b $90\sqrt{2}$ cm^2
10 a $-2\sqrt{2}$ b $2\sqrt{7} + 5\sqrt{5}$
c $2\sqrt{10} + 4\sqrt{5}$ d $3\sqrt{6} + 2 + 3\sqrt{5}$
11 a $2\sqrt{5} - 22\sqrt{6}$ b $4\sqrt{3} + 6\sqrt{2}$ c $15\sqrt{5}$
d $6\sqrt{2} + 24\sqrt{5} + 4\sqrt{10}$
12 a $6 - 8\sqrt{2} + 12\sqrt{3} - 16\sqrt{6}$ b $92 + 16\sqrt{15}$
13 a $\frac{5\sqrt{6}}{12}$ b $\frac{3\sqrt{3}}{2}$ c $\frac{8\sqrt{30} - 4\sqrt{15}}{21}$
d $\frac{-6\sqrt{7} - 10\sqrt{3} - 3\sqrt{42} - 15\sqrt{2}}{12}$
14 a i $x^{\frac{3}{2}}$ ii $\sqrt{x^3}$
b i $x^{\frac{5}{6}}$ ii $\sqrt[6]{x^5}$
c i $\frac{1}{9}x^{\frac{5}{12}}$ ii $\frac{1}{9}\sqrt[12]{x^5}$
d i $24^{\frac{1}{2}}x^2y^{\frac{3}{2}}$ ii $2x^2y\sqrt{6y}$
e i $3x^{\frac{1}{8}}y^{\frac{1}{5}}$ ii $3\sqrt[8]{x}\sqrt[5]{y}$
f i $\left(\frac{1}{5}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}}$ ii $\frac{1}{5}\sqrt{\frac{3x^3y}{8z}}$
15 a $\log_3 81 = 4$ b $\log_6 216 = 3$
c $\log_2 32 = 5$ d $\log_x w = y$
16 a $2^3 = 8$ b $4^3 = 64$
c $10^{-4} = \frac{1}{1000}$ d $x^b = z$
17 a 3 b 2 c 3 d 3
18 a 2 b $\frac{1}{125}$ c 4 d $\frac{1}{2}$ e $\frac{12}{5}$ f $\frac{625}{4}$

MIXED PRACTICE

1 C 2 1.385 3 B 4 D
5 D 6 $-4\sqrt{42}$ 7 $\frac{12}{5}$ 8 C
9 C 10 $-32\sqrt{15}$ 11 $\frac{9\sqrt{6}}{8}$
12 a $5\sqrt{5}$ m b $4\sqrt{10}$ cm
13 $3\sqrt{5}$ cm
14 C 15 $5\sqrt{3} + 2\sqrt{5} + 2$ 16 $27\sqrt{5}$
17 B 18 $89 - 24\sqrt{5}$
19 C 20 $(96 - 32\sqrt{2} + 4\sqrt{10} + 6\sqrt{3})$ m^2
21 $\frac{5\sqrt{3}}{3}$ 22 $3 + 2\sqrt{7}$ 23 D 24 D
25 A 26 $4\sqrt{6}$ 27 B 28 A
29 5 30 B 31 $\frac{1}{16}$ 32 B

ANALYSIS

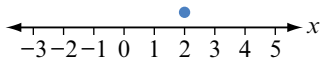
a 
$(7\sqrt{10} + 11\sqrt{7}$ cm
b $(1337 + 154\sqrt{70})$ cm^2 c irrational
d $(8022 + 924\sqrt{70})$ cm^2
e \$20.16 correct to nearest cent
g 94.5% correct to 1 decimal place
h diagonal of base 72.5 cm
i Yes, diagonally from bottom corner to opposite top corner.
j Interior measurements are the same as exterior.

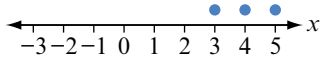
3 Connect

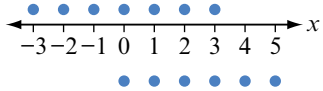
For feedback on this open-ended task, see your teacher.

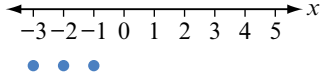
CHAPTER 4 LINEAR RELATIONSHIPS

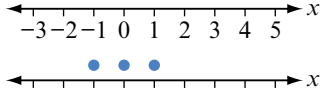
4 Are you ready?

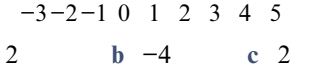
1 a $x = 13$ b $x = -8$ c $x = 5$ d $x = -45$
2 a 11 b -9 c -12 d 5
3 a $4x + 28$ b $-15x + 6$
4 a $11x + 8$ b $2 - 3x$ c $5x - 2$ d $3x - 9$
5 a $\frac{1}{4}$ b $\frac{1}{2}$ c $\frac{13}{10}$ or $1\frac{3}{10}$ d $\frac{1}{6}$
6 a true b true c true d false
7 a 

b 

c 

d 

e 

f 

8 a 2 b -4 c 2
9 a i $c = -4, m = 3$ ii $c = 2, m = -1$
iii $c = 0, m = 7$ iv $c = -1, m = -\frac{2}{3}$
b i rise = 3, run = 1 ii rise = -1, run = 1
iii rise = 7, run = 1 iv rise = -2, run = 3

10 **a** x-intercept = 1, y-intercept = 4
b x-intercept = 9, y-intercept = 3
c x-intercept = 10, y-intercept = −5
d x-intercept = −3, y-intercept = −6

11 **a** $6x + 3y = 12$ **b** $2x - 6y = 2$
c $-x + 8y = -6$ **d** $-16x + 20y = 0$

4A Solving linear equations

4A Start thinking!

1 a, c, f and g are linear equations

2 **a** subtract 9 **b** $x = 7$

3 **a** **i** add 4 **ii** divide by 3
b You would get an incorrect solution if you did not perform the 'undo' operations in the reverse order to that used to form the equation.
c $3x - 4 = 2$
$3x - 4 + 4 = 2 + 4$
$3x = 6$
$\frac{3x}{3} = \frac{6}{3}$
$x = 2$
d $3x - 4 = 2$
$3x = 6$
$x = 2$
e Substitute the solution into the original equation to check left side equals right side.
LS = 3 × 2 − 4 = 2 = RS

4 **a** multiply by 5 then add 2; $x = -18$
b subtract 7 then divide by −4; $x = -2$

Exercise 4A Solving linear equations

1 **a** $x = 5$ **b** $x = 1$ **c** $x = 15$ **d** $x = -5$
e $x = 8$ **f** $x = 10$ **g** $x = 6$ **h** $x = -4$
i $x = 9$ **j** $x = -7$ **k** $x = 3$ **l** $x = 2$
m $x = 8$ **n** $x = -1$ **o** $x = 0$

2 **a** $x = 10$ **b** $x = -4$ **c** $x = 5$ **d** $x = -8$
e $x = 7$ **f** $x = 6$ **g** $x = 8$ **h** $x = -2$
i $x = 3$

3 **a** $x = -8\frac{1}{2}$ **b** $x = 1.7$ **c** $x = 4$
d $x = -2.464$ **e** $x = 2.4$
f $x = -9\frac{1}{2}$ or −9.5

4 **a** $x = -5$ **b** $x = 3$ **c** $x = -2$ **d** $x = 6$
e $x = -\frac{1}{3}$ **f** $x = 7$ **g** $x = 3$ **h** $x = \frac{1}{9}$
i $x = 6$

5 **a** $x = 2$ **b** $x = -5$ **c** $x = -6$ **d** $x = -4$
e $x = 4$ **f** $x = 7.5$ **g** $x = 3\frac{1}{3}$ **h** $x = \frac{2}{35}$
i $x = 2$ **j** $x = -1$ **k** $x = \frac{1}{4}$ **l** $x = -2\frac{2}{5}$

6 **a** $x = 2$ **b** $x = 5$ **c** $x = -3$ **d** $x = 1$
e $x = -7$ **f** $x = -2$ **g** $x = 3$ **h** $x = 4$
i $x = \frac{3}{7}$ **j** $x = -4$ **k** $x = 2$ **l** $x = -5$

7 **a** $x = 5$ **b** $x = -2$ **c** $x = 3$ **d** $x = -1$
e $x = 6$ **f** $x = 2$ **g** $x = -4$ **h** $x = 8$
i $x = 4$ **j** $x = -1$ **k** $x = 5$ **l** $x = 9$

8 **a** $x = -8$ **b** $x = 4$ **c** $x = -2$ **d** $x = 2$
e $x = -1$ **f** $x = 4$

9 **a** $x = -8$ **b** $x = 31$ **c** $x = 8$ **d** $x = 1$
e $x = 17$ **f** $x = 13$ **g** $x = -4\frac{2}{3}$ **h** $x = 6\frac{1}{7}$
i $x = -2\frac{1}{8}$

10 **a** $x = b - a$ **b** $x = k + p$ **c** $x = \frac{d}{c}$
d $x = \frac{h - g}{3}$ **e** $x = \frac{y - 5}{4}$ **f** $x = mn - a$
g $x = \frac{5 + 2w}{7}$ **h** $x = a(b - c)$ **i** $x = \frac{h}{k}$
j $x = \frac{e + f}{2}$ **k** $x = \frac{v - 2y}{2}$ **l** $x = p(w - q)$
m $x = \frac{a}{b + c}$ **n** $x = \frac{2ny + m}{k}$ **o** $x = \frac{5c - 3d}{8}$

11 **a** let n = number of weeks of saving
b $24n + 105 = 219$ **c** $n = 4.75$
d It will take Tom 5 weeks to save for his new tennis racquet.

12 **a** \$89.50 **b** length: 28 m, width: 16 m
c Lisa: 14 goals, Nicole: 9 goals, Tania: 6 goals
d 51 people

13 **a** $6d + 3.8 = 4d + 9.2$, where d is cost of a dumpling
b \$2.70

14 **a** $A = 50\ \text{m}^2$ **b** $h = 7$ mm **c** $a = 7$ m
d $a = \frac{2A}{h} - b$
e **i** $a = 52$ cm **ii** $a = 7.6$ m

15 **a** $v = 34$ m/s **b** $t = 5.5$ s

16 15, 17, 19, 21, 23

17 $n - 25 - \frac{n - 25}{3} = 40$, where n is number of cupcakes made; 25 chocolate, 20 lemon, 40 without icing and 85 in total

18 **a** $x = -6$ **b** $x = 8$ **c** $x = -1$ **d** $x = 13$
e $x = \frac{1}{8}$ **f** $x = -2\frac{7}{10}$

4B Solving linear inequalities

4B Start thinking!

1 < (less than), ≤ (less than or equal to), > (greater than), ≥ (greater than or equal to)

2 **a** > **b** < **c** < **d** >

3 **a** x is greater than 2
b x is less than or equal to 3
c x is greater than or equal to 5
d x is less than −4

4 Each inequality represents a region on the number line and not a single point.

5 Some possible answers are given.
a 7, 30.5, 122 **b** 4, 0, −5
c −3, −1, 10.6 **d** −5, −10, −36.4

6 x is greater than 2, and less than 9. Some possible values are: 4, 5.7, 8.2.

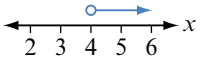
7 −4, −3, −2, −1, 0, 1, 2

Exercise 4B Solving linear inequalities

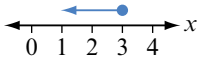
1 **a** 2.4, 3, 7, 8.3, $6\frac{4}{5}$
b −5, −1.2, 0, $-\frac{3}{4}$, −10, −4.9
c 7, 8.3, $6\frac{4}{5}$ **d** −5, −10, −4.9
e 2.4, 3, 7, $6\frac{4}{5}$ **f** 2.4, $\frac{1}{2}$, 3, $6\frac{4}{5}$
g −5, 2.4, −1.2, $\frac{1}{2}$, 0, $-\frac{3}{4}$, −4.9
h −5, −1.2, $-\frac{3}{4}$, −4.9

2 **a** $x \leq 5$ **b** $x > 1$ **c** $x \geq -4$
d $x < -2$ **e** $3 \leq x \leq 7$ **f** $-5 < x \leq 0$

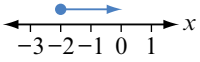
3 **a** $x > 4$



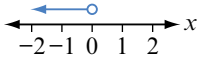
b $x \leq 3$



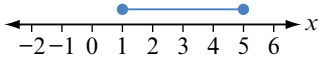
c $x \geq -2$



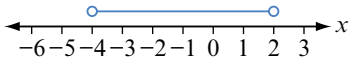
d $x < 0$



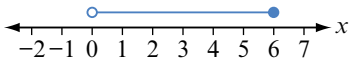
e $1 \leq x \leq 5$



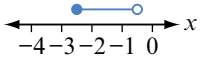
f $-4 < x < 2$



g $0 < x \leq 6$



h $-2.5 \leq x < -0.5$



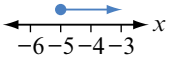
4 **a** **i** $3 < 7$; true **ii** $5 < 9$; true
iii $9 < 13$; true **iv** $-2 < 2$; true
b **i** $1 < 5$; true **ii** $-1 < 3$; true
iii $-5 < -1$; true **iv** $6 < 10$; true
c **i** $4 < 12$; true **ii** $-4 < -12$; not true
iii $6 < 18$; true **iv** $-2 < -6$; not true
d **i** $1 < 3$; true **ii** $-1 < -3$; not true
iii $\frac{2}{3} < 2$; true **iv** $-2 < -6$; not true
e Inequality sign stays the same when adding or subtracting any number. Also stays the same when multiplying or dividing by a positive number. However, direction of inequality sign is reversed when multiplying or dividing by a negative number.
f Some possible answers are given.
i $4 > -8$, $4 + 2 > -8 + 2$ $(6 > -6)$, $4 - 1 > -8 - 1$ $(3 > -9)$
ii $4 > -8$, $4 \times 2 > -8 \times 2$ $(8 > -16)$, $4 \div 2 > -8 \div 2$ $(2 > -4)$
iii $4 > -8$, $4 \times -2 < -8 \times -2$ $(-8 < 16)$, $4 \div -2 < -8 \div -2$ $(-2 < 16)$

5 **a** $x + 3 > 8$ **b** $-2x \geq 8$ **c** $x \leq 10$
d $x < 4$ **e** $-x \geq -9$ **f** $x > -16$
g $x \geq 7$ **h** $-x > 3$ **i** $-6x > -5$

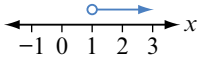
6 **a** $x < 5$ **b** $x \geq 7$ **c** $x \leq -6$
d $x > 6$ **e** $x \leq -5$ **f** $x < 50$
g $x \geq -20$ **h** $x \leq 2$ **i** $x > -5$

7 **a** $x \geq -6$ **b** $x < -2$ **c** $x \leq 3$
d $x > -24$ **e** $x < -7$ **f** $x \geq 1$
g $x > -24$ **h** $x < 7$ **i** $x \leq 10$

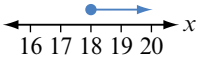
8 **a** $x \geq -5$



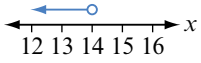
b $x > 1$



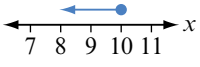
c $x \geq 18$



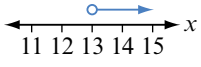
d $x < 14$



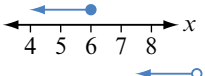
e $x \leq 10$



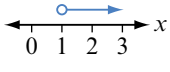
f $x > 13$



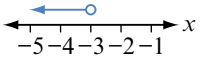
g $x \leq 6$



h $x > 1$



i $x < -3$



9 **a** $x > 4$ **b** $x \leq -2$ **c** $x \geq 3$
d $x > 4$ **e** $x < 1$ **f** $x \geq -1$
g $x \leq 5$ **h** $x < 2$ **i** $x > \frac{1}{2}$

10 **a** $x > 11$ **b** $x \leq 13$ **c** $x \geq -2$
d $x > -19$ **e** $x > -15$ **f** $x \leq 1$
g $x \leq 3$ **h** $x > -\frac{1}{4}$ **i** $x \geq -6.5$

11 **a** $x \leq -7$ **b** $x > 1$ **c** $x \leq -3$
d $x < 1$ **e** $x \geq 4$ **f** $x > -5$

12 **a** one
b Linear inequalities can have many solutions within a given range.

13 **a** $p \geq 650\,000$, where p is selling price in $
b $h > 97$, where h is height of a person in cm
c $s \leq 60$, where s is speed in km/h
d $125 \leq h \leq 196$, where h is height of a person in cm

14 **a** $3p \leq 20$, where p is cost of pack of gum in $; $p \leq 6\frac{2}{3}$
b Todd could buy 1, 2, 3, 4, 5 or 6 packs of gum.

15 **a** $m = \frac{1}{2}(20 - x)$, where m is number of watermelons they will each take home.
b $\frac{1}{2}(20 - x) \leq 3$; $x \geq 14$
c 14 or more watermelons

16 **a** 25 min **b** 16 min **c** 19 min to 29 min

17 **a** $x \geq -11$ **b** $x < 4$ **c** $x \geq 2$ **d** $x < -8$
e $x \leq 2$ **f** $x \leq -3$

18 **a** $1 \leq x \leq 6$ **b** $-4 > x > -7$

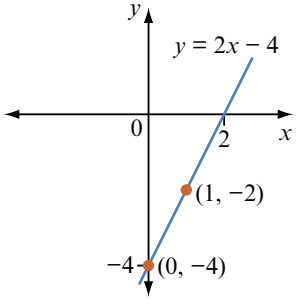
4C Sketching linear graphs

4C Start thinking!

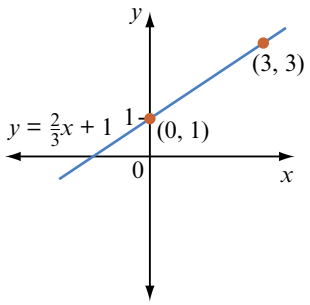
1 The graph of a linear relationship is a straight line.
2 A straight line connects two points, and this will define the linear relationship.
3 **a** **i** $m = 4$, $c = 1$ **ii** $m = -4$, $c = 1$
iii $m = \frac{1}{4}$, $c = 1$
b **i** (0, 1) **ii** (0, 1) **iii** (0, 1)
c **i** 4 **ii** -4 **iii** $\frac{1}{4}$
d **i** rise = 4, run = 1 **ii** rise = -4, run = 1
iii rise = 1, run = 4
e **i** $y = 4x + 1$ matches graph C
ii $y = -4x + 1$ matches graph A
iii $y = \frac{1}{4}x + 1$ matches graph B
4 Starting at y-intercept, use rise and run to locate a second point. Ruling a straight line passing through these two points will represent the graph of the linear relationship.

Exercise 4C Sketching linear graphs

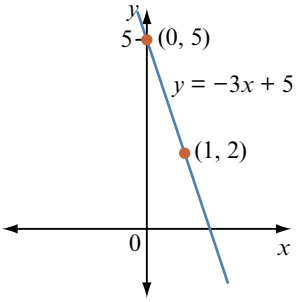
1 **a**



b



c



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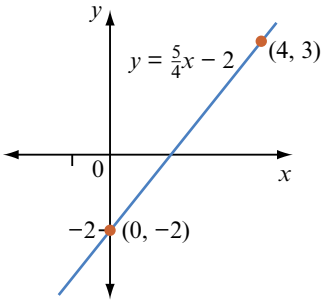

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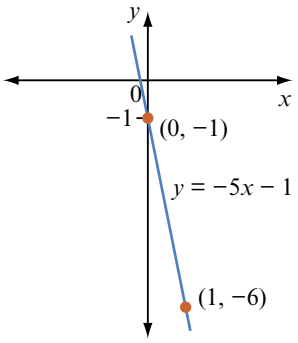

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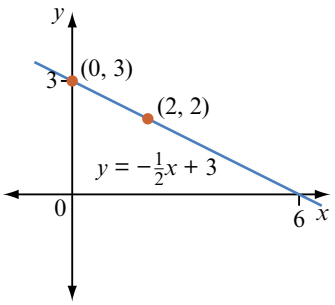

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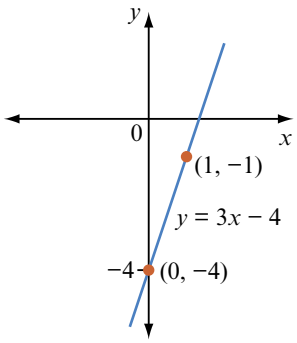

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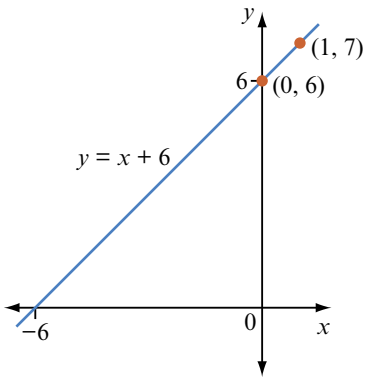

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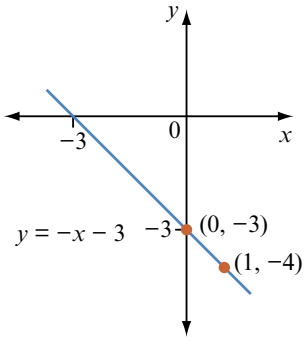


2 **a** $y = 2x + 1$, $m = 2$, $c = 1$

b $y = -3x + 4$, $m = -3$, $c = 4$

c $y = -\frac{3}{4}x - 3$, $m = -\frac{3}{4}$, $c = -3$

d $y = 3x - 6$, $m = 3$, $c = -6$

e $y = -2x + 6$, $m = -2$, $c = 6$

f $y = -5x - 2$, $m = -5$, $c = -2$

g $y = -3x + 2$, $m = -3$, $c = 2$

h $y = 4x - 5$, $m = 4$, $c = -5$

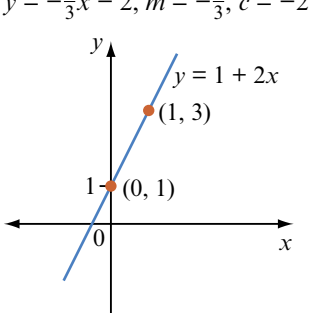
i $y = -2x + 3$, $m = -2$, $c = 3$

j $y = \frac{4}{5}x + 3$, $m = \frac{4}{5}$, $c = 3$

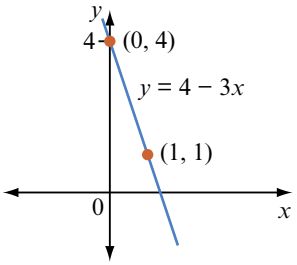
k $y = \frac{3}{2}x + 2$, $m = \frac{3}{2}$, $c = 2$

l $y = -\frac{1}{3}x - 2$, $m = -\frac{1}{3}$, $c = -2$

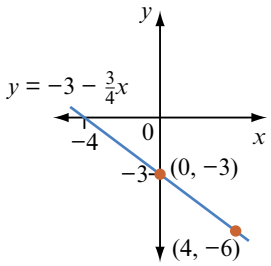
3 **a**



b



c



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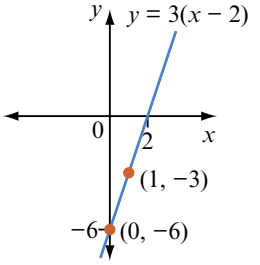

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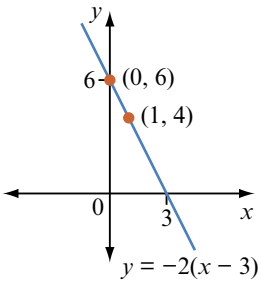

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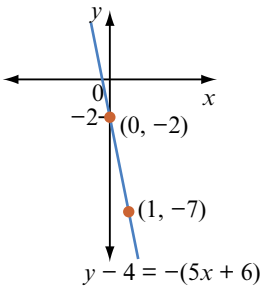

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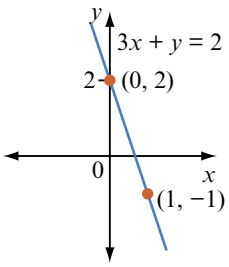

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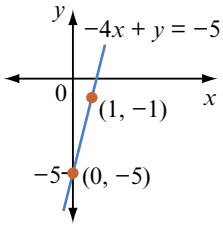

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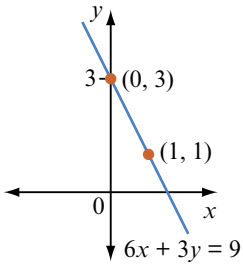

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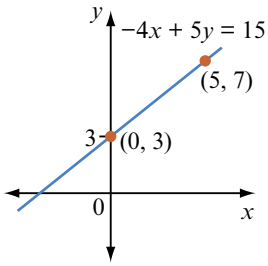

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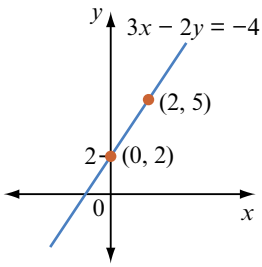

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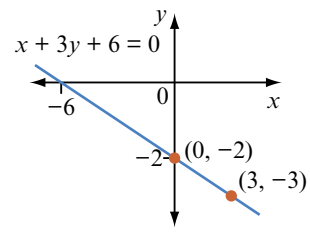
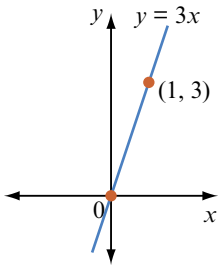


4 a $y = 3x + 0$

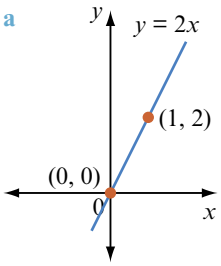
b **i** $m = 3$

ii $c = 0$

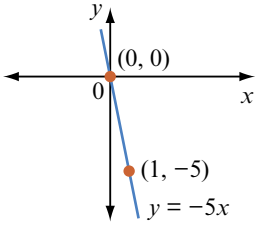
c



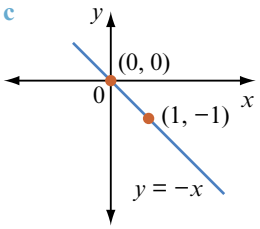
5 a



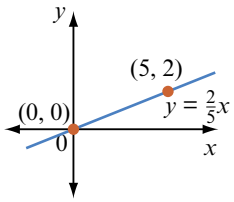
b



c



d



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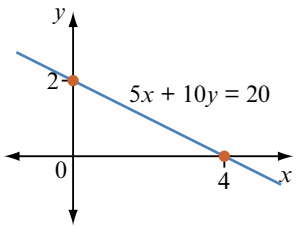
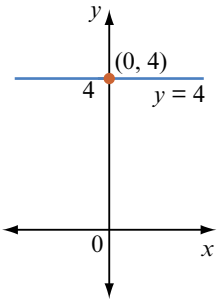


6 a $y = 0x + 4$

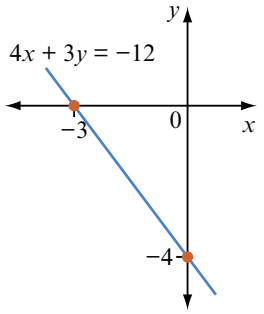
b **i** $m = 0$

ii $c = 4$

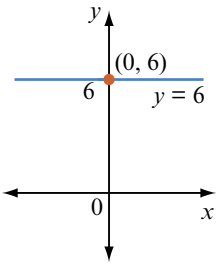
c



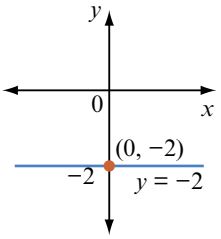
e



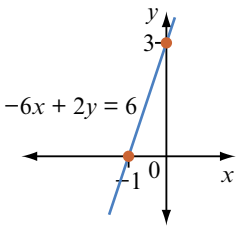
7 a



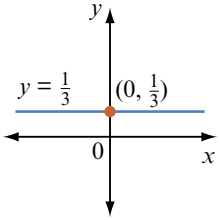
b



f



c



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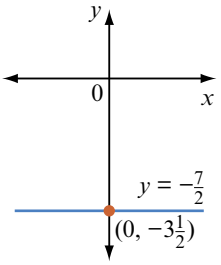
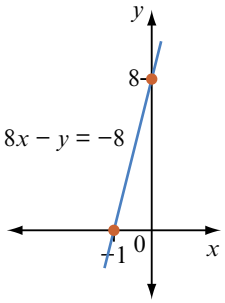
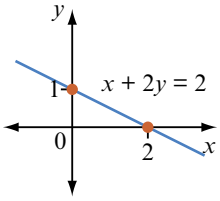


g

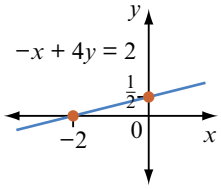


8 The rule $x = 4$ cannot be written in the form $y = mx + c$ since there is no y value specified.

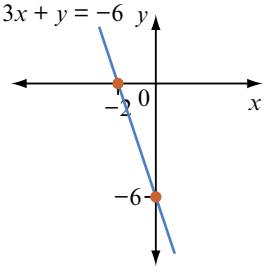
9 a



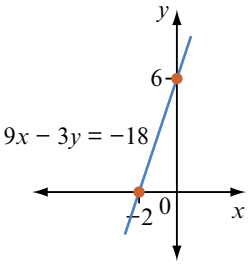
h



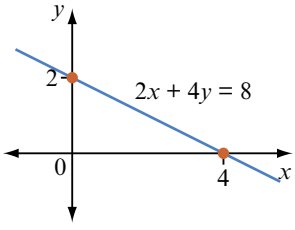
b $3x + y = -6$



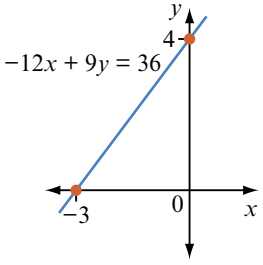
i



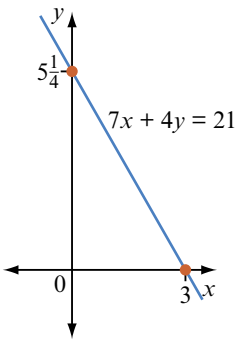
c



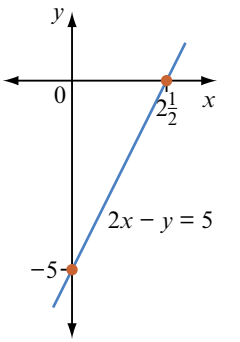
j



k

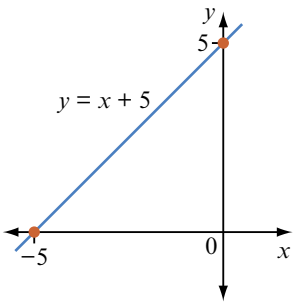


l

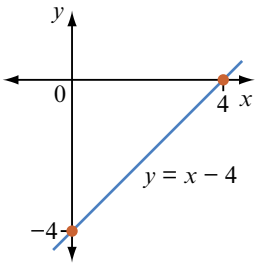


10 **a** x-intercept: $(-5, 0)$, y-intercept: $(0, 5)$
b x-intercept: $(4, 0)$, y-intercept: $(0, -4)$
c x-intercept: $(-2, 0)$, y-intercept: $(0, 6)$
d x-intercept: $(4, 0)$, y-intercept: $(0, -8)$
e x-intercept: $(2, 0)$, y-intercept: $(0, 2)$
f x-intercept: $(-\frac{1}{2}, 0)$, y-intercept: $(0, -3)$

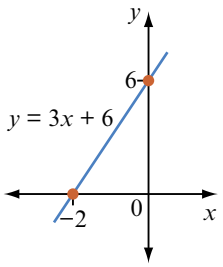
11 **a**



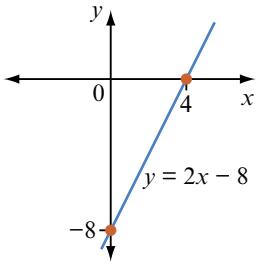
b



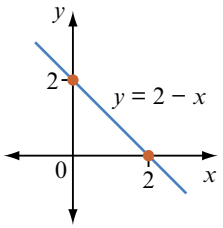
c



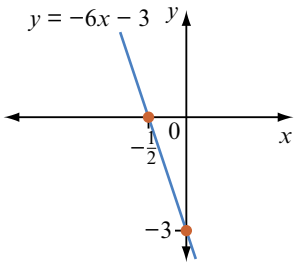
d



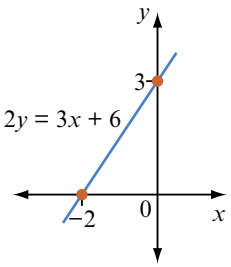
e



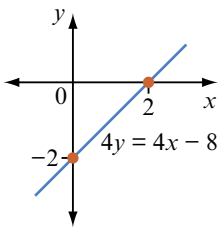
f



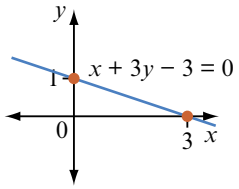
12 **a**



b



c



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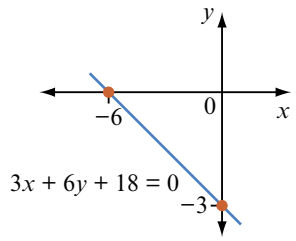
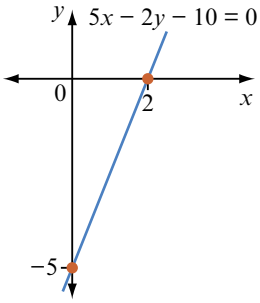
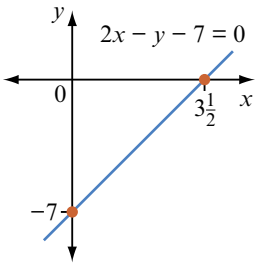


e



f



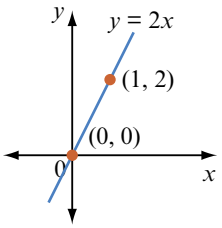
13 a i x-intercept: (0, 0) ii y-intercept: (0, 0)

b no; only have one point

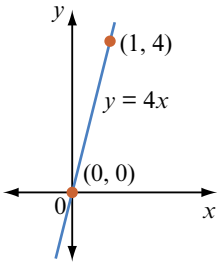
c coordinates of one other point on the line

d $y = 2$ **e** (0, 0) and (1, 2)

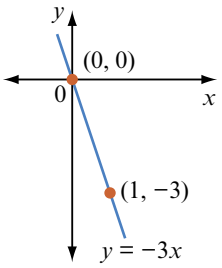
f



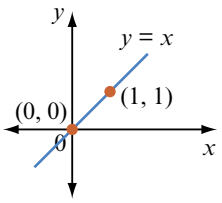
14 a



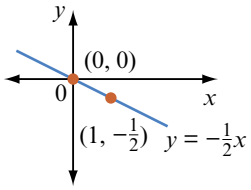
b



c

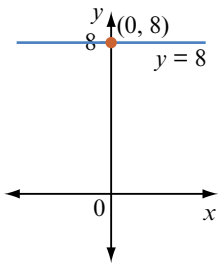


d



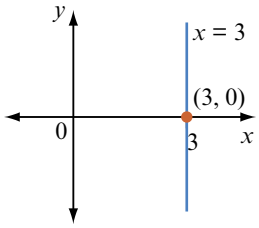
15 a i y-intercept: (0, 8), no x-intercept

ii



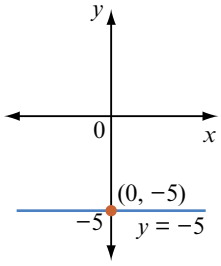
b i x-intercept: (3, 0), no y-intercept

ii



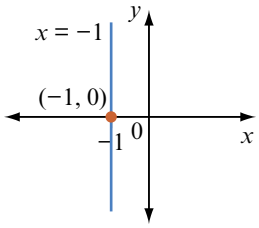
c i y-intercept: (0, −5), no x-intercept

ii



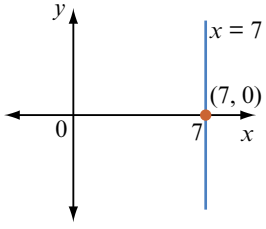
d i x-intercept: (−1, 0), no y-intercept

ii



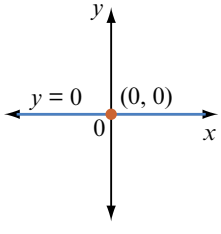
e i x-intercept: (7, 0), no y-intercept

ii

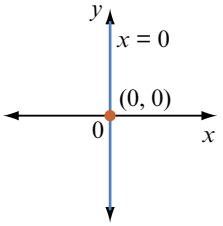


f i y-intercept: (0, 0), no x-intercept

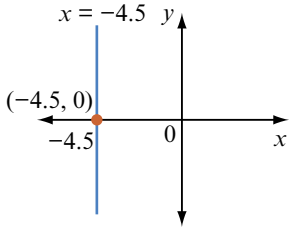
ii



g i x-intercept: (0, 0), no y-intercept

ii 

h i x-intercept: (−4.5, 0), no y-intercept

ii 

16 A linear graph can have one or two axis intercepts (or perhaps lie along the x-axis or y-axis).

a x-intercept: (2, 0), y-intercept: (0, −4), two intercepts

b no x-intercept, y-intercept: (0, −7), one intercept

c x-intercept: (3, 0), y-intercept: (0, 2), two intercepts

d x-intercept: (0, 0), y-intercept: (0, 0), one intercept

e x-intercept: (10, 0), y-intercept: (0, −50), two intercepts

f x-intercept: (−5, 0), y-intercept: (0, 3), two intercepts

g x-intercept: (28, 0), no y-intercept, one intercept

h x-intercept: (6, 0), y-intercept: (0, −2), two intercepts

i no x-intercept, y-intercept: (0, 5.6), one intercept

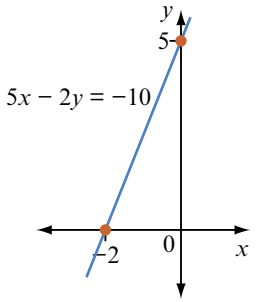
17 **a** gradient–intercept method

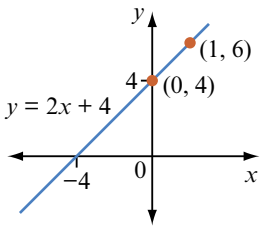
b x- and y-intercept method

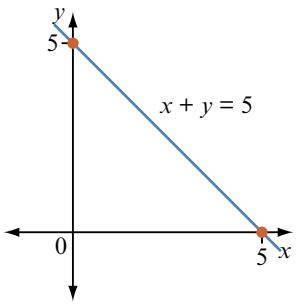
c Find the coordinates of another point, and draw the straight line through that point and the origin.

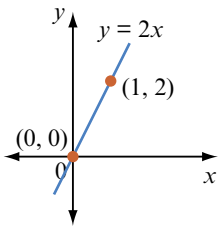
d Locate the y-intercept, and draw a line through that point parallel to the x-axis.

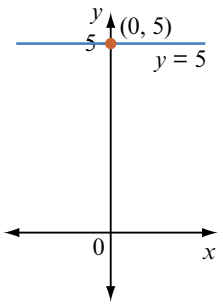
e Locate the x-intercept, and draw a line through that point parallel to the y-axis.

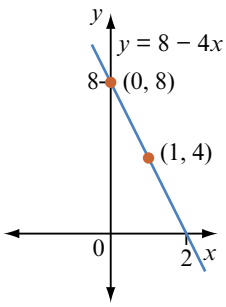
18 **a** 

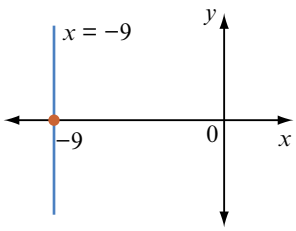
b 

c 

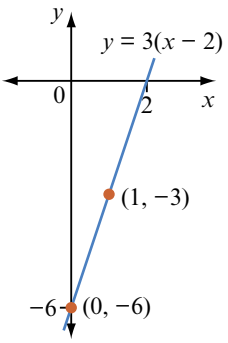
d 

e 

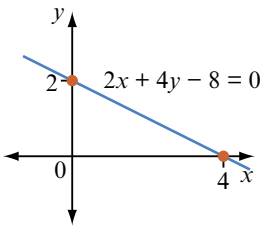
f 

g 

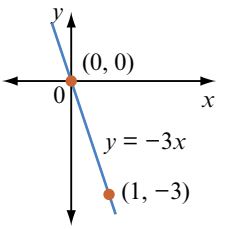
h



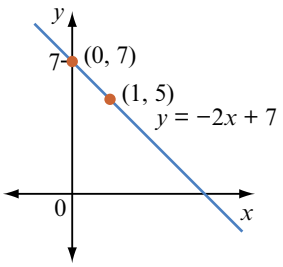
i



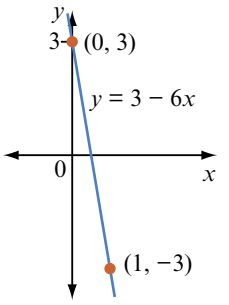
j



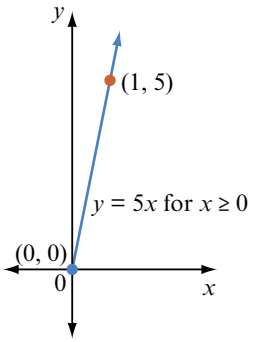
k



l



19 a



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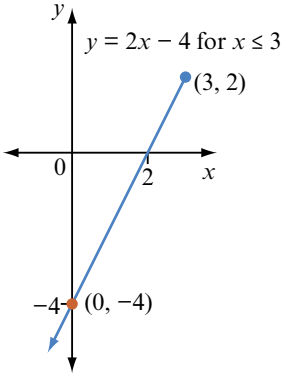

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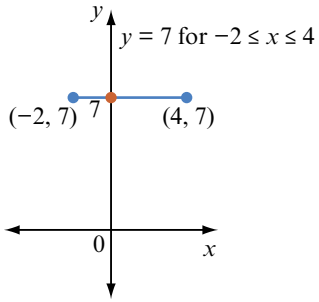

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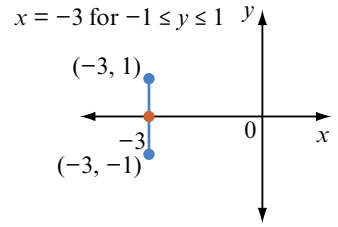

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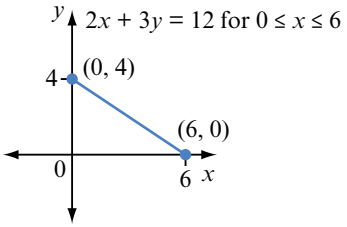

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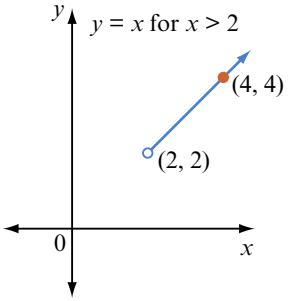

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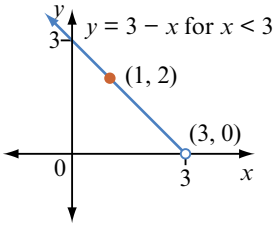

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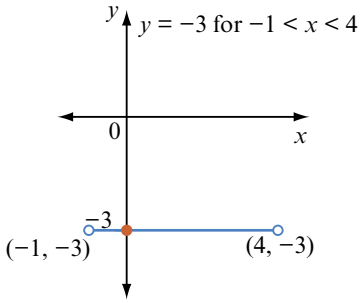

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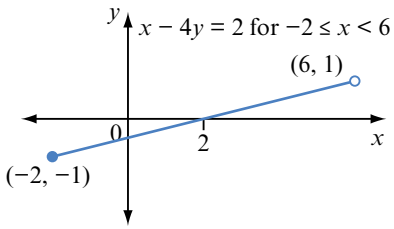

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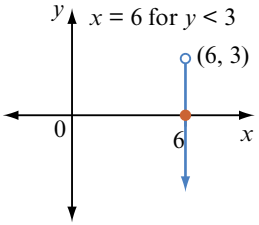

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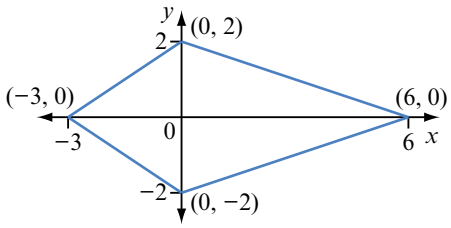


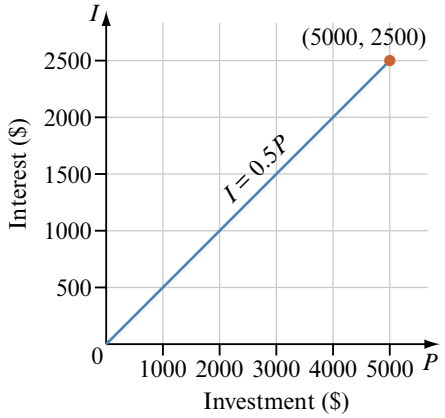
b kite

c **i** 19.9 units **ii** 18 square units

d coordinates of midpoints: (3, 1), (3, −1), (−1.5, −1) and (−1.5, 1)

e perimeter =13 units, area = 9 square units

21 a

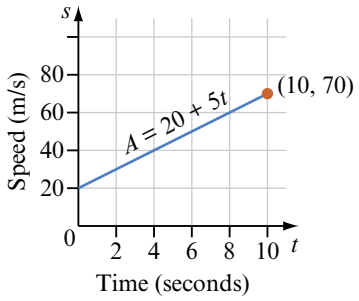


b **i** $500 **ii** $1250 **iii** $2125

c **i** $1500 **ii** $2000 **iii** $4500

d Gradient represents value of RT. This is the rate at which the investment grows over a 10-year period. $RT = 0.05 \times 10 = 0.5$.

22 a



b initial constant speed

c rate of increase in speed

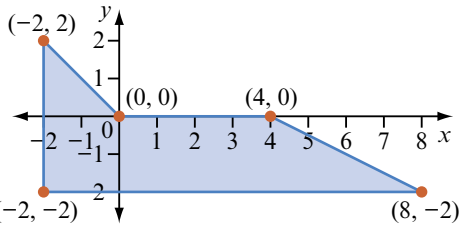
d **i** 35 m/s **ii** 45 m/s **iii** 57.5 m/s

e **i** 126 km/h **ii** 162 km/h **iii** 207 km/h

f **i** 2 s **ii** 7 s **iii** 9 s

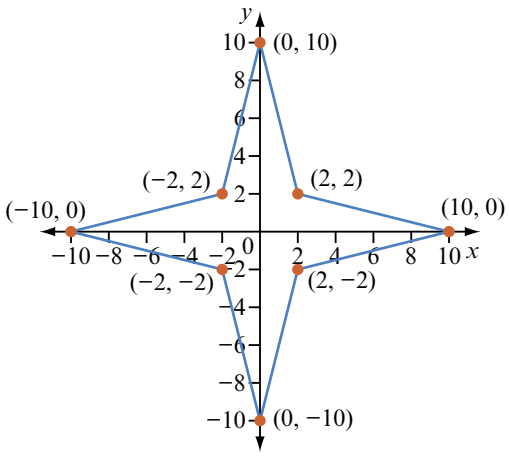
g 8 s **h** 252 km/h reached after 10 s

23 a



b 25.3 units **c** 18 square units

24 a



b 66 units **c** 80 square units

d $y = x$ **e** $y = -x$

4D Finding the rule for a linear relationship

4D Start thinking!

1 a Graph shows line drawn through two points on the Cartesian plane.

b Rule can be calculated from gradient and y-intercept.

c 3 **d** (0, 2) **e** $m = 3$, $c = 2$

f $y = 3x + 2$

2 c −2 **d** (0, 3) **e** $m = -2$, $c = 3$

f $y = -2x + 3$

3 General rule for straight line is $y = mx + c$. Gradient (m) is coefficient of x, and y-coordinate of y-intercept represents value of c.

4 a $y = 3x - 2$

b $y = -\frac{5}{3}x + 1$ or $5x + 3y = 3$

Exercise 4D Finding the rule for a linear relationship

1 **a** 4 **b** −2 **c** $\frac{1}{2}$ **d** −1 **e** 5 **f** −3

2 No. Consider the two points (2, 6) and (1, 4).
Taking (2, 6) as (x_1, y_1) and (1, 4) as (x_2, y_2),
$m = \frac{4-6}{1-2} = \frac{-2}{-1} = 2$.
Taking (1, 4) as (x_1, y_1) and (2, 6) as (x_2, y_2),
$m = \frac{6-4}{2-1} = \frac{2}{1} = 2$.

3 **a** $y = 2x + 1$ **b** $y = 4x - 3$ **c** $y = -3x + 5$
d $y = -x - 1$ **e** $y = \frac{3}{5}x$ **f** $y = -\frac{1}{2}x + 2$

4 **a** $y = 4x + 2$ **b** $y = x - 7$ **c** $y = \frac{1}{2}x + 5$
d $y = 10x + 6$ **e** $y = -3x - 5$ **f** $y = -\frac{1}{4}x + 4$

5 **a** $y = 5x - 3$ **b** $y = -2x + 4$ **c** $y = \frac{4}{5}x + 2$

6 **a** $y = 3x + 1$ **b** $y = 2x + 3$ **c** $y = -4x - 2$
d $y = -x + 9$ **e** $y = 5x + 9$ **f** $y = -3x - 1$

7 **a** $c = y_1 - mx_1$ **b** $y = mx + y_1 - mx_1$
c $y - y_1 = mx - mx_1$, $y - y_1 = m(x - x_1)$

8 **a** $y = 3x + 1$ **b** $y = 2x + 3$ **c** $y = -4x - 2$
d $y = -x + 9$ **e** $y = 5x + 9$ **f** $y = -3x - 1$

9 The answer is the same. Either of the two points can be used.

10 **a** $y = 2x + 1$ **b** $y = 6x - 28$ **c** $y = -3x + 10$
d $y = -2x + 4$ **e** $y = x - 1$ **f** $y = -x - 6$

11 **a** $y = 3x - 11$ **b** $y = -2x + 6$ **c** $y = x + 2$

12 $m = \frac{7-3}{3-1} = 2$ so $y = 2x + c$
Substituting the point (1, 3) gives $3 = 2 + c$ or $c = 1$.
So the rule is $y = 2x + 1$.

13 **a** $y = -2x + 4$ **b** $y = x + 3$ **c** $y = 2x - 6$
d $y = -5x + 5$

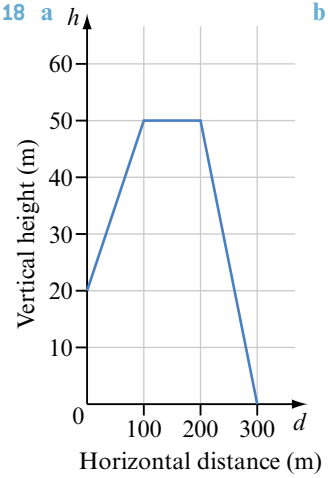
14 **a** $y = -3x + 5$ **b** $y = 2x + 4$ **c** $y = 8$
d $y = \frac{1}{3}x$ **e** $x = -\frac{5}{2}$ **f** $y = 2x - 1$

15 **a** $y = \frac{1}{200}x + 50$ **b** $y = -\frac{1}{15}x + 6$ **c** $y = \frac{1}{5}x$

16 **a** $a = 20n + 80$ **b** 21 weeks

17 **a** $w = -195n + 1560$ **b** \$975
c 8 months

18 **a**



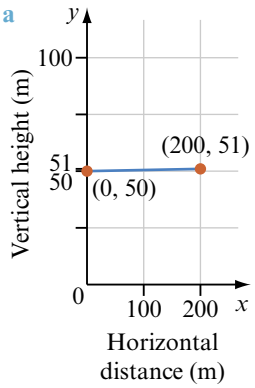
b In order of the flight:
$h = 0.3d + 20$, $h = 50$,
$h = -0.5d + 150$

c **i** 29 m **ii** 44 m **iii** 50 m **iv** 30 m

d $66\frac{2}{3}$ m and 220 m

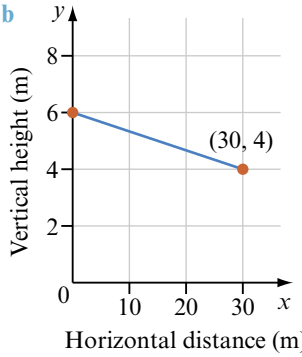
19 **a** $x - 2y = 4$ **b** $y = -23$
c $2x + 3y + 20 = 0$ **d** $12x - 17y = 0$

20 **a**



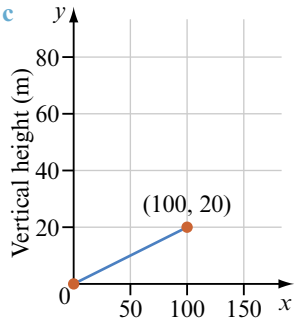
$y = \frac{1}{200}x + 50$ for $0 \le x \le 200$

b



$y = -\frac{1}{15}x + 6$ for $0 \le x \le 30$

c



$y = \frac{1}{5}x$ for $0 \le x \le 100$

21 In order of the flight:
$h = 0.3d + 20$ for $0 \le d \le 100$
$h = 50$ for $100 \le d \le 200$
$h = -0.5d + 150$ for $200 \le d \le 300$

4E Parallel and perpendicular lines

4E Start thinking!

1 A gradient: 2, y-intercept: 2
B gradient: 2, y-intercept: −2
C gradient: $-\frac{1}{2}$, y-intercept: 3

2 A $y = 2x + 2$ B $y = 2x - 2$ C $y = -\frac{1}{2}x + 3$

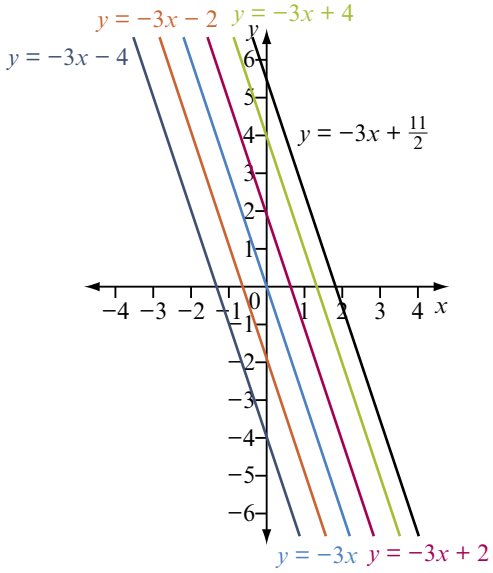
3 A and B are parallel. **4** gradient

5 90°; perpendicular **6** −1 **7** −1

8 Parallel lines have the same gradient. The product of the gradients of perpendicular lines is −1.

Exercise 4E Parallel and perpendicular lines

1 a



b All lines are parallel and have same gradient. They have different y-intercepts.

c All have same gradient (−3).

d **i** gradient = 3, so not parallel

ii gradient = −3, so parallel

iii gradient = 1, so not parallel

2 a 7 **b** −1 **c** $-\frac{3}{2}$ **d** 0 **e** −4 **f** −2

3 a $y = -2x + 3$ **b** $y = 6x - 4$

c $y = x$ **d** $y = -5x + 2$

4 a $y = -3x - 5$

b One possible answer is: $y = -3x + 20$.

5 a $y = 4x - 13$ **b** $y = -3x - 6$

c $y = -x - 2$ **d** $y = \frac{1}{2}x - 3$

6 a **i** gradients: 3 and $-\frac{1}{3}$

ii gradients: $\frac{3}{4}$ and $-\frac{4}{3}$

b Both products are −1.

c The product of gradients of perpendicular lines is always −1.

d Take negative reciprocal of known gradient.

7 a $-\frac{1}{5}$ **b** $\frac{1}{8}$ **c** 1 **d** −6 **e** $-\frac{5}{4}$ **f** $\frac{2}{3}$

8 a $-\frac{1}{2}$ **b** $\frac{1}{3}$ **c** −7 **d** $\frac{5}{2}$ **e** $\frac{1}{9}$ **f** −4

9 a $y = \frac{1}{2}x + 4$ **b** $y = -\frac{1}{5}x - 1$

c $y = -x$ **d** $y = \frac{2}{7}x + 3$

10 a $y = -\frac{1}{2}x + 3$ **b** $y = x + 3$

c $y = -\frac{2}{3}x + 3$

11 a **i** $y = 2x - 4$ **ii** $y = -\frac{1}{2}x - 4$

b **i** $y = -x - 4$ **ii** $y = x - 4$

c **i** $y = \frac{3}{2}x - 4$ **ii** $y = -\frac{2}{3}x - 4$

12 a $y = -\frac{1}{5}x - 7$ **b** $y = \frac{1}{7}x + 3$

c $y = -x - 3$ **d** $y = \frac{3}{2}x + 10$

13 a parallel; same gradient of 4

b perpendicular; gradients of $\frac{2}{3}$ and $-\frac{3}{2}$

c perpendicular; gradients of $\frac{3}{8}$ and $-\frac{8}{3}$

d neither; gradients of −2 and 2

e neither; gradients of −5 and $-\frac{1}{5}$

f parallel; same gradient of $\frac{2}{7}$

14 Some possible answers are:
$y - 5x = 6$ and $4y - 20x + 1 = 0$.

15 Some possible answers are:
$y = 8x + 2$ and $8x - y = 12$.

16 Gradient of line joining (2, −3) and (4, 5) is 4.
Gradient of $y = 4x - 7$ is 4.
So lines are parallel.

17 Gradient of line joining (−11, −7) and (−1, −2) is $\frac{1}{2}$.
Gradient of $y = -2x + 5$ is −2.
Product of gradients is −1, so lines are perpendicular.

18 a $\frac{2}{3}$ **b** **i** $y = \frac{2}{3}x - 5$ **ii** $y = -\frac{3}{2}x + 2$

19 a $y = -3x$ **b** $y = \frac{1}{3}x + 9$ **c** $y = \frac{1}{3}x - 1$

20 a Yes, both have gradient of $\frac{3}{7}$.

b No. AC has gradient of $\frac{5}{4}$, while BD has gradient of $-\frac{2}{3}$.

21 a gradient of AB: 3, gradient of BC: $-\frac{1}{3}$. So line segments AB and BC are perpendicular. This makes triangle ABC right-angled.

b length of AB: 9.5 units, length of BC: 6.3 units. So triangle is not isosceles right-angled.

c 27.2 units **d** 29.9 square units

22 a AD || BC, each with gradients of $\frac{1}{2}$, and AB || DC, each with gradients of 2.
length of AD = length of BC = length of AB = length of DC = 2.2 units

b EF || HG, each with gradients of $\frac{3}{7}$ and HE || GF, each with gradients of 2.
length of EF = length of HG = 7.6 units,
length of HE = length of GF = 8.9 units

23 gradient of KN = gradient of LM = $-\frac{1}{2}$, gradient of MN = gradient of LK = 2
So opposite sides are parallel, and adjacent sides are at right angles.
length of KL = length of LM = length of MN = length of NK = 4.5 units
perimeter: 17.9 units, area: 20.3 square units

24 gradient of PS = gradient of QR = 1, gradient of PQ = gradient of SR = −1
So opposite sides are parallel, and adjacent sides are at right angles.
length of PS = length of QR = 4.2 units,
length of PQ = length of SR = 5.7 units
perimeter: 19.8 units, area: 23.9 square units

25 $y = x - 5$

26 length of YX = length of YZ = 7.1 units and length of WX = length of WZ = 20.6 units
gradient of XZ = $-\frac{3}{4}$ and gradient of WY = $\frac{4}{3}$, so diagonals are at right angles.
midpoint of XZ is (3, 5). This point also lies on the diagonal WY, which has equation $y = \frac{4}{3}x + 1$.

27 Gradients of the line segments are: AB = $\frac{1}{2}$, BC = −3, CD = $\frac{1}{2}$, DA = $\frac{6}{5}$, AC = 0, BD = $\frac{9}{11}$. This indicates quadrilateral is a trapezium (one pair of parallel sides).

4F Solving linear simultaneous equations graphically

4F Start thinking!

1

x	−1	0	1	2	3	4	5
y_A	−6	−4	−2	0	2	4	6
y_B	6	5	4	3	2	1	0

2 $x = 3$ 3 (3, 2) 4 (3, 2)

5 This point lies on both lines, and satisfies both equations simultaneously.

6 a $y = 2x - 4$ b $y = -x + 5$

7 Line A: RS = 2 × 3 − 4 = 2 = LS
Line B: RS = −3 + 5 = 2 = LS

8 $x = 4$ 9 $x = 3, y = 2$

10 Identify coordinates of point of intersection.

Exercise 4F Solving linear simultaneous equations graphically

1 a solution b solution
c not a solution d solution
e not a solution f not a solution

2 a i (4, 3) ii $x = 4, y = 3$
b i (2, −4) ii $x = 2, y = -4$
c i (−5, −2) ii $x = -5, y = -2$
d i (−2, 3) ii $x = -2, y = 3$

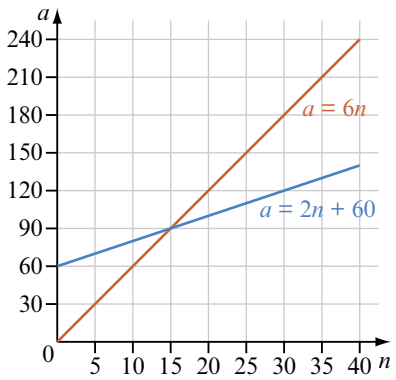
4 a $x = 5, y = 4$ b $x = 2, y = 6$
c $x = 6, y = 4$ d $x = -2, y = 5$
e $x = -1, y = 3$ f $x = -3, y = 1$
g $x = 3, y = 2$ h $x = 4, y = -2$
i $x = -2, y = -3$ j $x = -5, y = 6$
k $x = 2, y = -2$ l $x = 3, y = 5$

6 a $x = 3, y = -4$ b $x = 0, y = 0$
c $x = 2, y = 1$

7 a $x = \frac{1}{2}, y = \frac{1}{2}$ b $x = -3, y = -2\frac{1}{2}$
c $x = 1\frac{1}{2}, y = -4\frac{1}{2}$

8 Both graphs have gradient of 2. Lines are parallel so there is no point of intersection.

9 a To make n cards, cost of plain white cards is $2 \times n$ or $2n$. So, amount of money spent (in $) is cost of plain white cards plus start-up cost of $60.
b For selling n cards, the amount of money received (in $) is $6 \times n$ or $6n$.
c



d (15, 90); this shows number of cards to be sold (15) so that amount of money received is same as amount of money spent ($90).
e 15 cards f 16 to 40 cards

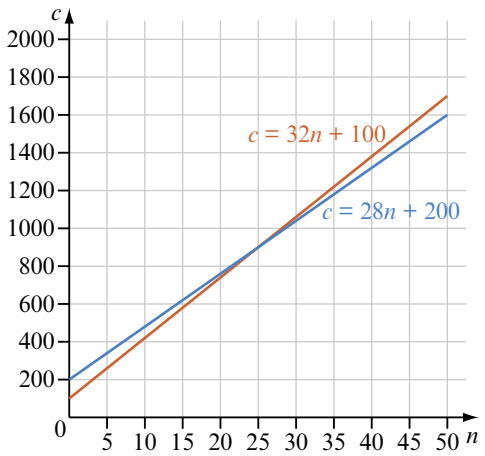
10 a $2x + 2y = 40$ or $x + y = 20$
b $y = x + 4$
c $x = 8, y = 12$; postcard has length of 12 cm and width of 8 cm.

11 a For x = Lachlan's age, y = Tia's age, equations are $y = 2x$ and $x + y = 51$.
b $x = 17, y = 34$
c Tia is 34 years old, Lachlan is 17 years old.

12 a For d = cost of drink, c = cost of choc top, equations are $d = c + 2$ and $5d + 7c = 70$
b Drink costs $7, choc top costs $5.

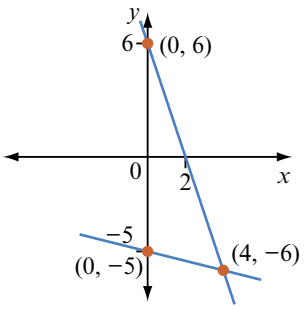
13 For l = length of land, w = width of land, equations are $l = w + 20$ and $2l + w = 124$. Land is 48 m long and 28 m wide.

14 Let n = number of people and c = catering cost.
a $c = 28n + 200$ b $c = 32n + 100$
c



d Cool Food Club; at $n = 18$, the line for Cool Food Club is lower than the line for Angie's Catering. This represents a lower cost for 18 people.
e Angie's Catering f 25 people at cost of $900
g When catering for less than 25 people, Cool Food Club is cheaper. When catering for more than 25 people, Angie's Catering is cheaper. Cost is same when catering for 25 people.

15 a



b $y = -\frac{1}{4}x - 5$, $y = -3x + 6$
c $x = 4$, $y = -6$
d For $x = 4$ and $y = -6$,
For $y = -\frac{1}{4}x - 5$, RS $= -\frac{1}{4} \times 4 - 5 = -6 =$ LS.
For $y = -3x + 6$, RS $= -3 \times 4 + 6 = -6 =$ LS.

16 One possible answer is: $y = 2x - 18$ and $y = 3 - x$.

17 They are equivalent equations. When graphed, the position of the lines is identical; that is, one line lies on top of the other. There are an infinite number of points where the lines intersect so there are infinite solutions.

4G Solving linear simultaneous equations algebraically

4G Start thinking!

1 $x = 3$, $y = 2$

2 a $x + x - 1 = 5$, $2x - 1 = 5$ **b** $x = 3$
c $x = 3$, $y = 2$

3 a $2x$ **b** 6 **c** y has been eliminated.
d $x = 3$; when $x = 3$, $x + y = 5$ becomes $3 + y = 5$ so $y = 2$
e $x = 3$, $y = 2$
f New equation is $2y = 4$; solution is the same.

Exercise 4G Solving linear simultaneous equations algebraically

1 a $x = 2$, $y = 7$ **b** $x = 4$, $y = 1$
c $x = -1$, $y = 3$ **d** $x = 6$, $y = 5$
e $x = -4$, $y = -2$ **f** $x = 2$, $y = 12$
g $x = 1$, $y = 3$ **h** $x = -3$, $y = 11$
i $x = 5$, $y = -5$ **j** $x = 5$, $y = 3$
k $x = -4$, $y = -1$ **l** $x = 5$, $y = 6$

2 a $x = 1$, $y = 8$ **b** $x = 6$, $y = 4$
c $x = -3$, $y = -5$ **d** $x = 5$, $y = -1$
e $x = -1$, $y = 4$ **f** $x = 4$, $y = -7$
g $x = 2$, $y = 1$ **h** $x = -2$, $y = 6$
i $x = -3$, $y = 19$

3 a $x = 2$, $y = 5$ **b** $x = -3$, $y = 1$
c $x = 1$, $y = -5$ **d** $x = -4$, $y = -2$
e $x = 5$, $y = 3$ **f** $x = -1$, $y = 2$

4 a $x = 5$, $y = 3$ **b** $x = -2$, $y = 9$
c $x = 3$, $y = -7$ **d** $x = -1$, $y = 5$
e $x = 6$, $y = 2$ **f** $x = -3$, $y = -4$

5 a $3x + 6y = 15$ **b** $-6x + 2y = 2$
c $-5x - 20y = 10$ **d** $20x + 8y = 12$
e $-4x + 6y = 14$ **f** $-12x + 6y = 9$

6 a $x = 3$, $y = 2$ **b** $x = 2$, $y = -1$
c $x = -4$, $y = 7$ **d** $x = 5$, $y = 3$
e $x = -3$, $y = -4$ **f** $x = -2$, $y = 6$

7 a No; ① + ② gives $11x + 5y = -7$ and ① − ② gives $3x - y = -9$.
b $21x + 6y = -24$ **c** $8x + 6y = 2$
d y; subtract the equations **e** $x = -2$, $y = 3$

8 a $x = 1$, $y = 4$ **b** $x = -3$, $y = 1$
c $x = 2$, $y = -1$ **d** $x = -4$, $y = 2$
e $x = 5$, $y = 5$ **f** $x = -3$, $y = -2$

9 Graphical method best when information given to draw graphs; substitution method best when one or both of the equations have a variable term as subject of the equation; elimination method best when equations are in the form $ax + by = d$.

10 a $x = 2$, $y = 0$ **b** $x = \frac{1}{3}$, $y = -3$
c $x = \frac{3}{2}$, $y = \frac{2}{5}$ or $x = 1.5$, $y = 0.4$
d $x = -9$, $y = -7$ **e** $x = 4$, $y = 2$
f $x = -2$, $y = -3$ **g** $x = 3$, $y = 10$
h $x = 5$, $y = -2$ **i** $x = 2$, $y = -1$
j $x = -2$, $y = 2$ **k** $x = 7$, $y = 4$
l $x = -4$, $y = 1$ **m** $x = -0.8$, $y = 2.3$
n $x = -\frac{1}{4}$, $y = 5$ or $x = -0.25$, $y = 5$
o $x = 28$, $y = 41$

11 a For two numbers x and y, equations are $x + y = 245$, $x - y = 91$. Solution is $x = 168$, $y = 77$.
b Two numbers are 77 and 168.

12 a $2l + 2w = 810$ or $l + w = 405$ **b** $l = 2w$
c 270 cm long and 135 cm wide

13 For x = width of pillow case, y = length of pillow case, equations are $y = 2x - 18$ and $2x + 2y = 240$ (or $x + y = 120$). Solution is $x = 46$, $y = 74$ so pillow case is 74 cm long and 46 cm wide.

14 For g = number of points for a goal, b = number of points for a behind, equations are $15g + 6b = 132$ and $13g + 12b = 128$. Solution is $g = 8$, $b = 2$ so number of points for a goal is 8 and number of points for a behind is 2.

15 For p = cost of a pie, r = cost of a sausage roll, equations are $7p + 6r = 63$ and $5p + 5r = 48$. Solution is $p = 5.4$, $r = 4.2$ so cost of a pie is $5.40 and cost of a sausage roll is $4.20.

16 a Strategy A: equation ③ is $10x - 6y = -2$.
① − ③: $-3x = -12$, $x = 4$
Substituting $x = 4$ in ②: $20 - 3y = -1$, $y = 7$
Solution is $x = 4$, $y = 7$.
Strategy B: equation ③ is $-10x + 6y = 2$.
① + ③: $-3x = -12$, $x = 4$
Substituting $x = 4$ in ②: $20 - 3y = -1$, $y = 7$
Solution is $x = 4$, $y = 7$.
Strategy C: equation ③ is $35x - 30y = -70$ and equation ④ is $35x - 21y = -7$.
③ − ④: $-9y = -63$, $y = 7$
Substituting $y = 7$ in ②: $5x - 21 = -1$, $x = 4$
Solution is $x = 4$, $y = 7$.

b Some possible answers are:
- **i** Multiply equation ① by 5 to form equation ③ and multiply equation ② by −7 for form equation ④. Add equations ③ and ④.
- **ii** Graph equations ① and ② and identify the coordinates of the point of intersection.
- **iii** Rearrange equation ② to obtain $3y = 5x - 1$ and substitute into equation ①. That is, substitute $5x - 1$ for $3y$ in $7x - 2(3y) = -14$.

17 Substituting $y = 11 - 4x$ into $8x + 2y = 5$ gives:
$8x + 2(11 - 4x) = 5$
$8x + 22 - 8x = 5$
There is no solution as there is no x value that makes this equation true.

18 Substituting $y = 2x - 3$ into $4x - 2y = 6$ gives:
$4x - 2(2x - 3) = 6$
$4x - 4x + 6 = 6$
There are an infinite number of solutions as all values of x make this equation true.

4 Chapter review

MULTIPLE-CHOICE

1 A	**2** C	**3** C	**4** B	**5** D
6 D	**7** A	**8** C	**9** D	**10** B
11 C	**12** D	**13** A	**14** A	

SHORT ANSWER

1 **a** $x = -1\frac{1}{2}$ **c** $x = 9\frac{2}{3}$

2 **a** $x = 9$ **b** $x = \frac{5}{13}$

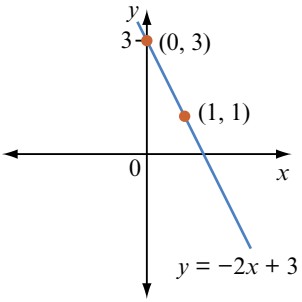
3 **a** $b = \frac{2A}{h} - a$ **b** $h = \frac{A}{2(l + w)}$

4 **a** $1\frac{5}{8}$, 2.6, 8.5, $\frac{3}{4}$ **b** −3.9, −8.5, $\frac{3}{4}$ **c** 2.6, 8.5

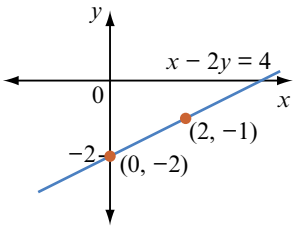
5 **a** $x \geq -3$ **b** $x < -10$ **c** $x \leq 2.5$

6 **a** $x > -1$ **b** $x < -2\frac{2}{3}$

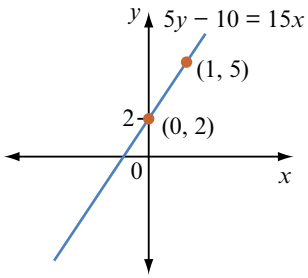
7 **a**



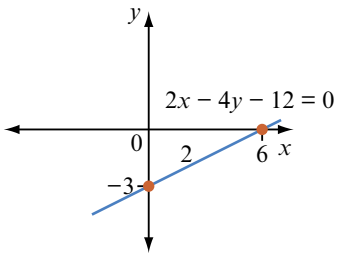
b



c



8 **a**



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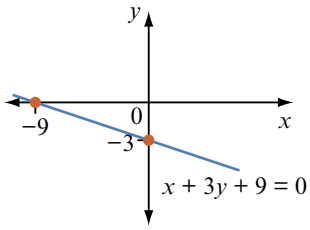

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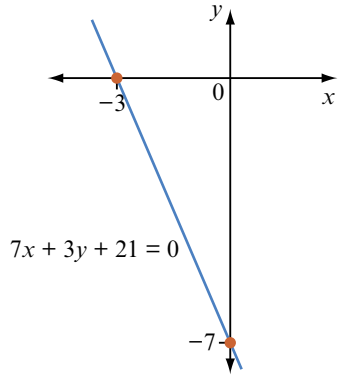


9 **a** $y = 2x - 4$ **b** $y = 6$ **c** $x = -3$

10 **a** $y = 2x + 3$ **b** $y = -2x + 7$

11 **a** $y = x + 2$ **b** $y = -3x + 2$ **c** $y = x$

12 **a** $y = -4x - 3$ **b** $y = -\frac{1}{2}x - 4$

13 **a** $y = -\frac{1}{3}x + 3$ **b** $y = 3x + 7$

14 **a** $x = 1, y = 2$ **b** $x = 6, y = 17$

15 **a** $x = 1, y = -1$ **b** $x = -2, y = -5$
c $x = 2, y = 10$

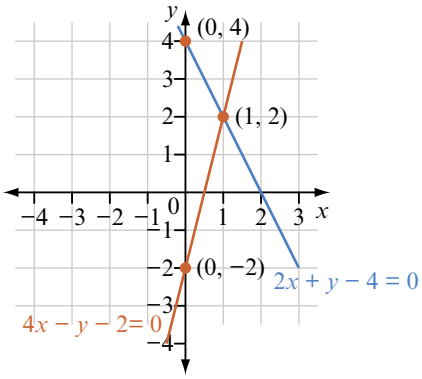
16 **a** $x = 2, y = 4$ **b** $x = -2, y = 3$ **c** $x = 3, y = -1$

MIXED PRACTICE

1 **a** $y = -x + 8$ **b** $y = x$

2 C

3 a



b (1, 2)

4 a perpendicular **b** perpendicular
c neither

5 length 35 m, width 7 m **6** D

7 $x = 1, y = -1$

8 A

9 a $z = 3V - x - y$ **b i** 10 **ii** 109

10 B

11 a $x = 1, y = 5$ **b** $x = -1, y = 2$ **c** $x = 2, y = 1$

12 (1, 5), (1, −3), (−4, −3) and (−4, 5)

13 hamburger $7, drink $5.50

14 a $x \leq 1$ **b** $x > -2$ **c** $x \geq -\frac{5}{9}$ **d** $x > \frac{7}{8}$

15 a (−1, −2) **b** (3, −2)

16 a 10.3 units **b** 10.6 units **c** 7.1 units

17 a true **b** true
c false; equation of x-axis is $y = 0$

18 a 2 **b** −10

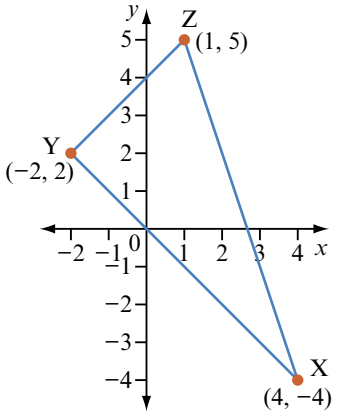
19 D **20** B

21 a $2x - 3 = 17$, $x = 10$; number is 10
b $7(x - 1) = 21$, $x = 4$; number is 4
c $5(3x - 12) = 15$, $x = 5$; number is 5

22 a $x > -\frac{1}{2}$ **b** $x \geq -7$

ANALYSIS

a



b line segment XY: $y = -x$, YZ: $y = x + 4$, ZX: $y = -3x + 8$

c gradient of XY = −1, gradient of YZ = 1. Product of the two gradients is −1, so XY is perpendicular to YZ. This means triangle XYZ is right-angled, with right angle at vertex Y. Hypotenuse is ZX.

d $x = 4.24$ units, $y = 9.49$ units, $z = 8.49$ units

e $4.24^2 + 8.49^2 = 90.06$
$9.49^2 = 90.06$
This shows that square of hypotenuse = sum of squares of other two sides.

f M is the point (2.5, 0.5)

g 4.74 units

h These are both acute-angled isosceles triangles, where MZ = MY = MX.

i $y = \frac{1}{3}x + 2\frac{2}{3}$

j $(1\frac{3}{5}, 3\frac{1}{5})$

k 3.79 units

l Using YX as the base and YZ as the perpendicular height:
area $= \frac{1}{2} \times 8.49 \times 4.24 = 18.0$ square units
Using XZ as the base and YH as the perpendicular height:
area $= \frac{1}{2} \times 9.49 \times 3.79 = 18.0$ square units

m $YH = \frac{4.24 \times 8.49}{9.49} = 3.79$ units

n With centre M, and radius MX = MY = MZ, a circle can be drawn to touch the vertices of the right-angled triangle. This is one of the geometry facts which states that the angle in a semi-circle is a right angle.

4 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 5 NON-LINEAR RELATIONSHIPS

5 Are you ready?

1 a $x(x - 3)$ **b** $(x + 5)(x - 5)$
c $(x - 3)(x - 4)$ **d** $-2x(2x + 9)$

2 a $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$
b $(x - \frac{3}{2} + \frac{\sqrt{13}}{2})(x - \frac{3}{2} - \frac{\sqrt{13}}{2})$

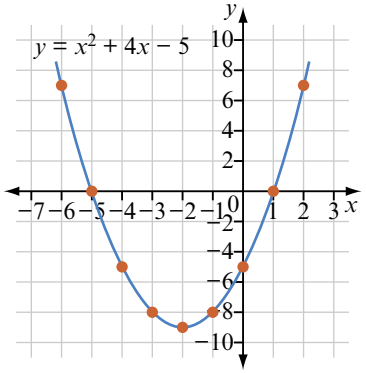
3 a 3 **b** $2\sqrt{6}$ **c** $14\sqrt{2}$ **d** $\sqrt{5}$

4 a 5 **b** 9 **c** −44 **d** $48\frac{1}{9}$

5 a

x	−6	−5	−4	−3	−2	−1	0	1	2
y	7	0	−5	−8	−9	−8	−5	0	7

b



c i (−5, 0) and (1, 0) **ii** (0, −5)
iii (−2, −9)

6 a (5, 2) **b** (3, −5) **c** (−2, 3)
d (3, −2) **e** (−3, 2)

7 (2, 1); 2 units
8 **a** $y = 1$ **b** $x = -3$ **c** $x = 2$ **d** $y = -2$
9 **a** 2^5 **b** 3^6 **c** 5^6 **d** 2^{12}
10 **a** **i** $\log_2 64 = 6$ **ii** $\log_{10} 10\,000 = 4$
b **i** $3^4 = 81$ **ii** $10^0 = 1$
11 **a** **i** undefined **ii** approaches ∞
b **i** 3 **ii** -8 **c** **i** 0 **ii** 0

5A Solving quadratic equations

5A Start thinking!

1 Quadratic equation has equals sign, quadratic expression does not.
2 i, iii; parts i and iii show equations where highest power of the pronumeral is 2; part ii shows linear equation, part iv shows quadratic expression.
3 **a** **i** 0 **ii** 0 **iii** 0 **iv** 0 **v** 0 **vi** 0
b result is zero
c **i** $x = 0$ **ii** $x = 0$ **iii** $x = 0$
iv $x = 4$ **v** $x = -1$ **vi** $x = 7$
d zero; zero
4 **a** $x - 2$ and $x - 7$
b One factor or the other must equal zero.
c $(x - 2)(x - 7) = 0$
$x - 2 = 0$ or $x - 7 = 0$
$x = 2$ or $x = 7$
d Substitute each x value into the original equation $(x - 2)(x - 7) = 0$ and show that each value makes the equation a true statement.
5 Express one side of quadratic equation in factor form and let each factor equal zero. Solution to each linear equation gives solution to the quadratic equation.

Exercise 5A Solving quadratic equations

1 **a** yes **b** no **c** yes **d** no **e** no **f** yes
2 **a** $(x + 4)(x - 1) = 0$
$x + 4 = 0$ or $x - 1 = 0$
$x = -4$ or $x = 1$
b $3x(x - 5) = 0$
$3x = 0$ or $x - 5 = 0$
$x = 0$ or $x = 5$
c $(x + \sqrt{3})(x - \sqrt{3}) = 0$
$x + \sqrt{3} = 0$ or $x - \sqrt{3} = 0$
$x = -\sqrt{3}$ or $x = \sqrt{3}$
$x = \pm\sqrt{3}$
3 **a** $x = -2$ or $x = 4$ **b** $x = -7$ or $x = -1$
c $x = 3$ or $x = 6$ **d** $x = \pm 9$
e $x = 0$ or $x = 5$ **f** $x = 0$ or $x = \frac{1}{2}$
g $x = 0$ or $x = -1$ **h** $x = 0$ or $x = -2$
i $x = 0$ or $x = \frac{3}{4}$ **j** $x = \pm\sqrt{2}$
k $x = \pm\sqrt{5}$ **l** $x = \pm 4\sqrt{3}$
m $x = -1\frac{1}{2}$ or $x = 4$ **n** $x = \pm\frac{1}{3}$
o $x = -\frac{2}{7}$ or $x = 1\frac{1}{4}$
4 **a** $x = -3$ or $x = -4$ **b** $x = -5$ or $x = 2$
c $x = 1$ or $x = 8$ **d** $x = \pm 5$
e $x = \pm 9$ **f** $x = \pm 2$
g $x = 0$ or $x = 3$ **h** $x = 0$ or $x = -1$
i $x = 0$ or $x = -4$ **j** $x = 0$ or $x = \frac{2}{5}$
k $x = 0$ or $x = 6$ **l** $x = 0$ or $x = -10$
6 **a** Dividing both sides of equation by 3 gives an identical equation.
b $x = -6$ or $x = 8$
7 **a** Dividing both sides of equation by -5 gives an identical equation.
b $x = -3$ or $x = -2$
8 Dividing both sides of first equation by 3 gives an equivalent equation of $x - 2 = 0$ and does not affect solution. Solution for both equations is $x = 2$. Solution to second equation is $x = 0$ or $x = 2$. Dividing both sides of second equation by x does not produce an equivalent equation. As $x = 0$ is a solution, we cannot divide by 0.
9 **a** $x = -5$ or $x = 6$ **b** $x = 2$ or $x = 4$
c $x = -8$ or $x = -1$ **d** $x = 3$ or $x = -9$
e $x = \pm 4$ **f** $x = 0$ or $x = -7$
g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -1\frac{2}{3}$
i $x = 1\frac{1}{2}$ or $x = -10$
10 **a** $x = -2$ or $x = -1$ **b** $x = -5$ or $x = 3$
c $x = 1$ or $x = 4$ **d** $x = -3$ or $x = 2$
e $x = -2$ or $x = 6$ **f** $x = -8$ or $x = 7$
g $x = \pm 2$ **h** $x = \pm 3$
i $x = 0$ or $x = \frac{1}{2}$
11 **a** $x = \pm\sqrt{3}$ **b** $x = \pm\sqrt{11}$ **c** $x = \pm\sqrt{7}$
d $x = \pm 2\sqrt{2}$ **e** $x = \pm 3\sqrt{3}$ **f** $x = \pm\sqrt{2}$
g $x = \pm\sqrt{5}$ **h** $x = \pm\sqrt{6}$ **i** $x = \pm 4\sqrt{2}$
j $x = \pm\sqrt{19}$ **k** $x = \pm 2\sqrt{5}$ **l** $x = \pm 2\sqrt{13}$
12 **a** $x = -1 \pm \sqrt{7}$ **b** $x = 2 \pm \sqrt{2}$
c $x = -4 \pm \sqrt{5}$ **d** $x = 6 \pm 2\sqrt{2}$
e $x = -10 \pm 5\sqrt{3}$ **f** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$
13 **a** $x = -3.65$ or $x = 1.65$
b $x = 0.59$ or $x = 3.41$
c $x = -6.24$ or $x = -1.76$
d $x = 3.17$ or $x = 8.83$
e $x = -18.66$ or $x = -1.34$
f $x = -0.62$ or $x = 1.62$
14 **a** two **b** one **c** two **d** one **e** two **f** none
15 **a** $x = \pm 3$ **b** $x = -11$ or $x = 7$
c $x = 4$ **d** no solutions
e $x = 0$ or $x = -4$ **f** $x = 3 \pm \sqrt{10}$
16 **a** $x = -6$ or $x = 2$ **b** $x = 1$ or $x = 4$
c $x = \pm 3$ **d** $x = -5$ or $x = 8$
e $x = 0$ or $x = 9$ **f** $x = 2 \pm \sqrt{3}$
g $x = 0$ or $x = 4$ **h** $x = 2$ or $x = 9$
i $x = -3$ or $x = 6$ **j** $x = -7$ or $x = 3$
k $x = 4$ or $x = 6$ **l** $x = -3$
18 **a** **i** 8 m **ii** 6 m **b** 6 m **c** 3 s
d Not possible to have negative time values.
19 **a** 25 kg **b** 9 kg **c** 5 weeks
d No, if amounts were the same, relationship would be linear.
20 **a** $x(x + 3) = 70$ or $x^2 + 3x - 70 = 0$
b $x = -10$ or $x = 7$. $x = 7$ is the feasible solution and $x = -10$ is not, as it is not possible to have a negative length.
c width 7 m, length 10 m

21 $x(x-5) = 66$ where x is length in cm.
Dimensions are length 11 cm, width 6 cm.

22 **a** $x = -3 - \sqrt{2}$ or $x = -3 + \sqrt{2}$
b $x = -1 - \sqrt{5}$ or $x = -1 + \sqrt{5}$
c $x = 2 - \sqrt{3}$ or $x = 2 + \sqrt{3}$
d $x = -4 - \sqrt{10}$ or $x = -4 + \sqrt{10}$
e $x = 3 - \sqrt{\frac{21}{2}}$ or $x = 3 + \sqrt{\frac{21}{2}}$
f $x = -\frac{1}{2} - \sqrt{\frac{13}{20}}$ or $x = -\frac{1}{2} + \sqrt{\frac{13}{20}}$

23 **a** $x = -5$ or $x = -\frac{3}{2}$ **b** $x = \frac{2}{5}$ or $x = 4$
c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ **d** $x = -7$ or $x = \frac{3}{4}$
e $x = -1$ or $x = \frac{1}{3}$ **f** $x = -\frac{3}{7}$ or $x = 2$

5B Solving quadratic equations using the quadratic formula

5B Start thinking!

1 $a = 2, b = 12, c = -4$ **3** yes; $x^2 + 6x - 2 = 0$
4 **a** no
b yes; $(x^2 + 6x + 9) - 9 - 2 = 0$;
$(x + 3)^2 - (\sqrt{11})^2 = 0$;
$(x + 3 + \sqrt{11})(x + 3 - \sqrt{11}) = 0$
5 **a** two
b $x = -3 - \sqrt{11}$ or $x = -3 + \sqrt{11}$
c $x = -6.32$ or $x = 0.32$
6 $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Exercise 5B Solving quadratic equations using the quadratic formula

1 **a** $a = 2, b = 7, c = 3$ **b** $a = 1, b = 5, c = 8$
c $a = 3, b = -6, c = 3$ **d** $a = 5, b = 1, c = 1$
e $a = -4, b = 12, c = -9$ **f** $a = -1, b = 4, c = 2$
2 **a** 25 **b** −7 **c** 0 **d** −19 **e** 0 **f** 24
3 **a** two **b** none **c** one **d** none **e** one **f** two
4 **a** none **b** two **c** two **d** none **e** one **f** two
5 **a** **i** 64 **ii** two **iii** $x = -3$ or $x = 5$
b **i** 0 **ii** one **iii** $x = -2$
c **i** 196 **ii** two **iii** $x = -7$ or $x = 7$
d **i** 324 **ii** two **iii** $x = 0$ or $x = 6$
e **i** 8 **ii** two
iii $x = 3 - \sqrt{2}$ or $x = 3 + \sqrt{2}$
f **i** −8 **ii** none **iii** no solutions

6 Step 3: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 4: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

Step 5: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Step 7: $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{4a^2}\right)^2 = 0$

Step 10: $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$

Step 11: $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$
or $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

Step 12: $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$
or $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

7 **a** **i** $x = \dfrac{7 \pm \sqrt{29}}{2}$ **ii** $x \approx 0.81$ or $x \approx 6.19$
b **i** $x = \dfrac{1 \pm \sqrt{13}}{6}$ **ii** $x \approx -0.43$ or $x \approx 0.77$
c **i** $x = \dfrac{-1 \pm \sqrt{33}}{4}$ **ii** $x \approx -1.69$ or $x \approx 1.19$
d **i** $x = \dfrac{-1 \pm \sqrt{109}}{6}$ **ii** $x \approx -1.91$ or $x \approx 1.57$
e **i** $x = \dfrac{-5 \pm \sqrt{97}}{12}$ **ii** $x \approx -1.24$ or $x \approx 0.40$
f **i** $x = 1 \pm \sqrt{2}$ **ii** $x \approx -0.41$ or $x \approx 2.41$
g **i** $x = -4 \pm 2\sqrt{3}$ **ii** $x \approx -7.46$ or $x \approx -0.54$
h **i** $x = \dfrac{3 \pm 2\sqrt{2}}{2}$ **ii** $x \approx 0.09$ or $x \approx 2.91$
i **i** $x = 2 \pm \sqrt{6}$ **ii** $x \approx -0.45$ or $x \approx 4.45$
j **i** $x = \dfrac{-5 \pm 2\sqrt{10}}{5}$ **ii** $x \approx -2.26$ or $x \approx 0.27$
k **i** $x = \dfrac{-3 \pm \sqrt{97}}{4}$ **ii** $x \approx -3.21$ or $x \approx 1.71$
l **i** $x = -1 \pm 2\sqrt{2}$ **ii** $x \approx -3.83$ or $x \approx 1.83$

8 **a**, **b** $x = 2$ or $x = 3$
d If solutions exist and are whole numbers, the method described in part a is usually quicker.
9 **a** \$620 **b** 59
10 length 81.4 m, width 61.4 m
11 length 369 mm, width 131 mm
12 **a** There is no real number that is the square root of a negative number so we cannot obtain a result using the quadratic formula. Hence there are no solutions.
b The square root of zero is zero so there will be only one distinct value obtained using the quadratic formula: $x = \dfrac{-b \pm 0}{2a} = \dfrac{-b}{2a}$.
c The square root of a positive number gives a positive result so there will be two distinct solutions: $x = \dfrac{-b - \text{positive}}{2a}$ or $\dfrac{-b + \text{positive}}{2a}$.
13 **a** $k = 3$ **b** **i** $k < 3$ **ii** $k > 3$
14 **a** $k = -4$ or 4
b $k < -4$ and $k > 4$. Some possible examples are:
$4x^2 - 5x + 1 = 0$, $4x^2 + 5x + 1 = 0$,
$4x^2 + 8x + 1 = 0$.
c $-4 < k < 4$. Some possible examples are:
$4x^2 - 3x + 1 = 0$, $4x^2 + x + 1 = 0$,
$4x^2 + 2x + 1 = 0$.
15 **a** $k = 1$ **b** $k < 2$ **c** $k < -9$ **d** $k = \frac{9}{4}$
e $k < -6$ and $k > 6$ **f** $-10 < k < 10$

5C Sketching parabolas using intercepts

5C Start thinking!

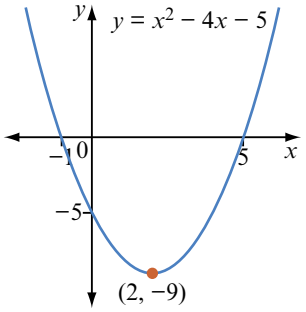
1 a inverted b 3 c −3, 1
d maximum e (−1, 4)

2 Coefficient is positive so parabola is upright.

3 a 0 b $y = 0 - 0 - 5 = -5$; −5

4 a 0 b $0 = x^2 - 4x - 5 = (x + 1)(x - 5)$
c $x = -1$ or 5; −1 and 5

5 (2, −9)

6 

Exercise 5C Sketching parabolas using intercepts

1 a i x-intercepts: −2, 4; y-intercept: −8
ii x-intercepts: −4, 2; y-intercept: −8
iii x-intercepts: −4, 2; y-intercept: 8
b i (1, −9) ii (−1, −9) iii (−1, 9)
c i minimum ii minimum iii maximum
d i C ii A iii B

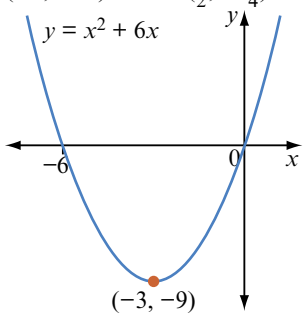
2 If coefficient of x^2 is positive, parabola is upright; for example, $y = x^2 - 2x - 8$.
If coefficient of x^2 is negative, parabola is inverted; for example, $y = -x^2 - 2x + 8$.

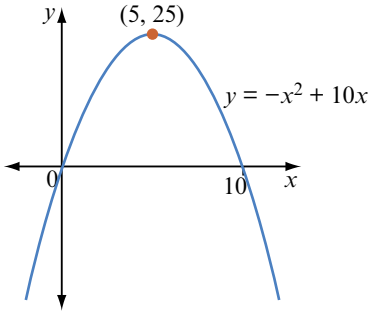
3 a i −6, 0 ii 0 b i 0, 10 ii 0
c i 1, 5 ii 5 d i −1, 1 ii −1
e i −6, 2 ii 12 f i −3, 3 ii 9

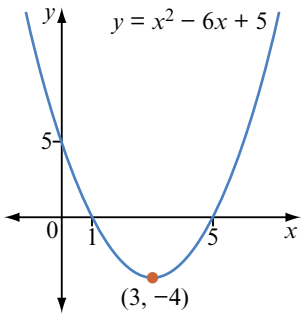
4 a upright b inverted c upright
d upright e inverted f inverted

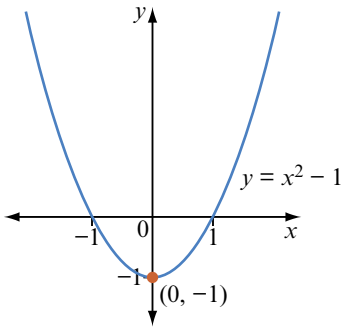
5 a (−3, −9) b (5, 25) c (3, −4)
d (0, −1) e (−2, 16) f (0, 9)

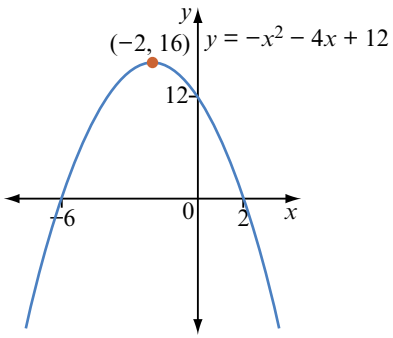
6 a (−1, −16) b $(\frac{1}{2}, 12\frac{1}{4})$

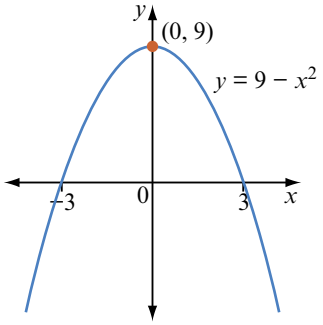
7 a 

b 

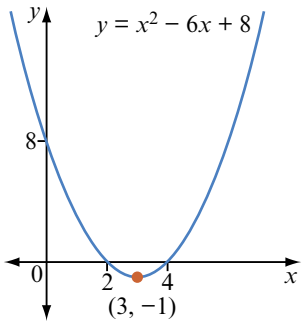
c 

d 

e 

f 

8 a



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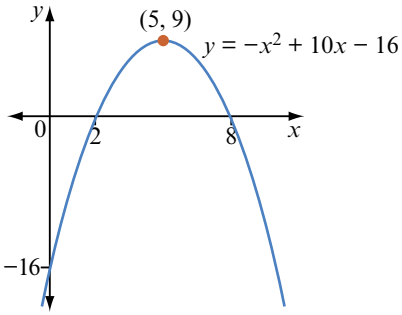

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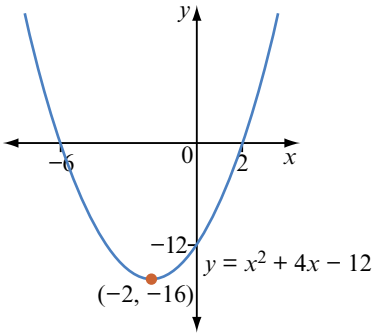

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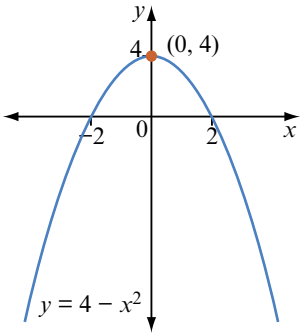

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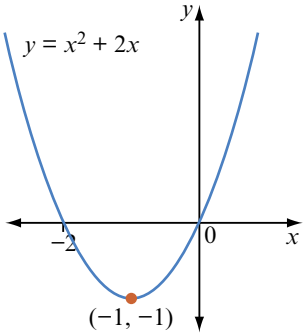

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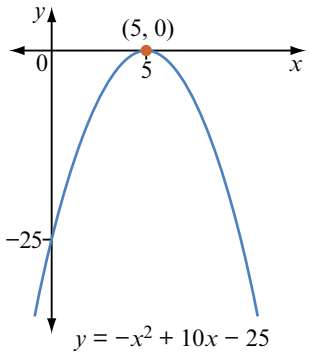
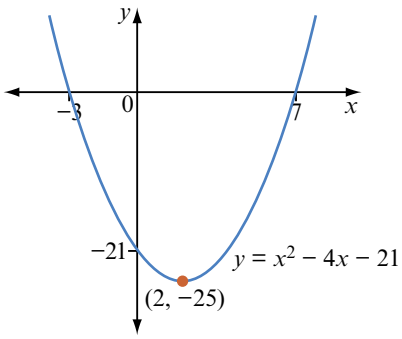
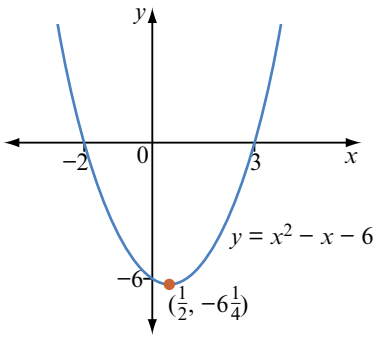


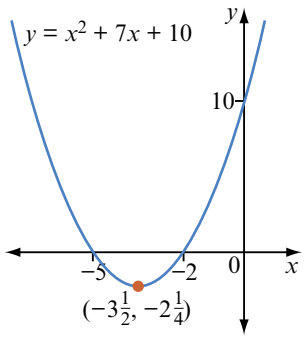
g



h



i

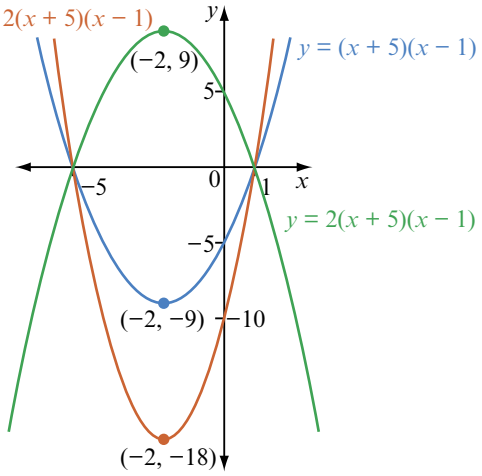


9 a −5 and 1

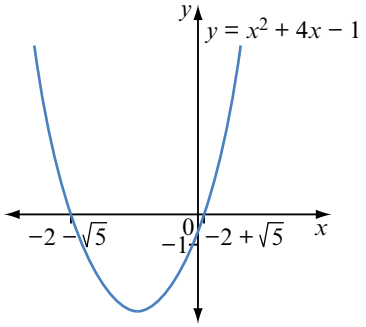
b For $y = 0$ at x-intercepts, the equations are equivalent and have the same solution. However, the y values differ for other x values.

c −5; −10; 5 **d** (−2, −9); (−2, −18); (−2, 9)

e $y = 2(x + 5)(x - 1)$



10 a



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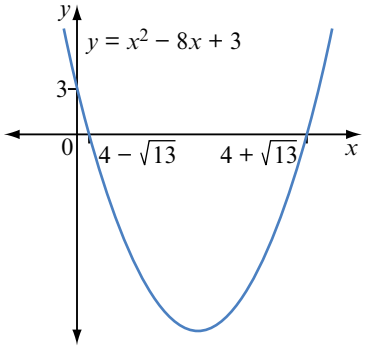
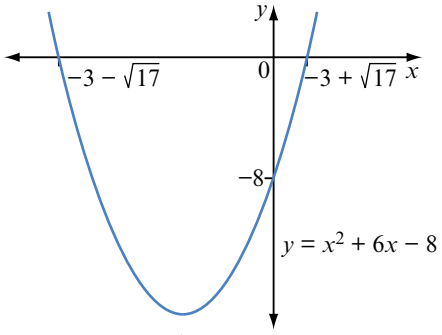


c



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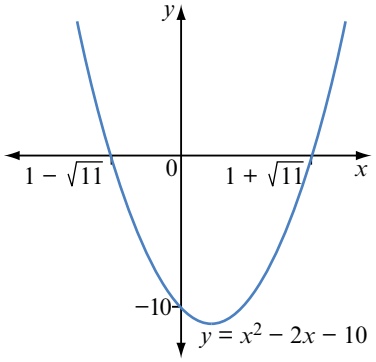

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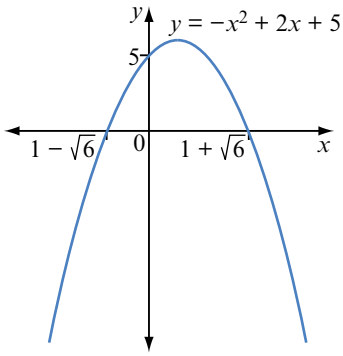

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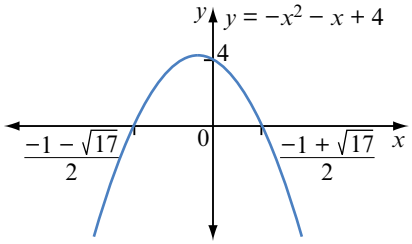


11 a **i** two **ii** one **iii** none

b **i** $x = -1$ or 2 **ii** $x = 2$ **iii** no solution

c **i** $\Delta = (-1)^2 - 4(1)(-2) = 9$. As $\Delta > 0$, there are two solutions to $x^2 - x - 2 = 0$ and hence two x-intercepts.

ii $\Delta = 4^2 - 4(-1)(-4) = 0$. As $\Delta = 0$, there is one solution to $-x^2 + 4x - 4 = 0$ and hence one x-intercept.

iii $\Delta = 0^2 - 4(1)(2) = -8$. As $\Delta < 0$, there are no solutions to $x^2 - x - 2 = 0$ and hence no x-intercepts.

d **i** $x = \dfrac{1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \dfrac{1 \pm \sqrt{9}}{2} = \dfrac{1 \pm 3}{2}$
$= -1$ or 2

ii $x = \dfrac{-4 \pm \sqrt{16 - 4(-1)(-4)}}{-2} = \dfrac{-4 \pm \sqrt{0}}{-2}$
$= \dfrac{-4 \pm 0}{-2} = 2$

iii $x = \dfrac{0 \pm \sqrt{0 - 4(1)(2)}}{2} = \dfrac{\pm\sqrt{-8}}{2}$ = undefined

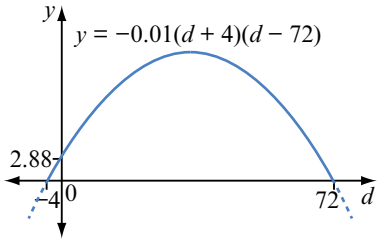
12 a **i** $\Delta = 0$, one x-intercept **ii** (2, 0)

b **i** $\Delta = -8$, no x-intercepts

c **i** $\Delta = 8$, two x-intercepts

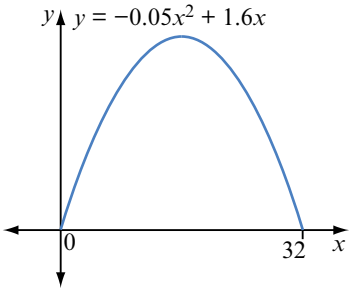
ii $(3 - \sqrt{2}, 0), (3 + \sqrt{2}, 0)$

13 a



b 2.88 m **c** 72 m **d** 14.44 m

14 a



b height of 12.8 m at horizontal distance of 16 m

c 32 m **d** 30.8 m

15 a $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

b Average of x-intercepts

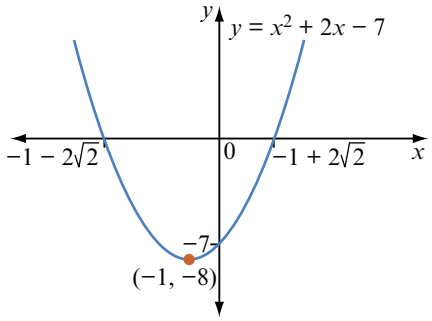
$= \frac{1}{2}\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$

$= \frac{1}{2}\left(\frac{-2b}{2a}\right) = \frac{-b}{2a}$

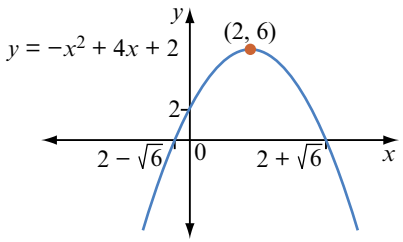
c When $x = \frac{-b}{2a}$, $y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$

$= \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c$ or $c - \frac{b^2}{4a}$

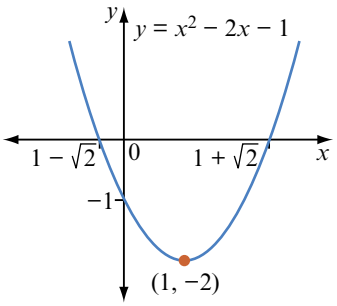
16 a



b



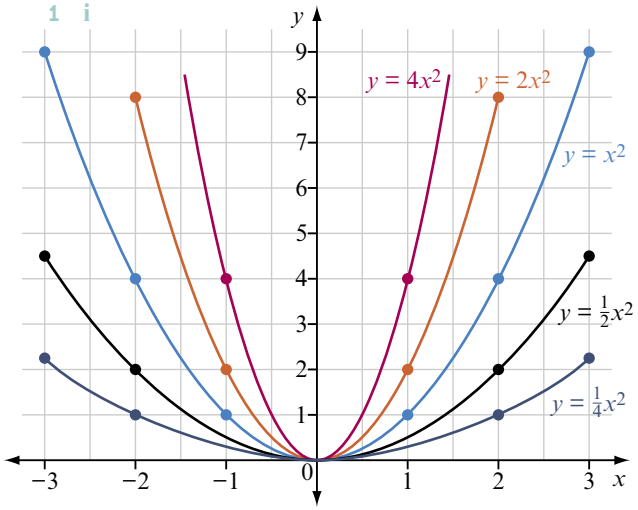
c



5D Sketching parabolas using transformations

5D Start thinking!

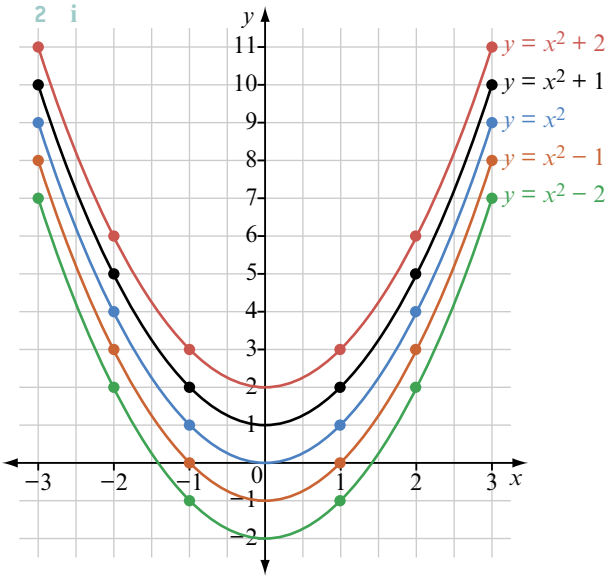
1 i



ii All graphs have same minimum turning point, axis of symmetry and x- and y-intercepts. Graphs of $y = 2x^2$ and $y = 4x^2$ are narrower than $y = x^2$; graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ are wider than $y = x^2$.

iii dilation; graph is narrower when coefficient of x^2 is larger than 1; graph is wider when the coefficient of x^2 is between 0 and 1

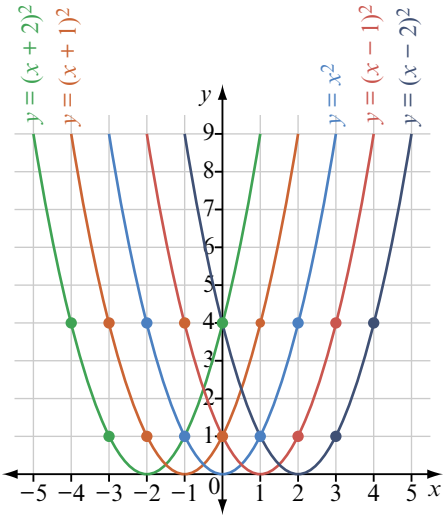
2 i



ii All graphs have same axis of symmetry and shape but different turning points.

iii vertical translation; graph of $y = x^2$ moves up when positive constant is added; graph of $y = x^2$ moves down when negative constant is added (or positive constant is subtracted).

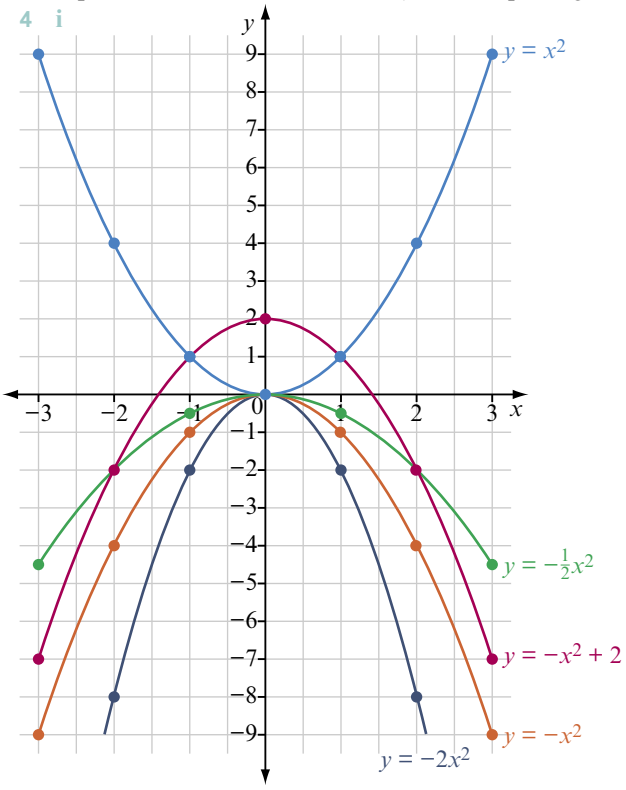
3 i



ii All graphs have same shape but different turning points.

iii horizontal translation; graph of $y = x^2$ moves to right when positive constant is subtracted from x before squaring; graph of $y = x^2$ moves to left when negative constant is subtracted from x (or positive constant is added to x) before squaring.

4 i



ii Graphs b–e are inverted compared to $y = x^2$. Graph b is inverted but has the same shape, axis of symmetry and coordinates of turning point as $y = x^2$. Graph c is inverted and moved 2 units up compared to the graph of $y = x^2$. Graph d is inverted and narrower compared to the graph of $y = x^2$. Graph e is inverted and wider compared to the graph of $y = x^2$.

iii Reflection in the x-axis for all graphs b–e since coefficient of x^2 is negative, also vertical translation for graph c as constant is added to x^2, and dilation for graph d (narrower as coefficient of x^2 is greater than 1) and graph e (wider as coefficient of x^2 is between 0 and 1)

Exercise 5D Sketching parabolas using transformations

1 a moves the graph vertically (translation)

b If k is positive, graph moves up and if k is negative, graph moves down.

c up, down

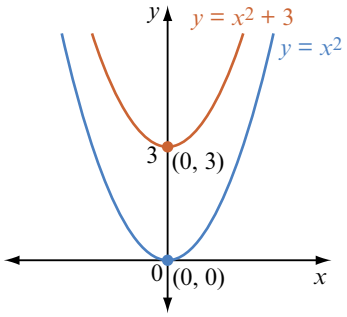
d $(0, k)$

2 a i $k = 3$ ii translated 3 units up

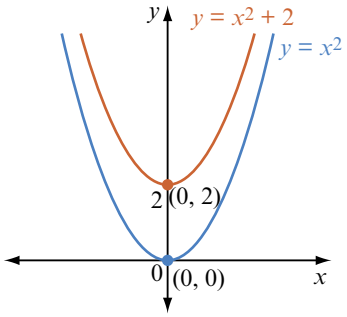
b i $k = -5$ ii translated 5 units down

c i $k = 4.5$ ii translated 4.5 units up

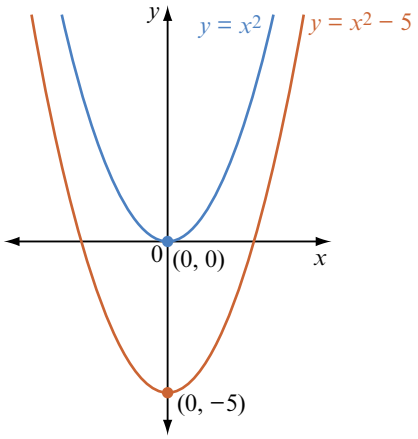
3 a



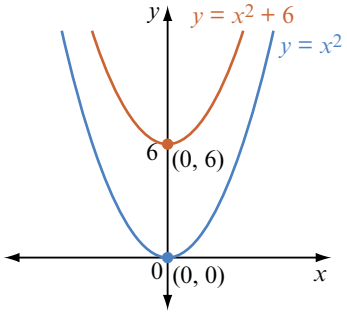
b



c



d



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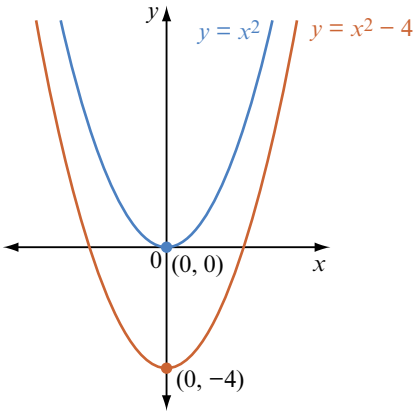

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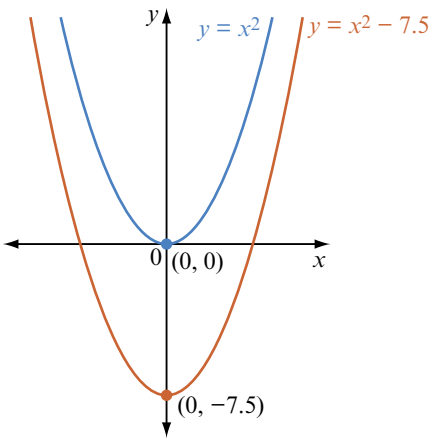


4 a moves the graph horizontally (translation)

b If h is positive, graph moves to right and if h is negative, graph moves to left.

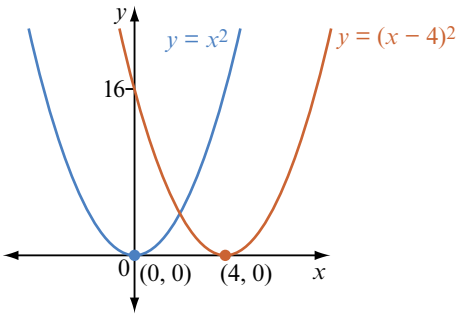
c right, left **d** $(h, 0)$

5 a **i** $h = 3$ **ii** translated 3 units right

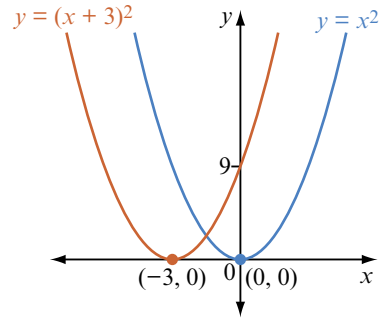
b **i** $h = -6$ **ii** translated 6 units left

c **i** $h = 4.5$ **ii** translated 4.5 units right

6 a



b



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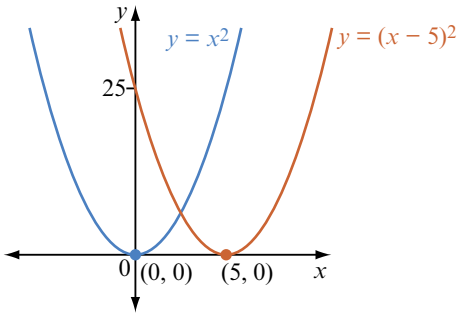

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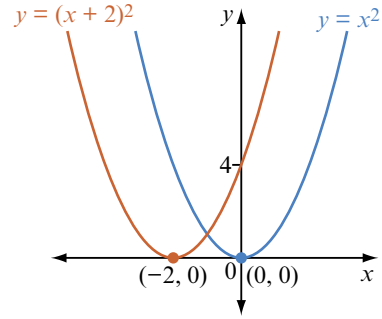

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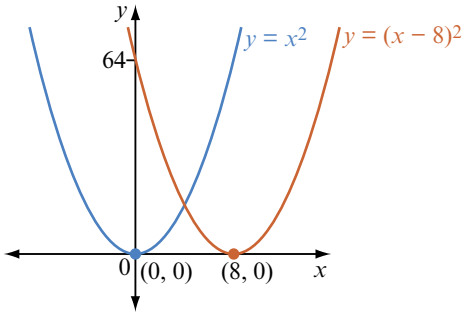

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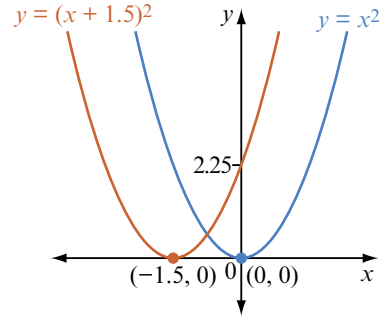


7 a Coefficient (a) of x^2 is a number other than 1 or −1. When $a > 1$ and $a < -1$, graph narrower than graph of $y = x^2$; when $0 < a < 1$ and $-1 < a < 0$, graph is wider than graph of $y = x^2$.

b Coefficient (a) of x^2 is negative. When $a = -1$, no dilation, only reflection in x-axis.

8 a B **b** D **c** A **d** F **e** E **f** C

9 a $y = 6x^2$ **b** $y = -x^2$ **c** $y = (x - 5)^2$

d $y = x^2 - 2$ **e** $y = -\frac{1}{4}x^2$ **f** $y = -x^2 + 3$

10 a $a = 1$, $h = 2$, $k = 5$

b no dilation or reflection; horizontal translation (2 units right) and vertical translation (5 units up)

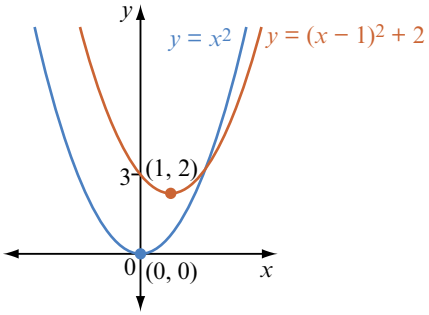
c (2, 5)

d turning point of (0, 0) is translated 2 units right and 5 units up

11 **a** **i** 4 units right, 3 units up **ii** (4, 3)
b **i** 7 units right, 2 units down **ii** (7, −2)
c **i** 3 units left, 1 unit up **ii** (−3, 1)
d **i** 2 units left, 4 units down **ii** (−2, −4)
e **i** 1 unit right, 8 units up **ii** (1, 8)
f **i** 6 units left, 5 units down **ii** (−6, −5)

12 **a** inverted **b** (3, −2) **c** C

13 **a**



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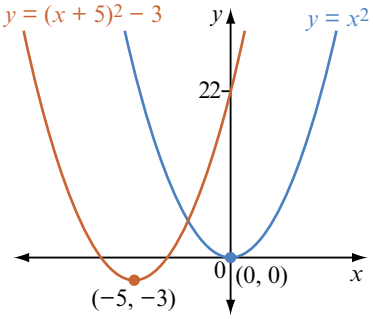

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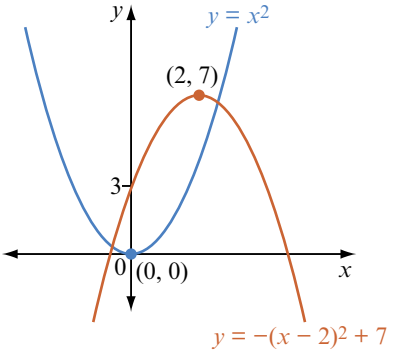

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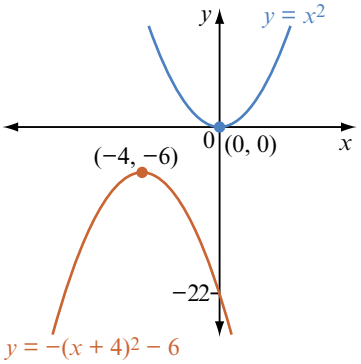

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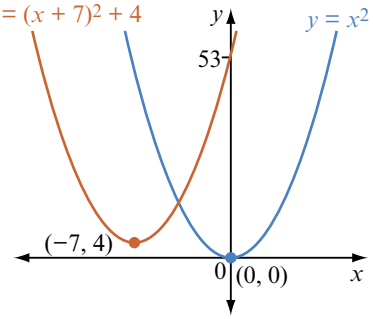

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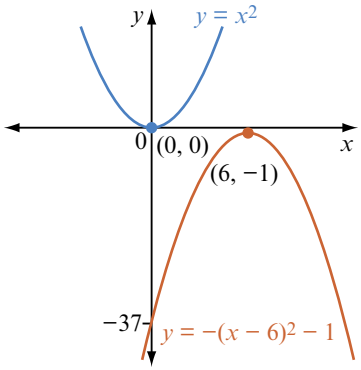


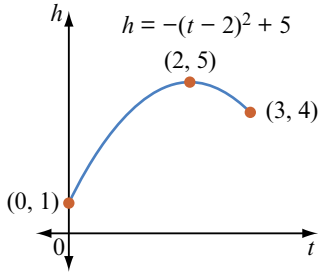
15 **a** **i** narrower **ii** upright **iii** (4, −8)
b **i** narrower **ii** inverted **iii** (−1, 12)
c **i** narrower **ii** upright **iii** (3, 0)
d **i** narrower **ii** inverted **iii** (0, 1)
e **i** wider **ii** upright **iii** (−6, −3)
f **i** wider **ii** inverted **iii** (2, −9)

16 **a** $y = 2(x - 5)^2$ **b** $y = -(x + 4)^2 - 3$
c $y = -\frac{1}{3}x^2$ **d** $y = -5x^2 + 7$

17 4, as graph is upright and has minimum turning point at (−2, 4)

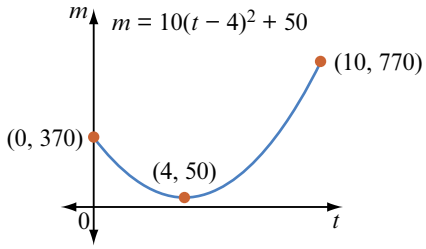
18 −1, as graph is inverted and has maximum turning point at (6, −1)

19 **a** (2, 5)
b



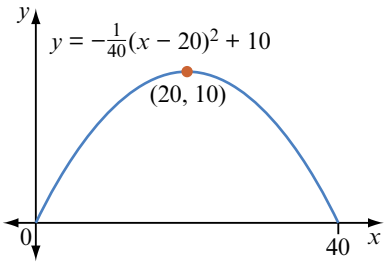
c **i** 1 m **ii** 5 m **iii** 4 m
d 4.2 s

20 **a**



b $370 **c** $50 **d** 8 weeks **e** $770

21 a



b 10 m **c** 40 m

22 a $y = (x - 2)^2 + 6$ **b** $y = 2(x + 4)^2 - 12$
c $y = -3(x + 1)^2 + 7$ **d** $y = -\frac{1}{3}(x - 3)^2 - 5$

23 a x-intercepts: 2, 6; y-intercept: 24
b x-intercepts: −3, 1; y-intercept: 9
c x-intercept: 3; y-intercept: 45
d x-intercepts: $-\frac{1}{2}, \frac{1}{2}$; y-intercept: 1
e x-intercepts: −9, −3; y-intercept: 9
f x-intercepts: none; y-intercept: −11

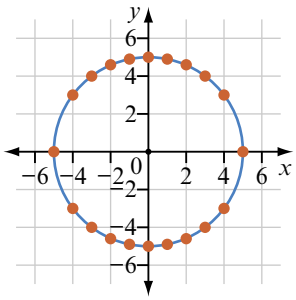
5E Graphs of circles

5E Start thinking!

1 (−5, 0), (5, 0) **2** (0, −5), (0, 5)
3 a −4 and 4 **b** −3 and 3
c no y values exist **d** no y values exist
4 $-5 \leq x \leq 5$; there are no y values that match $x < -5$ and $x > 5$ as $y^2 \neq$ negative number.
5

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	0	±3	±4	$\pm\sqrt{21}$ ≈ ±4.6	$\pm\sqrt{24}$ ≈ ±4.9	±5	$\pm\sqrt{24}$ ≈ ±4.9	$\pm\sqrt{21}$ ≈ ±4.6	±4	±3	0

6



7 (0, 0)
8 5 units; the value on the right side of the rule is the square of the radius ($5^2 = 25$)

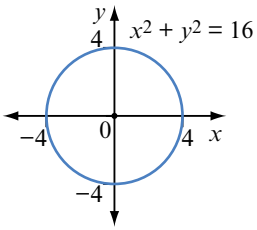
Exercise 5E Graphs of circles

1 a i (0, 0) **ii** (0, 0) **iii** (0, 0)
b i 2 units **ii** 7 units **iii** 10 units
c The left side of each rule is $x^2 + y^2$ and the centre of each circle is at (0, 0). The square of the radius is the same as the value on the right side of the rule.
d Take the positive square root of the value on the right side; that is, $\sqrt{r^2}$.

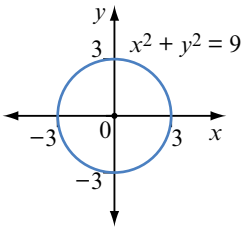
2 a (0, 0), 3 units **b** (0, 0), 1 unit
c (0, 0), 8 units

3 a (0, 0) **b** 4 units
c centre and points on circle directly above, below, left and right of centre

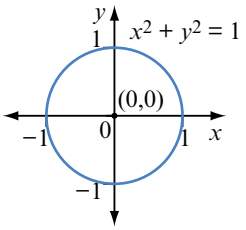
d, e (−4, 0), (0, 4), (4, 0), (0, −4)



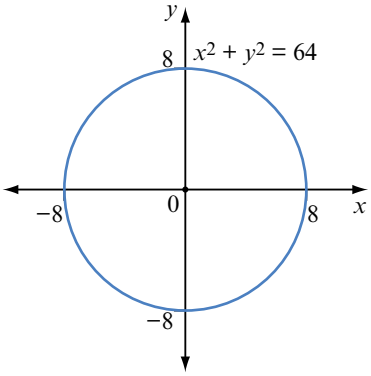
4 a



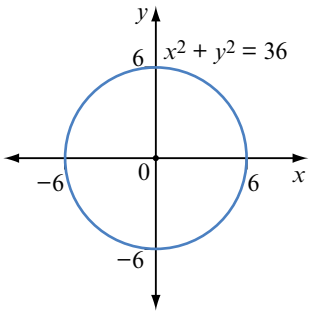
b



c



5 $x^2 + y^2 = 6^2$ or $x^2 + y^2 = 36$



6 $x^2 + y^2 = 9^2$ or $x^2 + y^2 = 81$
7 a $x^2 + y^2 = 144$ **b** $x^2 + y^2 = 225$
c $x^2 + y^2 = 400$ **d** $x^2 + y^2 = 6.25$
8 a i (1, 0) **ii** (0, 3) **iii** (1, 3)
b i translated 1 unit right
ii translated 3 units up
iii translated 1 unit right and 3 units up
c i 2 units **ii** 2 units **iii** 2 units

d The value subtracted from x shows the number of units the basic graph of $x^2 + y^2 = 4$ is translated to the right. The value subtracted from y shows the number of units the graph of $x^2 + y^2 = 4$ is translated up. The coordinates of the centre are translated by the same values.

9 **a** $h = 3, k = 7, r = 6$ **b** (3, 7) **c** 6 units

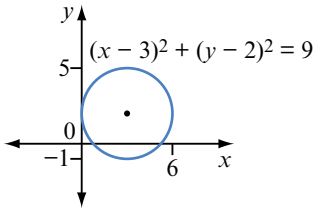
10 **i** **a** $h = 5, k = 2, r = 3$
b (5, 2) **c** 3 units
ii **a** $h = 4, k = -3, r = 1$
b (4, −3) **c** 1 unit
iii **a** $h = -6, k = 1, r = 5$
b (−6, 1) **c** 5 units
iv **a** $h = -2, k = -8, r = 10$
b (−2, −8) **c** 10 units

11 **a** (−3, 0) **b** 5 units
c Since $h = -3$ and $k = 0$, centre coordinates (h, k) are (−3, 0). Since $r = 5$, the radius is 5 units.
d translated 3 units left

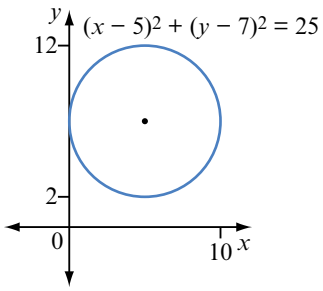
12 **a** (0, 4) **b** 3 units
c Since $h = 0$ and $k = 4$, centre coordinates (h, k) are (0, 4). Since $r = 3$, the radius is 3 units.
d translated 4 units up

13 **a** **i** (3, −5) **ii** (−9, 9)
b **i** 7 units **ii** 9 units
c **i** translated 3 units right and 5 units down
ii translated 9 units left and 9 units up

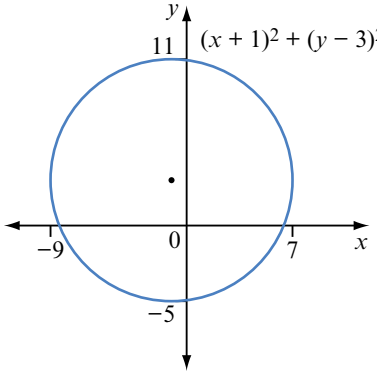
14 **a** centre (3, 2), radius 3 units



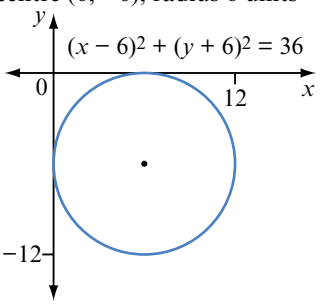
b centre (5, 7), radius 5 units



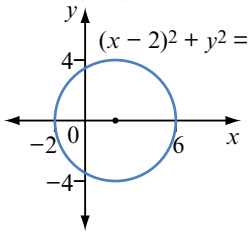
c centre (−1, 3), radius 8 units



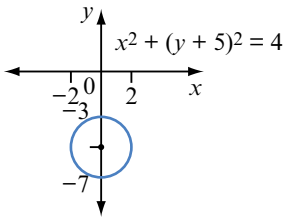
d centre (6, −6), radius 6 units



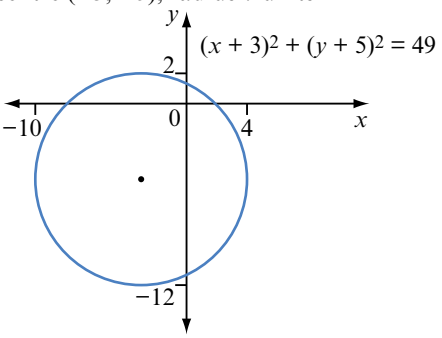
e centre (2, 0), radius 4 units



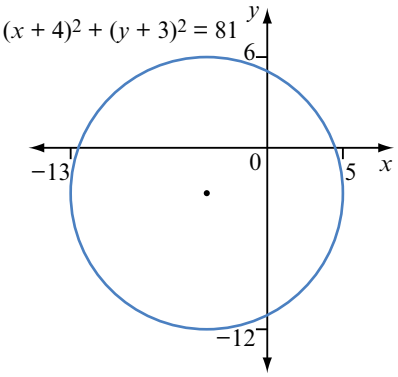
f centre (0, −5), radius 2 units



g centre (−3, −5), radius 7 units



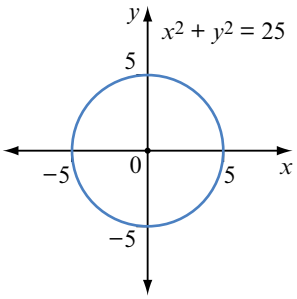
h centre (−4, −3), radius 9 units



16 **a** $(x - 4)^2 + (y + 2)^2 = 1$
b $(x + 5)^2 + (y + 6)^2 = 49$

17 **a** $(x - 7)^2 + (y - 3)^2 = 9$
b $(x + 1)^2 + (y + 4)^2 = 25$
c $(x + 6)^2 + (y - 5)^2 = 4$
d $(x - 2)^2 + (y + 8)^2 = 16$

18 a



b $y = \pm\sqrt{25 - x^2}$

c **i** B; as $y \leq 0$ for $-5 \leq x \leq 5$

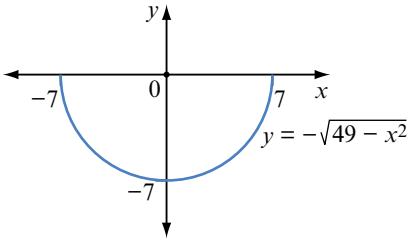
ii A; as $y \geq 0$ for $-5 \leq x \leq 5$

d $y = \sqrt{25 - x^2}$ gives upper half of circle and $y = -\sqrt{25 - x^2}$ gives lower half of circle. Together, $y = \pm\sqrt{25 - x^2}$ or $x^2 + y^2 = 25$ gives the whole circle.

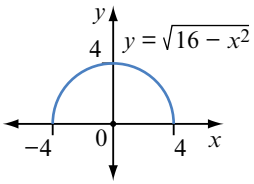
19 a $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{64 - x^2}$

c $y = \sqrt{1 - x^2}$

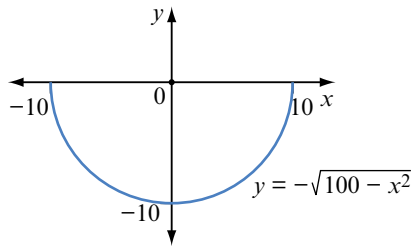
20 a



b



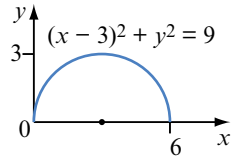
c



21 a $x = \pm\sqrt{25 - y^2}$

b For $x = \sqrt{25 - y^2}$, graph is right half of circle for $x^2 + y^2 = 25$.
Feasible values are $-5 \leq y \leq 5$ and $0 \leq x \leq 5$.
For $x = -\sqrt{25 - y^2}$, graph is left half of circle for $x^2 + y^2 = 25$. Feasible values are $-5 \leq y \leq 5$ and $-5 \leq x \leq 0$.

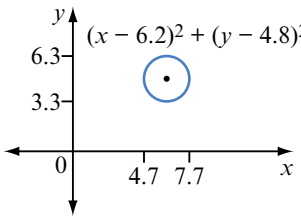
22 a



b 6 m **c** 3 m

d When $y = 1.5$, $x \approx 0.4$ and 5.6, so arch is at half its maximum height at a distance of 0.4 m from either end.

23 a



b 3.3 m **c** 7 m² **d** 4.7%

24 (−4, 3), (−3, 4)

25 a (0, −3), (3, 0) **b** (−3, 6), (−1, 8)

26 $(x + 4)^2 + (y - 5)^2 = 36$

27 Yes, but the reflected graph looks the same as the original. For example, $x^2 + y^2 = r^2$ can be written as two rules: $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. The reflection of $y = \sqrt{r^2 - x^2}$ (upper half circle) in the x-axis produces the graph of $y = -\sqrt{r^2 - x^2}$ (lower half circle) and the reflection of $y = -\sqrt{r^2 - x^2}$ in the x-axis produces the graph of $y = \sqrt{r^2 - x^2}$. Together they produce the whole circle $x^2 + y^2 = r^2$.

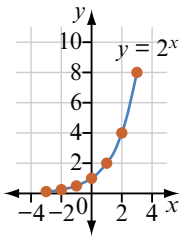
5F Graphs of exponential relationships

5F Start thinking!

1

x	−3	−2	−1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

2



3 a y values get bigger

b y values get smaller; they are positive values that get closer to 0

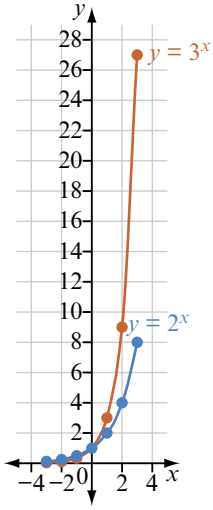
4 1

5 No, there are no x values that make $2^x = 0$.

6 $y = 0$

Exercise 5F Graphs of exponential relationships

1 a



b The curve for $y = 3^x$ becomes steeper as positive x values increase. That is, the y values for $y = 3^x$ are bigger than those for $y = 2^x$ for $x > 0$.
For $x < 0$, the y values for $y = 3^x$ are smaller than those for $y = 2^x$.

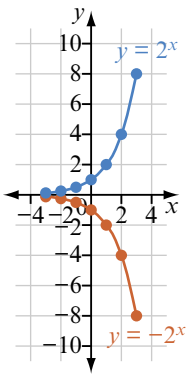
c same asymptote and y-intercept

d 1

e $y = 0$

f The curve for $y = 5^x$ becomes steeper as positive x values increase. That is, the y values for $y = 5^x$ are bigger than those for both $y = 2^x$ and $y = 3^x$ for $x > 0$. For $x < 0$, the y values for $y = 5^x$ are smaller than those for $y = 2^x$ and $y = 3^x$. However, the asymptote and y-intercept would be the same.

2 a



b $y = 2^x$ has positive y values only that increase as x increases. $y = -2^x$ has negative y values only that decrease as x increases.

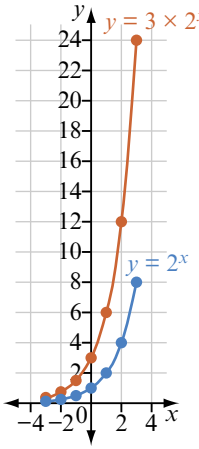
c same asymptote

d y-intercept for $y = 2^x$ is 1; y-intercept for $y = -2^x$ is -1

e $y = 0$

f reflection in x-axis

3 a



b The curve for $y = 3 \times 2^x$ is always above the curve for $y = 2^x$. That is, the y values for $y = 3 \times 2^x$ are three times as big as those for $y = 2^x$.

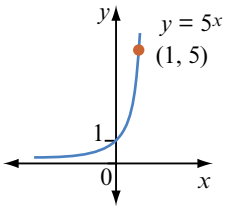
c same asymptote

d y-intercept for $y = 2^x$ is 1; y-intercept for $y = 3 \times 2^x$ is 3

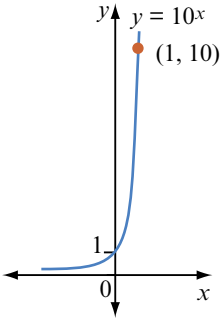
e $y = 0$

f dilation by a factor of 3

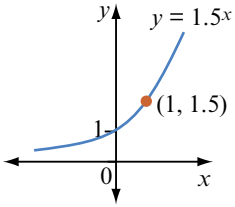
4 a



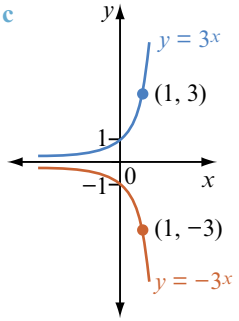
b



c

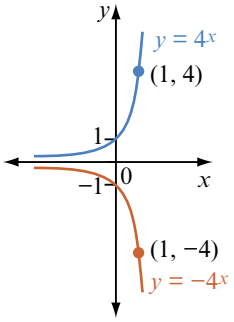


5 a, c

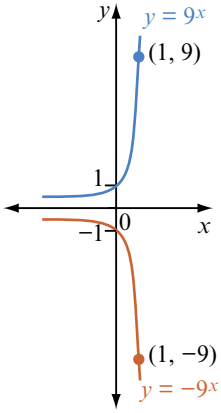


b reflect in x-axis

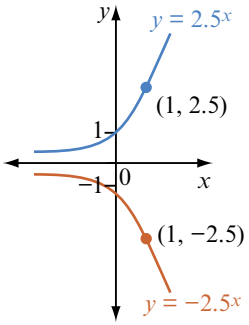
6 a



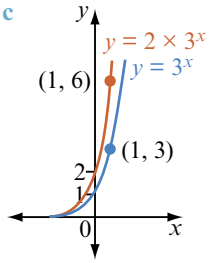
b



c

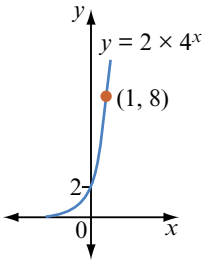


7 a, c

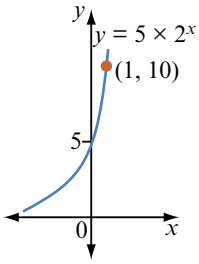


b dilate by factor of 2

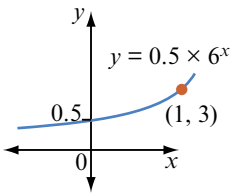
8 a



b



c



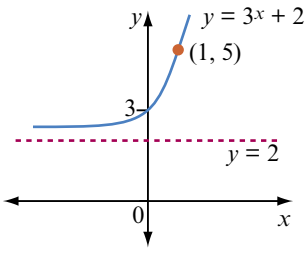
9 a **i** $y = 3$ **ii** $y = -5$

b **i** 4 **ii** -4

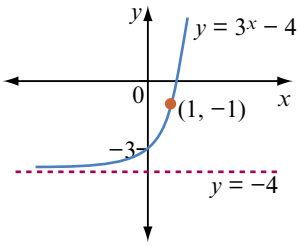
c **i** moved 3 units up **ii** moved 5 units down

d up, down **e** $y = k$; $(0, 1 + k)$

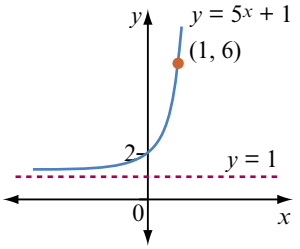
10 a



b



c



11 a i $y = 0$ ii $y = 0$

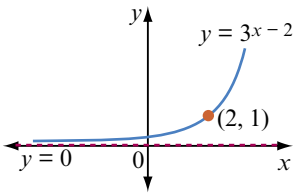
b i $\frac{1}{16}$ ii 8

c i (4, 1) ii (−3, 1)

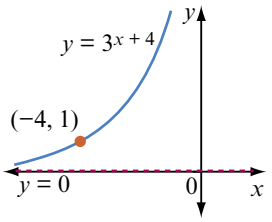
d i moved 4 units right ii moved 3 units left

e right, left **f** $y = 0$; 2^{-h} or $\frac{1}{2^h}$

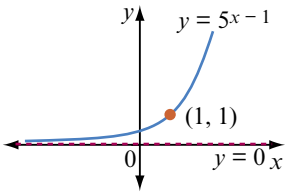
12 a



b



c



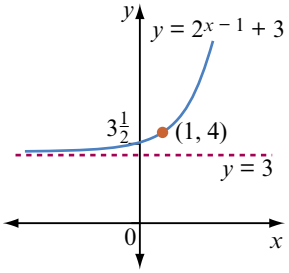
13 a i reflection in x-axis ii $y = -7^x$

b i dilation by factor of 3 ii $y = 3 \times 2^x$

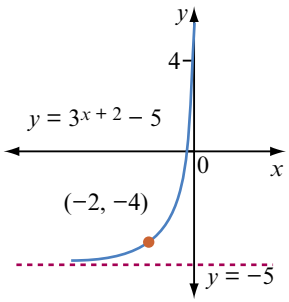
c i translation 2 units up ii $y = 4^x + 2$

d i translation 3 units right ii $y = 5^{x-3}$

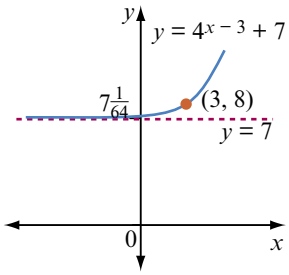
14 a



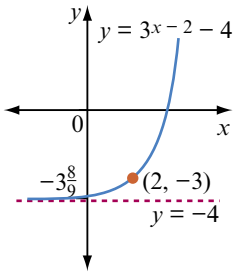
b



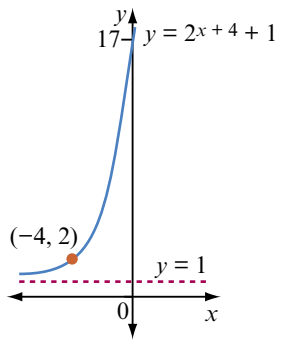
c



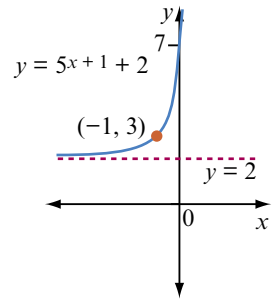
d



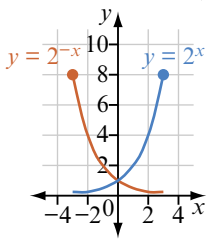
e



f



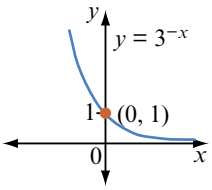
16



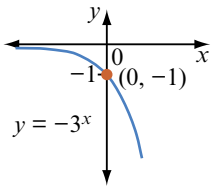
a $y = 2^x$ has positive y values that increase as x increases.
$y = 2^{-x}$ has positive y values that decrease as x increases.
b same asymptote **c** 1 **d** $y = 0$
e reflection in y-axis
f reflection in x-axis and then reflection in y-axis

17 **a** $y = -a^x$ **b** $y = a^{-x}$ **c** $y = -a^{-x}$

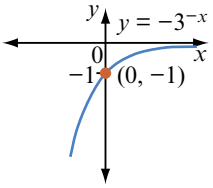
18 **a** **i** reflect $y = 3^x$ in y-axis
ii



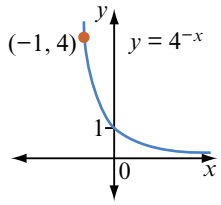
b **i** reflect $y = 3^x$ in x-axis
ii



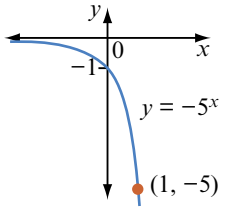
c **i** reflect $y = 3^x$ in x-axis and y-axis
ii



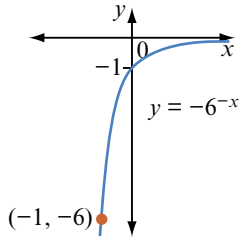
d **i** reflect $y = 4^x$ in y-axis
ii



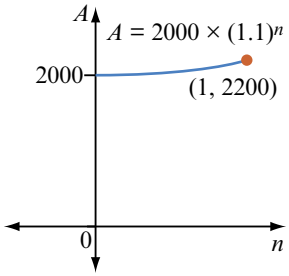
e **i** reflect $y = 5^x$ in x-axis
ii



f **i** reflect $y = 6^x$ in x-axis and y-axis
ii

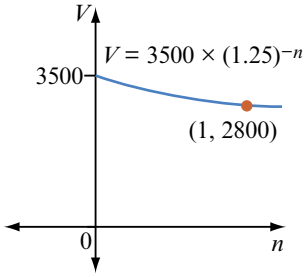


19 **a**



b A increases
c **i** approx. \$2600 (exact value is \$2662)
ii 8 years

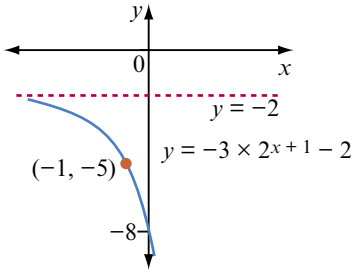
20 **a**



b \$3500 **c** V decreases **d** 6 years

21 **a** **i** dilation by factor of 2, translation 4 units up
ii A: $y = 5^x$; B: $y = 2 \times 5^x + 4$
b **i** translation of 2 units right and 5 units down
ii A: $y = 4^x$; B: $y = 4^{x-2} - 5$
c **i** dilation by factor of 4, reflection in x-axis and y-axis
ii A: $y = 3^x$; B: $y = -4 \times 3^{-x}$

22 dilation by factor of 3, reflection in x-axis, translation of 1 unit left and 2 units down



5G Solving exponential equations

5G Start thinking!

1 **a** $y = -4$ **b** -3
c At the x-intercept, $y = 0$ so substitute 0 for y in $y = 2^x - 4$ and solve for x.
d 2

2 **a** $y = 4$ **b** 5 **c** $2^x = -4$
d no; no; the graph of $y = 2^x$ (and its asymptote) has been translated 4 units up to produce $y = 2^x + 4$ and so $y > 4$.

3 **a** $y = -5$ **b** -4 **c** $2^x = 5$

d yes ($x \approx 2.3$), though it is more difficult to solve since x is not a whole number; yes; the graph of $y = 2^x$ (and its asymptote) has been translated 5 units down to produce $y = 2^x - 5$ and so the curve will cross the x-axis

Exercise 5G Solving exponential equations

1 **a** $x = 9$ **b** $x = 6$ **c** $x = -3$ **d** $x = 1$
e $x = 8$ **f** $x = 4$ **g** $x = 7$ **h** $x = -10$
i $x = 3$ **j** $x = 5$ **k** $x = 6$ **l** $x = 0$

2 **a** $x = 3$ **b** $x = 2$ **c** $x = 5$ **d** $x = 4$
e $x = 7$ **f** $x = 3$ **g** $x = 2$ **h** $x = -1$
i $x = \frac{7}{3}$ **j** $x = \frac{3}{2}$ **k** $x = \frac{3}{4}$ **l** $x = \frac{5}{4}$

3 **a** 2 and 3 **b** 5 and 6 **c** 1 and 2 **d** 3 and 4
e 1 and 2 **f** 2 and 3 **g** 1 and 2 **h** 4 and 5

4 **a** x **b** $x = \log_2 11$

5 **a** $x - 3$ **b** $x - 3 = \log_5 9$

6 **a** $x = \log_3 8$ **b** $x = \log_7 4$
c $x = \log_2 37$ **d** $x = \log_4 3$
e $x - 2 = \log_6 20$ **f** $x + 8 = \log_5 17$
g $5 - x = \log_8 9$ **h** $-x - 1 = \log_2 50$
i $3x = \log_7 11$ **j** $5x = \log_9 22$
k $2x - 7 = \log_3 19$ **l** $6x + 3 = \log_5 41$

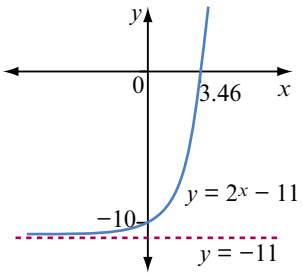
7 **a** $x = 3.46$ **b** $x = 2.81$ **c** $x = -0.63$
d $x = 1.68$ **e** $x = 2.71$ **f** $x = -2.93$
g $x = -2.58$ **h** $x = -3.86$ **i** $x = 0.88$
j $x = 0.63$ **k** $x = 2.65$ **l** $x = 0.77$

8 **a** $x = 1.77$ **b** $x = 3.17$ **c** $x = 2.02$
d $x = 3.31$ **e** $x = 5.13$ **f** $x = 0.36$
g $x = 4.24$ **h** $x = 0.18$

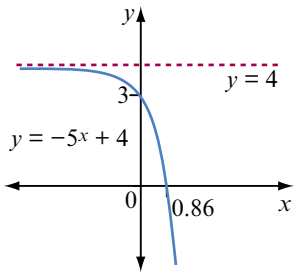
9 **a** $x = 1.58$ **b** $x = 0.47$ **c** $x = -0.54$
d $x = -0.05$ **e** $x = 7.25$ **f** $x = -4.05$
g $x = -3.46$ **h** $x = -5.44$ **i** $x = 3.70$
j $x = -8.39$ **k** $x = 2.76$ **l** $x = -1.87$

10 **a** 3.46 **b** 0.86 **c** 3.77
d −1.21 **e** 0.74 **f** 5.63

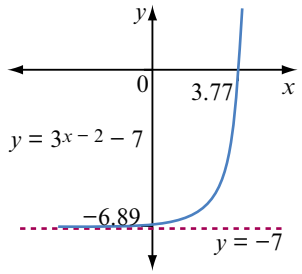
11 **a**



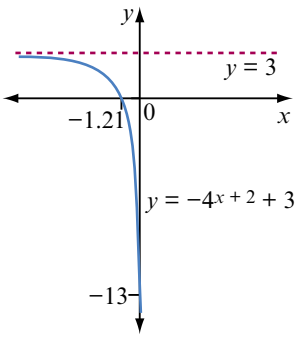
b



c



d



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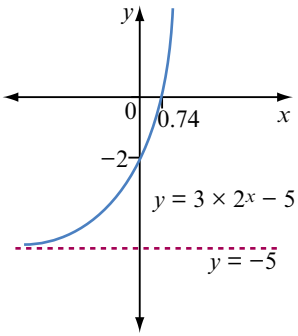

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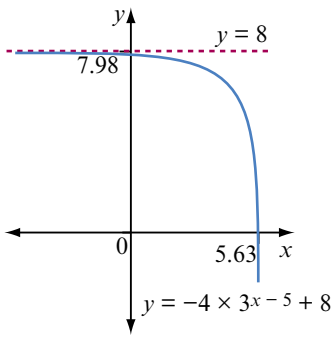
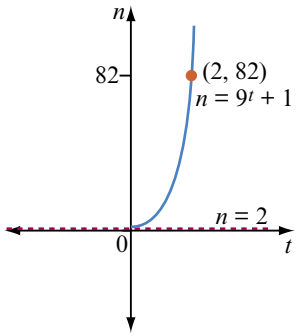
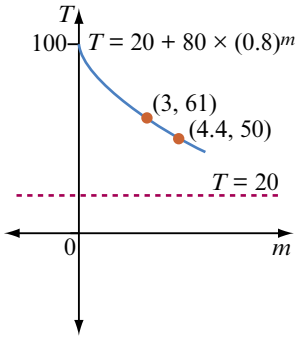


12 **a** 2 **b** 82 **c** 3.14
d population growth; population increases as amount of time increases



13 **a** 100°C **b** 61°C **c** 4.4 **d** 20°C
e decay; temperature decreases as amount of time increases



14 **a** **i** 8.5 billion **ii** 11.9 billion
b 2054
c model predicts 7.1 billion; changes to birth and death rates so overall growth rate has decreased since 1950

15 **a** **i** 7.7 billion **ii** 9.8 billion
b 2076
c model predicts 6.8 billion; this is a better model as it predicts a population of 6.8 billion in 2010, which is closer to the actual estimate of 6.9 billion. It uses a more recently calculated growth rate (pre-2000 compared to pre-1950).

17 one day; pond will be half covered after 29 days

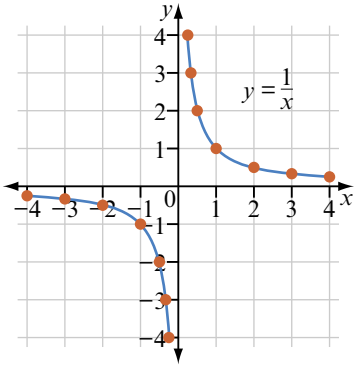
5H Graphs of hyperbolas

5H Start thinking!

1

x	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	-4	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

2



3 **a** y values are positive and decreasing (approaching 0)
b y values are positive and increasing (approaching ∞)
c y values are negative and increasing (approaching 0)
d y values are negative and decreasing (approaching $-\infty$)

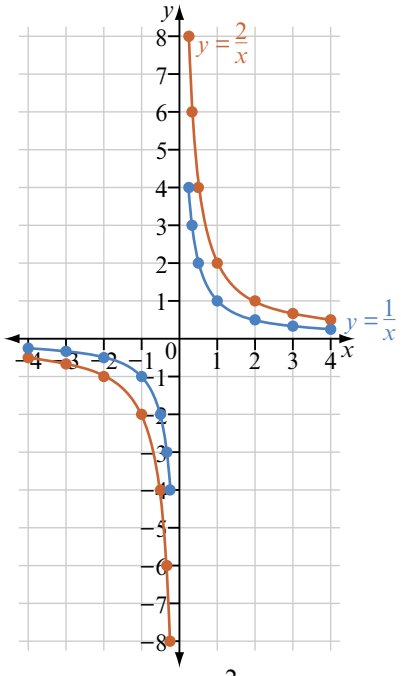
4 At any y-intercept, $x = 0$. However, $\frac{1}{0}$ is undefined so $x \neq 0$.

5 At any x-intercept, $y = 0$. However, there are no x values that make $\frac{1}{x} = 0$.

6 Vertical asymptote runs along the y-axis and has the rule $x = 0$. Horizontal asymptote runs along the x-axis and has the rule $y = 0$.

Exercise 5H Graphs of hyperbolas

1 **a**



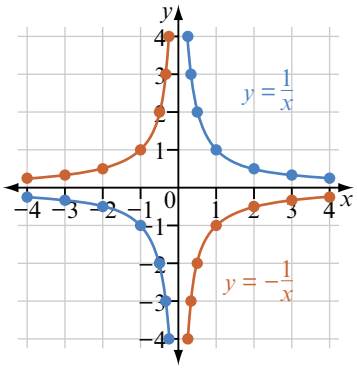
b The y values for $y = \frac{2}{x}$ are twice the size of those for $y = \frac{1}{x}$. For $x > 0$, the graph of $y = \frac{2}{x}$ sits above the graph of $y = \frac{1}{x}$ and its curve is steeper as x approaches 0. For $x < 0$, the graph of $y = \frac{2}{x}$ sits below the graph of $y = \frac{1}{x}$ and its curve is steeper as x approaches 0.

c curves appear in first and third quadrants of Cartesian plane, same asymptotes and no axis intercepts

d $x = 0$, $y = 0$ **e** dilation by factor of 2

f The y values for $y = \frac{5}{x}$ would be five times the size of those for $y = \frac{1}{x}$ (dilation by factor of 5). For $x > 0$, the graph of $y = \frac{5}{x}$ would sit above the graphs of $y = \frac{1}{x}$ and $y = \frac{2}{x}$ and its curve would be steeper as x approaches 0. For $x < 0$, the graph of $y = \frac{5}{x}$ would sit below the graphs of $y = \frac{1}{x}$ and $y = \frac{2}{x}$ and its curve would be steeper as x approaches 0. However, the asymptotes would be the same.

2 a

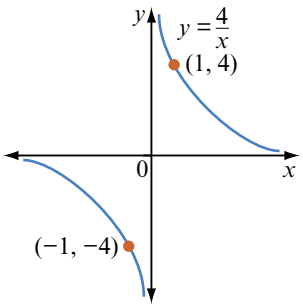


b The graph of $y = -\frac{1}{x}$ appears in the second and fourth quadrants of the Cartesian plane while the graph of $y = \frac{1}{x}$ appears in the first and third quadrants.

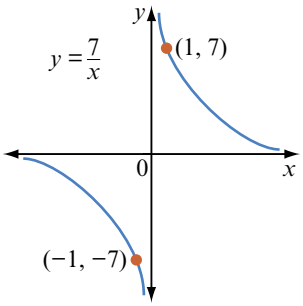
c curves have the same shape and asymptotes and no axis intercepts

d reflection in the x-axis (or reflection in the y-axis)

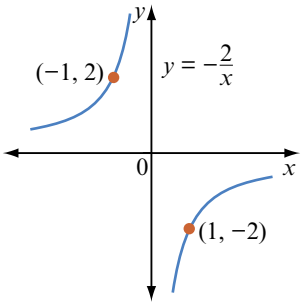
3 a



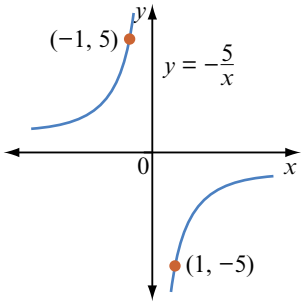
b



c



d

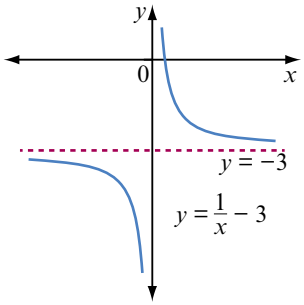


4 a $x = 0$, $y = 2$ **b** x-intercept at $-\frac{1}{2}$

c translation of 2 units up

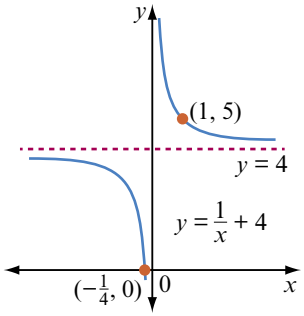
d translation of 3 units down

e

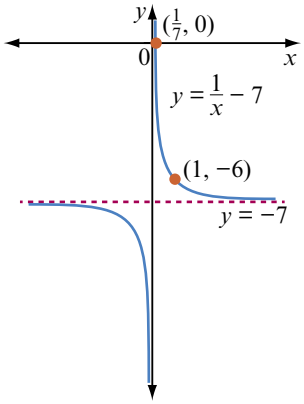


f up, down **g** $x = 0$, $y = k$; $(-\frac{1}{k}, 0)$

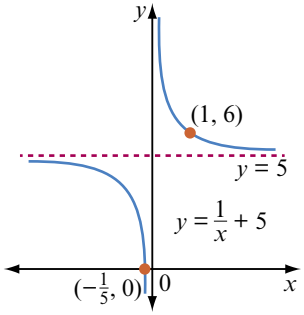
5 a



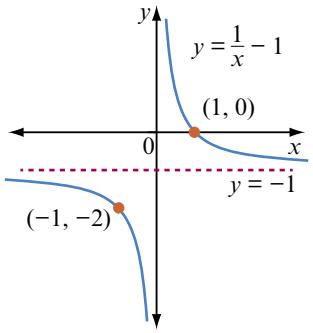
b



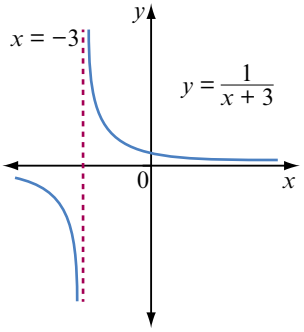
c



d

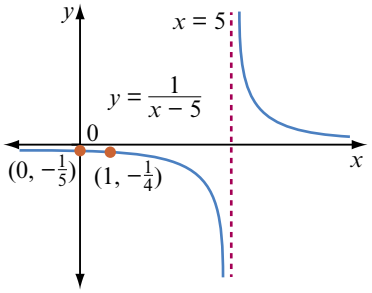


6 **a** $x = 2, y = 0$ **b** y-intercept at $-\frac{1}{2}$
c translation of 2 units right
d translation of 3 units left
e

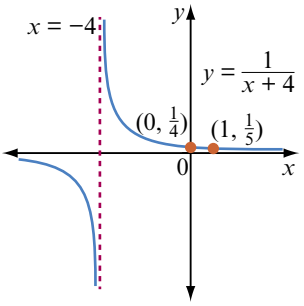


f right, left **g** $x = h, y = 0; (0, -\frac{1}{h})$

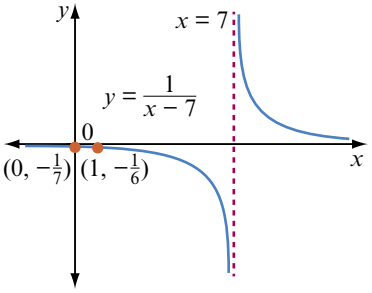
7 **a**



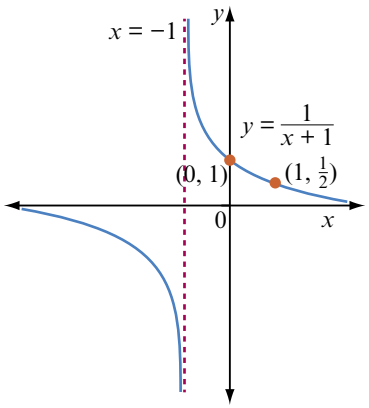
b



c

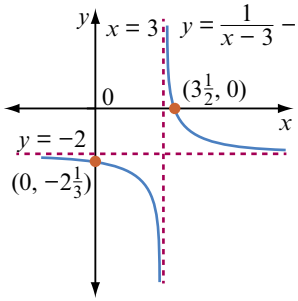


d

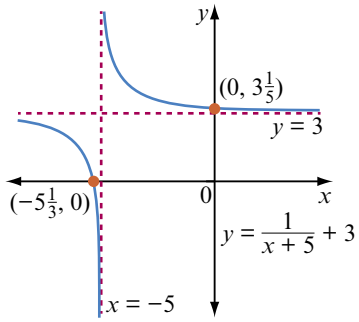


8 **a** move 2 units down **b** move 4 units right
c move 9 units up **d** move 6 units left
e move 3 units right, 8 units up
f move 6 units left, 2 units down
g move 4 units right, 5 units down
h move 9 units left, 11 units up

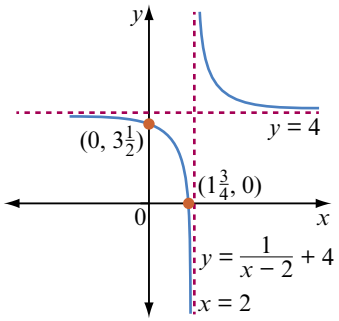
9 **a**



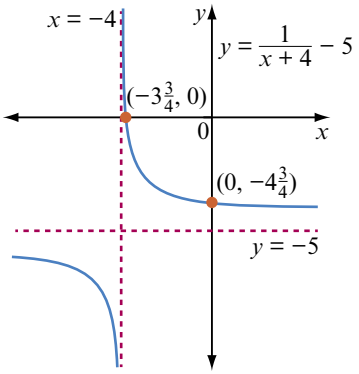
b



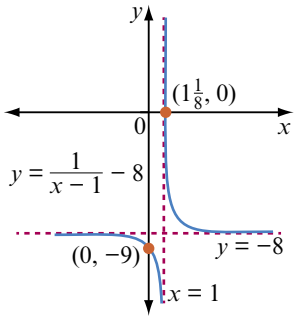
c



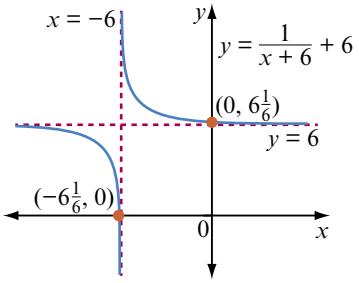
d



e

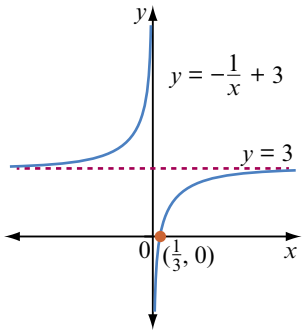


f



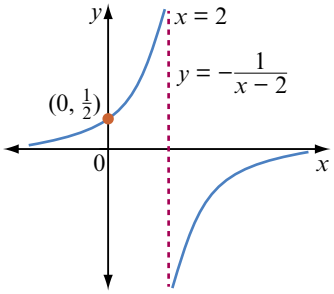
11 a i reflection in x-axis and translation 3 units up

ii



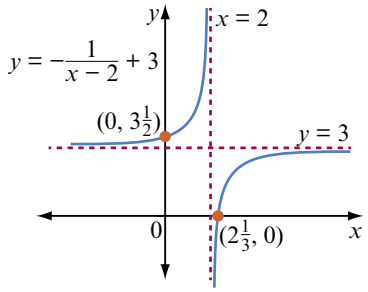
b i reflection in x-axis and translation 2 units right

ii

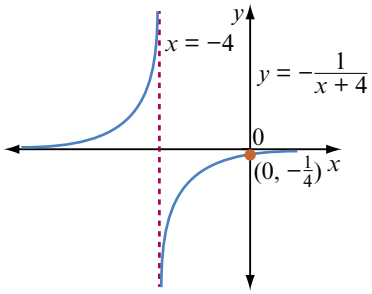


c i reflection in x-axis and translation 3 units up and 2 units right

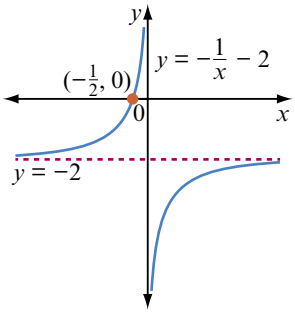
ii



12 a



b



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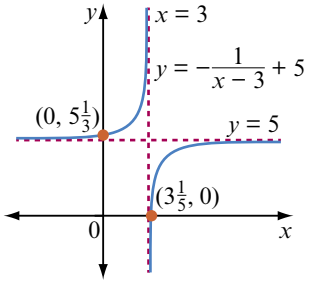

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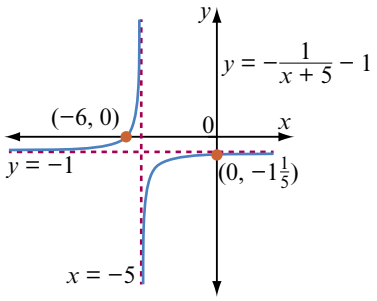

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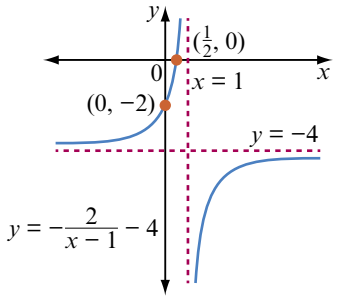

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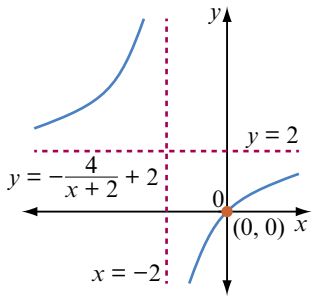


14 a i translation 2 units right and 2 units up

ii $y = \frac{1}{x-2} + 2$

b i reflection in x-axis, translation 1 unit down

ii $y = -\frac{1}{x} - 1$

c i dilation by factor of 4, translation 4 units left and 2 units down

ii $y = \frac{4}{x+4} - 2$

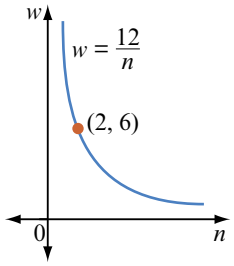
d i reflection in x-axis, dilation by factor of 3, translation 1 unit right and 3 units up

ii $y = -\frac{3}{x-1} + 3$

15 Reflecting in x-axis: curve in first quadrant reflects to fourth quadrant, curve in third quadrant reflects to second quadrant. Rule becomes $y = -\left(\frac{1}{x}\right)$ or $y = -\frac{1}{x}$. Reflecting in y-axis: curve in first quadrant reflects to second quadrant, curve in third quadrant reflects to fourth quadrant. Rule becomes $y = \frac{1}{(-x)}$ or $y = -\frac{1}{x}$.

16 a zero **b** infinity
c negative infinity **d** zero

17 a



b i 6 min **ii** 4 min **iii** 3 min
iv 2 min **v** 1 min **vi** 30 s

c 12 min when there is one bank teller on duty

d no, as $\frac{12}{n} \neq 0$; however, w approaches 0 as n increases

18 a i two **ii** $(-1, -1)$ and $(1, 1)$
b i none **ii** $(-4, -1)$ and $(3, 6)$
c i two **ii** $(-4, -1)$ and $(3, 6)$
d i one **ii** $(2, -\frac{1}{2})$

19 $y = \frac{8}{x-2}$ **20** $y = \frac{3}{x-1} + 2$ **21** $y = \frac{4}{x+3}$

5I Sketching non-linear relationships using transformations

5I Start thinking!

1 a C **b** D **c** B **d** A

2 a circle with centre at (0, 0) and radius of r units; x-intercepts $-r$, r; y-intercepts $-r$, r

b rectangular hyperbola; asymptotes at $y = 0$ and $x = 0$; no x- or y-intercepts; as $x \to \infty$, $y \to 0$ from positive side; as $x \to -\infty$, $y \to 0$ from negative side; as $x \to 0$ from positive side, $y \to \infty$; as $x \to 0$ from negative side, $y \to -\infty$

c parabola; minimum turning point at (0, 0); x-intercept 0; y-intercept 0

d exponential relationship; asymptote at $y = 0$; no x-intercept; y-intercept 1; as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to 0$

3 The graph of a linear relationship is a straight line; non-linear relationships are curves.

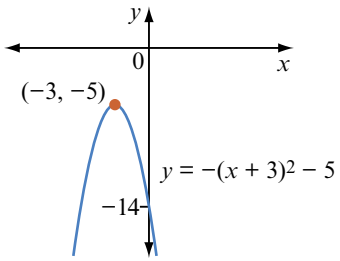
Exercise 5I Sketching non-linear relationships using transformations

1 a i parabola with maximum turning point

ii Reflection in x-axis and translation of 3 units left and 5 units down to be performed on the basic graph of $y = x^2$.

iii maximum turning point at $(-3, -5)$, no x-intercept, y-intercept of -14

iv

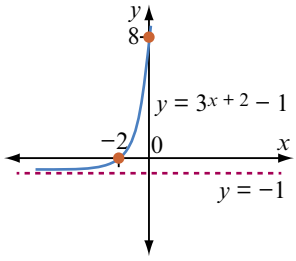


b i exponential relationship

ii Translation of 2 units left and 1 unit down to be performed on basic graph of $y = 3^x$.

iii asymptote at $y = -1$, x-intercept of -2, y-intercept of 8

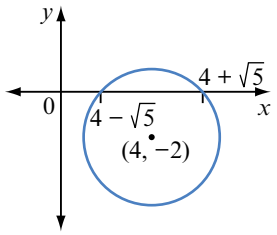
iv



c i circle with radius of 3 units

ii Translation of 4 units right and 2 units down to be performed on basic graph of $x^2 + y^2 = 9$.

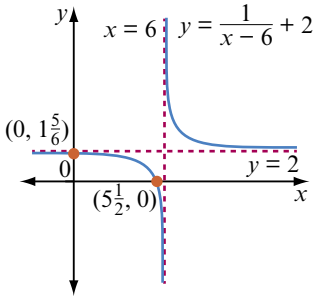
iii centre at (4, −2), radius of 3 units, x-intercepts of $4 - \sqrt{5}$ and $4 + \sqrt{5}$, no y-intercepts

iv 

d i Rectangular hyperbola with two asymptotes

ii Translation of 6 units right and 2 units up to be performed on basic graph of $y = \frac{1}{x}$.

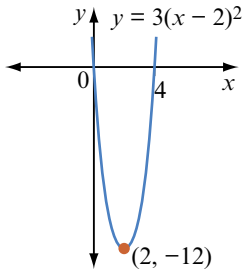
iii asymptotes at $y = 2$ and $x = 6$, x-intercept of $5\frac{1}{2}$, y-intercept of $1\frac{5}{6}$

iv 

e i parabola with minimum turning point

ii Dilation by factor of 3 and translation of 2 units right and 12 units down to be performed on the basic graph of $y = x^2$.

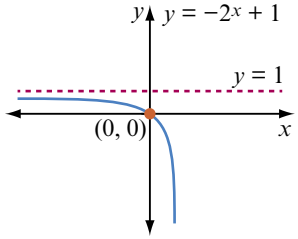
iii minimum turning point at (2, −12), x-intercepts of 0 and 4, y-intercept of 0

iv 

f i exponential relationship

ii Reflection in x-axis and translation of 1 unit up to be performed on basic graph of $y = 2^x$.

iii asymptote at $y = 1$, x-intercept of 0, y-intercept of 0

iv 

2 a i Hyperbola with two curves that approach asymptotes. As $x \to \infty$, $y \to -3^+$ and as $x \to -1^+$, $y \to \infty$. As $x \to -\infty$, $y \to -3^-$ and as $x \to -1^-$, $y \to -\infty$.

ii Basic graph is $y = \frac{1}{x}$ with asymptotes at $y = 0$ and $x = 0$ and no intercepts.

iii dilation by factor of 6, translation of 1 unit left and 3 units down

iv $y = \frac{6}{x + 1} - 3$

b i Circle with centre at (5, −3) and radius 5 units; x-intercepts of 1 and 9; y-intercept of −3.

ii Basic graph is $x^2 + y^2 = 25$ with centre at (0, 0) and radius 5 units; x-intercepts of −5 and 5; y-intercepts of −5 and 5.

iii translation of 5 units right and 3 units down

iv $(x - 5)^2 + (y + 3)^2 = 25$

c i Parabola with maximum turning point at (1, 8); x-intercepts of −1 and 3; y-intercept of 6.

ii Basic graph is $y = x^2$ with minimum turning point at (0, 0), x-intercept of 0 and y-intercept of 0.

iii reflection in x-axis, dilation by factor of 2, translation of 1 unit right and 8 units up

iv $y = -2(x - 1)^2 + 8$

d i Exponential relationship that approaches a minimum value of −8; x-intercept of 3 and y-intercept of −7. As x increases, y increases exponentially. As x decreases, y decreases but approaches a boundary or minimum value. That is, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -8^+$.

ii Basic graph is $y = a^x$ with asymptote at $y = 0$ and y-intercept of 1.

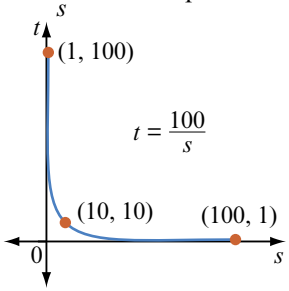
iii translation of 8 units down, base a is 2

iv $y = 2^x - 8$

3 a $x^2 + y^2 = 16$ b $(x + 7)^2 + (y - 5)^2 = 16$

c no

4 a $t = \frac{100}{s}$ b positive curve of hyperbola

c 

d as s increases, t decreases

5 Graph is inverted parabola with origin located where left edge of arch meets the ground. Turning point at (4.6, 7.1) is at the highest point of the arch. Arch is 7.1 m high and 9.2 m wide at the base.

6 Both graphs are the same; reflection in the y-axis of basic graph $y = 2^x$.

$y = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$

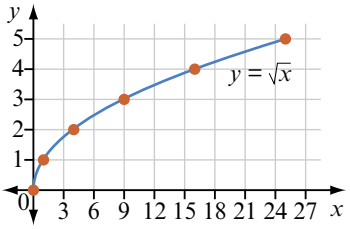
7 a Dilation must occur in both the x and y directions so the new graph is still a circle.

b circle with centre at (0, 0), radius of $\sqrt{\frac{4}{a}}$ units; when $a = 2$, radius is $\sqrt{2}$ units (≈ 1.4 units) so dilation produces a circle with a smaller radius

c circle with centre at (0, 0), radius of $\sqrt{\frac{4}{a}}$ units; when $a = \frac{1}{2}$, radius is $\sqrt{8}$ units (≈ 2.8 units) so dilation produces a circle with a larger radius

8 a, c

x	0	1	4	9	16	25
y	0	1	2	3	4	5

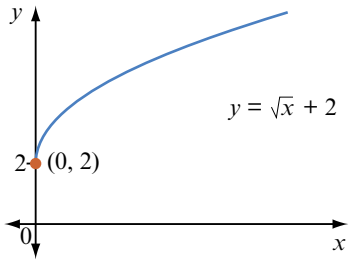


b curve with minimum value at (0, 0); no points for negative x as not possible to take square root of negative number

9 a (5, 0) **b** translate 5 units right

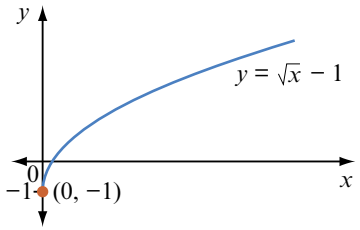
10 a i translate 2 units up **ii** (0, 2)

iii



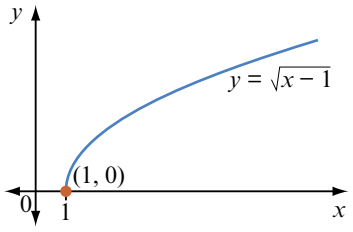
b i translate 1 unit down **ii** (0, −1)

iii



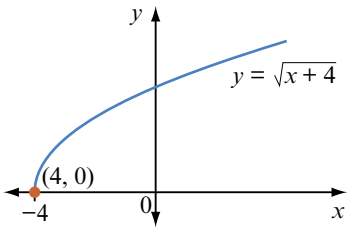
c i translate 1 unit right **ii** (1, 0)

iii



d i translate 4 units left **ii** (−4, 0)

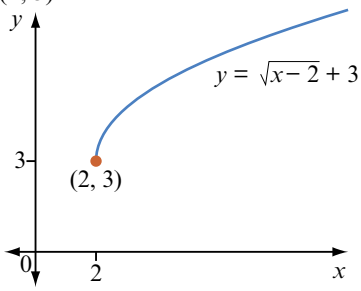
iii



e i translate 2 units right and 3 units up

ii (2, 3)

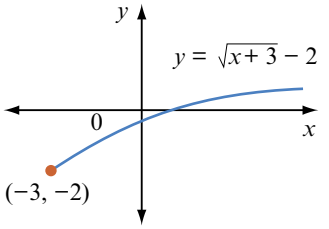
iii



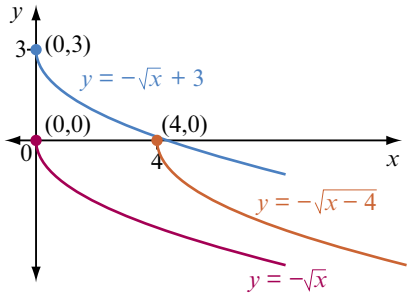
f i translate 3 units left and 2 units down

ii (−3, −2)

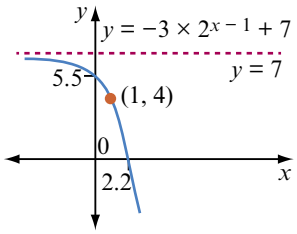
iii



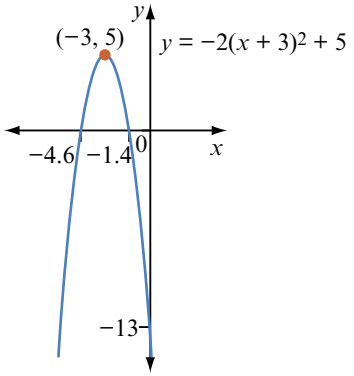
11



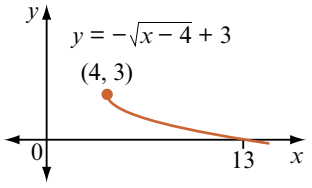
12 a



b



c



5 Chapter review

MULTIPLE-CHOICE

1 A 2 C 3 A 4 C 5 D 6 B
7 A 8 C 9 A 10 D 11 B

SHORT ANSWER

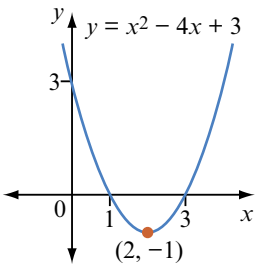
1 a $x = 2$ b $x = \frac{3}{4}$ or $x = -\frac{4}{3}$
c $x = \pm\sqrt{2}$ d $x = -2$ or $x = 4$
e $x = \pm\sqrt{\frac{7}{5}}$

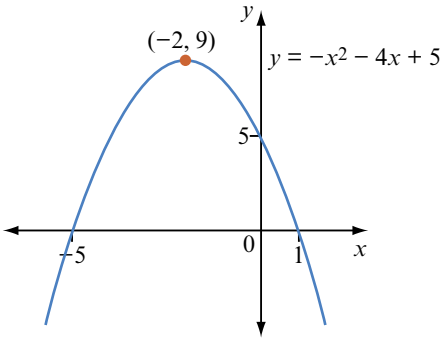
2 a $x = -3 \pm \sqrt{5}$ b $x = \frac{3 \pm \sqrt{21}}{2}$

3 a 1 b 0 c 0 d 2 e 2 f 0

4 a i $x = -5$ or $x = 2$
ii $x = -5.00$ or $x = 2.00$
b i $x = \frac{3 \pm \sqrt{29}}{10}$ ii $x = -0.24$ or $x = 0.84$
c i $x = \frac{5 \pm \sqrt{17}}{2}$ ii $x = 0.44$ or $x = 4.56$
d i $x = \frac{-5 \pm \sqrt{17}}{2}$ ii $x = -4.56$ or $x = -0.44$
e i $x = \frac{1 \pm \sqrt{17}}{4}$ ii $x = -0.78$ or $x = 1.28$
f i $x = \frac{-7 \pm \sqrt{13}}{6}$ ii $x = -1.77$ or $x = -0.57$

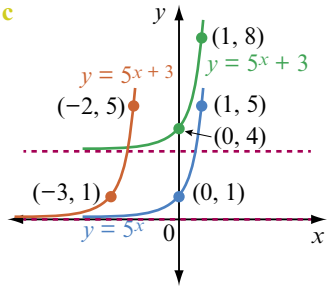
5 a i $(-5, 0)$ and $(5, 0)$ ii $(0, 25)$
b i $(-1, 0)$ and $(4, 0)$ ii $(0, -4)$
c i $(2, 0)$ ii $(0, 8)$
d i $(-3, 0)$ and $(2, 0)$ ii $(0, 6)$
e i $(1, 0)$ and $(10, 0)$ ii $(0, 10)$

6 a 

b 

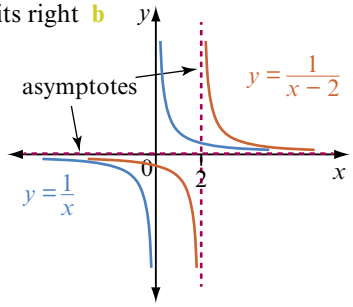
7 a i 5 units right, 2 units up ii (5, 2)
b i 6 units left, 4 units down ii $(-6, -4)$
c i 1 unit right, 1 unit down ii (1, −1)
d i 3 units left, 5 units up ii $(-3, 5)$
e i 4 units right, 2 units up ii (4, 2)
f i 7 units left, 3 units down ii $(-7, -3)$

8 $(x - 4)^2 + (y - 2)^2 = 4$

9 a, c 

b i translate 3 units left ii translate 3 units up
10 a $x = 0$ b $x = -\frac{1}{4}$ c $x \approx 0.56$ d $x \approx 0.54$
11 a translate 5 units down b reflect in the x-axis
c dilate by a factor of 4
d translate 5 units left and 6 units up
12 a circle b hyperbola
c exponential relationship d parabola
13 $y = \frac{1}{x + 4} - 6$

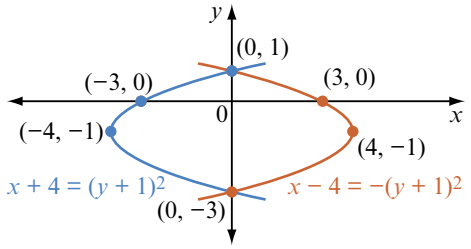
MIXED PRACTICE

1 D
2 a 2 b 1 c 1 d 2 e 2 f 0
3 A
4 a $(-1, -9)$ b (8, 64) c (2, 9)
d $(-5, -9)$ e $(-4, -1)$ f $(2\frac{1}{2}, -\frac{1}{4})$
5 a i narrower ii upright iii $(3, -1)$
b i wider ii upright iii $(-1, 2)$
c i narrower ii inverted iii (3, 0)
6 a reflection in x-axis
b dilation by a factor of 3
c translate 3 units right d translate 3 units down
7 C 8 $(-7.2, 0)$ and $(-2.8, 0)$
9 a $(-3, 2)$ b 5 units c $(1, -1)$
10 D
11 a $k = 25$ b $k > 25$ c $k < 25$
12 B 13 A 14 B
15 a $x = 4.64$ b $x = -0.38$ c $x = -1.89$
d $x = 0.13$
16 a $x = 1.46$ b x = 0.32 c $x = -0.18$
d $x = 1.79$
17 C
18 x-intercepts: $(-6.9, 0)$ and $(0.9, 0)$
y-intercepts: $(0, -1.6)$ and $(0, 3.6)$
19 C 20 B
21 a translate 2 units right b 

ANALYSIS

1 a i $a = 1, h = -4, k = -1$
ii $(-4, -1)$ iii $(-3, 0)$ iv $(0, -3)$ and $(0, 1)$
v See graph below.
vi translation of 4 units left and 1 unit down

b **i** $a = -1, h = 4, k = -1$
ii (4, −1) **iii** (3, 0) **iv** (0, −3) and (0, 1)
v



vi reflection in y-axis and translation of 4 units right and 1 unit down

c A positive value of a results in a parabola with its open end towards the right, while a negative value of a results in the open end being towards the left. Graphs in parts a and b are a reflection of each other in the y-axis.

2 a Using the first formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -3}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

$= -1.3$ or 2.3

Using the second formula:

$$x = \frac{2 \times -3}{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -3}}$$

$$= \frac{-6}{1 \pm \sqrt{13}}$$

$= -1.3$ or 2.3

The two formulas give the same answer.

b Start with the positive value of the square root:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Rationalise the numerator:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

Multiply the terms in the numerator, leaving the denominator in factorised form.

$$\frac{b^2 - (b^2 - 4ac)}{2a(-b - \sqrt{b^2 - 4ac})}$$

Simplify the numerator.

$$\frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})}$$

Cancel the common factor of $2a$ in the numerator and denominator.

$$\frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

A similar formula would result by taking the negative value of the square root in the original formula.

So, the two formulas are equivalent.

5 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 6 POLYNOMIALS

6 Are you ready?

1 a $6x^2 - 8x - 2$ **b** $2x^2 - 11x$
c $-3x^3 + 5x^2 + 3x + 1$ **d** $-7x^2 + 5x + 5$
2 a $3x^3$ **b** $14x^5$ **c** $5x$ **d** $6x^2$
3 a $3x^5 - 5x^2$ **b** $x^2 + 3x - 28$
c $2x^2 + 12x + 18$ **d** $6x^3 - x^2 - 2x$
4 a −2 **b** 18 **c** 3 **d** −17
5 a D **b** B **c** C **d** A
6 a quotient 25, remainder 6
b quotient 121, remainder 0
7 a $x = 0$ or 7 **b** $x = -3$ or 6
c $x = -4$ or 4 **d** $x = -2 - \sqrt{5}$ or $-2 + \sqrt{5}$
e $x = 1$ **f** no real solution
8 a x-intercept: 3; y-intercept: −6
b x-intercepts: −4, 3; y-intercept: −12
c x-intercepts: −3, 3; y-intercept: −9
d x-intercept: −2; y-intercept: 4
9 a translation 4 units up
b translation 2 units right
c dilation by factor 3 **d** reflection in x-axis
e translation 1 unit left and 2 units down
f dilation by factor 2, reflection in x-axis, translation 5 units right and 3 units up
10 a (0, 4) **b** (2, 0) **c** (0, 0) **d** (0, 0)
e (−1, −2) **f** (5, 3)

6A Understanding polynomials

6A Start thinking!

1 a x
b **i** 2 **ii** 1 **iii** 0
c yes; only variable is x and powers are non-negative integers (0, 1, 2)
d 2 **e** $3x^2$ **f** 3 **g** 7
2 a yes; only variable is x and powers are non-negative integers (0, 1, 2, 3)
b no; power of $\frac{1}{4}$ is not a whole number
c yes; only variable is x and powers are non-negative integers (1, 2, 4)
d no; $\frac{6}{x}$ is $6x^{-1}$ so power is not a positive integer
e no; $\sqrt{x}$ is $x^{\frac{1}{2}}$ so power is not a whole number
f no; two different variables (x and y) are used

Exercise 6A Understanding polynomials

1 a yes; cubic **b** no **c** yes; linear
d yes; quadratic **e** no **f** yes; constant
g no **h** yes; quartic **i** yes; cubic
2 a **i** 4 **ii** 3 **iii** 5 **iv** $2x^3$ **v** 2 **vi** 3
b **i** 6 **ii** 5 **iii** −3 **iv** $4x^5$ **v** 4 **vi** −2
c **i** 4 **ii** 4 **iii** 1 **iv** $-5x^4$ **v** −5 **vi** 5
d **i** 7 **ii** 7 **iii** 0 **iv** x^7 **v** 1 **vi** −1
e **i** 5 **ii** 6 **iii** 9 **iv** $-7x^6$ **v** −7 **vi** 2
f **i** 3 **ii** 10 **iii** 3 **iv** $-11x^{10}$
v −11 **vi** 0

3 **a** $5x^3 - 3x^2 - 4x + 2$ **b** $x^5 - 2x^4 + 4x^2$
c $2x^4 + 6x^3 + 2x^2 - 3$ **d** $5x^2 - 11x + 9$
e $3x^3 + 5x - 3$ **f** $-4x^4 - 4x^3 + 6x^2 + x$
4 **a** 2 **b** −8 **c** 16 **d** −68
5 **a** 127 **b** −5 **c** 15 **d** 97
6 **a** $2x^3 + 2x^2 - 4x - 4$ **b** $2x^3 - 4x^2 + 10x - 8$
c $-2x^3 + 4x^2 - 10x + 8$ **d** $6x^3 - 3x^2 + 9x - 18$
e $-6x^2 + 14x - 4$ **f** $4x^3 + x^2 - x - 10$
g $2x^3 - 13x^2 + 31x - 14$
h $6x^3 - 9x^2 + 23x - 22$
i −24 **j** 0 **k** 56 **l** −38
7 **a** $x^5 + 2x^4 + 7x^3$ **b** $4x^4 - 20x^3 + 8x^2$
c $-6x^7 + 24x^5 - 6x^3$ **d** $x^3 + x^2 - 2x + 12$
e $x^4 + 3x^3 - 8x^2 - 3x + 7$
f $x^5 - 6x^3 + 3x^2 + 5x - 15$
g $x^7 + 4x^6 - 4x^4 - 10x^3 + 8x^2 + 3x - 2$
h $2x^8 - 10x^6 + x^5 + 16x^3 - 6x^2 - 5x + 2$
8 **a** $3x^3 + 15x^2 + 18x$ **b** $-7x^4 - 21x^3 + 70x^2$
c $x^3 + 10x^2 + 27x + 18$ **d** $x^3 - 6x^2 + 5x + 12$
e $3x^3 - 23x^2 + 44x - 20$
f $8x^3 - 2x^2 - 113x - 28$
9 **a** $x^6 + 8x^3 + 16$ **b** $x^4 + 6x^3 + 9x^2$
c $2x^5 - 8x^3 + 8x$ **d** $x^6 - 10x^5 + 25x^4$
e $x^3 + 6x^2 + 12x + 8$ **f** $8x^3 - 36x^2 + 54x - 27$
g $x^4 + 12x^3 + 54x^2 + 108x + 81$
h $1 - 4x + 6x^2 - 4x^3 + x^4$
10 6 **11** 1
12 **a** $n + 1$ **b** 1 **c** n **d** $2n$
13 $k = -4$ **14** $k = 11$ **15** $a = -1, b = 4$
16 **a** $a^3 - 3a^2 + 2a + 1$ **b** $-27a^3 - 27a^2 - 6a + 1$
c $a^6 - 3a^4 + 2a^2 + 1$ **d** $-a^6 - 3a^4 - 2a^2 + 1$
e $a^3 + 3a^2 + 2a + 1$ **f** $8a^3 - 24a^2 + 22a - 5$
17 $(2x - 1)^4$
$= (2x - 1)(2x - 1)(4x^2 - 4x + 1)$
$= (2x - 1)(8x^3 - 8x^2 + 2x - 4x^2 + 4x - 1)$
$= (2x - 1)(8x^3 - 12x^2 + 6x - 1)$
$= 16x^4 - 24x^3 + 12x^2 - 2x - 8x^3 + 12x^2 - 6x + 1$
$= 16x^4 - 32x^3 + 24x^2 - 8x + 1$
18 **a** $x^6 - 6x^4 + 12x^2 - 8$
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$
c $16x^{12} + 32x^9 + 24x^6 + 8x^3 + 1$

6B Division of polynomials

6B Start thinking!

1 **a** 9 goes into 23 two times so 2 is written above 3 in the quotient line
b $2 \times 9 = 18$ **c** 5; written under 18
d bring down 7 from dividend to sit beside remainder of 5
e 6; written above 7 in quotient line
f $6 \times 9 = 54$ **g** $57 - 54 = 3$
2 dividend is 237, divisor is 9, quotient is 26, remainder is 3
3 **a** $237 \div 9 = 26$ remainder 3
b $237 = 9 \times 26 + 3$

Exercise 6B Division of polynomials

1 **a** $152 = 8 \times 19$ **b** $9475 = 25 \times 379$
c $351 = 7 \times 45 + 6$ **d** $2386 = 16 \times 149 + 2$
2 **a** $x^2 + 5x + 6 = (x + 2)(x + 3)$
b $x^3 + 2x^2 + 3x + 2 = (x + 1)(x^2 + x + 2)$
c $2x^2 - 3x - 4 = (x - 3)(2x + 3) + 5$.
d $x^3 + 4x^2 - x + 3 = (x - 1)(x^2 + 5x + 4) + 7$
3 **a**
$$\begin{array}{r} x + 4 \\ x + 2 \overline{)\, x^2 + 6x + 1} \\ \underline{x^2 + 2x} \\ 4x + 1 \\ \underline{4x + 8} \\ -7 \end{array}$$
Quotient is $x + 4$, remainder is −7.
b
$$\begin{array}{r} 4x - 3 \\ x - 1 \overline{)\, 4x^2 - 7x + 5} \\ \underline{4x^2 - 4x} \\ -3x + 5 \\ \underline{-3x + 3} \\ 2 \end{array}$$
Quotient is $4x - 3$, remainder is 2.
4 **a** $x + 4$; −4 **b** $x + 5$; 9 **c** $x - 2$; 11
d $x - 7$; 10 **e** $2x + 3$; −2 **f** $3x + 4$; 6
5 **a** $x^2 + 7x + 8 = (x + 3)(x + 4) - 4$
b $x^2 + 3x - 1 = (x - 2)(x + 5) + 9$
c $x^2 + 2x + 3 = (x + 4)(x - 2) + 11$
d $x^2 - 5x - 4 = (x + 2)(x - 7) + 10$
e $2x^2 - 3x - 11 = (x - 3)(2x + 3) - 2$
f $3x^2 + x + 2 = (x - 1)(3x + 4) + 6$
6 The right side should equal the left side.
7 **a** $x - 6$; 0
b $x^2 - 2x - 24 = (x + 4)(x - 6)$
c Divisor is a factor of $x^2 - 2x - 24$.
8 **a** quotient $x^2 + x - 2$, remainder 1
b quotient $x^2 + 2x + 5$, remainder −6
c quotient $x^2 + 8x + 12$, remainder 9
d quotient $x^2 - 7x + 8$, remainder −7
e quotient $2x^2 - x - 3$, remainder 2
f quotient $3x^2 + x - 6$, remainder −4
g quotient $5x^2 + 3x + 3$, remainder −6
h quotient $4x^2 + 5x - 13$, remainder 12
i quotient $x^3 + x^2 - 5x + 3$, remainder 5
j quotient $x^3 - 3x^2 - 4x - 9$, remainder −21
9 **a** quotient $x^2 - 3x + 10$, remainder −9
b quotient $2x^2 + x + 2$, remainder −2
c quotient $4x^3 + 4x^2 + x + 1$, remainder −4
d quotient $3x^3 - 6x^2 + 12x - 24$, remainder 7
10 **a** $k = 24$
b linear factor $(x - 2)$;
quadratic factor $(x^2 - x - 12)$
$P(x) = (x - 2)(x^2 - x - 12)$
$= (x - 2)(x - 4)(x + 3)$
11 **a** $k = 1$, 3 or 5
b $P(x) = (x + 1)(x + 3)(x + 5)$
12 **a** quotient $x - 3$, remainder $x - 1$
b quotient $x^2 + x - 4$, remainder $6x - 7$
c quotient $2x^2 + 2$, remainder −5

6C Remainder and factor theorems

6C Start thinking!

1 a 14 b 14; same answer
2 a 24 b 24; same answer
3 a $P(4) = 64 + 16 - 40 + 8 = 48$; remainder is 48
b Long division produces a quotient of $x^2 + 5x + 10$ and remainder of 48.
4 a $P(1) = 1 + 1 - 10 + 8 = 0$; remainder is 0
b Long division produces a quotient of $x^2 + 2x - 8$ and remainder of 0.
5 a $P(5) = 108$; remainder is 108
b $P(-1) = 18$; remainder is 18
c $P(2) = 0$; remainder is 0
6 Remainder of 0 means that $(x - 1)$ is a factor of $P(x)$.

Exercise 6C Remainder and factor theorems

1 a 0 b 14 c −16 d 0
2 a 12 b 15 c 15 d 0
3 a 12 b −48 c 0 d 630
4 Divisor is a factor if the remainder is zero. To find the remainder when $P(x)$ is divided by $(x - a)$, calculate $P(a)$.
5 a no b yes c no d no
e no f yes g no h yes
6 a no b no c yes d no
e no f yes g yes h no
7 a $(x - 1)$ or $(x + 3)$ or $(x + 6)$
b $(x - 2)$ or $(x - 3)$ or $(x + 4)$
c $(x - 3)$ or $(x + 3)$ or $(x - 4)$
d $(x + 2)$ or $(x + 3)$ or $(x - 5)$
e $(x - 1)$ or $(x - 3)$ f $(x + 2)$ or $(x - 4)$
g $(x - 2)$ h $(x + 3)$
8 a $(x - 6)(x + 3)(x + 4)$ b $(x + 4)(x - 7)(x - 2)$
c $(x + 3)(x + 5)(x - 6)$ d $(x - 2)(x + 4)(x - 4)$
e $(x + 1)(x + 2)(x + 2)$ f $(x - 7)(x + 8)(x - 3)$
9 a $(x - 1)(x + 2)(x + 5)$ b $(x - 2)(x + 2)(x + 3)$
c $(x - 1)(x - 2)(x + 7)$ d $(x - 2)(x + 2)(x + 5)$
e $(x + 3)(x - 4)(x - 4)$ f $(x - 2)(x - 3)(x + 3)$
g $(x - 2)(2x + 1)(x + 2)$ h $(x + 2)(4x - 5)(x - 3)$
i $(x - 4)(3x + 2)(2x - 1)$
j $(x - 1)(4x - 3)(3x + 1)$
10 a $(x - 1)(x + 3)(x + 6)$ b $(x - 2)(x - 3)(x + 4)$
c $(x - 3)(x + 3)(x - 4)$ d $(x + 2)(x + 3)(x - 5)$
e $(x - 1)(3x + 2)(x - 3)$
f $(x + 2)(2x - 1)(x - 4)$
g $(x - 2)(4x + 1)(2x - 3)$
h $(x + 3)(2x - 1)(2x - 3)$
11 a 2 b 3 c 4 d 7 e 12 f n
12 a 3 b 24; $2 \times -3 \times 4 = -24$
c −2, 3, −4; they are factors of −24
13 a $P(-1) = 0$ so factor is $(x + 1)$
b $P(2) = 0$ so factor is $(x - 2)$
c $P(-3) = 0$ so factor is $(x + 3)$
d $P(x) = (x + 1)(x - 2)(x + 3)$
e $(x + 1)(x - 2)(x + 3)$
$= (x + 1)(x^2 + x - 6)$
$= x^3 + x^2 - 6x + x^2 + x - 6$
$= x^3 + 2x^2 - 5x - 6$
14 a $(x + 1)$, $(x + 4)$ and $(x - 2)$
b $P(x) = (x + 1)(x + 4)(x - 2)$
c $(x + 1)(x + 4)(x - 2)$
$= (x + 1)(x^2 + 2x - 8)$
$= x^3 + 2x^2 - 8x + x^2 + 2x - 8$
$= x^3 + 3x^2 - 6x - 8$
15 a $(x - 1)$, $(x + 2)$ and $(x - 5)$
b $P(x) = (x - 1)(x + 2)(x - 5)$
16 $(x + 1)(x - 3)(x + 4)$
17 a need to substitute x values that are fractions ($\frac{1}{3}$ and $-\frac{3}{2}$); polynomial factorises to $(3x - 1)(2x + 3)(x + 1)$.
b only one linear factor; polynomial factorises to $(x + 2)(x^2 + 3x + 4)$
18 One possible answer is: $x^3 + x^2 + 2x - 11$
19 a $(x - 1)(x + 1)(x + 2)(x - 3)$
b $(x - 1)(x + 2)(x + 3)(x + 4)$
c $(x + 1)(x + 2)(2x - 1)(x + 4)$

6D Solving polynomial equations

6D Start thinking!

1 a 2 b parts iv and vii
c Factors of the constant term are possible solutions (parts i, ii, iii, iv, vii).
2 a parts i, iii, v, vi, vii; these values are factors of −6
b parts iii and vii
3 a parts i, iv, v, vii; these values are factors of −4
b parts i, v and vii
4 a $(x + 6)(x - 3) = 0$
b For the product of factors to equal zero, one or both factors must equal zero. So $x + 6 = 0$ or $x - 3 = 0$. That is, $x = -6$ or $x = 3$.
5 $x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$
$x + 3 = 0$ or $x - 2 = 0$
$x = -3$ or $x = 2$

Exercise 6D Solving polynomial equations

1 a $(x + 4)(x + 1) = 0$ b $(x - 3)(x - 6) = 0$
c $(x + 2)(x - 4) = 0$ d $(x + 3)(x + 3) = 0$
e $(x + 5)(x - 5) = 0$ f $2x(x - 7) = 0$
2 a $x = -4$ or -1 b $x = 3$ or 6
c $x = -2$ or 4 d $x = 3$
e $x = -5$ or 5 f $x = 0$ or 7
3 a $x = -5, -2$ or 4 b $x = -4, -1$ or 3
c $x = -3$, 2 or 6 d $x = -2$, 0 or 9
e $x = -1, \frac{1}{2}$ or 1 f $x = -5, -3\frac{1}{2}$ or $\frac{2}{3}$
4 a $x = -7, -3$, 1 or 4 b $x = -6$, 2, 3 or 5
c $x = -4, -2$ or -1 d $x = -5$, 0 or 6
e $x = -4, -\frac{1}{3}$, 0 or 4 f $x = -2\frac{1}{2}$ or $\frac{3}{4}$
5 a $(x + 1)(x + 2)(x + 6) = 0$; $x = -6, -2$ or -1
b $(x - 3)(x - 5)(x + 4) = 0$; $x = -4$, 3 or 5

c $(x-2)(x-4)(x-2)=0$ or $(x-2)^2(x-4)=0$; $x=2$ or 4
d $(x+4)(x+7)(x-1)=0$; $x=-7, -4$ or 1
e $(x+3)(x+4)(x-4)=0$; $x=-4, -3$ or 4
f $x(x-1)(x+1)(x-1)=0$ or $x(x-1)^2(x+1)=0$; $x=-1$, 0 or 1
g $(x-5)(x-5)(x-5)=0$ or $(x-5)^3=0$; $x=5$
h $(x+2)(x+3)(x+3)=0$ or $(x+2)(x+3)^2=0$; $x=-3$ or -2
i $(x-4)(x+2)(x-3)=0$; $x=-2$, 3 or 4
j $(x+1)(x^2+2x+2)=0$; $x=-1$
k $15x(x+3)(x+6)=0$; $x=-6, -3$ or 0
l $2(x-2)(x-1)(x+3)=0$; $x=-3$, 1 or 2

6 a $x=-4, -3$ or 1 b $x=-2$, 2 or 5
c $x=-6, -1$ or 6 d $x=-7, -3$ or 0
e $x=-2, -1$ or 3 f $x=-2$
g $x=-3$ or 3 h $x=-4$ or -1
i $x=4$ j $x=2$

7 a $x=-2, -1$ or 4 b $x=-1$, 1 or 5
c $x=-2$ or 2 d $x=-3, -2$ or -1

8 a $x=-3, -2$ or 3 b $x=-4$, 3 or 5
c $x=-1$, 1 or 4 d $x=-2$ or 3

9 a $x=1, -1-\sqrt{2}$ or $-1+\sqrt{2}$
b $x=-2, -2-\sqrt{3}$ or $-2+\sqrt{3}$
c $x=-3, 1-\sqrt{5}$ or $1+\sqrt{5}$
d $x=4, -4-\sqrt{11}$ or $-4+\sqrt{11}$

10 a $(x-1)$ is a factor.
b $P(x)=(x-1)(x^3+2x^2-5x-6)$
c linear factor is $(x+1)$, quadratic factor is (x^2+x-6)
d $(x+3)(x-2)$
e $P(x)=(x-1)(x+1)(x+3)(x-2)$; $x=-3, -1$, 1 or 2

11 a $x=-1$, 1, 2 or 3 b $x=-2, -1$, 3 or 4
c $x=-4, -3, -2$ or 2 d $x=-3, -2$, 1 or 5
e $x=-2$, 1, 2 or 4 f $x=-3, -2$, 2 or 3

12 a 1 b 2 c 3 d 4 e 6 f n

13 a $P(x)=(x-1)(x+2)(x-3)$
b $x=-2$, 1 or 3

14 $(x-2)(x^2+3x+4)=0$ has one solution $(x=2)$; quadratic factor has no solutions $(b^2-4ac=-7<0)$

15 $(x+1)(x-3)(x-3)=0$ has two solutions $(x=-1, 3)$; two of the linear factors are the same

16 a $x(2x)(x+2)=2x^3+4x^2$
b $2x^3+4x^2=192$; $x^3+2x^2-96=0$; $(x-4)(x^2+6x+24)=0$; $x=4$; height 4 cm, length 8 cm, width 6 cm

17 a $\frac{1}{2}x(5x)(x-1)=\frac{5}{2}x^3-\frac{5}{2}x^2$
b $\frac{5}{2}x^3-\frac{5}{2}x^2=450$; $x^3-x^2-180=0$; $(x-6)(x^2+5x+30)=0$; $x=6$; base 6 cm, height 5 cm, length 30 cm

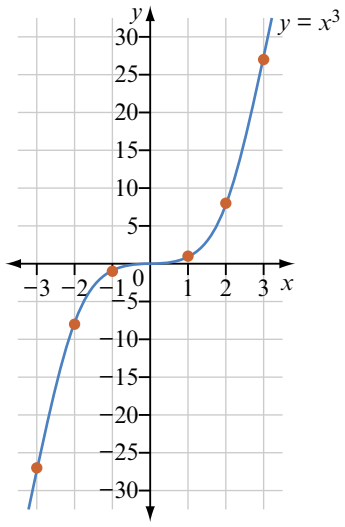
18 $P(x)=(x+1)(x-2)(x-4)(x+3)$; $x=-3, -1$, 2 or 4

19 a $x=-2, -1$, 1 or 5 b $x=-3, -2$, 1, 2 or 4

6E Graphs of polynomial relationships

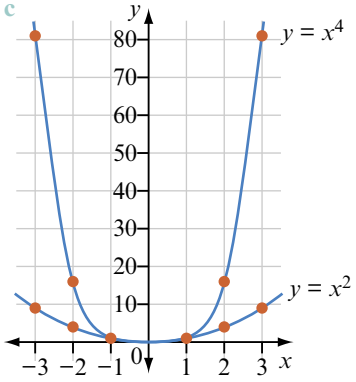
6E Start thinking!

1 a, c



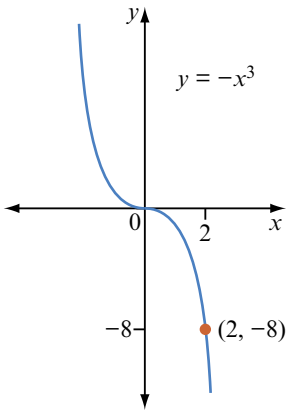
b x-intercept is 0, y-intercept is 0

2 a, c

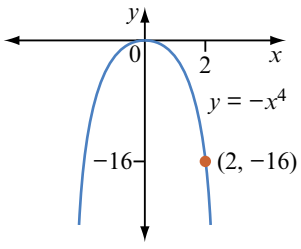


b x-intercept is 0, y-intercept is 0

3 a

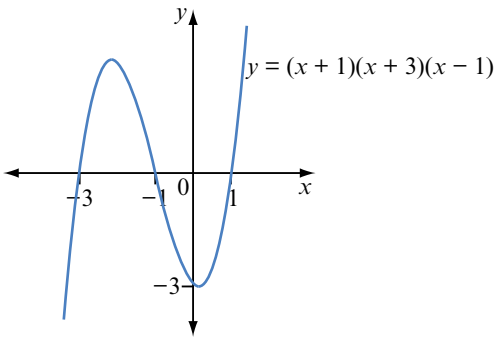


b

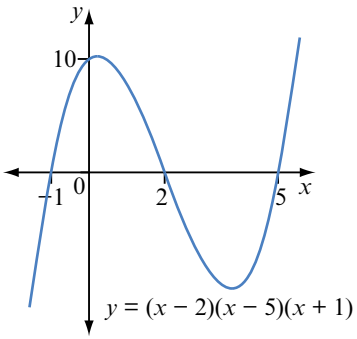


Exercise 6E Graphs of polynomial relationships

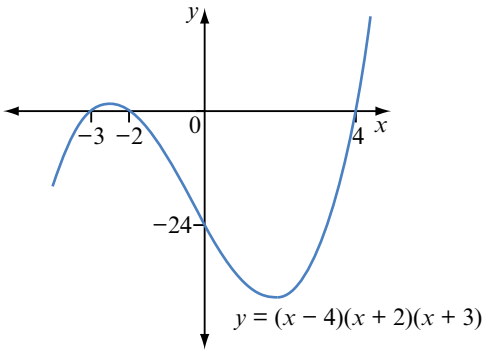
1 a



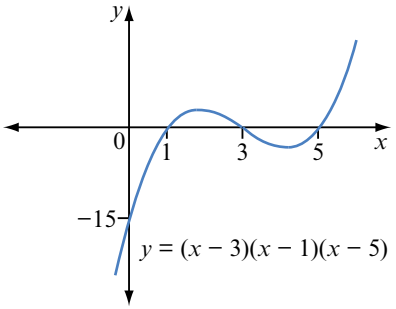
b



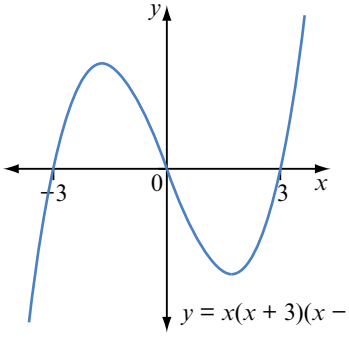
c



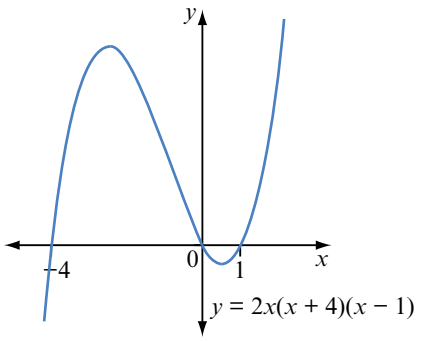
d



e



f



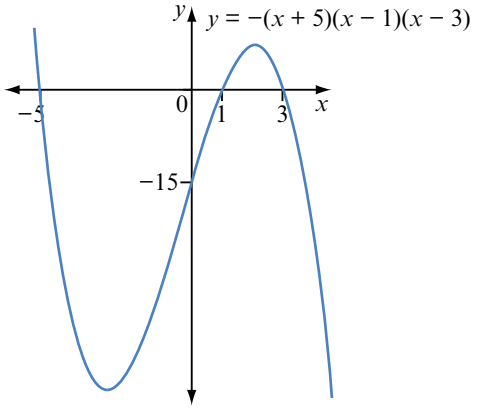
2 a i −1, 2 and 5 ii 10
iii $y = (x + 1)(x - 2)(x - 5)$

b i −6, −2 and 2 ii −24
iii $y = (x + 6)(x + 2)(x - 2)$

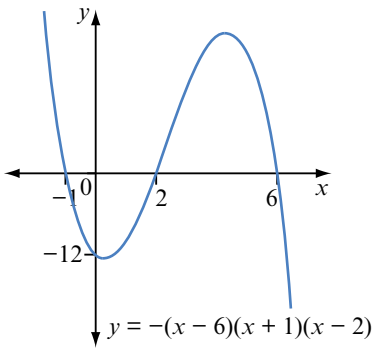
3 a i −7, 2 and 3 ii −7, 2 and 3

b reflection in x-axis; y values of graph for part ii are the negative of the y values of graph in part i

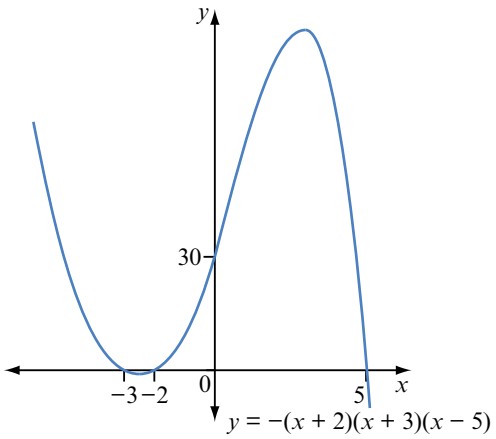
4 a



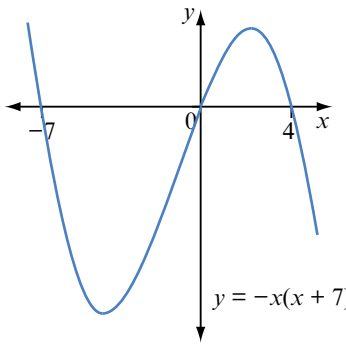
b



c

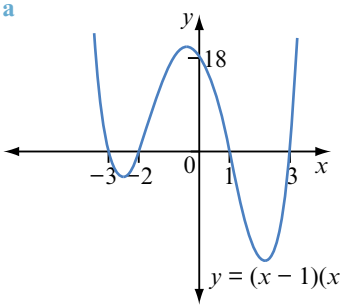


d



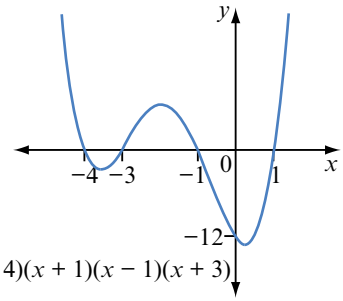
$y = -x(x + 7)(x - 4)$

5 a



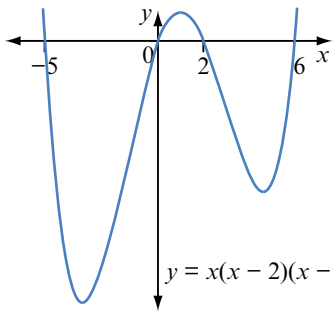
$y = (x - 1)(x + 2)(x + 3)(x - 3)$

b



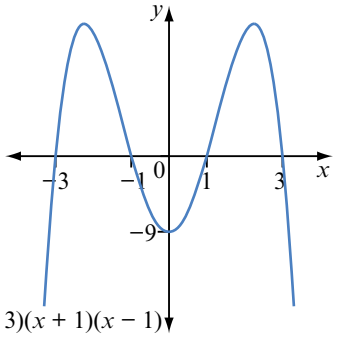
$y = (x + 4)(x + 1)(x - 1)(x + 3)$

c



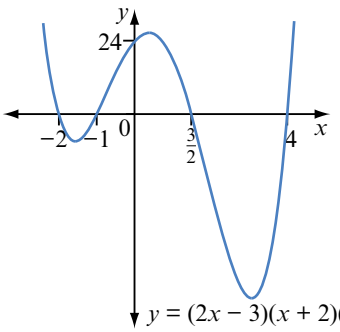
$y = x(x - 2)(x - 6)(x + 5)$

d



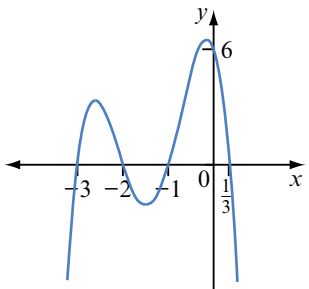
$y = -(x - 3)(x + 3)(x + 1)(x - 1)$

e



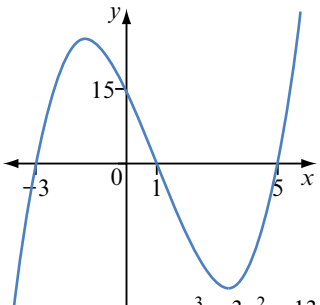
$y = (2x - 3)(x + 2)(x + 1)(x - 4)$

f



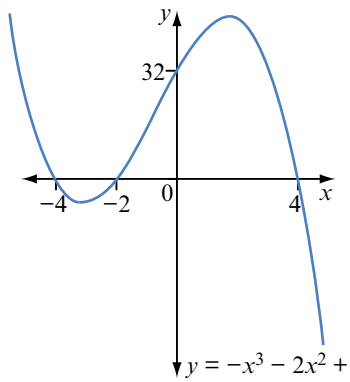
$y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 a



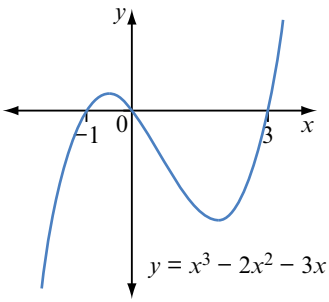
$y = x^3 - 3x^2 - 13x + 15$

b



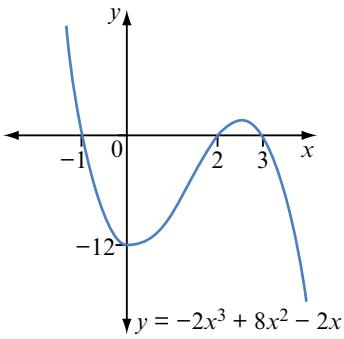
$y = -x^3 - 2x^2 + 16x + 32$

c



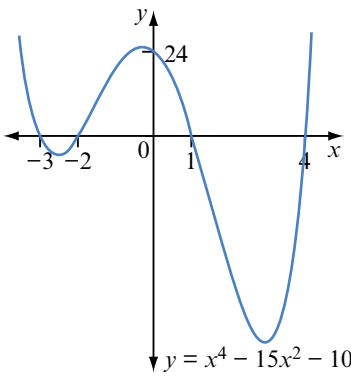
$y = x^3 - 2x^2 - 3x$

d



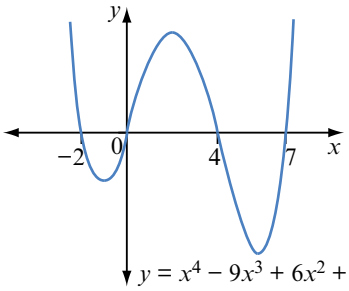
$y = -2x^3 + 8x^2 - 2x - 12$

e



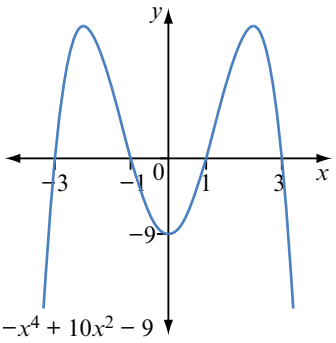
$y = x^4 - 15x^2 - 10x + 24$

f



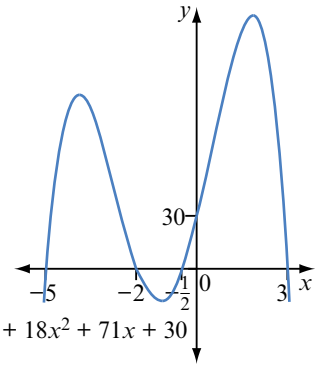
$y = x^4 - 9x^3 + 6x^2 + 56x$

g



$y = -x^4 + 10x^2 - 9$

h



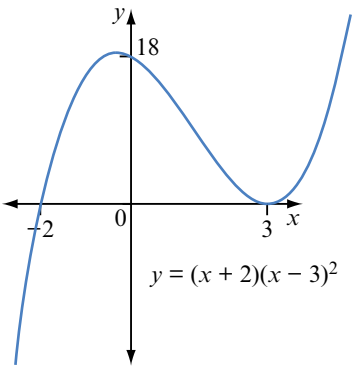
$y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

8 a cubic

b x-intercepts: -2 and 3; y-intercept: 18

c bottom left

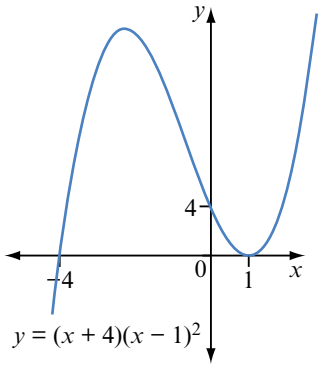
d



$y = (x + 2)(x - 3)^2$

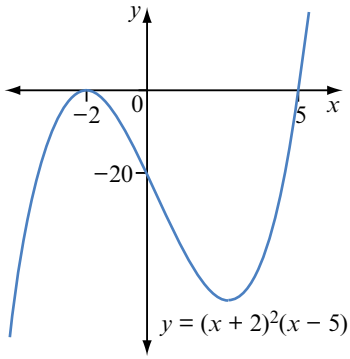
e The x-intercept at $x = 3$ is also a turning point.

9 a



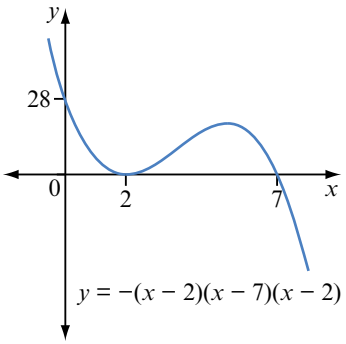
$y = (x + 4)(x - 1)^2$

b



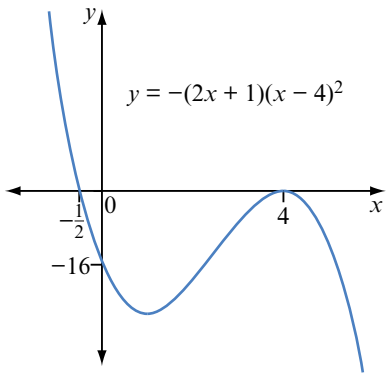
$y = (x + 2)^2(x - 5)$

c

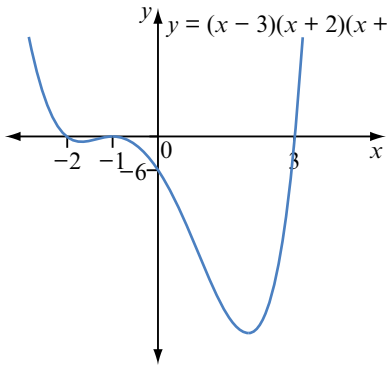


$y = -(x - 2)(x - 7)(x - 2)$

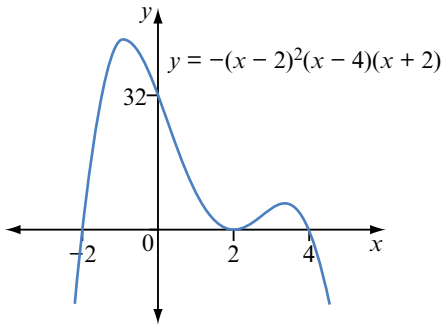
d



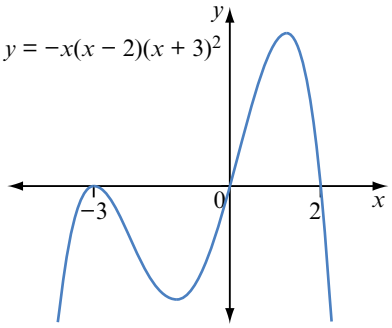
10 a



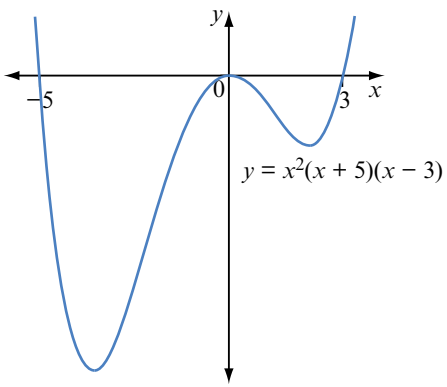
b



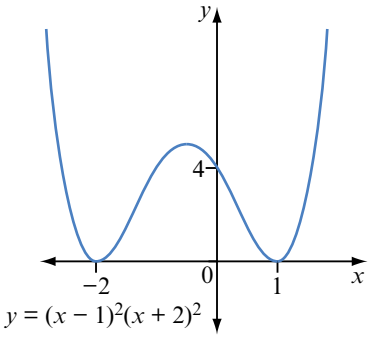
c



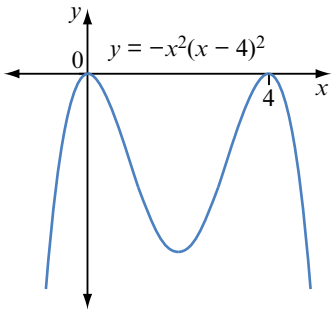
d



e

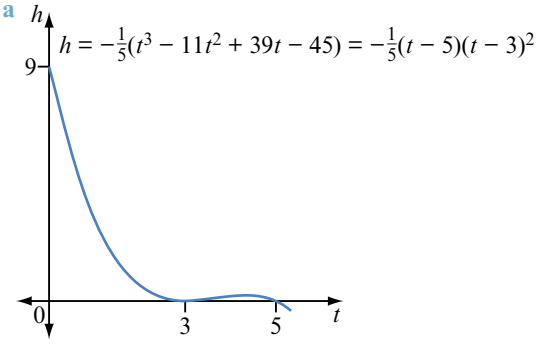


f



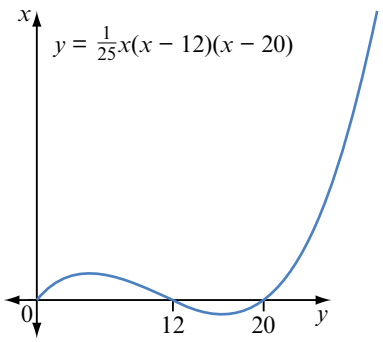
11 x-intercept at $(a, 0)$ is also a point of inflection

12 a



b 9 m **c** 3.2 m **d** 3 s **e** 2 s

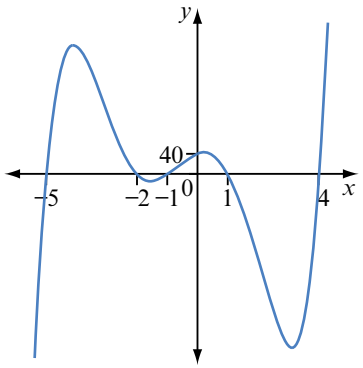
13 a



b $y = \frac{1}{25}x(x - 12)(x - 20)$ or
$y = 0.04x^3 - 1.28x^2 + 9.6x$

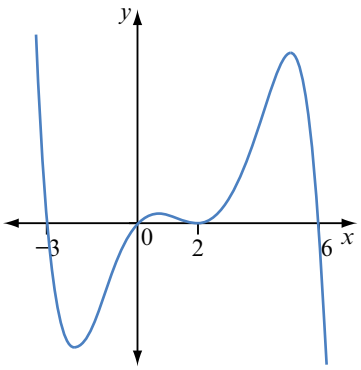
c $216

14 a



$y = (x - 1)(x + 2)(x + 1)(x - 4)(x + 5)$

b



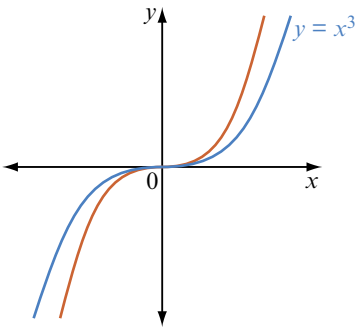
$y = -x(x - 6)(x + 3)(x - 2)^2$

6F Polynomials and transformations

6F Start thinking!

1 a i For each x value, the y value is doubled, curve is steeper

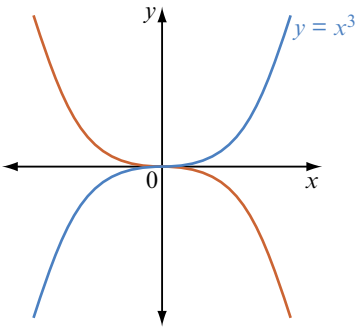
ii



iii $y = 2x^3$

b i For each x value, the y value is the negative of original y value; curve reflected in x-axis.

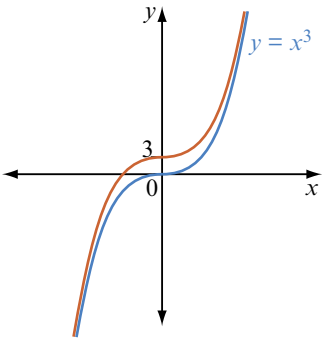
ii



iii $y = -x^3$

c i Each original coordinate point has 3 units added to the y-coordinate; curve moved 3 units up.

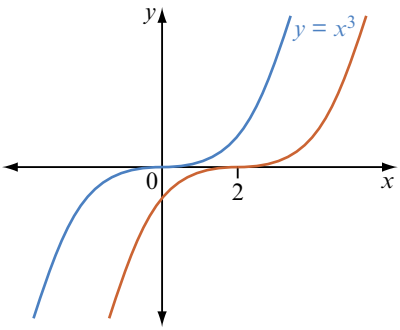
ii



iii $y = x^3 + 3$

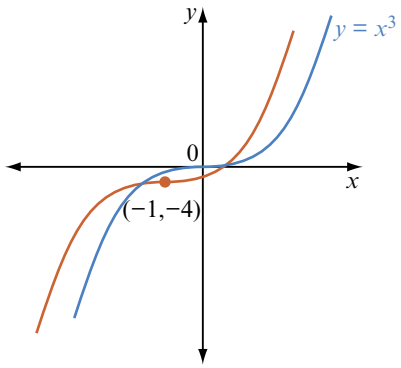
d i Each original coordinate point has 2 units added to the x-coordinate; curve moved 2 units right.

ii



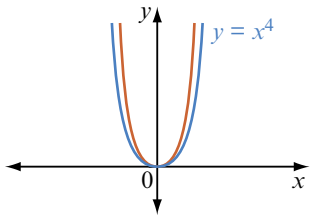
iii $y = (x - 2)^3$

e i Each original coordinate point has 1 unit subtracted from the x-coordinate and 4 units subtracted from the y-coordinate; curve moved 1 unit left and 4 units down

ii 

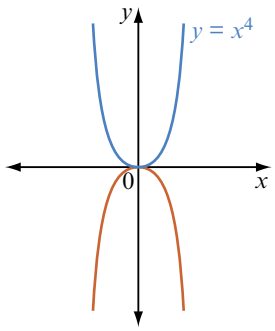
iii $y = (x + 1)^3 - 4$

2 a i For each x value, the y value is doubled, curve is steeper

ii 

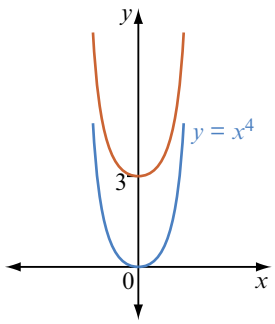
iii $y = 2x^4$

b i For each x value, the y value is the negative of original y value; curve reflected in x-axis.

ii 

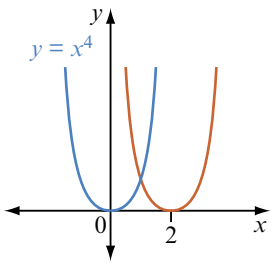
iii $y = -x^4$

c i Each original coordinate point has 3 units added to the y-coordinate; curve moved 3 units up.

ii 

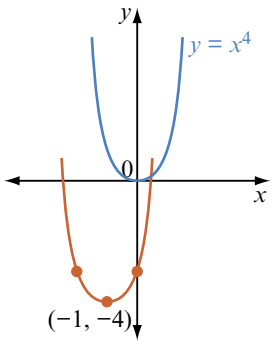
iii $y = x^4 + 3$

d i Each original coordinate point has 2 units added to the x-coordinate; curve moved 2 units right.

ii 

iii $y = (x - 2)^4$

e i Each original coordinate point has 1 unit subtracted from the x-coordinate and 4 units subtracted from the y-coordinate; curve moved 1 unit left and 4 units down

ii 

iii $y = (x + 1)^4 - 4$

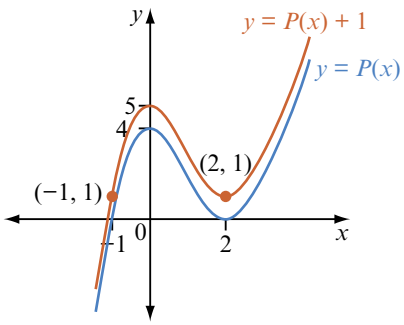
Exercise 6F Polynomials and transformations

1 a translation of 3 units up
b reflection in x-axis
c dilation by factor of 3
d translation of 2 units right

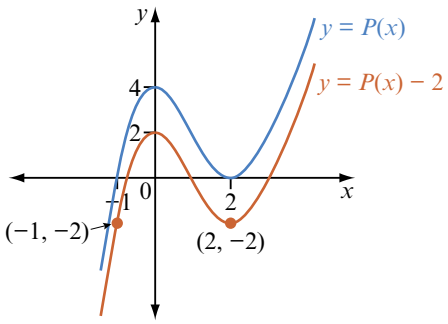
2 a $y = x^3 + 3$ b $y = -x^3$
c $y = 3x^3$ d $y = (x - 2)^3$

3 a reflection in x-axis b dilation by factor of 2
c translation of 1 unit down
d translation of 2 units left

4 a $y = -x^4$ b $y = 2x^4$
c $y = x^4 - 1$ d $y = (x + 2)^4$

5 a 

b



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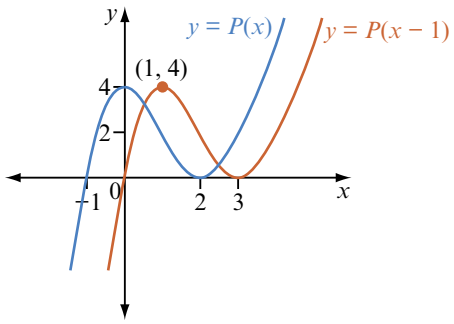

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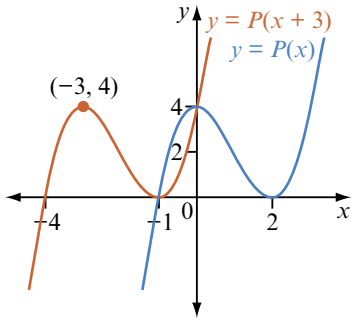

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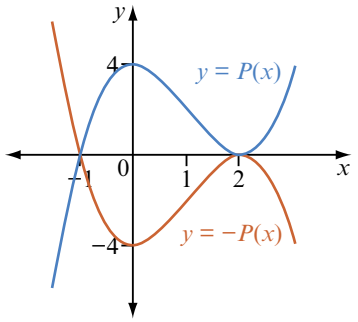

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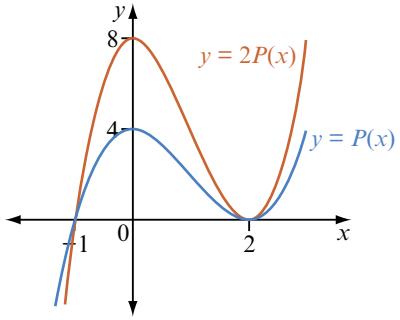
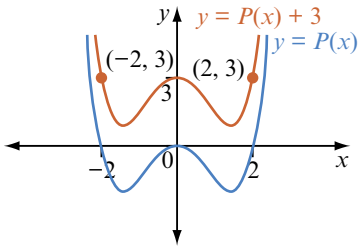
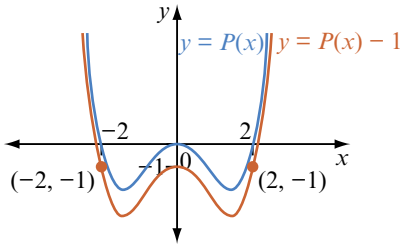


6 a



b



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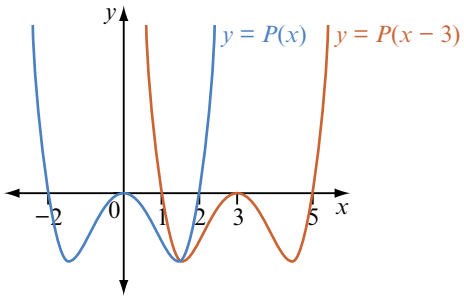

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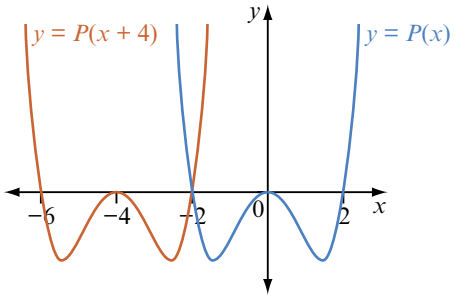

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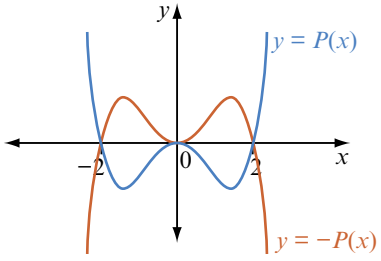

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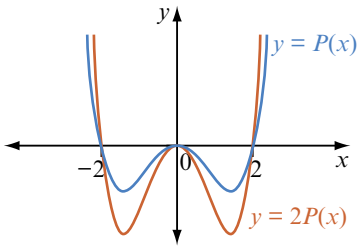


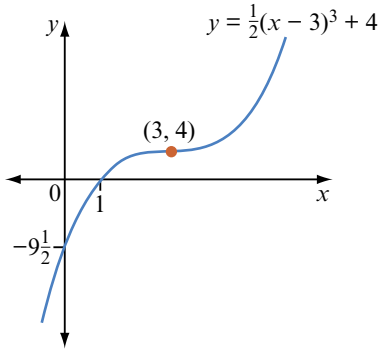
7 a $y = -P(x)$ **b** $y = P(x) - 5$
c $y = P(x - 3)$

8 same graph, $y = -x^3$

9 a i dilation by factor of $\frac{1}{2}$, translation of 3 units right and 4 units up
ii (3, 4)

iii x-intercept: 1; y-intercept: $-9\frac{1}{2}$

iv

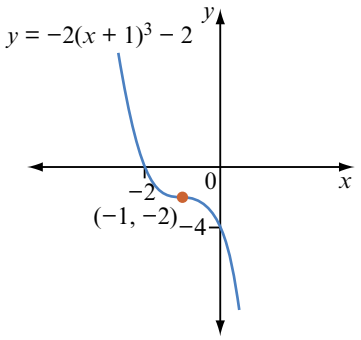


b i dilation by factor of 2, reflection in x-axis, translation of 1 unit left and 2 units down

ii $(-1, -2)$

iii x-intercept: -2; y-intercept: -4

iv



6 Chapter review

MULTIPLE-CHOICE

1 B	2 D	3 C	4 A	5 D	6 B
7 B	8 A	9 C	10 D	11 A	12 B

SHORT ANSWER

1 a yes, cubic b yes, linear
c yes, linear d not a polynomial
e yes, quadratic f yes, quartic

2 a 7 b 8
c there is no constant term
d $-5x^8$ e -5 f -7

3 a $x + 7$ remainder 12 b $3x - 4$ remainder 8
c $x + 3$ remainder 20 d $x + 2$ remainder $1 - 2x$

4 a true b false

5 a $x - 3$ b $x + 3$ c $x - 1$ d $x + 2$

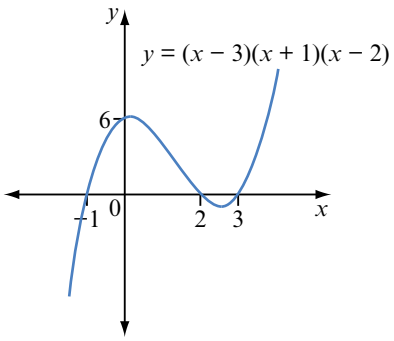
6 a $(x - 1)(x - 2)(x - 4)$ b $(x - 3)(x + 1)(x + 2)$

7 $x + 2$

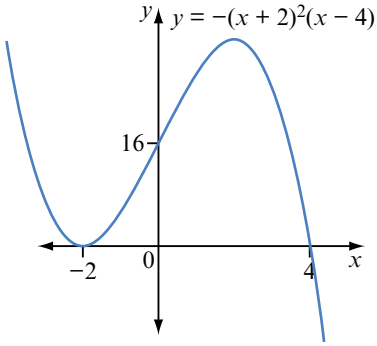
8 a $x = -2, -1, 1$ or 5 b $x = -3, 0, 2$ or 3
c $x = -3$ or 5 d $x = -4, 0$ or 4

9 a $x = -2, 0$ or 2 b $x = 4$

10 a

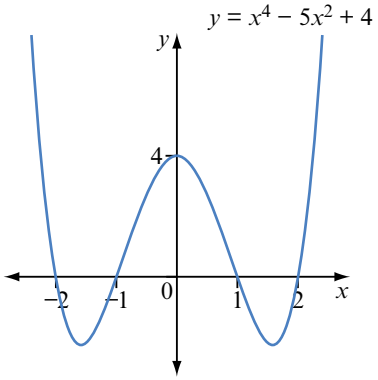


b



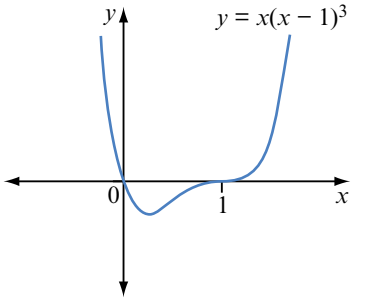
11 a -1, 1 and 5 b 5
c $y = (x - 1)(x + 1)(x - 5)$

12 a $y = (x - 1)(x + 2)(x + 1)(x - 2)$
b x intercepts: $-2, -1$, 1 and 2, y-intercept: 4
c



13 Graph of $y = x^3$ translated 2 units left and 4 units down.

14 a x-intercepts: 0 and 1; y-intercept: 0
b



15 a $y = x^2(x - 2)(x + 2)$
b $y = x^4 - 4x^2$

MIXED PRACTICE

1 A

2 a 10 b −14 c −42
d −100 e 96 f −4

3 B

4 a -8 b -1 c $2x^3 - 2x^2 + x - 4$
d $-4x^3 + 6x^2 - x - 4$ e 4 f 12

5 a quotient $x^2 - 2x + 1$, remainder -6
b quotient $x^2 - x - 4$, remainder 0
c quotient $x^2 + 6x + 26$, remainder 73

6 a $12x^3 - 24x^2$ b $x^6 - 10x^4 + 25x^2$
c $2x^8 - x^6 - 15x^4$
d $x^5 + x^4 - 2x^3 - 2x^2 + x + 1$

7 B

8 a $y = (x - 2)^2(x - 4)^2$
b $y = x^4 - 12x^3 + 52x^2 - 96x + 64$

9 a yes b yes c no

10 A 11 $k = -2$ 12 C 13 A

14 D 15 $x + 6$ and $x - 7$

16 a quartic
b x intercepts: -1 and 3, y-intercept: 9
c top left
d turning point at each x-intercept

17 -2, -1, 0, 2 and 4

18 $(x - 1)^2(x + 1)^2$

19 $x = -3$, 2 or 4

ANALYSIS

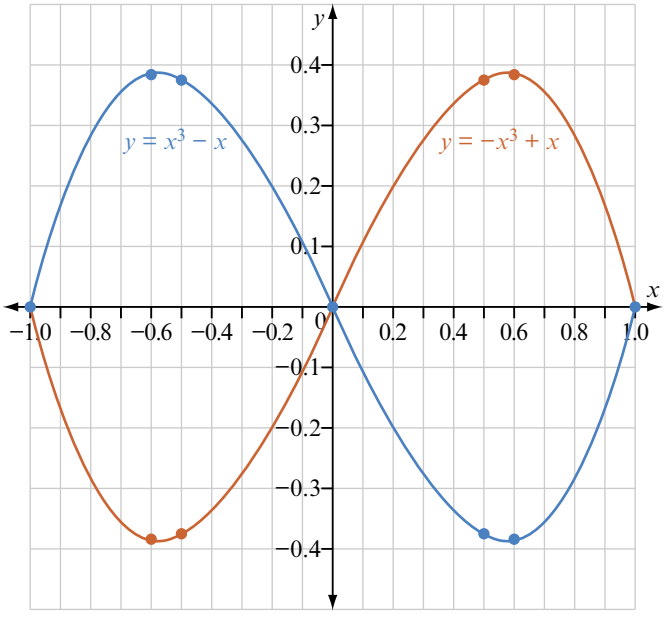
a i

x	−1	−0.6	−0.5	0	0.5	0.6	1
y	0	0.384	0.375	0	−0.375	−0.384	0

ii

x	−1	−0.6	−0.5	0	0.5	0.6	1
y	0	−0.384	−0.375	0	0.375	0.384	0

iii



iv $-1 \le x \le 1$

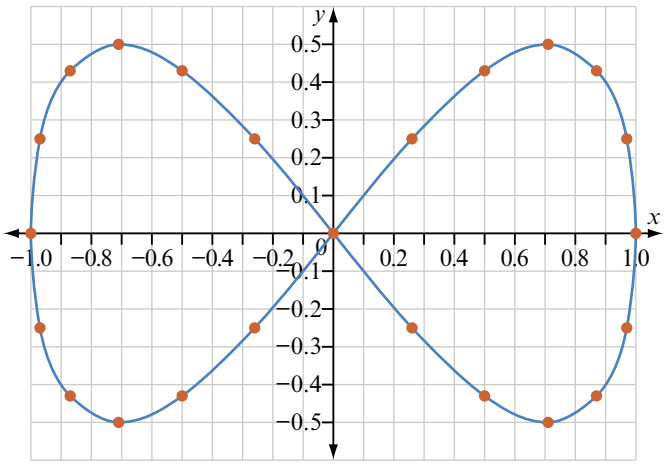
v No, the turning points do not occur midway between the x-intercepts. They occur at an x value between -0.6 and -0.5 and between 0.5 and 0.6.

vi Turning points at $(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9})$, $(-\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9})$, $(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9})$ and $(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9})$ or, to 3 decimal places, $(-0.577, 0.385)$, $(-0.577, -0.385)$, $(0.577, 0.385)$ and $(0.577, -0.385)$.

b i, ii

Angle (θ)	x $(\cos\theta)$	y $(\frac{\sin 2\theta}{2})$
0°	1	0
15°	0.97	0.25
30°	0.87	0.43
45°	0.71	0.5
60°	0.5	0.43
75°	0.26	0.25
90°	0	0
105°	−0.26	−0.25
120°	−0.5	−0.43
135°	−0.71	−0.5
150°	−0.87	−0.43
165°	−0.97	−0.25
180°	−1	0
195°	−0.97	0.25
210°	−0.87	0.43
225°	−0.71	0.5
240°	−0.5	0.43
255°	−0.26	0.25
270°	0	0
285°	0.26	−0.25
300°	0.5	−0.43
315°	0.71	−0.5
330°	0.87	−0.43
345°	0.97	−0.25
360°	1	0

iii



iv It looks like the infinity symbol.

v $-1 \le x \le 1$, $-0.5 \le y \le 0.5$

6 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 7 GEOMETRY

7 Are you ready?

1 180° 2 D
3 a 47° b C c $c = 133°$
4 A 5 35° 6 B 7 B
8 a 3.5 b 2
9 a radius b circumference
c diameter d sector

7A Geometry review

7A Start thinking!

1 a vertically opposite b supplementary
c complementary d co-interior
e alternate f corresponding
2 Some suggestions are: angle sum of triangle is 180°, opposite angles in parallelogram are equal, exterior angle of triangle is equal to sum of two interior opposite angles, parallel lines contain angles which are corresponding (equal angles), alternate (equal angles) and co-interior (supplementary angles), angles at a point add to 360°, vertically opposite angles (equal) are formed when two straight lines intersect.
3 Angle e is an obtuse angle within two intersecting straight lines. The angle which is corresponding to angle e on the parallel line is angle $(b + k)$. Angle k is acute.
4 acute, right-angled, obtuse, reflex, revolution
5 180°; they form the three interior angles of a triangle.
6 right-angled
7 acute-angled triangle (all internal angles acute)
right-angled triangle (contains one right angle)
obtuse-angled triangle (contains one obtuse angle)
scalene triangle (all sides have different length)
isosceles triangle (two side-lengths equal)
equilateral triangle (all side lengths equal)
8 The angle g is the exterior angle of the triangle.
9 360°; they are the four interior angles of a quadrilateral.
10 parallelogram
11 irregular quadrilateral (all sides and angles have different values)
parallelogram (opposite sides equal and parallel)
rhombus (parallelogram with all sides equal in length)
rectangle (parallelogram with all angles right angles)
square (rectangle with all sides lengths equal)
kite (two pairs of adjacent sides equal in length)
12 Angles can be found using various rules, for example: $a = b$ (vertically opposite), $b + d = 180°$ (on straight line), $e + l = 180°$ (on straight line), $e = g$ (opposite angles of parallelogram), $e + f = 180°$ (co-interior angles), $f = c$ (opposite angles of parallelogram), $g + j = 180°$ (on straight line), $h = 90°$ (right angle), $i = 180° - h - j$ (sum of angles in triangle), $k = 180° - a - c$ (on straight line). All unmarked angles can also be determined using similar procedures.

Exercise 7A Geometry review

1 a $a = 146°$ (supplementary)
b $b = 24°$ (complementary)
c $c = 57°$ (supplementary), $d = 57°$ (vertically opposite), $e = 123°$ (vertically opposite)
d $d = 75°$ (angles at a point)
e $e = 109°$ (supplementary), $f = 71°$ (vertically opposite), $g = 74°$ (supplementary)
f $f = 15°$ (angles at a point)
g $g = 71°$ (complementary), $h = 19°$ (vertically opposite), $i = 71°$ (complementary), $j = 90°$ (supplementary)
h $g = 42°$ (vertically opposite), $h = 73°$ (vertically opposite), $i = 21°$ (complementary), $j = 17°$ (complementary), $k = 48°$ (complementary), $l = 17°$ (vertically opposite)
i $i = 11°$ (vertically opposite), $j = 62°$ (complementary), $k = 44°$ (supplementary), $l = 44°$ (vertically opposite), $m = 108°$ (supplementary)
2 a $a = 117°$ b $b = 109°$ c $c = 49°$ d $d = 129°$
e $e = 63°$, $f = 117°$, $g = 117°$, $h = 63°$
f $f = 126°$
3 a $a = 97°$ b $b = 88°$, $c = 88°$, $d = 92°$
c $c = 58°$, $d = 302°$
d $d = 52°$, $e = 71°$, $f = 57°$
e $d = 180°$, $e = 30°$, $f = 115°$, $g = 51°$
f $f = 67°$, $g = 64°$, $h = 116°$, $i = 91°$
4 a $a = 76°$ b $b = 60°$ c $c = 32°$, $d = 116°$
d $d = 160°$ e $e = 115°$
f $f = 37°$, $g = 75°$, $h = 68°$
5 a $a = 55°$ b $b = 117°$, $c = 63°$, $d = 117°$
c $c = 30°$, $d = 118°$ d $d = 45°$
e $e = 119°$, $f = 25°$
f $f = 58°$, $g = 98°$, $h = 98°$, $i = 122°$
6 a 8 b 6 c 2
d angle sum = $180° \times 6 = 1080°$
e i pentagon: 5 sides, 3 triangles, difference of 2, angle sum = $180° \times 3 = 540°$
ii decagon: 10 sides, 8 triangles, difference of 2, angle sum = $180° \times 8 = 1440°$
iii heptagon: 7 sides, 5 triangles, difference of 2, angle sum = $180° \times 5 = 900°$
iv dodecagon: 12 sides, 10 triangles, difference of 2, angle sum = $180° \times 10 = 1800°$
f Number of triangles is 2 fewer than number of sides.

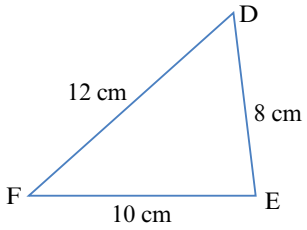
7 a 3240° b 8640° c 17 640°
8 a internal angle sum of polygon = $(n - 2) \times 180°$
A regular polygon with n equal sides has n equal angles.
So size of each angle = $\frac{(n - 2) \times 180°}{n}$
b Formula can only be used for regular polygons, because they have all angles equal in size.
9 a 128.6° b 140° c 120°
10 a 18 b 10 c 36 d 1000
11 a 18° b 27° c 22.5°
d 9.6875° e 21° f 17.63°
12 a true
b false; angles around a point add to 360°
c false; an isosceles triangle has one pair of equal angles
d false; corresponding angles are equal
e true
f false; a rectangle is a parallelogram, but not all parallelograms are rectangles
g false; complementary angles add to 90°
h false; the other two angles do not have to be 60° each
i true
13 a $a = 14°$ ($8 \times 301° + 8a = 180° \times 14$)
b $b = 30°$ (angle sum in quadrilateral is 360°),
$c = 44°$ (angle sum in triangle is 180°),
$d = 74°$ (supplementary angles on straight line),
$e = 29°$ (supplementary angles on straight line)
c $c = 72°$ (vertically opposite angles, alternate angles, corresponding angles, angle sum of triangle)
d $d = 109°$, $e = 8°$ (angle properties of parallel lines, angle sum in polygon, sum of angles at a point)
14 a $a = 60°$ (equilateral triangle, corresponding angles),
$b = 30°$ (equilateral triangle, corresponding angles, angle sum of triangle)
$c = 90°$ (angle sum of triangle)
b $d = 35°$ (alternate angles, co-interior angles, angles at a point)
c $e = 163°$
d $f = 79°$ (alternate angles)
e $e = 220°$ (co-interior angles, angle sum in isosceles triangle)
f $f = 154.5°$ (angle sum in polygon, angles at a point)

7B Congruence

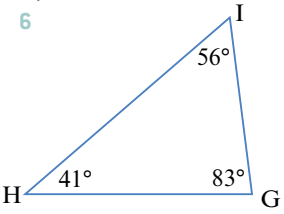
7B Start thinking!

1 AB = 12 cm, BC = 8 cm, CA = 10 cm,
∠BAC = 41°, ∠ABC = 56°, ∠BCA = 83°

2



3 The two triangles are congruent because they are the same shape and size.
4 ∠DEF = 83°, ∠EFD = 41°, ∠FDE = 56°
5 A corresponds to F,
B corresponds to D,
C corresponds to E so
ΔDEF ≅ ΔBCA
6



7 Perhaps; the corresponding angles are equal, but the corresponding sides may or may not be equal.
8 For two triangles to be congruent, corresponding sides and angles must be congruent. If only corresponding angles are known to be congruent, one triangle could be a dilation of the other. If the corresponding angles of the two triangles are not equal, there is no possibility of the triangles being congruent, as they will not be the same shape.

Exercise 7B Congruence

1 a PQ = NM, PR = NL, QR = ML
b TR = BD, TU = BC, RU = DC
c FG = OP, FH = ON, GH = PN
d BA= YZ, AC = ZX, BC = YX
e EF = JF, FG = FH, EG = JH
f YZ = XW, ZW = WZ, YW = XZ
2 a $a = 51°$, $b = 11$ cm, $c = 70°$, $d = 59°$, $e = 10$ cm
b $a = 48°$, $b = 25°$, $c = 4$ cm, $d = 25°$, $e = 107°$, $f = 7$ cm
c $a = 38°$, $b = 50°$, $c = 13$ cm, $d = 38°$, $e = 92°$, $f = 8$ cm, $g = 10$ cm
d $j = 25°$, $k = 7$ cm, $l = 3$ cm, $m = 5$ cm, $n = 115°$
3 a SAS b AAS c SSS d RHS
e SAS f AAS
4 a congruent (AAS) b congruent (RHS)
c congruent (SAS)
d not congruent (SSS does not apply)
e not congruent (AAS does not apply)
f not congruent (RHS does not apply)
5 a congruent; SAS
b not congruent; SAS does not apply
c congruent; SAS d congruent; AAS
6 a Consider a square and a rhombus which have equal side lengths. The angles in the square are all 90°, whereas those in the rhombus are not. This means the two figures do not have the same shape, so they are not congruent.

b A rhombus has four sides of equal length. If the SAS condition for congruence were to hold, the parallelogram would need to have adjacent sides equal in length. This would make it a rhombus.

c The AAS condition for congruence considers two angles (in this case, right angles) and one side. This may hold true for one section of the square and rectangle, but the other sides will not be equal lengths unless the rectangle is a square.

7 a The 9-cm side length is opposite the 85° angle in one figure, and adjacent to the 85° angle in the other figure.

b The 88° angle is not the angle between the two given side lengths.

c Three pairs of congruent angles is not a condition for congruence. One triangle could be a dilation of the other.

d The congruence condition RHS does not apply since the only side measurement given is the hypotenuse.

8 Congruent figure must be the same shape and size.

9 a False; the triangles may or may not be congruent, depending on their side lengths.

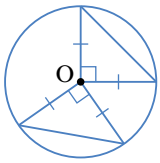
b False; the corresponding angle must be between the two corresponding sides for the triangles to be congruent in this case.

c False; if a pair of triangles meets one congruence condition, they will meet all congruence conditions (but perhaps further calculations are necessary to show these other congruence conditions are met).

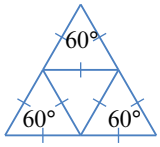
d True; figures which are congruent to each other all have the same shape and size.

e False; quadrilaterals which have all sides equal in length can be a square or a rhombus. A square and a rhombus are only congruent if all angles are right angles.

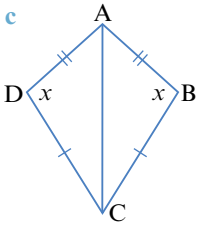
f False; all equilateral triangles are similar, but they are only congruent if their side lengths are the same.

10 a 

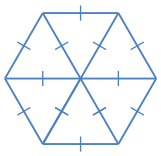
The two triangles are both right-angled triangles, with the two shorter sides being radii of the circle (so equal in length). The congruence condition SAS applies.

b 

Four congruent equilateral triangles are formed. The congruence conditions SSS and SAS apply.

c 

A kite has two pairs of adjacent sides equal, and one pair of opposite angles equal. The congruence condition SAS applies.

d 
The six equilateral triangles formed all have the same side length. The congruence condition SSS applies.

7C Similarity

7C Start thinking!

1 They have a similar shape, but different size.

2 Similar triangles have the same shape.

3 Their corresponding side lengths should be in the same ratio.

4 QS is 3 times the length of YX.

5 They are in the same ratio.

6 If image is larger than original, ratio calculation will be $\frac{\text{longer side}}{\text{shorter side}}$, which gives an answer greater than 1.

For smaller image, ratio will be $\frac{\text{shorter side}}{\text{longer side}}$, resulting in an answer less than 1.

Exercise 7C Similarity

1 a $\frac{5}{2}$ **b** $\frac{3}{2}$ **c** 2 **d** 3

2 a $x = 10.5$ cm, $y = 8$ cm **b** $c = 44$ cm, $d = 7$ cm

c $x = 4$ cm, $y = 9$ cm **d** $a = 9$ cm, $b = 4$ cm

e $x = 5\frac{7}{13}$ cm, $y = 17\frac{1}{3}$ cm

f $x = 52$ cm, $y = 20$ cm

3 a not similar; fails SSS test

b not similar; fails SAS test

c similar; passes AAA test

d similar; passes AAA test

e similar; passes SSS test

f similar; passes RHS test

4 a similar; passes SSS test

b similar; passes AAA test

c not similar; fails AAA test

d not similar; fails RHS test

5 a A rectangle and a parallelogram can have all corresponding sides in the same ratio without being similar. They are not the same shape unless the parallelogram has all right angles, which would make it a rectangle.

b A parallelogram and a rhombus may have corresponding angles all the same size, but they only have the same shape if the sides of the parallelogram are all equal. This makes the parallelogram a rhombus.

c If a rectangle meets the SAS property of a square, its adjacent sides would have to be the same length, making it a square.
d As in part b, the only way a parallelogram and a rhombus can have the same shape is if the parallelogram has all sides of equal length, making it a rhombus.

6 a It appears at first glance that the scale factor is $\frac{1}{2}$. However, the sides compared are not corresponding sides.
b The RHS test requires a right-angle, the hypotenuse and one of the shorter sides (which is not provided). This means that the pair of triangles fails the RHS test.

7 Similar figures must have the same shape. If corresponding sides and angles are not compared, the shapes will appear different.

8 4.9 m 9 3.6 m

11 Similar pairs: parts a and c (AAA), parts d and e (SAS)

12 a false; they must be similar if corresponding angles are equal
b true; the corresponding angles in these three triangles must be equal
c false; the angles in isosceles triangles are not fixed
d true; all squares have all angles equal to 90°, which makes them all the same shape
e true; if they pass SAS they must be similar, while if they fail SAS they are not similar
f true; consider a rhombus and a square with the same side lengths.

13 Enlargement; scale factor is greater than 1.

14 7.2 m

7D Understanding proofs

7D Start thinking!

2 $a = c$ and $b = d$. They are vertically opposite angles.

4 $a + b = 180°$ 5 $b + c = 180°$

6 They are both equal to 180°.

7 If $a + b = 180°$, then $b = 180° - a$. Also since $b + c = 180°$, $b = 180° - c$. The value of b must be fixed, so the values of a and c must be the same.

8 $a + b = 180°$
$b + c = 180°$
$\therefore a = c$

Exercise 7D Understanding proofs

1 $\angle MNQ + \angle QNP = 180°$
$\angle ONP + \angle QNP = 180°$
$\angle MNQ + \angle QNP = \angle ONP + \angle QNP$
$\angle MNQ = \angle ONP$

2 $\angle XYW + \angle WYV = 180°$
$\angle ZYV + \angle WYV = 180°$
$\angle XYW + \angle WYV = \angle ZYV + \angle WYV$
$\angle XYW = \angle ZYV$

3 a $\angle EOF + \angle FOB = 180°$
$\angle BOC + \angle FOB = 180°$
$\angle EOF + \angle FOB = \angle BOC + \angle FOB$
$\angle EOF = \angle BOC$
b $\angle EOF + \angle AOF + \angle AOB = 180°$
$\angle BOC + \angle AOF + \angle AOB = 180°$
$\angle EOF + \angle AOF + \angle AOB =$
$\angle BOC + \angle AOF + \angle AOB$
$\angle EOF = \angle BOC$

4 $\angle GKJ + \angle JKL = 180°$
$\angle NKL + \angle JKL = 180°$
$\angle GKJ + \angle JKL = \angle NKL + \angle JKL$
$\angle GKJ = \angle NKL$

5 $\angle TUQ = \angle PQR$
$\angle PQR + \angle PQU = 180°$
$\angle SQU + \angle PQU = 180°$
$\angle PQR + \angle PQU = \angle SQU + \angle PQU$
$\angle PQR = \angle SQU$
but $\angle PQR = \angle TUQ$
so $\angle TUQ = \angle SQU$

6 $\angle JLK = \angle LON$
$\angle JLK + \angle JLO = 180°$
$\angle MLO + \angle JLO = 180°$
$\angle JLK + \angle JLO = \angle MLO + \angle JLO$
$\angle JLK = \angle MLO$
but $\angle JLK = \angle LON$
so $\angle MLO = \angle LON$

7 $\angle XYW = \angle YUT$
$\angle XYW + \angle WYU = 180°$
$\angle VUY + \angle YUT = 180°$
$\angle XYW + \angle WYU = \angle VUY + \angle YUT$
but $\angle XYW = \angle YUT$
so $\angle WYU = \angle VUY$

8 $\angle ABC = \angle EFB$
$\angle ABC + \angle CBD = 180°$
$\angle DBF + \angle CBD = 180°$
$\angle ABC + \angle CBD = \angle DBF + \angle CBD$
$\angle ABC = \angle DBF$
$\angle EFB + \angle BFG = 180°$
but $\angle ABC = \angle DBF = \angle EFB$
so $\angle DBF + \angle BFG = 180°$

9 a Any even number is divisible by 2.
b Because x has a factor of 2, it can be written as 2 times another number.
c y can be written as $2b$.
d $x + y = 2a + 2b$ e $2(a + b)$
f If a number is a multiple of 2, it is even.
g $x = 2a$
$y = 2b$
$x + y = 2a + 2b = 2(a + b)$
Since $x + y$ has 2 as a factor, it is an even number.

10 Let the even number be represented by $2n$.
$(2n)^2 = 4n^2$
Since the (even number)^2 is a multiple of 4, it is also even.

11 a A rational number is one which can be expressed as the ratio of two integers.

b $\sqrt{2} = \frac{a}{b}$

c If a and b were both even, they would have a common factor of 2, so the fraction would not be in its simplest form. One of these could be even.

d $2 = \frac{a^2}{b^2}$

e Multiplying both sides by b^2 produces this result.

f a^2 is even because it is a multiple of 2.

g If the square of a number is even, the number itself is even.

h Since a is even, it can be expressed as a multiple of 2.

i $(2k)^2 = 2b^2$

j $4k^2 = 2b^2$ simplifies to $2k^2 = b^2$. Since the square of b is even, it follows that b itself must be even.

k Initial assumption was a and b have no common factors. If both are even, they have at least a common factor of 2.

l Let $\sqrt{2}$ be represented by the rational expression $\frac{a}{b}$, where a and b have no common factors.

$$\sqrt{2} = \frac{a}{b}$$
$$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$
$$2 = \frac{a^2}{b^2}$$
$$a^2 = 2b^2$$

This shows that a^2 is even and hence a is even.
Let a be represented by $2k$.
$(2k)^2 = 2b^2$
$4k^2 = 2b^2$
$2k^2 = b^2$
This shows that b^2 is even and hence b is even.
If a and b are both even, this violates the initial assumption that a and b have no common factors.
It follows that $\sqrt{2}$ is not a rational number.

12 Let $\sqrt[3]{2}$ be represented by the rational expression $\frac{a}{b}$, where a and b have no common factors.

$$\sqrt[3]{2} = \frac{a}{b}$$
$$(\sqrt[3]{2})^3 = \left(\frac{a}{b}\right)^3$$
$$2 = \frac{a^3}{b^3}$$
$$a^3 = 2b^3$$

This shows that a^3 is even and hence a is even.
Let a be represented by $2k$.
$a^3 = 2b^3$
$(2k)^3 = 2b^3$
$8k^3 = 2b^3$
$4k^3 = b^3$
This shows that b^3 is even and hence b is even.
If a and b are both even, this violates the initial assumption that a and b have no common factors.
It follows that $\sqrt[3]{2}$ is not a rational number.

13 a Parallel lines will never meet, no matter how far they extend. Lines which are not parallel will meet at some point.

b $\angle EAB$ and $\angle ACB$

c If AC is equal to 0, the points A and C would be the same point. A value less than zero is meaningless.

d $\angle EAB + \angle BAC = 180°$
But $\angle EAB = x$
So $x + \angle BAC = 180°$
That is, $\angle BAC = 180° - x$

e $(180° - x) + x + y = 180°$

f $180° + y = 180°$ simplifies to $y = 0°$

g If $\angle ABC$ is 0°, line BC would lie on top of line BA, so points A and C would lie on top of each other.

h This contradicts the assumption that $AC > 0$. In other words, the lines must be parallel.

i Assume that $\angle EAB$ and $\angle ACB$ are corresponding angles, and that DG is not parallel to HF.
Also assume length of AC is greater than zero.
In ΔABC, $\angle ABC + \angle ACB + \angle CAB = 180°$
But $\angle CAB = 180° - \angle EAB$
So $\angle ABC + \angle ACB + (180° - \angle EAB) = 180°$
It was assumed that $\angle EAB = \angle ACB$.
This relationship then reduces to $\angle ABC = 0°$.
This contradicts the statement that $AC > 0$.
So the lines DG and HF must be parallel.

14 a Contrapositive statement: if p is not odd, then p^2 is not odd.

b If p is not odd, then it is even, so will contain a factor of 2.
It follows that p^2 will be even, and so not odd.
This proves the contrapositive statement.

c It follows that 'if p^2 is odd, then p must be odd'.

15 Prove: if the product of a and b is odd, then both a and b must be odd.
Contrapositive statement: if both a and b are not odd, then the product of a and b is not odd.
If a and b are both not odd, they must both be even.
This means that a and b both have a factor of 2.
The product $a \times b$ must then be represented by (even) × (even) which results in an even answer.
So $a \times b$ is not odd.
This proves the contrapositive statement, which in turn proves the original statement.

16 Prove: If $a^2 - 2a + 3$ is even, then a is odd.
Contrapositive statement:
If a is even, then $a^2 - 2a + 3$ is odd.
If a is even there must be an integer k for which $a = 2k$.
$a^2 - 2a + 3$
$= (2k)^2 - 2(2k) + 3$
$= 4k^2 - 4k + 3$
$= 2(2k^2 - 2k + 1) + 1$
This means that $a^2 - 2a + 3$ must be odd.
This proves the contrapositive statement, which in turn proves the original statement.

17 a The pronumeral k can take on any integer value.
b i for $n = 1$, $2n + 1 = 3$ (odd)
ii for $n = 2$, $2n + 1 = 5$ (odd)
iii for $n = 99$, $2n + 1 = 199$ (odd)
iv for $n = 100$, $2n + 1 = 201$ (odd)
c for $n = k + 1$, $2n + 1 = 2(k + 1) + 1 = 2k + 2 + 1 = 2k + 3$
d Since $2k + 3$ = even + odd = odd, it follows that all numbers of this form are odd.

18 A base of 3, raised to any power that is a positive integer will result in an answer which is an odd number. Subtracting 1 from an odd number results in an even number.
So $3^n - 1$ is an even number, which means it is a multiple of 2.

19 If $a = b$, then $(a - b) = 0$. Dividing by $(a - b)$ is actually dividing by 0, which gives an answer which is undefined.

20 Let $\sqrt{5}$ be represented by the rational expression $\frac{a}{b}$, where a and b have no common factors.

$$\sqrt{5} = \frac{a}{b}$$
$$(\sqrt{5})^2 = \left(\frac{a}{b}\right)^2$$
$$5 = \frac{a^2}{b^2}$$
$$a^2 = 5b^2$$

This shows that a^2 is a multiple of 5 and a is a multiple of 5. That is, if 5 is a factor of a then 5 is also a factor of a^2.
Let a be represented by $5k$.

$$a^2 = 5b^2$$
$$(5k)^2 = 5b^2$$
$$25k^2 = 5b^2$$
$$5k^2 = b^2$$

This shows that b is also a multiple of 5.
If a and b are both multiples of 5, this violates the initial assumption that a and b have no common factors.
It follows that $\sqrt{5}$ is not a rational number.

7E Proofs and triangles

7E Start thinking!

1 180°
3 Showing something is true for 10 triangles doesn't prove it is true for every triangle.
4 ∠ABC, ∠ACB, ∠CAB
5 They are alternate angles within parallel lines.
6 ∠DBA
7 ∠DBA + ∠ABC + ∠CBE = 180°
8 ∠BAC + ∠ABC + ∠BCA = 180°
9 This has shown that the sum of the three angles of the triangle is 180°.
10 ∠DBA + ∠ABC + ∠CBE = 180° (angles on a straight line)
∠DBA = ∠BAC (alternate angles)
∠CBE = ∠BCA (alternate angles)
So ∠BAC + ∠ABC + ∠BCA = 180°

Exercise 7E Proofs and triangles

1 a ∠WXY + ∠YXZ = 180°
b ∠YXZ + ∠XZY + ∠ZYX = 180°
c ∠WXY + ∠YXZ = ∠YXZ + ∠XZY + ∠ZYX
d ∠WXY = ∠XZY + ∠ZYX

2 ∠JLM + ∠JLK = 180°
∠JLK + ∠JKL + ∠LJK = 180°
∠JLM + ∠JLK = ∠JLK + ∠JKL + ∠LJK
∠JLM = ∠JKL + ∠LJK

3 ∠EFG + ∠EFD = 180°
∠FDE + ∠DEF + ∠EFD = 180°
∠EFG + ∠EFD = ∠FDE + ∠DEF + ∠EFD
∠EFG = ∠FDE + ∠DEF

4 AB = CB (given)
AD = CD (given)
BD is common.
So ΔABD ≅ ΔCBD (using SSS)

5 QP = SP (given)
QR = SR (given)
PR is common.
So ΔPQR ≅ ΔPSR (using SSS)

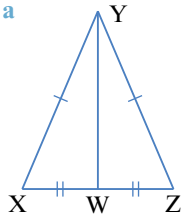
6 a AC = DC (given)
∠ACB = ∠DCE (given)
∠CBA = ∠CED (given)
b ∴ ∠CAB = ∠CDE (angle sum of triangle)
c ∴ ΔABC ≅ ΔDEC (using AAS)

7 QR = QT (given)
∠PQR = ∠SQT (given)
∠RPQ = ∠TSQ (given)
So ΔPQR ≅ ΔSQT (using AAS)

8 VX = YX (given)
WX = ZX (given)
∠VXW = ∠YXZ (vertically opposite)
So ΔVWX ≅ ΔYZX (using SAS)

9 ∠EFH + ∠DFE = 180° (straight line)
∠DFE + ∠DEF + ∠FDE = 180° (angle sum of triangle)
So ∠EFH + ∠DFE = ∠DFE + ∠DEF + ∠FDE
Simplifying, ∠EFH = ∠DEF + ∠FDE

10 a

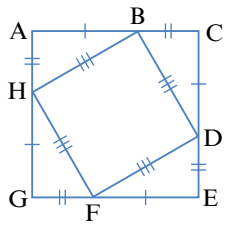


b ΔYWX and ΔYWZ
c YX = YZ and WX = WZ
d YW is common to both triangles
e Because ΔXYZ is an isosceles triangle, line YW drawn to midpoint of XZ will bisect angle Y, making ∠XYW = ∠ZYW.
f–h YX = YZ (given)
WX = WZ (by construction)
YW is common.
So ΔWXY ≅ ΔWZY (using SSS)
It follows that ∠WXY = ∠WZY, as these are corresponding angles in the two triangles.

11 Assume that an equilateral triangle has three equal sides. Follow procedure from previous question to prove the base angles equal (using figure, ∠YXZ = ∠YZX). Repeat procedure by drawing a line from one base angle to midpoint of opposite side. It can then be shown that, for example, ∠ZYX = ∠YZX. This proves the three angles are equal. Since they also add to 180°, each angle must be 60°.

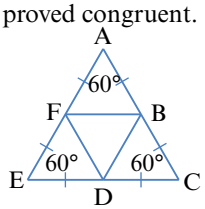
12 a A square has sides of equal length.
b length of side of larger square is $(a + b)$.
area of square = $(\text{side})^2$
So area of larger square = $(a + b)^2$
c area of smaller square = c^2
d area of triangle = $\frac{ab}{2}$
e area of larger square =
area of smaller square + area of four triangles
$= c^2 + 4 \times \frac{ab}{2}$
$= c^2 + 2ab$
f area of larger square is $(a + b)^2$
g, h $(a + b)^2 = c^2 + 2ab$
$a^2 + 2ab + b^2 = c^2 + 2ab$
$a^2 + b^2 = c^2$

13



Take ΔABH and ΔCDB.
AB = CD (given)
AH = CB (given)
BH = DB (given)
So ΔBAH ≅ ΔDCB (using SSS)
Hence, corresponding angles are equal.
By a similar procedure, all four triangles can be proved congruent.

14



Since ACE is an equilateral triangle, its side lengths are equal.
Points B, D and F lie at midpoints of these lines.
So it follows that AB = BC = CD = DE = EF = FA
Also since ΔACE is equilateral, ∠ACE = ∠CEA = ∠EAC = 60° (as proved in question 11)
Take ΔABF and ΔCDB.
AB = CD (midpoint lengths)
AF = CB (midpoint lengths)
∠FAB = ∠BCD = 60° (angles of equilateral triangle)
So ΔABF ≅ ΔCDB (using SAS)
It follows that FB = BD.
By considering ΔBCD and ΔDEF and following a similar proof, it can be shown that BD = DF.
So FB = BD = DF
This means that ΔFBD is equilateral, and ∠FBD = ∠BDF = ∠DFB = 60°
Consider ΔABF:
Since this triangle is isosceles (AF = AB) and ∠FAB = 60°, it follows that ∠ABF = ∠AFB = 60° (angle sum in triangle is 180°)
This means that ΔABF is actually equilateral, with three equal angles and three equal side lengths.
Similarly, ΔBCD and ΔDEF can be shown to be equilateral.
The side lengths of ΔABF, ΔBCD, ΔDEF and ΔFBD are all equal.
This means that the four triangles are congruent equilateral triangles.

7F Proofs and quadrilaterals

7F Start thinking!

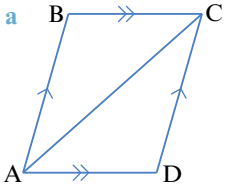
2 a true
b false; a trapezium has only one pair of parallel sides
c false; not all angles in a parallelogram measure 90°
d true **e** true
f false; not all angles in a rhombus measure 90°

3 Some suggested approaches:
a use parallel lines, congruent figures, similar figures, shapes of triangles
b use congruent triangles
c use congruent triangles

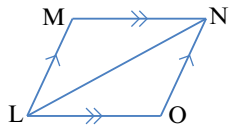
4 Congruence identifies figures which have the same properties.

Exercise 7F Proofs and quadrilaterals

1 a



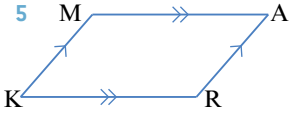
b ΔABC and ΔCDA
c AC is common.
d BC||AD,
so ∠BCA = ∠DAC
AB||DC,
so ∠BAC = ∠DCA
e ∠BCA = ∠DAC (alternate angles)
∠BAC = ∠DCA (alternate angles)
AC is common.
So ΔABC ≅ ΔCDA (using AAS)
f AB = CD and BC = DA

2 

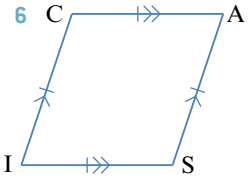
Take ΔMNL and ΔOLN.
∠MNL = ∠OLN (alternate angles)
∠MLN = ∠ONL (alternate angles)
LN is common.
ΔMNL ≅ ΔOLN (using AAS)
So MN = OL and LM = NO

3 Join EG.
Take ΔFGE and ΔHEG.
∠FGE = ∠HEG (alternate angles)
∠FEG = ∠HGE (alternate angles)
EG is common.
ΔFGE ≅ ΔHEG (using AAS)
So FG = HE and EF = GH
However, FG = EF (given)
So FG = EF = HE = GH
So the shape is a rhombus.

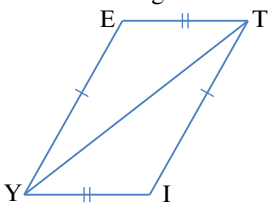
4 **a** ∠DNA + ∠NAY = 180° (co-interior angles)
∠DNA + ∠NDY = 180° (co-interior angles)
b ∠DNA + ∠NAY = ∠DNA + ∠NDY
So ∠NAY = ∠NDY
c ∠NAY + ∠AYD = 180° (co-interior angles)
∠NAY + ∠AND = 180° (co-interior angles)
∠NAY + ∠AYD = ∠NAY + ∠AND
So ∠AYD = ∠AND
d This proves that opposite angles of a parallelogram are equal.

5 

∠AMK + ∠MKR = 180° (co-interior angles)
∠ARK + ∠MKR = 180° (co-interior angles)
∠AMK + ∠MKR = ∠ARK + ∠MKR
So ∠AMK = ∠ARK
∠AMK + ∠MKR = 180° (co-interior angles)
∠AMK + ∠MAR = 180° (co-interior angles)
∠AMK + ∠MKR = ∠AMK + ∠MAR
So ∠MKR = ∠MAR
This proves that opposite angles of the parallelogram are equal.

6 

∠ACI + ∠CIS = 180° (co-interior angles)
∠ASI + ∠CIS = 180° (co-interior angles)
∠ACI + ∠CIS = ∠ASI + ∠CIS
So ∠ACI = ∠ASI
∠ICA + ∠CAS = 180° (co-interior angles)
∠ICA + ∠CIS = 180° (co-interior angles)
So ∠CAS = ∠CIS
This proves that opposite angles of the rhombus are equal.

7 **a** Join the diagonal YT.


b Take ΔYET and ΔTIY.
ET = IY (given)
EY = IT (given)
YT is common.
ΔYET ≅ ΔTIY (using SSS)
c So ∠ETY = ∠IYT and ∠EYT = ∠ITY
d Since ∠ETY = ∠IYT, and they are alternate angles, ET||IY.
Since ∠EYT = ∠ITY, and they are alternate angles, EY||IT.
e Since the opposite sides are equal and parallel, the figure is a parallelogram.

8 Join the diagonal CI.
Take ΔCHI and ΔIPC.
HI = PC (given)
CH = IP (given)
CI is common.
ΔCHI ≅ ΔIPC (using SSS)
So ∠HIC = ∠PCI and ∠HCI = ∠PIC
Since ∠HIC = ∠PCI, and they are alternate angles, HI||PC.
Since ∠HCI = ∠PIC, and they are alternate angles, HC||PI.
Since the opposite sides are equal and parallel, the figure is a parallelogram.

9 Draw quadrilateral ABCD with four equal sides and join diagonal DB.
Take ΔABD and ΔCDB.
AB = CD (given)
AD = CB (given)
DB is common.
ΔABD ≅ ΔCDB (using SSS)
So ∠ABD = ∠CDB and ∠ADB = ∠CBD.
Since ∠ABD = ∠CDB and they are alternate angles, AB||CD.
Since ∠ADB = ∠CBD and they are alternate angles, AD||CB.
Since all sides are equal and the opposite sides are parallel, the figure is a rhombus.

10 **a** ∠ETR **b** ∠TER
c Take ΔMIR and ΔTER.
∠MIR = ∠TER (alternate)
∠IMR = ∠ETR (alternate)
IM = ET (given)
So ΔMIR ≅ ΔTER (using AAS)
d It follows that IR = ER and MR = TR.
e IR and ER form the diagonal IE, while MR and TR form the diagonal TM. The diagonals bisect each other.

f Take ΔMIR and ΔTER.
∠MIR = ∠TER (alternate)
∠IMR = ∠ETR (alternate)
IM = ET (given)
So ΔMIR ≅ ΔTER (using AAS)
IR = ER and MR = TR
This means that the diagonals of the parallelogram bisect each other.

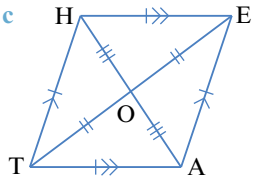
11 Proof is similar to that in question 10 part f.

12 Let the diagonals intersect at point K.
Take ΔOKL and ΔFKG.
OK = FK (given)
LK = GK (given)
∠OKL = ∠FKG (vertically opposite)
So ΔOKL ≅ ΔFKG (using SAS)
It follows that ∠LOK = ∠GFK and ∠OLK = ∠FGK.
Since these are pairs of alternate angles, OL||FG.
Similarly, by taking ΔOKG and ΔFKL, it can be proved that OG||FL.
Since opposite sides are parallel, the shape is a parallelogram.

13 a Take ΔHEO and ΔATO.
∠EHO = ∠TAO (alternate angles)
∠HEO = ∠ATO (alternate angles)
HE = AT (given)
So ΔHEO ≅ ΔATO (using AAS)
It follows that HO = AO and EO = TO.
This means that the diagonals bisect each other.

b ΔHEO ≅ ΔATO (proved previously)
Likewise, ΔAEO ≅ ΔHTO.
Now take ΔHEO and ΔAEO.
HE = AE (given)
HO = AO (proved previously)
EO is common.
ΔHEO ≅ ΔAEO (using SSS)
It follows then that ΔHEO ≅ ΔAEO ≅ ΔHTO ≅ ΔATO

c



d Since ΔHEO ≅ ΔAEO, ∠HOE corresponds to ∠AOE and so ∠HOE = ∠AOE.

e These are adjacent angles on a straight line (the diagonal HA).
Since ∠HOE + ∠AOE = 180° and ∠HOE = ∠AOE, it follows that they are both equal to 90°
Similarly, it can be shown that ∠HOT = ∠AOT = 90°.
This proves that the diagonals of the rhombus bisect each other at right angles.

14 The proof here is similar to that in question 13.

15 a Take ΔAUL and ΔMUL.
AL = ML (given)
∠ALU = ∠MLU (supplementary angles on diagonal AM)
UL is common.
So ΔAUL ≅ ΔMUL (using SAS)
This means that AU = MU.
Similarly, it can be proved that MU = MC, MC = AC and AC = AU
Take ΔALU and ΔMLC.
AL = ML (given)
UL = CL (given)
∠ALU = ∠MLC (vertically opposite)
So ΔALU ≅ ΔMLC (using SAS)
This means that ∠UAL = ∠CML and ∠AUL = ∠MCL.
These are alternate angles, so AU||MC.
Similarly, it can be proved that AC||MU.
Since opposite sides are parallel, and all sides are equal in length, the figure is a rhombus.

b Since AUMC is a parallelogram (opposite sides are parallel), opposite sides are equal.
So AU = MC and AC = MU
Take ΔALU and ΔMLC.
∠UAL = ∠CML (alternate angles)
∠AUL = ∠MCL (alternate angles)
AU = MC (opposite sides of parallelogram)
So ΔALU ≅ ΔMLC (using AAS)
This means that AL = ML and UL = CL.
Take ΔAUL and ΔMUL.
AL = ML (proved)
∠ULA = ∠ULM (supplementary angles on straight line)
UL is common.
So ΔAUL ≅ ΔMUL (using SAS)
This means that AU = MU.
Similarly, it can be proved that MU = MC and MC = AC
This proves the parallelogram has four sides of equal length, so it is a rhombus.

16 Since REMY is a parallelogram, opposite sides are equal and parallel.
RE||YM, so ∠REM + ∠EMY = 180° (co-interior angles)
But ∠REM = 90° (given)
So ∠EMY = 90°
Similarly, it can be shown that ∠MYR = 90° and ∠YRE = 90°
This proves that REMY is a rectangle.

17 a Take ΔEVI and ΔILE.
EV = IL (given)
EI is common.
∠VEI = ∠LIE (alternate angles)
So ΔEVI ≅ ΔILE (using SAS)
This means that ∠EVI = ∠ILE = 90°.
Since EV||IL, ∠EVI + ∠VIL = 180°, and
∠ILE + ∠LEV = 180°
This simplifies to
∠EVI = ∠VIL = ∠ILE = ∠ELV = 90°.

b Since ΔEVI ≅ ΔILE, EL = IV
Since ∠LEV = ∠EVI = 90° (proved previously)
∠LEV + ∠EVI = 180°
These are co-interior angles, so EL||IV.

c Pythagoras' Theorem states that $c^2 = a^2 + b^2$.
So $(EI)^2 = (EL)^2 + (IL)^2$ and
$(LV)^2 = (EL)^2 + (EV)^2$
But EV = IL (given)
$(LV)^2 = (EL)^2 + (IL)^2 = (EI)^2$
So LV = EI

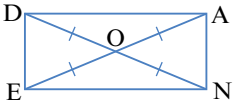
d Since EV = IL (given), LV = EI (proved) and EL is common, ΔLEV ≅ ΔELI (using SSS)

18 a Let the diagonals intersect at the point P.
Take ΔELP and ΔYKP.
PE = PY (given)
PL = PK (given)
∠EPL = ∠YPK (vertically opposite)
So ΔELP ≅ ΔYKP (using SAS)
It follows that
∠PEL = ∠PYK and ∠PLE = ∠PKY.
These are pairs of alternate angles, so EL||YK.
Similarly, it can be proved that EK||YL.
So the figure ELYK is a parallelogram.

b Take ΔELY.
∠LYE + ∠YEL + ∠ELK + ∠KLY = 180°
(angle sum of triangle)
Since ΔELP and ΔLPY are both isosceles triangles,
∠YEL = ∠ELK and ∠LYE = ∠KLY.
So ∠LYE + ∠YEL + ∠ELK + ∠KLY = 180°
becomes ∠KLY + ∠ELK + ∠ELK + ∠KLY
= 180°
or 2∠ELK + 2∠KLY = 180°
or ∠ELK + ∠KLY = 90°
So ∠ELY = 90°

c By a similar method, it can be shown that ∠LEK, ∠EKY and ∠KYL are all equal to 90°.
Since the figure is a parallelogram, and all angles are 90°, the figure is a rectangle.

19



OD = OA = ON = OE
∠DOA = ∠EON (vertically opposite angles)
Hence ΔDOA ≅ ΔEON (using SAS)
Hence DA = EN
Similarly, ∠DOE = ∠AON (vertically opposite angles)
Hence ΔDOE ≅ ΔAON (using SAS)
Hence DE = AN
Hence opposite sides are equal.
ΔDOA ≅ ΔEON, and they are isosceles triangles.
Hence ∠ODA = ∠DAO and ∠OEN = ∠ENO (isosceles triangle theorem)
Similarly, ΔDOE ≅ ΔAON and they are isosceles triangles.
Hence ∠ODE = ∠DEO and ∠OAN = ∠ANO (isosceles triangle theorem)
∠ODA + ∠DAO + ∠OEN + ∠ENO + ∠ODE + ∠DEO + ∠OAN + ∠ANO = 360°
But ΔDOA ≅ ΔEON
So ∠ODA = ∠DAO = ∠OEN = ∠ENO
And because ΔDOE ≅ ΔAON,
∠ODE = ∠DEO = ∠OAN = ∠ANO.
Hence 4∠ODA + 4∠ODE = 360°
So ∠ODA + ∠ODE = 90°.
Similarly, ∠DAO + ∠OAN = 90°,
∠ANO + ∠ONE = 90° and
∠NEO + ∠OED = 90°.
Hence all angles are equal and 90°.
If all angles are equal to 90°, it follows that DA||EN and DE||AN.
Hence DANE is a rectangle.

20 a Take ΔGUA and ΔGRS.
UA = RS (given)
∠AUG = ∠SRG (alternate)
∠UAG = ∠RSG (alternate)
ΔGUA ≅ ΔGRS (using AAS)
So UG = RG and AG = SG (the diagonals bisect each other)

b Take ΔUAG and ΔRAG.
UA = RA (given)
UG = RG (proved previously)
GA is common.
So ΔUAG ≅ ΔRAG (using SSS)

c This means that ∠UAG = ∠RAG.
Hence, ∠UAS = ∠RAS

d These two angles are the angles at vertex A, and the angle at A has been bisected by the diagonal SA to produce them.

21 Let the diagonals intersect at point P.
Take ΔEVP and ΔIVP.
EP = IP (given)
EV = IV (given)
PV is common.
So ΔEVP ≅ ΔIVP (using SSS)
This means that ∠VPE = ∠VPI.
Since they are supplementary angles,
∠VPE = ∠VPI = 90°.
Take ΔEVP.
∠EVP = ∠VEP (isosceles triangle)
∠VPE = 90° (proved previously)
So ∠EVP = ∠VEP = 45°
That is, ∠EVL = 45°

22 The diagonals bisect each other at right angles.

23 Since ACEG is a rectangle, the angle at each vertex is 90°.
The opposite sides of a rectangle are equal, and the vertices of HBDF lie at the midpoints of the sides, so AB = BC = GF = FE and AH = HG = CD = DE.
So ΔABH ≅ ΔCBD ≅ ΔEFD ≅ ΔFGH (using SAS)
It follows that BD = FD = FH = BH
So quadrilateral HBDF has four equal sides.
Since the four triangles are congruent, the corresponding angles within those triangles are equal. This means that
∠ABH = ∠CBD = ∠EFD = ∠GFH.
Also ∠ABH + ∠HBD + ∠CBD = 180° and
∠GFH + ∠HFD + ∠DFE = 180°
It follows that ∠HBD = ∠HFD.
These are opposite angles of the quadrilateral HBDF.
Similarly, it can be proved that ∠BHF = ∠BDF.
Since HBDF has four equal sides, and opposite angles equal, it can be classed as a rhombus.

7G Circle geometry: circles and angles

7G Start thinking!

1 a see glossary
b arc: *g*, centre: *h*, chord: *d*, circumference: *a*, diameter: *c*, radius: *b*, sector: *e*, segment: *f*

2 a AB and BC b AC c ∠ABC
d Arc AC subtends angle ∠ABC.
e Chord AC subtends angle ∠ABC.

3 A chord or arc that subtends an angle lies opposite the angle and joins the two lines that forms it.

4 Must understand terms or the wrong proof may be carried out.

Exercise 7G Circle geometry: circles and angles

2 a All radii of the circle.
b Equal angles in isosceles triangle ΔABO.
c ∠AOD = ∠OAB + ∠ABO
(exterior angle of ΔABO)
= ∠ABO + ∠ABO
= 2∠ABO
d ∠DOC = ∠OBC + ∠BCO
(exterior angle of ΔOBC)
= ∠OBC + ∠OBC
(equal angles in isosceles triangle)
= 2∠OBC
e ∠AOD + ∠DOC = 2∠ABO + 2∠OBC
= 2(∠ABO + ∠OBC)
= 2∠ABC

3 a 45° b 120° c 74°

4 a 180°
b Since the angle at the centre is twice the angle at the circumference subtended by the same arc, the angle at the circumference is equal to $\frac{1}{2}$ of 180°, or 90°.

5 a 90° b 73° c 18°

6 a ∠WOZ is an angle at the centre, while ∠WXZ is an angle at the circumference, both standing on the same chord WZ.
b Similarly, ∠WOZ and ∠WYZ both stand on the same chord WZ.
c It follows that ∠WYZ = ∠WXZ.

7 a $x = 32°$ b $x = 27°$ c $a = 9°$, $c = 6°$

8 In the following answers, CT stands for Circle Theorem.
a $x = 34.5°$, CT2 b $x = 109°$, CT1
c $x = 21°$, CT3 d $a = 14°$, $b = 7°$, CT3
e $x = 22.5°$, CT2
f $a = 26\frac{1}{3}°$, $e = 9.875°$, CT3
g $x = 43°$, CT1 h $x = 7.5°$, CT2
i $x = 23°$, CT1

9 a 12.2° b 76° c 32° d 68°
e 36° f 29°

10 a Obtuse ∠AOC is the angle at centre of circle, standing on same arc as angle x at the circumference.
Similarly, the reflex angle ∠AOC is the angle at centre of the circle, standing on same arc as angle y at the circumference.
b obtuse ∠AOC + reflex ∠AOC = 360°
That is, $2x$ + reflex ∠AOC = 360°
Or, reflex ∠AOC = 360° − $2x$
c Since reflex ∠AOC = $2y$
$2y = 360° - 2x$
Simplifying, $y = 180° - x$
d obtuse ∠BOD = 2∠BCD
reflex ∠BOD = 2∠BAD
obtuse ∠BOD + reflex ∠BOD = 360°
(angles at a point)
So 2∠BAD + 2∠BCD = 360°
Rearranging and simplifying,
∠BAD = 180° − ∠BCD
e i $x = 142°$, $y = 99°$ ii $a = 45°$, $b = 38°$
iii $x = 30°$, $y = 22.5°$

11 a ∠ADC + ∠ABC = 180°
(opposite angles of cyclic quadrilateral)
So ∠ADC = 180° − ∠ABC
b supplementary angles on a straight line
c both equal to same expression (180° − ∠ABC)
d i $a = 61°$
ii $a = 81°$, $b = 92°$, $c = 99°$, $d = 88°$
iii $x = 15°$, $z = 29.75°$

13 a ∠DBC = 66° (angle subtended by a diameter)
b ∠DEG = 19° c ∠ODC = 38°
d ∠OAB = 62°

7H Circle geometry: chords

7H Start thinking!

1 chords: AB and CD 2 radii: OB and OD
3 ∠BOD = 90° 4 arc BD
5 EF; it is a diameter. 6 AG = GB and CH = HD
7 equal distance

8 CD and AB are equidistant from centre of circle because GO (distance from centre of circle to AB) = OH (distance from centre of circle to CD).

Exercise 7H Circle geometry: chords

1 a They are all equal in length.
b isosceles triangles
c OA = OC (radii)
OB = OD (radii)
AB = CD (given)
So ΔOAB ≅ ΔOCD (using SSS)
d It follows that ∠COD = ∠AOB, since they are corresponding angles in congruent triangles.
2 a 101° b 38° c 58°
3 a They are radii of the circle.
b Since ∠OCA is given as 90°, ∠OCB is also equal to 90°, as they are supplementary angles on the chord AB.
c OA = OB (radii)
∠OCA = ∠OCB = 90° (supplementary angles)
OC is common.
So ΔACO ≅ ΔBCO (using SAS)
d AC and BC are corresponding sides in congruent triangles.
4 a 13 cm b 7.5 cm c 3.7 cm
5 a They are all radii of the circle.
b Since OG is perpendicular to AB, it bisects the chord, so AG = GB.
Similarly, since OH is perpendicular to chord CD, CH = HD.
c If chords AB and CD are same length, and are both bisected by perpendicular radii, it follows that AG = GB = CH = HD.
d OD = OB (radii)
HD = GB
∠OHD = ∠OGB = 90°
So ΔOHD ≅ ΔOGB (using RHS)
e OH = OG, since they are corresponding sides in congruent triangles.
f In a right-angled triangle, the longest side is the hypotenuse (in this case, the radius). The other two sides are shorter in length.
6 a $x = 11$ cm b $x = 4$ cm
c $x = 6$ cm, $y = 3.5$ cm
7 a vertically opposite angles
b equal angles at circumference subtended by same arc DB
c equal angles at circumference subtended by same arc CA
d The two triangles have three equal corresponding angles.
e In similar triangles, the ratio of the lengths of corresponding sides is constant: $\frac{AP}{CP} = \frac{BP}{DP}$
Rearranging these ratios, $\frac{AP}{BP} = \frac{CP}{DP}$ or $AP \times DP = CP \times BP$
8 a 2.3 cm b 3.5 cm c 7.6 cm
9 a $x = 2$ cm, $y = 13$ cm b $x = 18$ cm
c $x = 2$ cm d $x = 4$ cm, $y = 21$ cm
e $x = 17°$ f $x = 13.1$ cm
g $x = 4$ cm h $x = 2.75$ cm, $y = 4$ cm
i $x = 19°$
10 a $x = 4.3$ cm, $y = 8$ cm b $x = 2.7$ cm, $y = 4$ cm
c $x = 10°$ d $x = 3$ cm, $y = 5$ cm
e $x = 35.5°$ f $x = 2$ cm, $y = 4$ cm
11 a $x = 19°$ b $x = 12°$
c $x = 23.8°$, $y = 15.25°$
d $x = 6$ cm, $y = 4$ cm, $z = 1$ cm
e $x = 18°$, $y = 18°$
f x can take on any positive value, $y = 0.6$

7I Circle geometry: tangents and secants

7I Start thinking!

1 see glossary
2 tangent: GFE, secant: ABCD
3 point F 4 points B and C
5 chord: a straight line from one point on the circumference of a circle to another
secant: a chord that is extended beyond the circumference of a circle
tangent: a straight line that touches the circumference of a circle at one point only
arc: a portion of the circumference of a circle

Exercise 7I Circle geometry: tangents and secants

1 a OB and OC are radii, and they are equal because all radii of a given circle are equal in length.
b tangent ABDE c OB
d Because OB is a radius length and OD is clearly longer than this.
e It is not possible for any other angle in ΔODB to be 90°.
2 a 90° b 45° c 69°
3 a AB is a tangent to radius OB, and AD is a tangent to radius OD.
b They are radii of the circle.
c OB = OD (radii)
∠ABO = ∠ADO = 90°
AO is common.
So ΔABO ≅ ΔADO (using RHS)
d AB = AD (corresponding sides)
4 a $x = 6$ cm b $x = 2.75$ cm
c $x = 11.6$ cm, $y = 3$ cm
5 a ∠BOE is the angle at the centre of the circle, while ∠BAE is the angle at the circumference, standing on the same arc BDE.
b reflex ∠BOE = 360° − obtuse ∠BOE
c ∠BDE and ∠BAE are supplementary angles in a cyclic quadrilateral.
d ∠BAE + ∠BDE = 180° (cyclic quadrilateral)
∠BDC + ∠BDE = 180° (supplementary angles on straight line)
So ∠BDC = ∠BAE
e ∠BCD f ∠BDC = ∠BAE

g If two pairs of angles in a triangle are equal to corresponding angles in another triangle, the third angles in the triangle must also be equal.

h If ΔCEA is similar to ΔCBD, the ratio of corresponding sides is constant.

So $\frac{CA}{CD} = \frac{EC}{BC} = \frac{EA}{BD}$

i $\frac{CA}{CD} = \frac{EC}{BC}$

Cross multiplying, CA × BC = CD × EC or AC × BC = CE × CD

6 a 7.2 cm **b** 12.4 cm **c** 2 cm

7 a ∠BCD

b ∠BDO = ∠DBO (isosceles triangle)
so ∠BOD = 180° − 2∠BDO
Also ∠BOD = 2∠BAD (angles subtended by the same arc BD)
so 2∠BAD = 180° − 2∠BDO
∠BAD = 90° − ∠BDO
As ∠CDO = 90° (angle between radius and tangent)
∠CDB = 90° − ∠BDO
so ∠BAD = ∠CDB
or ∠CAD = ∠CDB

c Take ΔBCD and ΔDCA.
∠CDB = ∠CAD (proved previously)
∠BCD is common.
Therefore, the remaining angles in the two triangles are equal.
That is, ∠CBD = ∠CDA
This means that ΔBCD is similar to ΔDCA (using AAA).

d In similar triangles, the ratio of corresponding sides is constant.

So $\frac{CB}{CD} = \frac{CD}{CA} = \frac{DB}{AD}$

e Taking the first two ratios and cross multiplying,
CB × CA = CD × CD
or CA × CB = $(CD)^2$

8 a 14.1 cm **b** 2.7 cm **c** 4.5 cm

9 a x = 3 cm (tangents to a circle from an external point are equal in length)

b x = 72° (angle between radius and tangent, angle subtended by diameter, angle sum in triangle)

c x = 3 cm, y = 2 cm (equal tangent lengths from a point external to a circle)

d x = 1 cm (two secants intersecting outside circle)

e x = 2.5 cm (tangent and secant intersecting outside circle)

f x = 64.5° (angle sum in isosceles triangle, angle between radius and tangent)

10 a x = 20.125 cm, y = 12.5 cm (tangent and secant intersecting outside circle)

b x = 24° (angles at a point, angle between tangent and radius, angle sum of quadrilateral)

c x = 3.2 cm, y = 12.6 cm (tangent and secant intersecting outside circle, two tangents intersecting outside circle)

13 a x = 5.6 cm **b** x = 5.7 cm

c x = 17.7 cm, y = 4.5 cm

14 From question 3, ΔABO ≅ ΔADO.
That is, OB = OD (radii)
∠ABO = ∠ADO = 90°
AO is common.
So ΔABO ≅ ΔADO (using RHS).
This means ∠DAO = ∠BAO (corresponding angles in congruent triangles)
so OA bisects the angle formed by the two tangents.

7 Chapter review

MULTIPLE-CHOICE

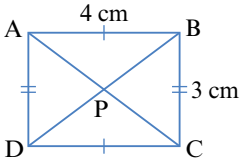
1 B **2** D **3** A **4** D **5** B
6 D **7** B **8** A **9** C

SHORT ANSWER

1 a true **b** true **c** true

d false. The 3-cm and 4-cm sides would lie flat on top of the 7-cm side.

2 a



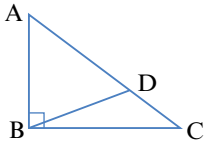
b alternate angles between parallel lines AB and CD

c AAS **d** 53°

3 a AAA (alternate and vertically opposite angles)

b $\frac{2}{5}$ **c** 7.5 cm

4 a



b ∠BAD + ∠ADB + ∠ABD = 180°
Adding the first two relationships,
∠ABD + ∠DBC + ∠BAD + ∠BCD
= 90° + 90° = 180°
So ∠BAD + ∠ADB + ∠ABD
= ∠ABD + ∠DBC + ∠BAD + ∠BCD
Simplifying, ∠ADB = ∠DBC + ∠BCD

5 a ∠AEC = ∠ADE + ∠DAE (exterior angle)
∠ADE = ∠DAE (isosceles triangle)
So ∠AEC = ∠ADE + ∠ADE = 2∠ADE

b Since ∠ABC + ∠BCE = 180°, and these are co-interior angles, it follows that AB||DC.
Therefore ∠ADE + ∠BAD = 180°
As ∠BAD = ∠BAE + ∠DAE and
∠DAE = ∠ADE
then ∠BAD = ∠BAE + ∠ADE
So ∠ADE + ∠BAE + ∠ADE = 180°
That is, 2∠ADE + ∠BAE = 180°

6 Take ΔPQT and ΔRST.
PQ = RS (given)
∠PQT = ∠RST (alternate angles)
∠PTQ = ∠RTS = 90° (vertically opposite)
So ΔPQT ≅ ΔRST (using AAS)
Therefore, TQ = TS
Take ΔTQR and ΔTSR.
TQ = TS (previously proved)
∠QTR = ∠STR = 90° (given)
TR is common.
So ΔTQR ≅ ΔTSR (using SAS)
This means that QR = SR.
Similarly, it can be proved that SR = SP, PS = PQ and QP = QR.
Since all the sides of PQRS are equal, and the opposite sides are parallel, the figure is a rhombus.
7 a 54° b 27° c 63°
8 ∠ROQ = 2∠RPQ (angle in centre)
That is, ∠ROQ = 2∠RPO + 2∠QPO
But ∠ROQ = 360° − ∠POR − ∠POQ
So 2∠RPO + 2∠QPO = 360° − ∠POR − ∠POQ
Rearranged, this becomes
2∠RPO + 2∠QPO + ∠POR + ∠POQ = 360°
9 46°
10 ∠DAC = ∠DBC (angle subtended by the same chord DC)
∠ACD = ∠ABD (angle subtended by the same chord AD)
But ∠DAC = ∠ACD (isosceles triangle)
So ∠DBC = ∠ABD
11 65°
12 ∠TSN = ∠KPT (exterior angle of cyclic quadrilateral)
∠KPT = ∠RNS (opposite angles of parallelogram)
So ∠TSN = ∠RNS

MIXED PRACTICE

1 ∠ABC = ∠EBD
2 a RZ = 5 cm b ∠WRZ = 90°
c WZ = 13 cm d ∠RWZ = 23°
3 D
4 a octagon b 135° c 45°
d They are similar (using AAA).
e They are congruent (using SSS).
5 a If the two similar figures also have their corresponding sides equal, then they are also congruent.
b Congruent figures have corresponding angles equal, so they must also be similar.
6 A
7 The reciprocal is also a rational number, provided that $x \neq 0$.
8 a internal angle $= \frac{(n-2) \times 180°}{n}$,
where n represents number of sides of polygon
b 12 c dodecagon
9 SSS 10 D 11 C
12 Take ΔADE and ΔBDC.
AE = BC (given)
ED = CD (given)
∠AED = ∠BCD = 90° (given)
So ΔADE ≅ ΔBDC
Therefore, ∠EDA = ∠CDB and ∠EAD = ∠CBD
But ΔADE and ΔBDC are isosceles right-angled triangles.
So ∠EDA = ∠CDB = ∠EAD = ∠CBD = 45°
Since ∠EDA + ∠ADB + ∠CDB = 180°
45° + ∠ADB + 45° = 180°
So ∠ADB = 90°
13 ∠OYZ = ∠WXY (alternate angles)
∠WOY = 2∠WXY (angles in centre and at circumference)
So ∠WOY = 2∠OYZ
14 C 15 30°
16 ∠POQ = 120° and ∠OQR = 60°,
so ∠POQ = 2∠OQR
17 AP = XP (intersecting tangents) and PB = PY (intersecting tangents)
So AP + PB = XP + PY
That is, AB = XY
18 Since the quadrilateral is cyclic, opposite angles are supplementary.
Since the quadrilateral is a parallelogram, opposite angles are equal.
If two angles are equal and supplementary, they must be 90° each.
With opposite sides equal and parallel, and all angles equal to 90°, the quadrilateral must be a rectangle.

ANALYSIS

1 a There are three sets of parallel lines (one horizontal and two oblique).
b corresponding and alternate angles
c Yes. The two shapes either side of the vertical line of symmetry are congruent.
d No. There are many shapes within the letters which have corresponding angles equal, but side lengths different, not necessarily with corresponding side lengths in a constant ratio.
2 c One 'L' shape is rotated 180° through a corner point, then translated, to form the position of the second 'L' shape. This is repeated once more to locate the position of the third 'L' shape. The three 'L' shapes fit neatly together to give the 3D triangular illusion.
3 a one congruent shape
b rhombus
c This figure extended forms a pattern known in patchwork circles as 'tumbling blocks'.

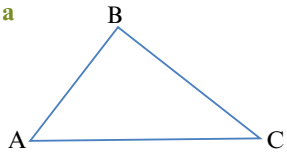
7 Connect

For feedback on this open-ended task, see your teacher.

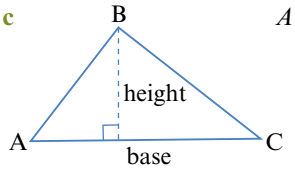
CHAPTER 8 PYTHAGORAS' THEOREM AND TRIGONOMETRY

8 Are you ready?

1 **a** 6.3 **b** 3.9 **c** 4.7 **d** 12.1
2 AC **3** 4 mm **4** B **5** D
6 C **7** A
8 **a** 2.34 **b** 14.24 **c** 9.59 **d** 0.70
9 **a** The angle measured up from horizontal to line of vision.
b The angle measured down from horizontal to line of vision.
c The direction from north or south towards east or west.
d The angle measured from north in a clockwise direction.
10 **a** **i** 045°T **ii** 157.5°T **iii** 067.5°T **iv** 292.5°T
b **i** 150°T **ii** 208°T **iii** 178°T **iv** 352°T
11 63°
12 **a** 6 **b** 4 **13** **a** 12 **b** 6
14 **a**

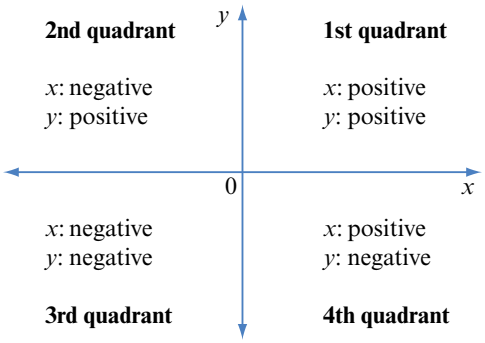


b Find height of triangle then use formula area = $\frac{1}{2}$ × base length × height to calculate area of triangle.
c

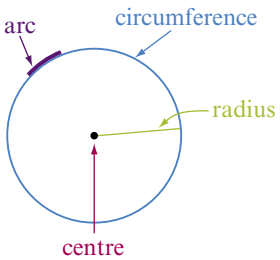


$A = \frac{1}{2} \times b \times h$

15 **a** $\frac{AB}{AC}$ **b** $\frac{BC}{AC}$
16



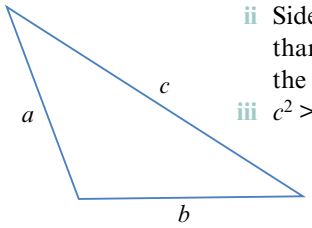
17



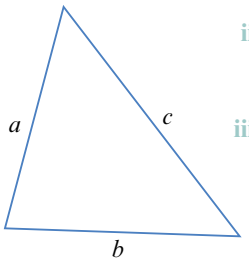
8A Finding lengths using Pythagoras' Theorem

8A Start thinking!

1 **a** In any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.
b hypotenuse
c It is the side opposite the largest angle in the triangle (90°).
2 **a** **i**



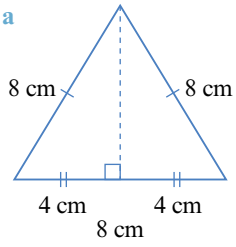
ii Side c is longer than it would be if the angle was 90°.
iii $c^2 > a^2 + b^2$
b **i**



ii Side c is shorter than it would be if the angle was 90°.
iii $c^2 < a^2 + b^2$
3 If $c^2 = a^2 + b^2$, the triangle is right-angled.
If $c^2 > a^2 + b^2$, the triangle is obtuse-angled.
If $c^2 < a^2 + b^2$, the triangle is acute-angled.
4 The ratio refers to one possible set of side lengths of a right-angled triangle. Since $5^2 = 3^3 + 4^2$, triangles with their sides in this ratio are right-angled.

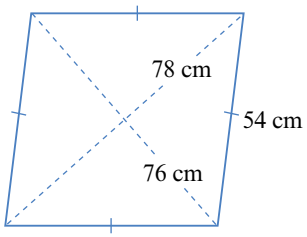
Exercise 8A Finding lengths using Pythagoras' Theorem

1 **a** 3.6 cm **b** 7.1 mm **c** 33.8 cm
d 7.3 cm **e** 2.0 cm **f** 75.0 mm
g 8.7 cm **h** 4.2 cm **i** 8.1 cm
2 **a** $5\sqrt{2}$ cm **b** $20\sqrt{2}$ mm **c** $8\sqrt{2}$ cm **d** $25\sqrt{2}$ cm
3 **a** 7.1 cm **b** 28.3 mm **c** 11.3 cm **d** 35.4 cm
4 **a** right-angled **b** right-angled **c** obtuse-angled
d right-angled **e** acute-angled **f** acute-angled
5 **a**



b **i** $4\sqrt{3}$ cm
ii 6.9 cm
6 **a** $78^2 > 54^2 + 54^2$, hence the frame is not truly square. The two sides form an obtuse angle.

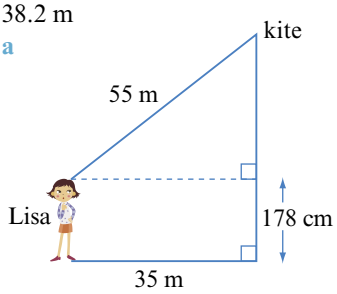
b



7 **a** 11 cm **b** 2 folds **c** 2 folds

8 38.2 m

9 **a**



b 44.21 m

10 **a** All sets of numbers satisfy Pythagoras' Theorem.

b All sets of numbers satisfy Pythagoras' Theorem.

c If the three numbers of a Pythagorean triad are multiplied or divided by the same number, the resulting numbers also are a Pythagorean triad.

d Each side is multiplied by the same scale factor, hence the corresponding sides are in the same ratio.

11 **a** $x^2 + y^2$; yes, always the case.

b Answers will vary. Can choose any value as long as $x > y$, for example $x = 3$, $y = 2$.

c Answers will vary; one possible answer is if $x = 3$ and $y = 2$:

i 5 **ii** 12 **iii** 13

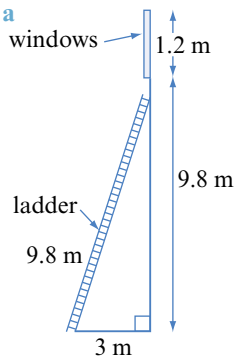
d Use Pythagoras Theorem: $13^2 = 5^2 + 12^2$ to verify.

e Ensures the first number, created using $x^2 - y^2$, is positive.

g $x = 2$, $y = 1$

12 $5\sqrt{5}$ cm, $10\sqrt{5}$ cm (11.2 cm, 22.4 cm)

13 **a**



b 9.3 m

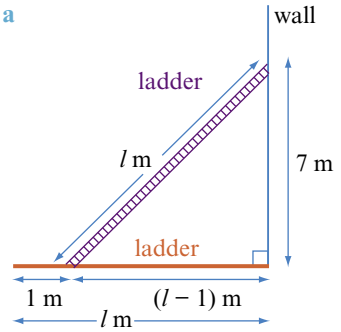
c The top of the window is 9.8 + 1.2 = 11 m above the ground. If Josh stands on the very top of the ladder he is the same height above the ground as the top of the window (9.3 m + 1.7 m = 11 m). He will be able to clean the windows.

14 **a** 6.4 m **b** 38.4 m **c** 74.6 m

d **i** 75.7 m **ii** 82 m

15 16.8 m

16 **a**



b 25 m

8B Finding lengths using trigonometry

8B Start thinking!

1 **a** $\sin \angle YXZ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{YZ}{XZ}$

b $\cos \angle YXZ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{XY}{XZ}$

c $\tan \angle YXZ = \frac{\text{opposite}}{\text{adjacent}} = \frac{YZ}{XY}$

d $\sin \angle XZY = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{XY}{XZ}$

e $\cos \angle XZY = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{YZ}{XZ}$

f $\tan \angle XZY = \frac{\text{opposite}}{\text{adjacent}} = \frac{XY}{YZ}$

2 **a** **i** $\angle BAD = 60°$ **ii** $\angle ABD = 30°$

b AD = 1 unit **c** BD = $\sqrt{3}$ units

d **i** $\sin \angle BAD = \sin 60° = \frac{\sqrt{3}}{2}$

ii $\cos \angle BAD = \cos 60° = \frac{1}{2}$

iii $\tan \angle BAD = \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

iv $\sin \angle ABD = \sin 30° = \frac{1}{2}$

v $\cos \angle ABD = \cos 30° = \frac{\sqrt{3}}{2}$

vi $\tan \angle ABD = \tan 30° = \frac{1}{\sqrt{3}}$

3 **a** $\sqrt{2}$ units

b **i** $\angle XYZ = 45°$ **ii** $\angle XZY = 45°$

c **i** $\sin \angle XYZ = \sin 45° = \frac{1}{\sqrt{2}}$

ii $\cos \angle XYZ = \cos 45° = \frac{1}{\sqrt{2}}$

iii $\tan \angle XYZ = \tan 45° = \frac{1}{1} = 1$

iv $\sin \angle XZY = \sin 45° = \frac{1}{\sqrt{2}}$

v $\cos \angle XZY = \cos 45° = \frac{1}{\sqrt{2}}$

vi $\tan \angle XZY = \tan 45° = \frac{1}{1} = 1$

4

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\frac{\sqrt{3}}{1} = \sqrt{3}$

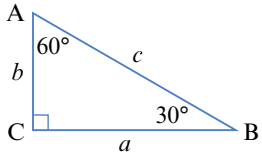
Exercise 8B Finding lengths using trigonometry

1 **a** PR = 5.3 cm, PQ = 4.3 cm
b TW = 6.3 mm, ST = 13.6 mm
c TX = 18.2 cm, AT = 16.1 cm
d EK = 41.2 mm, ME = 26.0 mm
e GR = 8.6 mm, GK = 9.2 mm
f SN = 6.8 cm, ST = 4.6 cm

2 **a** $x = \frac{24}{\sqrt{3}}$ cm **b** $y = 8\sqrt{3}$ mm
c $z = 5\sqrt{2}$ cm **d** $w = \frac{40}{\sqrt{3}}$ cm
e $k = 4\frac{1}{2}$ cm **f** $p = 5\sqrt{2}$ cm

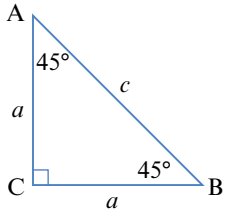
3 **a** $x = 13.9$ cm **b** $y = 13.9$ mm **c** $z = 7.1$ cm
d $w = 23.1$ cm **e** $k = 4.5$ cm **f** $p = 7.1$ cm

4 **a**



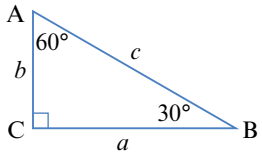
$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$
$= \frac{b}{c}$
$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}}$
$= \frac{b}{c}$
$\sin 30° = \cos 60°$

b



$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}}$
$= \frac{a}{c}$
$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}}$
$= \frac{a}{c}$
$\sin 45° = \cos 45°$

c



$\tan 30° = \frac{\text{opposite}}{\text{adjacent}}$
$= \frac{b}{a}$
$\tan 60° = \frac{\text{opposite}}{\text{adjacent}}$
$= \frac{a}{b}$
$\tan 30° = \frac{1}{\tan 60°}$

5 **a** $\cos 45° = \frac{1}{\sqrt{2}}$
$\cos 45° = \frac{x}{12}$
$\frac{1}{\sqrt{2}} = \frac{x}{12}$
$12 \times \frac{1}{\sqrt{2}} = 12 \times \frac{x}{12}$
$\frac{12}{\sqrt{2}} = x$
$x = 6\sqrt{2}$

b Multiply both sides by y (not 8).
$\sin 30° = \frac{8}{y}$
$\frac{1}{2} = \frac{8}{y}$
$\frac{1}{2} \times y = 8$
$y = 16$

c $\tan 60° = \sqrt{3}$
$\tan 60° = \frac{b}{7}$
$\sqrt{3} = \frac{b}{7}$
$7 \times \sqrt{3} = b$
$b = 7\sqrt{3}$

d Multiply both sides by z (not 4).
$\tan 20° = \frac{4}{z}$
$z \times \tan 20° = 4$
$z = \frac{4}{\tan 20°}$

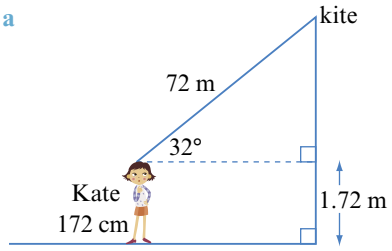
e Multiply (not divide) both sides by 12 and leave m in the numerator.
$\cos 78° = \frac{m}{12}$
$12 \times \cos 78° = m$
$m = 12 \cos 78°$

f Multiply both sides by 9 first.
$\sin 15° = \frac{a}{9}$
$9 \times \sin 15° = a$
$a = 9 \sin 15°$

6 **a** 8.1 cm
b **i** 32° **ii** 8.1 cm
iii Answers are the same.
c The procedure used in part b is easier to use when finding the length of the line segment as there is only one step required to make the unknown side the subject of the equation. There is less chance of making an error in the calculations when the unknown side is in the numerator.

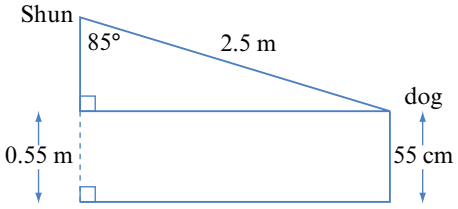
7 **a** paper: $l = 152$ mm, $w = 76$ mm,
polymer: $l = 130$ mm, $w = 65$ mm
b The dimensions of the $5 polymer note are smaller than the dimensions of the $5 paper note. Corresponding sides are in the same ratio.

8 **a**



b 38.15 m
c 39.87 m
d 61.06 m

9 **a**

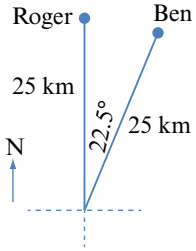


b 2.49 m **c** 0.77 m (77 cm)

10 **a** Two sides and no angles given.
b Only one side and one angle (other than the right-angle) given.
c Two sides and one angle (other than the right-angle) given.

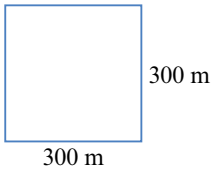
11 $l = 230$ mm, $w = 76$ mm

12 **a**



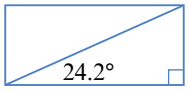
b 23.1 km
c 9.6 km
d 9.8 km

13 **a**



b 424.3 m
c 424.3 m
d Answers are identical.

14 **a**



b 8.99 m ≈ 9 m
c 58 m

15 **a** **i** 5.2 cm **ii** 5.2 cm **b** 30 cm
16 **a** 510 mm **b** 628.3 mm
17 **a** 1.7 m **b** $l = 100$ cm, $w = 80$ cm
18 20.78 cm
19 **a** 610 km **b** 558 km

8C Finding angles using trigonometry

8C Start thinking!

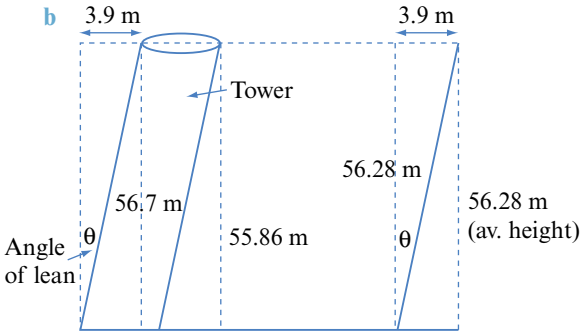
1 **a** 43°30′36″ **b** 78°27′36″ **c** 62°20′24″
d 1°55′48″

2 **a** 60′ = 1°. Minutes should be converted to degrees once 60′ are reached. Answer should be 29°3′17″.
b 60″ = 1′. Seconds should be converted to minutes once 60″ are reached. Answer should be 49°14′7″.
c 60′ = 1°. Minutes should be converted to degrees once 60′ are reached. Answer should be 34°11′17″.
d 60″ = 1′. Seconds should be converted to minutes once 60″ are reached. Answer should be 15°24′.

3 **a** 43°31′ **b** 78°28′ **c** 62°20′ **d** 1°56′
4 **a** **i** 9° **ii** 8°52′ **b** **i** 22° **ii** 22°11′
c **i** 47° **ii** 46°56′ **d** **i** 41° **ii** 40°50′
e **i** 2° **ii** 2°7′ **f** **i** 34° **ii** 34°00′

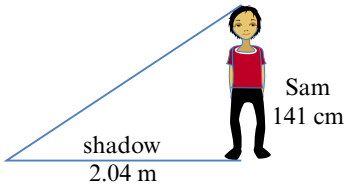
Exercise 8C Finding angles using trigonometry

1 **a** 23°28′ **b** 16°17′ **c** 82°8′ **d** 7°41′
e 46°5′ **f** 3°28′ **g** 1°2′ **h** 38°59′
2 **a** **i** 22° **ii** 22°13′ **b** **i** 32° **ii** 32°20′
c **i** 67° **ii** 67°10′ **d** **i** 6° **ii** 5°32′
e **i** 49° **ii** 49°23′ **f** **i** 76° **ii** 76°14′
g **i** 89° **ii** 88°44′ **h** **i** 1° **ii** 1°5′
i **i** 39° **ii** 39°1′
3 **a** 55°9′ **b** 53°45′ **c** 17°55′ **d** 38°22′
e 69°18′ **f** 80°55′ **g** 82°36′ **h** 63°37′
i 0°34′
4 **a** ∠ABC = 66°2′, ∠ACB = 23°58′
b ∠EDF = 39°48′, ∠EFD = 50°12′
c ∠JGH = 32°, ∠JHG = 58°
d ∠NKM = 29°3′, ∠NMK = 60°57′
e ∠STP = 28°30′, ∠SPT = 61°30′
f ∠WDR = 27°16′, ∠DRW = 62°44′
5 **a** 56.28 m
b



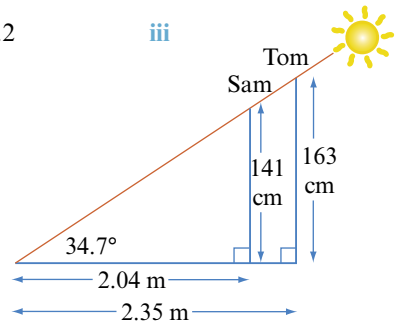
c $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ **d** **i** 4° **ii** 3°58′

6 **a**



b Tom: 34.7°, Sam: 34.7°. The angle of elevation from the ground to the sun at a particular time.
c **i** If the two triangles have the same base angle and a right angle then all three corresponding angles are equal. Hence, the triangles are similar.

ii 1.15 ≈ 1.2 iii



7 a All polymer notes have the same width; polymer notes are smaller in length and width that the paper notes; the length of the polymer notes increases uniformly (by 7 mm) as the value of each note increases.

b i

	$5	$10	$20	$50	$100
Paper note	26.6°	26.1°	26.9°	26.4°	25.6°
Polymer note	26.6°	25.4°	24.3°	23.3°	22.4°

ii $5 note

8 a Route 1: 1:2, 50%, Route 2: 5:12, 41.7%, Route 3: 13:30, 43.3%

b Route 1: 26.6°, Route 2: 22.6°, Route 3: 23.4°

c Route 2, Route 3, Route 1

9 a Answers may vary slightly.

i $\sin \angle BAC = \frac{BC}{AB} = \frac{16\text{ mm}}{32\text{ mm}} = 0.50$

ii $\sin \angle DAE = \frac{DE}{AD} = \frac{11\text{ mm}}{22\text{ mm}} = 0.50$

iii Answers are equal.

iv $\angle A = 30°$, $\sin A = 0.5$

Answer is the same as the answers found in parts i and ii.

b i $\cos \angle BAC = \frac{AC}{AB} = \frac{28\text{ mm}}{32\text{ mm}} = 0.88$

ii $\cos \angle DAE = \frac{AE}{AD} = \frac{19\text{ mm}}{22\text{ mm}} = 0.86$

c i $\tan \angle BAC = \frac{BC}{AC} = \frac{16\text{ mm}}{28\text{ mm}} = 0.57$

ii $\tan \angle DAE = \frac{DE}{AE} = \frac{11\text{ mm}}{19\text{ mm}} = 0.58$

d When using the same angle, the sine, cosine and tangent will be the same for both triangles.

e i Corresponding sides are in the same ratio.

ii Sin represents the ratio of opposite side to the hypotenuse in every right angle triangle irrespective of its size. Similar triangles have these sides in the same ratio and hence, sin 30° = 0.5 in all right-angled triangles.

iii All right-angled triangles with the same particular angle will have their corresponding sides in the same ratio.

10 a i 0 mm ii It becomes 0.

b i BC is equal in length to AB. ii 1

c The sine increases from 0 at 0° to 1 at 90°.

d The cosine ratio has a value of 1 at 0°, becoming 0 at 90°.

e The tangent is 0 at 0°, becoming infinitely large as the angle approaches 90°.

11 sine ratio: minimum value = 0, maximum value = +1
cosine ratio: minimum value = 0, maximum value = +1
tangent ratio: minimum value = 0, maximum value is ∞

13 a and b

Paper size	Length (mm)	Width (mm)	Angle
A0	1189	841	35.3°
A1	841	594	35.3°
A2	594	420	35.3°
A3	420	297	35.3°
A4	297	210	35.3°
A5	210	148	35.2°

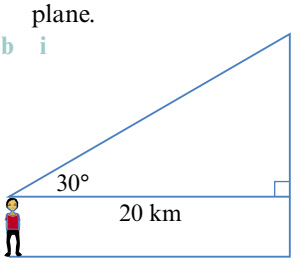
c All rectangular in shape. All have same angle where diagonal meets longest side. Hence, length and width of all notes are in same ratio, so they are similar in shape.

8D Applications of trigonometry

8D Start thinking!

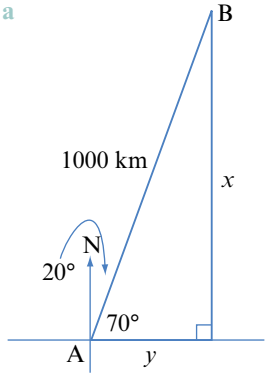
1 a Need to create a right-angled triangle to apply Pythagoras' Theorem. There is not enough information available: for example, don't have the height of the plane above the ground or the horizontal distance from viewing point to the plane.

b i



ii 11.5 km
iii 23.1 km
c 5.7°

2 a



b i 940 km
ii 342 km

3 a True bearings are angles measured from north in clockwise direction.

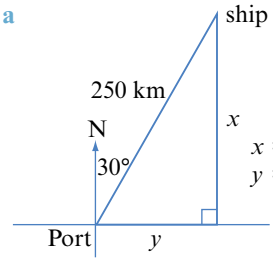
b Bearing to destination: 020°T, return bearing from destination: 200°T

Exercise 8D Applications of trigonometry

1 a 95.7 m b 27.0 m c 235.2 m d 148.2 m
e 121.7 m f 254.3 m

2 25 m **3** 144 m

4 a



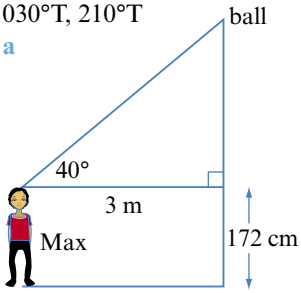
x = distance travelled north
y = distance travelled east

b i 125 km **ii** 217 km

5 Ship travels 250 km on bearing of S30°W to return to its original port.

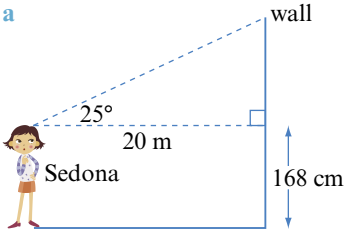
6 030°T, 210°T

7 a



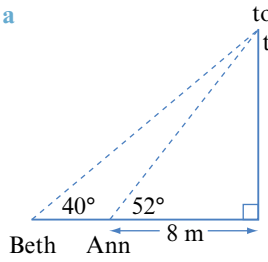
b 2.5 m
c 4.2 m
d The ball is above the person's horizontal line of sight.

8 a



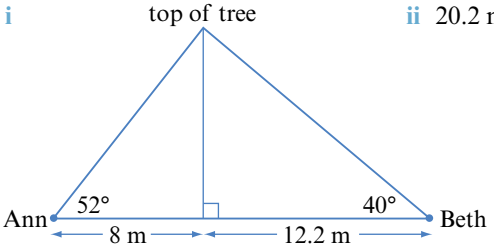
b 11 m

9 a



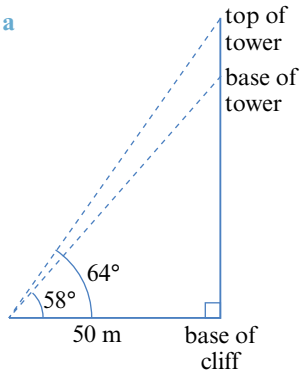
b 10.2 m
c 12.2 m
d 4.2 m

e i



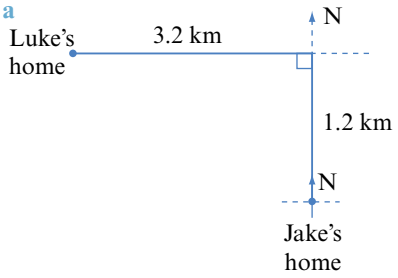
ii 20.2 m

10 a



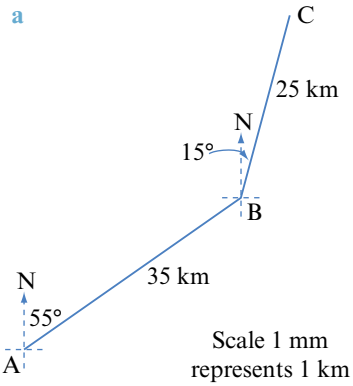
b 22.5 m

11 a



b 3.4 km
c 291°T
d 111°T
e The bearing and its back-bearing differ by 180°.
f i 3.4 km
ii Jake's house to Luke's house: 291°T; Luke's house to Jake's house: 111°T
g Answers are the same.

13 a

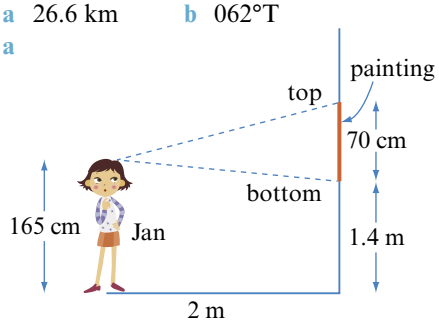


Scale 1 mm represents 1 km

b i 20.1 km
ii 24.1 km
iii 44.2 km
c i 28.7 km
ii 6.5 km
iii 35.2 km
d i 56.5 km
ii 039°T
iii 219°T
e i 56 km, bearing A to C 037°T, bearing C to A 220°T
ii They are almost the same.

14 a 26.6 km **b** 062°T

15 a



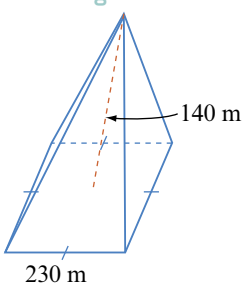
b 12°40′ **c** 7°7′ **d** 13°18′

16 139 m

8E Three-dimensional problems

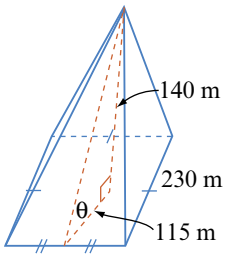
8E Start thinking!

1 a



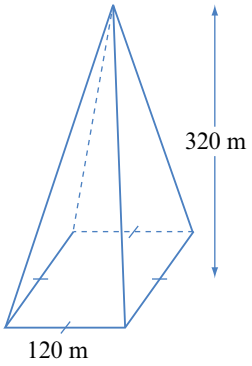
b 325.27 m
c 40°43′

d i



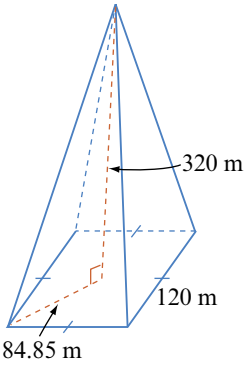
ii 50°36′

2 a



b 169.7 m

c i

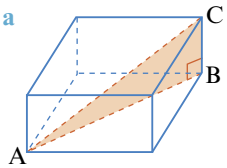


ii 331.1 m

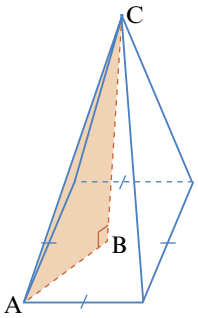
iii 75°9′

Exercise 8E Three-dimensional problems

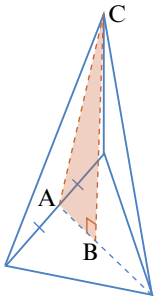
1 a



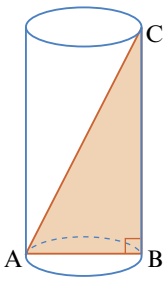
b



c



b



2 a 21.21 cm **b** 25.98 cm **c** 35°16′

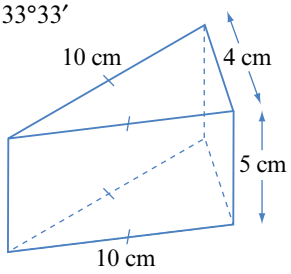
3 a

diagonal of base:	diagonal of box:
$d = \sqrt{20^2 + 15^2}$	$e = \sqrt{10^2 + 25^2}$
$d = \sqrt{625}$	$e = \sqrt{725}$
$d = 25$ cm	$e = 26.93$ cm

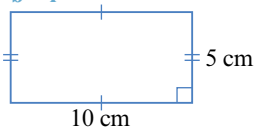
The length of the rod is greater than the length of the diagonal of the box, hence a 30-cm rod will not fit into the box.

b 33°33′

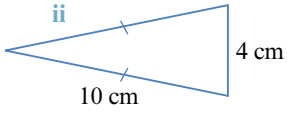
4 a



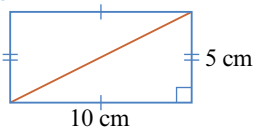
b i



ii



c



i 11.2 cm

ii 26°34′

iii Angles are complementary.
$90° - 26°34′ = 63°26′$

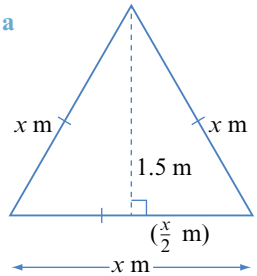
d i 9.8 cm **ii** 19.6 cm^2

e i

ii 98 cm^2

iii 11 cm

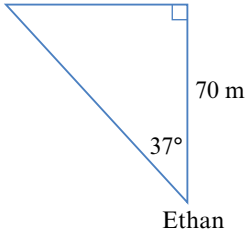
5 a



b 1.3 m

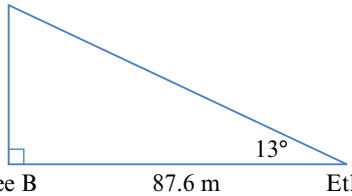
c There is sufficient material. Jack requires $2 \times 1.3 = 2.6$ m of material to cover the tent.

6 a



b 87.6 m

c top of tree

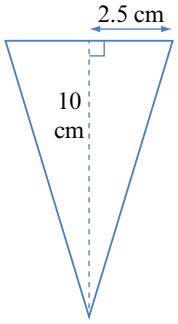


d 20.2 m

7 **a** 19.1 cm
b Internal diagonal = 22.6 cm. The spoon can fall in, as the internal diagonal is longer than the spoon.

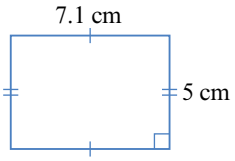
8 **a** 2.5 cm
c 10.3 cm

b



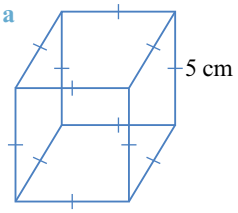
9 9 full streamers

10 **a** rectangle:



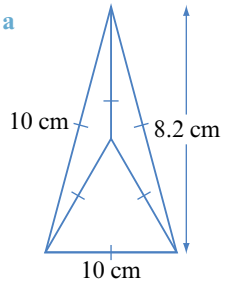
b Area of exposed face = 35.5 cm^2; area of cube face = 25 cm^2. Hence, area of exposed face is greater than area of one of the cube's faces.

11 **a**

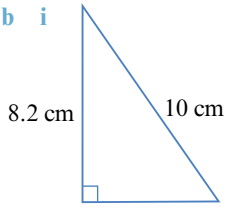


b 45 cm; the ant can only crawl along 9 edges before it comes to an edge which it has already crawled along.

12 **a**

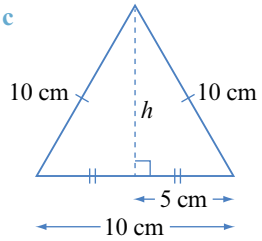


b **i**



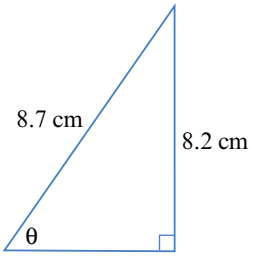
ii 55°

c

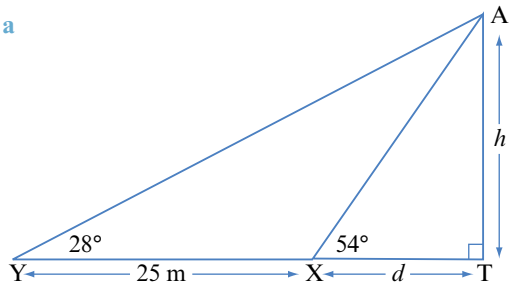


h = height of face

i 8.7 cm **ii** 70°

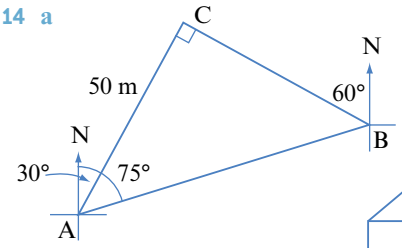


13 **a**



b $h = d \times \tan 54°$ **c** $h = (25 + d) \times \tan 28°$
d 15.7 m **e** 21.6 m

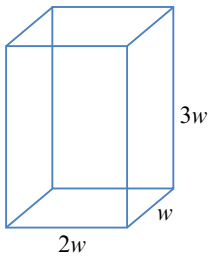
14 **a**



b **i** 50 m
ii 70.7 m
c 26°20′

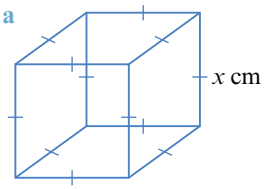
15 **a** $l = 2w$; $h = 3w$
c $\sqrt{5}w$
d $\sqrt{14}w$
e 53°

b

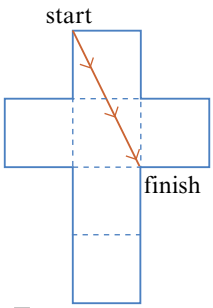


16 3.6 cm

17 **a**



b **i**

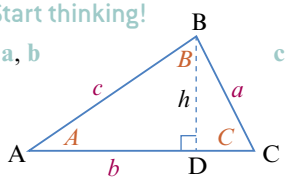


ii $\sqrt{5}x$ cm

8F Sine and area rules

8F Start thinking!

1 **a, b**



c **i** $h = c \sin A$
ii $h = a \sin C$
iii $a \sin C = c \sin A$

$$a = \frac{c \sin A}{\sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

2 **a** area $= \frac{1}{2}bh$

$= \frac{1}{2}b \times c \sin A$

$= \frac{1}{2}bc \sin A$

b lengths of two sides and size of angle between those two sides

Exercise 8F Sine and area rules

1 **a** 13.4 mm **b** 54° **c** 124°

2 **a** $b = 14$ cm, $C = 62°$, $c = 12.5$ cm
b $B = 64°$, $A = 31°$, $a = 19.3$ mm
c $C = 72°$, $B = 59°$, $b = 25.6$ cm
d $C = 87°$, $a = 12.2$ cm, $c = 13.2$ cm

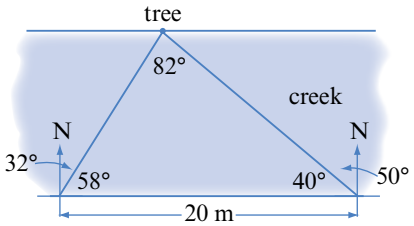
3 **a** 16.9 cm^2 **b** 12.4 mm^2 **c** 18.1 cm^2
d 39.2 cm^2

4 **a** 22.4 cm **b** 31.6 cm **c** 53.7 mm

5 **a** **i** 133° **ii** 20°
b 19.9 m **c** 14.6 m **d** 13.6 m **e** 28.6 m

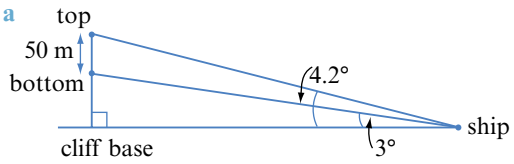
6 **a** 60° **b** 53° **c** 67° **d** 28.7 km
e **i** N37°E **ii** S37°W **iii** west (or 270°T)
f 310.6 km^2

7 **a**



b **i** 13 m **ii** 17.1 m
c 11 m

8 **a**



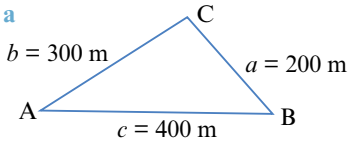
b **i** 2381.1 m **ii** 2384.2 m
c 2.4 km **d** 125 m

9 **a** 1 **b** $\sin B = \frac{b}{a}$, $\sin C = \frac{c}{a}$ **c** $\frac{1}{a}$
d In the right-angled triangle, the sine rule simplified to $\frac{1}{a}$, where a is the length of the hypotenuse.
e right angle at B: $\frac{1}{b}$, right angle at C: $\frac{1}{c}$

10 693 cm^2

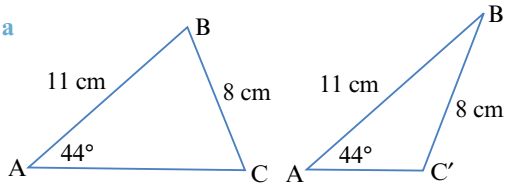
11 **a** $\sin C = \frac{c}{a}$ **b** $A = \frac{1}{2}bc$
c A represents area, b represents base of triangle, c represents height of triangle. It is written in same form as $A = \frac{1}{2}bh$, the formula for the area of any right-angled triangle.

12 **a**



b $s = 450$
c 29 047 m^2
d **i** 300, 400, 500 are multiples of the Pythagorean triad 3, 4, 5. Hence, they are also Pythagorean triads and form the sides of a right-angled triangle.
ii 60 000 m^2 **iii** 60 000 m^2
iv Answers are the same.

13 **a**



Only one angle is known. Remaining angles could be both acute or one acute and one obtuse. This makes it possible to draw two different triangles where only the third side varies in length.
b **i** isosceles triangle
ii ∠BCC′ and ∠BC′C are equal
iii They add up to 180° on the straight line.
iv Subtract ∠BC′A from 180° to find ∠BC′C. ∠BC′C = ∠BCC′.
c **i** 73° **ii** 63° **iii** 10.3 cm
d **i** 107° **ii** 29° **iii** 5.6 cm

14 $A = 45°$ or 135°

8G Cosine rule

8G Start thinking!

1 **a** $a^2 = h^2 + (b - x^2)$ **b** $h^2 = c^2 - x^2$
c $a^2 = c^2 - x^2 + (b - x)^2$
$a^2 = c^2 - x^2 + b^2 - 2bx + x^2$
$a^2 = b^2 + c^2 - 2bx$
d $x = c \cos A$
f $b^2 = a^2 + c^2 - 2ac \cos B$, $c^2 = a^2 + b^2 - 2ab \cos C$
h **i** two side lengths and the angle between them
ii three side lengths

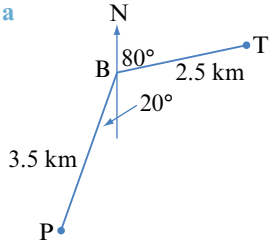
Exercise 8G Cosine rule

1 **a** $a = 3.6$ cm **b** $r = 8.8$ cm **c** $h = 22.5$ cm
d $d = 13.7$ cm **e** $m = 56.2$ mm **f** $t = 44.9$ cm
g $v = 55.4$ mm **h** $d = 14.7$ m **i** $z = 15.5$ cm

2 **a** $A = 47°$, $B = 78°$, $C = 55°$
b $A = 82°$, $B = 66°$, $C = 32°$
c $A = 41°$, $B = 88°$, $C = 51°$
d $A = 57°$, $B = 84°$, $C = 39°$

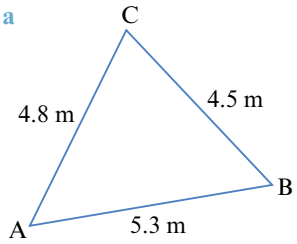
3 **a** $a = 18.5$ cm, $B = 94°$, $C = 38°$
b $e = 36.1$ mm, $D = 60°$, $F = 48°$
c $h = 48.2$ cm, $G = 53°$, $I = 68°$
d $j = 40.7$ m, $K = 50°$, $L = 67°$
e $n = 5.3$ m, $M = 42°$, $P = 59°$
f $s = 19.3$ mm, $Q = 51°$, $R = 60°$

4 **a**



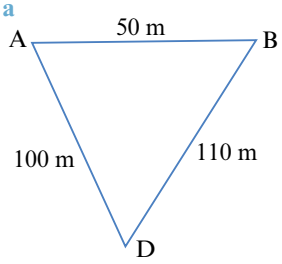
b 120°
c 5.2 km
d 25°
e N45°E
f 35°
g S45°W

5 **a**



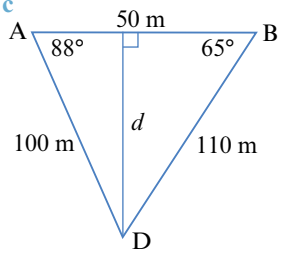
b C
c 69°
d Can use the formula: area $= \frac{1}{2}ab \sin C$ to calculate the area. $A = 10.1 \text{ m}^2$

6 **a**



b 65° and 88°

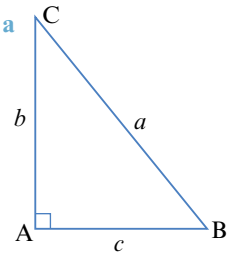
c



Use the sine ratio in either of the two right-angled triangles created.
d 99.9 m

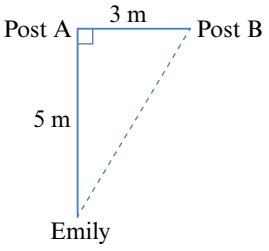
7 Third side = 35 cm. All sides are equal in length and hence, the triangle is equilateral.

8 **a**



b $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
c 0 **d** $a^2 = b^2 + c^2$; Pythagoras' Theorem

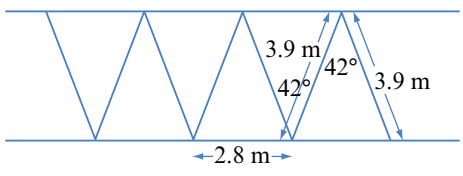
9 **a**



b 5.8 m
c 31°
d Emily has a greater chance of scoring a goal. She now has a larger angle of 35° within which she can shoot for goal.

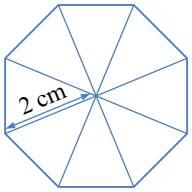
10 **a** 3.9 m **b** 3.7 m

11 **a** 42°
b



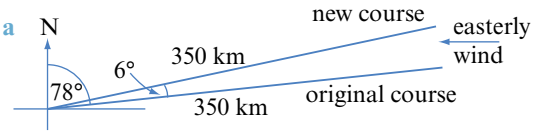
c 3.8 m

12 **a**



b **i** 45°
ii 67.5°
c 12.2 cm

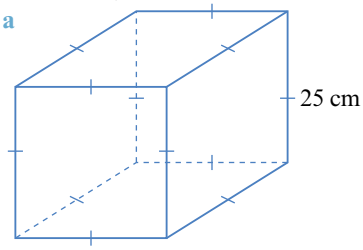
13 **a**



b 36.6 km

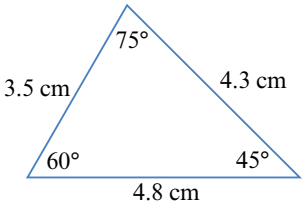
14 **a** Need to know the magnitude of a second angle, either B or C.
b 55.8 cm
c **i** Use $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ to find B. Subtract the sum of A and B from 180° to find C.
ii $B = 35°$, $C = 59°$
d $B = 35°$, $C = 59°$ **e** Answers are the same.

15 **a**

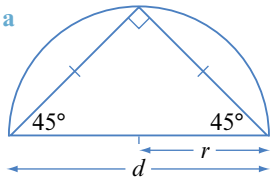


b $25\sqrt{2}$ cm
c $25\sqrt{3}$ cm

16 **a** 45°, 60°, 75° **b** 4.8 cm
c



17 **a**



b **i** and **ii** $\sqrt{2}r$ units
c Both methods give the same answer.

8H The unit circle and trigonometric graphs

8H Start thinking!

1 **a** $x = \cos\theta$, $y = \sin\theta$
b **i** $\cos\theta$ **ii** $\sin\theta$

2

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
sign of x-coordinate ($\cos\theta$)	+	−	−	+
sign of y-coordinate ($\sin\theta$)	+	+	−	−
sign of $\tan\left(\frac{\sin\theta}{\cos\theta}\right)$	+	−	+	−

Exercise 8H The unit circle and trigonometric graphs

1 **a** quadrant 4 **b** quadrant 2 **c** quadrant 3
d quadrant 3 **e** quadrant 1 **f** quadrant 2

2

	Angle	sin	cos	tan
a	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
b	300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
c	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
d	315°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
e	135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
f	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$

3

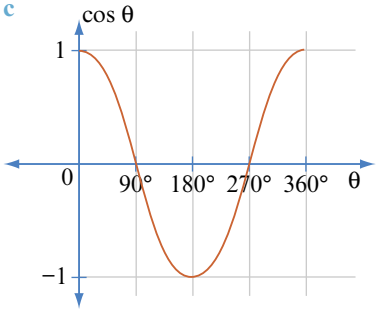
	Angle	sin	cos	tan
a	87°	0.999	0.052	19.081
b	285°	−0.966	0.259	−3.732
c	202°	−0.375	−0.927	0.404
d	340°	−0.342	0.940	−0.364
e	170°	0.174	−0.985	−0.176
f	95°	0.996	−0.087	−11.430

4 a i approaches −1 ii approaches 0 iii approaches 1

b

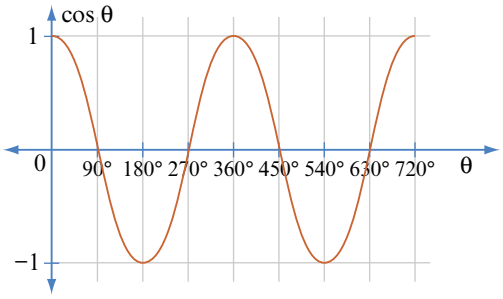
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0	0.5	0.9	1

c



The graph is like the trough of a wave. Starting value of cos θ is 1.

d



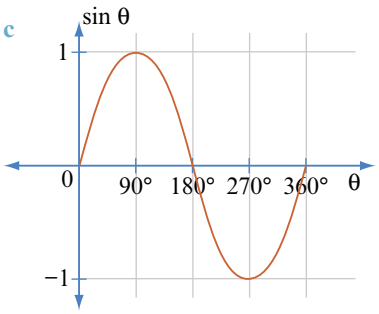
e i 0°; 360° ii 360°

f i 1 unit ii 1 unit iii 1

5 a The value of sine follows a cycle. It starts at 0, peaks at 1 when θ equals 90°, then drops down to −1 when θ equals 270° and returns to 0 when θ equals 360°.

b 0

c



d The cycle repeats itself.

e 360° **f** 1 unit **g** 1

6 Cosine graph starts at (0, 1) whereas sine graph starts at (0, 0).

7 a i 1 ii 0 iii 0 iv −1

b i 0.8 ii −0.2 iii 0.9 iv −0.9 v 0.2 vi −0.8

c i 0°, 360° ii 180° iii 90°, 270° iv 65°, 295° v 130°, 230° vi 35°, 325°

d There is often more than one answer.

8 a i 0 ii 1 iii −1 iv 0

b i 0.6 ii 0.9 iii −0.2 iv −0.3 v −0.9 vi 0.6

c i 90° ii 270° iii 0°, 180°, 360° iv 24°, 156° v 217°, 323° vi 53°, 127°

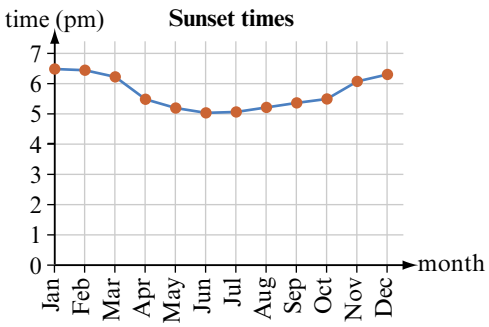
d There is more than one angle with the same sine value in most cases.

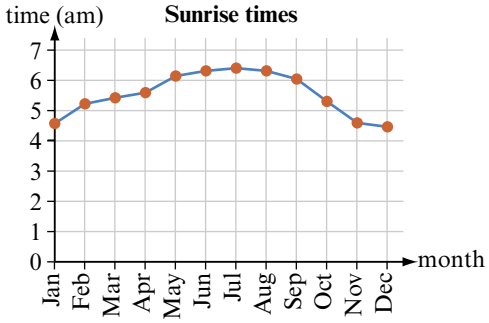
9 a A pendulum will swing back and forth the same distance. Its motion is repetitive and regular, hence it is periodic.

b i 2 s ii 2.5 s iii 2.8 s

c The longer the pendulum, the longer the period.

10 a





b Answers may vary. Comments could include: graphs are cyclic or periodic; graphs have peaks and troughs; graphs are U-shaped.

c Similarities: have repeating pattern; are cyclic or periodic; have peaks and troughs. Differences: do not start at 1 or 0; maximum point is not 1 and minimum is not −1; graph is completely above x-axis; resembles a cosine graph more than a sine graph; graph is not a perfect curve.

d Yes, the graphs repeat their values at regular intervals. In this case the interval is 1 year.

e Daylight hours are the hours which lie between the two graphs. These hours increase in summer and decrease in winter.

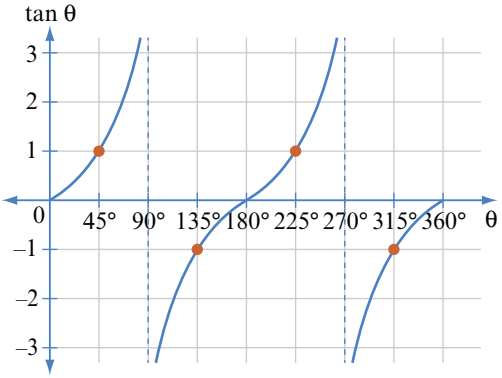
f Winter has the shortest days. These graphs indicate the least amount of daylight during the middle of the year. This is when winter occurs in the southern hemisphere.

11 a

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
tan θ	0	0.6	1	1.7	*	−1.7	−1	−0.6	0	0.6	1	1.7	*	−1.7	−1	−0.6	0

* undefined

b



c i A line which extends from −∞ to +∞ with a point of inflection on the x-axis.

ii It is not continuous; its cycle repeats after 180° (not 360°); it has values for tan θ which are undefined; the values approach ∞ and −∞.

iii At θ = 90° the value of tan θ is undefined so a vertical asymptote is shown at this point. An asymptote is a line which the graph approaches but never reaches. Next asymptote: θ = 270°.

iv The graph repeats after a certain period. Period = 180°.

v The graph has no amplitude. The tan θ values approach ∞ and −∞.

8I Solving trigonometric equations

8I Start thinking!

1 a i 2 angles ii 4 angles

b 1 angle

2 a 45° **b** [0°, 360°] **c** 360°; 1

d 2 **e** positive **f** quadrant 2

g 135° **h** 4 solutions: 45°, 135°, 405°, 495°

3 a 78° **b** quadrants 2 and 3 **c** 102°, 258°

Exercise 8I Solving trigonometric equations

1 a positive **b** negative **c** negative

d positive **e** positive **f** negative

g negative **h** positive **i** negative

j negative

2 a 180° − 40° **b** 180° − 88° **c** 180° − 10°

d 180° − 30° **e** 180° − 80°

3 a 180° + 20° **b** 180° + 80° **c** 180° + 10°

d 180° + 40° **e** 180° + 75°

4 a 360° − 80° **b** 360° − 45° **c** 360° − 15°

d 360° − 61° **e** 360° − 58°

5 a θ = 30°, 330° **b** θ = 225°, 315°

c θ = 120°, 240° **d** θ = 240°, 300°

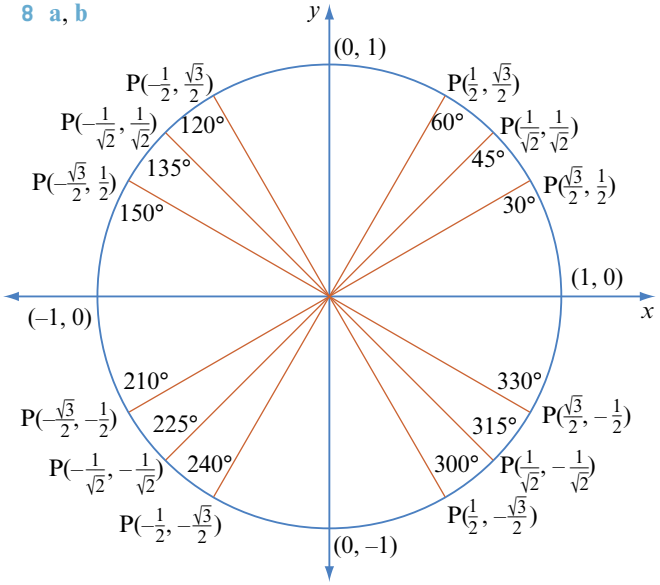
e θ = 45°, 315° **f** θ = 60°, 120°

6 a θ = 114°, 246° **b** θ = 197°, 343°

c θ = 26°, 334° **d** θ = 37°, 143°

e θ = 53°, 307° **f** θ = 224°, 316°

8 a, b



c i $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ ii $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ iii $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

iv $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ v $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ vi (0, 1)

9 a θ = 60°, 120° **b** θ = 30°, 330°

c θ = 240°, 300° **d** θ = 150°, 210°

e θ = 30°, 150° **f** θ = 45°, 315°

g θ = 225°, 315° **h** θ = 120°, 240°

i θ = 270°

10 a 90° **b** no **c** θ = 90°

d θ = 90°, 450°

11 a θ = 180° **b** θ = 180°, 540°

12 a, b, c When each of these equations is simplified the sine or cosine value is greater than 1. Sine and cosine can only take values in the range −1 to 1.

13 a yes **b** yes **c** yes **d** yes

e No, when the equation is simplified the cosine value is less than −1.

f yes

g No, when the equation is simplified the cosine value is greater than 1.

h yes
i No, when the expression is simplified the sine value is less than −1.

14 a 1 b 0 c 1 d 1 e 1 f 1

15 a θ = 90° b θ = 0° c θ = 135°, 225°
d θ = 240°

16 a θ = 90° b 90° c 90°
d [90°, 450°]

17 a [270°, 630°] b [0°, 180°] c [180°, 540°]
d [90°, 270°]

18 a 30° b 150° c [30°, 150°]
d [30°, 510°]

19 a 60° b 300° c [60°, 300°]
d [60°, 660°]

20 a [270°, 630°] b [90°, 270°] c [60°, 300°]
d [60°, 120°] e [45°, 135°] f [150°, 210°]

21 a i [210°, 330°] ii [210°, 690°]
b i [120°, 240°] ii [120°, 600°]
c i [45°, 135°] ii [45°, 495°]
d i [135°, 225°] ii [135°, 585°]

22 a The graph repeats itself continuously in cycles.
b There is more than one cycle between 0° and 360°.
c Sine curve: it starts and ends at $y = 0$ (on the θ axis).
d 180° e 4 f 4
g There are twice as many solutions.

23 a $y = \cos\theta + 3$ b $y = \sin\theta - 3$
c $y = -\cos\theta$ d $y = -\sin\theta$

8 Chapter review

MULTIPLE-CHOICE

1 C 2 B 3 A 4 B 5 A 6 C
7 B 8 B 9 A 10 D 11 A

SHORT ANSWER

1 a $\frac{5\sqrt{3}}{2}$ cm b 4.3 cm

2 length = 13.5 cm, width = 8.21 cm

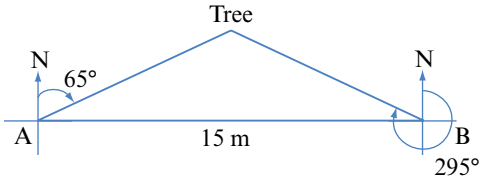
3 a 18°32′
b i $\frac{6.2}{18.5} = \frac{62}{185}$ or 62:185
ii 33.51%
c 19.51 km

4 40 m

5 a 16.30 cm b 19.72 cm c 43.30 cm

6 a 23° b 40° c 55°

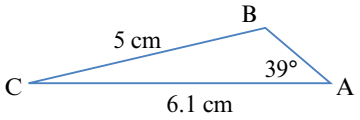
7 a



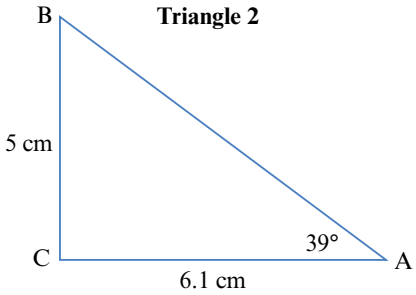
b 8.28 km in both positions.

8 a

Triangle 1



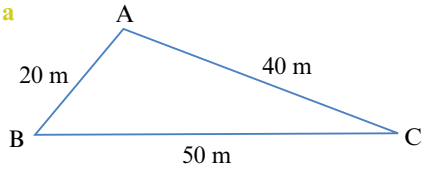
Triangle 2



Because the angle given is not between the two given sides, it is possible to draw two distinct diagrams.

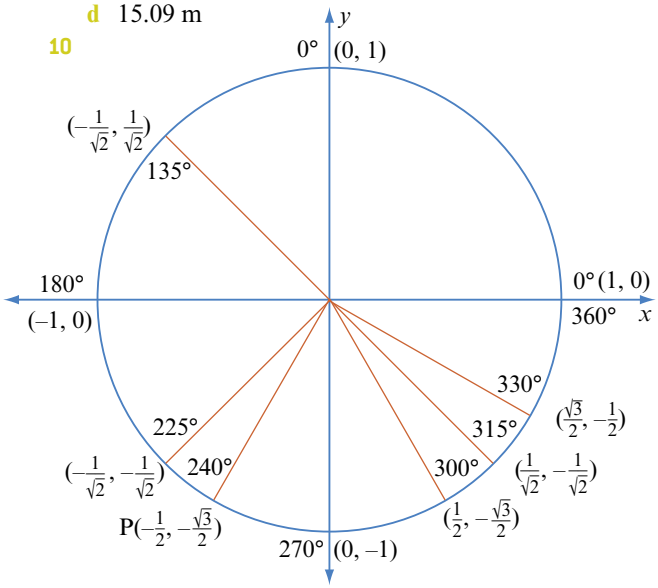
b Triangle 1: $B = 130°$, $C = 11°$, $c = 1.52$ cm;
Triangle 2: $B = 50°$, $C = 91°$, $c = 7.94$ cm

9 a



b 380 m^2 c $A = 108°$, $B = 49°$, $C = 23°$
d 15.09 m

10



a $-\frac{\sqrt{3}}{2}$ b $-\frac{1}{\sqrt{2}}$ c $\frac{1}{\sqrt{2}}$ d $\frac{1}{\sqrt{2}}$
e $-\frac{\sqrt{3}}{2}$ f $\frac{\sqrt{3}}{2}$

11 a θ = 180° b θ = 0°, 180°, 360°
c θ = 30°, 330° d θ = 210°, 330°
e θ = 135°, 225° f θ = 240°, 300°

MIXED PRACTICE

1 C 2 B 3 45° 4 4 cm
5 0.8 km 6 B 7 308°T 8 B
9 5.2 10 52°52′ 11 $\sin\angle QPR = \frac{32}{53}$

12 A **13** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

14 C **15** D

16 $d = \sqrt{x^2 + y^2 + z^2}$ **17** B **18** C

19 13.6 cm **20** 20.7 cm^2

21 The graphs of sin θ and cos θ have a period of 360° and an amplitude of 1. Both are cyclic graphs. The graph of cos θ is a translation of the sin θ graph, 90° along the x-axis.

22 Tan 90° is undefined. **23** A

ANALYSIS

1 a i $2R + G = 480$ mm; this is below the required range, hence the staircase does not fall within building regulation guidelines.

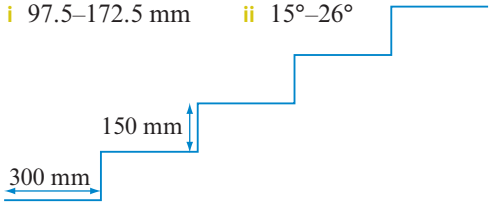
ii 25°

b i $2R + G = 735$ mm; this is above the required range, hence the staircase does not fall within building regulation guidelines.

ii 28°

c i 97.5–172.5 mm **ii** 15°–26°

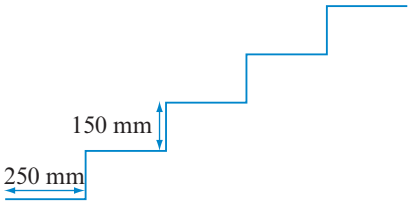
d



i 1500 mm **ii** 600 mm

iii 1615.5 mm **iv** 22°

e One possible solution is given.



115 mm ≤ 150 mm ≤ 190 mm

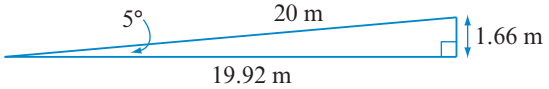
250 mm ≤ 250 mm ≤ 355 mm

$2R + G = 2 \times 150 + 250 = 300 + 250 = 550$ mm

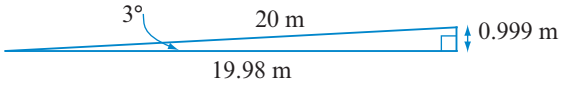
550 mm ≤ 550 mm ≤ 700 mm

f i

internal ramp 1:12

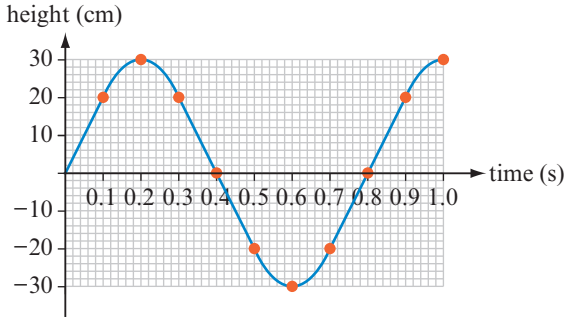


external ramp 1:20



ii Internal: rise = 1.66 m, going = 19.92 m
External: rise = 0.999 m, going = 19.98 m

2 a



b sine curve **c** 0.8 seconds

d Starting position was 0 cm. It takes 0.1 seconds for the tag to go from 20 cm to 0 cm in height.

e 30 cm **f** 188.5 cm **g** 2.4 m/s

h After 10 seconds, the wheel will complete 12.5 rotations. If the tag's starting position was at 0 cm, then the tag will be at the same height of 0 cm but on the opposite side of the wheel.

8 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 9 MEASUREMENT

9 Are you ready?

1 B **2** A **3** B **4** D **5** B
6 C **7** A **8** B **9** A **10** C
11 D **12** D **13** B **14** B

9A Length and perimeter

9A Start thinking!

1

Shape	Name	Perimeter formula
a	square	$P = 4l$
b	triangle	$P = a + b + c$
c	rectangle	$P = 2(l + w)$
d	parallelogram	$P = 2(l + w)$
e	rhombus	$P = 4l$
f	circle	$C = \pi D$ or $C = 2\pi r$

2 Square and rhombus share same formula because they both have four sides of equal length. Rectangle and parallelogram share same formula because they both have two pairs of opposite sides equal in length.

3 If diameter is given, $C = \pi D$ can be used, while $C = 2\pi r$ can be used if radius is given. The diameter of a circle (D) is twice the length of the radius (r).

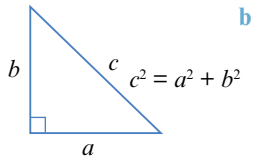
Exercise 9A Length and perimeter

1 a 430 mm **b** 12 500 m **c** 0.31 m
d 630 m **e** 0.3 cm **f** 270 cm
g 0.78 km **h** 8900 mm **i** 510 000 cm
j 5600 mm **k** 0.019 m **l** 0.000 997 km

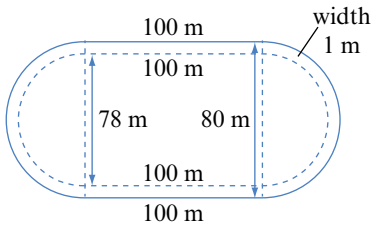
2 a 68 m **b** 45 cm **c** 60 mm **d** 85.6 mm

3 a 11 cm or 110 mm **b** 2.4 m or 240 cm

4 **a** 28.3 m **b** 144.5 mm **c** 204.2 cm
5 **a** 206.7 mm **b** 59.7 cm **c** 48.4 cm
6 **a** 30.8 cm **b** 42.4 mm **c** 38.6 cm
d 12.9 m **e** 175.8 mm **f** 71.0 cm
7 **a** 30 cm **b** 48 cm or 480 mm
c 31 m **d** 101.1 cm
e 232.8 m **f** 45.1 cm
8 19.6 cm
9 **a** 360° **b** 90° **c** $\frac{90°}{360°} = \frac{1}{4}$
d length of arc $= \frac{1}{4} \times 2\pi r$
e 11 cm **f** 25 cm
10 **a** 84° **b** $\frac{84°}{360°} = \frac{7}{30}$
c length of arc $= \frac{7}{30} \times 2\pi r = 20.5$ mm
d 48.5 mm
11 The length of the arc of a sector is a fraction of the circumference of a circle. This is equal to $\frac{\theta}{360°} \times 2\pi r$. If you add the two radii ($2r$) you have an expression for the total perimeter of the sector.
12 **a** 62.3 cm **b** 139.4 mm **c** 82.9 cm
d 66.7 cm **e** 283.8 mm **f** 163.8 cm
13 **a** 80.8 cm **b** 157.1 mm **c** 149.7 cm
d 404.2 mm **e** 44.6 cm **f** 45.7 cm
14 **a** 18° **b** 70.7 mm **c** 520.7 mm or 52.07 cm
d 1414 mm or 141.4 cm
e 1413.7 mm. Answers are close: rounding in part b accounts for difference.
15 **a**



b the lengths of the two equal sides
c Base of right-angled triangle is 4 cm and its height is 6 cm. Pythagoras' Theorem enables you to calculate the unknown side length, which is the hypotenuse in the right-angled triangle.
d unknown side length: 7.2 cm, perimeter: 22.4 cm
16 **a** 22.2 cm **b** 54.8 cm **c** 37.1 cm
17 **a** 451.3 m **b**



c 445 m
18 **a** 6.5 cm
b Since a square has four equal side lengths, side length = perimeter ÷ 4.
c $l = \frac{P}{4}$
19 **a** length: 13 cm, width: 10 cm
b 17 cm by 14 cm **c** 62 cm
20 **a** estimate: $126 \div 6 \approx 21$ cm
b Divide circumference by 2 and by π.
c $r = \frac{C}{2\pi}$ **d** 20 cm
21 **a** 64 cm **b** 875

9B Area

9B Start thinking!

1 Perimeter is distance around boundary of a 2D shape, while area is space enclosed within a 2D shape.
2 **a, b**

Shape	Name	Area formula
A	square	$A = l^2$
B	rectangle	$A = lw$
C	triangle	$A = \frac{1}{2}bh$
D	parallelogram	$A = bh$
E	kite	$A = \frac{1}{2}xy$
F	trapezium	$A = \frac{1}{2}(a + b)h$
G	circle	$A = \pi r^2$
H	rhombus	$A = bh$

Exercise 9B Area

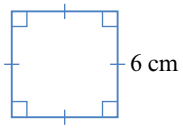
1 **a** 70 000 cm^2 **b** 0.0045 km^2
c 84000 m^2 **d** 0.29 cm^2
e 2.4 ha **f** 880 000 mm^2
g 0.003 42 km^2 **h** 102 200 000 000 cm^2
i 300 ha
2 **a** 42 m^2 **b** 76 cm^2 **c** 348.5 cm^2
d 242.1 cm^2 **e** 375 mm^2 **f** 164.4 m^2
3 **a** 272 cm^2 **b** 843.6 mm^2 **c** 216 cm^2
d 238.9 cm^2 **e** 28.6 cm^2 **f** 409.5 cm^2
4 **a, b** chimney: trapezium, 1.8 m^2;
roof: triangle, 6.5 m^2;
front wall: rectangle, 15.5 m^2;
front door: rectangle, 3.9 m^2;
windows: square, 0.49 m^2 each window
c multiply by 4
d area to be painted = area of chimney + area of roof + area of front wall − area of windows
e 21.84 m^2 **f** 1.5 L (or 2 L to nearest litre)
5 **a** 20 025 m^2 **b** 2 ha
c One possible set of dimensions is 225 m by 89 m.
6 **a** 168 cm^2 **b** 2 m^2 **c** 6.4 cm^2
d 348 cm^2 **e** 934.2 cm^2 **f** 220.4 cm^2
7 **a** Subtract area of inner circle from area of outer circle.
b 100.5 m^2
c **i** 115.5 m^2 **ii** 58.9 m^2 **iii** 125.7 m^2
8 **a** 32.2 cm^2
b Divide area of whole slice by 2.
c Divide area of whole slice by 4.
d $A_{\text{quadrant}} = \frac{1}{4}\pi r^2$. A quadrant has a central angle of 90°, and $\frac{90°}{360°} = \frac{1}{4}$.

9 **a** 36° **b** $\frac{36°}{360°} = \frac{1}{10}$

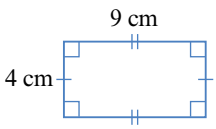
c $A = \frac{1}{10} \times \pi r^2$, 25.4 cm^2

10 **a** 59.4 cm^2 **b** 515.3 mm^2 **c** 2875.1 cm^2

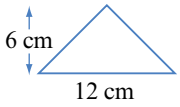
11 **a**



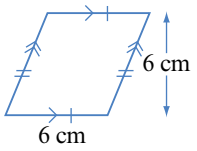
b



c



d



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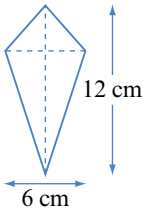
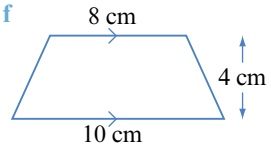
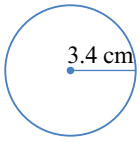


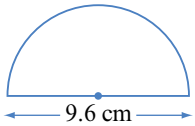
f



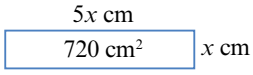
g



h



12 **a**



b Rectangle is 60 cm by 12 cm.

c 2880 cm^2 **d** 4 times the area

e 6480 cm^2, 9 times the area

f **i** When dimensions are quadrupled, area is 16 times as large.

ii When dimensions are multiplied by factor of n, the area becomes n^2 times as large.

13 **a** 75.4 cm^2 **b** 60°

c 12.6 cm^2 **d** 44 cm^2

14 **a** radius of full circle is twice the radius of the semi-circles

b 904.8 cm^2

c The area of the white section is the same as that of the black section: 904.8 cm^2.

9C Surface area of prisms and cylinders

9C Start thinking!

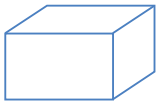
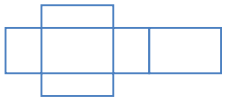
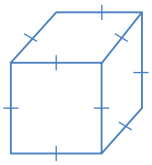
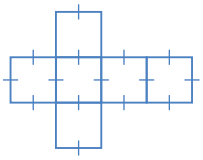
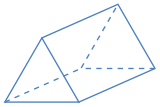
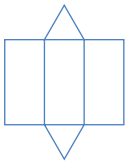
1 **a** rectangular prism, cube, triangular prism

2 **a** The cross-section of a prism is a shape with same shape and dimensions as its base.

b It is the same shape and size throughout the height (or length) of the prism.

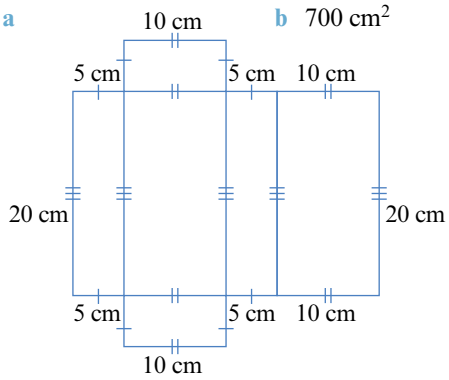
c rectangular prism: rectangle, cube: square, triangular prism: triangle

3 **a**

Prism name	Prism	Net
rectangular prism		
cube		
triangular prism		

b Total surface area is sum of areas of all surfaces.

4 **a**



b 700 cm^2

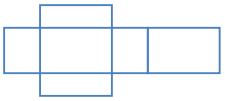
c There are three pairs of identical faces on the surface of a rectangular prism. Find the sum of one of each of these, then multiply by 2.

d TSA = $lh + lw + wh + lh + lw + wh$
$= 2(lh + lw + wh)$

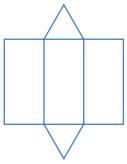
5 For a cube, w, l and h all have the same value.
For each of the pronumerals w and h, substitute l.
TSA = $2(l^2 + l^2 + l^2)$
TSA = $6l^2$

Exercise 9C Surface area of prisms and cylinders

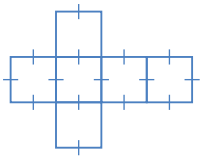
1 **a**



b



c



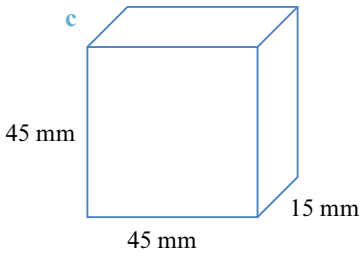
2 **a** 262 cm^2 **b** 23.88 m^2 **c** 6156 mm^2

3 **a** 1350 cm^2 **b** 34.56 m^2 **c** 6144 mm^2

4 **a** 876 mm^2 **b** 332 cm^2 **c** 53.7 m^2

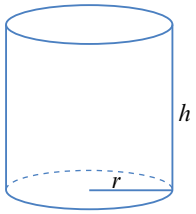
5 918 cm^2 **6** 23.7 m^2

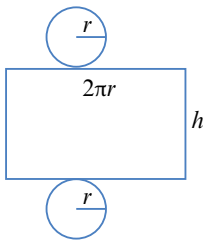
7 **a** 663 cm^2 **b** 2308.4 mm^2 **c** 117.4 cm^2

8 a 1350 mm^2
b 12 150 mm^2
d 6750 mm^2
c


e 5400 mm^2, 24 surfaces of dice are hidden within structure ($24 \times 15^2 = 5400$)

9 383 cm^2

10 a


b One face is curved.
c


d Length of rectangle is circumference of base circle.
e curved surface: $A = 2\pi rh$, bottom circle: $A = \pi r^2$, top circle: $A = \pi r^2$
f Adding these areas gives TSA = $2\pi rh + 2\pi r^2$.

11 a 1809.6 cm^2 b 883.4 cm^2 c 11 309.7 mm^2

12 a 7904.2 mm^2 b 23 712.6 mm^2

13 a 8 cm, divide TSA by 6 to find area of one face, then take square root to find length of side
b $l = \sqrt{\frac{\text{TSA}}{6}}$

14 a 5702 cm^2
b Pipe is open, so there is no top or bottom. This means $2\pi r^2$ is removed from formula.

15 279.9 cm^2

16 a 7191.3 cm^2
b No. The curved surface area will be double, but the one circular end which needs painting has same area for both poles.
c 14 259.9 cm^2 (7068.6 cm^2 + 122.7 cm^2 = 7191.3 cm^2 while 7068.6 × 2 + 122.7 cm^2 = 14 259.9 cm^2) This supports answer to part b.

17 1311 cm^2

18 a two rectangles
b 1200 cm^2 There are two faces of this shape.
c 32 cm × 65 cm = 2080 cm^2,
32 cm × 12 cm = 384 cm^2,
35 cm × 32 cm = 1120 cm^2,
14 cm × 32 cm = 448 cm^2,
32 cm × 30 cm = 960 cm^2,
32 cm × 26 cm = 832 cm^2
d 8224 cm^2

19 407.8 m^2

9D Volume of prisms and cylinders

9D Start thinking!

1 a rectangular base, area: 200 cm^2
b 1000 cm^3 c rectangular prism: $V = lwH$
2 a square b cube: $V = AH = l \times l \times l = l^3$
3 a $A = \frac{1}{2}bh$ b triangular prism: $V = \frac{1}{2}bhH$
4 trapezoidal prism: $V = \frac{1}{2}(a + b)h \times H$
5 cylinder: $V = \pi r^2 H$

Exercise 9D Volume of prisms and cylinders

1 a 12 500 000 cm^3 b 240 cm^3
c 34.2 m^3 d 550 000 000 mm^3
e 0.000 067 2 m^3 f 9000 mm^3
g 7.52 cm^3 h 8 740 000 cm^3
i 0.1429 m^3 j 0.073 m^3

2 a 2508 cm^3 b 5.8 m^3
c 96 140 mm^3 or 96.1 cm^3
d 43 264 cm^3 e 74 088 mm^3 f 53.5 m^3

3 a 16 128 mm^3 b 689.1 cm^3 c 6.2 m^3
d 2944 cm^3 e 1539 m^3 f 175 cm^3

4 a 1256.6 cm^3 b 172 240.2 mm^3
c 16 155.6 m^3 d 290.8 cm^3
e 178 623.1 cm^3 or 0.2 m^3 f 3698.5 cm^3

5 105 792 cm^3

6 a 0.064 m^3 b 64 000 cm^3

7 a 1102.5 cm^3 b 35 224 mm^3 c 15.58 m^3

8 5193.4 cm^3

9 a Volume is amount of space a container occupies, whereas capacity is amount of liquid that container can hold.
b 1 000 000 mL c 1 kL
d i 1200 L ii 15 800 cm^3
iii 1242 m^3 iv 0.000 008 8 kL

10 386 mL

11 a length: 9 cm, width: 8 cm, height: 13 cm, volume: 936 cm^3
b length: 9 cm, width: 13 cm, height: 8 cm, volume: 936 cm^3
c They are the same.
d Orientation of a prism has no effect on its volume. Height can also be regarded as length.

12 207.9 cm^3

13 a 163.2 cm^3 b 0.1632 kg c $3.43

14 a 45 000 cm^3 b 45 000 mL, 45 L
c 39.6 L

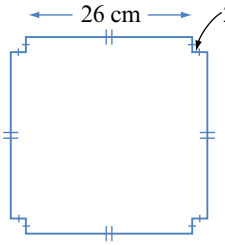
15 a 14 cm
b Two possible answers are: 14 cm by 49 cm by 4 cm, 8 cm by 49 cm by 7 cm.
c Two possible answers are:
triangular 14 cm base with 14 cm height, and prism height of 28 cm
triangular 8 cm base with 14 cm height, and prism height of 49 cm.

16 a 21 cm b 34 mm

17 a second tank b 15 316 L

18 **a** 3015.9 cm^3
b **i** 754 cm^3 **ii** $\frac{1}{4}$ its value
c **i** 12 063.7 cm^3 **ii** 4 times its value
d **i** 1508 cm^3 **ii** $\frac{1}{2}$ its value
e **i** 6031.9 cm^3 **ii** 2 times its value

19 **a**



b length: 26 cm, width: 26 cm, height: 2 cm
c 1352 cm^3
d Box would be 24 cm × 24 cm × 3 cm, giving volume of 1728 cm^3. Volume of box would increase.

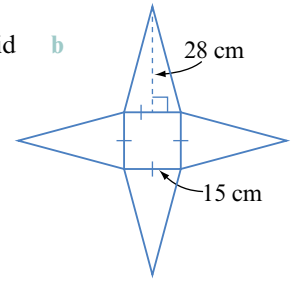
20 **a** annulus, area: 84.8 cm^2 **b** 2120 cm^3
c Take volume of inner cylinder from volume of outer cylinder.

21 **a** 12.48 m^3 **b** 10 560 mm^3 **c** 25 249 cm^3

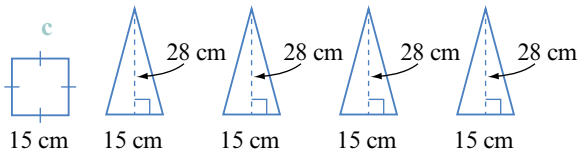
9E Surface area of pyramids and cones

9E Start thinking!

1 square-based pyramid
2 A prism has constant cross-section throughout its height. The sides of a pyramid converge to an apex, and its cross-section becomes smaller from base to apex, while retaining a similar shape.
3 Find sum of areas of all faces. Overall method is same as finding TSA for a prism; however, formula is different because number and type of faces differ.
4 **a** square-based pyramid **b**



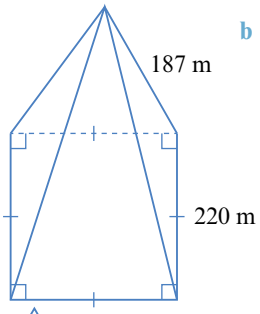
c



d base: 225 cm^2, each side face: 210 cm^2
e No. The four side faces all have same area.
f 1065 cm^2

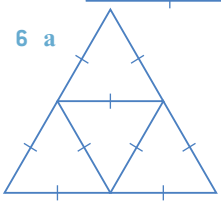
Exercise 9E Surface area of pyramids and cones

1 **a** 2744 mm^2 **b** 408 cm^2 **c** 7605 cm^2
2 **a** 118 cm^2 **b** 3234 m^2 **c** 4856 mm^2
3 **a** 809.2 cm^2 **b** 1580.4 mm^2 **c** 9262 cm^2
4 1890 m^2
5 **a**



b 66 528 m^2

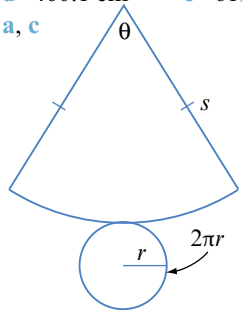
6 **a**



b 4 **c** 28 cm^2
d Since all faces are same shape and size, multiply area of one face by 4 to find TSA.

7 **a** 195.6 cm^2 **b** 914.3 mm^2 **c** 4877.4 cm^2
d 460.1 cm^2 **e** 619.5 mm^2 **f** 129.2 cm^2

8 **a**, **c**



b $A = \pi r^2$
d $A = \pi rs$
e Check with teacher

9 **a** 9123.2 mm^2 **b** 2801.5 cm^2 **c** 624.4 cm^2
d 282.7 cm^2 **e** 628.3 cm^2 **f** 1582.7 mm^2
10 20 cm
11 slant height: 18.2 cm, TSA: 1641 cm^2
12 339.3 cm^2
13 **a** 8 **b** 677.6 mm^2
c Each pyramid, including the base, has TSA of 534.8 mm^2. So, TSA of two pyramids is 1069.6 mm^2.
d Answers to parts b and c are not equal in this form. If you subtract the areas of the two bases of pyramids (which are not visible) answers are the same.
1069.6 mm^2 − 2 × 196 mm^2 = 677.6 mm^2
14 580.6 cm^2

9F Volume of pyramids and cones

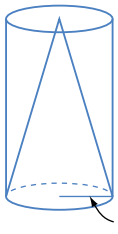
9F Start thinking!

2 The volume of a square-based pyramid = $\frac{1}{3}$ × volume of prism of same base area and height = $\frac{1}{3}l^3$
3 square-based pyramid: $V = \frac{1}{3}l^2H$
4 rectangular pyramid: $V = \frac{1}{3}lwH$
5 The two pyramids have same volume.

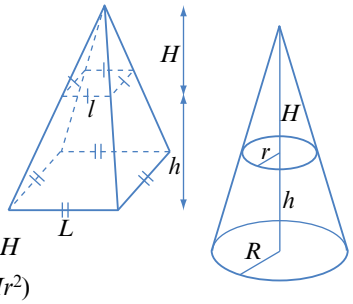
Exercise 9F Volume of pyramids and cones

1 **a** 506 mm^3 **b** 198 cm^3 **c** 149 mm^3
2 **a** 6700.7 cm^3 **b** 37 762.7 mm^3 **c** 38.2 m^3
d 11.4 cm^3 **e** 93.6 cm^3 **f** 5118.8 mm^3

3 **a** 1615 cm^3 **b** 28.9 cm^3 **c** 3033.3 cm^3
4 2 592 100 m^3
5 **a** 9 800 000 cm^3 **b** 0.0195 m^3 **c** 0.248 L
d 32 000 L **e** 250 mL **f** 0.54 m^3
g 145 mm^3 **h** 7.9 L **i** 0.0102 m^3
6 **a** 25.5 cm^2 **b** 93.5 cm^3, 93 500 mm^3
c triangular pyramid: $V = \frac{1}{3} \times \frac{1}{2}bhH$
7 **a** 65 m^3 **b** 1239.3 cm^3 **c** 15 519.2 cm^3
8 **a**



b Cylinder has same base radius and height as cone.
c 282.7 cm^3
d Volume of a cone is $\frac{1}{3}$ volume of a cylinder with same base radius and height; volume: 94.2 cm^3.
e cone: $V = \frac{1}{3}\pi r^3 H$
9 **a** **i** 823 825.1 mm^3 **ii** 0.8 L
b **i** 1747.7 cm^3 **ii** 1.7 L
c **i** 119.7 cm^3 **ii** 0.1 L
d **i** 5026.5 cm^3 **ii** 5.0 L
e **i** 5277.9 mm^3 **ii** 0.005 L
f **i** 53.1 m^3 **ii** 53 100 L
10 16 222.4 cm^3 **11** 74 mL
12 **a** 7 cm **b** 2.9 cm
13 5.2 L
14 **a** 47.1 cm^3 **b** **i** 6 cm **ii** 188.5 cm^3
c 754 cm^3
d When radius is doubled and height remains the same, volume increases by a factor of 4.
e 94.2 cm^3 **f** 188.5 cm^3 **g** 377 cm^3
h When height is doubled and radius remains the same, volume is doubled.
i cone: $V = \frac{1}{3}\pi r^3 H$ Formula for volume involves r^2, so when r is doubled, r^2 increases by factor of 2^2 (4). Formula also involves H, so when H is doubled, this doubles the volume.
15 **a** four are trapeziums, two are squares
b 7291.7 cm^3 **c** 466.7 cm^3 **d** 6825 cm^3
16 1.9 m^3
17 **a** 225 723.4 mm^3 **b** 225 mL
18 One possible answer is shown.
For a truncated square-based pyramid:
$V = \frac{1}{3}L^2(H + h) - \frac{1}{3}l^2H$
For a truncated cone:
$V = \frac{1}{3}\pi R^2(H + h) - \frac{1}{3}\pi r^2 H$
or $V = \frac{\pi}{3}((H + h)R^2 - Hr^2)$



9G Surface area and volume of spheres

9G Start thinking!

1 A sphere is a 3D circular object formed by the set of points in space all the same distance from a given fixed point.
2 radius (or diameter)
4 No, since volume is a cubic measurement, while surface area is a square measurement.
5 **a** half **b** TSA = $2\pi r^2$
c Area of circular disk (πr^2) is added to area of open hemisphere. TSA of a closed hemisphere is $3\pi r^2$.
d $V = \frac{2}{3}\pi r^3$
e No, volume of a hemisphere is same whether it is open or closed.

Exercise 9G Surface area and volume of spheres

1 **a** 1256.6 mm^2 **b** 8494.9 cm^2 **c** 3631.7 cm^2
d 7543.0 mm^2 **e** 2290.2 cm^2 **f** 8992.0 cm^2
2 **a** 4188.8 mm^3 **b** 73 622.2 cm^3 **c** 20 579.5 cm^3
d 61 600.9 mm^3 **e** 10 306.0 cm^3 **f** 80 178.9 cm^3
3 **a** **i** 508.9 cm^2 **ii** 1526.8 cm^3
b **i** 3041.1 mm^2 **ii** 22 301.1 mm^3
c **i** 4561.6 mm^2 **ii** 22 301.1 mm^3
d **i** 1490.1 cm^2 **ii** 7649.3 cm^3
e **i** 82.0 cm^2 **ii** 53.8 cm^3
f **i** 9954.9 mm^2 **ii** 71 896.7 mm^3
4 **a** 18 145.8 cm^2 **b** 229 847.3 cm^3
5 **a** 58.0 cm^2 **b** 41.5 cm^3
6 **a** TSA = $4\pi r^2$ **b** 6.7 cm
$\frac{TSA}{4\pi} = r^2$
$\sqrt{\frac{TSA}{4\pi}} = r$
7 **a** $V = \frac{4}{3}\pi r^3$ **b** 33 cm
$\frac{3V}{4\pi} = r^3$
$\sqrt[3]{\frac{3V}{4\pi}} = r$
8 **a**

Planet	Diameter (km)	Total surface area (km^2)	Volume (km^3)
Mercury	4878	7.5×10^7	6.1×10^{10}
Venus	12 104	4.6×10^8	9.3×10^{11}
Earth	12 756	5.1×10^8	1.1×10^{12}
Mars	6787	1.4×10^8	1.6×10^{11}
Jupiter	142 800	6.4×10^{10}	1.5×10^{15}
Saturn	120 000	4.5×10^{10}	9.0×10^{14}
Uranus	51 118	8.2×10^9	7.0×10^{13}
Neptune	49 528	7.7×10^9	6.4×10^{13}

b Jupiter **c** Mercury
d Yes. The only variable in the formulas for surface area and volume is the radius. This means that the larger the radius, the larger the area and the volume.
e Volume of Earth is about 18 times the volume of Mercury.

f Surface area of Jupiter is about 125 times the surface area of Earth.
g 3.621×10^8 km^2

9 a 8.6 cm b 75.3 cm^3

10 a 3591.4 cm^3 b 10 744.2 cm^3 c 10.7 L
d Approximately 4.3 s
e rate of inflation is an average rate; no time lost between filling one balloon and the next

11 a 508.9 cm^2 b 763.4 cm^2
c 1527 mL or 1.5 L d 452 mL
e 70.4%

12 a 33 510.3 mm^3 or 33.5 cm^3 b 0.08 g/cm^3

13 a i Doubling radius makes surface area 4 times as large.
ii Doubling radius makes volume 8 times as large.
b i Tripling radius makes surface area 9 times as large.
ii Tripling radius makes volume 27 times as large.
c i Halving radius makes surface area $\frac{1}{4}$ the size.
ii Halving radius makes volume $\frac{1}{8}$ the size.
d i Dividing radius by 3 makes surface area $\frac{1}{9}$ the size.
ii Dividing radius by 3 makes volume $\frac{1}{27}$ the size.

14 a 564.1 m^2 b \$25 384.50 c 1260 m^3
d i 840 m^3 ii 5.9 m e to change altitude

9H Surface area and volume of composite solids

9H Start thinking!

1 Capitol building in Washington, USA. Solids include: cylinder, prism, hemisphere.
2 hemisphere, cylinder
3 a cylinder: TSA = $2\pi r^2 + 2\pi rH$
b Top surface is covered by hemisphere, while bottom surface is sitting on ground. This eliminates $2\pi r^2$ from formula.
c 377.0 m^2
4 a Surface area of hemisphere is half surface area of sphere. Formula is $2\pi r^2$.
b 100.5 m^2
5 477.5 m^2 6 880 m^3

Exercise 9H Surface area and volume of composite solids

1 a 290 m^2 b 4815.8 m^2 c 3562.6 cm^2
d 169.2 cm^2 e 1313.2 cm^2 f 6578.5 mm^2

2 a 1406.4 cm^2 b 5277.9 mm^2 c 6927.2 cm^2

3 a 313.5 m^3 b 15 920.4 m^3 c 16 278.9 cm^3
d 141.4 cm^3 e 4308.2 cm^3 f 39 819.7 mm^3

4 a 676.2 cm^3 b 17 090.3 mm^3 c 1590.4 cm^3

5 a sphere: TSA = $4\pi r^2$, square-based pyramid: TSA = $l^2 + 4 \times \frac{1}{2}bh$, cylinder: TSA = $2\pi r^2 + 2\pi rH$, cube: TSA = $6l^2$, rectangular prism: TSA = $2(lw + lh + wh)$
b sphere: 12.6 cm^2, square-based pyramid: 14.1 cm^2, cylinder: 17.7 cm^2, cube: 13.5 cm^2, rectangular prism: 28 cm^2, total: 85.9 cm^2
c rectangular prism = 8 cm^3, sphere = 4.2 cm^3, cube = 3.375 cm^3, cylinder = 5.3 cm^3, square-based pyramid = 1.9 cm^3, total: 22.775 cm^3

6 a rectangular prism, triangular prism, cylinder
b total area of all surfaces showing
c 198 cm^2 d 97.4 cm^3

7 132 mL 8 334.2 cm^2

9 a 37 447.8 mm^2 b height of cone
c 566 827.1 mm^3

10 6898.9 cm^3 11 8078 cm^3 12 1.2%

13 a 164.6 cm^3 b 658.4 cm^3
c canister: diameter 6.8 cm, height 27.2 cm, volume: 987.8 cm^3
d 329.4 cm^3
e rectangular prism: length 6.8 cm, width 6.8 cm, height 27.2 cm, volume: 1257.7 cm^3. There would be more unused space (269.9 cm^3 more).

14 a 59 244 L b 40 097 L

15 208.9 cm^2

16 a Circular nut has surface area of 17.4 mm^2 and requires 107.88 mm^3 of steel.
Hexagonal nut has surface area of 23.8 mm^2 and requires 147.56 mm^3 of steel.
b hexagonal nut

9 Chapter review

MULTIPLE-CHOICE

1 B 2 D 3 B 4 A 5 C
6 B 7 B 8 A 9 B 10 C

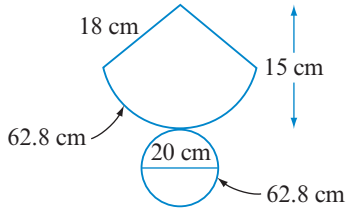
SHORT ANSWER

1 a 7.9 cm b 37.9 cm
2 29.7 cm
3 3.1 cm
4 a 80.4 cm^2 b 63.3 cm^2
5 3.9 cm^2
6 2664 mm^2
7 a 9600 cm^2, 38 400 cm^2, 153 600 cm^2
b 1 can
8 258.4 cm^2
9 64 000 cm^3, 512 000 cm^3, 4 096 000 cm^3
10 a 0.154 L b 6.5 m
11 225 cm^2
12 341.9 cm^2
13 6360 cm^3
14 Cylinder volume: 1024π cm^3
Sphere volume: $\frac{2048\pi}{3}$ cm^3
Volume of cylinder is 1.5 times that of sphere.
15 Surface area of cylinder is 1.5 times that of sphere.
16 a i 3769.9 cm^2 ii 9600 cm^2
b 10 856.7 cm^2

MIXED PRACTICE

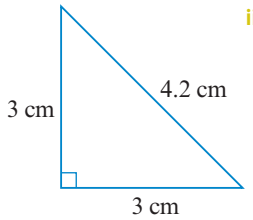
1 C 2 21 cm^2 3 32 cm
4 46 cm^2 5 D 6 148 cm^2

7 A 8 113 cm^2 9 33.75 L
10 C 11 its area is n^2 times as large
12 3054 cm^3 13 B 14 B
15 D 16 C 17 C
18 0.005 m^3 19 $3\pi r^2$ 20 A
21 22 808 mm^3 22 C
23 Its volume is 8 times as large.
24 Rectangular prism has constant cross-section, whereas rectangular pyramid has cross-section with shape that remains same, but decreases in size from base to apex.
25



ANALYSIS

a cube, rectangular prism, square pyramid, triangular prism, cone, cylinder, sphere
b i 54 cm^2 ii 27 cm^3
c i Rectangular prism has surface area of 90 cm^2, whereas surface area of cube is 54 cm^2. It is $1\frac{2}{3}$ times as large.
ii Volume of rectangular prism is 54 cm^3. It is twice volume of cube.
d i



ii 70.2 cm^2
iii No; TSA of triangular prism is less than TSA of rectangular prism. There is no direct relationship.
iv Volume of triangular prism is half that of rectangular prism. This is logical, as it is the rectangular prism cut in half.
e i 70.7 cm^2 ii 42.4 cm^3
f i cube, triangular prism, cylinder, rectangular prism
ii cube and triangular prism (equal), cylinder, rectangular prism
g i 46.2 cm^2 ii 18 cm^3
h i 36.3 cm^2 iii 14.1 cm^3
i i 28.3 cm^2 ii 14.1 cm^3
j i in increasing order of surface area: sphere, cone, square pyramid, cube, triangular prism, cylinder, rectangular prism
ii in increasing order of volume: sphere and cone (equal), square pyramid, cube and triangular prism (equal), cylinder, rectangular prism

9 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 10 STATISTICS

10 Are you ready?

1 5 2 6.2 3 B 4 74 5 14 6 6
7 (5, 7) 8 B 9 D 10 A

10A Measures of centre

10A Start thinking!

1 mean: 18.25, median: 16.5, mode: 14, average age: 16.5 years
2

Score (x)	Frequency (f)
8	1
14	4
15	2
16	3
17	1
18	3
21	3
22	2
45	1

3 Multiply score by frequency.
4 The $f \times x$ column represents totals for each particular score. Total of column gives total for all scores.
5, 6

Score (x)	Frequency (f)	$x \times f$
8	1	$8 \times 1 = 8$
14	4	$14 \times 4 = 56$
15	2	$15 \times 2 = 30$
16	3	$16 \times 3 = 48$
17	1	$17 \times 1 = 17$
18	3	$18 \times 3 = 54$
21	3	$21 \times 3 = 63$
22	2	$22 \times 2 = 44$
45	1	$45 \times 1 = 45$
Total	20	365

mean $= \frac{365}{20} = 18.25$; same as answer to question 1
7

cf
1
5
7
10
11
14
17
19
20

Numbers in cf column found by repeatedly adding on frequency value of each score.
8 median $= \frac{20 + 1}{2}$ score, or 10.5th score, which is $\frac{16 + 17}{2} = 16.5$

Exercise 10A Measures of centre

1 a mean: 5.7, median: 5, mode: 3, 5, 8 and 9
b mean: 13.2, median: 9, mode: 7 and 13
c mean: 13.8, median: 16, mode: 5, 17 and 18

d mean: 122.7, median: 113.5, mode: there is no mode
e mean: 5.6, median: 5.1, mode: 4.9 and 7.5
f mean: 21.3, median: 18, mode: 12, 13, 16, 18, 19 and 27

2 **a** 2.2 **b** 26.7 **c** 15.5 **d** 3.18 **e** 10.51 **f** 3.12

3 **a** 19 **b** 7.8 **c** 29.9 **d** 53.6 **e** 165.5 **f** 11.76

4 **a** 2 **b** 30 **c** 15.5 **d** 3 **e** 11 **f** 2

5 **a** 10–<20 **b** 5–<10 **c** 28–<32 **d** 60–<80 **e** 164–<166 **f** 10–<15

6 **a** 3 **b** 40 **c** 16 **d** 3 **e** 11 **f** 2

7 **a** 10–<20 **b** 10–<15 **c** 32–<36 **d** 60–<80 **e** 166–<168 **f** 5–<10

8 1 or 2

9 **a**, **b**

Score (x)	Frequency (f)	$x \times f$
1	5	$1 \times 5 = 5$
2	11	$2 \times 11 = 22$
3	19	$3 \times 19 = 57$
4	16	$4 \times 16 = 64$
5	7	$5 \times 7 = 35$
6	8	$6 \times 8 = 48$
Total	66	231

mean: 3.5, median: 3, mode 3

c 3

10 **a** iPhone, mode
b There are no actual data values to use to calculate mean or median. The data is categorical (qualitative), not numerical (quantitative).

11 **a** mean: 16, median: 16, mode 17
b 16

12 **a** mean: 48.1, median: 47, mode 13

b

Class interval	Midpoint (x)	Frequency (f)	$x \times f$
11–15	13	7	91
16–20	18	5	90
21–25	23	2	46
26–30	28	2	56
31–35	33	4	132
36–40	38	0	0
41–45	43	2	86
46–50	48	6	288
51–55	53	1	53
56–60	58	3	174
61–65	63	1	63
66–70	68	3	204
71–75	73	1	73
76–80	78	3	234
81–85	83	2	166
86–90	88	1	88
91–95	93	1	93
96–100	98	4	392
Total		48	2329

mean: 48.5, median class: 46–50, modal class: 11–15

c

Class interval	Midpoint (x)	Frequency (f)	$x \times f$
11–20	15.5	12	186
21–30	25.5	4	102
31–40	35.5	4	142
41–50	45.5	8	364
51–60	55.5	4	222
61–70	65.5	4	262
71–80	75.5	4	302
81–90	85.5	3	256.5
91–100	95.5	5	477.5
Total		48	2314

mean: 48.2, median class: 41–50, modal class: 11–20

d

Class interval	Midpoint (x)	Frequency (f)	$x \times f$
11–30	20.5	16	328
31–50	40.5	12	486
51–70	60.5	8	484
71–90	80.5	7	563.5
91–110	100.5	5	502.5
Total		48	2364

mean: 49.25, median class: 31–50, modal class: 11–30

e

	Mean	Median	Mode
Raw data	48.1	47	13
Class interval 5	48.5	46–50	11–15
Class interval 10	48.2	41–50	11–20
Class interval 20	49.25	31–50	11–30

The mean changes slightly with a change in class interval. The spread of scores around the midpoint affects its value. The median and mode are included in a wider band width as class interval increases, so its particular value is not as obvious.

13 **a** mean: 17.36, median: 15, mode: 16
b Mean is affected by outlier of 68. Values of median and mode lie quite close together and are not affected greatly by outlier.
c An extra large or extra small score (an outlier) affects value of mean. In these situations, the middle score is a better measure of centre. If mode varies greatly from median, it may be a better measure of a 'typical' score of data set.

14 **a** **i** mean: 8.35, median: 5, mode: 4
ii median
iii Fewer scores in centre of data set than towards ends of set.
b **i** mean: 45.15, median: 43, there is no mode
ii mean or median could be used
iii Scores are scattered widely around centre.
c **i** mean: 179.4, median: 183, mode: 184
ii median
iii Mean is affected by low outlier of 129. Scores generally scattered around mid-180s.

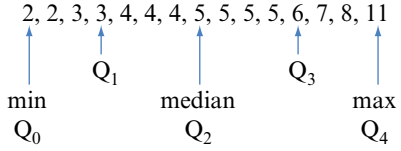
15 mean: 50, median class: 50–<60, modal class: 60–<70. Scores bunched towards higher end of distribution. Mean or median class could be used to describe centre.

16 Average price for seven items at Gray St Shop: $2.33; at Doey's: $2.73; at The Corner: $2.79. Average price per item of chocolate bar: $2.33, 2-L milk: $2.40, bottle of juice: $2.30, salad roll: $5.17, ice cream: $3.00, tin of corn: $2.30, fried dim sim: $0.80. Gray St Shop has five items below average price and two above. Doey's has two items below average, two equal to average and three above average prices. The Corner has three items below average price and four above.
The Gray St Shop is cheapest for that particular basket of goods, so would probably be the preferred shop.

10B Measures of spread

10B Start thinking!

1 range: 9
2 range = highest score − lowest score
3 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 7, 8, 11



4 Lower quartile in lower half of data, upper quartile in upper half of data.
5 7 6 3

Exercise 10B Measures of spread

1 a 16 b 45 c 87 d 67
e 4 f 25 g 20
2 a $Q_0 = 3, Q_1 = 3, Q_2 = 5, Q_3 = 7, Q_4 = 9$
b $Q_0 = 7, Q_1 = 12, Q_2 = 19, Q_3 = 46, Q_4 = 78$
c $Q_0 = 1, Q_1 = 3, Q_2 = 5, Q_3 = 7, Q_4 = 9$
d $Q_0 = 2, Q_1 = 7, Q_2 = 11, Q_3 = 19, Q_4 = 32$
3 a $Q_0 = 2, Q_1 = 4, Q_2 = 7, Q_3 = 9, Q_4 = 12$
b $Q_0 = 4, Q_1 = 13, Q_2 = 25.5, Q_3 = 46, Q_4 = 55$
c $Q_0 = 16, Q_1 = 20.5, Q_2 = 42, Q_3 = 72.5, Q_4 = 81$
d $Q_0 = 120, Q_1 = 177.5, Q_2 = 452.5, Q_3 = 755, Q_4 = 843$
e $Q_0 = 18, Q_1 = 19.5, Q_2 = 33, Q_3 = 42, Q_4 = 64$
f $Q_0 = 3, Q_1 = 8, Q_2 = 13.5, Q_3 = 19, Q_4 = 32$
4 a 10 b 9 c 10 d 19
5 a 32 b 9 c 56.5 d 36
e 9.5 f 253
6 range: 51, IQR: 23. Bakery sells 56–79 cupcakes on half the days of the month.
7 a Set A mean: 22, median: 19; set B mean: 22, median: 19. Sets have same mean and median.
b Both sets have range of 43.
c Set A IQR: 26, set B IQR: 10
d Because they have the same data range, mean and median.
8 a 6
b $Q_0 = 0, Q_1 = 2, Q_2 = 4, Q_3 = 5, Q_4 = 6$
c 3
9 a If IQR is small, 50% of scores are grouped closely around median.
b If IQR is small and range is large, 50% of scores are grouped closely around median, while the 25% of scores lying below the lower quartile and the 25% of scores above the upper quartile are spread out.
c If IQR is large, 50% of scores are spread widely around the median.
d If IQR is similar to range, 50% of scores are grouped around median, while the 25% of scores lying below the lower quartile and the 25% of scores above the upper quartile are grouped closely together.
10 a 14, 15, 15, 16, 18, 19, 19, 23, 26, 27, 27, 28, 29, 31, 67, IQR: 12
b 1, 1, 2, 2, 3, 4, 4, 5, 6, 6, 6, 7, 7, 7, 8, 9, 11, 11, 18, 19, IQR: 5
c 107, 108, 117, 123, 129, 144, 148, 149, 151, 156, 167, 168, 177, 189, 191, IQR: 45
d 63, 64, 65, 66, 67, 68, 72, 72, 72, 73, 73, 74, 75, 75, 75, 77, 77, 77, 78, 79, 79, 80, 81, 81, 769, IQR: 8.5
11 a range: $240, IQR: $45
b presence of an outlier
c Range includes 100% of the data, while IQR includes 50% of the data.
d IQR represents middle 50% of scores, those between Q_1 and Q_3.
e IQR shows region within which middle 50% of scores lie.
12 a i range: 17, IQR: 8.5
ii In this case, range is twice value of IQR, so either could be used to represent spread.
iii The data are spread fairly uniformly throughout the range.
b i range: 72, IQR: 30
ii Better to use IQR in this case, as it is less than half the value of the range.
iii The scores are packed more closely in the IQR. There appears to be an outlier of 11 in the lower quarter of the scores.
c i range: 73, IQR: 11
ii Better to use IQR in this case, as it is less than half the value of the range.
iii Scores are packed more closely in IQR. There appears to be an outlier of 191 in the upper quarter of the scores.
13 a 33.6 b 35.5 c 70 d 30
14 a–e The minimum and maximum scores are always Q_0 and Q_4 values in the five-number summaries. This table compares the quartiles Q_1 and Q_3 and medians (Q_2).

Number of data	All	Quartiles (Q_1, Q_3)	Median (Q_2)	None
5	✓			
6		✓		
7	✓			
8				✓
9			✓	
10		✓		
11	✓			
12				✓
13			✓	
14		✓		
15	✓			
16				✓

f The pattern repeats itself in groups of 4 from 6 data items onwards.

g If number of data is: multiple of 4, none; multiple of 4 + 1, median; multiple of 4 + 2, lower and upper quartiles; multiple of 4 + 3, all. Note that this is only true if all the data items have a different value.

15 a class A
i 171.3 **ii** 171.5 **iii** 27 **iv** 12
class B
i 170.9 **ii** 172.5 **iii** 48 **iv** 12.5

b The mean, median and IQR of heights in the two class is quite similar; however, the range of heights in class B is greater. Class B has the tallest and shortest members of the group, indicating it is more diverse.

16 a, b

Class interval	Midpoint (x)	Frequency (f)	cf
11–20	15.5	28	28
21–30	25.5	36	64
31–40	35.5	19	83
41–50	45.5	11	94
51–60	55.5	4	98
61–70	65.5	1	99
71–80	75.5	1	100

c 21–30 class interval, with midpoint 25.5
d 50 **e** 15.5 **f** 35.5 **g** 20
h It can give an approximate value for IQR.

10C Standard deviation

10C Start thinking!

1 Most values close to mean.
2 mean for set A and set B: 5; they are the same
3 set B
4 **a, b**

Score	Difference from mean
1	4
1	4
9	4
1	4
9	4
9	4
Total	24

c 4
5 average difference for set A: 4
6 **a, b**

Score	Difference from mean
2	3
3	2
1	4
7	2
2	3
15	10
Total	24

c average difference for set B: 4; same as for set A.
7 **a** 9 **b** 81
8 The square of a larger number is much bigger than the square of a smaller number.

Exercise 10C Standard deviation

1 **a** 15.17 **b** 5.13 **c** 23.85 **d** 2.19
e 10.52 **f** 105.69
2 **a** 3.81 **b** 13.26 **c** 1.30 **d** 1.30
e 6.67 **f** 136.29
3 **a**

Score	Frequency f
14	1
15	6
16	13
17	6
Total	26

b 0.78
c Ages of students are closely grouped, hence standard deviation is low.
4 First histogram has larger standard deviation than second, as scores are more spread out.
5 Standard deviation
a 3.51 **b** 29.63 **c** 3.22
6 **a** 68.51
b May be a discount pharmacy; or tablet may be generic brand rather than recognised brand name.
c 20.98
d Standard deviation is much lower with outlier excluded.
e IQR
7 **a** Use midpoint of class interval, together with frequency, to calculate mean.
b

Class interval	Midpoint (x)	Frequency (f)
0–100	50.5	3
101–200	150.5	5
201–300	250.5	8
301–400	350.5	11
401–500	450.5	19
501–600	550.5	32
601–700	650.5	22

c 158.48
d Midpoint of class interval represents score for all frequencies in that particular class interval.
8 **a** 41.7 **b** 9.75
c Spread of data is 13–58, with mean about 42. Average deviation of scores from mean is about 10.
9 **a** mean: 15.9, standard deviation: 8.99

b, c

Class interval	Midpoint (x)	Frequency (f)
1–10	5.5	9
11–20	15.5	9
21–30	25.5	5
31–40	35.5	2

mean: 15.5, standard deviation: 9.38

d Figures are not exactly the same, but reasonably close.

e Data are grouped towards lower end with most scores lying in range 1–20. Higher scores (above 20) increase value of mean in same way an outlier does.

10 a mean: line A: 5.4, line B: 4.5

b standard deviation: line A: 1.85, line B: 4.47

c Mean for line A is higher than that for line B, indicating trains are generally later on line A. Standard deviation for line A is lower than that for line B, indicating scores are bunched closely around mean. Line B, however, has two high scores of 15 and 19 minutes, and these can be classed as outliers. Without these two outliers, the mean is 3.3 and the standard deviation is 2.42.

d The outliers for Line B might have been caused by unusual circumstances. If these are disregarded, it could be said that the train on Line A runs late more often. If they are not disregarded, the train on Line B is on average the later one.

11 City A: mean: 34.3 years, standard deviation: 13.33
City B: mean: 25.8 years, standard deviation: 9.26
People auditioning in City B are generally younger than those auditioning in City A. Standard deviation in City B is also lower than for City A, indicating spread of ages in City B is lower than for City A. Generally, a young group of people auditioned in City B, and people with a wide range of ages auditioned in City A.

12 a **i** sample **ii** 0.86

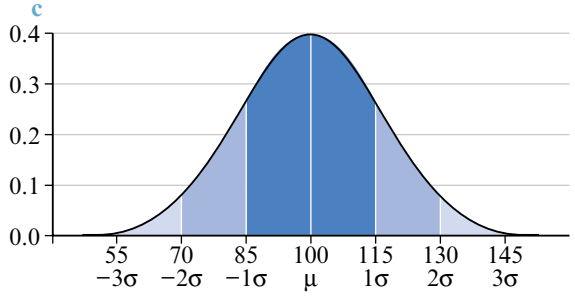
b **i** population **ii** 5.07

c **i** sample **ii** 0.54

13 a Adding percentages shown in figure:
2.12% + 13.6% + 34.13% + 34.13% + 13.6% + 2.12% = 99.7%
This covers region of three sigma either side of mean.

b **i** 68.26% **ii** 95.46%

c



d 55–145 **e** beyond 145 **f** below 70

g 2.27% **h** 15.87%

i mean: 136, standard deviation: 8

10D Box plots

10D Start thinking!

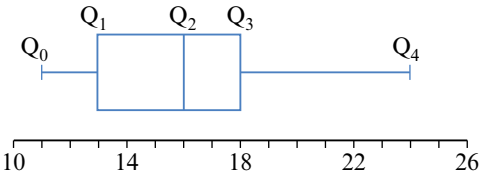
1 minimum, lower quartile, median, upper quartile, maximum

2 minimum: 11, maximum: 24

3 Vertical line in box represents median.

4 16 **5** Q_3 **6** Q_1 **7** $Q_1 = 13$, $Q_3 = 18$

8

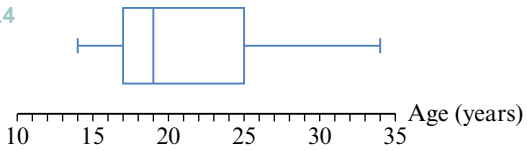


9 Lower whisker represents first quartile, lower part of box between Q_1 and Q_2 is second quartile, between Q_2 and Q_3 is third quartile, and upper whisker represents fourth quartile.

10 Interquartile range is region between upper and lower quartiles; the whole region of the box.

11 $Q_0 = 14$, $Q_1 = 17$, $Q_2 = 19$, $Q_3 = 25$, $Q_4 = 34$

12–14



15 In a box plot, median shown by line within box, IQR shown by length of box, and range shown by distance between maximum and minimum points (at ends of whiskers).

Exercise 10D Box plots

1 a **i** $Q_0 = 1$, $Q_1 = 2$, $Q_2 = 5$, $Q_3 = 7$, $Q_4 = 10$
ii 5

b **i** $Q_0 = 13$, $Q_1 = 20$, $Q_2 = 32$, $Q_3 = 40$, $Q_4 = 45$
ii 20

c **i** $Q_0 = 3$, $Q_1 = 4$, $Q_2 = 6$, $Q_3 = 8$, $Q_4 = 9$
ii 4

d **i** $Q_0 = 10$, $Q_1 = 16$, $Q_2 = 19$, $Q_3 = 24$, $Q_4 = 35$
ii 8

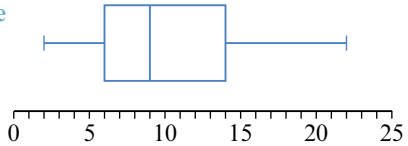
e **i** $Q_0 = 20$, $Q_1 = 45$, $Q_2 = 70$, $Q_3 = 75$, $Q_4 = 85$
ii 30

f **i** $Q_0 = 31$, $Q_1 = 33$, $Q_2 = 35$, $Q_3 = 37$, $Q_4 = 39$
ii 4

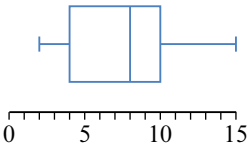
2 a 2, 3, 4, 4, 6, 7, 7, 8, 8, 9, 10, 10, 11, 13, 14, 17, 19, 19, 22

b $Q_0 = 2$, $Q_1 = 6$, $Q_2 = 9$, $Q_3 = 14$, $Q_4 = 22$

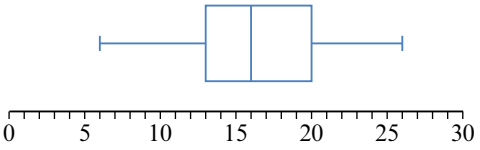
c–e



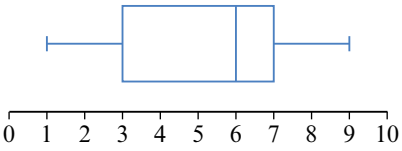
3 a



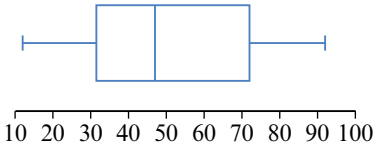
b



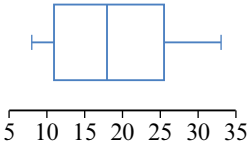
c



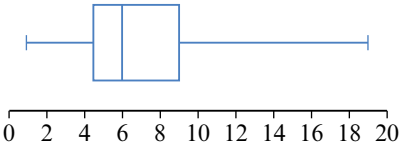
d



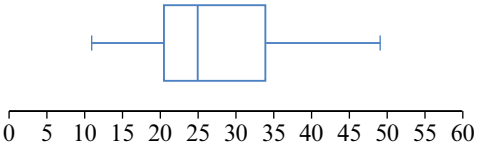
e



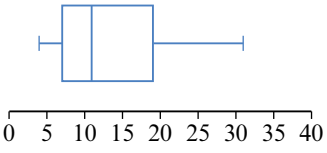
f



g

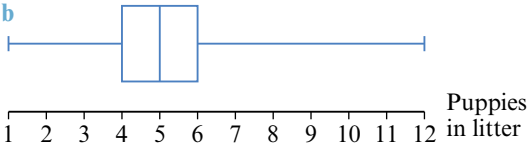


h

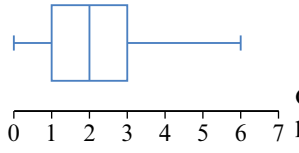


4 a $Q_0 = 1$, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6$, $Q_4 = 12$

b



5



Median number of visits per month was 2. Most people had 3 or fewer visits per month.

6 a roughly symmetrical

b Data are packed closely below median, and spread out above median. Distribution is positively skewed.

c

d Each of the four sections of a box plot represents 25% of the data. If, within any quarter, scores are packed closely together, length of that section will be smaller.

e **i** C **ii** A **iii** B **iv** D

7 a **i** Box plot appears negatively skewed.

ii Scores in first quarter are very closely packed, making lower whisker very short. In second quarter, scores are spread far apart, making lower half of box long. Scores in third and fourth quarters appear spread out to about the same degree: less closely packed than those in bottom whisker, but more densely packed than those in lower half of box.

iii Median score has value close to those of data at top end of distribution.

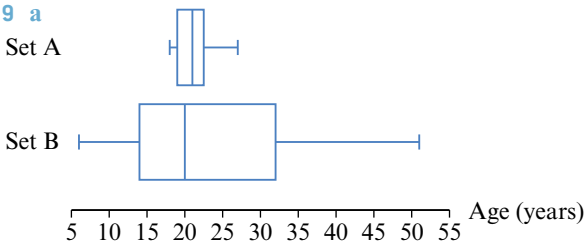
b There may be multiple scores of same value in lower whisker.

8 a Darwin temperatures range from 25°C to 36°C, with median 32°C.

b Canberra temperatures range from 7°C to 37°C, with median 20°C.

c Only one-quarter of Canberra temperatures lie in whole range of temperatures in Darwin as shown by upper whisker of Canberra temperatures. Minimum temperature in Darwin is equivalent to value of upper quartile temperature in Canberra. Range of temperatures in Darwin is only 11°C, whereas range in Canberra is 30°C. Clear that Darwin is the hotter city.

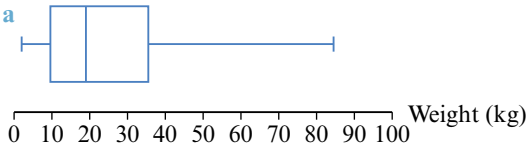
9 a



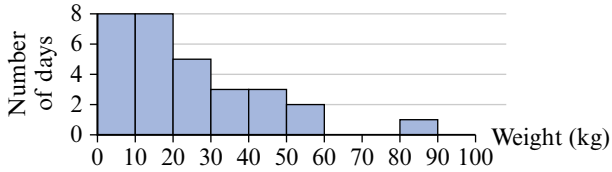
b Set A: centre: median 21, spread: 18–27 years, positively skewed. Range of ages is small, indicating that, during the week, centre is used mainly by young people.
Set B: centre: median 20, spread: 6–51 years old, positively skewed. Range of ages is much greater during weekend, indicating that families most probably visit then.

c Range is much greater in set B than in set A, although median ages are quite similar. More younger and older people visit during weekend than those attending on weekdays.

10 **a**



b



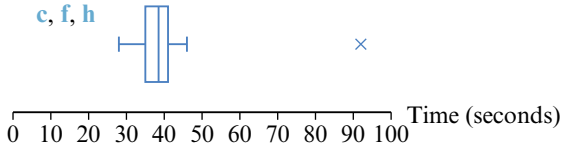
c positively skewed

d Histogram shows more detail than box plot. It also enables a mean to be calculated. Box plot shows the positive skew and the median, but not the fine detail.

11 **a** outlier

b Five-number summary: $Q_0 = 28$, $Q_1 = 35$, $Q_2 = 38.5$, $Q_3 = 41$, $Q_4 = 92$, IQR: 6

c, f, h



d 9 **e** 26

f No scores fall below 26: closest is 28.

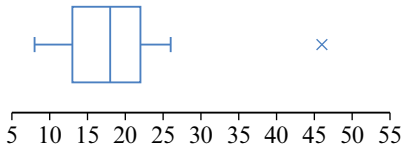
g 50

h Score of 92 falls above 50.

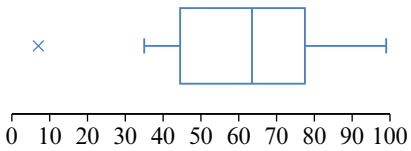
i 46 is an actual data value, whereas 50 is not.

j Scores outside this range are considered to be unusual scores.

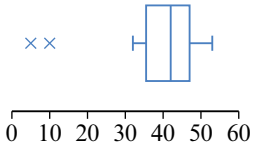
12 **a**



b



c



10E Scatterplots and bivariate data

10E Start thinking!

1 Some possible answers are: height and shoe size, age and height, latitude and temperature.

2 number of people at beach over a range of temperatures

3 Scatterplot shows correlation of moderate strength. General trend shows more people attend beach as temperature increases.

4 Positive correlation, because as one variable increases, the other also increases. Trend is for a positive gradient.

5 Temperature is on x-axis; number of people at the beach is on y-axis.

6 Number of people at the beach increases moderately with increasing temperature.

Exercise 10E Scatterplots and bivariate data

1 **a** **i** moderate to weak negative trend

ii With increasing height, time taken to run 10 km decreases moderately.

b **i** weak positive trend

ii Number of visits to doctor per year increases slightly as number of colds per year increases.

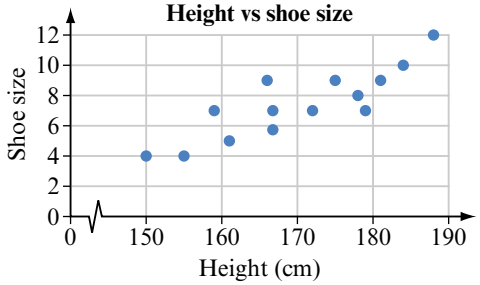
c **i** strong positive trend

ii Amount of petrol used increases strongly with increasing distance travelled.

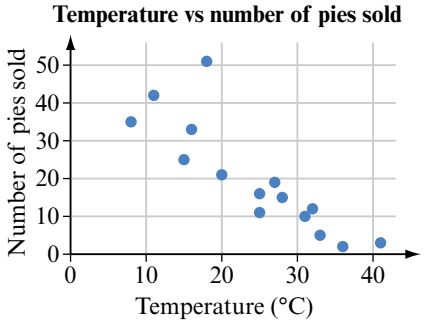
d **i** moderate negative trend

ii Exam marks decrease moderately with increasing hours of watching TV per week.

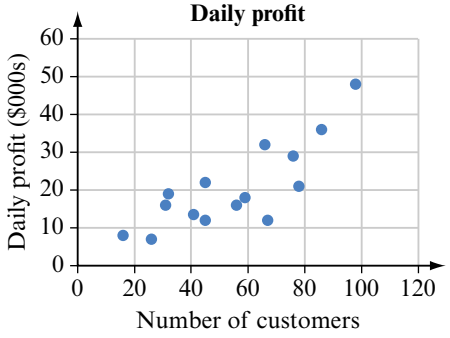
2 **a**



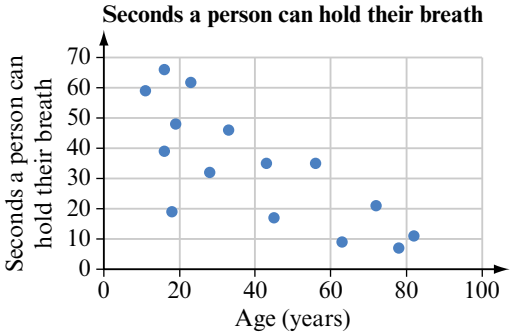
b



c

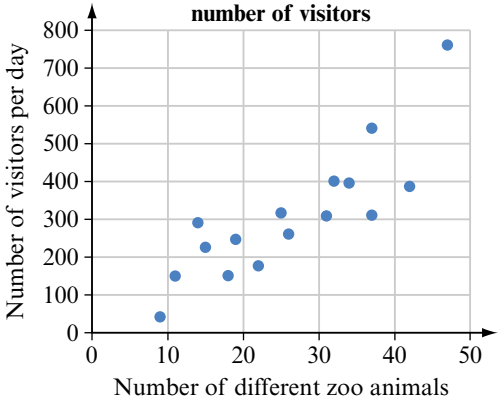


d



3 a Moderately strong increase in shoe size as height of a person increases.
b Number of pies sold decreases quite strongly as temperature increases.
c Moderate increase in daily profit as number of customers increases.
d Moderate decrease in time a person can hold their breath as age increases.

4 a moderate positive correlation
b

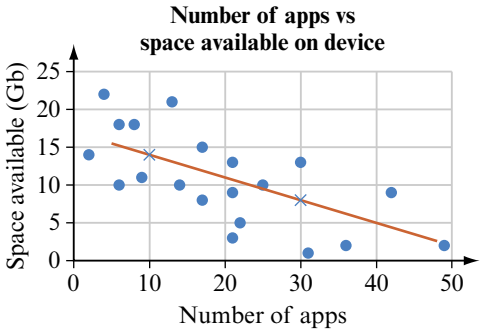


c Moderately strong increase in number of visitors per day as number of different zoo animals increases.

5 a As temperature increases, number of heaters purchased decreases strongly. Moderate negative correlation. Weak positive correlation between earnings and monthly phone bill: as earnings increase, phone bill also tends to increase.
b about 5
c about $100
d The prediction from the second scatterplot is not very reliable as there is not a strong relationship between the variables. With scatterplot A, relationship is much stronger, so prediction is more reliable.

6 a Trend is quite strong and positive: as height increases, weight also increases.
b i about 60 kg ii about 70 kg
iii about 75 kg
c Line represents trend of data and not individual points.
d 82 kg
e If points are spread far this minimises error in guesswork of drawing a line of best fit and errors due to points that may be slightly off the line.
f $y = 100x - 103$
g 82 kg: answer is the same.
h Tends to average out y-values.
i 67 kg

7 a, c



b Weak negative relationship: generally, as number of apps increases, space available on device decreases.
c $y = -0.3x + 17$ d 8 GB
e Relationship not very strong, so predictions are not reliable.
f Apps do not all require same space on a device, so relationship is not linear.
g i interpolation ii extrapolation
iii interpolation iv extrapolation
h There would be negative space (−43 GB) available.
i This is unrealistic as you cannot have negative space available. You cannot be confident about making a prediction outside this data set.

8 a weight depends on height
b As a person gets taller, they generally become heavier.
c i weight ii height
d Because time does not depend on anything.
e x-axis f x-axis g x h top
i Independent variable is in top row of table (or left column) and is graphed on horizontal axis (x-axis).

9 a i Up to a certain time (age of the tree) there is a relationship.
ii independent variable: time, dependent variable: height
iii Correlation is strong and positive in early years, but not so as tree ages.
b i Up to a certain age of car there is a relationship between value of a car and its age. If car becomes an antique, or collectable item, this relationship changes.
ii independent variable: age, dependent variable: value of car
iii Correlation is strong and negative, car losing value as it ages.

c No relationship between weight of a person and number of languages spoken.

d i There is a relationship between shoe size and height of a person.

ii independent variable: height of person, dependent variable: shoe size

iii Correlation is strong and positive. Generally, as person grows and gets taller, their shoe size increases.

10 a Correlation supports the view that more study time contributes to achieving a higher percentage on a test. There is an assumption that study time is 'effective learning' time, and does not include wasted time. However, extra study time does not guarantee a higher percentage.

b A high correlation does not imply that one variable 'caused' another to happen.

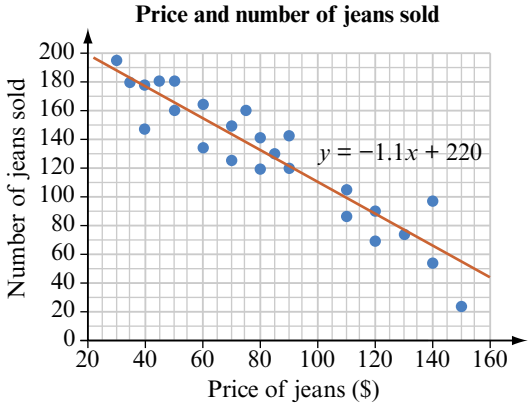
c It is commonly thought if the relationship between two variables is strong, one must have a strong influence on the other. Sometimes this is true, but the particular circumstances need to be examined to determine this.

10F Interpreting bivariate data

10F Start thinking!

1 quite strong and negative

2 One possible answer is shown.

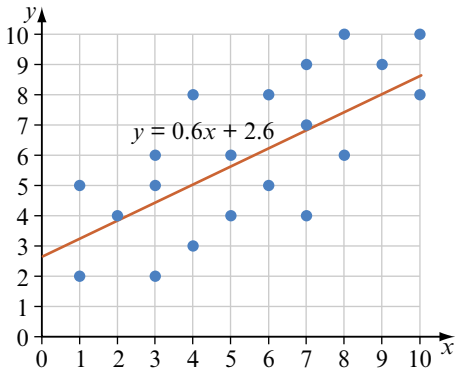


3 $y = -1.1x + 220$ 4 110

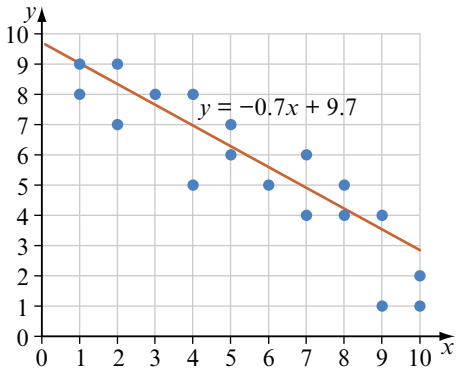
9 Position of line of best fit is such that points above the line are balanced by points below the line in terms of distance from the line.

Exercise 10F Interpreting bivariate data

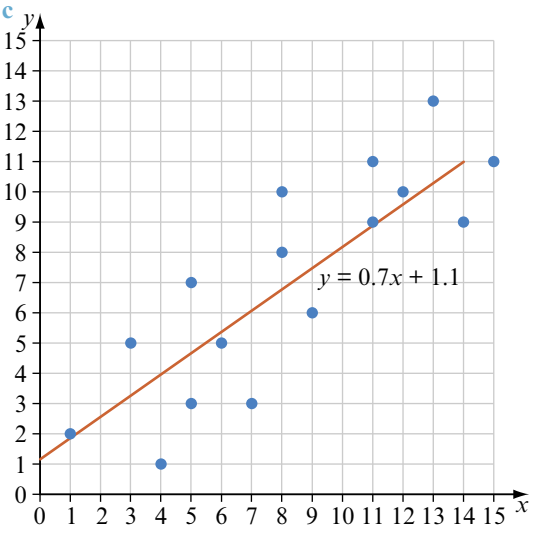
1 a



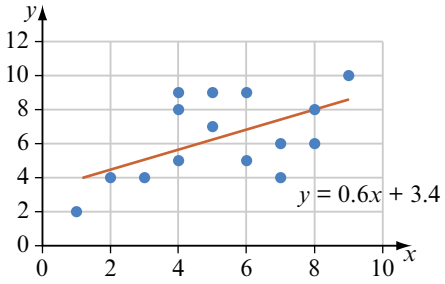
b

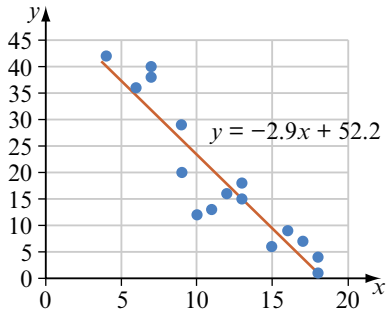


c



d



e


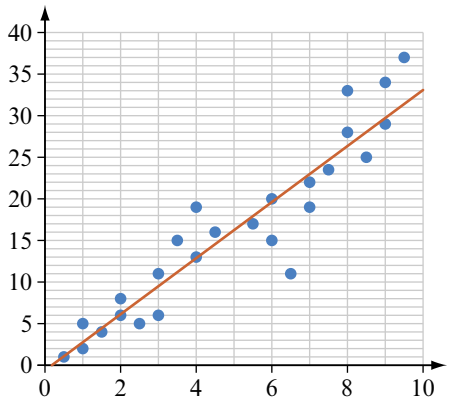
2 a 112.9 b −1.544 c 320.068 d −249.22

3 a 3.7 b 10.7 c 38 d 64

4 a 8.6 b 2.7 c 8.1 d 9.4

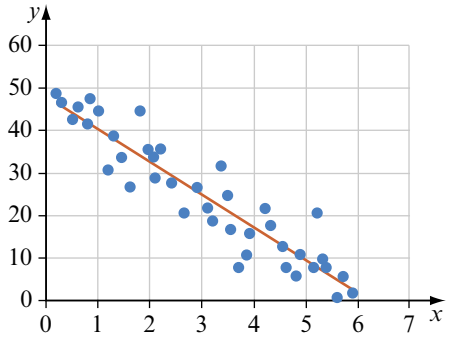
e 23.2

5 a i $y = 3.4x - 0.8$



ii 16.2

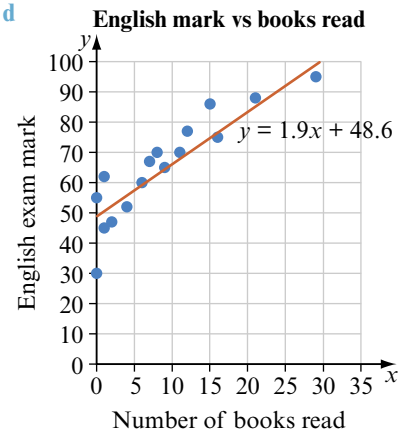
b i $y = -7.8x + 48.5$



ii 9.5

6 a strong, positive relationship

b independent variable: number of books read, dependent variable: English exam mark

c, d


e Data predicts about 11 books, but this does not guarantee an English exam mark of 70%.

7 b If there is one extra point, it is included in middle section; for two extra points, put one in each outer section.

e $y = 0.5x + 3.5$

f $y = 0.5x + 3.7$; equation is quite close.

g Gradient of line is formed from two outer median points. Line joining these points is moved one-third the way towards middle median point, but its gradient remains the same. This means gradient is calculated using $m = \frac{y_U - y_L}{x_U - x_L}$.

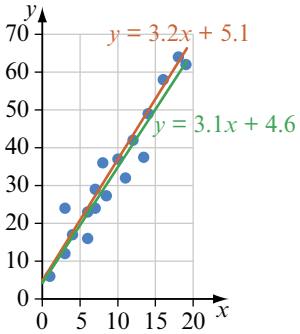
$c = \frac{1}{3}[(y_L + y_M + y_U) - m(x_L + x_M + x_U)]$

$= \frac{1}{3}$(sum of y-values of three median points) − $\frac{1}{3}m$(sum of x-values of three median points)

= mean y-value − m × mean x-value

This transforms to $y = mx + c$.

h The effect of extreme values is minimised by finding the median value of that particular third.

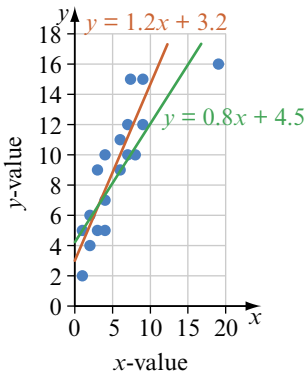
8 a i


ii $y = 3.1x + 4.6$ iii $y = 3.2x + 5.1$

iv 20.1, 21.1

v Values are fairly close. Least-squares regression may be more trustworthy in this case, as relationship between x and y is quite strong, and this calculation takes into account values of all points.

b i

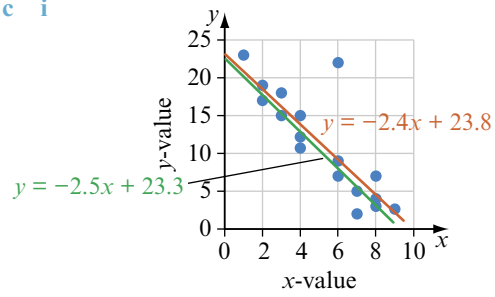


ii $y = 0.8x + 4.5$ **iii** $y = 1.2x + 3.2$

iv 8.5, 9.2

v There is an outlier, and its value greatly affects least-squares calculations. This is not the case with three-median regression calculations, so three-median calculation would be more trustworthy.

c i

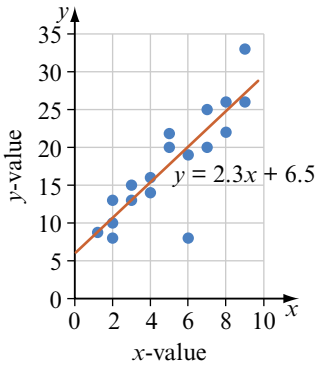


ii $y = -2.4x + 23.8$ **iii** $y = -2.5x + 23.3$

iv 11.8, 10.8

v There is an extreme value, and this greatly affects least-squares calculations. This is not the case with three-median regression calculations, so three-median calculation would be more trustworthy.

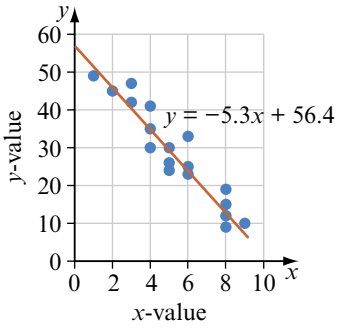
9 a i



ii Three-median regression chosen because of outlier. $y = 2.3x + 6.5$

iii 22.6

b i



ii Least-squares regression chosen because there was no outlier. $y = -5.3x + 56.4$

iii 19.3

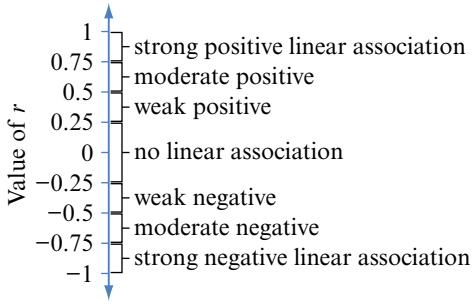
10 a Relationship between the two variables is strong and positive in first graph, but weak and positive in second graph.

b first graph: 15, second graph: 7

c More confidence in first prediction since points lie close to line of best fit.

d It calculates value of strength of relationship between two variables. A greater value of r indicates a stronger relationship.

e



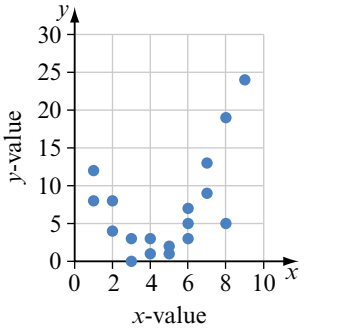
f i 0.5 **ii** −0.9 **iii** −0.5

g scatterplot A: $r \approx 0.98$, scatterplot B: $r \approx 0.69$

11 a Points do not generally lie in straight line.

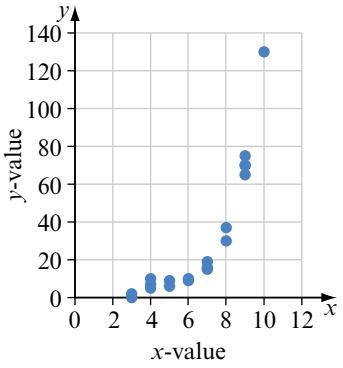
b The distribution of the points fits a logarithmic function.

c i



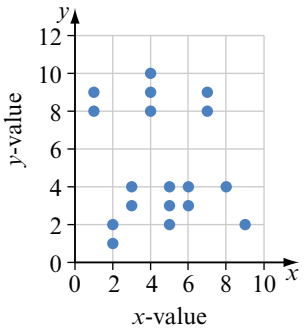
ii quadratic function.

d i



ii cubic function

e i



ii trigonometric function

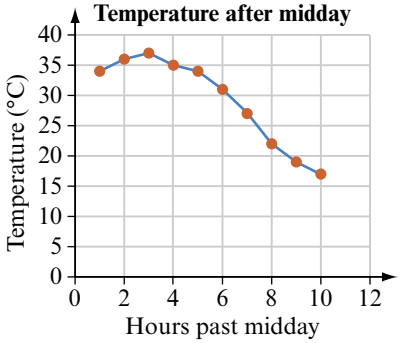
10G Time series

10G Start thinking!

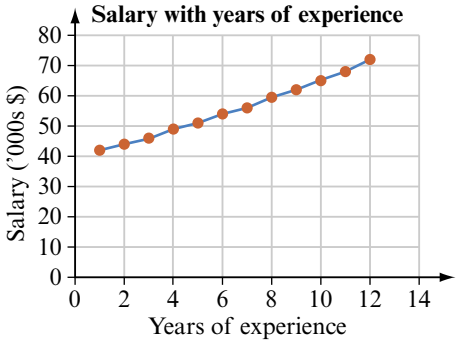
1 Price of camera is decreasing over 6-month period shown in graph.
2 time (months) 3 price ($)
4 Intervals are equal.
5 minutes, hours, days, years
6 stock market daily closing prices, monthly rainfall, maximum or minimum daily temperatures
7 Line graph shows movement from one time period to next.
8 a Non-linear, decreasing trend: points do not lie on straight line and are decreasing at a greater rate over time.
b Non-linear, seasonal trend: points don't all fall on straight line, but they increase and decrease in a regular pattern over time.
c Linear, increasing trend: points lie close to straight line, which is increasing over time
9 Very few occurrences over time have a perfect linear pattern.
10 Cycles appear to be moving upwards over time.
11 First series displays downward trend, while third series displays upward trend.
12 time
13 Over time, value of dependent variable decreases.

Exercise 10G Time series

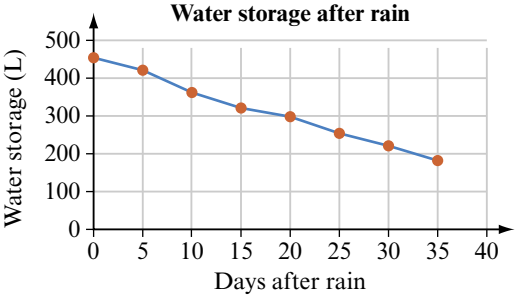
1 a



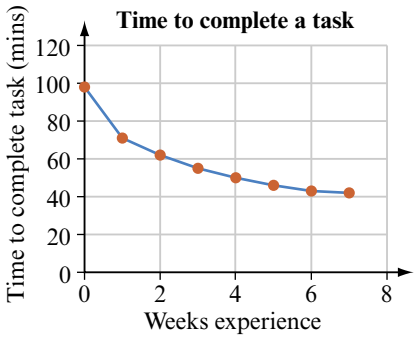
b



c

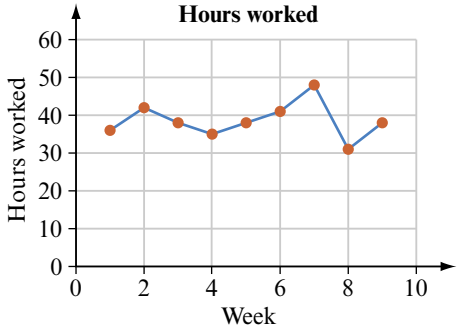


d



e

f



2 a **i** linear **ii** downward
b **i** seasonal **ii** stationary
c **i** non-linear **ii** upwards
d **i** linear **ii** upwards
e **i** non-linear **ii** downward
f **i** seasonal **ii** downward

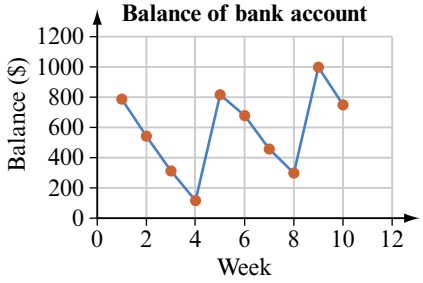
3 a Trend is non-linear and appears seasonal.
b Trend is linear and increasing over time.
c Trend is linear and decreasing over time.
d Trend is non-linear and decreasing over time.
e Trend is roughly linear and increasing over time.
f Appears to be no trend in relationship between week and hours worked.

4 a Price of milk per litre increases with roughly linear trend as time increases.
b Temperature of child decreases as time increases with roughly linear trend, with random fluctuations.
c Weight of baby increases with non-linear trend as time increases.
d Price of petrol shows seasonal trend which appears to be increasing as time increases.

5 a As time increases, height of candle decreases in linear fashion.
b Throughout the year, number of people at beach decreases with approach of winter, then increases as summer approaches; it reaches its minimum in middle of winter and maximum in middle of summer.
c Over 50 years, population of an outer city suburb has increased, with population growth being greater in later years.
d Over the 10 hours of trip, distance travelled increases steadily over time.
e Over the 8 years the shark population is monitored, it decreased with a non-linear trend.
f Over the 3 years sales were recorded, they appear to decrease with a seasonal trend.

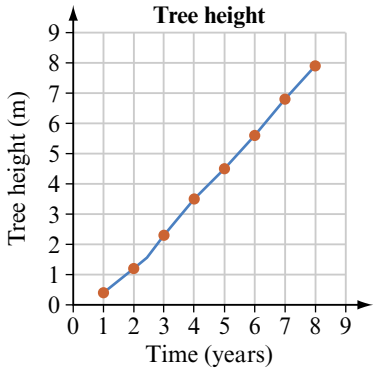
6 a Temperature increases slightly until it reaches a peak at 3 pm, then decreases steadily until 10 pm.
b With increasing years of experience, salary increases steadily.
c As days after rain increases, water storage decreases steadily.
d Time to complete a task decreases in a non-linear trend as time increases.
e Sales increase with a roughly linear trend as time increases.
f There does not appear to be a trend between week and hours worked

7 a



The series is seasonal with an upward trend.
b **i** $116 **ii** $998
c Increases in bank balance could be caused by periodic crediting of a work payment into account. Downward movement is probably caused by living expenses being withdrawn from account.

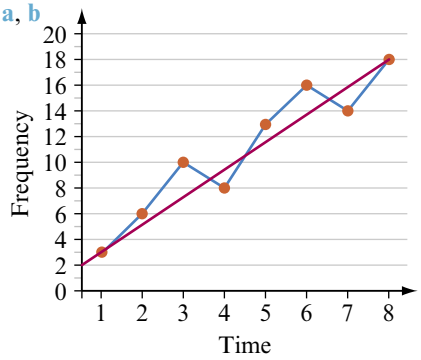
8 a



b Linear, increasing: as tree grows older (increasing time), it grows taller.
c **i** about 10 m
ii If continues to grow at same rate, tree should be about 21 m tall.
d Both of these readings are extrapolated values. In the first case, reading is just beyond recorded data (which makes prediction reasonably likely to be accurate), while predicted value in second case is a great distance from recorded values (making it very unreliable). The tree could, in fact, die before it reached 20 years of age!

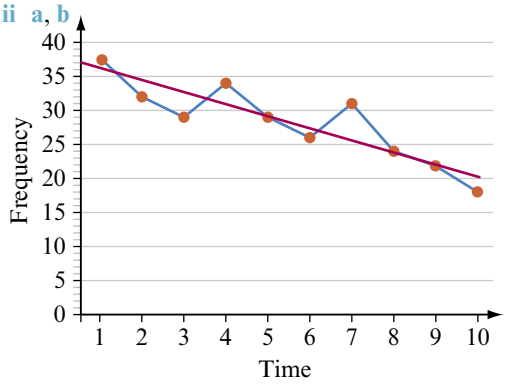
9 a Graph shows number of yearly enrolments in two courses, Italian and Spanish, over a period of 8 years.
b Trend shows course enrolments in Italian have fluctuated, increasing and decreasing over years, but general trend is for decrease. Enrolments in Spanish remained fairly stationary for first 5 years, then started to increase quite rapidly.
c If trends in both graphs continue, Spanish will have more enrolments in 2 years' time (at 10-year mark).

10 **i** **a, b**



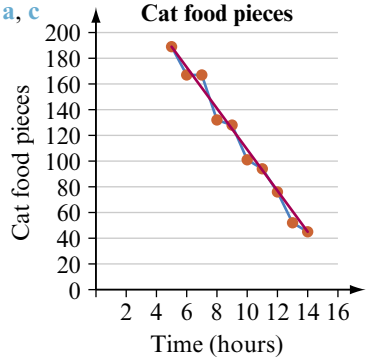
c **i** 12 **ii** 32

ii **a, b**



c **i** 29 **ii** 12

11 **a, c**



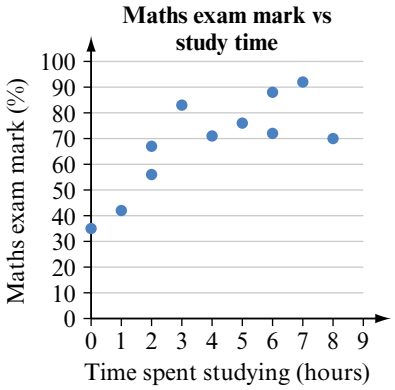
b Trend is linear, decreasing: as time increases, number of cat food pieces decreases.

d **i** 270 **ii** 16.6 hours

13 **a** Data represents a number of different individuals rather than changes in the performance of a single individual.

b It is not a time series, as time is not in equal intervals.

c



The trend is linear, increasing. As more time is spent studying, the Maths mark increases.

d A time series follows data of a particular category over time. These data are not continuous observations over a period of time.

10H Analysing reported statistics

10H Start thinking!

1 Some possible answers are: sample may have been selectively picked, unfavourable results may have been disregarded.

2 **a** Primary data are collected personally, while secondary data have been collected by someone else.

b A census is a questionnaire conducted on the whole population; a survey is conducted on a sample of a population.

c Stratified sampling is selecting a sample which has the same proportion of characteristics as the population. This generally ensures results reflect the population. One disadvantage is selection may not be truly random.
Systematic sampling is selecting according to some system, such as every 10th person. This does not always produce a sample which is representative of the population.
A random sample is one in which every item of the population has an equal chance of being chosen. This is considered the most satisfactory method of selecting a sample.

d Bias occurs when results of a survey are influenced by outside factors, such as a poorly-chosen sample, poor questionnaire design or invalid interpretation of results.

e Number chosen for a sample depends on number in population: a larger population generally requires a larger sample.

3 Questionnaire could list various music genres, with result showing which is most popular.

4 Generally a good sample size is approximately $\sqrt{\text{population size}}$.

5 **a** One possible answer is: Which of the following music genres do you prefer? ……..

b Wouldn't you agree that …… is the most popular music genre for our age group?

c Second question is suggesting an answer, whereas first one gives a free choice.

6 a A stratified sample, selected randomly, could be conducted on an appropriate number of people, and results then collated.

c Sample selected from the school wouldn't represent views of those in community outside school.

Exercise 10H Analysing reported statistics

1 a Biased: people at a football match are not an appropriate sample of whole country.

b Fair: this should provide answer representative of views of all people in school.

c Biased: question is not fair.

d Biased: sample of 20 is too small

e Biased: people outside community would probably not have appreciation of particular community issue, so their views could skew results.

f Biased: small sample of one section of population is not likely to be representative of entire population.

2 a Question is inappropriate and there is no indication of how people were chosen for survey.

b People in an Apple store are probably Apple supporters, so this is a biased sample.

c Sample is too small, and there is no indication of makeup of sample.

d Shareholders in a company have a vested interest in seeing a product succeed.

3 a i Data collection method is not fair; Tess only surveys her town, when population is entirety of NSW.

ii Interpretation is biased because sample is biased.

iii Tess should survey people across all of NSW rather than just in her town.

b i Sample is too small for any results to be reliable.

ii Seeing sample is so small, median would be best result to take.

iii Use a larger sample, and watch out for outliers.

c i Question only asks if they like Macs, not if they prefer them, so it's not a fair collection method.

ii Because question is biased no assumption can be made.

iii Question could be, 'Which brand of computer do you prefer?'

d i Data collection method is fair.

ii Poor interpretation: do not need to be above 50% to be the most popular.

iii She should conclude Coke is the most popular.

e i Sample only consists of people from the mall, so this could lead to biased results.

ii Poor method collection means interpretation could be biased.

iii Sample people from various locations to be representative of population. May also be better to use median as measure of centre.

f i Method is biased as his neighbourhood contains not only a small section of the population, but people that do not belong in the population.

ii Interpretation is not fair as sample is biased.

iii Sample in various locations and only people that belong in the population.

4 a One person made this claim. Most of the claims are far less than 18 hours, but since one person said 18 hours, the claim is 'up to 18 hours'.

b mean: 3.1, median: 2, mode: 2

c This claim is not too far from the truth. However, median or the mode would be a better representative value.

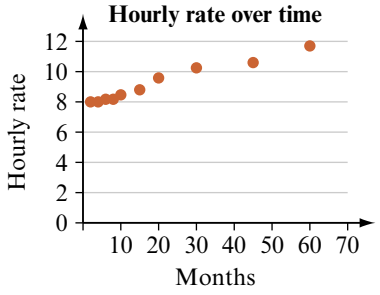
5 How was the data collected? How big was the sample? Was a survey or a census conducted? How was the questionnaire designed?

6 Data are manipulated to achieve desired outcome.

7 a Hourly overtime rate Fatima was paid over 60 months.

b Scale on x-axis is not uniform.

c



8 a If a phone is in stand-by mode, battery could last up to 7 days. But in reality, a phone would not be in stand-by mode for 7 days!

b These 42 people could have been specially chosen by interviewing those who purchased the toilet paper at a supermarket.

c This moisturising lotion was not compared with another moisturising lotion, but with soap.

9

Country	Number of refugees hosted per million population (2012)
Australia	1367
Germany	7191
Jordan	50 451
Pakistan	9416

Of these four countries, Australia hosts by far the lowest number of refugees.

10 a i 16th ii 11th

b Statistics were only provided for 26 countries.

c Which 15 countries were they?

d 9.2 deaths/100 000 population. Data not supplied for Italy, Luxembourg, Slovakia and Turkey. Also, OECD countries Korea and Mexico not included in list.

e Not possible to know where Australia sits with regard to these missing countries.

f More up-to-date figures could be sourced.

10 Chapter review

MULTIPLE-CHOICE

1 D **2** B **3** C **4** B **5** B
6 C **7** B **8** B

SHORT ANSWER

1 a 19.5 **b** 20–<30 **c** 20–<30

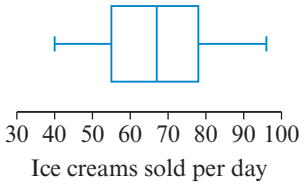
2 a 56 **b** 67 **c** 23

d $Q_0 = 40$, $Q_1 = 55$, $Q_2 = 67$, $Q_3 = 78$, $Q_4 = 96$

3 a Lily: **i** 72 **ii** 3.69
Cassy: **i** 71 **ii** 18.28

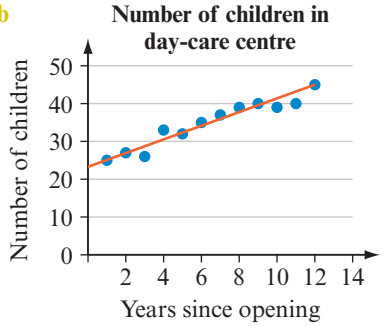
b The lower the standard deviation, the closer the scores lie to the mean. A low standard deviation indicates consistency, so Lily has performed more consistently than Cassy. This is also obvious from the raw scores, where it can be seen that Lily's scores are grouped closer together than Cassy's.

4



Box plot shows that 50% of the time 55–78 ice creams are sold each day, and at most, 67 ice creams are sold half the time.

5 a, b



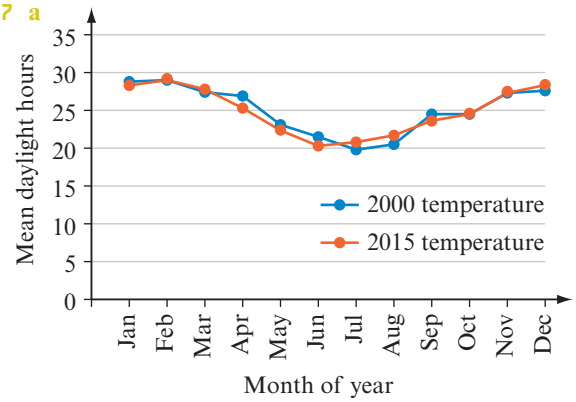
b about 58

c Predictions made using line of best fit assume that trend will continue in same way indefinitely. This is not always the case. The day care centre could, in fact, close within that time.

6 a, b $y = 1.7x + 23.8$

c About 58: answers are all quite close. The most reliable one is probably the linear regression equation.

7 a



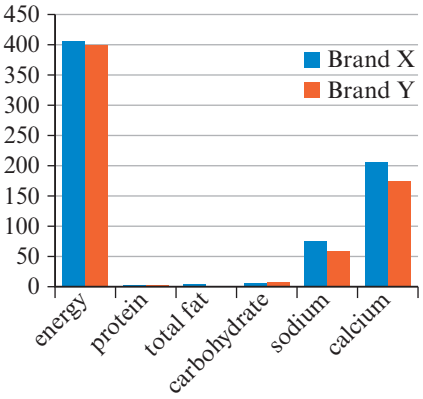
b Trend of both graphs is cyclical.

c Some of the temperatures are a little warmer in 2015 than those in the corresponding months of 2000, and some are a little cooler. Overall, there appears to be not too much change over the years.

8 a Brand X has more energy, protein, total fat, sodium and calcium than Brand Y, while it has less carbohydrate.

b Carbohydrate is the only category Brand X is lower in. It is quite a bit lower in that area than Brand Y. Would have to assume 'light' refers to amount of carbohydrate, but 'light' is more commonly used to mean low fat.

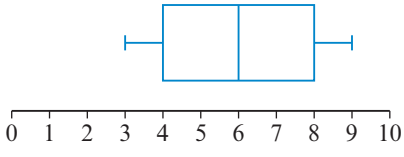
c



Graph is misleading. All figures (regardless of their units) are graphed on same axes so total fat content appears to be almost zero.

MIXED PRACTICE

1 C **2** B **3** B
4 3, 3, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9
5 6 **6** 4
7 $Q_0 = 3$, $Q_1 = 4$, $Q_2 = 6$, $Q_3 = 8$, $Q_4 = 9$
8

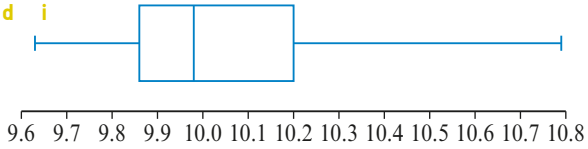


9 D **10** C **11** B
12 C **13** time **14** cyclical

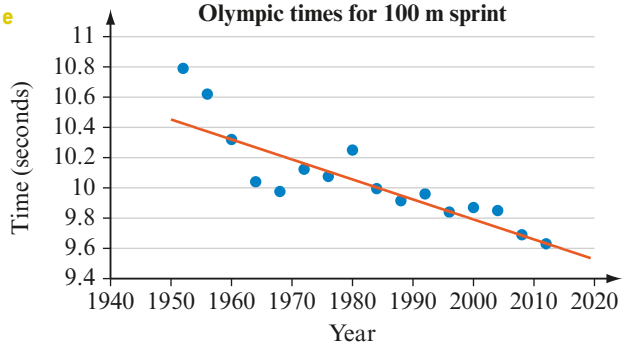
15 A time series graph records observations over regular time intervals of one particular event, whereas the time intervals in a scatterplot need not be regular, and could also involve observations of different events over time.
16 198
17 manipulate the scales in a graph, graph different units on the same graph, omit particular data
18 systematic 19 0.76 20 2.0
21 B 22 D 23 $y = 2.3x + 0.1$
24 0.66

ANALYSIS

a Over years, time for the event has generally decreased, although there has not always been a decrease from year to year.
b i 10.06 s, 9.98 s, 10.06 s
ii 1.16 s, 0.34 s
iii $Q_0 = 9.63$, $Q_1 = 9.86$, $Q_2 = 9.975$, $Q_3 = 10.195$, $Q_4 = 10.79$
c i 0.31
ii Value is quite low, indicating times are closely packed around mean value.
d i



ii Times are positively skewed, being packed more tightly below median than above median. This indicates it is becoming increasingly difficult to lower the time for the race.
iii Lower half of box plot will shrink in size as times become more densely packed in that region.
e



f $y = -0.014x + 38.3$
g three-median regression: $y = -0.011x + 31.7$
h They are slightly different, probably as result of time differences being very small, and measurements made to two decimal places.
i −0.87; there is a strong negative correlation between year and race times.
j If trend continues, perhaps could see 9.5-second barrier broken in 2024 Olympic Games. Only time will tell!

10 Connect

For feedback on this open-ended task, see your teacher.

CHAPTER 11 PROBABILITY

11 Are you ready?

1 B 2 B
3 a D b HH, HT, TH, TT
4 a C b C c B
5 a 18 b 16 c 10 d 4
6 a 18 b 5 c 38
7 a 65 b 62 c 18 d 9 e 127

11A Review of theoretical probability

11A Start thinking!

1 26
2 a 21 b 5 3 a $\frac{21}{26}$ b $\frac{5}{26}$
4 There are more consonants than vowels.
5 a i 0.81 ii 81% b i 0.19 ii 19%
6 The sum of both events makes one whole or 100%.
7 Pr(C) + Pr(V) = 1; together they cover all possible outcomes.
8 Still have same number of letters so answers to question 2 will be the same. Total number of items has increased, so this will change probabilities in question 3.
9 No. Drawing out a number is now also possible.
10 a $\frac{21}{36}$ b $\frac{5}{36}$ c $\frac{10}{36}$
11 $\frac{21}{36} + \frac{5}{36} + \frac{10}{36} = \frac{36}{36} = 1$

Exercise 11A Review of theoretical probability

1 a 11 b {M, A, T, H, E, I, C, S}
c seven consonants, not all different
d four vowels, not all different
2 a $\frac{7}{11}$ b $\frac{4}{11}$ c $\frac{7}{11}$
3 Parts a and b are complementary probabilities. 'Not selecting a vowel' is the complement of 'selecting a vowel', and is the same as 'selecting a consonant'.
4 a {1, 2, 3, 4, 5, 6, 7, 8}
b {A, U, S, T, R, L, I}
c {red, black} d {red, white, blue}
5 No. Some outcomes have a greater possibility of occurring.
6 a $\frac{5}{6}$ b $\frac{1}{13}$ c $\frac{5}{9}$ d $\frac{3}{5}$
7 a 37 b i $\frac{12}{37}$ ii $\frac{25}{37}$ c 1
d i picking a crane which is not red
ii picking a green crane
iii picking a crane which is not green and not red
e i Pr(not red) = $\frac{32}{37}$, Pr(red) = $\frac{5}{37}$
ii Pr(green) = $\frac{5}{37}$, Pr(not green) = $\frac{32}{37}$
iii Pr(not green or not red) = $\frac{27}{37}$, Pr(green or red) = $\frac{10}{37}$

f In each case sum of probabilities = 1.
g Sum of probability of complementary events = 1.

8 a Since Pr(A) + Pr(A′) = 1,
Pr(A′) = 1 − Pr(A)
b i $\frac{1}{4}$ ii $\frac{3}{4}$ iii $\frac{1}{4} + \frac{3}{4} = 1$

9 a T b F $\left(\frac{7}{13}\right)$ c F $\left(\frac{6}{13}\right)$ d F $\left(\frac{4}{11}\right)$ e T

10 a i {red, pink, purple, blue, yellow, orange}
ii 0 iii $\frac{5}{6}$
b i {red, orange, light green, dark green, blue, purple, pink}
ii $\frac{1}{7}$ iii $\frac{6}{7}$

11 a C b E c A d B e D

12 a $\frac{7}{13}$ b $\frac{6}{13}$ c $\frac{1}{26}$

13 a $\frac{5}{13}$ b $\frac{4}{13}$ c $\frac{2}{39}$

14 10 girls

15 a 28
b Tiles are stacked in order of increasing value of top half-tile. Within each group, value of bottom half-tile increases in order.
c 7
d i $\frac{1}{28}$ ii $\frac{1}{4}$ iii $\frac{3}{4}$ iv $\frac{3}{7}$ v $\frac{9}{14}$ vi $\frac{15}{28}$

16 five red decorations

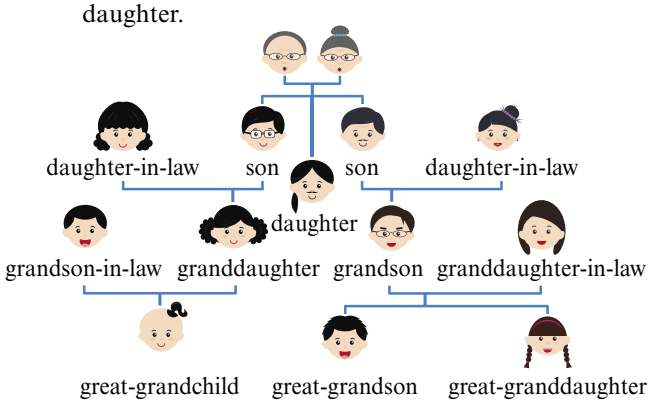
17 a No. There are more red bands than white.
b radii of circles from inner one out:
1 cm, 3 cm, 5 cm, 7 cm, 9 cm
c bullseye = 3.14 cm^2, white 1 = 25.13 cm^2, red 2 = 50.27 cm^2, white 2 = 75.4 cm^2, red 3 = 100.53 cm^2
d i 0.012 ii 0.40

18 a Move 1: move (2) and 3 to the left over 1.
Move 2: move 5 and (6) to the left, into the two vacant spaces.
Move 3: move (6) and (4) to the first and second places on the left.
b Minimum number of moves is 3.

11B Tree diagrams

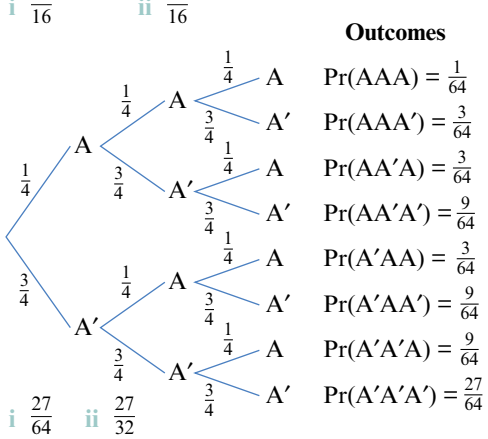
11B Start thinking!

1 First generation couple had two sons who both married and a daughter who did not marry. First son had a daughter who married and had one child. Second son married and had a son and a daughter.



2 a i A = {7, 8} ii A′ = {1, 2, 3, 4, 5, 6}
b A = {7, 8}, so Pr(A) = $\frac{2}{8} = \frac{1}{4}$,
Pr(A′) = 1 − Pr(A) = $1 - \frac{1}{4} = \frac{3}{4}$
c There are four outcomes but only three different possibilities.
d Pr(AA) = $\frac{1}{16}$, Pr(AA′) = $\frac{3}{16}$, Pr(A′A) = $\frac{3}{16}$, Pr(A′A′) = $\frac{9}{16}$
e i $\frac{1}{16}$ ii $\frac{7}{16}$

3 a

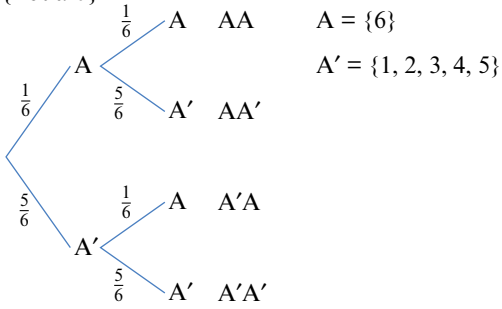


b i $\frac{27}{64}$ ii $\frac{27}{32}$

4 No. Conditions would not be different, so probabilities would be the same.

Exercise 11B Tree diagrams

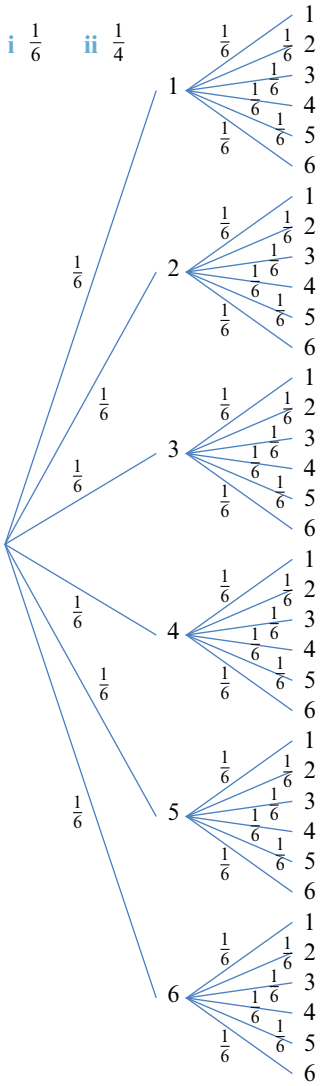
1 a {not a 6}
b



c 4
d No. AA′ and A′A contain same elements.
e i $\frac{1}{36}$ ii $\frac{11}{36}$

2 a Each outcome now has an equal chance of occurring.
b 36

c **i** $\frac{1}{6}$ **ii** $\frac{1}{4}$

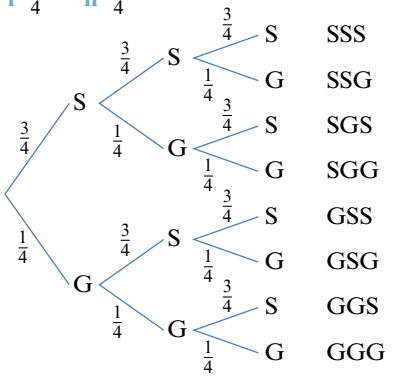


3 **a** {7, 8, 9, 10} **b** $\frac{2}{5}$ **c** $\frac{3}{5}$ **d** 40

4 **a** $\frac{3}{8}$ **b** $\frac{7}{8}$ **c** $\frac{1}{8}$ **d** $\frac{1}{2}$

5 **a** **i** $\frac{3}{4}$ **ii** $\frac{1}{4}$

b



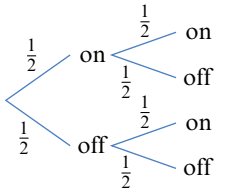
Outcomes
SSS
SSG
SGS
SGG
GSS
GSG
GGS
GGG

c **i** $\frac{27}{64}$ **ii** $\frac{1}{64}$ **iii** $\frac{9}{16}$

6 **a** 0.125 **b** 0.75 **c** 0.5

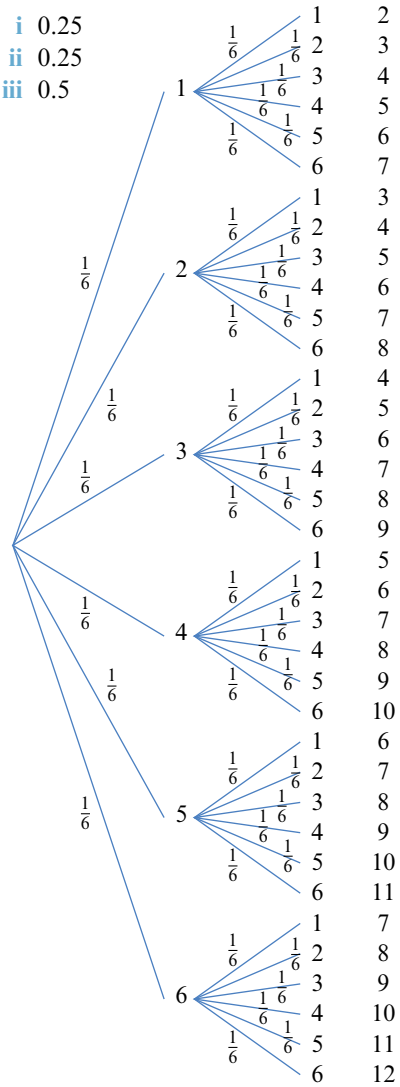
7 **a** 0.25 **b** 0.25 **c** 0.5

8 **a**



b **i** 0.25
ii 0.25
iii 0.5

9 **a**



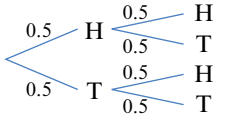
Outcomes
1 2
1 3
1 4
1 5
1 6
1 7
2 3
2 4
2 5
2 6
2 7
2 8
3 4
3 5
3 6
3 7
3 8
3 9
4 5
4 6
4 7
4 8
4 9
4 10
5 6
5 7
5 8
5 9
5 10
5 11
6 7
6 8
6 9
6 10
6 11
6 12

b **i** (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1), 15 outcomes
ii (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1); 6 outcomes
iii (2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6); 15 outcomes

c **i** $\frac{5}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{5}{12}$

d Bet on numbers greater than or less than 7.

10 **a**



Outcomes
HH
HT
TH
TT

b **i** 0.25 **ii** 0.5 **iii** 0.25

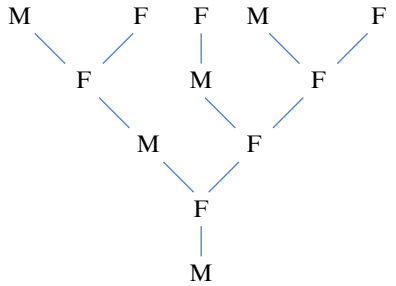
11 

12 **a** No. Outcome of one event is not affected by outcome of previous event.
b $\frac{1}{27}$ **c** $\frac{4}{27}$

13 **a** **i** 36
ii 6, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)
iii $\frac{1}{6}$
b **i** 64
ii 8, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)
iii $\frac{1}{8}$
c $\Pr(n) = \frac{1}{n}$

14 **a** 2 **b** 7
c **i** (6, 1),(5, 2), (4, 3), (3, 4), (2, 5), (1, 6)
ii 3
d **a** 1 **b** 4
c **i** (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)
ii 3

15 Answers will vary. One suggested response is: number the people in the chart, state the number of levels within the organisation, who is responsible for which employees, and number of people in each team.

16 

11C Experiments with and without replacement

11C Start thinking!

1 **a**

	Outcomes
1	11
2	12
3	13
4	14
5	15
1	21
2	22
3	23
4	24
5	25
1	31
2	32
3	33
4	34
5	35
1	41
2	42
3	43
4	44
5	45
1	51
2	52
3	53
4	54
5	55

b 25 **c** 5
d **i** $\frac{1}{5}$ **ii** $\frac{4}{5}$ **iii** $\frac{2}{5}$

2 **a** 14, 15, 21, 23, 24, 25, 31, 32, 34, 35, 41, 42, 43, 45, 51, 52, 53, 54
b 20
c Number of outcomes with repeated digits is 5. When remove this possibility, outcomes reduce by this total amount. So total number of possible outcomes becomes 20.
d none
e **i** 0 **ii** 1 **iii** $\frac{1}{2}$

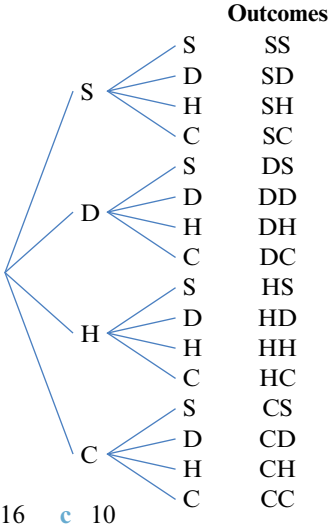
3 **a** When you did not replace the outcomes in experiment 1, it reduced the possibility of future outcomes by 5.
b Removing one of the outcomes eliminates the possibility of it occurring again, so reducing the probability.

Exercise 11C Experiments with and without replacement

1 **a** $\frac{466}{676}$ **b** $\frac{44}{65}$

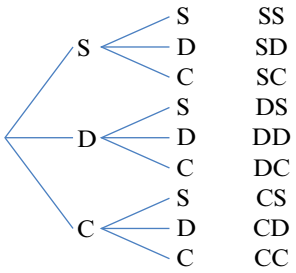
2 **a** $\frac{5}{18}$ **b** $\frac{4}{9}$ **c** $\frac{1}{2}$

3 **a**



b 16 **c** 10

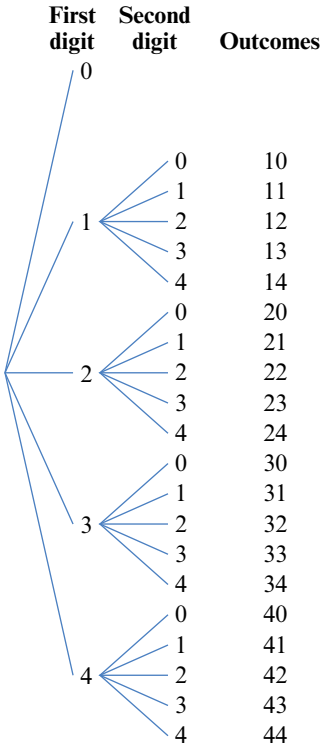
4 **a**



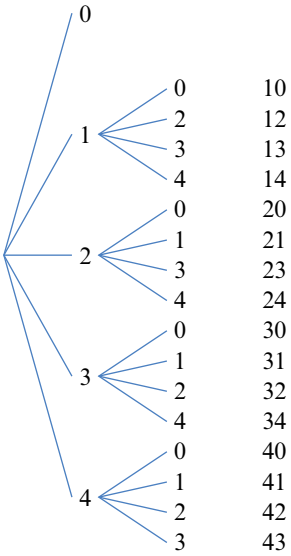
b 9 outcomes **c** 6

5 **a** $\frac{1}{3}$ **b** $\frac{8}{15}$ **c** $\frac{3}{5}$

6 **a** 20 outcomes



b 16 outcomes



c When digits cannot be repeated, possible number of outcomes decreases. This changes the probability of a particular outcome. For example, Pr(12 with replacement) = $\frac{1}{20}$, while Pr(12 without replacement) = $\frac{1}{16}$.

7 **a** same answer of 25 **b** same answer of 20

c **i** 125 **ii** 60

d **i** $4 \times 5 \times 5 = 100$ **ii** $4 \times 4 \times 3 = 48$

8 **a** **i** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

ii 10 **iii** 1 000 000

b **i** 17 576 000

ii You can form more possibilities with this system, as alphabet has 26 letters, as opposed to the 10 digits.

9 **a** 32

b HHHHH, HHHHT, HHHTH, HHHTT, HHTHH, HHTHT, HHTTH, HTHHH, HTHHT, HTTHH, HTHTH, THHHH, THHHT, THHTH, THTHH, TTHHH

10 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{8}$

11 **a** 0.109 **b** 0.509 **12** **a** 216 **b** 96

13 **a** $\frac{1}{4}$ **b** $\frac{23}{87}$ **14** 7

15 **a** **i** $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{7}{36}$

b **i** an odd/even number

ii a prime number

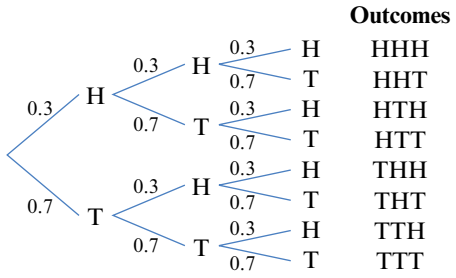
16 **a** 16 **b** 9

c **i** $\frac{1}{4}$ **ii** $\frac{3}{4}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{4}$

17 **a** No. Pr(double on four-sided die) = $\frac{1}{4}$, while Pr(double on six-sided die) = $\frac{1}{6}$. So this rule is not fair.

b Yes. Pr(odd, odd on four-sided die) = $\frac{2}{4} \times \frac{2}{4} = \frac{1}{4}$, while Pr(odd, odd on six-sided die) = $\frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$. So, this rule is fair.

18 a



b i TTT ii HHH

c i 0.441 ii 0.189 iii 0.216

19 a 0.04 **b** 0.03 **c** 0.308

11D Independent and dependent events

11D Start thinking!

1 a 1 **b** $\frac{1}{10}$

c No. They are from different containers.

d Outcome of selecting a pencil from container two is not dependent on selecting pencil from container one.

2 a i $\frac{1}{3}$ ii $\frac{1}{15}$

iii No, not if pencil is replaced.

iv independent

b i $\frac{2}{29}$ iii yes

iv Dependent; first selection will change probability of following outcomes.

Exercise 11D Independent and dependent events

1 a I **b** I **c** I **d** D **e** I **f** D

2 a Each outcome is not dependent on the one before.

b i $\frac{1}{8}$ ii $\frac{3}{8}$ iii $\frac{1}{2}$ iv $\frac{7}{8}$

3 a Coins weren't replaced, so each outcome will be dependent on one before.

b i $\frac{3}{10}$ ii $\frac{1}{120}$

iii $\frac{21}{120} = \frac{7}{40}$ iv $\frac{63}{120} = \frac{21}{40}$

4 a $\frac{5}{36}$ **b** $\frac{1}{4}$ **c** $\frac{1}{4}$ **d** $\frac{3}{4}$

5 a $\frac{1}{13}$ **b** $\frac{3}{13}$ **c** $\frac{10}{13}$

6 a i $\frac{9}{29}$ ii $\frac{2}{29}$ iii $\frac{3}{29}$ iv $\frac{3}{29}$ v $\frac{1}{435}$ vi $\frac{4}{435}$

b i $\frac{2}{87}$ ii $\frac{28}{87}$ iii $\frac{4}{435}$

c Undertaking the events in reverse order does not always give same answer. Depends on type of event.

7 a The larger the range of numbers used, the lower your chance of winning.

b i $\frac{1}{20}$ ii $\frac{1}{20}$ iii $\frac{1}{400}$

c i $\frac{1}{100}$ ii $\frac{1}{100}$ iii $\frac{1}{10\,000}$

d The greater the range of numbers, the smaller the chance of matching the number.

e The events are independent.

8 b i $\frac{1}{1000}$ ii $\frac{1}{10}$ iii $\frac{1}{100}$

c i 1, 1, 5: 5, 1, 1: 1, 5, 1

ii 3 iii $\frac{27}{1000}$ iv $\frac{27}{1000}$

d Chance of winning a prize is quite low.

9 a $\frac{3}{8}$ **b** $\frac{25}{247}$ **c** no **d** $\frac{225}{988}$ **e** $\frac{42}{185}$

10 a $\frac{7}{10}$ or 0.7

b i 0.343 ii 0.441 iii 0.189

11 a $\frac{1}{6}$ **b** $\frac{1}{4}$ **c** i 1 ii 24 iii $\frac{5}{24}$

12 nine games

13 a Meg knows only that her hat is one of the remaining hats, and there are one red and two blue hats left.

b Since Meg saw one red hat, while Nancy sees two blue hats, Nancy knows hers must be the red hat.

c Peta hears Nancy say that she sees two blue hats, so hers must be blue.

d Meg has a blue hat, Nancy has a red hat, and Peta has a blue hat.

11E Conditional probability with two-way tables and tree diagrams

11E Start thinking!

1 a There is 1 ace of hearts.

b 12 cards are hearts but not aces.

c 3 cards are aces but not hearts.

d 36 cards are neither hearts nor aces.

2 a {H′ and A} or {A and H′}

b {H′ and A′}

3 a $\frac{1}{4}$ **b** $\frac{1}{13}$

4 a ace (A) **b** not heart (H′)

c not ace (A′)

5 a 25% **b** 75%

Exercise 11E Conditional probability with two-way tables and tree diagrams

1 a 80 **b** 10 **c** 15 **d** 10 **e** 30

2 a 22% **b** 67% **c** 67% **d** 29%

3 a 92% **b** 92%

4 a 100 **b** 22 **c** 14 **d** 27 **e** 4

5 a 37% **b** 45% **c** 18% **d** 59% **e** 31%

6 Calculate missing entries in rows and columns by adding or subtracting existing entries, as appropriate.

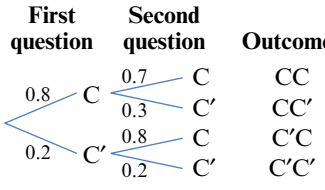
	Primary school	High school	Total
Catch public transport	22	18	40
Driven	37	11	48
Walk/ride	5	9	14
Total	64	38	102

7 a $\frac{15}{20}$ means there are 15 soft centres out of a total of 20 chocolates, while $\frac{5}{20}$ means 5 hard centres out of the 20.

b After 1 soft centre has been eaten, there are 14 left out of the remaining 19, and 5 hard centres left out of 19. After 1 hard centre has been eaten, there are 15 soft centres left out of the remaining 19, and 4 hard centres left out of 19.

c $\frac{4}{19}$ **d** $\frac{5}{19}$ **e** $\frac{15}{19}$

8 a



b i 0.7 ii 0.8 iii 0.3

9 a

	Male	Female	Total
Full-time employees	30	4	34
Part-time employees	5	11	16
Total	35	15	50

b i 4 ii 16 iii 35

c i 14% ii 31% iii 12% iv 27%

d i In part i, the selection is made from males (total of 35), whereas in part ii, the selection is made from part-time employees (total of 16).

ii In part iii, the selection is made from full time employees (total of 34), whereas in part iv, the selection is made from female employees (total of 15).

10 a i 48% ii 95% iii 95% iv 47%

b 52% of the unemployed were male ($\frac{140}{270} \times 100\%$), whereas 5% of the males were unemployed ($\frac{140}{2840} \times 100\%$). The resident has the claim incorrect.

11 a

	Small	Large	Total
Silver	27	35	62
Coloured	18	10	28
Total	45	45	90

b i T, 50% ii T, 31% iii F, 60%
iv F, 36% v T, 69%

12 0.78

13 a

	Year 7	Year 8	Year 9	Year 10	Total
Girls	37	32	38	40	147
Boys	45	47	40	45	177
Total	82	79	78	85	324

b i 25% ii 49% iii 59% iv 53% v 52%

c i No. Totals of students in Years 9 and 10 vary, which would change the percentage calculations.

ii Yes. The condition is different from the previous question. Both numbers are out of the total number of boys, which has not changed.

14 6

	Boys	Girls	Total
Left handed	5	6	11
Right handed	30	19	49
Total	35	25	60

15 a 0.208 b 0.208

11F Conditional probability and Venn diagrams

11F Start thinking!

1 i 8 ii 3 iii $\frac{3}{8}$ iv $\frac{3}{5}$

2 a i 1 ii 0 iii 0 iv 0

b The two events are mutually exclusive. No elements occur in B given A or vice versa.

3 a i 2 ii 2 iii 1 iv $\frac{2}{3}$

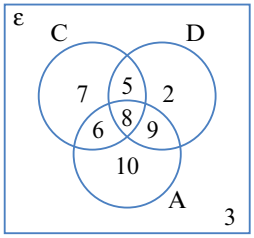
b B is a complete subset of A, so all the elements that occur in B also occur in A, making it one whole.

Exercise 11F Conditional probability and Venn diagrams

1 a i 13 ii 14 iii 12 iv 23 v 13

b i $\frac{2}{7}$ ii $\frac{4}{13}$ iii $\frac{5}{14}$ iv $\frac{5}{12}$ v $\frac{2}{3}$ vi $\frac{5}{13}$

2



3 a $\frac{13}{25}$ **b** $\frac{1}{2}$ **c** $\frac{11}{24}$ **d** $\frac{14}{17}$ **e** $\frac{47}{50}$ **f** $\frac{6}{13}$

4 a $\frac{1}{3}$ **b** $\frac{3}{17}$ **c** 0 **d** 0 **e** $\frac{3}{13}$

5 a i 13 ii 19 iii 13 iv 9 v 26 vi 6

b i $\frac{1}{2}$ ii $\frac{1}{2}$ iii $\frac{9}{26}$ iv $\frac{7}{26}$ v $\frac{7}{26}$ vi $\frac{2}{13}$
vii $\frac{3}{5}$ viii $\frac{9}{13}$ ix $\frac{9}{13}$ x $\frac{20}{26}$ xi $\frac{7}{11}$ xii 1

6 a i T; R lies completely within P.

ii T; R lies completely within Q.

iii F; R lies completely within the intersection of P and Q.

iv F; R lies completely within the union of P and Q.

v F; R lies within the universal set.

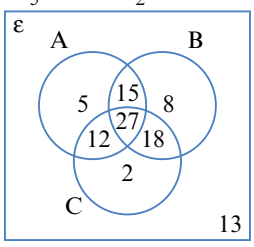
vi T; R lies within Q, which is within the universal set.

b i $\frac{1}{8}$ ii $\frac{1}{3}$ iii $\frac{1}{12}$ iv $\frac{1}{8}$ v $\frac{3}{16}$ vi $\frac{3}{16}$

7 a i 20 ii 12 iii 2 iv 10
v 18 vi 6 vii 0 viii 0

b i $\frac{1}{2}$ ii 1 iii $\frac{3}{10}$ iv 0
v $\frac{2}{5}$ vi $\frac{1}{2}$ vii 0 viii undefined

8 a



b i 85 ii 14 iii 72 iv 45
v 41 vi 15 vii 18 viii 73

9 Our solar system can be considered the universal set, with the orbits of the eight planets within it. With the Sun at the centre of each set, Mercury's orbit forms a subset of the orbit of Venus. The orbit of Venus, in turn, forms a subset of the orbit of the Earth, and so it continues with each planet in order outwards from the Sun (Mars, Jupiter, Saturn, Uranus and Neptune). The orbits do not overlap, so the orbits of the inner planets are complete subsets of the orbits of the outer planets.

10 a

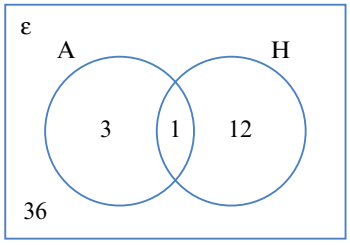
	X	X′	Total
Y	9	6	15
Y′	4	7	11
Total	13	13	26

b i 13 ii 11 iii 9

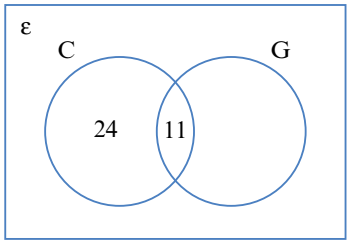
c i $\frac{3}{5}$ ii $\frac{9}{13}$ iii $\frac{7}{11}$ iv $\frac{7}{13}$

11 a Draw two overlapping circles. Start from the inside and work outwards. A ∩ B = 1, so fill this in first. The total number of aces is 4; 1 ace is accounted for in the overlap, so that leaves 3 for the region 'aces only'. There are 13 hearts; 1 heart is accounted for in the overlap, so that leaves 12 for the region 'hearts only'. 1 + 3 + 12 = 16, so this leaves 52 − 16 = 36 for the region outside the circles.

b



13 a



b $\frac{11}{59}$

c $\Pr(C) = \frac{35}{59}$

$\Pr(G|C) = \frac{11}{35}$

$\Pr(G|C) = \frac{\Pr(C \cap G)}{\Pr(C)}$

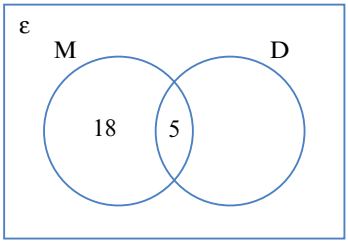
$\frac{11}{35} = \frac{\Pr(C \cap G)}{\frac{35}{59}}$

$\frac{11}{35} \times \frac{35}{59} = \Pr(C \cap G)$

$\frac{11}{59} = \Pr(C \cap G)$

d They give the same answer. Second method is handy when not all information is available.

14 a



b $\frac{5}{41}$

c $\Pr(M) = \frac{23}{41}$

$\Pr(D|M) = \frac{5}{23}$

$\Pr(D|M) = \frac{\Pr(M \cap D)}{\Pr(M)}$

$\frac{5}{23} = \frac{\Pr(M \cap D)}{\frac{23}{41}}$

$\frac{5}{23} \times \frac{23}{41} = \Pr(M \cap D)$

$\frac{5}{41} = \Pr(M \cap D)$

d They give same answer. Second method is handy when not all information is available.

15 P ∩ Q = {2, 5, 9, 13}
R ∩ S′ = {4, 5, 6, 7}
{2, 5, 9, 13} ∪ {4, 5, 6, 7}
= {2, 4, 5, 6, 7, 9, 13}
2 + 4 + 5 + 6 + 7 + 9 + 13 = 46

11G Sampling and reporting

11G Start thinking!

1 Students use data relevant to their own school circumstances.

a √all 7, 8, 9 and 10 students

b Make sure percentage from each level represents their proportion in the whole population. Make sure sample is randomly chosen.

c Take an equivalent percentage of each year level.

e Sample should be randomly chosen, and represent the proportion in the whole population.

f Bias could be introduced by: non-random samples, picking particular students to survey, asking leading questions, disregarding samples which are not in line with a particular opinion.

2 a Use a frequency table.

b Could be presented as a column graph.

c Make sure any display of data is correct; for example, correct scales of graphs.

d No. Undertake another survey with this in mind.

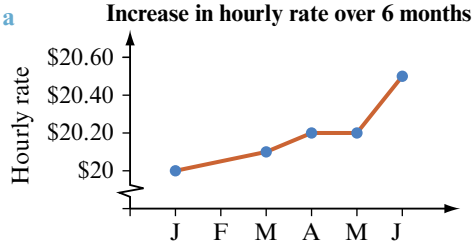
Exercise 11G Sampling and reporting

1 a 498 **b** 22

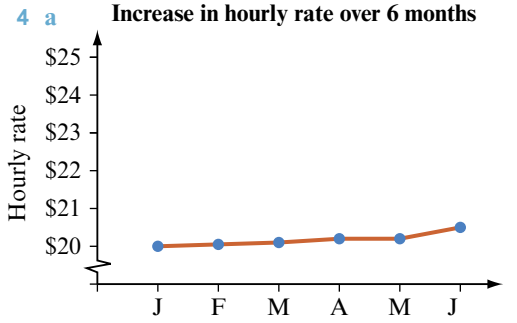
c 7 students from Year 7, 6 from Year 8 and 5 from Years 9 and 10.

d There are a different number of boys and girls from each year level.
e Year 7: 3 boys, 4 girls,
Year 8: 3 boys, 3 girls,
Year 9: 2 boys, 3 girls,
Year 10: 2 boys, 3 girls
f Draw names from a hat, for example.
g Hopefully the sample selection should not be biased.

2 a i People who go to the gym generally exercise more regularly
ii Survey has used results of one gym to represent entire population of Australia. It is extremely biased.
b i Four teenagers on a bus do not represent an entire population.
ii Information such as socio-economic backgrounds, ages, suburbs are not considered.
iii Age of boys is important due to age limit for employment in Australia.
c i Five males don't make an adequate sample to represent an entire population.
ii Males at an hardware store may generally be more handy around the house than other males.
d i Girls in one cheer squad do not represent an entire population.
ii It is not necessary for people to wear sunscreen indoors so it would be a biased result.

3 a


b i increased vertical scale.
ii started scale at zero
iii made both scales uniform
iv added title and labelled axes

4 a


b Scale on vertical axis has been increased far above maximum data value, giving the impression wages have hardly changed.

5 No. Success in school depends on far more variables than just whether or not a student eats breakfast.

6 a i They have provided no evidence to support this claim
ii no sample size given
iii Did all the women surveyed have diabetes, and if so, what type and to what degree?
iv Did they survey women who had coffee at other times of day?
b i They have provided no evidence to support this claim.
ii no sample size given
iii How did they conduct the survey? Depending on the advancement of the disease, can they be sure that their information is reliable?
iv There is no timeline given for this survey. Did they conduct it over many years and monitor the results?
c i Reports always need to follow up on source of survey, how it was conducted, sample size surveyed and group chosen in order to determine accuracy of information.

7 Source of information, sample used and funding for such surveys should be stated with advertised results.

9 Start by giving introductory questions that may provide specific information about the individual, such as age, gender, income bracket, marital status, residence etc. Then streamline remaining questions depending on answers to initial questions.

11 Not necessarily. Sample only includes people who watch the show and decide to call.

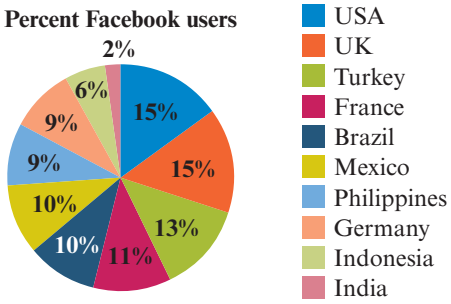
12 a not correct: $\frac{18}{60} \times 100\% = 30\%$
b correct: $\frac{4}{65} \times 100\% = 6\%$
c correct: $\frac{36}{125} \times 100\% = 29\%$
d not quite accurate: $\frac{31}{61} \times 100\% = 51\%$

13 a

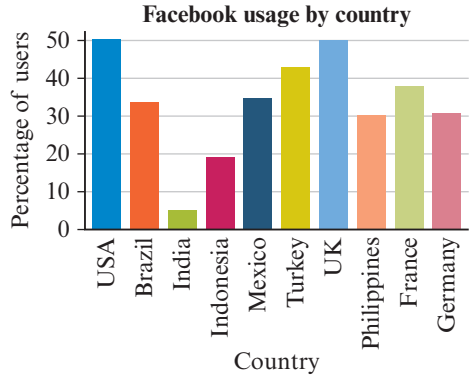
Country	Per cent users
USA	50.3
Brazil	33.8
India	5.1
Indonesia	19.2
Mexico	34.7
Turkey	42.9
UK	50.0
Philippines	30.3
France	37.9
Germany	30.9

b i USA ii India
c Many of their population may not own computers or have access to technology or the Internet.

d i



ii



e i Pie graph is best to show how population percentages of each country compared to other countries in top ten list.

ii Column graph is best because it shows overall population percentages of all countries surveyed.

14 a Only includes ages 14–64. It does not include people younger than 14, and groups ages 65 and over into one category. The age categories are not equal class intervals.

b If this sample truly represents the proportions in the Australian population, there must be more females than males in each age category, and the population appears to be ageing.

c You would need to know what proportion each age group makes of the total population. Figures would be available from the Australian Bureau of Statistics from census figures.

d Not necessarily. This figure represents the sample of 14–19-year-olds surveyed. It may not be precisely this figure but would indicate an averaged figure.

e Out of all people surveyed, 97% chose Facebook as their preferred social media network.

11 Chapter review

MULTIPLE-CHOICE

1 B **2** D **3** B **4** C **5** C
6 B **7** C **8** B **9** C **10** A

SHORT ANSWER

1 a $\frac{4}{15}$ **b** $\frac{11}{15}$ **c** $\frac{2}{3}$ **d** $\frac{7}{15}$

2 a $\frac{9261}{17\,576}$ **b** $\frac{8315}{17\,576}$ **c** $\frac{9261}{17\,576}$ **d** $\frac{8315}{17\,576}$

3 a 900

b You can't use zero as first outcome, as this will produce a two-digit number not a three-digit number.

c $\frac{1}{900}$ **d** $\frac{1}{100}$

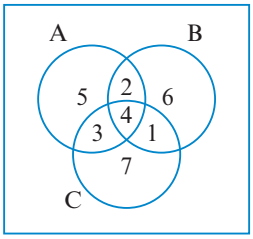
4 a 648 **b** First digit cannot be zero.

c $\frac{1}{648}$ **d** 360

5 a $\frac{4}{55}$ **b** $\frac{1}{11}$ **c** $\frac{17}{55}$ **d** $\frac{38}{55}$

6 a 43% **b** 36% **c** 44% **d** 74%

7 a



b i $\frac{6}{13}$ **ii** $\frac{3}{7}$ **iii** $\frac{7}{13}$ **iv** $\frac{8}{15}$ **v** $\frac{6}{13}$ **vi** $\frac{3}{7}$

8 a 26 **b** 8 females, 18 males

c Random sample of each gender.

MIXED PRACTICE

1 C **2** A **3** $\frac{18}{26}$ **4** 64
5 {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
6 A **7** $\frac{5}{9}$ **8** B **9** 20
10 60 **11** C **12** C **13** $(X \cup Y)'$
14 $\frac{1}{161\,700}$ **15** A **16** B **17** A
18 C **19** $\frac{1}{5}$ **20** $Q \not\subset R$ **21** $\frac{15}{36}$
22 $\frac{7}{15}$ **23** $\frac{1}{9}$ **24** $\frac{2}{9}$ **25** $\frac{2}{3}$

ANALYSIS

1 a A two-way table can only be used for experiments with two trials. In this case, there are three trials.

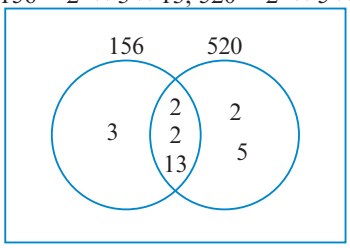
b 648

c Number cannot start with 0, and there can be no repeat digits.

d $\frac{1}{9}$ **e i** $\frac{1}{9}$ **ii** $\frac{1}{72}$ **f** $\frac{1}{8}$ **g** 6

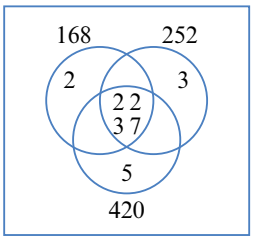
2 a i $156 = 2^2 \times 3 \times 13$, $520 = 2^3 \times 5 \times 13$

ii



iii HCF = $2 \times 2 \times 13 = 52$,
LCM = $3 \times 52 \times 2 \times 5 = 1560$

b **i** $168 = 2^3 \times 3 \times 7$, $252 = 2^2 \times 3^2 \times 7$,
$420 = 2^2 \times 3 \times 5 \times 7$



ii HCF $= 2 \times 2 \times 3 \times 7 = 84$,
LCM $= 2 \times 3 \times 5 \times 84 = 2520$

11 Connect

For feedback on this open-ended task, see your teacher.

Glossary

A

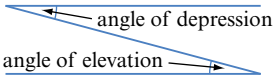
AAA abbreviation for angle-angle-angle. It describes the information needed to draw a triangle and also relates to a condition for similarity. Triangles are similar if the three angles they contain are the same. The condition AAA does not necessarily mean the triangles are congruent (more information is needed).

AAS abbreviation for angle-angle-side. It describes the information needed to draw a triangle and also relates to a condition for congruence. Triangles are congruent if two corresponding angles and a corresponding side are the same.

adjacent side (trigonometry) in a right-angled triangle, the side next to the reference angle that is not the hypotenuse.

alternate angles pair of angles not adjacent to (next to) each other on opposite sides of the transversal between two lines. If the two lines are parallel, the alternate angles are equal.

angle of depression angle between a horizontal line and the line of sight to an object below the horizontal.



annulus region formed between two circles of different radius but with the same centre.

arc the curved part of a sector.

area amount of space enclosed by a two-dimensional (2D) shape. Common units are mm^2, cm^2, m^2, km^2, ha.

asymptote a line which a graph approaches but never reaches.

axioms self-evident or assumed statements used in a proof; for example, the statement that there is 180° around a straight line can be used to prove that vertically opposite angles are equal.

axis of symmetry imaginary line that divides a symmetrical shape or graph so that one side is a reflection of the other.

B

back-bearing if the bearing from point A to point B is known, then the back-bearing is the direction from B to A. The bearing and the back-bearing differ by 180°.

back-to-back stem-and-leaf plot stem-and-leaf plot that displays two sets of data using the one set of stems.

balance method strategy for solving equations by using inverse operations to make equivalent equations. An equation remains balanced (or equivalent) by performing the same operation on both sides of the equation.

base **1** for a value expressed in index form, the base is the number which is repeatedly multiplied. For example, 2^4 has a base of 2 and can be written as $2 \times 2 \times 2 \times 2$.
2 in a 2D shape, it is a nominated side of the shape. The base and height are perpendicular to each other. *See* height for diagram.
3 in a 3D object, it is a nominated face of the object. The base and height are perpendicular to each other.

biased a biased sample is one where the method of collecting data produces a sample that does not accurately reflect the population.

binomial algebraic expression with two terms.

binomial product product of binomial expressions.

bivariate data data that shows the relationship between two variables; for example, height and weight.

box plots also known as box-and-whisker plots, are visual representations of data along a number line where the box represents the middle 50% of ordered data and the whiskers extend from the left of the box to the minimum value and from the right of the box to the maximum value. A box plot shows the five-number summary for the data set. *See* five-number summary.

C

capacity amount of liquid that a three-dimensional object (container) can hold. Common units are mL, L, kL, ML.

census survey of an entire population.

chord line segment between two points on the circumference of a circle.

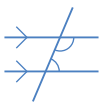
circumference perimeter of a circle.

class intervals groupings of data into equal-sized classes.

clinometer instrument used to measure angle of elevation or inclination.

coefficient number that appears in front of a variable showing that the number and the pronumeral are to be multiplied together. For example, the coefficient of $5m$ is 5.

co-interior angles pair of angles on the same side of the transversal between two lines. If the two lines are parallel, the co-interior angles are supplementary (add to 180°).



column graph graph where the frequency of categorical data is presented in columns.

commission payment to an employee calculated by finding a percentage of sales made.

compass bearing direction from one position to another and described by the number of degrees from north or south towards east or west. For example, S43°E represents a direction that is 43° east of due south.

complement the complement of a set is everything in the universal set that is not in the given set. For example, if the universal set contains whole numbers from 1 to 5 ($\varepsilon = \{1, 2, 3, 4, 5\}$) and set A contains even numbers ($A = \{2, 4\}$) then the complement of A contains odd numbers ($A' = \{1, 3, 5\}$).

complementary angles angles that add to make a right angle (90°).

complementary events two probability events are complementary when their probabilities add to 1.

completing the square method for factorising quadratic trinomial expressions that uses the perfect square rule and the difference of two squares rule.

composite solid object made up of more than one type of basic object.

compound interest interest that is added at regular intervals during an investment or loan period.

compounding periods the regular intervals at which interest is calculated during an investment or loan period.

concave type of polygon with at least one interior angle that is greater than 180°.

conditional probability the probability of an outcome given certain conditions.

cone 3D object that tapers from a circular base to a point.

congruent figures that are identical in shape and size but can be in any position or orientation.

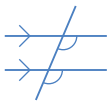
conjugate the conjugate of a binomial surd expression has the opposite operation sign between the two terms; for example, the conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$. The product of these two terms is a rational number.

constant term numerical value in an algebraic expression or equation. It is a term without any pronumeral component. For example, the constant in the expression $3b - 4c + 10$ is 10.

correlation describes the strength and direction of bivariate data displayed as a scatterplot.

correlation coefficient value between −1 and +1, denoted by r, that describes the correlation of a bivariate relationship. The closer r is to −1 or +1, the stronger the correlation. If $r = +1$, there is perfect positive correlation between the variables (the scatterplot of data shows points in a straight line).

corresponding angles pair of angles on the same side of the transversal and in corresponding positions on the two lines (both above a line or both below a line). If the two lines are parallel, the corresponding angles are equal.



cos (cosine) in trigonometry, the cosine of an angle in a right-angled triangle equals the ratio of the length of the adjacent side to the length of the hypotenuse; that is, the length of the adjacent side divided by the length of the hypotenuse.

cosine graph graph of the relationship between an angle and the cosine of the angle.

cosine rule a rule linking three side lengths a, b and c of a triangle ABC and one angle; for example, $a^2 = b^2 + c^2 - 2bc \cos A$.

cross-section a 'slice' cut parallel to the base of a prism that has the same shape and size as the base of the prism.

cube rectangular prism with equal length, width and height.

cumulative frequency running total of frequencies in a frequency table.

cylinder 3D object with a uniform circular cross-section.

D

degrees (of a polynomial) highest power of variable. For example, the polynomial $4x^3 - 2x^2 + 5x + 6$ has degree 3.

degrees–minutes–seconds (DMS) the convention of describing an angle using the three units of degrees, minutes and seconds. Each degree can be divided into 60 minutes and each minute can be divided into 60 seconds.

density mass per unit volume of a substance.

dependent events when the outcome of one probability event affects the outcome of another event.

dependent variable variable that depends on another variable (independent variable) in a relationship. On a Cartesian plane, the dependent variable is shown along the vertical axis.

diameter line segment from one side of a circle to the other through the centre of the circle. Also defined as the length of this line segment.

difference of two squares a pattern observable when expanding a factorised expression, represented as $(a + b)(a - b) = a^2 - b^2$.

dilation transformation that does not produce an identical figure. A dilation produces an enlargement or a reduction of the original figure.

dimensions measurements on a shape or object. For example, the length and width of a rectangle are the dimensions of the rectangle.

discount an amount off the selling price. New selling price = original selling price − discount.

distributive law same answer is obtained whether you add the numbers inside a pair of brackets first before multiplying by another number or multiply each number inside the brackets by the number outside the brackets before adding. For example, $3(2 + 5) = 3 \times 7 = 21$ and $3(2 + 5) = 3 \times 2 + 3 \times 5 = 6 + 15 = 21$. In algebraic terms, the distributive law can be written as $a(b + c) = ab + ac$.

dividend number that is being divided by another number (divisor). For example, in $15 \div 3 = 5$, the dividend is 15.

divisor number that divides into another number. For example, in $15 \div 3 = 5$, the divisor is 3.

domain set of x or θ values in an interval.

E

element member of a set. For example, 5 is an element of $\{1, 3, 5, 7, 9\}$.

elimination method algebraic method used to solve simultaneous equations where the aim is to form a new equation with one of the variables eliminated.

enlargement larger image produced after a figure has been dilated.

entire surd surd in which the entire number is written beneath the root sign. For example, $\sqrt{18}$ is an entire surd but $3\sqrt{2}$ is not.

equally likely when the outcomes of a event have the same probability of occurring; there could be two or more possible outcomes.

equation collection of two or more algebraic terms separated by mathematical operation symbols and containing an equals sign; for example, $2d + 5 = 10$.

equidistant at equal distances from a point.

equilateral describes a triangle with all three sides equal in length. All three angles are also equal in size.

equivalent equations equations that have the same solution. 646

event a specified result that may be obtained in a probability experiment.

exact values values exactly as measured or recorded, not rounded off.

expanded form **1** written as a repeated multiplication. For example, 4^3 written in expanded form is $4 \times 4 \times 4$.
2 an algebraic expression written without brackets (not in factor form).

expansion of a perfect square a pattern observable when expanding a factorised expression, represented as $(a + b)(a + b) = a^2 + 2ab + b^2$.

experiment (probability) where trials are performed to obtain data to predict the chances of an event occurring.

experimental probability probability of an event based upon the results of a probability experiment. Also known as relative frequency.

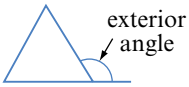
exponent power or index. *See* index (power).

exponential equation equation where the variable is an exponent (power); for example, $2^x = 8$.

exponential relationship relationship between two variables, one of which is an exponent (power); for example, $y = 2^x$.

expression collection of two or more algebraic terms separated by mathematical operation symbols like +, −, × and ÷. An expression does not have an equals sign. For example, $3b - 4c + 10$ is an expression.

exterior angle angle formed outside a triangle where a side has been extended. The exterior angle and the interior angle directly adjacent to it are supplementary (add to 180°).



F

faces flat surfaces of a prism or polyhedron. Each face is a polygon.

factor form expression made up of a product of factors.

factor theorem if a polynomial is divided by an expression and the remainder is 0, then the expression is a factor of the polynomial.

factorising to write an expression in factor form. For example, $6x + 3$ factorises to $3(2x + 1)$ where the two factors are 3 and $2x + 1$.

favourable outcome outcome within the sample space that we want to occur.

five-number summary five pieces of information related to a set of data: minimum value, first quartile, median, third quartile and maximum value.

formula relationship or rule between two or more variables. A formula contains an equals sign; for example, $y = 3x + 7$.

frequency table table organised to show data by recording the frequency of each value in the data set.

G

gradient the slope of a graph.

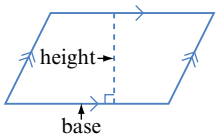
gradient–intercept method used for drawing a line graph, in which the gradient is used to find a second point from the y-intercept.

gross income amount earned before any deductions are made.

H

hectares unit of area equivalent to the area within a square of side length 100 m; that is, 1 ha = 10 000 m^2.

height length measurement from the base to the top or end of a shape or object. The height is perpendicular to the base.



highest common factor (HCF) largest factor that is common to two or more given numbers. For example, factors of 8 are 1, 2, 4, 8 and factors of 12 are 1, 2, 3, 4, 6, 12 so HCF of 8 and 12 is 4.

histogram graph used to display the frequency of grouped continuous data where the horizontal scale boundaries are located at the edges of the columns, rather than the centre.

hyperbola graph of relationship involving the reciprocal of a variable.

hypotenuse the longest side of a right-angled triangle; it is the side opposite the right angle.

I

image shape produced after a transformation.

independent events when the outcome of one probability event does not affect the outcome of another event.

independent variable one of two variables in a relationship. On a Cartesian plane, the independent variable is shown along the horizontal axis. It is listed first in a table of values and in pairs of coordinates.

index (power) for a value expressed in index form, the index (or power) indicates the number of times the base is written as a repeated multiplication. For example, 2^4 has an index of 4 and can be written as $2 \times 2 \times 2 \times 2$. (Plural of index is indices.)

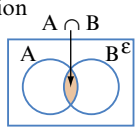
index form (index notation) shorter form of writing a repeated multiplication. A number in index form is written with a base and an index (power). For example, 2^4 is written in index form.

index laws set of laws that make calculations of numbers (or pronumerals) in index form easier to perform.

interest extra payment made for a loan or paid on an investment.

interquartile range measure of spread calculated by subtracting the lower quartile (Q_1) from the upper quartile (Q_3); that is, IQR = $Q_3 - Q_1$. It indicates the spread or range of the middle 50% of the data.

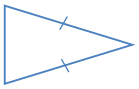
intersection the intersection of two sets ($A \cap B$) on a Venn diagram is where the two sets overlap.



investment money put into a bank, financial institution or other business for which interest is paid to the investor.

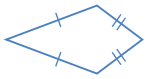
irrational numbers numbers that cannot be written in fractional form; that is, they are non-terminating decimals that are not recurring.

isosceles describes a triangle with two sides equal in length. The angles opposite the equal sides are also equal.



K

kite quadrilateral (four-sided shape) with two pairs of adjacent sides equal in length, one pair of opposite angles equal in size and no parallel sides.



L

leading coefficient coefficient of leading term in a polynomial. For example, the leading coefficient in the polynomial $4x^3 - 2x^2 + 5x + 6$ is 4. *See* leading term.

leading term term in a polynomial with the highest power. For example, the leading term in the polynomial $4x^3 - 2x^2 + 5x + 6$ is $4x^3$.

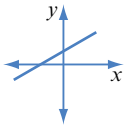
like surds surd terms that have the same number or pronumeral/s under the root sign. For example, $2\sqrt{5}$, $-\sqrt{5}$ and $7\sqrt{5}$ are like surds.

like terms algebraic terms that contain exactly the same pronumerals. For example, $2ab$, $-ab$ and $7ab$ are like terms.

line graph graph that displays the relationship between two variables where points are plotted and joined by a line. Often used to show data collected over time (horizontal axis labelled as time).

linear equations equations that contain a pronumeral term or terms where the highest power is one. For example, $2x + 3 = 6$.

linear graph graph of linear relationship. The graph is a straight line.



linear regression finding a line of best fit to data in a scatterplot so as to model a relationship between two variables.

linear relationship relationship between two variables where the coordinate points describing this relationship lie in a straight line when plotted on the Cartesian plane.

line segment part of a line between two points.

literal equation *see* formula.

loan amount of money given by a bank or other institution to a person for a limited time and on which they must pay interest.

log of a power logarithm of a power term; related logarithm law is $\log_a x^n = n \times \log_a x$.

log of a product logarithm of a product of two terms; related logarithm law is $\log_a(xy) = \log_a x + \log_a y$.

log of a quotient logarithm of a quotient of two terms; related logarithm law is $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

logarithm (log) power, index or exponent. For example, in $2^3 = 8$, the logarithm is 3 and can be written as $\log_2 8 = 3$.

logarithmic form alternate way of writing an relationship that is in index form. For example, $2^3 = 8$ (index form) is equivalent to $\log_2 8 = 3$ (logarithmic form).

loss the amount lost when the selling price is less than the original price.

lower quartile median of first or lower half of ordered data, denoted by Q_1.

lowest common denominator (LCD) lowest common multiple of the denominators in two or more fractions. For example, $\frac{1}{3}$ and $\frac{1}{2}$ have a LCD of 6.

M

mark-up amount added to an original price. For example, an original price has a 25% mark-up added to obtain the selling price.

maximum turning point a point at which a parabola changes direction and at which it has its maximum y value.

mean a measure of the centre of a data set. It is calculated by adding all of the data values together and dividing by the number of values. Also known as the average.

median a measure of the centre of a data set. For data placed in ascending order (smallest to largest), it is the middle number of the data set if there is an odd number of values, and the average of the two middle numbers if there is an even number of values.

median-median regression (three-median regression) type of linear regression where data in a scatterplot is divided into three sections. The median of each section is used to locate the line of best fit to model the relationship between two variables.

midpoint half way point or middle point in an interval.

minimum turning point a point at which a parabola changes direction and at which it has its minimum y value.

mode a measure of the centre of a data set. It is the value that occurs most frequently in the data set. There can be more than one mode or no mode.

N

natural logarithms logarithms to the base e ($\log_e$ or ln).

net two-dimensional plan that can be folded to form a three-dimensional object.

non-terminating decimal decimal number with an endless (infinite) number of decimal places.

Null Factor Law states that if the product of two factors is zero, then one or both of the factors must equal zero.

O

oblique pyramid pyramid with apex not directly above the centre of its base. *See* pyramid.

opposite side (trigonometry) in a right-angled triangle, the side opposite the reference angle.

origin point where the x- and y-axes cross on a Cartesian plane. Coordinates are (0, 0).

original price the price of an item before a discount is applied or a mark-up is added.

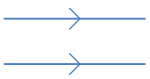
outcome possible result in a probability experiment.

outlier extreme piece of data that is much higher or lower than the rest of the data in the data set.

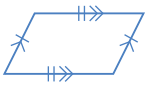
P

parabola the graph of a quadratic relationship.

parallel lines (or rays or segments) that are drawn side by side and are the same distance apart so they never touch are parallel. Matching arrowheads are used to indicate that lines are parallel.



parallelogram quadrilateral (four-sided shape) with two pairs of opposite sides that are parallel and equal in length, and two pairs of opposite angles that are equal in size. Rectangles, squares and rhombuses are also classified as parallelograms.



percentage a number expressed as parts out of one hundred.

percentage decrease percentage that an original amount has been decreased by. Found by calculating

$$\frac{\text{original amount} - \text{new amount}}{\text{original amount}} \times 100\%$$

percentage increase percentage that an original amount has been increased by. Found by calculating

$$\frac{\text{new amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

percentage of an amount when an amount is represented as a percentage of another amount.

perfect squares numbers that are the result of squaring a number; for example, 1 (1^2), 4 (2^2), 9 (3^2), etc.

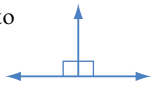
perimeter distance around the outside edge or boundary of a two-dimensional (2D) shape. Common units are mm, cm, m, km.

period horizontal distance covered by one full cycle of a trigonometric graph.

periodic function function or relationship that repeats itself continuously in cycles. For example, the sine, cosine and tangent functions are periodic functions.

perpendicular bisector line that is perpendicular to a line segment and intersects it at a point that divides the line segment in half.

perpendicular lines two lines (or rays or segments) that meet to form a right angle are perpendicular.



pie graph graph where the size of each sector in a circle represents the frequency of each data category as a fraction of the total number of pieces of data in the data set.

point of inflection point on a graph where the size of the gradient decreases to zero before increasing again.

polygon general name for a two-dimensional (2D) shape with straight sides.

polynomial expression with terms containing one variable only. Each term contains the variable raised to a power that is a positive integer or zero; for example, $4x^3 - 2x^2 + 5x + 6$.

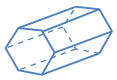
population complete set of individuals or things that we are seeking information about.

power (index) *see* index.

primary data data collected first hand, e.g. by survey or questionnaire.

principal sum of money invested or amount borrowed for a loan.

prism object with two ends that are identical polygons and joined by straight edges.



probability chance of an event occurring. The event can be very likely and have a high probability or not likely and have a low probability. Words or numbers can be used to describe the chance of an event occurring.

probability scale number line from 0 to 1 that can be used to indicate the probability of an event. An impossible event has a probability of 0. A certain event has a probability of 1.

profit amount made when the selling price is greater than the original price.

pronumerals letters or symbols that take the place of numbers.

proof series of statements which show that a theory is true in all cases.

pyramid object with an end that is a polygon and triangular sides (faces) that meet at a point (vertex).

Pythagoras' Theorem states that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the two other sides. For a right-angled triangle with hypotenuse of length c and other side lengths a and b, Pythagoras' Theorem states that $c^2 = a^2 + b^2$.

Pythagorean triad (Pythagorean triple) any set of three whole numbers that satisfy Pythagoras' theorem.

Q

quadratic equations equations that contain a pronumeral term or terms where the highest power is 2; for example, $x^2 + 3x - 2 = 10$.

quadratic formula used to solve equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. Solutions found by calculating $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

quadratic trinomial expression or polynomial with three terms where the highest power of the pronumeral is 2.

quadrilateral two-dimensional shape with four straight sides.

quotient result of a division; for example, in $28 \div 4 = 7$, the quotient is 7.

R

radius line segment from the centre of a circle to a point on the circumference of the circle. Also defined as the length of this line segment (and half the diameter of the circle).

random sampling selecting a sample randomly (such as pulling names out of a hat).

range difference between the highest and lowest value in a data set.

ratio comparison of two or more quantities of the same kind. For example, when comparing the number of red jellybeans to blue jellybeans, the ratio is 2:3.

ratio scale scale factor written as a ratio. For example, scale factor of $\frac{1}{10}$ is a ratio scale of 1:10.

rational denominator denominator of a surd fraction is a rational number. For example, $\frac{\sqrt{6}}{3}$ has a rational denominator but $\frac{\sqrt{2}}{\sqrt{3}}$ does not.

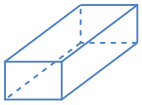
rational numbers numbers that can be written as fractions. This includes whole numbers, fractions, terminating decimals and recurring decimals.

real number system made up of rational numbers and irrational numbers. *See* rational numbers. *See* irrational numbers.

rectangle quadrilateral (four-sided shape) with two pairs of opposite sides that are parallel and equal in length, and four angles of 90°. A rectangle is also classified as a parallelogram.

rectangular hyperbola hyperbola with a vertical and a horizontal asymptote. For example, the graph of $y = \frac{1}{x}$ is a rectangular hyperbola.

rectangular prism prism with three pairs of identical faces (total of six faces). A cube is also a rectangular prism.



reduction smaller image produced after a figure has been dilated.

reflection transformation where a shape or object is reflected (flipped) in a mirror line to produce its exact mirror image.

regular polygon a polygon or shape with all sides equal in length and all angles equal in size.

regular tetrahedron tetrahedron with equilateral triangles for all four faces. *See* tetrahedron.

remainder leftover result of a division. For example, 30 ÷ 4 = 7 rem 2 which means there are 7 lots of 4 in 30 with a remainder of 2.

remainder theorem if a polynomial $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.

retail price *see* selling price.

retainer a fixed amount paid to a person who also earns commissions on sales.

rhombus quadrilateral (four-sided shape) with two pairs of opposite sides that are parallel, all four sides equal in length and two pairs of opposite angles that are equal in size. A rhombus can also be classified as a parallelogram.

RHS abbreviation for right-angled triangle-hypotenuse-side. It describes the information needed to draw a triangle and also relates to a condition for congruence. Right-angled triangles are congruent if their hypotenuses are the same length and one other corresponding pair of sides has the same length.

right cones cones with apex directly above the centre of the base. *See* cone.

right pyramids pyramids with apex directly above the centre of the base. *See* pyramid.

rise vertical distance between two points on a line.

rounding writing a number as an approximate value by rounding to the leading digit or a given number of decimal places. If the digit to the right of the leading digit or given decimal place is 0, 1, 2, 3 or 4, round down. If it is 5, 6, 7, 8 or 9, round up.

run horizontal distance between two points on a line.

S

sample small selection from a population.

sample space list of all the different outcomes possible for a probability experiment. For example, sample space when rolling a die is {1, 2, 3, 4, 5, 6}.

SAS abbreviation for side-angle-side. It describes the information needed to draw a triangle and also relates to a condition for congruence. Triangles are congruent if two corresponding sides are equal in length and the angle between them is the same.

scale factor indicates how many times larger or smaller an image is after a figure has been dilated. A scale factor larger than 1 produces an enlargement and a scale factor between 0 and 1 produces a reduction.

scatterplot graph that displays a relationship between two variables where points are plotted but not joined.

secant line that cuts a circle twice.

secondary data data that has been collected by someone else.

sector region of a circle between two radii. (Plural of radius is radii.)

segment region enclosed in a circle by a chord and the arc formed. *See* chord.

selling price price for which an item is sold.

semi-circle a half circle.

semi-perimeter (of a triangle) is half the perimeter of a triangle.

similar figures an image produced after dilation. The size of the figure has changed but not the shape.

simple interest an amount (usually a percentage) paid on a loan or investment.

simultaneous linear equations linear equations that are true for the same set of values for the variables.

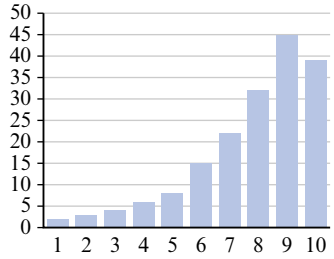
sin (sine) the sine of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the hypotenuse; that is, the length of the opposite side divided by the length of the hypotenuse.

sine graph graph of the relationship between an angle and the sine of the angle.

sine rule a rule linking side lengths a, b and c, and angles A, B and C of a triangle ABC; for example, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or alternatively as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. These ratios are used in pairs.

sketch graph a simple graph, not on graph paper, which has main features such as intercepts and turning points labelled.

skewed description of a data distribution where there is a higher proportion of the data displayed to the right or to the left of a graph. When there is more data to the right of the graph, it is said to be negatively skewed.



solution value of the pronumeral that makes an equation a true statement.

sphere 3D object shaped like a ball.

spread describes how widely distributed the data in a set is.

square quadrilateral (four-sided shape) with all four sides equal in length, two pairs of opposite sides that are parallel and four angles of 90°. A square is also classified as a rectangle.

SSS abbreviation for side-side-side. It describes the information needed to draw a triangle and also relates to a condition for congruence. Triangles are congruent if all three corresponding sides are equal in length.

standard deviation a measure of spread of a data set, describing how much data differs from the mean.

stem-and-leaf plot (stem plot) a display of data where each piece of data is split into two parts: the last digit becoming the leaf and the other digits becoming the stem. The stem is written once in the left column of the plot and the leaves are listed in numerical order beside the appropriate stem.

stratified sampling dividing the population into separate categories (such as gender or age) and then taking a random sample from each category that is proportional to its size.

subject of the equation a variable is the subject of the equation when it appears by itself on one side of the equation.

subset a set that lies entirely within another set.

substitution method algebraic method used to solve simultaneous equations by replacing a variable with an expression linking it to the other variable to form a new equation to be solved.

supplementary angles angles that add to make a straight angle (180°).

surds irrational numbers that are written with a root sign; for example, $\sqrt{2}$, $\sqrt[3]{2}$ or $\sqrt[5]{12}$. *See* irrational numbers.

symmetrical (symmetric) property of a shape, object or graph where it can be cut in half and each half is a reflection of the other.

systematic sampling selecting a sample at fixed intervals (such as every third person). The starting point should be random.

T

tangent in geometry, it is a line that touches a curve just once. A tangent to a circle touches the circumference at one point only and is perpendicular to its radius at that point.

tan (tangent) in trigonometry, the tangent of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the adjacent side; that is, the length of the opposite side divided by the length of the adjacent side.

tetrahedron pyramid with a triangular base. *See* pyramid.

theoretical probability probability calculated by considering the number of favourable outcomes compared to the total number of outcomes; that is, Pr(event) = $\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$. Each outcome must be equally likely to occur.

time in interest calculations, the period of the loan or investment.

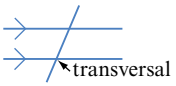
time series bivariate data where the independent variable is time. The data is recorded at regular time intervals. *See* bivariate data.

total surface area (TSA) total area of the surface of an object. Surface area of a prism is the sum of the areas of each face of the prism. Common units are mm^2, cm^2, m^2.

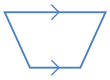
transformations general name for translations, reflections, rotations and dilations.

translation transformation where a shape or object is translated (moved) in a straight line without turning or changing size. Movement is often described as the number of units up or down and left or right.

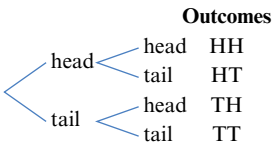
transversal line that crosses (intersects) a pair or set of lines.



trapezium quadrilateral (four-sided shape) with one pair of opposite sides that are parallel.



tree diagram display of outcomes for a multi-step probability experiment. For example, a tree diagram can be used to display the outcomes of flipping two coins.



triangular prism object with two ends that are identical triangles and joined by straight edges. The cross-section or base of the prism is a triangle.

trigonometric graphs graphs of trigonometric functions; for example, graphs of $y = \sin\theta$, $y = \cos\theta$ or $y = \tan\theta$.

trigonometry study of relationships between angles and side lengths of a triangle.

trinomial algebraic expression with three terms.

true bearing direction from one position to another and described by the number of degrees from north in a clockwise direction. The angle is written as three digits followed by T (or true). For example, 090°T is due east, 180°T is due south, 270°T is due west.

truncated cone the object remaining when the tip of a cone has been 'sliced' parallel to its base and removed. *See* cone.

truncated pyramid the object remaining when the tip of a pyramid has been 'sliced' parallel to its base and removed. *See* pyramid.

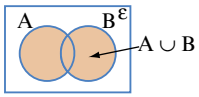
turning point the point where a parabola changes direction.

turning point form the form of a rule for a relationship where the coordinates of the turning point can be easily identified. For a quadratic relationship with rule $y = a(x - h)^2 + k$, the turning point of the parabola is at (h, k).

two-way table display of outcomes for a two-step probability experiment in a table.

U

union the union of two sets ($A \cup B$) on a Venn diagram is everything contained in both sets.



unit circle a circle drawn on the Cartesian plane with centre at the origin and a radius of 1 unit; used to define trigonometric functions.

unitary method a method of finding an original amount when the percentage of the original amount is known. We find 1% (one unit) and then multiply the result by 100 to find 100% (100 units or the original amount).

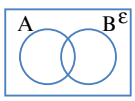
univariate data data related to one variable; for example, ages of people.

universal set the set of all elements. Everything inside the rectangle of a Venn diagram belongs to the universal set. Symbol for universal set is ε.

V

variable a quantity that can have varying or different values. A variable can be represented with words or a symbol.

Venn diagram consists of a number of circles contained in a rectangle to display the relationship between different sets of data. The rectangle represents the universal set, ε, and each set of data within the universal set is represented by a circle.



vertically opposite angles pair of equal angles not adjacent to (next to) each other, formed where two lines intersect.

vertices (plural of vertex) corner points where three or more edges of a prism or polyhedron meet.

volume amount of space that a three-dimensional (3D) object occupies. Common units are mm^3, cm^3, m^3.

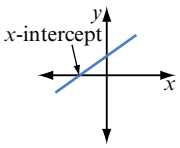
W

wholesale price original price.

X

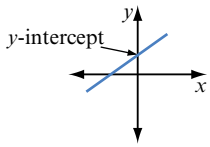
***x*- and *y*-intercept method** used for drawing a straight line graph, where the x- and y-intercepts are found and a straight line is drawn through them.

***x*-intercept** point where a line crosses the x-axis of a Cartesian plane. For example, the x-intercept is -3 or the coordinates of the x-intercept are $(-3, 0)$.



Y

***y*-intercept** point where a line crosses the y-axis of a Cartesian plane. For example, the y-intercept is 2 or the coordinates of the y-intercept are $(0, 2)$.



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