Essential Insight Exam Guide

Mathematics Methods

Year 12 WACE
Western Australian Curriculum

2025 Edition

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Essential Insight Exam Guide
Mathematics Methods
Year 12 WACE

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Acknowledgements
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Unit 3

Unit 3.1 – Further differentiation and applications

Section 1

2023 Section 1 Question 1 **Further differen-tiation and app-lications**	(a) Consider the function $f(x) = x^3e^{2x}$. (i) Differentiate $f(x)$. (2 marks) (ii) Determine the value of x for any stationary points of $f(x)$. (3 marks) (b) Evaluate $\int_0^{\frac{\pi}{4}} \sin(2x + \pi)dx$. (3 marks)

2023
Section 1
Question 5

Further differentiation and applications

The table below contains values of the polynomial function $f(x)$, its first and second derivatives, and the function $F(x)=\int_0^x f(t)\,dt$ for $x = 0, 1, 2, 3, 4, 5, 6$.

$f(x)$ has no stationary points at non-integer values of x, and the letters a, b, c, d and e represent unspecified constants.

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$f(x)$	a	b	4	c	0	d	e
$f'(x)$	16	0	–4	–2	0	–4	–20
$f''(x)$	–24	–9	0	3	0	–9	–24
$F(x)$	0	4.7	10.4	12.6	12.8	12.5	7.2

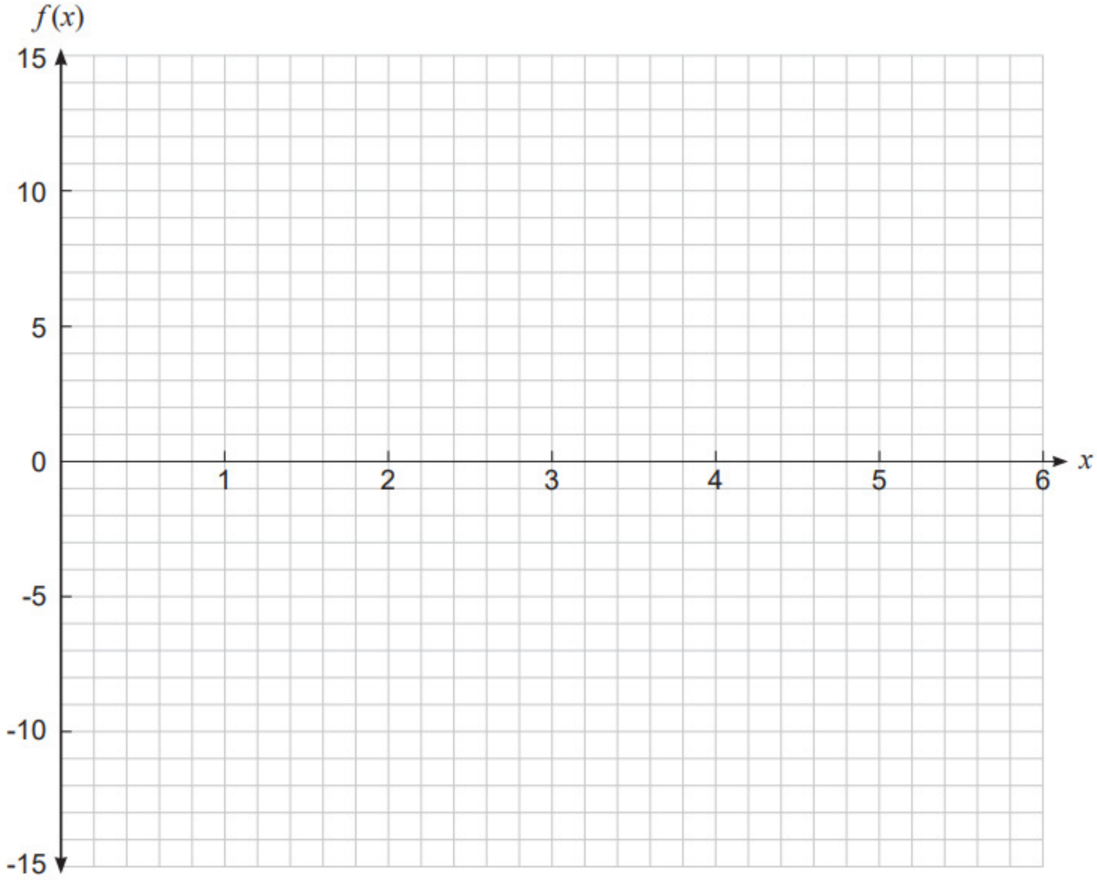
(a) Evaluate $\frac{d}{dx}\left(f(x)^2\right)$ when $x=2$. (2 marks)

(b) Evaluate $\int_2^4 \left(f(x)+2\right)\,dx$. (3 marks)

(c) Evaluate $\frac{d}{dx}\int_2^x f(t)\,dt$ when $x=2$. (2 marks)

(d) Determine the x-coordinate of any stationary points and whether they are local maxima, local minima or inflection points. Justify your answer. (3 marks)

(e) Sketch a possible graph of $f(x)$ for $0 \le x \le 6$ on the axes below. (3 marks)



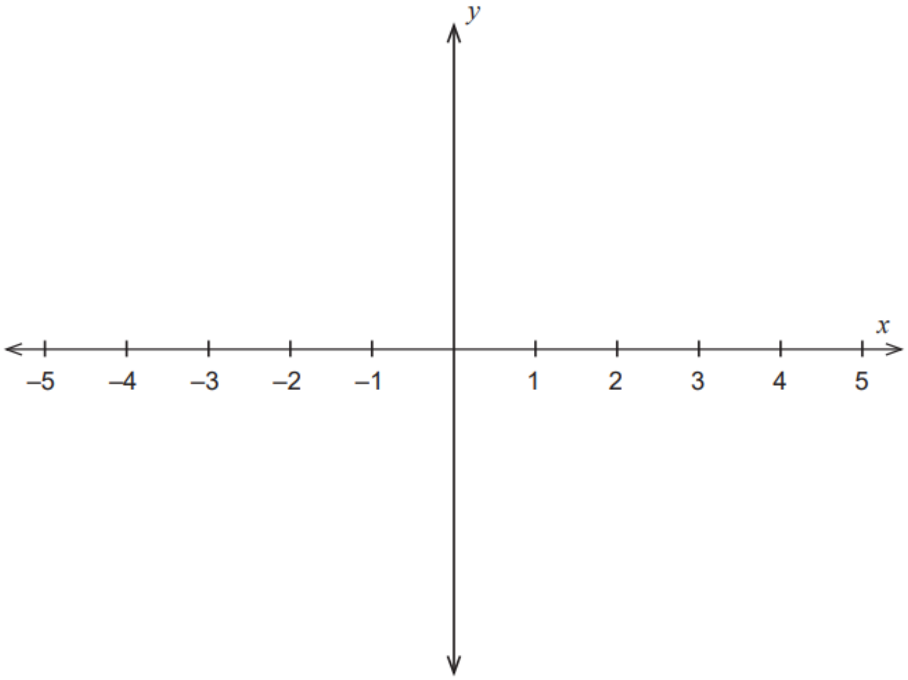
2022
Section 1
Question 5

Further differentiation and applications

A continuous function, f, satisfies the following conditions:

- $f(2) = 0$
- f has exactly 2 stationary points
- $f'(-1) = 0$ and $f'(1) = 0$
- $f''(-1) = 4$
- $f'(2) > 0$.

Sketch the function on the axes below. (5 marks)



2021
Section 1
Question 1

Further differentiation and applications

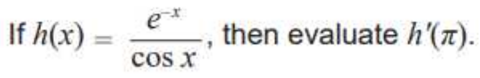
(a) Differentiate $\frac{3x+1}{x^3}$ and simplify your answer.

(3 marks)

(b) Let $f'(x) = x\ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$.

(3 marks)

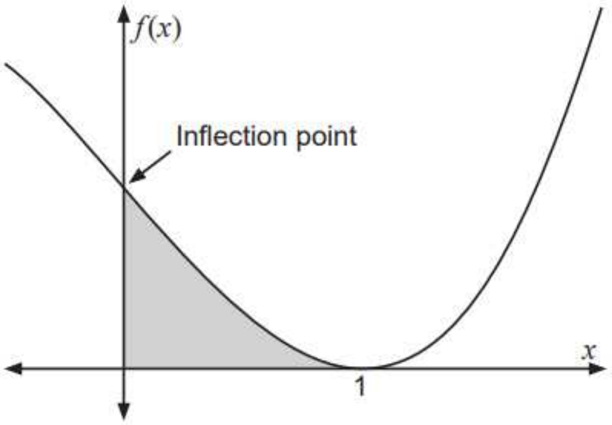
(c) Given that $g'(x) = 4e^{2x}$ and $g(1) = 0$, determine $g(5)$. (3 marks)

2020 **Section 1** **Question 2** **Further differentiation and applications**	If $h(x) = \dfrac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$. (4 marks)

2020
Section 1
Question 3

Further differentiation and applications

The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown below. A turning point is located at (1, 0) and the shaded region shown on the graph has an area of $\frac{3}{2}$ units^2.

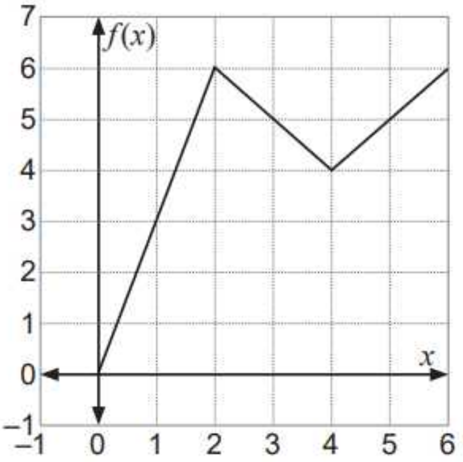


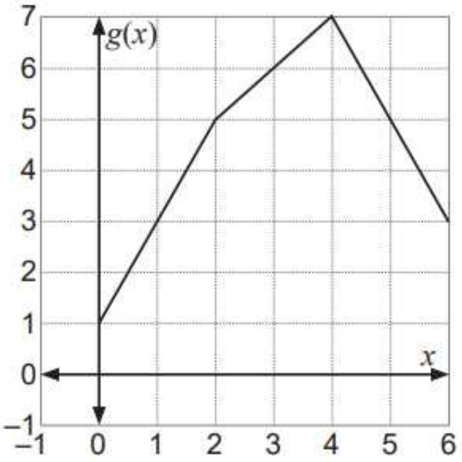
Use the above information to determine the values of a, b, c and d. (7 marks)

2020
Section 1
Question 5

Further differentiation and applications

The graphs of the functions f and g are displayed below.





(a) Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark)

(b) Evaluate the derivative of $f(x)g(x)$ at $x = 5$. (2 marks)

(c) Evaluate the derivative of $f(g(x))$ at $x = 1$. (2 marks)

2019 Section 1 Question 1 Further differentiation and applications	Consider the derivative function $f'(x) = xe^{x^2}$. (a) Determine $f''(1)$. (2 marks) (b) Explain the meaning of your answer to part (a). (1 mark) (c) Determine the expression for $y = f(x)$, given that it intersects the y-axis at the point (0,2). (3 marks)

2019
Section 1
Question 2

Further differentiation and applications

The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for $x = 1$, $x = 2$ and $x = 3$.

	$x = 1$	$x = 2$	$x = 3$
$g(x)$	3	5	–3
$h(x)$	2	–2	6
$g'(x)$	–4	1	4
$h'(x)$	0	–6	–5

(a) Evaluate the derivative of $\frac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

(b) Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

(c) If $h''(1) = -1$, describe with justification, what the graph of $h(x)$ looks like at this point. (2 marks)

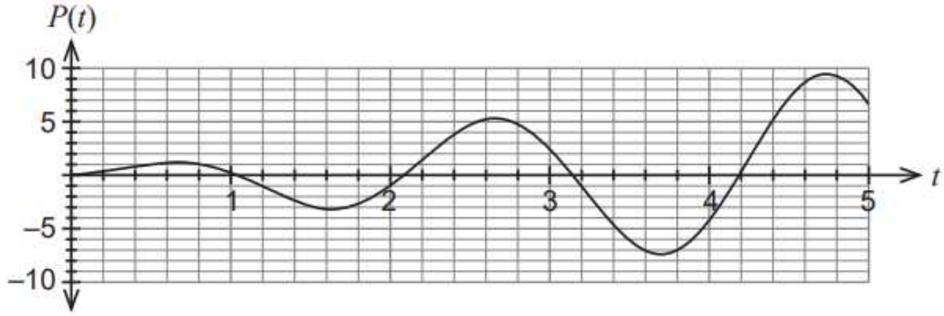
2019
Section 1
Question 7

Further differentiation and applications

A company's profit, in millions of dollars, over a five-year period can be modelled by the function:

$$P(t) = 2t\sin(3t) \quad 0 \le t \le 5 \text{ where } t \text{ is measured in years.}$$

The graph of $P(t)$ is shown below.



(a) Differentiate $P(t)$to determine the marginal profit function, $P'(t)$. (2 marks)

(b) Calculate the rate of change of the marginal profit function when $t = \dfrac{\pi}{18}$ years. (4 marks)

(c) Use the increments formula at $t = \dfrac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

Section 2

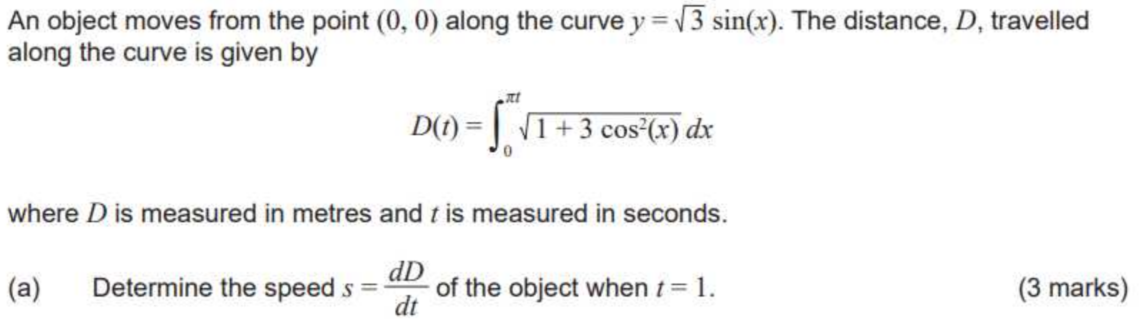
2023 Section 2 Question 6 **Further differentiation and applications**	A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by $$B(t) = 4e^{1.4t}$$ where t is the number of years since the establishment of the colony. (a) Determine the initial population of the bee colony. (1 mark) (b) Determine the increase in the population of the bee colony in the first six months. (2 marks) (c) Determine the rate of population growth two years after the establishment of the colony. (2 marks) (d) After how many years will the rate of population growth be 65 000 bees/year? (2 marks)

After three years, the beekeeper notices that the number of bees begins to decline. The declining population, b, in thousands, has the form $b(t) = Ae^{rt}$ where t is the number of years since the start of the decline.

(e) Determine A and r if one year after the start of the decline the bee population is 100 000. (4 marks)

2022 Section 2 Question 15

Further differentiation and applications



An object moves from the point $(0, 0)$ along the curve $y = \sqrt{3}\sin(x)$. The distance, D, travelled along the curve is given by

$$D(t) = \int_0^{\pi t} \sqrt{1 + 3\cos^2(x)}\,dx$$

where D is measured in metres and t is measured in seconds.

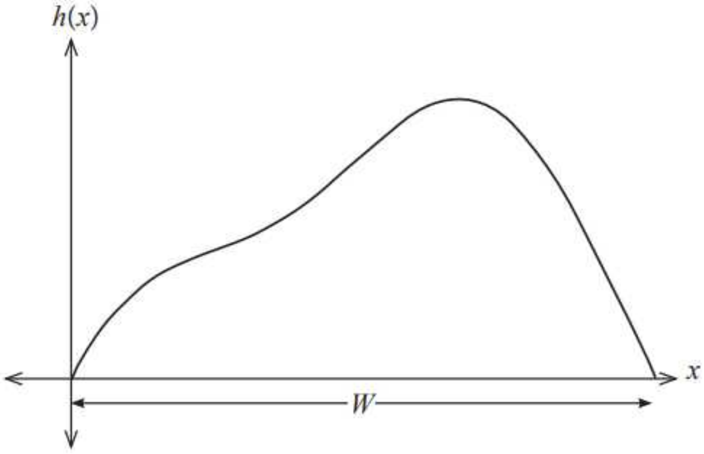
(a) Determine the speed $s = \frac{dD}{dt}$ of the object when $t = 1$. (3 marks)

(b) Use the increments formula to estimate the distance travelled by the object between $t = 1$ and $t = 1.02$. (2 marks)

2021
Section 2
Question 9

Further differentiation and applications

The Interesting Architecture company has designed a building with a constant cross-section shown in the figure below.



With reference to the figure, the height h(x) of the building at a point x along its width is given by

$$h(x) = 4\sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4, \text{ where } h \text{ and } 0 \le x \le W \text{ are measured in metres.}$$

(a) Determine the width W of the building to the nearest centimetre. (2 marks)

(b) Determine $h'(x)$. (1 mark)

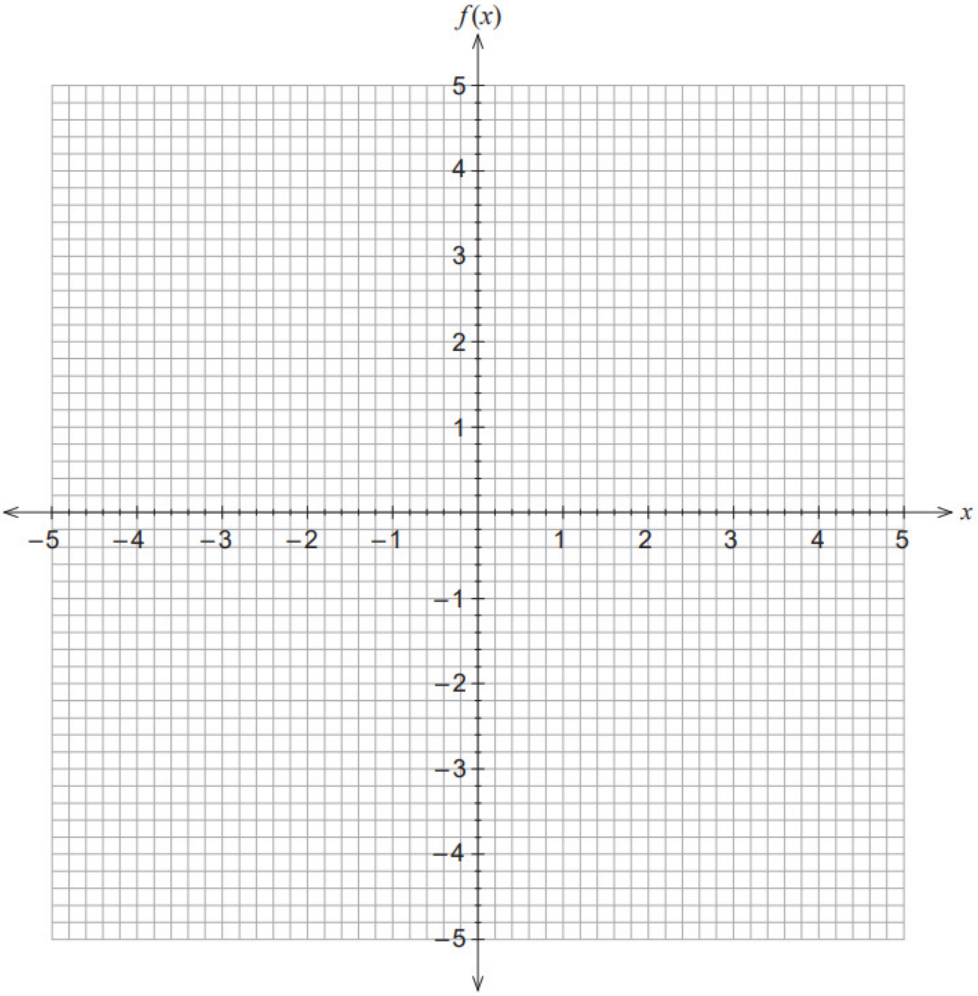
(c) Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

(d) An adventure company allows tourists to climb from the ground on the left of the building, then along the outside of the building to the top. The company installs a platform that allows climbers to rest on their way up to the top. The platform is located on the second half of the climb, at the point where it is the steepest. How high off the ground, to the nearest centimetre, is it positioned? (3 marks)

2021 Section 2 Question 12 **Further differentiation and applications**	Let $f(x)=x^2e^x$. (a) Show that $f'(x)=xe^x(x+2)$. (2 marks) (b) Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

(c) Determine the coordinates of any points of inflection. (2 marks)

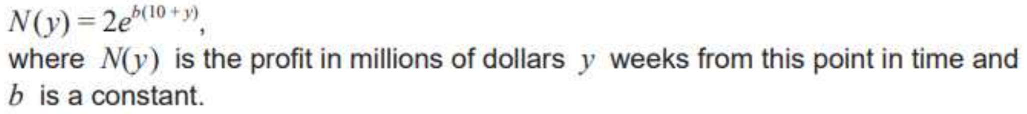
(d) Hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)



2021 Section 2 Question 16 **Further differentiation and applications**	An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company's profit was \$4 million. The model developed by the analyst was: $P(x) = \dfrac{20\ln(x+a)}{x+5}$, where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant. (a) Show that $a = e$. (2 marks) (b) What does the model predict the profit will be after five weeks? (1 mark) (c) Showing use of the quotient rule, determine an equation that, when solved, will give the time when the model predicts the profit will be maximised. (3 marks) (d) What is this maximum profit and during which week will it occur? (2 marks)

(e) According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:



$N(y) = 2e^{b(10+y)}$,
where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

(f) The company is projecting its profit to exceed \$5 million. During which week does the new model suggest this will happen? (3 marks)

2020
Section 2
Question 15

Further differentiation and applications

A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100 °C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T0 °C can be modelled by the equation below.

$$T(t) = T_0 - 175e^{-0.07t}$$

In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$ °C.

(a) What is the temperature of the water at the moment it is placed into the oven? (1 mark)

(b) What is the temperature of the water five minutes after being placed in the oven? (1 mark)

(c) What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks)

Assume that T_0 is still 200 °C.

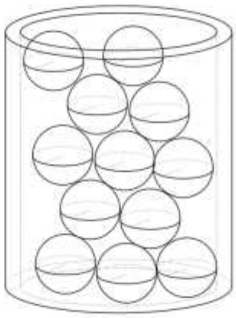
(d) Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)

(e) Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

2019
Section 2
Question
16

Further differentiation and applications

A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.



(a) Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks)

$$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$

	(b) Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks) (c) Can more than 20 of the spherical decorations fit inside the vase in part (b)? Use calculations to verify your answer. (3 marks)

Marking Guide – Section 1

**2023
Section 1
Question 1**

Further differentiation and applications

(a) Consider the function $f(x)=x^3e^{2x}$.

(i) Differentiate $f(x)$. (2 marks)

Solution
$f'(x)=\frac{d}{dx}(x^3)e^{2x}+x^3\frac{d}{dx}(e^{2x})$ $=3x^2e^{2x}+2x^3e^{2x}$
Specific behaviours
✓ demonstrates use of the product rule ✓ obtains correct derivative

(ii) Determine the value of x for any stationary points of $f(x)$. (3 marks)

Solution
Setting $f'(x)=0$ gives $0=3x^2e^{2x}+2x^3e^{2x}$ $\Rightarrow 0=x^2e^{2x}(3+2x)$ $\Rightarrow x=0,\ -\frac{3}{2}$
Specific behaviours
✓ sets $f'(x)=0$ ✓ solves to obtain stationary point at $x=0$ ✓ solves to obtain stationary point at $x=-\frac{3}{2}$

(b) Evaluate $\int_0^{\frac{\pi}{4}}\sin(2x+\pi)dx$. (3 marks)

Solution
$\int_0^{\frac{\pi}{4}}\sin(2x+\pi)\,dx=\left[-\frac{\cos(2x+\pi)}{2}\right]_0^{\frac{\pi}{4}}$ $=-\frac{\cos\left(\frac{3\pi}{2}\right)}{2}-\left(-\frac{\cos(\pi)}{2}\right)$ $=0-\left(\frac{1}{2}\right)$ $=-\frac{1}{2}$
Specific behaviours
✓ antidifferentiates correctly ✓ correctly substitutes integration bounds ✓ evaluates to obtain correct answer

2023
Section 1
Question 5

Further differentiation and applications

The table below contains values of the polynomial function $f(x)$, its first and second derivatives, and the function $F(x) = \int_0^x f(t)\,dt$ for $x = 0, 1, 2, 3, 4, 5, 6$.

$f(x)$ has no stationary points at non-integer values of x, and the letters a, b, c, d and e represent unspecified constants.

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$f(x)$	a	b	4	c	0	d	e
$f'(x)$	16	0	–4	–2	0	–4	–20
$f''(x)$	–24	–9	0	3	0	–9	–24
$F(x)$	0	4.7	10.4	12.6	12.8	12.5	7.2

(a) Evaluate $\frac{d}{dx}\left(f(x)^2\right)$ when $x = 2$.

(2 marks)

Solution

By the chain rule

$$\frac{d}{dx}\left(f(x)^2\right) = 2f(x)f'(x)$$

Substituting $x = 2$ gives

$$\left.\frac{d}{dx}\left(f(x)^2\right)\right|_{x=2} = 2f(2)f'(2)$$
$$= 2\times 4\times -4$$
$$= -32$$

Or

By the product rule

$$\frac{d}{dx}\left(f(x)^2\right) = \frac{d}{dx}\left(f(x)f(x)\right)$$
$$= f(x)f'(x) + f(x)f'(x)$$

Substituting $x = 2$ gives

$$\left.\frac{d}{dx}\left(f(x)^2\right)\right|_{x=2} = f(2)f'(2) + f(2)f'(2)$$
$$= 4\times -4 + 4\times -4$$
$$= -16 - 16$$
$$= -32$$

Specific behaviours

✓ correctly applies the chain rule or product rule
✓ calculates correct derivative

(b) Evaluate $\int_2^4 \left(f(x)+2\right)\,dx$. (3 marks)

Solution

$$\begin{aligned}\int_2^4 \left(f(x)+2\right)dx &= \int_2^4 f(x)\,dx+\int_2^4 2\,dx\\ &= F(4)-F(2)+\left[2x\right]_2^4\\ &= 12.8-10.4+(8-4)\\ &= 6.4\end{aligned}$$

Specific behaviours

✓ correctly applies linearity of definite integrals
✓ correctly applies fundamental theorem to first integral
✓ correctly evaluates $\int_2^4 2\,dx$ and obtains correct answer

(c) Evaluate $\frac{d}{dx}\int_2^x f(t)\,dt$ when $x=2$. (2 marks)

Solution

By the fundamental theorem of calculus

$$\frac{d}{dx}\int_2^x f(t)\,dt = f(x)$$

Substituting $x=2$ gives

$$\begin{aligned}\left.\frac{d}{dx}\int_2^x f(t)\,dt\right|_{x=2} &= f(2)\\ &= 4\end{aligned}$$

Specific behaviours

✓ correctly applies the fundamental theorem of calculus
✓ correctly evaluates for $x=2$

(d) Determine the x-coordinate of any stationary points and whether they are local maxima, local minima or inflection points. Justify your answer. (3 marks)

Solution

Stationary points are when $f'(x)=0$, hence $x=1$ and $x=4$ are the stationary points.

Since $f''(1)=-9$ it follows that $x=1$ is a local maximum by the second derivative test.

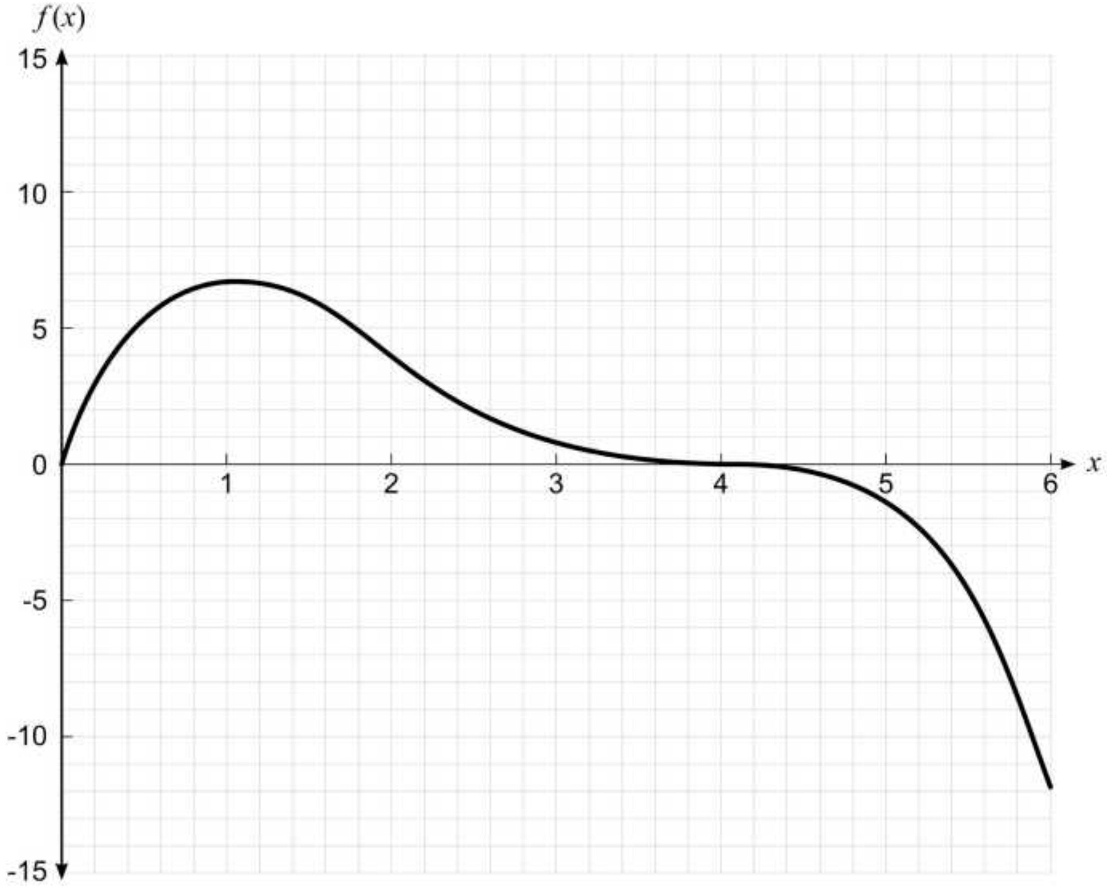
Since $f''(4)=0$ the second derivative test fails. Since the gradient of f is negative on both sides of $x=4$ (i.e. $f'(3)=-2<0$, $f'(5)=-4<0$, and there are no stationary points for non-integer values of x) it follows that $x=4$ is a horizontal point of inflection.

Specific behaviours

✓ correctly identifies the coordinates $x=1$ and $x=4$ as stationary points
✓ concludes that $x=1$ is a local maximum with correct justification
✓ concludes that $x=4$ is an inflection point with correct justification

(e) Sketch a possible graph of $f(x)$ for $0 \le x \le 6$ on the axes below. (3 marks)

Solution



Specific behaviours

- ✓ graph passes through the points $(2,4)$ and $(4,0)$
- ✓ graph includes local maximum at $x=1$ and a horizontal point of inflection at $x=4$
- ✓ concavity of graph is correct including inflection point at $x=2$

2023 Section 2 Question 6

Further differentiation and applications

A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by

$$B(t) = 4e^{1.4t}$$

where *t* is the number of years since the establishment of the colony.

(a) Determine the initial population of the bee colony. (1 mark)

Solution
$B = 4000$ bees
Specific behaviours
✓ correctly determines initial population

(b) Determine the increase in the population of the bee colony in the first six months. (2 marks)

Solution
$B(0.5) = 4e^{1.4(0.5)}$ $\approx 8.055...$ Population increase ≈ 8055 – 4000 = 4055
Specific behaviours
✓ correctly calculates $B(0.5)$ ✓ correctly calculates increase in bee population

(c) Determine the rate of population growth two years after the establishment of the colony. (2 marks)

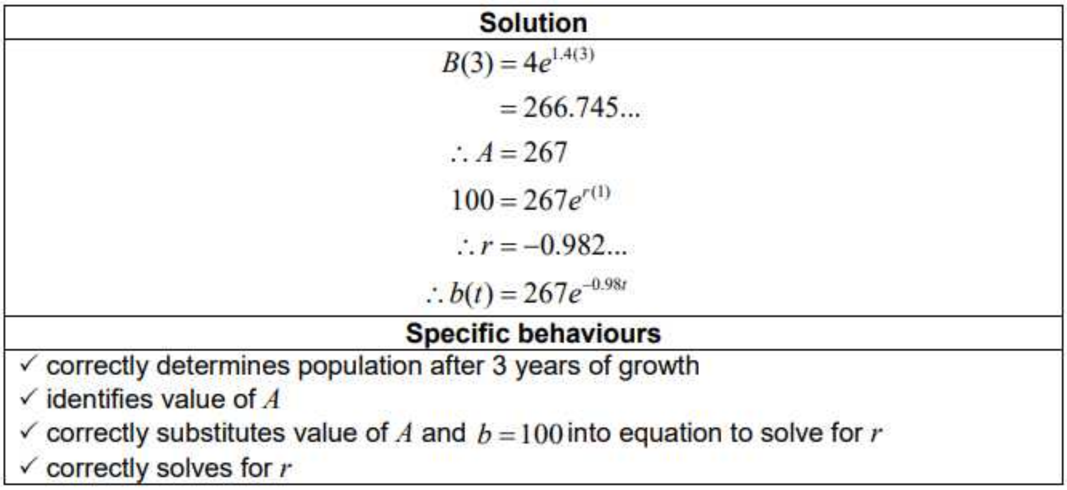
Solution
$\frac{dB}{dt} = 5.6e^{1.4t}$ $\left.\frac{dB}{dt}\right\|_{t=2} = 5.6e^{1.4(2)}$ $= 92.09...$ Hence the rate of population growth two years after the establishment of the colony is approximately 92 000 bees per year.
Specific behaviours
✓ correctly differentiates population equation ✓ correctly determines rate of population growth

(d) After how many years will the rate of population growth be 65 000 bees/year? (2 marks)

Solution
$65 = 5.6e^{1.4t}$ $t = 1.751...$ Hence the rate of population growth will be 65 000 bees/year after 1.75 years
Specific behaviours
✓ correctly substitutes $B = 65$ into growth rate equation ✓ correctly determines number of years

After three years, the beekeeper notices that the number of bees begins to decline. The declining population, b, in thousands, has the form $b(t) = Ae^{rt}$ where t is the number of years since the start of the decline.

(e) Determine *A* and *r* if one year after the start of the decline the bee population is 100 000. (4 marks)



Solution

$$B(3) = 4e^{1.4(3)}$$

$$= 266.745...$$

$$\therefore A = 267$$

$$100 = 267e^{r(1)}$$

$$\therefore r = -0.982...$$

$$\therefore b(t) = 267e^{-0.98t}$$

Specific behaviours

✓ correctly determines population after 3 years of growth
✓ identifies value of A
✓ correctly substitutes value of A and $b = 100$ into equation to solve for r
✓ correctly solves for r

2022
Section 1
Question 5

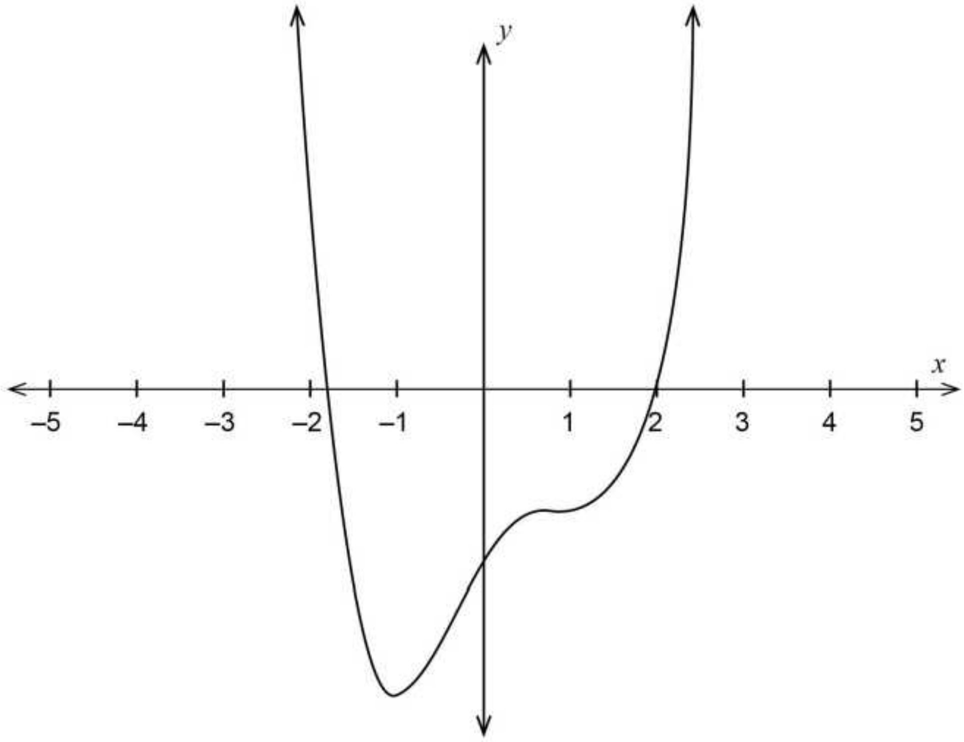
Further differentiation and applications

A continuous function, f, satisfies the following conditions:

- $f(2) = 0$
- f has exactly 2 stationary points
- $f'(-1) = 0$ and $f'(1) = 0$
- $f''(-1) = 4$
- $f'(2) > 0$.

Sketch the function on the axes below. (5 marks)

Solution



Specific behaviours

✓ locates an intercept at $x = 2$
✓ indicates a positive gradient for $x > 1$
✓ indicates a local minimum at $x = -1$
✓ locates a horizontal point of inflection at $x = 1$
✓ sketches a continuous curve with correct shape

2021
Section 1
Question 1

Further differentiation and applications

(a) Differentiate $\dfrac{3x+1}{x^3}$ and simplify your answer.

(3 marks)

Solution
$\dfrac{d}{dx}\left(\dfrac{3x+1}{x^3}\right)=\dfrac{x^3(3)-3x^2(3x+1)}{x^6}$ $=\dfrac{3x^3-9x^3-3x^2}{x^6}$ $=\dfrac{-6x-3}{x^4}$
Specific behaviours
✓ recognises the need for the quotient rule ✓ correctly differentiate the expression ✓ simplifies the result

(b) Let $f'(x)=x\ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$.

(3 marks)

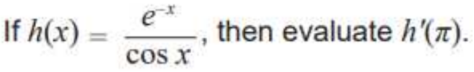
Solution
$f''(x)=x\times\dfrac{2}{2x}+1\times\ln(2x)$ $=1+\ln(2x)$
Specific behaviours
✓ identifies the rate of change as $f''(x)$ ✓ correctly determines $f''(x)$ ✓ simplifies the expression for $f''(x)$

(c) Given that $g'(x)=4e^{2x}$ and $g(1)=0$, determine $g(5)$. (3 marks)

Solution
$g(x)=2e^{2x}+c$ Since $g(1)=0$, $0=2e^2+c$ $\therefore c=-2e^2$ $\therefore g(x)=2e^{2x}-2e^2$ $\therefore g(5)=2e^{10}-2e^2$
Specific behaviours
✓ states an expression for $g(x)$, including the constant of integration ✓ correctly determines the constant ✓ correctly determines $g(5)$ as the final solution

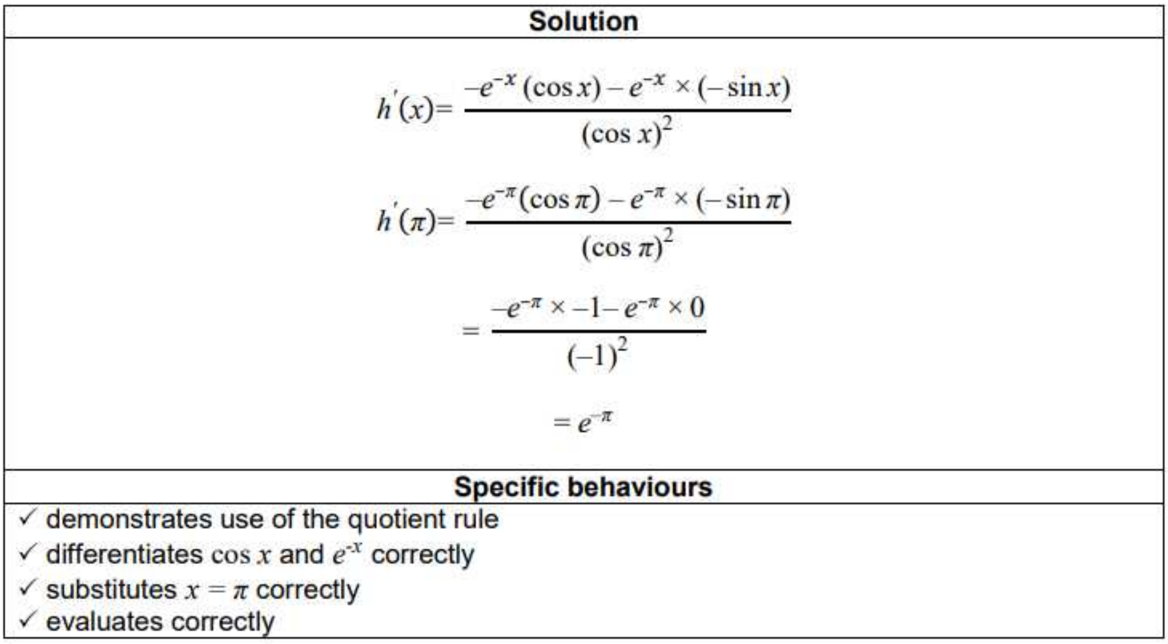
2020
Section 1
Question 2

Further differentiation and applications



If $h(x) = \dfrac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$.

(4 marks)



Solution

$$h'(x) = \frac{-e^{-x}(\cos x) - e^{-x} \times (-\sin x)}{(\cos x)^2}$$

$$h'(\pi) = \frac{-e^{-\pi}(\cos \pi) - e^{-\pi} \times (-\sin \pi)}{(\cos \pi)^2}$$

$$= \frac{-e^{-\pi} \times -1 - e^{-\pi} \times 0}{(-1)^2}$$

$$= e^{-\pi}$$

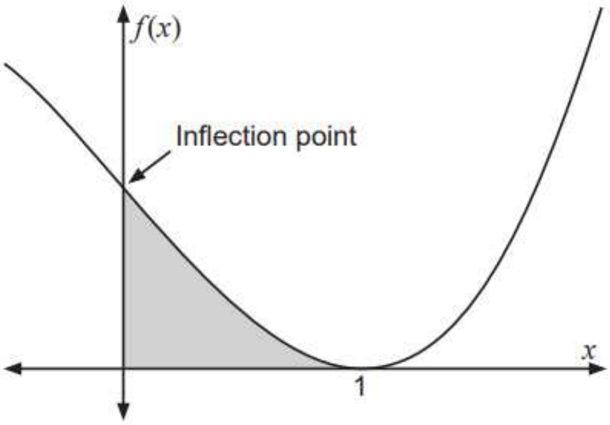
Specific behaviours

✓ demonstrates use of the quotient rule
✓ differentiates $\cos x$ and e^{-x} correctly
✓ substitutes $x = \pi$ correctly
✓ evaluates correctly

2020
Section 1
Question 3

Further differentiation and applications

The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown below. A turning point is located at (1, 0) and the shaded region shown on the graph has an area of $\frac{3}{2}$ units^2.



Use the above information to determine the values of a, b, c and d. (7 marks)

Solution

Firstly, note that

$$f'(x) = 3ax^2 + 2bx + c$$

and

$$f''(x) = 6ax + 2b$$

Given that there is an inflection point at $x = 0$ it follows that $f''(0) = 0$. Hence

$$0 = 2b$$
$$b = 0$$

Given that there is a turning point at $x = 1$ it follows that $f'(1) = 0$. Hence

$$0 = 3a + c$$
$$c = -3a$$

Given that there is an x-intercept at $x = 1$ it follows that $f(1) = 0$. Hence

$$0 = a - 3a + d$$
$$d = 2a$$

Finally, given that the area of the shaded region is $\frac{3}{2}$ it follows that $\int_0^1 f(x)dx = \frac{3}{2}$.

$$a\int_0^1 x^3 - 3x + 2\,dx = \frac{3}{2}$$
$$a\left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1 = \frac{3}{2}$$
$$\frac{3a}{4} = \frac{3}{2}$$
$$a = 2$$

Hence $a = 2$, $b = 0$, $c = -6$ and $d = 4$.

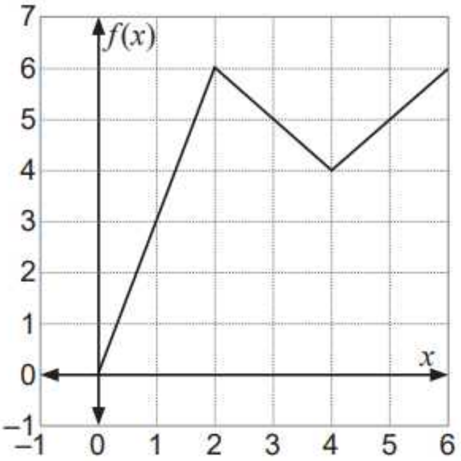
Specific behaviours

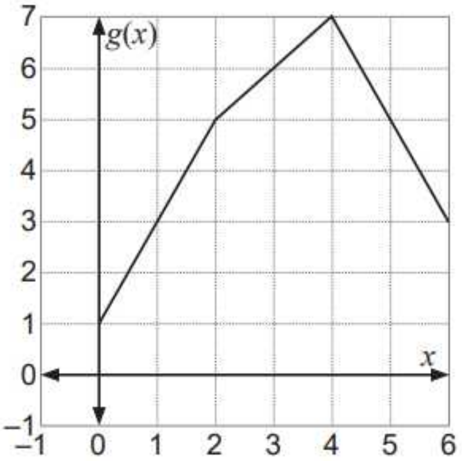
✓ states the first and second derivatives of f
✓ recognises that $f''(0) = 0$ and hence that $b = 0$
✓ recognises that $f'(1) = 0$ and hence that $c = -3a$
✓ recognises that $f(1) = 0$ and hence that $d = 2a$ (or $d = -a - c$)
✓ recognises that $\int_0^1 f(x)dx = \frac{3}{2}$
✓ evaluates definite integral to determine that $a = 2$
✓ solves for the values of c and d

2020
Section 1
Question 5

Further differentiation and applications

The graphs of the functions f and g are displayed below.





(a) Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark)

Solution
$f'(3) = -1$
Specific behaviours
✓ states correct derivative

(b) Evaluate the derivative of $f(x)g(x)$ at $x = 5$. (2 marks)

Solution
$(fg)'(5) = f'(5)g(5) + g'(5)f(5)$ $= (1)(5) + (-2)(5)$ $= -5$
Specific behaviours
✓ uses product rule to express derivative ✓ states correct derivative

(c) Evaluate the derivative of $f(g(x))$ at $x = 1$. (2 marks)

Solution
$f\left(g(x)\right)'\rvert_1 = f'\left(g(1)\right)g'(1)$ $= f'(3)2$ $= (-1)2$ $= -2$
Specific behaviours
✓ uses chain rule to express derivative ✓ states correct derivative

2019
Section 1
Question 1

Further differen-tiation and app-lications

Consider the derivative function $f'(x) = xe^{x^2}$.

(a) Determine $f''(1)$. (2 marks)

Solution
$f''(x) = x(2xe^{x^2}) + e^{x^2}$ $f''(1) = 3e$
Specific behaviours
✓ uses the chain rule to correctly differentiate $f'(x)$ ✓ evaluates at $x = 1$

(b) Explain the meaning of your answer to part (a). (1 mark)

Solution
$f''(1)$ is the rate of change of the derivative function when $x = 1$
Specific behaviours
✓ states it is the rate of change of the derivative AND includes when $x = 1$

(c) Determine the expression for $y = f(x)$, given that it intersects the y-axis at the point (0,2). (3 marks)

Solution
$\int xe^{x^2}\,dx = \frac{e^{x^2}}{2} + C$ $2 = \frac{e^0}{2} + C$ $C = \frac{3}{2}$ $\therefore f(x) = \frac{e^{x^2}}{2} + \frac{3}{2}$
Specific behaviours
✓ correctly integrates $f'(x)$ ✓ substitutes (0,2) into an expression involving C ✓ determines C and states the final expression for $y = f(x)$

2019
Section 1
Question 2

Further differentiation and applications

The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for x = 1, x = 2 and x = 3.

	$x = 1$	$x = 2$	$x = 3$
$g(x)$	3	5	–3
$h(x)$	2	–2	6
$g'(x)$	–4	1	4
$h'(x)$	0	–6	–5

(a) Evaluate the derivative of $\frac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

Solution
$\left(\frac{g}{h}\right)'(3) = \frac{g'(3)h(3) - g(3)h'(3)}{h(3)^2}$ $= \frac{4(6)-(-3)(-5)}{6^2}$ $= \frac{1}{4}$
Specific behaviours
✓ expresses the derivative using the quotient rule   ✓ evaluates the derivative

(b) Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

Solution
$h(g(1))' = h'(g(1))g'(1)$ $= h'(3)(-4)$ $= (-5)(-4)$ $= 20$
Specific behaviours
✓ expresses the derivative using the chain rule   ✓ evaluates the derivative

(c) If $h''(1) = -1$, describe with justification, what the graph of $h(x)$ looks like at this point. (2 marks)

Solution
Since $h'(1) = 0$ there is a stationary point at $x = 1$ Since 2nd derivative is negative the point is a maximum
Specific behaviours
✓ justifies stationary point   ✓ determines point is a maximum

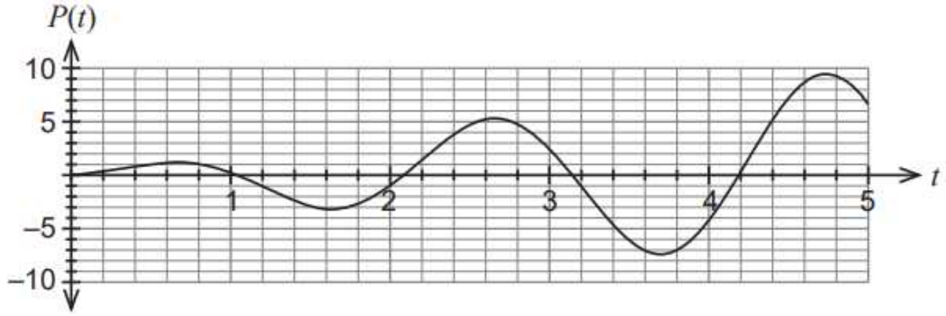
2019
Section 1
Question 7

Further differentiation and applications

A company’s profit, in millions of dollars, over a five-year period can be modelled by the function:

$$P(t) = 2t\sin(3t) \quad 0 \le t \le 5 \text{ where } t \text{ is measured in years.}$$

The graph of $P(t)$ is shown below.



(a) Differentiate $P(t)$to determine the marginal profit function, $P'(t)$. (2 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t)$ \$ / *year*
Specific behaviours
✓ uses the product rule ✓ determines correct derivative

(b) Calculate the rate of change of the marginal profit function when $t = \dfrac{\pi}{18}$ years. (4 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t)$ $P''(t) = 6\cos(3t) + 6\cos(3t) - 18t\sin(3t)$ $= 12\cos(3t) - 18t\sin(3t)$ $P''\left(\dfrac{\pi}{18}\right) = 12\cos\left(\dfrac{\pi}{6}\right) - \pi\sin\left(\dfrac{\pi}{6}\right)$ $= 6\sqrt{3} - \dfrac{\pi}{2}$ \$ / *year*^{2}
Specific behaviours
✓ uses product rule ✓ determines correct expression for the second derivative ✓ substitutes $\dfrac{\pi}{18}$ into second derivative expression ✓ calculates the exact rate of change

(c) Use the increments formula at $t = \frac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

Solution
$P'\left(\frac{7\pi}{6}\right) = 2\sin\left(\frac{7\pi}{2}\right) + 6\left(\frac{7\pi}{6}\right)\cos\left(\frac{7\pi}{2}\right)$ $= -2$ $\delta P \approx \frac{dP}{dt} \times \delta t$ $\approx -2 \times \frac{1}{12}$ $\approx -\frac{1}{6}$ The approximate change in profit is $-\frac{1}{6}$ million dollars. [$\frac{1}{6}$ million dollar loss]
Specific behaviours
✓ calculates the correct value of P' when $t = \frac{7\pi}{6}$ ✓ states an appropriate approximation for the change in profit using the increments formula ✓ substitutes and evaluates the change including units

Marking Guide – Section 2

<table>
<tr>
<td>2022
Section 2
Question
15

Further differentiation and applications</td>
<td>

An object moves from the point $(0, 0)$ along the curve $y = \sqrt{3}\sin(x)$. The distance, D, travelled along the curve is given by

$$D(t) = \int_0^{\pi t} \sqrt{1 + 3\cos^2(x)}\, dx$$

where D is measured in metres and t is measured in seconds.

(a) Determine the speed $s = \frac{dD}{dt}$ of the object when $t = 1$. (3 marks)

<table>
<tr><th>Solution</th></tr>
<tr><td>

Applying the fundamental theorem of calculus

$$\frac{dD}{dt} = \pi\sqrt{1 + 3\cos^2(\pi t)}$$

When $t = 1$

$$\frac{dD}{dt}(1) = \pi\sqrt{1 + 3\cos^2(\pi)} = 6.283 \text{ m/s}$$

</td></tr>
<tr><th>Specific behaviours</th></tr>
<tr><td>✓ applies fundamental theorem of calculus
✓ correctly applies chain rule
✓ evaluates $\frac{dD}{dt}$ at $t = 1$</td></tr>
</table>

(b) Use the increments formula to estimate the distance travelled by the object between $t = 1$ and $t = 1.02$. (2 marks)

<table>
<tr><th>Solution</th></tr>
<tr><td>

The increment in t is $\delta t = 1.02 - 1 = 0.02$. Hence

$$\begin{aligned} \delta D &\approx \frac{dD}{dt}(1) \times \delta t \\ &= 6.283 \times 0.02 \\ &= 0.126 \text{ m} \end{aligned}$$

</td></tr>
<tr><th>Specific behaviours</th></tr>
<tr><td>✓ determines correct increment for t
✓ applies increments formula to obtain correct answer</td></tr>
</table>

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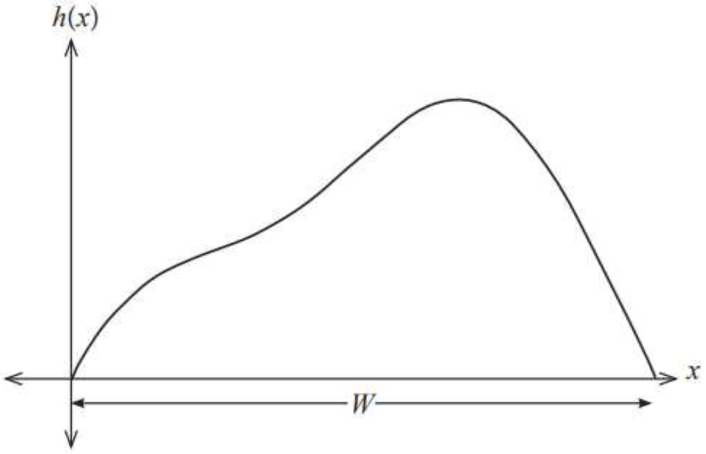
2021
Section 2
Question 9

Further differentiation and applications

The Interesting Architecture company has designed a building with a constant cross-section shown in the figure below.



With reference to the figure, the height h(x) of the building at a point x along its width is given by

$$h(x) = 4\sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4 \text{, where } h \text{ and } 0 \le x \le W \text{ are measured in metres.}$$

(a) Determine the width W of the building to the nearest centimetre. (2 marks)

Solution
$h(W) = 0$ $W = 8.64$ m (or 864 cm).
Specific behaviours
✓ sets $h\ (W) = 0$ ✓ solves for W

(b) Determine $h'(x)$. (1 mark)

Solution
$h'(x) = 4\cos\left(x - \frac{3\pi}{2}\right) - 2x + 3\pi$
Specific behaviours
✓ differentiates $h(x)$

(c) Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

Solution
Setting $h'(x) = 0$ gives $x = 5.74$ m. Hence the maximum height $h(5.74) = 20.57$ m.
Specific behaviours
✓ sets $h'(x) = 0$ and solves it to obtain the value of x for maximum height ✓ states the maximum height

(d) An adventure company allows tourists to climb from the ground on the left of the building, then along the outside of the building to the top. The company installs a platform that allows climbers to rest on their way up to the top. The platform is located on the second half of the climb, at the point where it is the steepest. How high off the ground, to the nearest centimetre, is it positioned? (3 marks)

Solution

The climb is steepest when the gradient is a maximum.

The second derivative is given by

$$h''(x) = -4\sin\left(x - \frac{3\pi}{2}\right) - 2$$

$h''(x) = 0$ at the points where $x = 2.09$ and $x = 4.19$

When $x = 4.19$ (second half of the climb), the platform is 15.93 metres off the ground.

Specific behaviours

✓ obtains the second derivative
✓ equates $h''(x)$ to zero and determines both x values
✓ identifies correct point and determines the height of the platform off the ground

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2021
Section 2
Question 12

Further differentiation and applications

Let $f(x) = x^2e^x$.

(a) Show that $f'(x) = xe^x(x+2)$. (2 marks)

Solution

$f'(x) = 2xe^x + x^2e^x = xe^x(x+2)$

Specific behaviours

✓ differentiates using product rule
✓ factorises correctly

(b) Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

Solution

$f'(x) = 0$
$\Rightarrow xe^x(x+2) = 0$
$\Rightarrow x = 0, -2$
$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$
$f''(0) = 2 > 0 \Rightarrow$ Local minima
$f(0) = 0$
$f''(-2) = -2e^{-2} < 0 \Rightarrow$ Local maxima
$f(-2) = 4e^{-2} \approx 0.54$

Specific behaviours

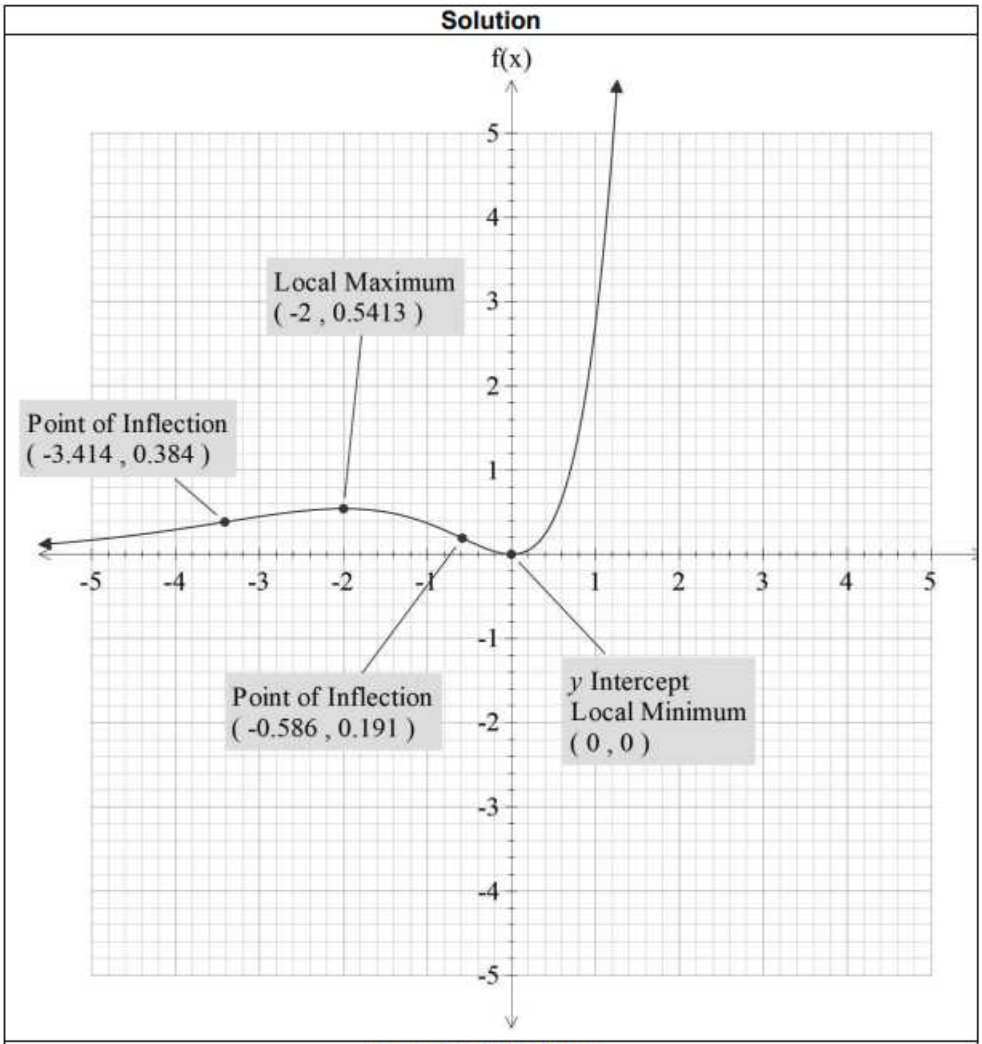
✓ sets first derivative equal to 0
✓ obtains the two solutions
✓ finds the second derivative
✓ evaluates the second derivative at $x = 0$ and concludes it is a local minima
✓ obtains y coordinate at minima
✓ evaluates the second derivative at $x = -2$ and concludes it is a local maxima
✓ obtains y coordinate at maxima

(c) Determine the coordinates of any points of inflection. (2 marks)

Solution
$f''(x) = 0$ $\Rightarrow e^x(2 + 4x + x^2) = 0$ $\Rightarrow x = -2 \pm \sqrt{2} \approx -3.4, -0.59$ $\Rightarrow$ points are $(-3.4, 0.38)$ and $(-0.59, 0.19)$
Specific behaviours
✓ sets second derivative equal to 0 ✓ obtains the two points

(d) Hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)

Solution



Specific behaviours

✓ indicates local minima
✓ indicates local maxima
✓ indicates points of inflections
✓ overall shape

2021
Section 2
Question
16

Further differentiation and applications

An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company's profit was $4 million. The model developed by the analyst was:

$$P(x) = \frac{20\ln(x+a)}{x+5},$$

where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant.

(a) Show that $a = e$. (2 marks)

Solution
Since $P(0) = 4$, we require $\ln(a) = 1$, giving $a = e$.
Specific behaviours
✓ recognises $P(0) = 4$ ✓ obtains $\ln(a) = 1$

(b) What does the model predict the profit will be after five weeks? (1 mark)

Solution
$P(5) = \dfrac{20\ln(5+e)}{5+5}$ $= 4.087 \text{ (3 d.p.)}$ Profit will be approximately $4 087 000
Specific behaviours
✓ states the profit

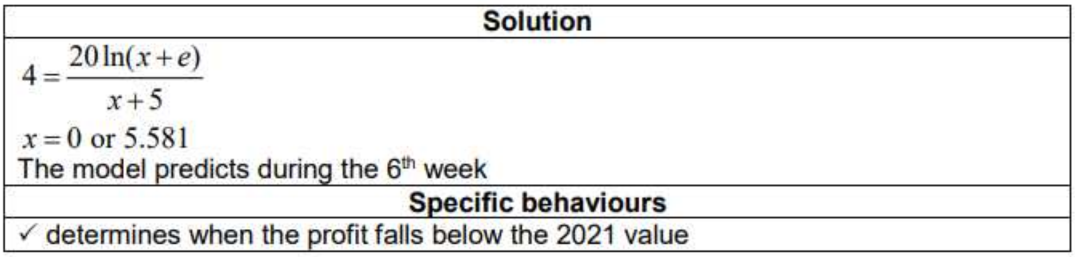
(c) Showing use of the quotient rule, determine an equation that, when solved, will give the time when the model predicts the profit will be maximised. (3 marks)

Solution
$P'(x) = \dfrac{(x+5)\dfrac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2}$ For maximum profit we require : $\dfrac{(x+5)\dfrac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2} = 0$
Specific behaviours
✓ demonstrates use of the quotient rule ✓ writes correct expression for $P'(x)$ ✓ equates $P'(x)$ to zero

(d) What is this maximum profit and during which week will it occur? (2 marks)

Solution
Maximum profit is approximately $4 436 000 This occurs when $x \approx 1.79$, so during the second week.
Specific behaviours
✓ states maximum profit ✓ states it occurs in the second week

(e) According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

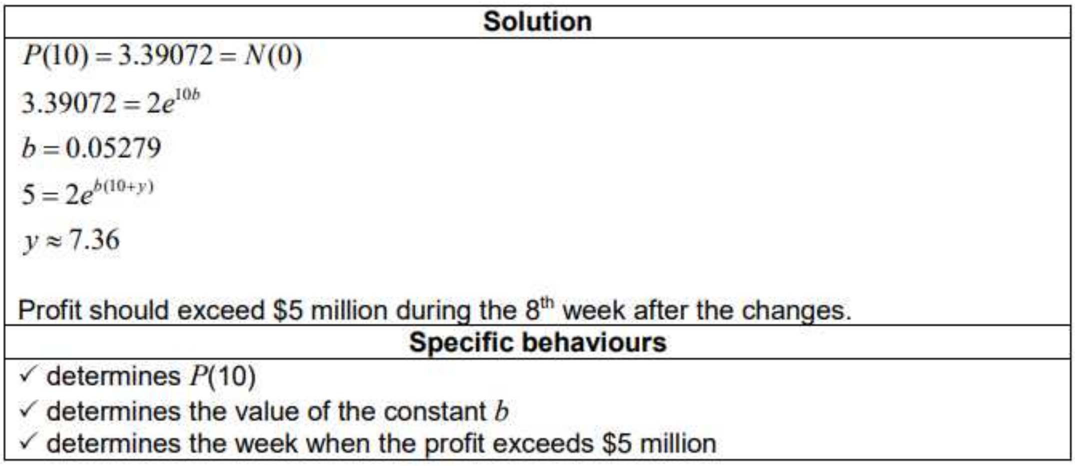


Solution
$4=\dfrac{20\ln(x+e)}{x+5}$ $x=0$ or 5.581 The model predicts during the 6th week
Specific behaviours
✓ determines when the profit falls below the 2021 value

The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:

$N(y)=2e^{b(10+y)}$,
where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

(f) The company is projecting its profit to exceed $5 million. During which week does the new model suggest this will happen? (3 marks)



Solution
$P(10)=3.39072=N(0)$ $3.39072=2e^{10b}$ $b=0.05279$ $5=2e^{b(10+y)}$ $y\approx 7.36$ Profit should exceed $5 million during the 8th week after the changes.
Specific behaviours
✓ determines $P(10)$ ✓ determines the value of the constant b ✓ determines the week when the profit exceeds $5 million

2020
Section 2
Question 15

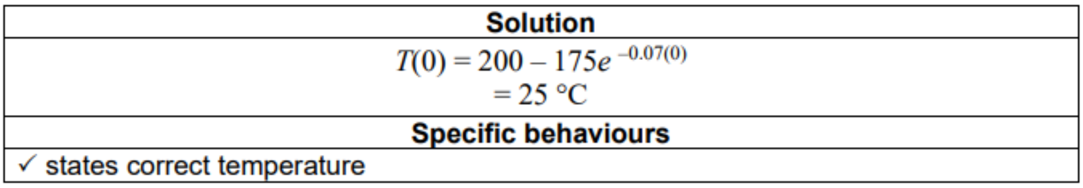
Further differentiation and applications

A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100 °C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T0 °C can be modelled by the equation below.

$$T(t) = T_0 - 175e^{-0.07t}$$

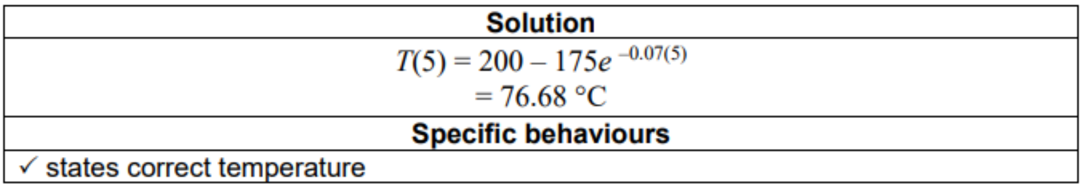
In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$ °C.

(a) What is the temperature of the water at the moment it is placed into the oven? (1 mark)



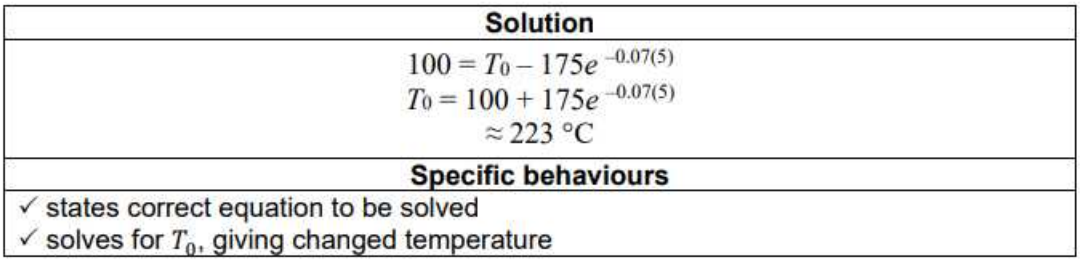
Solution
$T(0) = 200 - 175e^{-0.07(0)}$ $= 25$ °C
Specific behaviours
✓ states correct temperature

(b) What is the temperature of the water five minutes after being placed in the oven? (1 mark)



Solution
$T(5) = 200 - 175e^{-0.07(5)}$ $= 76.68$ °C
Specific behaviours
✓ states correct temperature

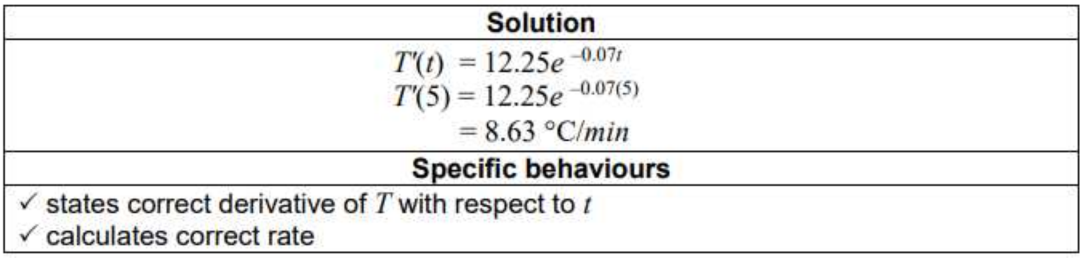
(c) What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks)



Solution
$100 = T_0 - 175e^{-0.07(5)}$ $T_0 = 100 + 175e^{-0.07(5)}$ ≈ 223 °C
Specific behaviours
✓ states correct equation to be solved ✓ solves for T_0, giving changed temperature

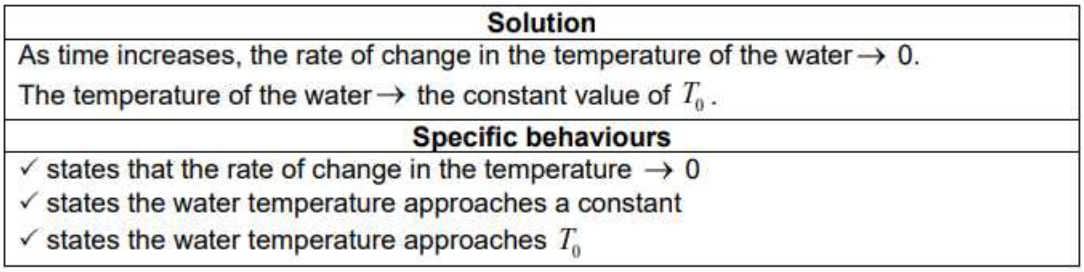
Assume that T_0 is still 200 °C.

(d) Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)



Solution
$T'(t) = 12.25e^{-0.07t}$ $T'(5) = 12.25e^{-0.07(5)}$ $= 8.63\ °C/min$
Specific behaviours
✓ states correct derivative of T with respect to t ✓ calculates correct rate

(e) Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

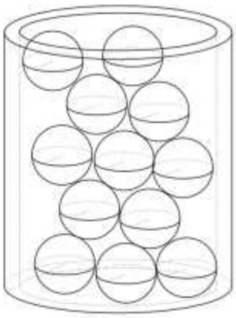


Solution
As time increases, the rate of change in the temperature of the water $\rightarrow$ 0. The temperature of the water $\rightarrow$ the constant value of T_0.
Specific behaviours
✓ states that the rate of change in the temperature $\rightarrow 0$ ✓ states the water temperature approaches a constant ✓ states the water temperature approaches T_0

2019
Section 2
Question 16

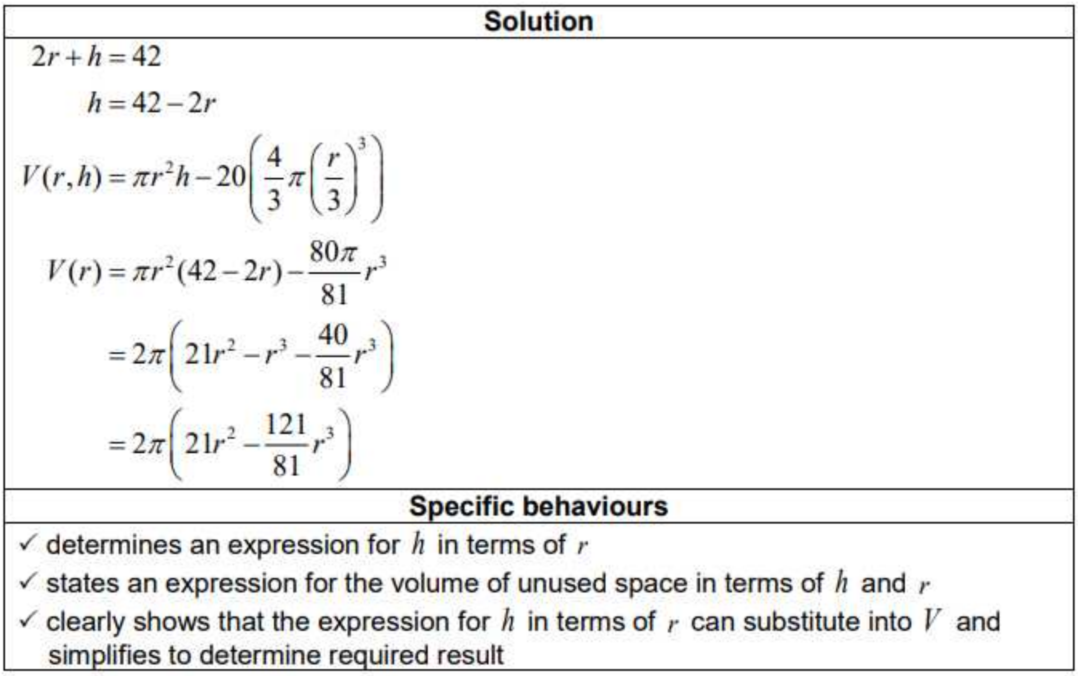
Further differentiation and applications

A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.



(a) Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks)

$$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$



Solution

$$2r + h = 42$$
$$h = 42 - 2r$$
$$V(r,h) = \pi r^2 h - 20\left(\frac{4}{3}\pi\left(\frac{r}{3}\right)^3\right)$$
$$V(r) = \pi r^2(42-2r) - \frac{80\pi}{81}r^3$$
$$= 2\pi\left(21r^2 - r^3 - \frac{40}{81}r^3\right)$$
$$= 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$

Specific behaviours

✓ determines an expression for h in terms of r
✓ states an expression for the volume of unused space in terms of h and r
✓ clearly shows that the expression for h in terms of r can substitute into V and simplifies to determine required result

(b) Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks)

Solution

$$V'(r) = 2\pi\left(42r - \frac{363r^2}{81}\right)$$

$$0 = 42r - \frac{363r^2}{81}$$

$$0 = r(42 - \frac{363r}{81})$$

$$r = 0 \text{ or } \frac{1134}{121}\{=9.372(3dp)\}$$

$$V''(r) = 2\pi\left(42 - \frac{726r}{81}\right)$$

$$V''(9.372) = -ve\{=-84\pi\} \Rightarrow \max$$

Dimensions are: $r = 9.4$ cm and $h = 23.3$ cm

Specific behaviours

- ✓ determines first derivative of $V(r)$
- ✓ equates to zero and determines 0 and 9.4 are solutions
- ✓ clearly shows the use of the second derivative or sign test to show that $r = 9.4$ is a maximum
- ✓ states the dimensions of the vase that maximise the unused space rounded to the nearest mm

(c) Can more than 20 of the spherical decorations fit inside the vase in part (b)? Use calculations to verify your answer. (3 marks)

Solution

$V(9.4) = 3863.1\text{ cm}^3$

$V(\text{decoration}) = 127.7\text{ cm}^3$

There is likely space for more decorations, but it is not certain as it would depend on the way the balls were packed into the vase.

Specific behaviours

- ✓ states the volume of unused space and the volume of one decoration
- ✓ infers likely to fit more
- ✓ states the limitation of packing

Unit 3.2 – Integrals

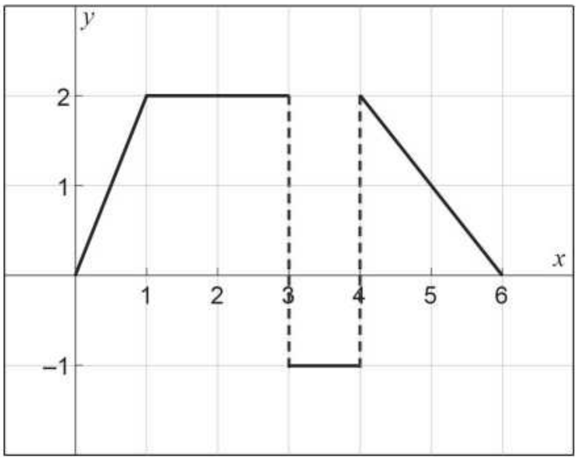
Section 1

2022 Section 1 Question 1 Integrals	Consider the derivative function $f'(x) = \frac{4x}{x^2+3}$. (a) Determine the rate of change of $f'(x)$ when $x = 1$. (3 marks) (b) Determine $f(x)$ given that $f(1) = \ln(32)$. (4 marks) (c) Determine $\frac{d}{dt}\int_t^3 f(x)\,dx$. (2 marks)

2022
Section 1
Question 2

Integrals

Consider the function $f(x)$ shown below.



Evaluate the following integrals.

(a) $\int_0^6 f(x)\,dx$ (2 marks)

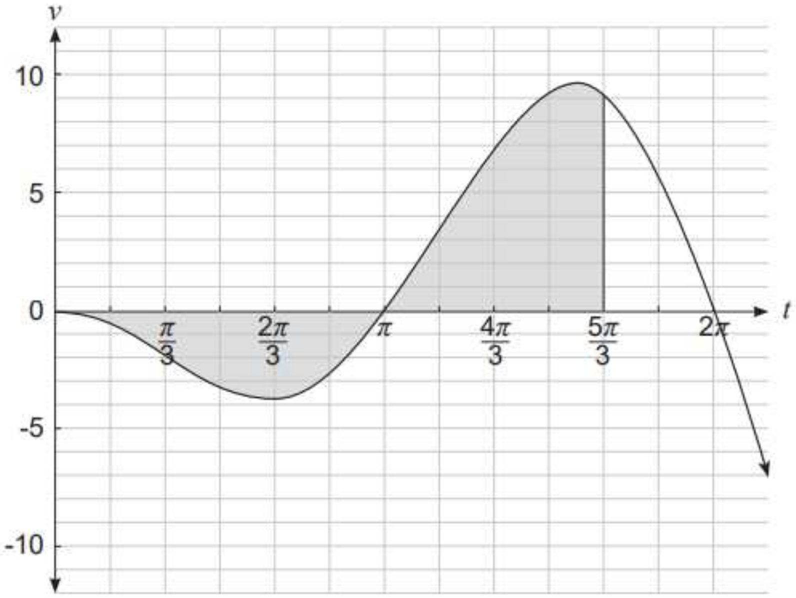
(b) $\int_0^4 f(x) - 2\,dx$ (2 marks)

(c) $\int_4^6 f'(x)\,dx$ (2 marks)

2021 **Section 1** **Question 4** **Integrals**	Determine the following: (a) $\int \left(2x^2 - x^3\right) dx$ (2 marks) (b) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3 - \cos(x)} dx$ (3 marks) (c) $\frac{d}{dy} \int_{-1}^{y} 3x^2 \cos(2x) dx$ (2 marks)

2021 **Section 1** **Question 5** **Integrals**	(a) Determine the area between the parabola $y = x2 - x + 3$ and the straight line $y = x + 3$. (4 marks) (b) The area between the parabola $y = x2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part (a). Explain why this is the case. (2 marks)

2019 **Section 1** **Question 5** **Integrals**	(a) Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks) (b) Hence, determine the area bound by the graph of $f\,(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks) (c) Determine the area bound by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

2023 **Section 2** **Question 8** **Integrals**	An oscillating mass has a velocity, v, given by $$v(t) = 2t\cos\left(t + \frac{\pi}{2}\right) \quad \text{for } t \geq 0.$$ The velocity is given in metres per second, and the time, t, is given in seconds. A graph of the velocity of the mass' motion is shown below.  (a) Determine the first two times, $t > 0$, at which the mass changes direction. State your answers exactly. (2 marks) (b) What does the signed area of the shaded region in the figure represent? (2 marks)

(c) Write an integral expression for the distance travelled from $t = \frac{\pi}{3}$ to $t = \frac{4\pi}{3}$. (3 marks)

(d) Determine the first time after $t = \pi$ that the acceleration of the object will be 0 m/s^2. (2 marks)

(e) The displacement of the mass is given by

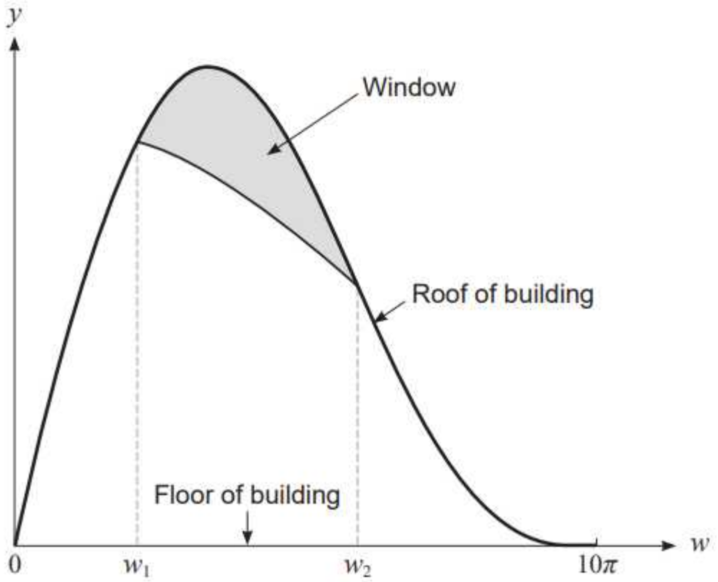
$$x(t) = A\sin\left(t + \frac{\pi}{2}\right) + B\cos\left(t + \frac{\pi}{2}\right) + 2t\sin\left(t + \frac{\pi}{2}\right)$$

where *A* and *B* are constants. Determine the value of *A* and *B*. (3 marks)

2023
Section 2
Question 9

Integrals

A new entertainment venue is being proposed. The preliminary design has a constant cross-section, as shown in the figure below.



The roof height $h(w)$ of the building at any point w along its width is given by

$$h(w) = 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)$$

where h and $0 \leq w \leq 10\pi$ are measured in metres.

(a) Determine the cross-sectional area of the building. (2 marks)

The designer would like to place a window, as shown in the figure above, that is bounded above by the roof of the building and below by the formula

$$g(w) = 7\cos\left(\frac{w}{20}\right).$$

(b) With reference to the figure

(i) determine the values of w_1 and w_2. (2 marks)

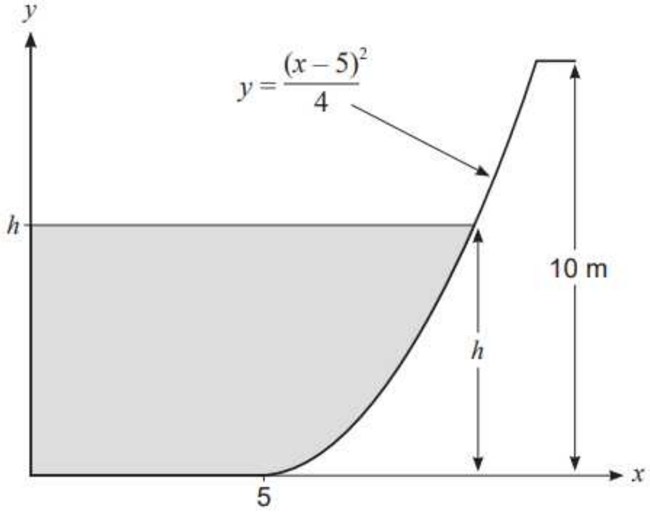
(ii) determine the area of the window. (2 marks)

(c) Use calculus techniques to determine the maximum height of the building. (4 marks)

2023
Section 2
Question
14

Integrals

A small dam on an agricultural property has a length of 20 m, and a uniform cross-section shown below where x and y are in metres. The base of the dam is flat for $0 \leq x \leq 5$, and the right side is given by $y = \frac{(x-5)^2}{4}$ for $5 < x \leq 11.325$. The shaded region on the graph below represents the cross-section of a volume of water V (m^3) in the dam with water level h (m).



(a) Using calculus, show that the volume of water in the dam is given by

$$V(h) = 100h + \frac{80}{3}h^{\frac{3}{2}}.$$

(5 marks)

(b) Use the increments formula to estimate the change in water volume if the water level rises from 6 m to 6.1 m. (3 marks)

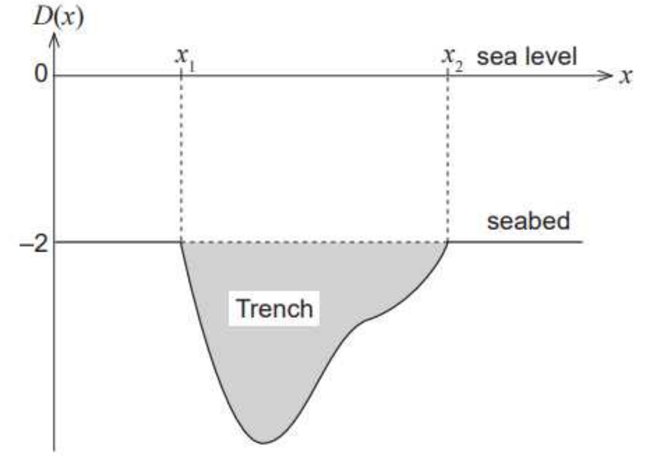
Suppose the water volume at the start of winter is 1000 m^3. On the basis of rainfall data from previous years, the volume of water V_R (m^3) that will flow into the dam over winter is normally distributed with a mean of 600 m3 and a standard deviation of 200 m^3.

(c) Assuming that there are no other sources of water and no losses, determine the probability that the dam will reach full capacity (i.e. depth of 10 m) during winter.(3 marks)

2022
Section 2
Question 7

Integrals

A team of oceanographers surveyed the depth of the ocean in a region populated by a particular endangered fish species. They discovered a large trench extending below the otherwise flat seabed as shown in the figure below.



The displacement, in kilometres, from sea level to the ocean floor is given by

$$D(x) = \begin{cases} (x-4)^2 + \cos(2x - 3\pi) - 5, & x_1 \le x \le x_2 \\ -2, & \text{otherwise} \end{cases}$$

where x (measured in kilometres) is the east–west horizontal displacement relative to a reference marker at sea level.

(a) With reference to the figure above:

(i) determine the values of x_1 and x_2. (2 marks)

(ii) use calculus to determine the cross-sectional area of the trench shaded in the figure above. (3 marks)

	(b) Using calculus, determine the maximum distance of the trench below sea level. (5 marks)

2022 **Section 2** **Question 10** **Integrals**	The displacement, x, of a mass on the end of a damped spring is given by $x(t) = 3e^{-t}\sin(t), \quad t \geq 0$ where x is in centimetres and t is in seconds (a) Determine when the mass first returns to its starting position at $x = 0$. (2 marks) (b) Determine an expression for the velocity of the mass. (2 marks)

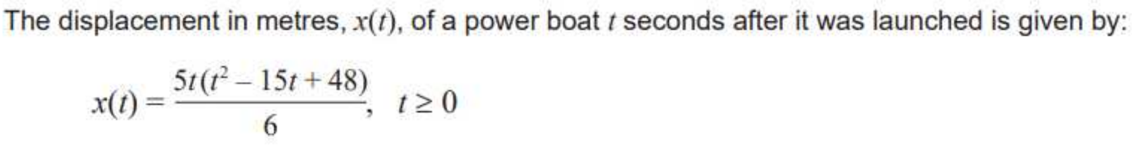
	(c) Determine the displacement of the mass when it first changes direction. (3 marks) (d) The mass is considered to have stopped oscillating when the oscillation amplitude $A(t) = 3e^{-t}$ drops to 0.01 cm. How long does it take for the spring to stop oscillating? (2 marks)

2022 **Section 2** **Question** **11** **Integrals**	The 100 m sprint is a race run on a straight section of track. During a race the velocity, v, measured in metres per second, of an athlete is given by $v(t) = -10e^{-0.8t} - 0.05e^{0.2t} + 10.05$ where t is the time, in seconds, measured from the moment the athlete starts to move from the start line. (a) Determine the acceleration of the athlete three seconds after moving from the start line. (2 marks)

(b) Using calculus, determine the maximum velocity of the athlete during the race, and the time, t, at which it is achieved. (4 marks)

(c) The displacement, x, of the athlete is 0 m at the start of the race. Determine an expression for the displacement of the athlete during the race. (3 marks)

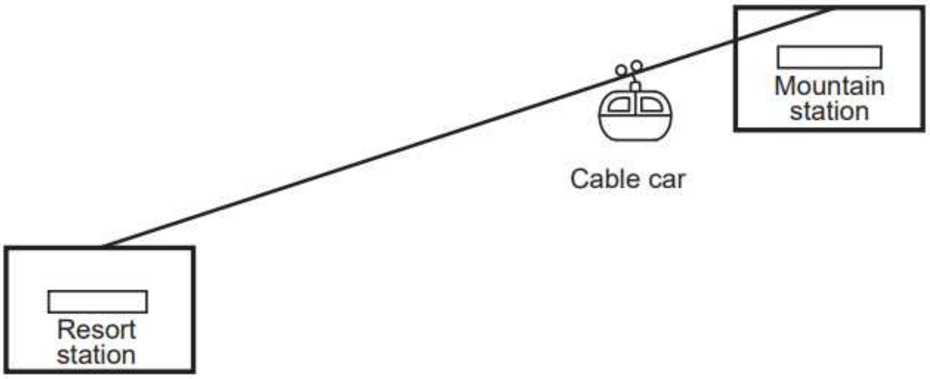
(d) Determine the time, t, at which the athlete finishes the 100 m race. (2 marks)

2021 Section 2 Question 14 Integrals	 The displacement in metres, $x(t)$, of a power boat t seconds after it was launched is given by: $x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$ How far has the power boat travelled before its acceleration is zero? (5 marks)

2021
Section 2
Question
17

Integrals

A resort in the Swiss Alps features a cable car that travels from the resort station to the mountain station. Engineers are fixing a cable car that unexpectedly stopped shortly before it reached the mountain station. The engineers are ready to test the cable car. For the purposes of the test, the cable car will initially be at rest in its current position, will head up the mountain, stop at the mountain station and immediately return to the resort station where it will stop, and the test will be complete.



The test begins and engineers believe that the acceleration, $a(t)$, of the cable car during the test will be: $a(t) = kt^2 - 23t + 20k$, measured in m/min^2. The variable t is the number of minutes from the moment the cable car leaves its position and k is a constant. After two minutes, the engineers expect that the cable car will be travelling with velocity 18 m/min and will not yet have reached the mountain station.

(a) Determine the value of the constant k. (3 marks)

(b) Once the cable car leaves the mountain station, how long should it take to return to the resort station? (3 marks)

	(c) Unfortunately, 10 minutes into the test, the cable car breaks down again. According to the engineers’ model, how far is the cable car from the mountain station at this time? (2 marks)

2020
Section 2
Question
11

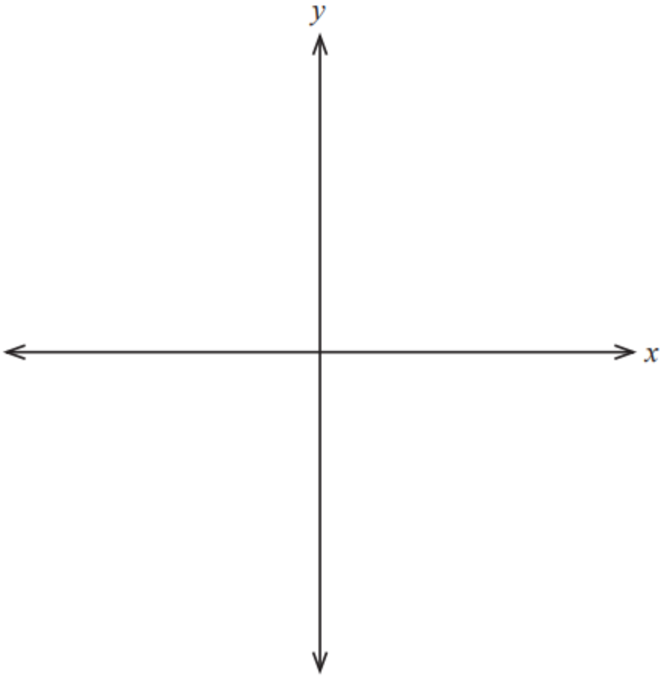
Integrals

The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

(a) Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

(b) What is the value of c? (1 mark)

(c) Sketch the graph of $f(x)$ and the tangent on the axes below. (1 mark)



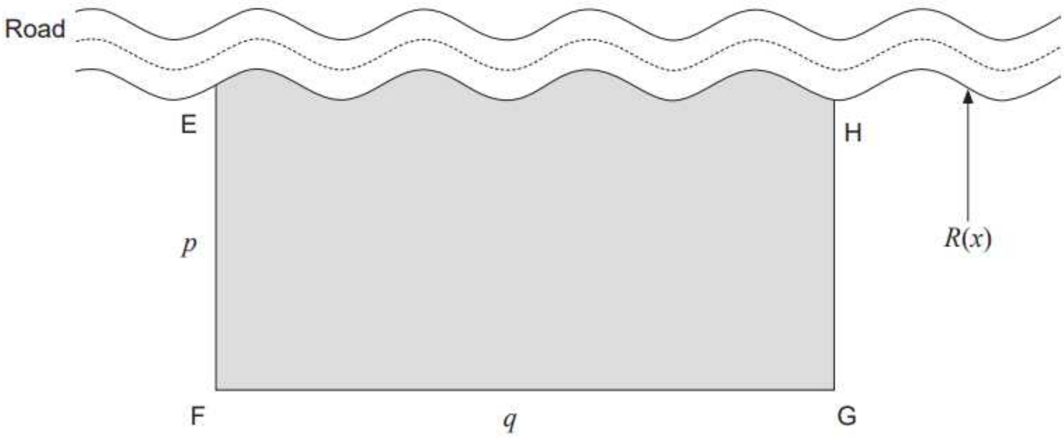
(d) Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

	(e) Given that g(x) is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

2020
Section 2
Question
17

Integrals

David and Katrina have a small farm and wish to fence off an area of their land so they can raise sheep. The area they have chosen has one border along a road as shown in the diagram below.



The enclosure is shown as the shaded area above and has right angles at points F and G. David and Katrina want the combined lengths of the fencing from E to F and F to G to equal 500 metres. Let the length of fence EF be equal to p metres and the length of fence FG be equal to q metres. If we locate the origin at point F and the x-axis along the line FG, the equation defining the fence along the road is given by:

$$R(x) = 10\sin\left(\frac{x}{15}\right) + p$$

(a) Show that the equation defining the area of the enclosure, $A(q)$, can be given in terms of q as follows:

$$A(q) = 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$

(4 marks)

(b) Determine, to the nearest metre, the value of q that will allow the sheep to graze over the maximum area and state this maximum area. (4 marks)

The length of the fence from E to H is given by the equation:

$$L_{EH} = \int_{0}^{q} \sqrt{1 + (R'(x))^2}\, dx$$, where $R'(x)$ is the first derivative of $R(x)$.

(c) (i) Determine $R'(x)$. (1 mark)

(ii) Hence determine the total length of fencing required by David and Katrina to enclose their sheep with maximum area for grazing. (3 marks)

2019
Section 2
Question 9

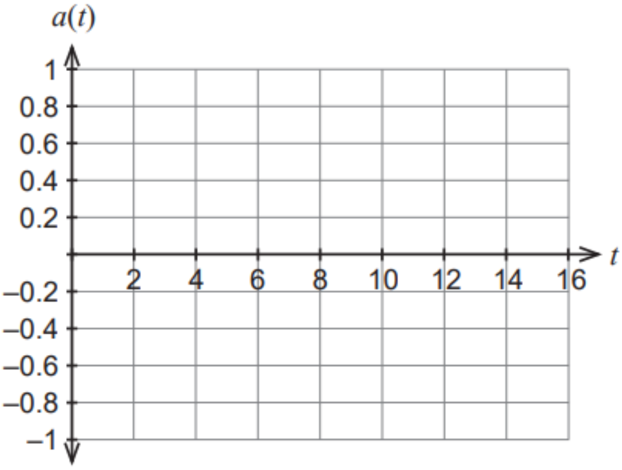
Integrals

It takes an elevator 16 seconds to ascend from the ground floor of a building to the sixth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{9\pi}{16}\sin\left(\frac{\pi t}{16}\right) \text{ m/s.}$$

The velocity, v, is measured in metres per second, while the time, t, is measured in seconds.

(a) Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \leq t \leq 16$. (2 marks)



(b) With reference to your answer from part (a), explain what is happening to the velocity of the elevator in the interval $0 < t < 8$ and in the interval $8 < t < 16$. (3 marks)

(c) Suppose that the ground floor has displacement $x = 0$ m. Determine the displacement function of the elevator and hence determine the height above the ground floor of the sixth floor. (3 marks)

2019
Section 2
Question
12

Integrals

Part of Josie's workout at her gym involves a 10 minute run on a treadmill. The treadmill's program makes her run at a constant 12.3 km/h for the first 2 minutes and then her speed, $s(t)$, is determined by the equation below, where t is the time in minutes after she began running.

$$s(t) = 10 - \frac{\ln(t - 1.99)}{t} \text{ km/h}$$

(a) Sketch the graph of her speed during this run versus time on the axes below. (3 marks)

(b) At what time(s) is Josie's speed 10 km/h? (1 mark)

(c) At what time(s) during her run is Josie's acceleration zero? (2 marks)

2019
Section 2
Question
15

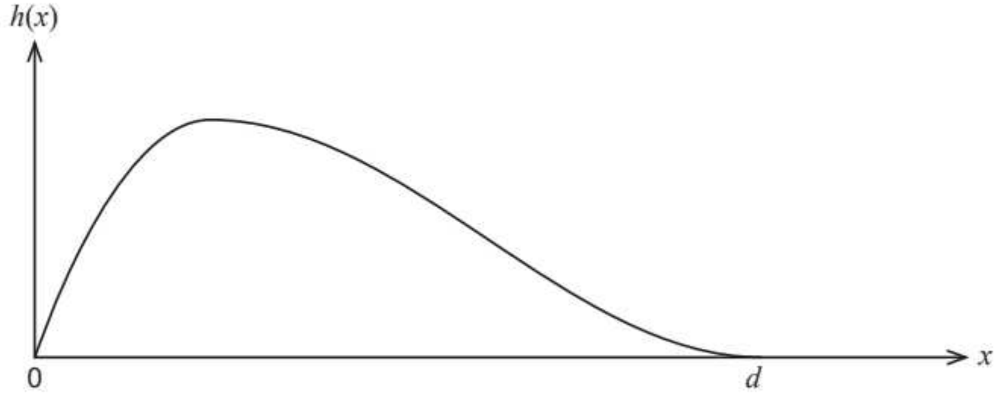
Integrals

A wall in a new Western Australian hotel is to feature a rolling, wave-shaped window. Engineers have modelled the top edge of the wave shape by joining together two functions,

$$h_1(x) = 4 - 4(x - 1)^2, \quad 0 \le x \le 1 \text{ and}$$

$$h_2(x) = a(\cos(x - 1) + 1), \quad 1 < x \le d \quad a, d \text{ constants.}$$

The functions give the height, h, above ground level of the top edge of the window measured in metres. The origin is defined as the leftmost point of the window which is at ground level and x is the horizontal distance to the right of the origin measured in metres. The graph of the two functions is shown below.

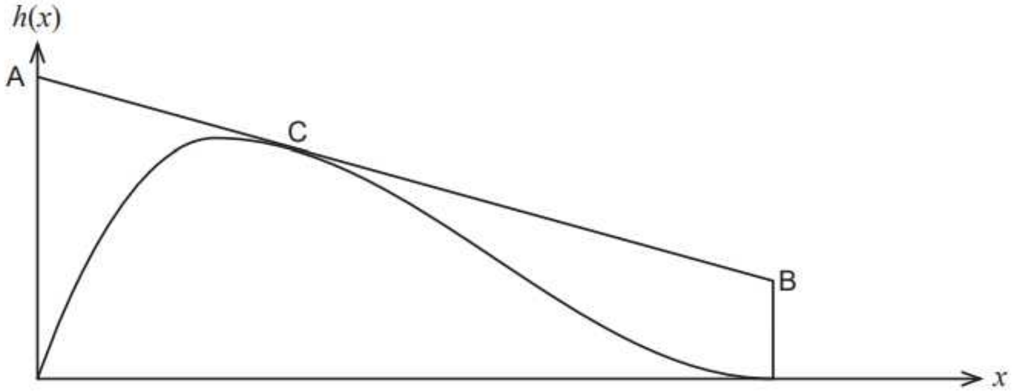


(a) Determine the value of the constant a in the function $h2\ (x) = a(\cos(x - 1) + 1)$. (3 marks)

(b) Determine the length of the bottom edge of the window. (2 marks)

(c) Determine the volume of glass required for the window if it has a uniform thickness of 3 cm. (5 marks)

The top edge of the wall, shown as the line AB below, is to just touch the window at the point C shown below. Point A is 1.39 m above the point B.



(d) How high is point C above the ground? (4 marks)

Marking Guide – Section 1

2022
Section 1
Question 1

Integrals

Consider the derivative function $f'(x) = \frac{4x}{x^2+3}$.

(a) Determine the rate of change of $f'(x)$ when $x = 1$. (3 marks)

Solution

The second derivative is

$$f''(x) = \frac{4(x^2+3) - 4x(2x)}{(x^2+3)^2}$$

Evaluating at $x = 1$ gives

$$f''(1) = \frac{(4)(4) - (4)(2)}{(4)^2}$$

$$= \frac{1}{2}$$

Specific behaviours

✓ correctly differentiates $f'(x)$
✓ indicates rate of change of $f'(x)$ when $x = 1$ is $f''(1)$
✓ correctly substitutes into $f''(x)$ and evaluates

(b) Determine $f(x)$ given that $f(1) = \ln(32)$. (4 marks)

Solution

$$f(x) = \int \frac{4x}{x^2+3} dx$$

$$= 2\int \frac{2x}{x^2+3} dx$$

$$= 2\ln(x^2+3) + c$$

Substituting $f(1) = \ln(32)$ gives

$$\ln(32) = 2\ln(1^2+3) + c$$

$$\Rightarrow c = \ln(32) - 2\ln(4)$$

$$= \ln(32) - \ln(16)$$

$$= \ln\left(\frac{32}{16}\right)$$

$$= \ln(2)$$

Hence

$$f(x) = 2\ln(x^2+3) + \ln(2)$$

Specific behaviours

✓ integrates correctly
✓ substitutes $f(1) = \ln(32)$
✓ correctly applies log laws to simplify
✓ evaluates c and states equation for $f(x)$

(c) Determine $\dfrac{d}{dt}\displaystyle\int_t^3 f(x)\,dx$.

(2 marks)

Solution

$$\frac{d}{dt}\int_t^3 f(x)\,dx = -\frac{d}{dt}\int_3^t f(x)\,dx$$

$$= -f(t)$$

$$= -2\ln(t^2+3) - \ln(2)$$

Specific behaviours

✓ uses properties of integrals to reverse integration bounds
✓ applies the fundamental theorem of calculus to obtain answer

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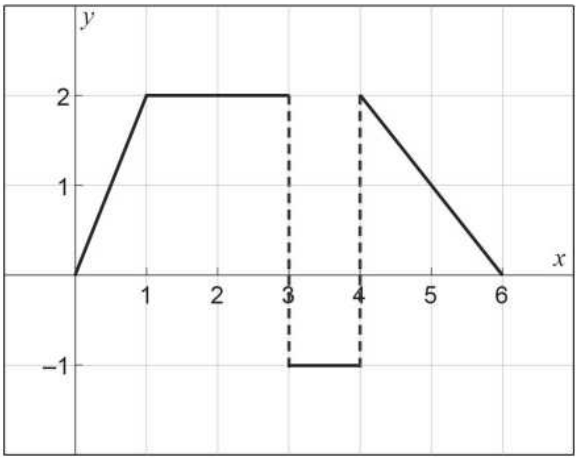
2022
Section 1
Question 2

Integrals

Consider the function *f(x)* shown below.



Evaluate the following integrals.

(a) $\displaystyle\int_0^6 f(x)\,dx$

(2 marks)

Solution

Summing signed areas

$$\int_0^6 f(x)dx = 1+4-1+2 = 6$$

Specific behaviours

✓ expresses integral in terms of signed areas
✓ evaluates integral correctly

(b) $\int_0^4 f(x) - 2\,dx$ (2 marks)

Solution
$\int_0^4 f(x) - 2\,dx = \int_0^4 f(x)\,dx - \int_0^4 2\,dx$ $= 4 - 2 \times 4$ $= -4$
Specific behaviours
✓ expresses integral as the difference between $\int_0^4 f(x)\,dx$ and $\int_0^4 2\,dx$ ✓ evaluates integral correctly

(c) $\int_4^6 f'(x)\,dx$ (2 marks)

Solution
$\int_4^6 f'(x)dx = f(6) - f(4) = 0 - 2 = -2$
Specific behaviours
✓ applies fundamental theorem of calculus ✓ evaluates correctly
Note: Accept other answers that apply the fundamental theorem of calculus.

2021
Section 1
Question 4

Integrals

Determine the following:

(a) $\int \left(2x^2 - x^3\right) dx$ (2 marks)

Solution
$\int \left(2x^2 - x^3\right) dx = \frac{2x^3}{3} - \frac{x^4}{4} + c$
Specific behaviours
✓ integrates correctly ✓ includes the constant of integration

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3 - \cos(x)} dx$ (3 marks)

Solution
$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3-\cos(x)} dx &= \left[\ln(3-\cos(x))\right]_0^{\frac{\pi}{2}} \\ &= \ln(3-\cos(\frac{\pi}{2})) - \ln(3-\cos(0)) \\ &= \ln 3 - \ln 2 \\ &= \ln\left(\frac{3}{2}\right) \end{aligned}$
Specific behaviours
✓ correctly integrates ✓ substitutes limits ✓ determines the correct simplified answer

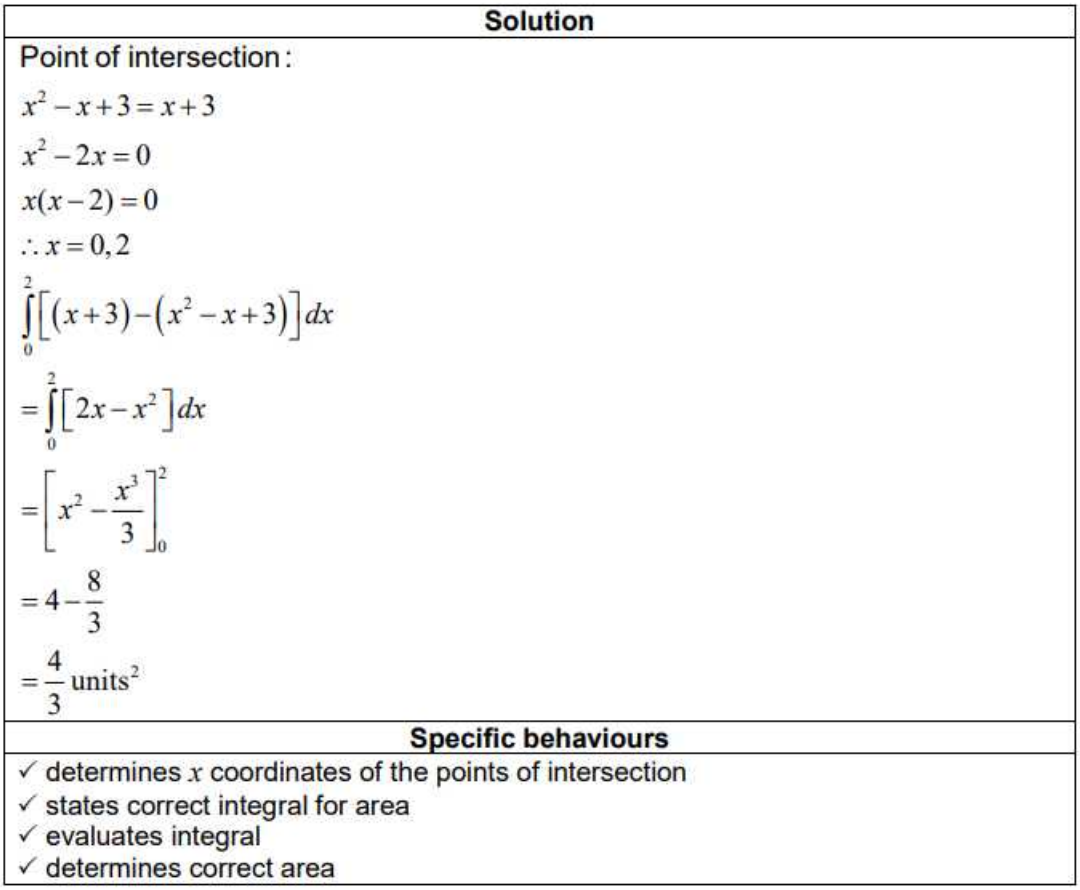
(c) $\frac{d}{dy} \int_{-1}^{y} 3x^2 \cos(2x) dx$ (2 marks)

Solution
$\frac{d}{dy} \int_{-1}^{y} 3x^2 \cos(2x) dx = 3y^2 \cos(2y)$
Specific behaviours
✓ identifies the need for the Fundamental Theorem of Calculus ✓ states the correct result

2021
Section 1
Question 5

Integrals

(a) Determine the area between the parabola $y = x2 - x + 3$ and the straight line $y = x + 3$. (4 marks)



Solution

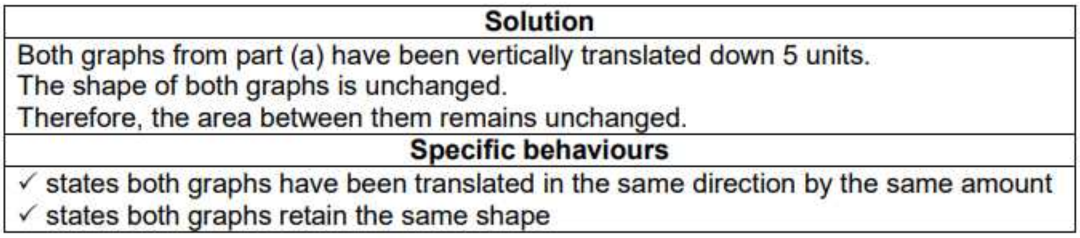
Point of intersection:

$$x^2 - x + 3 = x + 3$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$\therefore x = 0, 2$$
$$\int_0^2 \left[(x+3) - (x^2 - x + 3)\right] dx$$
$$= \int_0^2 \left[2x - x^2\right] dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= 4 - \frac{8}{3}$$
$$= \frac{4}{3} \text{ units}^2$$

Specific behaviours

✓ determines x coordinates of the points of intersection
✓ states correct integral for area
✓ evaluates integral
✓ determines correct area

(b) The area between the parabola $y = x2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part (a). Explain why this is the case. (2 marks)



Solution

Both graphs from part (a) have been vertically translated down 5 units.
The shape of both graphs is unchanged.
Therefore, the area between them remains unchanged.

Specific behaviours

✓ states both graphs have been translated in the same direction by the same amount
✓ states both graphs retain the same shape

2019
Section 1
Question 5

Integrals

(a) Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks)

Solution

First we obtain the area under the graph of $f(x)$between $x = 0$ and $x = \ln 2$. This is given by

$$A = \int_0^{ln2} e^x dx = e^x|_0^{ln2} = 2 - 1 = 1.$$

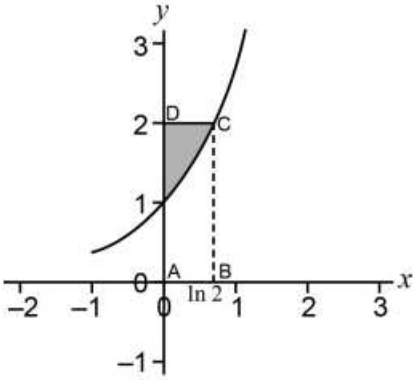
Specific behaviours

✓ writes down the correct integral
✓ integrates correctly
✓ simplifies to obtain final answer

(b) Hence, determine the area bound by the graph of $f(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks)

Solution

This is given by the area shown below.



That is,

$$Area = 2\ln 2 - 1$$

Specific behaviours

✓ correctly defines the area
✓ calculates the area correctly

(c) Determine the area bound by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

Solution

$$\int_0^{lna} e^x dx = e^x|_0^{lna} = a - 1$$

$$A = \ln(a) \times a - (a - 1)$$

$$= a\ln(a) - a + 1$$

Specific behaviours

✓ writes down the correct integral
✓ integrates correctly and simplifies to obtain $a–1$
✓ determines the correct expression for area

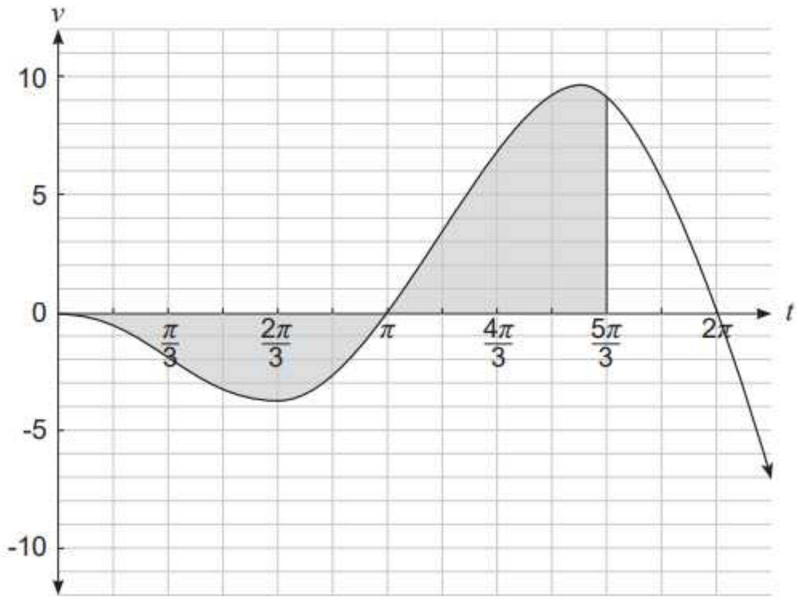
2023
Section 2
Question 8

Integrals

An oscillating mass has a velocity, *v*, given by

$$v(t) = 2t\cos\left(t + \frac{\pi}{2}\right) \quad \text{for } t \geq 0.$$

The velocity is given in metres per second, and the time, *t*, is given in seconds. A graph of the velocity of the mass' motion is shown below.



(a) Determine the first two times, $t > 0$, at which the mass changes direction. State your answers exactly. (2 marks)

Solution
The mass changes direction when $v(t) = 0$. From the graph it is clear that $v(t) = 0$ when $t = \pi$ seconds, and $t = 2\pi$ seconds.
Specific behaviours
✓ states that a change in direction occurs when $v(t) = 0$ ✓ correctly determines the first two times the mass changes direction

(b) What does the signed area of the shaded region in the figure represent? (2 marks)

Solution
The shaded region represents the change in the displacement of the mass from $t = 0$ to $t = \frac{5\pi}{3}$ seconds.
Specific behaviours
✓ states that the shaded region represents a change in displacement ✓ specifies the correct start and finish time

(c) Write an integral expression for the distance travelled from $t=\frac{\pi}{3}$ to $t=\frac{4\pi}{3}$. (3 marks)

Solution
distance = $\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left\lvert 2t\cos\left(t+\frac{\pi}{2}\right)\right\rvert dt$
Specific behaviours
✓ identifies distance as the integral of speed (absolute value of velocity) ✓ writes the correct integral (including bounds) ✓ *dt* included

Alternative solution one
distance = $-\int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t+\frac{\pi}{2}\right)dt+\int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t+\frac{\pi}{2}\right)dt$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ correct sign (+/-) in front of integrals ✓ *dt* included in both integrals

Alternative solution two
distance = $\left\lvert\int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t+\frac{\pi}{2}\right)dt\right\rvert+\left\lvert\int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t+\frac{\pi}{2}\right)dt\right\rvert$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ absolute value signs included (either inside the integral or outside) ✓ *dt* included in both integrals

(d) Determine the first time after $t=\pi$ that the acceleration of the object will be 0 m/s^2.
(2 marks)

Solution
The acceleration is given by $a(t)=v'(t)=2\cos\left(t+\frac{\pi}{2}\right)-2t\sin\left(t+\frac{\pi}{2}\right)$ or $a(t)=v'(t)=-2\sin(t)-2t\cos(t)$ Solving $v'(t)=0$ gives $t\approx 4.91$ seconds
Specific behaviours
✓ states correct expression for acceleration ✓ solves to obtain correct time

(e) The displacement of the mass is given by

$$x(t) = A\sin\left(t+\frac{\pi}{2}\right) + B\cos\left(t+\frac{\pi}{2}\right) + 2t\sin\left(t+\frac{\pi}{2}\right)$$

where *A* and *B* are constants. Determine the value of *A* and *B*. (3 marks)

Solution

$$v(t) = x'(t) = A\cos\left(t+\frac{\pi}{2}\right) - B\sin\left(t+\frac{\pi}{2}\right) + 2\sin\left(t+\frac{\pi}{2}\right) + 2t\cos\left(t+\frac{\pi}{2}\right)$$

Given that $v(t) = 2t\cos\left(t+\frac{\pi}{2}\right)$ it follows that $A = 0$ and

$$-B + 2 = 0$$
$$\Rightarrow B = 2$$

Specific behaviours

✓ correctly differentiates $x(t)$

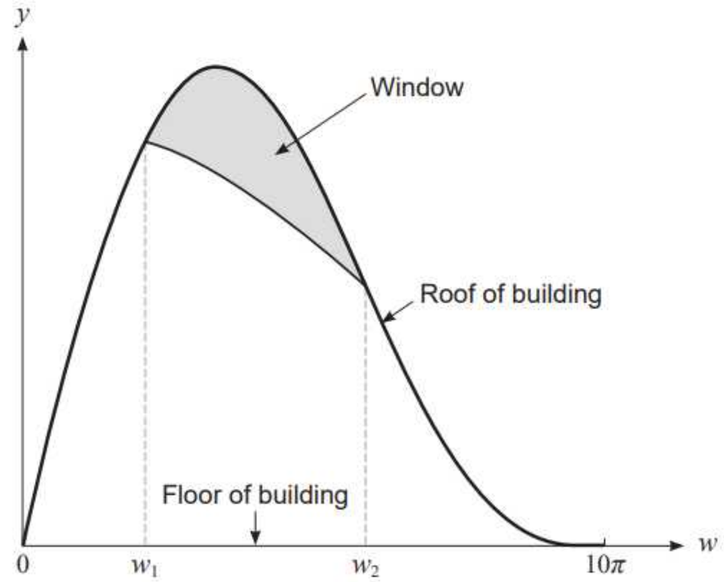
✓ compares $x'(t)$ and $v(t)$ to determine that $A = 0$

✓ compares $x'(t)$ and $v(t)$ to determine that $B = 2$

2023
Section 2
Question 9

Integrals

A new entertainment venue is being proposed. The preliminary design has a constant cross-section, as shown in the figure below.



The roof height $h(w)$ of the building at any point w along its width is given by

$$h(w) = 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)$$

where h and $0 \le w \le 10\pi$ are measured in metres.

(a) Determine the cross-sectional area of the building. (2 marks)

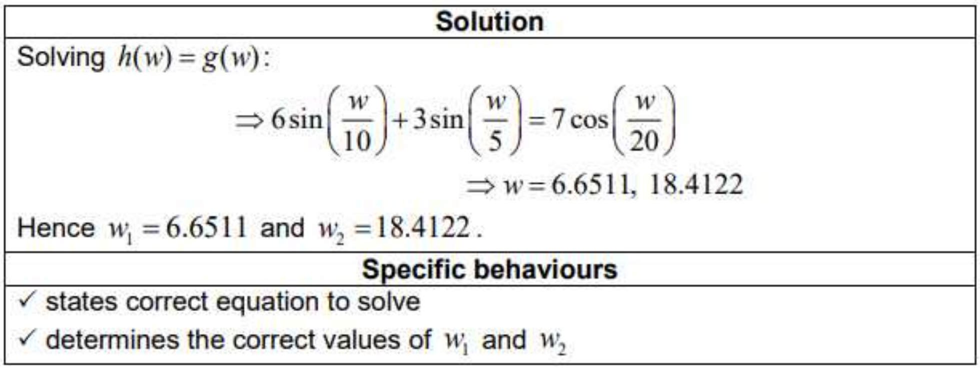
Solution
$$\text{Area} = \int_0^{10\pi}\left(6\sin\left(\frac{w}{10}\right)+3\sin\left(\frac{w}{5}\right)\right)dx$$ $$= 120 \text{ m}^2$$
Specific behaviours
✓ states a correct integral expression for the cross-sectional area ✓ correctly determines the cross-sectional area including units

The designer would like to place a window, as shown in the figure above, that is bounded above by the roof of the building and below by the formula

$$g(w) = 7\cos\left(\frac{w}{20}\right).$$

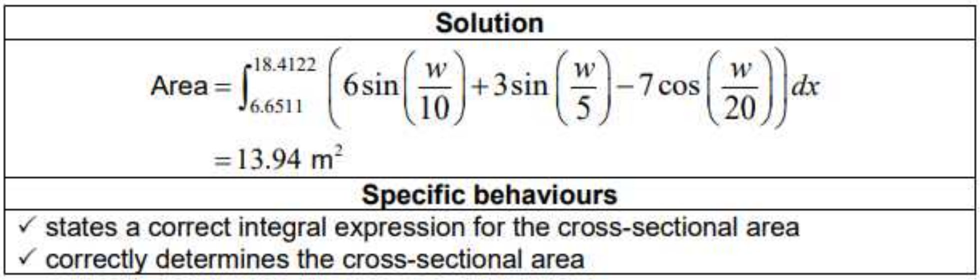
(b) With reference to the figure

(i) determine the values of w_1 and w_2. (2 marks)



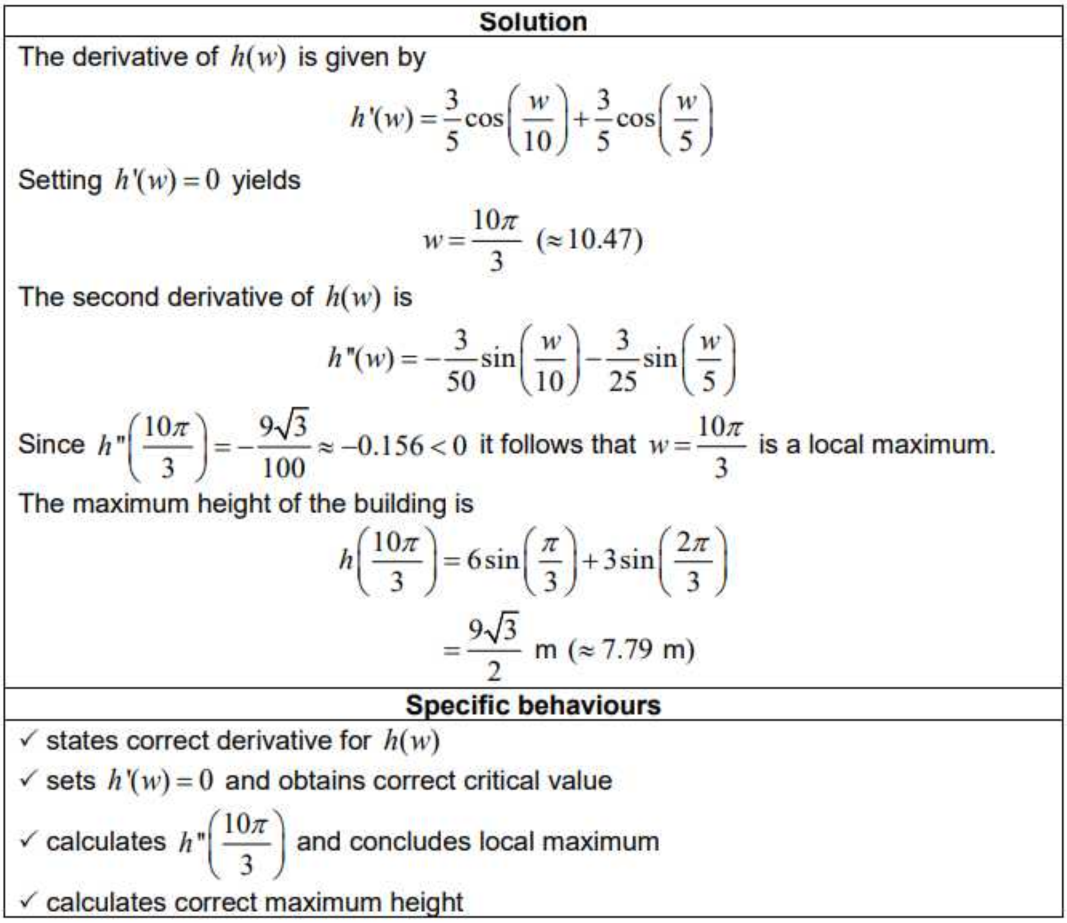
Solution
Solving $h(w) = g(w)$: $\Rightarrow 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right) = 7\cos\left(\frac{w}{20}\right)$ $\Rightarrow w = 6.6511,\ 18.4122$ Hence $w_1 = 6.6511$ and $w_2 = 18.4122$.
Specific behaviours
✓ states correct equation to solve ✓ determines the correct values of w_1 and w_2

(ii) determine the area of the window. (2 marks)



Solution
Area $= \int_{6.6511}^{18.4122} \left(6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right) - 7\cos\left(\frac{w}{20}\right)\right) dx$ $= 13.94 \text{ m}^2$
Specific behaviours
✓ states a correct integral expression for the cross-sectional area ✓ correctly determines the cross-sectional area

(c) Use calculus techniques to determine the maximum height of the building. (4 marks)

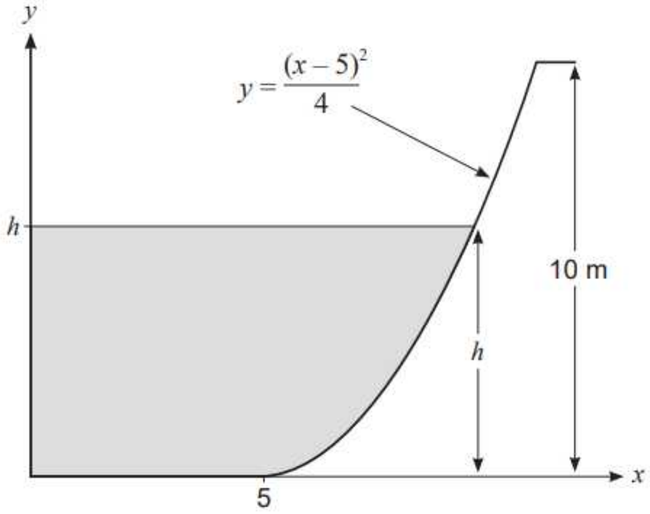


Solution
The derivative of $h(w)$ is given by $h'(w) = \frac{3}{5}\cos\left(\frac{w}{10}\right) + \frac{3}{5}\cos\left(\frac{w}{5}\right)$ Setting $h'(w) = 0$ yields $w = \frac{10\pi}{3}\ (\approx 10.47)$ The second derivative of $h(w)$ is $h''(w) = -\frac{3}{50}\sin\left(\frac{w}{10}\right) - \frac{3}{25}\sin\left(\frac{w}{5}\right)$ Since $h''\left(\frac{10\pi}{3}\right) = -\frac{9\sqrt{3}}{100} \approx -0.156 < 0$ it follows that $w = \frac{10\pi}{3}$ is a local maximum. The maximum height of the building is $h\left(\frac{10\pi}{3}\right) = 6\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{2\pi}{3}\right)$ $= \frac{9\sqrt{3}}{2} \text{ m } (\approx 7.79 \text{ m})$
Specific behaviours
✓ states correct derivative for $h(w)$ ✓ sets $h'(w) = 0$ and obtains correct critical value ✓ calculates $h''\left(\frac{10\pi}{3}\right)$ and concludes local maximum ✓ calculates correct maximum height

2023
Section 2
Question
14

Integrals

A small dam on an agricultural property has a length of 20 m, and a uniform cross-section shown below where x and y are in metres. The base of the dam is flat for $0 \le x \le 5$, and the right side is given by $y = \frac{(x-5)^2}{4}$ for $5 < x \le 11.325$. The shaded region on the graph below represents the cross-section of a volume of water V (m^3) in the dam with water level h (m).



(a) Using calculus, show that the volume of water in the dam is given by

$$V(h) = 100h + \frac{80}{3}h^{\frac{3}{2}}.$$

(5 marks)

Solution
Determine an expression for x in terms of h by solving $$h=\frac{(x-5)^2}{4}$$ $$\Rightarrow 4h=(x-5)^2$$ $$\Rightarrow x=5+2\sqrt{h}$$ Hence the volume is given by $$V(h)=20\left(5h+\int_5^{5+2\sqrt{h}} h-\frac{(x-5)^2}{4}dx\right)$$ $$=20\left(5h+\left[hx-\frac{(x-5)^3}{12}\right]_5^{5+2\sqrt{h}}\right)$$ $$=20\left(5h+\left(h\left(5+2\sqrt{h}\right)-\frac{\left(2\sqrt{h}\right)^3}{12}\right)-5h\right)$$ $$=20\left(5h+\frac{4}{3}h^{\frac{3}{2}}\right)$$ $$=100h+\frac{80}{3}h^{\frac{3}{2}}$$
Specific behaviours
✓ determines an expression for the upper bound of the volume integral ✓ states a correct integral expression for the volume or cross-sectional area of water ✓ states correct antiderivative of integrand ✓ correctly applies fundamental theorem of calculus by substituting integration bounds ✓ simplifies to give desired result

(b) Use the increments formula to estimate the change in water volume if the water level rises from 6 m to 6.1 m. (3 marks)

Solution

The derivative of V with respect to h is

$$\frac{dV}{dh} = 100 + 40\sqrt{h}$$

The change in h is

$$\delta h = 6.1 - 6 = 0.1$$

Hence the change in V is approximately

$$\begin{aligned}\delta V &\approx \frac{dV}{dh}\delta h \\ &= \left(100 + 40\sqrt{h}\right) \times 0.1 \\ &= 10 + 4\sqrt{h}\end{aligned}$$

When $h = 6$ we have

$$\begin{aligned}\delta V &\approx 10 + 4\sqrt{6} \\ &\approx 19.80 \text{ m}^3\end{aligned}$$

Specific behaviours

✓ determines correct expression for the derivative of V with respect to h
✓ determine the increment in h
✓ obtains correct estimate for the change in V (exact or decimal)

Suppose the water volume at the start of winter is 1000 m^3. On the basis of rainfall data from previous years, the volume of water V_R (m^3) that will flow into the dam over winter is normally distributed with a mean of 600 m3 and a standard deviation of 200 m^3.

(c) Assuming that there are no other sources of water and no losses, determine the probability that the dam will reach full capacity (i.e. depth of 10 m) during winter.(3 marks)

Solution

The full capacity of the dam is

$$\begin{aligned}V(10) &= 100(10) + \frac{80}{3}(10)^{\frac{3}{2}} \\ &\approx 1843.27 \text{ m}^3\end{aligned}$$

Hence the dam will reach capacity if

$$\begin{aligned}V_R &\geq V(10) - 1000 \\ &= 1843.27 - 1000 \\ &= 843.27 \text{ m}^3\end{aligned}$$

Since $V_R \sim N\left(600,\ 200^2\right)$ it follows that

$$P\left(V_R \geq 843.27\right) \approx 0.1119$$

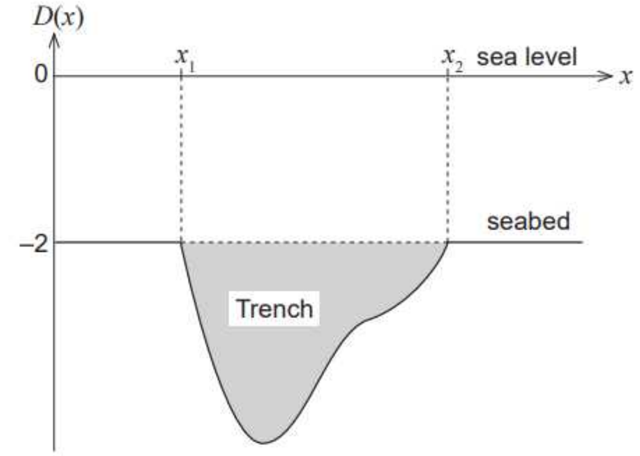
Specific behaviours

✓ determines the correct volume $V(10)$
✓ determines that we need $V_R \geq 843.27$
✓ obtains correct probability

2022
Section 2
Question 7

Integrals

A team of oceanographers surveyed the depth of the ocean in a region populated by a particular endangered fish species. They discovered a large trench extending below the otherwise flat seabed as shown in the figure below.



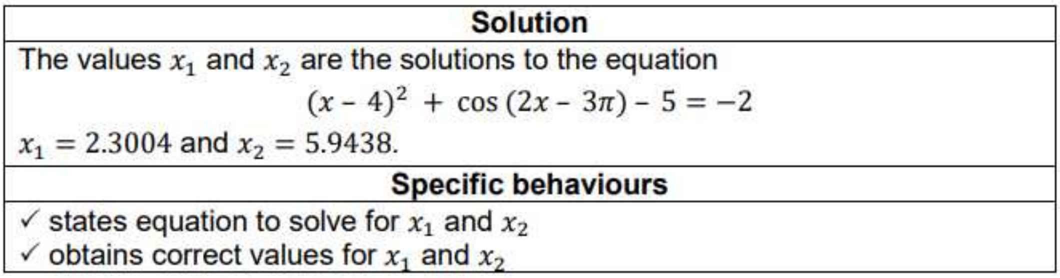
The displacement, in kilometres, from sea level to the ocean floor is given by

$$D(x) = \begin{cases} (x-4)^2 + \cos(2x - 3\pi) - 5, & x_1 \le x \le x_2 \\ -2, & \text{otherwise} \end{cases}$$

where x (measured in kilometres) is the east–west horizontal displacement relative to a reference marker at sea level.

(a) With reference to the figure above:

(i) determine the values of x_1 and x_2. (2 marks)



Solution
The values x_1 and x_2 are the solutions to the equation $(x - 4)^2 + \cos(2x - 3\pi) - 5 = -2$ $x_1 = 2.3004$ and $x_2 = 5.9438$.
Specific behaviours
✓ states equation to solve for x_1 and x_2 ✓ obtains correct values for x_1 and x_2

(ii) use calculus to determine the cross-sectional area of the trench shaded in the figure above. (3 marks)

Solution

The cross-sectional area of the trench A is given by

$$\begin{aligned} A &= \int_{2.3004}^{5.9438} -2 - \left((x-4)^2 + \cos(2x-3\pi) - 5\right) dx \\ &= \int_{2.3004}^{5.9438} (x-4)^2 + \cos(2x-3\pi) - 3\, dx \\ &= \left[\frac{(x-4)^3}{3} + \frac{1}{2}\sin(2x-3\pi) - 3x\right]_{2.3004}^{5.9438} \\ &\approx 7.0285 \text{ km}^2 \end{aligned}$$

Specific behaviours

✓ writes the correct definite integral for the trench cross-sectional area
✓ obtains correct antiderivative
✓ determines the correct cross-sectional area of the trench

(b) Using calculus, determine the maximum distance of the trench below sea level. (5 marks)

Solution

To determine the maximum depth of the trench we must minimise

$$D(x) = (x-4)^2 + \cos(2x-3\pi) - 5$$

Differentiating we have

$$D'(x) = 2(x-4) - 2\sin(2x-3\pi)$$

or

$$D'(x) = 2x + 2\sin(2x) - 8$$

Solving $D'(x) = 0$

$$\begin{aligned} 0 &= 2(x-4) - 2\sin(2x-3\pi) \\ x &\approx 3.4392 \end{aligned}$$

The second derivative of D is given by

$$D''(x) = -4\cos(2x-3\pi) + 2 > 0$$

Since $D''(3.4392) = 5.3121 > 0$ for all x it follows that there is a local minimum at $x \approx 3.4392$. Given that

$$\begin{aligned} D(3.4392) &= (3.4392-4)^2 + \cos(2(3.4392) - 3\pi) - 5 \\ &\approx -5.51 \text{ km} \end{aligned}$$

The maximum distance of the trench is 5.51 km below sea level.

Specific behaviours

✓ identifies maximum distance occurs when $D'(x) = 0$
✓ solve for the critical point
✓ verifies that the critical point is a local minimum
✓ evaluates the function at the critical point
✓ determines the maximum distance of the trench (converts distance to a positive number)

2022
Section 2
Question
10

Integrals

The displacement, x, of a mass on the end of a damped spring is given by

$$x(t) = 3e^{-t}\sin(t), \quad t \geq 0$$

where x is in centimetres and t is in seconds

(a) Determine when the mass first returns to its starting position at x = 0. (2 marks)

Solution

Setting $x = 0$ gives

$$0 = 3e^{-t}\sin(t)$$

$$\Rightarrow 0 = 3e^{-t} \text{ or } 0 = \sin(t)$$

The equation $0 = 3e^{-t}$ has no solution, while $0 = \sin(t)$ has solutions $t = 0, \pi, 2\pi, \ldots$
Hence the spring first returns to its starting position after 3.14 seconds.

Specific behaviours

✓ writes correct equation to solve for return to starting position
✓ solves for the first time returning to starting position

(b) Determine an expression for the velocity of the mass. (2 marks)

Solution

$$v(t) = 3e^{-t}\cos(t) - 3e^{-t}\sin(t)$$

Specific behaviours

✓ recognises that $v(t) = x'(t)$
✓ obtains correct expression for the velocity

(c) Determine the displacement of the mass when it first changes direction. (3 marks)

Solution

Mass changes direction when $v = 0$, so

$$0 = 3e^{-t}\cos(t) - 3e^{-t}\sin(t)$$

$$\Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$$

Hence the first change of direction is at $t = \frac{\pi}{4}$ seconds when

$$x\left(\frac{\pi}{4}\right) = 3e^{-\frac{\pi}{4}}\sin\left(\frac{\pi}{4}\right)$$

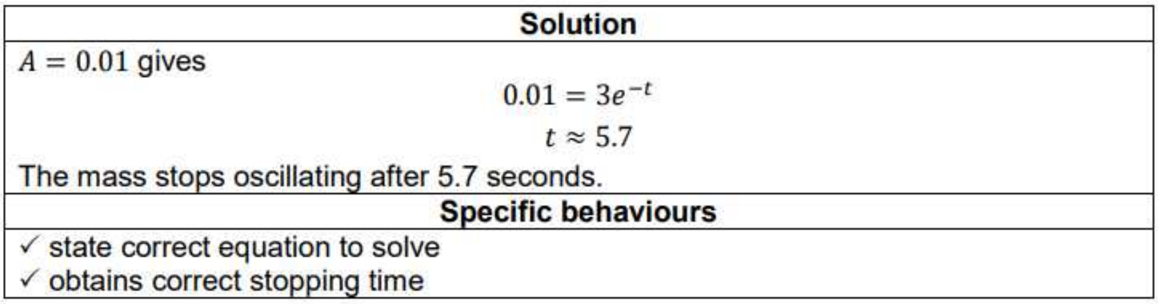
$$= \frac{3e^{-\frac{\pi}{4}}}{\sqrt{2}}$$

$$\approx 0.97 \text{ cm}$$

Specific behaviours

✓ states correct equation to solve
✓ solves to obtain correct value of t
✓ substitutes to obtain correct value of x

(d) The mass is considered to have stopped oscillating when the oscillation amplitude $A(t) = 3e^{-t}$ drops to 0.01 cm. How long does it take for the spring to stop oscillating? (2 marks)



Solution

$A = 0.01$ gives

$$0.01 = 3e^{-t}$$

$$t \approx 5.7$$

The mass stops oscillating after 5.7 seconds.

Specific behaviours

✓ state correct equation to solve
✓ obtains correct stopping time

2022
Section 2
Question
11

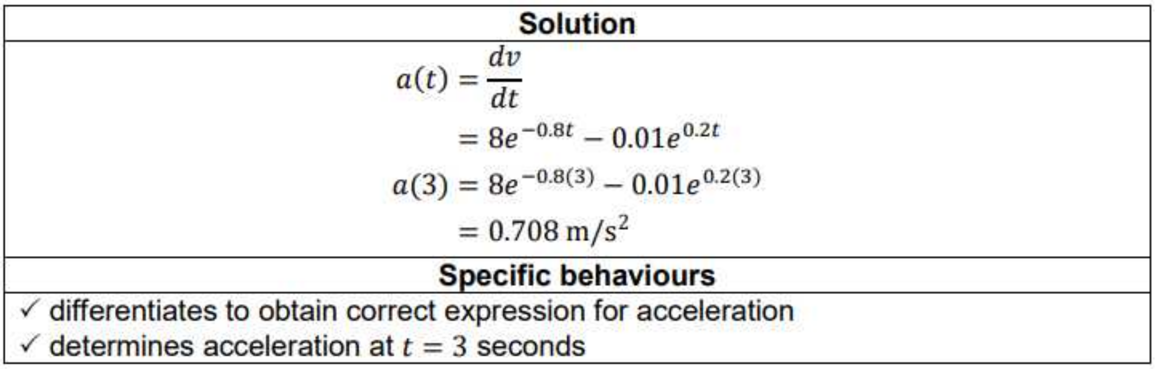
Integrals

The 100 m sprint is a race run on a straight section of track. During a race the velocity, v, measured in metres per second, of an athlete is given by

$$v(t) = -10e^{-0.8t} - 0.05e^{0.2t} + 10.05$$

where t is the time, in seconds, measured from the moment the athlete starts to move from the start line.

(a) Determine the acceleration of the athlete three seconds after moving from the start line. (2 marks)



Solution

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= 8e^{-0.8t} - 0.01e^{0.2t} \\ a(3) &= 8e^{-0.8(3)} - 0.01e^{0.2(3)} \\ &= 0.708 \text{ m/s}^2 \end{aligned}$$

Specific behaviours

✓ differentiates to obtain correct expression for acceleration
✓ determines acceleration at $t = 3$ seconds

(b) Using calculus, determine the maximum velocity of the athlete during the race, and the time, t, at which it is achieved. (4 marks)

Solution

Maximum velocity when $a(t) = 0$.

$$0 = 8e^{-0.8t} - 0.01e^{0.2t}$$

$$t = \ln(800) \approx 6.68 \text{ seconds}$$

Maximum velocity is then

$$v(6.68) \approx 9.81 \text{ m/s}$$

Since $v''(t) = -6.4e^{-0.8t} - 0.002e^{0.2t} < 0$ for all t it follows that we have found the maximum value of v.

Hence the maximum velocity of 9.81 m/s was achieved after 6.68 seconds.

Specific behaviours

✓ recognises $a(t) = 0$ gives time of maximum velocity
✓ solves for value of t
✓ determines value of v
✓ confirms that $v'' < 0$ to give a maximum

(c) The displacement, x, of the athlete is 0 m at the start of the race. Determine an expression for the displacement of the athlete during the race. (3 marks)

Solution

$$\begin{aligned} x(t) &= \int v(t)\, dt \\ &= \int -10e^{-0.8t} - 0.05e^{0.2t} + 10.05 \ dt \\ &= 12.5e^{-0.8t} - 0.25e^{0.2t} + 10.05t + c \end{aligned}$$

Given that $x(0) = 0$ it follows that

$$0 = 12.5 - 0.25 + c$$

$$c = -12.25$$

Hence

$$x(t) = 12.5e^{-0.8t} - 0.25e^{0.2t} + 10.05t - 12.25 \text{ m}$$

Specific behaviours

✓ recognises displacement as the integral of velocity
✓ determines correct anti-derivative (including $+c$)
✓ imposes initial condition to determine value for c and hence the solution

(d) Determine the time, t, at which the athlete finishes the 100 m race. (2 marks)

Solution

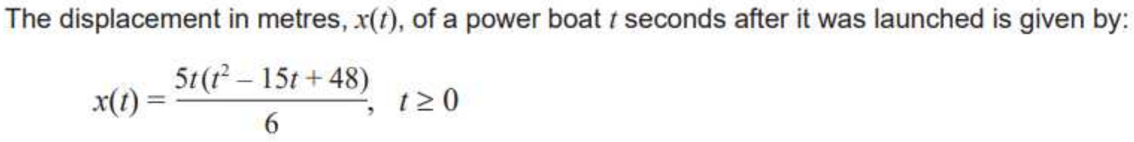
Need to solve $x(t) = 100$, $t = 11.41$ seconds.

Specific behaviours

✓ recognises need to solve $x(t) = 100$
✓ solves for correct value of t

2021
Section 2
Question
14

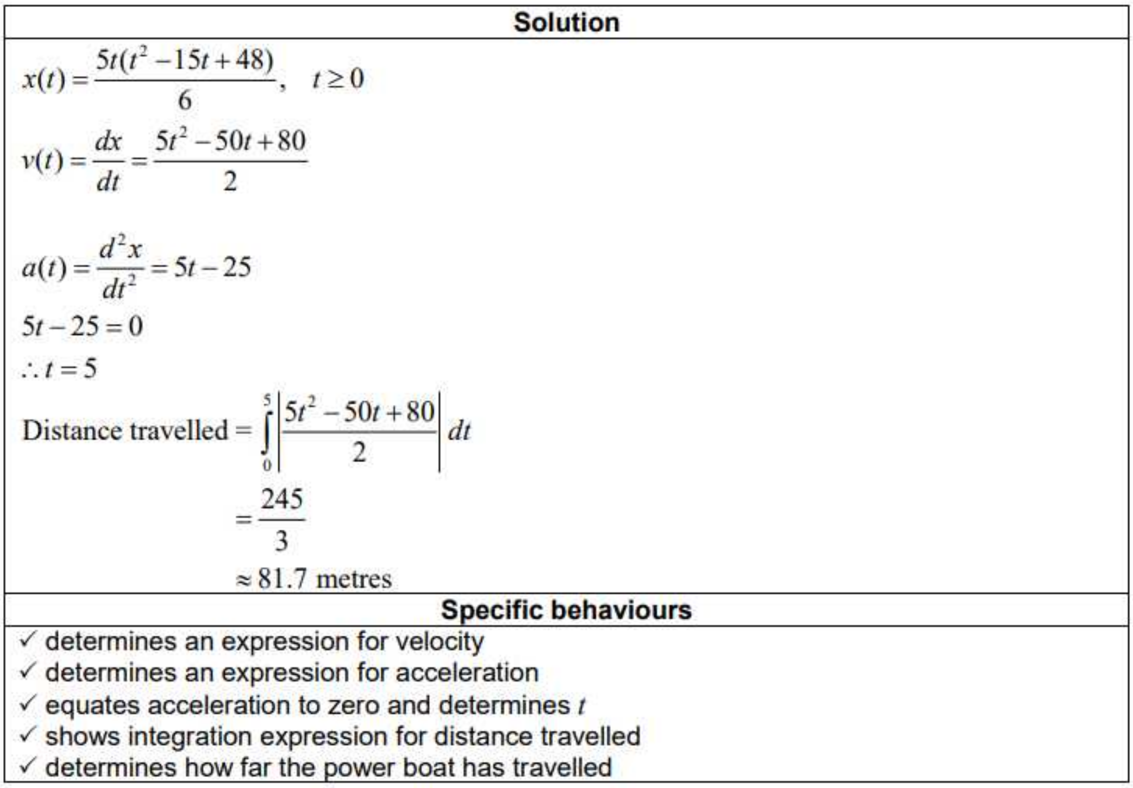
Integrals



The displacement in metres, $x(t)$, of a power boat t seconds after it was launched is given by:

$$x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$$

How far has the power boat travelled before its acceleration is zero? (5 marks)



Solution

$$x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$$

$$v(t) = \frac{dx}{dt} = \frac{5t^2 - 50t + 80}{2}$$

$$a(t) = \frac{d^2x}{dt^2} = 5t - 25$$

$$5t - 25 = 0$$

$$\therefore t = 5$$

$$\text{Distance travelled} = \int_0^5 \left| \frac{5t^2 - 50t + 80}{2} \right| dt$$

$$= \frac{245}{3}$$

$$\approx 81.7 \text{ metres}$$

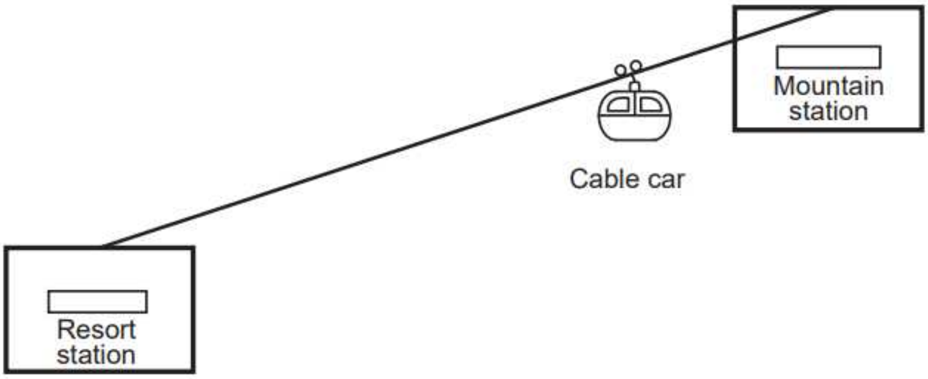
Specific behaviours

✓ determines an expression for velocity
✓ determines an expression for acceleration
✓ equates acceleration to zero and determines t
✓ shows integration expression for distance travelled
✓ determines how far the power boat has travelled

2021
Section 2
Question
17

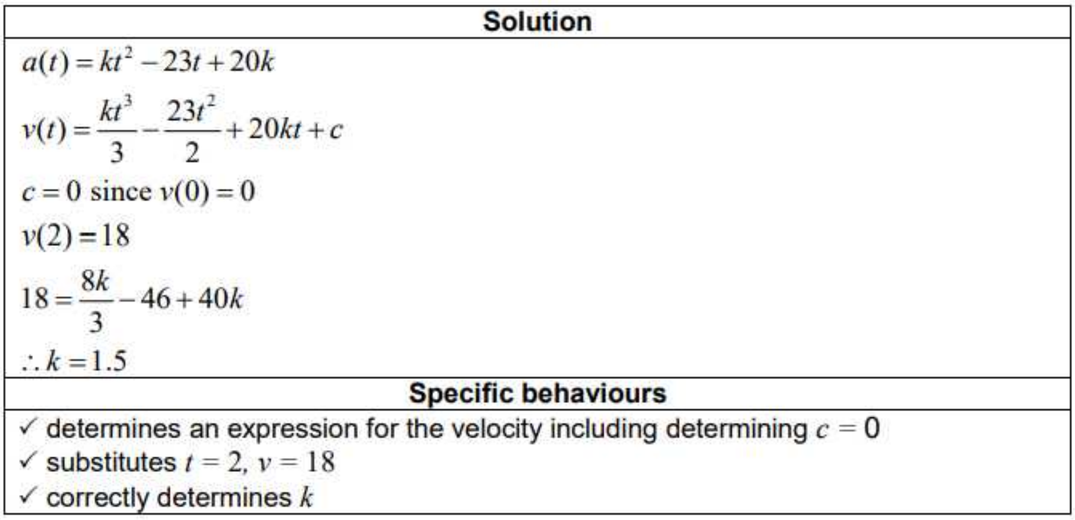
Integrals

A resort in the Swiss Alps features a cable car that travels from the resort station to the mountain station. Engineers are fixing a cable car that unexpectedly stopped shortly before it reached the mountain station. The engineers are ready to test the cable car. For the purposes of the test, the cable car will initially be at rest in its current position, will head up the mountain, stop at the mountain station and immediately return to the resort station where it will stop, and the test will be complete.



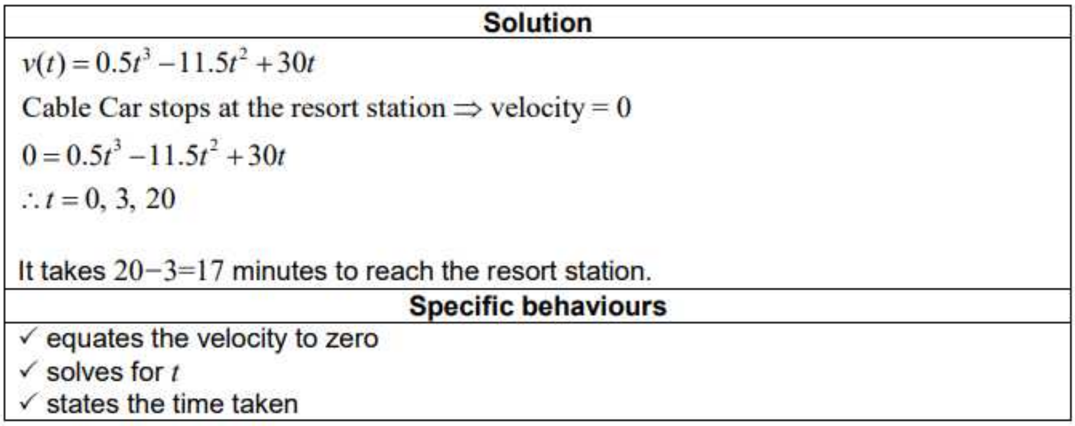
The test begins and engineers believe that the acceleration, $a(t)$, of the cable car during the test will be: $a(t) = kt^2 - 23t + 20k$, measured in m/min². The variable t is the number of minutes from the moment the cable car leaves its position and k is a constant. After two minutes, the engineers expect that the cable car will be travelling with velocity 18 m/min and will not yet have reached the mountain station.

(a) Determine the value of the constant k. (3 marks)



Solution
$a(t) = kt^2 - 23t + 20k$ $v(t) = \frac{kt^3}{3} - \frac{23t^2}{2} + 20kt + c$ $c = 0$ since $v(0) = 0$ $v(2) = 18$ $18 = \frac{8k}{3} - 46 + 40k$ $\therefore k = 1.5$
Specific behaviours
✓ determines an expression for the velocity including determining $c = 0$ ✓ substitutes $t = 2,\ v = 18$ ✓ correctly determines k

(b) Once the cable car leaves the mountain station, how long should it take to return to the resort station? (3 marks)



Solution

$v(t) = 0.5t^3 - 11.5t^2 + 30t$

Cable Car stops at the resort station $\Rightarrow$ velocity $= 0$

$0 = 0.5t^3 - 11.5t^2 + 30t$

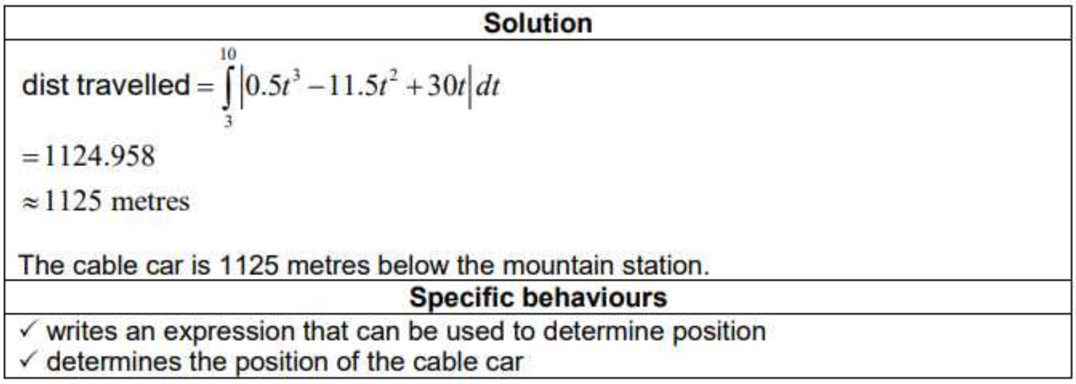
$\therefore t = 0,\ 3,\ 20$

It takes $20-3=17$ minutes to reach the resort station.

Specific behaviours

✓ equates the velocity to zero
✓ solves for t
✓ states the time taken

(c) Unfortunately, 10 minutes into the test, the cable car breaks down again. According to the engineers' model, how far is the cable car from the mountain station at this time? (2 marks)



Solution

dist travelled $= \int_3^{10} \left|0.5t^3 - 11.5t^2 + 30t\right| dt$

$= 1124.958$

≈ 1125 metres

The cable car is 1125 metres below the mountain station.

Specific behaviours

✓ writes an expression that can be used to determine position
✓ determines the position of the cable car

2020
Section 2
Question
11

Integrals

The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

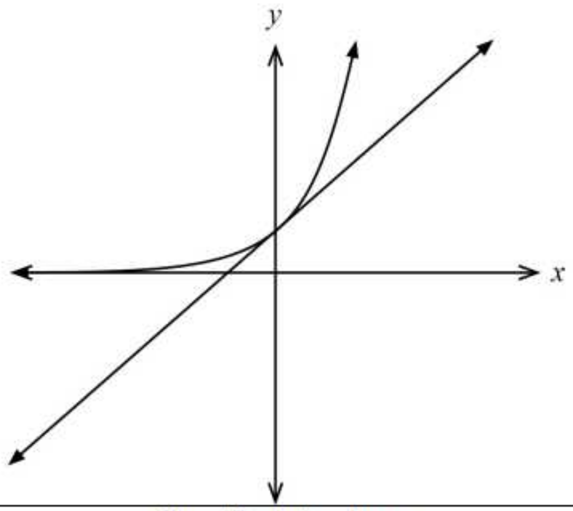
(a) Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

Solution
$f'(x) = e^x = 1$ So $x = 0, f(0) = 1$
Specific behaviours
✓ obtains equation to solve for the x coordinate ✓ states the coordinates of the point

(b) What is the value of c? (1 mark)

Solution
The point $(0,1)$ lies on the line, so $1 = 0 + c \Rightarrow c = 1$
Specific behaviours
✓ obtains the correct value of c

(c) Sketch the graph of $f(x)$ and the tangent on the axes below. (1 mark)

Solution

Specific behaviours
✓ sketches both functions showing tangent at (0,1) with shapes correct

(d) Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

Solution
$Area = \int_0^{\ln 2}(e^x - (x+1))dx$ $= e^x - \frac{x^2}{2} - x\Big\vert_0^{\ln 2}$ $= 2 - \frac{(\ln 2)^2}{2} - \ln 2 - 1$ $= 1 - \frac{(\ln 2)^2}{2} - \ln 2$
Specific behaviours
✓ writes down the correct integrand ✓ gives the correct limits for the integral ✓ evaluates correctly

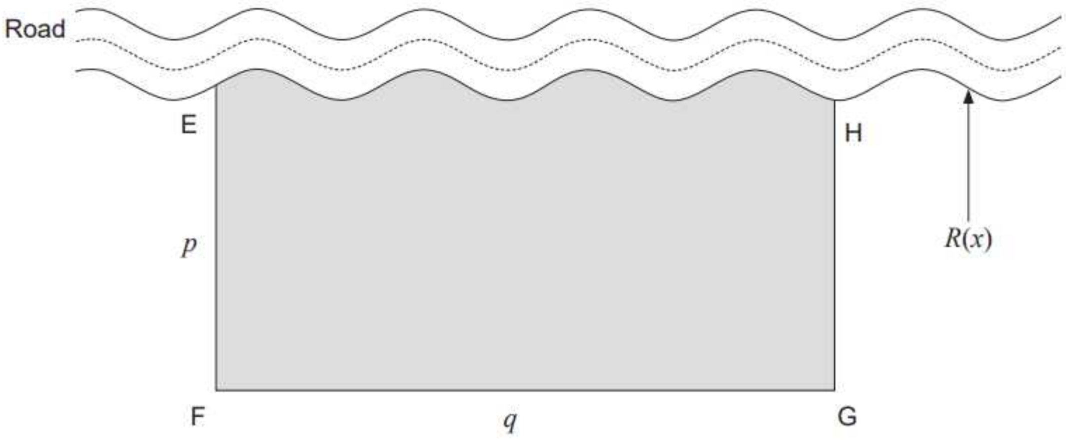
(e) Given that g(x) is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

Solution
$Area = -\int_{\ln 2}^{1} \ln(x)dx$
Specific behaviours
✓ recognises the inverse function of $f(x)$ ✓ states the correct definite integral

2020
Section 2
Question
17

Integrals

David and Katrina have a small farm and wish to fence off an area of their land so they can raise sheep. The area they have chosen has one border along a road as shown in the diagram below.

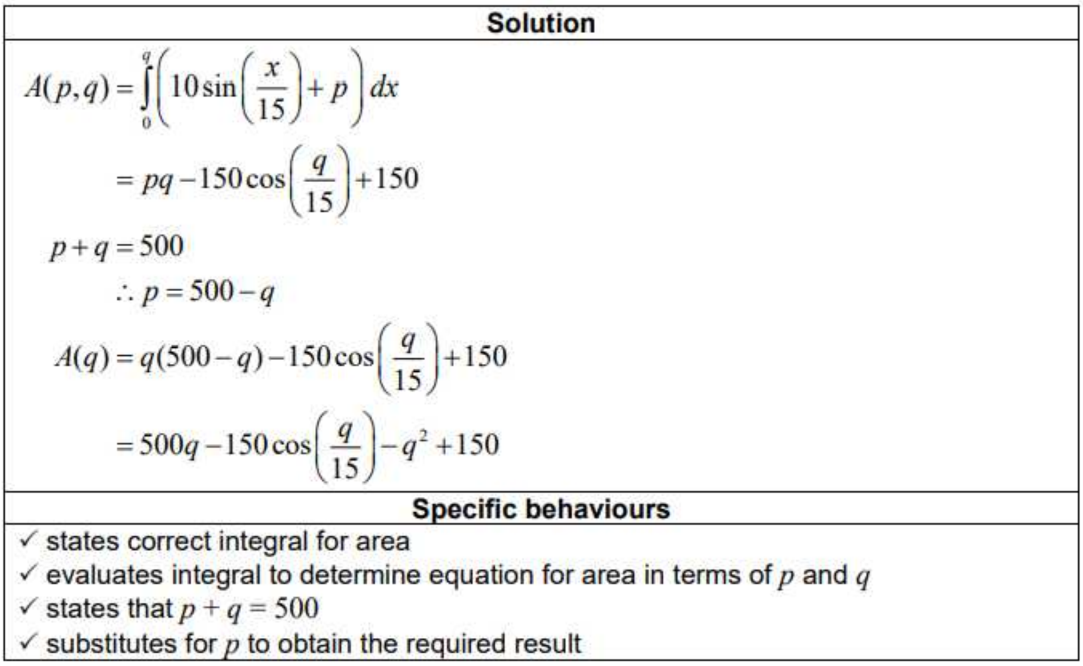


The enclosure is shown as the shaded area above and has right angles at points F and G. David and Katrina want the combined lengths of the fencing from E to F and F to G to equal 500 metres. Let the length of fence EF be equal to p metres and the length of fence FG be equal to q metres. If we locate the origin at point F and the x-axis along the line FG, the equation defining the fence along the road is given by:

$$R(x) = 10\sin\left(\frac{x}{15}\right) + p$$

(a) Show that the equation defining the area of the enclosure, $A(q)$, can be given in terms of q as follows:

$$A(q) = 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$ (4 marks)



Solution

$$A(p,q) = \int_0^q \left(10\sin\left(\frac{x}{15}\right) + p\right) dx$$

$$= pq - 150\cos\left(\frac{q}{15}\right) + 150$$

$$p + q = 500$$

$$\therefore p = 500 - q$$

$$A(q) = q(500 - q) - 150\cos\left(\frac{q}{15}\right) + 150$$

$$= 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$

Specific behaviours

✓ states correct integral for area
✓ evaluates integral to determine equation for area in terms of p and q
✓ states that $p + q = 500$
✓ substitutes for p to obtain the required result

(b) Determine, to the nearest metre, the value of q that will allow the sheep to graze over the maximum area and state this maximum area. (4 marks)

Solution
$A'(q) = 500 + 10\sin\left(\frac{q}{15}\right) - 2q$ $0 = 500 + 10\sin\left(\frac{q}{15}\right) - 2q$ $q \approx 247$ $A''(q) = \frac{2}{3}\cos\left(\frac{q}{15}\right) - 2$ $A''(247) = -ve\ \{-2.48\}$ $\therefore$ maximum $A(247) = 62750$ $\therefore$ maximum area = 62750 m^2
Specific behaviours
✓ differentiates the area equation   ✓ sets the derivative to 0 and solves it to obtain q ✓ obtains the second derivative (or draws a sign diagram for the derivative) to conclude that the point is a global maximum   ✓ states the maximum area

The length of the fence from E to H is given by the equation:

$$L_{EH} = \int_0^q \sqrt{1 + (R'(x))^2}\, dx$$, where $R'(x)$ is the first derivative of $R(x)$.

(c) (i) Determine $R'(x)$. (1 mark)

Solution
$R'(x) = \frac{2}{3}\cos\left(\frac{x}{15}\right)$
Specific behaviours
✓ correctly determines $R'(x)$

(ii) Hence determine the total length of fencing required by David and Katrina to enclose their sheep with maximum area for grazing. (3 marks)

Solution
$L_{EH} = \int_0^{247} \sqrt{1 + \left(\frac{2}{3}\cos\left(\frac{x}{15}\right)\right)^2}\, dx$ $= 273$ metres $\{272.86\}$ $p = 500 - 247 = 253$ $R(247) = 10\sin\left(\frac{247}{15}\right) + 253$ ≈ 247 Total length of fencing $\approx 253+247+273+247$ ≈ 1020 metres
Specific behaviours
✓ calculates L_{EH} ✓ calculates $R(247)$ ✓ states total length of fencing

2019
Section 2
Question 9

Integrals

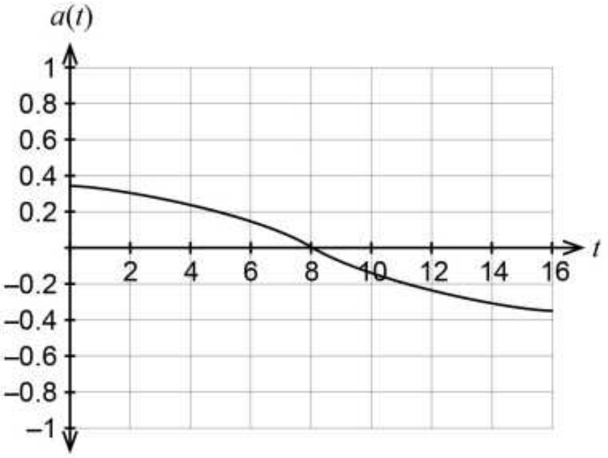
It takes an elevator 16 seconds to ascend from the ground floor of a building to the sixth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{9\pi}{16} \sin\left(\frac{\pi t}{16}\right) \text{ m/s.}$$

The velocity, v, is measured in metres per second, while the time, t, is measured in seconds.

(a) Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \leq t \leq 16$. (2 marks)

Solution



$$a(t) = \frac{9\pi^2}{256} \cos\left(\frac{\pi t}{16}\right) \text{m}^2\text{/s}$$

Specific behaviours

✓ differentiates the velocity function to obtain the acceleration
✓ provides a sketch clearly showing the peak acceleration values and correctly locating the zero acceleration point

(b) With reference to your answer from part (a), explain what is happening to the velocity of the elevator in the interval $0 < t < 8$ and in the interval $8 < t < 16$. (3 marks)

Solution

Since the acceleration is positive on the interval $0 < t < 8$, the velocity is increasing on the interval $0 < t < 8$,
Since the acceleration is negative on the interval $8 < t < 16$ the velocity is decreasing on the interval $8 < t < 16$.

Specific behaviours

✓ references acceleration graph or function
✓ recognises that the upward velocity is increasing on the interval $0 < t < 8$
✓ recognises that the upward velocity is decreasing on the interval $8 < t < 16$

(c) Suppose that the ground floor has displacement $x = 0$ m. Determine the displacement function of the elevator and hence determine the height above the ground floor of the sixth floor. (3 marks)

Solution

The displacement is the integral of the velocity

$$x(t) = -9\cos\left(\frac{\pi t}{16}\right) + c \text{ m}$$

Since $x(0) = 0$ it follows that

$$\begin{aligned} 0 &= -9\cos(0) + c \\ 0 &= -9 + c \\ c &= 9 \end{aligned}$$

Hence

$$x(t) = 9 - 9\cos\left(\frac{\pi t}{16}\right) \text{ m}$$

Evaluating $x(16)$

$$\begin{aligned} x(16) &= 9 - 9\cos(\pi) \\ &= 18 \text{ m} \end{aligned}$$

Specific behaviours

✓ integrates the velocity function to obtain the displacement including unknown integration constant
✓ determines integration constant
✓ evaluates $x(16)$

2019
Section 2
Question 12

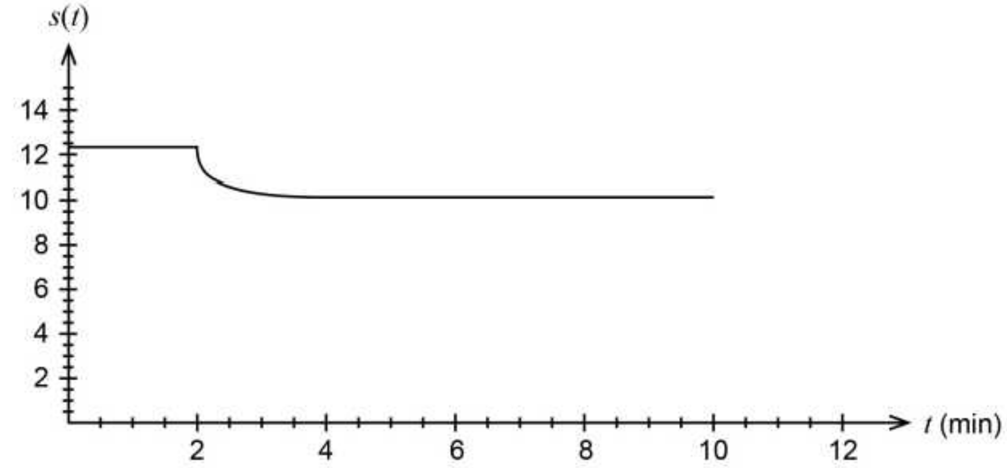
Integrals

Part of Josie's workout at her gym involves a 10 minute run on a treadmill. The treadmill's program makes her run at a constant 12.3 km/h for the first 2 minutes and then her speed, $s(t)$, is determined by the equation below, where t is the time in minutes after she began running.

$$s(t) = 10 - \frac{\ln(t - 1.99)}{t} \text{ km/h}$$

(a) Sketch the graph of her speed during this run versus time on the axes below. (3 marks)

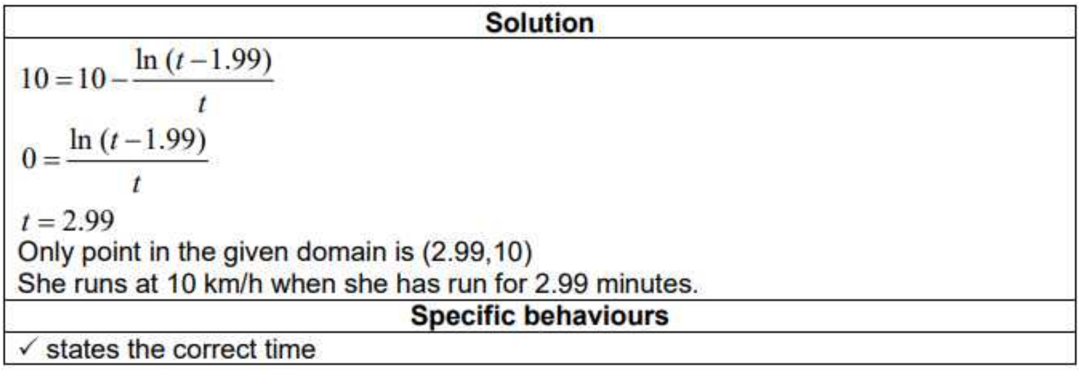
Solution



Specific behaviours

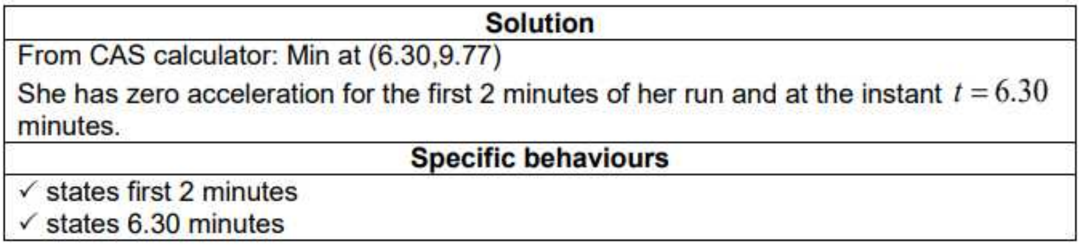
✓ correctly graphs $y = 12.3$
✓ correctly graphs $s(t)$
✓ shows scale and graphs do not exceed [0,10] domain

(b) At what time(s) is Josie’s speed 10 km/h? (1 mark)



Solution
$10 = 10 - \frac{\ln(t - 1.99)}{t}$ $0 = \frac{\ln(t - 1.99)}{t}$ $t = 2.99$ Only point in the given domain is (2.99,10) She runs at 10 km/h when she has run for 2.99 minutes.
Specific behaviours
✓ states the correct time

(c) At what time(s) during her run is Josie’s acceleration zero? (2 marks)



Solution
From CAS calculator: Min at (6.30,9.77) She has zero acceleration for the first 2 minutes of her run and at the instant $t = 6.30$ minutes.
Specific behaviours
✓ states first 2 minutes ✓ states 6.30 minutes

2019
Section 2
Question
15

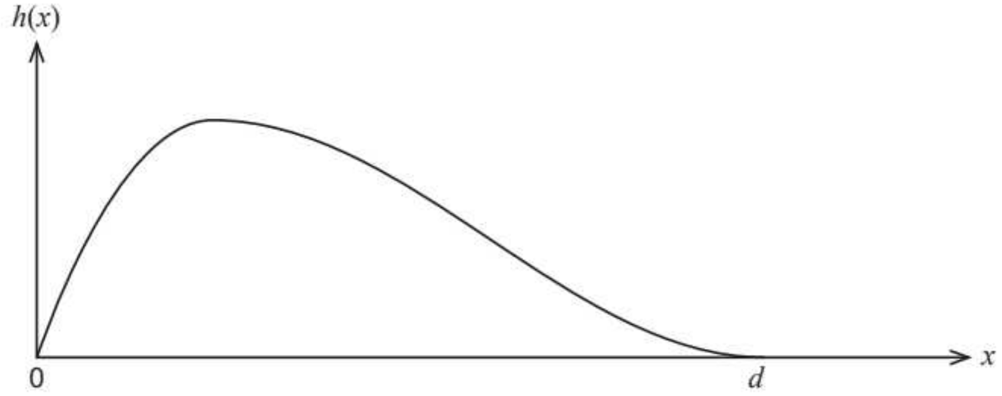
Integrals

A wall in a new Western Australian hotel is to feature a rolling, wave-shaped window. Engineers have modelled the top edge of the wave shape by joining together two functions,

$$h_1(x) = 4 - 4(x-1)^2, \quad 0 \le x \le 1 \text{ and}$$

$$h_2(x) = a(\cos(x-1)+1), \quad 1 < x \le d \quad a,d \text{ constants.}$$

The functions give the height, h, above ground level of the top edge of the window measured in metres. The origin is defined as the leftmost point of the window which is at ground level and x is the horizontal distance to the right of the origin measured in metres. The graph of the two functions is shown below.



(a) Determine the value of the constant a in the function $h2\ (x) = a(\cos(x - 1) + 1)$. (3 marks)

Solution

Functions meet at $x = 1$.

Highest point is $h_1(1) = 4 - 4(1-1)^2$

$= 4$

Functions meet at (1,4)

$h_2(1) = 4$

$4 = a(\cos(1-1)+1)$

$4 = 2a$

$a = 2$

Specific behaviours

✓ states highest point is $h_1(1)$

✓ recognises the need to solve $h_2(1) = h_1(1)$

✓ determines the value of a

(b) Determine the length of the bottom edge of the window. (2 marks)

Solution

$h_2(d) = 0$

$0 = 2(\cos(d-1)+1)$

$d = \pi + 1$

$= 4.14$

The bottom edge of the window is 4.14 metres long.

Specific behaviours

✓ recognises the need to solve $h_2(d) = 0$

✓ solves for d

(c) Determine the volume of glass required for the window if it has a uniform thickness of 3 cm. (5 marks)

Solution

$$A=\int_0^1 h_1(x)dx+\int_1^{\pi+1} h_2(x)dx$$

$$\int_0^1 h_1(x)dx=\frac{8}{3}$$

$$\int_1^{\pi+1} h_2(x)dx=2\pi$$

$$\begin{aligned}A&=\frac{8}{3}+2\pi\\&=8.95\end{aligned}$$

$$\begin{aligned}V&=\left(\frac{8}{3}+2\pi\right)\times 0.03\\&=0.27\end{aligned}$$

The window has a volume of 0.27 cubic metres

Specific behaviours

✓ recognises the required area is the sum of the areas under $h_1(x)$ and $h_2(x)$

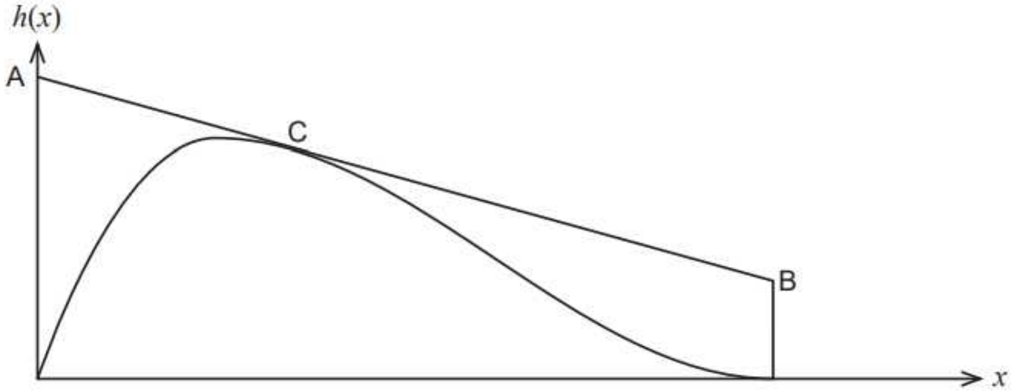
✓ determines the area under $h_1(x)$

✓ determines the area under $h_2(x)$

✓ determines the total area

✓ determines the volume including units

The top edge of the wall, shown as the line AB below, is to just touch the window at the point C shown below. Point A is 1.39 m above the point B.



(d) How high is point C above the ground? (4 marks)

Solution

$$m = \frac{1.39}{0-(\pi+1)}$$
$$= -0.3356$$

$$h_2'(x) = -2\sin(x-1)$$
$$-0.3356 = -2\sin(x-1)$$
$$x = 1.1686,\ 3.9730$$

Solution is 1.1686

$h_2(1.1686) = 3.9716$

C is 3.97 metres above the ground.

Specific behaviours

✓ determines the gradient, m , of the line segment AB

✓ differentiates $h_2(x)$

✓ equates $h_2'(x)$ to m and solves for x

✓ determines the height of point C

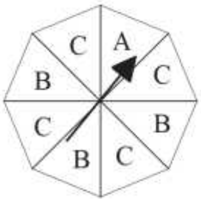
Unit 3.3 – Discrete random variables

Section 1

2020
Section 1
Question 1

Discrete
random
variables

Ashley and Xavier are playing a board game that requires them to use the spinner shown below.



The player spins the arrowhead and the result is where the arrowhead is pointing when it stops moving. The above diagram is showing a result of A.

(a) If the spinner is spun three times, what is the probability that B is never a result? (1 mark)

Let the random variable X be defined as the number of times B is the result when the spinner is spun three times.

(b) Complete the table below showing the probability distribution of X. (3 marks)

x	0	1	2	3
$P(X = x)$				

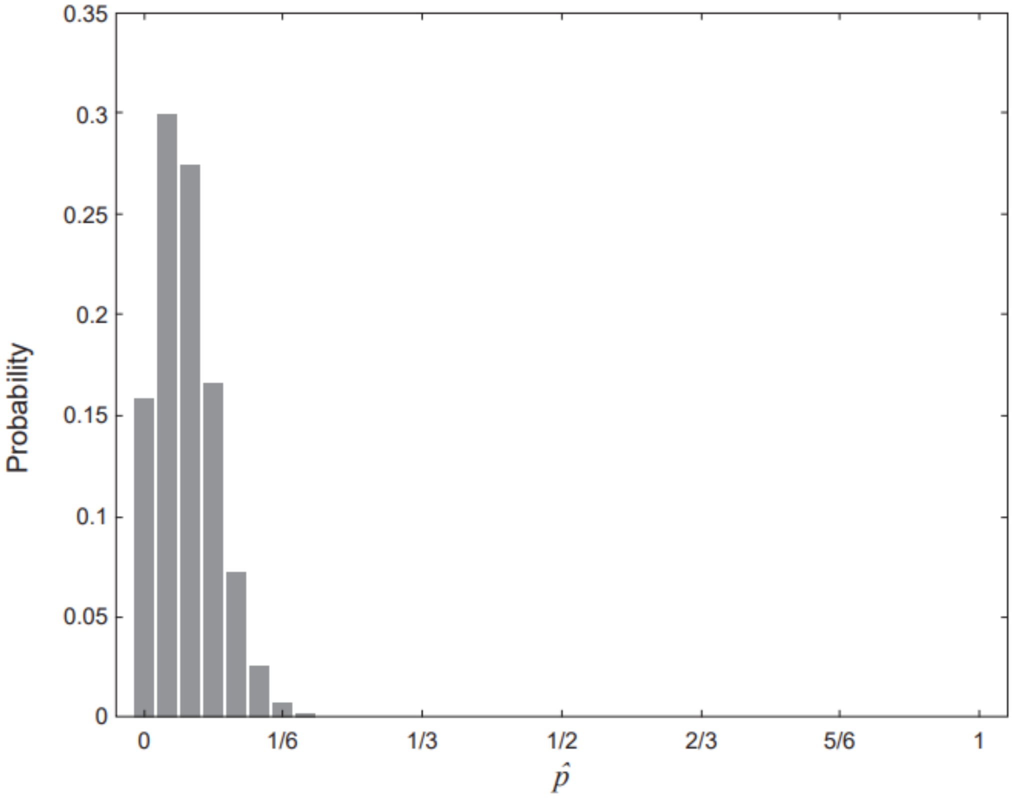
(c) Determine the mean and variance of the above distribution. (2 marks)

Section 2

2023 Section 2 Question 10

Discrete random variables

Fingerprints can be classified broadly as loop-shaped, whirl-shaped or arch-shaped. Only 5% of the world's population have arch-shaped fingerprints. Consider a random sample of 36 people and let $\hat{p}$ denote the sample proportion of people with arch-shaped fingerprints. The probability distribution for $\hat{p}$ is shown below.



(a) On the basis of the diagram above, is it appropriate to use the normal distribution to approximate the distribution of $\hat{p}$? Justify your answer. (2 marks)

A larger sample of 500 people is selected at random

(b) Determine the probability that more than 30 people in the sample have arch-shaped fingerprints. (3 marks)

(c) Use the approximate normality of the distribution to determine the probability that the sample proportion of people with arch-shaped fingerprints is greater than 0.06. (2 marks)

2022
Section 2
Question 9

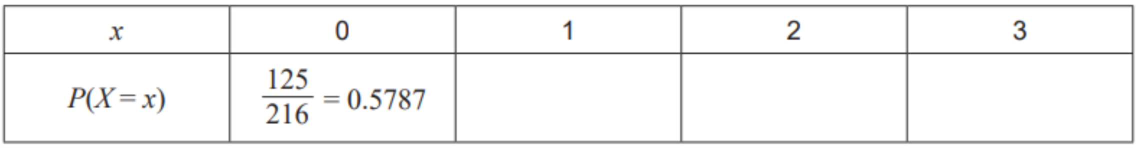
Discrete random variables

Andrew plans to run a game called Lucky Cup for a school fundraising event. All profits go toward the school's fundraising efforts. The game consists of three standard dice, each placed into a red cup. The red cup is shaken, the dice rolled, and the number of sixes recorded.

Let X be a random variable denoting the number of sixes rolled in a game of Lucky Cup.

(a) State the distribution of X. (2 marks)

An incomplete probability distribution for X is shown in the table below.



x	0	1	2	3
$P(X=x)$	$\frac{125}{216}=0.5787$			

(b) Complete the table above, providing the missing probabilities. (2 marks)

Lucky Cup costs $1 to play. If a player rolls one 6 they win $1, if they roll two 6s they win $2, and if they roll three 6s they win $3.

(c) Determine the school's expected profit/loss for each game of Lucky Cup. (3 marks)

(d) Determine the probability that a player will make a profit in a game of Lucky Cup. (2 marks)

Andrew wants to increase the attraction of the game by providing the opportunity for larger winnings. He modifies the rules of the game so that players only win money when two or more 6s are rolled, and the winnings for rolling three 6s is three times as much as the winnings for rolling two 6s. Each game still costs $1 to play. He estimates that he should be able to run 500 games and wants to make a profit of $200 for the school.

Andrew calls this game Lucky Cup II.

(e) Determine the winnings Andrew should set for rolling three 6s in Lucky Cup II. (3 marks)

Andrew decides to make the game even more dynamic and exciting. He adds a green cup with a die that he rolls at the beginning of each game. The value that Andrew rolls becomes the target value, and players must roll this target value to win. For example, if Andrew rolls a 2 from the green cup and a player rolls three 2s, the player wins the top prize.

Andrew calls this game Lucky Cup III.

(f) Explain how this change affects a player's chance of winning compared with Lucky Cup II. (2 marks)

2020
Section 2
Question 9

Discrete random variables

A cake shop makes birthday cakes. The probability distribution of the number of birthday cakes sold in a day, X, is given below.

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.25	0.35	0.1

(a) Calculate the mean number or birthday cakes sold in a day. (1 mark)

(b) On Monday, the cake shop makes four birthday cakes. If each birthday cake costs \$20 to make and sells for \$50, what is the expected profit or loss on that day? (3 marks)

On Tuesday, the shop makes three birthday cakes. Let the random variable Y denote the number of birthday cakes **not** sold on that day.

(c) Explain why $P(Y = 0) = 0.45$. (2 marks)

(d) Obtain the probability distribution of Y. (2 marks)

2019
Section 2
Question 10

Discrete random variables

A group of researchers conducted a study into the number of siblings of adult Australian citizens. They surveyed a total of 200 participants and recorded the number of siblings, X, of each participant.

A few days later the lead researcher discovered that the survey data had been misplaced. Fortunately, one of the research assistants had been doing some rough calculations on a whiteboard and the lead researcher was able to recover the following information about the probability distribution for X and the mean μ.

x	0	1	2	3
$P(X = x)$	0.2	a	b	0.1

$$\mu = 1.3$$

The letters a and b have been used to denote unknown probabilities.

(a) (i) Write **two** independent equations for a and b. (2 marks)

(ii) Hence solve for the unknown probabilities. (2 marks)

Later that day the research assistant found the complete probability distribution in their records and discovered that they had made an error in their original calculation of the mean. The correct probability distribution is given in the table below.

x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1

(b) (i) Given that there were 200 participants in the study, complete the table below to show the number of participants N with 0, 1, 2 and 3 siblings. (1 mark)

x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1
N	40			

(ii) Determine the correct mean and standard deviation of the number of siblings X. (2 marks)

2019
Section 2
Question
18

Discrete
random
variables

A building has five alarms configured in such a way that the system functions if at least two of the alarms work. The probability that an alarm fails overnight is 0.05. Let the random variable X denote the number of alarms that fail overnight.

(a) State the distribution of X. (2 marks)

(b) What assumptions are required for the distribution in part (a) to be valid? (2 marks)

(c) What is the probability that the alarm system fails overnight? (2 marks)

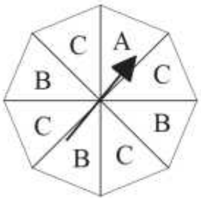
	One of the alarms is removed in the evening for maintenance and is not replaced. (d) What is the probability that the alarm system still works in the morning? (3 marks)

Marking Guide – Section 1

2020
Section 1
Question 1

Discrete random variables

Ashley and Xavier are playing a board game that requires them to use the spinner shown below.



The player spins the arrowhead and the result is where the arrowhead is pointing when it stops moving. The above diagram is showing a result of A.

(a) If the spinner is spun three times, what is the probability that B is never a result? (1 mark)

Solution

$$P(\text{not B}) = \left(\frac{5}{8}\right)^3$$
$$= \frac{125}{512}$$

Specific behaviours

✓ determines the correct probability

Let the random variable X be defined as the number of times B is the result when the spinner is spun three times.

(b) Complete the table below showing the probability distribution of X. (3 marks)

Solution

$$X \sim Bin\left(3, \frac{3}{8}\right)$$

x	0	1	2	3
$P(X = x)$	$\left(\frac{5}{8}\right)^3$ or $\frac{125}{512}$	$\binom{3}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)$ or $\frac{225}{512}$	$\binom{3}{2}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^2$ or $\frac{135}{512}$	$\left(\frac{3}{8}\right)^3$ or $\frac{27}{512}$

Specific behaviours

✓ recognises the distribution of X as binomial
✓ determines the correct probability for $x = 1$, 2 or 3
✓ determines the correct probability for remaining entries

(c) Determine the mean and variance of the above distribution. (2 marks)

Solution

$$\text{mean} = np$$
$$= 3 \times \frac{3}{8}$$
$$= \frac{9}{8} \quad \{1.125\}$$

$$\text{variance} = np(1-p)$$
$$= \frac{9}{8} \times \frac{5}{8}$$
$$= \frac{45}{64}$$

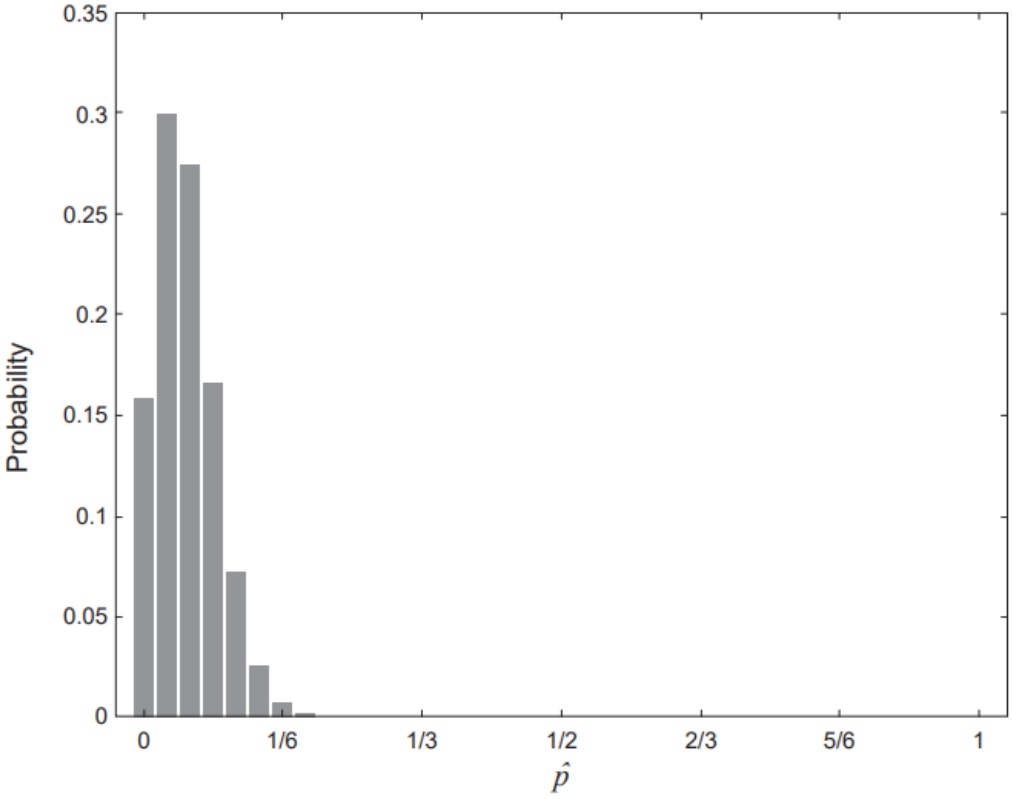
Specific behaviours

✓ determines the mean
✓ determines the variance

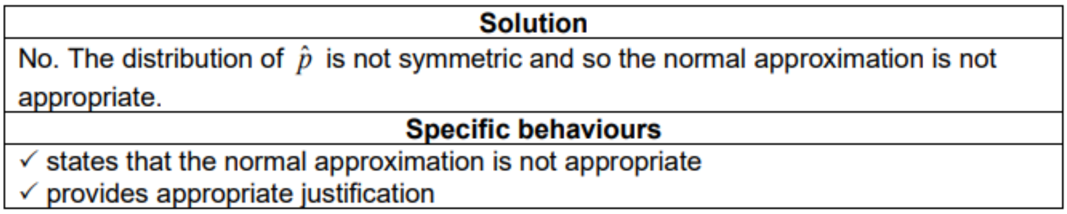
2023
Section 2
Question
10

Discrete
random
variables

Fingerprints can be classified broadly as loop-shaped, whirl-shaped or arch-shaped. Only 5% of the world's population have arch-shaped fingerprints. Consider a random sample of 36 people and let $\hat{p}$ denote the sample proportion of people with arch-shaped fingerprints. The probability distribution for $\hat{p}$ is shown below.



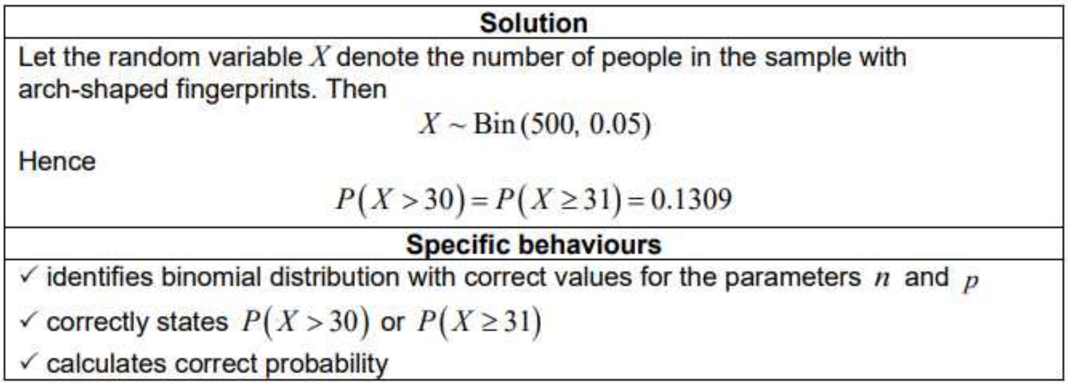
(a) On the basis of the diagram above, is it appropriate to use the normal distribution to approximate the distribution of $\hat{p}$? Justify your answer. (2 marks)



Solution
No. The distribution of $\hat{p}$ is not symmetric and so the normal approximation is not appropriate.
Specific behaviours
✓ states that the normal approximation is not appropriate ✓ provides appropriate justification

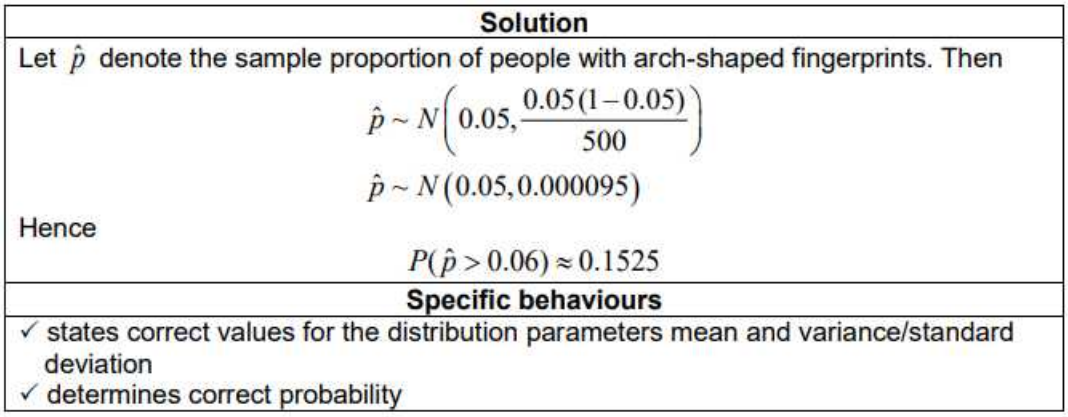
A larger sample of 500 people is selected at random

(b) Determine the probability that more than 30 people in the sample have arch-shaped fingerprints. (3 marks)



Solution
Let the random variable X denote the number of people in the sample with arch-shaped fingerprints. Then $X \sim \text{Bin}(500,\ 0.05)$ Hence $P(X>30)=P(X\geq 31)=0.1309$
Specific behaviours
✓ identifies binomial distribution with correct values for the parameters n and p ✓ correctly states $P(X>30)$ or $P(X\geq 31)$ ✓ calculates correct probability

(c) Use the approximate normality of the distribution to determine the probability that the sample proportion of people with arch-shaped fingerprints is greater than 0.06. (2 marks)



Solution

Let $\hat{p}$ denote the sample proportion of people with arch-shaped fingerprints. Then

$$\hat{p} \sim N\left(0.05, \frac{0.05(1-0.05)}{500}\right)$$

$$\hat{p} \sim N(0.05, 0.000095)$$

Hence

$$P(\hat{p} > 0.06) \approx 0.1525$$

Specific behaviours

✓ states correct values for the distribution parameters mean and variance/standard deviation
✓ determines correct probability

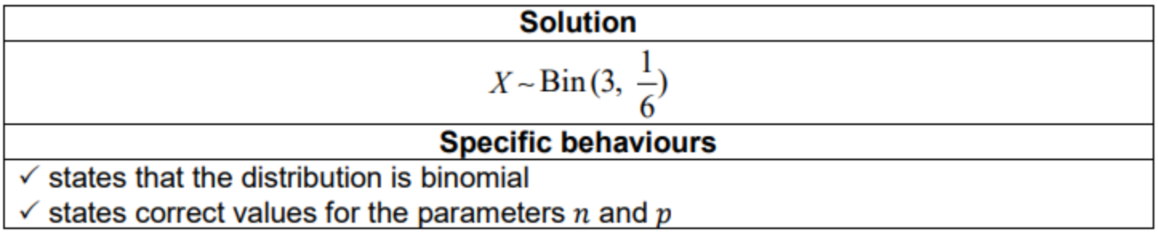
2022
Section 2
Question 9

Discrete random variables

Andrew plans to run a game called Lucky Cup for a school fundraising event. All profits go toward the school's fundraising efforts. The game consists of three standard dice, each placed into a red cup. The red cup is shaken, the dice rolled, and the number of sixes recorded.

Let X be a random variable denoting the number of sixes rolled in a game of Lucky Cup.

(a) State the distribution of X. (2 marks)



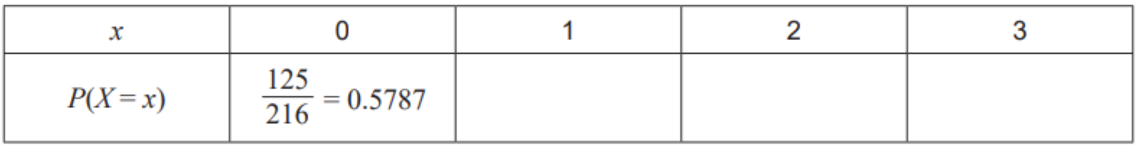
Solution

$$X \sim \text{Bin}(3, \frac{1}{6})$$

Specific behaviours

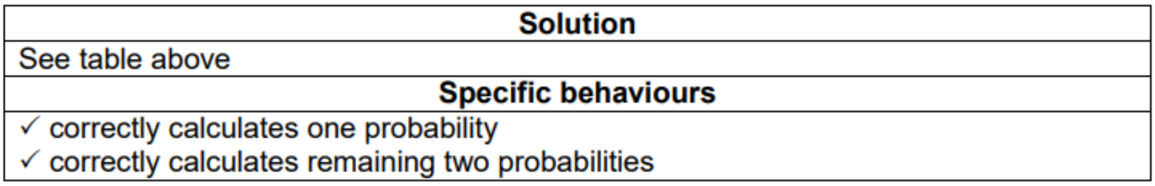
✓ states that the distribution is binomial
✓ states correct values for the parameters n and p

An incomplete probability distribution for X is shown in the table below.



x	0	1	2	3
$P(X=x)$	$\frac{125}{216} = 0.5787$			

(b) Complete the table above, providing the missing probabilities. (2 marks)



Solution

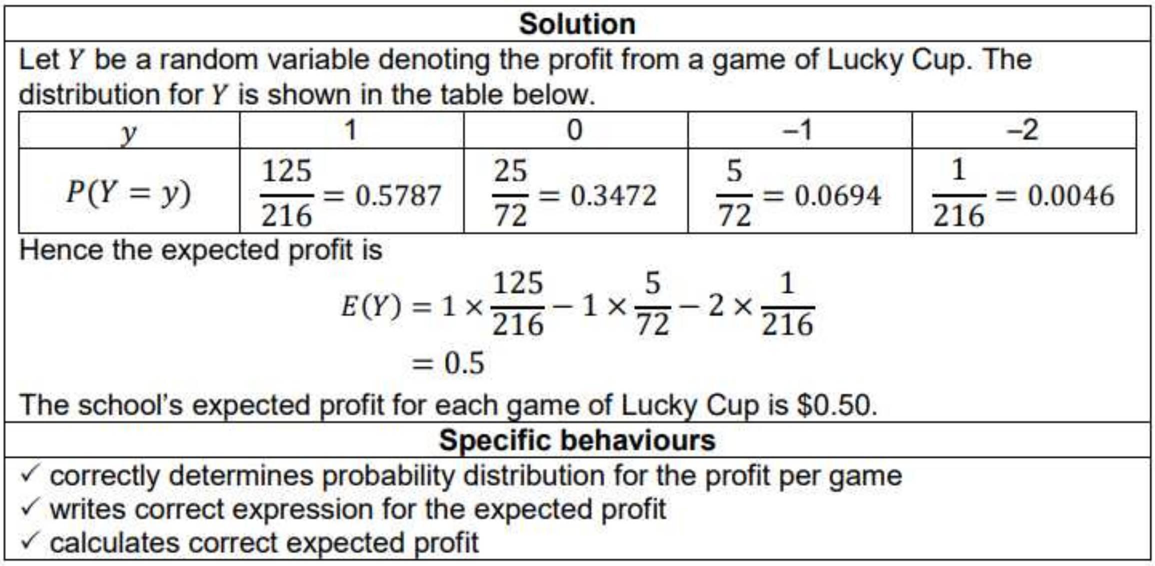
See table above

Specific behaviours

✓ correctly calculates one probability
✓ correctly calculates remaining two probabilities

Lucky Cup costs $1 to play. If a player rolls one 6 they win $1, if they roll two 6s they win $2, and if they roll three 6s they win $3.

(c) Determine the school's expected profit/loss for each game of Lucky Cup. (3 marks)



Solution

Let Y be a random variable denoting the profit from a game of Lucky Cup. The distribution for Y is shown in the table below.

y	1	0	–1	–2
$P(Y = y)$	$\frac{125}{216} = 0.5787$	$\frac{25}{72} = 0.3472$	$\frac{5}{72} = 0.0694$	$\frac{1}{216} = 0.0046$

Hence the expected profit is

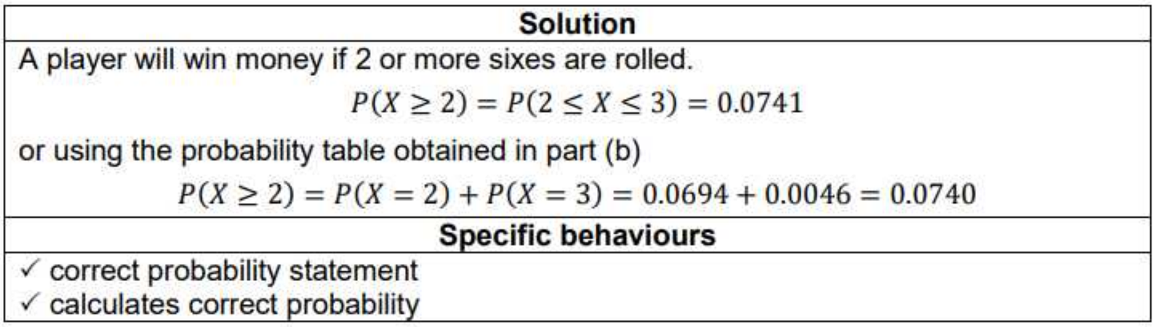
$$E(Y) = 1 \times \frac{125}{216} - 1 \times \frac{5}{72} - 2 \times \frac{1}{216}$$
$$= 0.5$$

The school's expected profit for each game of Lucky Cup is $0.50.

Specific behaviours

✓ correctly determines probability distribution for the profit per game
✓ writes correct expression for the expected profit
✓ calculates correct expected profit

(d) Determine the probability that a player will make a profit in a game of Lucky Cup. (2 marks)



Solution

A player will win money if 2 or more sixes are rolled.

$$P(X \geq 2) = P(2 \leq X \leq 3) = 0.0741$$

or using the probability table obtained in part (b)

$$P(X \geq 2) = P(X = 2) + P(X = 3) = 0.0694 + 0.0046 = 0.0740$$

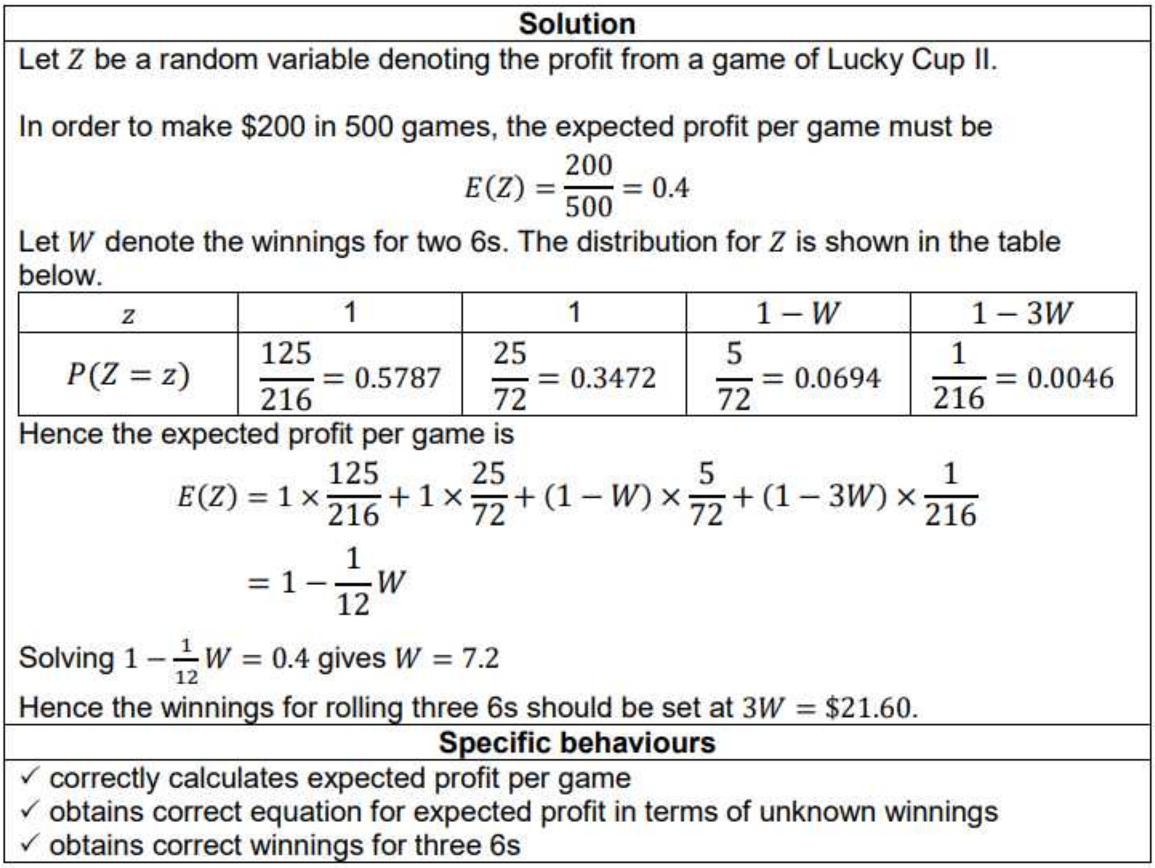
Specific behaviours

✓ correct probability statement
✓ calculates correct probability

Andrew wants to increase the attraction of the game by providing the opportunity for larger winnings. He modifies the rules of the game so that players only win money when two or more 6s are rolled, and the winnings for rolling three 6s is three times as much as the winnings for rolling two 6s. Each game still costs $1 to play. He estimates that he should be able to run 500 games and wants to make a profit of $200 for the school.

Andrew calls this game Lucky Cup II.

(e) Determine the winnings Andrew should set for rolling three 6s in Lucky Cup II. (3 marks)



Solution

Let Z be a random variable denoting the profit from a game of Lucky Cup II.

In order to make $200 in 500 games, the expected profit per game must be

$$E(Z) = \frac{200}{500} = 0.4$$

Let W denote the winnings for two 6s. The distribution for Z is shown in the table below.

z	1	1	$1 - W$	$1 - 3W$
$P(Z = z)$	$\frac{125}{216} = 0.5787$	$\frac{25}{72} = 0.3472$	$\frac{5}{72} = 0.0694$	$\frac{1}{216} = 0.0046$

Hence the expected profit per game is

$$\begin{aligned} E(Z) &= 1 \times \frac{125}{216} + 1 \times \frac{25}{72} + (1 - W) \times \frac{5}{72} + (1 - 3W) \times \frac{1}{216} \\ &= 1 - \frac{1}{12}W \end{aligned}$$

Solving $1 - \frac{1}{12}W = 0.4$ gives $W = 7.2$

Hence the winnings for rolling three 6s should be set at $3W = \$21.60$.

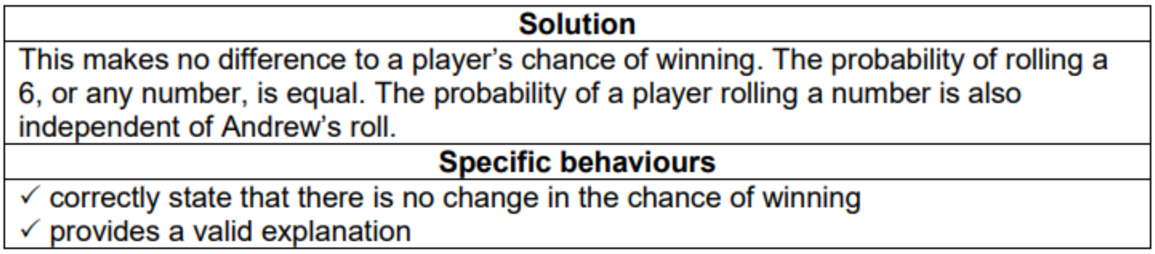
Specific behaviours

✓ correctly calculates expected profit per game
✓ obtains correct equation for expected profit in terms of unknown winnings
✓ obtains correct winnings for three 6s

Andrew decides to make the game even more dynamic and exciting. He adds a green cup with a die that he rolls at the beginning of each game. The value that Andrew rolls becomes the target value, and players must roll this target value to win. For example, if Andrew rolls a 2 from the green cup and a player rolls three 2s, the player wins the top prize.

Andrew calls this game Lucky Cup III.

(f) Explain how this change affects a player's chance of winning compared with Lucky Cup II. (2 marks)



Solution

This makes no difference to a player's chance of winning. The probability of rolling a 6, or any number, is equal. The probability of a player rolling a number is also independent of Andrew's roll.

Specific behaviours

✓ correctly state that there is no change in the chance of winning
✓ provides a valid explanation

2020
Section 2
Question 9

Discrete random variables

A cake shop makes birthday cakes. The probability distribution of the number of birthday cakes sold in a day, X, is given below.

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.25	0.35	0.1

(a) Calculate the mean number or birthday cakes sold in a day. (1 mark)

Solution
$E(X) = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.25 + 3 \times 0.35 + 4 \times 0.1 = 2.15$
Specific behaviours
✓ evaluates the mean correctly

(b) On Monday, the cake shop makes four birthday cakes. If each birthday cake costs \$20 to make and sells for \$50, what is the expected profit or loss on that day? (3 marks)

Solution
The cost of making four birthday cakes is $4 \times \$20 = \$80$ The expected number sold is $E(X) = 2.15$, so the expected income is $2.15 \times \$50 = \$107.50$ Thus, the expected profit for the shop is $\$107.50 - \$80 = \$27.50$
Specific behaviours
✓ calculates the cost of making the birthday cakes ✓ uses the expected number sold to calculate the expected income ✓ calculates the expected profit

On Tuesday, the shop makes three birthday cakes. Let the random variable Y denote the number of birthday cakes **not** sold on that day.

(c) Explain why $P(Y = 0) = 0.45$. (2 marks)

Solution
$Y = 0$ when all the birthday cakes are sold So, the number of birthday cakes requested for sale is 3 or 4 So $P(Y = 0) = P(X = 3) + P(X = 4) = 0.35 + 0.1 = 0.45$
Specific behaviours
✓ relates the value of Y to the values of X ✓ obtains the probability as a sum of the two probabilities

(d) Obtain the probability distribution of Y. (2 marks)

Solution

y	0	1	2	3
$P(Y = y)$	0.45	0.25	0.2	0.1

Specific behaviours

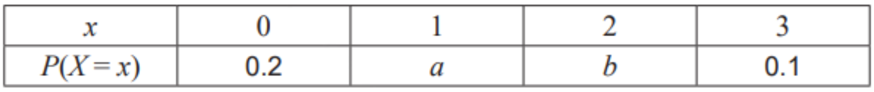
✓ obtains $P(Y = 1)$ or $P(Y = 2)$ or $P(Y = 3)$
✓ completes table correctly

2019
Section 2
Question 10

Discrete random variables

A group of researchers conducted a study into the number of siblings of adult Australian citizens. They surveyed a total of 200 participants and recorded the number of siblings, X, of each participant.

A few days later the lead researcher discovered that the survey data had been misplaced. Fortunately, one of the research assistants had been doing some rough calculations on a whiteboard and the lead researcher was able to recover the following information about the probability distribution for X and the mean μ.

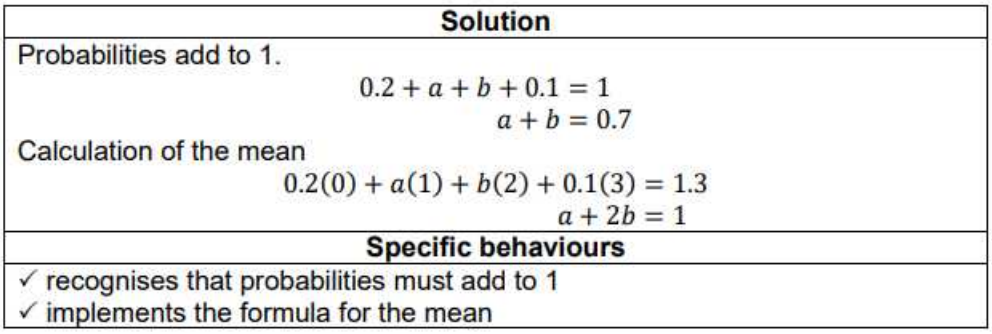


x	0	1	2	3
$P(X=x)$	0.2	a	b	0.1

$$\mu = 1.3$$

The letters a and b have been used to denote unknown probabilities.

(a) (i) Write **two** independent equations for a and b. (2 marks)



Solution

Probabilities add to 1.

$$0.2 + a + b + 0.1 = 1$$
$$a + b = 0.7$$

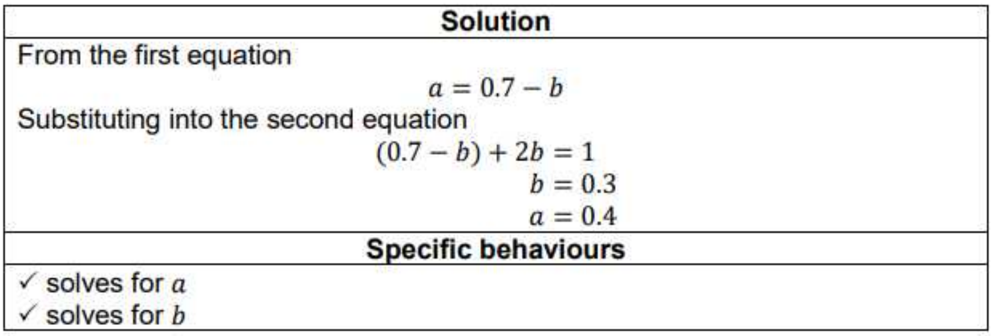
Calculation of the mean

$$0.2(0) + a(1) + b(2) + 0.1(3) = 1.3$$
$$a + 2b = 1$$

Specific behaviours

✓ recognises that probabilities must add to 1
✓ implements the formula for the mean

(ii) Hence solve for the unknown probabilities. (2 marks)



Solution

From the first equation

$$a = 0.7 - b$$

Substituting into the second equation

$$(0.7 - b) + 2b = 1$$
$$b = 0.3$$
$$a = 0.4$$

Specific behaviours

✓ solves for a
✓ solves for b

Later that day the research assistant found the complete probability distribution in their records and discovered that they had made an error in their original calculation of the mean. The correct probability distribution is given in the table below.

x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1

(b) (i) Given that there were 200 participants in the study, complete the table below to show the number of participants N with 0, 1, 2 and 3 siblings. (1 mark)

Solution				
x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1
N	40	60	80	20
Specific behaviours				
✓ determines correct number of participants in each response group				

(ii) Determine the correct mean and standard deviation of the number of siblings X. (2 marks)

Solution
$\mu=1.4$ $\sigma=0.9165$
Specific behaviours
✓ determines mean ✓ determines standard deviation

2019
Section 2
Question
18

Discrete
random
variables

A building has five alarms configured in such a way that the system functions if at least two of the alarms work. The probability that an alarm fails overnight is 0.05. Let the random variable X denote the number of alarms that fail overnight.

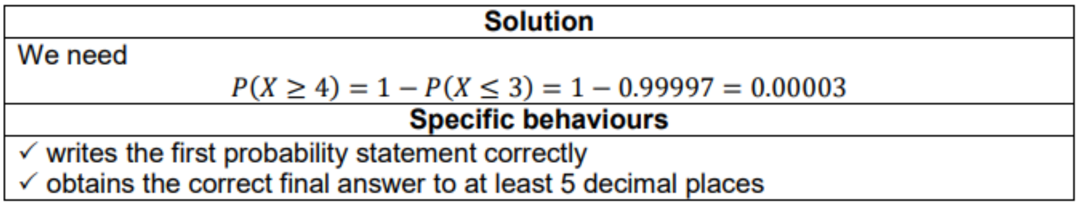
(a) State the distribution of X. (2 marks)

Solution
$X \sim Bin(5,0.05)$
Specific behaviours
✓ states the binomial distribution ✓ gives the correct parameters

(b) What assumptions are required for the distribution in part (a) to be valid? (2 marks)

Solution
1. The alarms fail independent of each other. 2. The probability that an alarm fails is constant/unchanging/same for all alarms.
Specific behaviours
✓ states one correct assumption ✓ states the second correct assumption

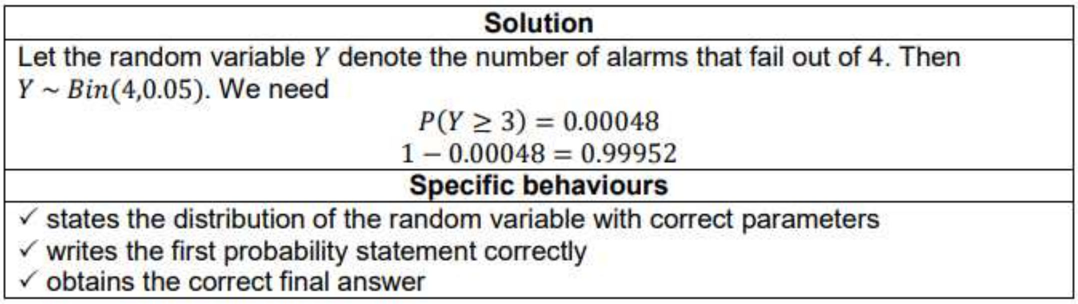
(c) What is the probability that the alarm system fails overnight? (2 marks)



Solution
We need $P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.99997 = 0.00003$
Specific behaviours
✓ writes the first probability statement correctly ✓ obtains the correct final answer to at least 5 decimal places

One of the alarms is removed in the evening for maintenance and is not replaced.

(d) What is the probability that the alarm system still works in the morning? (3 marks)



Solution
Let the random variable Y denote the number of alarms that fail out of 4. Then $Y \sim Bin(4,0.05)$. We need $P(Y \geq 3) = 0.00048$ $1 - 0.00048 = 0.99952$
Specific behaviours
✓ states the distribution of the random variable with correct parameters ✓ writes the first probability statement correctly ✓ obtains the correct final answer

Unit 4

Unit 4.1 – The logarithmic function

Section 1

2023 **Section 1** **Question 2** **The logarithmic function**	Let $p = \ln(2), q = \ln(3)$ and $r = \ln(5)$. (a) Express each of the following in terms of p, q and/or r. (i) ln(6) (2 marks) (ii) ln(6.25) (3 marks) (iii) $\int_2^3 \frac{d}{dx} \ln(x)\, dx$ (2 marks) (b) Evaluate e^{p+q}. (2 marks)

(c) (i) Determine $\frac{d}{dx}\left(x\ln(x)\right)$. (1 marks)

(ii) Hence show that $\int \ln(x)\,dx = x\ln(x) - x + c$ where c is a constant. (2 marks)

(iii) Evaluate $\int_1^3 \ln(x)\,dx$ in terms of p, q and/or r. (2 marks)

2023
Section 1
Question 4

The logarithmic function

An internet search engine uses a logarithmic scale to rank the importance of internet websites. If a website has *S* visits each week, the site rank, *R*, is given by

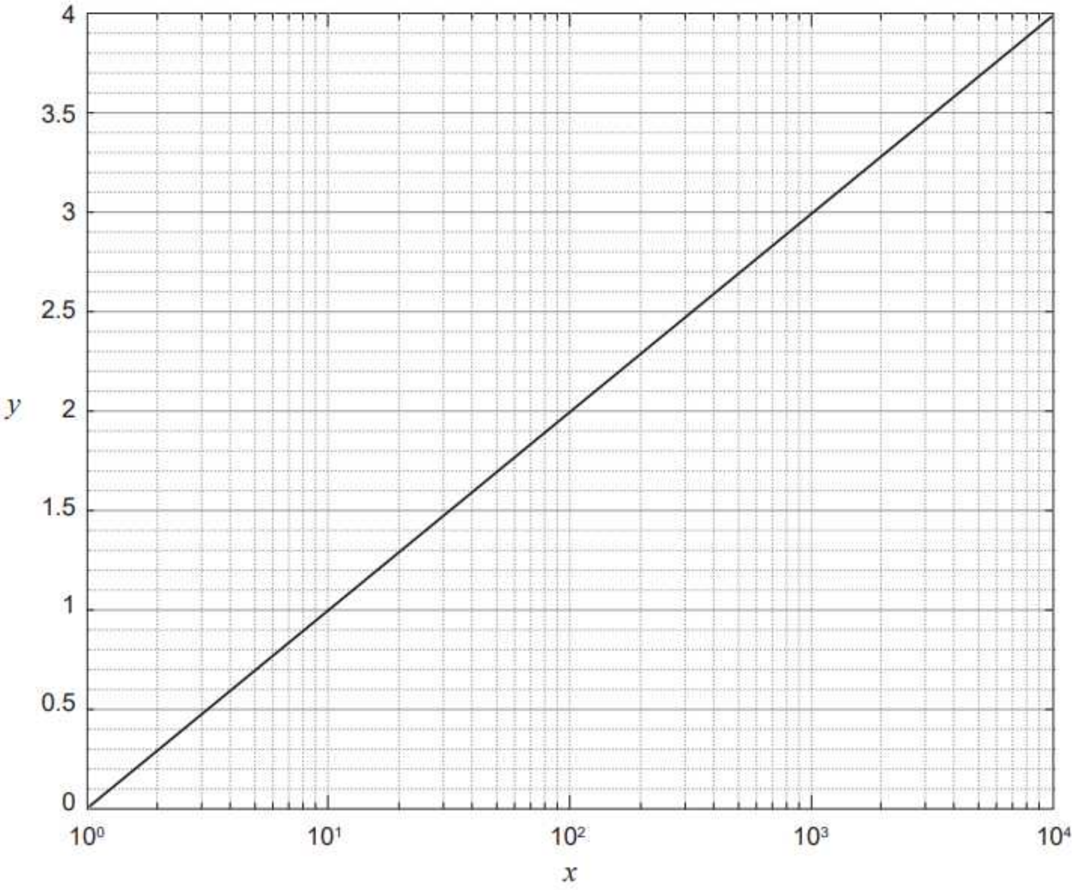
$$R = 2\log_{10}\left(\frac{S}{S_0}\right)$$

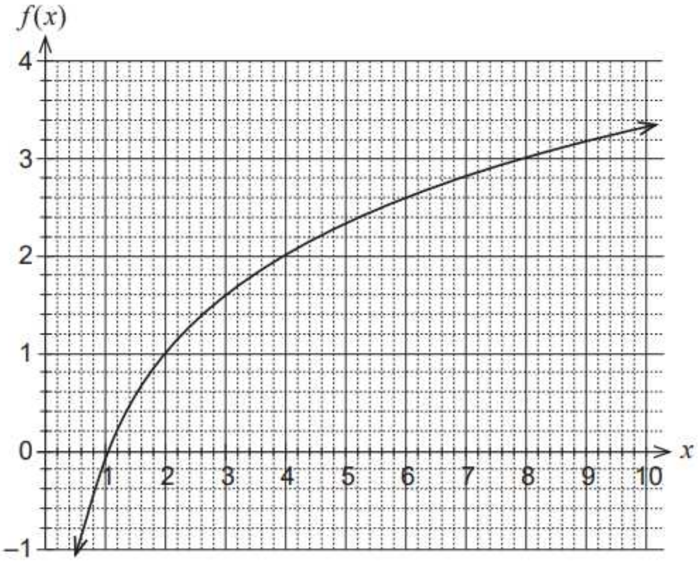
where S_0 is the reference value (the same for all websites). The reference value is the minimum number of visits per week required for a website to register on the site rank scale.

(a) Determine the site rank for a website whose weekly visits are one hundred times the reference value. (2 marks)

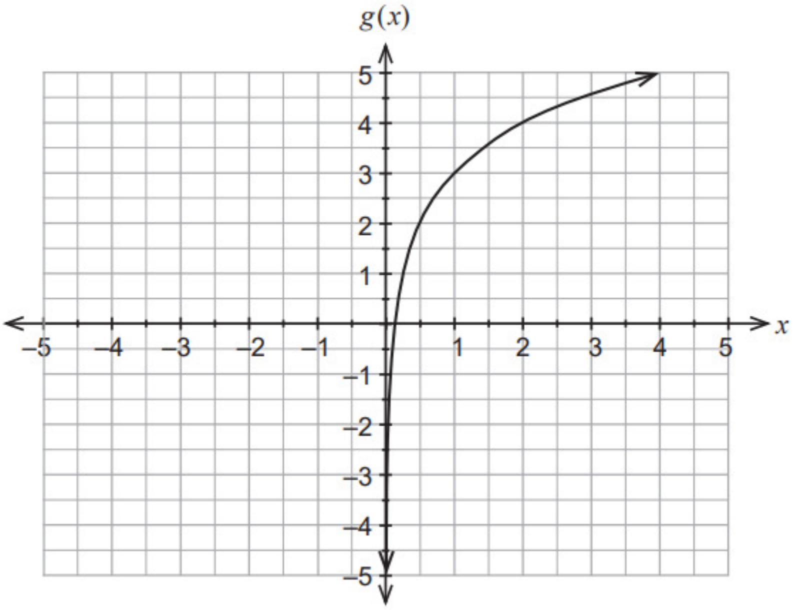
(b) Given that a site rank of 12 is assigned to a website with 1.5 billion (1.5×10^9) visits per week, determine the value of S_0. (3 marks)

(c) The plot of $y = \log_{10}(x)$ is shown below. If a website has a site rank of 3.2, use the plot and your answer from part (b) to approximate the website's number of weekly visits. (3 marks)



2022 **Section 1** **Question 4** **The logarithmic function**	The graph of the function $f(x) = \log_2(x)$ is shown below.  (a) Using the graph: (i) solve $\log_2 (x - 5) = 3$. (2 marks) (ii) determine $\sqrt{7}$, correct to one decimal place. (Hint: let $x = \sqrt{7}$.) (3 marks)

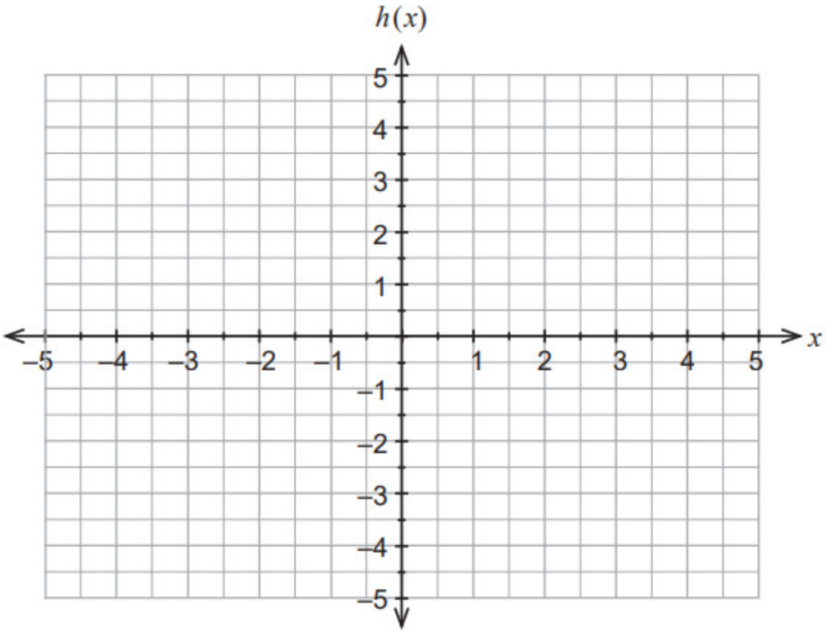
(b) The function $f(x) = \log_2(x)$ is translated to give the new function $g(x)$, which is shown in the graph below.



Determine the equation for $g(x)$. (2 marks)

(c) (i) Show that $\log_2\left(\frac{1}{x-1}\right) = -\log_2(x-1)$. (2 marks)

(ii) Hence sketch the graph of $h(x) = \log_2\left(\frac{1}{x-1}\right)$ on the axes below. (3 marks)



2021
Section 1
Question 3

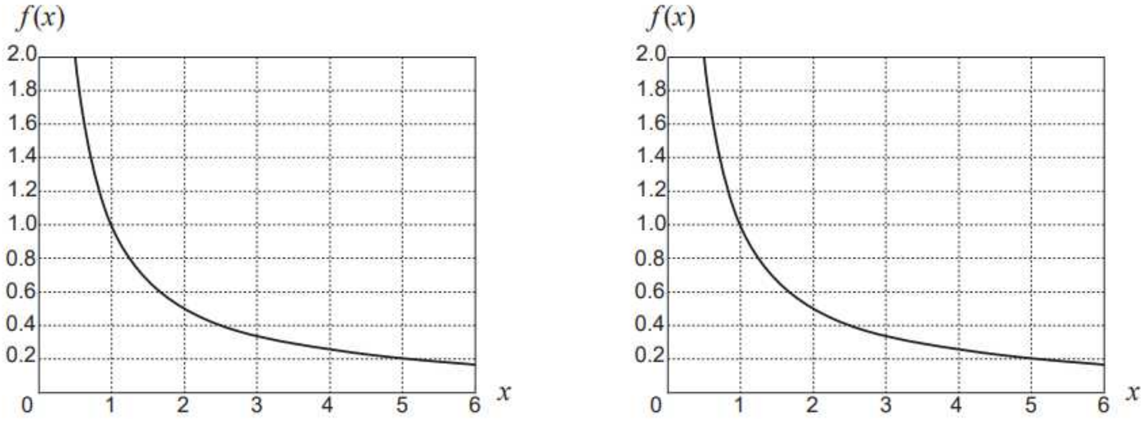
The logarithmic function

Given that $\ln(2) \approx 0.693$, use the increments formula to determine an approximation for $\ln(2.02)$. (3 marks)

2021
Section 1
Question 7

The logarithmic function

(a) Consider the function $f(x) = \dfrac{1}{x}$, graphed twice below.



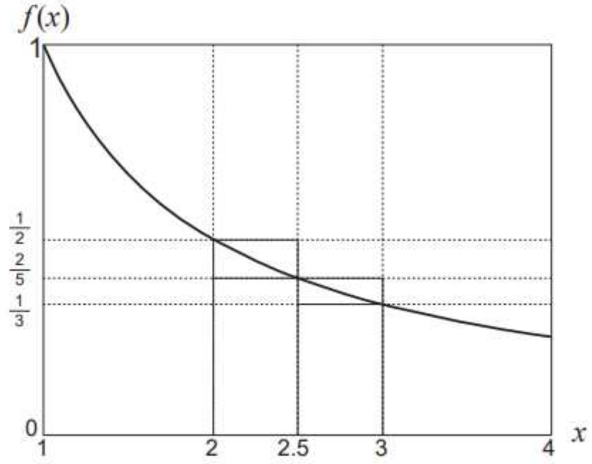
(i) Shade **two** different regions (one on each graph above) each with area exactly $\ln(2)$. (2 marks)

(ii) Given that

$$\int_a^b \frac{1}{x}\,dx = \ln(3)$$

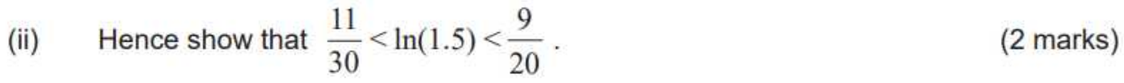
what is the relationship between a and b? (2 marks)

(b) Another graph of $f(x) = \frac{1}{x}$ is shown below.



(i) By considering the areas of the rectangles shown, demonstrate and explain why

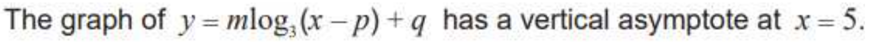
$$\frac{11}{30} < \int_2^3 \frac{1}{x}\,dx < \frac{9}{20}.$$ (3 marks)



(ii) Hence show that $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$. (2 marks)

2021
Section 2
Question
15

The
logarithmic
function



The graph of $y = m\log_3(x - p) + q$ has a vertical asymptote at $x = 5$.

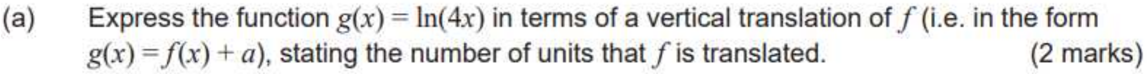
(a) Explain why p = 5. (2 marks)

(b) If this graph passes through the points ($6, 2$) and ($14, -6$), determine the values of m and q. (2 marks)

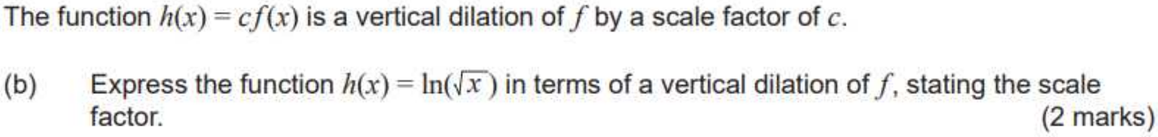
2020
Section 1
Question 6

The logarithmic function

Consider the function $f(x) = \ln(x)$. The function $g(x) = f(x) + a$ is a vertical translation of f by a units.

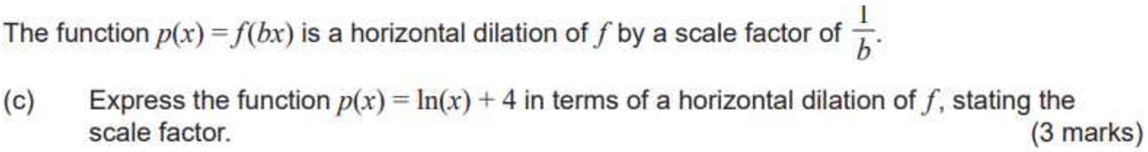


(a) Express the function $g(x) = \ln(4x)$ in terms of a vertical translation of f (i.e. in the form $g(x) = f(x) + a$), stating the number of units that f is translated. (2 marks)



The function $h(x) = cf(x)$ is a vertical dilation of f by a scale factor of c.

(b) Express the function $h(x) = \ln(\sqrt{x})$ in terms of a vertical dilation of f, stating the scale factor. (2 marks)

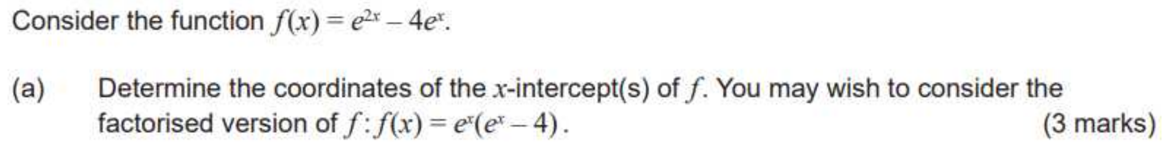


The function $p(x) = f(bx)$ is a horizontal dilation of f by a scale factor of $\frac{1}{b}$.

(c) Express the function $p(x) = \ln(x) + 4$ in terms of a horizontal dilation of f, stating the scale factor. (3 marks)

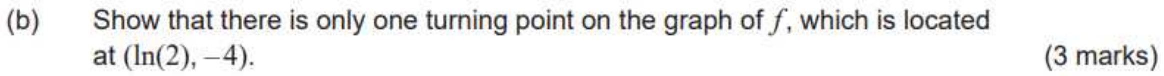
2020
Section 1
Question 7

The logarithmic function



Consider the function $f(x) = e^{2x} - 4e^{x}$.

(a) Determine the coordinates of the x-intercept(s) of f. You may wish to consider the factorised version of $f : f(x) = e^{x}(e^{x} - 4)$. (3 marks)

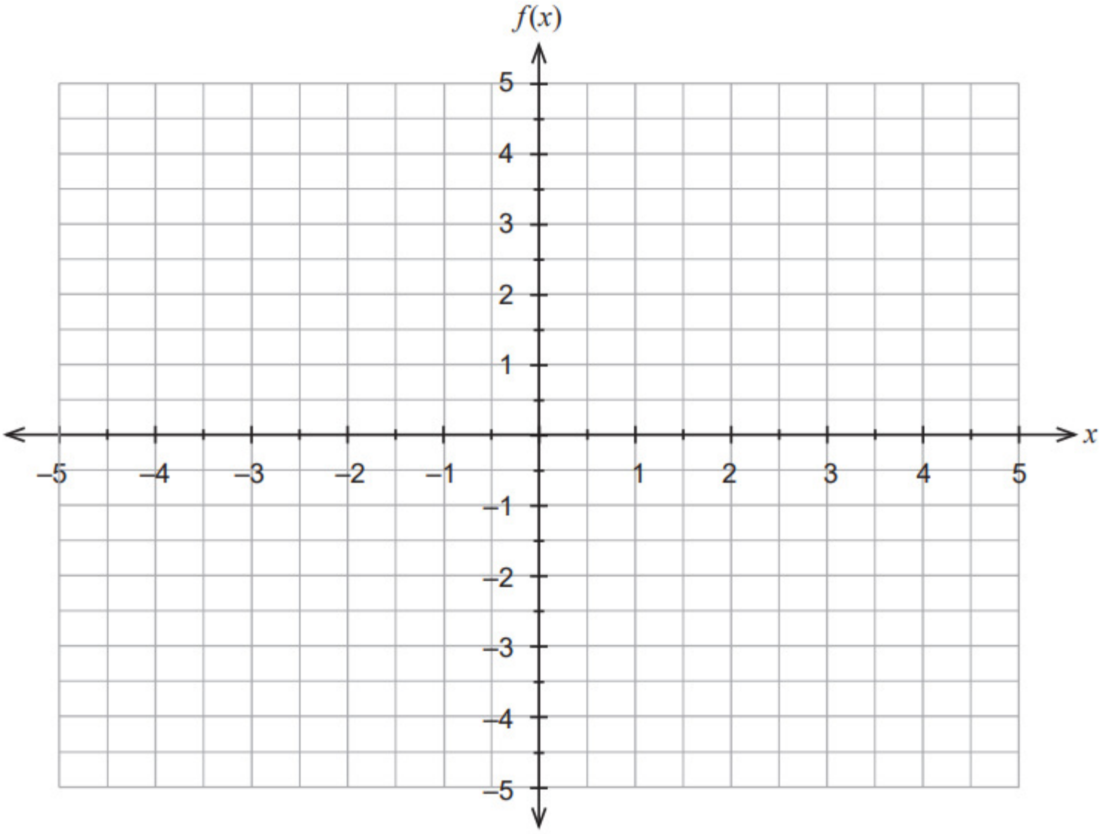


(b) Show that there is only one turning point on the graph of f, which is located at $(\ln(2), -4)$. (3 marks)

(c) Determine the coordinates of the point(s) of inflection of f. (3 marks)

(d) Sketch the function f on the axes below, labelling clearly all intercepts, the turning point and point(s) of inflection. Some approximate values of the natural logarithmic function provided in the table below may be helpful. (4 marks)

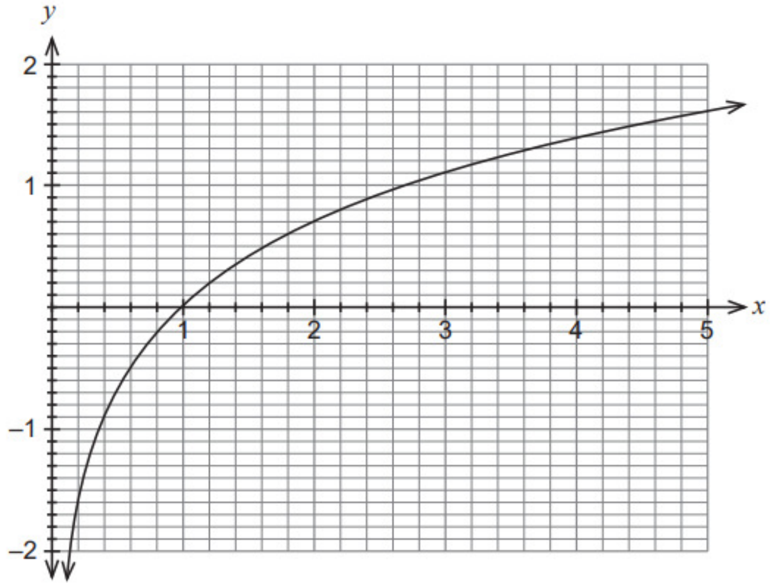
x	1	2	3	4
$\ln(x)$	0	0.7	1.1	1.4



2019
Section 1
Question 4

The logarithmic function

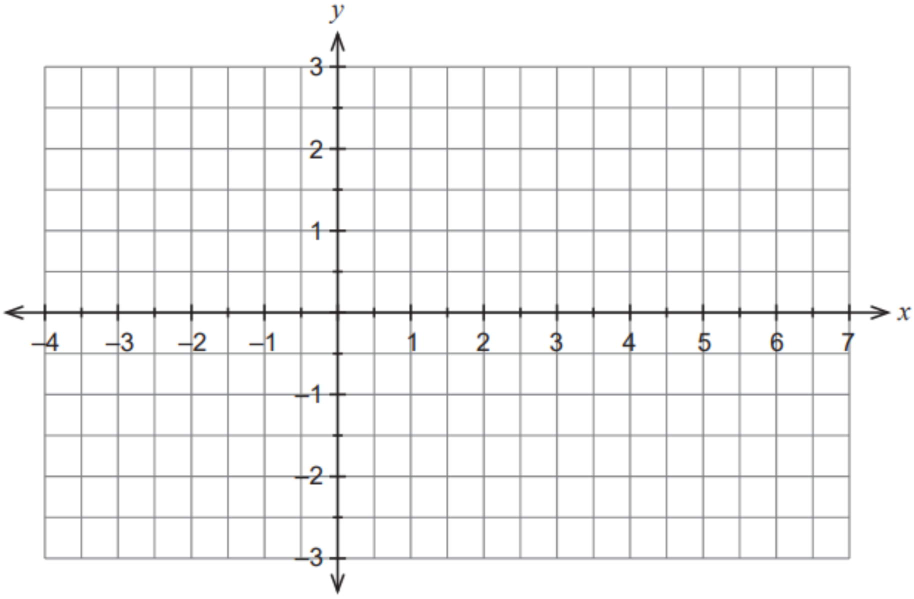
Consider the graph of $y = 1n\ (x)$ shown below.



(a) Use the graph to estimate the value of p in each of the following.

(i) $1.4 = 1n\ (p)$ (1 mark)

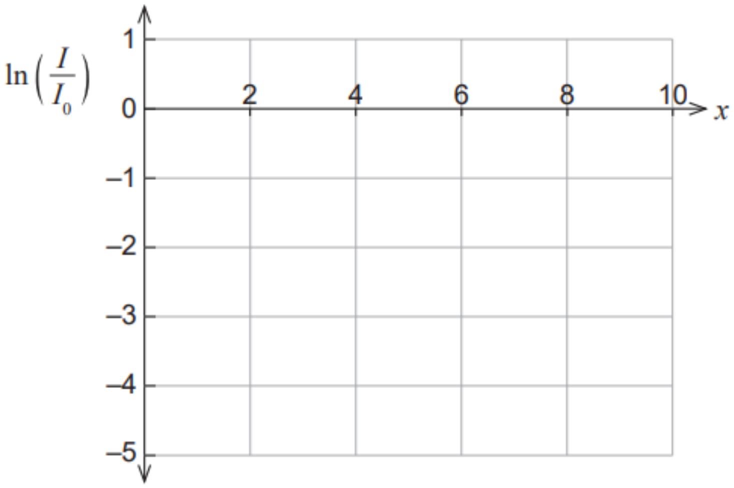
(ii) $e^{p+1} - 3 = 0$ (2 marks)

	(b) On the axes below, sketch the graph of $y = 1n\ (x - 2) + 1$. (3 marks) 

Section 2

2022 Section 2 Question 14 The logarithmic function	The intensity of light travelling through a medium decreases due to scattering and absorption. The intensity of light, I, after travelling a distance of x centimetres through a soft tissue sample is given by $$I = I_0 e^{-0.75x}$$ where I_0 is the initial light intensity. (a) What percentage of the initial light intensity remains after the light has travelled 1 cm through the soft tissue? (2 marks) (b) After how many centimetres will the light intensity have reached one quarter of its initial value? (2 marks)

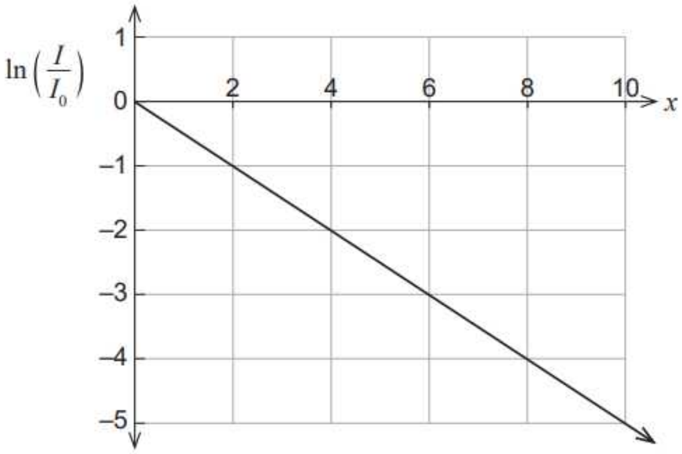
(c) Determine an expression for $\ln\left(\frac{I}{I_0}\right)$, and hence plot $\ln\left(\frac{I}{I_0}\right)$ versus x on the axes below. (3 marks)



The intensity of light passing through a different type of soft tissue satisfies the equation

$$I = I_0 e^{-\mu x}$$

where μ is the attenuation constant. Light intensity measurements were made on a sample of soft tissue, and the results plotted in the graph below.



(d) Use the graph to determine the value of the attenuation constant, μ. (1 mark)

(e) (i) Express the equation $I = I_0 e^{-0.75x}$ using base 10 (in the form $I = I_0 10^{-bx}$). State the value of b to three decimal places. (3 marks)

(ii) Describe the change in intensity over a distance of $\frac{1}{b}$ cm. (2 marks)

2020
Section 2
Question 10

The logarithmic function

Water flows into a bowl at a constant rate. The water level, h, measured in centimetres, increases at a rate given by

$$h'(t) = \frac{4t+1}{2t^2+t+1}$$

where the time t is measured in seconds.

(a) Determine the rate that the water level is rising when $t = 2$ seconds. (1 mark)

(b) Explain why $h(t) = \ln(2t^2 + t + 1) + c$. (2 marks)

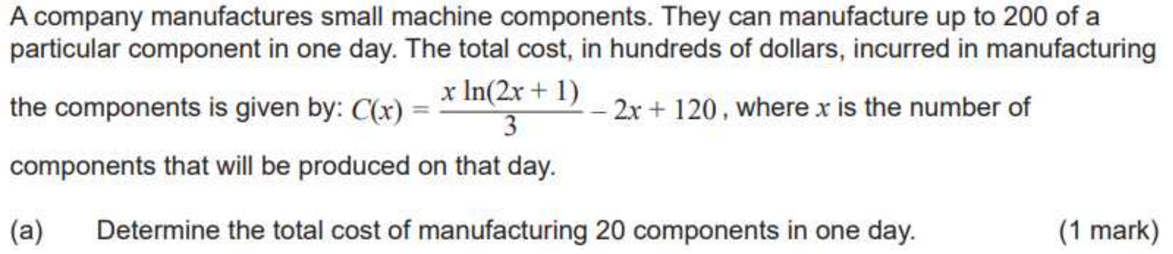
(c) Determine the total change in the water level over the first 2 seconds. (1 mark)

The bowl is filled when the water level reaches $\ln(56)$ cm.

(d) If the bowl is initially empty, determine how long it takes for the bowl to be filled. (3 marks)

2020
Section 2
Question 13

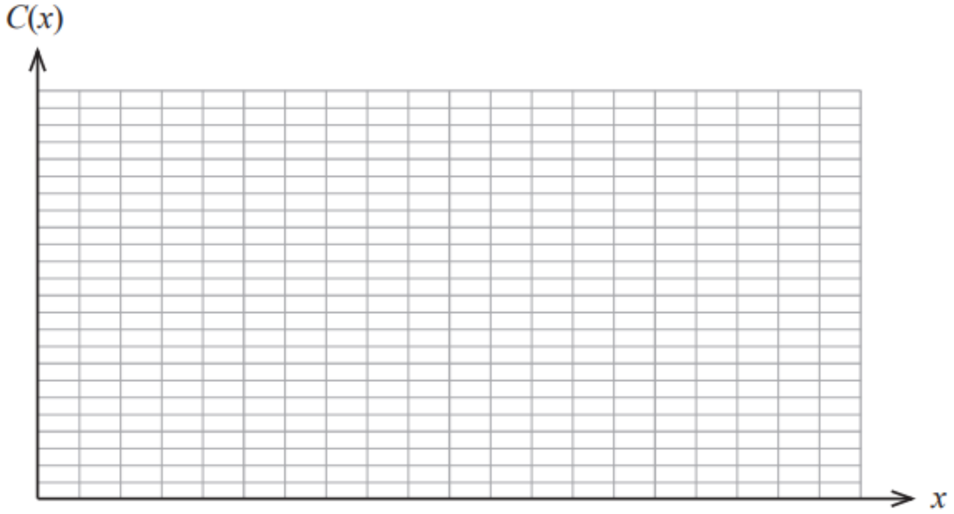
The logarithmic function



A company manufactures small machine components. They can manufacture up to 200 of a particular component in one day. The total cost, in hundreds of dollars, incurred in manufacturing the components is given by: $C(x) = \dfrac{x\ln(2x+1)}{3} - 2x + 120$, where x is the number of components that will be produced on that day.

(a) Determine the total cost of manufacturing 20 components in one day. (1 mark)

(b) On the axes below, sketch the graph of C(x). (3 marks)

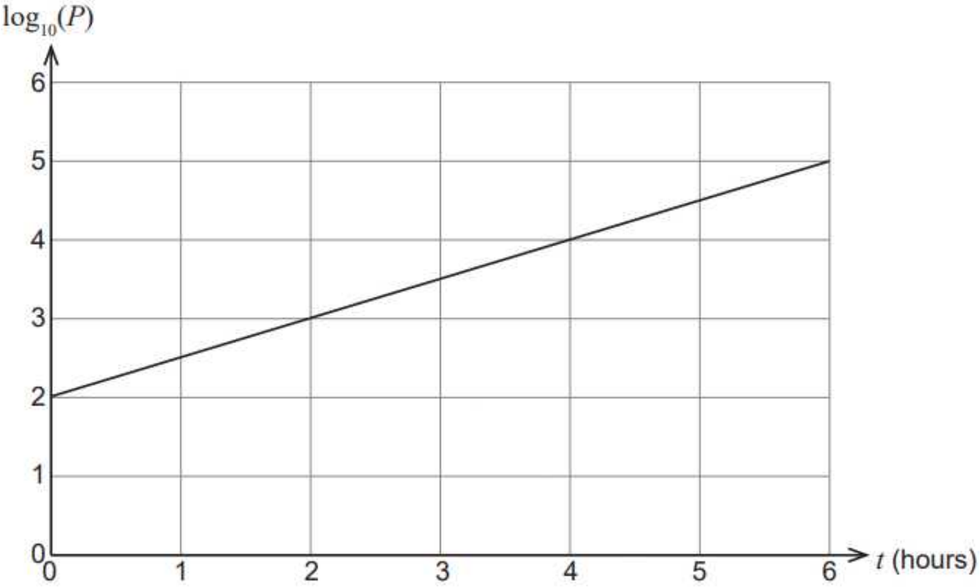


(c) With reference to your graph in part (b), explain how many components the company should manufacture per day if the total cost is to be as low as possible. (3 marks)

2019
Section 2
Question
17

The
logarithmic
function

A microbiologist is studying the effect of temperature on the growth of a certain type of bacteria under controlled laboratory conditions. A population of bacteria is incubated at a temperature of 30 °C and the size of the population measured at hourly intervals for six hours. The logarithm of the population size appears to lie on a straight line when plotted against time (measured in hours) and the line of best fit shown on the axes below.



(a) (i) On the basis of the graph above, what is the size of the bacteria population after two hours? (2 marks)

(ii) The equation of the line can be written in the form $\log_{10}(P) = At + B$. Use the graph to determine the values of A and B. (2 marks)

Another population of the same bacteria is cultured at 40 °C. The size of the population, P, after t hours satisfies the equation

$$\log_{10}(P) = \frac{1}{3}t + 2.$$

(b) (i) Express the above equation in the form $P = A(10)B^t$. (3 marks)

(ii) Determine the size of the population after exactly four hours to the nearest whole number. (1 mark)

(iii) Express the above equation in the form $t = C\log_{10}\left(\frac{P}{D}\right)$. (3 marks)

(iv) How many minutes does it take for the population to reach a size of 5000? Give your answer to the nearest minute. (2 marks)

(c) With reference to parts (a) and (b), describe the effect of temperature on the population growth of this type of bacteria. (2 marks)

Marking Guide – Section 1

2023
Section 1
Question 2

The logarithmic function

Let $p = \ln(2)$, $q = \ln(3)$ and $r = \ln(5)$.

(a) Express each of the following in terms of p, q and/or r.

(i) ln(6) (2 marks)

Solution

$$\begin{aligned}\ln(6) &= \ln(2\times 3)\\ &= \ln(2)+\ln(3)\\ &= p+q\end{aligned}$$

Specific behaviours

✓ expresses $\ln(6)$ as the sum of $\ln(2)$ and $\ln(3)$
✓ obtains correct expression in terms of p and q

(ii) ln(6.25) (3 marks)

Solution

$$\begin{aligned}\ln(6.25) &= \ln\left(\frac{25}{4}\right)\\ &= \ln(25)-\ln(4)\\ &= \ln\left(5^2\right)-\ln\left(2^2\right)\\ &= 2\ln(5)-2\ln(2)\\ &= 2r-2p\end{aligned}$$

Or

$$\begin{aligned}\ln(6.25) &= \ln\left(\frac{25}{4}\right)\\ &= 2\ln\left(\frac{5}{2}\right)\\ &= 2(\ln(5)-\ln(2))\\ &= 2(r-p)\end{aligned}$$

Specific behaviours

✓ expresses 6.25 as the fraction $\frac{25}{4}$ (or equivalent)
✓ applies log law to obtain a correct expression in terms of a difference of logs
✓ obtains correct expression in terms of p and r

(iii) $\int_2^3 \frac{d}{dx}\ln(x)\,dx$ (2 marks)

Solution
By the fundamental theorem of calculus $\int_2^3 \frac{d}{dx}\ln(x)\,dx = [\ln(x)]_2^3$ $= \ln(3) - \ln(2)$ $= q - p$
Specific behaviours
✓ evaluates the definite integral as $\ln(3) - \ln(2)$ ✓ correctly expresses answer in terms of p and q

(b) Evaluate e^{p+q}. (2 marks)

Solution
$e^{p+q} = e^p \times e^q$ $= e^{\ln(2)} \times e^{\ln(3)}$ $= 2 \times 3$ $= 6$
Specific behaviours
✓ correctly applies index law ✓ simplifies to obtain correct answer

(c) (i) Determine $\frac{d}{dx}(x\ln(x))$. (1 marks)

Solution
$\frac{d}{dx}(x\ln(x)) = \ln(x) + x\frac{1}{x}$ $= \ln(x) + 1$
Specific behaviours
✓ differentiates correctly

(ii) Hence show that $\int \ln(x)\, dx = x\ln(x) - x + c$ where c is a constant. (2 marks)

Solution
$\frac{d}{dx}\left(x\ln(x)\right) = \ln(x) + 1$ $\Rightarrow \int \frac{d}{dx}\left(x\ln(x)\right) dx = \int \ln(x)\, dx + \int 1\, dx$ $\Rightarrow x\ln(x) = \int \ln(x)\, dx + x + c$ $\Rightarrow \int \ln(x)\, dx = x\ln(x) - x + c$
Specific behaviours
✓ integrates both sides of the result from part (c)(i) and correctly evaluates $\int 1\, dx$ ✓ evaluates $\int \frac{d}{dx}\left(x\ln(x)\right) dx$ and applies valid mathematical operations to obtain required expression

(iii) Evaluate $\int_1^3 \ln(x)\, dx$ in terms of p, q and/or r. (2 marks)

Solution
$\int_1^3 \ln(x)\, dx = \left[x\ln(x) - x\right]_1^3$ $= 3\ln(3) - 3 - \left(\ln(1) - 1\right)$ $= 3q - 2$
Specific behaviours
✓ applies fundamental theorem of calculus to evaluate definite integral ✓ simplifies to obtain correct answer

2023
Section 1
Question 4

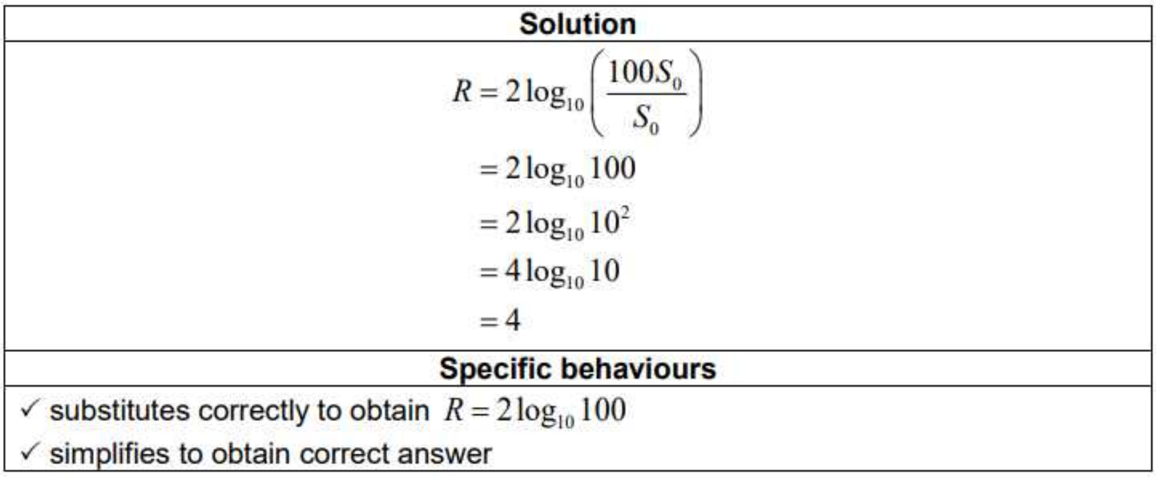
The logarithmic function

An internet search engine uses a logarithmic scale to rank the importance of internet websites. If a website has S visits each week, the site rank, R, is given by

$$R = 2\log_{10}\left(\frac{S}{S_0}\right)$$

where S_0 is the reference value (the same for all websites). The reference value is the minimum number of visits per week required for a website to register on the site rank scale.

(a) Determine the site rank for a website whose weekly visits are one hundred times the reference value. (2 marks)



Solution

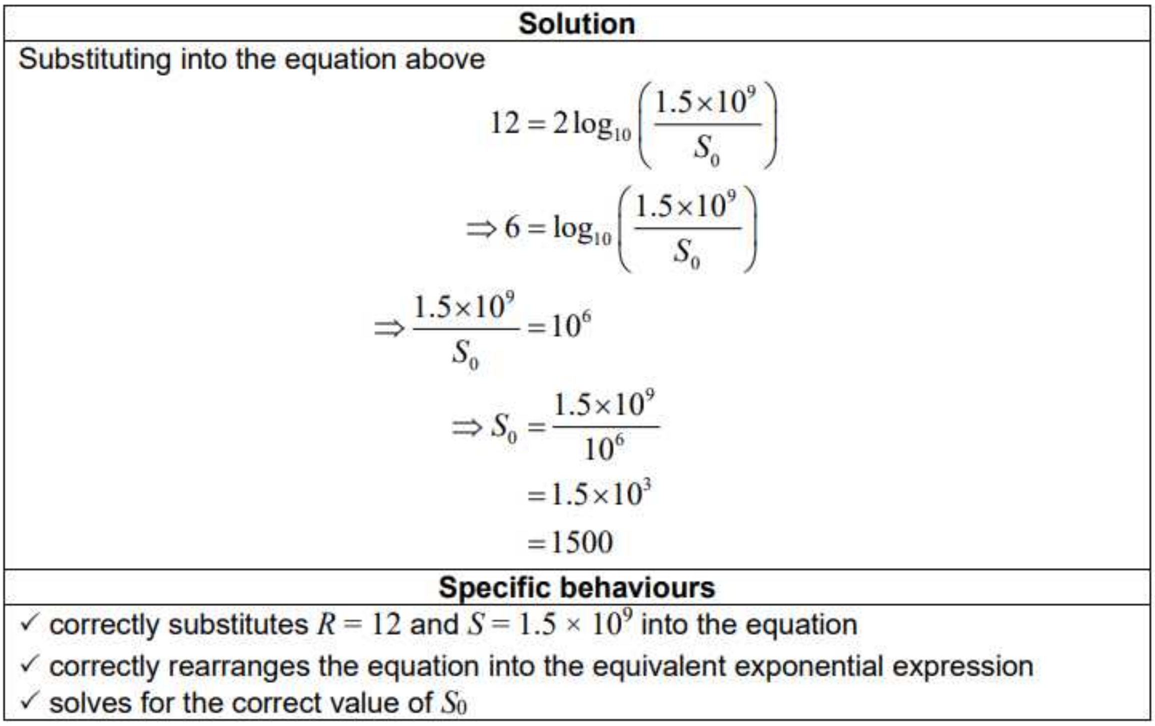
$$\begin{aligned} R &= 2\log_{10}\left(\frac{100S_0}{S_0}\right) \\ &= 2\log_{10}100 \\ &= 2\log_{10}10^2 \\ &= 4\log_{10}10 \\ &= 4 \end{aligned}$$

Specific behaviours

✓ substitutes correctly to obtain $R = 2\log_{10}100$

✓ simplifies to obtain correct answer

(b) Given that a site rank of 12 is assigned to a website with 1.5 billion (1.5×10^9) visits per week, determine the value of S_0. (3 marks)



Solution

Substituting into the equation above

$$12 = 2\log_{10}\left(\frac{1.5\times10^9}{S_0}\right)$$

$$\Rightarrow 6 = \log_{10}\left(\frac{1.5\times10^9}{S_0}\right)$$

$$\Rightarrow \frac{1.5\times10^9}{S_0} = 10^6$$

$$\begin{aligned} \Rightarrow S_0 &= \frac{1.5\times10^9}{10^6} \\ &= 1.5\times10^3 \\ &= 1500 \end{aligned}$$

Specific behaviours

✓ correctly substitutes $R = 12$ and $S = 1.5 \times 10^9$ into the equation

✓ correctly rearranges the equation into the equivalent exponential expression

✓ solves for the correct value of S_0

(c) The plot of $y = \log_{10}(x)$ is shown below. If a website has a site rank of 3.2, use the plot and your answer from part (b) to approximate the website's number of weekly visits. (3 marks)

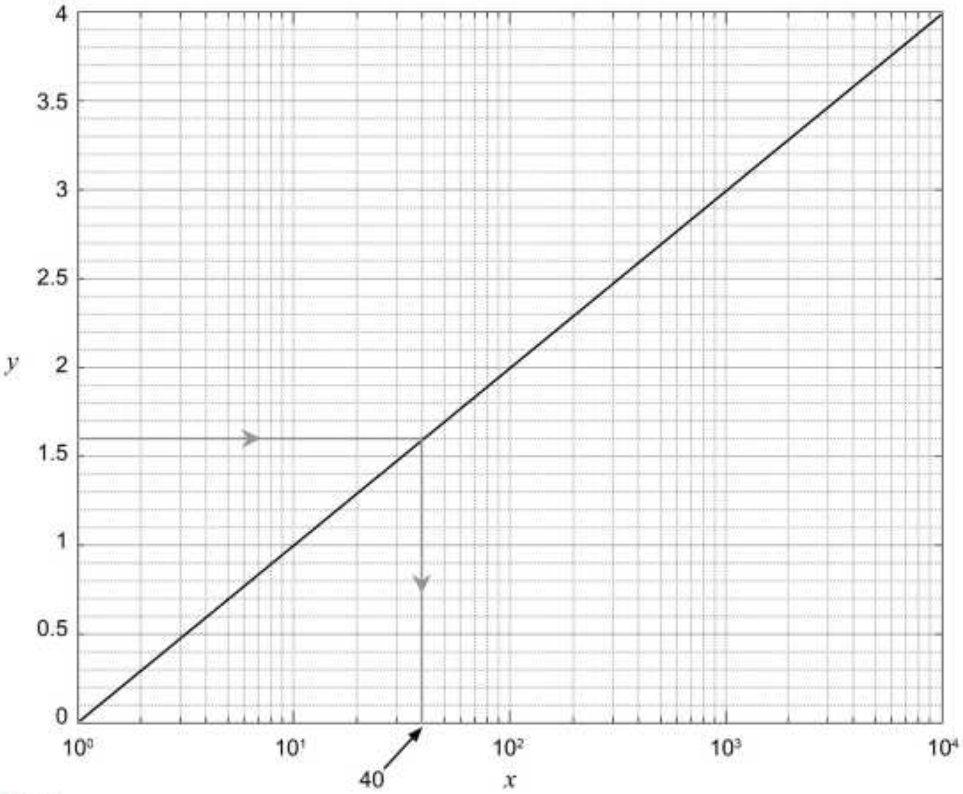
Solution

Substituting $R = 3.2$ into the site rank equation gives

$$3.2 = 2\log_{10}\left(\frac{S}{1500}\right)$$

$$\Rightarrow 1.6 = \log_{10}\left(\frac{S}{1500}\right)$$

From the graph, when $y = 1.6$, $x \approx 40$.



Hence

$$\frac{S}{1500} \approx 40$$

$$\Rightarrow S \approx 1500 \times 40$$

$$= 60\ 000$$

so the number of weekly visits is approximate 60 000.

Specific behaviours

✓ identifies the need to solve $1.6 = \log_{10}(x)$

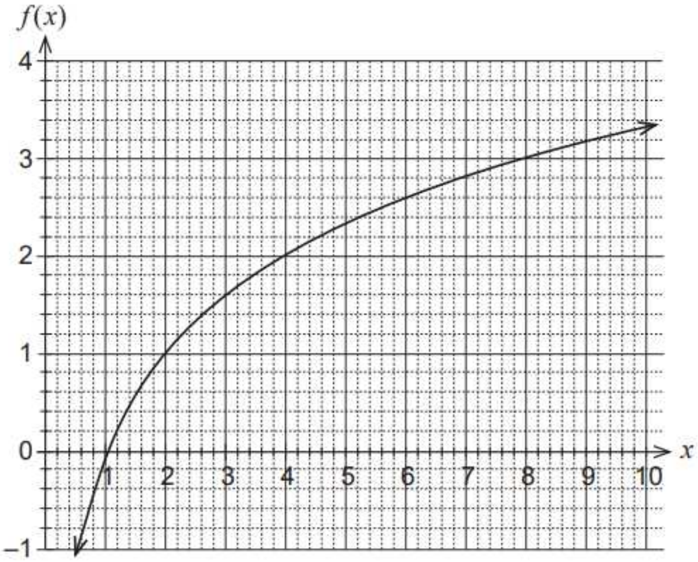
✓ uses the graph to determine that when $y = 1.6$, $x \approx 40$

✓ determines the correct number of weekly visits

2022
Section 1
Question 4

The logarithmic function

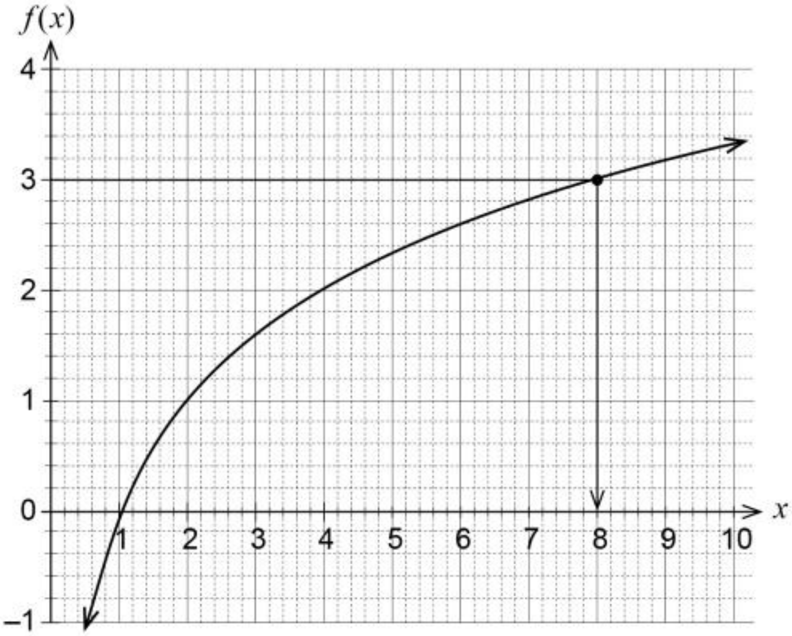
The graph of the function $f(x) = \log_2(x)$ is shown below.



(a) Using the graph:

(i) solve $\log_2 (x - 5) = 3$. (2 marks)

Solution



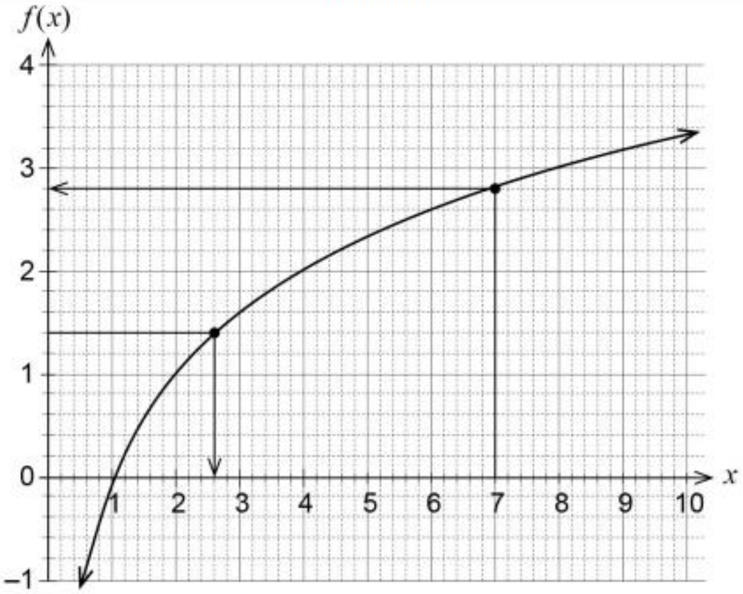
$$\begin{aligned} \log_2(x-5) &= 3 \\ \log_2 8 &= 3 \text{ (from graph)} \\ x - 5 &= 8 \\ x &= 13 \end{aligned}$$

Specific behaviours

✓ identifies $\log_2 8 = 3$
✓ solves correctly for x

(ii) determine $\sqrt{7}$, correct to one decimal place. (Hint: let $x = \sqrt{7}$.) (3 marks)

Solution



$$\begin{aligned} \log_2(x) &= \log_2(\sqrt{7}) \\ &= \log_2\left(7^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\log_2(7) \\ &\approx \frac{1}{2}\times 2.8 \\ &= 1.4 \end{aligned}$$

So $\log_2(x) = 1.4$ from graph

$x \approx 2.6$

Hence $\sqrt{7} \approx 2.6$

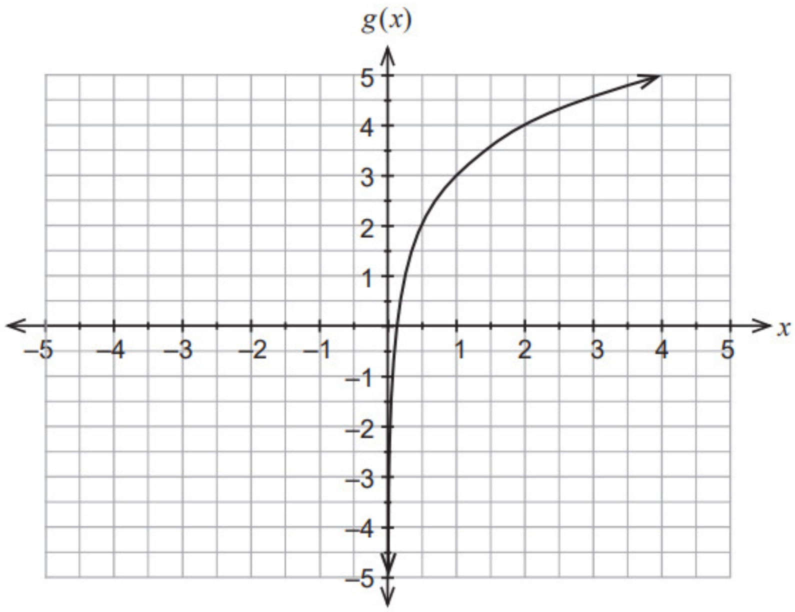
Specific behaviours

✓ applies log laws to express $\log_2(x)$ as $\frac{1}{2}\log_2(7)$

✓ approximates $\log_2(7)$ as 2.8

✓ obtains correct approximation to $\sqrt{7}$

(b) The function $f(x) = \log_2(x)$ is translated to give the new function $g(x)$, which is shown in the graph below.



Determine the equation for $g(x)$. (2 marks)

Solution
$g(x) = f(x) + 3$ $= \log_2(x) + 3$
Specific behaviours
✓ identifies a vertical translation of 3 units up ✓ states correct equation for $g(x)$

(c) (i) Show that $\log_2\left(\frac{1}{x-1}\right) = -\log_2(x-1)$. (2 marks)

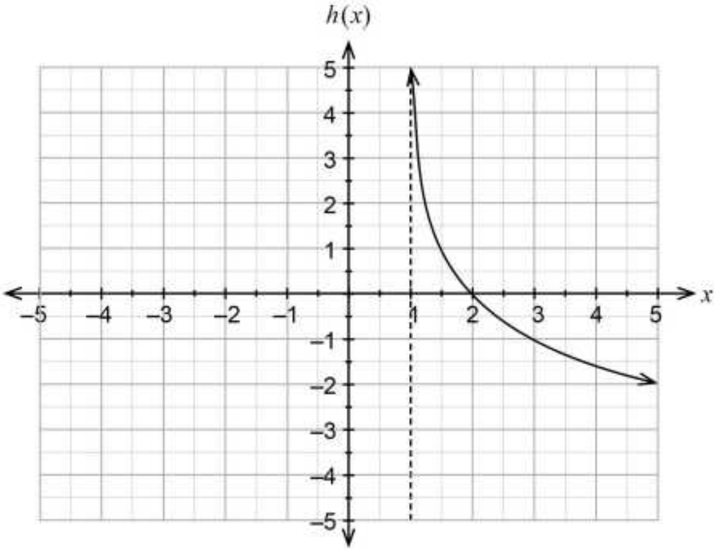
Solution
$\log_2\left(\frac{1}{x-1}\right)$ $= \log_2 1 - \log_2(x-1)$ $= -\log_2(x-1)$
Specific behaviours
✓ applies log laws to obtain $\log_2 1 - \log_2(x-1)$ ✓ recognises $\log_2 1 = 0$ and hence obtains required expression

Alternate solution
$\log_2\left(\frac{1}{x-1}\right)$ $= \log_2(x-1)^{-1}$ $= -\log_2(x-1)$
Specific behaviours
✓ recognises $\frac{1}{x-1}$ is equivalent to $(x-1)^{-1}$ ✓ applies log laws to obtain required expression

(ii) Hence sketch the graph of $h(x) = \log_2\left(\frac{1}{x-1}\right)$ on the axes below. (3 marks)

Solution

$h(x)$ is obtained by reflecting $f(x)$ vertically about the x-axis, and translating 1 unit to the right.



Specific behaviours

✓ draws asymptote at $x = 1$
✓ graph passes through $(2, 0)$
✓ graph passes through $(3, -1)$ and has correct shape

2021
Section 1
Question 3

The logarithmic function

Given that $\ln(2) \approx 0.693$, use the increments formula to determine an approximation for $\ln(2.02)$. (3 marks)

Solution

Let $y = \ln(x)$

Then $x = 2, \delta x = 0.02$

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\therefore \delta y \approx \frac{dy}{dx} \times \delta x$$

$$= \frac{1}{x} \times \delta x$$

$$= \frac{0.02}{2}$$

$$= 0.01$$

$$\therefore \ln(2.02) \approx \ln(2) + 0.01$$

$$= 0.703$$

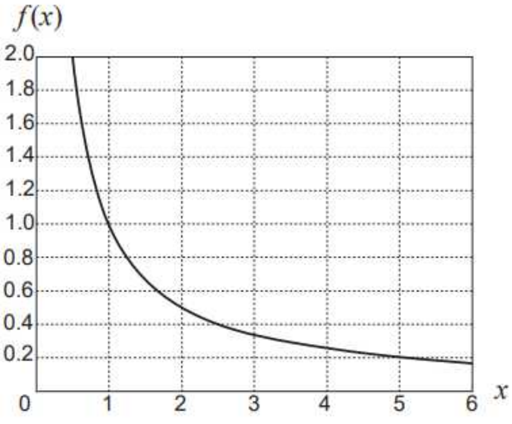
Specific behaviours

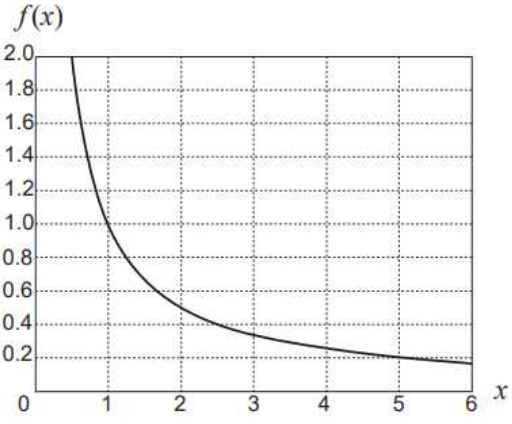
✓ correctly determines δx
✓ correctly determines δy
✓ determines correct approximation

2021
Section 1
Question 7

The logarithmic function

(a) Consider the function $f(x) = \frac{1}{x}$, graphed twice below.





(i) Shade **two** different regions (one on each graph above) each with area exactly $\ln(2)$. (2 marks)

Solution

Two distinct cases in which the upper bound is twice the lower bound. I would expect most to shade under the curve from $x = 1$ to $x = 2$, and then from $x = 2$ to $x = 4$.
Other possibilities would be $x = 1.5$ to $x = 3$, $x = 2.5$ to $x = 5$ or $x = 3$ to $x = 6$.

Specific behaviours

✓ shades a region under the curve corresponding to $\ln(2)$
✓ shades a second distinct region under the curve corresponding to $\ln(2)$

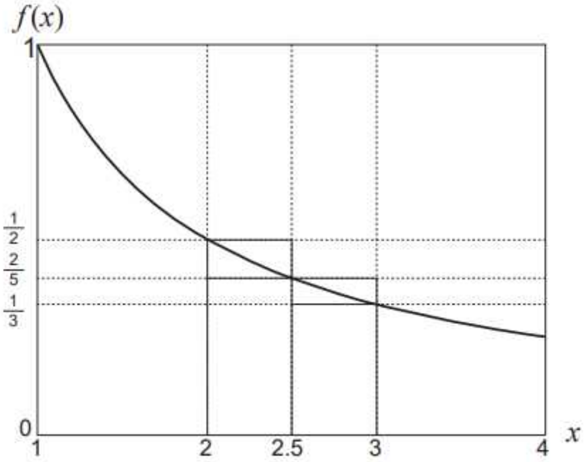
(ii) Given that

$$\int_a^b \frac{1}{x}\,dx = \ln(3)$$

what is the relationship between a and b? (2 marks)

Solution
$Area = \int_a^b \frac{1}{x}dx = \ln\left(\frac{b}{a}\right)$ $= \ln 3$ So, $b = 3a$.
Specific behaviours
✓ obtains the correct integral in terms of a and b ✓ states the relationship between a and b

(b) Another graph of $f(x) = \frac{1}{x}$ is shown below.



(i) By considering the areas of the rectangles shown, demonstrate and explain why

$$\frac{11}{30} < \int_2^3 \frac{1}{x}\,dx < \frac{9}{20}.$$ (3 marks)

Solution
Using the rectangles that estimate $y = \frac{1}{x}$ on the left side of each interval gives $$\int_2^3 \frac{1}{x}dx < \frac{1}{2}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{9}{20}$$ This is an overestimate of the integral as the top of the rectangles lie above the graph. Using the rectangles that estimate $y = \frac{1}{x}$ on the right side of each interval gives $$\int_2^3 \frac{1}{x}dx > \frac{2}{5}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}=\frac{11}{30}$$ This is an underestimate of the integral as the top of the rectangles lie below the graph. Hence, $$\frac{11}{30} < \int_2^3 \frac{1}{x}dx < \frac{9}{20}$$
Specific behaviours
✓ approximates the integral using $\frac{1}{2}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{9}{20}$ ✓ approximates the integral using $\frac{2}{5}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}=\frac{11}{30}$ ✓ explains why the first approximation is an overestimate and the second is an underestimate

(ii) Hence show that $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$. (2 marks)

Solution
$\int_2^3 \frac{1}{x}dx = [\ln(x)]_2^3 = \ln(3) - \ln(2) = \ln(1.5)$ Hence $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$
Specific behaviours
✓ correctly integrates $\frac{1}{x}$ and substitutes bounds to obtain $\ln(3) - \ln(2)$ ✓ applies log law to obtain $\ln(3) - \ln(2) = \ln(1.5)$

Marking Guide – Section 2

2021
Section 2
Question 15

The logarithmic function

The graph of $y = m\log_3(x-p)+q$ has a vertical asymptote at $x = 5$.

(a) Explain why p = 5. (2 marks)

Solution
The graph of $y=\log_3(x)$ has a vertical asymptote at $x=0$. The graph of $y=m\log_3(x-p)+q$ has been translated p units to the right. Since this graph has a vertical asymptote at $x=5$, p must equal 5.
Specific behaviours
✓ states the graph of $y=\log_3(x)$ has a vertical asymptote at $x=0$ ✓ identifies a horizontal translation and equates vertical asymptote to value of p

(b) If this graph passes through the points $(6, 2)$ and $(14, -6)$, determine the values of m and q. (2 marks)

Solution
Substituting the points into equation: $2 = m\log_3(1)+q$ (1) $-6 = m\log_3(9)+q$ (2) Equation (1) gives $q=2$ Equation (2) gives: $-6 = m\log_3(3^2)+2$ $-8 = 2m$ $m=-4$
Specific behaviours
✓ substitutes the points into the equation ✓ determines the values of q and m

2020
Section 1
Question 6

The logarithmic function

Consider the function $f(x) = \ln(x)$. The function $g(x) = f(x) + a$ is a vertical translation of f by a units.

(a) Express the function $g(x) = \ln(4x)$ in terms of a vertical translation of f (i.e. in the form $g(x) = f(x) + a$), stating the number of units that f is translated. (2 marks)

Solution
$g(x) = \ln(4x)$ $= \ln(4) + \ln(x)$ $= f(x) + \ln(4)$ f is translated vertically (upward) by $\ln(4)$ units.
Specific behaviours
✓ expresses $g(x)$ as a sum of logs ✓ recognises a vertical translation by $\ln(4)$ units

The function $h(x) = cf(x)$ is a vertical dilation of f by a scale factor of c.

(b) Express the function $h(x) = \ln(\sqrt{x})$ in terms of a vertical dilation of f, stating the scale factor. (2 marks)

Solution

$$\begin{aligned} h(x) &= \ln\left(\sqrt{\text{x}}\right) \\ &= \ln(x^{0.5}) \\ &= 0.5\ln(x) \\ &= 0.5f(x) \end{aligned}$$

f is scaled vertically by a factor of 0.5.

Specific behaviours

✓ expresses h as a product involving $\ln(x)$
✓ recognises a vertical scaling by a scale factor of 0.5

The function $p(x) = f(bx)$ is a horizontal dilation of f by a scale factor of $\frac{1}{b}$.

(c) Express the function $p(x) = \ln(x) + 4$ in terms of a horizontal dilation of f, stating the scale factor. (3 marks)

Solution

$$\begin{aligned} p(x) &= \ln(x) + 4 \\ &= \ln(x) + 4\ln(e) \\ &= \ln(x) + \ln(e^4) \\ &= \ln(e^4 x) \\ &= f(e^4 x) \end{aligned}$$

f is scaled horizontally by a scale factor of e^{-4}.

Specific behaviours

✓ expresses 4 as $4\ln(e)$
✓ expresses p using a single logarithm
✓ states horizontal scale factor

2020
Section 1
Question 7

The logarithmic function

Consider the function $f(x) = e^{2x} - 4e^x$.

(a) Determine the coordinates of the x-intercept(s) of f. You may wish to consider the factorised version of $f: f(x) = e^x(e^x - 4)$. (3 marks)

Solution

Solve $f(x) = 0$

$$\begin{aligned} 0 &= e^x(e^x - 4) \\ e^x &= 4 \\ x &= \ln(4) \end{aligned}$$

Hence x-intercept at $(\ln(4), 0)$.

Specific behaviours

✓ states correct equation to be solved
✓ solves for x
✓ states coordinates

(b) Show that there is only one turning point on the graph of f, which is located at $(\ln(2), -4)$. (3 marks)

Solution
$f'(x) = 2e^{2x} - 4e^{x}$ Solve $f'(x) = 0$ $0 = 2e^{2x} - 4e^{x}$ $= 2e^{x}(e^{x} - 2)$ $e^{x} = 2$ $x = \ln(2)$ Substitute $x = \ln(2)$ into $f(x)$ $f(\ln(2)) = e^{2\ln(2)} - 4e^{\ln(2)}$ $= e^{\ln(4)} - 4e^{\ln(2)}$ $= 4 - 8$ $= -4$ Turning point at $(\ln(2), -4)$.
Specific behaviours
✓ differentiates $f(x)$ correctly and equates to 0 ✓ shows the steps required to solve for x ✓ demonstrates the use of log laws to determine the y-coordinate

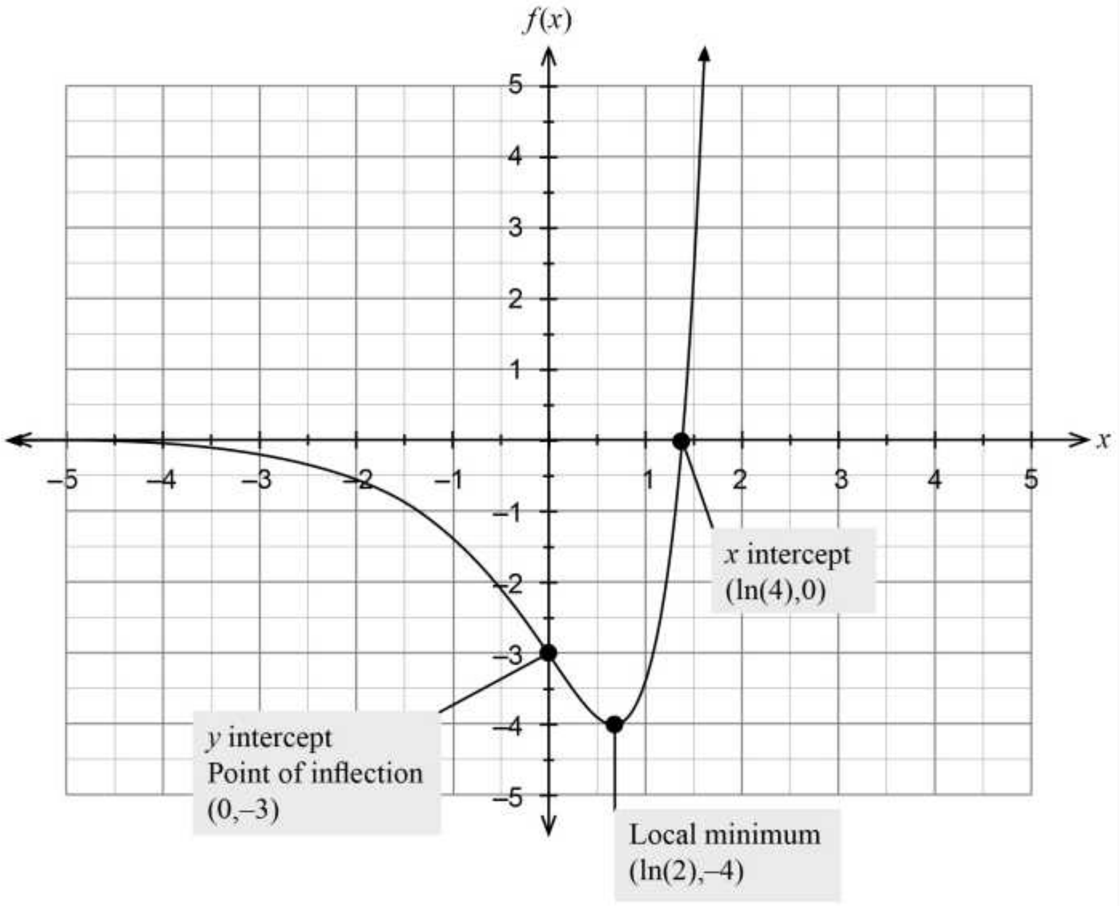
(c) Determine the coordinates of the point(s) of inflection of f. (3 marks)

Solution
$f''(x) = 4e^{2x} - 4e^{x}$ Solve $f''(x) = 0$ $0 = 4e^{2x} - 4e^{x}$ $= 4e^{x}(e^{x} - 1)$ $e^{x} = 1$ $x = \ln(1) = 0$ Substitute $x = 0$ into $f(x)$ $f(\ln(2)) = e^{2(0)} - 4e^{0}$ $= 1 - 4$ $= -3$ Inflection point at $(0, -3)$.
Specific behaviours
✓ differentiates $f'(x)$ correctly and equates to 0 ✓ solves for x ✓ determines y-coordinate of inflection point

	(d) Sketch the function f on the axes below, labelling clearly all intercepts, the turning point and point(s) of inflection. Some approximate values of the natural logarithmic function provided in the table below may be helpful. (4 marks)

x	1	2	3	4
$\ln(x)$	0	0.7	1.1	1.4

Solution



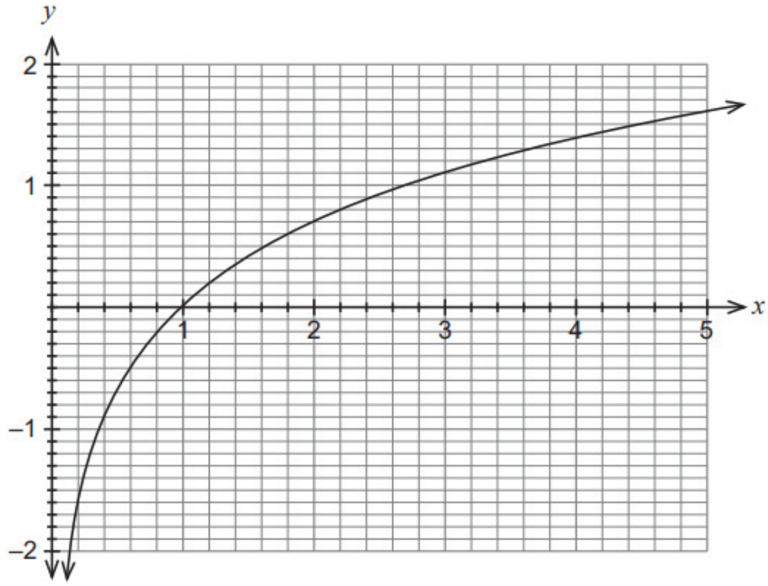
Specific behaviours

✓ intercepts correct and labelled
✓ turning point and inflection point correct and labelled
✓ concavity correct
✓ limiting behaviour correct

2019
Section 1
Question 4

The logarithmic function

Consider the graph of $y = 1n\,(x)$ shown below.



(a) Use the graph to estimate the value of p in each of the following.

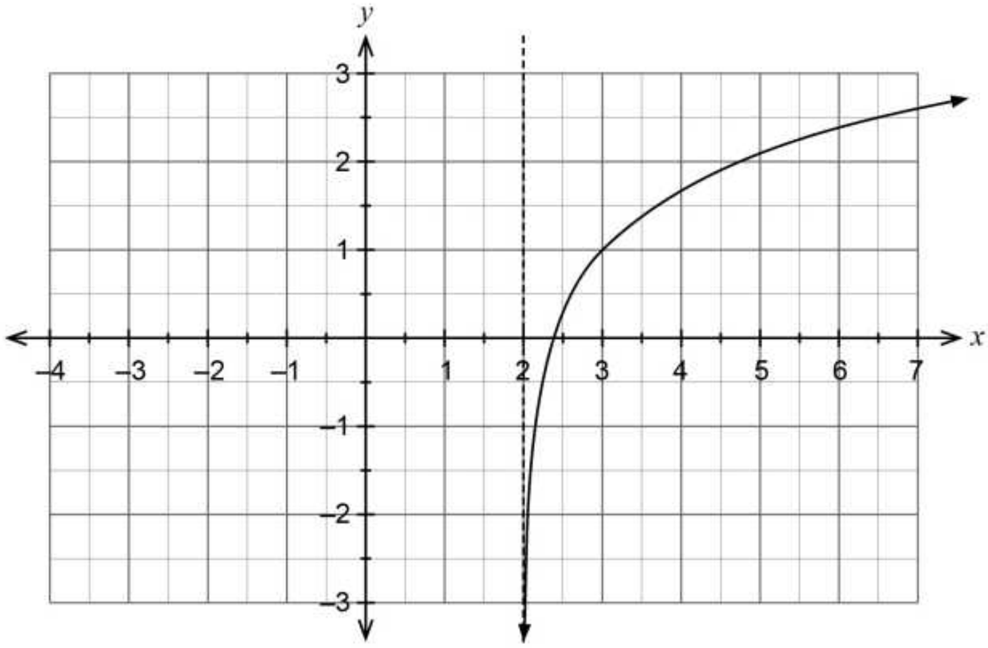
(i) $1.4 = 1n\,(p)$ (1 mark)

Solution
$p = 4$
Specific behaviours
✓ states the correct value of p

(ii) $e^{p+1} - 3 = 0$ (2 marks)

Solution
$e^{p+1} = 3$ $p+1 = \ln(3)$ $p+1 = 1.1$ $\therefore p = 0.1$
Specific behaviours
✓ rearranges to form a logarithmic equation ✓ states the correct value of p

(b) On the axes below, sketch the graph of $y = 1n\,(x - 2) + 1$. (3 marks)



Solution

Specific behaviours

✓ draws asymptote at $x = 2$

✓ the sketch passes through the point (3,1)

✓ the sketch has the correct shape and has a y-coordinate between 2.5 and 3 when $x = 7$

2022
Section 2
Question 14

The logarithmic function

The intensity of light travelling through a medium decreases due to scattering and absorption. The intensity of light, I, after travelling a distance of x centimetres through a soft tissue sample is given by

$$I = I_0 e^{-0.75x}$$

where I_0 is the initial light intensity.

(a) What percentage of the initial light intensity remains after the light has travelled 1 cm through the soft tissue? (2 marks)

Solution

$$\frac{I}{I_0} = e^{-0.75\times1} = 0.4724$$

Hence 47.24% of the initial light intensity remains after 1 cm.

Specific behaviours

✓ obtains correct ratio

✓ expresses answer as a percentage

(b) After how many centimetres will the light intensity have reached one quarter of its initial value? (2 marks)

Solution

$$\frac{1}{4} = e^{-0.75x}$$

$$\ln\left(\frac{1}{4}\right) = -0.75x$$

$$x = -\frac{1}{0.75}\ln\left(\frac{1}{4}\right)$$

$$\approx 1.85 \text{ cm}$$

Specific behaviours

✓ sets $\frac{I}{I_0} = \frac{1}{4}$ in equation
✓ solves for x

(c) Determine an expression for $\ln\left(\frac{I}{I_0}\right)$, and hence plot $\ln\left(\frac{I}{I_0}\right)$ versus x on the axes below. (3 marks)

Solution

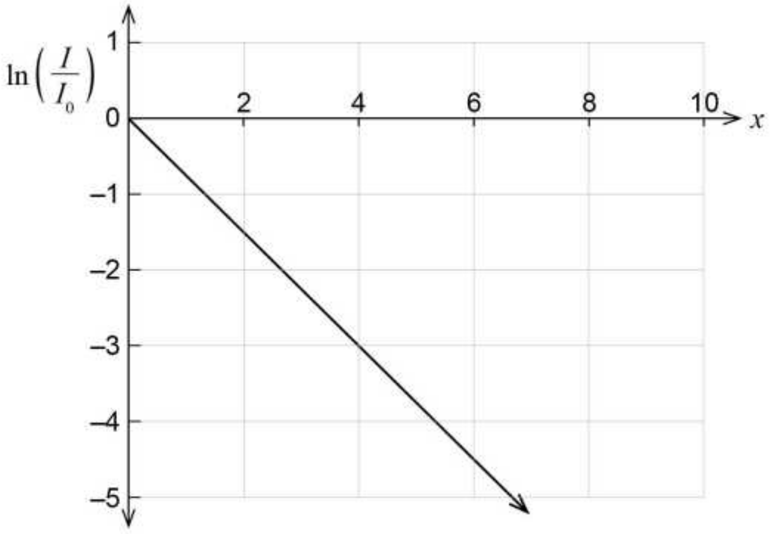
From the equation

$$\frac{I}{I_0} = e^{-0.75x}$$

$$\ln\left(\frac{I}{I_0}\right) = \ln e^{-0.75x}$$

$$\ln\left(\frac{I}{I_0}\right) = -0.75x$$

so the graph is a straight line passing through the origin with gradient -0.75.



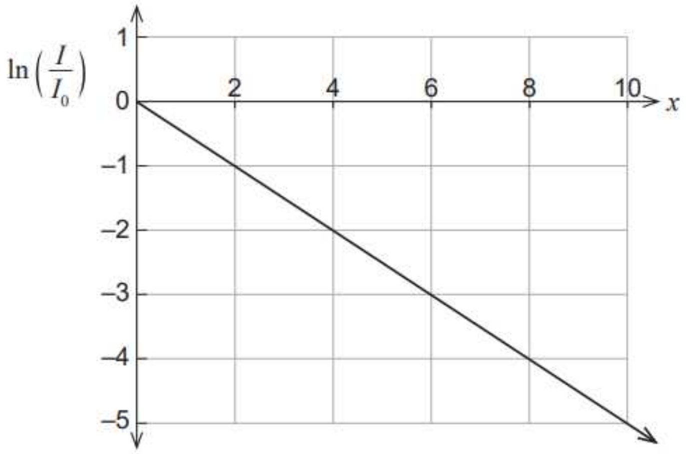
Specific behaviours

✓ rearranges the equation into a linear form using logs
✓ identifies that -0.75 is the gradient
✓ line plotted correctly (passes through origin and correct gradient)

The intensity of light passing through a different type of soft tissue satisfies the equation

$$I = I_0 e^{-\mu x}$$

where μ is the attenuation constant. Light intensity measurements were made on a sample of soft tissue, and the results plotted in the graph below.

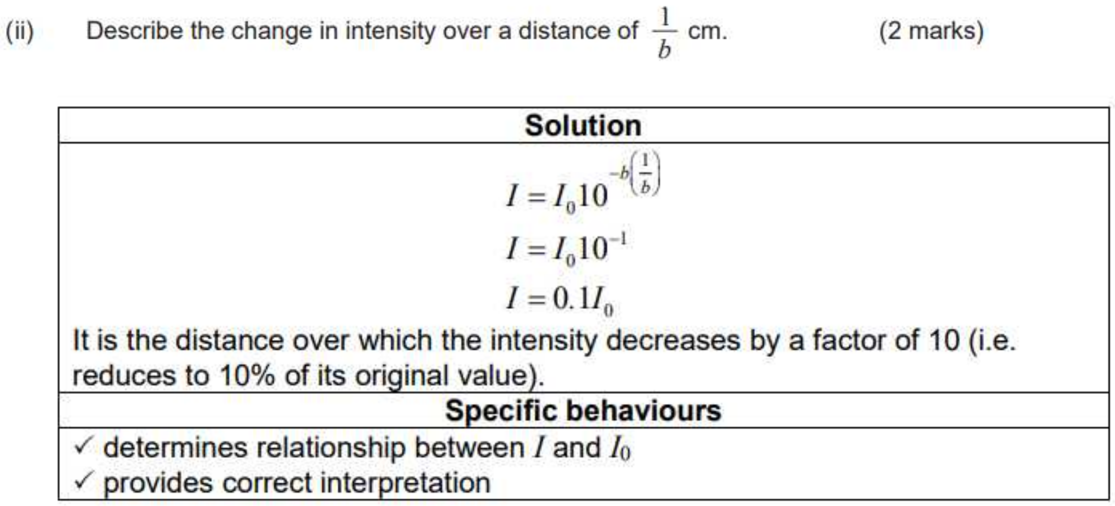


(d) Use the graph to determine the value of the attenuation constant, μ. (1 mark)

Solution
The form of the equation is $\ln\left(\frac{I}{I_0}\right) = -\mu x$ The gradient of the line is −0.5, which means that $\mu = 0.5$
Specific behaviours
✓ obtains correct value for μ

(e) (i) Express the equation $I = I_0 e^{-0.75x}$ using base 10 (in the form $I = I_0 10^{-bx}$). State the value of b to three decimal places. (3 marks)

Solution
$\frac{I}{I_0} = e^{-0.75x}$ $\Rightarrow \log_{10}\left(\frac{I}{I_0}\right) = \log_{10}(e^{-0.75x})$ $\Rightarrow \log_{10}\left(\frac{I}{I_0}\right) = -0.75x\log_{10}(e)$ $\Rightarrow \frac{I}{I_0} = 10^{-0.75\log_{10}(e)x}$ $\Rightarrow I \approx I_0 10^{-0.326x}$ Hence $b = 0.326$.
Specific behaviours
✓ takes logarithm base 10 of both sides of the equation ✓ applies logarithm laws to express the intensity (or intensity ratio) as a base 10 ✓ obtains correct value of b to three decimal places



(ii) Describe the change in intensity over a distance of $\frac{1}{b}$ cm. (2 marks)

Solution

$$I = I_0 10^{-b\left(\frac{1}{b}\right)}$$
$$I = I_0 10^{-1}$$
$$I = 0.1 I_0$$

It is the distance over which the intensity decreases by a factor of 10 (i.e. reduces to 10% of its original value).

Specific behaviours

✓ determines relationship between I and I_0
✓ provides correct interpretation

2020
Section 2
Question
10

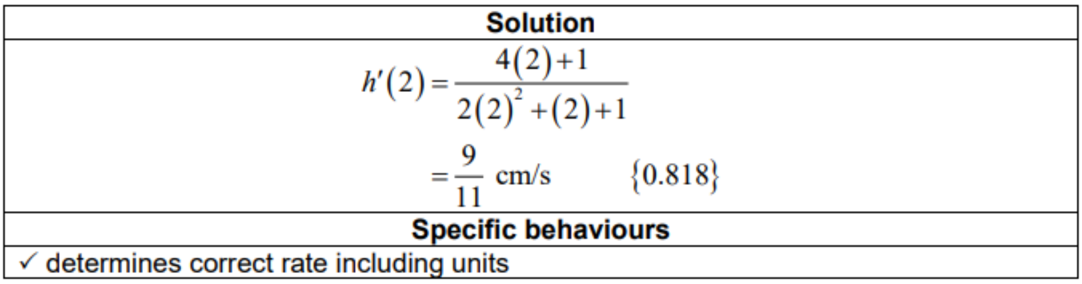
The logarithmic function

Water flows into a bowl at a constant rate. The water level, h, measured in centimetres, increases at a rate given by

$$h'(t) = \frac{4t+1}{2t^2+t+1}$$

where the time t is measured in seconds.

(a) Determine the rate that the water level is rising when $t = 2$ seconds. (1 mark)



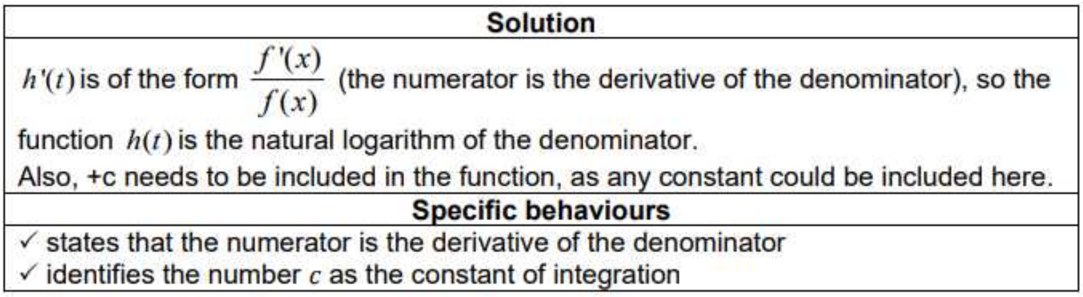
Solution

$$h'(2) = \frac{4(2)+1}{2(2)^2+(2)+1}$$
$$= \frac{9}{11} \text{ cm/s} \qquad \{0.818\}$$

Specific behaviours

✓ determines correct rate including units

(b) Explain why $h(t) = \ln(2t^2 + t + 1) + c$. (2 marks)



Solution

$h'(t)$ is of the form $\frac{f'(x)}{f(x)}$ (the numerator is the derivative of the denominator), so the function $h(t)$ is the natural logarithm of the denominator.
Also, +c needs to be included in the function, as any constant could be included here.

Specific behaviours

✓ states that the numerator is the derivative of the denominator
✓ identifies the number c as the constant of integration

(c) Determine the total change in the water level over the first 2 seconds. (1 mark)

Solution
$\Delta h = \int_0^2 \frac{4t+1}{2t^2+t+1}\,dt$ $= \ln(11)\text{ cm} \qquad \{2.398\}$
Specific behaviours
✓ determines total change

The bowl is filled when the water level reaches $\ln(56)$ cm.

(d) If the bowl is initially empty, determine how long it takes for the bowl to be filled. (3 marks)

Solution
Let the time taken for the bowl to be filled = a seconds $\ln(56) = \int_0^a \frac{4t+1}{2t^2+t+1}\,dt$ $= \left[\ln\left(2t^2+t+1\right)\right]_0^a$ $= \ln\left(2a^2+a+1\right)$ $56 = 2a^2+a+1$ $a = 5$ The bowl will take 5 seconds to completely fill.
Specific behaviours
✓ states a definite integral for depth of water ✓ equates definite integral to maximum water level ✓ determines time taken

2020
Section 2
Question 13

The logarithmic function

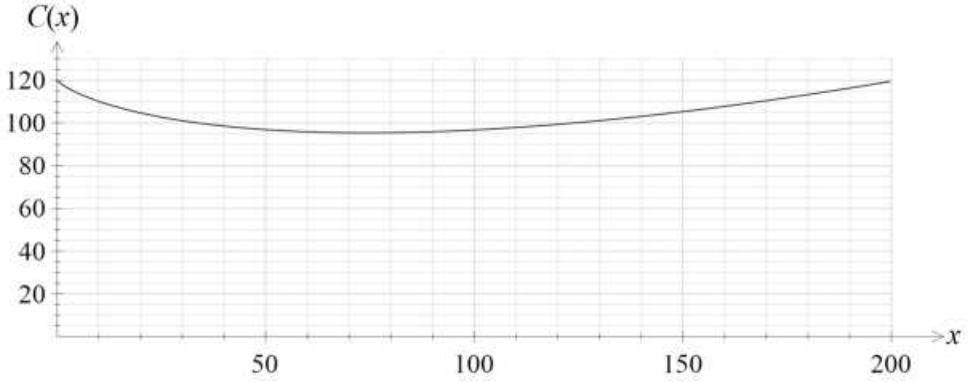
A company manufactures small machine components. They can manufacture up to 200 of a particular component in one day. The total cost, in hundreds of dollars, incurred in manufacturing the components is given by: $C(x) = \frac{x\ln(2x+1)}{3} - 2x + 120$, where x is the number of components that will be produced on that day.

(a) Determine the total cost of manufacturing 20 components in one day. (1 mark)

Solution
$C(20) = \frac{20\ln(41)}{3} - 40 + 120 = 104.7571$ i.e. $\$10475.71 \approx \$10476$
Specific behaviours
✓ determines the correct cost

(b) On the axes below, sketch the graph of C(x). (3 marks)

Solution



Specific behaviours

✓ graph covers correct domain
✓ $C(0)$ and $C(200)$ are correct
✓ minimum between 70 and 80

(c) With reference to your graph in part (b), explain how many components the company should manufacture per day if the total cost is to be as low as possible. (3 marks)

Solution

The minimum is at $x = 74.205$

$$C(74) = 95.4307 \quad \text{i.e. } \$9543.07$$

$$C(75) = 95.4320 \quad \text{i.e. } \$9543.20$$

The company should manufacture 74 components

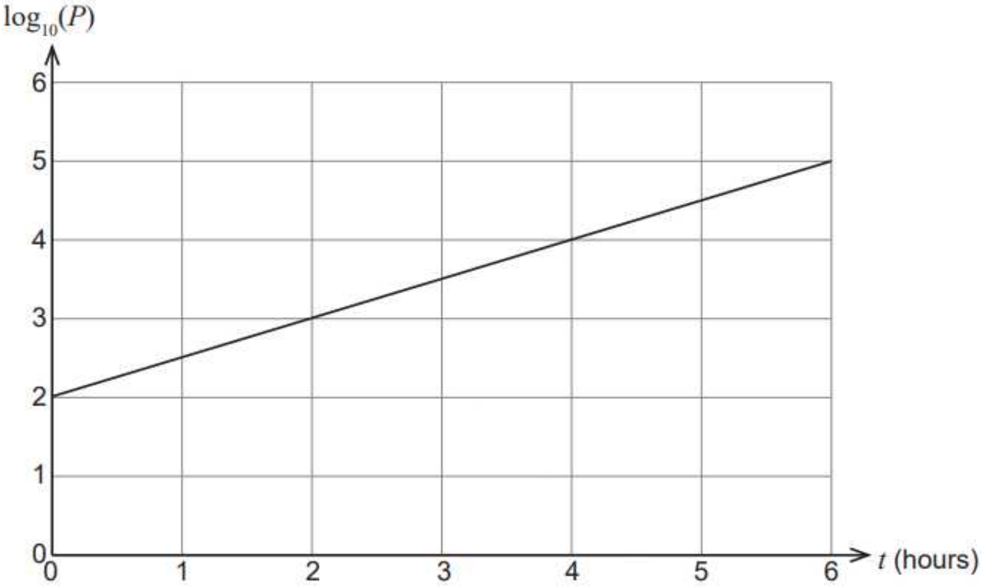
Specific behaviours

✓ states graph is a minimum at x =74.205
✓ determines cost values for x =74 and x =75
✓ states that 74 components should be manufactured per day

2019
Section 2
Question
17

The
logarithmic
function

A microbiologist is studying the effect of temperature on the growth of a certain type of bacteria under controlled laboratory conditions. A population of bacteria is incubated at a temperature of 30 °C and the size of the population measured at hourly intervals for six hours. The logarithm of the population size appears to lie on a straight line when plotted against time (measured in hours) and the line of best fit shown on the axes below.



(a) (i) On the basis of the graph above, what is the size of the bacteria population after two hours? (2 marks)

Solution
$\log_{10}(P) = 3$ $P = 10^3 = 1000$
Specific behaviours
✓ identifies the correct value of $\log_{10}(P)$ ✓ uses the inverse relationship between $\log_{10}(x)$ and 10^x to solve for P

(ii) The equation of the line can be written in the form $\log_{10}(P) = At + B$. Use the graph to determine the values of A and B. (2 marks)

Solution
Gradient of ½ and vertical axis intercept of 2 $\log_{10}(P) = \frac{1}{2}t + 2$
Specific behaviours
✓ determines the correct value for A ✓ determines the correct value for B

Another population of the same bacteria is cultured at 40 °C. The size of the population, P, after t hours satisfies the equation

$$\log_{10}(P) = \frac{1}{3}t + 2.$$

(b) (i) Express the above equation in the form $P = A(10)B^t$. (3 marks)

Solution
$\log_{10}(P) = \frac{1}{3}t + 2$ $P = 10^{t/3+2}$ $P = 10^{t/3}10^2$ $P = 100 \cdot 10^{t/3}$
Specific behaviours
✓ uses the inverse relationship between $\log_{10}(x)$ and 10^x ✓ uses the appropriate index law ✓ determines correct expression

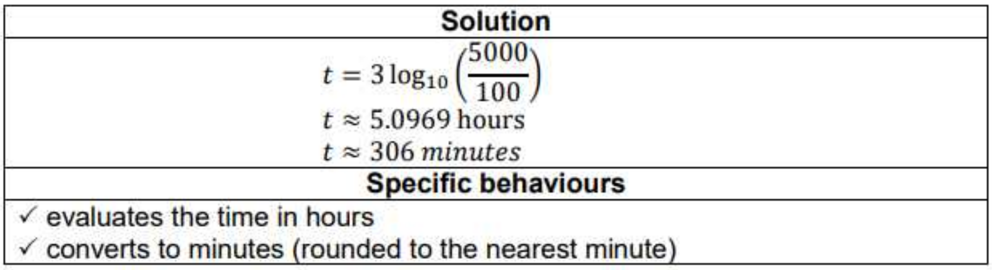
(ii) Determine the size of the population after exactly four hours to the nearest whole number. (1 mark)

Solution
$P = 100 \cdot 10^{4/3}$ $P \approx 2154$
Specific behaviours
✓ evaluates the population rounded to a whole number

(iii) Express the above equation in the form $t = C\log_{10}\left(\frac{P}{D}\right)$. (3 marks)

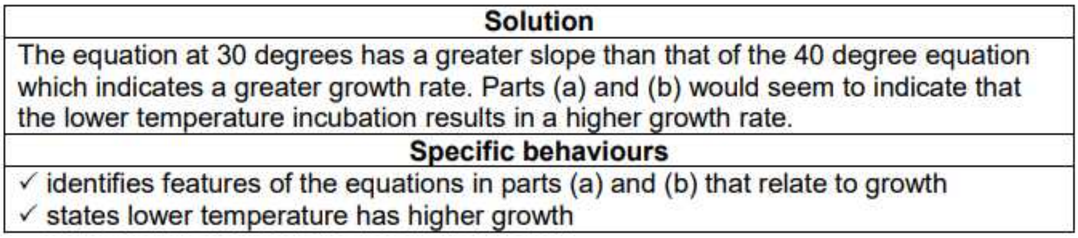
Solution
$\log_{10}(P) = \frac{1}{3}t + 2$ $\log_{10}(P) - 2 = \frac{1}{3}t$ $\log_{10}(P) - \log_{10}(100) = \frac{1}{3}t$ $\log_{10}\left(\frac{P}{100}\right) = \frac{1}{3}t$ $t = 3\log_{10}\left(\frac{P}{100}\right)$
Specific behaviours
✓ expresses 2 in terms of a log of base 10 ✓ applies appropriate log law to arrive at single log expression (second last line) ✓ determines correct expression

(iv) How many minutes does it take for the population to reach a size of 5000? Give your answer to the nearest minute. (2 marks)



Solution
$t = 3\log_{10}\left(\frac{5000}{100}\right)$ $t \approx 5.0969$ hours $t \approx 306\ minutes$
Specific behaviours
✓ evaluates the time in hours ✓ converts to minutes (rounded to the nearest minute)

(c) With reference to parts (a) and (b), describe the effect of temperature on the population growth of this type of bacteria. (2 marks)

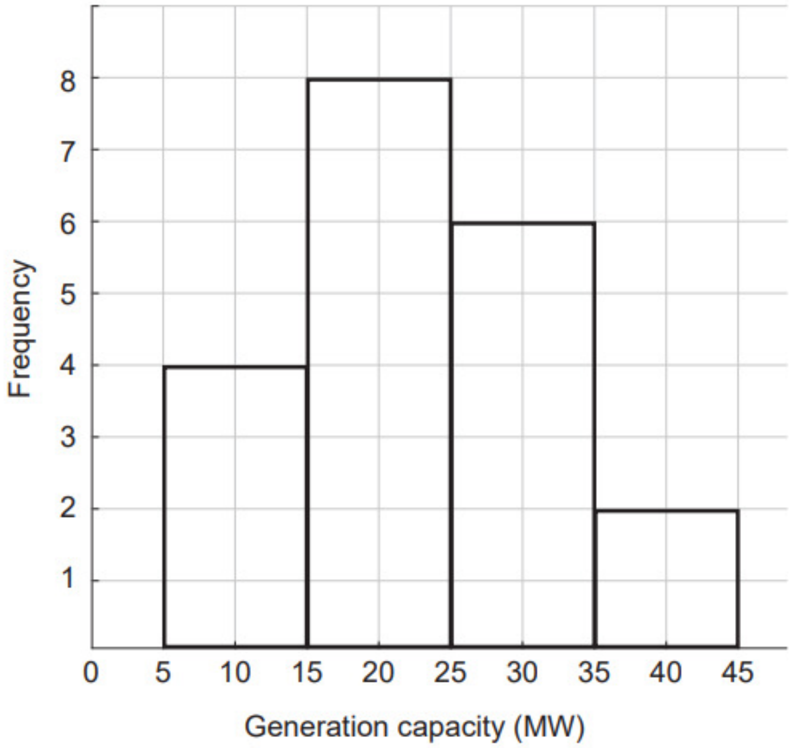


Solution
The equation at 30 degrees has a greater slope than that of the 40 degree equation which indicates a greater growth rate. Parts (a) and (b) would seem to indicate that the lower temperature incubation results in a higher growth rate.
Specific behaviours
✓ identifies features of the equations in parts (a) and (b) that relate to growth ✓ states lower temperature has higher growth

Unit 4.2 – Continuous random variables and the normal distribution

Section 1

2023 Section 1 Question 3 Continuous random variables and the normal distribution	Solcolwa is a green energy company that owns 20 solar farms across Western Australia. The generation capacities, in megawatts (MW), of the solar farms are displayed in the histogram below.



Suppose that one of the Solcolwa solar farms is selected at random. Let the random variable W denote the generation capacity of the randomly-selected solar farm.

(a) Complete the following table of cumulative probabilities for W. (2 marks)

w	5	15	25	35	45
$P(W \le w)$					

(b) Determine $P(W \ge 35)$. (1 mark)

(c) Assuming the solar farms are uniformly distributed within each interval:

(i) estimate $P(W \geq 20)$. (2 marks)

(ii) estimate the expected value $E(W)$. (2 marks)

To increase the generation capacity of its solar farms, Solcolwa decides to upgrade all its solar panels with the latest technology. A new random variable Y denotes the generation capacity of a randomly-selected upgraded solar farm. The random variables W and Y are related by

$$Y = aW$$

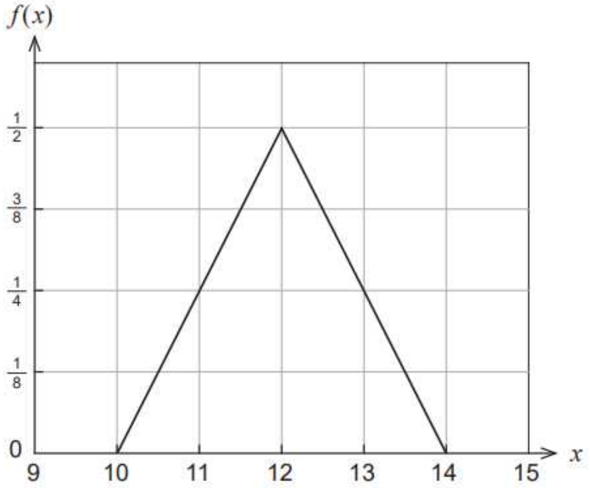
for some constant $a > 0$.

(d) Given that W and Y have variances $Var(W) = 81$ and $Var(Y) = 324$, determine the expected value $E(Y)$. (3 marks)

2022
Section 1
Question 3

Continuous random variables and the normal distribution

Arnold would like to purchase a toy for his child’s birthday. The Isosceles Toy Company claims that the number of weeks until delivery, X, is a random variable whose probability density function is displayed in the graph below.



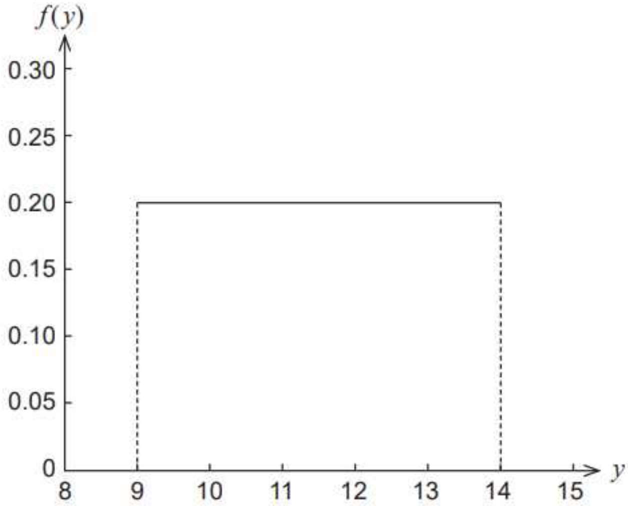
(a) What is the expected time for the toy to be delivered? (1 mark)

His child’s birthday is 13 weeks away.

(b) What is the probability that the Isosceles Toy Company will deliver the toy in time for his child’s birthday? (2 marks)

(c) Given that the toy arrives in time for his child’s birthday, what is the probability that it arrives at least one week early? (2 marks)

Uniform Toys, a rival toy company, claims that the number of weeks until delivery of the same toy, Y, is a random variable whose distribution is displayed in the graph below.



(d) Which toy company should Arnold choose if he would like to maximise the chance that the toy will be delivered in time for his child's birthday? Why? (2 marks)

Suppose that five people order the toy from Uniform Toys and let Z be a random variable that denotes the number of those people who receive the toy within 13 weeks.

(e) State the distribution for Z. (2 marks)

(f) What is the probability that four out of the five people receive the toy within 13 weeks? (2 marks)

2022
Section 1
Question 6

Continuous random variables and the normal distribution

The table of values below may be used to assist you in answering part (b) of this question.

$\sin(0) = 0$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$
$\cos(0) = 1$	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$\cos\left(\frac{\pi}{2}\right) = 0$

(a) (i) Determine $\frac{d}{dx}\left(x \sin\left(\frac{\pi x}{4}\right)\right)$. (2 marks)

(ii) Hence show that

$$\int \frac{\pi x}{4} \cos\left(\frac{\pi x}{4}\right) dx = x \sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi} \cos\left(\frac{\pi x}{4}\right) + \mathrm{c}$$

where c is a constant. (3 marks)

(b) The time in minutes, T, between incoming phone calls at a call centre is a random variable with probability density function

$$p(t) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right), & 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

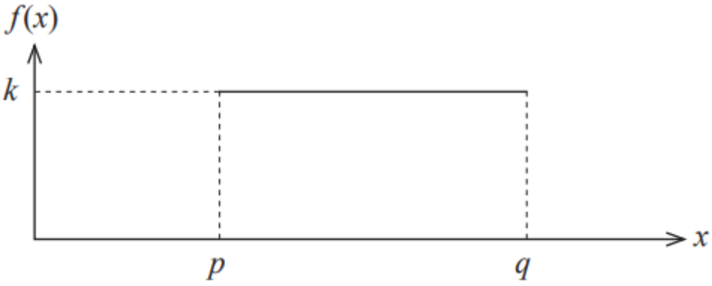
(i) Determine the probability that the time between two consecutive phone calls is less than 40 seconds. State your answer exactly. (3 marks)

(ii) Use the result from part (a)(ii) to determine the expected time between consecutive phone calls. (3 marks)

2021
Section 1
Question 2

Continuous random variables and the normal distribution

It takes Nahyun between 15 and 40 minutes to get to school each day, depending on traffic conditions. Nahyun leaves home for school at 8.00 am each school day. Let the random variable X be the time, in minutes after 8:00 am, that Nahyun arrives at school. The probability density function of X is shown below.



(a) What is the name of this type of distribution? (1 mark)

(b) Determine:

(i) the values of p, q and k (2 marks)

(ii) the expected value of X (1 mark)

(iii) the probability that Nahyun arrives at school before 8:25 am. (2 marks)

Nahyun will be late for her first class if she arrives at school after 8:28 am. Otherwise, she will not be late.

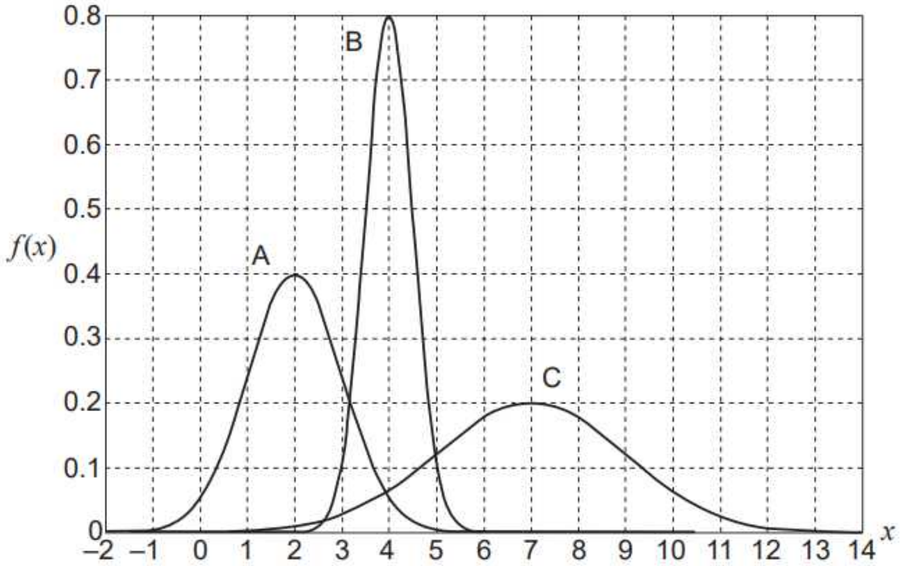
(c) If Nahyun is not late for her first class, what is the probability that she arrives after 8:25 am? (2 marks)

(d) If Nahyun only wants to be late for her first class at most 4% of the time, what time should she leave home, assuming the 15 to 40 minute travel time remains the same? (2 marks)

2021
Section 1
Question 6

Continuous random variables and the normal distribution

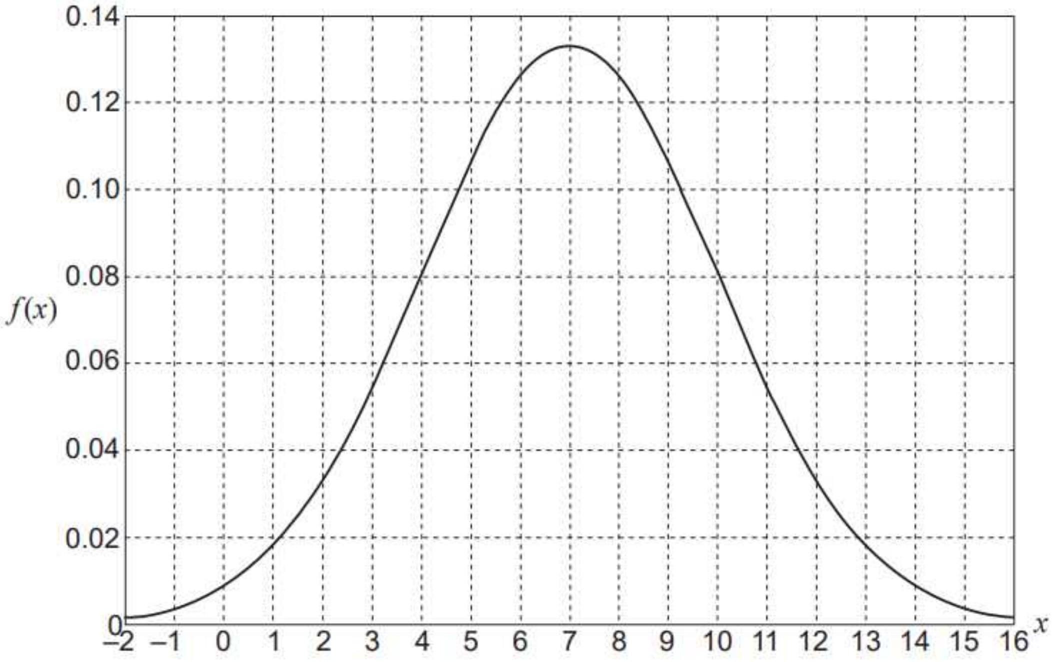
(a) The graphs of three normal distributions are displayed below. The distributions have been labelled A, B and C.



(i) What is the mean of distribution A? (1 mark)

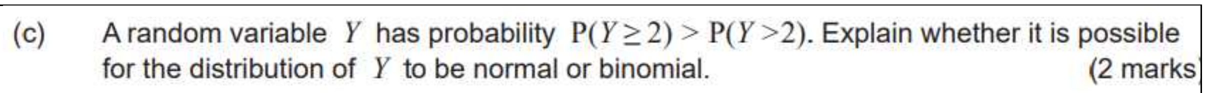
(ii) Which of the distributions has the largest standard deviation? Justify your answer. (1 mark)

(b) A random variable X is normally distributed. The distribution of X is graphed below.



(i) Shade the region with area corresponding to $P(6 \leq X \leq 9)$. (1 mark)

(ii) Is $P(6 \leq X \leq 9) \geq 0.5$? Justify your answer. (2 marks)

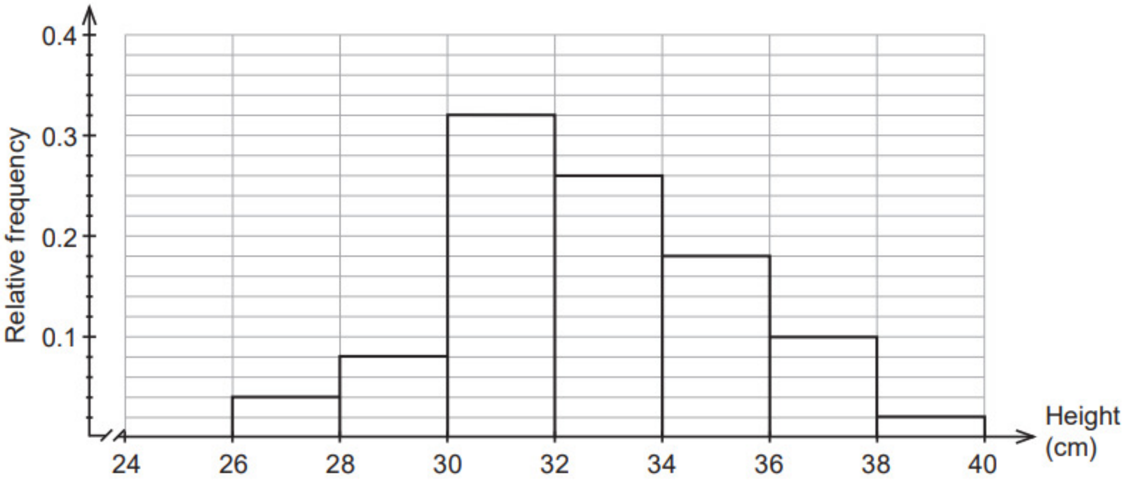


(c) A random variable Y has probability $\mathrm{P}(Y \geq 2) > \mathrm{P}(Y > 2)$. Explain whether it is possible for the distribution of Y to be normal or binomial. (2 marks

2020
Section 1
Question 4

Continuous random variables and the normal distribution

The heights reached by a species of small plant at maturity are measured by a team of biologists. The results are shown in the histogram of relative frequencies below.



(a) Determine the probability that a mature plant of this species reaches no higher than 30 cm. (1 mark)

(b) If a mature plant reaches a height of at least 32 cm, what is the probability that its height reaches above 38 cm? (2 marks)

Another team of biologists is studying the mature heights of a species of hedge. The height, h metres, has a probability density function, $d(h)$, as given below.

$$d(h) = \begin{cases} \dfrac{h-1}{5} & \text{for} \quad 1 \le h \le 2 \\ kh^2 & \text{for} \quad 2 < h \le 4 \\ 0 & \text{otherwise} \end{cases}$$

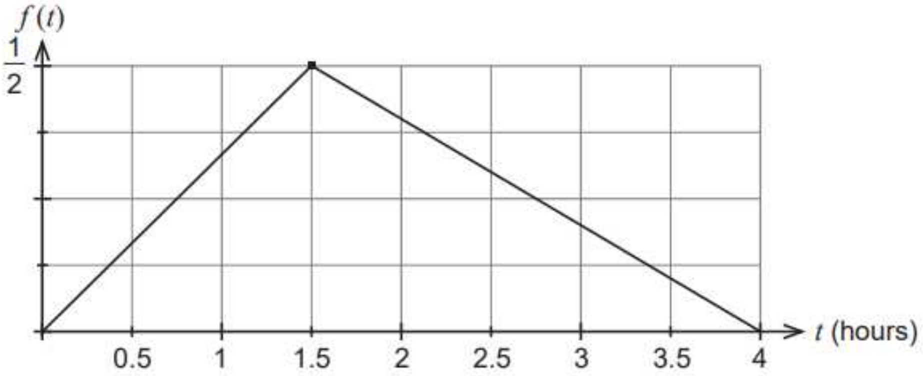
(c) What percentage of hedges from this study reaches a mature height less than 2 m? (3 marks)

(d) Determine the value of k. (3 marks)

2019
Section 1
Question 3

Continuous random variables and the normal distribution

Waiting times for patients at a hospital emergency department can be up to four hours. The associated probability density function is shown below.



(a) What is the probability a patient will wait less than one hour? (3 marks)

(b) What is the probability a patient will wait between one hour and three hours? (4 marks)

2019
Section 1
Question 6

Continuous random variables and the normal distribution

The error X in digitising a communication signal has a uniform distribution with probability density function given by

$$f(x) = \begin{cases} 1, & -0.5 < x < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$. (2 marks)

(b) What is the probability that the error is at least 0.35? (1 mark)

(c) If the error is negative, what is the probability that it is less than –0.35? (2 marks)

(d) An engineer is more interested in the square of the error. What is the probability that the square of the error is less than 0.09? (2 marks)

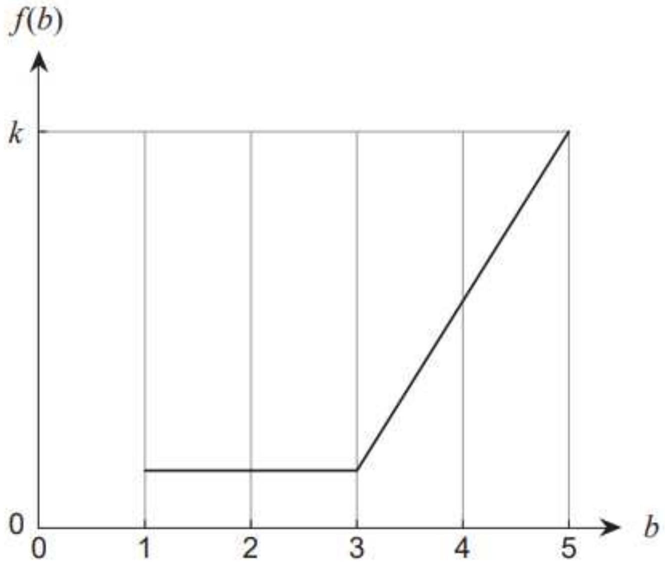
(e) Calculate the variance of the error. (3 marks)

2023 Section 2 Question 11

Continuous random variables and the normal distribution

Mrs Euler is having her car serviced at BIMDAS Mechanics. She drops her vehicle off at 8 am and is told that her car will be ready for collection at some time between 1 pm and 5 pm that day.

Let the random variable *B* denote the time after noon (12 pm) at which a vehicle is ready for collection at BIMDAS Mechanics. The probability density function for *B* is shown in the graph below.



The probability of a vehicle being ready for collection between 2 pm and 3 pm is 0.1.

(a) Determine the value of *k*. (2 marks)

(b) An incomplete expression for the probability density function of *B* is given below. Fill in the boxes to complete the missing parts of the expression. (2 marks)

$$f(b) = \begin{cases} 0.1, & \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad}, & 3 \le b \le 5 \\ 0, & \text{otherwise} \end{cases}$$

(c) Determine the expected time that Mrs Euler's vehicle will be ready for collection at BIMDAS Mechanics. (3 marks)

Mr Euler is also having his car serviced, but by Addition Autos. He drops his vehicle off at 8 am and is told that his car will be ready for collection at some time between 1 pm and 5 pm that day.

Let the random variable *A* denote the time after noon (12 pm) that a vehicle is ready for collection at Addition Autos. The cumulative distribution function for *A* is given by

$$P(A \leq a) = \begin{cases} 0, & a < 1 \\ \dfrac{10a - a^2 - 9}{16}, & 1 \leq a \leq 5 \\ 1, & a > 5 \end{cases}$$

(d) Determine the probability that Mr Euler's vehicle will be ready to collect

(i) by 3 pm. (1 mark)

(ii) between 3 pm and 4 pm. (2 marks)

(e) Determine the expected time at which Mr Euler's vehicle will be ready for collection at Addition Autos. (3 marks)

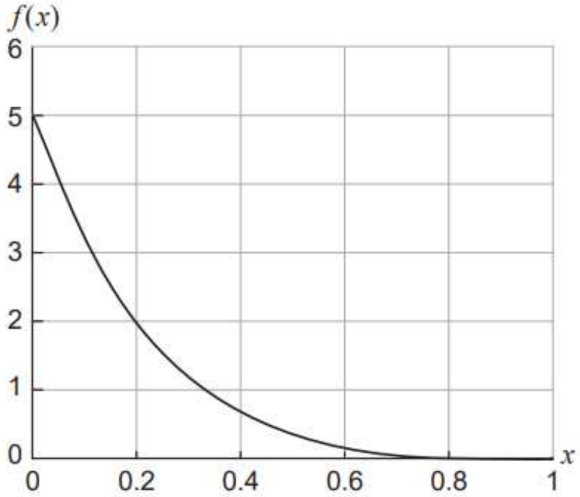
2022
Section 2
Question 8

Continuous random variables and the normal distribution

A small outback petrol station receives a weekly delivery of petrol. The volume of petrol sold in a week, X, (in units of 10 000 litres) is a random variable with probability density function

$$f(x) = 5(1 - x)^4, \quad 0 \le x \le 1$$

as shown in the graph below.



(a) Determine, using appropriate units, the expected value and variance of the amount of fuel sold in a week. (4 marks)

(b) What storage tank capacity will ensure that there is only a 1% chance of running out of petrol in a given week? State your answer to the nearest litre. (3 marks)

(c) When the petrol is delivered, it is pumped into the storage tank. The rate of change of the petrol level in the tank, $h(t)$, (measured in metres) at time t (measured in minutes) is given by

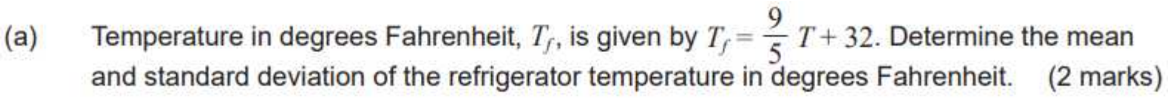
$$h'(t) = \frac{5}{2t+3}$$

Determine the height of the storage tank if it takes 20 minutes to fill. (3 marks)

2020
Section 2
Question
16

Continuous random variables and the normal distribution

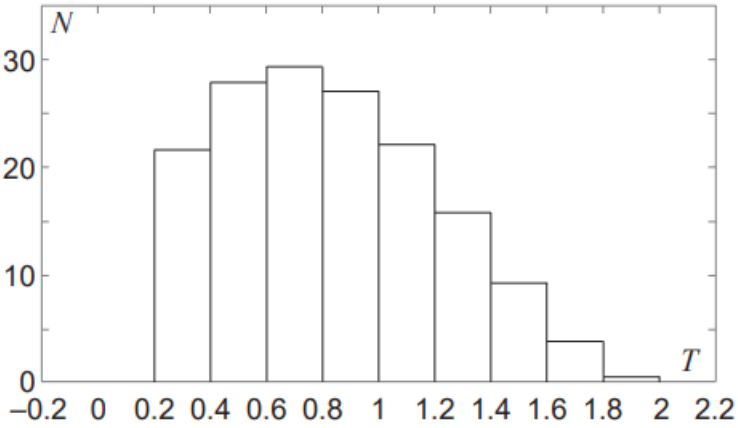
A large refrigerator in a scientific laboratory is always required to maintain a temperature between 0 °C and 1 °C to preserve the integrity of biological samples stored inside. A scientist working in the laboratory suspects that the refrigerator is not maintaining the required temperature and decides to record the temperature every hour for seven days. Based on these measurements, the scientist concludes that the temperature, T, in the refrigerator is normally distributed with a mean of 0.8 °C and a standard deviation of 0.4 °C.



(a) Temperature in degrees Fahrenheit, T_f, is given by $T_f = \frac{9}{5}T + 32$. Determine the mean and standard deviation of the refrigerator temperature in degrees Fahrenheit. (2 marks)

(b) Determine the probability that the refrigerator temperature is above 1 °C. Give your answer rounded to four decimal places. (1 mark)

The histogram of data gathered by the scientist is shown below. N denotes the number of observations in each temperature interval.



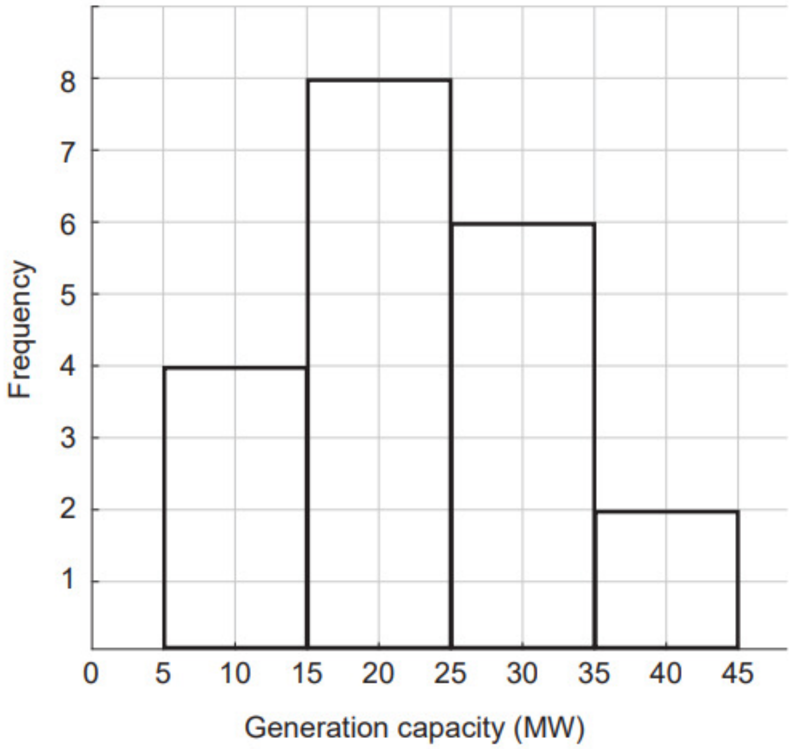
(c) Do you agree that the normal distribution was an appropriate model to use? Provide a reason to justify your response. (2 marks)

An alternative probability density function proposed to model the refrigerator temperature, in degrees Celcius, is given by:

$$p(t) = \frac{3}{4}t^3 - 3t^2 + 3t, \ 0 \leq t \leq 2$$

(d) Determine the probability that the refrigerator temperature is above 1 °C using the new model. (2 marks)

2023 Section 1 Question 3 **Continuous random variables and the normal distribution**	Solcolwa is a green energy company that owns 20 solar farms across Western Australia. The generation capacities, in megawatts (MW), of the solar farms are displayed in the histogram below.



Suppose that one of the Solcolwa solar farms is selected at random. Let the random variable W denote the generation capacity of the randomly-selected solar farm.

(a) Complete the following table of cumulative probabilities for W. (2 marks)

Solution

w	5	15	25	35	45
$P(W \le w)$	**0**	**0.2**	**0.6**	**0.9**	**1**

Specific behaviours

✓ correctly calculates at least three probabilities
✓ correctly calculates all probabilities

(b) Determine $P(W \geq 35)$. (1 mark)

Solution
$P(W \geq 35) = 1 - P(W \leq 35)$ $= 1 - 0.9$ $= 0.1$ Or $P(W \geq 35) = \frac{2}{20}$ $= 0.1$
Specific behaviours
✓ correctly calculates probability

(c) Assuming the solar farms are uniformly distributed within each interval:

(i) estimate $P(W \geq 20)$. (2 marks)

Solution
Using the table of cumulative probabilities and linear interpolation: $P(W \geq 20) = 1 - P(W \leq 20)$ $= 1 - \frac{0.2+0.6}{2}$ $= 1 - 0.4$ $= 0.6$
Specific behaviours
✓ uses linear interpolation to estimate $P(W \leq 20)$ ✓ calculates correct probability

Alternate solution
Using the histogram and linear interpolation: $P(W \geq 20) = \frac{2+6+\left(\frac{1}{2}\times 8\right)}{20}$ $= 0.6$
Specific behaviours
✓ determines number of solar farms with generating capacity between 20 and 25 ✓ calculates correct probability

(ii) estimate the expected value *E*(*W*). (2 marks)

Solution
$E(W) = 10\times 0.2 + 20\times 0.4 + 30\times 0.3 + 40\times 0.1$ $= 2+8+9+4$ $= 23$
Specific behaviours
✓ writes correct expression for the expected value ✓ calculates correct expected value

To increase the generation capacity of its solar farms, Solcolwa decides to upgrade all its solar panels with the latest technology. A new random variable Y denotes the generation capacity of a randomly-selected upgraded solar farm. The random variables W and Y are related by

$$Y = aW$$

for some constant $a > 0$.

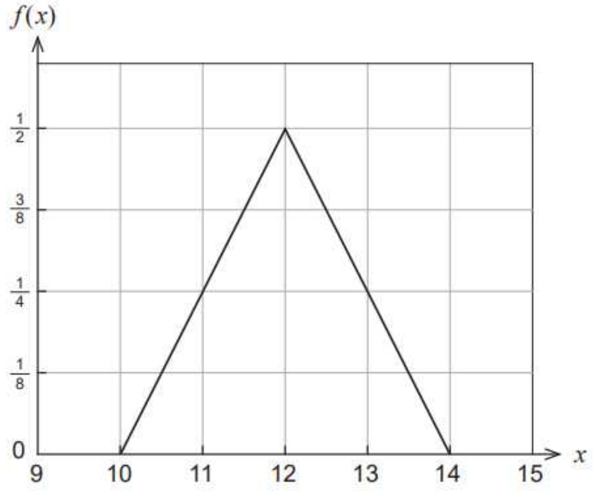
(d) Given that W and Y have variances $Var(W) = 81$ and $Var(Y) = 324$, determine the expected value $E(Y)$. (3 marks)

Solution
Given that $Y = aW$ it follows that $Var(Y) = a^2 Var(W)$ $\Rightarrow 324 = 81a^2$ $\Rightarrow a^2 = 4$ $a = 2$ Hence $E(Y) = aE(W)$ $= 2\times 23$ $= 46$
Specific behaviours
✓ states correct relationship between $Var(W)$ and $Var(Y)$ ✓ calculates correct value of a ✓ calculates correct expected value

2022
Section 1
Question 3

Continuous random variables and the normal distribution

Arnold would like to purchase a toy for his child's birthday. The Isosceles Toy Company claims that the number of weeks until delivery, X, is a random variable whose probability density function is displayed in the graph below.



(a) What is the expected time for the toy to be delivered? (1 mark)

Solution
$E(X) = 12$ weeks
Specific behaviours
✓ states correct expected time, including units

His child's birthday is 13 weeks away.

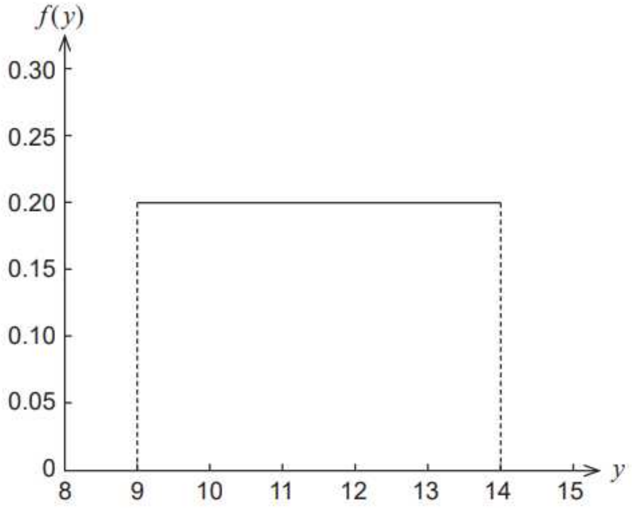
(b) What is the probability that the Isosceles Toy Company will deliver the toy in time for his child's birthday? (2 marks)

Solution
$P(X < 13) = 1 - \frac{1}{2} \times 1 \times \frac{1}{4}$ $= 1 - \frac{1}{8}$ $= \frac{7}{8}$
Specific behaviours
✓ identifies the probability as the area under the curve between 10 and 13 (or 1 minus the area under the curve between 13 and 14) ✓ calculates the correct probability

(c) Given that the toy arrives in time for his child’s birthday, what is the probability that it arrives at least one week early? (2 marks)

Solution
$$P(X < 12 \mid X < 13) = \frac{P(X < 12)}{P(X < 13)}$$ $$= \frac{\frac{1}{2}}{\frac{7}{8}}$$ $$= \frac{4}{7}$$
Specific behaviours
✓ determines correct conditional probability statement ✓ obtains correct probability

Uniform Toys, a rival toy company, claims that the number of weeks until delivery of the same toy, Y, is a random variable whose distribution is displayed in the graph below.



(d) Which toy company should Arnold choose if he would like to maximise the chance that the toy will be delivered in time for his child’s birthday? Why? (2 marks)

Solution
$$P(Y < 13) = 0.8$$ $$0.8 < \frac{7}{8}$$ Arnold should choose the Isosceles Toy Company because the probability of receiving the toy on time from the Isosceles Toy Company is greater than the probability of receiving the toy on time from Uniform Toys.
Specific behaviours
✓ determines the probability of delivering the toy within 13 weeks ✓ chooses correct toy company and provides justification

Suppose that five people order the toy from Uniform Toys and let Z be a random variable that denotes the number of those people who receive the toy within 13 weeks.

(e) State the distribution for Z. (2 marks)

Solution
$Z \sim \text{Bin}(5{,}0.8)$
Specific behaviours
✓ states distribution is binomial ✓ gives correct values for parameters n and p

(f) What is the probability that four out of the five people receive the toy within 13 weeks? (2 marks)

Solution
$P(Z=4)=\binom{5}{4}\times\left(\frac{4}{5}\right)^4\times\frac{1}{5}$ $=5\times\frac{256}{625}\times\frac{1}{5}$ $=\frac{256}{625}$
Specific behaviours
✓ writes correct expression for probability ✓ calculates correct probability

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2022
Section 1
Question 6

Continuous random variables and the normal distribution

The table of values below may be used to assist you in answering part (b) of this question.

$\sin(0)=0$	$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$	$\sin\left(\frac{\pi}{2}\right)=1$
$\cos(0)=1$	$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$	$\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$	$\cos\left(\frac{\pi}{2}\right)=0$

(a) (i) Determine $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)$. (2 marks)

Solution
Using the product rule $\frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right)=\sin\left(\frac{\pi x}{4}\right)+\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)$
Specific behaviours
✓ applies product rule to evaluate derivative ✓ obtains correct answer

(ii) Hence show that

$$\int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + \mathrm{c}$$

where c is a constant. (3 marks)

Solution

From part (i)

$$\frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right) = \sin\left(\frac{\pi x}{4}\right) + \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)$$

$$\Rightarrow \int \frac{d}{dx}x\sin\left(\frac{\pi x}{4}\right)dx = \int \sin\left(\frac{\pi x}{4}\right)dx + \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$$

$$\Rightarrow x\sin\left(\frac{\pi x}{4}\right) = -\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$$

$$\Rightarrow \int \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = x\sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right) + c$$

Specific behaviours

✓ integrates both sides of the result from (i) and correctly evaluates $\int \sin\left(\frac{\pi x}{4}\right)dx$

✓ applies the fundamental theorem of calculus to evaluate $\int \frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)dx$

✓ applies valid mathematical operations to obtain required expression

(b) The time in minutes, T, between incoming phone calls at a call centre is a random variable with probability density function

$$p(t) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right), & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Determine the probability that the time between two consecutive phone calls is less than 40 seconds. State your answer exactly. (3 marks)

Solution

$$P\left(T < \frac{2}{3}\right) = \int_0^{\frac{2}{3}} \frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right)dt$$

$$= \left[\sin\left(\frac{\pi t}{4}\right)\right]_0^{\frac{2}{3}}$$

$$= \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{2}$$

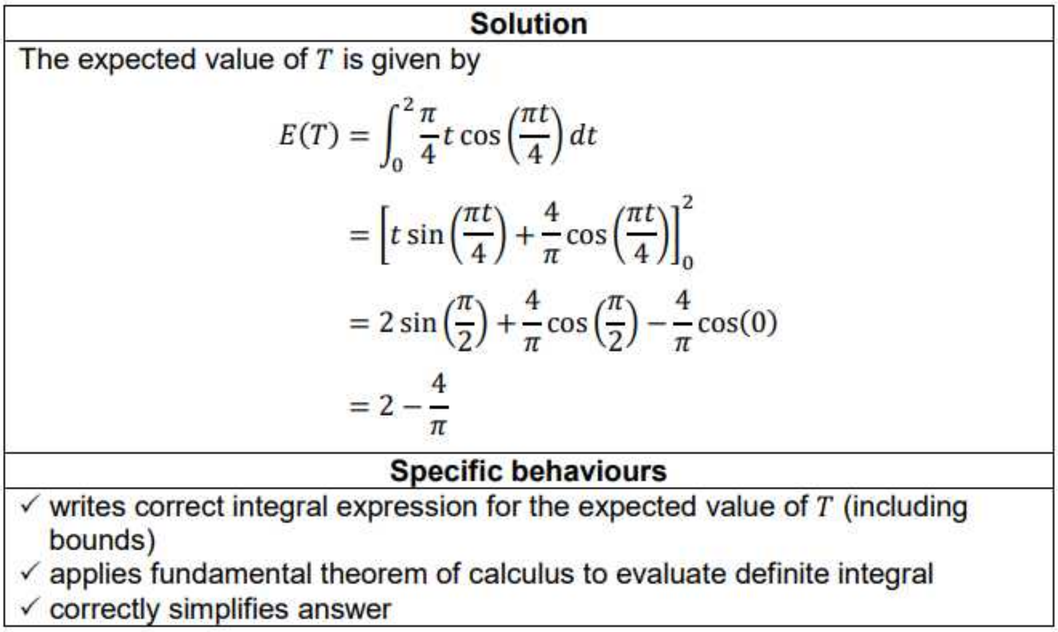
Specific behaviours

✓ writes correct integral (recognises that 40 seconds is $t = \frac{2}{3}$ minutes)

✓ anti-differentiates correctly

✓ obtains correct answer

(ii) Use the result from part (a)(ii) to determine the expected time between consecutive phone calls. (3 marks)



Solution

The expected value of T is given by

$$E(T) = \int_0^2 \frac{\pi}{4} t \cos\left(\frac{\pi t}{4}\right) dt$$

$$= \left[t \sin\left(\frac{\pi t}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi t}{4}\right)\right]_0^2$$

$$= 2\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{4}{\pi}\cos(0)$$

$$= 2 - \frac{4}{\pi}$$

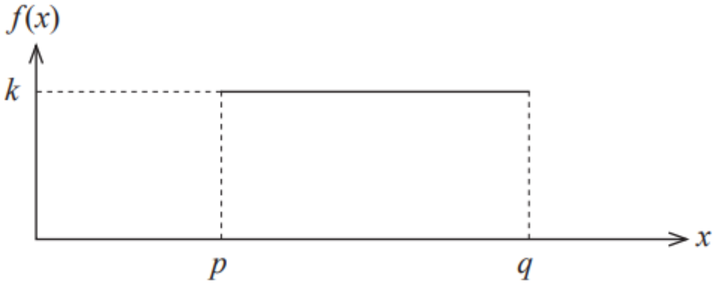
Specific behaviours

✓ writes correct integral expression for the expected value of T (including bounds)
✓ applies fundamental theorem of calculus to evaluate definite integral
✓ correctly simplifies answer

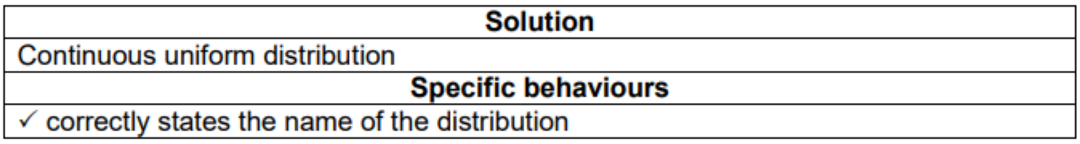
2021
Section 1
Question 2

Continuous random variables and the normal distribution

It takes Nahyun between 15 and 40 minutes to get to school each day, depending on traffic conditions. Nahyun leaves home for school at 8.00 am each school day. Let the random variable X be the time, in minutes after 8:00 am, that Nahyun arrives at school. The probability density function of X is shown below.



(a) What is the name of this type of distribution? (1 mark)



Solution

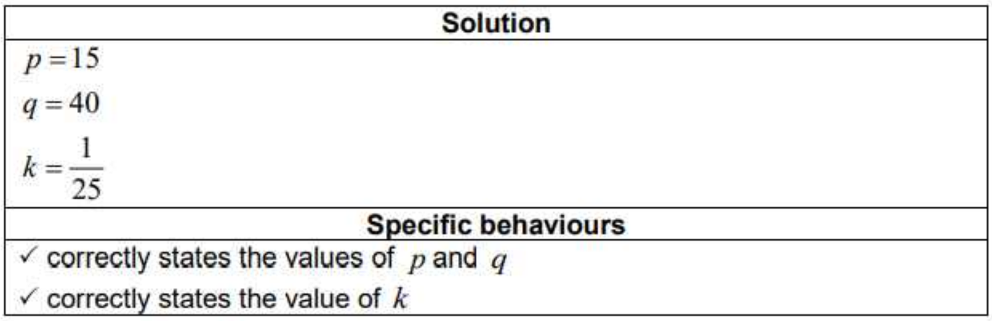
Continuous uniform distribution

Specific behaviours

✓ correctly states the name of the distribution

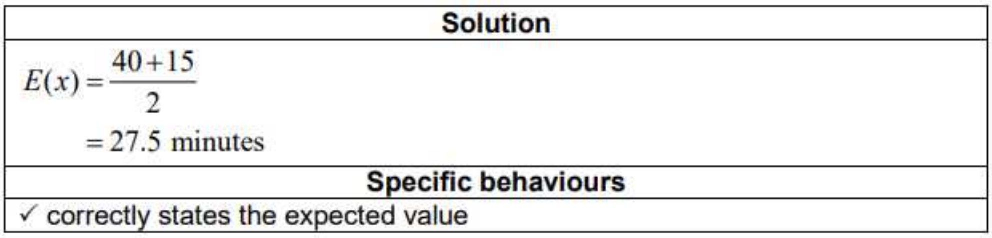
(b) Determine:

(i) the values of p, q and k (2 marks)



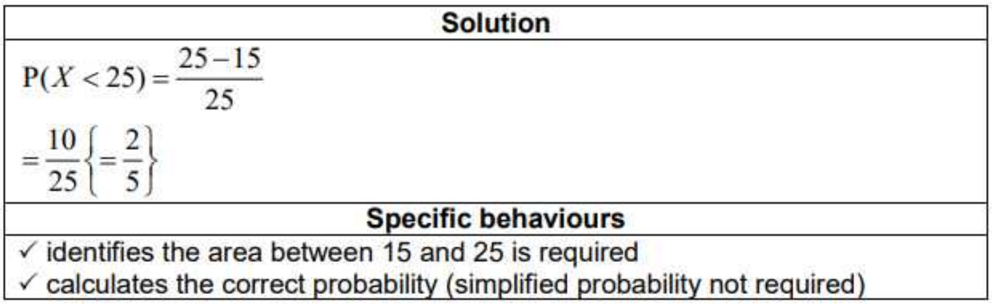
Solution
$p = 15$ $q = 40$ $k = \frac{1}{25}$
Specific behaviours
✓ correctly states the values of p and q ✓ correctly states the value of k

(ii) the expected value of X (1 mark)



Solution
$E(x) = \frac{40+15}{2}$ $= 27.5 \text{ minutes}$
Specific behaviours
✓ correctly states the expected value

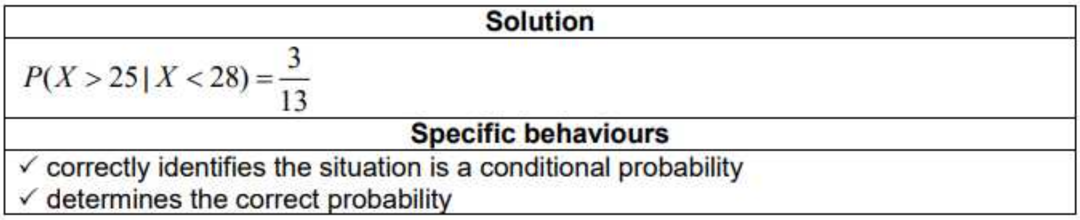
(iii) the probability that Nahyun arrives at school before 8:25 am. (2 marks)



Solution
$\mathrm{P}(X < 25) = \frac{25-15}{25}$ $= \frac{10}{25} \left\{ = \frac{2}{5} \right\}$
Specific behaviours
✓ identifies the area between 15 and 25 is required ✓ calculates the correct probability (simplified probability not required)

Nahyun will be late for her first class if she arrives at school after 8:28 am. Otherwise, she will not be late.

(c) If Nahyun is not late for her first class, what is the probability that she arrives after 8:25 am? (2 marks)



Solution
$P(X > 25 \mid X < 28) = \frac{3}{13}$
Specific behaviours
✓ correctly identifies the situation is a conditional probability ✓ determines the correct probability

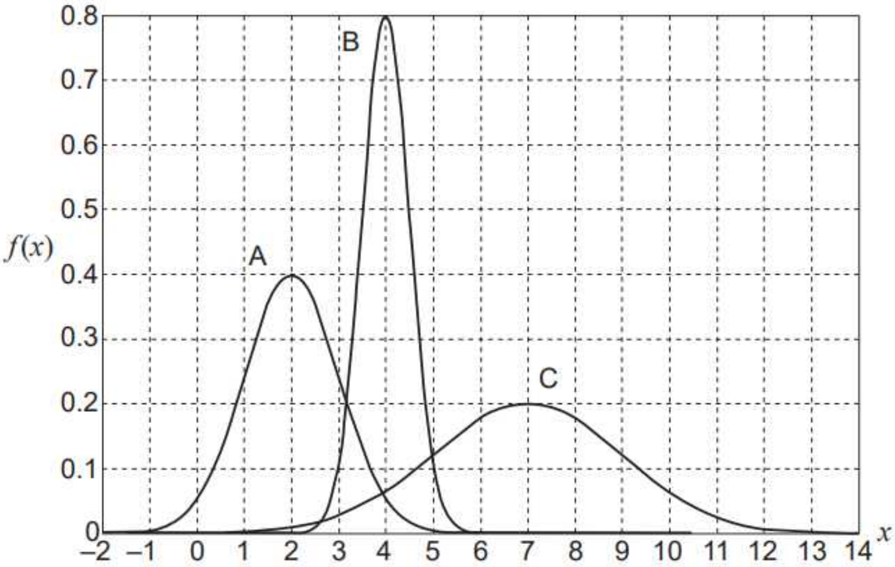
(d) If Nahyun only wants to be late for her first class at most 4% of the time, what time should she leave home, assuming the 15 to 40 minute travel time remains the same? (2 marks)

Solution
$4\% = \frac{4}{100} = \frac{1}{25}$ ∴ leaves 39 minutes before 8:28 am She should leave home at 7:49 am
Specific behaviours
✓ determines 4% = a probability of $\frac{1}{25}$ ✓ correctly determines the time

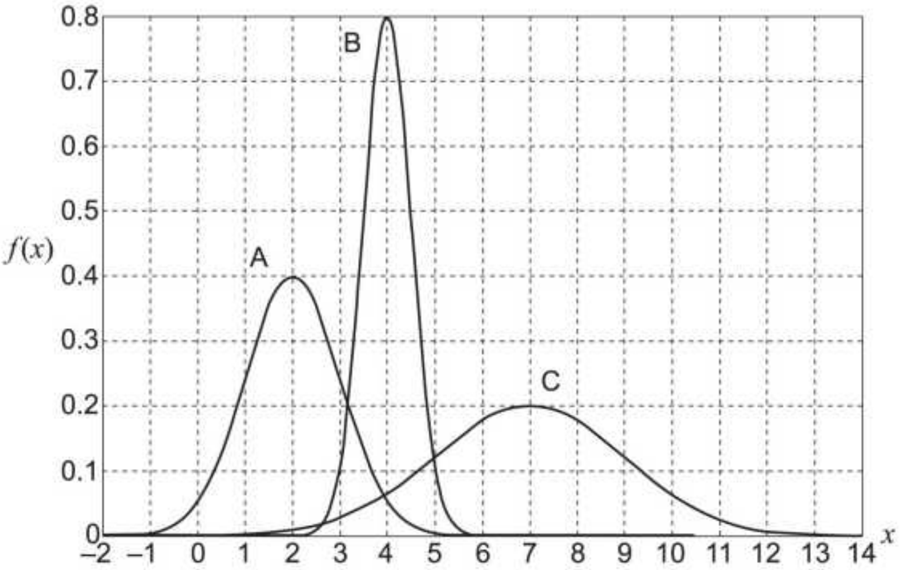
2021
Section 1
Question 6

Continuous random variables and the normal distribution

(a) The graphs of three normal distributions are displayed below. The distributions have been labelled A, B and C.



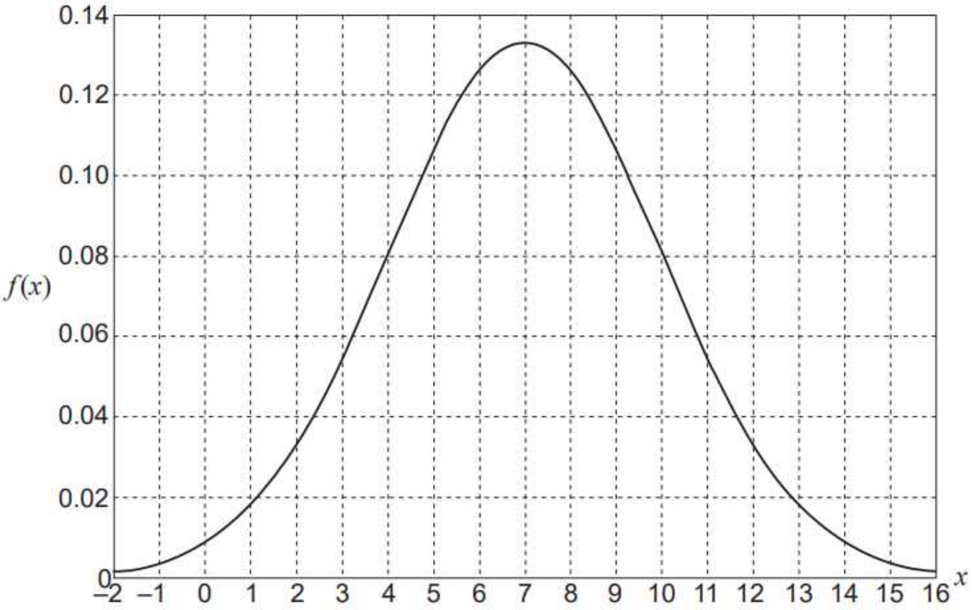
(i) What is the mean of distribution A? (1 mark)



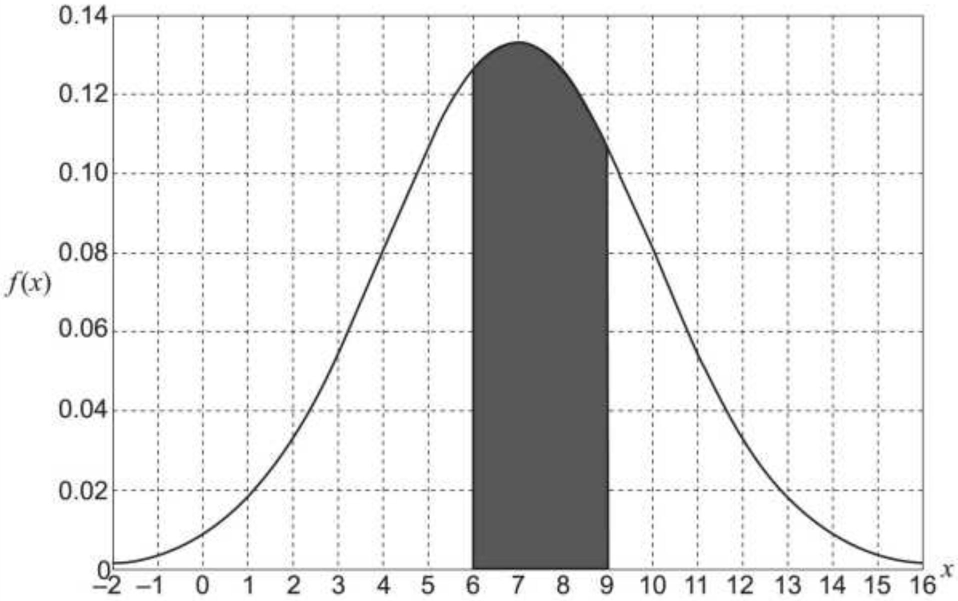
(ii) Which of the distributions has the largest standard deviation? Justify your answer. (1 mark)

Solution
C has the largest standard deviation as it is the widest distribution.
Specific behaviours
✓ states that C has the largest standard deviation and provides correct justification

(b) A random variable X is normally distributed. The distribution of X is graphed below.



(i) Shade the region with area corresponding to $\mathrm{P}(6 \leq X \leq 9)$. (1 mark)

Solution

Specific behaviours
✓ shades correct region

(ii) Is $\mathrm{P}(6 \le X \le 9) \ge 0.5$? Justify your answer. (2 marks)

Solution

No. The total area below the probability density function is 1, and the region shaded above is less than half of that area (i.e. area is less than 0.5). Hence, it corresponds to a probability that is less than 0.5.

Specific behaviours

✓ states that the probability is not greater than or equal to 0.5
✓ provides correct justification

(c) A random variable Y has probability $\mathrm{P}(Y \ge 2) > \mathrm{P}(Y > 2)$. Explain whether it is possible for the distribution of Y to be normal or binomial. (2 marks)

Solution

Not normal: a continuous random variable has $\mathrm{P}(Y \ge 2) = \mathrm{P}(Y > 2)$. Since a normally distributed random variable is continuous it follows that Y is not a normally distributed random variable.

Could be binomial: $\mathrm{P}(Y \ge 2) > \mathrm{P}(Y > 2)$ for a discrete random variable. Since the binomial distribution is discrete it follows that Y could be a binomially distributed random variable.

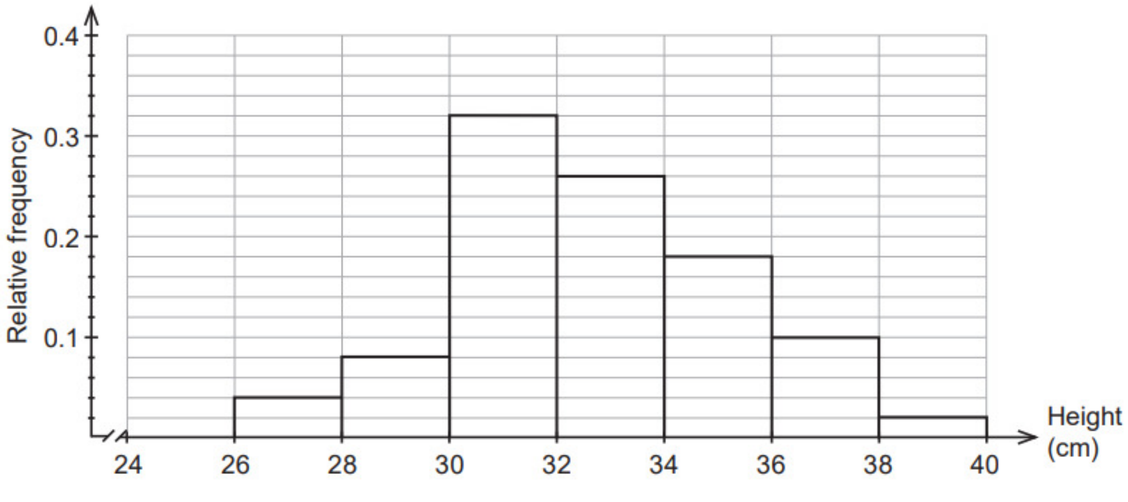
Specific behaviours

✓ states that Y could not be normal and provides a correct explanation
✓ states that Y could be binomial and provides a correct explanation

2020
Section 1
Question 4

Continuous random variables and the normal distribution

The heights reached by a species of small plant at maturity are measured by a team of biologists. The results are shown in the histogram of relative frequencies below.



(a) Determine the probability that a mature plant of this species reaches no higher than 30 cm. (1 mark)

Solution

$$P(h \le 30) = 0.04 + 0.08$$
$$= 0.12$$

Specific behaviours

✓ determines the correct probability

(b) If a mature plant reaches a height of at least 32 cm, what is the probability that its height reaches above 38 cm? (2 marks)

Solution
$P(h \geq 38 \mid h \geq 32) = \dfrac{0.02}{0.56} = \dfrac{1}{28}$
Specific behaviours
✓ recognises conditional probability and determines the correct denominator of the conditional probability ✓ determines the correct probability as a fraction

Another team of biologists is studying the mature heights of a species of hedge. The height, h metres, has a probability density function, $d(h)$, as given below.

$$d(h) = \begin{cases} \dfrac{h-1}{5} & \text{for} \quad 1 \leq h \leq 2 \\ kh^2 & \text{for} \quad 2 < h \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(c) What percentage of hedges from this study reaches a mature height less than 2 m? (3 marks)

Solution
Probability of $1 < h \leq 2$: $\displaystyle\int_1^2 \frac{h-1}{5}\,dh = \frac{1}{5}\left[\frac{h^2}{2} - h\right]_1^2$ $= \dfrac{1}{5}\left[2 - 2 - \dfrac{1}{2} + 1\right]$ $= \dfrac{1}{10}$ 10% reach a height of less than 2 m
Specific behaviours
✓ recognises the need to integrate the first equation of the PDF from 1 to 2 ✓ antidifferentiates the first equation correctly ✓ determines the correct percentage

(d) Determine the value of k. (3 marks)

Solution

$$\text{Probability of } 2 < h \leq 4 = 1 - \frac{1}{10} = \frac{9}{10}$$

$$\therefore \frac{9}{10} = \int_2^4 kh^2 \, dh$$

$$\frac{9}{10} = k\left[\frac{h^3}{3}\right]_2^4$$

$$\frac{9}{10} = k\left[\frac{64-8}{3}\right]$$

$$k = \frac{27}{560}$$

Specific behaviours

✓ recognises the need to integrate the second equation of the PDF from 2 to 4 and equates to the complement of part (c), $\frac{9}{10}$

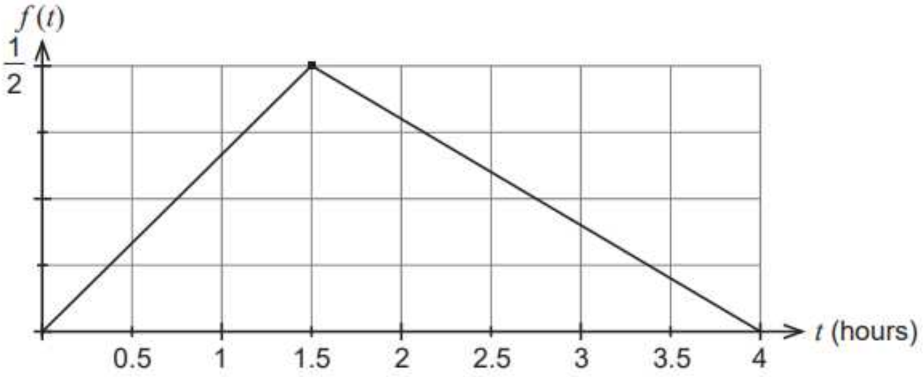
✓ antidifferentiates the second equation correctly

✓ determines the value of k

2019
Section 1
Question 3

Continuous random variables and the normal distribution

Waiting times for patients at a hospital emergency department can be up to four hours. The associated probability density function is shown below.



(a) What is the probability a patient will wait less than one hour? (3 marks)

Solution

For: $0 \le t \le 1.5$

$$f(t) = \frac{0.5}{1.5}t = \frac{t}{3}$$

$$P(T \le 1) = \int_0^1 \frac{t}{3}\,dt = \left[\frac{t^2}{6}\right]_0^1 = \frac{1}{6}$$

Specific behaviours

✓ determines equation for $f(t)$
✓ writes a correct statement for probability involving calculus
✓ evaluates integral to determine probability

OR

Alternate Solution

Required probability is the area of the triangle that has base 1 unit

The height of the triangle is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

$$P(T \le 1) = \frac{1}{2} \times 1 \times \frac{1}{3} = \frac{1}{6}$$

Specific behaviours

✓ recognises that the probability is the area of triangle with base length 1 unit
✓ determines the height of the triangle
✓ correctly calculates the area

(b) What is the probability a patient will wait between one hour and three hours? (4 marks)

Solution

$$P(1 \le T \le 3) = 1 - P(0 \le T \le 1) - P(3 \le T \le 4)$$

$$P(0 \le T \le 1) = \frac{1}{6}$$

For: $1.5 \le t \le 4$

$$f(t) = -\frac{0.5}{2.5}t + c = -\frac{t}{5} + c$$

$$f(4) = 0$$

$$0 = -\frac{4}{5} + c \Rightarrow \frac{4}{5}$$

$$f(t) = -\frac{t}{5} + \frac{4}{5}$$

$$P(3 \le T \le 4) = \frac{1}{5}\int_3^4 (-t+4)dt = \frac{1}{5}\left[-\frac{t^2}{2} + 4t\right]_3^4$$

$$= \frac{1}{5}\left[-8 + 16 + \frac{9}{2} - 12\right]$$

$$= \frac{1}{10}$$

$$P(1 \le T \le 3) = 1 - \frac{1}{10} - \frac{1}{6} = \frac{22}{30} = \frac{11}{15}$$

Specific behaviours

✓ determines the equation for $f(t)$ when $1.5 \le t \le 4$

✓ writes a correct statement for probability

✓ calculates $P(3 \le T \le 4)$ correctly

✓ calculates $P(1 \le T \le 3)$ correctly

2019
Section 1
Question 6

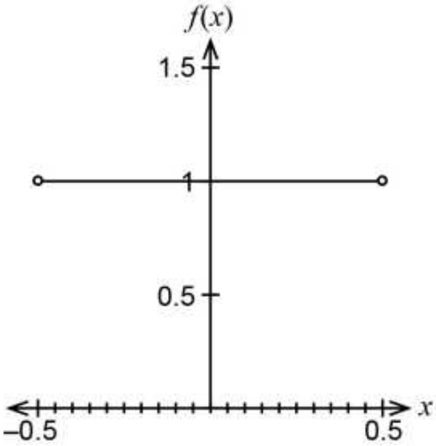
Continuous random variables and the normal distribution

The error X in digitising a communication signal has a uniform distribution with probability density function given by

$$f(x) = \begin{cases} 1, & -0.5 < x < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$. (2 marks)

Solution



Specific behaviours

✓ draws horizontal line at y=1
✓ indicates a domain of $-0.5 < x < 0.5$

(b) What is the probability that the error is at least 0.35? (1 mark)

Solution

$$P(X > 0.35) = Area = 0.15$$

Specific behaviours

✓ computes the correct probability

(c) If the error is negative, what is the probability that it is less than –0.35? (2 marks)

Solution

$$P(X < -0.35|X < 0) = \frac{P(X < -0.35 \cap X < 0)}{P(X < 0)} = \frac{P(X < -0.35)}{P(X < 0)} = \frac{0.15}{0.5} = 0.3$$

Specific behaviours

✓ writes the correct conditional probability statement
✓ computes the probability correctly

(d) An engineer is more interested in the square of the error. What is the probability that the square of the error is less than 0.09? (2 marks)

Solution

$$P(X^2 < 0.09) = P(-0.3 < X < 0.3) = 0.6$$

Specific behaviours

✓ correctly expresses the required probability in terms of X
✓ computes the probability correctly

(e) Calculate the variance of the error. (3 marks)

Solution
$E(X) = \int_{-0.5}^{0.5} x\,dx = 0$ So $Var(X) = \int_{-0.5}^{0.5} (x-0)^2(1)dx = \left.\frac{x^3}{3}\right\vert_{-0.5}^{0.5} = \frac{0.125+0.125}{3} = \frac{1}{12}$

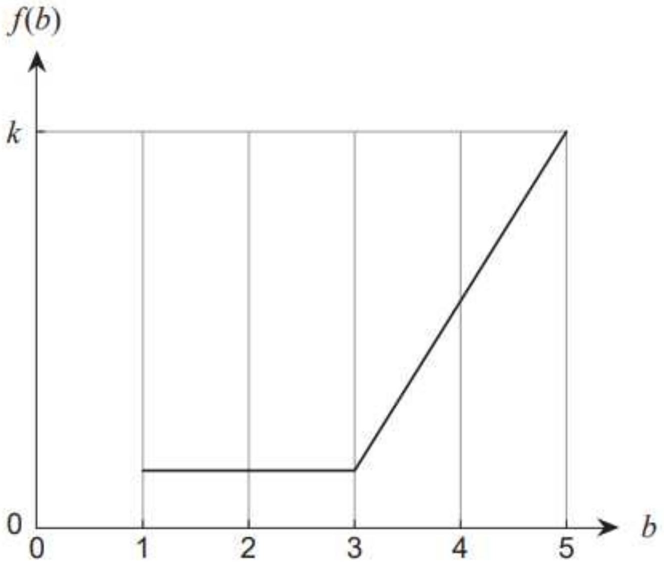
Specific behaviours
✓ computes mean correctly ✓ states an integral for the variance ✓ evaluates the integral to determine variance correctly

2023
Section 2
Question
11

Continuous random variables and the normal distribution

Mrs Euler is having her car serviced at BIMDAS Mechanics. She drops her vehicle off at 8 am and is told that her car will be ready for collection at some time between 1 pm and 5 pm that day.

Let the random variable *B* denote the time after noon (12 pm) at which a vehicle is ready for collection at BIMDAS Mechanics. The probability density function for *B* is shown in the graph below.



The probability of a vehicle being ready for collection between 2 pm and 3 pm is 0.1.

(a) Determine the value of *k*. (2 marks)

Solution

The area under the curve must be equal to 1. Hence

$$4\times 0.1+0.5\times 2\times (k-0.1)=1$$
$$\Rightarrow k+0.3=1$$
$$\Rightarrow k=0.7$$

Specific behaviours

✓ states that the area under the curve must equal 1
✓ obtains correct value of k

(b) An incomplete expression for the probability density function of *B* is given below. Fill in the boxes to complete the missing parts of the expression. (2 marks)

Solution

The probability density function for B is given by

$$f(b)=\begin{cases} 0.1, & \mathbf{1\le b<3} \\ \mathbf{0.3b-0.8,} & 3\le b\le 5 \\ 0, & \text{otherwise} \end{cases}$$

Specific behaviours

✓ correctly completes the interval
✓ correctly completes the linear function

(c) Determine the expected time that Mrs Euler's vehicle will be ready for collection at BIMDAS Mechanics. (3 marks)

Solution
$E(B)=\int_1^3 0.1b\,db+\int_3^5 b(0.3b-0.8)\,db$ $=3.8$ Therefore, the expected pickup time is 3:48 pm.
Specific behaviours
✓ states a correct integral expression for the expected value of B ✓ determines the correct expected value of B ✓ states the expected value as a time

Mr Euler is also having his car serviced, but by Addition Autos. He drops his vehicle off at 8 am and is told that his car will be ready for collection at some time between 1 pm and 5 pm that day.

Let the random variable *A* denote the time after noon (12 pm) that a vehicle is ready for collection at Addition Autos. The cumulative distribution function for *A* is given by

$$P(A\le a)=\begin{cases}0, & a<1\\ \dfrac{10a-a^2-9}{16}, & 1\le a\le 5\\ 1, & a>5\end{cases}$$

(d) Determine the probability that Mr Euler's vehicle will be ready to collect

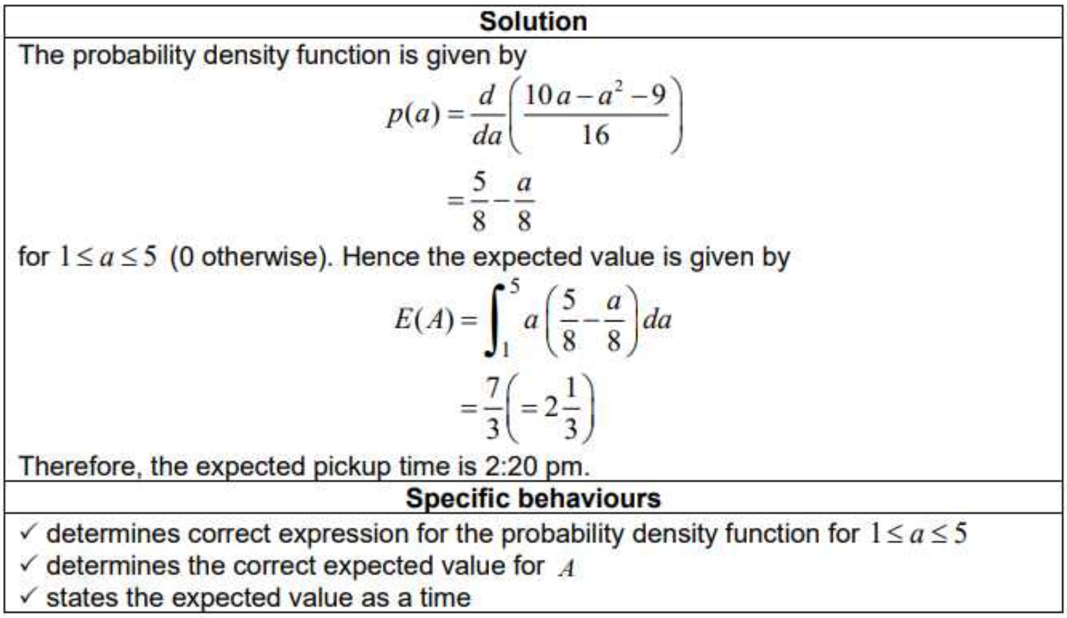
(i) by 3 pm. (1 mark)

Solution
$P(A\le 3)=\dfrac{10(3)-3^2-9}{16}$ $=0.75$
Specific behaviours
✓ calculates correct probability

(ii) between 3 pm and 4 pm. (2 marks)

Solution
$P(3\le A\le 4)=P(A\le 4)-P(A\le 3)$ $=\dfrac{10(4)-4^2-9}{16}-\dfrac{10(3)-3^2-9}{16}$ $=\dfrac{15}{16}-\dfrac{12}{16}$ $=\dfrac{3}{16}=0.1875$
Specific behaviours
✓ expresses the probability as the difference $P(A\le 4)-P(A\le 3)$ ✓ calculates correct probability

(e) Determine the expected time at which Mr Euler's vehicle will be ready for collection at Addition Autos. (3 marks)



Solution

The probability density function is given by

$$p(a) = \frac{d}{da}\left(\frac{10a - a^2 - 9}{16}\right)$$

$$= \frac{5}{8} - \frac{a}{8}$$

for $1 \le a \le 5$ (0 otherwise). Hence the expected value is given by

$$E(A) = \int_1^5 a\left(\frac{5}{8} - \frac{a}{8}\right)da$$

$$= \frac{7}{3}\left(= 2\frac{1}{3}\right)$$

Therefore, the expected pickup time is 2:20 pm.

Specific behaviours

✓ determines correct expression for the probability density function for $1 \le a \le 5$

✓ determines the correct expected value for A

✓ states the expected value as a time

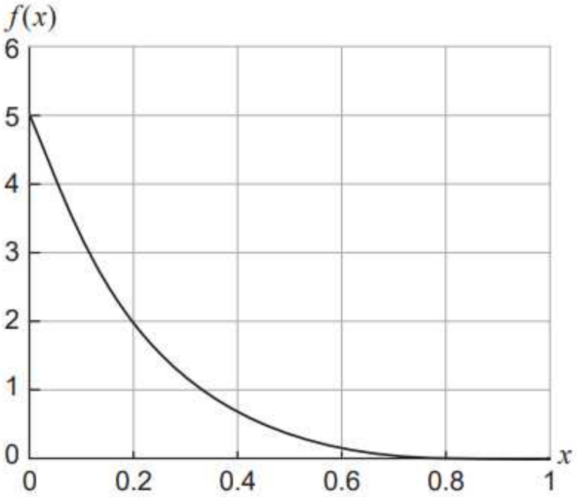
2022
Section 2
Question 8

Continuous random variables and the normal distribution

A small outback petrol station receives a weekly delivery of petrol. The volume of petrol sold in a week, X, (in units of 10 000 litres) is a random variable with probability density function

$$f(x) = 5(1-x)^4, \quad 0 \le x \le 1$$

as shown in the graph below.



(a) Determine, using appropriate units, the expected value and variance of the amount of fuel sold in a week. (4 marks)

Solution

The expected value is given by

$$E(X) = \int_0^1 5x(1-x)^4\,dx$$
$$= 0.16667$$

Hence the expected amount sold is 1667 litres.

The variance is given by

$$\sigma^2 = \int_0^1 5(x-0.1667)^2(1-x)^4\,dx$$
$$= 0.01984$$

Hence the variance is 198 litres^2.

Specific behaviours

✓ obtains the expected amount sold
✓ states correct integral expression for the variance
✓ obtains the variance of the amount sold
✓ uses units correctly for both expected value and variance

(b) What storage tank capacity will ensure that there is only a 1% chance of running out of petrol in a given week? State your answer to the nearest litre. (3 marks)

Solution

Let $x = c$ denote the required capacity of the storage tank. Then

$$\int_0^c 5(1-x)^4\,dx = 0.99$$

$$\Rightarrow [-(1-x)^5]_0^c = 0.99$$

$$\Rightarrow -(1-c)^5 + 1 = 0.99$$

$$c \approx 0.60189$$

Hence the required capacity of the storage tank is 6019 litres.

Or

Let $x = c$ denote the required capacity of the storage tank. Then

$$\int_c^1 5(1-x)^4\,dx = 0.01$$

$$\Rightarrow [-(1-x)^5]_c^1 = 0.01$$

$$c \approx 0.60189$$

Hence the required capacity of the storage tank is 6019 litres.

Specific behaviours

✓ state a correct integral expression for the capacity c
✓ solves for c
✓ states capacity to the nearest litre

(c) When the petrol is delivered, it is pumped into the storage tank. The rate of change of the petrol level in the tank, $h(t)$, (measured in metres) at time t (measured in minutes) is given by

$$h'(t) = \frac{5}{2t+3}$$

Determine the height of the storage tank if it takes 20 minutes to fill. (3 marks)

Solution

The height of the tank is given by

$$h(t) = \int_0^{20} \frac{5}{2t+3}\,dt$$

$$= \left[\frac{5}{2}\ln(2t+3)\right]_0^{20}$$

$$\approx 6.66 \text{ metres}$$

Specific behaviours

✓ states a definite integral for $h(t)$
✓ antidifferentiates correctly
✓ determines the height of the storage tank

2020
Section 2
Question 16

Continuous random variables and the normal distribution

A large refrigerator in a scientific laboratory is always required to maintain a temperature between 0 °C and 1 °C to preserve the integrity of biological samples stored inside. A scientist working in the laboratory suspects that the refrigerator is not maintaining the required temperature and decides to record the temperature every hour for seven days. Based on these measurements, the scientist concludes that the temperature, T, in the refrigerator is normally distributed with a mean of 0.8 °C and a standard deviation of 0.4 °C.

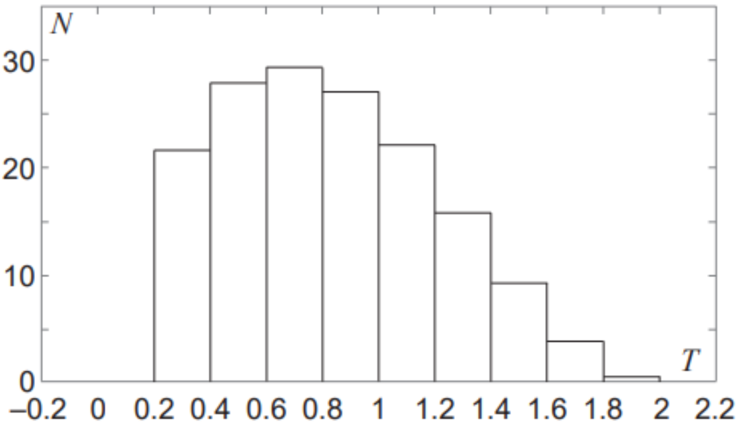
(a) Temperature in degrees Fahrenheit, T_f, is given by $T_f = \frac{9}{5}T + 32$. Determine the mean and standard deviation of the refrigerator temperature in degrees Fahrenheit. (2 marks)

Solution
The mean of T_f is $\mu_{T_f} = \frac{9}{5}\mu_T + 32$ $= \frac{9}{5}\left(\frac{4}{5}\right) + 32$ $= \frac{836}{25} = 33\frac{11}{25} = 33.44$ The standard deviation of T_f is $\sigma_{T_f} = \frac{9}{5}\sigma_T$ $= \frac{9}{5}\left(\frac{2}{5}\right)$ $= \frac{18}{25} = 0.72$
Specific behaviours
✓ determines correct mean ✓ determines correct standard deviation

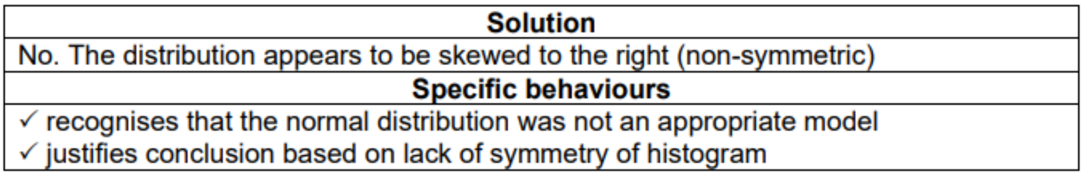
(b) Determine the probability that the refrigerator temperature is above 1 °C. Give your answer rounded to four decimal places. (1 mark)

Solution
$P(T > 1) = 0.3085$
Specific behaviours
✓ determines correct probability

The histogram of data gathered by the scientist is shown below. N denotes the number of observations in each temperature interval.



(c) Do you agree that the normal distribution was an appropriate model to use? Provide a reason to justify your response. (2 marks)

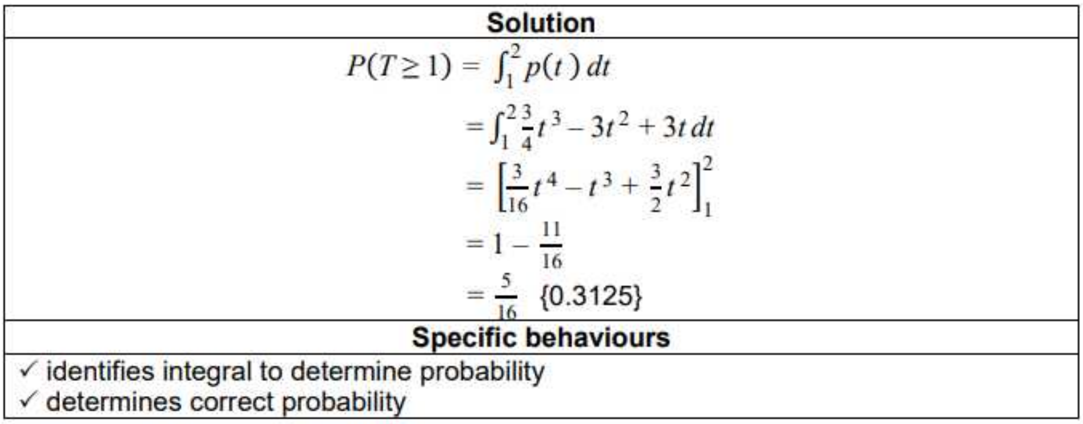


Solution
No. The distribution appears to be skewed to the right (non-symmetric)
Specific behaviours
✓ recognises that the normal distribution was not an appropriate model ✓ justifies conclusion based on lack of symmetry of histogram

An alternative probability density function proposed to model the refrigerator temperature, in degrees Celcius, is given by:

$$p(t) = \frac{3}{4}t^3 - 3t^2 + 3t, \quad 0 \le t \le 2$$

(d) Determine the probability that the refrigerator temperature is above 1 °C using the new model. (2 marks)



Solution
$P(T \geq 1) = \int_1^2 p(t)\,dt$ $= \int_1^2 \frac{3}{4}t^3 - 3t^2 + 3t\,dt$ $= \left[\frac{3}{16}t^4 - t^3 + \frac{3}{2}t^2\right]_1^2$ $= 1 - \frac{11}{16}$ $= \frac{5}{16}$ {0.3125}
Specific behaviours
✓ identifies integral to determine probability ✓ determines correct probability

Unit 4.3 – Interval estimates for proportions

Section 1

There have been no questions on this topic for this section in the exams of recent years.

Section 2

2023 Section 2 Question 7 Interval estimates for proportions	The Carnaby's Black Cockatoo is a bird native to southwest Australia. A birdwatcher is interested in estimating the proportion, *p*, of birds living in a Western Australian national park that are Carnaby's Black Cockatoos. The birdwatcher visited the national park one morning and, standing at their favourite bird-watching location, observed 200 birds flying past. Seventy-six of those birds were Carnaby's Black Cockatoos. (a) On the basis of the sample, determine a point estimate for *p*. (1 mark) (b) On the basis of the sample, determine a 95% confidence interval for *p*. (2 marks) (c) What is the minimum number of birds that would need to be sampled to ensure that the margin of error of the 95% confidence interval for *p* is at most 0.02? (2 marks)

	(d) Identify and explain **two** sources of bias in the birdwatcher's sampling method. (4 marks)

2023 Section 2 Question 12 **Interval estimates for proportions**	A factory produces pre-packed servings of udon noodles. The noodles are dispensed into packets of individual servings by a machine. However, there is variation in the serving sizes dispensed. The specifications attached to the side of the machine have been partially destroyed, so the only available information is that the mass in grams, X, of noodles dispensed is normally distributed, $P(X \leq 150) = 0.0228$ and $P(X \geq 165) = 0.1587$. (a) Determine the mean and standard deviation of the mass of noodles dispensed by the machine. (3 marks)

The factory sells trays containing 20 packets of individual servings of udon noodles to restaurants. A standard individual serving should have a mass of at least 150 g.

(b) Determine the probability that a tray of noodles contains no underweight servings. (3 marks)

Following some customer complaints about their serving sizes, the manager of the factory decides to investigate. They select a random sample of 200 individual servings of udon noodles and determine a confidence interval for the proportion p of underweight servings to be (0.0651, 0.1349).

(c) Determine the margin of error of the confidence interval. (1 mark)

(d) Determine the level of confidence that was used to calculate the confidence interval. (3 marks)

(e) On the basis of the above confidence interval, is the proportion of underweight servings of udon noodles different from what was claimed in the machine specifications? (2 marks)

(f) All else remaining equal, state how the margin of error would change if

(i) the confidence level was decreased. (1 mark)

(ii) the sample size was increased from 200 to 500. (1 mark)

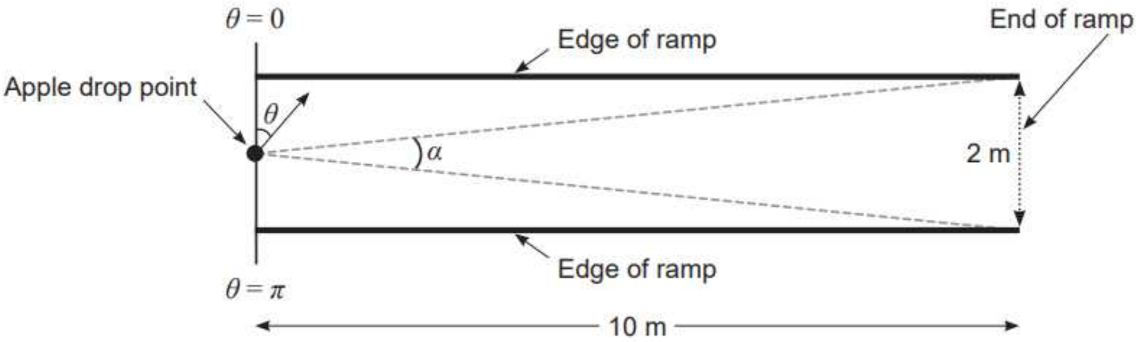
2023
Section 2
Question
13

Interval estimates for proportions

While leaving a shopping centre a mathematician accidentally drops a bag of apples at the top of a ramp of length 10 m and width 2 m. The diagram below shows the top view of the ramp. Four of the apples roll safely to the end of the ramp, while six roll off an edge and splatter on the ground below.

The mathematician decides to create a simple model by assuming that the:

- apples roll independently of one another along straight lines from the apple drop point
- direction each apple rolls, θ, is an angle measured about the apple drop point and is uniformly distributed over $0 \leq \theta \leq \pi$



Apples that roll along a line within the sector marked by α will arrive safely at the end of the ramp, while others will roll off the edge.

(a) (i) Determine the value of α. (2 marks)

(ii) Hence show that the probability, *p*, of an apple rolling safely to the end of the ramp is $p = 0.063$ (rounded to three decimal places). (1 mark)

(b) Determine the probability that, of the 10 apples, four or more make it safely to the end of the ramp. (2 marks)

The mathematician decides to purchase another 20 bags of apples, i.e. 200 apples, return to the top of the ramp, and break each bag open one at a time. After the experiment a total of 63 apples have rolled safely to the end of the ramp.

(c) Using the sample of 200 apples, calculate a 99% confidence interval for the population proportion of apples that will roll safely to the end of the ramp. (2 marks)

(d) What does the confidence interval from part (c) suggest about the validity of the model assumptions used to calculate the probability in part (a)(ii)? (2 marks)

2022 Section 2 Question 12 Interval estimates for proportions	The Larje Machine Co manufactures metal rods for large industrial equipment. Their standard manufacturing process produces rods whose lengths are normally distributed with a mean of 400 cm, and a standard deviation of 5 cm. A rod is considered 'useable' if its length is between 395 cm and 405 cm. Let X be a random variable denoting the length of a rod manufactured by the Larje Machine Co. (a) Determine the probability that a rod manufactured by the Larje Machine Co is useable. Round your answer to three decimal places. (3 marks) Recently the Larje Machine Co introduced a new manufacturing process that industry experts claim will improve the percentage of useable rods produced to 80%. The quality control department decides to investigate whether this standard is being achieved and plan to collect a random sample of rods manufactured using the new process. (b) What condition must the sample satisfy in order to use a normal distribution to model the sample proportion of useable rods? (1 mark) The quality control department collects a sample of 100 rods. (c) What is the approximate distribution of the sample proportion of useable rods? (2 marks) Upon measuring the sample of 100 rods, it is found that 75 are useable. (d) Calculate a 95% confidence interval for the population proportion of useable rods. (3 marks)

	(e) The quality control department would like to obtain a confidence interval with a smaller margin of error. State two methods that it could use to achieve this. (2 marks) (f) The quality control department decides to select a new sample for which the maximum possible margin of error for a 95% confidence interval is 0.05. What sample size will achieve this requirement? (3 marks) (g) The new sample yields the 95% confidence interval $(0.717, 0.803)$. On the basis of this sample, is the proportion of useable rods different from what was claimed by the industry experts? Justify your answer. (2 marks)

2022 Section 2 Question 13 **Interval estimates for proportions**	According to the Association of Poultry Farmers, 35% of people living in Melbourne purchase free-range eggs. (a) If a random sample of 100 people living in Melbourne is surveyed, what is the probability that the sample proportion of people who purchase free-range eggs will be less than 0.28? (3 marks)

A market research company wants to know whether the proportion p of people living in Perth who purchase free-range eggs is similar to that of Melbourne. A junior employee proposes that it gather a sample of shoppers by standing outside a particular shop between 9 am and 10 am on a Tuesday morning and asking all shoppers entering the shop if they purchase free-range eggs.

(b) Identify and explain **two** sources of bias in the proposed sampling method. (4 marks)

The company does not follow the suggestion of the junior employee and instead randomly samples 243 people living in Perth and asks them whether they purchase free-range eggs. On the basis of the results of their survey, a confidence interval for p is calculated to be (0.2520, 0.3488).

(c) Determine the number of people in the sample who purchase free-range eggs. (2 marks)

(d) Determine the level of confidence that was used to calculate the confidence interval. (3 marks)

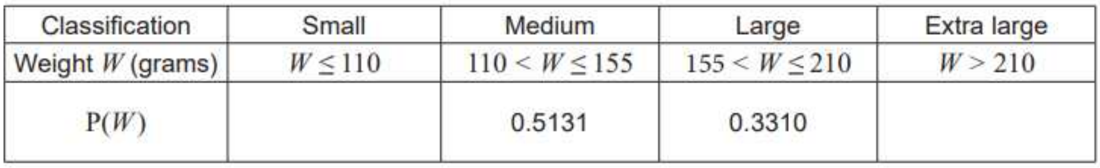
2021
Section 2
Question 8

Interval estimates for proportions

The weights W (in grams) of carrots sold at a supermarket have been found to be normally distributed with a mean of 142.8 g and a standard deviation of 30.6 g.

(a) Determine the percentage of carrots sold at the supermarket that weigh more than 155 g. (2 marks)

Carrots sold at the supermarket are classified by weight, as shown in the table below



Classification	Small	Medium	Large	Extra large
Weight W (grams)	$W \leq 110$	$110 < W \leq 155$	$155 < W \leq 210$	$W > 210$
$\mathrm{P}(W)$		0.5131	0.3310	

(b) Complete the table above, providing the missing probabilities. (2 marks)

(c) Of the carrots being sold at the supermarket that are not of medium weight, what proportion is small? (2 marks)

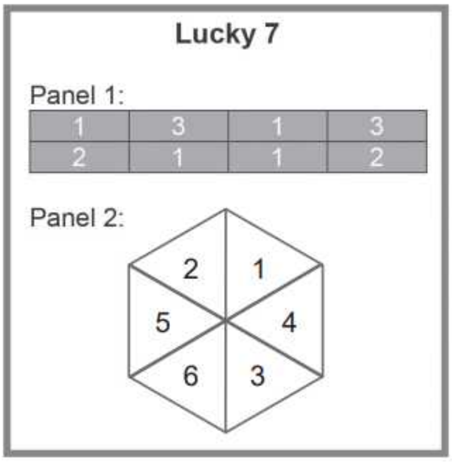
The supermarket sells bags of mixed-weight carrots, with 12 randomly-selected carrots placed in each bag.

(d) If a customer purchases a bag of mixed-weight carrots, determine the probability that there will be at most two small carrots in the bag. (3 marks)

2021
Section 2
Question 10

Interval estimates for proportions

A charity organisation has printed 'Lucky 7' scratchie tickets as a fundraiser for use at two special events. The tickets contain two panels. Each ticket has the same numbers as the sample ticket shown below, arranged randomly and hidden within each panel.



A player scratches one section of each panel to reveal a number. The two numbers revealed are then added together. If the total is seven or higher, the player wins a prize.

At the first event, 400 tickets are purchased, and a prize is won on 124 occasions. Let p denote the probability that a prize is won.

(a) Determine the sample proportion of times that a prize is won at the first event. (1 mark)

(b) Show that the probability p of winning a prize is $\frac{7}{24}$. (2 marks)

(c) Calculate the mean and standard deviation of the sample proportion of times a prize is won when 400 tickets are purchased. (2 marks)

(d) At a second event, 400 scratchie tickets are again purchased. If the sample proportion was 0.6 standard deviations from the population proportion, how many prizes were won at the second event? (3 marks)

2021
Section 2
Question
11

Interval
estimates
for
proportions

A new political party, the Sustainable Energy Party, is planning to have candidates run in the next election. Researchers have collected data that suggests the proportion of voters likely to vote for the party to be 23%.

One year before the next election, random samples of 400 voters were taken in a particular electorate. Let $\hat{p}$ denote the sample proportion of voters who indicated they would vote for the Sustainable Energy Party at the next election.

(a) State the distribution of $\hat{p}$. (3 marks)

(b) Calculate the probability that the proportion of voters likely to vote for the Sustainable Energy Party in a sample of 400 is less than 0.20. (3 marks)

One week before the election, researchers believed that the proportion of voters likely to vote for the party in that same electorate had increased. A random sample of 200 voters was taken at this time, and 55 of them indicated they would vote for the Sustainable Energy Party at the next election.

(c) Based on this sample, estimate the proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (1 mark)

(d) For a 99% confidence interval, what is the margin of error of the sample proportion of voters likely to vote for the Sustainable Energy Party in this electorate, based on this sample? (2 marks)

(e) Based on this sample, calculate a 95% confidence interval for the population proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (3 marks)

(f) Based on the research, did the proportion of voters likely to vote for the Sustainable Energy Party in this electorate increase in the year leading up to the election? Justify your answer. (2 marks)

	(g) The analysis above models the number of voters likely to vote for the Sustainable Energy Party as binomially distributed. State and discuss the validity of any assumptions for the binomial distribution in this context. (3 marks)

2021 Section 2 Question 13 Interval estimates for proportions	A carnival game involves five buckets, each containing 5 blue balls and 15 red balls. A player blindly selects a ball from each bucket and wins the game if they select at least 4 blue balls. Let X denote the number of blue balls selected. (a) State the distribution of X, including its parameters. (2 marks) (b) What is the probability of a player winning the game on any given attempt? (2 marks) (c) Players are charged \$2 for each attempt at the game and offered a \$150 prize if they win the game. By providing appropriate numerical justification, explain why this is not a good idea for the carnival organisers. (2 marks)

An observer records the outcome of 100 consecutive games and determines the 90% and 95% confidence intervals for the proportion of wins, p. The confidence intervals are (0.04, 0.16) and (0.05, 0.15).

(d) Which of these intervals is the 95% confidence interval for p? Justify your answer. (2 marks)

(e) How many wins were observed out of the 100 games? (2 marks)

(f) Determine what you would expect to happen to the width of the confidence intervals if 400 games had been observed. (2 marks)

(g) The true proportion of wins does not lie within either of the above confidence intervals. Does this suggest that a sampling error was made? Justify your answer. (2 marks)

2020 Section 2 Question 8 Interval estimates for proportions	The weight, X, of chicken eggs from a farm is normally distributed with mean 60 g and standard deviation 5 g. Eggs with a weight of more than 67 g are classed as ‘jumbo’. (a) What proportion of eggs from the farm are ‘jumbo’? (2 marks) (b) What proportion of ‘jumbo’ eggs are less than 75 g in weight? (3 marks) (c) The heaviest 0.05% of eggs fetch a higher price. What is the minimum weight of these eggs? (2 marks)

2020 Section 2 Question 12 Interval estimates for proportions	It is estimated that 20% of small businesses fail in the first year. A business advisory group takes a random sample of 500 new businesses that started in January 2018. An analyst employed by the group suggests the use of the binomial distribution is appropriate in this case. (a) What is the probability that at most 120 of the businesses fail in the first year? (2 marks)

(b) What is the approximate distribution of the sample proportion of small businesses that fail by the end of the year in this sample? Justify your answer. (3 marks)

(c) What is the probability that the sample proportion of businesses that fail by the end of the year is less than 0.18? (2 marks)

(d) By January 2019, 90 of the 500 new businesses had failed. Calculate a 95% confidence interval for the proportion of new businesses that fail in the first year. (2 marks)

The business advisory group believes that the proportion of new businesses that fail within a year can be reduced by providing financial advice. They took another random sample of 500 businesses that started in January 2019 and provided them with regular financial advice. In this random sample, at the end of the year 80 businesses had failed.

(e) Calculate the sample proportion and its margin of error at the 95% confidence level. (2 marks)

	(f) Calculate a 95% confidence interval for the proportion of businesses that failed. What do you conclude regarding the value of the financial advice provided to the new businesses? (4 marks) (g) If the sample size was reduced, what would be the effect on the confidence interval? Justify your answer. (2 marks) (h) State **two** assumptions that the analyst made in recommending the use of the binomial model in this case and discuss whether they are valid. (4 marks)

2020 Section 2 Question 14 **Interval estimates for proportions**	A suburban council hires a consultant to estimate the proportion of residents of the suburb who use its library. (a) The consultant decides to estimate a 95% confidence interval for the proportion to within an error of 0.01. What minimum sample size should be selected? (3 marks)

	(b) If resource limitations dictate that the maximum sample size that can be managed is 500, what is the maximum margin of error in estimating a 99% confidence interval? (3 marks) The consultant decides to select the sample by standing on the roadside outside the library at lunchtime and asking a random sample of the passers-by whether they use the library. (c) Identify and explain two possible sources of bias with this sampling scheme. (4 marks)

2019 **Section 2** **Question 8** **Interval estimates for proportions**	Big Foods is a large supermarket company. The manager of Big Foods wants to estimate the proportion of households that do the majority of their grocery shopping in their stores. A junior staff member at Big Foods conducted a survey of 250 randomly-selected households and found that 56 did the majority of their grocery shopping at a Big Foods store. (a) (i) Calculate the sample proportion of households who did the majority of their grocery shopping at Big Foods. (1 mark)

(ii) Determine the 95% confidence interval for the proportion of households who do the majority of their grocery shopping at Big Foods. Give your answer to four decimal places. (3 marks)

(iii) What is the margin of error of the 95% confidence interval? Give your answer to four decimal places. (1 mark)

An independent research company conducted a large-scale survey of household supermarket preferences and estimated that the true proportion of households that conduct most of their grocery shopping at Big Foods was 0.17 (assume that this is indeed the true proportion).

(b) With reference to your answer to part (a)(ii), does this result suggest that the junior staff member at Big Foods made a mistake? (2 marks)

2019
Section 2
Question
11

Interval estimates for proportions

A pizza company runs a marketing campaign based on the delivery times of its pizzas. The company claims that it will deliver a pizza in a radius of 5 km within 30 minutes of ordering or it is free. The manager estimates that the actual time, T, from order to delivery is normally distributed with mean 25 minutes and standard deviation 2 minutes.

(a) What it the probability that a pizza is delivered free? (1 mark)

(b) On a busy Saturday evening, a total of 50 pizzas are ordered. What is the probability that more than three are delivered free? (2 marks)

The company wants to reduce the proportion of pizzas that are delivered free to 0.1%.

(c) The manager suggests this can be achieved by increasing the advertised delivery time. What should the advertised delivery time be? (2 marks)

After some additional training the company was able to maintain the advertised delivery time as 30 minutes but reduce the proportion of pizzas delivered free to 0.1%.

(d) Assuming that the original mean of 25 minutes is maintained, what is the new standard deviation of delivery times? (3 marks)

2019
Section 2
Question 13

Interval estimates for proportions

The proportion of working adults who miss breakfast on week days is estimated to be 40%. A study takes a random sample of 400 working adults.

(a) For this sample:

(i) What is the (approximate) distribution of the sample proportion of workers who miss breakfast? (2 marks)

(ii) What is the probability that the sample proportion of workers who miss breakfast is greater than 44%? (2 marks)

Tom takes a random sample of 400 adults. He obtained his sample by selecting the first 400 workers he met in a busy mall in Perth city during lunch time.

(b) Discuss briefly two possible sources of bias in Tom's sample. (2 marks)

Amir suggests that a better sampling scheme is to obtain a random sample of 400 voters and contact them by telephone.

(c) (i) Outline one source of bias in Amir's sampling scheme. (1 mark)

(ii) Which of Tom's or Amir's sampling scheme is better? Provide a reason for your choice. (1 mark)

2019 Section 2 Question 14 Interval estimates for proportions	(a) What is the minimum sample size required to estimate a population proportion to within 0.01 with 95% confidence. (3 marks) (b) Identify **two** factors that affect the width of a confidence interval for a population proportion and describe the effect of each. (4 marks)

Marking Guide – Section 1

There have been no questions on this topic for this section in the exams of recent years.

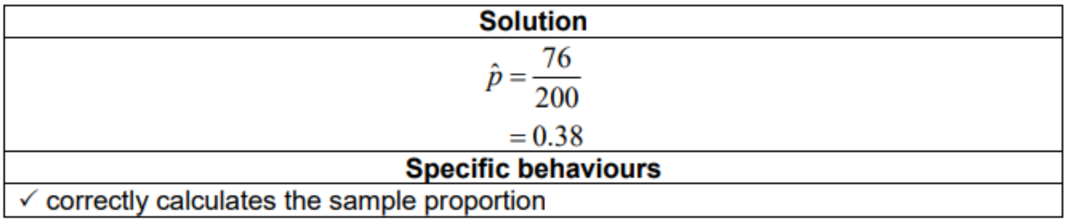
Marking Guide – Section 2

2023
Section 2
Question 7

Interval estimates for proportions

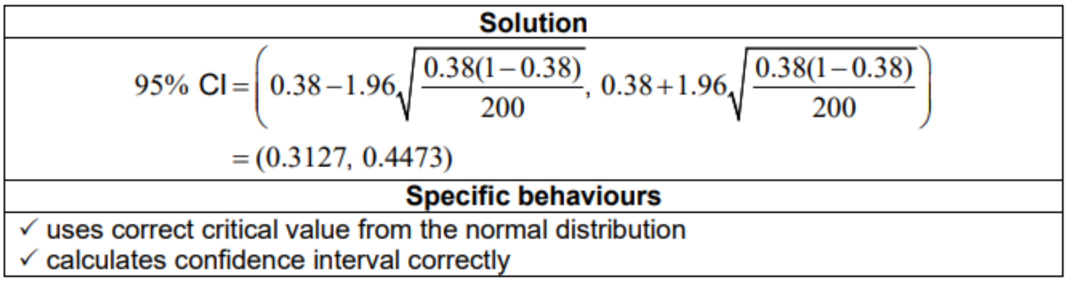
The Carnaby's Black Cockatoo is a bird native to southwest Australia. A birdwatcher is interested in estimating the proportion, *p*, of birds living in a Western Australian national park that are Carnaby's Black Cockatoos. The birdwatcher visited the national park one morning and, standing at their favourite bird-watching location, observed 200 birds flying past. Seventy-six of those birds were Carnaby's Black Cockatoos.

(a) On the basis of the sample, determine a point estimate for *p*. (1 mark)



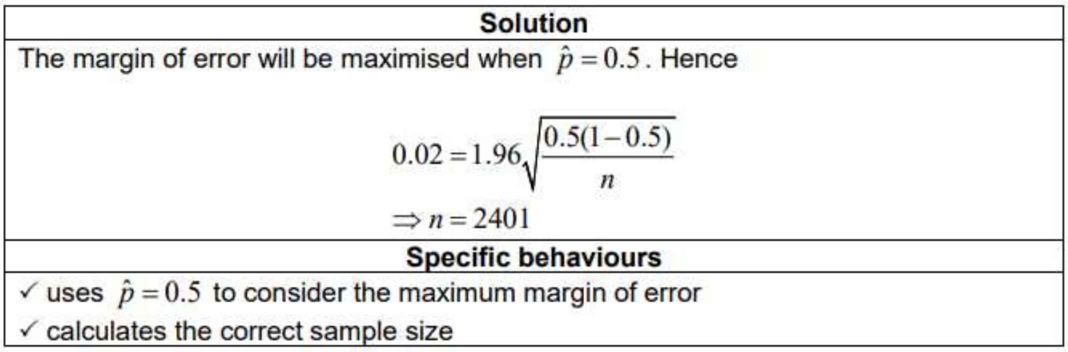
Solution
$\hat{p} = \frac{76}{200}$ $= 0.38$
Specific behaviours
✓ correctly calculates the sample proportion

(b) On the basis of the sample, determine a 95% confidence interval for *p*. (2 marks)



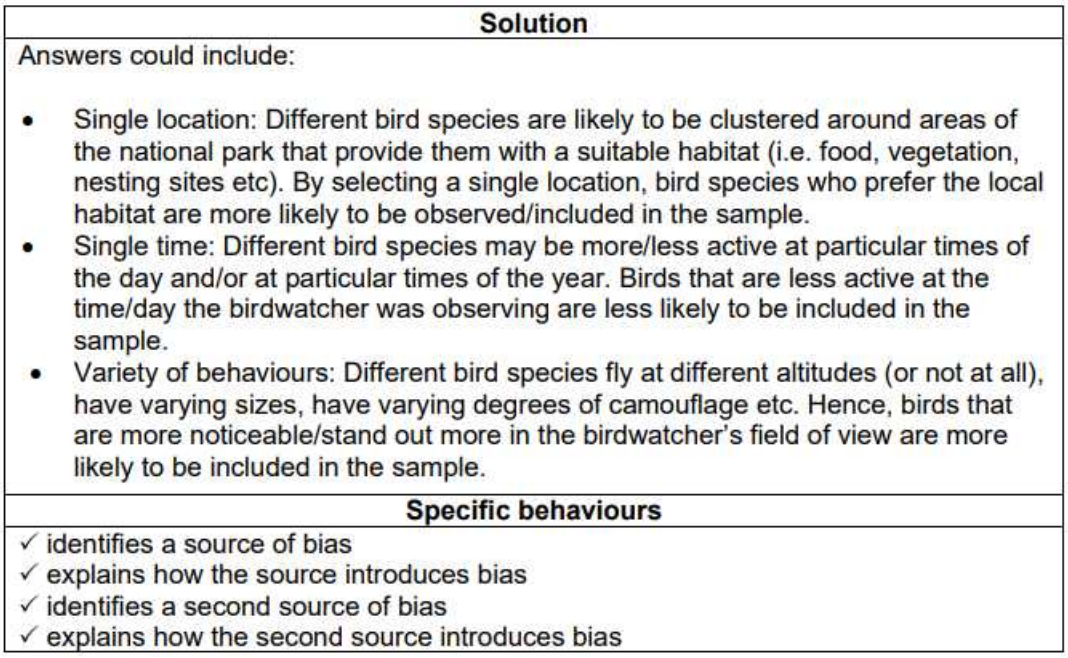
Solution
$95\%\ \text{CI} = \left(0.38 - 1.96\sqrt{\frac{0.38(1-0.38)}{200}},\ 0.38 + 1.96\sqrt{\frac{0.38(1-0.38)}{200}}\right)$ $= (0.3127,\ 0.4473)$
Specific behaviours
✓ uses correct critical value from the normal distribution ✓ calculates confidence interval correctly

(c) What is the minimum number of birds that would need to be sampled to ensure that the margin of error of the 95% confidence interval for *p* is at most 0.02? (2 marks)



Solution
The margin of error will be maximised when $\hat{p} = 0.5$. Hence $0.02 = 1.96\sqrt{\frac{0.5(1-0.5)}{n}}$ $\Rightarrow n = 2401$
Specific behaviours
✓ uses $\hat{p} = 0.5$ to consider the maximum margin of error ✓ calculates the correct sample size

(d) Identify and explain **two** sources of bias in the birdwatcher's sampling method. (4 marks)



Solution

Answers could include:

- Single location: Different bird species are likely to be clustered around areas of the national park that provide them with a suitable habitat (i.e. food, vegetation, nesting sites etc). By selecting a single location, bird species who prefer the local habitat are more likely to be observed/included in the sample.
- Single time: Different bird species may be more/less active at particular times of the day and/or at particular times of the year. Birds that are less active at the time/day the birdwatcher was observing are less likely to be included in the sample.
- Variety of behaviours: Different bird species fly at different altitudes (or not at all), have varying sizes, have varying degrees of camouflage etc. Hence, birds that are more noticeable/stand out more in the birdwatcher's field of view are more likely to be included in the sample.

Specific behaviours

✓ identifies a source of bias
✓ explains how the source introduces bias
✓ identifies a second source of bias
✓ explains how the second source introduces bias

2023
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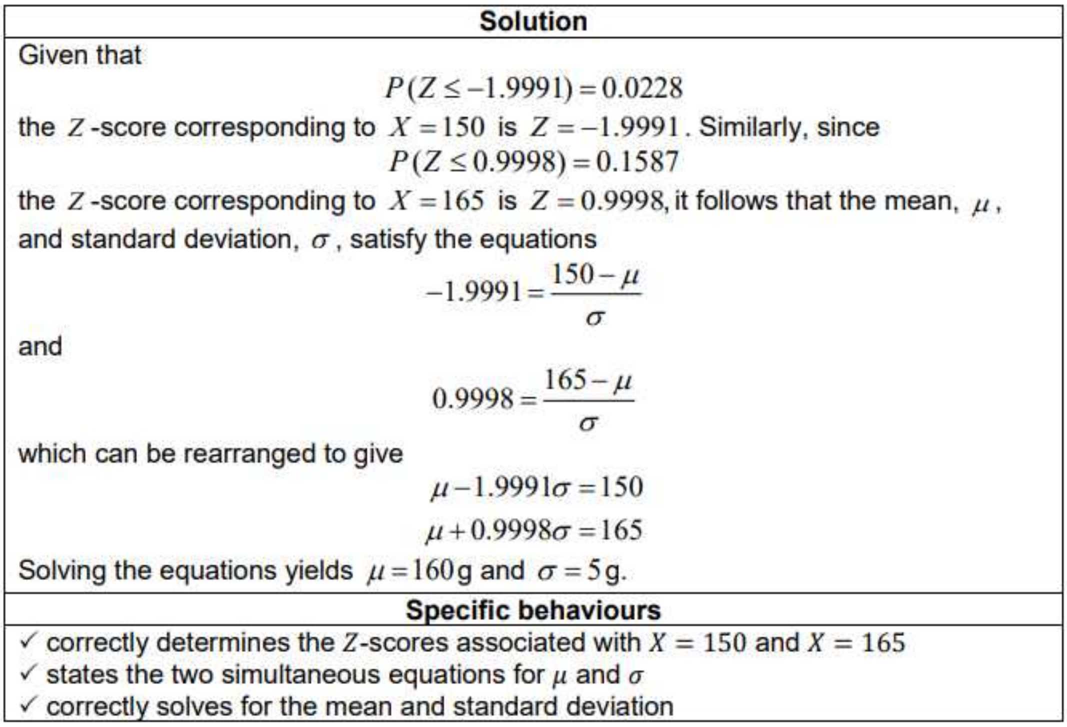
Question
12

Interval estimates for proportions

A factory produces pre-packed servings of udon noodles. The noodles are dispensed into packets of individual servings by a machine. However, there is variation in the serving sizes dispensed. The specifications attached to the side of the machine have been partially destroyed, so the only available information is that the mass in grams, X, of noodles dispensed is normally distributed, $P(X \leq 150) = 0.0228$ and $P(X \geq 165) = 0.1587$.

(a) Determine the mean and standard deviation of the mass of noodles dispensed by the machine. (3 marks)



Solution

Given that

$$P(Z \leq -1.9991) = 0.0228$$

the Z-score corresponding to $X = 150$ is $Z = -1.9991$. Similarly, since

$$P(Z \leq 0.9998) = 0.1587$$

the Z-score corresponding to $X = 165$ is $Z = 0.9998$, it follows that the mean, μ, and standard deviation, σ, satisfy the equations

$$-1.9991 = \frac{150 - \mu}{\sigma}$$

and

$$0.9998 = \frac{165 - \mu}{\sigma}$$

which can be rearranged to give

$$\mu - 1.9991\sigma = 150$$

$$\mu + 0.9998\sigma = 165$$

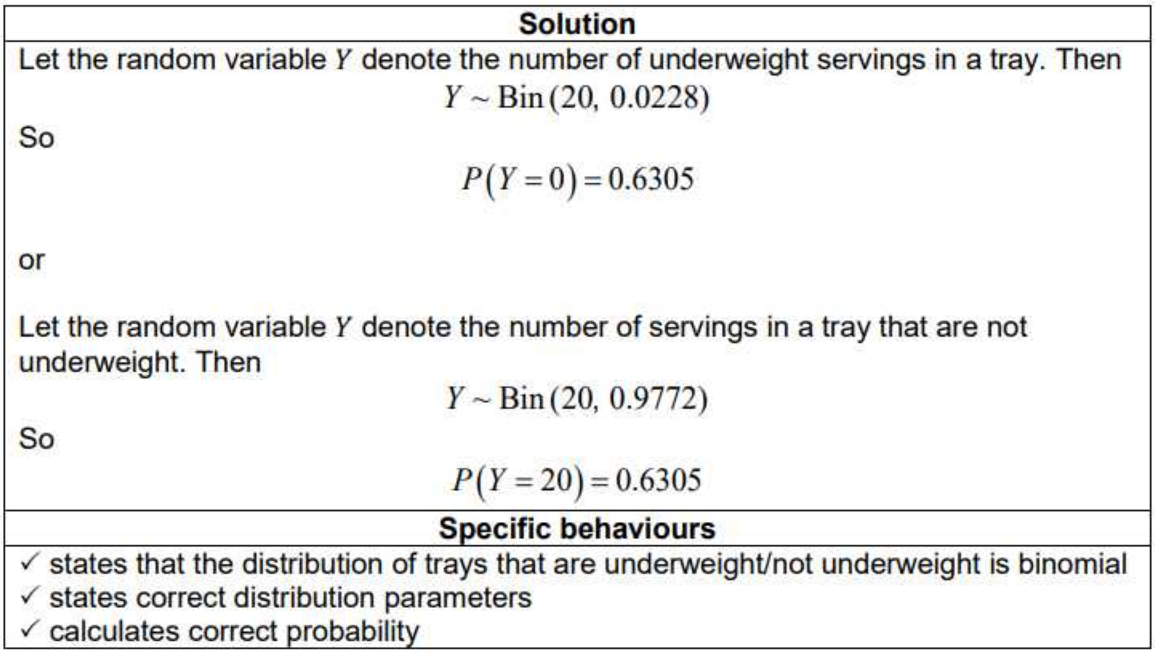
Solving the equations yields $\mu = 160$g and $\sigma = 5$g.

Specific behaviours

✓ correctly determines the Z-scores associated with $X = 150$ and $X = 165$
✓ states the two simultaneous equations for μ and σ
✓ correctly solves for the mean and standard deviation

The factory sells trays containing 20 packets of individual servings of udon noodles to restaurants. A standard individual serving should have a mass of at least 150 g.

(b) Determine the probability that a tray of noodles contains no underweight servings. (3 marks)



Solution

Let the random variable Y denote the number of underweight servings in a tray. Then

$$Y \sim \text{Bin}(20,\ 0.0228)$$

So

$$P(Y=0)=0.6305$$

or

Let the random variable Y denote the number of servings in a tray that are not underweight. Then

$$Y \sim \text{Bin}(20,\ 0.9772)$$

So

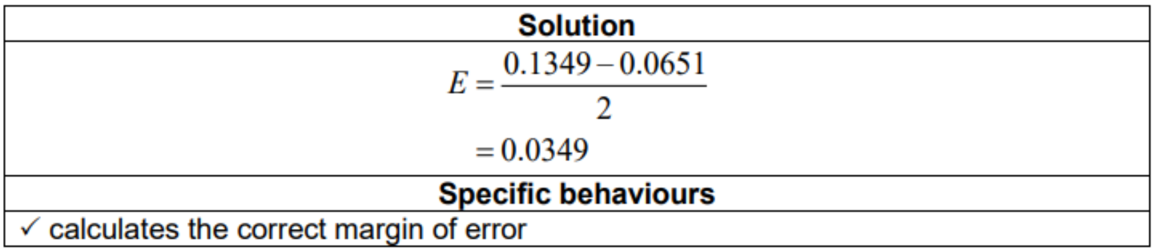
$$P(Y=20)=0.6305$$

Specific behaviours

✓ states that the distribution of trays that are underweight/not underweight is binomial
✓ states correct distribution parameters
✓ calculates correct probability

Following some customer complaints about their serving sizes, the manager of the factory decides to investigate. They select a random sample of 200 individual servings of udon noodles and determine a confidence interval for the proportion p of underweight servings to be (0.0651, 0.1349).

(c) Determine the margin of error of the confidence interval. (1 mark)



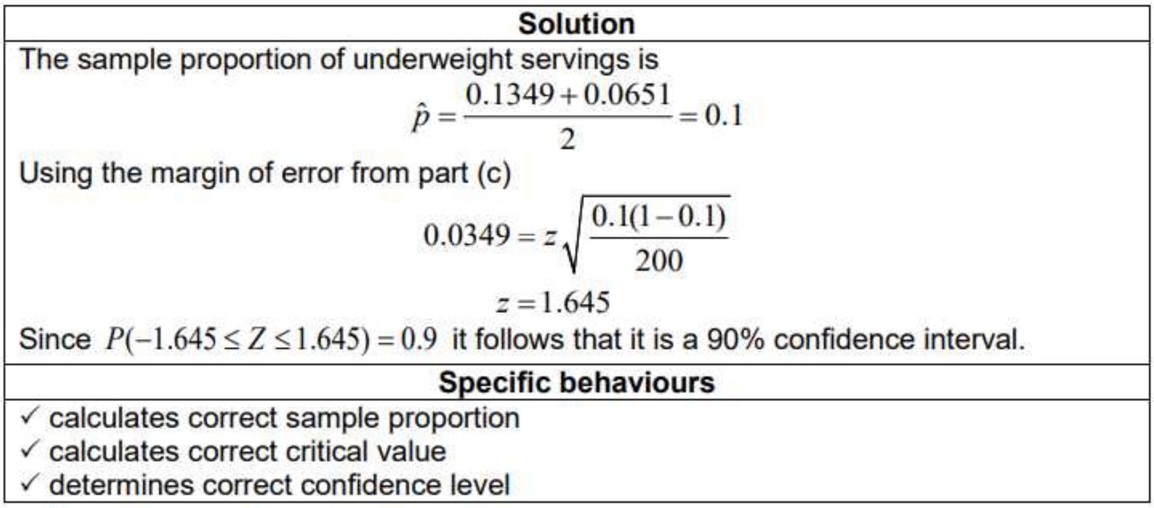
Solution

$$E=\frac{0.1349-0.0651}{2}$$
$$=0.0349$$

Specific behaviours

✓ calculates the correct margin of error

(d) Determine the level of confidence that was used to calculate the confidence interval. (3 marks)



Solution

The sample proportion of underweight servings is

$$\hat{p}=\frac{0.1349+0.0651}{2}=0.1$$

Using the margin of error from part (c)

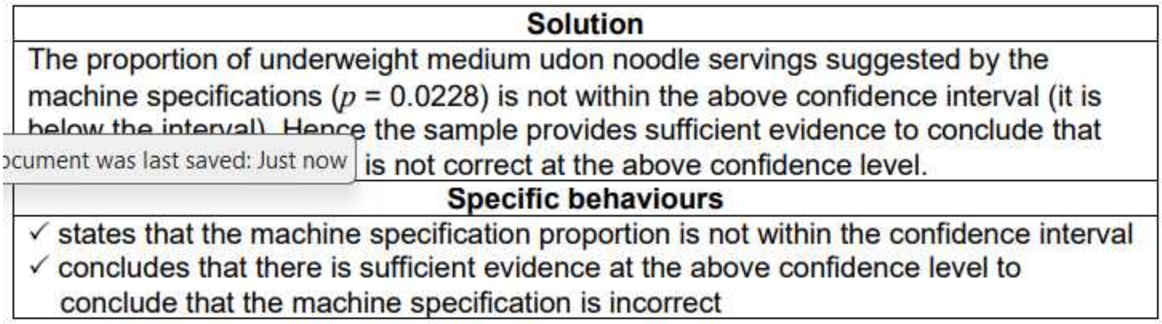
$$0.0349=z\sqrt{\frac{0.1(1-0.1)}{200}}$$
$$z=1.645$$

Since $P(-1.645 \le Z \le 1.645)=0.9$ it follows that it is a 90% confidence interval.

Specific behaviours

✓ calculates correct sample proportion
✓ calculates correct critical value
✓ determines correct confidence level

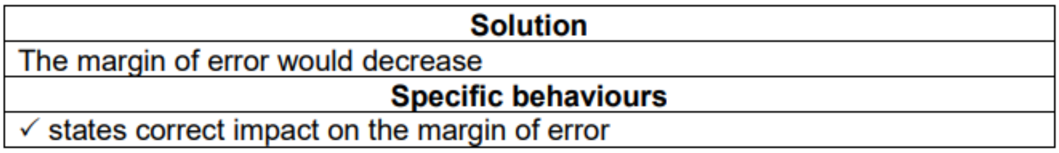
(e) On the basis of the above confidence interval, is the proportion of underweight servings of udon noodles different from what was claimed in the machine specifications? (2 marks)



Solution
The proportion of underweight medium udon noodle servings suggested by the machine specifications (p = 0.0228) is not within the above confidence interval (it is below the interval). Hence the sample provides sufficient evidence to conclude that ɔcument was last saved: Just now is not correct at the above confidence level.
Specific behaviours
✓ states that the machine specification proportion is not within the confidence interval ✓ concludes that there is sufficient evidence at the above confidence level to conclude that the machine specification is incorrect

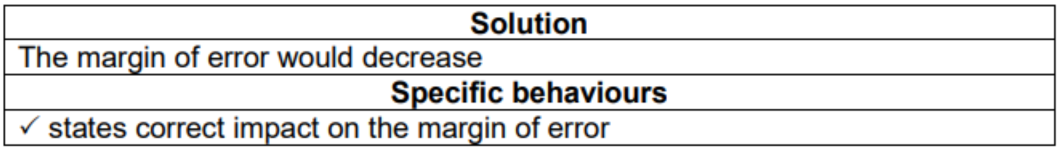
(f) All else remaining equal, state how the margin of error would change if

(i) the confidence level was decreased. (1 mark)



Solution
The margin of error would decrease
Specific behaviours
✓ states correct impact on the margin of error

(ii) the sample size was increased from 200 to 500. (1 mark)



Solution
The margin of error would decrease
Specific behaviours
✓ states correct impact on the margin of error

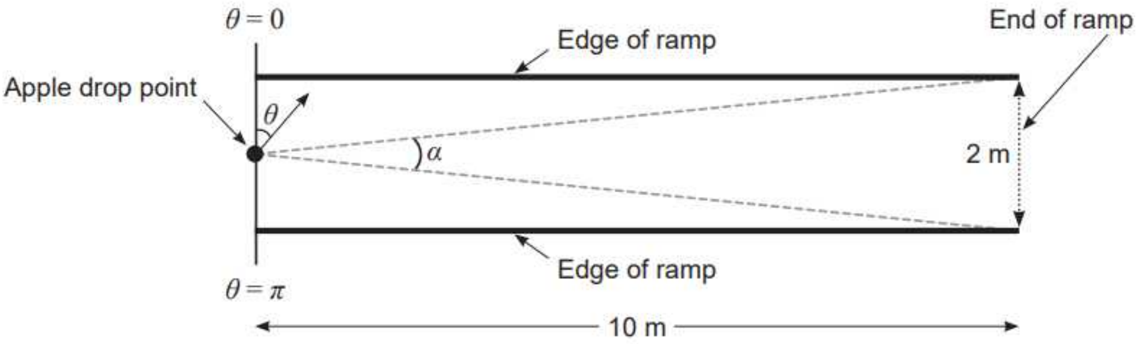
2023
Section 2
Question 13

Interval estimates for proportions

While leaving a shopping centre a mathematician accidentally drops a bag of apples at the top of a ramp of length 10 m and width 2 m. The diagram below shows the top view of the ramp. Four of the apples roll safely to the end of the ramp, while six roll off an edge and splatter on the ground below.

The mathematician decides to create a simple model by assuming that the:

- apples roll independently of one another along straight lines from the apple drop point
- direction each apple rolls, θ, is an angle measured about the apple drop point and is uniformly distributed over $0 \le \theta \le \pi$



Apples that roll along a line within the sector marked by α will arrive safely at the end of the ramp, while others will roll off the edge.

(a) (i) Determine the value of α. (2 marks)

Solution
From the diagram $$\tan\left(\frac{\alpha}{2}\right)=\frac{1}{10}$$ $$\Rightarrow \alpha = 2\tan^{-1}\left(\frac{1}{10}\right)$$ $$\approx 0.199$$
Specific behaviours
✓ writes correct expression for α ✓ obtains correct value for α

(ii) Hence show that the probability, *p*, of an apple rolling safely to the end of the ramp is *p* = 0.063 (rounded to three decimal places). (1 mark)

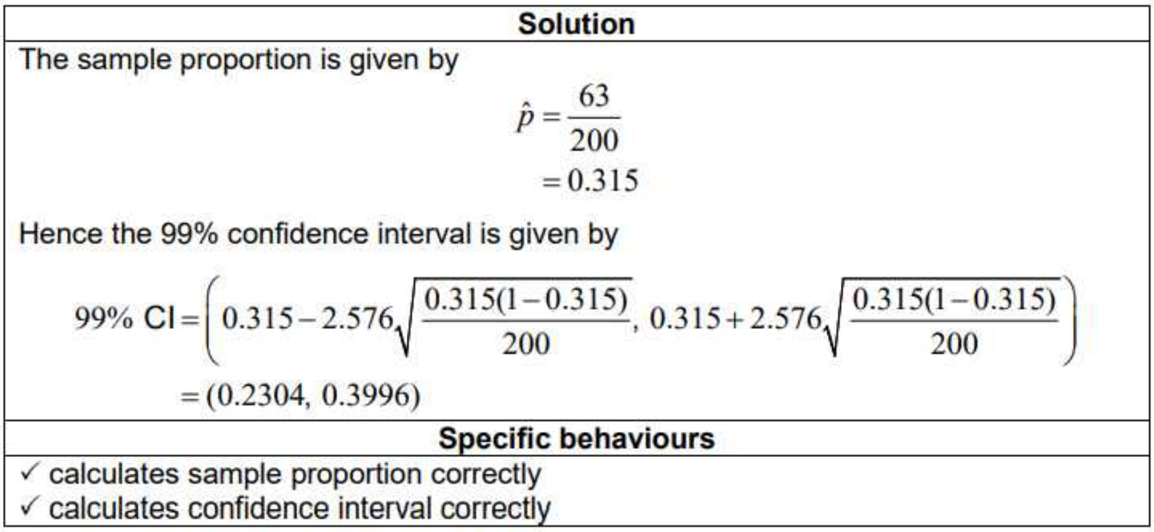
Solution
The probability of an apple rolling to the end of the ramp is given by $$p=\frac{\alpha}{\pi}$$ Hence $$p\approx\frac{0.199}{\pi}$$ $$=0.063$$
Specific behaviours
✓ demonstrates calculation of p as being $\frac{\alpha}{\pi}$

(b) Determine the probability that, of the 10 apples, four or more make it safely to the end of the ramp. (2 marks)

Solution
Based on the model assumptions, the number of apples that reach the end of the ramp X can be modelled as $$X \sim \text{Bin}(0.063,\ 10)$$ It follows that $$P(X \geq 4) = 0.0024$$
Specific behaviours
✓ states that the distribution is binomial and provides correct parameter values ✓ correctly calculates probability

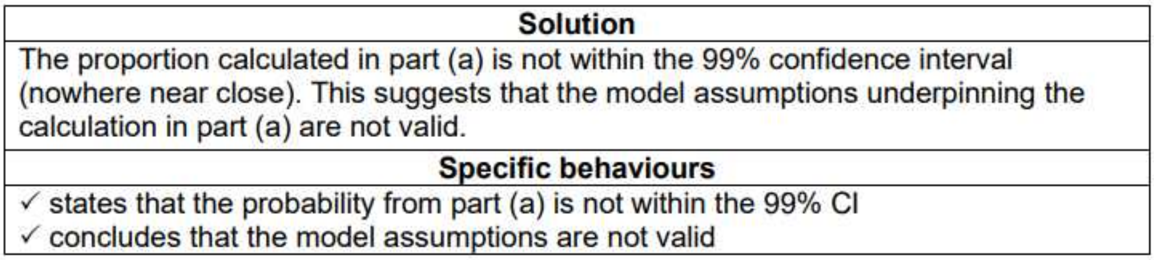
The mathematician decides to purchase another 20 bags of apples, i.e. 200 apples, return to the top of the ramp, and break each bag open one at a time. After the experiment a total of 63 apples have rolled safely to the end of the ramp.

(c) Using the sample of 200 apples, calculate a 99% confidence interval for the population proportion of apples that will roll safely to the end of the ramp. (2 marks)



Solution
The sample proportion is given by $\hat{p} = \frac{63}{200} = 0.315$ Hence the 99% confidence interval is given by $99\%\ \text{CI} = \left(0.315 - 2.576\sqrt{\frac{0.315(1-0.315)}{200}},\ 0.315 + 2.576\sqrt{\frac{0.315(1-0.315)}{200}}\right) = (0.2304,\ 0.3996)$
Specific behaviours
✓ calculates sample proportion correctly ✓ calculates confidence interval correctly

(d) What does the confidence interval from part (c) suggest about the validity of the model assumptions used to calculate the probability in part (a)(ii)? (2 marks)



Solution
The proportion calculated in part (a) is not within the 99% confidence interval (nowhere near close). This suggests that the model assumptions underpinning the calculation in part (a) are not valid.
Specific behaviours
✓ states that the probability from part (a) is not within the 99% CI ✓ concludes that the model assumptions are not valid

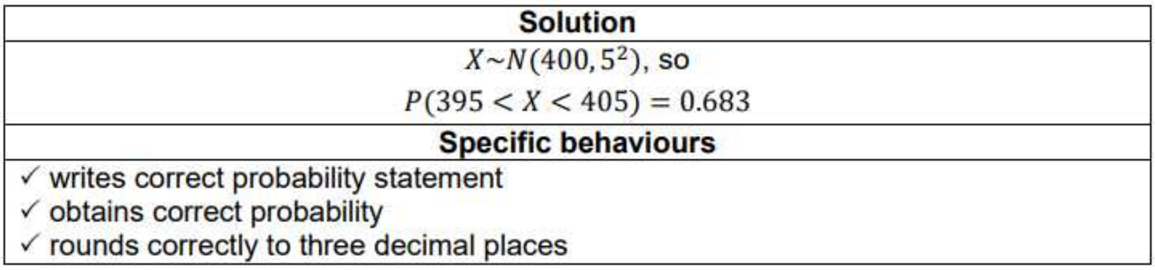
2022
Section 2
Question
12

Interval estimates for proportions

The Larje Machine Co manufactures metal rods for large industrial equipment. Their standard manufacturing process produces rods whose lengths are normally distributed with a mean of 400 cm, and a standard deviation of 5 cm. A rod is considered ‘useable’ if its length is between 395 cm and 405 cm.

Let X be a random variable denoting the length of a rod manufactured by the Larje Machine Co.

(a) Determine the probability that a rod manufactured by the Larje Machine Co is useable. Round your answer to three decimal places. (3 marks)



Solution
$X \sim N(400, 5^2)$, so $P(395 < X < 405) = 0.683$
Specific behaviours
✓ writes correct probability statement ✓ obtains correct probability ✓ rounds correctly to three decimal places

Recently the Larje Machine Co introduced a new manufacturing process that industry experts claim will improve the percentage of useable rods produced to 80%. The quality control department decides to investigate whether this standard is being achieved and plan to collect a random sample of rods manufactured using the new process.

(b) What condition must the sample satisfy in order to use a normal distribution to model the sample proportion of useable rods? (1 mark)

Solution
The sample must be large.
Specific behaviours
✓ states that the sample must be large

The quality control department collects a sample of 100 rods.

(c) What is the approximate distribution of the sample proportion of useable rods? (2 marks)

Solution

The mean of $\hat{p}$ is

$$E(\hat{p}) = 0.8$$

The variance of $\hat{p}$ is

$$\sigma^2 = \frac{0.8(1-0.8)}{100}$$

$$= 0.0016$$

(standard deviation $\sigma = 0.04$). Hence

$$\hat{p} \sim N(0.8, 0.0016)$$

Specific behaviours

✓ states that the distribution is normal with correct mean
✓ correctly determines the standard deviation or variance

Upon measuring the sample of 100 rods, it is found that 75 are useable.

(d) Calculate a 95% confidence interval for the population proportion of useable rods. (3 marks)

Solution

The sample proportion is given by

$$\hat{p} = \frac{75}{100} = 0.75$$

Hence the 95% confidence interval is given by

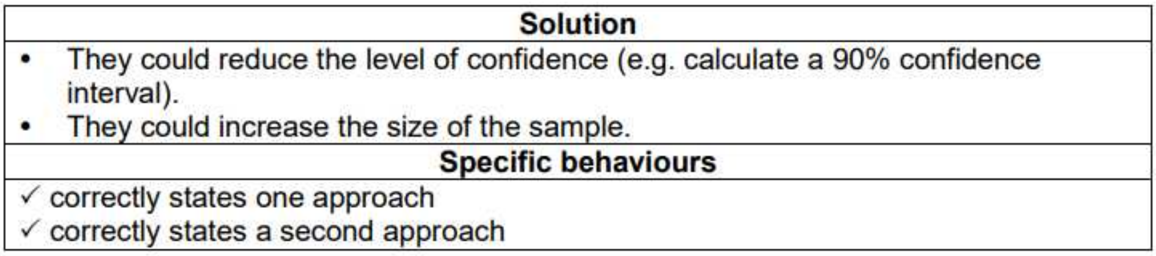
$$95\%\ \text{CI} = \left(0.75 - 1.96\sqrt{\frac{0.75(1-0.75)}{100}}, 0.75 + 1.96\sqrt{\frac{0.75(1-0.75)}{100}}\right)$$

$$= (0.665, 0.835)$$

Specific behaviours

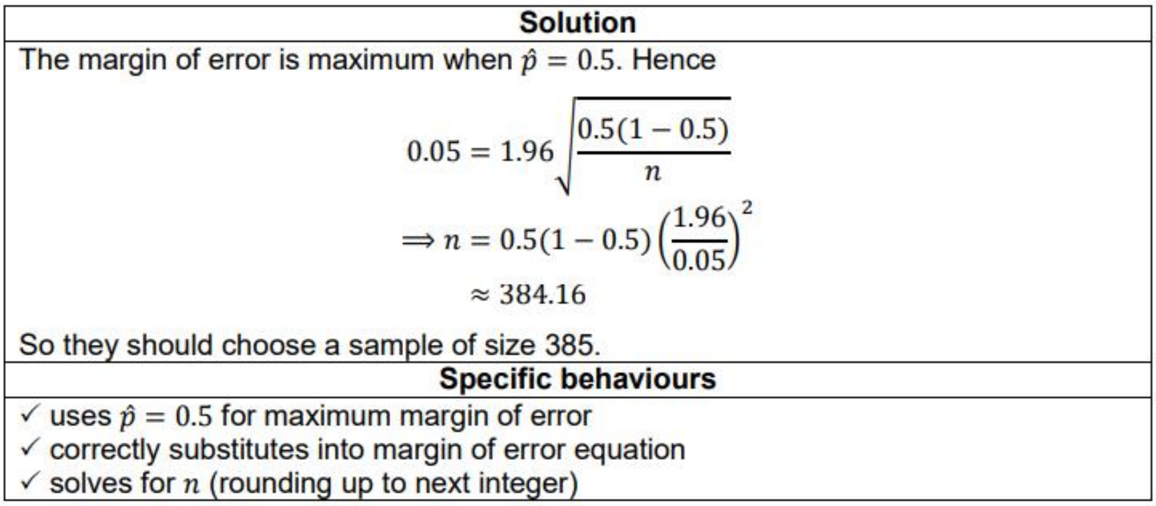
✓ calculates sample proportion correctly
✓ uses the correct critical value from the normal distribution
✓ calculates confidence interval correctly

(e) The quality control department would like to obtain a confidence interval with a smaller margin of error. State two methods that it could use to achieve this. (2 marks)



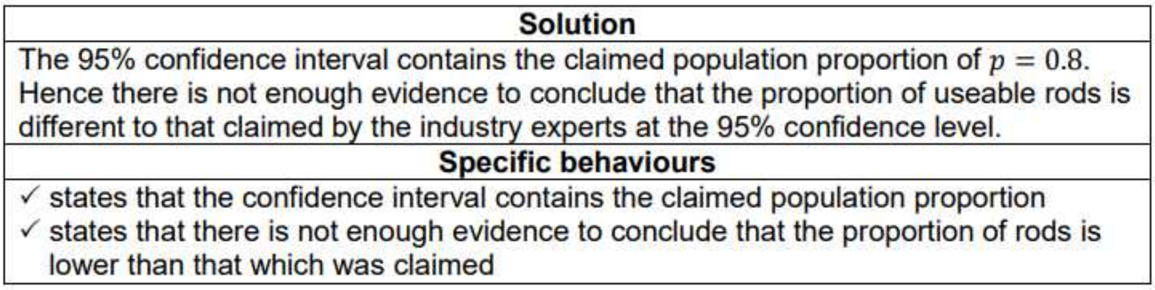
Solution
- They could reduce the level of confidence (e.g. calculate a 90% confidence interval). - They could increase the size of the sample.
Specific behaviours
✓ correctly states one approach ✓ correctly states a second approach

(f) The quality control department decides to select a new sample for which the maximum possible margin of error for a 95% confidence interval is 0.05. What sample size will achieve this requirement? (3 marks)



Solution
The margin of error is maximum when $\hat{p} = 0.5$. Hence $0.05 = 1.96\sqrt{\frac{0.5(1-0.5)}{n}}$ $\Rightarrow n = 0.5(1-0.5)\left(\frac{1.96}{0.05}\right)^2$ ≈ 384.16 So they should choose a sample of size 385.
Specific behaviours
✓ uses $\hat{p} = 0.5$ for maximum margin of error ✓ correctly substitutes into margin of error equation ✓ solves for n (rounding up to next integer)

(g) The new sample yields the 95% confidence interval $(0.717, 0.803)$. On the basis of this sample, is the proportion of useable rods different from what was claimed by the industry experts? Justify your answer. (2 marks)



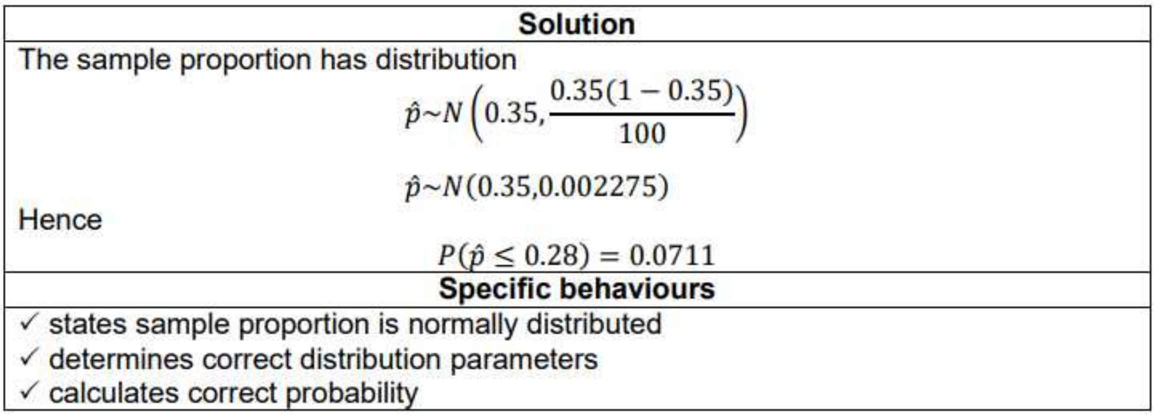
Solution
The 95% confidence interval contains the claimed population proportion of $p = 0.8$. Hence there is not enough evidence to conclude that the proportion of useable rods is different to that claimed by the industry experts at the 95% confidence level.
Specific behaviours
✓ states that the confidence interval contains the claimed population proportion ✓ states that there is not enough evidence to conclude that the proportion of rods is lower than that which was claimed

2022
Section 2
Question
13

Interval
estimates
for
proportions

According to the Association of Poultry Farmers, 35% of people living in Melbourne purchase free-range eggs.

(a) If a random sample of 100 people living in Melbourne is surveyed, what is the probability that the sample proportion of people who purchase free-range eggs will be less than 0.28? (3 marks)



Solution

The sample proportion has distribution

$$\hat{p} \sim N\left(0.35, \frac{0.35(1-0.35)}{100}\right)$$

$$\hat{p} \sim N(0.35, 0.002275)$$

Hence

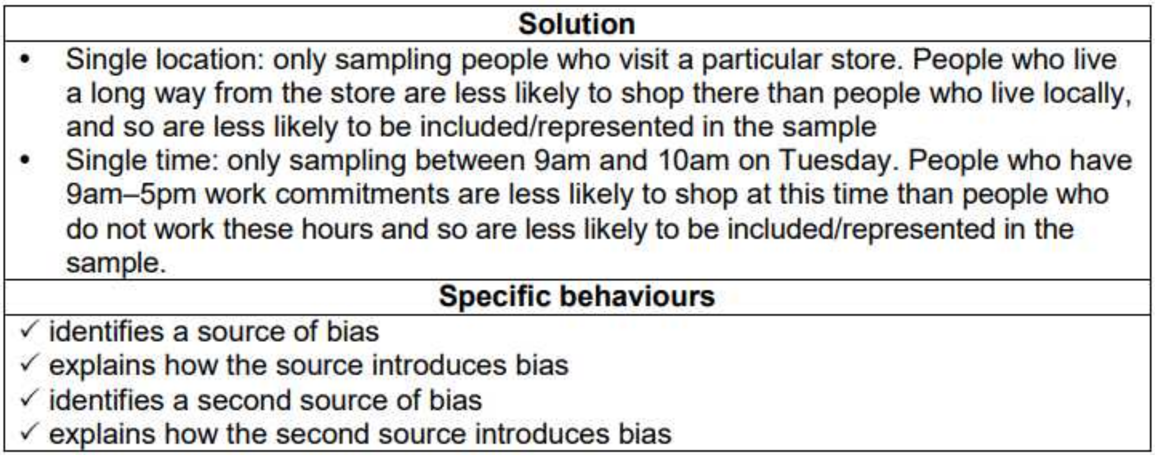
$$P(\hat{p} \leq 0.28) = 0.0711$$

Specific behaviours

✓ states sample proportion is normally distributed
✓ determines correct distribution parameters
✓ calculates correct probability

A market research company wants to know whether the proportion p of people living in Perth who purchase free-range eggs is similar to that of Melbourne. A junior employee proposes that it gather a sample of shoppers by standing outside a particular shop between 9 am and 10 am on a Tuesday morning and asking all shoppers entering the shop if they purchase free-range eggs.

(b) Identify and explain **two** sources of bias in the proposed sampling method. (4 marks)



Solution

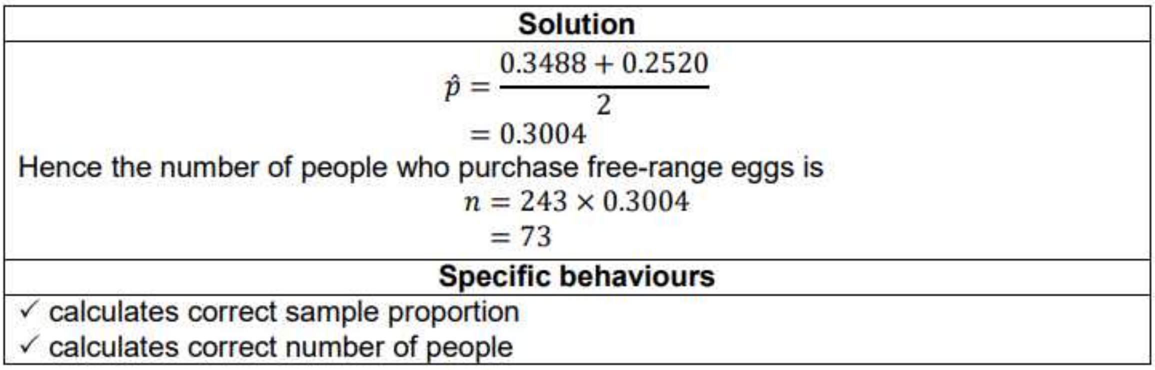
- Single location: only sampling people who visit a particular store. People who live a long way from the store are less likely to shop there than people who live locally, and so are less likely to be included/represented in the sample
- Single time: only sampling between 9am and 10am on Tuesday. People who have 9am–5pm work commitments are less likely to shop at this time than people who do not work these hours and so are less likely to be included/represented in the sample.

Specific behaviours

✓ identifies a source of bias
✓ explains how the source introduces bias
✓ identifies a second source of bias
✓ explains how the second source introduces bias

The company does not follow the suggestion of the junior employee and instead randomly samples 243 people living in Perth and asks them whether they purchase free-range eggs. On the basis of the results of their survey, a confidence interval for p is calculated to be (0.2520, 0.3488).

(c) Determine the number of people in the sample who purchase free-range eggs. (2 marks)



Solution

$$\hat{p} = \frac{0.3488 + 0.2520}{2}$$
$$= 0.3004$$

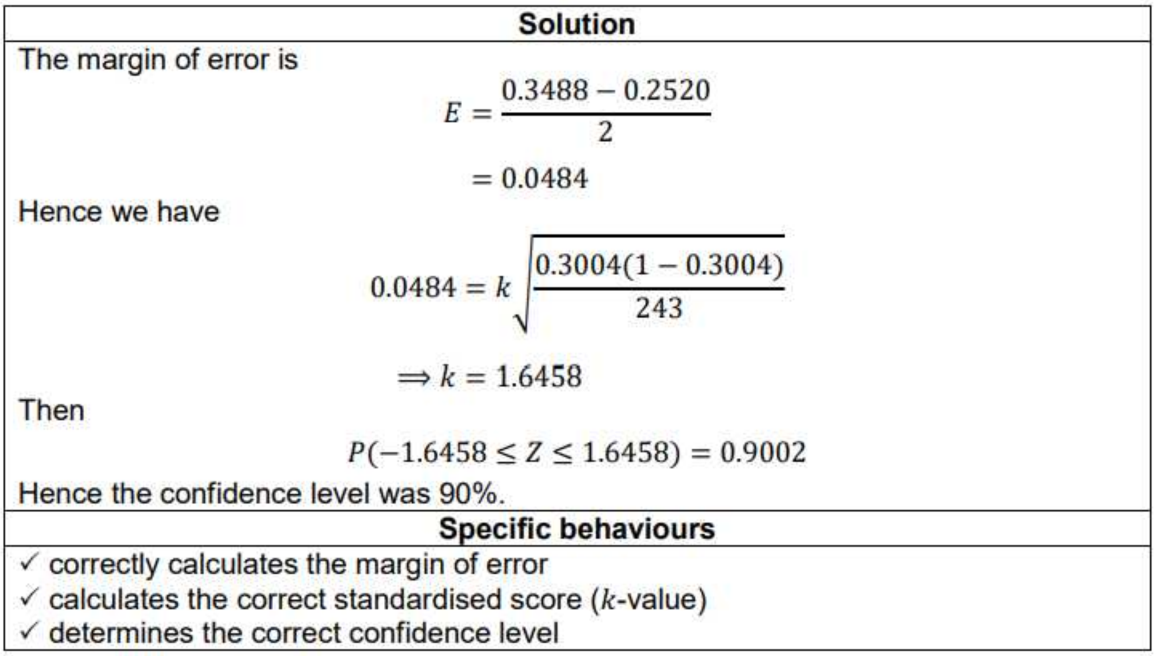
Hence the number of people who purchase free-range eggs is

$$n = 243 \times 0.3004$$
$$= 73$$

Specific behaviours

✓ calculates correct sample proportion
✓ calculates correct number of people

(d) Determine the level of confidence that was used to calculate the confidence interval. (3 marks)



Solution
The margin of error is $$E = \frac{0.3488 - 0.2520}{2}$$ $$= 0.0484$$ Hence we have $$0.0484 = k\sqrt{\frac{0.3004(1 - 0.3004)}{243}}$$ $$\Rightarrow k = 1.6458$$ Then $$P(-1.6458 \leq Z \leq 1.6458) = 0.9002$$ Hence the confidence level was 90%.
Specific behaviours
✓ correctly calculates the margin of error ✓ calculates the correct standardised score (k-value) ✓ determines the correct confidence level

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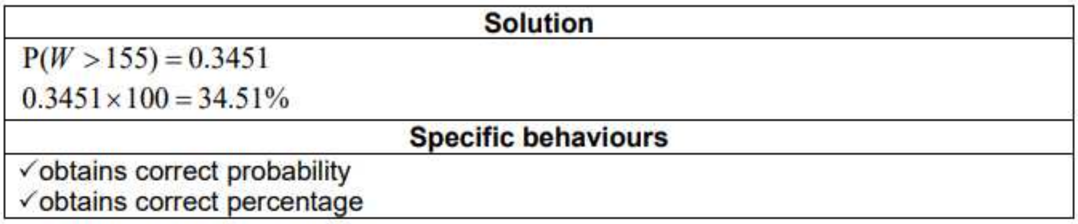
2021
Section 2
Question 8

Interval estimates for proportions

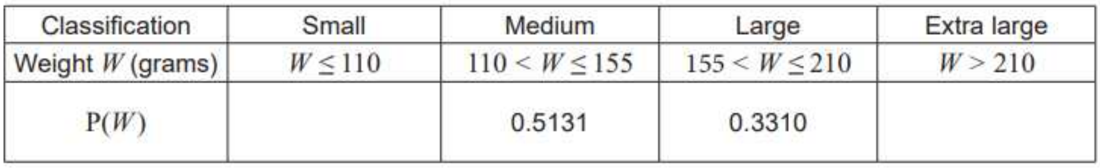
The weights W (in grams) of carrots sold at a supermarket have been found to be normally distributed with a mean of 142.8 g and a standard deviation of 30.6 g.

(a) Determine the percentage of carrots sold at the supermarket that weigh more than 155 g. (2 marks)



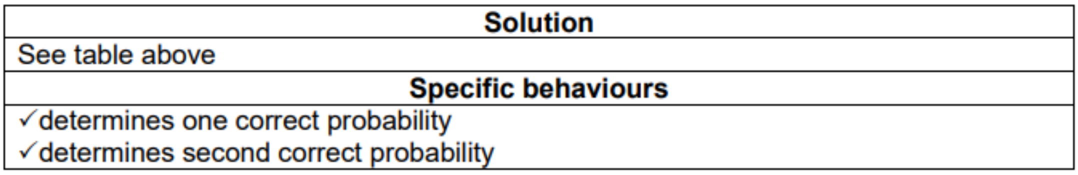
Solution
$\text{P}(W > 155) = 0.3451$ $0.3451 \times 100 = 34.51\%$
Specific behaviours
✓obtains correct probability ✓obtains correct percentage

Carrots sold at the supermarket are classified by weight, as shown in the table below



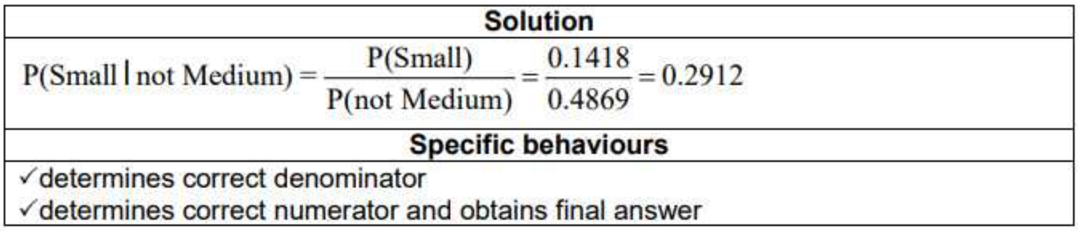
Classification	Small	Medium	Large	Extra large
Weight W (grams)	$W \leq 110$	$110 < W \leq 155$	$155 < W \leq 210$	$W > 210$
$\text{P}(W)$		0.5131	0.3310	

(b) Complete the table above, providing the missing probabilities. (2 marks)



Solution
See table above
Specific behaviours
✓determines one correct probability ✓determines second correct probability

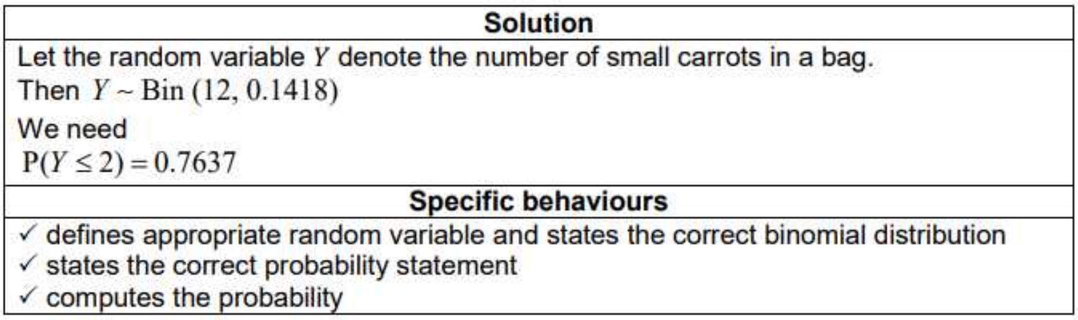
(c) Of the carrots being sold at the supermarket that are not of medium weight, what proportion is small? (2 marks)



Solution
$\text{P(Small} \mid \text{not Medium)} = \dfrac{\text{P(Small)}}{\text{P(not Medium)}} = \dfrac{0.1418}{0.4869} = 0.2912$
Specific behaviours
✓determines correct denominator ✓determines correct numerator and obtains final answer

The supermarket sells bags of mixed-weight carrots, with 12 randomly-selected carrots placed in each bag.

(d) If a customer purchases a bag of mixed-weight carrots, determine the probability that there will be at most two small carrots in the bag. (3 marks)

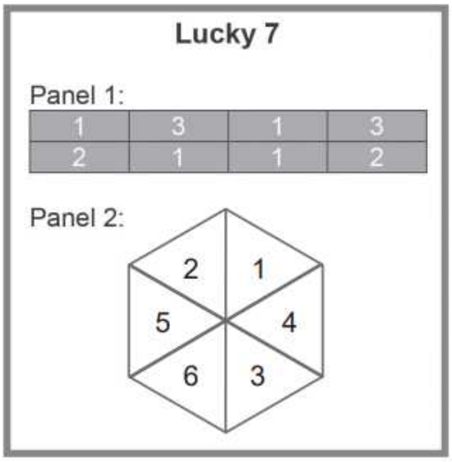


Solution
Let the random variable Y denote the number of small carrots in a bag. Then $Y \sim \text{Bin}\ (12, 0.1418)$ We need $\text{P}(Y \leq 2) = 0.7637$
Specific behaviours
✓ defines appropriate random variable and states the correct binomial distribution ✓ states the correct probability statement ✓ computes the probability

2021
Section 2
Question
10

Interval estimates for proportions

A charity organisation has printed 'Lucky 7' scratchie tickets as a fundraiser for use at two special events. The tickets contain two panels. Each ticket has the same numbers as the sample ticket shown below, arranged randomly and hidden within each panel.



A player scratches one section of each panel to reveal a number. The two numbers revealed are then added together. If the total is seven or higher, the player wins a prize.

At the first event, 400 tickets are purchased, and a prize is won on 124 occasions. Let p denote the probability that a prize is won.

(a) Determine the sample proportion of times that a prize is won at the first event. (1 mark)

Solution

$$\hat{p} = \frac{124}{400} = 0.31$$

Specific behaviours

✓ correctly determines the sample proportion

(b) Show that the probability p of winning a prize is $\frac{7}{24}$. (2 marks)

Solution

score	combinations	probability
7	3,4 or 2,5 or 1,6	$\frac{2}{8}\times\frac{1}{6}+\frac{2}{8}\times\frac{1}{6}+\frac{4}{8}\times\frac{1}{6}=\frac{8}{48}$
8	3,5 or 2,6	$\frac{4}{48}$
9	3,6	$\frac{2}{48}$

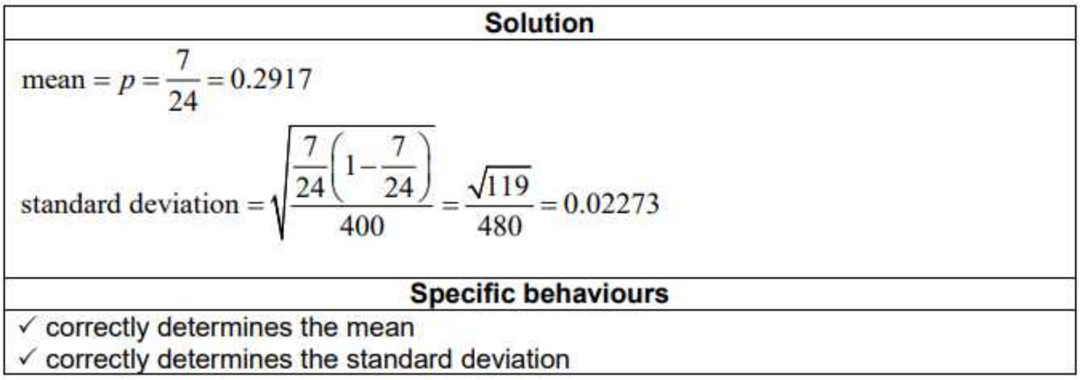
$$\text{Probability of a prize} = \frac{8}{48}+\frac{4}{48}+\frac{2}{48}=\frac{14}{48}=\frac{7}{24}$$

Specific behaviours

✓ shows how to determine at least one probability

✓ correctly shows all three probabilities and shows they sum to $\frac{7}{24}$

(c) Calculate the mean and standard deviation of the sample proportion of times a prize is won when 400 tickets are purchased. (2 marks)



Solution

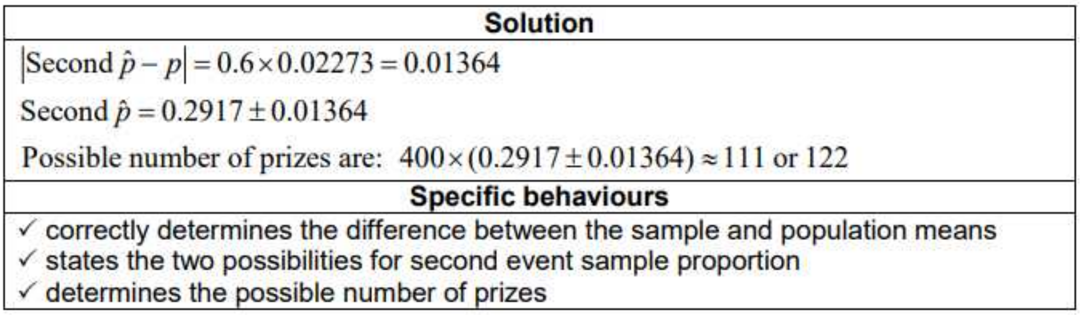
$$\text{mean} = p = \frac{7}{24} = 0.2917$$

$$\text{standard deviation} = \sqrt{\frac{\frac{7}{24}\left(1-\frac{7}{24}\right)}{400}} = \frac{\sqrt{119}}{480} = 0.02273$$

Specific behaviours

✓ correctly determines the mean
✓ correctly determines the standard deviation

(d) At a second event, 400 scratchie tickets are again purchased. If the sample proportion was 0.6 standard deviations from the population proportion, how many prizes were won at the second event? (3 marks)



Solution

$$|\text{Second } \hat{p} - p| = 0.6 \times 0.02273 = 0.01364$$

$$\text{Second } \hat{p} = 0.2917 \pm 0.01364$$

$$\text{Possible number of prizes are: } 400 \times (0.2917 \pm 0.01364) \approx 111 \text{ or } 122$$

Specific behaviours

✓ correctly determines the difference between the sample and population means
✓ states the two possibilities for second event sample proportion
✓ determines the possible number of prizes

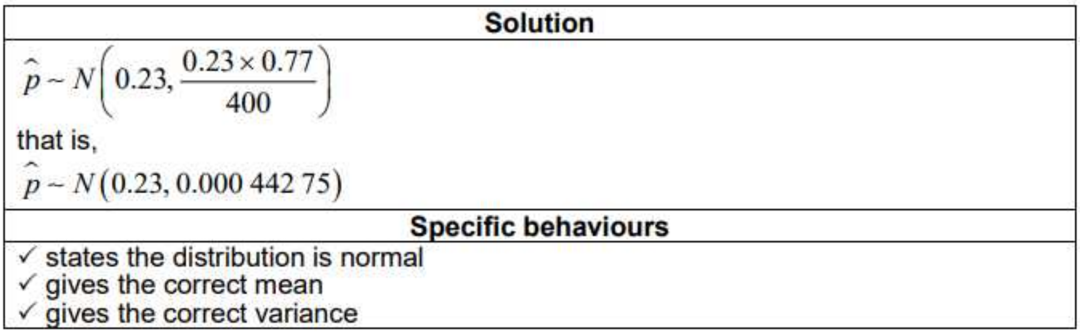
2021
Section 2
Question
11

Interval estimates for proportions

A new political party, the Sustainable Energy Party, is planning to have candidates run in the next election. Researchers have collected data that suggests the proportion of voters likely to vote for the party to be 23%.

One year before the next election, random samples of 400 voters were taken in a particular electorate. Let $\hat{p}$ denote the sample proportion of voters who indicated they would vote for the Sustainable Energy Party at the next election.

(a) State the distribution of $\hat{p}$. (3 marks)



Solution

$$\hat{p} \sim N\left(0.23, \frac{0.23 \times 0.77}{400}\right)$$

that is,

$$\hat{p} \sim N(0.23, 0.000\,442\,75)$$

Specific behaviours

✓ states the distribution is normal
✓ gives the correct mean
✓ gives the correct variance

(b) Calculate the probability that the proportion of voters likely to vote for the Sustainable Energy Party in a sample of 400 is less than 0.20. (3 marks)

Solution
$P\left(\hat{p} < 0.20\right) = 0.07697$
Specific behaviours
✓ writes correct probability statement ✓ uses correct mean and standard deviation ✓ obtains final answer

One week before the election, researchers believed that the proportion of voters likely to vote for the party in that same electorate had increased. A random sample of 200 voters was taken at this time, and 55 of them indicated they would vote for the Sustainable Energy Party at the next election.

(c) Based on this sample, estimate the proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (1 mark)

Solution
$\hat{p} = \frac{55}{200} = 0.275$
Specific behaviours
✓ calculates sample proportion correctly

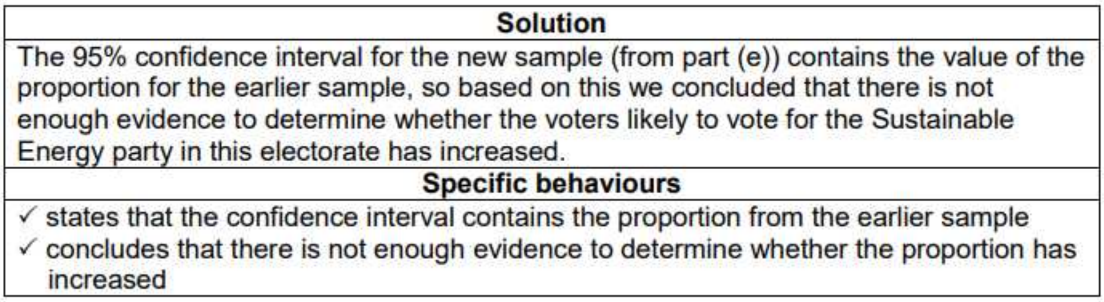
(d) For a 99% confidence interval, what is the margin of error of the sample proportion of voters likely to vote for the Sustainable Energy Party in this electorate, based on this sample? (2 marks)

Solution
$E = 2.576\sqrt{\frac{0.275 \times 0.725}{200}} = 0.08133$
Specific behaviours
✓ substitutes correct values in the formula for margin of error ✓ calculates margin of error correctly

(e) Based on this sample, calculate a 95% confidence interval for the population proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (3 marks)

Solution
$95\%\ \text{CI} = \left(0.275 - 1.96 \times \sqrt{\frac{0.275(0.725)}{200}}, 0.275 + 1.96 \times \sqrt{\frac{0.275(0.725)}{200}}\right)$ $95\%\ \text{CI} = (0.2131, 0.3369)$
Specific behaviours
✓ uses the correct critical value from the normal distribution ✓ substitutes correct values in the expression for the confidence interval ✓ calculates the confidence interval correctly

(f) Based on the research, did the proportion of voters likely to vote for the Sustainable Energy Party in this electorate increase in the year leading up to the election? Justify your answer. (2 marks)



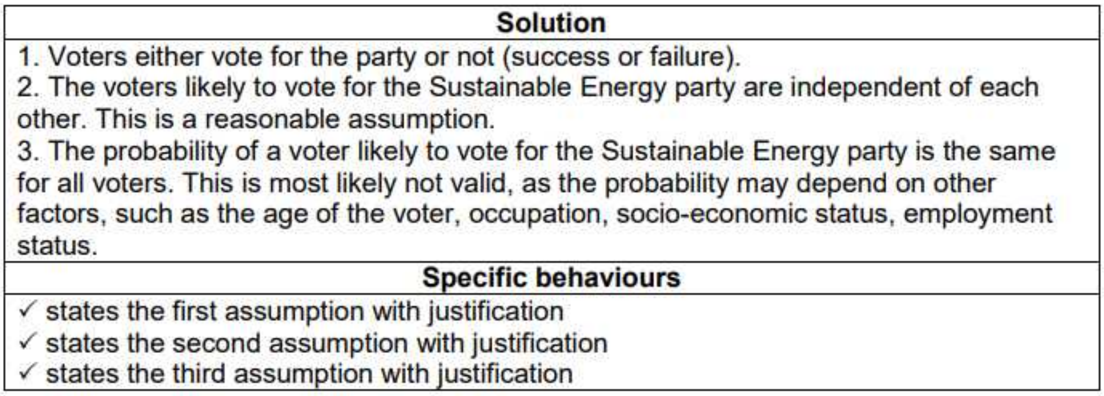
Solution

The 95% confidence interval for the new sample (from part (e)) contains the value of the proportion for the earlier sample, so based on this we concluded that there is not enough evidence to determine whether the voters likely to vote for the Sustainable Energy party in this electorate has increased.

Specific behaviours

✓ states that the confidence interval contains the proportion from the earlier sample
✓ concludes that there is not enough evidence to determine whether the proportion has increased

(g) The analysis above models the number of voters likely to vote for the Sustainable Energy Party as binomially distributed. State and discuss the validity of any assumptions for the binomial distribution in this context. (3 marks)



Solution

1. Voters either vote for the party or not (success or failure).
2. The voters likely to vote for the Sustainable Energy party are independent of each other. This is a reasonable assumption.
3. The probability of a voter likely to vote for the Sustainable Energy party is the same for all voters. This is most likely not valid, as the probability may depend on other factors, such as the age of the voter, occupation, socio-economic status, employment status.

Specific behaviours

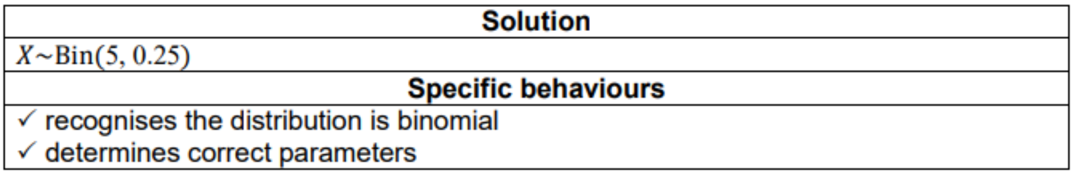
✓ states the first assumption with justification
✓ states the second assumption with justification
✓ states the third assumption with justification

2021
Section 2
Question
13

Interval
estimates
for
proportions

A carnival game involves five buckets, each containing 5 blue balls and 15 red balls. A player blindly selects a ball from each bucket and wins the game if they select at least 4 blue balls. Let X denote the number of blue balls selected.

(a) State the distribution of X, including its parameters. (2 marks)



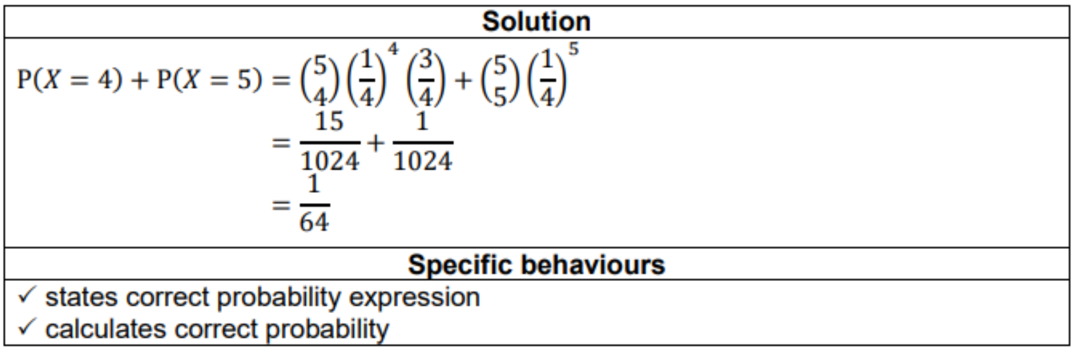
Solution

$X \sim \text{Bin}(5, 0.25)$

Specific behaviours

✓ recognises the distribution is binomial
✓ determines correct parameters

(b) What is the probability of a player winning the game on any given attempt? (2 marks)



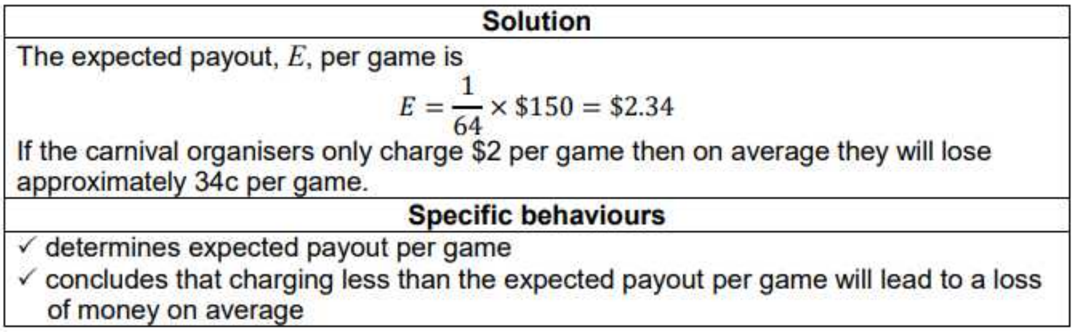
Solution

$$\begin{aligned}\text{P}(X=4)+\text{P}(X=5) &= \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)+\binom{5}{5}\left(\frac{1}{4}\right)^5 \\ &= \frac{15}{1024}+\frac{1}{1024} \\ &= \frac{1}{64}\end{aligned}$$

Specific behaviours

✓ states correct probability expression
✓ calculates correct probability

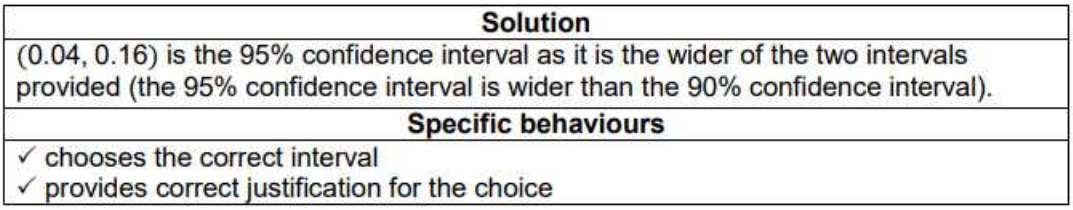
(c) Players are charged \$2 for each attempt at the game and offered a \$150 prize if they win the game. By providing appropriate numerical justification, explain why this is not a good idea for the carnival organisers. (2 marks)



Solution
The expected payout, E, per game is $E = \frac{1}{64} \times \$150 = \$2.34$ If the carnival organisers only charge \$2 per game then on average they will lose approximately 34c per game.
Specific behaviours
✓ determines expected payout per game ✓ concludes that charging less than the expected payout per game will lead to a loss of money on average

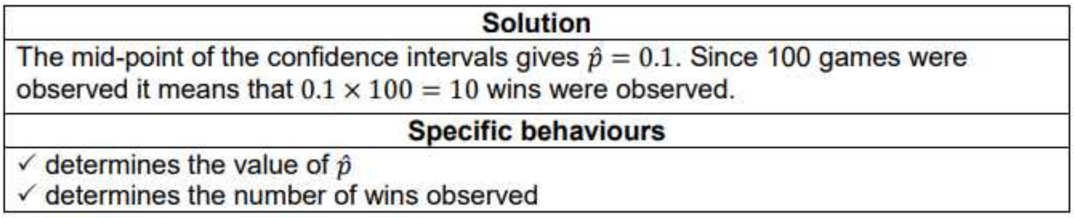
An observer records the outcome of 100 consecutive games and determines the 90% and 95% confidence intervals for the proportion of wins, p. The confidence intervals are (0.04, 0.16) and (0.05, 0.15).

(d) Which of these intervals is the 95% confidence interval for p? Justify your answer. (2 marks)



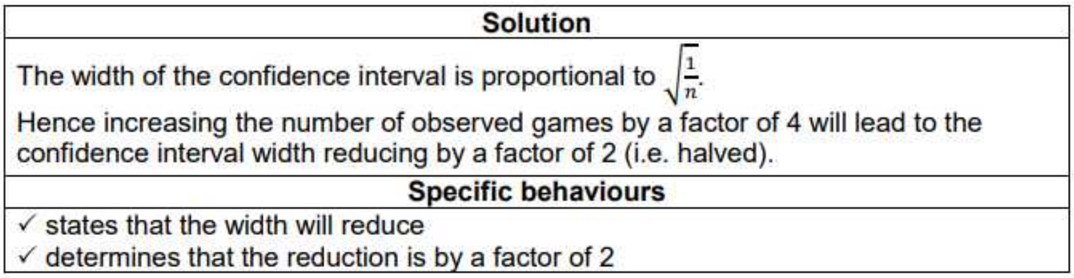
Solution
(0.04, 0.16) is the 95% confidence interval as it is the wider of the two intervals provided (the 95% confidence interval is wider than the 90% confidence interval).
Specific behaviours
✓ chooses the correct interval ✓ provides correct justification for the choice

(e) How many wins were observed out of the 100 games? (2 marks)



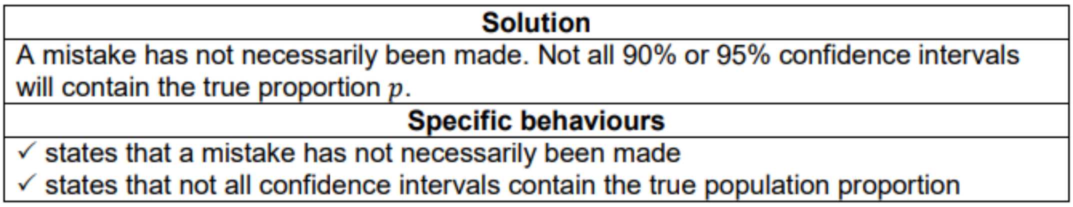
Solution
The mid-point of the confidence intervals gives $\hat{p} = 0.1$. Since 100 games were observed it means that $0.1 \times 100 = 10$ wins were observed.
Specific behaviours
✓ determines the value of $\hat{p}$ ✓ determines the number of wins observed

(f) Determine what you would expect to happen to the width of the confidence intervals if 400 games had been observed. (2 marks)



Solution
The width of the confidence interval is proportional to $\sqrt{\frac{1}{n}}$. Hence increasing the number of observed games by a factor of 4 will lead to the confidence interval width reducing by a factor of 2 (i.e. halved).
Specific behaviours
✓ states that the width will reduce ✓ determines that the reduction is by a factor of 2

(g) The true proportion of wins does not lie within either of the above confidence intervals. Does this suggest that a sampling error was made? Justify your answer. (2 marks)



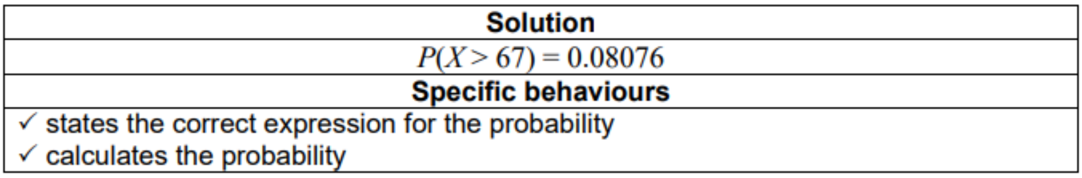
Solution
A mistake has not necessarily been made. Not all 90% or 95% confidence intervals will contain the true proportion p.
Specific behaviours
✓ states that a mistake has not necessarily been made ✓ states that not all confidence intervals contain the true population proportion

2020
Section 2
Question 8

Interval estimates for proportions

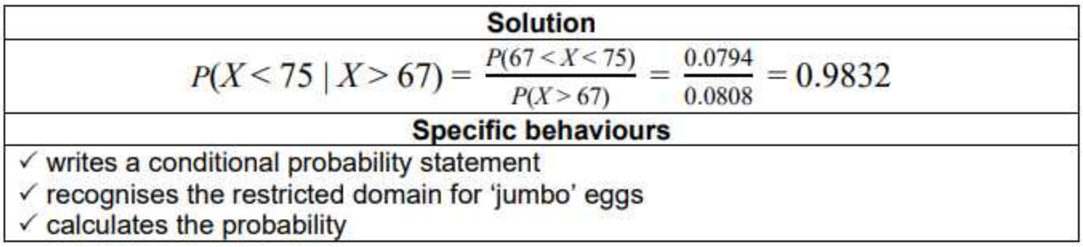
The weight, X, of chicken eggs from a farm is normally distributed with mean 60 g and standard deviation 5 g. Eggs with a weight of more than 67 g are classed as ‘jumbo’.

(a) What proportion of eggs from the farm are ‘jumbo’? (2 marks)



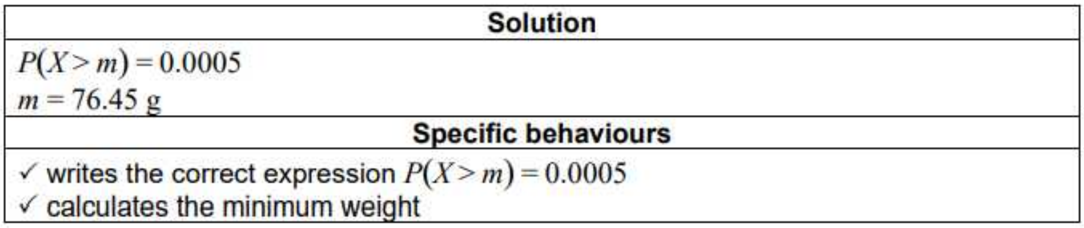
Solution
$P(X > 67) = 0.08076$
Specific behaviours
✓ states the correct expression for the probability ✓ calculates the probability

(b) What proportion of ‘jumbo’ eggs are less than 75 g in weight? (3 marks)



Solution
$P(X < 75 \mid X > 67) = \frac{P(67 < X < 75)}{P(X > 67)} = \frac{0.0794}{0.0808} = 0.9832$
Specific behaviours
✓ writes a conditional probability statement ✓ recognises the restricted domain for ‘jumbo’ eggs ✓ calculates the probability

(c) The heaviest 0.05% of eggs fetch a higher price. What is the minimum weight of these eggs? (2 marks)



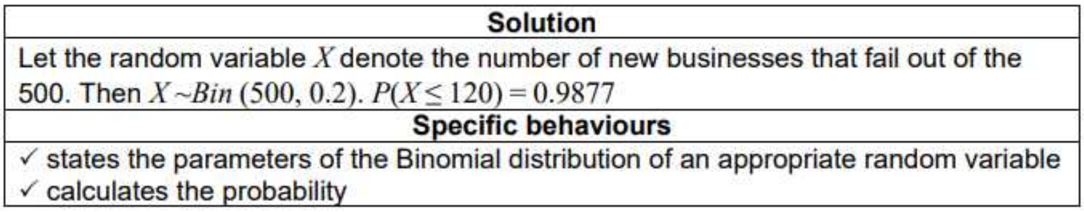
Solution
$P(X > m) = 0.0005$ $m = 76.45$ g
Specific behaviours
✓ writes the correct expression $P(X > m) = 0.0005$ ✓ calculates the minimum weight

2020
Section 2
Question 12

Interval estimates for proportions

It is estimated that 20% of small businesses fail in the first year. A business advisory group takes a random sample of 500 new businesses that started in January 2018. An analyst employed by the group suggests the use of the binomial distribution is appropriate in this case.

(a) What is the probability that at most 120 of the businesses fail in the first year? (2 marks)



Solution
Let the random variable X denote the number of new businesses that fail out of the 500. Then $X \sim Bin\,(500,\ 0.2)$. $P(X \leq 120) = 0.9877$
Specific behaviours
✓ states the parameters of the Binomial distribution of an appropriate random variable ✓ calculates the probability

(b) What is the approximate distribution of the sample proportion of small businesses that fail by the end of the year in this sample? Justify your answer. (3 marks)

Solution

Sample proportion $\hat{p} = N\left(0.2, \frac{0.2 \times 0.8}{500}\right)$,

That is, $\hat{p} = N(0.2, 0.00032)$, as the sample size is large

Specific behaviours

✓ states the distribution is normal as the sample size is large

✓ gives the value of the mean

✓ gives the value of the variance

(c) What is the probability that the sample proportion of businesses that fail by the end of the year is less than 0.18? (2 marks)

Solution

$$P(\hat{p} < 0.18) = 0.1318$$

Specific behaviours

✓ uses the correct value of mean and standard deviation

✓ obtains the correct probability

(d) By January 2019, 90 of the 500 new businesses had failed. Calculate a 95% confidence interval for the proportion of new businesses that fail in the first year. (2 marks)

Solution

Sample proportion $\hat{p} = \frac{90}{500} = 0.18$

95% confidence interval $\left(0.18 - 1.96 \times \sqrt{\frac{0.18 \times 0.82}{500}},\ 0.18 + 1.96 \times \sqrt{\frac{0.18 \times 0.82}{500}}\right)$

$= (0.1463, 0.2136)$

Specific behaviours

✓ calculates the lower bound of the interval correctly

✓ calculates the upper bound of the interval correctly

The business advisory group believes that the proportion of new businesses that fail within a year can be reduced by providing financial advice. They took another random sample of 500 businesses that started in January 2019 and provided them with regular financial advice. In this random sample, at the end of the year 80 businesses had failed.

(e) Calculate the sample proportion and its margin of error at the 95% confidence level. (2 marks)

Solution

Sample proportion $\hat{p} = \frac{80}{500} = 0.16$

$$E = 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}} = 0.0321$$

Specific behaviours

✓ calculates the sample proportion correctly

✓ calculates E correctly

(f) Calculate a 95% confidence interval for the proportion of businesses that failed. What do you conclude regarding the value of the financial advice provided to the new businesses? (4 marks)

Solution

Sample proportion $\hat{p} = \frac{80}{500} = 0.16$

95% confidence interval $\left(0.16 - 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}},\ 0.16 + 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}}\right)$

$= (0.1279, 0.1921)$

Comparing this confidence interval with the previous one $(0.1463, 0.2136)$, we can see that they overlap.
Therefore, it does not appear that the financial advice has reduced the proportion of businesses that fail in the first year.

Specific behaviours

✓ calculates the lower bound of the interval correctly
✓ calculates the upper bound of the interval correctly
✓ conclusion refers to the confidence intervals overlapping
✓ states the correct conclusion

(g) If the sample size was reduced, what would be the effect on the confidence interval? Justify your answer. (2 marks)

Solution

The width of the confidence interval would be increased, as the margin of error of the sample proportion will increase, thus increasing the error.

Specific behaviours

✓ claims the width of the interval would increase
✓ refers to the increase in margin of error **or** increase in error

(h) State **two** assumptions that the analyst made in recommending the use of the binomial model in this case and discuss whether they are valid. (4 marks)

Solution

1. We assume that the businesses fail independently of each other. This is unlikely to be valid because:
   (a) two similar businesses may both fail or both survive
   (b) if two similar businesses in an area then one may dominate and the other fails.
2. We assume that the probability of a business failing is the same for all businesses. This is unlikely to be valid, as businesses of different types are expected to have different probabilities of failing.
3. The probability of failure is fixed (for the year). This is unlikely to be true, and the probability will depend on changing conditions over time.

Specific behaviours

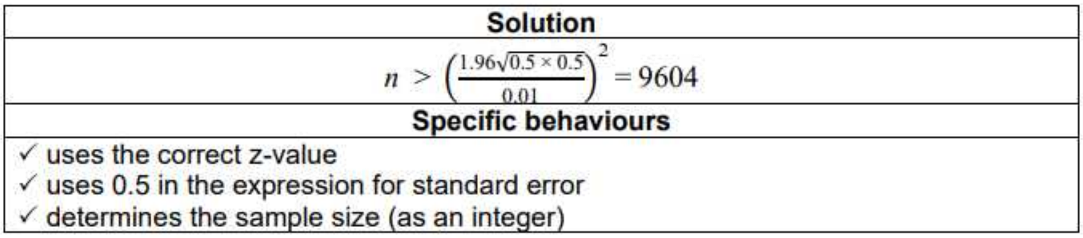
✓ states one assumption
✓ states with reasons that assumption is unlikely to be valid
✓ states second assumption
✓ states with reasons that assumption is unlikely to be valid

2020
Section 2
Question
14

Interval
estimates
for
proportions

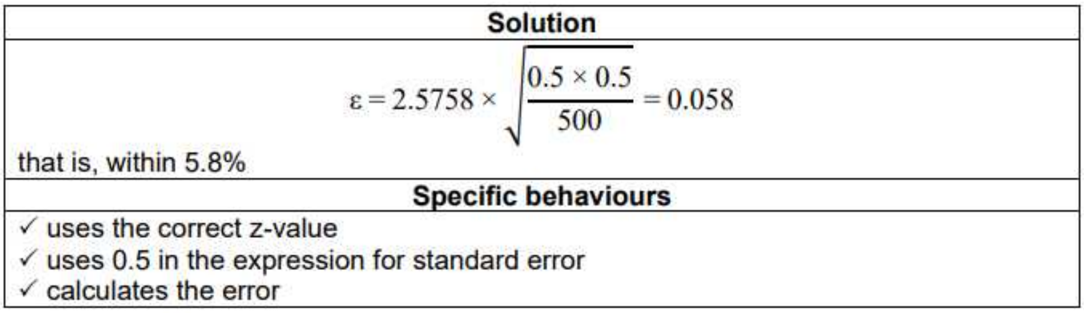
A suburban council hires a consultant to estimate the proportion of residents of the suburb who use its library.

(a) The consultant decides to estimate a 95% confidence interval for the proportion to within an error of 0.01. What minimum sample size should be selected? (3 marks)



Solution
$n > \left(\frac{1.96\sqrt{0.5 \times 0.5}}{0.01}\right)^2 = 9604$
Specific behaviours
✓ uses the correct z-value ✓ uses 0.5 in the expression for standard error ✓ determines the sample size (as an integer)

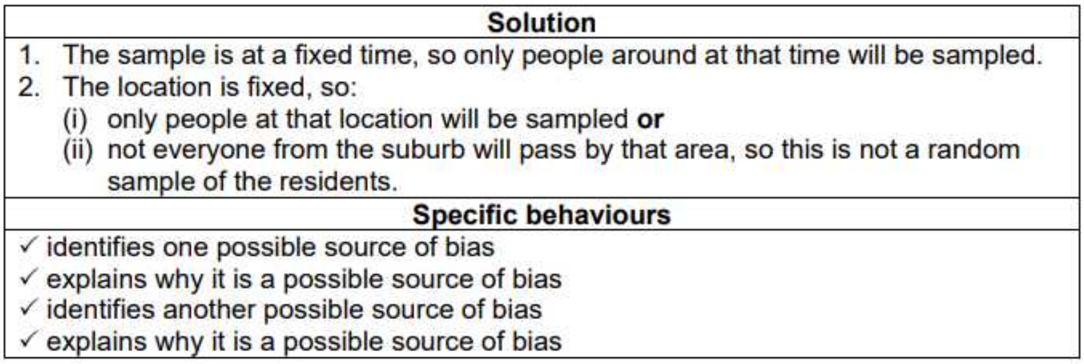
(b) If resource limitations dictate that the maximum sample size that can be managed is 500, what is the maximum margin of error in estimating a 99% confidence interval? (3 marks)



Solution
$\varepsilon = 2.5758 \times \sqrt{\frac{0.5 \times 0.5}{500}} = 0.058$ that is, within 5.8%
Specific behaviours
✓ uses the correct z-value ✓ uses 0.5 in the expression for standard error ✓ calculates the error

The consultant decides to select the sample by standing on the roadside outside the library at lunchtime and asking a random sample of the passers-by whether they use the library.

(c) Identify and explain two possible sources of bias with this sampling scheme. (4 marks)



Solution
1. The sample is at a fixed time, so only people around at that time will be sampled. 2. The location is fixed, so: (i) only people at that location will be sampled **or** (ii) not everyone from the suburb will pass by that area, so this is not a random sample of the residents.
Specific behaviours
✓ identifies one possible source of bias ✓ explains why it is a possible source of bias ✓ identifies another possible source of bias ✓ explains why it is a possible source of bias

2019
Section 2
Question 8

Interval estimates for proportions

Big Foods is a large supermarket company. The manager of Big Foods wants to estimate the proportion of households that do the majority of their grocery shopping in their stores.

A junior staff member at Big Foods conducted a survey of 250 randomly-selected households and found that 56 did the majority of their grocery shopping at a Big Foods store.

(a) (i) Calculate the sample proportion of households who did the majority of their grocery shopping at Big Foods. (1 mark)

Solution
$\hat{p} = \frac{56}{250}$ $= 0.224$
Specific behaviours
✓ calculates correct proportion

(ii) Determine the 95% confidence interval for the proportion of households who do the majority of their grocery shopping at Big Foods. Give your answer to four decimal places. (3 marks)

Solution
$\left(0.224 - 1.96\sqrt{\frac{0.224(1-0.224)}{250}}, 0.224 + 1.96\sqrt{\frac{0.224(1-0.224)}{250}}\right)$ $(0.1723, 0.2757)$
Specific behaviours
✓ uses $Z = 1.96$ ✓ calculates confidence interval ✓ rounds to four decimal places

(iii) What is the margin of error of the 95% confidence interval? Give your answer to four decimal places. (1 mark)

Solution
Either $E = 1.96\sqrt{\frac{0.224(1-0.224)}{250}}$ $= 0.0517$ or $E = \frac{0.2757 - 0.1723}{2}$ $= 0.0517$
Specific behaviours
✓ calculates margin of error

An independent research company conducted a large-scale survey of household supermarket preferences and estimated that the true proportion of households that conduct most of their grocery shopping at Big Foods was 0.17 (assume that this is indeed the true proportion).

(b) With reference to your answer to part (a)(ii), does this result suggest that the junior staff member at Big Foods made a mistake? (2 marks)

Solution
No. Only 95% of 95% confidence intervals are expected to contain the true proportion. It is possible that the survey and calculation by the junior staff member was performed appropriately, but happened to yield one of the 5% of confidence intervals that do not contain the true proportion.
Specific behaviours
✓ answers 'No' with a reference to part (a) ✓ justifies answer by saying that only 95% of intervals are expected to contain the true proportion

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2019
Section 2
Question
11

Interval estimates for proportions

A pizza company runs a marketing campaign based on the delivery times of its pizzas. The company claims that it will deliver a pizza in a radius of 5 km within 30 minutes of ordering or it is free. The manager estimates that the actual time, T, from order to delivery is normally distributed with mean 25 minutes and standard deviation 2 minutes.

(a) What it the probability that a pizza is delivered free? (1 mark)

Solution
$P(T > 30) = P\left(Z > \frac{30-25}{2}\right) = P(Z > 2.5) = 1 - 0.9938 = 0.0062$
Specific behaviours
✓ gives the correct value of the probability

(b) On a busy Saturday evening, a total of 50 pizzas are ordered. What is the probability that more than three are delivered free? (2 marks)

Solution
Let Xdenote the number of pizzas out of 50 that are delivered free. Then $X \sim Bin(50,0.0062)$ $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.9997 = 0.0003$
Specific behaviours
✓ states the distribution of the number of pizzas delivered free ✓ computes the probability correctly

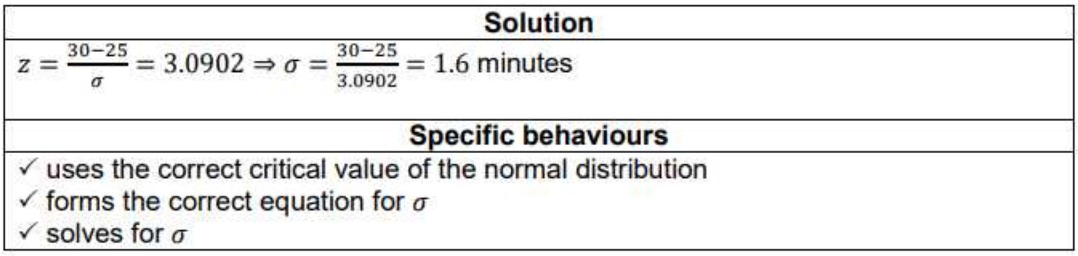
The company wants to reduce the proportion of pizzas that are delivered free to 0.1%.

(c) The manager suggests this can be achieved by increasing the advertised delivery time. What should the advertised delivery time be? (2 marks)

Solution
$P(Z > z) = 0.001 \Rightarrow z = 3.0902 \Rightarrow t = 25 + 3.0902 \times 2 = 31.2$ minutes
Specific behaviours
✓ uses a tail probability of 0.001 ✓ calculates the correct value of the delivery time

After some additional training the company was able to maintain the advertised delivery time as 30 minutes but reduce the proportion of pizzas delivered free to 0.1%.

(d) Assuming that the original mean of 25 minutes is maintained, what is the new standard deviation of delivery times? (3 marks)



Solution
$z = \frac{30-25}{\sigma} = 3.0902 \Rightarrow \sigma = \frac{30-25}{3.0902} = 1.6$ minutes
Specific behaviours
✓ uses the correct critical value of the normal distribution ✓ forms the correct equation for σ ✓ solves for σ

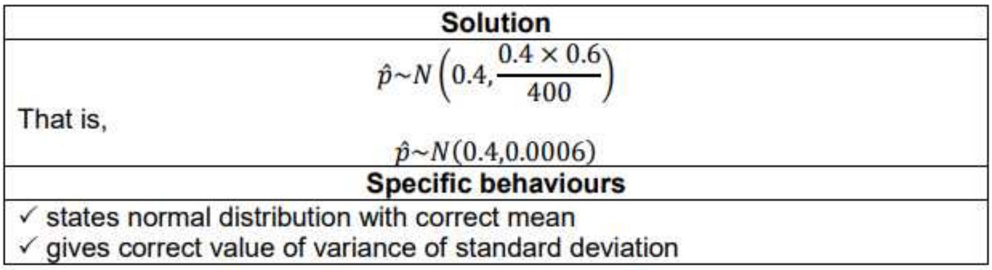
2019
Section 2
Question
13

Interval
estimates
for
proportions

The proportion of working adults who miss breakfast on week days is estimated to be 40%. A study takes a random sample of 400 working adults.

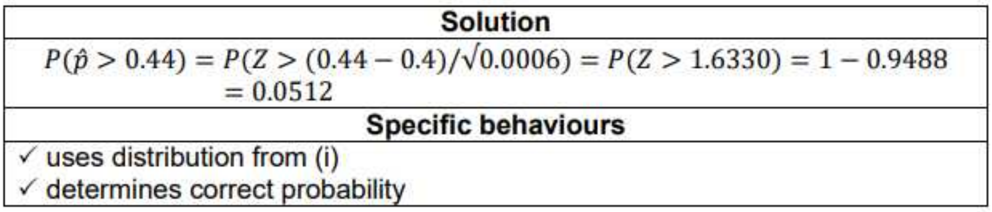
(a) For this sample:

(i) What is the (approximate) distribution of the sample proportion of workers who miss breakfast? (2 marks)



Solution
$\hat{p} \sim N\left(0.4, \frac{0.4 \times 0.6}{400}\right)$ That is, $\hat{p} \sim N(0.4, 0.0006)$
Specific behaviours
✓ states normal distribution with correct mean ✓ gives correct value of variance of standard deviation

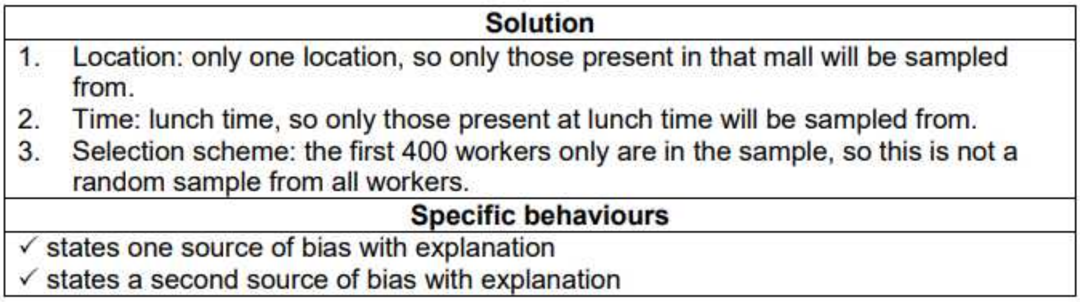
(ii) What is the probability that the sample proportion of workers who miss breakfast is greater than 44%? (2 marks)



Solution
$P(\hat{p} > 0.44) = P(Z > (0.44 - 0.4)/\sqrt{0.0006}) = P(Z > 1.6330) = 1 - 0.9488 = 0.0512$
Specific behaviours
✓ uses distribution from (i) ✓ determines correct probability

Tom takes a random sample of 400 adults. He obtained his sample by selecting the first 400 workers he met in a busy mall in Perth city during lunch time.

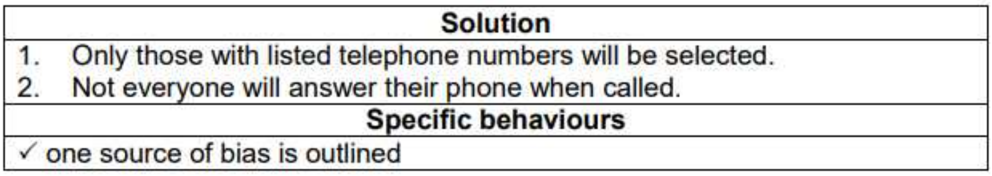
(b) Discuss briefly two possible sources of bias in Tom's sample. (2 marks)



Solution
1. Location: only one location, so only those present in that mall will be sampled from. 2. Time: lunch time, so only those present at lunch time will be sampled from. 3. Selection scheme: the first 400 workers only are in the sample, so this is not a random sample from all workers.
Specific behaviours
✓ states one source of bias with explanation ✓ states a second source of bias with explanation

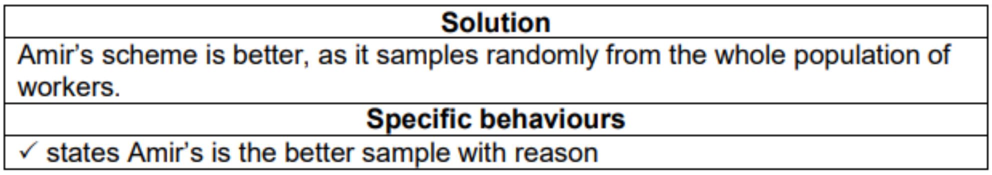
Amir suggests that a better sampling scheme is to obtain a random sample of 400 voters and contact them by telephone.

(c) (i) Outline one source of bias in Amir's sampling scheme. (1 mark)



Solution
1. Only those with listed telephone numbers will be selected. 2. Not everyone will answer their phone when called.
Specific behaviours
✓ one source of bias is outlined

(ii) Which of Tom's or Amir's sampling scheme is better? Provide a reason for your choice. (1 mark)



Solution
Amir's scheme is better, as it samples randomly from the whole population of workers.
Specific behaviours
✓ states Amir's is the better sample with reason

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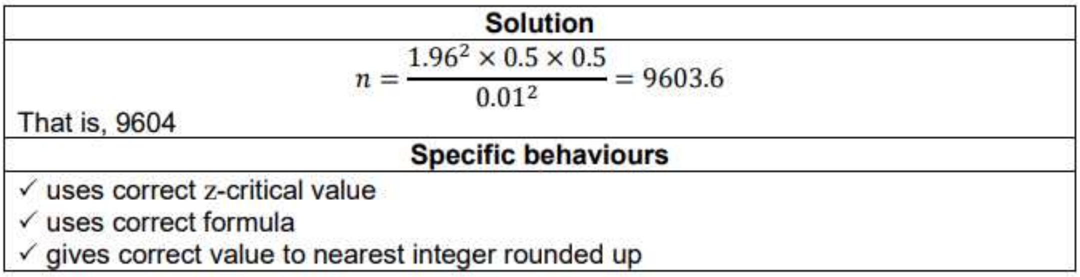
14

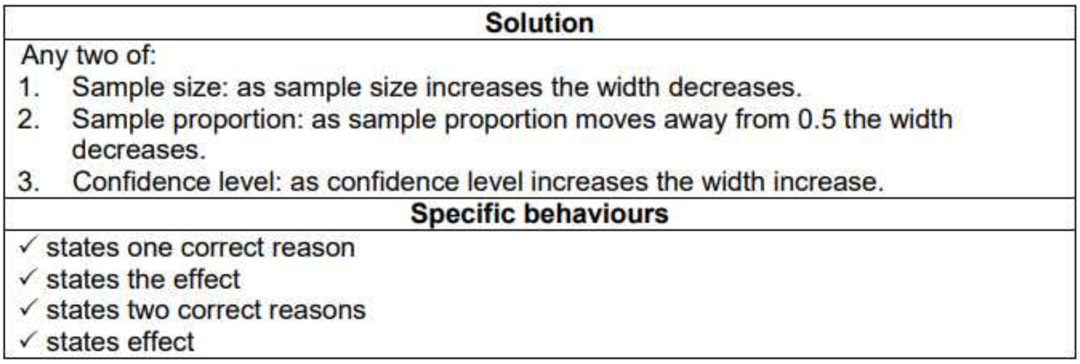
Interval
estimates
for
proportions

(a) What is the minimum sample size required to estimate a population proportion to within 0.01 with 95% confidence. (3 marks)



Solution
$n = \dfrac{1.96^2 \times 0.5 \times 0.5}{0.01^2} = 9603.6$ That is, 9604
Specific behaviours
✓ uses correct z-critical value ✓ uses correct formula ✓ gives correct value to nearest integer rounded up

(b) Identify **two** factors that affect the width of a confidence interval for a population proportion and describe the effect of each. (4 marks)



Solution
Any two of: 1. Sample size: as sample size increases the width decreases. 2. Sample proportion: as sample proportion moves away from 0.5 the width decreases. 3. Confidence level: as confidence level increases the width increase.
Specific behaviours
✓ states one correct reason ✓ states the effect ✓ states two correct reasons ✓ states effect