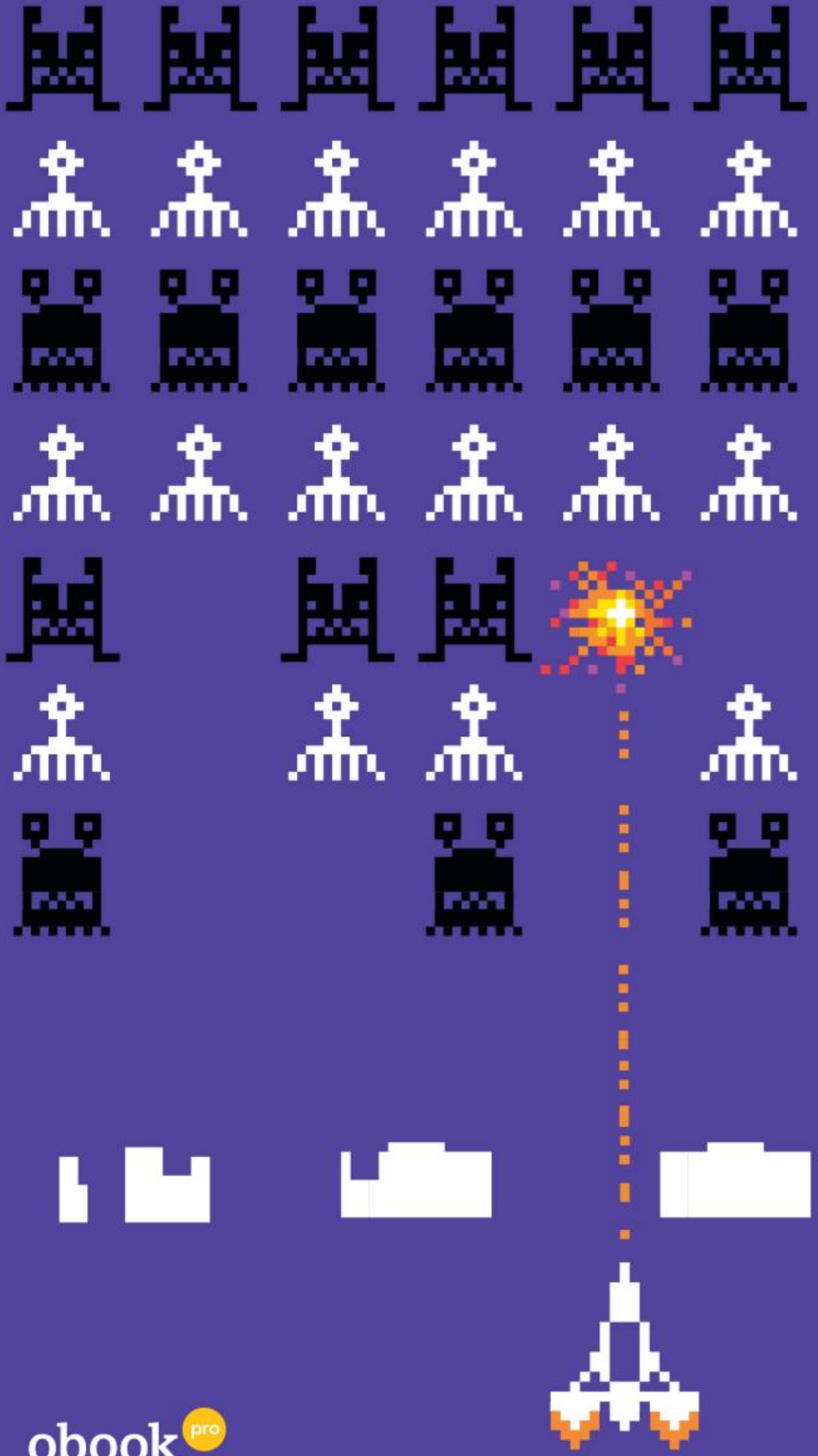
MATHEMATICS

OXFORD MATHS 10

STAGE 5



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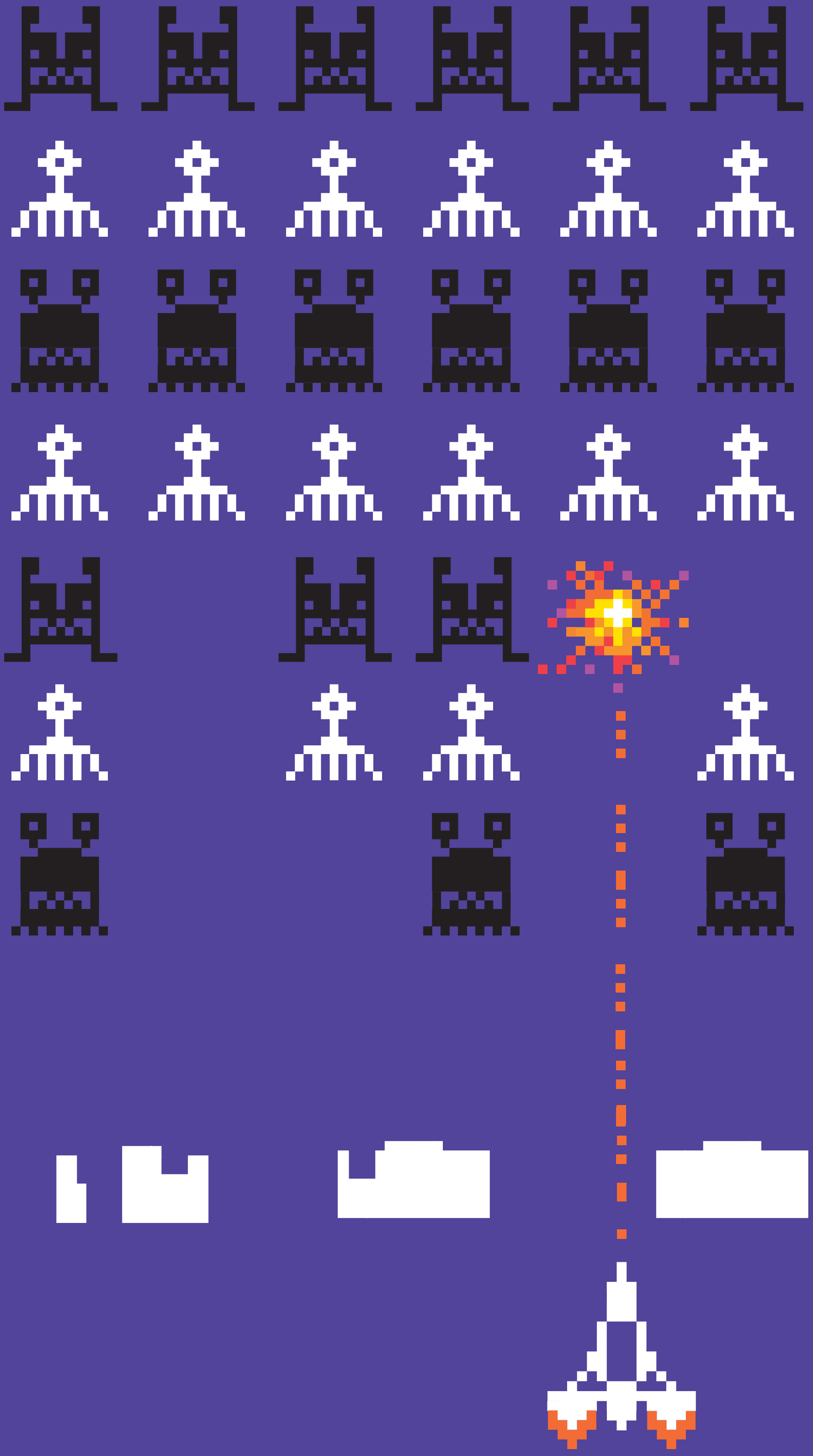

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Warning to First Nations Australians
Aboriginal and Torres Strait Islander peoples are advised that this publication may include images or names of people now deceased.

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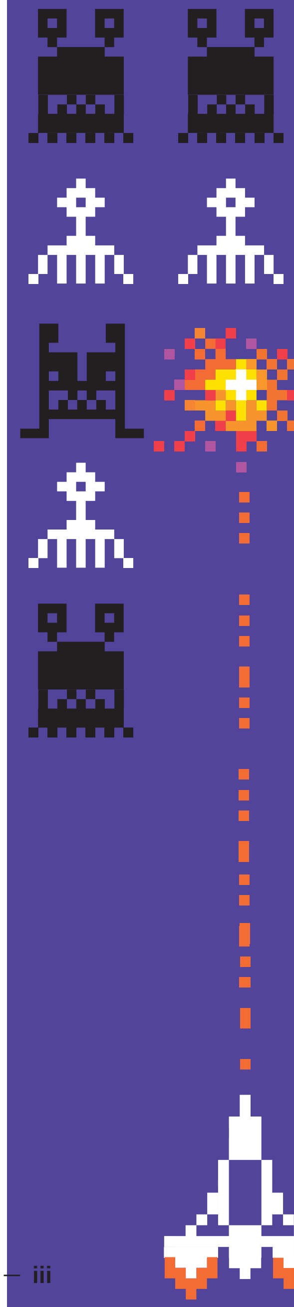

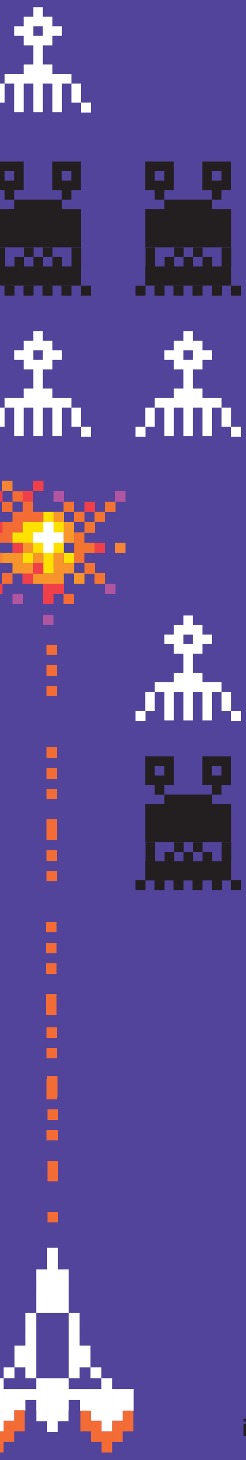

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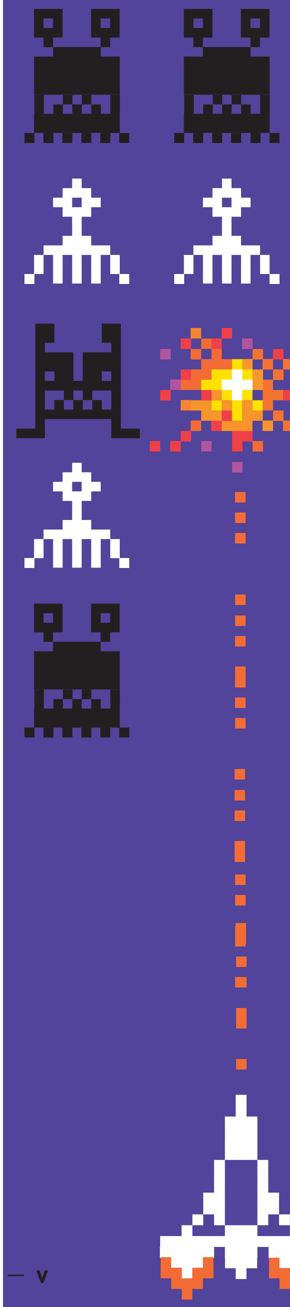
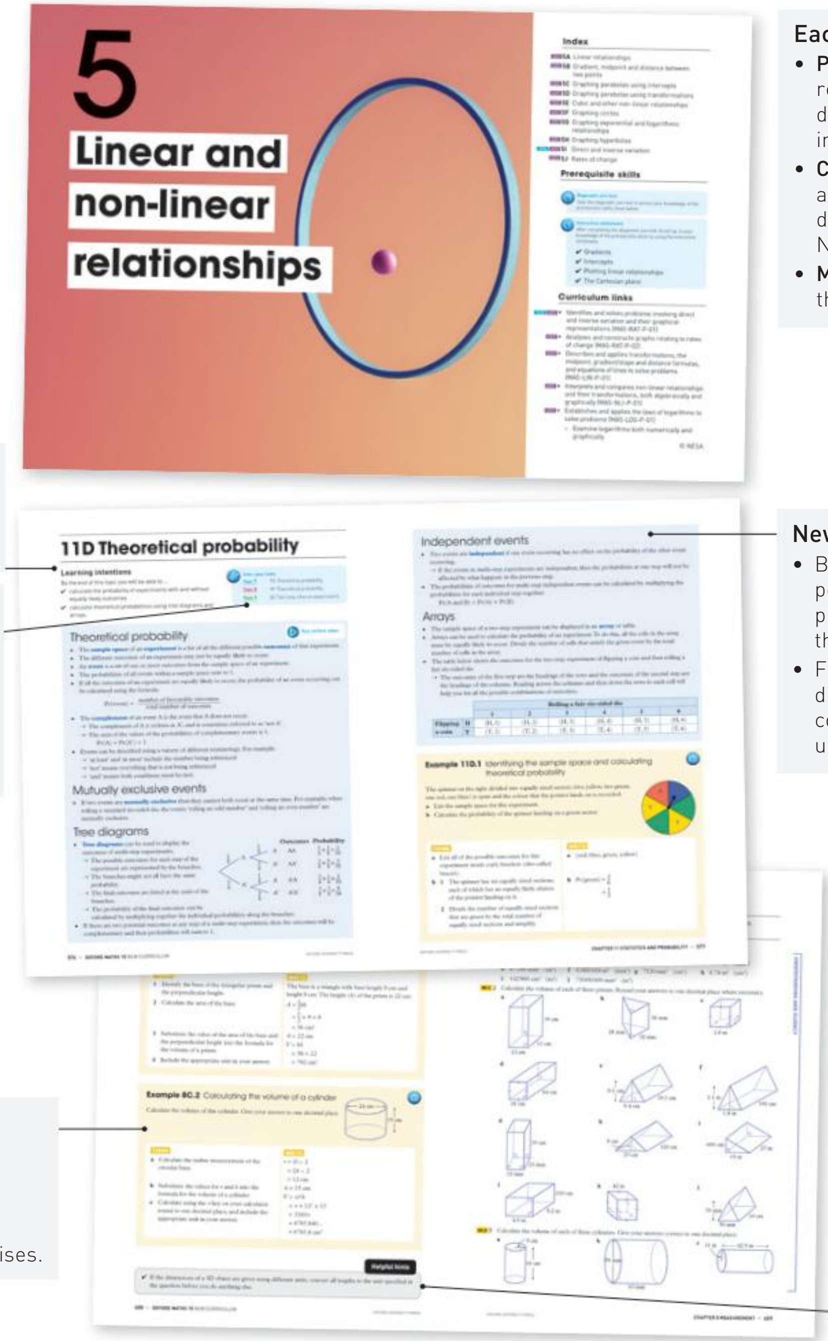


Introducing Oxford Maths 7–10

Oxford Maths 7–10 NSW Curriculum utilises an innovative suite of print and digital resources to guide students on a focused mathematics journey. The series makes maths accessible to students with differing levels of understanding, increasing engagement by giving learners the opportunity to achieve success at their own skill level while also providing comprehensive syllabus coverage.

Key features of Student Books

- Complete access to all digital resources available on Student obook pro.
- Australian Maths Trust (AMT) spreads offer unique questions designed to challenge students and build engagement.
- STEAM projects encourage inter-disciplinary thinking.
- Semester reviews provide an opportunity to revise key concepts from each semester.
- An illustrated glossary of newly introduced mathematical terms.



Each chapter opens with:

- **Prerequisite skills** with reference to an online diagnostic pre-test and interactive skillsheets.
- **Curriculum links** to all relevant content descriptions in the NSW Curriculum.
- **Materials** used to complete the exercises.

Learning intentions

- Signpost the foundational skills being developed in each section.

Inter-year links

- Provide easy access to support and extension material from each of the 7–10 Student Books as students build knowledge year on year.

New theory

- Backed by the latest pedagogical research to promote engagement with the material.
- Filled with precise diagrams that bring key concepts to life, and aid understanding.

Worked examples

- Outline a step-by-step thought process for solving essential questions with direct reference to the exercises.

Helpful hints

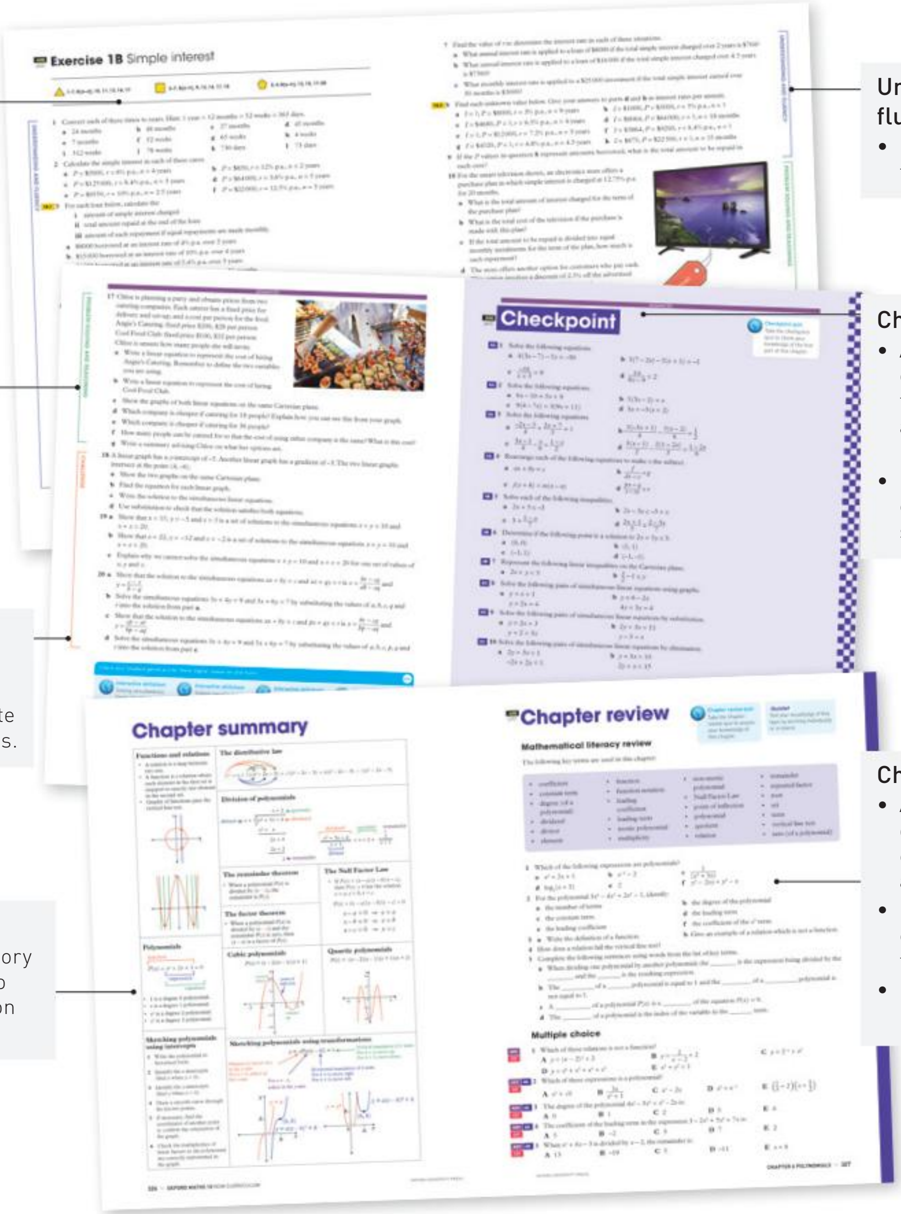
- Provide additional strategies for tackling problems.
- Highlight important elements of the theory.
- Point out common misconceptions.

Differentiated learning pathways

- Each exercise is separated into three pathways, tailoring for students of all skill levels.
- Each pathway can be assigned based on results of the diagnostic pre-tests that are recommended at the beginning of every chapter.

Understanding and fluency

- Basic exercises dedicated to practising key concepts.



Exercise 1B Simple interest

Checkpoint

Chapter summary

Chapter review

Mathematical literacy review

Multiple choice

Problem solving and reasoning

- Comprehensive exercises bring together new ideas and provide engaging contexts from real-world problems.

Checkpoint

- A section in the middle of each chapter dedicated to summarising key skills and encouraging memory retention.
- Reference to an online checkpoint quiz to gauge student progress.

Challenge

- Advanced exercises designed to build engagment and anticipate future learning outcomes.

Chapter review

- Additional practice questions to further consolidate understanding at the end of each chapter.
- Reference to an online chapter review quiz to track results.
- Reference to Quizlet test to revise new terminology.

Chapter summary

- Condenses all the theory from each section into one accessible revision page.

[STEAM project 1]

How can we use technology so that we improve the lives of people in the world's poorest nations?

Your task

The digital divide

HSIE

MATHS

SCIENCE

Integrated STEAM projects

- Take the hard work out of cross-curricular learning with engaging STEAM projects. Two fully integrated projects are included at the end of each book in the series, and are scaffolded and mapped to the Science, Maths and HSIE curricula.

Problem solving through design thinking

- Each STEAM project investigates a real-world problem that students are encouraged to problem-solve using design thinking.

Full digital support

- Each STEAM project is supported by a wealth of digital resources, including student booklets (to scaffold students through the design-thinking process of each project), videos to support key concepts and skills, and implementation and assessment advice for teachers.

Key features of Student obook pro

obook pro

> Student obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.

> It offers a complete digital version of the Student Book with interactive note-taking, highlighting and bookmarking functionality, allowing students to revisit points of learning.

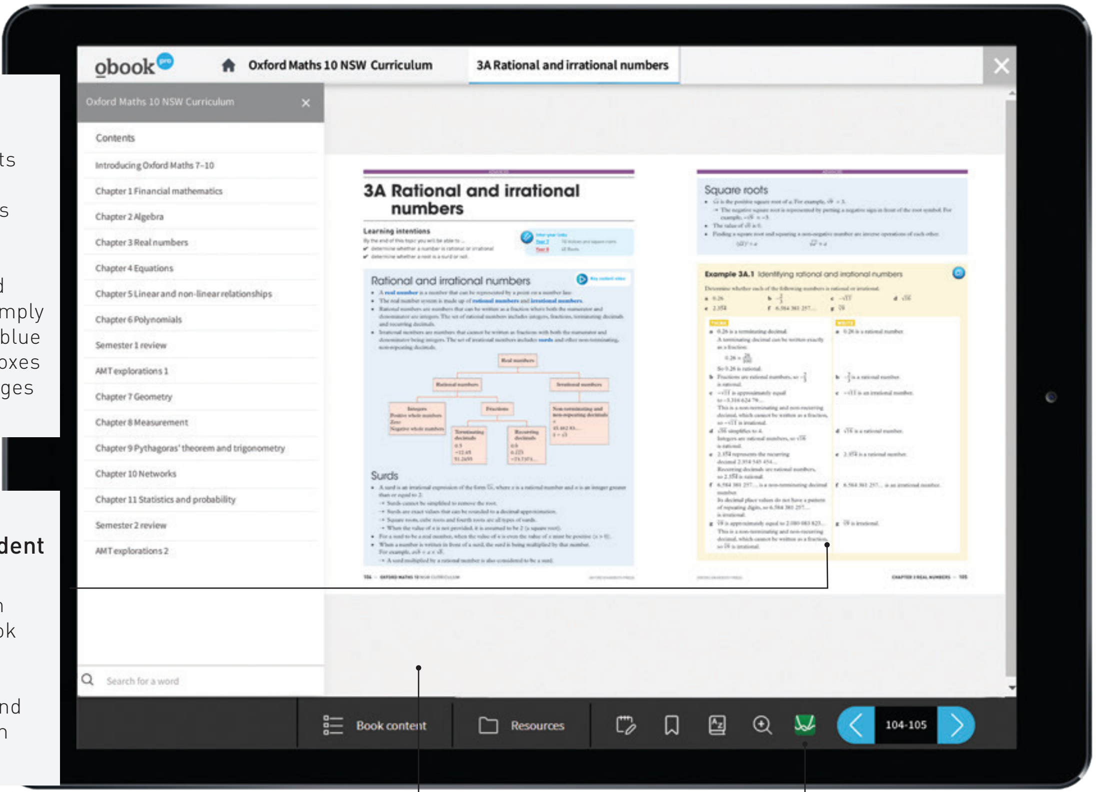
> A complete ePDF of the Student Book is also available for download for offline use and read-aloud functionality.

Integrated digital resources

- Integrated hotspots allow students to access diagnostic tests, quizzes, interactive skill sheets, videos and inter-year links simply by clicking on the blue digital resource boxes throughout the pages of the book.

Complete digital version of the Student Book

- The digital version of the Student Book is true to the print version, making it easy to navigate and transition between print and digital.



Toolbar features

- Notes can be added and saved to the text by simply selecting and highlighting.
- Bookmarks can be saved to any page.
- *Australian Concise Oxford Dictionary* can provide immediate definitions to any word within the text.

Desmos integration

- Our partership with Desmos allows students to access a suite of calculator tools as they read through the text, providing convenient graphical support as well as the opportunity to investigate plane geometry and Cartesian coordinates.

Benefits for students

> Integrated *Australian Concise Oxford Dictionary* look-up feature

> Targeted instructional videos for every worked example question

> Groundwork resources to support assumed knowledge

> Interactive assessments to consolidate understanding

> Auto-marked practice exam question sets

> Integrated Quizlet sets, including real-time online quizzes with live leaderboards

> Access to online assessment results to track progress.

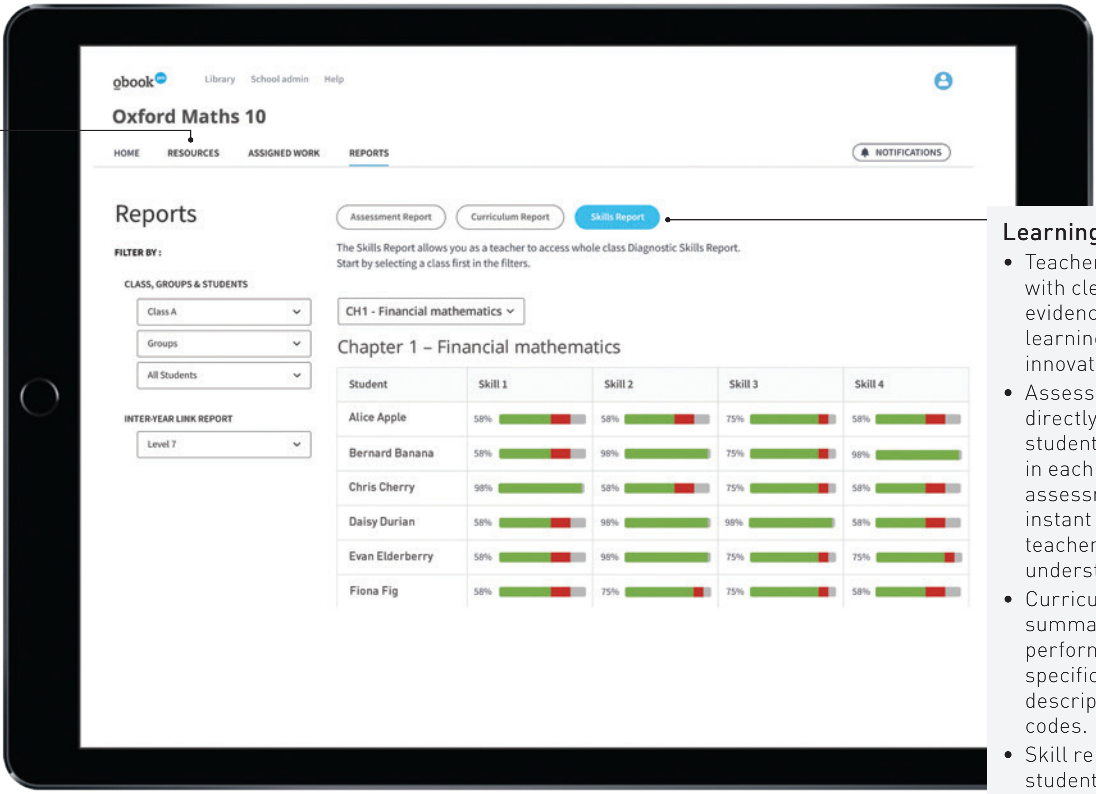
Key features of Teacher obook pro

obook pro

> Teacher obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.

> Each chapter and topic of the Student Book is accompanied by full teaching support, including assessment reporting, worked solutions, chapter tests, detailed teacher notes and lesson plans.

> Teachers can use their Teacher obook pro to share notes and easily assign resources or assessments to students, including due dates and email notifications.



The Skills Report allows you as a teacher to access whole class Diagnostic Skills Report. Start by selecting a class first in the filters.

Chapter 1 – Financial mathematics

Student	Skill 1	Skill 2	Skill 3	Skill 4
Alice Apple	58%	58%	75%	58%
Bernard Banana	58%	98%	75%	98%
Chris Cherry	98%	58%	75%	58%
Daisy Durian	58%	98%	98%	58%
Evan Elderberry	58%	98%	75%	75%
Fiona Fig	58%	75%	75%	58%

Learning pathway reports

- Teachers are provided with clear and tangible evidence of student learning progress through innovative reports.
- Assessment reports directly show how students are performing in each online interactive assessment, providing instant feedback for teachers about areas of understanding.
- Curriculum reports summarise student performance against specific curriculum content descriptors and curriculum codes.
- Skill reports indicate the students' understanding of a specific skill in mathematics.

Additional resources

- Each chapter of the Student Book is accompanied by additional interactive skillsheets, worksheets, investigations and topic quizzes to help students progress.

Benefits for teachers

> Diagnostic pre-tests and chapter tests that track students' progress against Study Design key knowledge, providing detailed learning pathway reports that differentiate each student's ability in each skill

> Assign reading and assessments to students either individually, or in groups – administration is taken care of!

> Ability to set-up classes, monitor student progress and graph results

> Worked solutions for every Student Book question

> Detailed teacher notes, teaching programs and lesson plans.

1

Financial mathematics

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Fractions, decimals and percentages
✔ Percentages
✔ Multiplying and dividing decimals

Curriculum links

- Solves financial problems involving simple interest, earning money and spending money (MA5-FIN-C-01)
  - Solve problems involving earning money
  - Solve problems involving simple interest
- Solves financial problems involving compound interest and depreciation (MA5-FIN-C-02)
  - Solve problems involving compound interest and depreciation

Materials

✔ Calculator

1A Earning money

Learning intentions

By the end of this topic you will be able to ...

- ✔ calculate earnings for different periods of time
- ✔ calculate an individual's taxable income
- ✔ calculate an individual's net income.

Inter-year links

Year 8	3C Mark-ups and discounts
Year 9	1A Wages and salaries

Key content video

Wages and salaries

- **Income** is money received in exchange for work or investment.
- A **wage** is a fixed rate of income for a period of time, for example $30/hour.
- **Penalty rates** are higher rates of pay given for working overtime, on weekends or on public holidays.
  - → Time-and-a-half means 1.5 × standard wage.
  - → Double time means 2 × standard wage.
- **Leave loading** is an extra payment on top of annual leave pay. Leave loading is usually calculated at 17.5% of normal pay for the period of annual leave.
- A **salary** is a fixed income paid on a regular basis.
  - → When calculating weekly or fortnightly salaries, assume there are 52 weeks and 26 fortnights in a year.
- A **bonus** can be paid as an incentive or to reward good performance. Bonuses are typically given as a percentage of an employee's salary.

Commissions, piece work and royalties

- **Piece work** is income which is paid for each item (or groups of items) produced or action performed.
- A **commission** is a proportion of the value of an employee's sales that is paid to them.
- A **royalty** is a payment made to a person or company for the use of their assets.

Income tax and the Medicare levy

- **Tax deductions** are expenses that can be subtracted from an individual's total income to determine their **taxable income**.
  - → Tax deductions include work-related expenses, gifts and donations, the cost of managing tax affairs, and self-education expenses.
- **Income tax** is tax which is paid on an individual's taxable income.
- The table on the right details the income tax rates for the 2023–2024 financial year:

Taxable income	Tax on this income
0–$18 200	Nil
$18 201–$45 000	19 cents for each $1 over $18 200
$45 001–$120 000	$5092 plus 32.5 cents for each $1 over $45 000
$120 001–$180 000	$29 467 plus 37 cents for each $1 over $120 000
$180 001 and over	$51 667 plus 45 cents for each $1 over $180 000

- The **Medicare levy** is a 2% tax on taxable income.
  - → Some individuals are exempt from paying the Medicare levy.
- An individual's net income can be calculated by subtracting their income tax and Medicare levy from their taxable income.

net income = taxable income – income tax – Medicare levy

Example 1A.1 Calculating earnings for different time intervals

Ryu is paid a standard rate of $34.40 per hour. Each week he works 26 hours on weekdays and 8 hours on weekends (for which he is paid time-and-a-half).

a Calculate Ryu's weekly earnings.

b Calculate Ryu's yearly earnings.

c Calculate Ryu's monthly earnings, giving your answer to the nearest cent.

THINK	WRITE
a 1 Multiply the standard pay rate by the penalty rate (1.5) to determine the weekend rate.	**a** weekend rate = $34.40 × 1.5 = $51.60
2 Calculate the weekday earnings by multiplying the standard rate by the hours worked.	weekday earnings = $34.40 × 26 = $894.40
3 Calculate the weekend earnings by multiplying the weekend rate by the hours worked.	weekend earnings = $51.60 × 8 = $412.80
4 Add the weekday earnings to the weekend earnings to get the weekly earnings.	weekly earnings = $894.40 + $412.80 = $1307.20
b Multiply the weekly earnings by 52 to calculate the yearly earnings.	**b** yearly earnings = $1307.20 × 52 = $67 974.40
c 1 Divide the yearly earnings by 12 to calculate the monthly earnings.	**c** monthly earnings = $67 974.40 ÷ 12 = $5644.533...
2 Round the answer to the nearest cent.	≈ $5644.53

Example 1A.2 Calculating taxable income

In the last financial year, Kaitlyn earned a salary of $8600 per month with a 3% bonus. Kaitlyn also earned $2351 in interest on her investments. Her tax deductions for the year were $720 of work-related expenses, $550 in donations to charity and $1500 for self-education expenses. Calculate Kaitlyn's taxable income.

THINK	WRITE
1 Calculate the annual salary by multiplying the monthly salary by 12.	annual salary = $8600 × 12 = $103 200
2 Calculate the bonus figure by multiplying the annual salary by 3%.	bonus = $103 200 × 3% = $103 200 × 0.03 = $3096
3 Calculate the total income by adding the salary, bonus and interest together.	total income = $103 200 + $3096 + $2351 = $108 647
4 Calculate the total deductions by adding the work-related expenses, charity donations and self-education expenses together.	total deductions = $720 + $550 + $1500 = $2770
5 Subtract the tax deductions from the total income to determine the taxable income.	taxable income = $108 647 – $2770 = $105 877

Example 1A.3 Calculating net income

In the 2023–2024 financial year, Khan earned $62 900 and had tax deductions of $1050.

a What is Khan's taxable income?

b How much income tax must Khan pay?

c How much does Khan have to pay for the Medicare levy?

d What is Khan's net income?

THINK	**WRITE**
a Subtract the tax deductions from the total income to determine the taxable income.	**a** $\text{taxable income} = \$62\,900 - \$1050$ $= \$61\,850$
b **1** Identify the row in the tax table into which the taxable income falls. **2** To calculate '$5092 plus 32.5 cents for each $1 over $45 000', start by calculating the income over $45 000. Then, multiply this figure by 0.325. Finally, add the result to $5092.	**b** $\$61\,850 - \$45\,000 = \$16\,850$ $\text{income tax} = \$5092 + \$16\,850 \times 0.325$ $= \$5092 + \$5476.25$ $= \$10\,568.25$
c Multiply the taxable income by 2% to determine the Medicare levy.	**c** $\text{Medicare levy} = \$61\,850 \times 2\%$ $= \$61\,850 \times 0.02$ $= \$1237$
d Subtract the income tax and the Medicare levy from the taxable income to determine the net income.	**d** $\text{net income} = \$61\,850 - \$10\,568.25 - \$1237$ $= \$50\,044.75$

Helpful hints

✔ Monthly income should be calculated by dividing the annual income by 12. Do not multiply the weekly income by 4 to calculate the monthly income.

ANS p644

Exercise 1A Earning money

△ 1–6, 8, 9(a–c), 10, 12, 15, 16 ■ 2–5, 7, 8, 9(d–f), 10, 13, 15, 17, 19 ⬠ 4, 7, 9(d–f), 10, 11, 14–20

1 For each of the following standard rates of pay, calculate the:

i double time rate of pay

ii time-and-a-half rate of pay.

a $38 **b** $29 **c** $31.60 **d** $19.80

e $42.10 **f** $36.30 **g** $25.50 **h** $28.90

1A.1 2 Svetlana is paid a standard rate of $27.70 per hour. Each week she works 30 hours on weekdays and 6 hours on weekends (for which she is paid time-and-a-half).

a Calculate Svetlana's weekly earnings.

b Calculate Svetlana's yearly earnings.

c Calculate Svetlana's monthly earnings.

3 Convert each of the following wages into an annual salary.

a $1145 per week **b** $1702 per week **c** $1999 per fortnight

d $3174 per fortnight **e** $5464 per month **f** $8212 per month

4 **a** Ji receives an annual salary of $90 896. What is his weekly wage?

b Min receives an annual salary of $57 590. What is her fortnightly wage?

c Bai receives an annual salary of $125 064. What is her monthly wage?

5 Calculate the total monthly income for the following workers who earn a base salary plus a commission.

a Shane receives a base salary of $800/month plus a 2.5% commission on sales. In June, he sells $106 000 worth of product.

b Mila receives a base salary of $450/month plus a 1.5% commission on sales. In August, she sells $620 000 in property.

c Ki-woo receives a base salary of $1100/month plus a 5% commission on sales. In December, she sells $73 400 of product.

6 Calculate the income for the following piece work tasks.

a Cleaning 32 cars at a rate of $11.20 per car

b Editing 45 pages of manuscript at a rate of $13.50/page

c Writing subtitles for a 28-minute video at a rate of $8.90/minute

7 Drew takes 4 weeks of annual leave, for which he receives additional leave loading of 17.5%. If Drew works 34 hours per week and receives a wage of $37.75/hour, how much income does he receive for his 4 weeks of annual leave?

1A.2 8 In the last financial year, Sara earned a salary of $2250 per fortnight with a 5% bonus. Sara also earned $1122 in interest on her investments. Her tax deductions for the year were $250 of work-related expenses, $330 in donations to charity and $100 for an accountant to manage her tax affairs. Calculate Sara's taxable income.

9 Use the 2023–2024 tax table to calculate the annual income tax payable, to the nearest dollar, on the following salaries.

a $72 200 per year **b** $49 950 per year **c** $6880 per month

d $4764 per month **e** $1332 per week **f** $3110 per week

1A.3 10 In the 2023–2024 financial year, Jude earned $85 335 and had tax deductions of $1429.

a What is Jude's taxable income?

b How much income tax, to the nearest dollar, must Jude pay?

c How much, to the nearest dollar, does Jude have to pay for the Medicare levy?

d What is Jude's net income?

11 In the 2023–2024 financial year, Kannika earned $98 920 and had tax deductions of $775. What is Kannika's net income if she is not exempt from paying the Medicare levy? Give your answer correct to the nearest dollar.

12 Rose earned a salary of $5380 per month. Her sister stated that this meant her weekly salary was $1345, as 5380 ÷ 4 = 1385. Is her sister correct? If not, what is Rose's weekly salary correct to the nearest dollar?

13 In each case, which salary earns more on an annual basis? Show workings to justify your answers.

a $77 230/year vs $6635/month

b $103 448/year vs $3995/fortnight

c $6540/month vs $1645/week

14 Panya receives a monthly wage of $4849.

a What is Panya's weekly wage?

b Panya receives a leave loading of 17.5% for her four weeks of annual leave each year. What is Panya's total annual income from her job? Give your answer correct to the nearest dollar.

c Last year Panya had an additional income of $7500, as well as tax deductions of $2400. What was Panya's net income for the year if she also has to pay the Medicare levy?

15 Kaia earned an income of $3248.64 for 3 weeks of annual leave, which included an additional leave loading payment of 17.5% of her wage for this period. If she works 36 hours per week, what is her wage per hour?

16 Gem works as a real-estate agent. She receives a base salary of $500 per month plus a 1.8% commission on sales up to $300 000. If her sales exceed $300 000, she receives an additional 2.5% commission on sales above this amount.

If Gem earns $9000 in a month, what were her sales for the month?

17 Tyreek receives a base salary of $1800 plus a 5.2% commission, while Tyla receives a base salary of $1200 plus a 6% commission.

a If they both make $42 000 worth of sales in a month, who earns more money?

b If they both make $54 000 worth of sales in a month, who earns more money?

c At what values of sales will Tyreek and Tyla earn the same amount?

18 In September Fredrick earned $5702 from his job. He received a base salary of $2150 and made $48 000 worth of sales, for which he earned a commission. What % commission does Fredrick receive on his sales?

19 Elise earns $1792 each week, of which $435 is withheld for PAYG income tax. Elise receives an additional 17.5% leave loading for her 4 weeks of annual leave. In the past financial year Elise earned $430 in interest on her savings and had $780 of expenses, which were tax deductible.

a Calculate the amount of income tax that was withheld using the PAYG method.

b Calculate the amount of income tax, to the nearest dollar, that Elise is due to pay for the financial year.

c State whether Elise receives a tax refund or has to pay more tax, and by how much. Give your answer to the nearest dollar.

20 For his new role as a salesperson, Bo is given a choice of three different options for his salary.

- Option 1: Base salary of $2000/month plus a 4.5% commission on all sales
- Option 2: Base salary of $1600/month plus a 5% commission on all sales
- Option 3: 6.5% commission on sales up to $50 000, plus a 6% commission on sales thereafter

Identify the level of monthly sales for which each option will be the most profitable.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Wages and salaries	Interactive skillsheet Commission, piece work and royalties	Interactive skillsheet Income tax	Topic quiz 1A

1B Simple interest

Learning intentions

By the end of this topic you will be able to ...

- ✔ calculate simple interest
- ✔ calculate the principal, interest rate or number of time periods of a simple interest investment or loan.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	1E Simple interest

Calculating simple interest

Key content video

- **Interest** is a sum of money added to an initial amount that was either borrowed or invested.
  - → If the initial sum of money was borrowed then it is a **loan** and the borrower is charged interest. The borrower must repay the interest as well as the initial amount borrowed.
  - → If the initial money was invested then it is an **investment** and the investor is paid the interest.
- **Simple interest** can be calculated using the formula:

$$\underbrace{I}_{\text{interest}} = \underbrace{P}_{\text{principal}}\underbrace{r}_{\text{interest rate}}\underbrace{n}_{\text{number of time periods}}$$

  - → I = interest, the amount a borrower is charged by a lender, or the amount that an investor is paid for their investment
  - → P = **principal**, the initial amount of money borrowed or invested
  - → r = **interest rate**, the amount of interest due per time period, expressed as a proportion of the principal
  - → n = number of **time periods** since the principal was borrowed or invested
  - → The length of the time period for the interest rate (r) and the number of time periods (n) must be the same.

Solving for other quantities using the simple interest formula

- The simple interest formula can also be used to find any one of the values (principal, interest rate or number of time periods) when the other three quantities are known.

$I = Prn$ Solve for principal

$I = Prn$ Solve for interest rate

$I = Prn$ Solve for number of time periods

- To find P, r or n, if the other three values are known, use inverse operations to solve the equation by dividing both sides of the formula by the two known quantities on the right-hand side.

→ For example, if the simple interest was \$300, the interest rate was 5% (0.05) and the number of time periods was 5, the principal could be calculated as follows:

$$\begin{aligned} I &= Prn \\ 300 &= P \times 0.05 \times 5 \\ 300 &= P \times 0.25 \\ \frac{300}{0.25} &= P \\ 1200 &= P \end{aligned}$$

Example 1B.1 Calculating simple interest

For a loan of \$18 000, borrowed at an interest rate of 6.8% p.a. for 21 months, calculate:

a the amount of simple interest charged

b the total amount repaid at the end of the loan

c the amount of each repayment if equal repayments are made monthly.

THINK	WRITE
a **1** Identify the known values. Write the interest rate as a decimal and the number of time periods in years (as the interest rate is per annum).	**a** $P = \$18\,000$ $r = 6.8\%$ $= 0.068$ $n = 21$ months $= \frac{21}{12}$ years $= 1.75$ years
2 Substitute the values for the pronumerals into the formula and calculate I.	$I = Prn$ $= \$18\,000 \times 0.068 \times 1.75$ $= \$2142$
b The total repaid at the end of the loan is the interest added to the principal.	**b** Total amount repaid = $\$18\,000 + \$2142$ $= \$20\,142$
c The amount of each repayment is found by dividing the total amount to be paid by the number of instalments.	**c** Number of instalments = 21 Amount of each repayment = $\frac{\$20\,142}{21}$ $\approx \$959.14$

Example 1B.2 Solving for other quantities in the simple interest formula

Use the simple interest formula, $I = Prn$, to help you find each unknown value below.

a $I = \$800$, $P = \$4000$, $r = 1\%$ p.a., $n = ?$

b $I = \$3279.60$, $P = ?$, $r = 4.5\%$ p.a., $n = 4$ years

c $I = \$20\,150$, $P = \$124\,000$, $r = ?$, $n = 15$ months

THINK	WRITE
a 1 Identify the known values. Write the interest rate as a decimal.	**a** $I = \$800$ $P = \$4000$ $r = 1\%$ $= 0.01$
2 Substitute the known values into the formula and simplify.	$I = Prn$ $\$800 = \$4000 \times 0.01 \times n$ $= \$40 \times n$
3 Use inverse operations to solve the equation for *n*.	$\frac{\$800}{\$40} = \frac{\$40 \times n}{\$40}$ $n = 20$
4 Write the answer.	The number of time periods is 20.
b 1 Identify the known values. Write the interest rate as a decimal and the number of time periods in years (as the interest rate is per annum).	**b** $I = \$3279.60$ $r = 4.5\%$ $= 0.045$ $n = 4$ years
2 Substitute the known values into the formula and simplify.	$I = Prn$ $\$3279.60 = P \times 0.045 \times 4$ $= P \times 0.18$
3 Use inverse operations to solve the equation for *P*.	$\frac{\$3279.60}{0.18} = \frac{P \times 0.18}{0.18}$ $P = 18\,220$
4 Write the answer in dollars.	The principal is $18 220.
c 1 Identify the known values. Write the number of time periods in years.	**c** $I = \$20\,150$ $P = \$124\,000$ $n = 15$ months $= \frac{15}{12}$ years $= 1.25$ years
2 Substitute the values in the formula and simplify.	$I = Prn$ $\$20\,150 = \$124\,000 \times r \times 1.25$ $= \$155\,000 \times r$
3 Use inverse operations to solve the equation for *r*.	$\frac{\$20\,150}{\$155\,000} = \frac{\$155\,000 \times r}{\$155\,000}$ $r = 0.13$
4 Write the answer as a percentage per annum.	The interest rate is 13% p.a.

Helpful hints

✔ Instead of substituting known values into the formula and using inverse operations to find the unknown value, try rearranging the simple interest formula to create a new formula with the unknown value on the left-hand side, and the known values on the right-hand side. This is called changing the subject of the formula. If you do this, you will be able to substitute the known values into your new formula and calculate the unknown value immediately!

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Exercise 1B Simple interest

▲ **1–7, 8(a–d), 10, 11, 13, 16, 17** ■ **3–7, 8(e–h), 9, 12, 14, 17, 18** ⬟ **3, 4, 8(e–h), 12, 15, 17–20**

UNDERSTANDING AND FLUENCY

1 Convert each of these times to years. Hint: 1 year = 12 months = 52 weeks = 365 days.

a 24 months
b 48 months
c 27 months
d 45 months
e 7 months
f 52 weeks
g 65 weeks
h 4 weeks
i 312 weeks
j 78 weeks
k 730 days
l 73 days

2 Calculate the simple interest in each of these cases.

a $P = \$5000$, $r = 6\%$ p.a., $n = 4$ years
b $P = \$850$, $r = 12\%$ p.a., $n = 2$ years
c $P = \$125\,000$, $r = 8.4\%$ p.a., $n = 3$ years
d $P = \$64\,000$, $r = 3.6\%$ p.a., $n = 5$ years
e $P = \$9550$, $r = 10\%$ p.a., $n = 2.5$ years
f $P = \$22\,000$, $r = 12.5\%$ p.a., $n = 3$ years

1B.1 **3** For each loan below, calculate the:

i amount of simple interest charged
ii total amount repaid at the end of the loan
iii amount of each repayment if equal repayments are made monthly.

a \$8000 borrowed at an interest rate of 4% p.a. over 2 years
b \$15 000 borrowed at an interest rate of 10% p.a. over 4 years
c \$1250 borrowed at an interest rate of 5.4% p.a. over 3 years
d \$14 000 borrowed at an interest rate of 8.2% p.a. over 30 months
e \$25 000 borrowed at an interest rate of 6.8% p.a. over 18 months
f \$6550 borrowed at an interest rate of 4.5% p.a. over 104 weeks

4 For each investment below, calculate the:

i amount of simple interest earned
ii value of the investment at the end of the term.

a \$12 000 invested at an interest rate of 3% p.a. for 3 years
b \$66 000 invested at an interest rate of 5% p.a. for 2 years
c \$125 000 invested at an interest rate of 6% p.a. for 2.5 years
d \$5500 invested at an interest rate of 4.2% p.a. for 12 months
e \$8250 invested at an interest rate of 8.5% p.a. for 15 months
f \$4580 invested at an interest rate of 3.6% p.a. for 36 months

5 Find the value of n to determine the number of time periods required for each of these investments.

a How long will it take an investment of \$10 000 with a simple interest rate of 3.5% p.a. to earn \$1400?
b How long does \$1600 invested at an interest rate of 8% p.a. take to earn \$320 in simple interest?
c How long will it take an investment of \$8500 with a simple interest rate of 5.6% p.a. to earn \$1428?

6 Find the value of P to determine the principal in each of these situations.

a How much needs to be invested for 4 years at an interest rate of 7% p.a. to earn \$4200 in simple interest?
b How much was borrowed at an interest rate of 8.8% p.a. over 2.5 years if it cost the borrower \$1430 in simple interest?
c How much was borrowed at an interest rate of 4.75% p.a. over 3 years if it cost the borrower \$2850 in simple interest?

UNDERSTANDING AND FLUENCY

7 Find the value of r to determine the interest rate in each of these situations.

- **a** What annual interest rate is applied to a loan of \$8000 if the total simple interest charged over 2 years is \$704?
- **b** What annual interest rate is applied to a loan of \$16 000 if the total simple interest charged over 4.5 years is \$7560?
- **c** What monthly interest rate is applied to a \$25 000 investment if the total simple interest earned over 30 months is \$3000?

1B.2 8 Find each unknown value below. Give your answers to parts **d** and **h** as interest rates per annum.

- **a** $I = ?$, $P = \$8000$, $r = 3\%$ p.a., $n = 9$ years
- **b** $I = \$1000$, $P = \$5000$, $r = 5\%$ p.a., $n = ?$
- **c** $I = \$4680$, $P = ?$, $r = 6.5\%$ p.a., $n = 4$ years
- **d** $I = \$8064$, $P = \$64\,000$, $r = ?$, $n = 18$ months
- **e** $I = ?$, $P = \$12\,000$, $r = 7.2\%$ p.a., $n = 3$ years
- **f** $I = \$3864$, $P = \$9200$, $r = 8.4\%$ p.a., $n = ?$
- **g** $I = \$4320$, $P = ?$, $r = 4.8\%$ p.a., $n = 4.5$ years
- **h** $I = \$675$, $P = \$22\,500$, $r = ?$, $n = 15$ months

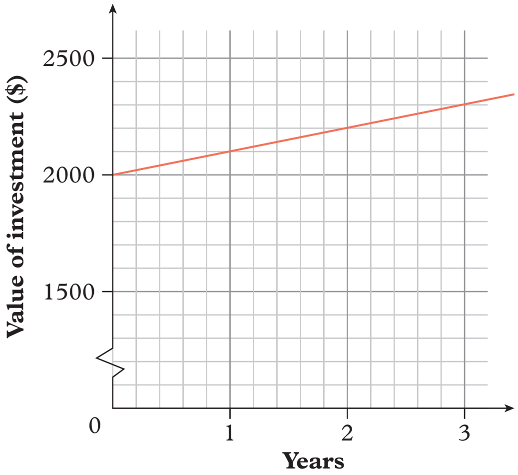
9 If the P values in question **8** represent amounts borrowed, what is the total amount to be repaid in each case?

PROBLEM SOLVING AND REASONING

10 For the smart television shown, an electronics store offers a purchase plan in which simple interest is charged at 12.75% p.a. for 20 months.

- **a** What is the total amount of interest charged for the term of the purchase plan?
- **b** What is the total cost of the television if the purchase is made with this plan?
- **c** If the total amount to be repaid is divided into equal monthly instalments for the term of the plan, how much is each repayment?
- **d** The store offers another option for customers who pay cash. This option involves a discount of 2.5% off the advertised price. What is the price for cash customers?
- **e** Calculate the difference in the price of the same television purchased using each of the two options.

11 The following graph details the value of a simple interest investment.

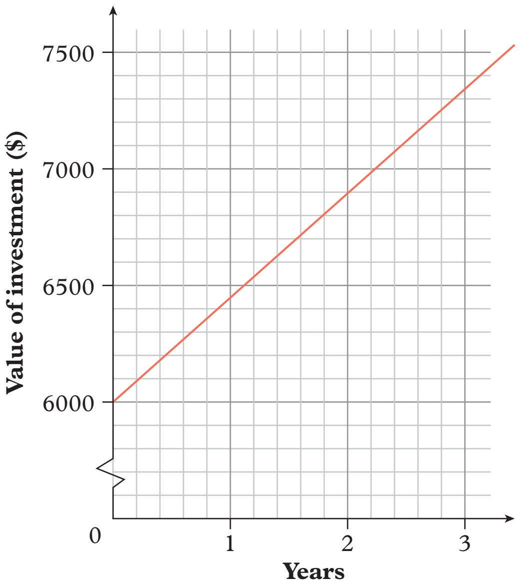


- **a** What is the value of the principal?
- **b** What is the value of the investment after 4 years?
- **c** What is the annual interest rate of the investment?
- **d** What would be the value of the investment if it remained in the account for 7 years?
- **e** After how many months will the investment be worth \$2250?

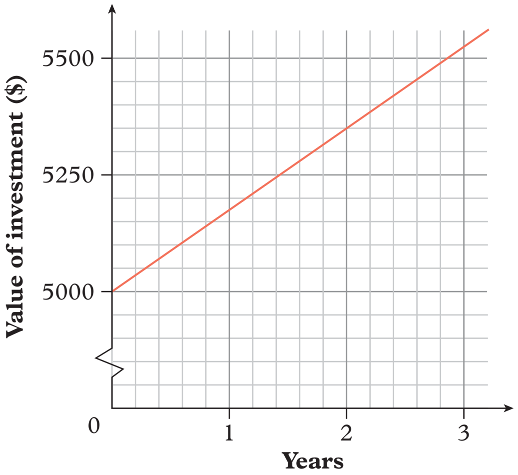
12 For each of the following graphs, which detail a simple interest investment:

- **i** determine the annual interest rate
- **ii** determine what the value of an investment which stays in the account for 5 years would be.

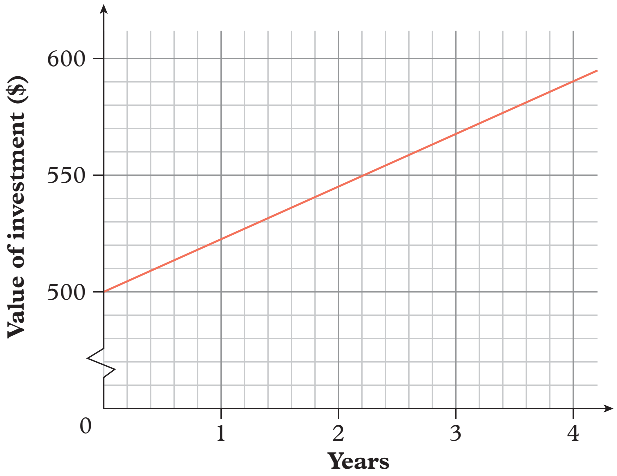
a



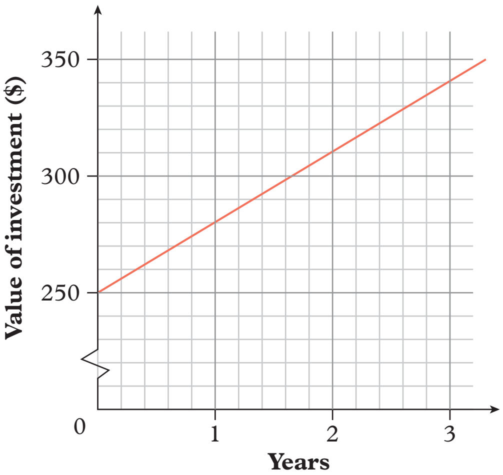
b



c



d



13 A bank offers an interest rate of 1.4% p.a. on its everyday accounts, with interest calculated on the daily balances. Consider the account balances shown on the right.

Date	Transaction	Amount	Balance
01/05	Opening balance		$985.90
09/05	Withdrawal at ATM	$60.00	$925.90
11/05	Deposit (Pay)	$875.55	$1801.45
20/05	EFTPOS Purchase	$185.95	
22/05	Gym membership	$69.50	
25/05	Deposit (Pay)	$875.55	
31/05	Interest for May		

- **a** The account balances following three of the transactions are not shown. State the missing account balances.
- **b** The opening balance of $985.90 applies for the first eight days of the month before each new balance applies on the date the transaction is made and until the next transaction. How many days does each balance on this account apply for?
- **c** For each new balance on the account, calculate the simple interest based on the number of days for which each balance applies. Remember that 8 days is equivalent to $\frac{8}{365}$ years.
- **d** Add all the amounts from part **c** to calculate the total interest for the month.
- **e** What is the account balance at the end of May, if the total interest is added to the account at the end of the last day of the month?

14 A bank is offering the following interest rates to attract investment in term deposits. Interest is calculated at the end of the investment term.

Bank interest rates per annum for different investment amounts and terms				
Terms	**Amounts invested**			
	$5000 to <$10 000	**$10 000 to <$20 000**	**$20 000 to <$75 000**	**$75 000 to <$250 000**
1 to <2 months	2.5%	2.5%	2.5%	2.5%
2 to <6 months	6.25%	6.25%	6.25%	6.25%
6 to <8 months	3.25%	3.25%	3.25%	3.25%
8 to <12 months	5.35%	5.35%	5.35%	5.35%
12 to <24 months	5.4%	5.4%	5.4%	5.4%

a Answer these questions using the information in the table.
  i What is the minimum amount that can be invested in a term deposit?
  ii Do the interest rates change as the amount invested increases?
  iii The bank appears to have a special rate for one period of time. What investment term receives this rate?

b What interest rate will be offered for a 12-month investment of $45 000?

c Calculate the amount of simple interest earned on $45 000 invested for 12 months.

d Compare your answer to part **c** with the option of investing the same amount for three 4-month terms. (Assume that the interest rate will stay the same for the second and third term.)

e Reconsider the option of investing $45 000 for three 4-month terms, as calculated in part **d**. This time, at the end of the each term, the interest earned is added to the investment amount and the interest for the second term is calculated using that new total. How does the amount of interest for the second term change?

15 Some savings accounts offered by banks include bonus interest rates if certain conditions are met. These bonus rates are seen as a reward to encourage people to save and earn more interest.

Consider Saul's account balance for November, shown on the right. His account offers interest at 1.2% p.a. (calculated daily and added into the account at the end of the month), plus a bonus 4.55% p.a. (calculated daily, without considering the 1.2% interest, and only added once at the end of the month) if no withdrawals are made in the month and the balance increases by at least $250 each month.

Date	Transaction	Amount	Balance
01/11	Opening balance		$8400.90
15/11	Deposit – Pay	$655.80	
19/11	Deposit – at branch	$240.00	
29/11	Deposit – Pay	$655.80	
30/11	Interest		

a Will this month's account receive the bonus interest rate? Provide a reason to support your answer.

b Calculate the total interest earned for the month. Hint: First determine the account balances following each transaction.

c State the final account balance for the month.

d On the final day of the month, Saul made an EFTPOS purchase at a clothing store for $100 and the transaction did not appear on his statement. How would the balances and the interest calculations have differed if the withdrawal had appeared on the statement?

16 A term deposit advertised on radio offers a fixed interest rate of 6.8% p.a. for 3 months.

a What do you think the expression 'fixed interest rate' means for the 3 months?

b How much interest is earned on an investment of $12 000 in this term deposit?

c What amount of money must be invested to earn $500 in interest?

d At the end of the term, the rate of interest decreases to a new fixed interest amount of 6.15% p.a. for 100 days. What amount of money must be invested to earn the same amount of interest as would be earned in part **b**?

17 Jacqui is saving money to purchase a racing bike like the one shown on the right. She has saved $1980 and hopes to invest it to earn the remaining amount from interest on that investment. The best simple interest rate she can find is 7.2% p.a.

a For how long must Jacqui invest the money in order to earn the extra amount she needs? Give your answer in years, correct to one decimal place.

b Jacqui wants to investigate ways to earn the money quicker with a shorter investment term. What interest rate does she need if she wishes to have the money in 18 months, correct to one decimal place?

c What will the interest rate need to be if Jacqui wishes to have the money in 8 months, correct to the nearest per cent?

d Consider the rates you calculated for parts **b** and **c**. What problems do you think Jacqui may find in trying to achieve the return she wants on her investment?

18 The statement shown on the right is for a credit card. Interest on purchases made with this card is charged from the day of purchase. To avoid extra charges, the total amount spent plus interest must be paid each month.

Date	Description	Amount
05/09	Clothing store purchase	$120.00
09/09	BPAY to Electricity provider	$356.00
12/09	ABC Bookstore	$39.90
19/09	Ticketmaster	$185.00
25/09	Party hire	$150.00
30/09	Petrol	$65.85
	Interest charge for the month of September	$8.94

a Determine the amounts owing throughout the month and the total number of days that interest is charged on those amounts. Remember that interest is charged from the day of purchase.

b How much needs to be paid at the end of the month to avoid any extra charges?

c What is the annual interest rate (% p.a.) that is charged on this credit card account? Hint: Use the total interest shown and remember that 4 days is equivalent to $\frac{4}{365}$ years.

19 For each short-term investment below, calculate the total amount of interest earned if interest is calculated:

i at maturity

ii monthly, with interest added to the investment at the end of each month.

a $10 000 invested at 3.6% p.a. for 2 months

b $15 000 invested at 8.4% p.a. for 3 months

c $35 000 invested at 12.75% p.a. for 4 months

d $8400 invested at 21.5% p.a. for 3 months

20 Reconsider the amount of interest earned from each investment in question **19** when the monthly interest amounts are added to the investment (part **ii**). If, instead, the interest is calculated at maturity, what interest rate would have to be applied to the principal to earn that same amount of interest over the term of the investment? Give your answers correct to two decimal places.

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Calculating simple interest

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Investigation
Deferred payments

Topic quiz
1B

Interactive skillsheet
Calculating the principal by simple interest formula

Interactive skillsheet
Calculating the number of time periods for an investment

1C Compound interest

Learning intentions

By the end of this topic you will be able to ...

- ✔ understand how compound interest works
- ✔ use the compound interest formula to calculate the future value
- ✔ calculate the interest rate for a time period as well as the number of time periods.

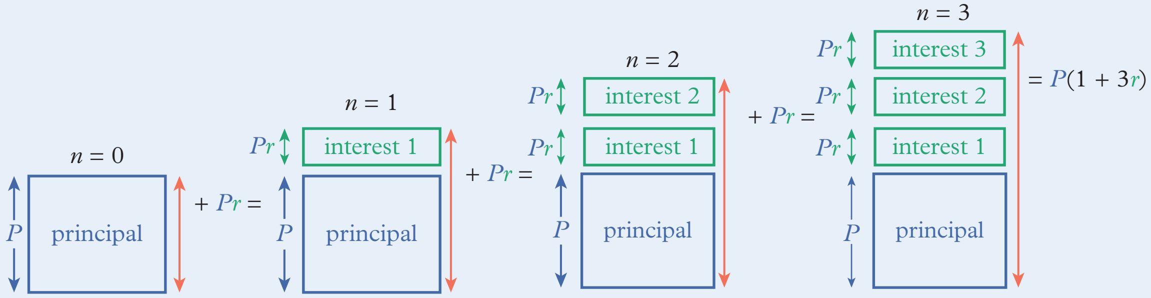
Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	1E Simple interest

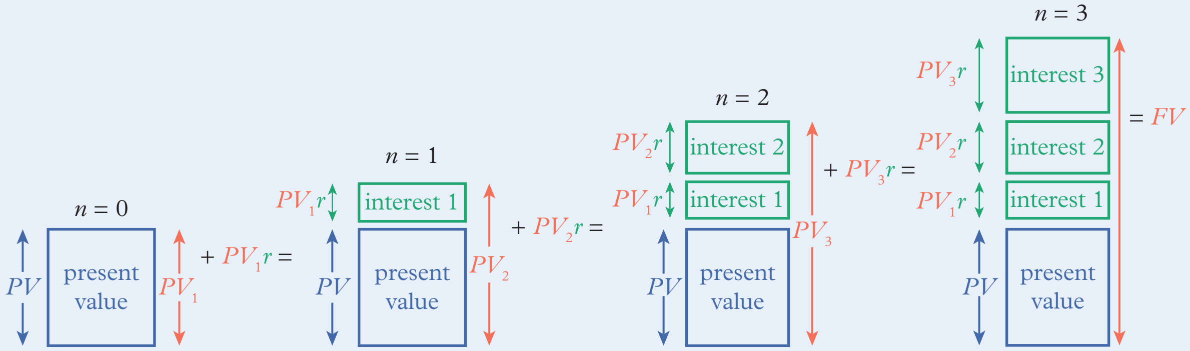
Compound interest

Key content video

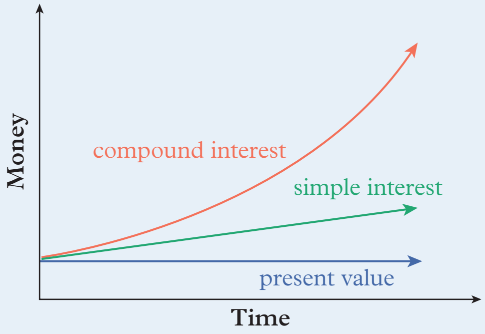
- With simple interest the total value of an investment or loan increases by the same amount of interest each time period. Consider the diagram below to see how an amount of money increases over 3 years as a result of simple interest.



- **Compound interest** is generated when simple interest is added to the **present value** (PV) at regular intervals, so that the amount of money used for calculating the interest continues to increase over time.
- The **future value** (FV) of an investment or loan is the final total after a given number of compounding periods have elapsed: $FV = PV + I$
- A time period (n) is the regular interval of time at which interest is calculated and added to the present value. Consider the diagram below to see how the compounded value increases over 3 years with a time period of 1 year.



- When the interest rates are the same, an amount of money that is subjected to compound interest increases faster than the same amount of money subjected to simple interest. As time passes, the difference between the amount generated by compound interest and that generated by simple interest increases.



Compound interest formula

- The future value of a compound interest investment or loan can be calculated using the formula:

$$\underset{\substack{\text{future}\\\text{value}}}{FV} = \underset{\substack{\text{present}\\\text{value}}}{PV}(1 + \underset{\substack{\text{interest rate}\\\text{per time period}}}{r})^{\overset{\text{number of time periods}}{n}}$$

  → FV = the future value of the investment or loan (the total amount after n time periods)
  → PV = the present value of the investment or loan
  → R = the interest rate per time period, written as a decimal
  → n = the number of time periods.
- You can divide the interest rate per annum by the number of time periods in a year to get the interest rate per time period.
- The index in the compound interest formula indicates that it is a repeated multiplication of $(1 + r)$.
- Compound interest is the difference between the future value and the present value (that is, the difference between the final amount and the initial amount): $I = FV - PV$

  For example, if interest of 6% p.a. is compounded every 2 months on an investment of $500 that lasts for half a year, then the value of r and n are:

$$\begin{aligned} r &= \text{interest rate per annum} \times \frac{\text{time period}}{1\text{ year}} \\ &= 6\% \times \frac{2\text{ months}}{12\text{ months}} \\ &= 6\% \times \frac{1}{6} \\ &= 1\% \\ &= 0.01 \end{aligned}$$

$$\begin{aligned} n &= \frac{\text{time of investment}}{\text{time period}} \\ &= \frac{\text{half a year}}{2\text{ months}} \\ &= \frac{6\text{ months}}{2\text{ months}} \\ &= 3 \end{aligned}$$

  The future value will be:

$$\begin{aligned} FV &= \$500 \times (1 + 0.01)^3 \\ &= \$500 \times 1.01^3 \\ &= \$500 \times 1.030\,301 \\ &\approx \$515.15 \end{aligned}$$

  and the amount of compound interest will be:

$$\begin{aligned} I &= FV - PV \\ &= \$515.15 - \$500 \\ &= \$15.15 \end{aligned}$$

Example 1C.1 Understanding compound interest

An amount of $20 000 is invested at 4% p.a. with interest calculated at the end of each year and added to the present value. Use the simple interest formula to calculate the following amounts:

a the future value after 1 year

b the future value after 2 years

c the compound interest earned after 3 years.

THINK	**WRITE**
a **1** Identify the known values for use in the simple interest formula.	**a** $PV_1 = \$20\,000$ $r = 4\%$ $= 0.04$ $n = 1$

2 Calculate the simple interest and add that amount to the present value to find the future value of the investment after 1 year.	$I_1 = PV_1rn$ $= \$20\,000 \times 0.04 \times 1$ $= \$800$ $FV_{1\text{ year}} = \$20\,000 + \$800$ $= \$20\,800$
b **1** Use the present value after 1 year as the new principal for use in the simple interest formula.	**b** $PV_2 = \$20\,800$
2 Calculate the interest again and add the future value after 1 year to find the future value after 2 years.	$I_2 = PV_2rn$ $= \$20\,800 \times 0.04 \times 1$ $= \$832$ $FV_{2\text{ years}} = \$20\,800 + \$832$ $= \$21\,632$
c **1** Use the future value after 2 years as the new principal for use in the simple interest formula.	**c** $PV_3 = \$21\,632$
2 Calculate the interest again and add this to the future value after two years to find the future value after 3 years.	$I_3 = PV_3rn$ $= \$21\,632 \times 0.04 \times 1$ $= \$865.28$ $FV_{3\text{ years}} = \$21\,632 + \$865.28$ $= \$22\,497.28$
3 Subtract the present value from the future value after 3 years to find the amount of compound interest earned after 3 years.	$I = FV_{3\text{ years}} - PV$ $= \$22\,497.28 - \$20\,000$ $= \$2497.28$

Example 1C.2 Using the compound interest formula

For an amount of \$14 000 invested at a compound interest rate of 3.5% p.a. for 5 years with interest compounded annually, calculate:

a the value of the investment at the end of the 5 years

b the total amount of compound interest earned.

THINK	**WRITE**
a **1** Identify the variables for the compound interest formula. Convert r to a decimal.	**a** $PV = \$14\,000$ $r = 3.5\%$ $= 0.035$ $n = 5$
2 Substitute the known values into the formula and calculate the result. Give your answer to the nearest cent by rounding the answer to two decimal places.	$FV = PV(1 + r)^n$ $= \$14\,000 \times (1 + 0.035)^5$ $= \$14\,000 \times 1.035^5$ $\approx \$16\,627.61$
b The amount of compound interest earned is the difference between the future value and the present value.	**b** $I = FV - PV$ $= \$16\,627.61 - \$14\,000$ $= \$2627.61$

Example 1C.3 Calculating r and n

For an amount of \$11 000 invested at 6.9% p.a. for 8 years, with interest compounded every 4 months:

a write the given interest rate per annum in a form that matches the time period

b state the values for r and n that are to be used in the compound interest formula.

THINK	WRITE
a **1** Identify the interest rate per annum and the time period.	**a** interest rate per annum = 6.9% time period = 4 months
2 To find the interest rate per time period, r, express the time period as a fraction of a year by writing the time period as the numerator and 1 year as the denominator. Then multiply this fraction by the interest rate per annum.	$r = \dfrac{4\text{ months}}{1\text{ year}} \times 6.9\%$
3 Simplify the fraction by converting 1 year to the appropriate unit of time. Then evaluate to find the interest rate per time period.	$= \dfrac{4\text{ months}}{12\text{ months}} \times 6.9\%$ $= \dfrac{1}{3} \times 6.9\%$ $= 2.3\%$
b **1** Convert the interest rate per time period to a decimal value.	**b** $r = 2.3\%$ $= 0.023$
2 Divide the time of the investment by the time period to find the number of time periods.	$n = \dfrac{8\text{ years}}{4\text{ months}}$ $= \dfrac{8 \times 12\text{ months}}{4\text{ months}}$ $= \dfrac{96}{4}$ $= 24$

Helpful hints

- ✔ You can also calculate n using the unitary method. To do this, you need to figure out how many time periods occur in 1 year and then multiply that number by the number of years of the investment or loan. For instance, in Example 1C.3 the number of time periods in a year is 3, so: $n = 3 \times 8 = 24$

ANS p645

Exercise 1C Compound interest

▲ 1–4, 5(a, c, e), 6, 7, 10, 11, 13, 16 ■ 1, 4, 5(b, d), 6, 7, 10, 11, 14, 15, 19, 21 ⬟ 4, 8, 9, 11, 12, 17, 18, 20–22

1 For each investment below, state the:

i length of each time period **ii** number of time periods.

a \$5000 is invested at 4.5% p.a. for 2 years with interest compounded annually.

b \$12 000 is invested at 9% p.a. for 3 years with interest compounded annually.

c \$8500 is invested at 5% p.a. for 4 years with interest compounded every 6 months.

d \$25 000 is invested at 8.4% p.a. for 2 years with interest compounded every month.

2 Calculate the missing values in the table below for a compound interest investment of $100 at 10% p.a over the course of 6 years, with interest compounded annually.

	Investment of $100 at 10% p.a.				
	Years	**Simple interest**	**Future value for simple interest**	**Compound interest**	**Future value for compound interest**
	1	$10	$110	$10	$110
	2	$10	$120	$I = \$110 \times 10\%$ $= \$11$	$FV = \$110 + \$11$ $= \$121$
a	3			$I = \$121 \times 10\%$ $= \$12.10$	$FV = \$121 + \$12.10$ $= \$133.10$
b	4			$13.31	
c	5				
d	6				

1C.1 3 For each of the following investments, where interest is calculated at the end of each year and added to the present value, use the simple interest formula to calculate the following amounts:

i the future value after 1 year **ii** the future value after 2 years

iii the compound interest earned after 3 years.

a $40 000 is invested at 5% p.a.

b $6000 is invested at 2% p.a.

c $1000 is invested at 4% p.a.

d $20 000 is invested at 3.5% p.a.

e $6500 is invested at 7% p.a.

f $15 800 is invested at 2.4% p.a.

4 Mac is investigating ways to earn more interest on an investment. His bank offers him an interest rate of 5.2% p.a. with interest compounded annually. Mac has $8500 to invest for 3 years.

a From the given information:

i how long is each time period **ii** how many time periods will there be in Mac's investment?

b How much interest will Mac earn in the first year?

c Add the interest for the first year that you calculated in part **b** to the present value. This will be the new amount you will use for the principal for the second year.

d How much interest will Mac earn in the second year?

e Add the interest from part **d** to the future value calculated in part **c**. This will be the new amount for the principal for the third (final) year.

f How much interest will Mac earn in the final year of his investment?

g What will be the final value of Mac's investment?

h Compare the result you found in part **g** with the future value that would be obtained after 3 years of earning simple interest on a $8500 investment at a rate of 5.2% p.a.

5 For each of the investments shown in the table below, calculate the amount of interest earned if interest is compounded annually using the same method as outlined in question **4**, parts **a–g**.

	Principal	Interest rate (p.a.)	Term
a	$8400	12%	3 years
b	$12 000	5.4%	4 years
c	$25 000	3.75%	3 years
d	$4590	9.5%	4 years
e	$150 000	6.25%	2 years
f	$35 500	15.8%	5 years

6 Use the compound interest formula to find the value of FV in each of these situations.

a $FV = ?$, $PV = \$10\,000$, $r = 0.05$ p.a., $n = 4$ years

b $FV = ?$, $PV = \$5000$, $r = 0.06$ p.a., $n = 3$ years

c $FV = ?$, $PV = \$600$, $r = 0.08$ p.a., $n = 4$ years

d $FV = ?$, $PV = \$44\,000$, $r = 0.038$ p.a., $n = 5$ years

e $FV = ?$, $PV = \$100\,000$, $r = 0.015$ per quarter, $n = 8$ quarters

f $FV = ?$, $PV = \$19\,500$, $r = 0.0025$ per month, $n = 10$ months

1C.2 **7** For each investment below, calculate:

i the value of the investment at the end of the given period

ii the total amount of compound interest earned.

a $7500 is invested at 5% p.a. for 3 years, with interest compounded annually

b $9000 is invested at 4.5% p.a. for 3 years, with interest compounded annually

c $22 500 is invested at 6.8% p.a. for 5 years, with interest compounded annually

d $40 000 invested at 7.3% p.a. for 4 years, with interest compounded annually

8 For each investment listed in the table below, use the compound interest formula to calculate the final value of the investment if interest is compounded annually.

	Principal	Interest rate (p.a.)	Term
a	$3 200	6.4%	4 years
b	$4 590	9.8%	5 years
c	$16 000	2.85%	2 years
d	$25 900	10.5%	3 years
e	$100 000	4.99%	5 years
f	$255 000	12.15%	6 years

9 Calculate the total amount of interest earned by each investment listed in the table in question **8**.

1C.3 **10** For each investment below:

i write the given interest rate so that it matches the time period

ii state the values for r and n that are to be used in the compound interest formula.

a $15 000 is invested at 4.8% p.a. for 3 years with interest compounded quarterly.
Hint: ‘Quarterly’ means every quarter of a year, or every three months.

b $22 000 is invested at 5.9% p.a. for 4 years with interest compounded every 6 months.

c $3500 is invested at 8.4% p.a. for 2 years with interest compounded monthly.

d $14 500 is invested at 9.6% p.a. for 5 years with interest compounded monthly.

11 Use the compound interest formula to determine the value of FV for each set of information provided in the table below.

	Present value	Interest rate (p.a.)	Term	Time period
a	$4 800	8.15%	3 years	annually
b	$9 500	8.8%	4 years	every 6 months
c	$18 000	4.84%	5 years	quarterly
d	$35 000	16.8%	4 years	monthly
e	$50 000	6.95%	3 years	every 6 months
f	$150 000	8.16%	1 year	quarterly
g	$500 000	4.98%	2 years	every 4 months
h	$750 000	10.26%	3 years	every 2 months

12 The compound interest formula can be found by continuously increasing an amount of money by a percentage of the current amount, rather than by a percentage of the initial amount. In this question you will derive the compound interest formula by generalising from an example.

Consider the following table, which shows information about a \$10 000 investment that receives 5% p.a. interest compounded annually.

Years since investment	Calculation using previous year's value	Calculation using initial investment value	Value of investment (\$)
0			10 000
1	$10\,000 \times (1.05)$	$10\,000 \times (1.05)$	10 500
2	$10\,500 \times (1.05)$	$10\,000 \times (1.05)^2$	11 025
3	$11\,025 \times (1.05)$	$10\,000 \times (1.05)^3$	11 576.25
4	$11\,576.25 \times (1.05)$	$10\,000 \times (1.05)^4$	12 155.06

a Explain why multiplying a value by 1.05 increases that value by 5%.

b The table shows the value of the investment for the first four years. Write an expression for the value of the investment after n years.

c Write an expression for the value after n years if the interest rate is r (as a decimal), rather than 5%.

d Write an expression for the value of the investment after n years if the interest rate is r (as a decimal), rather than 5%, and the present value is PV, rather than \$10 000.

13 Maria and her family have begun long-term planning for an overseas holiday. They have invested their savings of \$16 000 for 3 years in a bank account, which offers them 5.75% p.a. interest compounded annually.

a How many time periods will there be in the family's investment?

b What is the length of each time period?

c Identify the values for PV, r and n.

d Use the compound interest formula to calculate the future value of the investment after the 3-year investment period.

e How much interest will the family have earned on this investment?

f Maria estimates that she will need \$20 000 to fund the entire overseas holiday. If she invests the money in the same account for a fourth year, will she achieve this goal?

14 Carolyn borrows \$45 000 to set up a photography studio for her daughter. The loan is for 4 years with interest charged at 8.95% p.a. compounding annually. The terms of the loan require Carolyn to repay the total amount at the end of the 4-year term.

a Use the compound interest formula to calculate the total amount that Carolyn must repay at the end of the 4-year term.

b How much of the amount in part **a** represents the interest charged?

15 While investigating different investment options offered by his bank, Bailey decided he wanted to explore the following two options:

- Option 1: Interest rate of 5.35% p.a. with interest paid at maturity (at the end of the investment)
- Option 2: Interest rate of 4.95% p.a. with interest compounded annually

He has $9000 to invest and, initially, he plans to invest the money for 3 years.

a For each option, what would be the value of Bailey's investment at the end of the 3-year term?

b Which option should Bailey take for his 3-year investment? Provide a reason for your answer.

c How do the answers to parts **a** and **b** change if Bailey invests the money for 5 years?

16 Sheetal has saved $8400 and is exploring the benefits of investing with an online bank. She finds that the bank will pay 8.6% p.a. interest on amounts invested for at least 36 months, with interest compounded quarterly.
She decides to invest her $8400 for 5 years.

a Write the annual interest rate as a rate per quarter.

b Use your answer from part **a** to state the value of r in the compound interest formula.

c How many time periods, n, will there be in Sheetal's investment?

d Use the compound interest formula to determine the future value of the investment.

e What is the total amount of interest Sheetal can expect to earn on her investment?

17 An investment of $10 000 is made for 4 years with a bank that offers an annual interest rate of 8.4%. Answer the following questions to explore how the length of the time periods affects the future value of the investment.

a Calculate the future value of the investment if the interest is calculated at the end of the term; that is, at maturity.

b Calculate the future value of the investment if the interest is compounded:

i every 6 months
ii quarterly
iii every 2 months
iv monthly.

c Use your results from part **b** to briefly explain what happens to the future value of an investment as the number of time periods increases. Will this always be the case?

18 Through a store purchase plan, Troy is informed that interest is calculated as compound interest based on the amount borrowed. In this situation, would Troy be better off with a plan involving a higher frequency of time periods or fewer time periods?

19 Erica knows she can earn 6.8% p.a. from her bank for her investments. She plans to invest some money for 2 years with interest compounded quarterly.

a For Erica's investment, state:

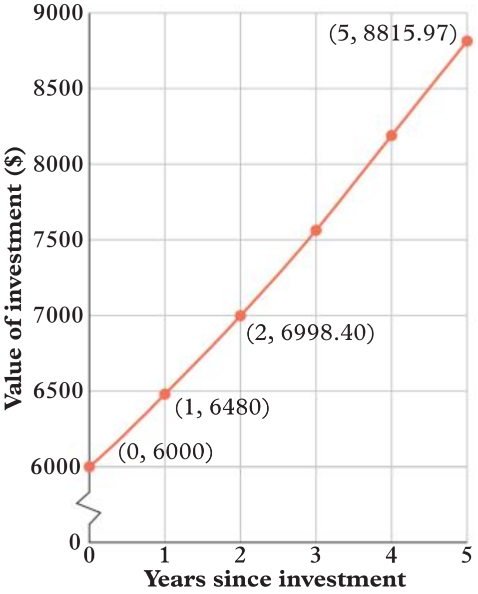
i the number of time periods, n
ii the value for r to be used in the compound interest formula.

b At the end of her 2-year investment term, Erica hopes her investment will have grown to $6500. Which variable of the compound interest formula does this $6500 represent?

c Substitute all the known values into the compound interest formula and solve the equation for the unknown variable.

d To the nearest dollar, how much should Erica invest at the start of her investment if she is aiming to have an amount of $6500 after 2 years?

20 Consider the following graph, which shows the value of Raj's investment in a high interest savings account.



a How much money did Raj invest?

b If interest is compounded annually, what is the interest rate per annum for the savings account?

c Find the coordinates of the two points on the graph that are not already labelled.

d What will be the balance in Raj's account 8 years after investment?

21 Thao is investigating different options to invest $50 000, which he inherited from his aunt. He decides to pick one of the following three options.

- Option 1: a 6.9% p.a. simple interest account
- Option 2: a 6.4% p.a. compound interest account, with interest compounding monthly
- Option 3: a 6.3% p.a. compound interest account, with interest compounding daily

a How much money will be in the account for each option at the end of 1 year? Give your answers correct to the nearest dollar where necessary.

b How much money will be in the account for each option at the end of 2 years? Give your answers correct to the nearest dollar where necessary.

c How much money will be in the account for each option at the end of 3 years? Give your answers correct to the nearest dollar where necessary.

d Which option would you advise Thao to invest the money into?

22 a Calculate the simplest interest rate, correct to two decimal places, that is equivalent to a compound interest rate of 5.9% p.a. if the interest is compounded:

i quarterly **ii** weekly **iii** daily.

b Use a 'guess, check and improve method' to determine the compound interest rate, correct to two decimal places, that is equivalent to a simple interest rate of 6.2% p.a. if the interest is compounded:

i quarterly **ii** monthly **iii** daily.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Calculating compound interest

Topic quiz
1C

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Checkpoint

Checkpoint quiz Check your knowledge of the first part of this chapter.

1A **1** Kisha is paid a standard rate of \$34.50 per hour. Each week she works 25 hours on weekdays and 12 hours on weekends (for which she is paid time-and-a-half).

a Calculate Kisha's weekly earnings. **b** Calculate Kisha's yearly earnings.

c Calculate Kisha's monthly earnings.

1A **2** Calculate the total monthly earnings for the following workers who earn a base salary plus a commission.

a Damo receives a base salary of \$2000/month plus a 4.5% commission on sales. In June he sells \$52 000 worth of product.

b Olivia receives a base salary of \$1350/month plus a 2.2% commission on sales. In February she sells \$92 000 worth of product.

1A **3** Use the 2023–2024 tax table to calculate the annual income tax payable, to the nearest dollar, on the following salaries.

a \$45 400 per year **b** \$71 330 per year

c \$11 024 per month **d** \$2905 per week

Taxable income	Tax on this income
0–\$18 200	Nil
\$18 201–\$45 000	19 cents for each \$1 over \$18 200
\$45 001–\$120 000	\$5092 plus 32.5 cents for each \$1 over \$45 000
\$120 001–\$180 000	\$29 467 plus 37 cents for each \$1 over \$120 000
\$180 001 and over	\$51 667 plus 45 cents for each \$1 over \$180 000

1A **4** In the 2023–2024 financial year, Leo earned \$59 202 and had tax deductions of \$819.

a What is Leo's taxable income?

b How much income tax must Leo pay?

c How much is Leo's Medicare levy?

d What is Leo's net income?

1B **5** Calculate the simple interest in each case.

a $P = \$5000$, $r = 4\%$ p.a., $n = 3$ years **b** $P = \$12\,000$, $r = 2.5\%$ p.a., $n = 7$ years

c $P = \$8\,500$, $r = 3.8\%$ p.a., $n = 5$ years

1B **6** For each investment below, calculate:

i the amount of interest earned **ii** the value of the investment at the end of the term.

a \$5000 is invested at a simple interest rate of 4% p.a. for 3 years

b \$12 000 is invested at a simple interest rate of 3.5% p.a. for 2 years

c \$132 800 is invested at a simple interest rate of 2.75% p.a. for 8 years

1B **7** William takes out a loan of \$4500 at a simple interest rate of 5% p.a. He plans to pay the loan back over 4 years.

a Find the total amount of simple interest changed over 4 years.

b Calculate the total amount of money William will have to repay.

c If William pays back an equal amount each month for the four years, how much will he pay per month?

1C **8** Calculate the future value of each investment, to the nearest dollar, with interest compounded annually.

a $PV = \$11\,000$, $r = 0.08$ p.a., $n = 3$ years **b** $PV = \$1700$, $r = 0.034$ p.a., $n = 7$ years

c $PV = \$5200$, $r = 0.057$ p.a., $n = 4$ years

1C **9** For each investment below, state the:

i length of each time period **ii** number of time periods.

a \$3600 is invested at 7.5% p.a. for 2 years with interest compounded each month.

b \$12 000 is invested at 9.2% p.a. for 3 years with interest compounded each quarter.

1C **10** Calculate the future value of each of the investments in question **10** correct to the nearest dollar.

1C **11** Elle borrows \$26 000 to purchase a new car. The loan is for 5 years with interest charged at 8.45% p.a. compounding daily. The term of the loan requires Elle to repay the total amount at the end of the 5-year term.

a What is the total amount that Elle must repay at the end of the 5-year loan?

b How much interest is charged on the loan?

Give your answers correct to the nearest dollar.

1D Compound interest calculations

Learning intentions

By the end of this topic you will be able to ...

✔ use the compound interest formula to calculate the present value

✔ use the compound interest formula to calculate the interest rate.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	1F Simple interest calculations

Key content video

Using the compound interest formula to find other unknown quantities

- As well as calculating the compounded value of a loan or investment, the compound interest formula can also be used to find the present value or the interest rate, provided the other values are known.

$FV = PV(1 + r)^n$ Solve for present value

$FV = PV(1 + r)^n$ Solve for interest rate per time period

Calculating the present value using the compound interest formula

- To find PV using $FV = PV(1 + r)^n$, use inverse operations to solve the equation by dividing both sides of the formula by the value of $(1 + r)^n$.

  For example, if an 8% p.a. investment is compounded annually and grows to the value of \$2459 after 3 years, the present value can be calculated as follows:

$$\begin{aligned} \$2459 &= PV(1 + 0.08)^3 \\ \$2459 &= PV(1.08)^3 \\ \$2459 &= PV \times 1.259\,712 \\ \frac{\$2459}{1.259\,712} &= PV \\ PV &\approx \$1952.03 \end{aligned}$$

Calculating the interest rate using the compound interest formula

- To find r using $FV = PV(1 + r)^n$, we can also use inverse operations.
  1. Divide both sides of the equation by the value of PV.
  2. Find $\sqrt[n]{\ }$ of both sides of the equation.
  3. Subtract 1 from both sides of the equation.
- To find the annual interest rate, multiply r by the number of compounding periods in a year.

For example, if an investment of \$500 is compounded every 6 months and grows to have a value of \$653.48 after 2 years, then the annual interest rate, correct to one decimal place, can be calculated using the following method.

$$\begin{aligned} n &= \frac{2 \text{ years}}{6 \text{ months}} \\ &= \frac{24 \text{ months}}{6 \text{ months}} \\ &= 4 \end{aligned}$$

$$\begin{aligned} FV &= PV(1 + r)^n \\ \$653.48 &= \$500 \times (1 + r)^4 \\ \frac{\$653.48}{\$500} &= (1 + r)^4 \\ 1.306\,96 &= (1 + r)^4 \\ \sqrt[4]{1.306\,96} &= 1 + r \\ 1.069\,216\,31 &= 1 + r \\ r &= 0.069\,216\,31 \\ r &\approx 6.9\% \text{ (every 6 months)} \end{aligned}$$

compounding periods in a year = 2

$$\begin{aligned} \text{interest rate} &= 2 \times 6.9\% \text{ p.a.} \\ &= 13.8\% \text{ p.a.} \end{aligned}$$

Example 1D.1 Calculating the present value for compound interest

How much must be invested at an interest rate of 8.4% p.a. for the future value to be \$9000 after 2 years with interest compounded quarterly?

THINK	**WRITE**
1 Identify the known values for use in the compound interest formula.	$FV = \$9000$ $r = \frac{8.4\%}{4}$ $= 2.1\%$ per quarter $= 0.021$ $n = \frac{2 \text{ years}}{\text{one-quarter of 1 year}}$ $= \frac{24 \text{ months}}{3 \text{ months}}$ $= 8$
2 Substitute the values into the formula and simplify where possible.	$\$9000 = PV(1 + 0.021)^8$ $\$9000 = PV \times 1.021^8$
3 Solve for PV by dividing both sides of the equation by $(1 + r)^n$.	$\frac{\$9000}{1.021^8} = \frac{PV \times 1.021^8}{1.021^8}$ $PV \approx \$7621.43$

Example 1D.2 Calculating the interest rate for compound interest

For an investment of \$6000 that increases in value to \$6900 after 5 months, with interest compounded monthly:

a find the value of *r* as a decimal, correct to four decimal places

b state the annual interest rate, correct to one decimal place.

THINK	WRITE
a 1 Identify the known values for the compound interest formula.	**a** $FV = \$6900$ $PV = \$6000$ $n = \frac{5 \text{ months}}{1 \text{ month}}$ $= 5$
2 Substitute the values into the formula.	$FV = PV(1 + r)^n$ $\$6900 = \$6000 \times (1 + r)^5$
3 Solve for *r* by first dividing both sides of the equation by *PV*. Then, swap both sides of the equation.	$\frac{\$6900}{\$6000} = \frac{\$6000 \times (1 + r)^5}{\$6000}$ $(1 + r)^5 = 1.15$
4 Find $\sqrt[n]{}$ of both sides of the equation. Round to four decimal places.	$\sqrt[5]{(1 + r)^5} = \sqrt[5]{1.15}$ $1 + r = 1.028\,346\ldots$
5 Subtract 1 from both sides to find *r* correct to four decimal places.	$r = 0.0283$ (to four decimal places)
b 1 Find the number of compounding periods per year.	time periods per year $= \frac{1 \text{ year}}{1 \text{ month}}$ $= 12$
2 Multiply *r* by this number.	interest rate $= 0.0283 \times 12$ $= 0.3396$
3 Convert to a percentage correct to one decimal place.	$\approx 34.0\%$ p.a. (to one decimal place)

Helpful hints

✔ In most instances you will need to use a calculator to find square roots, cube roots and roots of a higher order. However, sometimes you can use the index laws and your knowledge of square roots to simplify roots yourself.

For example,
$$\begin{aligned}\sqrt[4]{16} &= \sqrt[4]{4^2}\\ &= \sqrt[4]{(2^2)^2}\\ &= \sqrt[4]{2^{2\times 2}}\\ &= \sqrt[4]{2^4}\\ &= 2\end{aligned}$$

✔ Remember to avoid rounding errors when calculating interest rates. Only round your calculations to the appropriate number of decimal places at the very last step.

For example, in Example 1D.2 if we rounded *r* to two decimal places at the end of part **a**, we would get $r = 0.03$ instead of 0.0283.

The answer to part **b** would then be different from 34.0%: annual interest rate $= 12 \times r$
$$\begin{aligned}&= 0.36\\ &= 36.0\%\end{aligned}$$

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Exercise 1D Compound interest calculations

△ 1–6, 8(a, c, d), 9–11 ■ 2, 3, 5, 6, 8(b, e, f), 10, 12–14 ⬟ 2, 6, 8(b, e, f), 10, 12, 13, 15–17

UNDERSTANDING AND FLUENCY

1D.1 **1** Find the present value (PV) for each of these investments.

- **a** How much must be invested at an interest rate of 10.5% p.a. for the value to increase to \$30 000 over 4 years with interest compounded annually?
- **b** How much must be invested at an interest rate of 5.4% p.a. for the value to increase to \$10 000 over 3 years with interest compounded every 6 months?
- **c** How much must be invested at an interest rate of 6.4% p.a. for the value to increase to \$5500 over 5 years with interest compounded quarterly?
- **d** How much must be invested at an interest rate of 7.2% p.a. for the value to increase to \$8000 over 2 years with interest compounded every month?

2 Use the compound interest formula to determine the value of PV, to the nearest dollar, for each set of investment information provided in the table below.

	Future value	**Interest rate (p.a.)**	**Term**	**Time period**
a	\$5200	6.75%	3 years	annually
b	\$10 000	4.2%	6 years	monthly
c	\$22 000	12.5%	4 years	every 6 months
d	\$16 650	15.5%	3 years	quarterly
e	\$225 000	9.95%	$4\frac{1}{2}$ years	every 6 months

3 How much is the total interest earned by each investment in question **2**?

4 Consider an investment of \$12 000 that increased in value to \$15 000 over a 3-year period, with interest compounded annually.

- **a** From this information, which variable from the compound interest formula don't you know the value of?
- **b** Substitute the known values into the compound interest formula and show that it simplifies to $\frac{5}{4} = (1 + r)^3$.
- **c** Copy and complete the calculation shown on the right to solve the equation for r by performing inverse operations to both sides of the equation. Note: the inverse operation to cubing a number is finding the cube root of a number.

$$\begin{aligned}\frac{5}{4} &= (1 + r)^3\\ 1.25 &= (1 + r)^3\\ (1 + r)^3 &= 1.25\\ \sqrt[3]{(1 + r)^3} &= \sqrt[3]{\underline{\qquad\qquad}}\\ 1 + r &= \sqrt[3]{\underline{\qquad\qquad}}\\ 1 + r &= \underline{\qquad\qquad}\\ r &= 0.077\,217\ldots\end{aligned}$$

- **d** The value for r in part **c** is the rate per compounding period expressed as a decimal. Write this as an annual interest rate, correct to one decimal place.

1D.2 **5** For each investment below:

- **i** find the value of r as a decimal, correct to four decimal places
- **ii** state the annual interest rate, correct to one decimal place.

- **a** After 1 year an investment of \$1000 increased in value to \$1170, with interest compounded every 6 months.
- **b** After 2 years an investment of \$2500 increased in value to \$2800, with interest compounded annually.
- **c** After 6 months an investment of \$3750 increased in value to \$4050, with interest compounded quarterly.

6 For each investment below:

   i find the value of r as a decimal, correct to four decimal places

   ii state the annual interest rate, correct to one decimal place.

a After 3 years an investment of $1600 increased in value to $2180, with interest compounded annually.

b After 2 years an investment of $7500 increased in value to $8200, with interest compounded every 6 months.

c After 18 months an investment of $10 000 increased in value to $11 000, with interest compounded every 6 months.

7 Jade's parents invested $5500 for her in a savings account, where it increased in value to $8000 in 4 years, with interest compounding annually.

a Write the values for FV, PV and n.

b Use the compound interest formula to find the value of r. Give your answer correct to three decimal places.

c What annual interest rate, correct to one decimal place, was applied to Jade's savings account?

8 Use the compound interest formula with the information in the table on the right to determine the following for each set of investment values (**a–f**):

   i the value for r (to three decimal places)

   ii the annual interest rate (to one decimal place).

	Present value	Future value	Term	Time period
a	$1000	$1500	3 years	annually
b	$8000	$12 000	4 years	annually
c	$16 000	$19 000	2 years	every 6 months
d	$3500	$4000	1 year	quarterly
e	$12 000	$16 500	3 years	every 6 months
f	$25 000	$29 500	2 years	quarterly

9 Riley aims to have $15 000 by the time he turns 18 so that he can buy a car. He opens a bank savings account that pays interest at a rate of 7.2% p.a. compounded monthly. Riley plans to keep the money invested under these terms for 3 years, but he needs to know how much he should invest to be sure he has $15 000 at the end of those 3 years.

a How many time periods, n, will there be in Riley's 3-year investment?

b What is the interest rate, R, per time period?

c From the given information, which variable in the compound interest formula don't you know the value of?

d Use the compound interest formula to calculate the amount of money Riley must invest for it to increase to $15 000 over the 3-year term of the investment.

10 An investment of $25 000 increased in value to just over $30 000 over a period of time in a savings account that earns 5.0% p.a. with interest compounded annually.

a From the given information, which variable in the compound interest formula don't you know the value of?

b Substitute the known values into the compound interest formula and show that it simplifies to $1.2 = (1.05)^n$.

c One strategy for solving the equation for n is the 'guess and refine' method. Substitute different values for n into the equation and evaluate the right-hand side until the value on that side is slightly greater than the value on the left-hand side. That value for n represents the solution to the equation.

Value for n	Left-hand side	Right-hand side	Comment
1	1.2	$1.05^1 = 1.05$	Value on the right-hand side is too small.
2	1.2	$1.05^2 = 1.1025$	
3	1.2	$1.05^3 =$	
4	1.2	$1.05^4 =$	

d The value for n obtained in part **c** is the number of time periods for the investment. Briefly explain how this relates to this investment.

11 Yousef watched his savings grow from \$850 to just over \$1100 in a savings account that earned 4.2% p.a. with interest compounded annually.

a Use the given information to determine the values for FV, PV and r.

b Substitute the given values into the compound interest formula and simplify.

c Use the 'guess and refine' method from question **10** to solve the equation for n.

d How many compounding periods did it take for Yousef's savings to grow from \$850 to just over \$1100?

12 For each set of investment information (**a–f**) given in the table below, state the number of compounding periods, n, needed for the initial principal to exceed the given future value.

	Present value	Future value	Interest rate (p.a.)	Time period
a	\$900	\$1200	5.0%	annually
b	\$12 000	\$14 000	3.5%	annually
c	\$15 000	\$18 500	4.8%	every 6 months
d	\$650	\$900	10.5%	quarterly
e	\$35 000	\$42 000	6.8%	every 6 months
f	\$50 000	\$55 590	7.2%	quarterly

13 For each investment in question **12** (**a–f**), write the term of the investment in years.

14 Shannon is saving money to buy a car. She wants to have \$10 000 in 4 years.

a If her savings account pays 4% p.a. compounded yearly, how much does she need to have in her account now in order to have \$10 000 in four years?

b If her savings account pays 4% p.a. compounded monthly, how much does she need to have in her account now in order to have \$10 000 in four years?

c If she only has \$8000 in her account, what annual rate of interest does Shannon need, as a percentage rounded to two decimal places, to have \$10 000 in 4 years if interest is compounded:

i yearly

ii quarterly

iii monthly

iv daily?

15 Consider the table on the right, which shows the balance of an account earning compound interest.

Time period	Amount (\$)
0	5000
1	5045
2	5090.41
3	5136.22
4	5182.44
5	5229.09
6	5276.15

a If the time periods are quarters, and interest is compounded quarterly, find the interest rate per annum. Give your answer as a percentage, correct to one decimal place.

b If the time periods are months, and interest is compounded monthly, find the interest rate per annum. Give your answer as a percentage, correct to one decimal place.

c If the time periods are weeks, and interest is compounded weekly, find the interest rate per annum. Give your answer as a percentage, correct to one decimal place.

d If the table represents the value in a regular savings account, do you think the time periods are quarters, months or weeks? Explain your reasoning.

e If the table represents a credit card debt, do you the think the time periods are quarters, months or weeks? Explain your reasoning.

CHALLENGE

16 **a** If an investment has a present value of PV, an interest rate of $R\%$ p.a., a compounding period of C years, and an investment term of T years, write the compound interest formula in terms of PV, R, C and T.
Hint: Determine equations for r and n first.

Compare your formula to $FV = PV(1 + r)^n$.

b Explain what happens to the value of r as C decreases.

c Explain what happens to the value of n as C decreases.

d Use your calculator to investigate what happens to $(1 + r)^n$ as the compounding period decreases for a fixed investment term, T, and a fixed annual interest rate, R. Use your observations to determine what happens to FV in your formula as C decreases. Explain your answer.

e What happens when the compounding period is equal to the investment term? What kind of interest does your formula now represent? Explain your answer.

17 Samantha deposits \$30 000 in a bank account that receives 4.8% p.a. compounded quarterly.

a Use the ‘guess, check and improve’ method to find the number of compounding periods required for Samantha’s account to grow to at least:

i \$32 000

ii \$40 000.

b It is possible to solve for n in the compound interest formula, but this requires a special kind of function called a logarithm. In algebra, a logarithm is written as $\log_a$, where a is a number referred to as the ‘base’ of the logarithm.

Scientific calculators usually only include logarithms with bases that are either 10 or Euler’s number ($e \approx 2.72$), so the following calculation uses a base 10 logarithm to determine a formula for n.

$$\begin{aligned}
FV &= PV(1 + r)^n \\
\frac{FV}{PV} &= (1 + r)^n \\
\log_{10}\left(\frac{FV}{PV}\right) &= \log_{10}[(1 + r)^n] \\
\log_{10}\left(\frac{FV}{PV}\right) &= n\log_{10}(1 + r) \\
n &= \frac{\log_{10}\left(\frac{FV}{PV}\right)}{\log_{10}(1 + r)}
\end{aligned}$$

Use a scientific calculator and the formula provided to solve the questions in part **a**. Give your answers to two decimal places.

c Explain why your answers for **b** must be rounded up to the nearest whole number, rather than rounded down.

d Use the formula provided to find the number of compounding periods required for Samantha’s account balance to grow to at least:

i \$35 000

ii \$45 000

iii \$50 000.

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Interactive skillsheet Calculating compound interest rate	**Interactive skillsheet** Calculating the present value by compound interest formula	**Worksheet** Compound interest calculations	**Investigation** Rain, rain and more rain!	**Topic quiz** 1D

1E Depreciation

Learning intentions

By the end of this topic you will be able to …

- ✔ create depreciation tables
- ✔ solve problems involving straight-line depreciation
- ✔ solve problems involving diminishing-value depreciation.

Inter-year links

Year 7	4I Percentage increases and decreases
Year 9	1F Simple interest calculations

Depreciation

Key content video

- A **depreciating asset** is an asset with a limited life expectancy. Depreciating assets are expected to decline in value over the time they are used.
  - → Examples of depreciating assets include machinery, motor vehicles, furniture and computers.
- **Depreciation** is the decline in the value of a depreciating asset over time. The **salvage value** of an asset is the value of an asset after it has been depreciated.
  - → Businesses can claim tax deductions for depreciating assets.
  - → The decline in value of a depreciating asset is based on its effective life. After the effective life is over, no more tax deductions can be claimed.

Straight-line depreciation

- **Straight-line depreciation** reduces the value of a depreciating asset uniformly over its effective life.
  - → Straight-line depreciation is also known as prime cost depreciation.
- The salvage value of an asset using straight-line depreciation can be calculated using the formula:

  $$S = V_0 - Dn$$

  where S = salvage value, V_0 = initial value of the asset, D = depreciation per time period and n = number of time periods.
  - → The straight-line depreciation formula can be rearranged to find the values of V_0, D or n if you are given the other values.

Diminishing-value depreciation

- **Diminishing-value depreciation** reduces the value of a depreciating asset by a constant percentage each time period (usually per year).
  - → Diminishing-value depreciation is also known as declining-balance depreciation.
- The salvage value of an asset using diminishing-value depreciation can be calculated using the formula:

  $$S = V_0 (1 - r)^n$$

  where S = salvage value, V_0 = initial value of the asset, r = depreciation rate per time period and n = number of time periods.
  - → The diminishing-value depreciation formula can be rearranged to find the values of V_0, r or n if you are given the other values.

Example 1E.1 Creating a depreciation schedule (straight-line depreciation)

Construct a depreciation schedule to show the value of a piece of machinery at the end of the first three years of the machine's life if it cost \$22 000 and it depreciates in value by \$2200 each year.

THINK

1 Create a table with the headings 'Year', 'Value at start of year', 'Depreciation amount' and 'Value at end of year'.

2 Enter the value at the start of the first year and the depreciation amount each year into the table.

3 Calculate the value at the end of each year by subtracting the depreciation amount from the value at the start of each year.

4 The value at the end of a year can be copied across to the value at the start of the next year.

WRITE

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$22 000	\$2200	\$19 800
2	\$19 800	\$2200	\$17 600
3	\$17 600	\$2200	\$15 400

Value at end of year 1:
\$22 000 – \$2200 = \$19 800
Value at end of year 2:
\$19 800 – \$2200 = \$17 600
Value at end of year 3:
\$17 600 – \$2200 = \$15 400

Example 1E.2 Creating a depreciation schedule (diminishing-value depreciation)

Construct a depreciation schedule to show the value of a piece of furniture at the end of the first three years of the machine's life if it cost \$6000 and it depreciates in value by 10% each year.

THINK

1 Create a table with the headings 'Year', 'Value at start of year', 'Depreciation amount' and 'Value at end of year'.

2 Calculate the depreciation amount for each year by multiplying the value at the start of the year by the depreciation percentage.

3 Calculate the value at the end of each year by subtracting the depreciation amount from the value at the start of each year.

4 The value at the end of a year can be copied across to the value at the start of the next year.

WRITE

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$6000	\$600	\$5400
2	\$5400	\$540	\$4860
3	\$4860	\$486	\$4374

Depreciation amount for year 1:
\$6000 × 10% = \$6000 × 0.1
= \$600
Value at end of year 1: \$6000 – \$600 = \$540
Depreciation amount for year 2:
\$5400 × 10% = \$5400 × 0.1
= \$540
Value at end of year 2: \$5400 – \$540 = \$4860
Depreciation amount for year 3:
\$4860 × 10% = \$4860 × 0.1
= \$486
Value at end of year 3: \$4860 – \$486 = \$4374

Example 1E.3 Using the straight-line depreciation formula

A car purchased for $25 000 is depreciated by $3125 per year.

a What is the salvage value of the car after 3 years?

b After how many years is the salvage value of the car $9375?

THINK	WRITE
a **1** State the known values in the formula for straight-line depreciation.	**a** $V_0 = 25\,000, D = 3125, n = 3$
2 Substitute the values into the formula and solve for *S*.	$S = V_0 - Dn$ $= 25\,000 - 3125 \times 3$ $= 25\,000 - 9375$ $= \$15\,625$
b **1** State the known values in the formula for straight-line depreciation.	**b** $S = 9375, V_0 = 25\,000, D = 3125$
2 Substitute the values into the formula and solve for *n*.	$S = V_0 - Dn$ $9375 = 25\,000 - 3125n$ $3125n = 25\,000 - 9375$ $3125n = 15\,625$ $n = \frac{15\,625}{3125}$ $n = 5$ years

Example 1E.4 Using the diminishing-value depreciation formula

A computer purchased for $2800 is depreciated by 25% per year. Use the diminishing-value formula to calculate the salvage value after 4 years correct to the nearest dollar.

THINK	WRITE
1 State the known values in the formula for diminishing-value depreciation, converting the depreciation rate to a decimal.	$V_0 = 25\,000, r = 0.25, n = 4$
2 Substitute the values into the formula and solve for *S*.	$S = V_0\,(1 - r)^n$ $= 25\,000 \times (1 - 0.25)^4$ $= 25\,000 \times 0.75^4$ $= 7910.15...$
3 Round the answer to the nearest dollar.	$= \$7910$

Helpful hints

- ✔ For diminishing-value depreciation, the amount of depreciation in any given year will always be less than in all previous years.
- ✔ When compared to straight-line depreciation, diminishing-value depreciation will decrease the value by more in the early years and by less in later years.

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Exercise 1E Depreciation

△ 1–6, 8, 9, 12, 14 □ 1, 3, 5, 7, 8, 10, 13–16 ⬠ 5, 7–11, 13, 16–18

1E.1 **1** Construct a depreciation schedule to show the value of a piece of machinery at the end of the first three years of the machine's life if it cost \$36 000 and it depreciates in value by \$3600 each year.

2 Given the following depreciation schedule, what will be the salvage value of this asset at the end of the fifth year?

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$30 000	\$2400	\$27 600
2	\$27 600	\$2400	\$25 200
3	\$25 200	\$2400	\$22 800

1E.2 **3** Construct a depreciation schedule to show the value of a piece of furniture at the end of the first three years of the furniture's life if it cost \$8000 and it depreciates in value by 10% each year.

4 Given the following depreciation schedule, what will be the salvage value of this asset at the end of the fourth year?

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$20 000	\$4000	\$16 000
2	\$16 000	\$3200	\$12 800
3	\$12 800	\$2560	\$10 240

1E.3 **5** A truck purchased for \$72 000 is depreciated by \$4800 per year.

a What is the salvage value of the truck after 3 years?

b After how many years is the salvage value of the truck \$38 400?

6 A motorbike purchased for \$18 200 has an effective life of 7 years. If the salvage value of the bike at the end of its effective life is \$0, by how much was it depreciated each year?

7 An EFTPOS machine purchased for \$1800 is depreciated by \$300 each year.

a What is the value of the machine after 2 years?

b What is the effective life of the machine?
Hint: When will the salvage value be \$0?

1E.4 **8** An oven purchased for \$2200 is depreciated by 10% per year. Use the diminishing-value formula to calculate the salvage value after 4 years correct to the nearest dollar.

9 A dishwasher purchased for \$850 is depreciated by 12.5% per year. Use the diminishing-value formula to calculate the salvage value after 3 years correct to the nearest dollar.

10 Carpet covering a small room was depreciated in value from \$4000 to \$2560 in two years using diminishing-value depreciation. What was the annual rate of depreciation? Hint: Calculate the value of $(1 - r)^2$ and take the square root of both sides of the equation.

11 Window blinds for an office were purchased for \$8000. After 3 years the salvage value of the blinds, using diminishing-value depreciation, was \$6859. What was the rate of depreciation?

12 Explain why diminishing the value of an asset by 25% for two years does not result in the same salvage value as diminishing the value of an asset by 50%.

13 Explain the relationship between the two different methods of depreciation and simple and compound interest.

PROBLEM SOLVING AND REASONING

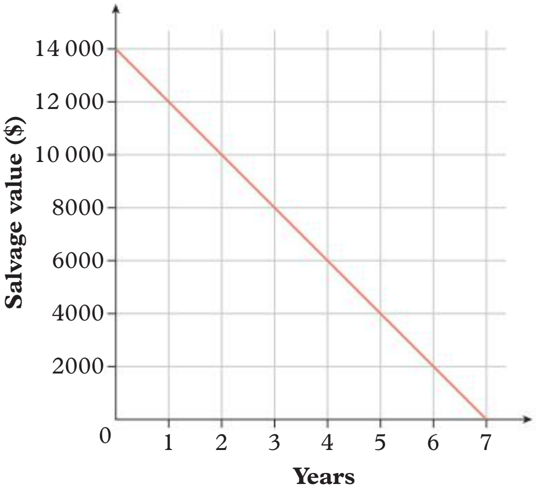
14 The graph on the right shows the salvage value of an asset using straight-line depreciation.

a What was the purchase price of the asset?

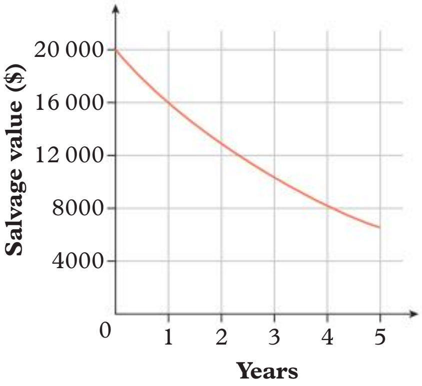
b What is the amount of depreciation each year?

c What is the salvage value of the asset after 4 years?

d What is the effective life of the asset?



15 The following graph shows the salvage value of an asset using diminishing-value depreciation.



a What was the purchase price of the asset?

b What is the salvage value of the asset after 1 year?

c What is the rate of depreciation?

16 Kumar is depreciating machinery for his business, which cost $40 000. Using the straight-line method of depreciation, he can depreciate the machinery by 6% of the initial cost per year. Using the diminishing-value method of depreciation, he can depreciate the machine by 12% each year.

a By how much will the machinery be depreciated in the first year using each form of depreciation?

b Create a depreciation schedule for the first three years of the machine's life for both methods of depreciation. Round your answers to the nearest dollar where necessary.

c In what year will the depreciation amount be greater using the straight-line method of depreciation than the diminishing-value method of depreciation for the first time?

d Considering that depreciation can be used as a tax deduction, give a reason why Kumar might choose the diminishing-value method of depreciation.

17 In general, the diminishing-value method of depreciation produces twice the amount of depreciation as the straight-line method of depreciation in the first year.

How does the percentage of diminishing-value depreciation relate to the amount of time it takes for the diminishing-value method of depreciation to result in a lower amount of depreciation than the straight-line method of depreciation? Use an example to support your answer.

18 What rate of diminishing-value depreciation is required to make the salvage value half of the initial value in:

a 2 years

b 3 years

c 4 years

d 5 years?

Give your answers correct to two decimal places.

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Interactive skillsheet
Depreciation

Topic quiz
1E

Chapter summary

Earning money

- Income: money received in exchange for work or investment
- Wage: a fixed rate of income for working a given period
- Salary: fixed income paid on a regular basis
- Commission: being paid a proportion of sales
- Royalties: payments made for the use of assets
- Piece work: income paid for each item produced or action performed.
- Time-and-a-half: 1.5 × standard wage
- Double time: 2 × standard wage
- Leave loading: 17.5% additional pay for annual leave

Hourly wage = \$28

$$\begin{aligned}\text{Time-and-a-half} &= \$28 \times 1.5\\ &= \$42\end{aligned}$$

$$\begin{aligned}\text{Double time} &= \$28 \times 2\\ &= \$56\end{aligned}$$

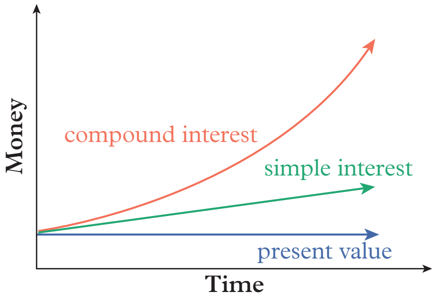
Income tax

Taxable income	Tax on this income
0–\$18 200	Nil
\$18 201–\$45 000	19 cents for each \$1 over \$18 200
\$45 001–\$120 000	\$5092 plus 32.5 cents for each \$1 over \$45 000
\$120 001–\$180 000	\$29 467 plus 37 cents for each \$1 over \$120 000
\$180 001 and over	\$51 667 plus 45 cents for each \$1 over \$180 000

- Taxable income = total income – tax deductions
- Medicare levy: 2% tax on taxable income
- Net income = taxable income – income tax – Medicare levy

Interest

- Interest is a sum of money added to a principal amount that was either invested or borrowed.



Simple interest

$$I = Prn$$

I interest
P principal
r interest rate
n number of time periods

Compound interest

$$FV = PV(1 + r)^n$$

FV future value
PV present value
r interest rate of time period
n number of time periods

Straight-line depreciation

$S = V_0 - Dn$
S = salvage value
V_0 = initial value of the asset
D = depreciation per time period
n = number of time periods

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$22 000	\$2200	\$19 800
2	\$19 800	\$2200	\$17 600
3	\$17 600	\$2200	\$15 400

Diminishing-value depreciation

$S = V_0(1 - r)^n$
S = salvage value
V_0 = initial value of the asset
r = depreciation rate per time period
n = number of time periods

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$6000	\$600	\$5400
2	\$5400	\$540	\$4860
3	\$4860	\$486	\$4374

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Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- bonus
- commission
- compound interest
- depreciating asset
- depreciation
- diminishing-value depreciation
- future value
- income
- income tax
- interest
- interest rate
- investment
- leave loading
- loan
- Medicare levy
- penalty rates
- piece work
- present value
- principal
- royalty
- salary
- simple interest
- straight-line depreciation
- tax deduction
- taxable income
- time period
- wage

1 Explain when the interest rate in the compound interest formula will be expressed per annum.

2 Explain how an individual's net income is calculated.

3 Explain the difference between straight-line depreciation and diminishing-value depreciation.

4 Identify the key terms being referenced in each of these definitions.

a an extra payment on top of annual leave pay that is usually calculated as 17.5% of normal pay for this period

b an asset with a limited life expectancy which declines in value over its effective life

5 Using an example, provide a definition in your own words for the following key terms.

a principal

b salvage value

6 Complete the following sentences using words from the key term list.

a ________ are higher rates of pay given for working overtime, on weekends or on public holidays.

b A ________ is the regular interval of time at which ________ is calculated and added to the present value.

Multiple choice

1A **1** In the last financial year, Pete earned a salary of \$1735 per week with a 4% bonus. Pete also earned \$725 in interest on his investments. His tax deductions for the year were \$420 in donations to charity and \$120 for an accountant to manage his tax affairs. What is Pete's taxable income correct to the nearest dollar?

A \$86 796 **B** \$90 405 **C** \$92 564 **D** \$94 014 **E** \$95 094

1A **2** Trina receives a base salary of \$3200/month plus a 6% commission on sales. In January she sells \$35 000 worth of product. What is her monthly income?

A \$1908 **B** \$2100 **C** \$2292 **D** \$3410 **E** \$5300

1B **3** An amount of \$1500 is borrowed at 6.75% p.a. simple interest for 24 months. Which set of values should be substituted into the simple interest formula for this loan?

A $P = 1500, r = 6.75, n = 24$ **B** $P = 1500, r = 6.75, n = 2$ **C** $P = 1500, r = 0.0675, n = 2$

D $P = 1500, r = 0.0675, n = 24$ **E** $P = 1500, r = 0.675, n = 24$

1B **4** What is the annual interest rate applied to a loan of \$20 000 if the amount of simple interest charged on the loan over 3 years is \$2550?

A 0.0425% **B** 0.354% **C** 4.25% **D** 14.875% **E** 3.54%

1C **5** An amount of \$10 000 invested for 4 years earns interest at 8.4% p.a. compounded annually. What is the value of the investment after 2 years?

A \$11 750.56 **B** \$11 680 **C** \$12 737.61 **D** \$13 807.57 **E** \$11 822.44

1C **6** If an amount of \$4800 is invested for 3 years at 3.6% p.a. with interest compounded monthly, what is the number of time periods?

A 3 **B** 6 **C** 12 **D** 36 **E** 152

1D **7** Over a 4-year period, the value of an investment grew from \$18 500 to \$27 000 with interest compounded quarterly. Given this information, which variable in the compound interest formula don't you know the value of?

A FV **B** PV **C** r **D** n **E** None of them

1D **8** The annual interest rate on an investment compounded quarterly is 8.25% p.a. What is the interest rate per time period?

A 8.25% **B** 2.0625% **C** 0.0825% **D** 0.020 625% **E** 4.125%

1E **9** A car purchased for \$35 000 is depreciated by \$4375 per year. What is the salvage value of the car after 4 years?

A \$13 125 **B** \$17 500 **C** \$21 875 **D** \$30 625 **E** \$52 500

1E **10** A piece of machinery purchased for \$13 500 is depreciated by 7.5% per year using diminishing-value depreciation. Which of the following best represents the salvage value after 4 years?

A \$12 488 **B** \$10 649 **C** \$9883 **D** \$9142 **E** \$1013

Short answer

1A **1** Lloyd takes 3 weeks of annual leave, for which he receives additional leave loading of 17.5%. If Lloyd works 32 hours per week and receives a wage of \$41.20/hour, how much income does he receive for his 3 weeks of annual leave? Give your answer correct to the nearest dollar.

1A **2** **a** Poppy receives an annual salary of \$85 592. What is her weekly wage?

b Hui receives an annual salary of \$107 042. What is her fortnightly wage?

c Hiram receives an annual salary of \$58 620. What is his monthly wage?

1A **3** Use the 2023–2024 tax table to calculate the income tax payable, to the nearest dollar, on the following taxable incomes.

Taxable income	Tax on this income
0–\$18 200	Nil
\$18 201–\$45 000	19 cents for each \$1 over \$18 200
\$45 001–\$120 000	\$5092 plus 32.5 cents for each \$1 over \$45 000
\$120 001–\$180 000	\$29 467 plus 37 cents for each \$1 over \$120 000
\$180 001 and over	\$51 667 plus 45 cents for each \$1 over \$180 000

a \$57 724 **b** \$90 130 **c** \$36 892 **d** \$103 359

1A **4** Lucy receives a monthly wage of \$7345.

a What is Lucy's weekly wage?

b Lucy receives a leave loading of 17.5% for her four weeks of annual leave each year. What is Lucy's total annual income from her job? Give your answer correct to the nearest dollar.

c Last year Lucy had an additional income of \$3550, as well as tax deductions of \$1100. What was Lucy's net income for the year?

1B **5** Calculate the simple interest if:

a $P = \$2500$, $r = 6\%$ p.a. and $n = 3$ years **b** $P = \$28\,000$, $r = 4.25\%$ p.a. and $n = 5$ years

1B **6** For each investment below, calculate:

i the amount of simple interest earned

ii the value of the investment at the end.

a \$150 000 invested at an interest rate of 4.8% p.a. for $3\frac{1}{2}$ years.

b \$9450 invested at an interest rate of 2.95% p.a. for 5 months.

1B 7 Find the unknown value in each simple interest situation below. Write the value for r as a percentage and the value for n in years.

a $I = \$308$, $P = \$3850$, $r = 6.4\%$ p.a., $n = ?$

b $I = \$765$, $P = ?$, $r = 4.25\%$ p.a., $n = 12$ months

c $I = \$9843.75$, $P = \$35\,000$, $r = ?$, $n = 2.5$ years

1C 8 For each investment below, interest is calculated at the end of each year and added to the present value. Calculate the total amount of interest compounded each year during the term of the investment, as well as the value of the investment at the end of each year.

a \$15 000 is invested at 4.5% p.a. for 3 years.

b \$6850 is invested at 6.4% p.a. for 4 years.

1C 9 Given the information provided in the table below, use the compound interest formula to determine the future value (FV) for each investment.

	Present value	Interest rate (p.a.)	Term	Time period
a	\$2250	7.5%	2 years	annually
b	\$18 750	9.2%	4 years	quarterly
c	\$150 000	6.24%	4 years	every 6 months

1D 10 With reference to the compound interest formula, find each unknown value, correct to two decimal places.

a $FV = ?$, $PV = \$5650$, $r = 0.058$, $n = 3$

b $FV = \$20\,000$, $PV = ?$, $r = 0.08$, $n = 4$

c $FV = \$10\,000$, $PV = \$8000$, $r = ?$, $n = 6$

1D 11 Tahlia deposited \$6000 in a compound interest savings account and the amount in the account grew to \$6054.16 after three compounding time periods.

a Find the value of r (as a decimal), rounded to three decimal places.

b Find the interest rate per year (as a percentage to one decimal place) if interest is compounded:

i yearly **ii** quarterly **iii** monthly **iv** weekly.

c Which of the interest rates per year from part **b** do you think are realistic for a regular savings account at a bank?

1E 12 Construct a depreciation schedule to show the value of a commercial refrigerator at the end of the first three years of the refrigerator's life if it cost \$4450 and it depreciates in value by \$371 each year.

1E 13 Construct a depreciation schedule to show the value of a dry ice machine at the end of the first three years of the machine's life if it cost \$12 000 and it depreciates in value by 10% each year using the diminishing-value method of depreciation.

1E 14 A car used for a ride-sharing service was depreciated in value from \$32 000 to \$24 500 in two years using diminishing-value depreciation. What was the annual rate of depreciation?

Analysis

1 Susan invested \$12 000 with her bank for 4 years with interest compounded quarterly throughout the term of the investment. The interest rate earned on the investment varied at different stages.

In the first year, the interest rate was $5\frac{1}{2}\%$ p.a.; for the next 2 years the interest rate remained fixed at 5.7% p.a.; and the rate increased to 6.2% p.a. for the entire final year.

a How many compounding time periods were there for the term of the investment?

b Consider the annual interest rates stated above. What value for r would be used in the compound interest formula in the:

i first year **ii** middle two years **iii** final year?

c Determine the value of the investment at the end of the first year.

d State the values for PV, r and n you would use to calculate the growth in the investment from the start of the second year to the end of the third year.

e What was the value of the investment at the end of the third year?

- **f** What was the value of the investment at the end of the whole 4-year term?
- **g** What was the total amount of interest Susan earned on her investment?
- **h** Write the interest from part **g** as a percentage of the original amount invested, correct to two decimal places.
- **i** Susan had hoped that her investment would grow to $15 500 by the end of the 4-year term. Keeping the compounding time periods the same, what constant annual interest rate would Susan need to have received to achieve her desired final value? Give your answer correct to two decimal places.
- **j** If the interest was compounded monthly, how does the answer to part **i** change?

2 Dani is preparing her tax return for the 2023–2024 financial year.

- **a** Dani earns a weekly salary of $1780 from her job. What is her annual salary?
- **b** Dani also receives a 6% royalty from a book she co-authored. The book sales were $32 500 for the year. How much income did Dani receive from the royalty?
- **c** Dani purchased a set of books for work purposes for which she can depreciate as a tax deduction. The books cost $1200 and can be depreciated using straight-line depreciation by 10% of the initial cost each year. What tax deduction can Dani claim from the depreciation of the books?
- **d** Dani is also claiming a tax deduction for the depreciation of a desk she purchased for working from home the previous year. The desk cost $800 and is depreciated by 10% each year. What tax deduction can Dani claim from the depreciation of the desk?
- **e** For the year, Dani earns $2535 in interest on her investments and has $858 in additional tax deductions. What is Dani's taxable income for the year?
- **f** How much income tax, to the nearest dollar, does Dani pay for the year?
- **g** Calculate the Medicare levy payable on Dani's taxable income, correct to the nearest dollar.
- **h** Calculate Dani's net income, to the nearest dollar, for the year.

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Calculate earnings for different periods of time ☐ Calculate an individual's taxable income ☐ Calculate an individual's net income	☐ Go back to Topic 1A Earning money
☐ Calculate simple interest ☐ Calculate the principal, interest rate or number of time periods of a simple interest investment or loan	☐ Go back to Topic 1B Simple interest
☐ Understand how compound interest works ☐ Use the compound interest formula to calculate the future value ☐ Calculate the interest rate for a time period as well as the number of time periods	☐ Go back to Topic 1C Compound interest
☐ Use the compound interest formula to calculate the present value ☐ Use the compound interest formula to calculate the interest rate	☐ Go back to Topic 1D Compound interest calculations
☐ Create depreciation tables ☐ Solve problems involving straight-line depreciation ☐ Solve problems involving diminishing-value depreciation	☐ Go back to Topic 1E Depreciation

2

Algebra

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Indices
✔ Roots
✔ Highest common factor
✔ Like terms
✔ Equivalent fractions

Curriculum links

- Simplifies algebraic fractions with numerical denominators and expands algebraic expressions (MA5-ALG-C-01)
- Simplifies algebraic expressions involving positive-integer and zero indices, and establishes the meaning of negative indices for numerical bases (MA5-IND-C-01)
- ADV Simplifies algebraic fractions involving indices, and expands and factorises algebraic expressions (MA5-ALG-P-01)
- ADV Selects and applies appropriate algebraic techniques to operate with algebraic fractions, and expands, factorises and simplifies algebraic expressions (MA5-ALG-P-02)
- ADV Applies the index laws to operate with algebraic expressions involving negative-integer indices (MA5-IND-P-01)

2A Indices

Learning intentions

By the end of this topic you will be able to …

✔ use the index laws to simplify algebraic expressions.

Inter-year links

Year 7	1G Indices and square roots
Year 8	4A Indices
Year 9	2A Indices

Index notation

Key content video

- **Index notation**, or **index form**, is used for indicating the number of times a **base** number is multiplied by itself repeatedly. The number of repeated multiplications corresponds to the **index** (also called the **exponent**).

$$\underset{\text{index form}}{\text{base} \longrightarrow \overset{\substack{\text{index/exponent}\\ \downarrow}}{3^4}} = \underset{\text{expanded form}}{3 \times 3 \times 3 \times 3} = \underset{\text{basic numeral}}{81}$$

$$\underset{\text{index form}}{\text{base} \longrightarrow \overset{\substack{\text{index/exponent}\\ \downarrow}}{a^3}} = \underset{\text{expanded form}}{a \times a \times a}$$

Index laws

- The **index laws** are used for simplifying expressions that include terms written in index form.
- The index laws for multiplying or dividing terms in index form only apply when both terms have the same base.
- Any non-zero base raised to the power of zero is equal to one.

Index law	Algebraic form	Example
Product of powers law	$a^m \times a^n = a^{m+n}$	$2^3 \times 2^5 = 2^{3+5} = 2^8$
Quotient of powers law	$a^m \div a^n = a^{m-n}$	$2^5 \div 2^3 = 2^{5-3} = 2^2$
Power of a power law	$(a^m)^n = a^{m \times n}$	$(2^3)^5 = 2^{3 \times 5} = 2^{15}$
The zero index	$a^0 = 1$	$2^0 = 1$

- We can remove indices outside brackets using the associative property of multiplication and use a combination of index laws to simplify complex expressions.

$$(a \times b)^m = \underbrace{a \times b \times \ldots \times a \times b}_{m \text{ times}} = a^m \times b^m \qquad (2 \times 3)^5 = 2^5 \times 3^5$$

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \times \ldots \times \frac{a}{b}}_{m \text{ times}} = \frac{a^m}{b^m} \qquad \left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$$

Example 2A.1 Using the index laws

Use an appropriate index law to simplify each of these expressions.

a $(g^3)^4$ **b** $\left(\frac{w}{3}\right)^3$ **c** $3k^2 \times 6k^5$ **d** $8a^{12} \div 16a^7$

THINK	WRITE
a Apply the power of a power law to expand the brackets by multiplying the indices.	**a** $(g^3)^4 = g^{3 \times 4}$ $= g^{12}$

b	**1**	Expand the brackets by applying the index to the numerator and the denominator.	**b** $\left(\frac{w}{3}\right)^3 = \frac{w^3}{3^3}$
	2	Simplify.	$= \frac{w^3}{27}$
c	**1**	Multiply 3 and 6.	**c** $3k^2 \times 6k^5 = 18 \times k^2 \times k^5$
	2	Since the indices have the same base, apply the product of powers law by adding the indices.	$= 18 \times k^{2+5}$
	3	Simplify.	$= 18k^7$
d	**1**	Write the expression as a fraction.	**d** $8a^{12} \div 16a^7 = \frac{8a^{12}}{16a^7}$
	2	Divide 8 and 16 by their highest common factor (HCF), which is 8.	$= \frac{\cancel{8}^{1}a^{12}}{\cancel{16}^{2}a^7}$
	3	Since the indices have the same base, apply the quotient of powers law by subtracting the indices.	$= \frac{a^{12}}{2a^7}$ $= \frac{a^{12-7}}{2}$
	4	Simplify.	$= \frac{a^5}{2}$

Example 2A.2 Combining index laws to simplify algebraic expressions

Use the appropriate index laws to simplify these expressions.

a $3a^4 \times 10a \div 6a^2$ **b** $(2z^7)^3$ **c** $\frac{x^6 \times 6x^2}{8(x^4)^2}$

		THINK	WRITE
a	**1**	Write the expression as a fraction.	**a** $3a^4 \times 10a \div 6a^2 = \frac{3a^4 \times 10a}{6a^2}$
	2	Simplify the numerator by multiplying 3 and 10 and using the product of powers law to add the indices.	$= \frac{30a^5}{6a^2}$
	3	Divide 30 and 6 by their HCF, which is 6.	$= \frac{\cancel{30}^{5}a^5}{\cancel{6}^{1}a^2}$ $= \frac{5a^5}{a^2}$
	4	Apply the quotient of powers law to subtract the indices.	$= 5a^{5-2}$ $= 5a^3$
b	**1**	Expand the brackets by applying the index to the numeral and the variable.	**b** $(2z^7)^3 = 2^3 \times (z^7)^3$
	2	Apply the power of a power law to multiply the indices and simplify.	$= 8 \times z^{7 \times 3}$ $= 8z^{21}$
c	**1**	Simplify the numerator by applying the product of powers law. Simplify the denominator by applying the power of a power law.	**c** $\frac{x^6 \times 6x^2}{8(x^4)^2} = \frac{6x^8}{8x^8}$
	2	Divide 6 and 8 by their HCF, which is 2.	$= \frac{\cancel{6}^{3}x^8}{\cancel{8}^{4}x^8}$ $= \frac{3x^8}{4x^8}$
	3	Divide the numerator and denominator by the common factor x^8.	$= \frac{3\cancel{x^8}}{4\cancel{x^8}}$ $= \frac{3}{4}$

Helpful hints

- ✔ Remember, the product and quotient of powers laws only apply if the expressions have the same base.
- ✔ Expanding brackets raised to an index only works if the brackets contain division and multiplication. You cannot apply the same method to expand brackets containing addition and subtraction!

  $(a + b)^m \neq a^m + b^m$ and $(a - b)^m \neq a^m - b^m$

  In order to expand these expressions, you will need to use the distributive law covered in section **2D**.
- ✔ Avoid errors in your working out by applying the index laws one at a time.

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Exercise 2A Indices

△ 1–7, 8(a, b), 10(a–e), 11 □ 1(e–h), 3(d, g–i), 4–6(2nd, 3rd columns), 7, 8(b, d), 9–10(a–e), 12, 17 ⬠ 6–7(f–i), 8, 9–10(f–i), 13–17

UNDERSTANDING AND FLUENCY

1 Use an appropriate index law to simplify these expressions.

a $a^7 \div a^3$ **b** $(b^4)^5$ **c** $c^9 \times c^6$ **d** $d^3 \div d^2$

e $y^5 \times y^5$ **f** $(m^7)^2$ **g** $n^{15} \div n$ **h** $h \times h^6$

i $\frac{x^{11}}{x^5}$ **j** $(p^6)^3$ **k** $\frac{w^4}{w}$ **l** $a^2 \times a \times a^7$

2 **a** What is the value of a non-zero term expressed in index form raised to the power of zero?

b Write the value of each of the following.

i 3^0 **ii** 100^0 **iii** m^0 **iv** $5^9 \div 5^9$

3 Use the property $a^0 = 1$ to simplify each of these expressions.

a $3a^0$ **b** $(3a)^0$ **c** $8m^0$ **d** $(8m)^0$

e $(-5x)^0$ **f** $a^0 + b^0$ **g** $j^0 - k^0$ **h** $p^0 + p^0$

i $2c^0 + 4g^0$ **j** $w^0 + x^0 - y^0$ **k** $(m + n)^0$ **l** x^0

4 Use the property $(a \times b)^m = a^m \times b^m$ to write each of these expressions without brackets.

a $(a \times b)^5$ **b** $(k \times p)^9$ **c** $(4 \times c)^3$

d $(7 \times y)^{13}$ **e** $(xy)^4$ **f** $(cd)^7$

g $(3w)^2$ **h** $(6g)^6$ **i** $(2p)^8$

5 Use the property $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ to write each of these expressions without brackets.

a $\left(\frac{a}{b}\right)^4$ **b** $\left(\frac{f}{g}\right)^7$ **c** $\left(\frac{k}{2}\right)^6$

d $\left(\frac{5}{x}\right)^6$ **e** $\left(\frac{a}{bc}\right)^2$ **f** $\left(\frac{m}{3n}\right)^5$

2A.1 **6** Use the appropriate index laws to simplify these expressions.

a $5x^2 \times x^3$ **b** $2a^2 \times 3a^5$ **c** $(3x)^2$

d $(2x^4)^3$ **e** $\frac{4a^5}{a^2}$ **f** $\frac{12x^8}{6x^3}$

g $\frac{4x^3}{2x^3}$ **h** $3x^6 \times 2x^2 \times 5x^4$ **i** $\frac{(6a^4)^2}{12a^5}$

2A.2 **7** Use the appropriate index laws to simplify these expressions.

a $\frac{4y^2 \times 12y^5}{6y^3}$ **b** $3(2x^3)^3$ **c** $(3k)^2 \times (2k)^3$

d $(3b^5)^2 \times 5b^2$ **e** $3a^3 \times (2a^4)^2 \times 5a^9$ **f** $15c^4 \times 6c^3 \div (3c)^2$

g $\frac{(2n^2)^2 \times 5n^3}{10n^5}$ **h** $\frac{3v^2}{4} \times \frac{8v^4}{6}$ **i** $\left(\frac{5m^5}{2}\right)^2 \times 4m^3$

PROBLEM SOLVING AND REASONING

8 Use the index laws to simplify each expression. Write your answers in index form.

a $\dfrac{p^4 \times (p^5)^3}{p^{10}}$ **b** $\dfrac{2(t^2)^8 \times (3t^5)^3}{3t^{11}}$

c $\dfrac{(5m^{11}n^{10})^8 \times (5mn^6)^2}{(5m^9n^7)^5 \times (5m^2n^3)^3}$ **d** $\dfrac{(8j^5p)^0 \times (j^0p^4)^3}{(jp^2)^6}$

9 Use the index laws to decide whether each of these statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $x^7 \times x \times x^7 = x^{14}$ **b** $(5a)^3 = 5 \times a^3$ **c** $-k^0 = -1$

d $a^2b^3 \times a^3b^4 = a^{12}b^{12}$ **e** $x^7y^4 \div y^6 = \dfrac{x^7}{y^2}$ **f** $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n}$

g $\dfrac{w^6 \times w^6}{w^{12}} = 0$ **h** $\dfrac{(b^5)^4 \times b^2}{(b^3)^7} = b$ **i** $\dfrac{a^8b^5}{a^4b^3} \times \dfrac{a^2b^2}{a^9b^4} = a^3$

10 Find the value of x that makes the statement true.

a $a^x \times a^3 = a^{12}$ **b** $b^x = 1$ **c** $c^x \div c^6 = c^4$

d $(d^3)^x = d^{12}$ **e** $\left(\dfrac{m}{n}\right)^x = \dfrac{m^7}{n^7}$ **f** $\dfrac{w^x \times w^4}{w^7} = w^6$

g $(2a^x)^5 = 32a^{30}$ **h** $\dfrac{(n^3)^x \times n^4}{(n^2)^6} = n^7$ **i** $\dfrac{a^3b^6}{a^4b^x} \times \dfrac{a^7(b^2)^3}{a^5b^4} = a$

11 Write $\dfrac{8^5 \times 2^4}{4^7}$ in simplest index form with a base of 2.

12 Write $\dfrac{3^{4x} \times 9^{2x}}{27^x \times 3^{3x}}$ in simplest index form with a base of 3.

13 Use the appropriate index laws to simplify $\dfrac{25^{3x} \times 625^x}{125^{2x} \times 3^{3x}}$.

14 Find the value of x and y to make each of these statements true.

a $a^xb \times a^y(b^4)^y = a^{12}b^{33}$ **b** $\dfrac{a^xb^y}{a^2b^3} \times \dfrac{a^{x+y}b^y}{a^3b^7} = a^3b^{14}$

CHALLENGE

15 A square number is a number that can be written in the form n^2, where n is a whole number.

a Show that $\dfrac{2^{13} \times 3^{11} \times 5^9}{30^7}$ is a square number.

b Find the possible values of a if $\dfrac{5^5 \times 7^7}{a}$ is a square number.

c Describe the prime factorisation of a square number.

16 Evaluate $\dfrac{6^5 \times 5^7}{2^4 \times 15^5}$ without a calculator.

17 Many students mistakenly believe for non-zero numbers a and b and positive integer n that $(x + y)^n = x^n + y^n$. This false equation is sometimes called the ‘Freshman’s dream’.

a Expand the expression $(x + y)^2$ to show the above equation is false for $n = 2$.

b Substitute the following values into both sides of the equation above to show the equation is false. You may need to use a calculator.

i $x = 2,\ y = 3,\ n = 3$ **ii** $x = 4,\ y = 5,\ n = 4$ **iii** $x = 1,\ y = 5,\ n = 5$

It is true, however, for a prime number p and non-zero numbers x and y that $(x + y)^p = x^p + y^p + mp$, where m is an integer.

c For the sets of values in part **b** where n is prime, check that this equation holds and identify the corresponding values of m.

d Prove this equation in the case $p = 2$.

e The following identity holds for all values of x and y.

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Use this identity to prove the equation holds in the case $p = 3$.

f Find a set of values x, y, n, where n is composite, such that $(x + y)^n \neq x^n + y^n + mn$ for all integers m.

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Index laws

Topic quiz
2A

2B Negative indices

Learning intentions

By the end of this topic you will be able to …

- ✔ write algebraic terms with only positive indices
- ✔ use the index laws to simplify expressions with negative indices.

Inter-year links

Year 8	4D Combining the index laws
Year 9	2D Negative indices

Negative indices

Key content video

- The **reciprocal** of a number or a term can be expressed using an index of −1. The algebraic properties on the right apply for all non-zero values *a* and *b*.
- A base with a **negative index** is the reciprocal of the base with a positive index. It indicates the number of times that the base is divided repeatedly. This can be shown by extending the same pattern that is applied to positive indices in the reverse direction.

multiply by the base

$$\dots,\ 3^{-3}=\frac{1}{27},\ 3^{-2}=\frac{1}{9},\ 3^{-1}=\frac{1}{3},\ 3^{0}=1,\ 3^{1}=3,\ 3^{2}=9,\ 3^{3}=27,\ \dots$$

divide by the base

- The index laws apply to negative indices in the same way they do for positive indices.
- These properties allow for expressions containing negative indices to be expressed using only positive indices. For example,

$$3hr^{-2}=\frac{3h}{r^2}$$

- They can also be used to write fractions in index form. For example,

$$\frac{2b^3}{a}=2a^{-1}b^3$$

Algebraic property	Example
$a^{-1}=\frac{1}{a}$	$2^{-1}=\frac{1}{2}$
$\frac{1}{a^{-1}}=a$	$\frac{1}{3^{-1}}=3$
$\left(\frac{a}{b}\right)^{-1}=\frac{b}{a}$	$\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$
$a^{-m}=\frac{1}{a^m}$	$3^{-2}=\frac{1}{3^2}$
$\frac{1}{a^{-m}}=a^m$	$\frac{1}{4^{-3}}=4^3$
$\left(\frac{a}{b}\right)^{-m}=\left(\frac{b}{a}\right)^m$	$\left(\frac{2}{3}\right)^{-4}=\left(\frac{3}{2}\right)^4$

Algebraic form	Example
$a^{-m}\times a^{-n}=a^{(-m)+(-n)}$ $=a^{-m-n}$	$5^{-3}\times 5^{-2}=5^{-3+(-2)}$ $=5^{-5}$
$a^{-m}\div a^{-n}=a^{(-m)-(-n)}$ $=a^{-m+n}$	$5^{-3}\div 5^{-2}=5^{-3-(-2)}$ $=5^{-1}$
$(a^{-m})^{-n}=a^{(-m)\times(-n)}$ $=a^{mn}$	$(5^{-3})^{-2}=5^{(-3)\times(-2)}$ $=5^6$

Example 2B.1 Determining the reciprocal of a term

Determine the reciprocal of each of the following.

a $2x$ **b** $\frac{2m}{n}$

THINK

1 Write the base with a negative index as a fraction if it is not already.

2 Find the reciprocal of the fraction. Swap the numerator and denominator.

3 Simplify the result.

WRITE

a $(2x)^{-1}=\left(\frac{2x}{1}\right)^{-1}$

$=\frac{1}{2x}$

b $\left(\frac{2m}{n}\right)^{-1}=\frac{n}{2m}$

Example 2B.2 Using the index laws with negative indices

Use the appropriate index laws to simplify each of these expressions.

a $(a^5)^{-2}$ **b** $3b^{-2} \times b^{-3}$ **c** $9c^{-3} \div 3c^5$

THINK	WRITE
a Apply the power of a power law, multiply the index of *a* by the index outside the brackets.	**a** $(a^5)^{-2} = a^{5 \times (-2)}$ $= a^{-10}$
b Apply the product of powers law, add the index of *b* and multiple the result by the coefficient 3.	**b** $3b^{-2} \times b^{-3} = 3 \times b^{-2 + (-3)}$ $= 3 \times b^{-2-3}$ $= 3 \times b^{-5}$ $= 3b^{-5}$
c 1 Write the expression as a fraction.	**c** $9c^{-3} \div 3c^5 = \frac{9c^{-3}}{3c^5}$
2 Simplify the fraction by dividing both the numerator and denominator by the highest common factor 3. Apply the quotient of the powers law to subtract the index of *c*.	$= 3 \times c^{-3-5}$
3 Simplify.	$= 3c^{-8}$

Example 2B.3 Writing terms with positive indices

Write an equivalent term with only positive indices for each of the following.

a x^{-3} **b** $x^{-5}y^3$ **c** $\left(\frac{z}{3}\right)^{-2}$

THINK	WRITE
a Apply $a^{-m} = \frac{1}{a^m}$ to write the expression with a positive index.	**a** $x^{-3} = \frac{1}{x^3}$
b 1 Write the term as a product of two factors.	**b** $x^{-5}y^3 = x^{-5} \times y^3$
2 Apply $a^{-m} = \frac{1}{a^m}$ to write x^{-5} with a positive index.	$= \frac{1}{x^5} \times y^3$
3 Simplify.	$= \frac{y^3}{x^5}$
c 1 Apply $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ to write the expression with a positive index.	**c** $\left(\frac{z}{3}\right)^{-2} = \left(\frac{3}{z}\right)^2$
2 Expand the brackets by applying the index to the numerator and the denominator.	$= \frac{3^2}{z^2}$
3 Simplify.	$= \frac{9}{z^2}$

ADVANCED

Example 2B.4 Combining index laws to simplify negative indices

Use the appropriate index laws to simplify these expressions. Write your answers using only positive indices.

a $\left(\frac{x^2}{x^{-1}}\right)^{-3}$ **b** $6g^{-4} \times 2g \div 3g^{-2}$ **c** $\frac{3}{m^{-4}} \times (m^2)^{-2}$ **d** $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2$

THINK	WRITE
a 1 Apply the quotient of powers law to simplify the fraction inside the bracket. 2 Apply the power of a power law to remove the brackets. 3 Apply $a^{-m} = \frac{1}{a^m}$ to write the expression with a positive index.	**a** $\left(\frac{x^2}{x^{-1}}\right)^{-3} = (x^{2-(-1)})^{-3}$ $= (x^{2+1})^{-3}$ $= (x^3)^{-3}$ $= x^{3\times(-3)}$ $= x^{-9}$ $= \frac{1}{x^9}$
b 1 Write the expression as a fraction. 2 Simplify the numerator by multiplying the coefficients, 6 and 2, in the numerator and applying the product of powers law to add the indices. 3 Divide 12 and 3 by their HCF of 3. 4 Apply the quotient of powers law to subtract the indices. 5 Apply $a^{-1} = \frac{1}{a}$ to write the number as a fraction with a positive index.	**b** $6g^{-4} \times 2g \div 3g^{-2} = \frac{6g^{-4} \times 2g}{3g^{-2}}$ $= \frac{\cancel{12}^{4}g^{-4+1}}{\cancel{3}^{1}g^{-2}}$ $= \frac{4g^{-3}}{g^{-2}}$ $= 4g^{-3-(-2)}$ $= 4g^{-3+2}$ $= 4g^{-1}$ $= 4 \times \frac{1}{g}$ $= \frac{4}{g}$
c 1 Separate the denominator from the numerator and write each factor with a positive index by applying: $\frac{1}{a^{-m}} = a^m$ and $a^{-m} = \frac{1}{a^m}$. 2 Apply the power of a power law to remove the brackets, then the quotient of powers law to subtract the indices. 3 Simplify. Remember that any non-zero base raised to a zero index is equal to one.	**c** $\frac{3}{m^{-4}} \times (m^2)^{-2} = 3 \times \frac{1}{m^{-4}} \times (m^2)^{-2}$ $= 3 \times m^4 \times \frac{1}{(m^2)^2}$ $= 3 \times m^4 \times \frac{1}{m^4}$ $= 3 \times m^{4-4}$ $= 3m^0$ $= 3$
d 1 Apply $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ and $\frac{1}{a^{-m}} = a^m$ to write the expression using positive indices. 2 Apply the associative property of multiplication and the power of a power law to remove the brackets and write the expression as one fraction. 3 Simplify the expression using the product and quotient of powers laws.	**d** $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2 = \left(\frac{x}{y}\right)^2 \times (xy^2)^2$ $= \frac{x^2 \times x^2y^{2\times2}}{y^2}$ $= \frac{x^{2+2}y^4}{y^2}$ $= x^4y^{4-2}$ $= x^4y^2$

Example 2B.5 Writing fractions in index form

Write the following fractions in index form.

a $\frac{5y^{-4}}{x^{-7}}$ **b** $\frac{1}{(2x)^{-2}}$ **c** $\left(\frac{3x}{y}\right)^{-2}$

THINK	**WRITE**
a 1 Apply $\frac{1}{a^{-m}} = a^m$ to write the denominator of the fraction with a positive index.	**a** $\frac{5y^{-4}}{x^{-7}} = 5y^{-4} \times x^7$
2 Simplify by removing the multiplication sign between variables and arrange variables in alphabetical order if not already.	$= 5x^7y^{-4}$
b 1 Apply $\frac{1}{a^{-m}} = a^m$ to write the denominator of the fraction with a positive index.	**b** $\frac{1}{(2x)^{-2}} = (2x)^2$
2 Simplify by applying the index to both the coefficient and the variable.	$= 4x^2$
c 1 Appy the index to both the numerator and the denominator of the fraction.	**c** $\left(\frac{3x}{y}\right)^{-2} = \frac{(3x)^{-2}}{y^{-2}}$
2 Remove the brackets in the numerator by applying the negative index to both the coefficient 3 and the variable *x*.	$= \frac{3^{-2} \times x^{-2}}{y^{-2}}$
3 Rewrite the fraction as the product of the numerator and the reciprocal of the denominator.	$= 3^{-2} \times x^{-2} \times \frac{1}{y^{-2}}$
4 Use $\frac{1}{a^{-m}} = a^m$ to write $\frac{1}{y^{-2}}$ in index form.	$= 3^{-2} \times x^{-2} \times y^2$
5 Calculate the coefficient of the term 3^{-2}.	$= \frac{1}{3^2} \times x^{-2} \times y^2$
6 Simplify by removing the multiplication signs between variables and arrange variables in alphabetical order if not already.	$= \frac{1}{9}x^{-2}y^2$

Helpful hints

✔ When converting the denominator of a fraction into index form, use brackets to make sure the index is applied to both the coefficient and all variables in the denominator.
For example,

$$\frac{1}{2xy^2z} = (2xy^2z)^{-1}, \frac{1}{2xy^2z} \neq 2xy^2z^{-1}.$$

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Exercise 2B Negative indices

△ 1–9(1st, 2nd columns), 10(a–c), 11(a, b) ■ 1–4(3rd column), 5–9(3rd, 4th columns), 10(c, e, f), 11, 14(a), 18 ⬟ 5–9(4th column), 10(d, g, h), 11–17

UNDERSTANDING AND FLUENCY

2B.1 **1** Determine the reciprocal of each of the following.

a 1 **b** -2 **c** $\frac{1}{3}$ **d** $-5d$

e $2w$ **f** $\frac{1}{2n}$ **g** $\frac{z}{xy}$ **h** $\frac{b}{a^2}$

2B.2 **2** Simplify each of these expressions.

a $x^5 \times x^{-2}$ **b** $a^{-2} \times a^{-3}$ **c** $n^2 \times n^{-2}$ **d** $y^{-6} \times y^{10}$

e $a^{-4} \div a^{-9}$ **f** $m^4 \div m^{-3}$ **g** $n^{-3} \div n^{-2}$ **h** $n^0 \div n^{-5}$

i $(k^{-6})^{-2}$ **j** $(d^{-5})^3$ **k** $y^3 \times y^{-4} \times y^2$ **l** $n^{-7} \times n^{-3} \times n^{12}$

3 Find the reciprocal of each of your answers in **2**.

4 Simplify each of these expressions.

a $a^{-1}b \times ab^{-2}$ **b** $mn \times nm^{-2}$ **c** $a^{-3}b^{-2} \times a^3b^2$ **d** $-x^3y^{-2} \times y^{-3}x^2$

e $a^{-1}bc \times cab^{-2}$ **f** $cab^{-2} \times a^{-1}bc \times cab^{-2}$ **g** $-x^{-1}y \times xy^{-2}$ **h** $x^{-1}yz \times xz^{-2}$

2B.3 **5** Write each of these terms using only positive indices.

a x^{-5} **b** $c^{-9}d^4$ **c** j^5k^{-3} **d** $x^{-1}y^8$

e $2a^3b^{-6}$ **f** $3p^{-4}q^{-5}$ **g** $w^4x^{-5}y^7$ **h** $k^{-3}m^5n^{-8}$

i $6d^2e^{-7}f$ **j** $11a^{-3}b^{-5}c^{-9}$ **k** $\frac{1}{x^{-3}}$ **l** $\frac{b^4}{c^{-6}}$

m $\frac{4m^3}{p^{-7}}$ **n** $\frac{a^{-4}}{b^{-2}}$ **o** $\frac{8x^{-2}}{2y^{-5}}$ **p** $\frac{c^{-1}d^8}{e^{-4}}$

6 Write each of these expressions without brackets and using only positive indices.

a $(x \times y)^{-4}$ **b** $(4 \times a)^{-6}$ **c** $(9 \times n)^{-1}$ **d** $(5p)^{-3}$

e $(3p)^{-2}$ **f** $(km)^{-5}$ **g** $\left(\frac{x}{4}\right)^{-3}$ **h** $\left(\frac{7}{d}\right)^{-1}$

2B.4 **7** Use the appropriate index laws to simplify each of these expressions. Write your answers using only positive indices.

a $(x^3)^{-2}$ **b** $(2y^{-3})^2$ **c** $(w^{-3})^{-4}$ **d** $\left(\frac{2x^4}{x^{-3}}\right)^2$

e $\left(\frac{a^4}{3}\right)^{-2}$ **f** $(2m)^{-2} \times (2m)^3$ **g** $\left(\frac{h^{-4}}{2}\right)^{-3}$ **h** $3c^{-4} \times 5c^{-8} \times 2c^3$

i $(2d)^{-2} \times 8d^5$ **j** $(6t^2)^{-2} \times 72(t^4)^{-3}$ **k** $(4x^{-2})^2 \times (2x^{-3})^{-3} \div (2x^4)$ **l** $\left(\frac{a^2}{3}\right)^{-3} \times \left(\frac{2a^{-1}}{3}\right)^2$

8 Simplify each of these expressions. Write your answers using only positive indices.

a $x^3 \times x^{-5}$ **b** $2x^{-4} \times x^{-1}$ **c** $7x^{-3} \times 3x^6 \times 2x^{-2}$ **d** $4x^{-8} \times 5x^3 \times x$

e $x^{-11} \div x^{-9}$ **f** $6x^4 \div (3x^{-5})$ **g** $3x^{-7} \div (12x^2)$ **h** $9x^6 \div (15x^{10})$

i $(x^{-3})^2 \times x^{-5}$ **j** $(x^{-4})^6 \times 10x^{18}$ **k** $3x^{-6} \times (x^{-1})^7 \times (x^3)^2$ **l** $(x^{-2})^7 \times (x^{-5})^{-1} \times (x^{-3})^4$

2B.5 **9** Write the following fractions in index form.

a $\frac{1}{x^2}$ **b** $\frac{1}{y^{-2}}$ **c** $\frac{a^{-4}}{b^{-2}}$ **d** $\frac{n^{-4}}{m^{-2}n^2}$

e $\frac{1}{(2x)^6}$ **f** $\frac{1}{(pq)^{-2}}$ **g** $\frac{123}{g^{-234}}$ **h** $\frac{t}{(2t)^{-2}}$

i $\left(\frac{u}{10}\right)^{-10}$ **j** $\left(\frac{3q^2}{p}\right)^{-1}$ **k** $\left(\frac{3y}{xy}\right)^{-2}$ **l** $3\left(\frac{2m}{3n}\right)^{-3}$

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PROBLEM SOLVING AND REASONING

10 Use the index laws to help you write all fractions in index form.

a $(ab)^4 \times a^7b^3$ **b** $(4p)^3 \times (6p)^2$ **c** $(2y^3)^5$ **d** $3(m^2n)^6$

e $\frac{d^5}{h^4} \times \left(\frac{d}{h}\right)^7$ **f** $\left(\frac{5w}{y}\right)^3$ **g** $\left(\frac{a^5}{b^{-2}}\right)^6$ **h** $\left(\frac{a^{-4}b^2}{c^7}\right)^4$

i $\left(\frac{wx^5}{u^3}\right)^4 \times \left(\frac{u^5}{wx}\right)^3$ **j** $\left(\frac{c^6}{a^6n^2}\right)^2 \times \frac{(5a^4)^3}{c^8n^{-5}}$ **k** $\frac{(m^5n^4)^3 \times (m^6n)^4}{(mn^3)^5}$ **l** $\frac{(2x^2)^6(y^4)^2}{(4x^4y^5)^3}$

11 Use the index laws to decide whether each of the following statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $\frac{a^5}{a^9} = a^4$ **b** $\frac{x^3}{x^6} = x^{\frac{1}{2}}$ **c** $\frac{y^3}{y^9} = \frac{1}{y^6}$

d $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{6a^4}{b^2c^6}$ **e** $(5m^4n^{-3})^{-2} = \frac{n^6}{25m^8}$ **f** $\frac{2a^{-4}b^3}{5^{-1}ab^2} = \frac{10b}{a^5}$

g $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^5z}$ **h** $(6abc)^{-2} \times (3a^{-1}b)^2 \div (4a^4c^2)^{-1} = 1$

12 Find the value of x that will make each of these statements true.

a $\frac{a^x}{a^4} = \frac{1}{a}$ **b** $\frac{3}{a^x} = 3a^4$ **c** $\left(\frac{2a^{-2}b}{c^3}\right)^x = \frac{a^4c^6}{4b^2}$

d $\frac{(ab)^x}{a^2b^4} = \frac{a}{b}$ **e** $\frac{a^3b^x}{a^{3x}b^2} = \frac{1}{b}$ **f** $\frac{y^7 \times y^x}{y^{2x}} = y^2$

13 Simplify each of these expressions.

a $x^{2a}y^{b+3} \times x^5y^{3b-2}$ **b** $\frac{m^{3x+4}n^{x-2}}{m^xn^{x-5}}$

14 Simplify each of these expressions.

a $x^{2a+1} \times x^{a-5}$ **b** $(y^m)^2 \times y^{m+5}$ **c** $\frac{a^{3x+3}}{a^{2x}}$ **d** $\frac{b^{2x}}{b^y} \times b^3 \times \frac{b^{y+1}}{b^x}$

15 Use index laws to simplify each of the following expressions. Write your answers with positive indices only.

a $\frac{3a^2b^4c}{8a^5b^3} \times \frac{4a^5b^4}{b^3c^2}$ **b** $\frac{10ac^5}{5a^5b^9c^2} \times \frac{3a^4b^3c}{(2bc^3)^2}$ **c** $\frac{4(x^2y^{-1}z)^3}{10x^4y^2} \div \frac{2x^5z^2}{5x^3y^4z^{-1}}$

CHALLENGE

16 a Express $\left(\frac{3a^2b^3}{2c^4}\right)^{-3}$ in expanded form, using positive indices only, by:

i raising each numeral and pronumeral to the power of −3 and then simplifying

ii flipping the fraction and then raising each numeral and pronumeral to the power of +3.

b Which method did you prefer and why?

17 Determine which of the following expressions is not equivalent to $\left(\frac{c^7d^3}{4a^3b^{-6}}\right)^{-2} \times \left(\frac{2a^2b^{-4}}{c^5d^2}\right)^{-3}$.

A $\left(\frac{4a^3}{b^6c^7d^3}\right)^2 \times \left(\frac{b^4c^5d^2}{2a^2}\right)^3$ **B** $\frac{c^{14}d^6}{(2a^2b^{-4})^3} \times \frac{(4a^3b^{-6})^2}{c^{15}d^6}$ **C** $\frac{(4a^3b^{-6})^2}{(2a^2b^{-4})^3c^{14}d} \times \frac{d}{c^{-15}}$

D $\left(\frac{4a^3b^{-6}}{c^7d^3}\right)^2 \times \frac{c^{15}d^6}{(2a^2b^{-4})^3}$ **E** $\left(\frac{4a^3b^{-6}}{c^7d^3}\right)^2 \times \left(\frac{c^5d^2}{2a^2b^{-4}}\right)^3$

18 Write $a^m \div a^n$ in two different forms.

a $a^m \div a^n = a^{\square} \times a^{\square}$

$= a^{\square + \square}$

b $a^m \div a^n = \frac{1}{a^{\square} \times a^{\square}}$

$= \frac{1}{a^{\square + \square}}$

$= a^{-(\square + \square)}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Negative indices

Topic quiz
2B

2C Simplifying

Learning intentions

By the end of this topic you will be able to ...

✔ evaluate algebraic expressions using substitution
✔ identify like terms to simplify algebraic expressions
✔ multiply and divide algebraic terms.

Inter-year links

Year 7	6E Substitution
Year 8	5C Adding and subtracting algebraic terms
Year 9	3A Simplifying

Key content video

Evaluating algebraic expressions

- When pronumerals are given numerical values, an algebraic expression can be evaluated by **substitution**.

  For example, if $x = 3$ and $y = 2$:

$$\begin{aligned} x^2 + 5y - \frac{x}{3} &= 3^2 + 5(2) - \frac{3}{3} \\ &= 9 + 10 - 1 \\ &= 18 \end{aligned}$$

Adding and subtracting terms

- **Like terms** are the terms of an algebraic expression that contain the same pronumerals raised to the same power. The order of the pronumerals does not matter.

  For example, the following are sets of like terms:

  → x^2 and $3x^2$
  → 3, 16 and 100
  → $2y^2z$ and $8zy^2$
  → $4w^3h^4t$, $2w^3th^4$ and $10th^4w^3$

- Like terms can be added or subtracted to simplify an algebraic expression.

  For example,

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Multiplying and dividing terms

- To multiply algebraic terms, multiply the coefficients and combine the pronumerals, using index laws if necessary.

  For example,

$$\begin{aligned} 3a^3b \times -2a^2b^2 &= 3 \times -2 \times a^{3+2} \times b^{1+2} \\ &= -6a^5b^3 \end{aligned}$$

- To divide algebraic terms, divide the coefficients and combine the pronumerals, using index laws if necessary.

  For example,

$$\begin{aligned} \frac{6a^5b^3}{3a^3b} &= \frac{{}^{2}\cancel{6} \times a^5 \times b^3}{{}^{1}\cancel{3} \times a^3 \times b} \\ &= 2 \times a^{5-3} \times b^{3-1} \\ &= 2a^2b^2 \end{aligned}$$

Example 2C.1 Using substitution to evaluate algebraic expressions

If $a = -2$, $b = 5$ and $c = -1$, evaluate:

a a^3bc^2

b $2b - ab + ac$

c $\frac{a^2}{4} + \frac{ab}{2} - c$

THINK	**WRITE**
a Replace a with −2, b with 5 and c with −1. If the index of a negative number is odd, the power is negative.	**a** $a^3bc^2 = (-2)^3(5)(-1)^2$ $= -8 \times 5 \times 1$ $= -40$
b Replace a with −2, b with 5 and c with −1. Put negative numbers in brackets to avoid errors.	**b** $2b - ab + ac = 2(5) - (-2)(5) + (-2)(-1)$ $= 10 + 10 + 2$ $= 22$
c Replace a with −2, b with 5 and c with −1. Remember the fraction line acts like a pair of brackets and groups the expression in the numerator together.	**c** $\frac{a^2}{4} + \frac{ab}{2} - c = \frac{(-2)^2}{4} + \frac{-2 \times 5}{2} - (-1)$ $= \frac{4}{4} + (-5) + 1$ $= 1 - 5 + 1$ $= -3$

Example 2C.2 Adding and subtracting algebraic terms

Simplify each of these expressions by collecting like terms.

a $9a + 12b - 3 + 3a + 2a - 1$

b $2m^2n - 8n - 10n + n^{-2} + 4nm^2$

c $\frac{3a}{4} + \frac{b}{2} - \frac{a}{4} - \frac{b}{4}$

THINK	**WRITE**
a 1 Rearrange the expression, grouping like terms together. 2 Simplify by combining like terms.	**a** $9a + 12b - 3 + 3a + 2a - 1$ $= (9a + 3a + 2a) + (12b) + (-3 - 1)$ $= 14a + 12b - 4$
b 1 Rearrange the expression by grouping like terms together. Write each term with its pronumerals in alphabetical order so that the like terms can be seen clearly. 2 Simplify by combining like terms.	**b** $2m^2n - 8n - 10n + n^{-2} + 4nm^2$ $= (2m^2n + 4m^2n) + (n^{-2}) + (-8n - 10n)$ $= 6m^2n + n^{-2} - 18n$
c 1 Rearrange the expression by grouping like terms together. 2 Simplify by combining like terms. 3 Write the expression as a single fraction by adding the numerators.	**c** $\frac{3a}{4} + \frac{b}{2} - \frac{a}{4} - \frac{b}{4} = \left(\frac{3a}{4} - \frac{a}{4}\right) + \left(\frac{b}{2} - \frac{b}{4}\right)$ $= \frac{2a}{4} + \frac{b}{4}$ $= \frac{2a + b}{4}$

Example 2C.3 Multiplying and dividing algebraic terms

Simplify these expressions by writing each one as a single term with positive indices only.

a $2ab^2 \times 6a^3c$ **b** $12pqr^2 \div (4qr)$ **c** $\frac{ab}{12} \div \frac{3b}{5}$

THINK	WRITE
a Multiply the coefficients and apply the product of powers law to multiply the terms with the same base by adding their indices.	**a** $2ab^2 \times 6a^3c = 12 \times a^{1+3}b^2c$ $= 12a^4b^2c$
b 1 Write the expression as a fraction.	**b** $12pqr^2 \div (4qr) = \frac{12pqr^2}{4qr}$
2 Simplify the coefficients by dividing them both by their HCF of 4.	$= \frac{\cancel{12}^{3}pqr^2}{\cancel{4}^{1}qr}$
3 Apply the quotient of powers law to divide the terms with the same base by subtracting their indices.	$= \frac{3pqr^2}{qr}$ $= 3pq^{1-1}r^{2-1}$
4 Simplify. Remember that any non-zero base raised to a zero index is equal to 1.	$= 3pq^0r^1$ $= 3pr$
c 1 Write the expression as the product of two fractions.	**c** $\frac{ab}{12} \div \frac{3b}{5} = \frac{ab}{12} \times \frac{5}{3b}$
2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.	$= \frac{a\cancel{b} \times 5}{12 \times 3\cancel{b}}$
3 Look for common factors. Divide both the numerator and the denominator by any common factors.	$= \frac{a \times 5}{12 \times 3}$ $= \frac{5a}{36}$

Helpful hints

✔ The order in which you write the coefficients and pronumerals in an algebraic term doesn't matter. But it is easier to identify like terms if you write the pronumerals in alphabetical order. For example,

$$acb + cb - ba + 12bca + 8ab - 6bac = abc + 12abc - 6abc + bc - ab + 8ab$$
$$= 7abc + 7ab + bc$$

✔ Recall the following rules for algebraic notation.

→ To simplify products, leave out the multiplication sign and place the number first. For example,

$$7 \times x = 7x$$

and:

$$7 \times (x + 2) = 7(x + 2)$$

→ When a pronumeral is multiplied by 1, the 1 does not need to be included. For example,

$$1 \times x = x$$

→ Division is represented using fractions. For example,

$$x \div 7 = \frac{x}{7}$$

→ Terms with coefficients that are fractions can be written in two ways. For example,

$\frac{1}{7}x$ is the same as $\frac{x}{7}$

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Exercise 2C Simplifying

△ 1, 2–4(1st column), 5–6, 7(a, c, e), 9–13, 15(1st column)

□ 2, 4, 7, 8, 10, 12, 13, 15(a–d)

⬠ 4, 7–9, 14–16

UNDERSTANDING AND FLUENCY

1 For each of the following terms:

i identify the coefficient **ii** write an example of a like term.

a $3abcd$ **b** $-4mn$ **c** xy^2 **d** $9k^2m^4p$

2C.1 **2** If $a = 1$, $b = 3$ and $c = -2$, evaluate:

a abc **b** $ab + bc$ **c** $ac - 2bc$ **d** a^4b^2

e $3ab^2c - ac$ **f** $7ac + 3bc - 4ac$ **g** $a^3b^2 + bc^3$ **h** $2a^2b + 3ac^2 - b^3$

i $\frac{a^2b}{4} + \frac{ab^2}{2}$ **j** $\frac{b}{6} + \frac{a}{3} + \frac{c}{2}$ **k** $\frac{a^2b}{4} + \frac{b^2c}{2} - \frac{ac^2}{2}$ **l** $\frac{abc - ab - ac - bc}{2}$

2C.2 **3** Simplify each of these expressions by collecting like terms.

a $8x - 5x + 6x$ **b** $3ab - 4ab - 9ab$ **c** $4y^2 + y^2 + 5y^2$

d $7m^2n + 2m^2n - m^2n$ **e** $4a + 6d + 8a + 3d$ **f** $k - 3m + 5m + 7k$

g $4 + 8y - 2 - 11y$ **h** $a^2 + 6a - 5a^2 + a$ **i** $3xy + 2x^2 - xy + x^2$

j $\frac{7a}{8} - \frac{3ba}{4} + \frac{ab}{2} - \frac{a}{12}$ **k** $\frac{nm^2}{2} - \frac{mn^2}{3} + \frac{m^2n}{4} - \frac{n^2m}{5}$ **l** $xy - \frac{4xy}{5} + \frac{3x}{4} - \frac{2y}{3} + \frac{xy}{2} - x + y$

4 Simplify each of these expressions by collecting like terms.

a $5a + 2b - a + 3b + 7a - 5b$ **b** $6mn - 3n - n + 2n^2 + m - 7mn$

c $x + 4x^2 - 8x + 1 + 5x - x^2$ **d** $2x^2y - 5x^2y - 4xy + 3 + 7xy - 8$

e $9p^2 - 3p + 2p^3 + 6p^2 - 9 + 3p$ **f** $a^2b + 4a^2 + 8ba^2 - b^2 + 3ab^2 - 5a^2$

g $5ab + 2ba$ **h** $3abc + 4bac$

i $6x^2y - 2xy + 5yx - yx^2$ **j** $14map + 2amp + 3pam - 19mpa$

5 Simplify each of the following expressions by writing each one as a single term.

a $5ab \times 2cd$ **b** $-7xy \times 3mn$ **c** $4kp \times k$

6 Simplify each of the following expressions by writing each one as a single term with positive indices only.

a $abc \div b$ **b** $6mnp \div mp$ **c** $8wx \div 4wy$

2C.3 **7** Simplify each of the following expressions by writing each one as a single term with positive indices only.

a $4x^2y^2 \times x^3y$ **b** $3ab^2 \times 2a^5b$ **c** $4m^5n^{-3} \times 6m^2n$

d $\frac{5abc}{2} \div \frac{ab}{5}$ **e** $\frac{x^5y^3z^2}{2} \div \frac{2xyz^2}{5}$ **f** $\frac{m^5n^3}{2} \times \frac{2m^2n^2}{3}$

g $6x^2y \times (xy)^2$ **h** $\frac{4xy}{5} \times \frac{3x}{4} \times \frac{2y}{3}$ **i** $\frac{ab}{5} \times \left(\frac{3a}{4}\right)^2 \div \frac{2b}{3}$

8 If $a = 3$, $b = -2$ and $c = 4$, evaluate each of the following expressions. Hint: Simplify each expression first.

a $4a + 3b + 2c - 2a - c + b$ **b** $9ab + 7a - 10a + ab$ **c** $a^2b + ab^2 + a^2c - 2a^2b + 5a^2c$

d $3abc \times b^2c \times 6a$ **e** $12a^2b^2c \div 3bc^2$ **f** $ac^2 \times 2ab^3 \div 8a^2bc$

9 The surface area of this box can be calculated using the expression $2lb + 2lh + 2bh$. Calculate the surface area of a box with each of the following sets of dimensions.

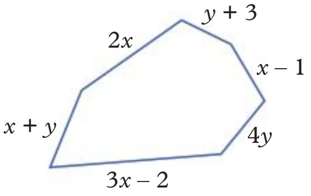
a $l = 25$ cm, $b = 15$ cm, $h = 10$ cm

b $l = 0.5$ m, $b = 0.3$ m, $h = 0.2$ m

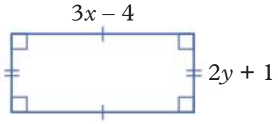
PROBLEM SOLVING AND REASONING

10 Write the perimeter of each of these shapes as an algebraic expression in simplest form.

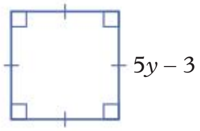
a



b



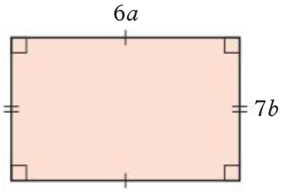
c



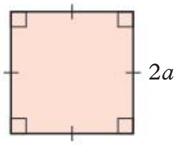
11 Calculate the perimeter of each shape in question **10** if $x = 7$ cm and $y = 2$ cm.

12 Write the area of each shaded shape below as an algebraic expression in simplest form.

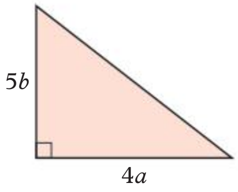
a



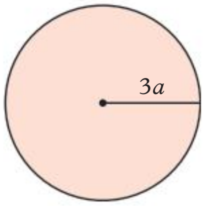
b



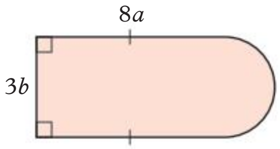
c



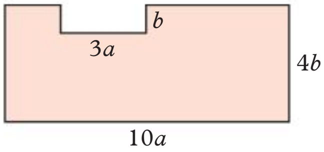
d Leave your answer as an exact value containing π.



e Leave your answer as an exact value containing π.

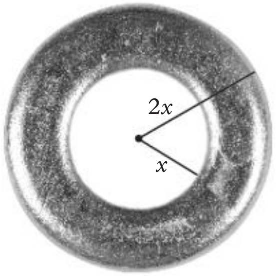


f



13 Calculate the area of each shape in question **12** if $a = 4$ m and $b = 3$ m. Leave your answers as exact values containing π.

14 Consider the top surface of a metal washer shaped like an annulus as shown on the right.



a Write an algebraic expression for the area of the top surface of one washer.

b List three possible values for x that will result in an area of between 400 mm^2 and 500 mm^2 for the surface area of the top of the washer.

c Write an algebraic expression for the total length of the inner and outer circular edges of one washer.

d Use the values you wrote for part **b** to calculate the total length of the inner and outer circular edges of one washer, correct to one decimal place.

CHALLENGE

15 Use the order of operations and index laws to simplify each of the following expressions.

a $2(3a + 5a) - 4a$

b $4b \times 3a + 5(ab + 3ab)$

c $2a \div 2 + 5a \times 3b - 4a$

d $19xy^2 + 8x \div 4 + 5x^2y^2 \div x - 3x$

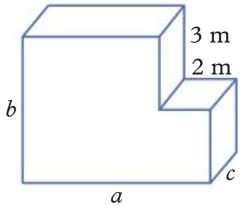
e $3(2a)^2 + (5ab^3)^2 - 2(3a)^2$

f $(-6xy)^2 + 4x^2y^2 \div x^2 + (3y)^2 - 5xy^2 \times 4x$

g $2a^2b \times 4b + 3ab^2 \times 5a$

h $\frac{6xy^2z}{2xz} + \frac{5xyz}{y} - \frac{7x^2y^2z}{14x^2z} + \frac{xz(2y)^2}{2y^2}$

16 Consider the object on the right. Hint: All angles are right angles, meaning all opposite sides are parallel and of equal length.



a Write an expression for the volume of the shape in cubic metres (m^3) in terms of a, b and c in its simplest form.

b Write an expression for the surface area of the shape in square metres (m^2) in terms of a, b and c in its simplest form.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Multiplying terms

Interactive skillsheet
Dividing terms

Topic quiz
2C

ADVANCED

2D Expanding

Learning intentions

By the end of this topic you will be able to ...

✔ expand algebraic expressions with one pair of brackets

✔ expand binomial products.

Inter-year links

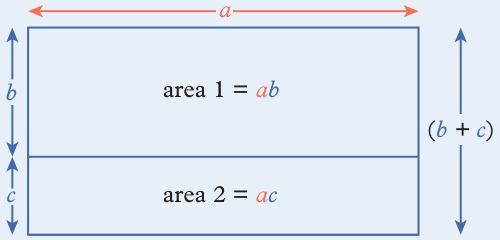
Year 7	6D Order of operations in algebra
Year 8	5F Expanding
Year 9	3D Expanding

Key content video

Expanding one pair of brackets

- The **distributive law** is used to distribute or expand products over addition and subtraction.

$$a(b + c) = ab + ac$$



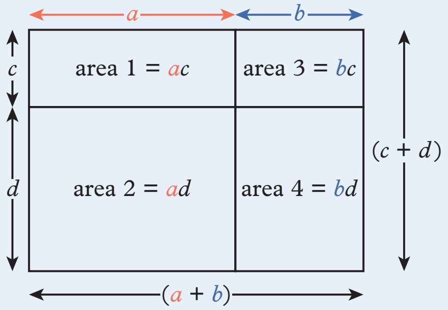
total area = $a(b + c) = ab + ac$

Expanding binomial products

- A **binomial** is an algebraic expression that contains two terms. Examples are:

  $x + 1$ $\quad a^3 - 3a^4$ $\quad 21m^2 + n$
- The distributive law can be used to expand the product of two binomials.

$$(a + b)(c + d) = ac + ad + bc + bd$$



total area = $(a + b)(c + d) = ac + ad + bc + bd$

- Expanding binomial products often requires simplification using the index laws and the identification of like terms.
- The **difference of two squares** is a specific form of expansion with the simplified rule:

  $(a + b)(a - b) = a^2 - b^2$
- The **expansion of a perfect square** is a specific form of expansion with the simplified rule:

  $(a + b)^2 = a^2 + 2ab + b^2$ $\quad (a - b)^2 = a^2 - 2ab + b^2$

Example 2D.1 Expanding one pair of brackets

Expand each of these algebraic expressions to remove the brackets.

a $4(a + 11)$ **b** $5k(k - 6)$ **c** $6g^3(g^4 + 3g)$ **d** $cd(3d - c^2)$

THINK	WRITE
a 1 Multiply each term inside the brackets by the term outside the brackets.	**a** $4(a + 11) = 4 \times a + 4 \times 11$
2 Simplify by performing the multiplications.	$= 4a + 44$
b 1 Multiply each term inside the brackets by the term outside the brackets.	**b** $5k(k - 6) = 5k \times k + 5k \times (-6)$
2 Simplify by performing the multiplications. Remember to take care with the + and – signs when simplifying.	$= 5k^2 - 30k$
c 1 Multiply each term inside the brackets by the term outside the brackets.	**c** $6g^3(g^4 + 3g) = 6g^3 \times g^4 + 6g^3 \times 3g$
2 Simplify by performing the multiplications. Apply the product of powers law, multiplying the terms with the same base by adding their indices.	$= 6g^7 + 18g^4$
d 1 Multiply each term inside the brackets by the term outside the brackets.	**d** $cd(3d - c^2) = cd \times 3d + cd \times (-c^2)$
2 Simplify by performing the multiplications. Apply the quotient of powers law, multiplying the terms with the same base by adding their indices.	$= 3cd^2 - c^3d$

Example 2D.2 Expanding a binomial product

Expand each of these algebraic expressions to remove the brackets.

a $(x + 4)(y + 9)$ **b** $(-f + 3)(4f - 6)$ **c** $(2z^2 + z)(z + 8)$ **d** $(h^2g - 2)(h + 4g^3)$

THINK	WRITE
a 1 Multiply each term inside the second pair of brackets, *y* and 9, by the first term in the first pair of brackets, *x*. Then multiply them by the second term in the first pair of brackets, 4.	**a** $(x + 4)(y + 9)$ $= x \times y + x \times 9 + 4 \times y + 4 \times 9$
2 Simplify each term.	$= xy + 9x + 4y + 36$
b 1 Multiply each term inside the second pair of brackets, 4*f* and –6, by the first term in the first pair of brackets, –*f*. Then multiply them by the second term in the first pair of brackets, 3.	**b** $(-f + 3)(4f - 6)$ $= (-f) \times 4f + (-f) \times (-6) + 3 \times 4f + 3 \times (-6)$
2 Simplify each term.	$= -4f^2 + 6f + 12f - 18$
3 Simplify any like terms.	$= -4f^2 + 18f - 18$

ADVANCED

c **1** Multiply each term inside the second pair of brackets, z and 8, by the first term in the first pair of brackets, $2z^2$. Then multiply them by the second term in the first pair of brackets, z.	**c** $(2z^2 + z)(z + 8)$ $= 2z^2 \times z + 2z^2 \times 8 + z \times z + z \times 8$
2 Use the index laws to simplify each term.	$= 2z^3 + 16z^2 + z^2 + 8z$
3 Simplify any like terms.	$= 2z^3 + 17z^2 + 8z$
d **1** Multiply each term inside the second pair of brackets, h and $4g^3$, by the first term in the first pair of brackets, h^2g. Then multiply them by the second term in the first pair of brackets, –2.	**d** $(h^2g - 2)(h + 4g^3)$ $= h^2g \times h + h^2g \times 4g^3 + (-2) \times h + (-2) \times 4g^3$
2 Use the index laws to simplify each term.	$= gh^3 + 4g^4h^2 - 2h - 8g^3$

Helpful hints

- ✔ It's important to be able to recognise when an expression is the difference of two squares, or a perfect square. These are both very useful in algebra!
- ✔ Remember to pay close attention to minus signs when you are expanding an expression! Using brackets around negative terms can make it clearer.
- ✔ To make sure you correctly expand the product of two binomials, it may help to include the following step:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

ANS p649

Exercise 2D Expanding

△ 1–2(1st, 2nd columns), 3, 5–6(1st column), 7, 8

□ 3, 4, 5–6(2nd column), 7, 8, 10, 12

⬠ 4, 5–6(3rd column), 7–9, 11, 13, 14

1 Expand each of these algebraic expressions to remove the brackets.

a $3(a + 7)$ **b** $5(b - 4)$ **c** $2(6 - c)$ **d** $4(1 + d)$

e $-6(e + 3)$ **f** $-7(f - 2)$ **g** $-3(g - 1)$ **h** $-9(5 - h)$

i $x(x + 2)$ **j** $2y(y - 4)$ **k** $-k(k + 5)$ **l** $-3p(p - 1)$

m $m(3n + k)$ **n** $4a(3b + 6c)$ **o** $-w(w - 8x)$ **p** $-2p(2 + 5p)$

2D.1 **2** Expand each of these algebraic expressions to remove the brackets.

a $3(a - 4)$ **b** $-5(b - 3)$ **c** $4(2x + 3)$ **d** $-7(2y + 4)$

e $5a(2 - a)$ **f** $2m(m^2 + 3)$ **g** $8ab(3a + 1)$ **h** $4w(2x - w)$

i $5k^2(3 - 2k^2)$ **j** $6x^2y(2x - y^3)$ **k** $5cd(3c - 2d)$ **l** $-2a^2c(4a - 3b)$

3 Expand and simplify each of these algebraic expressions.

a $2(x + 5) + 7x$ **b** $4(y - 7) + 15$ **c** $m(n + 6) - 3m$

d $-3x(2 - x) + 5x^2 + x$ **e** $7p^3 - 8p + 5p(1 - p^2)$ **f** $x^4(x^3 - 2x) - x^5$

4 Expand and simplify each of these algebraic expressions.

a $4(x - 2) + 2(x + 5)$ **b** $m(m + 8) + 6(m - 3)$ **c** $2a(3a + 1) - 3(2 - 4a)$

d $k^2(k^3 + 2) + 5(k^2 + 1)$ **e** $3p^4(p^2 - 2) + 2p(p^3 + 6)$ **f** $5y^3(2y^5 + 3) - 2y(7y^2 - 4)$

UNDERSTANDING AND FLUENCY

ADVANCED

UNDERSTANDING AND FLUENCY

2D.2 **5** Expand each of these algebraic expressions to remove the brackets.

a $(a + 4)(b + 2)$ **b** $(c + 3)(d + 6)$ **c** $(m + 7)(n + 5)$

d $(x + 6)(y + 8)$ **e** $(k + 5)(p - 1)$ **f** $(f - 2)(g + 3)$

g $(x - 9)(y - 4)$ **h** $(2m + 3)(n + 1)$ **i** $(4x + 5)(3y - 7)$

j $(3a - 2b)(5c - 4d)$ **k** $(x + 2)(x + 5)$ **l** $(y + 9)(y - 1)$

m $(m - 3)(m - 11)$ **n** $(p - 4)(p - 2)$ **o** $(5 - k)(6 - k)$

p $(4x + 7)(x + 3)$ **q** $(3a + 2)(a - 5)$ **r** $(8w - 7)(w - 2)$

s $(6p + 5)(2p + 7)$ **t** $(2x + 3)(4x - 1)$ **u** $(5y - 2)(3y - 4)$

6 Expand each of these algebraic expressions to remove the brackets.

a $(2a - b)(a + 3)$ **b** $(4t + u)(t - u)$ **c** $(2x + 3y)(x + 4y)$

d $(5 - x)(x + 4)$ **e** $(6x^2 + y)(x + 3y)$ **f** $(ab^2 + 1)(b^2 - 2a)$

g $(a + b)(b + c)$ **h** $(b + 3c)(b + d)$ **i** $(a + 2b)(3 - c)$

j $(a^3 + b^2)(3ab + 2)$ **k** $(x^4 - 3z)(5x - 2y)$ **l** $(c^2 + 4a^3b)(c^2 - 4a^3b)$

7 **a** Expand each of these binomial products.

i $(x + 2)(x - 2)$ **ii** $(k + 7)(k - 7)$ **iii** $(m + 8)(m - 8)$

iv $(w - 6)(w + 6)$ **v** $(y - 1)(y + 1)$ **vi** $(a + b)(a - b)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c The binomial products in this question are all examples of the 'difference of two squares'. Why do you think the rule is called the 'difference of two squares'?

d Does it matter whether the product is $(a + b)(a - b)$ or $(a - b)(a + b)$? Explain.

e Use the 'difference of two squares' rule to expand each of these binomial products.

i $(a - 3)(a + 3)$ **ii** $(x + 10)(x - 10)$ **iii** $(m + n)(m - n)$

iv $(3 + x)(3 - x)$ **v** $(1 + d)(1 - d)$ **vi** $(2x - 5)(2x + 5)$

vii $(2 - 3k)(2 + 3k)$ **viii** $(4g + h)(4g - h)$ **ix** $(5y + 2w)(5y - 2w)$

8 **a** Expand each of these binomial products.

i $(x + 3)(x + 3)$ **ii** $(y + 8)(y + 8)$ **iii** $(a + b)(a + b)$

iv $(k - 5)(k - 5)$ **v** $(p - 6)(p - 6)$ **vi** $(x - y)(x - y)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c These are all examples of 'perfect squares'. Why do you think the rule is called the 'perfect square' rule?

d Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a + 2)^2$ **ii** $(x + 5)^2$ **iii** $(p + 4)^2$

iv $(y + 10)^2$ **v** $(k + 9)^2$ **vi** $(m + n)^2$

vii $(5 + x)^2$ **viii** $(1 + 2d)^2$ **ix** $(3w + 7)^2$

e Why is $(a - b)^2 = a^2 - 2ab + b^2$ also a perfect square?

f Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a - 3)^2$ **ii** $(b - 4)^2$ **iii** $(x - 9)^2$

iv $(n - 11)^2$ **v** $(2 - x)^2$ **vi** $(g - h)^2$

vii $(4w - 1)^2$ **viii** $(5 - 2y)^2$ **ix** $(3m - 2p)^2$

9 Use the 'difference of two squares' and 'perfect square' rules to expand and simplify each of these expressions.

a $(x^2 + 2)(x^2 - 2)$ **b** $(x^2 - 5)(x^2 + 5)$ **c** $(x^2 + y^2)(x^2 - y^2)$

d $(x^2 + 4)^2$ **e** $(x^2 - 3)^2$ **f** $(x^3 - 1)^2$

10 Expand each of these algebraic expressions to remove the brackets.

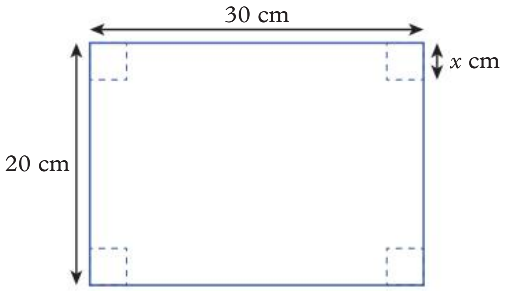
a $(2x + 3)(x^2 + 3x - 1)$ **b** $(x + 2)(2x^2 + x - 1)$ **c** $(2 - x)(1 - x^2 + 2x)$

d $(x^2 + 2)(2x^2 + x - 1)$ **e** $(2x^2 - 3)(x^2 - 3x + 1)$ **f** $(x^2 + 2x)(3x^2 - x - 3)$

ADVANCED

PROBLEM SOLVING AND REASONING

11 A rectangular piece of cardboard is 30 cm long and has a breadth of 20 cm. A square of side length x cm is cut from each corner so that the cardboard can be folded to form an open box.



a Write an expression containing x for:

i the length of the box **ii** the breadth of the box **iii** the height of the box.

b Draw a labelled diagram of the open box.

c Write an expression for:

i the volume of space contained in the box **ii** the inner surface area of the box.

Simplify each of your expressions by expanding them to remove any brackets.

d If $x = 3$ cm, calculate:

i the volume of space contained in the box **ii** the inner surface area of the box.

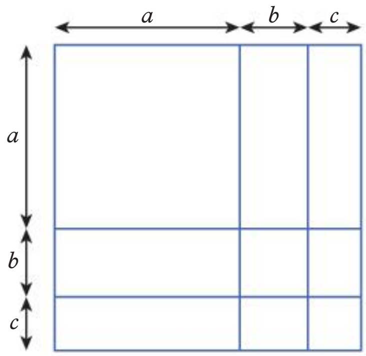
CHALLENGE

12 It is possible to expand brackets with three terms in each bracket.

a Use the diagram on the right to help you expand $(a + b + c)^2$.

b Expand $(x + 2y + 3)^2$.

c Expand $(2a - 3b + c)(2a + 3b - c)$.



13 It is possible to expand products of more than two brackets.

For example,

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= (a^2 + 2ab + b^2)(a + b)\\ &= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Expand each of the following:

a $(x + 1)^3$ **b** $(x - 3)^3$ **c** $(x + 2)^4$ **d** $(x + y)^2(x - y)^2$

14 In the diagram on the right, the area of the shaded rectangle is $(x - 2)(x - 5)$, which can be expanded to be expressed as $x^2 - 7x + 10$.

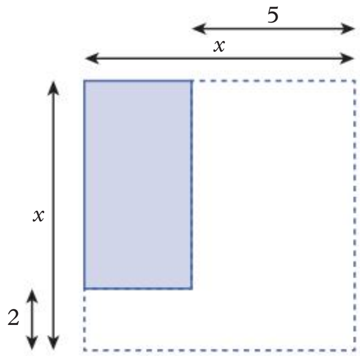
a Copy the diagram into your workbook and draw:

i the outline of a rectangle with area $2 \times x$ square units

ii the outline of a rectangle with area $5 \times x$ square units

iii the outline of a rectangle with area 10 square units.

b Explain why $+10$ appears in the expansion of $(x - 2)(x - 5)$, even though the rectangle with area 10 square units is not shaded.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Expanding over one pair of brackets

Interactive skillsheet
Expanding binomial products

Investigation
Binomial expansion and Pascal's Triangle

Topic quiz
2D

2E Algebraic fractions

Learning intentions

By the end of this topic you will be able to …

✔ simplify algebraic fractions by identifying common factors

✔ multiply and divide algebraic fractions

✔ add and subtract algebraic fractions.

Inter-year links

Year 7	3B Equivalent fractions
Year 8	5E Dividing algebraic terms
Year 9	3B Algebraic fractions with numerical denominators

Key content video

Simplifying algebraic fractions

- **Algebraic fractions** are fractions that contain at least one pronumeral. They can be simplified just like any other fractions, by identifying common factors of the numerator and the denominator.
- The fastest way to simplify a fraction is to divide the numerator and the denominator by their highest common factor (HCF).

Multiplying and dividing algebraic fractions

- The product of two algebraic fractions is the product of the numerators over the product of the denominators. For example,

$$\begin{aligned}\frac{3x}{2}\times\frac{x}{x+1}&=\frac{3x\times x}{2\times(x+1)}\\&=\frac{3x^2}{2(x+1)}\end{aligned}$$

- The **quotient** of two algebraic fractions is found by multiplying the fraction before the division sign by the reciprocal of the fraction after the division sign. For example,

$$\begin{aligned}\frac{2x}{y}\div\frac{5}{x}&=\frac{2x}{y}\times\frac{x}{5}\\&=\frac{2x^2}{5y}\end{aligned}$$

Adding and subtracting algebraic fractions

- Adding and subtracting algebraic fractions is no different from adding and subtracting any other fractions.
  1. Identify the lowest common denominator (LCD) by identifying the lowest common multiple (LCM) of the denominators.
  2. Write each fraction as an equivalent fraction with the LCD.
  3. The sum of the algebraic fractions with a common denominator is then found by adding the numerators. The difference of algebraic fractions with a common denominator is found by subtracting the numerators.

  For example,

  The LCD of $\frac{x+1}{4}$ and $\frac{x}{6}$ is 12, so:

$$\begin{aligned}\frac{x+1}{4}+\frac{x}{6}&=\frac{x+1}{4}\times\frac{3}{3}+\frac{x}{6}\times\frac{2}{2}\\&=\frac{3(x+1)}{12}+\frac{2x}{12}\\&=\frac{3x+3+2x}{12}\\&=\frac{5x+3}{12}\end{aligned}$$

$$\begin{aligned}\frac{x+1}{4}-\frac{x}{6}&=\frac{x+1}{4}\times\frac{3}{3}-\frac{x}{6}\times\frac{2}{2}\\&=\frac{3(x+1)}{12}-\frac{2x}{12}\\&=\frac{3x+3-2x}{12}\\&=\frac{x+3}{12}\end{aligned}$$

Example 2E.1 Simplifying algebraic fractions

Simplify each of these algebraic fractions.

a $\frac{8(2x+1)}{16}$ **b** $\frac{(x+9)(3x+1)}{(x+2)(x+9)}$

THINK	WRITE
a **1** Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is 8.	**a** $\frac{8(2x+1)}{16}=\frac{\cancel{8}^{1}(2x+1)}{\cancel{16}_{2}}$
2 Divide both the numerator and the denominator by 8.	$=\frac{2x+1}{2}$
b **1** Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is $(x+9)$.	**b** $\frac{(x+9)(3x+1)}{(x+2)(x+9)}=\frac{\cancel{(x+9)}^{1}(3x+1)}{(x+2)\cancel{(x+9)}_{1}}$
2 Divide both the numerator and the denominator by $(x+9)$.	$=\frac{3x+1}{x+2}$

Example 2E.2 Adding and subtracting algebraic fractions with binomial numerators

Write each of these expressions as one fraction in simplest form.

a $\frac{2x+3}{7}+\frac{x-1}{3}$ **b** $\frac{7x-1}{10}-\frac{3x+1}{6}$

THINK	WRITE
a **1** The LCM of 7 and 3 is 21. So write each fraction as an equivalent fraction with an LCD of 21.	**a** $\frac{2x+3}{7}+\frac{x-1}{3}=\frac{2x+3}{7}\times\frac{3}{3}+\frac{x-1}{3}\times\frac{7}{7}$ $=\frac{3(2x+3)}{21}+\frac{7(x-1)}{21}$
2 Write the expression as a single fraction by adding the numerators.	$=\frac{3(2x+3)+7(x-1)}{21}$
3 Expand the brackets.	$=\frac{6x+9+7x-7}{21}$
4 Simplify the numerator by adding and subtracting like terms.	$=\frac{13x+2}{21}$
b **1** The LCM of 10 and 6 is 30. So write each fraction as an equivalent fraction with an LCD of 30.	**b** $\frac{7x-1}{10}-\frac{3x+1}{6}=\frac{7x-1}{10}\times\frac{3}{3}-\frac{3x+1}{6}\times\frac{5}{5}$ $=\frac{3(7x-1)}{30}-\frac{5(3x+1)}{30}$
2 Write the expression as a single fraction by subtracting the numerators.	$=\frac{3(7x-1)-5(3x+1)}{30}$
3 Expand the brackets.	$=\frac{21x-3-15x-5}{30}$
4 Simplify the numerator by adding and subtracting like terms.	$=\frac{6x-8}{30}$
5 Simplify by dividing the numerator and the denominator by their HCF of 2.	$=\frac{3x-4}{15}$

ADVANCED

Example 2E.3 Multiplying and dividing algebraic fractions

Simplify the following expressions.

a $\dfrac{x(x-2)}{(x+3)(2x+1)} \times \dfrac{(x+1)(2x+1)}{(x-2)(x+3)}$ **b** $\dfrac{x+4}{x^2} \div \dfrac{3(x+4)}{x-9}$

THINK	**WRITE**
a **1** Write the expression as a single fraction by multiplying the numerators together and the denominators together.	**a** $\dfrac{x(x-2)}{(x+3)(2x+1)} \times \dfrac{(x+1)(2x+1)}{(x-2)(x+3)}$ $= \dfrac{x(x-2) \times (x+1)(2x+1)}{(x+3)(2x+1) \times (x-2)(x+3)}$
2 Look for common factors. Divide both the numerator and the denominator by $(x-2)$ and $(2x+1)$.	$= \dfrac{x\cancel{(x-2)}^1(x+1)\cancel{(2x+1)}^1}{(x+3)\cancel{(2x+1)}^1\cancel{(x-2)}^1(x+3)}$ $= \dfrac{x(x+1)}{(x+3)(x+3)}$
3 Simplify.	$= \dfrac{x(x+1)}{(x+3)^2}$
b **1** Write the expression as the product of two fractions.	**b** $\dfrac{x+4}{x^2} \div \dfrac{3(x+4)}{x-9} = \dfrac{x+4}{x^2} \times \dfrac{x-9}{3(x+4)}$
2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.	$= \dfrac{(x+4) \times (x-9)}{x^2 \times 3(x+4)}$
3 Look for common factors. Divide both the numerator and the denominator by $(x+4)$.	$= \dfrac{\cancel{(x+4)}^1(x-9)}{3x^2\cancel{(x+4)}^1}$ $= \dfrac{x-9}{3x^2}$

Example 2E.4 Adding and subtracting algebraic fractions

Simplify the following expressions.

a $\dfrac{4}{x(x+1)} + \dfrac{3}{(x+1)(x-1)}$ **b** $\dfrac{7x}{x+1} - \dfrac{x-3}{x}$

THINK	**WRITE**
a **1** The LCM of $x(x+1)$ and $(x+1)(x-1)$ is $x(x+1)(x-1)$. Write each fraction as an equivalent fraction with the denominator $x(x+1)(x-1)$.	**a** $\dfrac{4}{x(x+1)} + \dfrac{3}{(x+1)(x-1)}$ $= \dfrac{4}{x(x+1)} \times \dfrac{(x-1)}{(x-1)} + \dfrac{3}{(x+1)(x-1)} \times \dfrac{x}{x}$ $= \dfrac{4(x-1)}{x(x+1)(x-1)} + \dfrac{3x}{x(x+1)(x-1)}$
2 Write the expression as a single fraction by subtracting the numerators.	$= \dfrac{4(x-1)+3x}{x(x+1)(x-1)}$
3 Expand the brackets in the numerator.	$= \dfrac{4x-4+3x}{x(x+1)(x-1)}$
4 Simplify the numerator by adding and subtracting like terms.	$= \dfrac{7x-4}{x(x+1)(x-1)}$

b 1 The LCM of $(x+1)$ and x is $x(x+1)$. So write each fraction as an equivalent fraction with a denominator of $x(x+1)$.	**b** $\frac{7x}{x+1}-\frac{x-3}{x}=\frac{7x}{x+1}\times\frac{x}{x}-\frac{x-3}{x}\times\frac{x+1}{x+1}$ $=\frac{7x\times x}{x(x+1)}-\frac{(x-3)(x+1)}{x(x+1)}$
2 Write the expression as a single fraction by subtracting the numerators.	$=\frac{7x^2-(x-3)(x+1)}{x(x+1)}$
3 Expand the binomial product in the numerator.	$=\frac{7x^2-(x^2-3x+x-3)}{x(x+1)}$ $=\frac{7x^2-(x^2-2x-3)}{x(x+1)}$
4 Simplify the numerator by adding and subtracting like terms.	$=\frac{7x^2-x^2+2x+3}{x(x+1)}$ $=\frac{6x^2+2x+3}{x(x+1)}$

Helpful hints

✔ You don't need to keep the brackets around a binomial factor if the other factors have been removed. For example,

$$\frac{2(x+3)}{4}=\frac{x+3}{2}$$

✔ Remember, if you want to turn an algebraic fraction into a sum of algebraic terms, you must divide *every* term in the numerator by the denominator.

$$\frac{x+9}{3}\neq x+3$$

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Exercise 2E Algebraic fractions

△ 1–5, 6(a, c, e, g, i, j), 7(a, c), 8–10(1st, 2nd columns), 13(a–c)

■ 4–5(2nd column), 6(b, d, f, h, j, k), 7(b, d), 8–10, 11, 12, 13–15, 16(a–c)

⬟ 3–6(3rd column), 7(h–l), 8-11, 13, 16–17

1 Simplify each of these algebraic fractions.

a $\frac{3x}{6}$ **b** $\frac{9x}{9xy}$ **c** $\frac{8x}{2x}$ **d** $\frac{4xy}{6y}$

e $\frac{2(x+3)}{10}$ **f** $\frac{8(x-5)}{4}$ **g** $\frac{x(x-4)}{x}$ **h** $\frac{3x(x+1)}{3x}$

2 Simplify each of these algebraic fractions.

a $\frac{(x-2)(x+5)}{(x+1)(x-2)}$ **b** $\frac{(x-6)(x+4)}{(x-2)(x+4)}$ **c** $\frac{(x-1)(x+3)}{(x+3)(x+1)}$

d $\frac{(x-7)(x+2)}{(x-7)(x-8)}$ **e** $\frac{(x+4)(x-5)}{(x+4)(x-5)}$ **f** $\frac{(x-1)(x+1)}{(x+1)(x-1)}$

2E.1 **3** Simplify each of these algebraic fractions.

a $\frac{8x}{4xy}$

b $\frac{14abc}{7ac}$

c $\frac{5(a+1)}{a(a+1)}$

d $\frac{x(x+3)}{2(x+3)}$

e $\frac{5(m-2)}{15m(m-2)}$

f $\frac{(n+3)(n+7)}{2(n+3)}$

g $\frac{3y^2(y+x)}{y(x+y)}$

h $\frac{x^2(x-4)}{x(x+1)(x-4)}$

i $\frac{5(y-3)(y+4)}{(y+4)(y-4)}$

j $\frac{(t+2)(r+3)}{(r+3)(r+2)}$

k $\frac{(x+2)(x-1)(2x+7)}{(2x+7)(x-4)(x+2)}$

l $\frac{(p^2+2)(p+6)(p-2)}{3(p+6)(p+2)(p^2+2)}$

4 Simplify each of these expressions.

a $\frac{3x}{x+2} \times \frac{x+2}{6x}$

b $\frac{x-3}{x} \times \frac{5x}{x-3}$

c $\frac{x+4}{x+1} \times \frac{x+1}{x+4}$

d $\frac{6x}{x-5} \div \frac{2x}{x-5}$

e $\frac{x+7}{4x} \div \frac{x+7}{20x}$

f $\frac{8x}{2x+1} \div \frac{8}{2x+1}$

2E.2 **5** Write each of these expressions as a single fraction in simplest form.

a $\frac{c+1}{4} + \frac{c}{6}$

b $\frac{m-2}{3} + \frac{m+1}{4}$

c $\frac{x-2}{2} - \frac{x}{5}$

d $\frac{4k+1}{4} - \frac{2k}{5}$

e $\frac{5x+1}{8} + \frac{x-3}{10}$

f $\frac{x}{6} - \frac{x-1}{3}$

g $\frac{b+4}{8} - \frac{b+1}{12}$

h $\frac{2a-1}{4} - \frac{3a+1}{6}$

6 Write each of these expressions as a single fraction in simplest form.

a $\frac{x+1}{2} + \frac{x+5}{3}$

b $\frac{x+2}{5} + \frac{x-4}{6}$

c $\frac{x-3}{4} + \frac{x-1}{8}$

d $\frac{x+4}{3} - \frac{x+2}{4}$

e $\frac{x-6}{8} - \frac{x-3}{12}$

f $\frac{x+1}{2} - \frac{x+4}{10}$

g $\frac{2x+3}{3} + \frac{x-7}{5}$

h $\frac{x+5}{6} + \frac{2x+1}{9}$

i $\frac{4x+7}{3} + \frac{5x-3}{2}$

j $\frac{3x-1}{4} - \frac{2x-5}{5}$

k $\frac{4-x}{7} - \frac{x-4}{3}$

l $\frac{x^2+2}{5} - \frac{x^2+3}{8}$

2E.3 **7** Simplify the following expressions.

a $\frac{a+2}{a-4} \times \frac{2a+1}{a+2}$

b $\frac{x(x+1)}{x+5} \times \frac{x+5}{x(x-3)}$

c $\frac{t+3}{4(t+5)} \div \frac{t(t+3)}{8(t+5)}$

d $\frac{m(m+2)}{(m-3)(3m+1)} \times \frac{3m+1}{m(m+7)}$

e $\frac{(x+3)(x-2)}{x(x-2)} \times \frac{x(x+5)}{(x-3)}$

f $\frac{x+4}{(x-2)(x+4)} \div \frac{x+5}{x-2}$

g $\frac{2y+3}{y-4} \div \frac{y+3}{3(y-4)}$

h $\frac{(m+1)(m-2)}{(m+4)(m-3)} \times \frac{m(m+4)}{(m-1)(m-2)}$

i $\frac{(c^2+6)(c-4)}{(c+2)(c^2+3)} \times \frac{(2c+1)(c^2+3)}{(c-4)(2c+1)}$

j $\frac{n^2(n+p)}{n+2p} \times \frac{(n-p)(n+2p)}{n^3(n+p)}$

k $\frac{x^2y^3(x+2y)}{y(x-2y)(x+y)} \div \frac{x^4(x+2y)(x+1)}{(x+y)(x-2y)}$

l $\frac{5(2a+b)}{(4-3a)(b+2)} \times \frac{(a+b)(b+2)}{(a-b)(a+b)} \div \frac{b+2a}{2(4-3a)(a^2+b^2)}$

8 Simplify the following expressions.

a $\frac{6(x+5)(x-5)}{(x-2)(x+2)} \times \frac{(x-1)(x+2)}{3(x+5)(x-1)}$

b $\frac{(x+3)(x+8)}{4x(x+2)(x-3)} \times \frac{4x(x+2)(x-6)}{(x+3)(x+6)}$

c $\frac{(2x-1)(3x+2)}{(3x-2)(x+1)} \div \frac{(x+2)(2x-1)}{(5x+1)(3x-2)}$

d $\frac{5x(3x-5)}{(2x+5)(7x-4)} \div \frac{10(3x-5)(x+8)}{(7x-4)(2x+5)}$

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PROBLEM SOLVING AND REASONING

2E.4 **9** Simplify the following expressions.

a $\frac{2}{m+1}+\frac{3}{m}$ **b** $\frac{k}{k^2+1}+\frac{3}{k}$ **c** $\frac{3}{y+2}+\frac{1}{y-3}$ **d** $\frac{x}{x+3}+\frac{2}{x}$

e $\frac{t}{t+5}+\frac{2}{t-1}$ **f** $\frac{5}{x}-\frac{2}{2x+1}$ **g** $\frac{5}{x+2}-\frac{3}{x+1}$ **h** $\frac{4}{2h-1}-\frac{3}{h+2}$

i $\frac{m}{m-3}-\frac{1}{2m+1}$ **j** $\frac{a^2}{3a+1}+\frac{4}{a}$ **k** $\frac{x}{2}-\frac{2x+3}{x+1}$ **l** $\frac{x+1}{x}-\frac{x}{x+1}$

10 Simplify the following expressions.

a $\frac{1}{(x+2)}+\frac{3}{(x+2)(x-2)}$ **b** $\frac{2}{(x+1)(x-1)}+\frac{2(x-1)}{(x+1)}$

c $\frac{1}{(x+1)}+\frac{2}{(x+2)(x+1)}$ **d** $\frac{4x}{(x-1)}+\frac{3}{x(x-1)(x-2)}$

e $\frac{4x+2}{x(x-2)}-\frac{3x}{(x+2)(x-2)}$ **f** $\frac{x}{x+1}+\frac{2x}{(x+1)(x+2)}-\frac{3x}{(x+1)(x+2)(x+3)}$

g $\frac{4-2x}{x(x+2)}-\frac{3}{(x+2)(x-2)}$ **h** $\frac{4-2x}{x(x+2)}-\frac{3(x-1)}{x}$

11 Find the algebraic expression that is equal to a if $\frac{a}{2}+\frac{x+1}{4}=\frac{3x-5}{4}$.

12 Find the algebraic expression that is equal to b if $\frac{b}{5}-\frac{x-2}{2}=\frac{x+12}{10}$.

13 For each of the following, expand the brackets and write the final answer as a single fraction in simplest form. For some, it may be helpful to simplify before expanding.

a $\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)$ **b** $\left(\frac{1}{x}+x\right)^2$

c $\frac{x}{x+1}\left(\frac{2}{x-1}-\frac{1}{x}\right)$ **d** $\left(\frac{2}{x+1}-\frac{1}{x}\right)\left(\frac{1}{x-1}-\frac{1}{2x}\right)$

14 Consider the expression $\frac{x}{x+1}-\frac{x-1}{x}$.

a Write this expression as a single fraction in simplest form.

b Use your answer to part **a** to evaluate.

i $\frac{6}{7}-\frac{5}{6}$ **ii** $\frac{9}{10}-\frac{8}{9}$ **iii** $\frac{99}{100}-\frac{98}{99}$ **iv** $\frac{1000}{1001}-\frac{999}{1000}$

15 Consider the expression $\frac{x}{y}-\frac{x}{y+1}$.

a Write this expression as a single fraction in simplest form.

b Use your answer to part **a** to evaluate.

i $\frac{5}{6}-\frac{5}{7}$ **ii** $\frac{7}{10}-\frac{7}{11}$ **iii** $\frac{67}{100}-\frac{67}{101}$ **iv** $\frac{11}{999}-\frac{11}{1000}$

CHALLENGE

16 Consider the following algebraic property: $x-y=-(y-x)$.

Use this property to simplify the following algebraic fractions.

a $\frac{a-b}{b-a}$ **b** $\frac{4(m-n)}{n-m}$

c $\frac{(x-1)(5-x)(x-3)}{(x+4)(1-x)(x-5)}$ **d** $\frac{(x+1)^2}{x-1}\times\frac{1-x}{x+1}$

e $\frac{x-5}{x+5}\div\frac{5-x}{x-5}$ **f** $\frac{2(2-x)}{x^2-4}\div\frac{x-2}{x+2}$

17 If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and $a^2+b^2+c^2=1$, what is the value of $a+b+c$?

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Interactive skillsheet Adding and subtracting algebraic fractions

Worksheet Simplifying algebraic fractions

Investigation The shoemaker's knife

Topic quiz 2E

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Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.
✔ Core
✔ Advanced

2A **1** Use the appropriate index law to simplify each of these expressions.

a $x^2 \times x^7$ **b** $y^6 \div y^2$ **c** $\frac{2m^4}{m^3}$

d $(a^2)^5$ **e** $(3xy^2)^2$ **f** $\left(\frac{2a}{b^2}\right)^3$

2B **2** Use the property $a^0 = 1$ to simplify each of the following.

a x^0 **b** $2y^0$ **c** $(3a^2)^0$ **d** $x^0 + y^0 + z^0$

ADV 2B **3** Rewrite each of these terms with positive indices.

a y^{-3} **b** $4x^{-2}$ **c** $\frac{a^{-7}}{6}$

d $\frac{2}{b^{-3}}$ **e** $\frac{a^{-2}}{2b^5}$ **f** $5c^{-3}d^4$

ADV 2B **4** Simplify the following expression, writing your answer using only positive indices.

$$\frac{3a^2b^3}{a^3} \times \left(\frac{2a^{-3}}{b^2}\right)^2$$

2C **5** Simplify each of these expressions.

a $4x + 3x - x$ **b** $5m^2 + 3m^2$

c $3a + 5b + 4a$ **d** $c + 3d - 4c + 2d$

e $3a^2 - 4a + 2a^2 + 2a$ **f** $2x^2y + 3xy - 4x^2y + xy^2$

g $\frac{2x}{5} + \frac{y}{2} - \frac{y}{4} - \frac{x}{4}$ **h** $m + \frac{m^2}{4} - \frac{m}{2} + \frac{m^2}{6}$

2C **6** For each of the following, find the missing term(s) that will make the equation true.

a $4ab + ? = 7ab$ **b** $5x^2y + 3xy + ? = 11xy + 5x^2y$

c $a^2b - b^2a + ? = 2ab^2 + a^2b$ **d** $2x^2 + 3x - 6 + ? = 2x^2 + 6$

ADV 2B **7** Expand and simplify each of these algebraic expressions.

a $3(a + 7)$ **b** $3(2y + 1) + 4$

c $2(x - 5) + x + 2$ **d** $-3(x + 2) - 4(x - 4)$

ADV 2B **8** Expand and simplify each of the following.

a $(a + 1)(a + 2)$ **b** $(x - 4)(x + 6)$

c $(k - 5)(k + 5)$ **d** $(2x - 3)(x + 2)$

ADV 2B **9** Expand the following expressions:

a $(x^2 - y)(x + 2y)$ **b** $(2x + 3)(3x + \frac{x^2}{4} - 2)$

ADV 2E **10** Write each of these expressions as a single fraction in simplest form.

a $\frac{x}{4} - \frac{5x - 2}{12}$ **b** $\frac{y + 1}{3} + \frac{5y}{2}$

c $\frac{m^2 + 1}{4} - \frac{m}{5}$ **d** $\frac{2x + 1}{3} - \frac{x + 2}{7}$

ADV 2E **11** Simplify each of these algebraic fractions.

a $\frac{a(a + 4)}{3(a + 4)}$ **b** $\frac{(x + 4)(x - 3)}{5(x - 3)}$

c $\frac{(x + 4)(x - 3)}{(x + 7)(2x - 3)} \div \frac{(x - 3)(x + 4)}{2x - 3}$ **d** $\frac{m^2(m - 1)(m + 3)}{(m + 3)(m + 1)} \times \frac{(m + 1)(m - 4)}{m^3(m - 1)}$

ADV 2E **12** Simplify each of these expressions.

a $\frac{x}{x - 2} - \frac{3}{x + 1}$ **b** $\frac{x}{4(x + 2)} - \frac{12}{x(x + 2)}$

2F Factorising

Learning intentions

By the end of this topic you will be able to ...

✔ factorise algebraic expressions by identifying the highest common factor

✔ factorise algebraic expressions by grouping terms

✔ factorise algebraic expressions using the 'difference of two squares' and the 'perfect square' rules.

Inter-year links

Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	3E Factorising using the HCF

Key content video

Factorising using the highest common factor

- The distributive law can be used to factorise algebraic expressions by identifying the HCF of the terms in the expression.

$$ab + ac = a(b + c)$$

For example, the HCF of $2x^2$ and $-6x$ is $2x$, so:

$$\underbrace{2x^2 - 6x}_{\text{expanded form}} = \underbrace{2x(x - 3)}_{\text{factorised form}}$$

Factorising by grouping terms

- It is possible to find binomial factors of large algebraic expressions by grouping pairs of terms and factorising each pair using their HCF.

$$\begin{aligned} ac + ad + bc + bd &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

For example, $x^2 + 2x - 3x + 6$ can be factorised into a binomial product by grouping the terms $x^2 + 2x$ and $-3x - 6$, and then factorising each of them with a common factor of $x + 2$:

$$\begin{aligned} x^2 + 2x - 3x - 6 &= x(x + 2) - 3(x + 2) \\ &= (x + 2)(x - 3) \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Binomials that have the same terms but a different sign, such as $a + b$ and $a - b$, are conjugates of each other.
- If the expression is the difference of two squares, you can factorise it into the product of conjugate pairs using the 'difference of two squares' rule:

$$a^2 - b^2 = (a + b)(a - b)$$

- If the terms of a binomial expression have opposite signs, but both terms are not perfect squares, you can still factorise the expression using surds. For example,

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$$

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Factorising using the 'perfect square' rule

- If the expression is of the same form as a perfect square, then you can factorise it using the 'perfect square' rule.

$$a^2 + 2ab + b^2 = (a + b)^2$$

For example,

$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(x)(3) + 3^2 \\ &= (x + 3)^2. \end{aligned}$$

Example 2F.1 Factorising using the highest common factor

Factorise each of these expressions.

a $4n + 36$ **b** $t^2 - 7t$ **c** $5x^5y - 13x^2y^3$ **d** $3klm + 9km - 6k$

THINK	WRITE
a 1 Identify that the HCF is 4 and write each term as a product of the HCF and its other factor.	**a** $4n + 36 = 4 \times n + 4 \times 9$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 4(n + 9)$
b 1 Identify that the HCF is t and write each term as a product of the HCF and its other factor.	**b** $t^2 - 7t = t \times t + t \times (-7)$
2 Use the distributive law to factorise the expression.	$= t(t - 7)$
c 1 Identify that the HCF is x^2y and write each term as a product of the HCF and its other factor.	**c** $5x^5y - 13x^2y^3 = x^2y \times 5x^3 + x^2y \times (-13y^2)$
2 Use the distributive law to factorise the expression.	$= x^2y(5x^3 - 13y^2)$
d 1 Identify that the HCF is $3k$ and write each term as a product of the HCF and its other factor.	**d** $3klm + 9km - 6k = 3k \times lm + 3k \times 3m + 3k \times (-2)$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 3k(lm + 3m - 2)$

Example 2F.2 Factorising by grouping terms

Factorise $a^2 + 3a - 15b - 5ab$ by grouping terms.

THINK	WRITE
1 Check for the HCF of all four terms. In this case there isn't one. Pair the terms in groups of two so that each pair has a common factor.	$a^2 + 3a - 15b - 5ab = (a^2 + 3a) + (-15b - 5ab)$
2 Use the distributive law to factorise each pair of terms.	$= a(a + 3) - 5b(a + 3)$
3 Factorise using the common binomial factor, which is $(a + 3)$. Put the common binomial factor in front of a new pair of brackets containing the sum of the other factors.	$= (a + 3)(a - 5b)$

Example 2F.3 Factorising using the 'difference of two squares' rule

Factorise each of these expressions.

a $16h^2 - 49$ **b** $12a^4 - 27b^2$ **c** $(r + 11)^2 - 1$

THINK	WRITE
a 1 Check for the HCF of all the terms. In this case there isn't one. Write each term as a square in index form.	**a** $16h^2 - 49 = (4h)^2 - 7^2$
2 Factorise using the 'difference of two squares' rule.	$= (4h + 7)(4h - 7)$
b 1 Check for the HCF of all the terms. In this case, 3 is the HCF. Using the distributive law, factorise using the HCF.	**b** $12a^4 - 27b^2 = 3(4a^4 - 9b^2)$
2 Write each term in the binomial factor as a square in index form.	$= 3([2a^2]^2 - [3b]^2)$
3 Factorise the binomial factor using the 'difference of two squares' rule.	$= 3(2a^2 + 3b)(2a^2 - 3b)$
c 1 Check for the HCF of all the terms. In this case, there isn't one. Write each term as a square in index form.	**c** $(r + 11)^2 - 1 = (r + 11)^2 - 1^2$
2 Factorise using the 'difference of two squares' rule.	$= (r + 11 + 1)(r + 11 - 1)$
3 Simplify each of the bracketed expressions.	$= (r + 12)(r + 10)$

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Example 2F.4 Factorising using the 'difference of two squares' rule and surds

Factorise each of these expressions.

a $25x^2 - 7$ **b** $(y + 5)^2 - 12$

THINK	WRITE
a 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in index form.	**a** $25x^2 - 7 = (5x)^2 - (\sqrt{7})^2$
2 Factorise using the 'difference of two squares' rule.	$= (5x + \sqrt{7})(5x - \sqrt{7})$
b 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in index form.	**b** $(y + 5)^2 - 12 = (y + 5)^2 - (\sqrt{12})^2$
2 Simplify the surd by factorising 12 so it is a product of 3 and 4, where 4 is a perfect square.	$= (y + 5)^2 - (\sqrt{4 \times 3})^2$ $= (y + 5)^2 - (\sqrt{4} \times \sqrt{3})^2$ $= (y + 5)^2 - (2\sqrt{3})^2$
3 Factorise using the 'difference of two squares' rule.	$= (y + 5 + 2\sqrt{3})(y + 5 - 2\sqrt{3})$

Example 2F.5 Factorising using the 'perfect square' rule

Factorise each of these expressions.

a $t^2 - 18t + 81$ **b** $16x^2 - 56x + 49$

THINK	WRITE
a 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of t^2 is t, the negative square root of 81 is –9, and $-18t$ is twice the product of both these roots: $2 \times (-9) \times t$.	**a** $t^2 - 18t + 81 = t^2 + 2(-9)t + (-9)^2$
2 Factorise using the 'perfect square' rule.	$= (t - 9)^2$
b 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of $16x^2$ is $4x$, the negative square root of 49 is –7, and $-56x$ is twice the product of both these roots: $2 \times (-7) \times (4x)$.	**b** $16x^2 - 56x + 49 = (4x)^2 + 2 \times (-7) \times (4x) + (-7)^2$
2 Factorise using the 'perfect square' rule.	$= (4x - 7)^2$

Helpful hints

✔ It's important that you are able to recognise when an expression can be factorised by grouping terms, or by identifying it as the difference of two squares, or as a perfect square.

✔ Factorising an expression is the reverse process of expanding an expression. You can check if your factorised solution is correct by expanding it to see if you get the same expression that you started with.

expanding →

$$7(a + 2) = 7a + 14$$

← factorising

✔ Always check to see if the terms of an expression have any common factors. It is always best to start with common factors before you move on with any other factorising methods.

For example, it is easier to identify that $3x^2 + 18x + 27$ can be solved using the 'perfect square' rule once you take out 3 as a common factor:

$$3(x^2 + 6x + 9) = 3(x + 3)^2$$

✔ Remember that the difference of two squares requires one term to be positive and the other to be negative. For example, the following are NOT the difference of two squares:

$x^2 + y^2$ and $-a^2 - 4$

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Exercise 2F Factorising

△ 1–2(1st, 2nd columns), 3, 4(a, c, e, g), 5, 6(a, d, g, j), 7(a–f), 8, 9(1st, 2nd columns), 10(a, d, g), 11, 12, 13(a, e, g, h, j), 14, 15(a–c)

■ 1–2(3rd column), 4(b, e, h, i), 6–7(2nd column), 9(2nd column), 10, 13(3rd, 4th columns), 15(d, g, i), 17, 18

⬟ 2(4th column), 4(g–i), 6–7(3rd column), 9(j–r), 10, 13(4th column), 15(i–l), 16, 17, 19–21

2F.1 **1** Factorise each of these expressions.

a $5a + 35$ **b** $30b + 6$ **c** $24 + 33c$ **d** $12d + 8$

e $8e - 18$ **f** $35f - 21$ **g** $24 + 16g$ **h** $10 - 40h$

i $9j + 9k$ **j** $16w - 28x$ **k** $m^2 + 2m$ **l** $n^2 - 6n$

m $5a + a^2$ **n** $9p - p^2$ **o** $3q^2 + 3q$ **p** $2r^2 - 8r$

2 Factorise each of these expressions.

a $3x - 12$ **b** $6y - 15$ **c** $8x^2 - 16x$ **d** $4k^2 + 6k$

e $5c + 11c^2$ **f** $8u - 4u^2$ **g** $6m^2 - 3m$ **h** $y^2 + 5y^3$

i $14t - 8t^2$ **j** $9p^3 + 3p^2$ **k** $4ab + 3a$ **l** $5t^2u - 3tu$

m $4m^2n + 2mn^2$ **n** $6x^2y + 3xy$ **o** $15c^2 + 21cd$ **p** $18a^2b^3c - 12a^4bc^2$

3 Complete the following factorisations so each expression is factorised using a negative HCF.

a $-2ab - 4a^2 = (-2a) \times \_\_ + (-2a) \times 2a$
$= -2a(\_\_ + 2a)$

b $-10x^3 + 35x = \_\_ \times 2x^2 + \_\_ \times (-7)$
$= \_\_(2x^2 - 7)$

4 Factorise each of these expressions using a negative HCF.

a $-6mn - 18$ **b** $-3ab - 3bc$ **c** $-y^2 - 10y$

d $-18x^2 - 9x$ **e** $-4k + 8k^2$ **f** $-20a^3 - 30a$

g $-12x^2y + 14xy$ **h** $-w^5 + w^7$ **i** $-4b^3 + 2b^2 - 6b^4$

5 **a** Using the binomial factor $(x - 5)$ as the HCF, show that $y(x - 5) + 2(x - 5)$ is written in factorised form as $(x - 5)(y + 2)$.

b In the same way, show that $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2)(4a - 9)$.

ADVANCED

UNDERSTANDING AND FLUENCY

6 Factorise each of these expressions using an HCF that is a binomial factor.

a $a(x + 3) + 5(x + 3)$ **b** $m(k - 2) + 4(k - 2)$ **c** $y(y + 5) + 2(y + 5)$

d $x(x - 1) + 9(x - 1)$ **e** $k(k + 6) - 3(k + 6)$ **f** $p(p - 9) - 6(p - 9)$

g $7(4 - a) + a(4 - a)$ **h** $3n(2n - 5) + 4(2n - 5)$ **i** $x(y^2 + 2) + 8(y^2 + 2)$

j $a^2(a - 4) + 3(a - 4)$ **k** $2x(y^2 + 1) - 5(y^2 + 1)$ **l** $5x(7 - x^3) + 2(7 - x^3)$

2F.2 **7** Factorise each of these expressions by grouping terms.

a $ab + 4b + 3a + 12$ **b** $x^2 - 7x + 2xy - 14y$ **c** $mn + 5m - 6n - 30$

d $a^2 + 2a + 5a + 10$ **e** $x^2 - x + 4x - 4$ **f** $c^2 + 8c - 3c - 24$

g $6y - 3y^2 + 8 - 4y$ **h** $p^3 + 2p^2 + 5p + 10$ **i** $x^4 + x^2 + 3x^2 + 3$

j $2m^3 + 4m - 3m^2 - 6$ **k** $4 + 4k^2 + k^5 + k^3$ **l** $xy - 15 + 5y - 3x$

8 **a** Expand $(x + 5)(x - 5)$ using the 'difference of two squares' rule.

b Explain how you can factorise $x^2 - 25$.

2F.3 **9** Factorise each of these expressions.

a $x^2 - 4$ **b** $a^2 - 36$ **c** $100 - y^2$

d $64b^2 - 9$ **e** $25 - 49p^2$ **f** $4a^2 - 81b^2$

g $3m^2 - 3$ **h** $8k^2 - 18$ **i** $(xy)^2 - 16w^2$

j $m^6 - n^2$ **k** $4a^{10} - 36b^6$ **l** $(x^2y)^2 - w^4$

m $(h + 3)^2 - 25$ **n** $(c - 4)^2 - 9$ **o** $(2 - x)^2 - x^2$

p $1 - (a + b)^2$ **q** $4 - (y - 5)^2$ **r** $(x + 3)^2 - (x - 6)^2$

2F.4 **10** Factorise each of these expressions. Hint: Where needed, use a surd to write a term as a square.

a $x^2 - 7$ **b** $a^2 - 13$ **c** $19 - y^2$

d $16k^2 - 5$ **e** $(p + 8)^2 - 2$ **f** $(m - 4)^2 - 6$

g $(x + 7)^2 - 3$ **h** $(y - 1)^2 - 19$ **i** $(d - 17)^2 - 17$

11 **a** Expand $(x + 5)^2$ using the 'perfect square' rule.

b Explain how you can factorise $x^2 + 10x + 25$.

12 **a** Expand $(y - 8)^2$ using the 'perfect square' rule.

b Explain how you can factorise $y^2 - 16y + 64$.

2F.5 **13** Factorise each of these expressions using what you know from the 'perfect square' rule.

a $x^2 + 12x + 36$ **b** $m^2 + 8m + 16$ **c** $p^2 + 20p + 100$ **d** $y^2 - 6y + 9$

e $x^2 - 14x + 49$ **f** $g^2 - 4g + 4$ **g** $1 + 2a + a^2$ **h** $25 - 10x + x^2$

i $81 - 18b + b^2$ **j** $w^2 + 2wx + x^2$ **k** $k^2 - 2km + m^2$ **l** $9x^2 + 12x + 4$

PROBLEM SOLVING AND REASONING

14 **a** Write an expression for the missing side length of each rectangular item below.

i The area of the welcome mat is $(x^2 - 100)$ cm^2.

ii The area of the stained glass panel is $(2x^2 + 3x)$ cm^2.

b Write an expression in factorised form for the perimeter of each of the items above.

PROBLEM SOLVING AND REASONING

15 Simplify each of these algebraic fractions by factorising the numerator and denominator when possible.

a $\dfrac{a^2 + 2a}{ab + 2b}$ **b** $\dfrac{xy + 2x}{3y^2 + 6y}$ **c** $\dfrac{abc^2 - 4ab^2}{ab}$ **d** $\dfrac{2mn + m^2n^2}{mn + 2}$

e $\dfrac{x^2 - y^2}{2x + 2y}$ **f** $\dfrac{3x + 15}{x^2 + 5x}$ **g** $\dfrac{9x^2 - 15x}{12x - 20}$ **h** $\dfrac{m^2 - 49}{5m^2 - 35m}$

i $\dfrac{a^2 + 2ab + b^2}{a^2 - b^2}$ **j** $\dfrac{x^2 + 6x + 9}{x^2 - 9}$ **k** $\dfrac{16a^2 - 25}{8a^2 - 10a}$ **l** $\dfrac{4x^2 + 28x + 49}{49 - 4x^2}$

16 Simplify each of the following.

a $\dfrac{3x + 21}{2x^2 - 10x} \times \dfrac{4x^2 - 20x}{x^2 - 49}$ **b** $\dfrac{x^2 - 1}{10x^2 + 30x} \times \dfrac{15x + 45}{x^2 + 2x + 1} \times \dfrac{2x^2 + 2x}{3x - 3}$

17 Consider an odd number represented by the pronumeral n.

a Write expressions for the next two odd numbers.

b Write an expression for the sum of n and the next two odd numbers.

c Factorise your expression from part **b**. Explain how this relates to one of the three odd numbers.

d Use the shortcut you found in part **c** to find the sum of 423, 425 and 427 without adding.

e Investigate a shortcut for adding five consecutive odd numbers. Show your working.

f Use the shortcut you found in part **e** to find the sum of 2671, 2673, 2675, 2677 and 2679 without adding.

CHALLENGE

18 In question **17**, you found expressions for the sum of three consecutive odd numbers and five consecutive odd numbers.

a Investigate shortcuts for finding the sum of an even number of consecutive odd terms. Start with two consecutive odd numbers, then investigate for four consecutive odd numbers, then six consecutive odd numbers, where the first term is n each time.

b Without adding, find the sum of the numbers listed below.

73, 75, 77, 79, 81, 83, 85, 87, 89, 91

19 The difference between two numbers is 7 and the difference between the squares of the two numbers is 105. What is the sum of the two numbers?

20 Factorising using the 'difference of two squares' method can be useful for simplifying certain multiplications. Evaluate each of the following without a calculator by using the 'difference of two squares' factorisation method.

a 19×21 **b** 28×32 **c** 83×77 **d** 54×46

21 Legend has it that the famous mathematician Carl Friedrich Gauss was once punished for misbehaving in primary school by being told he had to find the sum of all the numbers from 1 to 100 before leaving class. Much to his teacher's surprise, he did so in seconds.

The shortcut Gauss probably used was to start by pairing the highest and lowest numbers together, like so:

$$(1 + 100) + (2 + 99) + (3 + 98) + (4 + 97) + \ldots$$

a What is the sum of each binomial pair in Gauss's method?

b How many binomial pairs will there be once all of the numbers from 1 to 100 are grouped together?

c Use your answers from parts **b** and **c** to write the sum of all the numbers from 1 to 100 as the product of two numbers. Evaluate this product to find Gauss's solution.

d This method can be generalised to write the sum of all the numbers from 1 to n, for any positive integer n. Write the sum of all the numbers from 1 to n as a factorised expression in terms of n.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Factorising using the HCF

Interactive skillsheet
Factorising by grouping terms

Interactive skillsheet
Factorising the difference of two squares

Topic quiz
2F

ADVANCED

2G Factorising quadratic expressions

Learning intentions

By the end of this topic you will be able to ...

✔ factorise quadratic expressions by considering each term.

Inter-year links

Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	3F Factorising monic quadratic expressions

Quadratic trinomials

Key content video

- A quadratic expression is an algebraic expression that contains a squared pronumeral, with no exponents greater than 2 in the expression. The following are all quadratic expressions:

$$6t^2 \qquad x^2+5 \qquad k^2+14k+30 \qquad 4b^2-a^2$$

- A **trinomial** is an algebraic expression that contains three terms.
- A **quadratic trinomial** is an algebraic expression of the form ax^2+bx+c, where a, b and c are constants and $a \neq 0$.

$$ax^2+bx+c$$

  a is the **leading coefficient**, b is the coefficient of the linear term, and c is the constant term that can be viewed as the coefficient of x^0.

- A **monic quadratic** is a quadratic in which the leading coefficient is equal to 1.

$$x^2+bx+c$$

Factorising monic quadratic trinomials

- Expanding a binomial product of the form $(x+m)(x+n)$ results in a monic quadratic trinomial:

$$\begin{aligned}(x+m)(x+n) &= x^2+mx+nx+mn\\ &= x^2+(m+n)x+mn\\ &= x^2+bx+c \qquad \text{where } b=m+n \text{ and } c=m\times n\end{aligned}$$

- The process can be reversed to factorise a monic quadratic trinomial, x^2+bx+c, by finding two numbers (m and n) that add to give b and multiply to give c.

  For example, in the quadratic trinomial x^2+4x+3, the linear coefficient is 4 (that is, $b=4$) and the constant is 3 (that is, $c=3$). So the values $m=3$ and $n=1$ satisfy the conditions $b=m+n$ and $c=m\times n$.

$$\begin{aligned}x^2+4x+3 &= x^2+(3+1)x+(3\times 1)\\ &= (x+3)(x+1)\end{aligned}$$

expanding ←

$$x^2+4x+3=(x+1)(x+3)$$

quadratic trinomial (expanded form) binomial product (factorised form)

factorising →

Example 2G.1 Factorising simple quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 6x + 5$ **b** $x^2 + 9x + 18$

THINK	WRITE
a 1 For $x^2 + 6x + 5$, list the factor pairs of the constant term, 5. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are two combinations of factor pairs with the same numerals.	**a** $1 \times 5 = 5 \rightarrow 1 + 5 = 6$ $(-1) \times (-5) = 5 \rightarrow -1 - 5 = -6$
2 Add the factor pairs together and identify which pair add to the linear coefficient, 6.	
3 Write the expression in factorised form.	$x^2 + 6x + 5 = (x + 1)(x + 5)$
b 1 For $x^2 + 9x + 18$, list the factor pairs of the constant term, 18.	**b** $1 \times 18 = 18 \rightarrow 1 + 18 = 19$ $(-1) \times (-18) = 18 \rightarrow -1 - 18 = -19$ $2 \times 9 = 18 \rightarrow 2 + 9 = 11$ $(-2) \times (-9) = 18 \rightarrow -2 - 9 = -11$ $3 \times 6 = 18 \rightarrow 3 + 6 = 9$ $(-3) \times (-6) = 18 \rightarrow -3 - 6 = -9$
2 Add the factor pairs together and identify which pair add to the linear coefficient, 9.	
3 Write the expression in factorised form.	$x^2 + 9x + 18 = (x + 3)(x + 6)$

Example 2G.2 Factorising more complex quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 2x - 3$ **b** $x^2 - 7x - 8$ **c** $x^2 - 5x + 6$

THINK	WRITE
1 List the factor pairs of the constant term. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals. **2** Add the factor pairs together and identify which pair add to the linear coefficient. **3** Write the expression in factorised form.	**a** $1 \times -3 = -3 \rightarrow 1 - 3 = -2$ $(-1) \times 3 = -3 \rightarrow -1 + 3 = +2$ $x^2 + 2x - 3 = (x + 3)(x - 1)$ **b** $1 \times (-8) = -8 \rightarrow 1 - 8 = -7$ $(-1) \times 8 = -8 \rightarrow -1 + 8 = 7$ $2 \times (-4) = -8 \rightarrow 2 - 4 = -2$ $(-2) \times 4 = -8 \rightarrow -2 + 4 = 2$ $x^2 - 7x - 8 = (x - 8)(x + 1)$ **c** $1 \times 6 = 6 \rightarrow 1 + 6 = 7$ $(-1) \times (-6) = 6 \rightarrow -1 - 6 = -7$ $2 \times 3 = 6 \rightarrow 2 + 3 = 5$ $(-2) \times (-3) = 6 \rightarrow -2 - 3 = -5$ $x^2 - 5x + 6 = (x - 3)(x - 2)$

ADVANCED

Example 2G.3 Factorising quadratic trinomials by first taking out a common factor

Factorise these by first taking out a common factor.

a $2x^2 - 10x - 12$ **b** $-3x^2 + 12x - 9$

THINK

1 If the coefficient of x^2 is not 1, look for a common factor of the three terms. Write the HCF before the brackets. Inside the brackets, write the results of dividing each term by the HCF.

2 List the factor pairs of the constant term each time. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.

3 Add the factor pairs together and identify which pair add to the linear coefficient.

4 Write the expression in factorised form.

WRITE

a $2x^2 - 10x - 12 = 2(x^2 - 5x - 6)$

$-1 \times 6 = -6 \rightarrow -1 + 6 = 5$

$1 \times (-6) = -6 \rightarrow 1 - 6 = -5$

$-2 \times 3 = -6 \rightarrow -2 + 3 = 1$

$2 \times (-3) = -6 \rightarrow 2 - 3 = -1$

$$\begin{aligned} 2x^2 - 10x - 12 &= 2(x^2 - 5x - 6) \\ &= 2(x - 6)(x + 1) \end{aligned}$$

b $-3x^2 + 12x - 9 = -3(x^2 - 4x + 3)$

$1 \times 3 = 3 \rightarrow \quad 1 + 3 = 4$

$(-1) \times (-3) = 3 \rightarrow -1 - 3 = -4$

$$\begin{aligned} -3x^2 + 12x - 9 &= -3(x^2 - 4x + 3) \\ &= -3(x - 3)(x - 1) \end{aligned}$$

Helpful hints

✔ You can always check your factorised quadratic by expanding your solution to see if you get the same expression that you started with.

✔ You don't need to list all the factor pairs of the constant term. Just write each new pair as you go and stop when you find the pair that add to give the linear term.

✔ You don't always need to consider all the factor pairs of the constant term, c, to factorise a quadratic of the form $x^2 + bx + c$.

If the constant term, c, is positive, then both factors have the same sign. So:

→ if b is also positive, both factors are positive

→ if b is negative, both factors are negative.

✔ Remember, always look for common factors! You should start by checking to see if the leading coefficient is a common factor of all the terms.

ANS p653

Exercise 2G Factorising quadratic expressions

△ 1, 2, 3(1st column), 4, 5, 6(1st column), 7(a, d, g, h, k), 8(a, b, e, f), 9(a, c), 10

□ 3(2nd, 3rd columns), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 12, 15, 17

⬠ 3(3rd column), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 11, 13–14, 16, 17

1 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a 7, 8 **b** 16, 10 **c** 45, 14 **d** 60, 16 **e** 36, 13 **f** 24, 11

2 Use your results from question **1** to factorise each of these quadratic trinomials.

a $x^2 + 8x + 7$
b $x^2 + 10x + 16$
c $x^2 + 14x + 45$
d $x^2 + 16x + 60$
e $x^2 + 13x + 36$
f $x^2 + 11x + 24$

2G.1 **3** Factorise each of these quadratic trinomials.

a $x^2 + 3x + 2$
b $x^2 + 5x + 6$
c $x^2 + 11x + 10$
d $x^2 + 12x + 27$
e $x^2 + 7x + 10$
f $x^2 + 9x + 14$
g $x^2 + 6x + 9$
h $x^2 + 12x + 11$
i $x^2 + 11x + 18$
j $x^2 + 12x + 32$
k $x^2 + 16x + 63$
l $x^2 + 14x + 40$
m $x^2 + 11x + 30$
n $x^2 + 13x + 22$
o $x^2 + 15x + 36$

4 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a $-8, 7$
b $6, -7$
c $-35, -2$
d $-15, 2$
e $-8, -2$
f $30, -11$

5 Use your results from question **4** to factorise each of these quadratic trinomials.

a $x^2 + 7x - 8$
b $x^2 - 7x + 6$
c $x^2 - 2x - 35$
d $x^2 + 2x - 15$
e $x^2 - 2x - 8$
f $x^2 - 11x + 30$

2G.2 **6** Factorise each of these quadratic trinomials.

a $x^2 + 4x - 12$
b $x^2 - 8x - 20$
c $x^2 - 5x + 6$
d $x^2 + 6x - 27$
e $x^2 - 13x + 40$
f $x^2 - x - 20$
g $x^2 + 4x - 21$
h $x^2 - 14x + 33$
i $x^2 - 3x - 28$
j $x^2 + x - 12$
k $x^2 - 7x - 18$
l $x^2 - 15x + 56$

2G.3 **7** Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 18x + 16$
b $3x^2 + 15x + 18$
c $7x^2 + 14x - 56$
d $-5x^2 - 35x - 60$
e $-4x^2 + 28x - 40$
f $6 - 5x - x^2$
g $2x^2 + 14x - 36$
h $19x^2 - 38x - 57$
i $-3x^2 + 33x + 78$
j $6x - 2x^2 - 4$
k $\frac{1}{2}x^2 + \frac{3}{2}x - 2$
l $\frac{1}{3}x^2 + x - 6$

8 Simplify each of the following algebraic fractions.

a $\dfrac{x^2 + 6x + 8}{x + 2}$
b $\dfrac{x + 2}{x^2 - 4}$
c $\dfrac{x^2 + 6x}{x^2 + 4x - 12}$
d $\dfrac{x^2 - 8x + 15}{x^2 - 9}$
e $\dfrac{2x^2 - 50}{x^2 + 7x + 10}$
f $\dfrac{11x^2 - 22x}{11x^2 - 44}$
g $\dfrac{x^2 + 2x + 1}{1 - x^2}$
h $\dfrac{4x^2 + 8x - 60}{2x^2 + 18x + 40}$
i $\dfrac{3x^2 - 9x - 30}{6x^2 + 42x + 60}$

9 Simplify each of these expressions using the most appropriate method.

a $\dfrac{x^2 + x - 20}{x^2 - 6x + 8} \times \dfrac{x^2 - 4}{x^2 - 25} \times \dfrac{8x^2 - 40x}{4x^2 + 8x}$
b $\dfrac{x}{x + 7} \times \dfrac{x^2 - 49}{3x} \div \dfrac{x^2 - 5x - 14}{x^2 - 4x - 12}$
c $\dfrac{3x^2 + 3x}{x - 3} \times \dfrac{x^2 - 7x + 12}{6x^2 + 6x} + \dfrac{x + 5}{3}$
d $\dfrac{x^2 + 9x + 18}{2x + 6} \div \dfrac{3x + 18}{5x - 1} + \dfrac{3x - 5}{8}$

10 Choose the most appropriate method to factorise each of these expressions. Then factorise them.

a $x^2 + 15x + 44$
b $x^2 - 2x - 15$
c $x^2 + 4x + 4$
d $3x^2 - 18x + 24$
e $5x^2 - 45$
f $2x^2 - 8x$

11 Explain why each of these expressions cannot be factorised using the methods covered so far.

a $x^2 + 9$
b $(x - 3)^2 + 16$
c $(x + 1)^2 + 4$
d $x^2 - 4x + 5$
e $(x + 2)^2 - 4x$
f $x^2 + 12x + 40$

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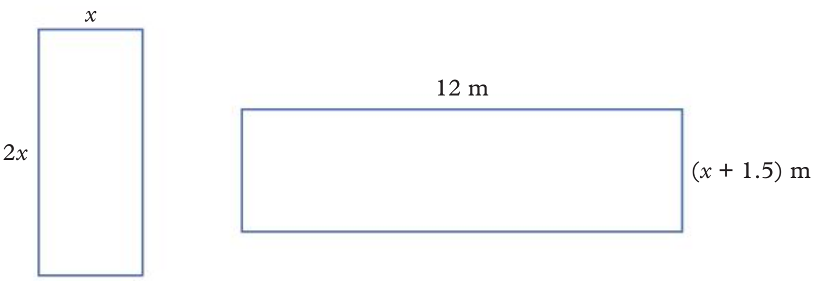
PROBLEM SOLVING AND REASONING

12 Consider the following two rectangles:



a Write an expression for the area of each rectangle in terms of x.

b Write an expression for the total area of the two rectangles in terms of x.

c The total area of the two rectangles can be combined and reshaped to form two equally sized squares. What is the side length of these squares, in terms of x?

13 The triangle below has a base of $(x + 2)$ metres and an area of $\left(\frac{1}{2}x^2 + 4x + 6\right)$ square metres. Find an expression, in terms of x, for the height of the triangle.

CHALLENGE

14 It is possible for an expression to be a quadratic with two variables.

a Expand each of the following:

i $(x + y)(x + y)$ **ii** $(x + 2y)(x + 3y)$

b Factorise each of the following:

i $x^2 + 10x + 16$ **ii** $x^2 + 10xy + 16y^2$

c Factorise each of the following:

i $x^2 + 7x + 10$ **ii** $x^2 + 7xy + 10y^2$

d Factorise each of the following:

i $x^2 - 6xy + 8y^2$ **ii** $y^2 - 9xy + 14x^2$ **iii** $4x^2 + 5xy + y^2$

15 If possible, factorise each of the following quadratic expressions, using a method already covered in this chapter.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$

c $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$

g $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

16 What values can k have for each quadratic trinomial below to be able to be factorised using the methods already covered in this chapter?

a $x^2 - 6x + k$ **b** $x^2 + 20x + k$

17 Factorise the following expressions.

a $x^4 - 16$ **b** $64a^2 - 4b^2$ **c** $a^2 - 2ax + x^2 - y^2$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Factorising quadratic expressions

Worksheet Factorising algebraic expressions

Topic quiz 2G

2H Completing the square

Learning intentions

By the end of this topic you will be able to ...

- ✔ add a term to a quadratic binomial to make a perfect square
- ✔ factorise quadratic expressions by completing the square.

Inter-year links

Year 8	5G Factorising
Year 9	3F Factorising monic quadratic equations

Key content video

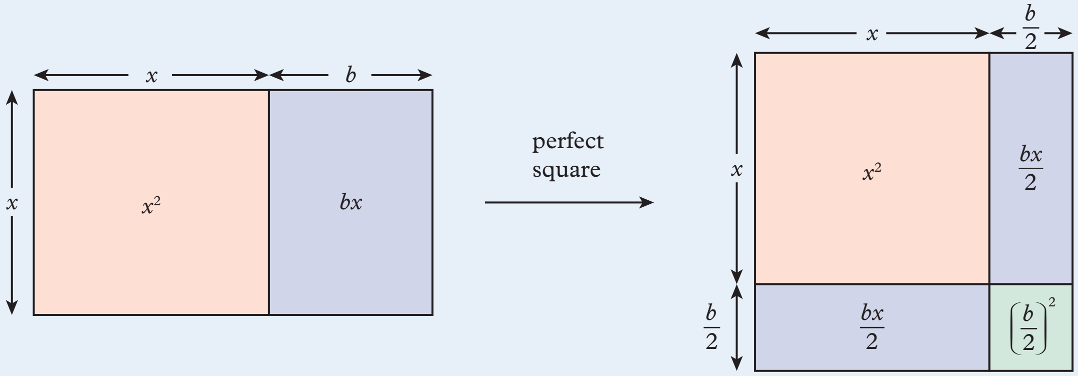
Adding a term to make a perfect square

- For any quadratic of the form $x^2 + bx$, a constant term of the form $\left(\frac{b}{2}\right)^2$ can be added to make it a perfect square.

$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

‘perfect square’ rule

$$a^2 + 2ab + b^2 = (a + b)^2$$

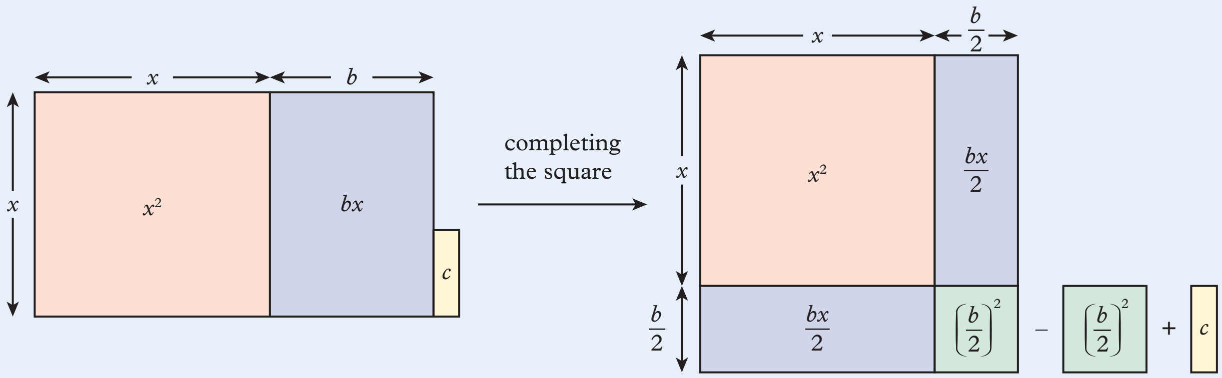


For example, consider the expression $x^2 + 6x$. The coefficient of the linear term is 6, so we can add $\left(\frac{6}{2}\right)^2 = 9$ to the binomial to make it a perfect square:

$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(x)(3) + 3^2 \\ &= (x + 3)^2 \end{aligned}$$

Completing the square

- **Completing the square** is the process of changing a quadratic expression from the general form, $ax^2 + bx + c$, to the form $a(x - h)^2 + k$ by applying the ‘perfect square’ rule.
- To complete the square for a monic quadratic, $x^2 + bx + c$, the term $\left(\frac{b}{2}\right)^2$ must be *added and subtracted* from the expression to change it to the form $(x - h)^2 + k$.



- Just as zero can be added to any number without changing its value, an expression equivalent to zero can be added to an expression without changing its value. In this case the expression is:

$$+\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = 0$$

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + 0 + c \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= (x - h)^2 + k \end{aligned}$$

$$\text{when } h = -\frac{b}{2} \quad \text{and} \quad k = -\left(\frac{b}{2}\right)^2 + c$$

For example, to complete the square for $x^2 + 8x + 1$, first find the constant term needed to make this a perfect square: $\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 16$

We then add and subtract this value to complete the square:

$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + 0 + 1 \\ &= x^2 + 8x + 16 - 16 + 1 \\ &= (x^2 + 8x + 16) - 16 + 1 \\ &= (x + 4)^2 - 15 \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Once a monic quadratic is written in the form $(x - h)^2 + k$, the value of k determines whether it can be factorised and written as the product of two linear factors.
  - → If $k = 0$, then the quadratic is already factorised and is written as the square of a linear expression.
  - → If $k > 0$, then the quadratic cannot be factorised using real numbers.
  - → If $k < 0$, the quadratic can be factorised using the 'difference of two squares' rule.
- For example, to factorise the expression $(x + 4)^2 - 15$, from above:

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ (x + 4)^2 - 15 &= (x + 4)^2 - (\sqrt{15})^2 \\ &= (x + 4 + \sqrt{15})\,(x + 4 - \sqrt{15}). \end{aligned}$$

Example 2H.1 Adding a term to make a perfect square

Add a term to each of these expressions to make a new expression that is a perfect square in expanded form.

a $x^2 + 10x$ **b** $x^2 - 8x$

THINK	**WRITE**
a **1** For $x^2 + 10x$, substitute the coefficient of the linear term, 10, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.	**a** $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2$ $= 5^2$ $= 25$
2 Add the term to make a perfect square.	$x^2 + 10x + 25$
b **1** For $x^2 - 8x$, substitute the coefficient of the linear term, −8, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.	**b** $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2$ $= (-4)^2$ $= 16$
2 Add the term to make a perfect square.	$x^2 - 8x + 16$

Example 2H.2 Factorising by completing the square

Factorise each of these quadratic trinomials by first completing the square.

a $x^2 + 12x + 15$ **b** $x^2 - 2x - 7$ **c** $x^2 + x - 3$

THINK

1. Substitute the coefficient of the linear term into the expression $\left(\frac{b}{2}\right)^2$ each time to find the term to make a perfect square.
2. Add and subtract the term from the quadratic so that the overall expression remains unchanged.
3. Factorise the perfect square expression using the ‘perfect square’ rule and simplify the constant terms at the end of the expression.
4. Write as the difference of two squares. Simplify the square root if possible.
5. Factorise using the ‘difference of two squares’ rule.

WRITE

a The coefficient of the linear term in $x^2 + 12x + 15$ is 12.

$$\left(\frac{12}{2}\right)^2 = 6^2$$
$$= 36$$

$$\begin{aligned} x^2 + 12x + 15 &= x^2 + 12x + 36 - 36 + 15 \\ &= (x^2 + 12x + 36) - 36 + 15 \\ &= (x^2 + 2[6]x + 6^2) - 21 \\ &= (x + 6)^2 - 21 \\ &= (x + 6)^2 - (\sqrt{21})^2 \\ &= (x + 6 + \sqrt{21})(x + 6 - \sqrt{21}) \end{aligned}$$

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b The coefficient of the linear term in $x^2 - 2x - 7$ is -2.

$$\left(\frac{-2}{2}\right)^2 = (-1)^2$$
$$= 1$$

$$\begin{aligned}
x^2 - 2x - 7 &= x^2 - 2x + 1 - 1 - 7 \\
&= (x^2 - 2x + 1) - 1 - 7 \\
&= (x^2 + 2[-1]x + [-1]^2) - 8 \\
&= (x - 1)^2 - 8 \\
&= (x - 1)^2 - (\sqrt{8})^2 \\
&= (x - 1)^2 - (2\sqrt{2})^2 \\
&= (x - 1 + 2\sqrt{2})(x - 1 - 2\sqrt{2})
\end{aligned}$$

c The coefficient of the linear term in $x^2 + x - 3$ is 1.

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\begin{aligned}
x^2 + x - 3 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 3 \\
&= (x^2 + x + \frac{1}{4}) - \frac{1}{4} - 3 \\
&= \left(x^2 + 2\left[\frac{1}{2}\right]x + \left[\frac{1}{2}\right]^2\right) - \frac{1}{4} - \frac{12}{4} \\
&= \left(x + \frac{1}{2}\right)^2 - \frac{13}{4} \\
&= \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2 \\
&= \left(x + \frac{1 + \sqrt{13}}{2}\right)\left(x + \frac{1 - \sqrt{13}}{2}\right)
\end{aligned}$$

Helpful hints

- ✔ Remember to always simplify surds by factorising any square numbers.
- ✔ Remember that you must always add and also subtract the $\left(\frac{b}{2}\right)^2$ term for the expression to remain unchanged.
- ✔ Factorising using the ‘difference of two squares’ rule only works if $k < 0$. If $k > 0$, then the expression cannot be written as the product of two linear factors.

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Exercise 2H Completing the square

△ 1–5, 6(1st column), 7, 8, 11

■ 3–5, 6(2nd column), 7, 9, 10, 12, 13

⬠ 3, 4, 6(2nd column), 7, 10, 12, 14–16

1 **a** **i** Expand $(x + 3)^2$ using the ‘perfect square’ rule. **ii** Use your answer for part **i** to factorise $x^2 + 6x + 9$.

b **i** Expand $(x - 9)^2$ using the ‘perfect square’ rule. **ii** Use your answer for part **i** to factorise $x^2 - 18x + 81$.

c Factorise each of these quadratic trinomials.

i $x^2 + 8x + 16$ **ii** $x^2 + 4x + 4$ **iii** $x^2 - 12x + 36$

iv $x^2 + 16x + 64$ **v** $x^2 - 10x + 25$ **vi** $x^2 - 20x + 100$

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UNDERSTANDING AND FLUENCY

2 Copy and complete each of these expressions to make a perfect square.

a $x^2 + 6x + \_\_ = (x + 3)^2$

b $x^2 + 10x + \_\_ = (x + 5)^2$

c $x^2 - 8x + \_\_ = (x - 4)^2$

d $x^2 - 4x + \_\_ = (x - 2)^2$

e $x^2 + 2x + \_\_ = (x + 1)^2$

f $x^2 - 16x + \_\_ = (x - 8)^2$

2H.1 3 Add a term to each of these expressions to make a new perfect square in expanded form.

a $x^2 + 14x$

b $x^2 - 6x$

c $x^2 + 18x$

d $x^2 + 20x$

e $x^2 - 12x$

f $x^2 + 22x$

4 Factorise each perfect square expression you wrote for question **3**.

5 The quadratic trinomial $x^2 + 6x + 7$ can be written as $(x^2 + 6x + 9) - 2$. Explain how knowledge of the 'perfect square' rule and the 'difference of two squares' rule can be used to factorise it.

2H.2 6 Factorise each of these quadratic trinomials by first completing the square.

a $x^2 + 8x + 9$

b $x^2 - 10x + 14$

c $x^2 + 4x - 6$

d $x^2 + 2x - 5$

e $x^2 - 6x - 4$

f $x^2 - 12x + 13$

g $x^2 + 10x - 4$

h $x^2 - 14x + 46$

i $x^2 + 8x + 7$

j $x^2 + 3x + 1$

k $x^2 - 5x + 3$

l $x^2 + 11x + 9$

7 Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 8x + 2$

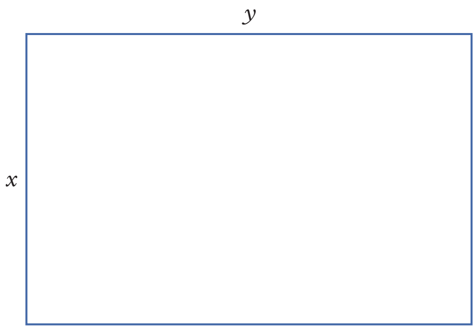
b $-x^2 - 6x + 2$

c $-5x^2 + 10x + 5$

d $3x^2 - 15x + 6$

PROBLEM SOLVING AND REASONING

8 To create a square from the rectangle on the right, other rectangles can be cut out or attached.



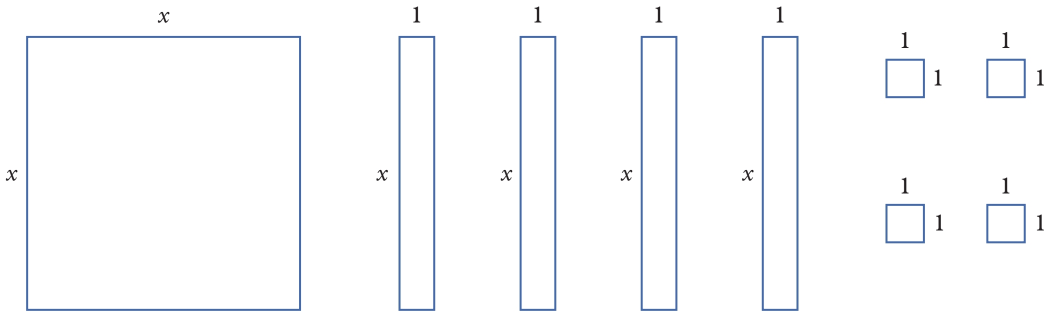
a What are the dimensions of the rectangle needing to be cut out to turn the rectangle into a square?

b If a rectangle is cut out to turn the rectangle into a square, write an expression for the area of the resulting square.

c What are the dimensions of the rectangle needing to be attached to turn the rectangle into a square?

d If a rectangle is attached to turn the rectangle into a square, write an expression for the area of the resulting square.

9 **a** Show how the following shapes can be arranged to form a square.



b Express the area of the square you arranged in part **a** as an expanded quadratic expression.

c Express the area of the square you arranged in part **a** as a factorised quadratic expression.

ADVANCED

PROBLEM SOLVING AND REASONING

10 In the challenge section of Exercise **2G**, you were asked to factorise as many of the following quadratic expressions as possible using the methods that had been covered. Now that you know how to complete the square, factorise each of the following quadratics using the most appropriate method.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$

c $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$

g $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

11 a Factorise $x^2 + 7x + 6$ by first completing the square.

b Look at your answer to part **a**. Is there an easier way of obtaining it? Explain.

12 Not all quadratic trinomials can be factorised using real numbers.

If a quadratic cannot be factorised by completing the square, then it cannot be factorised at all. For example, a quadratic like $x^2 + 2x + 5$ becomes $(x + 1)^2 + 3$ after completing the square, and this expression cannot be factorised using real numbers because it cannot be expressed as the difference of two squares. Complete the square for each of the following expressions to find the specified values.

a Find the values of c for which $x^2 + 8x + c$ cannot be factorised using real numbers.

b Find the values of c for which $x^2 - 10x + c$ cannot be factorised using real numbers.

c Find the values of b for which $x^2 + bx + 1$ cannot be factorised using real numbers.

13 Explain why each of these expressions cannot be factorised using the methods covered in this chapter so far.

a $x^2 + 6x + 10$ **b** $x^2 - 8x + 21$ **c** $3x^2 + 7x + 2$

d $12x^2 + 64x + 45$ **e** $2x^2 + 4x + 4$

CHALLENGE

14 Factorise each of the following expressions, if possible.

a $2x^2 + 6x - 56$ **b** $x^2 - 3x + 5$ **c** $x^3 + 3x^2 - 18x$

d $12 + x - x^2$ **e** $98 - 2x^2$ **f** $4x^2 + 25$

15 Non-monic quadratics can also be factorised by completing the square. For example:

$$\begin{aligned}2x^2 + 4x - 9 &= 2\left(x^2 + 2x - \frac{9}{2}\right)\\ &= 2\left[(x^2 + 2x + 1) - 1 - \frac{9}{2}\right]\\ &= 2\left[\left(x + 1\right)^2 - \frac{11}{2}\right]\\ &= 2\left(x + 1 - \sqrt{\frac{11}{2}}\right)\left(x + 1 + \sqrt{\frac{11}{2}}\right)\\ &= 2\left(x + 1 - \frac{\sqrt{22}}{2}\right)\left(x + 1 + \frac{\sqrt{22}}{2}\right)\end{aligned}$$

Factorise each of the following quadratics by completing the square:

a $2x^2 + 12x + 5$ **b** $2x^2 + 7x + 1$ **c** $3x^2 + 11x - 1$

16 a Complete the square to factorise the general monic quadratic trinomial $x^2 + bx + c$.

b Considering the fact that the square root of a negative number is not a real number, write an algebraic fraction in terms of b and c that must be *greater than or equal to zero* for $x^2 + bx + c$ to be factorised using real numbers.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Factorising by completing the square

Topic quiz
2H

2I Factorising non-monic quadratic expressions

Learning intentions

By the end of this topic you will be able to ...

✔ factorise non-monic quadratic expressions by splitting the middle term

✔ factorise non-monic quadratic expressions by completing the square.

Inter-year links

Year 8	5G Factorising
Year 9	3F Factorising monic quadratic equations

Key content video

Non-monic quadratic trinomials

- A **non-monic quadratic** trinomial is a quadratic trinomial in which the leading coefficient is not equal to one. These trinomials are often more difficult to factorise.

  $ax^2 + bx + c$ when $a \neq 1$ and $a \neq 0$
- When a non-monic quadratic trinomial $ax^2 + bx + c$ can be factorised, it has the general form $(mx + n)(px + q)$ where m, n, p and q are non-zero constants.

  $ax^2 + bx + c = (mx + n)(px + q)$

Factorising by splitting the middle term

- Expanding the general form of the factorised non-monic quadratic trinomial gives:

  $$\begin{aligned}(mx + n)(px + q) &= mpx^2 + mqx + npx + nq \\ &= mpx^2 + (mq + np)x + nq \\ &= ax^2 + bx + c\end{aligned}$$

  So: $a = m \times p$ $\quad b = mq + np$ $\quad c = n \times q$

  This means there are two numbers, mq and np, which add to give b and multiply to give $a \times c$.

  $b = mq + np$ and $a \times c = (mq)(np)$
- If the middle term, bx, of a quadratic trinomial, $ax^2 + bx + c$, is split into the sum of two terms whose coefficients multiply to give $a \times c$, the quadratic can be factorised by grouping terms. This method of factorisation is called splitting the middle term.

  For example, $3x^2 + 10x + 8$ is in the form $ax^2 + bx + c$. We have $a = 3$, $b = 10$ and $c = 8$. To split the middle term, $10x$, we must find two numbers that add to give b (which is 10) and multiply to give $a \times c$ (which is $3 \times 8 = 24$).

  The numbers 4 and 6 satisfy these conditions:

  $4 + 6 = 10$ and $4 \times 6 = 24$

  So: $$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 4x + 6x + 8 \\ &= x(3x + 4) + 2(3x + 4) \\ &= (3x + 4)(x + 2)\end{aligned}$$
- If you choose the correct values of mq and np, the order you write the two new terms when you split the middle term does not matter. The resulting factorisation is the same.

  For example, switching the $4x$ and the $6x$ above gives:

  $$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 6x + 4x + 8 \\ &= 3x(x + 2) + 4(x + 2) \\ &= (x + 2)(3x + 4)\end{aligned}$$

Factorising by completing the square

- A non-monic quadratic trinomial can also be factorised by completing the square. This is done by taking out the leading coefficient as a factor.

  $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a} + \frac{c}{a}\right)$$

  Then we can complete the square using the normal method.

Example 2I.1 Factorising by splitting the middle term

Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 + 8x + 4$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - x - 2$

THINK

a
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = 12$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = 8$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

b
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = -18$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = 7$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

c
1. Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.
2. List the factor pairs of the product $a \times c = -20$. Remember to consider both positive and negative factors.
3. Add the factor pairs together and identify which pair add to the linear coefficient ($b = -1$).
4. Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.
5. Factorise by grouping terms.

WRITE

a $3x^2 + 8x + 4$

$b = 8$ and $a \times c = 3 \times 4 = 12$

$1 \times 12 = 12 \rightarrow 1 + 12 = 13$

$(-1) \times (-12) = 12 \rightarrow -1 - 12 = -13$

$2 \times 6 = 12 \rightarrow 2 + 6 = 8$

$(-2) \times (-6) = 12 \rightarrow -2 - 6 = -8$

$3 \times 4 = 12 \rightarrow 3 + 4 = 7$

$(-3) \times (-4) = 12 \rightarrow -3 - 4 = -7$

$$\begin{aligned}3x^2 + 8x + 4 &= 3x^2 + 2x + 6x + 4\\ &= x(3x + 2) + 2(3x + 2)\\ &= (3x + 2)(x + 2)\end{aligned}$$

b $6x^2 + 7x - 3$

$b = 7$ and $a \times c = 6 \times (-3) = -18$

$1 \times (-18) = -18 \rightarrow 1 - 18 = -17$

$(-1) \times 18 = -18 \rightarrow -1 + 18 = 17$

$2 \times (-9) = -18 \rightarrow 2 - 9 = -7$

$(-2) \times 9 = -18 \rightarrow -2 + 9 = 7$

$3 \times (-6) = -18 \rightarrow 3 - 6 = -3$

$(-3) \times 6 = -18 \rightarrow -3 + 6 = 3$

$$\begin{aligned}6x^2 + 7x - 3 &= 6x^2 - 2x + 9x - 3\\ &= 2x(3x - 1) + 3(3x - 1)\\ &= (3x - 1)(2x + 3)\end{aligned}$$

c $10x^2 - x - 2$

$b = -1$ and $a \times c = 10 \times (-2) = -20$

$1 \times (-20) = -20 \rightarrow 1 - 20 = -19$

$(-1) \times 20 = -20 \rightarrow -1 + 20 = 19$

$2 \times (-10) = -20 \rightarrow 2 - 10 = -8$

$(-2) \times 10 = -20 \rightarrow -2 + 10 = 8$

$4 \times (-5) = -20 \rightarrow 4 - 5 = -1$

$(-4) \times 5 = -20 \rightarrow -4 + 5 = 1$

$$\begin{aligned}10x^2 - x - 2 &= 10x^2 + 4x - 5x - 2\\ &= 2x(5x + 2) - 1(5x + 2)\\ &= (5x + 2)(2x - 1)\end{aligned}$$

Example 21.2 Factorising non-monic quadratic expressions by completing the square

Factorise $5x^2 - 10x + 1$ by completing the square.

THINK	**WRITE**
1 Factorise the trinomial by writing the expression as a product of the leading coefficient, which is 5, and a monic quadratic.	$5x^2 - 10x + 1 = 5\left(x^2 - 2x + \frac{1}{5}\right)$
2 The coefficient of the linear term in the monic quadratic is -2, so add and subtract the term $\left(\frac{-2}{2}\right)^2 = 1$ from the quadratic so that the overall expression remains unchanged.	$= 5\left(x^2 - 2x + 1 - 1 + \frac{1}{5}\right)$ $= 5\left([x^2 - 2x + 1] - \frac{5}{5} + \frac{1}{5}\right)$
3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.	$= 5\left(\left[x - 1\right]^2 - \frac{4}{5}\right)$
4 Write as the difference of two squares. Simplify the square root by identifying any perfect square factors and rationalising the denominator.	$= 5\left(\left[x - 1\right]^2 - \left[\sqrt{\frac{4}{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2\sqrt{5}}{5}\right]^2\right)$
5 Factorise using the 'difference of two squares' rule.	$= 5\left(x - 1 + \frac{2\sqrt{5}}{5}\right)\left(x - 1 - \frac{2\sqrt{5}}{5}\right)$

Helpful hints

✔ Factorising non-monic quadratic trinomials by completing the square often results in an expression containing fractions with surds.

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Exercise 2I Factorising non-monic quadratic expressions

△ **1, 2, 3–5(1st, 2nd columns), 6(a, c, e), 7(a, b), 8(a, c, e, g), 12**

□ **3–6(2nd, 3rd columns), 7(c, d), 8(d, f, g, i), 9, 10, 13**

⬠ **4–6(2nd, 3rd columns), 7(e, f), 8(f–i), 11, 14–16**

1 Each quadratic trinomial below is in the form $ax^2 + bx + c$. For each one:

i identify ac and b

ii find two numbers that multiply together to give ac and add together to give b

iii use the two numbers you found in part **ii** to split the middle term into two terms.

a $2x^2 + 5x + 2$ **b** $3x^2 + 11x + 6$ **c** $4x^2 + 17x + 4$

d $3x^2 + 10x - 8$ **e** $6x^2 - 7x - 3$ **f** $5x^2 - 13x + 6$

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UNDERSTANDING AND FLUENCY

2 Factorise each of these expressions by grouping terms.

a $3x^2 + 2x + 9x + 6$
b $4x^2 + x + 16x + 4$
c $2x^2 + x + 4x + 2$
d $6x^2 - 9x + 2x - 3$
e $3x^2 - 2x + 12x - 8$
f $5x^2 - 3x - 10x + 6$

21.1 **3** Factorise each of these quadratic trinomials by first splitting the middle term.

a $2x^2 + 7x + 6$
b $3x^2 + 5x + 2$
c $4x^2 + 12x + 5$
d $7x^2 + 9x + 2$
e $5x^2 + 14x + 8$
f $2x^2 + 9x + 9$
g $6x^2 + 7x + 2$
h $10x^2 + 17x + 3$
i $8x^2 + 14x + 5$

4 Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 - 8x - 3$
b $4x^2 + 5x - 6$
c $2x^2 - 9x + 10$
d $5x^2 - x - 4$
e $8x^2 + 10x - 3$
f $4x^2 + 4x - 15$
g $7x^2 - 18x + 8$
h $12x^2 + x - 1$
i $3x^2 - 11x - 20$

5 Factorise each of these quadratic trinomials. Where appropriate, take out any common factors first. Hint: The common factor can be negative.

a $4x^2 + 10x + 6$
b $12x^2 + 27x + 6$
c $10x^2 + 38x - 8$
d $8x^2 + 22x + 15$
e $6x^2 + 3x - 30$
f $12x^2 - 8x - 4$
g $-4x^2 - 6x - 2$
h $-12x^2 - 21x + 6$
i $-6x^2 - 17x - 12$

6 Simplify each of these expressions. Remember to factorise first.

a $\frac{2x^2 + 15x + 18}{x + 6}$
b $\frac{3x + 4}{3x^2 + 13x + 12}$
c $\frac{4x^2 + 18x + 14}{2x + 2}$
d $\frac{6x^2 - 5x + 1}{3x^2 + 14x - 5}$
e $\frac{5x^2 - 28x - 12}{2x^2 - 15x + 18}$
f $\frac{-8x^2 + 4x + 40}{2x^2 - x - 10}$

7 Simplify each of these expressions.

a $\frac{5x^2 - 8x + 3}{x^2 - 1} \times \frac{2x^2 + x - 1}{2x^2 - 7x + 3}$
b $\frac{2x^2 + 13x + 21}{6x^2 - x - 2} \times \frac{3x^2 - 14x + 8}{x^2 - x - 12}$
c $\frac{4x^2 + 13x - 12}{2x^2 + 13x + 20} \div \frac{12x^2 - 5x - 3}{6x^2 + 17x + 5}$
d $\frac{4x^2 + 28x + 49}{4x^2 - 49} \div \frac{2x^2 - x - 28}{2x^2 + 15x + 28}$
e $\frac{3x^2 + 6x - 24}{3x^2 - 7x + 2} \times \frac{15x^2 + x - 2}{6x + 24}$
f $\frac{2x^3 - x^2 - 3x}{2x^2 + 9x - 18} \div \frac{x^3 - 9x}{x^2 + 9x + 18}$

21.2 **8** Factorise each of these quadratic trinomials by completing the square.

a $3x^2 + 18x + 21$
b $2x^2 - 20x + 40$
c $5x^2 + 10x - 30$
d $3x^2 + 12x + 3$
e $2x^2 - 12x - 3$
f $4x^2 + 8x - 1$
g $6x^2 - 24x - 5$
h $3x^2 + 24x + 1$
i $9x^2 - 18x + 2$

9 Simplify the following expression by first factorising each fraction.

$$\frac{2x^2 - 3x - 20}{x^3 - 16x} \times \frac{3x^2 + 16x + 16}{4x^2 - 25} \div \frac{9x^2 + 24x + 16}{70x - 28x^2}$$

PROBLEM SOLVING AND REASONING

10 a Factorise $4x^2 + 7x - 2$.

b Show that $-4x^2 - 7x + 2$ factorises to $(1 - 4x)(x + 2)$.

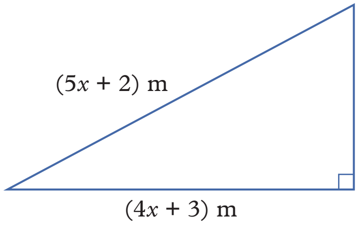
c Factorise each of these quadratic trinomials.

i $-2x^2 + x + 3$
ii $-3x^2 + 16x - 5$
iii $-5x^2 - 3x + 14$
iv $-4x^2 + 5x + 6$
v $-6x^2 - x + 2$
vi $-25x^2 + 10x - 1$

11 Consider the right-angled triangle shown on the right.

a Find an expression for the length of the missing side, in terms of x.

b The expression you wrote for part **a** should include a square root symbol. Factorise the expression appearing under that square root symbol.



12 Factorise each of the following by first completing the square.

a $x^2 + 8x + 5$ **b** $2x^2 + 8x + 5$ **c** $3x^2 + 8x + 5$

13 Decide whether each of these expressions can be factorised using the methods covered in this chapter so far. Write those that can be factorised in factorised form.

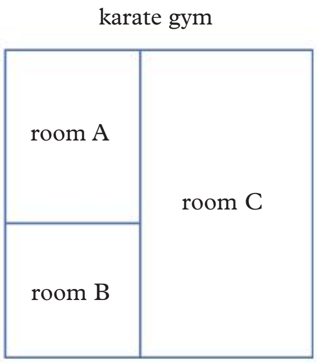
a $2x^2 + 4x + 6$ **b** $15x^2 - 2x - 8$

c $3x^2 - 12x + 3$ **d** $4x^2 - 8x + 16$

e $-2x^2 + 12x - 22$ **f** $-6x^2 + 69x - 189$

14 A karate instructor bought a gym with three training rooms: A, B and C. To make each room a safe place for training, the instructor plans to set down some large square mats of length x metres.

The dimensions of all the rooms are a number of mats plus or minus an integer number of metres. Training room A has an area of $(8x^2 + 2x - 15)$ m². The length of the square floor of training room B is the same as the smaller floor dimension of training room A. Assume $x < 4$ m.



Answer the questions about the karate gym.

a Write expressions for the dimensions of the floors of training rooms A and B in terms of the length of the mats.

b Write an expression for the total floor area of training rooms A and B combined, in terms of the length of the mats.

c If the shape of the karate gym is a square, what are the dimensions of the gym?

d Write an expression for the total area of training rooms A and C combined, as the difference of two squares.

e Factorise your answer for part **d**.

f Divide the total area of the gym by the area of one mat to determine an expression for the maximum number of mats that the instructor will need to cover the floor of the gym. Give your answer as a trinomial.

g If $x = 3$ m, what is the maximum number of mats that can fit on the floor of the gym?

15 A quartic is an expression of the form $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are constants. If a quartic does not include any indices that are odd numbers (so the b and d coefficients are zero), it can be factorised using the same methods for factorising quadratics.

For example, $x^4 + 5x^2 + 6$ can be factorised to $(x^2 + 3)(x^2 + 2)$.

a Factorise:

i $x^4 + 7x^2 + 10$ **ii** $x^4 + 14x^2 + 24$ **iii** $x^4 + 4x^2 + 3$

b Write each of the following quartics as the product of one quadratic factor and two linear factors, by using the ‘difference of two squares’ method of factorisation.

i $x^4 - 1$ **ii** $x^4 - 16$ **iii** $x^4 - 81$

c Write each of the following quartics as the product of four linear factors.

i $x^4 - 13x^2 + 36$ **ii** $x^4 - 26x^2 + 25$ **iii** $x^4 - 11x^2 + 10$

16 a Complete the square for the general quadratic trinomial $ax^2 + bx + c$.

b Use your answer for part **a** to factorise the general quadratic trinomial $ax^2 + bx + c$.

c Considering the fact that you cannot find the square root of a negative number, write a simplified expression in terms of a, b and c that *must be non-negative* for the quadratic to be factorised.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Factorising non-monic quadractic expressions

Topic quiz
2I

Chapter summary

Index laws and properties

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{m \times n}$	$(a \times b)^m = a^m \times b^m$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
$a^0 = 1$	$a^{-1} = \frac{1}{a}$

Expanding

- Use the distributive law to expand brackets. $a(b + c) = ab + ac$

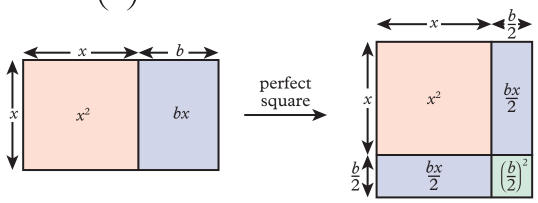
$$(a + b)(c + d) = ac + ad + bc + bd$$

Factorising by grouping terms

$$x^2 + 2x - 3x - 6 = x(x + 2) - 3(x + 2)$$
$$= (x + 2)(x - 3)$$

Adding a term to make a perfect square

- Add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to make it a perfect square.



$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Simplifying terms and expressions

- Like terms contain the same pronumerals raised to the same exponents.
- Like terms can be added and subtracted by adding and subtracting their cooefficients.

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Factorising using the HCF

expanding ←

$$2x^2 - 6x = 2x(x - 3)$$

expanded form factorised form

factorising →

Factorising monic quadratics

$$x^2 + bx + c$$

- Find numbers, m and n, that add to give b and multiply to give c.

$$x^2 + (m + n)x + mn = (x + m)(x + n)$$

- Substitute those two numbers into the factorised form (the binomial product).

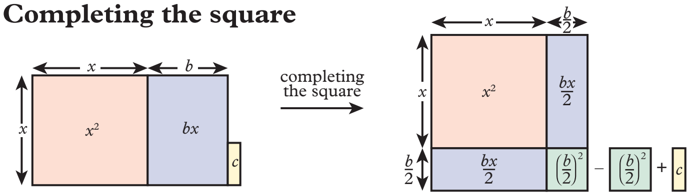
Factorising by splitting the middle term

$$ax^2 + bx + c = mpx^2 + mqx + npx + nq$$

- Find the unknowns that add to give b and multiply to give $a \times c$.
- Use these numbers to split the linear term.
- Factorise by grouping terms.

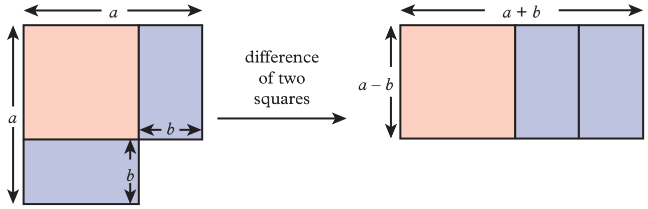
$$mpx^2 + mqx + npx + nq = (mx + n)(px + q)$$

Completing the square



$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Factorising using the difference of two squares



$$a^2 - b^2 = (a + b)(a - b)$$
$$(x - 1)^2 - 2 = (x - 1)^2 - (\sqrt{2})^2$$
$$= (x - 1 + \sqrt{2})(x - 1 + \sqrt{2})$$

ANS
p655

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.
✓ Core
✓ Advanced

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- algebraic fraction
- base
- binomial
- coefficient
- completing the square
- constant term
- difference of two squares
- distributive law
- expanding
- exponent form
- factorising
- highest common factor (HCF)
- index
- index form
- negative index
- perfect square
- pronumeral
- reciprocal
- simplifying
- splitting the middle term
- term
- variable

1 Identify the following from these given expressions.

$6x^3 - 2x + 9, x + 5, \dfrac{4}{x + 1} - 7x$

a a binomial
b a trinomial
c a constant term
d an algebraic fraction
e a negative coefficient
f leading coefficient of the trinomial
g an index

2 Match the expressions in the left-hand column with a key term in the right-hand column.

a $a^2 - b^2$
b $(a + b)^2$
c $a(b + c) = ab + ac$
d $\dfrac{4a}{8(a + b)} = \dfrac{a}{2(a + b)}$
e $x^2 + 2x + 1$
f $3x^2 - x + 5$
g $x^2 + 6x + 8 = (x + 3)^2 - 1$
h $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3$

I monic quadratic
II splitting the middle term
III non-monic quadratic
IV completing the square
V difference of squares
VI simplifying
VII perfect square
VIII expanding

3 Complete the following sentence using words from the list of key terms.

a If a term in the denominator has a ________________, this is the same as multiplying by the same term in the numerator but with a positive ________________.

b Factorisation is the reverse process to ________________ and involves dividing each term by the ________________.

4 Select the best option(s) for this statement: A factor of a factorised quadratic could be ________________.

A a constant
B a single term involving the pronumeral
C a binomial expression
D a quadratic expression
E all of the above

Multiple choice

2A **1** What is the simplified form of $x^2 \times (x^3)^4$?

A x^9 **B** x^{24} **C** x^{20} **D** x^{14} **E** x^5

2A **2** Which of the following expressions is not equivalent to the others?

A $(x^3 \times x^3 \times x^3)^2$ **B** $(x^4)^3$ **C** $(x^3)^2 \times (x^3)^2$ **D** $x^2 \times (x^2)^5$ **E** $(x^3)^2 \times x^6$

ADV 2B **3** What is the value of the missing index in $x^{-1}yz \times yz^{\square} = \frac{y^2}{xz}$

A 2 **B** 1 **C** 0 **D** −1 **E** −2

ADV 2B **4** What is $\frac{36x^2y}{4x^3y^{-2}}$ when it is simplified?

A $\frac{9}{xy^3}$ **B** $\frac{9y^3}{x}$ **C** $9x^{-1}y^{-3}$ **D** $\frac{9}{xy}$ **E** $9xy^3$

2C **5** Which of the following expressions is equivalent to $a^2b - 2ab - ab^2 + 5ab$?

A $7ab$ **B** $a^2b + 3ab + ab^2$ **C** $3ab$

D $3a^2b^2$ **E** $a^2b + 3ab - ab^2$

ADV 2D **6** What is the expanded form of $6t^2w(4w^3 + 3t^4)$?

A $24t^2w^3 + 18t^6w$ **B** $24t^2w^4 + 18t^8w$ **C** $24t^2w^4 + 18t^6w$

D $24t^2w^3 + 18t^8w$ **E** $10t^2w^4 + 9t^6w$

ADV 2D **7** What is the expanded form of $(x - 4)(x + 3)$?

A $x^2 + x + 12$ **B** $x^2 + x - 12$ **C** $x^2 - x + 12$ **D** $x^2 - x - 12$ **E** $x^2 + 7x - 12$

ADV 2E **8** What is the lowest common denominator of $\frac{2x}{(x+2)}$, $\frac{xy}{(x+1)(x+2)}$ and $\frac{2y}{x(x+2)}$?

A $2xy$ **B** $(x + 1)(x + 2)$ **C** $x(x + 1)(x + 2)$

D $(x + 2)$ **E** $x(x + 1)(x + 2)^3$

ADV 2E **9** Which of the following expressions is equivalent to $\frac{2x}{3} + \frac{x+1}{4}$?

A $\frac{3x+1}{7}$ **B** $\frac{11x+3}{12}$ **C** $\frac{3x+1}{12}$ **D** $\frac{11x+3}{7}$ **E** $\frac{x^2}{6}$

ADV 2F **10** What is the fully factorised form of $8x^2y^3 + 12x^3y$?

A $8x^2y(y^2 + 4x)$ **B** $4x^2y(2y^2 + 3x)$ **C** $4xy(2xy^2 + 3x^2)$

D $4xy(2y^2 + 3x)$ **E** $8x^3y^3(x + y^2)$

ADV 2F **11** What is the factorised form of $3(a - 2) - b(a - 2)$?

A $(a - 2)(3 - b)$ **B** $(a - 2)(b - 3)$ **C** $(a - 2)(3 + b)$

D $(a + 2)(3 + b)$ **E** $-3b(a - 2)$

ADV 2G **12** The quadratic trinomial $x^2 + 7x + c$ *cannot* be factorised for which of the following values of c?

A −18 **B** −8 **C** 10 **D** 12 **E** 18

ADV 2G **13** Which of these is a quadratic trinomial?

A $5x^2 + 4$ **B** $2x^3 + 4x^2 - 6$ **C** $x + 5x - 3$ **D** $4x - 2 + 3x^2$ **E** $x^2 - 3$

ADV 2H **14** Which of the following expressions is equivalent to $2x^2 + 6x + 7$?

A $2(x + 3)^2 - 2$ **B** $2\left(x + \frac{3}{2}\right)^2 - 2$ **C** $2(x + 3)^2 + 5$

D $2\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}$ **E** $2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$

ADV 2I **15** What is the factorised form of $9x^2 + 21x + 10$?

A $(9x + 5)(x + 2)$ **B** $(9x + 10)(x + 1)$ **C** $(3x + 5)(3x + 2)$

D $(3x + 10)(3x + 1)$ **E** $(9x + 2)(x + 5)$

Short answer

2A **1** Simplify each of the following.

a $a^2 \times a^5$ **b** $(2x^2y)^3 \times 3xy^2$ **c** $3(x^2y)^0$ **d** $\dfrac{12ab^2c^5}{(2bc)^2}$

ADV **2B** **2** Write each of these terms in simplified form with positive indices.

a $\dfrac{xy^{-3}}{x^{-2}y}$ **b** $\dfrac{4a^2b^{-1}}{12a^{-2}b^4}$ **c** $\dfrac{18p^3q^{-1}r^5}{15p^{-1}q^4r^{-2}}$

2C **3** Simplify each of these expressions, writing the answers with positive indices.

a $(x^2)^3 \times x^{-2} \div x^{-4}$ **b** $[(p^2q^{-3})^2 \div p^4q^{-3}] \times p^{-1}q$ **c** $(2a^2)^{-3} \times (3ab^{-1})^{-2} \div 12b^5$

2C **4** Simplify each of these expressions.

a $4xy - 10xy + x^2 - 4x^2 + 2x^2y - 5xy^2$ **b** $ab^2 + (ab)^2 - 3a^2b^2 + 2a^2b - 4b^2a$

c $pq - 5pq + p^2 - q^2 - (3p)^2 + (2q)^2$

2C **5** Name the like terms in each of these lists.

a $3abc^2,\ a^2b^2c^2,\ -2a^2b^2c,\ (2abc)^2$ **b** $5xyz^2,\ -2z^2xy,\ 3yz^2x,\ 4x^2y^2y^2$

c $2p^2q^2,\ -(pq)^2,\ p^2 \times q \times -q,\ (p \div q)^2$

2C **6** If $a = -2$, $b = -1$ and $c = 3$, evaluate each of these expressions.

a $a^2 + b^2 + c^2$ **b** $abc - ab^2c - c^3 \div b$ **c** $(a + b)^2 - (c - a)^2$

ADV **2D** **7** Expand and simplify each of these expressions.

a $(x - 3)(x + 7)$ **b** $(x + 2)^2$ **c** $(x - 4)^2$

d $(2x + 1)(x - 4)$ **e** $(a - b)(a + b)$ **f** $(a^2 + b)(a + b^2)$

ADV **2D** **8** Expand each of these expressions, giving your answers in simplest form.

a $(2x^2 - 4)(3x^2 + 5)$ **b** $(2x + 3y)(2x - 3y)$ **c** $(3x^2 - 4)^2$

ADV **2D** **9** Expand and simplify each of these expressions.

a $3(x^2 - 2) - x(x + 5)$ **b** $-2x(3 - x) + 5x(2 - 3x)$ **c** $x(x + 1) - x(x - 1) + x(x - 3)$

ADV **2E** **10** Simplify each of these expressions.

a $\dfrac{3x}{4} + \dfrac{4x}{3}$ **b** $\dfrac{x - 2}{4} - \dfrac{3 - x}{5}$ **c** $\dfrac{4x - 1}{12} - \dfrac{3 - 2x}{8}$ **d** $\dfrac{4}{3x} + \dfrac{5}{4x}$

ADV **2F** **11** Factorise each of these expressions by grouping terms.

a $x^2 - 3x - 4x + 12$ **b** $x^3 + x^2 - x - 1$ **c** $16x + 6x^2 - 32 - 12x$

ADV **2F** **12** Use the 'difference of two squares' rule to factorise each of these expressions. Write your answers in simplest form.

a $18x^2 - 8y^2$ **b** $(x - 3)^2 - x^2$ **c** $(x + 2)^2 - (x - 2)^2$

ADV **2F** **13** Factorise each of these expressions, including surds in your answer if necessary.

a $x^2y^2 - 13$ **b** $(x + 1)^2 - 3$ **c** $7 - (a - 1)^2$

ADV **2G** **14** Factorise each of these expressions.

a $x^2 + 8x + 12$ **b** $x^2 - 2x - 63$ **c** $-x^2 + 6x - 8$ **d** $-6x^2 - 6x + 36$

ADV **2G** **15** Simplify each of these expressions.

a $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{x^2 + x - 12}{x^2 + x - 6}$ **b** $\dfrac{x^2 - 5x + 6}{x^2 - 16} \div \dfrac{x^2 + x - 6}{x^2 - 3x - 4}$

ADV **2H** **16** Factorise each of these expressions by first completing the square.

a $x^2 + 10x + 20$ **b** $x^2 - 8x - 1$

ADV **2I** **17** Factorise each of these expressions.

a $3x^2 + 29x + 18$ **b** $8x^2 + 10x - 3$ **c** $2x^2 - 7x - 22$ **d** $12x^2 - 17x + 6$

ADV **2I** **18** Factorise each of these expressions, if possible.

a $2x^2 + 12x - 4$ **b** $5x^2 - 20x + 30$ **c** $6x^2 + 21x - 90$ **d** $-4x^2 + 12x - 2$

Analysis

ADV 1 Sameera wants to know about factorising quadratic trinomials of the form $x^2 + bx + 12$, where b is an integer.

a Suppose $b = 0$. Explain why Sameera can't use the 'difference of two squares' rule.

b Is there a value of b that would allow the 'perfect square' rule to be used? If not, why not?

Sameera wants to find all values of b that allow the quadratic to be factorised into a product of two linear factors in the form $(x + c)$ where c is a non-zero integer.

c Complete the square of $x^2 + bx + 12$ and use your answer to show that $b^2 \geq 49$.

d Find all possible values of b by considering the factor pairs of 12 and the corresponding factorisations.

e Sameera next considers quadratics of the form $2x^2 + bx + 12$. Find all values of b for which the quadratic can be written in the form $(x + n)(px + q)$.

ADV 2 **a** Find values of a and b such that $(ax - 3)(6x + 5) = 12x^2 - 8x + b$.

b Find values of a, b and c such that $(4x - b)(ax + 2) = 12x^2 + cx - 2$.

c Find values of a, b and c such that $(ax - 3)(bx + 5) = 12x^2 - 31x + c$.

d Find values of a, b and c such that $(ax + 1)(-2x + \mathrm{b}) = 12x^2 + cx - 15$.

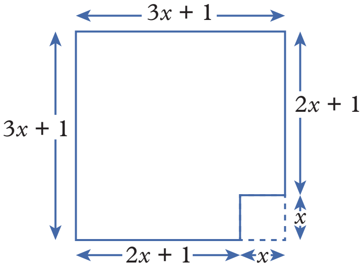
e How many possible sets of values of a, b, c and d are there, such that $(ax - 3)(bx + \mathrm{c}) = 12x^2 + dx - 15$?

f For $(ax + b)(cx + \mathrm{d}) = ex^2 + fx + \mathrm{g}$:

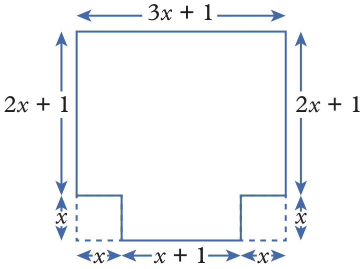
- **i** Determine the rule to calculate the value of f.
- **ii** How many possible values of f are there given e and g are prime integers?
- **iii** How many possible values of f are there if e is a composite integer with 4 distinct factors and g is a prime integer? Verify your answer with an example.
- **iv** How many possible values of f are there if both e and g are composite integers with 4 and 6 distinct factors, respectively?
- **v** How many possible values of f are there if both e and g are composite integers with m and n distinct factors, respectively?
- **vi** If the equation was $(ax + b)(cx + d) = ex^2 + fx - \mathrm{g}$, how would this change your answer to part **v**?

ADV 3 Write the area of the following shapes as a difference of squares and then in factored form.

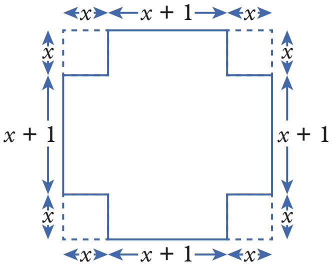
a



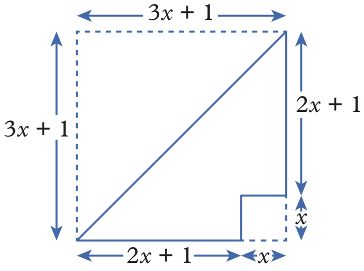
b



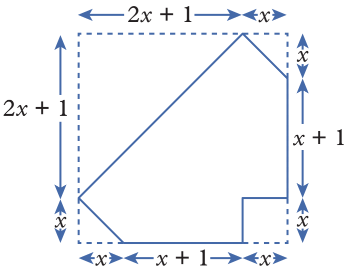
c



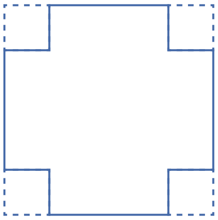
d



e



f Four identical little squares have been cut off from a big square as shown below. Determine the side length of the big square in terms of x, if the shape has an area of $A = (9x + 3)(x + 3)$.



g If the shape above has an area of $A = (mx + a)(nx + a)$. Determine the side length of the big square and the side length of the small cut out squares.

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Use the index laws to simplify algebraic expressions	☐ Go back to Topic 2A Indices
☐ Write algebraic terms with only positive indices ☐ Use the index laws to simplify expressions with negative indices	☐ Go back to Topic 2B Negative indices
☐ Evaluate algebraic expressions using substitution ☐ Identify like terms to simplify algebraic expressions ☐ Multiply and divide algebraic terms	☐ Go back to Topic 2C Simplifying
☐ Expand algebraic expressions with one pair of brackets ☐ Expand binomial products	☐ Go back to Topic 2D Expanding
☐ Simplify algebraic fractions by identifying common factors ☐ Multiply and divide algebraic fractions ☐ Add and subtract algebraic fractions	☐ Go back to Topic 2E Algebraic fractions
☐ Factorise algebraic expressions by identifying the highest common factor ☐ Factorise algebraic expressions by grouping terms ☐ Factorise algebraic expressions using the ‘difference of two squares’ and the ‘perfect square’ rules	☐ Go back to Topic 2F Factorising
☐ Factorise quadratic expressions by considering each term	☐ Go back to Topic 2G Factorising quadratic expressions
☐ Add a term to a quadratic binomial to make a perfect square ☐ Factorise quadratic expressions by completing the square	☐ Go back to Topic 2H Completing the square
☐ Factorise non-monic quadratic expressions by splitting the middle term ☐ Factorise non-monic quadratic expressions by completing the square	☐ Go back to Topic 2I Factorising non-monic quadratic expressions

3

Real numbers

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Indices
✔ Roots
✔ Multiplying terms
✔ Dividing terms
✔ Equivalent fractions

Curriculum links

- ADV Describes and performs operations with surds and fractional indices (MA5-IND-P-02)
  - Describe surds
  - Apply knowledge of surds to solve problems
  - Describe and use fractional indices
- ADV Establishes and applies the laws of logarithms to solve problems (MA5-LOG-P-01)
  - Examine logarithms both numerically and graphically
  - Establish and apply the laws of logarithms to solve problems

Materials

✔ Calculator

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3A Rational and irrational numbers

Learning intentions

By the end of this topic you will be able to …

- ✔ determine whether a number is rational or irrational
- ✔ determine whether a root is a surd or not.

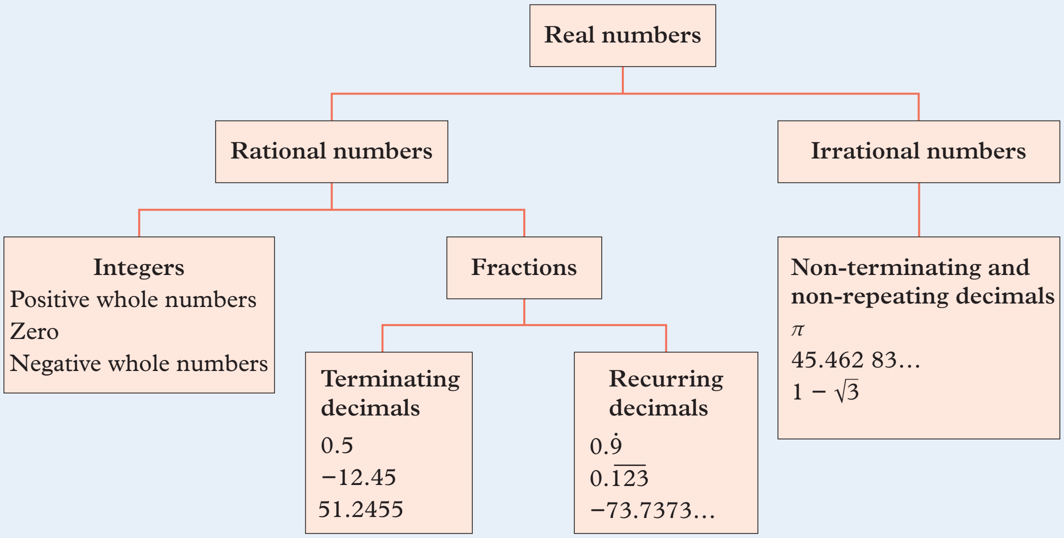
Inter-year links

Year 7	1G Indices and square roots
Year 8	4E Roots

Rational and irrational numbers

Key content video

- A **real number** is a number that can be represented by a point on a number line.
- The real number system is made up of **rational numbers** and **irrational numbers**.
- Rational numbers are numbers that can be written as a fraction where both the numerator and denominator are integers. The set of rational numbers includes integers, fractions, terminating decimals and recurring decimals.
- Irrational numbers are numbers that cannot be written as fractions with both the numerator and denominator being integers. The set of irrational numbers includes **surds** and other non-terminating, non-repeating decimals.



Surds

- A surd is an irrational expression of the form $\sqrt[n]{x}$, where x is a rational number and n is an integer greater than or equal to 2.
  - → Surds cannot be simplified to remove the root.
  - → Surds are exact values that can be rounded to a decimal approximation.
  - → Square roots, cube roots and fourth roots are all types of surds.
  - → When the value of n is not provided, it is assumed to be 2 (a square root).
- For a surd to be a real number, when the value of n is even the value of x must be non-zero ($x \geq 0$).
- When a number is written in front of a surd, the surd is being multiplied by that number. For example, $a\sqrt{b} = a \times \sqrt{b}$.
  - → A surd multiplied by a rational number is also considered to be a surd.

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Square roots

- $\sqrt{a}$ is the positive square root of a. For example, $\sqrt{9} = 3$.
  - → The negative square root is represented by putting a negative sign in front of the root symbol. For example, $-\sqrt{9} = -3$.
- The value of $\sqrt{0}$ is 0.
- Finding a square root and squaring a non-negative number are inverse operations of each other.

$$(\sqrt{a})^2 = a \qquad \sqrt{a^2} = a,$$

Example 3A.1 Identifying rational and irrational numbers

Determine whether each of the following numbers is rational or irrational.

a 0.26 **b** $-\frac{2}{3}$ **c** $-\sqrt{11}$ **d** $\sqrt{16}$

e $2.3\overline{54}$ **f** 6.584 381 257… **g** $\sqrt[3]{9}$

THINK	**WRITE**
a 0.26 is a terminating decimal. A terminating decimal can be written exactly as a fraction: $0.26 = \frac{26}{100}$ So 0.26 is rational.	**a** 0.26 is a rational number.
b Fractions are rational numbers, so $-\frac{2}{3}$ is rational.	**b** $-\frac{2}{3}$ is a rational number.
c $-\sqrt{11}$ is approximately equal to –3.316 624 79… This is a non-terminating and non-recurring decimal, which cannot be written as a fraction, so $-\sqrt{11}$ is irrational.	**c** $-\sqrt{11}$ is an irrational number.
d $\sqrt{16}$ simplifies to 4. Integers are rational numbers, so $\sqrt{16}$ is rational.	**d** $\sqrt{16}$ is a rational number.
e $2.3\overline{54}$ represents the recurring decimal 2.354 545 454… Recurring decimals are rational numbers, so $2.3\overline{54}$ is rational.	**e** $2.3\overline{54}$ is a rational number.
f 6.584 381 257… is a non-terminating decimal number. Its decimal place values do not have a pattern of repeating digits, so 6.584 381 257… is irrational.	**f** 6.584 381 257… is an irrational number.
g $\sqrt[3]{9}$ is approximately equal to 2.080 083 823… This is a non-terminating and non-recurring decimal, which cannot be written as a fraction, so $\sqrt[3]{9}$ is irrational.	**g** $\sqrt[3]{9}$ is irrational.

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Example 3A.2 Identifying surds

Which of the following numbers are surds?

a $\sqrt{9}$ **b** $2\sqrt{6}$ **c** $4\sqrt{25}$

THINK

a $\sqrt{9}$ can be simplified to 3, so $\sqrt{9}$ is not a surd.

b $2\sqrt{6} = 2 \times \sqrt{6}$

$\sqrt{6} = 2.449\,489\,743\ldots$, so $2\sqrt{6}$ is a surd.

c $4\sqrt{25} = 4 \times \sqrt{25}$

$\sqrt{25}$ can be simplified to 5.

$4\sqrt{25} = 4 \times 5 = 20$, which is rational.

So $4\sqrt{25}$ is not a surd.

WRITE

a $\sqrt{9}$ is not a surd.

b $2\sqrt{6}$ is a surd.

c $4\sqrt{25}$ is not a surd.

Helpful hints

✔ Some people think that decimals are more 'exact' than fractions. However, mathematicians prefer fractions because it is never possible to tell if a decimal has been rounded, truncated, or if it has only been measured to a certain number of decimal places. This uncertainty is not an issue with a fraction which has integers for both its numerator and denominator.

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Exercise 3A Rational and irrational numbers

△ 1–12, 16, 17, 18, 21 ■ 2, 3, 5–9, 11(a, c, f, i, k), 13, 14, 17–21, 23 ⬟ 2, 3, 6, 7, 9, 13–15, 22–27

UNDERSTANDING AND FLUENCY

1 Match each number below with the correct term from the list **I–VI**.

a $\frac{3}{4}$

b $45.710\,326\,58\ldots$

c -9

d $\sqrt{23}$

e $0.154\,715\,471\,547\ldots$

f $6.213\,764\,29$

I recurring decimal

II integer

III terminating decimal

IV fraction

V surd

VI non-terminating and non-recurring decimal

3A.1 **2** Determine whether each of the following numbers is rational or irrational.

a $\frac{4}{5}$ **b** -5 **c** $\sqrt{3}$ **d** $\sqrt{64}$ **e** $2.548\,247\,001\ldots$

f 5.684 **g** $-\frac{4}{9}$ **h** $-3\sqrt{5}$ **i** $5\frac{1}{6}$ **j** -1.5

k $\sqrt{9}$ **l** $\sqrt{8}$ **m** $\sqrt[3]{125}$ **n** $0.001\,002\,142\ldots$ **o** $\sqrt[4]{32}$

3A.2 **3** Which of the following numbers are surds?

a $\sqrt{21}$ **b** $-\sqrt{16}$ **c** $\sqrt[4]{64}$ **d** $-\sqrt{6}$

e $\frac{1}{2}\sqrt{9}$ **f** $5\sqrt{5}$ **g** $-3\sqrt{27}$ **h** $-\sqrt[3]{27}$

i $\sqrt[3]{3}$ **j** $3\sqrt{3}$ **k** $\frac{\sqrt{100}}{5}$ **l** $\frac{\sqrt{6}}{5}$

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UNDERSTANDING AND FLUENCY

4 Use your calculator to help you write a decimal approximation, rounded to four decimal places, for each of the surds in question **3**.

5 Which of the irrational numbers in question **2** are surds?

6 Calculate the value of each of these without using a calculator.

a $\sqrt{121}$ **b** $5\sqrt{36}$ **c** $-4\sqrt{64}$ **d** $-\frac{\sqrt{16}}{4}$

e $4\sqrt{16} + \sqrt{64}$ **f** $\sqrt{36} + \sqrt{64}$ **g** $\sqrt{36 + 64}$ **h** $\sqrt{9 + 16}$

i $\sqrt{25 + 144}$ **j** $-\sqrt{9} - \frac{\sqrt{36}}{2}$ **k** $2\sqrt{100} - \frac{3\sqrt{144}}{4}$ **l** $\sqrt{9^2}$

7 Which of the following are rational numbers and which are irrational numbers?

a $-\sqrt{81}$ **b** $\frac{2}{3}$ **c** $\sqrt{0.64}$ **d** $\sqrt{0.036}$

e 3.358 294 721… **f** 8.4 **g** $-\frac{1}{7}$ **h** $-3\sqrt{0.004}$

i $\sqrt{1.369}$ **j** 4.435 353 535… **k** $\sqrt{20.25}$ **l** $\frac{3}{4}\sqrt{100}$

8 Which of the irrational numbers in question **7** are surds?

9 State whether each of the following numbers is rational or irrational.

a $\sqrt{\frac{4}{16}}$ **b** $\sqrt{\frac{1}{12}}$ **c** $\frac{\sqrt{16}}{\sqrt{5}}$ **d** $\sqrt{\frac{1}{16}}$

e $\frac{2\sqrt{6}}{\sqrt{4}}$ **f** $\sqrt{\frac{1}{2}}$ **g** $\frac{\sqrt{6}}{\sqrt{25}}$ **h** $\sqrt{\frac{7}{49}}$

10 The Greek letter π (pi) is used as the symbol for a number commonly used in mathematics (equal to 3.141 592…). Is π a rational or irrational number? Provide a reason for your answer.

11 Use your calculator to help you find an approximation for the value of each of the following expressions, correct to four decimal places.

a $\sqrt{5} + \sqrt{6}$ **b** $2\sqrt{3} + \sqrt{11}$ **c** $\sqrt{30} - \sqrt{15}$ **d** $4\sqrt{8} - 2\sqrt{10}$

e $\sqrt{7} \times \sqrt{5}$ **f** $4\sqrt{3} \times 3\sqrt{7}$ **g** $2\sqrt{24} \div \sqrt{8}$ **h** $(2\sqrt{7} - 4\sqrt{3})^2$

i $\frac{5\sqrt{6}}{\sqrt{7}}$ **j** $\frac{5}{4\sqrt{11}}$ **k** $\frac{4\sqrt{2}}{2 + \sqrt{3}}$ **l** $\frac{3 - \sqrt{5}}{\sqrt{3} + 5}$

PROBLEM SOLVING AND REASONING

12 Consider the two fractions $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$.

a Use your calculator to help you determine the approximate value for each of $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$, correct to six decimal places.

b Are the two fractions equivalent? Explain.

c Use your results to comment on the relationship between fractions written in the forms $\sqrt{\frac{a}{b}}$ and $\frac{\sqrt{a}}{\sqrt{b}}$.

d Is $\sqrt{\frac{5}{8}}$ a rational number or an irrational number? Provide a reason for your answer.

e Write each of these fractions in the form $\sqrt{\frac{a}{b}}$ to help you decide if it is a rational or irrational number.

i $\frac{\sqrt{32}}{\sqrt{2}}$ **ii** $\frac{\sqrt{15}}{\sqrt{3}}$ **iii** $\frac{\sqrt{7}}{\sqrt{28}}$

13 a Calculate:

i $\sqrt{4}$ **ii** $\sqrt{9}$

b Between which two consecutive integers does $\sqrt{5}$ lie?

c How does knowledge of perfect squares help you decide which two consecutive integers a square root lies between?

d Between which two consecutive integer values do each of these square roots lie?

i $\sqrt{11}$ **ii** $-\sqrt{20}$ **iii** $\sqrt{131}$

iv $-\sqrt{48}$ **v** $\sqrt{180}$ **vi** $\sqrt{300}$

PROBLEM SOLVING AND REASONING

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14 Knowing which two integers a surd lies between can help us approximate values when more calculations are involved. We can apply the same operations to the two consecutive integers as are applied to the surd.

a Use the fact that $\sqrt{26}$ lies between 5 and 6 to determine which two integers the results of the following calculations lie between.

i $\sqrt{26} + 3$ **ii** $\sqrt{26} - 10$ **iii** $10 - \sqrt{26}$

iv $3\sqrt{26}$ **v** $3\sqrt{26} - 10$ **vi** $3(\sqrt{26} - 10)$

b Determine which two rational numbers each of the following lies between.

i $\frac{\sqrt{14}}{5}$ **ii** $\frac{5}{\sqrt{14}}$ **iii** $\frac{6 + \sqrt{51}}{2}$

iv $\frac{6 - \sqrt{51}}{2}$ **v** $-5 + \frac{\sqrt{91}}{6}$ **vi** $-5 + \frac{6}{\sqrt{91}}$

15 Determine which two rational numbers the value of each of the following expressions lies between.

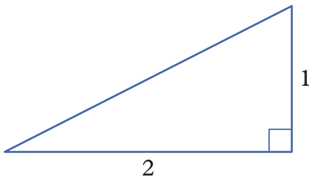
a $\sqrt[3]{11}$ **b** $\sqrt[4]{11}$ **c** $\sqrt[3]{36}$ **d** $\sqrt[5]{400\,000}$

e $7\sqrt[3]{25}$ **f** $7 - \sqrt[3]{70}$ **g** $3 - 5\sqrt[3]{144}$ **h** $\frac{6 + 2\sqrt[3]{300}}{3}$

16 Surds represent exact values, whereas the decimal value for a surd that is obtained by a calculation is only an approximation.

a Use Pythagoras' theorem to show that the exact length of the hypotenuse of the triangle in this diagram is $\sqrt{5}$.

b Write a decimal approximation for the length of the hypotenuse, correct to two decimal places.



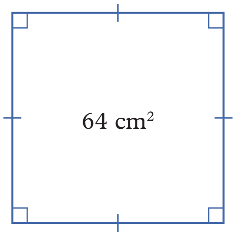
17 A square has an area of 64 cm².

a What is the length of each side of the square?

b Calculate the length of the diagonal of the square:

i as an exact value

ii rounded to two decimal places.



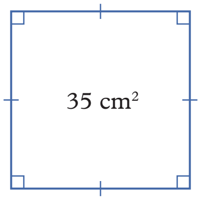
18 Find the side length of each of the following squares:

i as an exact value

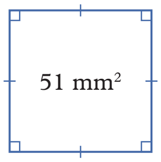
ii rounded to two decimal places

iii truncated to a number with two decimal places.

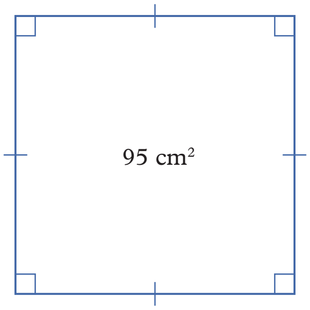
a



b



c



19 Find the perimeter of each of the squares in question **18** using the side length:

i expressed as an exact value, rounding your final answer to two decimal points

ii rounded to two decimal places

iii truncated to a number with two decimal places.

20 Explain why the perimeters you found for each square in question **19** varied.

21 Show that each of these decimals is a rational number by writing it as a fraction in simplest form.

a 0.4 **b** 0.65 **c** 0.49 **d** 1.6 **e** 2.57 **f** 0.851

22 Given a surd in the form $\sqrt[n]{x}$, explain why the value of x must be non-negative when the value of n is even, but the value of x can be negative when the value of n is odd.

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PROBLEM SOLVING AND REASONING

23 You will have seen by now that recurring decimals are rational numbers. For example: $0.\dot{6}$ is equivalent to $\frac{2}{3}$.

You can verify that $\frac{2}{3} = 0.666\,666\ldots$ by using a calculator to find $2 \div 3$.

To write a recurring decimal as a fraction (and to show it is rational), a series of steps can be followed.

a In the calculation below, match each step (**1–6**) with its corresponding line of working (**I–VI**). Then write the lines of working in the correct order.

Steps:

1 Let x represent the recurring decimal.

2 Multiply the recurring decimal by 10 to write an equation for $10x$.

3 Form a new subtraction equation from the two equations for x and $10x$. Then carry out the subtraction on each side of that equation.

4 Write the simplified equation that results from the subtraction.

5 Divide both sides of the equation by 9 to obtain x on the left-hand side of the equation and a fraction on the right-hand side.

6 Write the recurring decimal as a fraction, in simplest form.

Lines of working:

I $0.\dot{6} = \frac{2}{3}$

II $x = \frac{6}{9}$

III $10x = 6.666\,666\ldots$

IV $x = 0.666\,666\ldots$

V $10x - x = 6.666\,666\ldots - 0.666\,666\ldots$

VI $9x = 6$

b Why did you multiply x by 10 in the calculation above? How did this help you obtain a whole number on the right-hand side of the equation in step **4**?

24 Use the method shown in question **23a** to write each of these recurring decimals as a fraction.

a $0.\dot{2}$ **b** $0.\dot{7}$ **c** $2.\dot{3}$ **d** $5.\dot{6}$

CHALLENGE

25 a Complete the working below to write $0.\overline{36}$ as a fraction.

$$\begin{aligned} \text{Let: } x &= 0.363\,636\ldots \\ 100x &= \underline{\hspace{3cm}} \\ 100x - x &= \underline{\hspace{3cm}} - 0.363\,636\ldots \\ 99x &= \underline{\hspace{3cm}} \\ x &= \frac{\square}{99} \\ \text{So: } \quad 0.\overline{36} &= \frac{\square}{11} \end{aligned}$$

b Explain why multiplying x by 100 was helpful in this case.

26 Write each of these recurring decimals as a fraction.

a $0.\overline{14}$ **b** $0.\dot{2}\dot{1}$ **c** $0.\overline{48}$ **d** $1.\dot{7}\dot{3}$

27 Write each of these recurring decimals as a fraction. Show all your working.

a $0.333\,333\,333\ldots$ **b** $0.\overline{24}$ **c** $0.\overline{785}$ **d** $0.\dot{4}\dot{1}$

e $0.5\dot{6}$ **f** $3.\overline{18}$ **g** $4.2\dot{5}$ **h** $3.5\overline{126}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Terminating, non-terminating and recurring decimals

Topic quiz
3A

ADVANCED

3B Simplifying surds

Learning intentions

By the end of this topic you will be able to …

- ✔ simplify surds in $\sqrt{a}$ form
- ✔ simplify surds in $a\sqrt{b}$ form.

Inter-year links

Year 7	1G Indices and square roots
Year 8	4E Roots

Key content video

Square roots

- For all positive real numbers, finding the square root of a number and squaring a number are inverse operations.
  $$(\sqrt{a})^2 = a = \sqrt{a^2}$$
  → The square root of a negative real number is not a real number.
- When a real number is multiplied by a square root, it can be written in front of the root without using a multiplication sign.
  $$a \times \sqrt{b} = a\sqrt{b}$$
- The square root of a product can be written as the product of the square roots of its positive factors.
  $$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

Simplifying surds

- A square root can be simplified if the number under the square root sign has a perfect square factor.
- To simplify a square root (in this case, $\sqrt{12}$):
  1. Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.
  2. Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
  3. Simplify the square roots of any perfect squares.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= \sqrt{4} \times \sqrt{3}\\ &= 2 \times \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

Example 3B.1 Simplifying a surd in $\sqrt{a}$ form

Write $\sqrt{54}$ in its simplest form.

THINK	**WRITE**
1 Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.	$\sqrt{54} = \sqrt{9 \times 6}$
2 Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{9} \times \sqrt{6}$
3 Simplify the square roots of any perfect squares.	$= 3 \times \sqrt{6}$ $= 3\sqrt{6}$

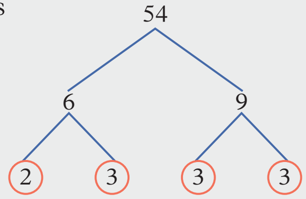
Example 3B.2 Simplifying a surd in $a\sqrt{b}$ form

Write $4\sqrt{72}$ in its simplest form.

THINK	**WRITE**
1 Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.	$4\sqrt{72} = 4\sqrt{36 \times 2}$ $= 4 \times \sqrt{36} \times \sqrt{2}$
2 Simplify the square roots of any perfect squares.	$= 4 \times 6 \times \sqrt{2}$
3 Multiply the coefficients and simplify the surd.	$= 24 \times \sqrt{2}$ $= 24\sqrt{2}$

Helpful hints

- ✔ Factor trees can be helpful in determining perfect square factors. Prime factors that appear more than once in a prime factorisation can be combined to give a perfect square factor. For example, in the factor tree shown, $3 \times 3 = 3^2 = 9$ is a perfect square factor of 54.
- ✔ If there are no repeated prime factors of a root, then the root cannot be simplified.



Exercise 3B Simplifying surds

ANS p657

△ 1, 2, 3–4(1st–3rd columns), 5, 6(1st–3rd columns), 8, 9, 13, 14

□ 3–4(3rd–5th columns), 6(3rd–5th columns), 8, 9(b, d, j, l), 10, 11, 13, 14, 15(b, d, h, j, m)

⬠ 3–4(4th, 5th columns), 6(4th, 5th columns), 8, 9(i–l), 10(a), 11, 12, 16, 17, 18(c, d)

UNDERSTANDING AND FLUENCY

1 Calculate the value of each of these.

a $\sqrt{16}$ **b** $\sqrt{81}$ **c** $\sqrt{49}$ **d** $\sqrt{4}$ **e** $\sqrt{100}$

f $\sqrt{25}$ **g** $3\sqrt{36}$ **h** $4\sqrt{121}$ **i** $\sqrt{1}$ **j** $\frac{1}{2}\sqrt{16}$

2 Simplify each of these expressions.

a $\sqrt{81} \times \sqrt{7}$ **b** $\sqrt{16} \times \sqrt{3}$ **c** $\sqrt{4} \times \sqrt{6}$

d $\sqrt{25} \times \sqrt{10}$ **e** $\sqrt{9} \times \sqrt{2}$ **f** $\sqrt{5} \times \sqrt{36}$

g $2 \times \sqrt{16} \times \sqrt{11}$ **h** $3 \times \sqrt{100} \times \sqrt{5}$ **i** $\frac{1}{2}\sqrt{144} \times \sqrt{10}$

3B.1 3 Simplify the following square roots.

a $\sqrt{28}$ **b** $\sqrt{50}$ **c** $\sqrt{24}$ **d** $\sqrt{27}$ **e** $\sqrt{48}$

f $\sqrt{75}$ **g** $\sqrt{63}$ **h** $\sqrt{60}$ **i** $\sqrt{56}$ **j** $\sqrt{45}$

k $\sqrt{128}$ **l** $\sqrt{108}$ **m** $\sqrt{180}$ **n** $\sqrt{147}$ **o** $\sqrt{162}$

p $\sqrt{125}$ **q** $\sqrt{450}$ **r** $\sqrt{288}$ **s** $\sqrt{270}$ **t** $\sqrt{567}$

3B.2 4 Write each of the following surds in simplest form.

a $5\sqrt{20}$ **b** $3\sqrt{8}$ **c** $6\sqrt{12}$ **d** $9\sqrt{18}$ **e** $4\sqrt{80}$

f $3\sqrt{90}$ **g** $12\sqrt{150}$ **h** $-3\sqrt{48}$ **i** $7\sqrt{96}$ **j** $-2\sqrt{245}$

k $8\sqrt{192}$ **l** $10\sqrt{252}$ **m** $-5\sqrt{112}$ **n** $7\sqrt{99}$ **o** $-12\sqrt{68}$

p $100\sqrt{300}$ **q** $\frac{1}{2}\sqrt{320}$ **r** $\frac{1}{2}\sqrt{216}$ **s** $8\sqrt{248}$ **t** $\frac{3}{4}\sqrt{176}$

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5 Show that the surd $5\sqrt{27}$ simplifies to $15\sqrt{3}$. Note: 'Show that' means to show all the necessary calculations.

6 Write each of the following surds in simplest form.

a $\sqrt{44}$ **b** $\sqrt{125}$ **c** $\sqrt{120}$ **d** $\sqrt{72}$ **e** $\sqrt{150}$

f $-\sqrt{216}$ **g** $\sqrt{126}$ **h** $\sqrt{240}$ **i** $-3\sqrt{18}$ **j** $\sqrt{245}$

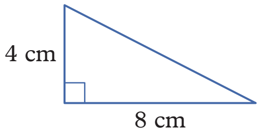
k $\frac{1}{2}\sqrt{192}$ **l** $\frac{\sqrt{108}}{3}$ **m** $-2\sqrt{48}$ **n** $15\sqrt{80}$ **o** $3\sqrt{84}$

p $\frac{3}{5}\sqrt{450}$ **q** $-8\sqrt{800}$ **r** $-\frac{2}{3}\sqrt{900}$ **s** $-\sqrt{1000}$ **t** $\frac{\sqrt{288}}{4}$

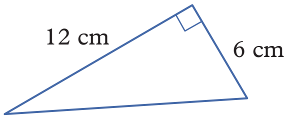
PROBLEM SOLVING AND REASONING

7 Use Pythagoras' theorem to calculate the exact length of the unknown side in each triangle below. Where appropriate, write the answer in simplest surd form.

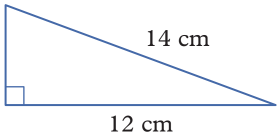
a



b



c



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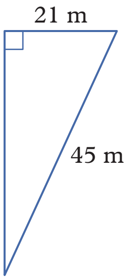


8 A surd in which the entire number is written beneath the root sign is called an entire surd. For example, $\sqrt{6}$ is an entire surd, whereas $3\sqrt{6}$ is not an entire surd.

a Copy and complete the following to write $3\sqrt{6}$ as an entire surd.

$$\begin{aligned} 3\sqrt{6} &= \sqrt{3^2} \times \sqrt{6} \\ &= \sqrt{\underline{}} \times \sqrt{6} \\ &= \sqrt{\underline{} \times 6} \\ &= \sqrt{\underline{}} \end{aligned}$$

b Consider the first line of working in part **a**. Why has the coefficient been written as a perfect square beneath the root sign?

c How are the surd we started with ($3\sqrt{6}$) and the answer (the entire surd) related to each other?

d Briefly explain how you could check that your answer is correct when you write a surd as an entire surd.

e Follow the working shown in part **a** to write $4\sqrt{3}$ as an entire surd.

9 Write each of the following as an entire surd.

a $3\sqrt{5}$ **b** $2\sqrt{3}$ **c** $4\sqrt{2}$ **d** $6\sqrt{7}$

e $4\sqrt{10}$ **f** $5\sqrt{11}$ **g** $8\sqrt{6}$ **h** $5\sqrt{5}$

i $4\sqrt{15}$ **j** $-8\sqrt{2}$ **k** $12\sqrt{7}$ **l** $-5\sqrt{3}$

10 **a** Write $a\sqrt{b}$ as an entire surd.

b Use the general form of the entire surd you wrote in part **a** to check your answers to question **9**.

c In your own words, explain how a simplified surd can be expressed as an entire surd.

11 Write each of the following as an entire surd.

a $m\sqrt{n}$ **b** $7\sqrt{a}$ **c** $x\sqrt{5}$ **d** $10\sqrt{xy}$

e $d\sqrt{2c}$ **f** $3\sqrt{4abc}$ **g** $a\sqrt{ab^2}$ **h** $xy\sqrt{x^2yz^2}$

12 Write each of the following in simplest form.

a $\sqrt{x^2y}$ **b** $\sqrt{x^2y^2}$ **c** $\sqrt{ab^2}$ **d** $\sqrt{x^2yz^2}$

e $\sqrt{x^4}$ **f** $\sqrt{x^4y}$ **g** $\sqrt{x^4y^2}$ **h** $\sqrt{a^4b^2c}$

PROBLEM SOLVING AND REASONING

13 An alternative method for simplifying a surd involves writing the number appearing under the root sign as a product of its prime factors. Consider the surd $\sqrt{12}$.

a Write 12 as a product of its prime factors in index form.

b Write the product of the prime factors of 12 under the root sign and show that it simplifies to $2\sqrt{3}$.

14 When using the method described in question **13**, you will notice that not all numbers under the root sign result in perfect squares. But, often, they can be written differently so that they do contain perfect squares. Consider $\sqrt{54}$.

a Write 54 as a product of its prime factors in index form.

b Use your answer to part **a** to show that $\sqrt{3^2 \times 3 \times 2}$ is an equivalent form of $\sqrt{54}$.

c Simplify the expression in part **b**.

15 Use the methods from questions **13** and **14** to fully simplify each of the following.

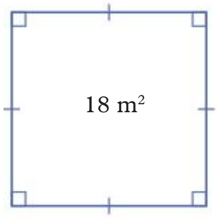
a $\sqrt{18}$ **b** $\sqrt{28}$ **c** $\sqrt{32}$ **d** $\sqrt{27}$ **e** $\sqrt{24}$ **f** $\sqrt{50}$

g $3\sqrt{20}$ **h** $2\sqrt{128}$ **i** $5\sqrt{63}$ **j** $8\sqrt{40}$ **k** $7\sqrt{48}$ **l** $6\sqrt{45}$

m $\sqrt{180}$ **n** $5\sqrt{147}$ **o** $6\sqrt{162}$ **p** $\sqrt{125}$ **q** $2\sqrt{450}$ **r** $\sqrt{288}$

CHALLENGE

16 Consider the square on the right with an area of 18 m².



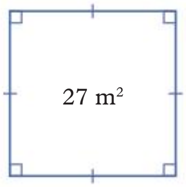
a Write the exact side length of the square as an entire surd and a simplified surd.

b The 18 m² square can be divided into nine 2 m² squares. Write the exact side length of each of the 2 m² squares.

c Describe the relationship between the side lengths of the two different sized squares in parts **a** and **b** and the number of squares the 18 m² square is divided into.

d Determine the side length of each of the following blue squares. Note: In part **iii**, the area of each small square is 2 m²; in part **iv**, the area of each small square is 3 m². Write your answer as an entire surd and a simplified surd each time.

i



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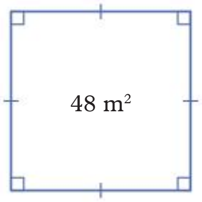

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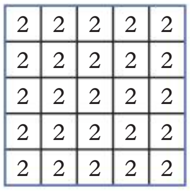
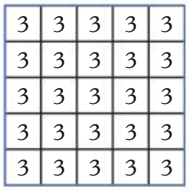


2	2	2	2	2
2	2	2	2	2
2	2	2	2	2
2	2	2	2	2
2	2	2	2	2

iv



3	3	3	3	3
3	3	3	3	3
3	3	3	3	3
3	3	3	3	3
3	3	3	3	3

17 a Simplify each of the following surds.

i $\sqrt{8}$ **ii** $\sqrt{27}$ **iii** $\sqrt{125}$ **iv** $\sqrt{a^3}$, where $a > 0$

b Simplify each of the following surds.

i $\sqrt{32}$ **ii** $\sqrt{243}$ **iii** $\sqrt{3125}$ **iv** $\sqrt{a^5}$, where $a > 0$

c Simplify each of the following surds. Hint: Look for perfect cube factors.

i $\sqrt[3]{16}$ **ii** $\sqrt[3]{81}$ **iii** $\sqrt[3]{625}$ **iv** $\sqrt[3]{a^4}$

d Simplify each of the following surds.

i $\sqrt[4]{32}$ **ii** $\sqrt[4]{243}$ **iii** $\sqrt[4]{3125}$ **iv** $\sqrt[4]{a^5}$, where $a > 0$

18 Arrange the surds in each of the following lists in ascending order.

a $\sqrt{46}, \sqrt{22}, \sqrt{42}, \sqrt{10}, \sqrt{27}, \sqrt{5}, \sqrt{21}, \sqrt{30}$

b $2\sqrt{3}, 3, 6, \sqrt{6}, \sqrt{39}, 2\sqrt{6}, \sqrt{11}, 4\sqrt{3}$

c $3\sqrt{3}, 2\sqrt{5}, 3\sqrt{6}, \sqrt{83}, \sqrt{22}, \sqrt{43}, 4, 2\sqrt{11}$

d $7\sqrt{10}, 12\sqrt{2}, 3\sqrt{43}, 2\sqrt{57}, 3\sqrt{14}, 20, 4\sqrt{5}, 2\sqrt{6}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Simplifying surds

Topic quiz
3B

3C Multiplying and dividing surds

Learning intentions

By the end of this topic you will be able to ...

✔ multiply two or more surds together

✔ divide a surd by a surd.

Inter-year links

Year 8	5D Multiplying algebraic terms
Year 9	3A Simplifying

Multiplying surds

Key content video

- The product of two surds written in the form $\sqrt{a}$ is equal to the square root of the product of the numbers under the root signs.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

- To multiply surds written in the form $a\sqrt{b}$, multiply the surd terms and multiply the coefficients.

$$a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$$

Dividing surds

- A square root divided by a square root can be written as the square root of the quotient.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- To divide surds written in the form $a\sqrt{b}$, divide the surd terms and divide the coefficients.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \times \sqrt{\frac{b}{d}}$$

Example 3C.1 Multiplying surds

Simplify the following expressions.

a $\sqrt{6} \times \sqrt{5}$ **b** $\sqrt{15} \times \sqrt{6}$ **c** $5\sqrt{3} \times 8\sqrt{11}$

THINK	WRITE
a Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.	**a** $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times 5}$ $= \sqrt{30}$
b **1** Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. **2** Simplify the square root.	**b** $\sqrt{15} \times \sqrt{6} = \sqrt{15 \times 6}$ $= \sqrt{90}$ $= \sqrt{9 \times 10}$ $= \sqrt{9} \times \sqrt{10}$ $= 3 \times \sqrt{10}$ $= 3\sqrt{10}$

c **1** Multiply the square roots together and multiply the coefficients together.	**c** $5\sqrt{3} \times 8\sqrt{11} = 5 \times 8 \times \sqrt{3} \times \sqrt{11}$
2 Write the product of the coefficient first. Then write the product of the surd terms under the root sign.	$= 40 \times \sqrt{3 \times 11}$ $= 40 \times \sqrt{33}$ $= 40\sqrt{33}$

Example 3C.2 Dividing surds

Simplify the following expressions.

a $\sqrt{27} \div \sqrt{3}$ **b** $\frac{\sqrt{35}}{\sqrt{15}}$ **c** $\frac{36\sqrt{60}}{9\sqrt{12}}$

THINK	WRITE
a **1** Divide one square root by the other. Remember: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**a** $\sqrt{27} \div \sqrt{3} = \frac{\sqrt{27}}{\sqrt{3}}$
2 Simplify the fraction.	$= \sqrt{\frac{27}{3}}$ $= \sqrt{9}$
3 Simplify the square root.	$= 3$
b **1** Divide one square root by the other. Remember: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**b** $\frac{\sqrt{35}}{\sqrt{15}} = \sqrt{\frac{35}{15}}$
2 Simplify the fraction.	$= \sqrt{\frac{7}{3}}$
3 Write in the form $\frac{\sqrt{a}}{\sqrt{b}}$. $\sqrt{7}$ and $\sqrt{3}$ cannot be further simplified	$= \frac{\sqrt{7}}{\sqrt{3}}$
c **1** Divide the surd terms and divide the coefficients.	**c** $\frac{36\sqrt{60}}{9\sqrt{12}} = \frac{36 \times \sqrt{60}}{9 \times \sqrt{12}}$ $= \frac{36}{9} \times \frac{\sqrt{60}}{\sqrt{12}}$
2 Divide one square root by the other.	$= \frac{36}{9} \times \sqrt{\frac{60}{12}}$
3 Simplify the fractions. $\sqrt{5}$ cannot be simplified further.	$= 4 \times \sqrt{5}$ $= 4\sqrt{5}$

Helpful hints

- ✔ Make sure that the horizontal line that is part of the root symbol covers all relevant numbers or pronumerals.
- ✔ Your skills with fractions will be very useful to you in this chapter. Simplifying when dividing roots is similar to multiplying and dividing fractions.

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Exercise 3C Multiplying and dividing surds

△ 1(e–l), 2, 3, 4(e–l), 5(e–l), 6–9, 10(a, c, e, g), 11, 12(a, d, e, f), 13, 14, 16, 21(a, b)

■ 1(e–l), 2(a, b, e, f), 3, 4(g–l), 5(i–l), 7(e–h), 8, 10(c, d, g, h), 11(f, h, i, k), 12(d, f, g, h), 17, 18(c–f), 21(a, d, e), 23

⬟ 4(k–p), 5(i–l), 10(f–i), 11(h–k), 12(g–i), 15, 19, 20, 21, 22, 24

UNDERSTANDING AND FLUENCY

Do not use a calculator for the questions in this exercise.

3C.1 **1** Simplify the following expressions.

a $\sqrt{7} \times \sqrt{5}$ **b** $\sqrt{3} \times \sqrt{11}$ **c** $\sqrt{2} \times (-\sqrt{7})$ **d** $-\sqrt{15} \times \sqrt{7}$

e $\sqrt{13} \times \sqrt{5}$ **f** $-\sqrt{2} \times \sqrt{11}$ **g** $\sqrt{7} \times \sqrt{3}$ **h** $\sqrt{19} \times \sqrt{5}$

i $\sqrt{7} \times 3\sqrt{11}$ **j** $5\sqrt{3} \times 4\sqrt{2}$ **k** $-4\sqrt{2} \times 7\sqrt{11}$ **l** $12\sqrt{6} \times (-3\sqrt{5})$

m $11\sqrt{2} \times 6\sqrt{17}$ **n** $8\sqrt{6} \times 8\sqrt{7}$ **o** $4\sqrt{3} \times (-6\sqrt{11})$ **p** $12\sqrt{3} \times 11\sqrt{5}$

2 Simplify the following expressions.

a $\sqrt{7} \times \sqrt{7}$ **b** $\sqrt{3} \times \sqrt{3}$ **c** $\sqrt{2} \times \sqrt{2}$ **d** $\sqrt{6} \times \sqrt{6}$

e $\sqrt{5} \times \sqrt{5}$ **f** $-\sqrt{11} \times (-\sqrt{11})$ **g** $\sqrt{10} \times \sqrt{10}$ **h** $\sqrt{9} \times \sqrt{9}$

3 **a** What do you notice about each result in question **2**?

b Simplify $\sqrt{a} \times \sqrt{a}$, where a is a positive real number.

4 Simplify each of these expressions, writing your answers in surd form where appropriate.

a $3\sqrt{3} \times \sqrt{3}$ **b** $-\sqrt{11} \times 7\sqrt{11}$ **c** $4\sqrt{2} \times 5\sqrt{2}$ **d** $6\sqrt{5} \times 2\sqrt{5}$

e $\sqrt{5} \times 3\sqrt{2}$ **f** $15\sqrt{6} \times 2\sqrt{5}$ **g** $-4\sqrt{2} \times 8\sqrt{13}$ **h** $12\sqrt{3} \times \frac{1}{4}\sqrt{11}$

i $2\sqrt{12} \times 7\sqrt{3}$ **j** $\frac{1}{2}\sqrt{5} \times 3\sqrt{20}$ **k** $\frac{3}{4}\sqrt{13} \times -8\sqrt{2}$ **l** $\frac{3}{5}\sqrt{7} \times \frac{5}{3}\sqrt{15}$

m $-12\sqrt{6} \times (-6\sqrt{7})$ **n** $\frac{1}{2}\sqrt{15} \times \frac{3}{4}\sqrt{13}$ **o** $15\sqrt{10} \times \left(-\frac{1}{3}\sqrt{7}\right)$ **p** $-10\sqrt{17} \times 1.1\sqrt{5}$

3C.2 **5** Simplify the following expressions.

a $\sqrt{34} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{8}$ **c** $\sqrt{15} \div \sqrt{15}$ **d** $\sqrt{15} \div (-\sqrt{3})$

e $\dfrac{\sqrt{42}}{\sqrt{7}}$ **f** $\dfrac{\sqrt{56}}{\sqrt{8}}$ **g** $4\sqrt{18} \div \sqrt{3}$ **h** $22\sqrt{40} \div 11\sqrt{8}$

i $\dfrac{-24\sqrt{35}}{8\sqrt{7}}$ **j** $\dfrac{16\sqrt{50}}{8\sqrt{10}}$ **k** $-12\sqrt{60} \div 3\sqrt{6}$ **l** $8\sqrt{55} \div 16\sqrt{11}$

6 Determine the value of each of these expressions.

a $\sqrt{32} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{6}$ **c** $\sqrt{54} \div \sqrt{6}$ **d** $\sqrt{40} \div \sqrt{10}$

7 Simplify each of these expressions, writing your answers in surd form.

a $12\sqrt{15} \div (-\sqrt{3})$ **b** $35\sqrt{24} \div 7\sqrt{8}$ **c** $-12\sqrt{48} \div 4\sqrt{24}$ **d** $\dfrac{18\sqrt{35}}{9\sqrt{7}}$

e $\dfrac{-32\sqrt{42}}{-8\sqrt{6}}$ **f** $\dfrac{35\sqrt{5}}{-15}$ **g** $\dfrac{-24}{-8\sqrt{3}}$ **h** $\sqrt{240} \div 5\sqrt{48}$

8 Show that the result of the calculation $6\sqrt{3} \times 3\sqrt{2}$ is $18\sqrt{6}$. Note: 'Show that' means to show all the necessary calculations.

9 Consider $\dfrac{24\sqrt{40}}{8\sqrt{20}}$. By dividing the coefficients and then dividing the surd terms, show that the result is $3\sqrt{2}$.

10 Simplify each of these expressions. Write your answers in fractional surd form where appropriate.

a $6\sqrt{15} \div \sqrt{15}$ (Remember: $6\sqrt{15} \div \sqrt{15}$ is equivalent to $\dfrac{6\sqrt{15}}{\sqrt{15}}$.)

b $\dfrac{12\sqrt{30}}{16\sqrt{6}}$ **c** $44\sqrt{34} \div 28\sqrt{42}$ **d** $\dfrac{-20\sqrt{75}}{25\sqrt{45}}$ **e** $\dfrac{36\sqrt{35}}{16\sqrt{7}}$

f $\dfrac{22\sqrt{5}}{-33\sqrt{5}}$ **g** $\dfrac{8\sqrt{3} \times 6}{-\sqrt{3}}$ **h** $\frac{1}{2}\sqrt{15} \div 4\sqrt{8}$ **i** $9\sqrt{32} \div 18\sqrt{8}$

ADVANCED

UNDERSTANDING AND FLUENCY

11 Simplify each of these expressions.

a $\dfrac{\sqrt{54}}{\sqrt{6}}$ **b** $\sqrt{\dfrac{81}{100}}$ **c** $\sqrt{32}\times3\sqrt{2}$

d $\dfrac{\sqrt{3}}{\sqrt{27}}$ **e** $5\sqrt{12}\times-2\sqrt{3}$ **f** $\sqrt{\dfrac{36}{9}}$

g $7\sqrt{8}\times\sqrt{18}$ **h** $4\sqrt{6}\times\sqrt{8}\times6\sqrt{12}$ **i** $8\sqrt{45}\div5\sqrt{5}$

j $\dfrac{\sqrt{32}}{16\sqrt{8}}$ **k** $\dfrac{-4\sqrt{75}}{15\sqrt{3}}$ **l** $\dfrac{-5\sqrt{343}}{15\sqrt{7}}$

12 Simplify each of these fractions, writing your answer in fraction form where appropriate.

a $\dfrac{3\sqrt{5}\times4\sqrt{6}}{6\sqrt{3}}$ **b** $\dfrac{5\sqrt{7}\times\sqrt{6}}{25\sqrt{14}}$ **c** $\dfrac{12\sqrt{48}}{3\sqrt{2}\times5\sqrt{6}}$

d $\dfrac{11\sqrt{132}}{4\sqrt{11}\times(-2\sqrt{6})}$ **e** $\dfrac{8\sqrt{3}\times2\sqrt{8}}{6\sqrt{6}\times\sqrt{2}}$ **f** $\dfrac{4\sqrt{5}\times-3\sqrt{24}}{5\sqrt{10}\times6\sqrt{2}}$

g $\dfrac{\sqrt{3}\times\sqrt{15}\times\sqrt{8}}{\sqrt{6}\times9\sqrt{12}}$ **h** $\dfrac{4\sqrt{12}\times3\sqrt{10}\times(-6\sqrt{7})}{5\sqrt{8}\times8\sqrt{15}}$ **i** $\dfrac{-5\sqrt{6}\times2\sqrt{28}\times(-\sqrt{44})}{3\sqrt{21}\times(-4\sqrt{18})}$

PROBLEM SOLVING AND REASONING

13 To square a number, we multiply the number by itself. Find the value of each of these expressions.

a $(\sqrt{5})^2$ **b** $(\sqrt{8})^2$ **c** $(\sqrt{2})^2$

d $(\sqrt{7})^2$ **e** $(\sqrt{9})^2$ **f** $(4\sqrt{5})^2$

g $(3\sqrt{6})^2$ **h** $(8\sqrt{3})^2$ **i** $(9\sqrt{11})^2$

14 Study your answer for each part of question **13**.

a Does the square of a surd expressed in the form $\sqrt{a}$ or $a\sqrt{b}$ result in a rational answer or an irrational answer?

b Show that your answer to part **a** will always be true.

c Write the surds in parts **f–i** as entire surds, with the entire number beneath a single root sign.

d Describe the connection between the number beneath the square root of an entire surd and the square of that surd.

15 Simplify each of these expressions.

a $(\sqrt{x})^2$ **b** $(\sqrt{ab})^2$

c $\sqrt{6y}\times\sqrt{6y}$ **d** $\sqrt{x+y}\times\sqrt{x+y}$

16 Find the value of each of these expressions.

a $\left(\dfrac{5}{\sqrt{7}}\right)^2$ **b** $\left(\dfrac{1}{\sqrt{5}}\right)^2$ **c** $\left(-\dfrac{3}{\sqrt{6}}\right)^2$ **d** $\left(\dfrac{\sqrt{10}}{\sqrt{13}}\right)^2$

17 Consider $3\sqrt{392}\times5\sqrt{192}$.

a Calculate the value of the expression using each of these methods:

Method 1: Multiply the coefficients, then multiply the surd terms and write the surd in simplest form.

Method 2: Simplify each surd first and then perform the multiplication.

b Do both methods from part **a** produce the same answer?

c When do you think it is best to use:

i method 1 **ii** method 2?

18 Simplify each of the following, writing each answer as a fraction in simplest surd form where appropriate.

a $3\sqrt{48}\div\sqrt{6}$ **b** $\dfrac{2\sqrt{54}}{6\sqrt{3}}$ **c** $15\sqrt{104}\div25\sqrt{8}$

d $\dfrac{54\sqrt{75}}{63\sqrt{15}}$ **e** $\dfrac{-21\sqrt{84}}{6\sqrt{7}}$ **f** $\dfrac{4\sqrt{300}}{3\sqrt{25}}$

g $\dfrac{8\sqrt{960}}{-\sqrt{48}}$ **h** $\sqrt{384}\div5\sqrt{64}$ **i** $\dfrac{-\sqrt{504}}{\sqrt{54}}$

ADVANCED

PROBLEM SOLVING AND REASONING

19 Simplify each of these expressions, writing your answer in simplest surd form.

a $\dfrac{22\sqrt{48}}{8\sqrt{2}\times 4\sqrt{3}}$ **b** $\dfrac{4\sqrt{10}\times 6\sqrt{24}}{2\sqrt{5}\times 3\sqrt{2}}$ **c** $\dfrac{3\sqrt{8}\times 7\sqrt{12}}{12\sqrt{6}\times(-8\sqrt{2})}$

d $\dfrac{2\sqrt{15}\times 4\sqrt{12}\times\sqrt{10}}{\sqrt{2}\times 5\sqrt{12}}$ **e** $\dfrac{3\sqrt{14}\times(-5\sqrt{12})}{4\sqrt{2}\times\sqrt{7}}$ **f** $\dfrac{2\sqrt{35}\times 3\sqrt{2}\times 6\sqrt{8}}{5\sqrt{5}\times 12\sqrt{14}}$

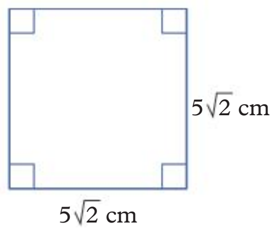
g $\dfrac{-4\sqrt{8}\times 3\sqrt{7}\times 12\sqrt{45}}{2\sqrt{14}\times 6\sqrt{15}\times 3\sqrt{3}}$ **h** $\dfrac{-6\sqrt{26}\times 12\sqrt{2}\times 8\sqrt{12}\times 16\sqrt{33}}{2\sqrt{11}\times 4\sqrt{13}\times 18\sqrt{3}\times 22\sqrt{8}}$

20 Simplify $\dfrac{a\sqrt{bc^2}\times b^2\sqrt{ac}}{c\sqrt{ab^2}}$, where a, b and c are positive real numbers.

21 For each of these shapes, calculate:

i the area, correct to two decimal places **ii** the exact area.

a



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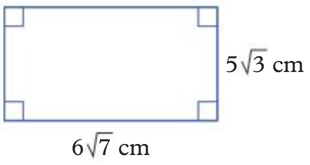

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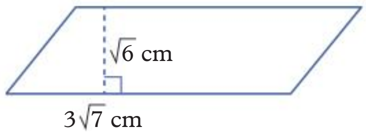

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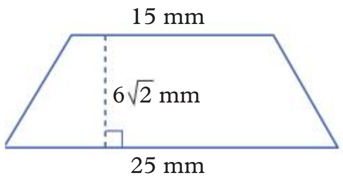

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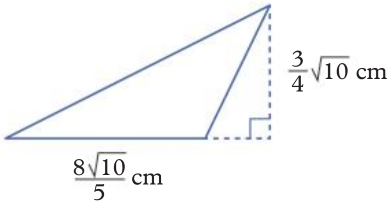

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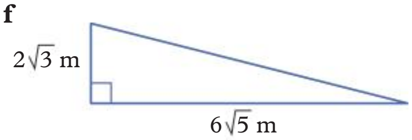

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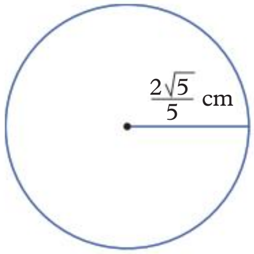

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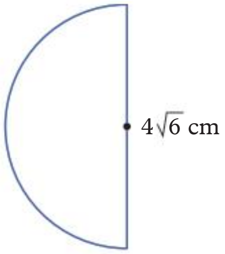
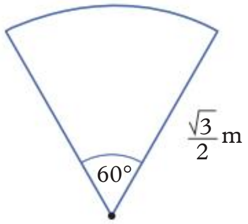


i



CHALLENGE

22 Consider a square with side length $\sqrt{a}$, where $\sqrt{a}$ is a surd in its simplest form. Is it possible for the value representing the perimeter to be the same as the value representing the area of this square? Explain your answer.

23 Determine the exact (simplified) perimeter of a square with an area of:

a 5 m^2 **b** 17 m^2 **c** 18 m^2

d 20 m^2 **e** $\frac{3}{4}$ m^2 **f** $\frac{175}{36}$ m^2

24 For each of the following, determine the smallest surd, x, that will make the product an integer.

a $x\times\sqrt{180}$ **b** $x\times\sqrt{5400}$ **c** $x\times\sqrt{2835}$

d $x\times\sqrt{8064}$ **e** $x\times\sqrt{1911}$ **f** $x\times\sqrt{11\,154}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Multiplying and dividing surds

Worksheet
Multiplying and dividing surds

Topic quiz
3C

3D Adding and subtracting surds

Learning intentions

By the end of this topic you will be able to ...

- ✔ identify like surds
- ✔ add and subtract like surds
- ✔ add and subtract surds after first simplifying.

Inter-year links

Year 8	5C Adding and subtracting algebraic terms
Year 9	3A Simplifying

Adding and subtracting surds

Key content video

- **Like surds** are surds that in simplest form contain exactly the same surd term.
  For example, $2\sqrt{2}$ and $3\sqrt{2}$ are like surds, whereas $2\sqrt{5}$and $2\sqrt{3}$ are not like surds.
- To add or subtract like surds, add or subtract the coefficients.

$$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$

- Sometimes like surds need to be simplified to have the same surd term. Simplify each surd in an expression by rewriting the number under each root sign as a product of its factors, with the first factor being the greatest possible perfect square. Then, if the simplified surds are like surds, they can be added or subtracted.
  For example,

$$\begin{aligned} 2\sqrt{27} - \sqrt{12} &= 2\sqrt{9 \times 3} - \sqrt{4 \times 3} \\ &= 2 \times \sqrt{9} \times \sqrt{3} - \sqrt{4} \times \sqrt{3} \\ &= 2 \times 3\sqrt{3} - 2\sqrt{3} \\ &= 6\sqrt{3} - 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

Example 3D.1 Identifying like surds

State whether the surds in each of these pairs are like surds or not.

a $2\sqrt{6}$ and $5\sqrt{6}$ **b** $7\sqrt{6}$ and $7\sqrt{3}$ **c** $5\sqrt{2}$ and $\sqrt{8}$

THINK	WRITE
a The surds are like surds because they have exactly the same surd term.	**a** Yes, they are like surds.
b The surds are not like surds because they do not have exactly the same surd term.	**b** No, they are not like surds.
c $\sqrt{8}$ can be simplified first. $\sqrt{8} = \sqrt{4} \times \sqrt{2}$ $= 2\sqrt{2}$ $\sqrt{8}$ and $2\sqrt{2}$ are like surds because $2\sqrt{2}$ and $5\sqrt{2}$ have exactly the same surd term.	**c** Yes, they are like surds.

Example 3D.2 Adding and subtracting surds

Simplify each of these expressions.

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3}$ **b** $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

THINK	WRITE
a 1 Group the like surds together. All the surds in this expression are like surds.	**a** $7\sqrt{3} + 4\sqrt{3} - \sqrt{3} = 7\sqrt{3} + 4\sqrt{3} - 1\sqrt{3}$
2 Simplify the expression by adding and subtracting the coefficients.	$= 10\sqrt{3}$
b 1 Group the like surds together. There are two groups of like surds in this expression: $\sqrt{5}$ and $\sqrt{2}$	**b** $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5} = 6\sqrt{5} + 5\sqrt{5} - 3\sqrt{2} - 8\sqrt{2}$
2 Simplify the like terms in the expression by adding and subtracting the coefficients.	$= 11\sqrt{5} - 11\sqrt{2}$

Example 3D.3 Adding and subtracting surds after first simplifying

Simplify the following expressions.

a $3\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{48} - 2\sqrt{75}$

THINK	WRITE
a 1 Simplify each surd in the expression. The first term, $3\sqrt{2}$, is already in its simplest form. Rewrite the number under the other root sign as a product of its factors, with the first factor being the greatest possible perfect square.	**a** $3\sqrt{2} + 4\sqrt{8} = 3\sqrt{2} + 4\sqrt{4 \times 2}$ $= 3\sqrt{2} + 4 \times \sqrt{4} \times \sqrt{2}$ $= 3\sqrt{2} + 4 \times 2 \times \sqrt{2}$
2 Simplify the like terms in the expression by adding the coefficients.	$= 3\sqrt{2} + 8\sqrt{2}$ $= 11\sqrt{2}$
b 1 Simplify each surd in the expression, rewriting the number under each root sign as a product of its factors, with the first factor being the greatest possible perfect square.	**b** $4\sqrt{48} - 2\sqrt{75} = 4\sqrt{16 \times 3} - 2\sqrt{25 \times 3}$ $= 4 \times \sqrt{16} \times \sqrt{3} - 2 \times \sqrt{25} \times \sqrt{3}$ $= 4 \times 4 \times \sqrt{3} - 2 \times 5 \times \sqrt{3}$
2 Simplify the like terms in the expression by subtracting the coefficients.	$= 16\sqrt{3} - 10\sqrt{3}$ $= 6\sqrt{3}$

Helpful hints

- ✔ Pay attention to your surds. Do not attempt to simplify the sum or difference of unlike surds!
- ✔ Starting every question by asking yourself ‘Can I simplify these surds?’ is a great way to avoid common mistakes.

ANS p659

Exercise 3D Adding and subtracting surds

△ 1, 2(1st column), 3–5, 6, 7(1st column), 8(a, c, e, g), 9(a–f), 10(a–c), 11, 12,15 □ 2(2nd column), 4, 5, 6–10(2nd column), 13, 14, 16(a, b), 18 ⬠ 6–10(2nd column), 16, 18–22

UNDERSTANDING AND FLUENCY

3D.1 **1** State whether the terms in each of these pairs are like surds or not.

a $3\sqrt{5}$ and $\sqrt{5}$ **b** $4\sqrt{2}$ and $2\sqrt{4}$ **c** 3 and $\sqrt{3}$ **d** $-3\sqrt{6}$ and $6\sqrt{6}$

e $5\sqrt{5}$ and $2\sqrt{2}$ **f** $3\sqrt{7}$ and $3\sqrt{10}$ **g** $2\sqrt{2}$ and $-\sqrt{2}$ **h** $2\sqrt{2}$ and $\sqrt{8}$

3D.2 **2** Simplify each of these expressions by adding (or subtracting) the like terms, if possible. Hint: Look for like surds.

a $2\sqrt{3} + \sqrt{2} + 5\sqrt{3}$ **b** $4\sqrt{3} - 2\sqrt{2} + \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $6\sqrt{13} + 4\sqrt{39} - \sqrt{3} + 2\sqrt{13}$

e $9\sqrt{2} - \sqrt{2} - 6\sqrt{5} + 5\sqrt{2}$ **f** $3\sqrt{6} + 4\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6} - 4$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $7\sqrt{21} - 6\sqrt{3} + 5\sqrt{3} - 6\sqrt{21}$

i $6\sqrt{6} + 3\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6}$ **j** $-3\sqrt{15} - 2\sqrt{7} - \sqrt{7} - 4\sqrt{15}$

3 Write each surd in simplest form and then identify any like surds in each list.

a $2\sqrt{3}$ $\sqrt{2}$ $4\sqrt{2}$ $2\sqrt{5}$ $3\sqrt{2}$ **b** $3\sqrt{7}$ $14\sqrt{2}$ $7\sqrt{3}$ $2\sqrt{14}$

c $-3\sqrt{6}$ $3\sqrt{12}$ $\sqrt{3}$ $\sqrt{18}$ $2\sqrt{27}$ **d** $9\sqrt{32}$ $4\sqrt{8}$ $\sqrt{128}$ $5\sqrt{200}$

3D.3 **4** Simplify each of these expressions.

a $5\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{36} - 2\sqrt{12}$ **c** $3\sqrt{24} + \sqrt{45}$

d $2\sqrt{28} + 8\sqrt{63}$ **e** $4\sqrt{72} + 5\sqrt{98}$ **f** $-2\sqrt{8} + 7\sqrt{16}$

5 Simplify each of these expressions.

a $5\sqrt{27} + \sqrt{12} - 8\sqrt{48}$ **b** $2\sqrt{20} - 4\sqrt{80} + 2\sqrt{45}$ **c** $\sqrt{28} - 3\sqrt{112} + 2\sqrt{63}$

d $5\sqrt{12} + \sqrt{27} - 8\sqrt{3}$ **e** $2\sqrt{8} + 4\sqrt{32} + 3\sqrt{2}$ **f** $\sqrt{7} - 3\sqrt{28} + 5\sqrt{7}$

6 Simplify each of these expressions.

a $3\sqrt{45} - \sqrt{5} - 2\sqrt{80} + 2\sqrt{20}$ **b** $\sqrt{108} - 4\sqrt{27} - 2\sqrt{12} + 2\sqrt{75}$

c $3\sqrt{24} - 4\sqrt{54} + 5\sqrt{24}$ **d** $3\sqrt{18} - 4\sqrt{50} - \sqrt{8} + 2\sqrt{98}$

e $3\sqrt{24} - 7\sqrt{54} - \sqrt{96} + 4\sqrt{20}$ **f** $4\sqrt{108} - 5\sqrt{75} - 6\sqrt{48} + \sqrt{8}$

g $\sqrt{128} - 5\sqrt{27} + 2\sqrt{32} - 3\sqrt{75}$ **h** $2\sqrt{75} - 6\sqrt{3} + 2\sqrt{48} - 6\sqrt{12}$

i $3\sqrt{12} - \sqrt{28} + 5\sqrt{8} - 2\sqrt{20}$ **j** $6\sqrt{54} + \sqrt{45} - 5\sqrt{96} - 2\sqrt{27}$

7 Simplify each of these expressions.

a $\sqrt{180} + 4\sqrt{125} - 3\sqrt{245}$ **b** $4\sqrt{243} + 3\sqrt{300} - 2\sqrt{432}$

c $3\sqrt{338} - 12\sqrt{147} - 2\sqrt{50} + \sqrt{48}$ **d** $3\sqrt{216} - 4\sqrt{125} - 2\sqrt{147} + 5\sqrt{45}$

e $\sqrt{63} + 7\sqrt{567} - 2\sqrt{96} + 8\sqrt{588}$ **f** $\sqrt{200} - 4\sqrt{180} + 5\sqrt{864} - 3\sqrt{160}$

g $-4\sqrt{320} - 25 + \sqrt{245} - 36 + \sqrt{27}$ **h** $12\sqrt{72} - 6\sqrt{192} + 2\sqrt{128} - 5\sqrt{507}$

8 Simplify each of these surd expressions containing pronumerals.

For example:
$$\begin{aligned}\sqrt{27x} + \sqrt{12x} &= \sqrt{9 \times 3 \times x} + \sqrt{4 \times 3 \times x}\\ &= 3\sqrt{3x} + 2\sqrt{3x}\\ &= 5\sqrt{3x}\end{aligned}$$

a $\sqrt{x} + \sqrt{16x} - \sqrt{25x}$ **b** $3\sqrt{4x} + 2\sqrt{x} - 5\sqrt{36x}$

c $\sqrt{9x} + \sqrt{12x} + \sqrt{16x} - 2\sqrt{48x}$ **d** $\sqrt{8xy} - 2\sqrt{32xy} + \frac{1}{3}\sqrt{18xy}$

e $\sqrt{150xy} + 2\sqrt{384xy} - \sqrt{6xy}$ **f** $\sqrt{24x^2y} + 3x\sqrt{96y} - 5\sqrt{54x^2y}$

g $\sqrt{3xy^2} - 5y\sqrt{12x} + 7\sqrt{27xy^2}$ **h** $3\sqrt{100x^2y} - \sqrt{147x^2y} + \frac{1}{2}\sqrt{48x^2y} + \sqrt{16x^2y}$

i $2x\sqrt{80xy} + 5\sqrt{12xy} - \sqrt{125x^2y} - 2\sqrt{432x^2y}$

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PROBLEM SOLVING AND REASONING

9 The distributive law is used to expand expressions, as shown below.

$$a(b + c) = ab + ac$$

The distributive law can be applied to expressions involving surds.

Expand each of the expressions below. Where necessary, write all surds in simplest form.

a $6(2 + \sqrt{3})$

b $8(\sqrt{5} - 4)$

c $5(4 - 6\sqrt{2})$

d $4(3\sqrt{2} + 2\sqrt{3})$

e $2(5\sqrt{6} - 3\sqrt{5})$

f $-5\sqrt{7}(2\sqrt{6} - 3\sqrt{10} + 2)$

g $3\sqrt{6}(5\sqrt{3} + 2\sqrt{10})$

h $5\sqrt{7}(2\sqrt{7} - 12)$

i $6\sqrt{11}(2\sqrt{3} - 5\sqrt{7} + \sqrt{11})$

j $(3 - 2\sqrt{5} + 5\sqrt{15})\sqrt{5}$

k $-3\sqrt{3}(4\sqrt{6} - 2\sqrt{10} + 9\sqrt{15})$

10 Binomial products can be expanded as shown below.

$$(a + b)(c + d) = ac + ad + bc + bd$$

Binomial products can also contain surds. Expand and simplify each of these expressions.

a $(5 + 3\sqrt{3})(2 + 4\sqrt{6})$

b $(3\sqrt{2} + 6)(5\sqrt{5} - 8)$

c $(7\sqrt{5} - 3\sqrt{2})(4\sqrt{2} + 8\sqrt{6})$

d $(\sqrt{6} + 2\sqrt{3})(6\sqrt{2} + 5\sqrt{3})$

e $(4\sqrt{8} - 5\sqrt{5})(3\sqrt{3} - 9\sqrt{8})$

f $(4\sqrt{10} + 3\sqrt{5})(7\sqrt{2} - 6\sqrt{12})$

11 Expand and simplify each of these perfect squares.

a $(5 + \sqrt{6})^2$

b $(\sqrt{7} - 3)^2$

c $(3\sqrt{5} + 2\sqrt{3})^2$

d $(4\sqrt{6} - 3\sqrt{7})^2$

12 The results of the expansions in question **11** followed a pattern that you've seen before; the 'perfect square' rule.

a Show that the expansion of $(a + b)^2$ will always produce the expression $a^2 + 2ab + b^2$.

b Show that the expansion of $(a - b)^2$ will always produce the expression $a^2 - 2ab + b^2$.

c Use these rules to help you check your answers to question **11**.

13 Expand and simplify each of these expressions.

a $(4 + \sqrt{6})(4 - \sqrt{6})$

b $(5\sqrt{5} - 3)(5\sqrt{5} + 3)$

c $(2\sqrt{3} + \sqrt{2})(2\sqrt{3} - \sqrt{2})$

d $(4\sqrt{5} + 7\sqrt{3})(4\sqrt{5} - 7\sqrt{3})$

14 Consider each expression in question **13**.

a What do you notice about the terms inside each pair of brackets?

b Are your answers to question **13** rational numbers or irrational numbers? Why do you think this is the case?

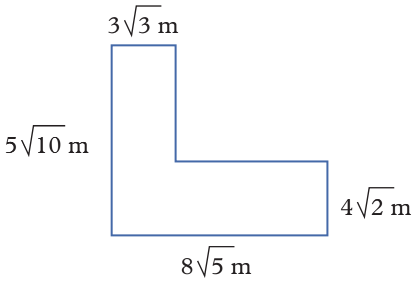
c The results of the expansions in question **13** follow a pattern that you've seen before; the 'difference of two squares' rule. Show that the expansion of binomial products in the form $(a + b)(a - b)$ or $(a - b)(a + b)$ will always be $a^2 - b^2$.

d Use this rule to check your answers for question **13**.

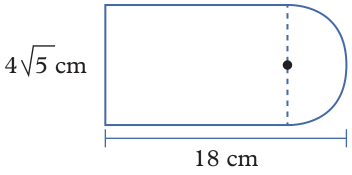
15 Use two non-zero numbers to demonstrate $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$.

16 Calculate the exact area of each of the following composite shapes.

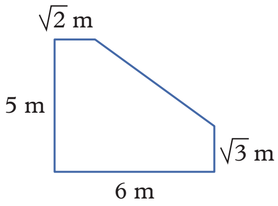
a



b

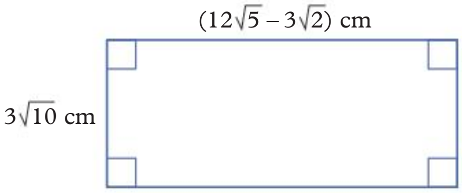


c

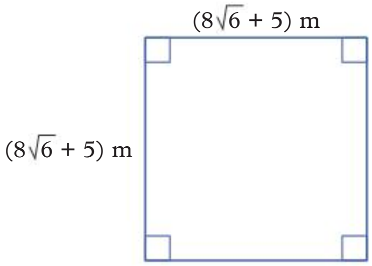


17 Calculate the exact area of each of these shapes. Write your answers in simplest surd form.

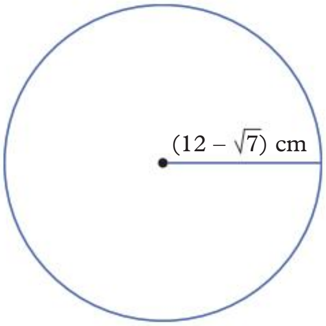
a



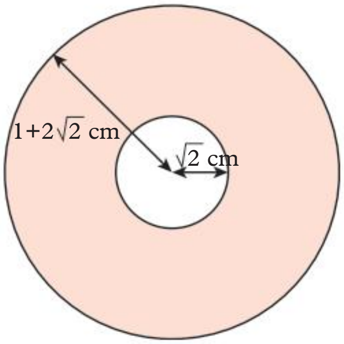
b



c



d



18 Calculate the exact perimeter of each shape in parts **a** and **b** of question **16**.

19 Evaluate each of the following.

a $\sqrt{3}\left(\sqrt{8} + \sqrt{72}\right)$

b $-7\sqrt{15} + 5\sqrt{3} \times 2\sqrt{5}$

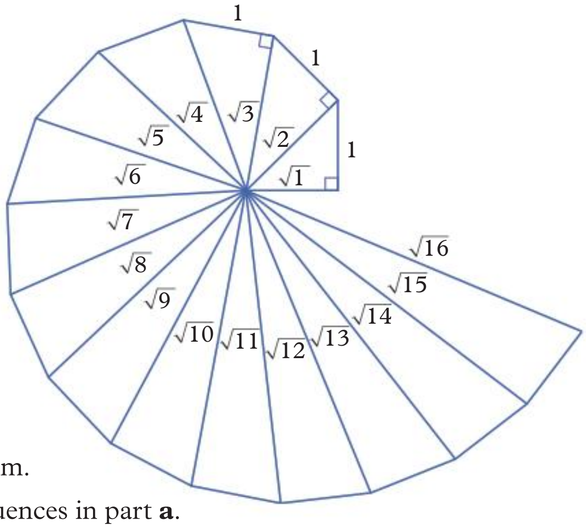
c $\frac{8\sqrt{30}}{3\sqrt{10}} + \frac{5\sqrt{42}}{6\sqrt{14}} + \frac{5\sqrt{105}}{9\sqrt{35}}$

d $\sqrt{50}\left(\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{5}}\right) + \frac{\sqrt{10}}{2}(\sqrt{8} + \sqrt{18} + \sqrt{4})$

e $\frac{3\sqrt{55} - 5\sqrt{35}}{\sqrt{5}} + \frac{2\sqrt{77} - 2\sqrt{121}}{\sqrt{11}}$

f $\frac{6}{\sqrt{2}} \times \frac{6}{\sqrt{3}} + \left(\frac{4}{\sqrt{2}} + 2\sqrt{3}\right)\left(3\sqrt{2} - \frac{9}{\sqrt{3}}\right)$

20 The spiral of Theodorus is created by consecutively forming a right-angled triangle on the hypotenuse of the previous right-angled triangle, so that the length of one of the shorter sides is always 1. Determine the total length of line segments drawn for this section of the spiral of Theodorus. Write your answer as a simplified expression.



21 **a** Describe the rule for generating the next term in each of the following sequences.

- **i** $3\sqrt{5},\ 8\sqrt{5},\ 13\sqrt{5},\ 18\sqrt{5}, \ldots$
- **ii** $2\sqrt{3},\ 3\sqrt{3} + 7,\ 4\sqrt{3} + 14,\ 5\sqrt{3} + 21, \ldots$
- **iii** $3\sqrt{2},\ 6,\ 6\sqrt{2},\ 12, \ldots$
- **iv** $\sqrt{5},\ 10,\ 20\sqrt{5},\ 200, \ldots$

b Continue each of the sequences in part **a** to the tenth term.

c Evaluate the sum of the first 10 terms in each of the sequences in part **a**.

22 For each of the following lists, calculate:

i the mean

ii the median.

a $\sqrt{2},\ \sqrt{4},\ \sqrt{8},\ \sqrt{16},\ \sqrt{32},\ \sqrt{64},\ \sqrt{128},\ \sqrt{256}$

b $\sqrt{5},\ \sqrt{3},\ 5\sqrt{2},\ 2\sqrt{5},\ 3\sqrt{3},\ \sqrt{2}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Adding and subtracting surds

Worksheet
Adding and subtracting surds

Investigation
Tangrams and surds

Topic quiz
3D

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3E Rationalising the denominator

Learning intentions

By the end of this topic you will be able to ...

- ✔ rationalise the denominator of simple fractions
- ✔ identify the conjugate of a binomial expression with a surd
- ✔ rationalise the denominator of a fraction using the conjugate surd.

Inter-year links

Year 8	5D Multiplying algebraic terms
Year 9	3B Algebraic fractions with numerical denominators

Key content video

Rationalising the denominator of a simple fraction

- Fractions that contain surds are conventionally written with a rational denominator. In most cases, this will be an integer denominator.
- To write a fraction containing a surd with a rational denominator:

  Step 1: Identify the surd in the denominator.

  Step 2: Multiply the numerator and the denominator of the fraction by that surd.

  Step 3: Simplify the resulting fraction.

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Conjugate of a binomial expression

- The **conjugate** of a binomial expression with one surd term has the same two terms but the surd term has a different sign. For example, the conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$ and the conjugate of $a - \sqrt{b}$ is $a + \sqrt{b}$.
- The product of conjugate expressions is the difference of the squares of the two terms.

$$(a + b)(a - b) = a^2 - b^2$$

$$\underbrace{(a + \sqrt{b})(a - \sqrt{b})}_{\text{conjugates}} = \underbrace{a^2 - b}_{\text{rational number}}$$

For example, the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ and the product of the conjugate expressions is:

$$\begin{aligned}(2 + \sqrt{3})(2 - \sqrt{3}) &= 2^2 + \sqrt{3} \times 2 - \sqrt{3} \times 2 - \sqrt{3}^2 \\ &= 2^2 - \sqrt{3}^2 \\ &= 4 - 3 \\ &= 1 \qquad \text{which is a rational number.}\end{aligned}$$

Rationalising the denominator using the conjugate surd

- When the denominator of a fraction is a binomial including one surd term and one integer term, the denominator can be rationalised by multiplying the numerator and denominator of the fraction by the conjugate of the denominator.

$$\frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

Example 3E.1 Rationalising the denominator of a fraction

Rewrite each of these fractions with a rational denominator.

a $\frac{7}{2\sqrt{5}}$ **b** $\frac{2\sqrt{3}}{\sqrt{2}}$ **c** $\frac{1-\sqrt{3}}{\sqrt{2}}$

THINK	WRITE
a **1** Multiply the numerator and the denominator by the surd term of the denominator. It is not necessary to include the coefficient of the denominator as it is already rational.	**a** $\frac{7}{2\sqrt{5}} = \frac{7}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
2 Simplify the fraction.	$= \frac{7\sqrt{5}}{2\sqrt{25}}$ $= \frac{7\sqrt{5}}{2 \times 5}$ $= \frac{7\sqrt{5}}{10}$
b **1** Multiply the numerator and the denominator by the surd term of the denominator.	**b** $\frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{2\sqrt{6}}{\sqrt{4}}$
2 Simplify the fraction.	$= \frac{\cancel{2}^{1}\sqrt{6}}{\cancel{2}^{1}}$ $= \sqrt{6}$
c **1** Multiply the numerator and the denominator by the surd term of the denominator.	**c** $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{1-\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
2 Expand the expression using the distributive law.	$= \frac{(1-\sqrt{3})\sqrt{2}}{\sqrt{4}}$ $= \frac{\sqrt{2}-\sqrt{6}}{\sqrt{4}}$
3 Simplify the fraction.	$= \frac{\sqrt{2}-\sqrt{6}}{2}$

Example 3E.2 Identifying the conjugate of an expression

Identify the conjugate for each of these expressions.

a $7 + \sqrt{2}$ **b** $1 - \sqrt{5}$

THINK	WRITE
a To form a conjugate of a binomial expression with one surd term, change the sign of the surd term. So $7 - \sqrt{2}$ is the conjugate of $7 + \sqrt{2}$.	**a** $7 - \sqrt{2}$
b To form a conjugate of a binomial expression with one surd term, change the sign of the surd term. So $1 + \sqrt{5}$ is the conjugate of $1 - \sqrt{5}$.	**b** $1 + \sqrt{5}$

ADVANCED

Example 3E.3 Rationalising the denominator using the conjugate

Write $\frac{2}{3+\sqrt{6}}$ with a rational denominator.

THINK

1 Multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of $3+\sqrt{6}$ is $3-\sqrt{6}$.

2 Simplify the fraction by first expanding the expression in the numerator and then simplify the expression in the denominator using the 'difference of two squares' rule: $(a+b)(a-b)=a^2-b^2$.

3 Check that your answer has a rational denominator.

WRITE

$$\frac{2}{3+\sqrt{6}}=\frac{2}{3+\sqrt{6}}\times\frac{3-\sqrt{6}}{3-\sqrt{6}}$$

$$=\frac{2(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}$$

$$=\frac{6-2\sqrt{6}}{9-6}$$

$$=\frac{6-2\sqrt{6}}{3}$$

Helpful hints

- ✔ It can be helpful to work out the binomial multiplication before you multiply the numerator. Then you can easily see if your multiplication will rationalise the denominator. For example, $(4+\sqrt{3})(4-\sqrt{3})=16-3=13$

  then: $\frac{1}{4+\sqrt{3}}\times\frac{4-\sqrt{3}}{4-\sqrt{3}}=\frac{4-\sqrt{3}}{13}$
- ✔ To write a fraction containing a surd in the denominator in simplest form you need to rationalise the denominator.

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Exercise 3E Rationalising the denominator

△ 1–3, 4–6(1st, 2nd columns), 7, 8(a), 9, 10(a, c), 13

■ 3–6(3rd, 4th columns), 7–9, 11, 12(a, b, c), 13(e, f), 14

⬟ 5–7(3rd, 4th columns), 7–9, 10(e, f), 11, 12(e, f), 15

UNDERSTANDING AND FLUENCY

1 **a** Determine the answer for each of these multiplications. Write your answers in simplest form.

i $\frac{1}{2}\times 1$ **ii** $\frac{3}{4}\times\frac{1}{1}$ **iii** $\frac{3}{4}\times\frac{4}{4}$ **iv** $\frac{2}{5}\times\frac{5}{5}$

b What did you notice about each of your results in part **a**?

c What is special about the second value in each of the multiplications in this question?

2 **a** Calculate results for these expressions by multiplying the numerators together and multiplying the denominators together.

i $\frac{2}{\sqrt{3}}\times\frac{1}{1}$ **ii** $\frac{2}{\sqrt{3}}\times\frac{3}{3}$ **iii** $\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$ **iv** $\frac{2}{\sqrt{2}}\times\frac{2}{2}$

b Which result from part **a** is a fraction with a rational denominator?

c What operation was performed so that a rational denominator was obtained?

3E.1 **3** Rewrite each of these fractions with a rational denominator.

a $\frac{3}{\sqrt{6}}$ **b** $\frac{2}{\sqrt{7}}$ **c** $\frac{3}{\sqrt{3}}$ **d** $\frac{4\sqrt{5}}{\sqrt{7}}$

e $\frac{5}{3\sqrt{2}}$ **f** $\frac{7}{4\sqrt{5}}$ **g** $\frac{2\sqrt{3}}{\sqrt{5}}$ **h** $-\frac{5}{2\sqrt{5}}$

i $\frac{1}{7\sqrt{7}}$ **j** $\frac{4\sqrt{11}}{\sqrt{10}}$ **k** $\frac{8\sqrt{5}}{3\sqrt{2}}$ **l** $\frac{9\sqrt{6}}{4\sqrt{5}}$

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UNDERSTANDING AND FLUENCY

4 Rewrite each of these fractions with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{-\sqrt{6}}{\sqrt{2}}$ **b** $\frac{-\sqrt{15}}{\sqrt{6}}$ **c** $\frac{2\sqrt{8}}{3\sqrt{7}}$ **d** $\frac{2\sqrt{6}}{\sqrt{8}}$

e $\frac{6\sqrt{15}}{7\sqrt{3}}$ **f** $-\frac{3\sqrt{20}}{\sqrt{5}}$ **g** $\frac{5}{\sqrt{x}}$ **h** $\frac{x}{x\sqrt{y}}$

i $\frac{4x}{\sqrt{2y}}$ **j** $\frac{\sqrt{x}}{x\sqrt{y}}$ **k** $\frac{\sqrt{2y}}{2y\sqrt{x}}$ **l** $\frac{-3\sqrt{x}}{6x\sqrt{yz}}$

3E.2 5 State the conjugate for each of these expressions.

a $3+\sqrt{2}$ **b** $2-\sqrt{3}$ **c** $-\sqrt{6}+5$ **d** $-4+3\sqrt{5}$

e $3\sqrt{5}-2$ **f** $-4\sqrt{7}-1$ **g** $1+2\sqrt{3}$ **h** $\sqrt{2}+\sqrt{3}$

i $5-\sqrt{3}$ **j** $-5\sqrt{6}+2$ **k** $3\sqrt{7}-2$ **l** $-5-2\sqrt{5}$

3E.3 6 Rewrite each of these fractions with a rational denominator.

a $\frac{1}{2+\sqrt{3}}$ **b** $\frac{3}{3-\sqrt{2}}$ **c** $\frac{5}{6+\sqrt{5}}$ **d** $\frac{3}{3-4\sqrt{5}}$

e $\frac{\sqrt{2}}{5+3\sqrt{2}}$ **f** $\frac{2\sqrt{3}}{1+2\sqrt{3}}$ **g** $\frac{3}{\sqrt{5}-1}$ **h** $\frac{3\sqrt{7}}{2\sqrt{6}+5}$

i $\frac{-4\sqrt{2}}{\sqrt{3}-2}$ **j** $\frac{\sqrt{3}}{\sqrt{5}+2}$ **k** $\frac{2\sqrt{5}}{3\sqrt{6}+10}$ **l** $\frac{6\sqrt{10}}{5\sqrt{2}-5}$

7 Rewrite each of these fractions with a rational denominator.

a $\frac{\sqrt{2}+1}{2\sqrt{3}+3}$ **b** $\frac{5+2\sqrt{3}}{4+\sqrt{2}}$ **c** $\frac{1+\sqrt{3}}{3-\sqrt{5}}$ **d** $\frac{3+\sqrt{5}}{\sqrt{6}+1}$ **e** $\frac{2\sqrt{3}-2}{3\sqrt{5}+2}$ **f** $\frac{5\sqrt{2}-1}{3+2\sqrt{5}}$

8 Write each of these fractions in simplest form, with a rational denominator.

a $\frac{x}{x-\sqrt{y}}$ **b** $\frac{\sqrt{x}}{\sqrt{x}-y}$ **c** $\frac{x+y}{x\sqrt{x}+y}$ **d** $\frac{x-\sqrt{y}}{x+\sqrt{y}}$

PROBLEM SOLVING AND REASONING

9 You can perform additions and subtractions with fractions containing surds in a similar way as you do with regular fractions.

Consider $\frac{3}{\sqrt{2}}+\frac{1}{2}$.

a Rationalise the denominator of the first fraction.

b Write the result of the addition in simplest form.

10 Find the exact value of each of these expressions.

a $\frac{3}{\sqrt{5}}+\frac{1}{5}$ **b** $\frac{4}{\sqrt{3}}-\frac{1}{3}$ **c** $\frac{3\sqrt{2}}{2}+\frac{2}{\sqrt{3}}$

d $\frac{1}{4\sqrt{3}}+\frac{3}{2\sqrt{2}}$ **e** $\frac{5}{\sqrt{5}}-\frac{2}{3\sqrt{6}}$ **f** $\frac{2\sqrt{3}}{5\sqrt{2}}+\frac{\sqrt{5}}{4\sqrt{3}}$

11 Consider the expression $\frac{1}{\sqrt{5}-1}+\frac{3}{\sqrt{5}+1}$.

a Write each fraction in the expression as an equivalent fraction with a rational denominator.

b Add your answers from part **a** together, writing your answer in simplest form.

12 Simplify each of these expressions after first expressing each fraction with a rational denominator.

a $\frac{2}{\sqrt{6}-2}+\frac{2}{\sqrt{6}+2}$ **b** $\frac{2}{\sqrt{5}+2}-\frac{2}{\sqrt{5}-2}$ **c** $\frac{2}{\sqrt{6}-1}+\frac{2}{\sqrt{3}+1}$

d $\frac{5}{\sqrt{2}+1}-\frac{4}{\sqrt{5}-2}$ **e** $\frac{2\sqrt{3}}{2\sqrt{2}+5}+\frac{\sqrt{3}}{3\sqrt{2}+2}$ **f** $\frac{\sqrt{3}+1}{\sqrt{3}+4}-\frac{\sqrt{3}-1}{\sqrt{3}+1}$

13 Write the reciprocal of each of the following. Write your answers with rational denominators and simplified surds.

a $\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\frac{\sqrt{28}}{7}$ **d** $\frac{\sqrt{27}}{4}$ **e** $\sqrt{6}+2$ **f** $\frac{3+\sqrt{5}}{3-\sqrt{5}}$

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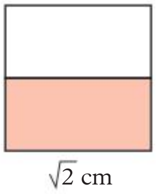
14 Consider the square on the right, which has been divided into congruent rectangles.



a State the area of the whole square.

b State the area of the shaded rectangle.

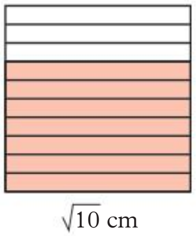
c The breadth of a rectangle can be determined by dividing its area by its length. Determine the breadth of the shaded rectangle. Leave your answer with an irrational denominator.

d The breadth of the shaded rectangle can also be determined by considering the fraction that the breadth of the shaded rectangle is of the square's side length. State what fraction the shaded rectangle's breadth is of the square's side length.

e State the breadth of the shaded rectangle in the form $\frac{a}{b}\sqrt{c}$, where a, b and c are integers.

f Describe the connection between the answers from parts **c** and **e** referring to 'rationalising the denominator'.

g Use a similar method to determine the breadth of the shaded region in the square on the right with a rational denominator.



CHALLENGE

15 When an expression contains two surd terms, the expression has two conjugates resulting from changing the sign of one of the surd terms. For example, the conjugates of $\sqrt{a} + \sqrt{b}$ are $\sqrt{a} - \sqrt{b}$ and $-\sqrt{a} + \sqrt{b}$. The conjugates of $-\sqrt{6} - \sqrt{5}$ are $\sqrt{6} - \sqrt{5}$ and $-\sqrt{6} + \sqrt{5}$.

a Write all of the conjugates of the following expressions.

i $\sqrt{2} + \sqrt{3}$ **ii** $\sqrt{5} - \sqrt{3}$ **iii** $-2\sqrt{5} - 5\sqrt{6}$ **iv** $-2\sqrt{5} + 3\sqrt{7}$

b Evaluate the following products.

i $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **ii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} + \sqrt{3})$ **iii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3})$

iv $(-\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **v** $(\sqrt{2} - \sqrt{3})(-\sqrt{2} - \sqrt{3})$

c We can rationalise a denominator involving multiple surd terms by making use of the 'difference of two squares' rule. It is important that only the sign of one term is changed to create suitable conjugates.

Write the two conjugates which could be used to rationalise the denominator of each of the following.

i $\frac{1}{\sqrt{5} - \sqrt{3}}$ **ii** $\frac{1}{2\sqrt{2} + 5\sqrt{5}}$ **iii** $\frac{1}{-\sqrt{11} + 3\sqrt{2}}$ **iv** $\frac{1}{-5\sqrt{13} - 4\sqrt{3}}$

d Use the conjugates from part **c** to rationalise the denominator for each fraction in part **c**.

e Lucy was rationalising the denominator of $\frac{4}{-\sqrt{2} + \sqrt{3}}$. Her working is shown below right.

i Explain why Lucy did not get a rational denominator.

ii Explain what Lucy could do next to rationalise the denominator.

$$\frac{4}{-\sqrt{2} + \sqrt{3}} \times \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} = \frac{4(\sqrt{2} - \sqrt{3})}{-2 + \sqrt{6} + \sqrt{6} - 3}$$
$$= \frac{4\sqrt{2} - 4\sqrt{3}}{2\sqrt{6} - 5}$$

f Rationalise the denominator of each of the following fractions.

i $\frac{2}{\sqrt{3} + \sqrt{5}}$ **ii** $\frac{-4\sqrt{2}}{\sqrt{3} - \sqrt{2}}$ **iii** $\frac{\sqrt{3}}{\sqrt{5} + \sqrt{2}}$ **iv** $\frac{2\sqrt{5}}{3\sqrt{6} + 2\sqrt{10}}$

v $\frac{6\sqrt{10}}{5\sqrt{2} - 2\sqrt{5}}$ **vi** $\frac{5\sqrt{2} - 1}{3\sqrt{3} + 2\sqrt{5}}$ **vii** $\frac{\sqrt{7} - \sqrt{5}}{\sqrt{5} - \sqrt{7}}$ **viii** $\frac{x + y}{x\sqrt{x} + \sqrt{y}}$

g Evaluate $(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

h Write $\frac{4}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Rationalising the denominator

Interactive skillsheet
Rationalising the denominator using conjugates

Worksheet
Rationalising the denominator

Topic quiz
3E

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Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

3A **1** State if the following numbers are rational or irrational.

a $\frac{4}{3}$
b -12.01
c $19.279\,102\,865\,725\ldots$
d $-2\frac{5}{7}$
e π
f $\sqrt{2}$

3A **2** State if the following roots are surds or not.

a $\sqrt{15}$
b $\sqrt{16}$
c $\sqrt{100}$
d $\sqrt[3]{8}$
e $\sqrt{160}$
f $\sqrt{256}$

3B **3** Simplify the following surds.

a $\sqrt{8}$
b $\sqrt{45}$
c $\sqrt{175}$
d $\sqrt{600}$

3B **4** Simplify the following surds.

a $5\sqrt{12}$
b $7\sqrt{54}$
c $25\sqrt{72}$
d $5\sqrt{1280}$

3C **5** Simplify the following products.

a $-\sqrt{5} \times (-\sqrt{7})$
b $\sqrt{14} \times (-\sqrt{22})$
c $-6\sqrt{3} \times 5\sqrt{6}$
d $\sqrt{18} \times \sqrt{112}$

3C **6** Simplify the following quotients.

a $\frac{-\sqrt{60}}{-\sqrt{10}}$
b $\frac{-8\sqrt{34}}{12\sqrt{2}}$
c $\frac{\sqrt{864}}{-\sqrt{384}}$
d $\frac{9\sqrt{525}}{3\sqrt{252}}$
e $\frac{-5\sqrt{72}}{6\sqrt{15}}$
f $\frac{12\sqrt{338}}{13\sqrt{148}}$

3C **7** Simplify the following fractions.

a $\frac{\sqrt{54} \times \sqrt{12}}{\sqrt{18}}$
b $\frac{8\sqrt{5} \times 5\sqrt{6}}{10\sqrt{3} \times 6\sqrt{15}}$

3D **8** Simplify the following sums and differences.

a $3\sqrt{2} + 9\sqrt{2}$
b $3\sqrt{2} - 9\sqrt{2}$
c $5\sqrt{3} + 3\sqrt{5} + 15\sqrt{5} + 7\sqrt{3}$
d $21\sqrt{7} - 15\sqrt{11} - 23\sqrt{7} + 32\sqrt{11}$

3D **9** Simplify the following sums and differences.

a $\sqrt{8} + \sqrt{128} + \sqrt{18}$
b $\sqrt{75} - \sqrt{27}$
c $\sqrt{150} + \sqrt{108} + \sqrt{54} + \sqrt{192}$
d $\sqrt{4000} - \sqrt{242} - \sqrt{90} - \sqrt{72}$

3D **10** Simplify the following.

a $\frac{5\sqrt{6} + 13\sqrt{6}}{3\sqrt{2}}$
b $3\sqrt{3}(6\sqrt{2} - 2\sqrt{6})$
c $(\sqrt{5} - 2)(3 - \sqrt{7})$
d $(\sqrt{8} + \sqrt{27})(\sqrt{125} + \sqrt{48})$
e $(7\sqrt{12} + 6\sqrt{5})(5\sqrt{6} - 3\sqrt{40})$
f $(4\sqrt{32} + 2\sqrt{7})(5\sqrt{22} - \sqrt{72})$

2E **11** Rationalise the denominators of the following fractions.

a $\frac{5}{\sqrt{2}}$
b $\frac{12}{\sqrt{3}}$
c $\frac{\sqrt{20}}{\sqrt{7}}$
d $\frac{\sqrt{55}}{\sqrt{30}}$

3E **12** Rationalise the denominators of the following fractions.

a $\frac{5}{1 + \sqrt{2}}$
b $\frac{\sqrt{3}}{3 - \sqrt{3}}$
c $\frac{2\sqrt{5}}{3\sqrt{5} - 2}$
d $\frac{2 + \sqrt{7}}{-\sqrt{63} + 7}$
e $\frac{4 - \sqrt{5}}{-13 + \sqrt{5}}$
f $\frac{\sqrt{10} + 2\sqrt{3}}{2 - \sqrt{128}}$

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3F Fractional indices

Learning intentions

By the end of this topic you will be able to ...

✔ apply the index laws to fractional indices.

Inter-year links

Year 8	4B Products and quotients of powers
Year 9	2B Products and quotients of powers

Fractional indices

Key content video

- Fractional indices can be used to express powers of roots.

base ⟶ $a^{\frac{1}{3}}$ ⟵ fractional index

  → The numerator of the fractional index is the number the base is raised to.

  → The denominator of the fractional index refers to the type of root applied to the base.

- Fractional indices have an equivalent surd form:

$$a^{\frac{1}{3}} = \sqrt[3]{a} \qquad a^{\frac{3}{4}} = \sqrt[4]{a^3}$$

fractional index form surd form fractional index form surd form

  → $\sqrt[4]{a^3}$ is equivalent to $(\sqrt[4]{a})^3$.

- The index laws also apply to fractional indices.

	Integer indices	Fractional indices
Zero index law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$\left(a^5\right)^3 = a^{5\times 3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$

- Zero index law: Excluding 0, any number or variable with an index of 0 is equal to 1.
- Product of powers rule: When multiplying numbers or variables in index form that have the same base, add the indices.
- Quotient of powers rule: When dividing numbers or variables in index form that have the same base, subtract the second index from the first index.
- Power of a power rule: When raising a number or variable to two indices, multiply the indices. Every base number or variable inside brackets should have its index multiplied by the index outside the brackets.
- Fractional indices outside the brackets can also be applied to each factor inside the brackets like integer indices.

Integer indices	Fractional indices
$(ab)^3 = a^3b^3$	$(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$
$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$	$\left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

Example 3F.1 Applying the index laws to fractional indices

Simplify each of these expressions.

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}}$ **c** $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$

THINK	WRITE
a Add the indices of the common base, *x*. To add fractional indices, convert them to equivalent fractions with a common denominator.	**a** $\begin{aligned} x^{\frac{1}{4}} \times x^{\frac{2}{3}} &= x^{\frac{1}{4}+\frac{2}{3}} \\ &= x^{\frac{3}{12}+\frac{8}{12}} \\ &= x^{\frac{11}{12}} \end{aligned}$
b Subtract the indices of the common base, *x*. To subtract fractional indices, convert them to equivalent fractions with a common denominator.	**b** $\begin{aligned} \frac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}} &= x^{\frac{3}{5}-\frac{1}{3}} \\ &= x^{\frac{9}{15}-\frac{5}{15}} \\ &= x^{\frac{4}{15}} \end{aligned}$
c Multiply the index of the base inside the brackets by the index outside the brackets.	**c** $\begin{aligned} \left(x^{\frac{1}{3}}\right)^{\frac{1}{2}} &= x^{\frac{1}{3}\times\frac{1}{2}} \\ &= x^{\frac{1}{6}} \end{aligned}$

Example 3F.2 Raising a term to a fractional index

Simplify $4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}}$.

THINK	WRITE
1 Raise each term inside the brackets to the index outside the brackets.	$4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}} = 4\left(x^{\frac{2}{3}}\right)^{\frac{5}{6}}\left(y^{3}\right)^{\frac{5}{6}}$
2 Multiply the index of every base inside the brackets by the index outside the brackets.	$= 4 \times x^{\frac{2}{3}\times\frac{5}{6}} \times y^{3\times\frac{5}{6}}$
3 Simplify the fractional indices.	$\begin{aligned} &= 4 \times x^{\frac{10}{18}} \times y^{\frac{15}{6}} \\ &= 4x^{\frac{5}{9}}y^{\frac{5}{2}} \end{aligned}$

Example 3F.3 Multiplying terms with fractional indices

Simplify $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}}$.

THINK	WRITE
1 Multiply the coefficients and multiply the terms with the same bases. Add the indices of the common bases, *x* and *y*.	$\begin{aligned} 5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}} &= 5 \times 2 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}} \\ &= 10 \times x^{\frac{1}{2}+\frac{1}{5}} \times y^{\frac{1}{3}+\frac{3}{5}} \end{aligned}$
2 Simplify the fractional indices.	$\begin{aligned} &= 10 \times x^{\frac{7}{10}} \times y^{\frac{14}{15}} \\ &= 10x^{\frac{7}{10}}y^{\frac{14}{15}} \end{aligned}$

Helpful hints

✔ Take care when writing fractional indices. They should be smaller than the base and sit high up on the shoulder of the base to avoid confusion between the base and the index. For example, $x^{\frac{1}{3}}$ not $x\frac{1}{3}$.

✔ Take care not to mix up the index laws.

- → When multiplying numbers with the same base, add the indices.
- → When dividing numbers with the same base, subtract the indices.
- → If a number raised to an index is in brackets and raised to another index, multiply the indices.

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Exercise 3F Fractional indices

△ 1–11, 13, 14, 15(a–e), 16

□ 1–2(d–f), 3–6(e–h), 7–11, 13, 14, 15(g–l), 17, 20, 21, 23

⬠ 1–2(d–f), 7, 8, 12, 13, 17–19, 21–24

UNDERSTANDING AND FLUENCY

1 Write each of these fractional indices in the form $\sqrt{a}$, $\sqrt[n]{a}$ or $\sqrt[n]{a^m}$.

a $5^{\frac{1}{2}}$ **b** $12^{\frac{2}{3}}$ **c** $x^{\frac{3}{4}}$ **d** $32^{\frac{2}{3}}$ **e** $100^{\frac{1}{4}}$ **f** $x^{\frac{4}{5}}$

2 Write each of the surds in fractional index form.

a $\sqrt{6}$ **b** $\sqrt[3]{x^2}$ **c** $\sqrt{5^3}$ **d** $\sqrt{x^3}$ **e** $\sqrt{x^5}$ **f** $\sqrt[4]{x^3}$

3F.1 **3** Use an appropriate index law to simplify each of these expressions.

a $x^{\frac{1}{5}} \times x^{\frac{2}{5}}$ **b** $x^{\frac{3}{10}} \times x^{\frac{1}{5}}$ **c** $x^{\frac{3}{4}} \times x^{\frac{1}{8}}$ **d** $x^{\frac{2}{5}} \times x^{\frac{1}{6}}$

e $x^{\frac{5}{6}} \times x^{\frac{1}{6}}$ **f** $x^{\frac{2}{3}} \times x^{\frac{1}{4}}$ **g** $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **h** $x^{\frac{1}{3}} \times x^{\frac{1}{6}}$

4 Use an appropriate index law to simplify each of these expressions.

a $x^{\frac{13}{11}} \div x^{\frac{9}{11}}$ **b** $x^{\frac{3}{8}} \div x^{\frac{1}{8}}$ **c** $\dfrac{x^{\frac{1}{2}}}{x^{\frac{3}{7}}}$ **d** $\dfrac{x^{\frac{4}{5}}}{x^{\frac{3}{4}}}$

e $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ **f** $x^{\frac{2}{5}} \div x^{\frac{3}{10}}$ **g** $\dfrac{x^{\frac{9}{4}}}{x^{\frac{1}{2}}}$ **h** $x^{\frac{6}{7}} \div x^{\frac{6}{7}}$

5 Use an appropriate index law to simplify each of these expressions.

a $\left(x^{\frac{3}{7}}\right)^{\frac{5}{4}}$ **b** $\left(x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **c** $\left(x^{\frac{3}{7}}\right)^{0}$ **d** $\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}$

e $\left(x^{\frac{1}{8}}\right)^{\frac{4}{9}}$ **f** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ **g** $\left(x^{\frac{27}{2}}\right)^{\frac{8}{9}}$ **h** $\left(x^{\frac{8}{45}}\right)^{\frac{9}{56}}$

6 Simplify each of these expressions.

a $5x^{\frac{1}{3}} \times 2x^{\frac{1}{4}}$ **b** $6x^{\frac{2}{5}} \times 5x^{\frac{3}{8}}$ **c** $9x^{\frac{3}{7}} \times -4x^{\frac{2}{9}}$ **d** $4x^{\frac{1}{4}} \div x^{\frac{1}{6}}$

e $12x^{\frac{3}{5}} \div 6x^{\frac{3}{8}}$ **f** $\dfrac{32x^{\frac{5}{6}}}{8x^{\frac{4}{5}}}$ **g** $3x^{\frac{1}{2}} \times 2x^{\frac{1}{3}}$ **h** $\dfrac{x^{\frac{3}{5}}}{4x^{\frac{1}{2}}}$

3F.2 **7** Simplify each of these expressions.

a $\left(5x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(16x^{\frac{2}{5}}\right)^{\frac{1}{4}}$ **c** $\left(27x^{\frac{3}{8}}\right)^{\frac{2}{3}}$ **d** $2\left(x^{\frac{2}{5}}y^{\frac{3}{4}}\right)^{\frac{1}{2}}$

e $4\left(x^{\frac{1}{5}}y^{\frac{3}{7}}\right)^{\frac{2}{3}}$ **f** $5\left(x^{\frac{2}{3}}y^{\frac{4}{5}}\right)^{\frac{3}{8}}$ **g** $5\left(x^{2}y^{\frac{3}{5}}\right)^{\frac{1}{2}}$ **h** $7\left(x^{\frac{1}{5}}y^{\frac{4}{11}}\right)^{\frac{5}{3}}$

3F.3 **8** Simplify each of these expressions.

a $4x^{\frac{1}{4}}y^{\frac{1}{5}} \times 7x^{\frac{1}{3}}y^{\frac{3}{4}}$ **b** $-6x^{\frac{2}{3}}y^{\frac{4}{7}} \times 12x^{\frac{5}{6}}y^{\frac{3}{7}}$ **c** $9x^{\frac{1}{6}}y^{\frac{2}{5}} \times 5x^{\frac{5}{8}}y$

d $25x^{\frac{3}{5}}y^{\frac{1}{4}} \div 5x^{\frac{1}{5}}y^{\frac{1}{12}}$ **e** $12x^{\frac{3}{8}}y^{\frac{5}{11}} \div 6x^{\frac{1}{5}}y^{\frac{1}{3}}$ **f** $\dfrac{7x^{\frac{1}{2}}y^{\frac{3}{7}}}{7x^{\frac{3}{7}}y^{\frac{2}{5}}}$

9 Consider the numerical expression $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.

a Which one of the index laws would you use to simplify it?

b Use the index law you chose to simplify the expression.

c Now consider the surd expression $\sqrt{3} \times \sqrt{3}$. Multiply the two surds and write your answer in simplest form.

d Compare your answers for parts **b** and **c**. What do you notice?

e Repeat parts **b–d** for each of these pairs of expressions.

i $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt{2} \times \sqrt{2}$
ii $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$ and $\sqrt{5} \times \sqrt{5}$
iii $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$ and $\sqrt{10} \times \sqrt{10}$
iv $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ and $\sqrt{x} \times \sqrt{x}$

f Using your results from part **e**, explain what you notice about the relationship between the fractional index of $\frac{1}{2}$ and the square root sign.

10 Now consider the numerical expression $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$.

a Use the appropriate index law to simplify the expression.

b Calculate $\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}$.

c Compare your answers for parts **a** and **b**. What do you notice?

d Repeat parts **a–c** for each of these pairs of expressions.

i $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}}$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$
ii $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$
iii $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$ and $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10}$
iv $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ and $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$

e What are the differences between the surd expression in question **10b** and the surd expression in question **9c**?

f Use your results to explain what you notice about the relationship between the fractional index of $\frac{1}{3}$ and the cube root sign.

11 Consider the term $x^{\frac{2}{3}}$.

a Show that this term has the equivalent forms of $\left(x^{2}\right)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^{2}$.

b Show that both expressions in part **a** are equivalent to $\sqrt[3]{x^{2}}$ and $(\sqrt[3]{x})^{2}$.

c Use a calculator to substitute each of the values below for x in $x^{\frac{2}{3}}$. Check that it produces the same results as substituting those values into each of the expressions from parts **a** and **b**.

i $x = 5$ **ii** $x = 2$ **iii** $x = 3$ **iv** $x = 10$ **v** $x = 12$ **vi** $x = 27$

12 Show that any fractional index of the form $\frac{m}{n}$ is equivalent to $\sqrt[n]{(\ )^{m}}$ or $\left(\sqrt[n]{\ }\right)^{m}$. That is, show that $x^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{x^{m}}$ or $(\sqrt[n]{x})^{m}$.

13 Calculate each of these values, without using a calculator.

a $25^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $1^{\frac{1}{2}}$ **d** $144^{\frac{1}{2}}$ **e** $100^{\frac{1}{2}}$ **f** $196^{\frac{1}{2}}$
g $8^{\frac{1}{3}}$ **h** $27^{\frac{1}{3}}$ **i** $125^{\frac{1}{3}}$ **j** $1^{\frac{1}{3}}$ **k** $1^{\frac{1}{5}}$ **l** $32^{\frac{1}{5}}$

14 Look at each part of question **13**.

i If each fractional index was negative, briefly explain how the answers would change.
ii What would the answer for each part of question **13** be if the indices were negative?
iii Which parts can have a negative base? Explain why.
iv What would the value of each of the parts you identified in part **iii** be if the bases were negative?

15 Write each of these values in simplest surd form.

a $18^{\frac{1}{2}}$ **b** $8^{\frac{1}{2}}$ **c** $24^{\frac{1}{2}}$ **d** $108^{\frac{1}{2}}$ **e** $152^{\frac{1}{2}}$ **f** $54^{\frac{1}{3}}$
g $128^{\frac{1}{3}}$ **h** $320^{\frac{1}{2}}$ **i** $243^{\frac{1}{2}}$ **j** $432^{\frac{1}{2}}$ **k** $5^{\frac{3}{2}}$ **l** $6^{\frac{3}{2}}$

16 Write each of these expressions using fractional indices.

a $\sqrt{x^{3}y}$ **b** $\sqrt[3]{x^{2}y}$ **c** $\sqrt{x^{3}y^{5}}$ **d** $\sqrt{3xy^{3}}$ **e** $\sqrt{16x^{-1}y}$ **f** $\sqrt[3]{x^{4}y^{2}}$

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PROBLEM SOLVING AND REASONING

17 Simplify each of these expressions, writing your answers with fractional indices.

a $12x^{\frac{1}{4}}y^{\frac{1}{5}} \times 9x^{\frac{1}{5}}y^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{1}{3}}y^{\frac{3}{4}}}{5x^{\frac{1}{9}}y^{\frac{2}{5}}}$ **c** $\left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$ **d** $\sqrt{4x^3y} \times 3\sqrt{x^5y^3}$

e $3\sqrt{x^4y^3} \div \sqrt{9xy^2}$ **f** $(\sqrt{x^3y^5})^{\frac{1}{3}}$ **g** $\sqrt{8x^3yz^5} \times 2\sqrt{3xz^3}$ **h** $\left(\sqrt[3]{x^6y^{\frac{1}{3}}z}\right)^2$

18 Write each of these expressions in simplest form.

a $\sqrt{16x^2y}$ **b** $\sqrt{12x^{-3}y^7}$ **c** $\sqrt{\dfrac{72x^4y^3}{24xy}}$

d $\sqrt[3]{27xy^2}$ **e** $\sqrt[4]{16x^3y^2z}$ **f** $\sqrt{343x^{-1}y^5}$

19 Simplify each of these expressions, writing your answers with positive indices where appropriate.

a $3x^{-\frac{1}{3}}y^{\frac{2}{3}} \times 6x^{\frac{3}{4}}y^{-\frac{1}{5}}$ **b** $12x^{\frac{1}{4}}y^{\frac{1}{3}} \times 6x^{-\frac{2}{3}}y^{-\frac{1}{4}}$ **c** $\dfrac{xy^2}{x^{\frac{1}{3}}y^{\frac{5}{2}}}$ **d** $\dfrac{3x^{\frac{1}{3}}y^{\frac{2}{5}}}{12x^{\frac{2}{5}}y^{\frac{3}{4}}}$

e $\left(\dfrac{x^{-\frac{1}{3}}}{y^3}\right)^{\frac{1}{4}}$ **f** $\left(\dfrac{2x^{-\frac{2}{5}}y^{\frac{3}{4}}}{y^2}\right)^4$ **g** $\dfrac{5x^{-\frac{1}{2}} \times 4x^{\frac{1}{3}}}{2x^{-\frac{2}{5}}}$ **h** $\sqrt{\dfrac{x^{-\frac{1}{3}}y^{\frac{2}{3}}}{3x^{-\frac{1}{4}}y^{-\frac{1}{5}}}}$

20 Consider the expression $\dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$.

a Use an appropriate combination of the index laws to simplify the expression.

b Use your answer to part **a** to help you solve this equation for x: $9^x = \dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$

21 Solve each of these equations for x.

a $25^{x+2} = \dfrac{1}{5}$ **b** $27^{x-3} = 9^{x+1}$ **c** $8^x = \dfrac{16^{-\frac{1}{4}} \times 2^{\frac{1}{2}}}{64^{\frac{1}{6}}}$

CHALLENGE

22 Consider the diagrams on the right.

a The diagrams compare multiplying by a fraction and raising a number to a fractional index. Describe what is similar and what is different.

b Evaluate each of the following.

i $81^{\frac{1}{4}}$ **ii** $81^{\frac{3}{4}}$ **iii** $81^{\frac{5}{4}}$

iv $343^{\frac{7}{3}}$ **v** $81^{\frac{5}{2}}$ **vi** $256^{\frac{3}{8}}$

c Write the values appearing in the questions in part **b** in surd form.

d Describe how the denominator of an index and the root of a surd are related, as well as how the numerator of an index and the index of a surd are related. Your description should refer to the number of repeated factors a base has.

$$6 = \boxed{2 + 2 + 2}$$
$$\frac{1}{3} \times 6 = \boxed{2}$$
$$\frac{2}{3} \times 6 = \boxed{2 + 2}$$
$$\frac{4}{3} \times 6 = \boxed{2 + 2 + 2} + \boxed{2}$$

$$8 = \boxed{2 \times 2 \times 2}$$
$$8^{\frac{1}{3}} = \boxed{2}$$
$$8^{\frac{2}{3}} = \boxed{2 \times 2}$$
$$8^{\frac{4}{3}} = \boxed{2 \times 2 \times 2} \times \boxed{2}$$

23 a Use indices to show that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, where a and b are positive integers.

b Use indices to show that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$, where a and b are positive integers.

24 Write each of the following in the form $\sqrt[a]{x^b}$, where a and b are positive integers.

a $10^{2.5}$ **b** $2^{4.25}$ **c** $3^{0.16}$ **d** $5^{5.555}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Fractional indices

Worksheet
Fractional indices

Topic quiz
3F

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3G Logarithms

Learning intentions

By the end of this topic you will be able to ...

✔ convert between exponential form and logarithmic form

✔ evaluate logarithms

✔ solve simple logarithmic equations.

Inter-year links

Year 8	4A Indices
Year 9	2A Indices

Logarithms and exponentials

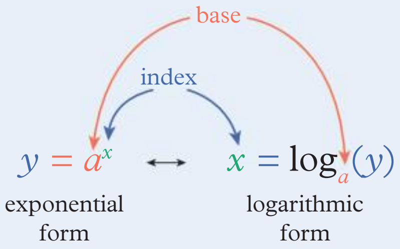
Key content video

- Any equation in which the pronumeral appears as an index is called an exponential equation. For example, $2^x = 8$ is an exponential equation with the solution $x = 3$.
- A **logarithm** is a function that asks, 'What index applied to base a gives y?' It is the number to which the base is raised to produce a specific number.

  For example, in the equation above, the logarithm is $x = \log_2(8) = 3$. It is read as 'the logarithm, base 2, of 8 is equal to 3'.

  → The base of a logarithm must be a positive real number and cannot be equal to 1: $a > 0$, $a \neq 1$

  The diagram on the right shows the general relationship between exponential form and logarithmic form.



- To evaluate a logarithm, ask yourself, 'What index applied to a gives y?' In the diagram above, the answer is x.
- Logarithms and indices are inverse operations of each other.
- Logarithmic identities include:

$$\log_a(a) = 1 \qquad \log_a(1) = 0 \qquad \log_a(a^x) = x \qquad a^{\log_a(x)} = x$$

Example 3G.1 Converting between exponential form and logarithmic form

For each of these, write an equivalent equation in the form specified.

a $3^4 = 81$ (in logarithmic form)

b $\log_2(16) = 4$ (in exponential form)

THINK	WRITE
a Remember that $a^x = y$ is the same as $\log_a(y) = x$. Identify the value of each variable and rewrite the equation.	**a** $\log_3(81) = 4$
b Recall that $\log_a(y) = x$ is the same as $a^x = y$. Identify the value of each variable and restate the equation.	**b** $2^4 = 16$

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Example 3G.2 Calculating the value of logarithms

Find the value of each of these logarithms.

a $\log_3(9)$ **b** $\log_2(64)$

THINK	WRITE
a Determine the value of the logarithm. The term $\log_3(9)$ is asking, 'What index applied to 3 gives 9?' Since $3^2 = 9$, the value of the logarithm is 2.	**a** $\log_3(9) = 2$
b Determine the value of the logarithm. The term $\log_2(64)$ is asking, 'What index applied to 2 gives 64?'. Since $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$, the value of the logarithm is 6.	**b** $\log_2(64) = 6$

Example 3G.3 Converting from exponential form with a negative index to logarithmic form

For each part:

i find the value of the term

ii write an equation in logarithmic form.

a 5^{-1} **b** $\left(\frac{1}{4}\right)^{-2}$ **c** 3^{-3}

THINK	WRITE
a **1** Write an equivalent value with a positive index and simplify.	**a** $5^{-1} = \frac{1}{5^1}$ $= \frac{1}{5}$
2 Use the answer to part **a** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)	$\log_5\left(\frac{1}{5}\right) = -1$
b **1** Write an equivalent value with a positive index and simplify.	**b** $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^{2}$ $= 16$
2 Use the answer to part **a** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)	$\log_{\frac{1}{4}}(16) = -2$
c **1** Write an equivalent value with **a** positive index and simplify.	**c** $3^{-3} = \frac{1}{3^3}$ $= \frac{1}{27}$
2 Use the answer to part **a** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)	$\log_3\left(\frac{1}{27}\right) = -3$

Helpful hints

✔ When the base of the logarithm and the number in the brackets are the same, the result is always 1. It doesn't matter how big or small the values are! For example, $\log_{238\,042}(238\,042) = 1$.

✔ Be careful when writing the base of the logarithms. The base needs to be written in a smaller size than the number in the brackets. For example, write $\log_2(3)$ rather than $\log 2(3)$.

ANS p662

Exercise 3G Logarithms

△ 1–9

■ 1–5(3rd, 4th columns), 6, 7, 10–13, 14(a), 18 (a-d)

⬟ 1–5(3rd, 4th columns), 12–18

UNDERSTANDING AND FLUENCY

3G.1 **1** Write each of these equations in logarithmic form.

a $3^3 = 27$ **b** $4^1 = 4$ **c** $5^4 = 625$ **d** $10^3 = 1000$

e $6^2 = 36$ **f** $9^3 = 729$ **g** $3^4 = 81$ **h** $2^7 = 128$

2 Write each of these equations in exponential form.

a $\log_2(32) = 5$ **b** $\log_3(9) = 2$ **c** $\log_5(25) = 2$ **d** $\log_7(49) = 2$

e $\log_4(16) = 2$ **f** $\log_2(1024) = 10$ **g** $\log_6(216) = 3$ **h** $\log_{10}(100) = 2$

3G.2 **3** Calculate each of these values.

a $\log_2(16)$ **b** $\log_4(256)$ **c** $\log_5(125)$ **d** $\log_{10}(1000)$

e $\log_8(64)$ **f** $\log_3(27)$ **g** $\log_9(81)$ **h** $\log_2(32)$

i $\log_{10}(10\,000)$ **j** $\log_5(625)$ **k** $\log_6(36)$ **l** $\log_3(81)$

m $\log_2(8)$ **n** $\log_4(64)$ **o** $\log_7(49)$ **p** $\log_3(243)$

3G.3 **4** **i** Find the value of each of the terms below.

ii Write an equation in logarithmic form for each term.

a 6^{-1} **b** 4^{-1} **c** 2^{-2} **d** 2^{-5}

e 5^{-2} **f** 10^{-1} **g** 10^{-3} **h** 3^{-3}

i $\left(\frac{1}{2}\right)^{-1}$ **j** $\left(\frac{1}{3}\right)^{-2}$ **k** $\left(\frac{1}{5}\right)^{-3}$ **l** $(0.01)^{-1}$

5 Calculate the value of each of these.

a $\log_2\left(\frac{1}{8}\right)$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_{10}(0.01)$ **d** $\log_4\left(\frac{1}{4}\right)$

e $\log_3\left(\frac{1}{81}\right)$ **f** $\log_2\left(\frac{1}{64}\right)$ **g** $\log_4\left(\frac{1}{64}\right)$ **h** $\log_8\left(\frac{1}{64}\right)$

6 **a** Evaluate the following.

i 2^4 **ii** 2^{-4}

b Is there a real number such that 2 to the power of that number is -16? (What is the value of $\log_2(-16)$?)

c Repeat part **b** for each of the following values.

i $\log_5(-25)$ **ii** $\log_3(-27)$ **iii** $\log_{10}(-1000)$

iv $\log_6(-36)$ **v** $\log_4(-64)$ **vi** $\log_8(-8)$

d What can you conclude about the logarithm of any negative number to any base?

7 **a** Consider $\log_1(5)$. Is it possible for this to be written in an equivalent unique exponential form?

b Is it possible for any of the following to be written in an equivalent exponential form?

i $\log_1(3)$ **ii** $\log_1(10)$ **iii** $\log_1(6)$ **iv** $\log_1(4)$ **v** $\log_1(1)$

c What can you conclude about the logarithm of any number to the base 1?

PROBLEM SOLVING AND REASONING

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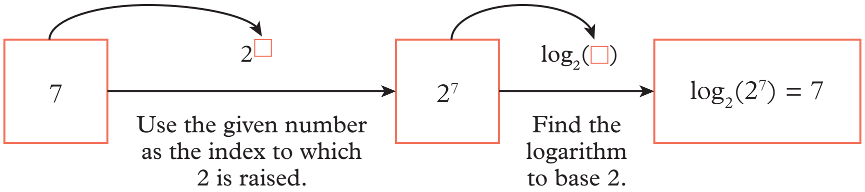
8 Since indices and logarithms are inverse operations, they 'undo' each other. If you raise a base to an index (which is called 'exponentiating'), then you find the logarithm to the same base of that result, you will get the original number you raised the base to.

a Describe the process shown below.



b Evaluate each of the following.

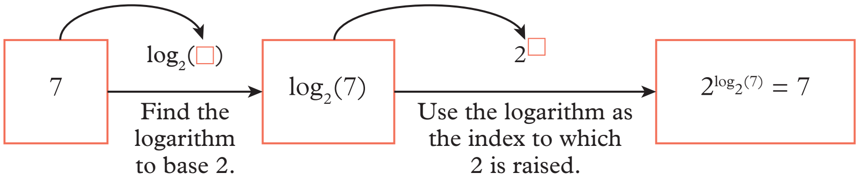
i $\log_5(5^7)$ **ii** $\log_7(7^5)$ **iii** $\log_{12}(12^9)$

iv $\log_2(2^{140})$ **v** $\log_{19}(19^{-6})$ **vi** $\log_{456}(456^{123})$

c Describe, in terms of how we evaluate logarithms, why $\log_a(a^x) = x$ for $a > 0$.

9 If we find the logarithm of a number to a given base, then exponentiate that with the same base, we get the original number back. (See question **8** if you don't know what 'exponentiate' means.)

a Describe the process shown below.



b Evaluate each of the following.

i 5^x, where $x = \log_5(7)$ **ii** 7^x, where $x = \log_7(5)$

iii 21^x, where $x = \log_{21}(11)$ **iv** 2^x, where $x = \log_2(531)$

v 101^x, where $x = \log_{101}(8)$ **vi** 963^x, where $x = \log_{963}(147)$

c Describe, in terms of how we evaluate exponential expressions, why $a^{\log_a(x)} = x$, when $a > 0$ and $x > 0$.

10 A knowledge of the inverse relationship between logarithms and indices is useful when solving logarithmic equations.

Consider the equation $\log_x(64) = 2$, when $x > 0$.

a Copy and complete the following, using both sides of the equation as indices to which x is raised.

$\log_x(64) = 2$

$x^{\square} = x^{\square}$

$\_\_\_\_\_ = x^{\square}$

$x = \_\_\_\_\_$

b What is the value of x? Explain why there is only one solution.

c Write the exponential equation in the last line of your answer for part **a** in the form $x^a = b^a$.

d Notice that, in the equation shown in part **c**, the two indices are the same. If those indices are the same, equating the bases will produce the solution. How does the answer found using this method compare with your answer to part **b**?

11 Use the methods discussed in question **10** to solve each of these logarithmic equations for x.

a $\log_x(64) = 6$ **b** $\log_x(125) = 3$

c $\log_x(16) = 2$ **d** $\log_x(8) = 3$

e $\log_x(128) = 7$ **f** $\log_x(81) = 4$

g $\log_x\left(\frac{1}{4}\right) = -1$ **h** $\log_x\left(\frac{1}{8}\right) = -3$

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PROBLEM SOLVING AND REASONING

12 Consider the equation $\log_4(x) = 3$.

a Copy and complete the following, using both sides of the equation as indices to which 4 is raised.

$$\begin{aligned} \log_4(x) &= 3 \\ 4^{\square} &= 4^{\square} \\ \underline{\qquad} &= 4^{\square} \end{aligned}$$

b Find the value of x.

c Briefly explain how using both sides of the equation as indices to which the base, 4, is raised expresses the logarithmic equation in exponential form.

13 Use the method from question **12** to solve each of these logarithmic equations for x.

a $\log_3(x) = 3$

b $\log_5(x) = 2$

c $\log_6(x) = 1$

d $\log_{10}(x) = 4$

e $\log_2(x) = 5$

f $\log_4(x) = 3$

g $\log_3(x) = -3$

h $\log_2(x) = -2$

14 a Use the index laws to show $(a^2)^{\frac{1}{2}} = a$ and $(a^{\frac{1}{2}})^2 = a$.

b Use the rule in part **a** to complete the following:

$$\begin{aligned} 12^{\square} &= 144 \\ 12 &= 144^{\square} \\ \log_{144}(12) &= \log_{144}(144^{\square}) \\ \log_{144}(12) &= \square \end{aligned}$$

c Simplify $\log_{a^x}(a)$.

15 Use the method from question **14** to find the value of each of the following:

a $\log_{729}(9)$

b $\log_{729}(3)$

c $\log_{625}(5)$

d $\log_{512}(2)$

16 Use the findings from question **8** and question **14** to find the value of the following:

a $\log_6(36) \times \log_{36}(6)$

b $\log_4(256) \times \log_{256}(4)$

c $\log_{\frac{1}{2}}(8) \times \log_8\left(\frac{1}{2}\right)$

d $\log_a(b) \times \log_b(a)$

CHALLENGE

17 Evaluate the following.

a $\log_2(\sqrt{2})$

b $\log_{\sqrt{2}}(2)$

c $\log_2\left(\frac{1}{\sqrt{8}}\right)$

d $\log_2\left(\sqrt[3]{128}\right)$

e $\log_3(3\sqrt{3})$

f $\log_5\left(\sqrt[8]{125}\right)$

g $\sqrt{\log_3(81)}$

h $\log_{\sqrt{125}}\left(\frac{\sqrt{5}}{5}\right)$

18 Solve the following equations.

a $2^x = \frac{1}{\sqrt{2}}$

b $2^x = \frac{1}{8\sqrt{2}}$

c $4^x = \frac{1}{8\sqrt{2}}$

d $4^{x+1} = \frac{1}{8\sqrt{2}}$

e $3^{x-1} = \frac{1}{9\sqrt{3}}$

f $3^{2x} = \frac{1}{\sqrt{27}}$

g $4^{1-x} = \frac{1}{\sqrt{32}}$

h $4^{2x-1} = \frac{4}{\sqrt{8}}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Converting between exponential form and logarithmic form

Interactive skillsheet
Calculating the value of logarithms

Topic quiz
3G

ADVANCED

3H Logarithm laws

Learning intentions

By the end of this topic you will be able to ...

- ✔ simplify logarithmic expressions involving addition and subtraction
- ✔ simplify logarithmic expressions involving indices
- ✔ simplify expressions using the logarithm laws.

Inter-year links

Year 8	4A Indices
Year 9	2C Raising indices and the zero index

Logarithm laws

Key content video

- The logarithm laws can be applied to expressions in logarithmic form:

  $y = \log_a(x)$, where $a > 0$, $a \neq 1$, $x > 0$.
- The logarithm laws include:
  - → The **logarithm of a product** is equal to the sum of the logarithms of the factors.

    $$\log_a(mn) = \log_a(m) + \log_a(n)$$
  - → The **logarithm of a quotient** is equal to the difference of the logarithm of the dividend and the logarithm of the divisor.

    $$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$$
  - → The **logarithm of a power** is equal to the product of the index of x and the logarithm of x.

    $$\log_a(m^n) = n \times \log_a(m)$$
- Logarithmic identities include:

  $$\log_a(a) = 1$$

  $$\log_a(1) = 0$$

Example 3H.1 Calculating logarithmic expressions

Calculate each of these values.

a $\log_3(1)$ **b** $\log_4(4)$

THINK	WRITE
a The logarithm of 1 is always equal to 0, no matter what the base is. $3^0 = 1$, so $\log_3(1) = 0$.	**a** $\log_3(1) = 0$
b The logarithm of a number to a base equal to the number itself is always 1. $4^1 = 4$, so $\log_4(4) = 1$.	**b** $\log_4(4) = 1$

Example 3H.2 Adding and subtracting logarithms

Use an appropriate logarithm law to calculate:

a $\log_4(32) + \log_4(2)$ **b** $\log_5(50) - \log_5(2)$

THINK	WRITE
a **1** Both logarithms in the sum contain the same base, so use the 'logarithm of a product' law: $\log_a(x) + \log_a(y) = \log_a(xy)$	**a** $\log_4(32) + \log_4(2) = \log_4(32 \times 2)$
2 Which index of base 4 will give an answer of 64?	$= \log_4(64)$ $= \log_4(4^3)$ $= 3$
b **1** Both logarithms in the expression contain the same base, so use the 'logarithm of a quotient' law: $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$	**b** $\log_5(50) - \log_5(2) = \log_5\left(\frac{50}{2}\right)$
2 Which index of base 5 will give an answer of 25?	$= \log_5(25)$ $= \log_5(5^2)$ $= 2$

Example 3H.3 Determining the logarithm of a number in exponential form

Use an appropriate logarithm law to calculate:

a $\log_2(4^3)$ **b** $\log_5\left(\frac{1}{5}\right)$

THINK	WRITE
a **1** Use an 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n)= n \times \log_a(x)$	**a** $\log_2(4^3) = 3 \times \log_2(4)$
2 Calculate the logarithm and simplify.	$= 3 \times \log_2(2^2)$ $= 3 \times 2$ $= 6$
b **1** Write the fraction as an integer with a negative index.	**b** $\log_5\left(\frac{1}{5}\right) = \log_5(5^{-1})$
2 Use the 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n) = n \times \log_a(x)$	$= -1 \times \log_5(5)$
3 Use the logarithm identity $\log_a(a) = 1$ to simplify the expression.	$= -1 \times 1$ $= -1$

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Example 3H.4 Simplifying a logarithmic expression

Calculate $\log_6(48) + \log_6(3) - 2\log_6(2)$.

THINK	WRITE
1 Use the 'logarithm of a power' law to rewrite the third term.	$\log_6(48) + \log_6(3) - 2\log_6(2)$ $= \log_6(48) + \log_6(3) - \log_6(2^2)$
2 Use the 'logarithm of a product' law to simplify the first two terms.	$= \log_6(48 \times 3) - \log_6(2^2)$
3 Simplify the product.	$= \log_6(144) - \log_6(4)$
4 Use the 'logarithm of a quotient' law to simplify the expression to one term.	$= \log_6\left(\frac{144}{4}\right)$ $= \log_6(36)$
5 Work out the value of the logarithm. (Remember: $6^2 = 36$)	$= \log_6(6^2)$ $= 2$

Helpful hints

- ✔ The order of operations is important when solving logarithms. Recall your knowledge of BIDMAS. The I stands for indices, which can also be called exponents. Logarithms and exponential equations are in the same category as indices. After brackets, you'll need to simplify or apply logarithms and indices.
- ✔ Remember that a logarithm cannot be applied to a negative number. This is important when solving logarithmic equations for which there might be two solutions.

  For example, $2\log_2(x) = \log_2(9)$

  $$\log_2(x^2) = \log_2(9)$$
  $$x^2 = 9$$
  $$x = \pm 3$$

  Except, x cannot be negative, so $x = 3$ only.

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Exercise 3H Logarithm laws

△ 1–5, 6(a–f), 7–11, 12–13(1st column), 14(1st, 2nd columns), 15, 16

■ 3, 5, 6(g–l), 9–11, 12–13(2nd column), 14(3rd, 4th columns), 15, 17(a, c, e), 18, 19(a, c, e), 20, 22

⬠ 6(g–l), 9, 11, 13(2nd column), 14(3rd, 4th columns), 17, 18, 19(d–f), 20, 21, 23

3H.1 **1** Evaluate each of the following.

a $\log_5(5)$ **b** $\log_3(3)$ **c** $\log_{10}(10)$ **d** $4\log_5(5)$

e $3\log_3(3)$ **f** $\log_3(1)$ **g** $\log_2(1)$ **h** $4\log_5(1)$

2 Use the 'logarithm of a product' law, $\log_a(mn) = \log_a(m) + \log_a(n)$, to write each of the following expressions as the sum of two logarithms.

a $\log_{10}(3 \times 5)$ **b** $\log_2(6 \times 3)$ **c** $\log_5(2.5 \times 10)$ **d** $\log_2(xy)$

3H.2 **3** Write each of these expressions as the logarithm of a product.

a $\log_3(6) + \log_3(2)$ **b** $\log_{10}(5) + \log_{10}(3)$ **c** $\log_2(10) + \log_2(5)$

d $\log_4(6) + \log_4(12)$ **e** $\log_a(5) + \log_a(3)$ **f** $\log_2(x) + \log_2(y)$

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UNDERSTANDING AND FLUENCY

4 Use the 'logarithm of a quotient' law, $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$, to write each of the following expressions as the difference of two logarithms.

a $\log_{10}\left(\frac{5}{7}\right)$ **b** $\log_2\left(\frac{5}{3}\right)$ **c** $\log_5\left(\frac{11}{20}\right)$ **d** $\log_2\left(\frac{a}{b}\right)$

5 Write each of these expressions as the logarithm of a quotient.

a $\log_3(12) - \log_3(5)$ **b** $\log_{10}(50) - \log_{10}(15)$ **c** $\log_2(25) - \log_2(10)$

d $\log_4(36) - \log_4(9)$ **e** $\log_a(3) - \log_a(15)$ **f** $\log_2(x) - \log_2(y)$

6 Use an appropriate logarithm law to evaluate each of the following.

a $\log_3(9) + \log_3(9)$ **b** $\log_6(108) - \log_6(3)$ **c** $\log_2(8) + \log_2(4)$

d $\log_{10}(500) + \log_{10}(20)$ **e** $\log_4(256) - \log_4(16)$ **f** $\log_3(243) - \log_3(9)$

g $\log_6(9) + \log_6(4)$ **h** $\log_5(175) - \log_5(7)$ **i** $\log_2(32) + \log_2(16)$

j $\log_{10}(25) + \log_{10}(4)$ **k** $\log_3(54) - \log_3(6)$ **l** $\log_4(768) - \log_4(3)$

7 Use the logarithm laws to calculate these values.

a $\log_6(27) + \log_6(4) - \log_6(3)$ **b** $\log_{10}(60) - \log_{10}\left(\frac{3}{5}\right)$ **c** $\log_3\left(\frac{3}{2}\right) + \log_3(18)$

8 Use the 'logarithm of a power' law, $\log_a(x^n) = n \times \log_a(x)$, to rewrite each of the following.

a $\log_3(7^2)$ **b** $\log_5(2^9)$ **c** $\log_4(8^3)$ **d** $\log_9(11^4)$

3H.3 **9** Use an appropriate logarithm law to evaluate each of the following.

a $\log_3(3^5)$ **b** $\log_4(4^2)$ **c** $\log_6(6^5)$ **d** $\log_3(9^4)$

e $3\log_3(27)$ **f** $4\log_4(64)$ **g** $5\log_5(5^2)$ **h** $\frac{1}{2}\log_4(4^4)$

10 Use an appropriate logarithm law to evaluate each of the following.

a $\log_{15}\left(\frac{1}{15^2}\right)$ **b** $\log_{12}\left(\frac{1}{12}\right)$ **c** $\log_{19}\left(\left[\frac{1}{19}\right]^2\right)$ **d** $\log_{103}\left(\frac{1}{103^{14}}\right)$

e $5\log_2\left(\frac{1}{64}\right)$ **f** $7\log_6\left(\frac{1}{216}\right)$ **g** $\frac{1}{9}\log_3\left(\frac{4}{108}\right)$ **h** $\frac{5}{2}\log_9\left(\frac{3}{243}\right)$

11 Write each of these expressions in a simpler form and evaluate.

a $2\log_9(3)$ **b** $8\log_4(2)$ **c** $2\log_3(9)$

d $\log_a(\sqrt{a})$ **e** $\log_3(5) - \log_3\left(\frac{5}{3}\right)$ **f** $2\log_x(\sqrt{xy}) - \log_x(y)$

3H.4 **12** Use the logarithm laws to evaluate each of the following.

a $\log_6(12) + \log_6(9) + \log_6(2)$ **b** $\log_2(24) + \log_2(6) - \log_2(36)$

c $\log_5(40) + \log_5(10) - \log_5(16)$ **d** $\log_3(15) - \log_3(45)$

e $\log_4(2) - \log_4(32)$ **f** $\log_{10}(20) + \log_{10}(45) + \log_{10}\left(\frac{10}{9}\right)$

13 Evaluate each of the following.

a $3\log_4(2) + \log_4(8)$ **b** $\log_5(225) - 2\log_5(3)$

c $2\log_3(9) + 2\log_3(3)$ **d** $4\log_{10}(5) + 4\log_{10}(2)$

e $\log_2(72) - 2\log_2(6)$ **f** $5\log_2(4) - \log_2(2^3)$

g $2\log_2(4) + \log_2(14) - \log_2(7)$ **h** $4\log_3(2) - \log_3(32) + \log_3(54)$

i $\log_5(15) + 2\log_5(5) + 3\log_5(2) - \log_5(24)$ **j** $3\log_{10}(5) + 2\log_{10}(3) + \log_{10}\left(\frac{8}{9}\right)$

14 Solve each of these logarithmic equations for x.

a $\log_x(27) = 3$ **b** $\log_4(64) = x$ **c** $\log_3(x) = 2$ **d** $\log_7(2401) = x$

e $\log_6(x) = 3$ **f** $\log_2(x) = 6$ **g** $\log_5\left(\frac{1}{125}\right) = x$ **h** $\log_x(6) = \frac{1}{2}$

i $\log_3(x) = 0$ **j** $\log_x\left(\frac{1}{16}\right) = 4$ **k** $\log_2(\sqrt{2}) = x$ **l** $\log_x\left(\frac{1}{16}\right) = -2$

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PROBLEM SOLVING AND REASONING

15 Fractions containing logarithmic expressions in the numerator and the denominator can be simplified in a similar way to common fractions; that is, by cancelling any common factors in the numerator and denominator.

Consider $\frac{\log_{10}(3)}{\log_{10}(9)}$.

- **a** Use a calculator to find the value of this fraction.
- **b** Show that an equivalent logarithmic expression for the denominator of the fraction is $2\log_{10}(3)$.
- **c** Show that the fraction simplifies to the same answer you obtained in part **a**.

16 Three students attempt to simplify the fraction $\frac{\log_6(8)}{\log_6(2)}$.

Hailey says the correct answer is 3, Danielle thinks the answer is 4 and Joyce is certain that it must be $\log_6\left(\frac{8}{2}\right)$ or $\log_6(4)$. Which student do you think is correct and why?

17 Simplify each of these expressions.

- **a** $\frac{\log_{10}(16)}{\log_{10}(2)}$
- **b** $\frac{\log_6(216)}{\log_6(36)}$
- **c** $\frac{\log_3(8)}{\log_3(2)}$
- **d** $\frac{\log_4(64)}{\log_4(256)}$
- **e** $\frac{\log_{10}(16)}{\log_{10}(64)}$
- **f** $\frac{\log_5(125)}{\log_5(5)}$

18 You can use the logarithm laws to solve more logarithmic equations.

Consider the equation $\log_5(2x) + \log_5(3) = \log_5(12)$. The following parts of this question highlight two techniques that can be used to solve equations like this. Work through both parts to solve the equation in two different ways.

- **a** **i** Subtract $\log_5(12)$ from both sides of the equation and show that the use of the logarithm laws can help you to simplify the equation to $\log_5\left(\frac{x}{2}\right) = 0$.
  - **ii** Write the simplified equation from part **i** in exponential form and solve it for x.
- **b** **i** Look at the original equation. Apply an appropriate law to the left-hand side of the equation to show that it simplifies to $\log_5(6x) = \log_5(12)$.
  - **ii** If the same base appears on both sides, the number parts of each logarithm can be equated; that is, $6x = 12$. Show that this method produces the same value for x as you found in part **a**.

19 Use either method from question **18** to solve each of these logarithmic equations for x.

- **a** $\log_3(x) + \log_3(5) = \log_3(15)$
- **b** $\log_4(2x) + \log_4(8) = 3$
- **c** $\log_5(5x) - \log_5(4) = 2$
- **d** $\log_5(10) - \log_5(x) = \log_5(15)$
- **e** $\log_2(2x + 1) - \log_2(x) = 4$
- **f** $\log_{10}(2x) + \log_{10}(3) = 4 + \log_{10}(5)$

20 Apply the laws of logarithms to evaluate the following expressions given $\log_a 3 = 0.683$ and $\log_a 4 = 0.862$.

- **a** $\log_a 12$
- **b** $\log_a 2$
- **c** $\log_a 6$

CHALLENGE

21 Solve this logarithmic equation for x: $\log_5(x^2 - 9x + 21) = 0$

22 Show that the solution to $\log_{10}(2^{2x+1}) = \log_{10}(4)$ is $\frac{1}{2}$.

23 If $2\log_x\left(\frac{7}{5}\right) + \log_x\left(\frac{704}{35}\right) - 3\log_x\left(\frac{4}{5}\right) = \log_x(y)$, find the value of y.

Check your Student obook pro for these digital resources and more:

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Logarithm of a product

Interactive skillsheet
Logarithm of a quotient

Interactive skillsheet
Logarithm of a power

Interactive skillsheet
Simplifying a logarithmic expression

Worksheet
Logarithms

Investigation
Before calculators … there were slide rules!

Topic quiz
3H

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3I Logarithmic scales

Learning intentions

By the end of this topic you will be able to ...

- ✔ determine the order of magnitude of base-ten numbers
- ✔ determine the position of values on a base-ten logarithmic scale
- ✔ interpret values displayed on a logarithmic scale.

Inter-year links

Year 8	1A Rounding and estimating
Year 9	2E Scientific notation

Key content video

Scientific notation and orders of magnitude

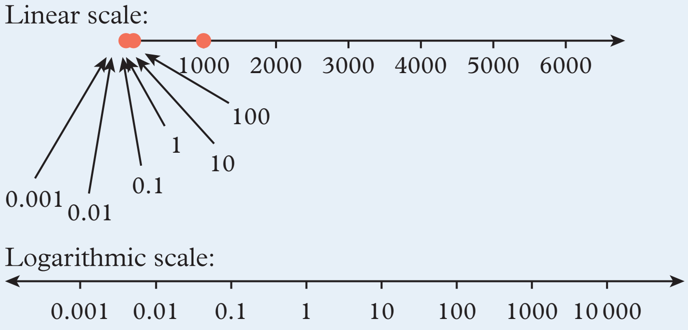
- **Base-ten** refers to the traditional number system we use, where each digit in a number is between 0 and 9 (comprised of ten different possible digits).
- **Scientific notation** is a simple way of writing very large or very small numbers in base-ten. A number written in scientific notation is in the form $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$, and m is an integer.
- The **order of magnitude** of a number in base-ten is the equal to the value of m when the number is written in scientific notation.

Base-ten number	0.001	0.01	0.1	1	10	100	1000
Scientific notation	1×10^{-3}	1×10^{-2}	1×10^{-1}	1×10^{0}	1×10^{1}	1×10^{2}	1×10^{3}
Order of magnitude	−3	−2	−1	0	1	2	3

- In base-ten, if a number is increased by 1 order of magnitude, it is multiplied by 10. Similarly, if a number is decreased by 1 order of magnitude, it is divided by 10.

Logarithmic scales

- **Logarithmic scales** can be used to display numerical data, which covers a wide range of values of differing orders of magnitude.
- A logarithmic scale is non-linear, which means the distance between consecutive numbers on the scale is not equal. Instead, in a logarithmic scale the distance between the orders of magnitude of the numbers is equal.
- To determine the position of a value on a logarithmic scale, the logarithm of the value should be determined using a calculator. Use 1 as the reference point and move the specified number of units right or left on the scale.



  - → When working with a base-ten logarithmic scale, the base-ten logarithm of a value must be taken to determine its position on the scale. For example, to find the position of 500 on the above scale, $\log_{10}(500) = 2.70$, so 500 is located 2.70 units to the right of 1 on the scale.

- If the range of values in a numerical data set are all within one order of magnitude of each other, it is more appropriate to use a linear scale than a logarithmic scale.
- Logarithmic scales should not be used if any of the quantities do not have a real-valued logarithm. Since the logarithm of zero is undefined and the logarithm of a negative value is not a real number, they cannot be displayed on a logarithmic scale.

Example 3I.1 Determining the order of magnitude of base-ten numbers

Determine the order of magnitude of the following base-ten numbers.

a 6.29×10^8 **b** $0.001\,24$

THINK	WRITE
a For numbers written in scientific notation, the order of magnitude is the index of 10 in the scientific notation.	**a** The order of magnitude is 8.
b **1** Write the number in scientific notation.	**b** $0.001\,24 = 1.24 \times 10^{-3}$
2 The order of magnitude is the index of 10 in the scientific notation.	The order of magnitude is -3.

Example 3I.2 Determining the position of values on a logarithmic scale

Determine the position of the following values on a base-ten logarithmic scale, relative to 1.

a 1 000 000 **b** 0.0423 (correct to two decimal places)

THINK	WRITE
a **1** Evaluate the base-ten logarithm of the number.	**a** $\log_{10}(1\,000\,000) = 6$
2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.	1 000 000 lies 6 units to the right of 1.
b **1** Evaluate the base-ten logarithm of the number to two decimal places.	**b** $\log_{10}(0.0423) = -1.373...$ ≈ -1.37
2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.	0.0423 lies 1.37 units to the left of 1.

Example 3I.3 Determining values on a logarithmic scale

Determine the value of the quantity which is 3.3 units to the right of 1 on a base-ten logarithmic scale. Give your answer correct to the nearest whole number.

THINK	WRITE
1 Evaluate 10 to the power of the given (logarithmic) value.	$10^{3.3} = 1995.2\ldots$
2 Round your answer to the nearest whole number.	≈ 1995

Helpful hints

- ✔ An increase of 1 unit on a base-ten logarithmic scale corresponds to a quantity that is one order of magnitude (ten times) greater.
- ✔ A logarithmic scale is not appropriate if the range of quantities do not span multiple orders of magnitude or if any of the values do not have a real-valued logarithm.

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Exercise 3I Logarithmic scales

△ 1–3, 4–6(a, b, e), 11, 13(a–c), 14 □ 2, 3, 4–7(c, d, f), 8, 9(c–f), 11, 12, 13(d–f), 16 ⬠ 2, 3, 7, 8, 10, 11, 13(d–f), 14–18

3I.1 **1** Determine the order of magnitude of the following base-ten numbers.

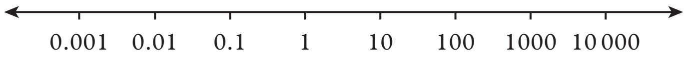
a 8.2751×10^{12} **b** 2.1582×10^{-5} **c** 6.014×10^{3}

d 190 124 098 **e** 0.000 000 281 9 **f** 5217.241

3I.2 **2** Determine the position of the following values on a base-ten logarithmic scale, relative to 1. Round your answers correct to two decimal places where necessary.

a 1000 **b** 0.000 001 **c** 10 **d** 200 **e** 0.02 **f** 0.0008

3I.3 **3** Determine the value of the quantities in the following positions on a base-ten logarithmic scale. Give your answers correct to three significant figures where necessary.



a 5 units to the right of 1 **b** 1 unit to the left of 1 **c** 3.4 units to the right of 1

d 2.7 units to the left of 1 **e** 0.1 units to the right of 1 **f** 4.6 units to the left of 1

4 **a** Increase 4.215×10^{3} by 1 order of magnitude.
b Increase 4.215×10^{3} by 2 orders of magnitude.
c Increase 4.215×10^{3} by 4 orders of magnitude.
d Increase 0.0729 by 1 order of magnitude.
e Increase 0.0729 by 2 orders of magnitude.
f Increase 0.0729 by 4 orders of magnitude.

5 **a** Decrease 4.215×10^{3} by 1 order of magnitude.
b Decrease 4.215×10^{3} by 2 orders of magnitude.
c Decrease 4.215×10^{3} by 5 orders of magnitude.
d Decrease 0.0729 by 1 order of magnitude.
e Decrease 0.0729 by 2 orders of magnitude.
f Decrease 0.0729 by 5 orders of magnitude.

6 For each pair of numbers, determine how many orders of magnitude higher or lower the first number is than the second.
a 4.12×10^{8} and 4.12×10^{2}
b 9.0210×10^{-4} and 9.0210×10^{-1}
c 3.14×10^{-8} and 3.14×10^{7}
d 42 000 and 420
e 0.000 88 and 0.88
f 72 400 000 and 0.000 072 4

UNDERSTANDING AND FLUENCY

PROBLEM SOLVING AND REASONING

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7 What number would you need to multiply or divide by to move the following number of units on a base-ten logarithmic scale? Give your answers correct to the nearest integer where appropriate.

a 3 units to the left
b 4 units to the right
c 11 units to the right
d 6.3 units to the right
e 2.1 units to the left
f 11.1 units to the left

8 **a** Write each of the following numbers in scientific notation.
b State the order of magnitude of each number.
c State the two consecutive integers the base-ten logarithm of each number will lie between.
d Determine the base-ten logarithm of each number, correct to three decimal places.

i 7293.85
ii 105 235.79
iii 0.002 135
iv 50 123
v 61.29
vi 0.000 003 279 3

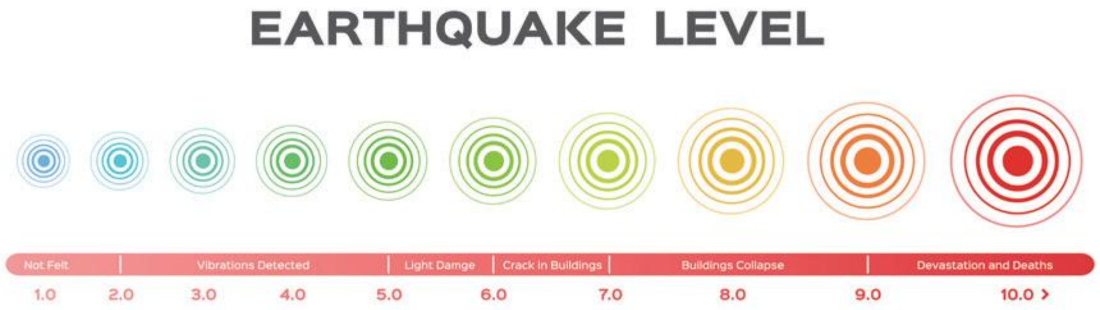
9 Complete the following sentences. Write the missing values correct to three significant figures.

a Increasing a value's base-ten logarithm by 0.5 corresponds to the value becoming ________ times greater.
b Increasing a value's base-ten logarithm by 1.5 corresponds to the value becoming ________ times greater.
c Increasing a value's base-ten logarithm by 2.5 corresponds to the value becoming ________ times greater.
d Increasing a value's base-ten logarithm by 3.5 corresponds to the value becoming ________ times greater.
e Multiplying a value by 2 increases its base-ten logarithm by _______.
f Multiplying a value by 4 increases its base-ten logarithm by _______.
g Multiplying a value by 8 increases its base-ten logarithm by _______.
h Multiplying a value by 16 increases its base-ten logarithm by _______.

10 Use the approximate logarithmic values provided to determine the positions of the following quantities on a base-ten logarithmic scale, relative to 1.

a 5.291×10^{4}, given that $\log_{10}(5.291) \approx 0.724$
b 5.291×10^{-7}, given that $\log_{10}(5.291) \approx 0.724$
c 2.11×10^{-7}, given that $\log_{10}(2.11) \approx 0.324$
d 9.78×10^{8}, given that $\log_{10}(9.78) \approx 0.990$
e 7.02×10^{-6}, given that $\log_{10}(7.02) \approx 0.846$
f 1.05×10^{-1}, given that $\log_{10}(1.05) \approx 0.021$

11 The Richter scale measures the magnitude (size) of an earthquake by taking the base-ten logarithm of the amplitude of the largest seismic wave measured in micrometres (μm).



a Determine the magnitude of earthquakes with the following amplitudes of their largest seismic waves. Give your answers correct to one decimal place.
Hint: 1000 μm = 1 mm

i 320 000 μm
ii 1200 μm
iii 75 600 μm
iv 600 μm
v 21 460 mm
vi 146 m

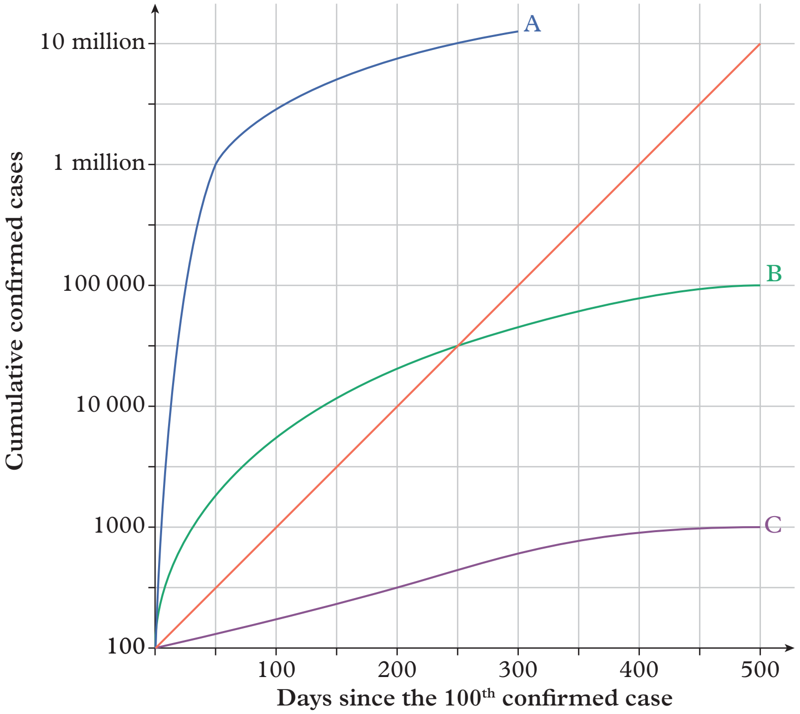
b Determine the amplitudes, in micrometres, of the largest seismic waves of earthquakes with the following magnitudes Give your answers correct to the nearest micrometre.

i 5.0
ii 6.0
iii 6.5
iv 1.5
v 7.8
vi 4.3

c Each increase of one unit on the Richter scale corresponds to an earthquake that is 10 times stronger. For each pair of earthquake magnitudes, determine how many times stronger the larger magnitude is than the smaller magnitude. Give your answers correct to four significant figures where necessary.

i 4.2 and 5.2
ii 4.2 and 4.7
iii 4.2 and 5.7
iv 6.1 and 7.9
v 1.2 and 7.3
vi 5.1 and 5.2

12 When change occurs at an exponential rate, the change is often displayed on a logarithmic scale. The number of people infected by a virus in three countries has been plotted against the number of days since the 100th infection on the following base-ten logarithmic scale.



a An orange line has been drawn at a 45° angle onto the graph so that it passes through the points (100, 1000), (200, 10 000), (300, 100 000). Describe what this line represents.

b Use the graph to determine:

i the number of days it took for Country A to go from its 100th infection to its one millionth infection
ii the number of days it took for Country A to go from its 1 millionth infection to its ten millionth infection
iii the number of days it took for Country B to go from its 100th infection to its hundred thousandth infection
iv correct to the nearest integer, the number of cumulative confirmed cases in Country B after 250 days
v the number of cumulative confirmed cases in Country C after 500 days.

c Describe the importance of the point where country B crosses the orange 45° line.

13 For each set of quantities, state if using a linear or base-ten logarithmic scale would be more appropriate to display the values.

a 790, 392, 597, 598, 800, 748, 828, 831, 814, 967
b 194 112, 42 461, 697 139, 17 186, 5599, 680 398, 67 184, 7065, 365 973
c 0.08, 0.8, 0.06, 0.09, 0.1, 0.05, 0.03, 0.7, 0.006
d 0.514, 0.64, 0.39, 0.421, 0.717, 0.49, 0.851, 0.874, 0.525, 0.33, 0.832
e 5.042, 0.118, 3.386, 4.4847, 0.096, 1.373, 7.717, 2.644, 5.005, 7.117, 0.476
f 0.000 173 4, 0.000 623 15, 0.000 513 54, 0.000 417 34, 0.0006, 0.000 961, 0.000 615 23, 0.000 252 31, 0.000 551 32

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14 For each of the following sets of values, give a reason why a logarithmic scale would not be appropriate to display the data.

a –8.2, 62, 0.09, 1.2, –57, 0.1, –0.57, –6.23, 600, –7.92, –9.7, 4.31

b 2.4, 8.6, 6.5, 6.7, 3.2, 0, 0, 3.3, 4.4, 8.3, 7.3, 2.3

15 Logarithmic scales are also useful for plotting data in histograms that involve very small and very large values, where otherwise the smaller values would all be grouped in the same class interval. The average masses of 47 breeds of animals, in kilograms, are listed below.

1390	1100	725	475	386	190	180	154.25	117.5	100	90	64.475
60	44.984	28.25	26.625	25.5	20	16.5	13.25	9.3	8.6	7.875	6.81
6	5.765	4.25	3.1	2.25	2.06	1.25	0.425	0.397	0.336	0.15	0.29
0.2	0.186	0.173	0.115	0.1	0.07	0.06	0.055	0.05	0.028	0.018	

a Complete the grouped data tables using the average masses listed above.

i

Class interval	Frequency
0–<200	
200–<400	
400–<600	
600–<800	
800–<1000	
1000–<1200	
1200–<1400	

ii

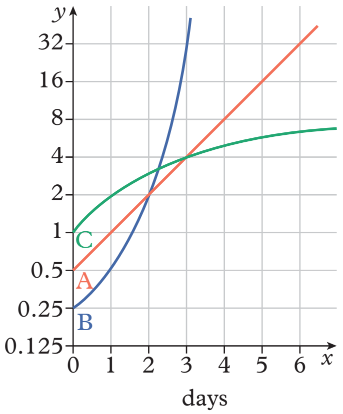
Class interval	Frequency
0.01–<0.1	
0.1–<1	
1–<10	
10–<100	
100–<1000	
1000–<10 000	

b Draw a histogram for each grouped data table in part **a**. Use a horizontal, base-ten logarithmic scale for the table in part **ii**, as shown here.

c Describe the shape of the two histograms as either symmetric, positively skewed or negatively skewed.

d Explain why using a logarithmic scale helps to better understand the distribution of average masses than a linear scale.

16 A base-two logarithmic scale can be used when we want to consider the time it takes for a quantity to double in value. In a base-two logarithmic scale, the quantities double in value for each unit on the scale. Consider the following graph.



a How long does it take for A to double the first time?

b How long does it take for the A to double the second time?

c For the data shown, is the doubling time for A constant, increasing, or decreasing?

d For the data shown, is the doubling time for B constant, increasing, or decreasing?

e For the data shown, is the doubling time for C constant, increasing, or decreasing?

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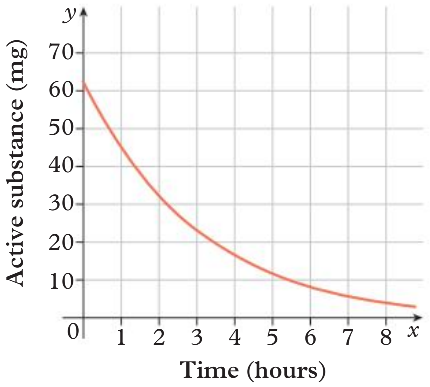
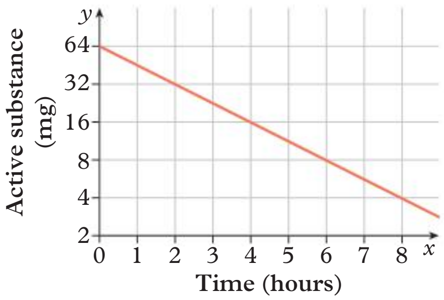
17 The half-life of a drug is the time it takes for the amount of active substance of the drug in a body to reduce by 50%. The two graphs below show the amount of active amount of drug remaining using two different scales.





a What is the half-life of the drug?

b Discuss the advantages and disadvantages of using a base-two logarithmic scale instead of a linear scale to display this data.

18 Bels (B) are a unit of measurement for loud noise, named in honour of Alexander Graham Bell, the inventor of the telephone. Bels are measured on a logarithmic scale, on which an increase in 1 B is an increase of 10 times in loudness. More frequently, decibels (dB) are used where 10 dB = 1 B.

a Determine how many times louder the larger sound is in each of these pairs, correct to four significant figures.

i 15 dB and 25 dB
ii 60 dB and 70 dB
iii 50 dB and 90 dB
iv 95 dB and 145 dB
v 10 dB and 100 dB
vi 99 dB and 100 dB

b The gain, G, in decibels is 10 times the base-ten logarithm of the ratio, $\frac{P_{\text{out}}}{P_{\text{in}}}$, of the power in watts of the output power level, P_{out}, and the input power level, P_{in}.

That is, $G = 10\log_{10}\left(\frac{P_{\text{out}}}{P_{\text{in}}}\right)$.

Determine the gain, correct to the nearest decibel, given the following input and output power levels.

i output = 5 W, input = 50 W
ii output = 50 W, input = 5 W
iii output = 50 W, input = 5000 W
iv output = 2 W, input = 8 W
v output = 240 W, input = 240 W
vi output = 100 W, input = 1 W

c Determine the output power level in watts, correct to one decimal place, given the following input power levels and gains.

i input = 5 W, gain = 30 dB
ii input = 5 W, gain = 3 dB
iii input = 50 W, gain = 3 dB
iv input = 500 W, gain = 30 dB
v input = 240 W, gain = 60 dB
vi input = 240 W, gain = 20 dB

CHALLENGE

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Logarithmic scales

Topic quiz
3I

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Chapter summary

Surds

$\sqrt[n]{x}$ x is rational, n is an integer, $n \geq 2$

- Surds cannot be simplified to remove the root
- Surds are exact values
- When n is even, x must be positive

$a\sqrt{b} = a \times \sqrt{b}$

Square roots

- $(\sqrt{a})^2 = a$
- $\sqrt{a^2} = a$

Multiplying surds

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$

Dividing surds

- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $a\sqrt{b} \div c\sqrt{d} = \dfrac{a\sqrt{b}}{c\sqrt{d}} = \dfrac{a}{c} \times \sqrt{\dfrac{b}{d}}$

Adding and subtracting surds

- $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$
- $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

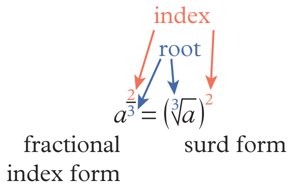
Rationalising the denominator

$$\underbrace{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})}_{\text{conjugates}} = \underbrace{a - b}_{\text{rational number}}$$

- $\dfrac{a}{\sqrt{b}} = \dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$
- $\dfrac{1}{a + \sqrt{b}} = \dfrac{1}{a + \sqrt{b}} \times \dfrac{a - \sqrt{b}}{a - \sqrt{b}} = \dfrac{a - \sqrt{b}}{a^2 - b}$

Fractional indices

$\text{base} \rightarrow a^{\frac{1}{3}} \leftarrow \text{fractional index}$



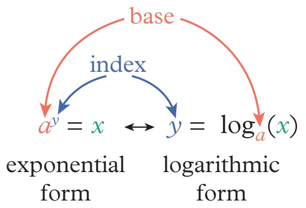
$a^{\frac{2}{3}} = (\sqrt[3]{a})^2$

$a^{\frac{1}{3}} = \sqrt[3]{a}$ $\quad a^{\frac{3}{4}} = \sqrt[4]{a^3} = (\sqrt[4]{a})^3$

	Integer indices	Fractional indices
Zero index law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$(a^5)^3 = a^{5\times 3}$	$(a^{\frac{1}{5}})^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$
	$(ab)^3 = a^3b^3$	$(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$
	$\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$	$\left(\dfrac{a}{b}\right)^{\frac{1}{3}} = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

Logarithms and exponentials

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(mn) = \log_a(m) + \log_a(n)$
- $\log_a(m^n) = n \times \log_a(m)$
- $\log_a\left(\dfrac{m}{n}\right) = \log_a(m) - \log_a(n)$

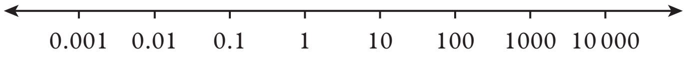


$a^y = x \leftrightarrow y = \log_a(x)$

Scientific notation & orders of magnitude

- $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$ and m is an integer
- Order of magnitude in base-ten = m
- To increase a number by 1 order of magnitude: × 10
- To decrease a number by 1 order of magnitude: ÷ 10

Logarithmic scales



- Position of value on scale relative to 1 = logarithm of number
- Should not be used if the data set has a zero or negative values

ANS
p665

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- base
- base-ten
- conjugate
- cube root
- exponential form
- index
- index form
- irrational number
- like surds
- logarithm
- logarithmic form
- logarithmic scale
- order of magnitude
- power
- product
- quotient
- rational number
- rationalise
- recurring decimal
- scientific notation
- square root
- surds

1 State whether the following statements are true or false. If false, write a corresponding true statement and give an example.
  a All fractions with the denominator and numerator being integers are rational numbers.
  b All recurring decimals are irrational numbers.
  c All terminating decimals are irrational numbers.
  d All surds are irrational numbers.
  e A number with a fractional index will always be irrational.
  f A square root is always a rational number.
  g A cube root can be a rational number.

2 Select the best option to complete this sentence: The logarithm of a number is …
  A the power of 10 needed to make the number.
  B the quotient of the number and the base.
  C the number raised to the power of the base.
  D the number multiplied by the base.
  E the power of the base needed to make the number.

3 Identify the key terms being referenced in each of these definitions.
  a the traditional number system we use which uses the digits 0 to 9
  b the result of a multiplication

4 Using an example, provide a definition in your own words for the following key terms.
  a rational number
  b like surds

5 Complete the following sentences using words from the list of key terms.
  a $9 = 3^2$ is written in ________________, and the corresponding $\log_3 9 = 2$ is written in ________________.
  b $2 + \sqrt{3}$ is the ________________ of $2 - \sqrt{3}$ and is used to ________________ the denominator of $\frac{5}{2-\sqrt{3}}$.

Multiple choice

3A **1** Which of these numbers is irrational?
  A $\sqrt{120}$ **B** $-\sqrt{16}$ **C** $3\sqrt{9}$ **D** $\sqrt[3]{8}$ **E** $\log_2(8)$

3B **2** Which of these numbers is the simplest form of $4\sqrt{32}$?
  A $8\sqrt{8}$ **B** $4\sqrt{2}$ **C** $8\sqrt{2}$ **D** $16\sqrt{2}$ **E** $\sqrt{512}$

3C **3** Which of these surd expressions is equivalent to $\sqrt{35}$?

A $\sqrt{30} \times \sqrt{5}$ **B** $\sqrt{7} \times 5$ **C** $\sqrt{7} \times \sqrt{5}$ **D** $\sqrt{5} \times 5\sqrt{5}$ **E** $\sqrt{30} + \sqrt{5}$

3D **4** Which of these surd expressions does $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ simplify to?

A $8\sqrt{2}$ **B** $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ **C** $80\sqrt{2}$
D $16\sqrt{2}$ **E** $28\sqrt{2}$

3D **5** After expanding, what does $2\sqrt{6}\left(4\sqrt{3} + 5\sqrt{10}\right)$ simplify to?

A $8\sqrt{18} + 10\sqrt{60}$ **B** $24\sqrt{2} + 20\sqrt{15}$ **C** $24\sqrt{2} + 5\sqrt{10}$
D $72\sqrt{2} + 40\sqrt{15}$ **E** $24\sqrt{2} + 5\sqrt{10}$

3E **6** Which of these is equivalent to $\frac{3\sqrt{5}}{2\sqrt{2}}$ written with a rational denominator?

A $\frac{3\sqrt{10}}{2}$ **B** $\frac{3\sqrt{10}}{4}$ **C** $\frac{3\sqrt{7}}{4}$ **D** $\frac{3\sqrt{10}}{8}$ **E** $\frac{15}{2\sqrt{10}}$

3F **7** Which of these expressions is *not* equivalent to $x^{\frac{3}{4}}$?

A $\sqrt[4]{x^3}$ **B** $\left(\sqrt[4]{x}\right)^3$ **C** $\left(x^6\right)^{\frac{1}{8}}$ **D** $\left(x^3\right)^{\frac{1}{4}}$ **E** $\left(\sqrt[3]{x}\right)^4$

3G **8** What is $x^y = z$ equivalent to in logarithmic form?

A $\log_x(z) = y$ **B** $\log_x(y) = z$ **C** $\log_z(x) = y$ **D** $\log_z(y) = x$ **E** $\log_y(z) = x$

3H **9** Which of these is an equivalent logarithmic expression for $\log_{10}(42)$?

A $\log_{10}(20) + \log_{10}(22)$ **B** $\log_{10}(6) + \log_{10}(7)$ **C** $\log_{10}(6) \times \log_{10}(7)$
D $\frac{\log_{10}(7)}{\log_{10}(6)}$ **E** $\frac{\log_{10}(84)}{\log_{10}(2)}$

3I **10** Which of the following best represents the position of 0.005 on a base-ten logarithmic scale?

A 2 units to the left of 1
B 2.3 units to the left of 1
C 1.3 units to the left of 1
D 2 units to the right of 1
E 2.3 units to the right of 1

Short answer

3A **1** Determine whether each of these numbers is rational or irrational.

a $\frac{3}{7}$ **b** $0.256\,371\ldots$ **c** $-\sqrt{25}$
d $\frac{3\sqrt{5}}{2}$ **e** $4\sqrt{2} + 16$ **f** $0.\overline{215}$

3A **2** Determine which of these numbers are surds.

a $3\sqrt{8}$ **b** $\frac{1}{\sqrt{5}}$ **c** $6\sqrt{4}$ **d** $\sqrt{6} - 1$ **e** $\sqrt{\frac{9}{25}}$ **f** $\frac{\sqrt{6}}{4}$

3A **3** Use a calculator to find the value of each of these expressions, correct to three decimal places.

a $2\sqrt{6}$ **b** $\frac{\sqrt{5} - 2}{4}$

3A **4** Show that each of these numbers is rational.

a 0.65 **b** 0.2175 **c** $0.5\dot{2}$ **d** $1.\overline{23}$

3B **5** Write each of these terms in simplest form.

a $2\sqrt{24}$ **b** $3\sqrt{125}$ **c** $\frac{1}{2}\sqrt{108}$

3C **6** Calculate the value of each of these expressions, writing your answers in simplest form.

a $3\sqrt{12} \times 4\sqrt{6}$ **b** $-\sqrt{20} \times 5\sqrt{45}$

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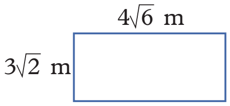
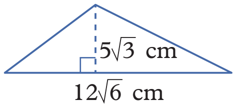
3C **7** Calculate the exact area of each of these shapes, giving your answers in simplest surd form.

a



b



3C **8** Calculate the value of each of these expressions.

a $\sqrt{11} \times \sqrt{7}$ **b** $3\sqrt{6} \times 9\sqrt{5}$

3C **9** Calculate the value of each of these expressions.

a $\sqrt{24} \div \sqrt{8}$ **b** $\dfrac{4\sqrt{45}}{\sqrt{15}}$

3D **10** Simplify each of these expressions.

a $3\sqrt{2} - 6\sqrt{2} + \sqrt{2}$ **b** $3\sqrt{7} + 2\sqrt{5} - \sqrt{7} + 3\sqrt{5}$

c $2\sqrt{10} - \sqrt{2} + 4\sqrt{5} + \sqrt{2}$ **d** $5\sqrt{6} - 2\sqrt{6} + 5 + 3\sqrt{5} - 3$

3D **11** Simplify each of these expressions.

a $2\sqrt{24} - 2\sqrt{54} - 5\sqrt{96} + \sqrt{20}$ **b** $\sqrt{108} - 2\sqrt{75} + 2\sqrt{48} + 3\sqrt{8}$

c $4\sqrt{180} - 6\sqrt{125} + 3\sqrt{245}$ **d** $3\sqrt{200} + 4\sqrt{180} + \sqrt{160} - 3\sqrt{128}$

3D **12** Expand and simplify each of these expressions.

a $(2 + 4\sqrt{3})(3 - 2\sqrt{8})$ **b** $(2\sqrt{3} + 4\sqrt{5})^2$

3E **13** Write each of these fractions with a rational denominator, writing the numerator in simplest surd form where appropriate.

a $\dfrac{5}{2\sqrt{6}}$ **b** $\dfrac{3\sqrt{15}}{2\sqrt{5}}$ **c** $\dfrac{4\sqrt{5}}{\sqrt{3} + 2\sqrt{6}}$ **d** $\dfrac{2 + \sqrt{6}}{3\sqrt{7} - 5\sqrt{3}}$

3F **14** Use an appropriate index law to simplify each of these expressions. Write each answer:

i in fractional index form **ii** in an appropriate surd form.

a $\left(x^{\frac{3}{5}}\right)^{\frac{5}{2}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$ **c** $\dfrac{x^{\frac{2}{3}}}{9x^{\frac{1}{4}}}$

d $\sqrt{8x^3y^2} \times \sqrt{3xy}$ **e** $\left(9x^{\frac{1}{4}}y^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f** $\dfrac{\sqrt{3x^5y^2z}}{5\sqrt{8x^2yz^2}}$

3G **15** Write each of these equations in logarithmic form.

a $3^4 = 81$ **b** $6^3 = 216$ **c** $2^5 = 32$ **d** $x^y = w$

3G **16** Write each of these equations in exponential form.

a $\log_2(8) = 3$ **b** $\log_4(64) = 3$ **c** $\log_{10}\left(\dfrac{1}{10\,000}\right) = -4$ **d** $\log_x(z) = b$

3H **17** Use the logarithm laws to calculate the value of each of these expressions.

a $\log_5(5^3)$ **b** $\log_4(96) - \log_4(6)$

c $3\log_9(3) + \log_9(27)$ **d** $2\log_3(4) - \log_3(32) + \log_3(54)$

3H **18** Solve each of these logarithmic equations for x.

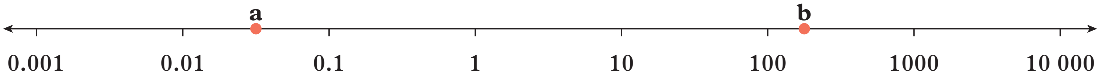
a $\log_x(32) = 5$ **b** $\log_5(x) = -3$ **c** $\log_4(256) = x$

d $\log_x\left(\dfrac{1}{8}\right) = 3$ **e** $\log_2(x) + \log_2(5) = \log_2(12)$ **f** $\log_5(4x) - \log_5(5) = 3$

3I **19** **a** Increase 2.65×10^2 by 2 orders of magnitude. **b** Increase 0.084 by 4 orders of magnitude.

c Decrease 7.51×10^4 by 4 orders of magnitude. **d** Decrease 9091 by 5 orders of magnitude.

3I **20** Consider the following base-ten logarithmic scale.



a A quantity is placed halfway between 0.01 and 0.1. What is its value correct to three significant figures?

b A different quantity is placed one quarter of the way between 100 and 1000. What is its value correct to the nearest integer?

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Analysis

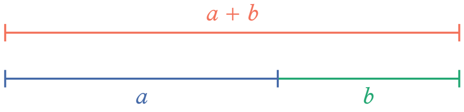
1 A storage chest is in the shape of a cube with edge lengths each measuring $7\sqrt{10} + 11\sqrt{7}$ cm.

- **a** Draw a labelled diagram to represent the chest.
- **b** Calculate the exact area of one of the faces of the chest.
- **c** Is the area you calculated in part **b** a rational number or an irrational number?
- **d** All the external faces of the chest need to be covered with a special coating to protect it against wear and tear. What is the total surface area that needs to be covered?
- **e** If the special coating is sold for $12.80 per square metre, how much will it cost to cover the chest?
- **f** An alternative protective paint offers the best protection when three coats are applied. It is advertised that 1 L of this protective paint will cover an area of 5 m^2. Show that a 1 L can of the protective paint is enough to provide the best protection for the chest.
- **g** Calculate the percentage of the paint in the 1 L can from part **f** that will be used to cover the chest with three coats.
- **h** A narrow 80 cm long pole needs to fit inside the chest. Show that it is not possible for the pole to lie along the bottom of the chest.
- **i** Is there any way that the pole from part **h** can fit inside the chest? Use appropriate mathematics to show your working and to support your answer.
- **j** What assumptions did you need to consider for the calculations you have performed in this task?

2 The Golden Ratio, φ (the Greek letter phi), is an irrational number that is believed to make the most pleasing shapes or proportions. It is equal to 1.618 033… and can be calculated from the ratios:

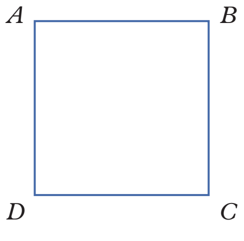
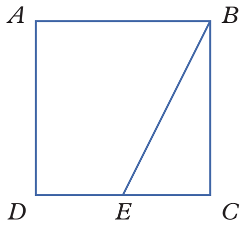
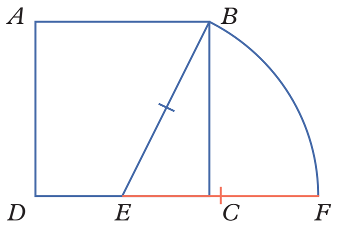
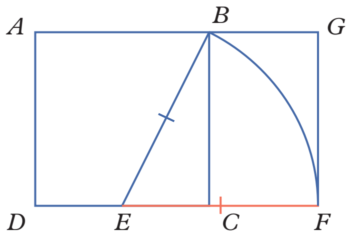
$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

- **a** Measure the length of the lines below and calculate the fractions $\frac{a+b}{a}$ and $\frac{a}{b}$. Are the fractions equal to each other?



- **b** Try different lengths of a and b until you find values that make the two fractions close to equal.

The exact value of φ can be found by completing the following construction. Follow the steps outlined below.

Step 1	**Step 2**	**Step 3**	**Step 4**
Draw a square with an integer side length.	Join the midpoint of CD to vertex B. Label the midpoint E.	Using the midpoint, E, as the centre of a circle, draw an arc so that a new radius extends along EC to F.	Add lines BG and FG to create rectangle $AGFD$.
			

- **c** Let $a = AB = AD$ and $b = BG$. Measure the lengths AB and BG and calculate the fractions $\frac{a+b}{a}$ and $\frac{a}{b}$.
- **d** Rectangle $BGFC$ is also a golden rectangle. Which side corresponds to $(a + b)$ and which side corresponds to a?
- **e** If $AD = 1$, calculate the exact values for EB, AG and BG.
- **f** Hence, calculate the exact value for φ, rationalising the denominator where necessary.

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3 Sound is commonly measured using a logarithmic scale with units of decibels, dB. The following formula is used to calculate the number of decibels of a noise given the pressure or sound intensity of the noise, P, divided by a reference pressure for no noise, P_{ref}.

$$\text{dB} = 10\log_{10}\left(\frac{P}{P_{\text{ref}}}\right)$$

- **a** If a room is silent, so the noise level is the same as the reference loudness, P_{ref}, what is the dB level?
- **b** If the sound intensity is doubled to $2 \times P_{\text{ref}}$, what is the decibel level?
- **c** If a sound is 100 times more intense than 0 dB, what is the decibel level?
- **d** What change in sound intensity increases the decibel level from 70 dB to 90 dB?
- **e** What change in sound intensity decreases the decibel level from 70 dB to 40 dB?
- **f** If you are listening to a song at 80 dB and the sound intensity triples, what is the new decibel level? Give your answer correct to one decimal place.
- **g** Using earplugs reduces noise by 15–30 dB. What reduction in noise levels are these equivalent to?

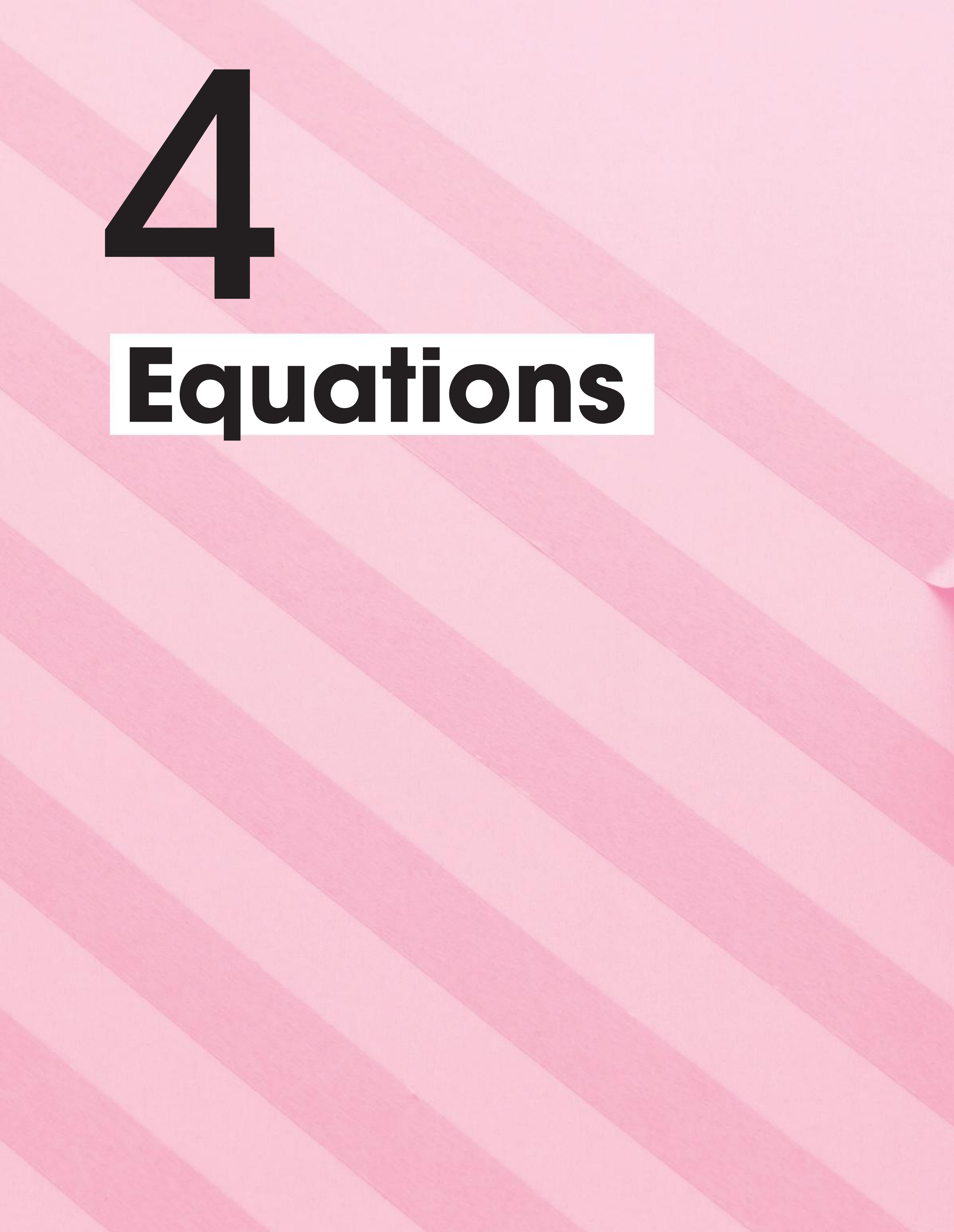
Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Determine whether a number is rational or irrational ☐ Determine whether a root is a surd or not	☐ Go back to Topic 3A Rational and irrational numbers
☐ Simplify surds in $\sqrt{a}$ form ☐ Simplify surds in $a\sqrt{b}$ form	☐ Go back to Topic 3B Simplifying surds
☐ Multiply two or more surds together ☐ Divide a surd by a surd	☐ Go back to Topic 3C Multiplying and dividing surds
☐ Identify like surds ☐ Add and subtract like surds ☐ Add and subtract surds after first simplifying	☐ Go back to Topic 3D Adding and subtracting surds
☐ Rationalise the denominator of simple fractions ☐ Identify the conjugate of a binomial expression with a surd ☐ Rationalise the denominator of a fraction using the conjugate surd	☐ Go back to Topic 3E Rationalising the denominator
☐ Apply the index laws to fractional indices	☐ Go back to Topic 3F Fractional indices
☐ Convert between exponential form and logarithmic form ☐ Evaluate logarithms ☐ Solve simple logarithmic equations	☐ Go back to Topic 3G Logarithms
☐ Simplify logarithmic expressions involving addition and subtraction ☐ Simplify logarithmic expressions involving indices ☐ Simplify expressions using the logarithm laws	☐ Go back to Topic 3H Logarithm laws
☐ Determine the order of magnitude of base-ten numbers ☐ Determine the position of values on a base-ten logarithmic scale ☐ Interpret values displayed on a logarithmic scale	☐ Go back to Topic 3I Logarithmic scales

4

Equations



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Solving equations using inverse operations
- ✔ The Cartesian plane
- ✔ Plotting linear relationships
- ✔ Substitution
- ✔ Squares and square roots

Curriculum links

- ADV Solves monic quadratic equations, linear inequalities and cubic equations of the form $ax^3 = k$ (MA5-EQU-P-01)
  - – Solve monic quadratic equations
  - – Solve cubic equations
- ADV Solves linear equations of more than 3 steps, monic and non-monic quadratic equations, and linear simultaneous equations (MA5-EQU-P-02)
  - – Solve linear equations involving algebraic fractions and equations of more than 3 steps
  - – Rearrange literal equations
  - – Solve quadratic equations using a variety of methods
  - – Solve linear simultaneous equations, both algebraically and graphically
- ADV Uses function notation to describe and graph functions of one variable and graphs inequalities in one and 2 variables (MA5-FNC-P-01)
  - – Graph regions corresponding to linear inequalities in one and 2 variables

Materials

- ✔ Ruler
- ✔ Graph paper

4A Solving linear equations

Learning intentions

By the end of this topic you will be able to ...

✔ solve linear equations with an unknown on both sides

✔ solve linear equations involving algebraic fractions

✔ rearrange literal equations.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6C Solving equations with the unknown on both sides
Year 9	4A Solving linear equations

Linear equations

Key content video

- A **linear equation** is an equation featuring pronumerals that are only raised to the power of 1. These pronumerals represent variable or unknown values.

Linear equations	Non-linear equations
$4x - 1 = 6$	$2x^2 = 32$
$y = x + 3$	$y = \frac{1}{x}$
$2a - \frac{3}{5}b = 10$	$a^3 + 6b = 0$

- If a linear equation has only one pronumeral, algebraic methods can be used to find a solution for the unknown value.
- If a linear equation has more than one pronumeral, then it has more than one solution for each variable. It cannot be solved for numerical solutions by itself and can only be rearranged.

Solving linear equations using inverse operations

- If the same operation is performed on both sides of an equation, the result is an equation equivalent to the original equation and the value(s) of the unknown(s) remains the same.
- Any operation can be undone by applying its inverse operation.
- Operations performed on an unknown in an equation can be undone by performing the appropriate inverse operation to both sides of the equation.

For example,
$$\begin{aligned} \frac{5x-4}{2} &= 8 && (\times 2)\\ 5x - 4 &= 16 && (+ 4)\\ 5x &= 20 && (\div 5)\\ x &= 4 \end{aligned}$$

Operation	Inverse operation
+ 3	− 3
− 3	+ 3
× 3	÷ 3
÷ 3	× 3

Rearranging literal equations

- A **literal equation** is an equation containing two or more variables. For example, $d = st$.
- A pronumeral is the **subject of an equation** if it is by itself on one side of the equation (usually the left-hand side) and does not appear on the other side of the equation.
- Inverse operations can be used to rearrange a literal equation to make any pronumeral the subject.

For example,
$$\begin{aligned} \frac{x}{a} - b &= c && (+ b)\\ \frac{x}{a} &= b + c && (\times a)\\ x &= a(b + c) \end{aligned}$$

Example 4A.1 Solving linear equations with the unknown on both sides

Solve each of these equations by finding the value of x.

a $9x + 2 = 7x - 8$ **b** $5 - 2x = 6x - 1$

THINK	WRITE
a 1 Remove the pronumeral term from one side of the equation by subtracting the pronumeral that has the smaller coefficient ($7x$) from both sides of the equation.	**a** $9x + 2 = 7x - 8 \quad (-\,7x)$
2 Solve the equation using inverse operations.	$2x + 2 = -8 \quad (-\,2)$ $2x = -10 \quad (\div\,2)$ $x = -5$
b 1 Remove the negative pronumeral term by adding $2x$ to both sides of the equation.	**b** $5 - 2x = 6x - 1 \quad (+\,2x)$
2 Solve the equation using inverse operations.	$5 = 8x - 1 \quad (+\,1)$ $6 = 8x \quad (\div\,8)$ $\frac{6}{8} = x$
3 Simplify the solution and write x on the LHS.	$x = \frac{3}{4}$

Example 4A.2 Solving linear equations with an unknown in the denominator

Solve $\frac{24}{x + 2} - 1 = 5$ by finding the value of x.

THINK	WRITE
1 Use inverse operations to rearrange the equation so that the fraction is by itself on one side.	$\frac{24}{x + 2} - 1 = 5 \quad (+\,1)$
2 Multiply both sides of the equation by the denominator of the fraction.	$\frac{24}{x + 2} = 6 \quad (\times\,(x + 2))$
3 Identify and apply the correct inverse operations to both sides of the equation.	$24 = 6(x + 2) \quad (\div\,6)$ $4 = x + 2 \quad (-\,2)$ $2 = x$
4 Write the solution to the equation with x on the LHS.	$x = 2$

Example 4A.3 Solving linear equations that involve two or more fractions

Solve the following linear equations.

a $\frac{2x-5}{3}+\frac{x-7}{5}=2$ **b** $\frac{2x-5}{3}-\frac{x}{5}=\frac{2x}{30}$

THINK	WRITE
a 1 The LCD of the two fractions is 15. Multiply both sides of the equation by the LCD.	**a** $\frac{2x-5}{3}+\frac{x-7}{5}=2$ (× 15)
2 Cancel the 3 and 5 in the denominators and expand the brackets.	$\left(\frac{2x-5}{3}+\frac{x-7}{5}\right)\times 15=2\times 15$ $5(2x-5)+3(x-7)=30$
3 Simplify by combining like terms.	$10x-25+3x-21=30$
4 Apply the correct inverse operation to solve the linear equation.	$13x-46=30$ (+ 46) $13x=76$ (÷ 13) $x=\frac{76}{13}$
b 1 The LCD of the three fractions is 30. Multiply both sides of the equation by the LCD.	**b** $\frac{2x-5}{3}-\frac{x}{5}=\frac{2x}{30}$ (× 30)
2 Cancel the numbers in the denominators and expand the brackets.	$\left(\frac{2x-5}{3}-\frac{x}{5}\right)\times 30=\frac{2x}{30}\times 30$ $10(2x-5)-6x=2x$ $20x-50-6x=2x$
3 Rearrange the equation to have terms with the unknown on one side of the equation.	$20x-6x-2x=50$
4 Simplify by combining like terms.	$12x=50$ (÷ 12)
5 Apply the correct inverse operation to solve the linear equation.	$x=\frac{50}{12}$
6 Simplify the result.	$x=\frac{25}{6}$

Example 4A.4 Rearranging literal equations

Rearrange each equation to make x the subject on the left-hand side.

a $2x-m=4n$ **b** $\frac{h+1}{x}=\frac{a}{b}$

THINK	WRITE
a Identify the correct inverse operation and apply it to both sides of the equation until x is by itself on one side of the equation.	**a** $2x-m=4n$ (+ m) $2x=4n+m$ (÷ 2) $x=\frac{4n+m}{2}$
b 1 Multiply both sides of the equation by the denominator containing x.	**b** $\frac{h+1}{x}=\frac{a}{b}$ (× x)
2 Identify the inverse operation and apply it to both sides of the equation until x is by itself on one side of the equation.	$h+1=\frac{ax}{b}$ (× b) $b(h+1)=ax$ (÷ a) $\frac{b(h+1)}{a}=x$

ADVANCED

3 Rewrite the equation with x on the LHS.	$x = \frac{b(h+1)}{a}$ or, in expanded form: $x = \frac{bh+b}{a}$

Helpful hints

- ✔ Writing the inverse operation beside the appropriate line of working is a great way to keep track of your calculations.

  $\frac{x}{3} - 5 = 7 \qquad (+\ 5)$

- ✔ Writing the pronumeral on the LHS of your solution is only a convention. Once you have isolated the unknown on one side of the equation, you have the solution.
  For example, $3 = x$ is the same as $x = 3$.
- ✔ When given a problem in words, define your pronumerals before writing the equation to represent the problem.
  For example, if you are told that a community garden grows twice as many zucchinis as tomatoes, you could define the pronumerals like this: t = the number of tomatoes, z = the number of zucchinis, and the formula can be written as: $2t = z$.

ANS p666

Exercise 4A Solving linear equations

△ 1(1st column), 2–4(1st, 2nd columns), 5–6(d, f, h, j, l), 7, 8(b, d, f, h), 9(a, c, e), 10(a, c, e), 11–13, 14(a)

■ 1–4(3rd column), 5–10(2nd column), 13, 14, 16, 17–19

⬠ 1–7(2nd column), 8–10(3rd column), 15–20

UNDERSTANDING AND FLUENCY

1 Solve each of these equations.

a $3x - 4 = 15$
b $\frac{x}{5} + 6 = -2$
c $3 = 5 - 7x$
d $8 = \frac{5x}{3}$
e $\frac{x+3}{10} = -5$
f $6 = 3(x+5)$
g $3(x-2) = -11$
h $1 - \frac{x}{2} = 5$
i $-4(5-3x) = 9$
j $\frac{2x-5}{8} = -3$
k $\frac{7-6x}{5} = 10$
l $6 = \frac{5(x-4)}{7}$
m $-4 = \frac{9x}{4} + 4$
n $12 - \frac{8x}{15} = 8$
o $8 = \frac{4(11-3x)}{9}$

2 Solve each of these equations by finding the value of x. Hint: You will need to perform operations that involve fractions and decimals.

a $x - \frac{1}{2} = -9$
b $2x + 3.6 = 7$
c $8.5x = 34$
d $\frac{x}{0.28} + 3.2 = -5.6$
e $\frac{3-0.5x}{4} = 0.45$
f $\frac{11}{3}x = \frac{5}{7}$
g $\frac{4}{3}x + \frac{5}{6} = \frac{3}{2}$
h $\frac{3}{4}\left(x + \frac{9}{5}\right) = \frac{11}{10}$
i $\frac{8}{7}\left(\frac{21}{2}x + \frac{56}{5}\right) = \frac{48}{35}$

ADVANCED

3 Solve each of these equations.

a $2(x + 4) - 5 = -7$ **b** $3(x - 1) + 2 = 8$ **c** $4(x + 3) + 2x = 0$

d $9(x - 5) - x = 3$ **e** $2(3x + 4) + 7 = 13$ **f** $7(2x - 11) - 4x = -7$

g $5(2 - x) - 1 = -6$ **h** $4(1 - 3x) + 3x = 3$ **i** $-3(4 - x) - 2x + 5 = -1$

4A.1 **4** Solve each of these equations.

a $5x + 3 = 2x + 9$ **b** $6x - 7 = 4x + 3$ **c** $3x + 4 = x - 2$

d $4x - 5 = 2x - 3$ **e** $3x + 8 = 2x + 1$ **f** $5x + 6 = 2 + 3x$

g $4x - 5 = 13 - 2x$ **h** $2x - 1 = 11 - x$ **i** $2x + 3 = 6 - 5x$

j $3x - 5 = 7x + 11$ **k** $2 - x = 4x - 8$ **l** $8x + 5 = 11x + 20$

5 Solve each of these equations by first expanding to remove the brackets.

a $3(x + 2) = 2x + 11$ **b** $8x - 9 = 5(x - 3)$ **c** $7x + 3 = 6(x + 1)$

d $5(2x + 1) = 7x + 2$ **e** $4(x - 7) = 2 - x$ **f** $x - 4 = 2(3 - 2x)$

g $3(x + 6) = 2(x + 7)$ **h** $9(x - 3) = 5(x + 1)$ **i** $4(x - 1) = 2(x + 2)$

j $6(x + 5) = 4(5 - x)$ **k** $2(3x - 4) = 11(x - 3)$ **l** $7(x - 12) = -3(x - 2)$

6 Solve each of these equations. Start by multiplying both sides of the equation by the common denominator.

a $\frac{2x + 1}{5} = \frac{x - 7}{5}$ **b** $\frac{4x - 3}{3} = \frac{2x + 5}{3}$ **c** $\frac{7x - 2}{4} = \frac{3x - 10}{4}$

d $\frac{x + 5}{6} = \frac{11 - 2x}{6}$ **e** $\frac{9 - 2x}{11} = \frac{4 - 7x}{11}$ **f** $\frac{3(x - 4)}{2} = \frac{4 - x}{2}$

7 Solve each of these equations. Start by writing each fraction as an equivalent fraction with the same denominator as the other. Hint: Some solutions may not be integers.

a $\frac{x - 2}{5} = \frac{x + 4}{2}$ **b** $\frac{x - 4}{3} = \frac{x + 5}{4}$ **c** $\frac{x + 2}{2} = \frac{x + 7}{3}$

d $\frac{x + 3}{4} = \frac{x + 6}{7}$ **e** $\frac{x - 8}{3} = \frac{x + 1}{6}$ **f** $\frac{3x + 5}{4} = \frac{x + 9}{2}$

g $\frac{2x + 1}{5} = \frac{x - 2}{4}$ **h** $\frac{x - 7}{3} = \frac{5 - x}{4}$ **i** $\frac{2x - 1}{6} = \frac{3x + 2}{5}$

4A.2 **8** Solve each of these equations.

a $\frac{8}{x} = 4$ **b** $\frac{45}{x} = -9$ **c** $\frac{6}{x} = -1$

d $\frac{-12}{x} = 3$ **e** $\frac{-20}{x} = -5$ **f** $\frac{15}{x} = 2$

g $\frac{10}{3x} = 1$ **h** $\frac{-2}{5x} = -7$ **i** $\frac{6}{x} + 5 = 8$

j $\frac{11}{x} + 7 = -4$ **k** $\frac{7}{2x} - 1 = 13$ **l** $\frac{-24}{5x} - 2 = 0$

m $\frac{5}{x + 2} = 8$ **n** $2 = \frac{21}{x - 7}$ **o** $\frac{10}{2x - 5} = -3$

9 Solve each of these equations.

a $\frac{5x}{6} + \frac{4x}{3} = \frac{13}{2}$ **b** $\frac{3x}{4} - \frac{4x}{3} = \frac{21}{12}$ **c** $\frac{7x}{8} = \frac{1}{2} + \frac{5x}{6}$

d $\frac{4x}{9} + \frac{7}{3} = \frac{7}{4} - \frac{11x}{12}$ **e** $\frac{2}{5}\left(\frac{3x}{4} + \frac{7}{2}\right) = \frac{9x}{5}$ **f** $\frac{55}{6}\left(\frac{14x}{3} - \frac{2}{5}\right) = \frac{22}{15}\left(\frac{5}{6} - \frac{15x}{2}\right)$

4A.3 **10** Solve each of these equations.

a $\frac{3x}{4} + \frac{2x + 1}{3} = 1$ **b** $\frac{1 - 3x}{4} + \frac{2x}{3} = 3$ **c** $\frac{2x - 3}{5} + \frac{x + 7}{3} = 1$

d $\frac{x + 5}{2} - \frac{x - 1}{6} = 8$ **e** $\frac{3x}{4} + \frac{1 - x}{3} = \frac{5x}{2}$ **f** $\frac{2x + 3}{3} + \frac{1 - 2x}{6} = \frac{5x - 2}{2}$

11 Solve each of these equations.

a $\frac{3(4 - 2x)}{4} + \frac{2(3x + 9)}{3} = 1$ **b** $\frac{3(-4x - 4)}{4} + \frac{2x + 10}{2} = 3$ **c** $\frac{5x + 6}{5} + \frac{1 - 2x}{3} = 3x + 4$

d $\frac{3(2 - x)}{4} + \frac{3(x - 2)}{2} = \frac{x - 1}{8}$ **e** $\frac{1 - 3x}{4} + \frac{1 - x}{3} = 2(x - 1)$ **f** $2x + \frac{1 - 2x}{6} = \frac{5x}{2} + 3$

ADVANCED

4A.4 **12** Rearrange each of these equations to make x the subject on the left-hand side.

a $x + a = b$ **b** $k = x - p$ **c** $cx = d$

d $g + 3x = h$ **e** $y = 4x + 5$ **f** $\frac{a + x}{m} = n$

g $7x - 2w = 5$ **h** $b - \frac{x}{a} = c$ **i** $k = \frac{h}{x}$

j $6x - e = 4x + f$ **k** $2(x + y) = v$ **l** $\frac{x}{p} + q = w$

m $\frac{a}{x} - b = c$ **n** $y = \frac{kx - m}{2n}$ **o** $x + 3d = 5c - 7x$

PROBLEM SOLVING AND REASONING

13 Liam is saving to buy a tennis racquet that costs \$219. He is able to save \$24 per week and currently has \$105. Follow the steps below to work out how many weeks it will take Liam to save enough for the racquet.

a Choose a pronumeral to represent the unknown quantity in the problem.

b Write an equation to represent the problem, using your chosen pronumeral for the unknown value.

c Solve the equation using inverse operations.

d Write your answer to the problem, in weeks.

14 For each of these problems, set up an equation and solve it using inverse operations.

a Edith bought four makeup kits online for a total cost of \$387, which included the delivery charge of \$29. What was the cost of each makeup kit?

b The perimeter of a rectangular car park is 88 m. If the length of the car park is 12 m more than its breadth, what are the car park's dimensions?

c Lisa, Sara and Bindi scored a combined total of 29 goals in a netball match. Lisa scored five more goals than Sara, and Bindi scored three goals less than Sara. How many goals did each girl score?

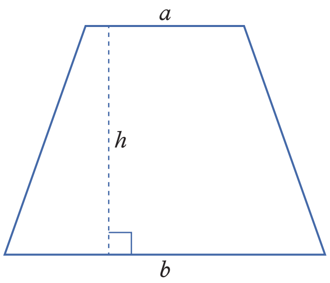
d Oliver has a budget of \$1800 for his birthday party. The cost of hiring a local hall is \$400 and catering is \$27 per person. What is the maximum number of people that Oliver can afford to attend his party?

15 Shana and Paulo have the same amount of money. Shana buys six dumplings and has \$3.80 left over. Paulo buys four dumplings and has \$9.20 left over.

a Write an equation to represent this situation.

b Solve the equation to find the cost of one dumpling.

16 The formula $A = \frac{h}{2}(a + b)$ links the area, A, of a trapezium with its three length measurements a, b and h, as shown in the diagram below.



a Calculate the area, A, of a trapezium for which $a = 4$ cm, $b = 6$ cm and $h = 10$ cm.

b Calculate the height, h, of a trapezium for which $a = 15$ mm, $b = 19$ mm and $A = 119$ mm^2. Hint: Substitute the values for a, b and A into the formula and then solve the resulting equation to find h.

c Calculate a when $b = 5$ m, $h = 3$ m and $A = 18$ m^2.

d Rearrange the formula to make a the subject.

e Calculate a when:

i $b = 37$ cm, $h = 24$ cm and $A = 1068$ cm^2

ii $b = 2.4$ m, $h = 6.1$ m and $A = 30.5$ m^2.

PROBLEM SOLVING AND REASONING

ADVANCED

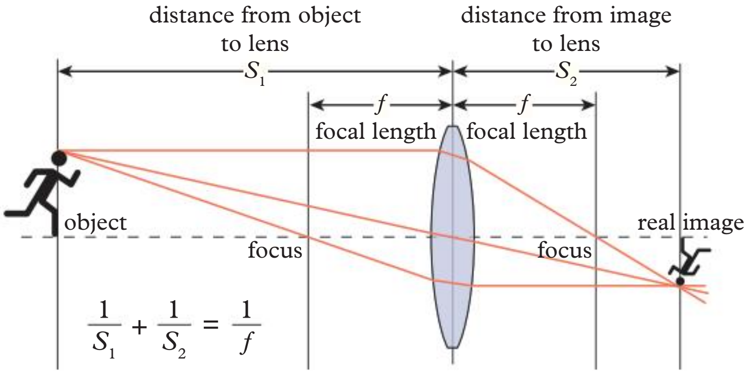
17 The optic equation, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, equates the sum of the reciprocals of two numbers with the reciprocal of a third. The optic equation is used for dealing with many systems, such as:

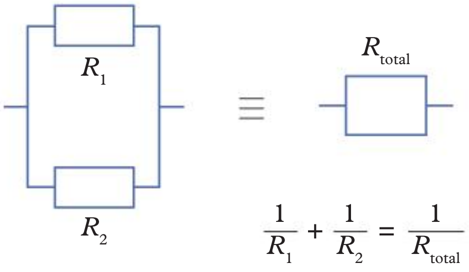
- the distances between an object and its image through a thin lens,

$$\frac{1}{\text{distance from object to lens}} + \frac{1}{\text{distance from image to lens}} = \frac{1}{\text{focal length}}$$

- the resistances in a parallel circuit,

$$\frac{1}{\text{resistance}_1} + \frac{1}{\text{resistance}_2} = \frac{1}{\text{total resistance}}$$





a Show that, for the optic equation, $c = \frac{ab}{a + b}$.

b Calculate:

i the focal length when the distance from the object to the lens is 4 cm and the distance from the image to the lens is 12 cm

ii the total resistance in a parallel circuit when the two resistances are 500 ohms and 2000 ohms.

c Rearrange the optic equation to make a the subject.

d Determine:

i the distance from the lens to the object when the distance from the lens to the image is 10 cm and the focal length is 8 cm

ii the resistance of a resistor that has been connected in parallel with a 16 ohm resistor, where the total resistance is 8 ohms.

CHALLENGE

18 The sum of five consecutive odd numbers is 50 more than triple the smallest number. Write an equation you can solve to find the five numbers. Hint: Let the smallest number be x.

19 Solve each of these equations.

a $\frac{2x + 5}{x - 4} = 7$

b $\frac{9 - 4x}{3x + 5} = 10$

c $\frac{5}{6} = \frac{7x - 3}{3 - 2x}$

20 Make x the subject of each of the following.

a $\frac{mx + p}{nx + q} = r$

b $\frac{a + by}{b + ax} = \frac{b - ay}{a - bx}$

c $\frac{a + by}{b - ax} = \frac{b + ay}{a - bx}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving equations using inverse operations

Interactive skillsheet
Solving equations with the unknown on both sides

Interactive skillsheet
Solving equations with the unknown in the denominator

Interactive skillsheet
Rearranging literal equations

Investigation
Calendar capers

Topic quiz
4A

4B Solving linear inequalities

Learning intentions

By the end of this topic you will be able to ...

- ✔ solve linear inequalities using inverse operations
- ✔ test if a point on the Cartesian plane is a solution to a linear inequality
- ✔ graph regions corresponding to linear inequalities on the Cartesian plane.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	4B Solving linear inequalities

Solving linear inequalities

Key content video

- A **linear inequality** is an inequality containing pronumerals only raised to the power of 1.
- If the same operation is performed on both sides of an inequality sign, this results in an equivalent inequality of a different form. The possible values of the unknown remains the same.
- Inverse operations can be used to solve linear inequalities. However, the inequality sign needs to be reversed when both sides of the inequality are multiplied or divided by a negative number.

  For example, $5 - 2x \le 1 \quad (-5)$

  $-2x \le -4 \quad (\div (-2))$

  $x \ge 2$
- A linear inequality can also have two variables. For example, $y > 5 - 2x$ and $3x + 4y < 5$.
- Linear inequalities with two variables have infinitely many solutions that can be written as ordered pairs in the form (x, y).
- Each ordered pair would produce a true statement when the values of x and y are substituted into the inequality.

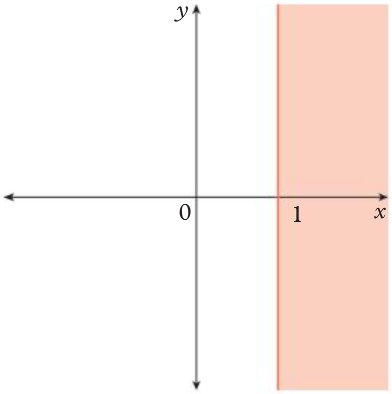
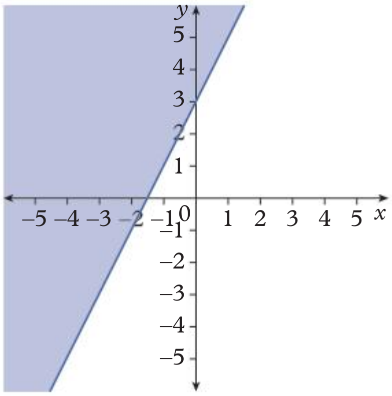
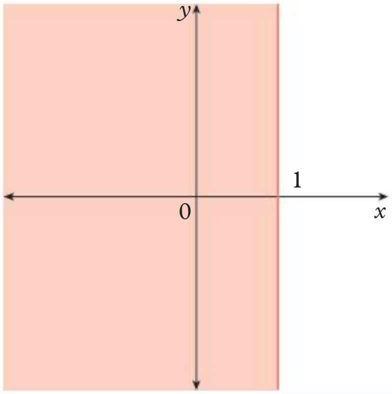
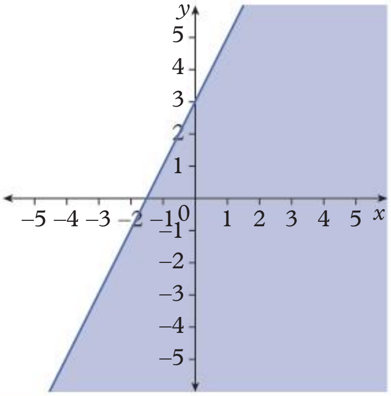
  For example, $(0, 0)$ is one of the solutions to $3x + 4y < 5$.

  Because $3 \times 0 + 4 \times 0 = 0$, $0 < 5$ is a true statement.

Representing linear inequalities with two variables on the Cartesian plane

- Linear inequalities with two variables can be represented on the Cartesian plane using a boundary line and shading.
  - → a solid boundary line indicates that the points on the boundary line are included, meaning one side of the inequality is either ‘less than or equal to’ (≤) or ‘greater than or equal to’ (≥) the other side of the inequality. All the points on the boundary line and in the shaded area are solutions to the inequality.
  - → a dashed line indicates that one side of the inequality is either ‘less than’ (<) or ‘greater than’ (>) the other side of the inequality. All the points in the shaded area are solutions to the linear inequality.
- The boundary line of linear inequalities on the Cartesian plane can be found by changing the inequality sign in the inequality to an equals sign.

  For example:
  - → the boundary line of $2x + 3 < 1$ can be found by solving $2x + 3 = 1$
  - → the boundary line of $2x + 3 < y$ is $2x + 3 = y$.
- The boundary line divides the Cartesian plane into two halves; points on either side of the boundary line can be used as test points to determine which half should be shaded.

<table>
<tr><th>Inequality with one variable</th><th>Inequality with two variables</th></tr>
<tr><td>$x \ge 1$
 </td><td>$y \ge 2x + 3$
 </td></tr>
<tr><td>$x \le 1$
 </td><td>$y \le 2x + 3$
 </td></tr>
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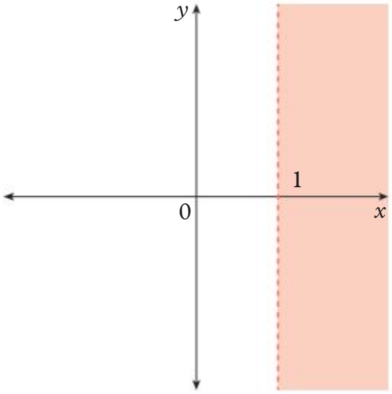
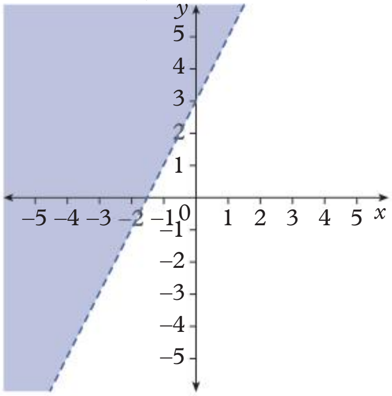
 </td><td>$y > 2x + 3$
 </td></tr>
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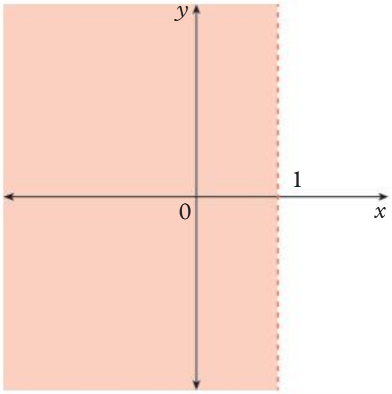
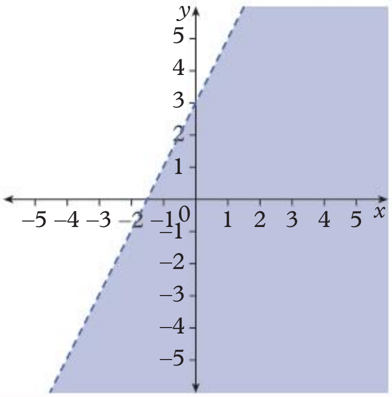
 </td><td>$y < 2x + 3$
 </td></tr>
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ADVANCED

Example 4B.1 Solving linear inequalities

Solve each of these inequalities.

a $5x + 3 < -2$ **b** $3 > -1 - \frac{x}{7}$ **c** $7x - 2 \geq 3x - 10$

THINK	WRITE
a 1 Apply the inverse operations to both sides in the correct order.	**a** $5x + 3 < -2 \quad (-\ 3)$ $5x < -5 \quad (\div\ 5)$
2 Write the solution with x on the LHS.	$x < -1$
3 Check your solution. Substitute $x = -1$ into the LHS of the inequality. The result should equal the RHS. Then substitute any value for x that is less than -1 into the original inequality to check the inequality sign. The result should be less than the RHS.	Check by substituting $x = -1$ into $5x + 3 < -2$. $\text{LHS} = 5(-1) + 3$ $= -2$ $= \text{RHS}$ (For $x = -1$, LHS = RHS as required.) Check by substituting a value less than -1 (for example, $x = -2$). $\text{LHS} = 5(-2) + 3$ $= -7$ $-7 < -2$ (For $x < -1$, LHS < RHS as required.)
b 1 Apply the inverse operations to both sides in the correct order. Reverse the inequality sign when both sides of the inequality are multiplied or divided by a negative number.	**b** $3 > -1 - \frac{x}{7} \quad (+\ 1)$ $4 > -\frac{x}{7} \quad (\times\ (-7))$ $-28 < x$
2 Write the solution with x on the LHS, reverse the inequality sign if needed.	$x > -28$
3 Check your solution. Substitute $x = -28$ into the RHS of the inequality. The result should equal the LHS. Then substitute any value for x that is greater than -28 into the original inequality to check the inequality sign. The result should be less than the LHS.	Check by substituting $x = -28$ into $3 > -1 - \frac{x}{7}$. $\text{RHS} = -1 - \frac{-28}{7}$ $= -1 + 4$ $= 3$ $= \text{LHS}$ (For $x = -28$, LHS = RHS as required.) Check by substituting a value greater than -28 (for example, $x = 0$). $\text{RHS} = -1 - \frac{0}{7}$ $= -1$ $-1 < 3$ (For $x > -28$, LHS > RHS as required.)
c 1 Use inverse operations to remove the pronumeral on the side where the coefficient is less than the coefficient on the other side.	**c** $7x - 2 \geq 3x - 10 \quad (-\ 3x)$
2 Identify and apply the correct inverse operations to both sides of the inequality in the correct order.	$4x - 2 \geq -10 \quad (+\ 2)$ $4x \geq -8 \quad (\div\ 4)$ $x \geq -2$
3 Check your solution by substituting appropriate values for x into the original inequality to see if the result is a true statement.	Check by substituting $x = -2$ into $7x - 2 \geq 3x - 10$. $\text{LHS} = 7(-2) - 2$ $= -16$ $\text{RHS} = 3(-2) - 10$ $= -16$ (For $x = -2$, LHS = RHS as required.)

Check by substituting a value greater than −2 (for example, $x = 0$).

$$\begin{aligned}\text{LHS} &= 7(0) - 2\\ &= -2\\ \text{RHS} &= 3(0) - 10\\ &= -10\end{aligned}$$

(For $x > -2$, LHS > RHS as required.)

Example 4B.2 Determining if a point is a solution to a linear inequality

Determine if the following number pairs are solutions of $3x + 2y \geq 23$.

a (0, 0) **b** (5, 4) **c** (4, 5) **d** (4, 6)

THINK	**WRITE**
a **1** Substitute $x = 0$ and $y = 0$ into the LHS of the linear inequality.	**a** $3x + 2y = 3(0) + 2(0)$
2 Calculate the LHS of the linear inequality.	LHS = 0
3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.	$0 \geq 23$ is a false statement (0, 0) is not a solution of $3x + 2y \geq 23$.
b **1** Substitute $x = 5$ and $y = 4$ into the LHS of the linear inequality.	**b** $3x + 2y = 3(5) + 2(4)$
2 Calculate the LHS of the linear inequality.	LHS = 15 + 8 = 23
3 Compare the LHS and RHS of the linear inequality. The LHS = RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.	$23 \geq 23$ is a true statement (5, 4) is a solution of $3x + 2y \geq 23$.
c **1** Substitute $x = 4$ and $y = 5$ into the LHS of the linear inequality.	**c** $3x + 2y = 3(4) + 2(5)$
2 Calculate the LHS of the linear inequality.	LHS = 12 + 10 = 22
3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.	$22 \geq 23$ is a false statement (4, 5) is not a solution of $3x + 2y \geq 23$.
d **1** Substitute $x = 4$ and $y = 6$ into the LHS of the linear inequality.	**d** $3x + 2y = 3(4) + 2(6)$
2 Calculate the LHS of the linear inequality.	LHS = 12 + 12 = 24
3 Compare the LHS and RHS of the linear inequality. The LHS > RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.	$24 \geq 23$ is a true statement (4, 6) is a solution of $3x + 2y \geq 23$.

Example 4B.3 Representing linear inequalities on the Cartesian plane

Represent the following linear inequalities on the Cartesian plane.

a $y > 2x + 4$ **b** $2y + x \le 2$

THINK

a **1** Sketch $y = 2x + 4$ on the Cartesian plane by finding the x- and y-intercepts.

WRITE

a $y = 2x + 4$

When $x = 0$,

$y = 2(0) + 4$

$= 4$

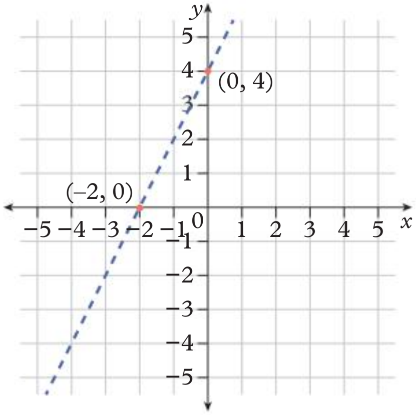
When $y = 0$,

$0 = 2x + 4$

$x = -2$

2 Connect the intercepts with a dashed line as the inequality sign is > not ≥.

The x- and y- intercepts are (0, 4) and (–2, 0).

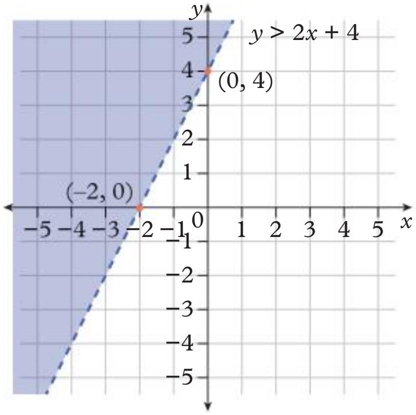


3 Use (0, 0) as a test point to find out which side should be shaded.

Substitute (0, 0) into the inequality.

$0 > 2(0) + 4$ is a false statement. (0, 0) is not a solution of $y > 2x + 4$.

4 As (0, 0) is not a solution to $y > 2x + 4$**,** shade the area on the side of the line that does not include (0, 0).



b **1** Sketch $2y + x = 2$ on the Cartesian plane by finding the x- and y- intercepts.

b $2y + x = 2$

When $x = 0$,

$2y + 0 = 2$

$y = 1$

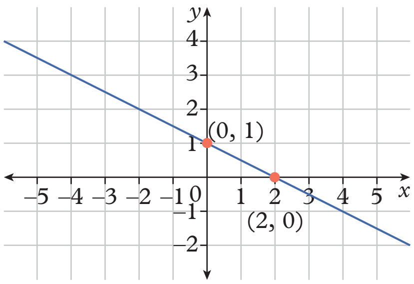
When $y = 0$,

$0 + x = 2$

$x = 2$

The x- and y-intercepts are (0, 1) and (2, 0).

2 Connect the intercepts with a solid line.

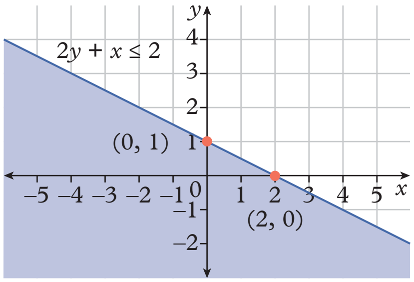


3 Use (0, 0) as a test point to find out which side should be shaded.

Substitute (0, 0) into the inequality.

$2(0) + 0 \leq 2$ is a true statement. (0, 0) is a solution of $2y + x \leq 2$.

4 As (0, 0) is a solution to $2y + x \leq 2$, shade the area on the side of the line that includes (0, 0).



Helpful hints

- ✔ Be careful with the direction of the inequality sign when multiplying and dividing inequalities by negative numbers.
- ✔ If you need to swap the LHS and RHS of an inequality, remember to reverse the inequality sign so that it is still pointing towards the correct direction.
- ✔ When sketching the boundary line of a linear inequality, you can use either two intercepts or one intercept and the gradient.
- ✔ You can use any point on the the Cartesian plane as the test point to find out which area should be shaded.

ANS p667

Exercise 4B Solving linear inequalities

△ 1, 2–4(1st, 2nd columns), 6–8, 13, 14

■ 4(2nd column) 6, 7, 8–11, 15

⬠ 4(3rd column), 7, 8, 12, 16

UNDERSTANDING AND FLUENCY

1 Determine which of the following are linear inequalities.

a $x = 1$
b $y = 2x + 1$
c $y + 2 < 2x + 1$
d $x < 2x$
e $x + y + x \geq 0$
f $y < x^2 + 1$
g $x^2 < 1$
h $y > \frac{1}{2}$
i $y < \frac{1}{x}$
j $y \leq 2x$
k $xy > 2$
l $y < \frac{x}{2}$

4B.1 2 Solve the following linear inequalities.

a $x + 5 < 9$
b $2 \geq 2x + 1$
c $12 < 2x - 4$
d $2x < -20$
e $x + 12 \geq 0$
f $23 < x + 1$
g $11(x + 1) \leq 121$
h $\frac{(x-3)}{2} > 72$
i $12(2x + 2) \leq 144$
j $100 \leq \frac{3}{5}x$
k $\frac{2x}{5} - 10 \geq 0$
l $21 < \frac{x}{7} + 7$

3 Solve the following linear inequalities.

a $-x + 3 < 9$ **b** $2 \geq -2x + 1$ **c** $-12x - 10 < 110$ **d** $-2x < -20$

e $-2x + 12 \geq 0$ **f** $23 < -x + 1$ **g** $11(2 - x) \leq 121$ **h** $\frac{(3 - x)}{2} > 72$

i $12(2 - 2x) \leq 144$ **j** $12 \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3$ **l** $21 < 7 - \frac{x}{7}$

4 Solve the following linear inequalities.

a $x + 3 < 9x - 2$ **b** $2x \geq 5x + 1$ **c** $12x - 10 < 2x + 110$

d $2 - 2x < x - 20$ **e** $-3x + 12 \geq 2x$ **f** $x + 23 < -x + 1$

g $2(2 - x) \leq 12x$ **h** $\frac{(3 - x)}{2} > 7$ **i** $12(2 - 2x) \leq 2x + 5$

j $12x \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3x$ **l** $2x - 1 < 7 - \frac{x}{7}$

5 Solve the following linear inequalities.

a $\frac{x + 3}{12} < \frac{9}{12}$ **b** $\frac{x + 1}{3} < \frac{2}{3}$ **c** $\frac{3 - x}{2} < \frac{9 - x}{12}$

d $\frac{1 - x}{6} > \frac{x - 2}{12}$ **e** $\frac{x - 3}{21} < \frac{9 - x}{7}$ **f** $\frac{x + 3}{5} \leq \frac{9 - x}{7}$

4B.2 **6** Determine if the following points are solutions of the given inequalities.

a $2x + 1 < y$ **b** $y \geq 3 - x$ **c** $x - 5 \leq y$ **d** $y < 8 + 2x$

i (0, 0) **ii** (1, 1) **iii** (3, 5)

7 Determine if the following points are solutions of the given inequalities.

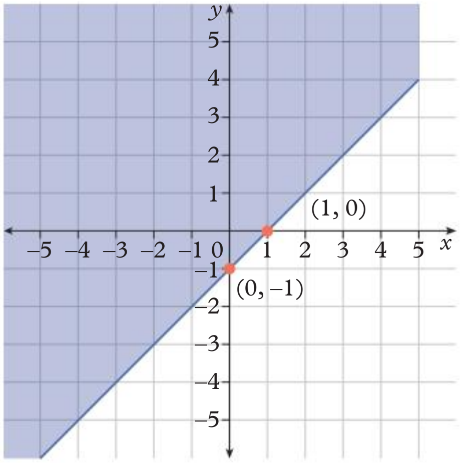
a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

i (0, 0) **ii** (2, 3) **iii** (1, 9)

8 Use your answer to question **7** to match the inequalities with the graphs below.

a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

I



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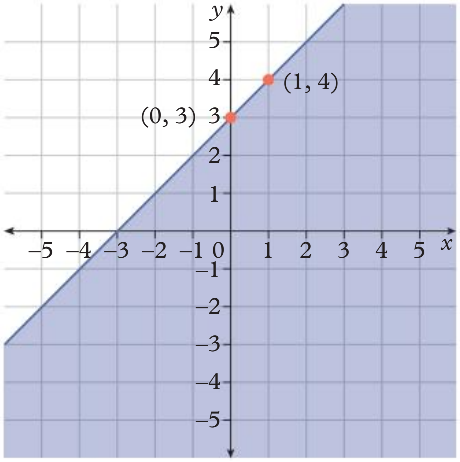

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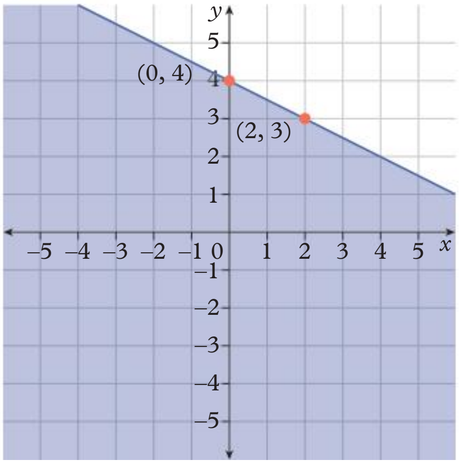

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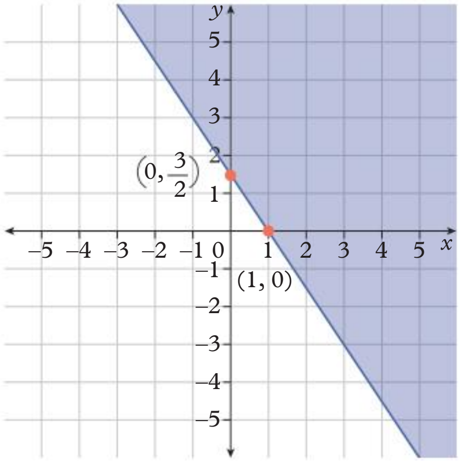

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UNDERSTANDING AND FLUENCY

4B.3 **9** Represent the linear inequalities in question **6** on the Cartesian plane.

10 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5$ **b** $2y + 4x > 2$ **c** $3y + 2x < 2$

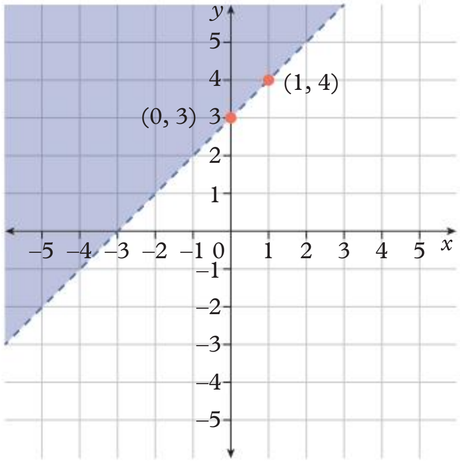
11 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5 - x$ **b** $y + 4x > 2 + 2y$ **c** $x + y + 1 \leq 2x$

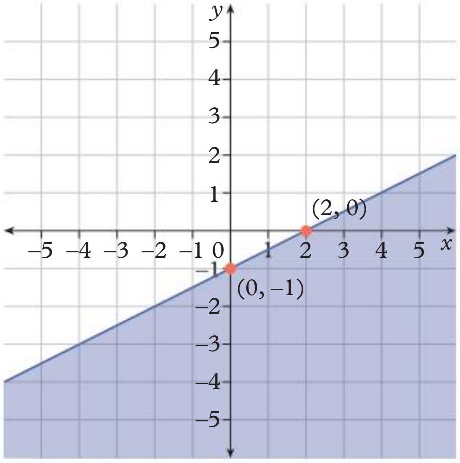
d $\frac{1 - x}{6} < \frac{-2y}{12}$ **e** $\frac{x - 3}{21} \geq \frac{9 - y}{7}$ **f** $\frac{y + 3}{5} > \frac{9 - x}{7}$

12 Write the inequalities represented by the following graphs.

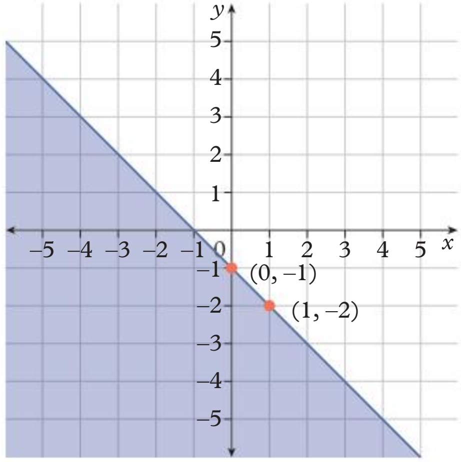
a



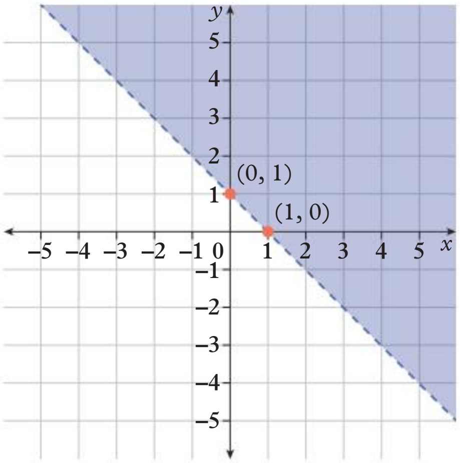
b



c



d



PROBLEM SOLVING AND REASONING

13 Write a linear inequality to represent each of the following situations.

a The breadth and length of a rectangle are b and l, and the perimeter of the rectangle is greater than 20 cm.

b Eric has $100 to spend on fertilisers and pesticides, each bag of fertiliser costs $22 and each bag of pesticide costs $6.

c The capacity of a water tank is 2145 L, each shower uses 55 L of water, each bath uses 80 L of water.

d Emily has decided to buy p pizzas for g guests. Each pizza is cut into 8 pieces. Emily wants to eat 4 pieces and ensure each guest can have at least 3 pieces.

e Mark is making plans for a triathlon. He plans to swim for s hours and bike for b hours. The fastest he can complete the running course is in 3 hours 45 minutes and he aims to finish the whole race in 12 hours.

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PROBLEM SOLVING AND REASONING

14 Ken runs a small business selling products A and B. He buys product A for \$10 and product B for \$20, respectively, and resells them later at a higher price.

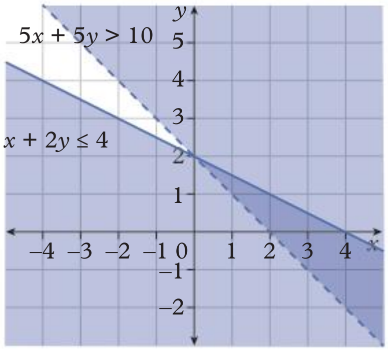
- **a** Write an inequality to represent the relationship between the numbers of product A, x, and product B, y, Ken can stock up on if he has \$400.
- **b** Ken prices product A and B at \$15 and \$25, respectively. Ken wants to make at least \$100 in profit. Ken wrote the inequality $15x + 25y \geq 400 + 100$ to represent the amount of money paid by customers versus his costs and desired profit. Explain why this is incorrect.
- **c** Write the correct inequality to represent Ken's profit.
- **d** What is Ken's profit if he only stocks up on product B and manages to sell all the stock?
- **e** Ken is confident that he can sell all the products in stock. How many of product A and product B should Ken buy if he wants to maximise his profit?

15 Ada is buying souvenirs for her friends in the airport. Each snow globe costs \$10 and weighs 1 kg and each hoodie costs \$20 and weighs 0.7 kg.

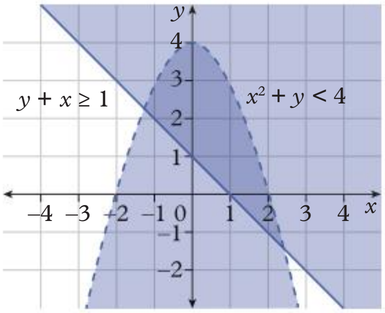
- **a** Write an inequality to represent the relationship between the number of snow globes x, and the number of hoodies, y, Ada could afford if she has \$50.
- **b** Graph the inequality on the Cartesian plane.
- **c** Explain why you only need to include the first quadrant in the graph for part **b**.
- **d** Use your graph to find out if Ada can buy 2 snow globes and 2 hoodies.
- **e** List all the combinations of snow globes and hoodies Ada could buy with \$50.
- **f** The weight limit of Ada's carry-on luggage is 7 kg. Her laptop weighs 3 kg. Write an inequality to represent the relationship between the number of snow globes, x, and the number of hoodies, y, if Ada wants to fit all the souvenirs and her laptop into the luggage.
- **g** Which combination(s) in part **e** satisfy the inequality in **f**?

CHALLENGE

16 When two or more inequalities are sketched on the same set of axes, the shaded areas overlap and form an intersecting region. Points in the intersecting region satisfy both inequalities. For example, the graph on the right shows $x + 2y \leq 4$ and $5x + 5y > 10$ on the same set of axes.



- **a** Sketch the following inequalities on the same set of axes.
  - **i** $y \geq 2x + 4$ and $y < -x$
  - **ii** $y + 2x \geq 4$ and $y + x < 1$
  - **iii** $2x - y \leq 2$ and $y < 3 - x$
  - **iv** $y + 2x < 2$ and $y + x < 2$
- **b** Use the graph on the right to find out if the given value pairs satisfy $x^2 + y < 4$ and $y + x \geq 1$
  - **i** $x = 1, y = 1$
  - **ii** $x = 2, y = 1$
  - **iii** $x = 2, y = -2$
  - **iv** $x = -1, y = 2$
  - **v** $x = 0, y = 4$
  - **vi** $x = -2, y = 3$



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Representing linear inequalities on the Cartesian plane

Interactive skillsheet
Solving linear inequalties

Worksheet
Graphing linear inequalities

Topic quiz
4B

4C Simultaneous linear equations

Learning intentions

By the end of this topic you will be able to …

- ✔ solve simultaneous linear equations graphically
- ✔ solve simultaneous linear equations using substitution
- ✔ solve simultaneous linear equations using elimination.

Inter-year links

Year 8	6G Solving linear equations using graphs
Year 9	4E Sketching linear graphs

Key content video

Simultaneous linear equations

- If linear equations contain the same variables, we call them **simultaneous linear equations**.
- The solutions to a pair of simultaneous linear equations are the values for the variables that are solutions to both equations.

  For example, to check whether $x = 2$ and $y = 7$ are solutions to $y = 5x - 3$ and $y = -x + 9$, substitute the value $x = 2$ into both equations.

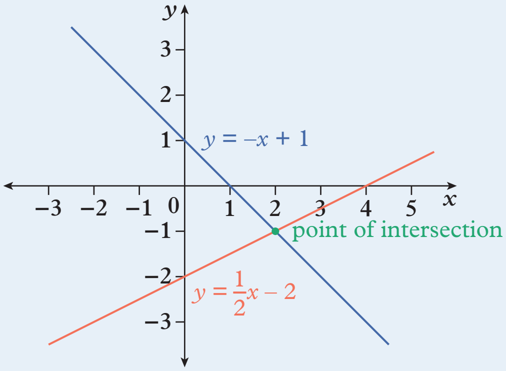
  $$\begin{aligned} y &= 5x - 3 & \qquad y &= -x + 9 \\ y &= 5(2) - 3 & y &= -1(2) + 9 \\ y &= 10 - 3 & y &= -2 + 9 \\ y &= 7 & y &= 7 \end{aligned}$$

  When $x = 2$, the result of both equations is $y = 7$, showing that the solution to the pair of simultaneous linear equations is $x = 2$ and $y = 7$.

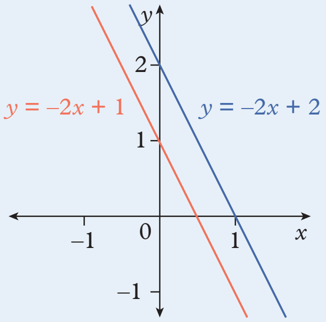
Solving simultaneous linear equations graphically

- If two linear graphs are not parallel, then they have exactly one **point of intersection**. The x and y value at the point of intersection is the solution to the simultaneous linear equations.

  For example, the pair of simultaneous linear equations $y = -x + 1$ and $y = \frac{1}{2}x - 2$ can be represented by the pair of linear graphs shown on the right.



- The point of intersection is on both linear graphs, so the x and y value at the point of intersection is a solution to both linear equations describing the pair of intersecting linear graphs.
- If the linear graphs are parallel, the simultaneous linear equations have no solution. The linear graphs do not intersect. For example, $y = -2x + 1$ and $y = -2x + 2$ are parallel as their gradients are both -2.



- If two linear graphs line up exactly with each other, they are **coincident lines**. For example, $x + 2y = 3$ and $2x + 4y = 6$ are coincident lines, because the second equation can be obtained by multiplying 2 on both sides of the first equation. If two linear graphs are coincident lines, there are infinitely many solutions to the simultaneous linear equations.

Solving simultaneous linear equations by substitution

- The **substitution method** involves substituting an expression from one equation into another equation.
- When at least one of the equations already has one variable as the subject, using the substitution method is the simplest.

  For example, when solving the simultaneous linear equations $y = 4 - 2x$ and $y + 11 = 3x$, substituting y from the first equation into the second equation will give us a value for x.

  Substitute $y = 4 - 2x$ into $y + 11 = 3x$:

$$\begin{aligned} (4 - 2x) + 11 &= 3x \\ 4 - 2x + 11 &= 3x \qquad (+\ 2x) \\ 4 + 11 &= 5x \\ 15 &= 5x \qquad (\div\ 5) \\ x &= 3 \end{aligned}$$

  Substituting the value $x = 3$ into either equation gives the result $y = -2$:

$$\begin{aligned} y &= 4 - 2x \\ y &= 4 - 2(3) \\ y &= 4 - 6 \\ y &= -2 \end{aligned} \qquad\qquad \begin{aligned} y + 11 &= 3x \\ y + 11 &= 3(3) \\ y + 11 &= 9 \qquad (-\ 11) \\ y &= -2 \end{aligned}$$

  So, the solution to the pair of simultaneous equations is $x = 3$ and $y = -2$.

Solving simultaneous linear equations by elimination

- The LHS and RHS of two equations can be added or subtracted from each other to create a new equation.

  For example, the LHS and RHS of $6 = 1 + 5$ and $3 + 4 = 7$ can be added to get $6 + (3 + 4) = (1 + 5) + 7$.
- The **elimination method** uses this idea to ‘add’ or ‘subtract’ simultaneous linear equations to create a new equation in which one of the variables is eliminated.

  For example, the pair of simultaneous linear equations $x + y = 4$ and $-3x + y = -8$ can be solved by elimination when the second equation is ‘subtracted’ from the first.

$$x + y = 4 \qquad ①$$
$$-3x + y = -8 \qquad ②$$

① – ②:
$$\begin{aligned} (x + y) - (-3x + y) &= (4) - (-8) \\ x + 3x + y - y &= 4 + 8 \\ 4x &= 12 \qquad (\div\ 4) \\ x &= 3 \end{aligned}$$

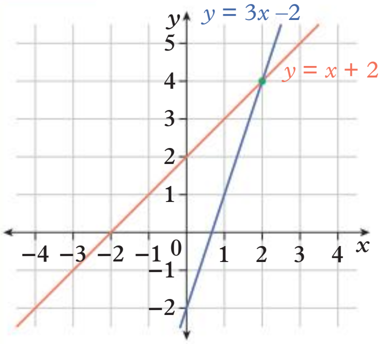
  Substituting this value for x into either equation gives the result $y = 1$. The pair of simultaneous equations has been solved.
- One or both of the equations might need to be multiplied by a constant so that the coefficient of one of the variables becomes zero when the equations are added/subtracted.

ADVANCED

Example 4C.1 Solving simultaneous linear equations graphically

Solve this pair of simultaneous linear equations using a graph.

$y = x + 2 \qquad y = 3x - 2$

THINK	**WRITE**
1 Sketch the graphs of $y = x + 2$ and $y = 3x - 2$ on the same Cartesian plane, using graph paper. Use a ruler to extend both lines until they intersect.	
2 Use the coordinates of the point of intersection to determine a solution to the simultaneous linear equations.	The point of intersection is at (2, 4), so the solution to the simultaneous linear equations is $x = 2, y = 4$.
3 Check the solution by substituting the value for x into both equations. If both equations result in the same y value, the solution is correct.	Check: $y = x + 2$ $y = 3x - 2$ $y = (2) + 2$ $y = 3(2) - 2$ $y = 4$ $y = 6 - 2$ $y = 4$

Example 4C.2 Solving simultaneous linear equations by substitution

Solve this pair of simultaneous linear equations using the substitution method.

$5x + 2y = 2 \qquad y = 7x + 20$

THINK	**WRITE**
1 Number each equation for reference.	$5x + 2y = 2$ ① $y = 7x + 20$ ②
2 Equation ② has one variable (y) as its subject. Substitute the expression for y from equation ② into equation ①. Use brackets when substituting the expression.	Substituting $y = 7x + 20$ into ①: $5x + 2y = 2$ $5x + 2(7x + 20) = 2$
3 Solve for x.	$5x + 14x + 40 = 2$ $19x + 40 = 2$ $(- 40)$ $19x = -38$ $(\div 19)$ $x = -2$
4 Substitute the solution for x into either equation and solve it for y.	Substitute $x = -2$ into ②: $y = 7x + 20$ $y = 7(-2) + 20$ $y = 6$ The solution is $x = -2, y = 6$.

ADVANCED

Example 4C.3 Solving simultaneous linear equations by elimination

Solve this pair of simultaneous linear equations using the elimination method.

$y - 3x = -10 \qquad 3x + 2y = 16$

THINK	WRITE
1 Number each equation for reference.	$y - 3x = -10$ ① $3x + 2y = 16$ ②
2 The coefficient for *x* in equation ① is the negative value of the coefficient for *x* in equation ②. So, the equations can be added to remove *x*.	① + ②: $(y - 3x) + (3x + 2y) = (-10) + (16)$
3 Solve the resulting equation for *y*.	$y - 3x + 3x + 2y = -10 + 16$ $3y = 6$ (÷ 3) $y = 2$
4 Substitute the solution for *y* into either equation to solve it for *x*.	Substitute $y = 2$ into ①: $y - 3x = -10$ $(2) - 3x = -10$ (− 2) $-3x = -12$ (÷ −3) $x = 4$ The solution is $x = 4$, $y = 2$.

Example 4C.4 Solving simultaneous linear equations by elimination (multiplying by a constant)

Solve each of these pairs of simultaneous linear equations using the elimination method.

a $y - x = 11$
$3y - 2x = 27$

b $4x - 3y = -5$
$2y - 5x = 1$

THINK	WRITE
a 1 Number each equation for reference.	**a** $y - x = 11$ ① $3y - 2x = 27$ ②
2 The coefficient of *x* in equation ① is −1 and the coefficient of *x* in equation ② is −2. Equation ② can stay the same, multiply equation ① by 2.	2 × ①: $y - x = 11$ (× 2) $2y - 2x = 22$
3 Equation '2 × ①' and equation ② have the same coefficients of *x*, subtract the equations to remove *x*.	(2 × ①) − ②: $(2y - 2x) - (3y - 2x) = (22) - (27)$ $2y - 2x - 3y + 2x = 22 - 27$
4 Solve the equation for *y*.	$-y = -5$ $y = 5$

5 Substitute the solution for y into either equation and solve it for x.	Substituting $y = 5$ into ①: $y - x = 11$ $(5) - x = 11 \quad (-5)$ $-x = 6$ $x = -6$ The solution is $x = -6$, $y = 5$.
b **1** Number each equation for reference.	**b** $4x - 3y = -5$ ① $2y - 5x = 1$ ②
2 The coefficient of y in equation ① is -3 and the coefficient of y in equation ② is 2. Multiply equation ① by 2 and equation ② by 3.	2 × ①: $4x - 3y = -5 \quad (\times 2)$ $8x - 6y = -10$ 3 × ②: $2y - 5x = 1 \quad (\times 3)$ $6y - 15x = 3$
3 The coefficients of y in equation '2 × ①' and equation '3 × ②' can be added to remove y.	(2 × ①) + (3 × ②): $(8x - 6y) + (6y - 15x) = (-10) + (3)$ $8x - 6y + 6y - 15x = -10 + 3$
4 Solve for x.	$-7x = -7$ $x = 1$
5 Substitute the solution for x into either equation to solve it for y.	Substituting $x = 1$ into ①: $4x - 3y = -5$ $4(1) - 3y = -5$ $4 - 3y = -5 \quad (-4)$ $-3y = -9 \quad (\div 3)$ $y = 3$ The solution is $x = 1$, $y = 3$.

Helpful hints

- ✔ Always number your equations for easy reference.
- ✔ While parallel lines have no intersection, that does not mean that only perpendicular lines do. Any pair of non-parallel lines on the Cartesian plane will have an intersection.
- ✔ Remember, you can always check your solution to a pair of simultaneous equations by substituting one of the values back into each equation to see if the result matches your solution for the other variable.
- ✔ When solving simultaneous linear equations graphically, make sure to choose a suitable size for the units of your x- and y-axes and use graph paper so that you can identify the intersection point with accuracy.
- ✔ When solving simultaneous linear equations using the substitution method, always make sure you substitute algebraic expressions enclosed in brackets, then remove the brackets by expanding in the next step.
- ✔ Remember, you can't eliminate a variable by adding or subtracting equations unless the magnitude of the coefficients are the same.

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Exercise 4C Simultaneous linear equations

△ 1, 2, 3–5(1st, 2nd columns), 6–8, 9(1st, 2nd columns), 11(a, b, g, h), 13

□ 2, 3–5(3rd, 4th columns), 8, 9 (2nd, 3rd columns), 10, 11(e, f, h, i, o), 12, 14, 15, 18, 19

⬠ 3(k, l, o, p), 4–5(3rd, 4th columns), 9(3rd column), 11, 12, 16, 17, 19, 20

1 Check if the given x and y values are the solution to each pair of simultaneous linear equations.

a $x = 1, y = 5$ Equations: $y = x + 4$ and $y = -2x + 7$

b $x = -2, y = 1$ Equations: $y = x + 3$ and $y = -3x - 5$

c $x = 5, y = -2$ Equations: $y = x - 9$ and $y = -2x + 7$

d $x = 3, y = -7$ Equations: $2x + y = -1$ and $y = -x - 4$

e $x = 4, y = 3$ Equations: $y = 2x - 5$ and $y = -x + 1$

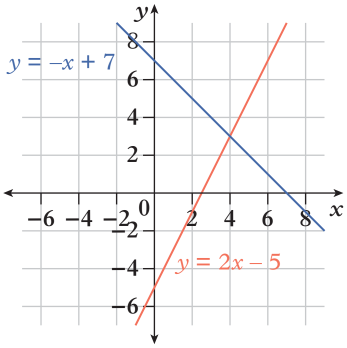
f $x = -6, y = 0$ Equations: $y = x - 6$ and $x + y = -6$

2 For each of the following pairs of simultaneous linear equations and their graphs:

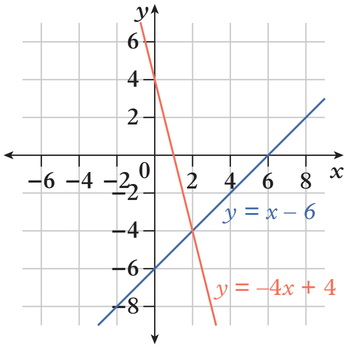
i write the coordinates of the point of intersection of the two lines

ii use the coordinates of the point of intersection to help you find the solution to the simultaneous linear equations. Use substitution to check your solution.

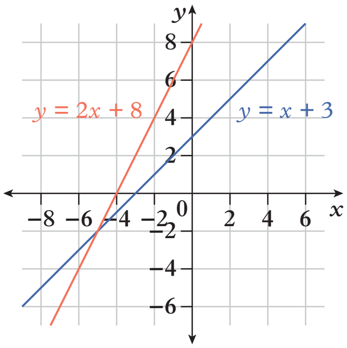
a $y = 2x - 5$
$y = -x + 7$



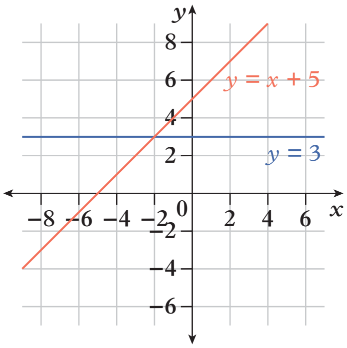
b $y = -4x + 4$
$y = x - 6$



c $y = 2x + 8$
$y = x + 3$



d $y = x + 5$
$y = 3$



4C.1 **3** Use graphs to help you solve these pairs of simultaneous linear equations (**a–p**). Use substitution to check your solutions.

a $y = 2x - 6$
$y = x - 1$

b $y = 3x$
$y = x + 4$

c $y = x - 2$
$y = 10 - x$

d $y = x + 7$
$y = -3x - 1$

e $y = 3x + 6$
$2x + y = 1$

f $y = -2x - 5$
$y = 2x + 7$

g $y = 2x - 4$
$y = x - 1$

h $2x + y = 6$
$y = x - 6$

i $y = 3x + 3$
$y = -x - 5$

j $2x + y = -4$
$x + y = 1$

k $x - y = 4$
$4x + y = 6$

l $3y = x + 12$
$y = x + 2$

m $y = \frac{7}{2}x + 13$
$-2x + 3y = 5$

n $y = -\frac{5}{4}x - \frac{11}{4}$
$3x - 2y = 11$

o $y = \frac{8}{5}x + \frac{23}{5}$
$9x + 16y = -134$

p $y = \frac{5}{7}x + \frac{19}{7}$
$7x + 4y = 1$

UNDERSTANDING AND FLUENCY

UNDERSTANDING AND FLUENCY

4C.2 **4** Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $x + y = 9$, $y = x + 5$

b $x + 2y = 6$, $y = x - 3$

c $2x + y = 1$, $y = x + 4$

d $y = x - 1$, $x - 3y = -9$

e $3x - y = -10$, $y = x + 2$

f $y = 6x$, $5x - y = -2$

g $6x + 5y = 21$, $y = 2x + 1$

h $y = 2 - 3x$, $4x + y = -1$

i $2x + 3y = -5$, $y = 5 - 2x$

j $x = y + 2$, $2x - 3y = 1$

k $x = 4y$, $y = 3x + 11$

l $x = 5$, $4x - 3y = 2$

5 Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $y = 2x + 6$, $y = x + 7$

b $y = x - 2$, $y = 4x - 20$

c $y = 7x + 16$, $y = 2x + 1$

d $y = x - 6$, $y = 4 - x$

e $y = 5x + 9$, $y = 3 - x$

f $y = 1 - 2x$, $y = x - 11$

g $y = 3x - 5$, $y = 5 - 2x$

h $y = 4x + 14$, $y = -4x - 2$

6 Solve each of these pairs of simultaneous linear equations by adding the equations together to remove one variable.

a $2x + y = 9$, $3x - y = 1$

b $x + 2y = -1$, $-x + 5y = 8$

c $x - 2y = 11$, $5x + 2y = -5$

7 Solve each of these pairs of simultaneous linear equations by subtracting one equation from the other to remove one variable.

a $x + 4y = 17$, $x + 2y = 11$

b $3x + y = 3$, $x + y = 7$

c $4x - 3y = 33$, $4x + y = 5$

4C.3 **8** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $-3x + y = 10$, $3x - 4y = -4$

b $x - 2y = 2$, $3x - 2y = 14$

c $6x - y = -8$, $-6x + 5y = 16$

d $6x + 5y = 19$, $4x + 5y = 21$

e $4x + 3y = 29$, $2x - 3y = 1$

f $-5x + y = 11$, $-5x - 2y = 23$

4C.4 **9** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $5x + 3y = 21$, $2x - y = 4$

b $x + 4y = -2$, $4x - 3y = 11$

c $3x + y = -5$, $7x + 2y = -14$

d $6x - 7y = 9$, $-2x + 5y = 5$

e $4x + 3y = -24$, $5x - 12y = 33$

f $2x + 5y = 26$, $10x - 3y = -38$

g $4x + 3y = 16$, $3x - 2y = -5$

h $3x + 7y = -2$, $5x + 11y = -4$

i $-8x + 5y = -21$, $5x + 2y = 8$

j $3x - 10y = -32$, $-7x + 3y = 34$

k $-9x + 2y = -35$, $8x - 9y = -5$

l $-4x - 5y = 22$, $-3x + 7y = -5$

10 Consider the three methods you have used for solving simultaneous linear equations:

- **i** graphical
- **ii** substitution
- **iii** elimination.

a For each method, describe the situation for which its use is most appropriate.

b For each method, describe the situation for which its use is least appropriate.

c What could you do to make the least appropriate situation for each method more appropriate for its use?

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PROBLEM SOLVING AND REASONING

11 Use the most appropriate method to solve each of these pairs of simultaneous equations. Hint: Not all solutions are integers.

a $y = 3x - 6$
$4x + 11y = 8$

b $y = 6x - 5$
$y = 3x - 4$

c $4x - 5y = 4$
$2x + 5y = 5$

d $y = -7$
$y = 4x + 29$

e $x = 5y - 6$
$x - y = 2$

f $2y = x - 4$
$3x - 2y = 0$

g $2x - y = -4$
$4x + y = 22$

h $x + 5y = -5$
$x - 2y = 9$

i $-x - 7y = 5$
$-2x - 7y = 3$

j $-3x + 4y = 14$
$3x - y = -8$

k $x + y = 11$
$x - y = 3$

l $5x + 2y = -18$
$-5x + 2y = 22$

m $y = 2.5x + 4.3$
$y = 7.5x + 8.3$

n $4x = y - 6$
$8x + 3y = 13$

o $6x - 7y = -119$
$8x - 5y = 19$

12 The perimeter around the top of a rectangular billiard table is 810 cm.

a Use l and b to write a linear equation for the perimeter of the table.

b Use l and b to write a linear equation describing the relationship between the length and breadth of the top of the table if the length is twice the breadth.

c Solve the simultaneous linear equations you wrote for parts **a** and **b** and find the dimensions of the top of the billiard table.

13 Sophie has decided to make hand-painted cards. She invests \$60 in a set of paintbrushes and enough paint to make 40 cards. The cost of each plain white card is \$2 and she plans to sell the painted cards for \$6 each.

a Explain how the amount of money, a, to produce n cards (where $0 \le n \le 40$) can be written as $a = 2n + 60$.

b Explain how the amount of money, a, received for selling n cards can be written as $a = 6n$.

c Draw graphs of the two equations on the same Cartesian plane. Label the horizontal axis as n and the vertical axis as a.

d What are the coordinates of the point of intersection of the two graphs? What does this tell you?

e How many cards should Sophie produce and then sell to break even (recover her costs but nothing extra)?

f How many cards should Sophie produce and sell to make a profit?

14 Explain why there is no solution to $2x + 5 = 2x - 1$ by graphing $y = 2x + 5$ and $2x - 1 = y$ on the same Cartesian plane.

15 Explain why the pair of simultaneous linear equations $y = 4 - 3x$ and $6x + 2y = 8$ has an infinite number of solutions.

16 Consider this pair of simultaneous equations:

$7x - 6y = -14$ ①
$5x - 3y = -1$ ②

a Demonstrate that the same solution is obtained using each of the strategies described below.

Strategy A: Multiply equation ② by 2 to form equation ③ and then subtract equation ③ from equation ①.
Strategy B: Multiply equation ② by −2 to form equation ④ and then add equations ① and ④.
Strategy C: Multiply equation ① by 5 to form equation ⑤ and multiply equation ② by 7 to form equation ⑥. Subtract equation ⑥ from equation ⑤.

b Are there other strategies you could use? Explain.

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PROBLEM SOLVING AND REASONING

17 Chloe is planning a party and obtains prices from two catering companies. Each caterer has a fixed price for delivery and set-up, and a cost per person for the food.

Angie's Catering: fixed price \$200, \$28 per person

Cool Food Club: fixed price \$100, \$32 per person

Chloe is unsure how many people she will invite.

a Write a linear equation to represent the cost of hiring Angie's Catering. Remember to define the two variables you are using.

b Write a linear equation to represent the cost of hiring Cool Food Club.

c Show the graphs of both linear equations on the same Cartesian plane.

d Which company is cheaper if catering for 18 people? Explain how you can see this from your graph.

e Which company is cheaper if catering for 36 people?

f How many people can be catered for so that the cost of using either company is the same? What is this cost?

g Write a summary advising Chloe on what her options are.

CHALLENGE

18 A linear graph has a y-intercept of –5. Another linear graph has a gradient of –3. The two linear graphs intersect at the point (4, –6).

a Show the two graphs on the same Cartesian plane.

b Find the equation for each linear graph.

c Write the solution to the simultaneous linear equations.

d Use substitution to check that the solution satisfies both equations.

19 a Show that $x = 15$, $y = -5$ and $z = 5$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

b Show that $x = 22$, $y = -12$ and $z = -2$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

c Explain why we cannot solve the simultaneous equations $x + y = 10$ and $x + z = 20$ for one set of values of x, y and z.

20 a Show that the solution to the simultaneous equations $ax + by = c$ and $ax + qy = r$ is $x = \dfrac{br - cq}{ab - aq}$ and $y = \dfrac{c - r}{b - q}$.

b Solve the simultaneous equations $3x + 4y = 9$ and $3x + 6y = 7$ by substituting the values of a, b, c, q and r into the solution from part **a**.

c Show that the solution to the simultaneous equations $ax + by = c$ and $px + qy = r$ is $x = \dfrac{br - cq}{bp - aq}$ and $y = \dfrac{cp - ar}{bp - aq}$.

d Solve the simultaneous equations $3x + 4y = 9$ and $5x + 6y = 7$ by substituting the values of a, b, c, p, q and r into the solution from part **c**.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving simultaneous linear equations graphically

Interactive skillsheet
Solving simultaneous linear equations by substitution

Interactive skillsheet
Solving simultaneous linear equations by elimination

Worksheet
Solving simultaneous linear equations

Investigation
Drawing a circumference around a triangle

Topic quiz
4C

ANS p673

Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

4A **1** Solve the following equations.

a $4(3x - 7) - 5x = -30$ **b** $3(7 - 2x) - 5(x + 1) = -1$

c $\dfrac{-18}{x + 5} = 9$ **d** $\dfrac{14}{8x - 9} = 2$

4A **2** Solve the following equations.

a $6x - 10 = 5x + 9$ **b** $5(3x - 2) = x$

c $9(4 - 7x) = 3(9x + 11)$ **d** $3x = -3(x + 2)$

4A **3** Solve the following equations.

a $\dfrac{-2x - 3}{4} + \dfrac{2x + 7}{3} = 1$ **b** $\dfrac{3(-3x + 1)}{4} - \dfrac{5(x - 2)}{6} = \dfrac{1}{2}$

c $\dfrac{3x - 1}{4} - \dfrac{x}{6} = \dfrac{1 - x}{2}$ **d** $\dfrac{3(x - 1)}{2} - \dfrac{2(1 - 2x)}{3} = \dfrac{1 - 2x}{6}$

4A **4** Rearrange each of the following equations to make x the subject.

a $ax + by = c$ **b** $\dfrac{f}{dx - e} = g$

c $j(x + k) = m(x - n)$ **d** $\dfrac{px - q}{s - tx} = r$

4B **5** Solve each of the following inequalities.

a $2x + 5 \leq -3$ **b** $2x - 3x \leq -3 + x$

c $3 > \dfrac{1 - x}{7}$ **d** $\dfrac{2x + 1}{5} \geq \dfrac{2 - 3x}{7}$

4B **6** Determine if the following point is a solution to $2x + 5y \leq 3$.

a $(0, 0)$ **b** $(1, 1)$

c $(-1, 1)$ **d** $(-1, -1)$

4B **7** Represent the following linear inequalities on the Cartesian plane.

a $2x + y < 3$ **b** $\dfrac{x}{2} - 1 \leq y$

4C **8** Solve the following pairs of simultaneous linear equations using graphs.

a $y = x + 1$
$y + 2x = 4$

b $y = 6 - 2x$
$4y + 3x = 4$

4C **9** Solve the following pairs of simultaneous linear equations by substitution.

a $y = 2x + 3$
$y + 2 = 3x$

b $2y + 3x = 11$
$y - 3 = x$

4C **10** Solve the following pairs of simultaneous linear equations by elimination.

a $2y + 3x = 1$
$-2x + 2y = 1$

b $y + 3x = 10$
$2y + x = 15$

ADVANCED

4D Solving quadratic equations

Learning intentions

By the end of this topic you will be able to ...

- ✔ solve quadratic equations by factorisation
- ✔ solve quadratic equations by completing the square
- ✔ verify the solution to a quadratic equation by substitution.

Inter-year links

Year 7	6C Terms, expressions and equations
Year 8	6A Equations
Year 9	5A Solving quadratic equations

Key content video

Quadratic equations

- The general form of a **quadratic equation** is: $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- Non-monic quadratic equations in the form $ax^2 + bx + c = 0$ can be turned into a monic quadratic equation by dividing both sides of the equation by the coefficient of the leading term a.

The Null Factor Law

- The **Null Factor Law** states that, if the product of factors is 0, one or more of the factors must be 0; that is:

  if $p \times q = 0$, then $p = 0$ or $q = 0$, or both $p = 0$ and $q = 0$.
- Some quadratic equations can be solved by making one side equal to zero and factorising the non-zero expression on the other side.
- When we apply the Null Factor Law to quadratic equations in factorised form, $(x - p)(x - q) = 0$, we get:

  $x - p = 0$ or $x - q = 0$ then $x = p$ or $x = q$.
- Some monic quadratic equation in the form $x^2 + bx + c = 0$ can be factorised to $(x - p)(x - q) = 0$ where $b = -p - q$ and $c = p \times q$.
- Special products $(a \pm b)^2 = a^2 \pm 2ab + b^2$ and $(a + b)(a - b) = a^2 - b^2$ can also be used to factorise quadratic equations.

Solving quadratic equations by completing the square

- For any quadratic equation of the form $ax^2 + bx + c = 0$, the same operations can be performed on both sides of the equation to create a perfect square on one side of the equation.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \end{aligned}$$

- If the RHS of the equation is zero or positive after completing the square, you can take the square root of both sides of the equation to get the value of $x + \frac{b}{2a}$, then use inverse operations to find the value of x.
- A positive number has two square roots, so there may be two values for $x + \frac{b}{2a}$.
- Solutions can be written in a shorthand way using a ± ('plus or minus') sign.

  For example:

  $x = -2$ or $x = 2$ can be written as $x = \pm 2$.

ADVANCED

Example 4D.1 Solving quadratic equations by factorising

Solve each of these equations.

a $(x + 5)(x - 4) = 0$ **b** $x^2 - 4x - 12 = 0$ **c** $x^2 - 16 = 0$ **d** $5x^2 = -35x$

THINK	WRITE
a 1 Check that the quadratic equation is in factorised form and one side of the equation is equal to 0.	**a** $(x + 5)(x - 4) = 0$
2 Apply the Null Factor Law by equating each factor to 0.	$x + 5 = 0$ or $x - 4 = 0$
3 Solve each linear equation.	$x = -5$ or $x = 4$
b 1 Check that one side of the quadratic equation is equal to 0.	**b** $x^2 - 4x - 12 = 0$
2 Factorise the quadratic equation.	$(x - 6)(x + 2) = 0$
3 Apply the Null Factor Law.	$x - 6 = 0$ or $x + 2 = 0$
4 Solve each linear equation.	$x = 6$ or $x = -2$
c 1 Check that one side of the equation is equal to 0.	**c** $x^2 - 16 = 0$
2 Factorise the quadratic expression using the 'difference of two squares' rule.	$x^2 - 4^2 = 0$ $(x + 4)(x - 4) = 0$
3 Apply the Null Factor Law.	$x + 4 = 0$ or $x - 4 = 0$
4 Solve each linear equation. Use ± to write the solutions.	$x = -4$ or $x = 4$ $x = \pm 4$
d 1 Rearrange the equation so one side of the equation is equal to 0.	**d** $5x^2 = -35x$ $5x^2 + 35x = 0$
2 Factorise the quadratic expression by taking out a common factor.	$5x(x + 7) = 0$
3 Apply the Null Factor Law.	$5x = 0$ or $x + 7 = 0$
4 Solve each linear equation.	$x = 0$ or $x = -7$

Example 4D.2 Solving quadratic equations by completing the square

Solve each of these equations. Write the solutions as exact values.

a $4(x - 2)^2 - 1 = 0$ **b** $x^2 - 12x - 3 = 0$ **c** $-2x^2 + 8x + 8 = 0$

THINK	WRITE
a 1 Use inverse operations to rearrange the equation, leave the LHS as a perfect square.	**a** $4(x - 2)^2 - 1 = 0 \quad (+\ 1)$ $4(x - 2)^2 = 1 \quad (\div\ 4)$ $(x - 2)^2 = \frac{1}{4}$
2 Take the square root of both sides of the equation, use ± to show the two roots.	$x - 2 = \pm\frac{1}{2} \quad (+\ 2)$
3 Use inverse operations on both sides of the equation to solve for x.	$x = 2 \pm \frac{1}{2}$

b 1 Use inverse operations to move the constant term to the RHS of the equation.	**b** $x^2 - 12x - 3 = 0$ (+ 3)
2 Make the LHS of the equation a perfect square by adding $\left(\frac{-12}{2}\right)^2 = 36$ to both sides of the equation.	$x^2 - 12x = 3$ (+ 36)
3 Write the LHS as a perfect square, simplify the RHS.	$x^2 - 12x + 36 = 3 + 36$
4 Take the square root of both sides of the equation, use ± to show the two roots.	$(x - 6)^2 = 39$
5 Use inverse operations on both sides of the equation to solve for x.	$x - 6 = \pm\sqrt{39}$ (+ 6) $x = 6 \pm \sqrt{39}$
c 1 Divide all terms by the coefficient of the x^2 term to turn the quadratic equation into a monic quadratic equation.	**c** $-2x^2 + 8x + 8 = 0$ (÷ −2)
2 Use inverse operations to move the constant term to the RHS of the equation.	$x^2 - 4x - 4 = 0$ (+ 4)
3 Make the LHS of the equation a perfect square by adding $\left(\frac{-4}{2}\right)^2 = 4$ to both sides of the equation.	$x^2 - 4x = 4$ (+ 4) $x^2 - 4x + 4 = 4 + 4$
4 Write the LHS as a perfect square, simplify the RHS.	$(x - 2)^2 = 8$
5 Take the square root of both sides of the equation, use ± to show the two roots. Simplify the surd.	$x - 2 = \pm\sqrt{8}$
6 Use inverse operations on both sides of the equation to solve for x.	$x - 2 = \pm 2\sqrt{2}$ (+ 2) $x = 2 \pm 2\sqrt{2}$

Example 4D.3 Verifying the solution to a quadratic equation by substitution

Verify if the following values for x are solutions to $x^2 + 4x - 12 = 0$.

a $x = -2$ **b** $x = 2$

THINK	WRITE
1 Substitute the given values into the LHS and the RHS of the equation.	**a** LHS $= x^2 + 4x - 12$ $= (-2)^2 + 4(-2) - 12$
2 Calculate the value of the LHS and RHS.	$= 4 - 8 - 12$ $= -16$
3 Compare the value of LHS and RHS. If LHS ≠ RHS, the value is not a solution to the equation.	RHS = 0 LHS ≠ RHS, −2 is not a solution to $x^2 + 4x - 12 = 0$
	b LHS $= x^2 + 4x - 12$ $= (2)^2 + 4(2) - 12$ $= 4 + 8 - 12$ $= 0$ RHS = 0 LHS = RHS, 2 is a solution to $x^2 + 4x - 12 = 0$

Helpful hints

- ✔ Watch for changes in sign when solving quadratic equations for x.

$$(x + 6)(x - 2) = 0$$
$$x + 6 = 0 \text{ or } x - 2 = 0$$
$$x = -6 \text{ or } x = +2$$

- ✔ Although it is possible to add, subtract and multiply by x, you must not divide both sides of an equation by x, in the case of $x = 0$.

$$\frac{1}{0} = \text{undefined}$$

- ✔ The Null Factor Law can only be used to solve equations if one side of the equation is equal to zero. You cannot use the rule if the equation is equal to any other number.
  For example, if $x(x - 2) = 1$, it does not imply that $x = 1$ or $x - 2 = 1$.
  But, if $x(x - 2) = 0$, two possible solutions are $x = 0$ or $x - 2 = 0$.
- ✔ If a value is verified to be a solution to a quadratic equation, it does not mean it is the only solution to the quadratic equation. In Example 4D.3, $x = 2$ is verified as a solution to the given quadratic equation, but $x = -6$ is also a solution to the given equation.

ANS p673

Exercise 4D Solving quadratic equations

▲ 1–3(1st, 2nd column), 4, 5–11 (1st, 2nd column), 12–14, 16, 19

■ 1–3(2st column), 5–11(2nd column), 15–17, 19, 20

⬟ 1–3(3rd column), 5–11(3rd column), 16–21

UNDERSTANDING AND FLUENCY

4D.1 **1** Solve each of the following equations. Show all steps of your working.

a $(x + 2)(x - 4) = 0$ **b** $(x + 7)(x + 1) = 0$ **c** $(x - 3)(x - 6) = 0$

d $(x + 9)(x - 9) = 0$ **e** $x(x - 5) = 0$ **f** $x(2x - 1) = 0$

g $2x(x + 1) = 0$ **h** $7x(2x + 4) = 0$ **i** $\frac{1}{2}x(4x - 3) = 0$

j $(x + \sqrt{2})(x - \sqrt{2}) = 0$ **k** $(x - \sqrt{5})(x + \sqrt{5}) = 0$ **l** $(x + 4\sqrt{3})(x - 4\sqrt{3}) = 0$

m $(2x + 3)(x - 4) = 0$ **n** $(3x - 1)(3x + 1) = 0$ **o** $(7x + 2)(4x - 5) = 0$

2 Solve each of these equations.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$ **c** $x^2 - 9x + 8 = 0$

d $x^2 - 25 = 0$ **e** $x^2 - 81 = 0$ **f** $4 - x^2 = 0$

g $x^2 - 3x = 0$ **h** $x^2 + x = 0$ **i** $2x^2 + 8x = 0$

j $10x^2 - 4x = 0$ **k** $3x^2 = 18x$ **l** $7x^2 = -70x$

3 Solve each of these quadratic equations, leave your answer as exact values. Hint: First rearrange the equation into the form $ax^2 + bx + c = 0$.

a $x^2 + 4x = 12$ **b** $x^2 + 4 = 5x$ **c** $6x^2 = 54$

d $x^2 - 3x - 20 = 20$ **e** $x^2 - 6x = 3x$ **f** $x^2 + 1 = 4x$

g $5x - x^2 = x$ **h** $x^2 + 21 = 11x + 3$ **i** $2x^2 - 3x = x^2 + 18$

j $x(x + 4) = 21$ **k** $(x - 5)^2 = 1$ **l** $x^2 + 3x + 10 = 1 - 3x$

UNDERSTANDING AND FLUENCY

4 Consider $3(x + 6)(x - 8) = 0$.

a Why is this equivalent to $(x + 6)(x - 8) = 0$?

b Solve $(x + 6)(x - 8) = 0$. Then write the solutions for $3(x + 6)(x - 8) = 0$.

c Use substitution to check that you have the correct solutions.

5 Solve each of these quadratic equations.

a $2(x + 5)(x - 6) = 0$ **b** $7(x - 2)(x - 4) = 0$ **c** $-5(x + 8)(x + 1) = 0$

d $-4(x + 9)(x - 3) = 0$ **e** $-11(x - 4)(x + 4) = 0$ **f** $-x(x + 7) = 0$

g $-6x(x - 2) = 0$ **h** $-2x(3x + 5) = 0$ **i** $-8(2x - 3)(x + 10) = 0$

6 Solve each of these quadratic equations. Hint: Take out a common factor first.

a $2x^2 + 6x + 4 = 0$ **b** $3x^2 + 6x - 45 = 0$ **c** $6x^2 - 30x + 24 = 0$

d $-5x^2 - 5x + 30 = 0$ **e** $-4x^2 + 16x + 48 = 0$ **f** $-x^2 - x + 56 = 0$

g $11x^2 - 44 = 0$ **h** $-7x^2 + 63 = 0$ **i** $-16x^2 + 8x = 0$

7 Solve each of these equations. Write the solutions as exact values.

a $4(x - 2)^2 = 0$ **b** $3(x - 3)^2 - 9 = 0$ **c** $(2 - 2x)^2 - 7 = 9$

d $x^2 - 12x = 4$ **e** $x^2 - 2x - 1 = 0$ **f** $-x^2 + 12x - 3 = 0$

4D.2 **8** Solve each of these quadratic equations by completing the square.

a $x^2 + 2x - 6 = 0$ **b** $x^2 - 4x + 2 = 0$ **c** $x^2 + 8x + 11 = 0$

d $x^2 - 12x + 28 = 0$ **e** $x^2 + 20x + 25 = 0$ **f** $x^2 - x - 1 = 0$

g $3x^2 + 18x + 21 = 0$ **h** $4x^2 + 8x - 16 = 0$ **i** $-x^2 + 4x - 1 = 0$

j $-6x^2 - 48x - 36 = 0$ **k** $2x^2 - 12x - 3 = 0$ **l** $5x^2 + 5x - 2 = 0$

9 Solve each of these quadratic equations.

a $2x^2 + 13x + 15 = 0$ **b** $5x^2 - 22x + 8 = 0$ **c** $6x^2 + x - 2 = 0$

d $4x^2 + 25x - 21 = 0$ **e** $3x^2 + 2x = 1$ **f** $7x^2 = 11x + 6$

4D.3 **10** Verify if the following values for x are solutions to the given quadratic equations by comparing the value of the LHS and RHS.

i $x = 3$ **ii** $x = 5$

a $(x - 8)(x - 3) = 0$ **b** $x^2 - 8x + 15 = 0$

c $x^2 - 25 = 0$ **d** $x^2 - 8x = 15$

11 Verify if the following values for x are solutions to the given quadratic equations by comparing the value of the LHS and RHS.

i $x = -5$ **ii** $x = 5$ **iii** $x = \sqrt{5}$

a $x^2 + 2x - 12 = 3$ **b** $x^2 + x + 15 = 9x$

c $2x^2 + x - 2 = 3x^2 + x - 7$ **d** $2x^2 + 35 = -3x^2 + 40x - 40$

12 Simplify **a–d** in question **11** to quadratic equations in the form $ax^2 + bx + c = 0$.

13 Substitute the x values from question **11** to verify if they are solutions to the simplified quadratic equations in question **12**.

14 Explain why you can divide both sides of $3x(x - 2) = 0$ by 3 but you cannot divide both sides of $3x(x - 2) = 0$ by $3x$.

15 How many real solutions does each of these quadratic equations have?

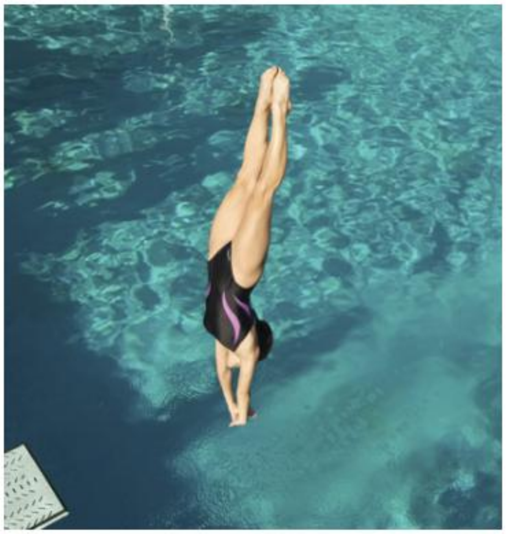
a $(x - 3)(x - 5) = 0$ **b** $(x + 2)(x + 2) = 0$ **c** $(x - 4)(x + 4) = 0$

d $(x - 7)^2 = 0$ **e** $x(x - 6) = 0$ **f** $x^2 + 9 = 0$

16 Find solution/s for each of these equations, if real solutions exist. Hint: Some of these equations cannot be solved.

a $x^2 - 9 = 0$ **b** $x^2 + 4x - 77 = 0$ **c** $x^2 - 8x + 16 = 0$

d $x^2 + 4 = 0$ **e** $x^2 + 4x = 0$ **f** $x^2 - 6x - 1 = 0$



17 Nutsa dives with her arms outstretched from a platform into a diving pool. The height of her fingertips above the pool surface can be represented by the quadratic relationship $h = -2t^2 + 4t + 6$, where h is the height of her fingertips above the water, in metres, after t seconds.

- **a** What is the height of Nutsa's fingertips above the water after:
  - **i** 1 second
  - **ii** 2 seconds?
- **b** What is the height of Nutsa's fingertips when she starts the dive?
- **c** How long does the dive take? Hint: How long does it take for Nutsa's fingertips to hit the surface of the water?
- **d** Explain why there is only one time value for your answer to part **c** even though you have solved a quadratic equation that has two solutions.

18 Oliver buys rice to serve with stir-fried meals over the coming weeks. As time passes, the amount of rice Oliver has left can be represented by the quadratic relationship $y = x^2 - 10x + 25$, where y is the amount left, in kilograms, after x weeks.

- **a** How much rice did Oliver buy?
- **b** How much rice is left after 2 weeks?
- **c** How long does it take for all of the rice to be used?
- **d** Did Oliver use the same amount of rice each week? Explain.

19 The area of a rectangular vegetable plot is 70 m². The length is 3 m longer than the breadth.

- **a** Write a quadratic equation to represent this. Hint: Let x represent the width of the vegetable plot, in metres.
- **b** Solve the quadratic equation.
- **c** Write the dimensions of the vegetable plot.

20 The breadth of a mobile phone screen is 5 cm less than its length. Write an equation to represent the situation if the area of the screen is 66 cm². Then solve your equation to find the dimensions of the screen.

21 Solve each of the following equations to find the value(s) of x by substituting the relation given in the brackets into the equation.

- **a** $x^4 - 12x^2 + 36 = 0$ $(p = x^2)$
- **b** $\frac{1}{x^2} + \frac{4}{x} + 4 = 0$ $\left(q = \frac{1}{x}\right)$

22 If $x - \frac{1}{x} = 3$, find the value of $x^2 + \frac{1}{x^2}$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving quadratic equations

Investigation
Beware of irreversible operations

Topic quiz
4D

ADVANCED

4E The quadratic formula

Learning intentions

By the end of this topic you will be able to ...

- ✔ identify the number of solutions to a quadratic equation using the discriminant
- ✔ solve quadratic equations using the quadratic formula.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution

The quadratic formula

Key content video

- The **quadratic formula** can be used to solve equations of the general form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- The solutions for $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This can be written more simply as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant

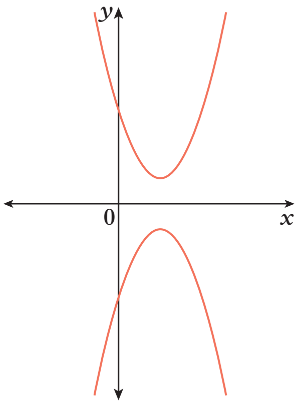
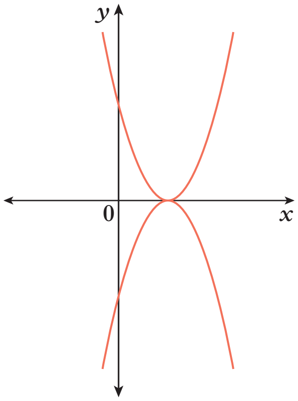
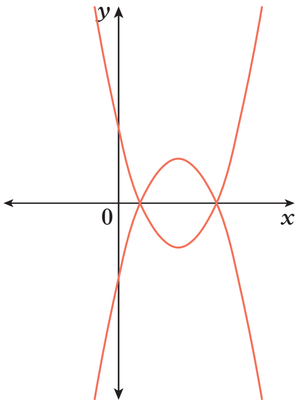
- The **discriminant** is $\Delta = b^2 - 4ac$.

Evaluating this expression after substituting values for a, b and c allows us to identify the number of real solutions for the quadratic equation.

→ If $\Delta < 0$, there are no real solutions as the square root of a negative number is not a real number.

→ If $\Delta = 0$, there is one real solution. It is $x = \frac{-b}{2a}$ and the quadratic is a perfect square.

→ If $\Delta > 0$, there are two real solutions. They are $x = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Discriminant	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)
Number of solutions	No real solutions	One real solution	Two real solutions
Graph			

Example 4E.1 Identifying the number of solutions to a quadratic equation

Using the discriminant, identify the number of solutions to each of these quadratic equations.

a $3x^2 + 7x + 2 = 0$ **b** $x^2 + 16 = 8x$ **c** $-2x^2 + 3x - 5 = 0$

THINK

1 If necessary, rearrange each equation so that the right-hand side is 0.

2 Identify a, b and c.

3 Write the discriminant formula and substitute values for a, b and c into it. Simplify the result.

4 Identify the number of solutions.

WRITE

a $3x^2 + 7x + 2 = 0$

$a = 3, b = 7, c = 2$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 7^2 - 4 \times 3 \times 2\\ &= 49 - 24\\ &= 25\end{aligned}$$

Since $\Delta > 0$, the equation has two solutions.

b $x^2 + 16 = 8x$

$x^2 - 8x + 16 = 0$

$a = 1, b = -8, c = 16$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-8)^2 - 4 \times 1 \times 16\\ &= 64 - 64\\ &= 0\end{aligned}$$

Since $\Delta = 0$, the equation has one solution.

c $-2x^2 + 3x - 5 = 0$

$a = -2, b = 3, c = -5$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 3^2 - 4 \times (-2) \times (-5)\\ &= 9 - 40\\ &= -31\end{aligned}$$

Since $\Delta < 0$, the equation has no solutions.

Example 4E.2 Using the quadratic formula to solve quadratic equations

Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values

ii approximate values, correct to two decimal places.

a $2x^2 - 5x + 1 = 0$ **b** $9x^2 + 12x + 4 = 0$ **c** $4x^2 = 3x - 1$

THINK

1 If necessary, rearrange the equation so that one side of the equation equals 0.

2 Identify a, b and c.

3 Determine the discriminant and identify the number of solutions.

4 Write the quadratic formula and substitute values for a, b and c into it. Simplify the result.

5 Write the solution as exact values.

6 Use a calculator to help you rewrite the values correct to two decimal places.

WRITE

a $2x^2 - 5x + 1 = 0$

$a = 2, b = -5, c = 1$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-5)^2 - 4 \times 2 \times 1\\ &= 25 - 8\\ &= 17\end{aligned}$$

$\Delta > 0$. The equation has two solutions.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-(-5) \pm \sqrt{17}}{2(2)}\\ &= \frac{5 \pm \sqrt{17}}{4}\end{aligned}$$

i $x = \frac{5 - \sqrt{17}}{4}$ or $\frac{5 + \sqrt{17}}{4}$

ii $x \approx 0.22$ or 2.28

c $4x^2 = 3x - 1$

$4x^2 - 3x + 1 = 0$

$a = 4, b = -3, c = 1$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - 4 \times 4 \times 1\\ &= 9 - 16\\ &= -7\end{aligned}$$

$\Delta < 0$. The equation has no solutions.

b $9x^2 + 12x + 4 = 0$

$a = 9, b = 12, c = 4$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (12)^2 - 4 \times 9 \times 4\\ &= 144 - 144\\ &= 0\end{aligned}$$

$\Delta < 0$. The equation has one solution.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-12 \pm \sqrt{0}}{2(9)}\\ &= \frac{-12}{18}\\ &= -\frac{2}{3}\end{aligned}$$

i $x = -\frac{2}{3}$

ii $x \approx -0.67$

Helpful hints

- ✔ If the discriminant is positive, don't forget to evaluate both possible solutions using the quadratic formula.

  Remember that the quadratic formula is equal to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- ✔ Be careful when identifying the values of a, b and c. Make sure one side of the equation equals zero before identifying a, b and c.
- ✔ Be careful when substituting the values of a, b and c into the formula. You need to include the sign of each number.

  For example, if $-2x^2 - 6x - 4 = 0$, then $a = -2$, $b = -6$ and $c = -4$.

  So, the solutions are $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-2)(-4)}}{2(-2)}$.

- ✔ If you have already calculated the discriminant of a quadratic equation, you can substitute it into the quadratic formula.

  For example, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$.

ADVANCED

ANS p674

Exercise 4E The quadratic formula

△ 1, 2, 3–4(1^{st}, 2^{nd} columns), 5, 6, 8, 11, 13

■ 2, 3–4(2^{nd} column), 5–10, 14, 15, 18

⬟ 2–4(3^{rd} column), 5–7, 11, 14–18

UNDERSTANDING AND FLUENCY

4E.1 **1** Using the discriminant, identify the number of solutions to each of these quadratic equations.

a $x^2 + 5x + 7 = 0$ **b** $3x^2 - 8x - 4 = 0$ **c** $x^2 - 9 = 0$

d $2x^2 + 3x + 4 = 0$ **e** $12x - x^2 = 36$ **f** $6x^2 - 5 = 7x$

2 Complete the following for each quadratic equation below.

i Calculate the discriminant.

ii Identify the number of solutions.

iii Solve, by first factorising and then using the Null Factor Law.

iv Check whether your answer for part **iii** confirms your result for part **ii** each time.

a $x^2 - 2x - 15 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 - 49 = 0$

d $3x^2 - 18x = 0$ **e** $x^2 - 6x + 7 = 0$ **f** $x^2 + 2x + 3 = 0$

4E.2 **3** Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values

ii approximate values, correct to two decimal places.

a $x^2 - 7x + 5 = 0$ **b** $3x^2 - x - 1 = 0$ **c** $2x^2 + x - 4 = 0$

d $-3x^2 - x + 9 = 0$ **e** $-6x^2 - 5x + 3 = 0$ **f** $x^2 - 2x - 1 = 0$

g $x^2 + 8x + 4 = 0$ **h** $4x^2 - 12x + 1 = 0$ **i** $-x^2 + 4x + 2 = 0$

j $5x^2 + 10x - 3 = 0$ **k** $2x^2 = 11 - 3x$ **l** $x^2 + 2x = 7$

4 Use the quadratic formula to find any solutions to each of the following quadratic equations. If solutions exist, write them as exact values.

a $x^2 + 3x = 1$ **b** $x^2 + 6x + 12 = 0$ **c** $\frac{1}{2}x^2 + 3x + 2 = 0$

d $x^2 + 3x + 5 = 0$ **e** $2x^2 + 5x + 1 = 0$ **f** $x^2 + 16 = 8x$

g $x^2 + 2x = 4$ **h** $x^2 = 3 - 7x$ **i** $10 = x^2 + 8x$

j $x^2 - 5 = x$ **k** $x^2 + 5 = x$ **l** $x^2 + \sqrt{5}x = 1$

5 Use the quadratic formula to find the solutions to each of the following quadratic equations.

a $m^2 + 3m - 7 = 0$ **b** $u^2 + 4u = 1$ **c** $r^2 + 9r + 1 = 0$

d $y^2 - 3 = 2y$ **e** $3h^2 + 7h = 1$ **f** $w^2 = 5 + 6w$

6 Solve $\frac{1}{x} = x - 3$ by first multiplying both sides by x to create a quadratic equation.

7 Consider the quadratic equation $x^2 - 5x + 6 = 0$.

a Solve the equation by factorising the left-hand side and using the Null Factor Law.

b Now solve the equation again, using the quadratic formula.

c Which method was quicker or easier to use?

d The solutions to any quadratic equation can be found using the quadratic formula. Why might you try using the method described in part **a** first?

8 When answering each of these, consider $\sqrt{b^2 - 4ac}$ in the quadratic formula.

a Explain why there are no real solutions to a quadratic equation when the discriminant is negative.

b Explain why there is only one real solution when the discriminant is zero.

c Explain why there are two real solutions when the discriminant is positive.

ADVANCED

PROBLEM SOLVING AND REASONING

9 Copy and complete the lines of working below to derive the quadratic formula. You will complete the square to factorise the left-hand side and then use the Null Factor Law to find the solutions for $ax^2 + bx + c = 0$ (where $a \neq 0$).

THINK	**WRITE**
1 Write the quadratic equation.	$ax^2 + bx + c = 0$
2 Factorise by taking out the coefficient of x^2.	$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$
3 Divide both sides by a.	$x^2 + \frac{b}{a}x + \frac{\square}{\square} = 0$
4 Add a new third term to complete the square and compensate by subtracting the same term.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{\square} = 0$
5 Form a perfect square using the first three terms and simplify the fourth term.	$\left(x + \frac{b}{2a}\right)^2 - \frac{\square}{4a^2} + \frac{c}{a} = 0$
6 Group the last two terms together.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0$
7 Combine the last two terms using a common denominator.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\square - 4ac}{4a^2}\right) = 0$
8 Write the left-hand side as the difference of two squares.	$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$
9 Simplify the second square.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$
10 Factorise the left-hand side using the 'difference of two squares' rule.	$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \square\right) = 0$
11 Use the Null Factor Law.	$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \square = 0$
12 Solve each linear equation.	$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \square$
13 Simplify each solution.	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula for finding the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

10 a Copy and complete the lines below to derive the solution of a quadratic equation in the form $a(x - h)^2 + k = 0$ by completing the square.

$a(x - h)^2 + k = 0$

$a(x - h)^2 = \_\_\_\_\_$

$(x - h)^2 = \_\_\_\_\_$

$x - h = \_\_\_\_\_$

$x = \_\_\_\_\_$

b How can you use the value of a, h and k to determine if a quadratic equation in the form $a(x - h)^2 + k = 0$ would have zero, one or two real solutions?

c Determine how many real solutions each of the following quadratic equations have using your conclusion in part **b**.

i $4(x - 5)^2 - 1 = 0$

ii $5(x + 4)^2 = 0$

iii $-3(x - 1)^2 - 2 = 0$

d Find the solutions of the quadratic equations in part **c** using the formula in part **a** if real solutions exist.

ADVANCED

PROBLEM SOLVING AND REASONING

11 The profit, p (in dollars), made from selling n soft toys is given by the relationship $p = -0.1n^2 + 17n + 200$.

a What is the profit when 30 toys are sold?

b What is the smallest number of toys that need to be sold to make a profit of at least \$850?

12 A rectangular paddock has an area of 5000 m^2; the length is 20 m greater than the width. Solve a quadratic equation to find the dimensions of the paddock, correct to one decimal place.

13 The perimeter of a framed painting is 100 cm. Within the 2 cm frame, the area of the painting that can be viewed is 300 cm^2. Find the dimensions of the framed painting, correct to the nearest millimetre.

14 The discriminant of a quadratic equation can be used to find the number of solutions, but it can also be used to find the nature of the solutions, if they exist. If a quadratic equation has integer coefficients and the discriminant is a square number, then the solutions are rational (that is, whole numbers or fractions).

a Consider each of the following equations with integer coefficients. Find the discriminant and use it to determine whether the solutions are rational or irrational.

i $0 = x^2 + 5x - 2$

ii $0 = 2x^2 + 3x - 2$

iii $0 = x^2 - 7x + 10$

iv $0 = 5x^2 + 11x + 4$

v $0 = 3x^2 - x - 4$

b Explain with reference to the quadratic formula why, if the discriminant is a square number, the solutions are guaranteed to be rational, assuming all coefficients are integers.

15 a Find the coordinates of the x-intercepts, if they exist, of the graphs of each of the following equations. You do not need to sketch the graph.

i $y = x^2 + 7x + 10$

ii $y = x^2 - 6x + 8$

iii $y = x^2 + 3x + 5$

iv $y = x^2 - 8x + 16$

v $y = -2x^2 + 6x - 7$

b Explain how the discriminant of a quadratic relationship relates to the number of x-intercepts of its graph.

CHALLENGE

16 a Find the value of k so that $kx^2 + 6x + 3 = 0$ has only one real solution.

b For what values of k does $kx^2 + 6x + 3 = 0$ have:

i two real solutions

ii no real solutions?

17 For each of these equations, find the value of k that will give the number of real solutions shown in square brackets.

a $x^2 + 2x + k = 0$ [one real solution]

b $2x^2 - 4x + k = 0$ [two real solutions]

c $kx^2 + 6x - 1 = 0$ [no real solutions]

d $kx^2 - 3x + 1 = 0$ [one real solution]

e $3x^2 + kx + 3 = 0$ [two real solutions]

f $x^2 + kx + 25 = 0$ [no real solutions]

18 A quadratic equation is in the form $ax^2 + bx + 5 = 0$.

a Find the discriminant of the equation in terms of a and b.

b If the equation has one and only one solution, write the solution in terms of a and b.

c If $x = -1$ is the only solution to the equation, use your answer to parts **a** and **b** to find the value of a and b.

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pro

Interactive skillsheet
The quadratic formula

Worksheet
Solving quadratic equations

Topic quiz
4E

4F Solving cubic equations

Learning intentions

By the end of this topic you will be able to ...

- ✔ solve simple cubic equations of the form $ax^3 = k$
- ✔ write the answer of cubic equations as exact and approximated values.

Inter-year links

Year 8 4E Roots

Solving cubic equations

Key content video

- A **cubic equation** is a single-variable equation in which the highest index of the variable is 3.

Cubic equations	Non-cubic equations
$a^3 = 8$	$3^x = 1$
$2x^3 - x^2 + x = 3$	$x^2 + x^4 = 2$
$1 + m + m^2 + m^3 = 0$	$1 + a^3 + b^3 = 0$

- When a positive number is cubed, the result is positive. $3^3 = 3 \times 3 \times 3 = 27$

  When 0 is cubed, the result is 0. $0^3 = 0 \times 0 \times 0 = 0$

  When a negative number is cubed, the result is negative. $(-3)^3 = (-3) \times (-3) \times (-3) = -27$
- The inverse operation of taking the cube root is an index of 3.
- The inverse operation of an index of 3 is taking the cube root.

$\sqrt[3]{2^3} = 2$ $\sqrt[3]{a^3} = a$

$(\sqrt[3]{2})^3 = 2$ $(\sqrt[3]{a})^3 = a$

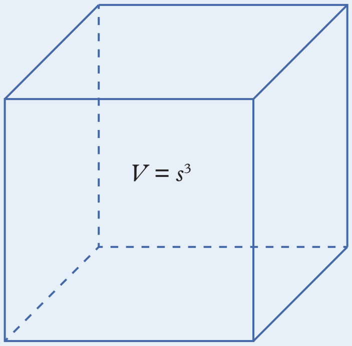
- $ax^3 = k$ has one and only one solution: $x = \sqrt[3]{\frac{k}{x}}$.
- The negative sign under a cube root can be moved in front of the cube root sign.

  For example, $\sqrt[3]{-5} = -\sqrt[3]{5}$
- The cube root of a fraction can be found by taking the cube root of the numerator and denominator separately.

  For example, $\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$

  $= \frac{2}{3}$.
- If the solution to a cubic equation is irrational, it can be written as a surd or rounded to a given accuracy to approximate the exact solution.
- We can substitute values into formulas that involve cubed variables to solve for the unknown.

  For example, we can substitute $V = 8$ into $V = s^3$ then solve for the possible values of s.



Example 4F.1 Solving cubic equations

Solve the following cubic equations.

a $x^3 = -125$ **b** $8x^3 = 27$ **c** $x^3 - 0.025 = 0.1$

THINK	WRITE
a **1** Take the cube root of both sides of the equation.	**a** $x^3 = -125$ $\sqrt[3]{x^3} = \sqrt[3]{-125}$
2 Evaluate the cube root. The cube root of a negative number is a negative number.	$x = -5$
b **1** Divide both sides of the equation by 8.	**b** $8x^3 = 27$ $(\div 8)$ $x^3 = \frac{27}{8}$
2 Take the cube root of both sides of the equation.	$\sqrt[3]{x^3} = \sqrt[3]{\frac{27}{8}}$
3 Evaluate the cube root of the fraction by finding the cube root of both the denominator and the numerator.	$x = \frac{\sqrt[3]{27}}{\sqrt[3]{8}}$ $x = \frac{3}{2}$
c **1** Use inverse operations to rearrange the equation, leave only the x^3 term on one side of the equation.	**c** $x^3 - 0.025 = 0.1$ $(+ 0.025)$ $x^3 = 0.125$
2 Take the cube root of both sides of the equation.	$\sqrt[3]{x^3} = \sqrt[3]{0.125}$
3 Evaluate the cube root.	$x = 0.5$

Example 4F.2 Writing the answer of a cubic equation in exact and approximated form

Solve the following cubic equations. Write your answers:

i in exact form **ii** correct to two decimal places.

a $x^3 = 25$ **b** $m^3 + 2 = -3$

THINK	WRITE
a **i** Confirm the cube root is irrational and, if it is, leave the answer with the cube root notation.	**a** **i** $\sqrt[3]{25}$
ii Evaluate the cube root using a calculator, round to the given accuracy.	**ii** $\sqrt{25} = 2.924...$ ≈ 2.92
b **1** Use inverse operations to rearrange the equation, leave only the m^3 term on one side of the equation.	**b** $m^3 + 2 = -3$ $m^3 = -5$
2 Take the cube root of both sides of the equation. **i** Confirm the cube root is irrational and, if it is, leave the answer with the cube root notation. The negative sign under a cube root can be moved in front of the cube root sign.	**i** $m = \sqrt[3]{-5}$ $m = -\sqrt[3]{5}$
ii Evaluate the cube root using a calculator, round to the given accuracy.	**ii** $m \approx -1.71$

Helpful hints

- ✔ Be careful when evaluating the root of a number, make sure you check if it is asking for the cube root $\sqrt[3]{\quad}$ or the square root $\sqrt[2]{\quad}$.
- ✔ A positive number has two square roots but has only one real cube root.
- ✔ The cube root and square root of 0 are both 0.
- ✔ A negative number has one real cube root but does not have a square root.

	negative number	**0**	**positive number**
square root	do not exist	0	one positive square root, one negative square root
cube root	one negative cube root	0	one positive cube root

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Exercise 4F Solving cubic equations

△ 1, 2–5(1st, 2nd columns), 6(a), 7(a,b), 10, 12, 13

■ 2–5(2nd columns), 6, 7(c, d), 8–13

⬟ 2–5(3rd column), 6(c), 7(2nd column), 9, 11–14

UNDERSTANDING AND FLUENCY

1 Apply appropriate inverse operations to make x^3 the only term left on one side of the equation.

a $2x^3 = 10$ **b** $-x^3 = 12$ **c** $\frac{x^3}{10} = 10$

d $x^3 - 5 = 10$ **e** $12 - x^3 = 10$ **f** $\frac{x^3}{10} = \frac{1}{1000}$

4F.1 **2** Solve the following equations for the unknown.

a $a^3 = 1000$ **b** $x^3 = 27$ **c** $y^3 = -64$

d $m^3 = \frac{216}{1000}$ **e** $t^3 = \frac{512}{343}$ **f** $z^3 = -\frac{125}{1331}$

g $n^3 = 0$ **h** $-27x^3 = 1$ **i** $64p^3 = -8$

j $\frac{s^3}{4} = 16$ **k** $\frac{-x^3}{8} = 1$ **l** $0.1m^3 = 100$

m $q^3 = 1.728$ **n** $0.331 - x^3 = -1$ **o** $n^3 - 0.016 = 0.2$

4F.2 **3** Solve the following equations for the unknown. Write your answers in exact form.

a $a^3 = 10$ **b** $b^3 = 9$ **c** $c^3 = -3$

d $d^3 = 12$ **e** $e^3 = -15$ **f** $f^3 = -36$

4 Write your answers to question **3** correct to two decimal places.

5 Solve the following equations for the unknown. Write your answers in exact form.

a $-20a^3 = 60$ **b** $64b^3 = 81$ **c** $21c^3 = 7$

d $\frac{d^3}{3} = 3$ **e** $-\frac{e^3}{3} = 12$ **f** $\frac{f^3}{2} = -12$

g $n^3 - 1 = 0.2$ **h** $x^3 + 0.1 = 0.2$ **i** $m^3 - 2 = 0.2$

j $\frac{x^3}{3} = \frac{1}{6}$ **k** $\frac{y^3}{64} = \frac{1}{4}$ **l** $-\frac{s^3}{8} = -\frac{1}{4}$

6 Substitute the values given into the formula and solve for the unknown. Write your answers correct to two decimal places.

a Solve $V = s^3$ for s where $V = 10$ cm³.

b Solve $V = \frac{\pi h^3}{3}$ for h where $V = 20$ mm³ and use the exact value of π on your calculator.

c Solve $V = \frac{4\pi r^3}{3}$ for r where $V = 100$ m³ and use the exact value of π on your calculator.

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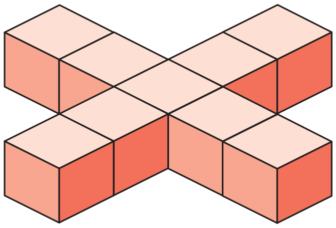
PROBLEM SOLVING AND REASONING

7 Cubes used in each of the following composite solids are identical.

i Write an equation to represent the relationship between the side lengths of the small cube, s, and the volume of the composite solid.

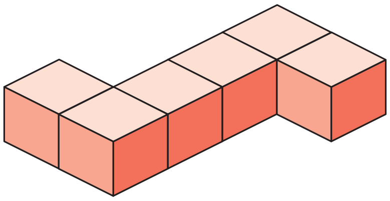
ii Determine the side lengths of the cubes.

a



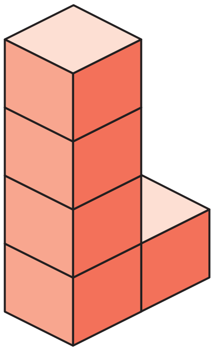
$V = 72\text{ cm}^3$

b



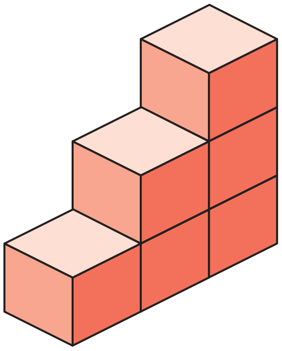
$V = 162\text{ cm}^3$

c



$V = 320\text{ mm}^3$

d

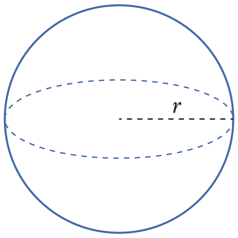


$V = 6000\text{ m}^3$

8 The density of an object is defined as the ratio between its mass and volume and can be calculated using the formula density = $\frac{\text{mass}}{\text{volume}}$. Density often has units of grams per cubic meter(g/cm^3). Determine the side length of a Rubik's cube if it has a density of 1.5 g/cm^3 and a mass of 264 g. Correct your answer to two decimal places.

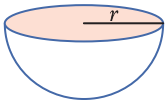
9 The volume of a sphere can be calculated using the formula $V = \frac{4\pi r^3}{3}$, determine the radius of the following sphere and hemisphere. Write your answer correct to two decimal places.

a



$V = 128\text{ m}^3$

b



$V = 66\text{ cm}^3$

10 **a** Determine the value of x in each of the following equations.

i $x^3 = 0.000\,001$ **ii** $x^3 = 0.001$ **iii** $x^3 = 1000$ **iv** $x^3 = 1\,000\,000$

b Determine which of the following equations have a rational solution then calculate the value of x in these equations without using a calculator.

i $x^3 = 27\,000$ **ii** $x^3 = 4000$ **iii** $x^3 = 0.000\,064$

iv $x^3 = -0.000\,002$ **v** $x^3 = -8\,000\,000$ **vi** $x^3 = 6\,400\,000$

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PROBLEM SOLVING AND REASONING

11 a Determine the value of x in each of the following equations.

i $x^3 = 1.331$ **ii** $x^3 = 1331$ **iii** $x^3 = 1\,331\,000$ **iv** $x^3 = 0.001\,331$

b Use your answer from part **a** to deduce the value of x in the following equations. Use a calculator to verify your answers.

i $x^3 = -1331$ **ii** $x^3 = 0.000\,001\,331$

iii $x^3 = 1\,331\,000\,000$ **iv** $x^3 = -0.000\,000\,001\,331$

c Given the solution of $x^3 = 12.167$ is $x = 2.3$, determine the value of x in each of the following equations without using a calculator.

i $x^3 = -12.167$ **ii** $x^3 = 0.000\,012\,167$

iii $x^3 = 12\,167\,000\,000$ **iv** $x^3 = 12\,167\,000$

12 a Copy and complete the lines below to show $\sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b}$.

Let $a = m^3$, $b = n^3$

$$
\begin{aligned}
\sqrt[3]{ab} &= \sqrt[3]{\_\_\_^3\; n^3}\\
&= \sqrt[3]{m \times m \times \_\_\_ \times \_\_\_\_ \times n \times n}\\
&= \sqrt[3]{m \times \_\_\_\_ \times m \times \_\_\_\_ \times m \times \_\_\_\_}\\
&= \sqrt[3]{(m \times \_\_\_\_)^3}\\
&= m \times \_\_\_\\
&= \sqrt[3]{a} \times \_\_\_
\end{aligned}
$$

b Use the property from part **a** to write the solution of the following equations in the form $a\sqrt[3]{b}$.

i $x^3 = 24$ **ii** $x^3 = 54$ **iii** $x^3 = 250$ **iv** $x^3 = 16$

13 Copy and complete the lines below to show $\sqrt[3]{-x} = -\sqrt[3]{x}$.

$$
\begin{aligned}
\sqrt[3]{-x} &= \sqrt[3]{\_\_\_\_ \times x}\\
&= \sqrt[3]{\_\_\_\_ \times 1 \times x}\\
&= \sqrt[3]{\_\_\_\_ \times (-1 \times -1) \times x}\\
&= \sqrt[3]{\_\_\_\_^3 \times x}\\
&= \sqrt[3]{\_\_\_\_^3} \times \sqrt[3]{x}\\
&= \_\_\_\_ \times \sqrt[3]{x}\\
&= -\sqrt[3]{x}
\end{aligned}
$$

CHALLENGE

14 A cubic equation in the form $(x + h)^3 = k$ can be solved for x by first finding the cube root of k, then subtracting h from both sides.

For example,

$$
\begin{aligned}
(x + 3)^3 &= 8\\
\sqrt[3]{(x + 3)^3} &= \sqrt[3]{8}\\
x + 3 &= 2\\
x &= -1
\end{aligned}
$$

Solve the following equations for the unknown. Write your answer correct to two decimal places where required.

a $(x + 2)^3 = -8$ **b** $(x - 3)^3 = 1$ **c** $(x + 0.2)^3 = 0.125$

d $\left(x - \frac{1}{2}\right)^3 = -1$ **e** $(x + 0.6)^3 = 0.064$ **f** $\left(x - \frac{1}{3}\right)^3 = \frac{1}{27}$

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Interactive skillsheet
Solving simple cubic equations

Topic quiz
4F

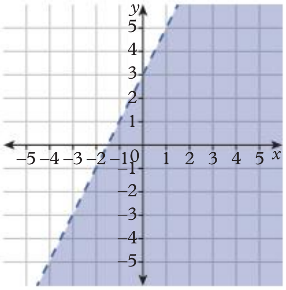
Chapter summary

Solving linear equations

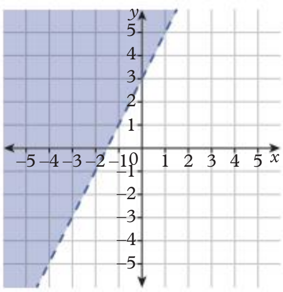
1. Use the LCD of all the fractions to simplify linear equations involving algebraic fractions.
2. Expand the brackets then simplify the equation by combining like terms.
3. Use inverse operations to solve the linear equation.

Linear inequalities on the Cartesian planes

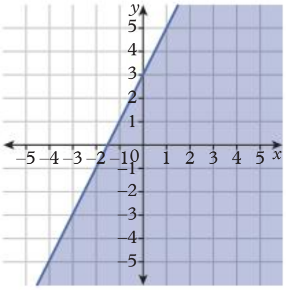
$y < 2x + 3$



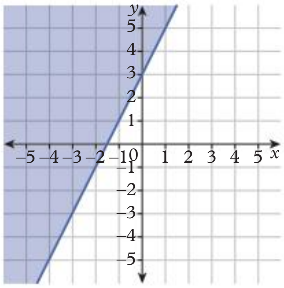
$y > 2x + 3$



$y \leq 2x + 3$



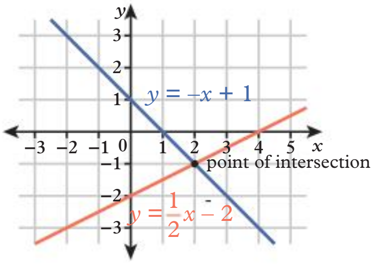
$y \geq 2x + 3$



Simultaneous linear equations

Solving graphically

- The coordinates for the point of intersection of the two graphs give the solution to the simultaneous linear equations.



- If two linear relationships are parallel on the graph, there's no solution to the simultaneous linear equations, because there's no point of intersection.

Solving by substitution

- Substitute an expression from one equation into the other equation to create a new equation containing only one variable.
  For example:
  Substitue $y = 4 - 2x$ into
  $y + 11 = 3x$
  to get $4 - 2x + 11 = 3x$
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

Solving by elimination

- Multiply one or both of the equations by a constant so that the magnitudes of the coefficients for one of the variables are the same in both equations.
- Add or subtract the LHS and RHS of one equation from the other equation to eliminate one of the variables.

  For example, $2x - y = 4$ ①
  $-3x - 2y = -6$ ②

  2 × ① – ②:
  $\text{LHS} = 2 \times (2x - y) - (-3x - 2y)$
  $\text{RHS} = 2 \times (4) - (-6)$
  $7x = 14$
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

Solving quadratic equations

- Using the Null Factor Law

  If $(x - p)(x - q) = 0$, then $x - p = 0$ and $x - q = 0$, $x = p$, $x = q$
- Using the quadratic formula

$$ax^2 + bx + c = 0,\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \Delta = b^2 - 4ac$$

- $\Delta < 0$, there is no real solutions.
- $\Delta = 0$, there is one real solution.
- $\Delta > 0$, there are two real solutions.

Solving cubic equations

- $ax^3 = k$ has one and only one solution $x = \sqrt[3]{\frac{k}{a}}$
- $\sqrt[3]{\frac{k}{a}} = \frac{\sqrt[3]{k}}{\sqrt[3]{a}}$
- $\sqrt[3]{-x} = -\sqrt[3]{x}$
- $\sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b}$

ANS p676

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- boundary line
- coincident lines
- cubic equation
- discriminant
- elimination method
- factorised form
- linear equation
- linear inequality
- literal equation
- non-linear equation
- Null Factor Law
- point of intersection
- quadratic equation
- quadratic formula
- simultaneous equations
- subject of an equation
- substitution method

1 Match the following equations with the key term below. There may be no or more than one equation for each key term.

a linear equation
b linear inequality
c non-linear equation
d quadratic equation
e cubic equation
f literal equation

I $y = 3x - 1$
II $y = ax + b$
III $y = x^2 - 2x - 15$
IV $y = 2x^3 - 16$
V $y = x^2 + c$
VI $2y \geq 3x - 1$

2 Which key term involves a method for solving a quadratic equation that cannot be factorised?

3 Which key term is the name for an expression that can be used to determine the number of solutions to a quadratic equation?

4 Complete the following sentences using words from the list of key terms.

a If we are trying to find a common solution(s) to a group of equations, they are referred to as ________________.

b If we are trying to solve this pair of simultaneous equations, $y = 2x + 1$ and $3x - y = 10$, we could use the ________________ because one of the equations already has one variable on its own as the ________________.

c If we are trying to solve this pair of simultaneous equations, $2x + y = 5$ and $3x - y = 10$, we could use the ________________ because each equation has the same or opposite coefficient for one of the variables.

d If we are trying to solve a factorised quadratic equation, we apply the ________________ to get the exact answers.

e A ________________ with two variables has an infinite number of points that satisfy the equation, and they lie on a straight line. A ________________ has an infinite number of solutions that lie in a region. The ________________ is either solid or dashed.

Multiple choice

4A **1** The solution to $\frac{x-4}{2} = -3$ is:

A $x = 2$ **B** $x = -2$ **C** $x = 10$ **D** $x = -10$ **E** $x = -14$

4A **2** Which of these equations has the solution $x = -5$?

A $\frac{12}{x} = -2.4$ **B** $\frac{2x+3}{13} = 1$ **C** $3x - 7 = 8$ **D** $\frac{3}{2-x} = 1$ **E** $3x + 1 = 8(x + 6)$

4B **3** If $3x + 2 > 5x - 6$, which of the following could be a value for x?

A 10 **B** 8 **C** 6 **D** 4 **E** 2

4B **4** Which of the following value pairs could not be a solution to $3x + 2y < 11$?

A (0, 5) **B** (1, 2) **C** (2, 3) **D** (−1, 6) **E** (−2, 5)

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4C **5** The pair of linear equations $y = x + 3$ and $y = -2x + 6$ intersect at which of these points?

A $(-3, 0)$ **B** $(3, 6)$ **C** $(1, 8)$ **D** $(1, 4)$ **E** $(0, 6)$

4C **6** The pair of linear equations $y = 3x + 1$ and $6x - 2y + 2 = 0$ have how many simultaneous solutions?

A none **B** one **C** two **D** three **E** many

4C **7** When $y = 3x - 4$ is substituted into the equation $x - 2y = 3$, the resulting equation is:

A $-5x - 8 = 3$ **B** $-5x + 8 = 3$ **C** $5x + 8 = 3$ **D** $5x - 8 = 3$ **E** $7x + 8 = 3$

4C **8** Consider the two equations:

$2x - y + 3 = 0$ ①
$4x - y + 5 = 0$ ②

If equation ② is subtracted from equation ①, the resulting equation will be:

A $-2x - 2 = 0$ **B** $-2x - 8 = 0$ **C** $2x + 2 = 0$ **D** $2x + 8 = 0$ **E** $2x - 2 = 0$

4D **9** Which of these is a quadratic equation?

A $-2x = x^2 + 4$ **B** $-2x = -\frac{1}{x^2}$ **C** $2^x = 4$ **D** $\frac{x}{2} = \frac{3x}{4} - 2$ **E** $12 = 2x^3$

4E **10** For the quadratic equation, $2x - 4x^2 = 3$, which of these is the value of the discriminant?

A 52 **B** -8 **C** -44 **D** 44 **E** 0

4F **11** Which of the following equations has more than one answer?

A $3x + x^2 = 0$ **B** $8 = -x^3$ **C** $2x^2 = -x^2$ **D** $\frac{5x}{2} = 3x - 1$ **E** $\frac{2}{x - 5} = 1$

4F **12** What is the solution of $-\frac{x^3}{9} = 1$?

A 3 **B** -3 **C** $\frac{-1}{3}$ **D** $\sqrt[3]{9}$ **E** $-\sqrt[3]{9}$

Short answer

4A **1** Solve each of these equations. Leave your answer as a fraction where appropriate.

a $3 - 2(4 - 5x) = -20$ **b** $4(3x + 5) = 3(5x - 3)$

4A **2** Solve each of these equations, leaving your answer as a fraction if appropriate.

a $\frac{4 + x}{3} = \frac{2x - 5}{3}$ **b** $\frac{5 - 3x}{2} = \frac{2x + 5}{3}$

4A **3** Rearrange each of these equations so that the coloured pronumeral in the brackets is the subject.

a $A = \frac{h(a + b)}{2}$ (b) **b** $A = 2(l + b)h$ (h)

4B **4** Solve each of these inequalities.

a $5 - 2x < 7$ **b** $\frac{2 - 3x}{5} > 2$

4B **5** Represent the following linear inequalities on the Cartesian plane with boundary lines and shading.

a $y + x \leq 3$ **b** $3y + 6x > 3$

c $x + 2y \leq 5 - y$ **d** $x + 7y \leq y + 2x$

4C **6** Use a graph to help you solve each of these pairs of simultaneous equations.

a $y = 4x - 2$
$y = -3x + 5$

b $2x - y + 5 = 0$
$3x - y = 1$

4C **7** Use the substitution method to solve each of these pairs of simultaneous equations.

a $3x + 4y = -1$
$y = x - 2$

b $y = 4x + 3$
$x - 2y = 8$

c $7x - y = 4$
$y = 4x + 2$

4C **8** Use the elimination method to solve each of these pairs of simultaneous equations.

a $2x + y = 8$
$4x - y = 4$

b $x + 5y = 13$
$x + 2y = 4$

c $2x - y = 7$
$3x + 5y = 4$

4D **9** Solve each of these equations. Write you answers as exact values.

a $(x - 2)^2 = 0$ **b** $(4x - 3)(3x + 4) = 0$ **c** $(\sqrt{2} - x)(\sqrt{2} + x) = 0$

d $3x^2 - 6x - 24 = 0$ **e** $5x^2 - 7 = 0$ **f** $x^2 - 6x + 9 = 0$

4D **10** Solve each of these equations by completing the square. Write your solutions as exact values.

a $x^2 + 6x + 4 = 0$ **b** $x^2 - 3x - 3 = 0$

4E **11** Use the discriminant to identify the number of real solutions of each of these quadratic equations.

a $x^2 - 4x + 4 = 0$ **b** $3x^2 = 2x - 4$ **c** $5x - 4x^2 = 2$

d $4 - 3x^2 = 8x$ **e** $1 - x - x^2 = 0$ **f** $6x^2 + 4x = -1$

4E **12** Use the quadratic formula to solve each of these equations. Write your solutions as:

- **i** exact values
- **ii** approximate values, to two decimal places.

a $x^2 + 3x = 11$ **b** $5x^2 = 3x + 1$ **c** $x^2 = 5x - 2$

d $x^2 + 5x + 2 = 0$ **e** $2x^2 = x + 2$ **f** $3x^2 + 7x + 3 = 0$

4F **13** Solve each of these equations. Write your solutions as:

- **i** exact values
- **ii** approximate values, to two decimal places.

a $x^3 = 10$ **b** $0.5x^3 = 0.125$ **c** $5x^3 = 125$

d $x^3 - 1 = 0.5$ **e** $\frac{x^3}{8} = \frac{1}{2}$ **f** $\frac{x^3}{15} = -\frac{1}{5}$

Analysis

1 In a local cafe, Jack purchases and then sells muffins and croissants. The muffins cost \$1.60 each and the croissants cost \$0.80 each. Jack sells the muffins for \$3.30 each and the croissants for \$2.50 each. Jack has a budget of \$100 per day. Let *m* be the number of muffins Jack buys and *c* be the number of croissants Jack buys.

a Write an inequality that describes how Jack can spend his budget each day.
b If Jack buys 20 muffins, write an inequality that describes the number of croissants he can buy.
c If Jack buys 30 croissants, write an inequality that describes the number of muffins he can buy.
d Assume Jack is able to sell all the muffins and croissants he buys on a particular day. Let *P* be the profit Jack will make that day from muffins and croissants. Write an equation that describes the profit Jack will make.

On another day, Jack made \$170 profit.

e Sketch the graph of the number of muffins against the number of croissants Jack could have bought and sold. Hint: Put muffins on the vertical axis.
f On the day Jack made \$170 profit, the price he paid for the muffins and croissants was \$99.20. Determine the number of muffins and croissants Jack bought and sold that day.

2 **a** Solve the pair of simultaneous equations $y = 2x - 1$ and $y = x + 5$.
b Sketch the following linear inequalities on the same set of axes: $y \geq 2x - 1$ and $y < x + 5$. Shade the region where the points satisfy both inequalities.
c Does the point (0, –1) lie in the region? Explain.
d Does the point (0, 5) lie in the region? Explain.
e Does your solution to part **a** lie in the region? Explain.
f Does the point (0, 0) lie in this region?
g Given $y \geq 2x - a$ and $y < x + b$, for what values of a and b does (0, 0) lie in the region?
h Given $y \leq 2x - a$ and $y \geq x + b$, for what values of a and b does (0, 0) lie in the region?
i Given $y \geq 2x - a$ and $y < x + b$, for what values of a and b does (–2, 3) lie in the region?
j Solve the pair of simultaneous equations $y = 2x - 1$ and $y = x^2$.
k Use graphing software to draw the region where $y \geq 2x - 1$, $y < x + 5$ and $y \geq x^2$.
l Given $y \geq 2x - a$, $y < x + b$ and $y \geq x^2 + c$ for what values of a, b and c does (–2, 3) lie in the region?

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3 If $ax^2 + bx + c = 2x^2 - 5x + 3$, then the corresponding terms must be the same and so $a = 2$, $b = -5$ and $c = 3$.

- **a** Find the value of the pronumerals that make the following equations true.
  - **i** $ax^2 + bx + c = (x + 3)(2x + 1)$
  - **ii** $(ax + 4)(bx - 1) = 3x^2 + x - 4$
  - **iii** $(ax + b)(cx - 5) = -12x^2 - 15x$
  - **iv** $(ax + 1)(bx + c) = 4x^2 - 1$
  - **v** $(ax - 3)(bx + c) = 12x^2 + 11x - 15$, where $b > 0$
- **b** If $(ax + b)(cx + d) = x^2 + fx + 1$, find:
  - **i** the value of f in terms of c and d.
  - **ii** the solutions to $x^2 + fx + 1 = 0$ in terms of c and d.
  - **iii** the solutions to $x^2 + fx + 1 = 0$ in terms of f.
- **c** For what values of f do the solutions in part **b** exist?

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	**I need to review this**
☐ Solve linear equations with an unknown on both sides ☐ Solve linear equations involving algebraic fractions ☐ Rearrange literal equations	☐ Go back to Topic 4A Solving linear equations
☐ Solve linear inequalities using inverse operations ☐ Test if a point on the Cartesian plane is a solution to a linear inequality ☐ Graph regions corresponding to linear inequalities on the Cartesian plane	☐ Go back to Topic 4B Solving linear inequalities
☐ Solve simultaneous linear equations graphically ☐ Solve simultaneous linear equations using substitution ☐ Solve simultaneous linear equations using elimination	☐ Go back to Topic 4C Simultaneous linear equations
☐ Solve quadratic equations by factorisation ☐ Solve quadratic equations by completing the square ☐ Verify the solution to a quadratic equation by substitution	☐ Go back to Topic 4D Solving quadratic equations
☐ Identify the number of solutions to a quadratic equation using the discriminant ☐ Solve quadratic equations using the quadratic formula	☐ Go back to Topic 4E The quadratic formula
☐ Solve simple cubic equations of the form $ax^3 = k$ ☐ Write the answer of cubic equations as exact and approximated values	☐ Go back to Topic 4F Solving cubic equations

5

Linear and non-linear relationships

Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✓ Gradients
- ✓ Intercepts
- ✓ Plotting linear relationships
- ✓ The Cartesian plane

Curriculum links

- STN ADV Identifies and solves problems involving direct and inverse variation and their graphical representations (MA5-RAT-P-01)
- ADV Analyses and constructs graphs relating to rates of change (MA5-RAT-P-02)
- ADV Describes and applies transformations, the midpoint, gradient/slope and distance formulas, and equations of lines to solve problems (MA5-LIN-P-01)
- ADV Interprets and compares non-linear relationships and their transformations, both algebraically and graphically (MA5-NLI-P-01)
- ADV Establishes and applies the laws of logarithms to solve problems (MA5-LOG-P-01)
  - Examine logarithms both numerically and graphically

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5A Linear relationships

Learning intentions

By the end of this topic you will be able to ...

✔ rearrange linear equations from gradient–intercept form to general form and vice versa

✔ find the *x*-and *y*-intercepts and graph the equation of a straight line in any form

✔ use point–gradient form to find a linear equation.

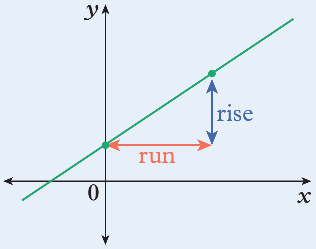
Inter-year links

Support	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6F Plotting linear relationships
Year 9	4E Sketching linear graphs

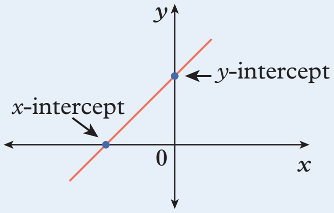
Linear graphs

Key content video

- A linear equation with two pronumerals describes a **linear relationship** between two variables.
- A linear relationship can be represented on the Cartesian plane as a straight line called a **linear graph**.
- An intercept is the point where a graph crosses an axis.
  - → The x-intercept is the point at which the graph crosses the x-axis. That is, where $y = 0$.
  - → The y-intercept is the point at which the graph crosses the y-axis. That is, where $x = 0$.
  - → The values $x = 0$ and $y = 0$ can be substituted into a linear equation in order to find the line's x- and y-intercepts and sketch the graph.
- The **gradient** is a numerical measure of the slope or steepness of a graph. It can be calculated as the ratio of the change in the y-values (the **rise**) to the change in the x-values (the **run**).



$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$



Gradient–intercept form

- The equations for all linear relationships can be expressed in general **gradient–intercept form** shown on the right, where:
  - → m is the gradient of the line
  - → c is the y-coordinate of the y-intercept (the point where the line crosses the y-axis).
- Gradient–intercept form can be used to identify the gradient and y-intercept, enabling us to sketch the graph.

$$y = \underset{\text{gradient}}{m}x + \underset{y\text{-intercept}}{c}$$

General form

- All linear relationships in the gradient–intercept form can be rearranged to **general form** $ax + by + c = 0$ by performing the same operations on both sides of the equation and vice versa.

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For example,

$$\begin{aligned} y &= -\frac{3}{4}x + 2 && \left(+\frac{3}{4}x\right) \\ \frac{3}{4}x + y &= 2 && (\times 4) \\ 3x + 4y &= 8 && (-8) \end{aligned}$$

$3x + 4y - 8 = 0$ where $a = 3, b = 4, c = -8$

- In general form $ax + by + c = 0$, a, b, c can be any real number. The gradient of a linear relationship in the form $ax + by + c = 0$ is $-\frac{a}{b}$. The coordinate of the x-intercept is $\left(-\frac{c}{a}, 0\right)$, and the y-intercept is $\left(0, -\frac{c}{b}\right)$.
  - → When $a = 0$, $by + c = 0$ is a horizontal graph, the gradient of a horizontal graph is 0.
  - → When $b = 0$, $ax + c = 0$ is a vertical graph, the gradient of a vertical graph is not defined.
  - → a and b can't both be 0 at the same time.
  - → When $c = 0$, $ax + by = 0$ passes though the origin (0, 0).

Point–gradient form

- The equation of a linear graph with gradient m passing through a point (x_1, y_1) can be found by substituting the values of m, x_1 and y_1 into **point–gradient form** $y - y_1 = m(x - x_1)$.
- The gradient of a linear graph between any point on the line and the given point (x_1, y_1) on the line is constant. This means $m = \frac{y - y_1}{x - x_1}$.

Example 5A.1 Rearranging linear equations

a Rearrange $\frac{y}{3} = \frac{x}{2} - 5$ into general form.

b Rearrange $2x + 4y = 5$ into gradient–intercept form.

THINK	WRITE
a **1** Perform the same operations on both sides of the equation to move all terms to one side of the equals sign.	**a** $\frac{y}{3} = \frac{x}{2} - 5 \quad (+5)$ $\frac{y}{3} + 5 = \frac{x}{2} \quad \left(-\frac{x}{2}\right)$ $\frac{y}{3} + 5 - \frac{x}{2} = 0$
2 Apply the commutative law of addition to arrange the equation in the form $ax + by + c = 0$. If needed, multiply both sides of the equation with the lowest common denominator 6 to make all the coefficients integers.	$-\frac{x}{2} + \frac{y}{3} + 5 = 0$ or $-3x + 2y + 30 = 0$
b **1** Perform the same operations on both sides of the equation to have the y term on one side of the equation and all the other terms on the other side of the equation.	**b** $2x + 4y = 5 \quad (-2x)$
2 Perform the same operations on both sides to remove the coefficient of y.	$4y = 5 - 2x \quad (\div 4)$ $y = \frac{5}{4} - \frac{2x}{4}$
3 Apply the commutative law of addition to arrange the equation in the form $y = mx + c$. Simplify the fractions where possible.	$y = -\frac{x}{2} + \frac{5}{4}$

Example 5A.2 Determining the gradient and intercepts of a linear relationship

Identify the gradient and the coordinates of the intercepts of $y - 2x + 1 = 0$

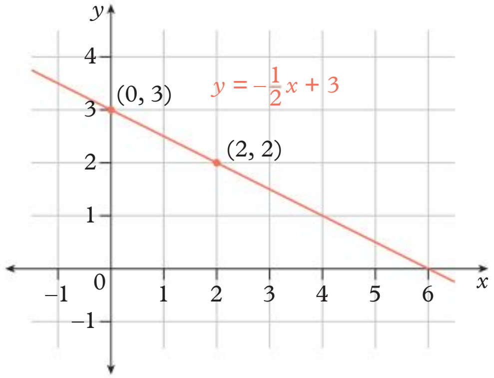
THINK	**WRITE**
1 Rearrange the equations into gradient–intercept form of $y = mx + c$ to identify the value of m.	**a** $y - 2x + 1 = 0$ $(+\ 2x)$ $y + 1 = 2x$ $(-\ 1)$ $y = 2x - 1,$ $m = 2, c = -1$ The gradient of $y - 2x + 1 = 0$ is 2.
2 Determine the x-intercept by substituting $y = 0$ into the linear equation and solving for x.	When $y = 0, 0 - 2x + 1 = 0$ $x = \frac{1}{2}$ The coordinates of the x-intercept is $\left(\frac{1}{2}, 0\right)$.
3 Determine the x-intercept by substituting $x = 0$ into the linear equation and solving for y.	When $x = 0, y = 2(0) - 1$ $y = -1$ The coordinates of the y-intercept is $(0, -1)$.

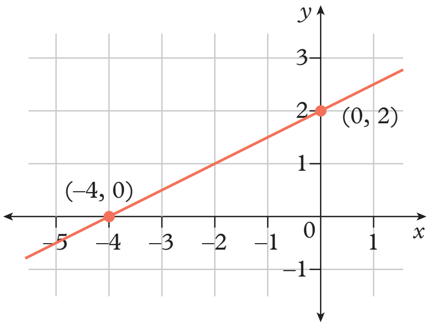
Example 5A.3 Sketching linear graphs

a Sketch the graph of $y = -\frac{1}{2}x + 3$ using gradient–intercept method.

b Sketch the graph of $-2x + 4y - 8 = 0$ by first finding the x- and y-intercepts.

THINK	**WRITE**
a 1 Identify the gradient, m, and the y-intercept, c, by comparing the equation to the gradient–intercept form of the equation: $y = mx + c$.	**a** $y = -\frac{1}{2}x + 3$ From $y = mx + c$: $m = -\frac{1}{2}$ $c = 3$
2 Write the gradient, m, and identify appropriate values for the rise and the run.	$\frac{\text{rise}}{\text{run}} = -\frac{1}{2}$ So, rise $= -1$, run $= 2$
3 Plot a point at the y-intercept and label the coordinates $(0, 3)$. 4 Add the rise to the y-coordinate and the run to the x-coordinate of the y-intercept to find a second point on the graph. Move two units to the right and one unit down to the point $(2, 2)$. 5 Rule a straight line through the points. Label the graph with its equation.	



b **1** Determine the x-intercept by substituting $y = 0$ into the equation and solving for x. Determine the y-intercept by substituting $x = 0$ into the equation and solving for y.	**b** For x-intercept, let $y = 0$: $-2x + 4(0) - 8 = 0$ $-2x - 8 = 0$ $x = -4$ x-intercept: $(-4, 0)$ For y-intercept, let $x = 0$: $-2(0) + 4y - 8 = 0$ $4y - 8 = 0$ $y = 2$ y-intercept: $(0, 2)$
2 Plot and label the x-and y-intercepts on the Cartesian plane. **3** Rule a straight line through the points. Label the graph with its equation.	

Example 5A.4 Finding the equation of a linear relationship given the gradient and a point on the graph

Find the equation of a linear relationship with a gradient of 2 which passes through the point (4, 3).

THINK	**WRITE**
1 Substitute gradient $m = 2$, $x_1 = 4$, $y_1 = 3$ into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$ $y - 3 = 2(x - 4)$
2 Simplify the equation and write the equation in either gradient–intercept form or general form.	$y - 3 = 2x - 8$ $y = 2x - 5$ or $y - 2x + 5 = 0$

Helpful hints

- ✔ The c in general form $ax + by + c = 0$ and gradient–intercept form $y = mx + c$ are both constants. When converting a linear relationship from one form to the other, the value of c is not the same.
- ✔ When writing a linear relationship in general form where the coefficients of one or both terms are fractions, the denominator or the lowest common denominator can be multiplied on both sides of the equation to make all the coefficients in the relationship integers.
- ✔ To find the x- and y- intercepts of a linear relationship, you can substitute $x = 0$ and $y = 0$ into any form of the equation – the result is always the same!

ANS p678

Exercise 5A Linear relationships

△ 1–2(a, c, e, g, i), 3–6, 7–8(1st, 2nd column), 9(a, c, d, e), 10(b, f, j), 11, 12 (1st column), 16

□ 1–2(b, c, f, g), 4, 6, 7(3rd, 4th columns), 8(a, d, f), 9, 10(1st, 2nd columns), 12–17

⬠ 1–2(j–l), 9, 10(3rd, 4th columns), 13, 16–18

UNDERSTANDING AND FLUENCY

5A.1 **1** Rearrange each of these equations to be in the form:

i $y = mx + c$ **ii** $ax + by + c = 0$.

a $y = 1 + 2x$ **b** $y = 4 - 3x$ **c** $y = -3 - \frac{3}{4}x$

d $y = 3(x - 2)$ **e** $y = -2(x - 3)$ **f** $y - 4 = -(5x + 6)$

g $3x + y = 2$ **h** $-4x + y = -5$ **i** $6x + 3y = 9$

j $-4x + 5y = 15$ **k** $3x - 2y = -4$ **l** $3y + 6 = -x$

5A.2 **2** For each of the relationships in question **1**, determine:

i the gradient

ii the coordinates of the x- and y-intercepts.

3 Consider the linear equation $y = 2x$.

a Find:

i the x-intercept

ii the y-intercept.

b Do you have enough information from part **a** to sketch the graph? Explain.

c What other information do you need? Discuss this with a classmate.

d Another way of identifying a second point to plot is to find the matching y value for a chosen x value. Any x value can be used, but we often choose $x = 1$.

Find the value of y in $y = 2x$ when x is 1. Hint: Substitute $x = 1$ into the equation.

e List the coordinates of the two points you can now use to sketch the graph of $y = 2x$.

f Plot the two points and rule a straight line through them to produce the graph of $y = 2x$. Label your graph with its equation.

4 Use the approach used in question **3** to sketch the following linear relationships.

a $y = 4x$ **b** $y = -3x$ **c** $y = x$ **d** $y = -\frac{1}{2}x$

5 **a** If $y = 4$ is compared to the gradient–intercept form, $y = mx + c$, identify the value of:

i m **ii** c

b Sketch the graph of $y = 4$ using the gradient–intercept method.

6 Sketch each of these linear relationships using the gradient–intercept method.

a $y = 6$ **b** $y = -2$ **c** $y = \frac{1}{3}$ **d** $y = -\frac{7}{2}$

5A.3 **7** Sketch the graph of each of these linear relationships.

a $x + 2y = 2$ **b** $3x + y = -6$ **c** $2x + 4y = 8$ **d** $5x + 10y = 20$

e $4x + 3y = -12$ **f** $-6x + 2y = 6$ **g** $8x - y = -8$ **h** $-x + 4y = 2$

i $9x - 3y = -18$ **j** $-12x + 9y = 36$ **k** $7x + 4y = 21$ **l** $2x - y = 5$

8 For each of the linear relationships below:

i find the x-intercept **ii** find the y-intercept **iii** sketch the graph.

a $y = x + 5$ **b** $y = x - 4$ **c** $y = 3x + 6$

d $y = 2x - 8$ **e** $y = 2 - x$ **f** $y = -6x - 3$

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PROBLEM SOLVING AND REASONING

9 Describe which method you think would be the best to use to sketch:

a linear graphs with equations in the form $y = mx + c$ (for example, $y = 3x + 2$)

b linear graphs with equations in the form $ax + by = d$ (for example, $4x + 2y = 8$)

c linear graphs that pass through the origin (for example, $y = 3x$)

d linear graphs that are horizontal lines (for example, $y = 7$)

e linear graphs that are vertical lines (for example, $x = -2$).

10 Use the most appropriate method to sketch the graph of each of these linear relationships.

a $5x - 2y = -10$ **b** $y = 2x + 4$ **c** $x + y = 5$ **d** $y = 2x$

e $y = 5$ **f** $y = 8 - 4x$ **g** $x = -9$ **h** $y = 3(x - 2)$

i $2x + 4y - 8 = 0$ **j** $y = -3x$ **k** $y = -2x + 7$ **l** $y = 3 - 6x$

11 Consider the equation $y = mx + c$.

a For a line passing through the point (x_1, y_1), the coordinates can be substituted into the equation $y = mx + c$ to produce $y_1 = mx_1 + c$. Rearrange this equation to make c the subject.

b Substitute the equation you wrote for c into $y = mx + c$.

c Rearrange your answer to part **b** and show that the equation can be expressed in point–gradient form: $y - y_1 = m(x - x_1)$.

5A.4 **12** Determine the equations of the lines with the features described in **a**–**j** below:

i expressed in the point–gradient form $y - y_1 = m(x - x_1)$ derived in question **11**, where m is the gradient and (x_1, y_1) is a point on the line

ii expressed in the gradient–intercept form $y = mx + c$.

a gradient of 3; passes through (1, 4) **b** gradient of 2; passes through (3, 9)

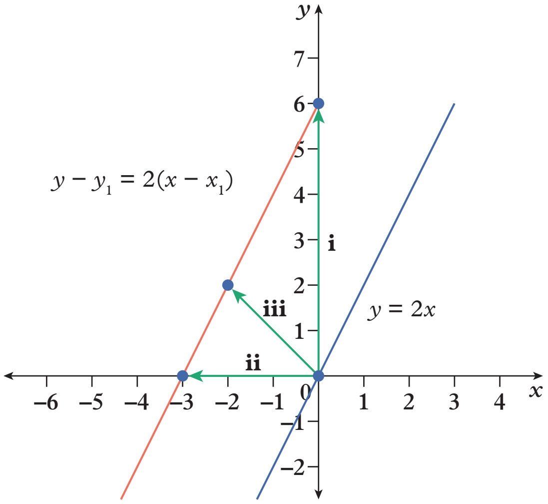
c gradient of −4; passes through (−1, 2) **d** gradient of −1; passes through (2, 7)

e gradient of 5; passes through (−3, −6) **f** gradient of −3; passes through (−4, 11)

g gradient of $\frac{6}{5}$; passes through (15, 4) **h** gradient of $-\frac{3}{4}$; passes through (8, −5)

i gradient of $\frac{5}{2}$; passes through (−3, −6) **j** gradient of $-\frac{7}{9}$; passes through (−6, 4)

13 Consider the linear graphs $y = 2x$ and $y - y_1 = 2(x - x_1)$ shown on the right. The line $y - y_1 = 2(x - x_1)$ can be considered as a translation of the line $y = 2x$. Three possible translations are shown as arrows on the diagram.



a State the translation each arrow describes.

b Write the equation of the translated line in the form $y - y_1 = 2(x - x_1)$ using the three points at the end of the arrows.

c Describe the translation required for the graph of the first equation to become the graph of the second equation each time:

i $y = 3x \rightarrow y + 2 = 3(x + 1)$

ii $y = 3x \rightarrow y - 2 = 3(x - 1)$

iii $y = -3x \rightarrow y - 2 = -3(x - 1)$

iv $y = \frac{5}{2}x \rightarrow y - 6 = \frac{5}{2}(x + 4)$

v $y = \frac{2}{5}x \rightarrow y - \frac{4}{3} = \frac{2}{5}\left(x - \frac{3}{2}\right)$

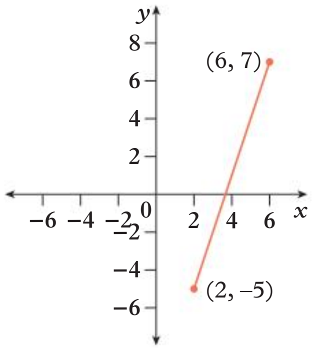
vi $y = 3x + 6 \rightarrow y - 8 = 3(x - 4)$

ADVANCED

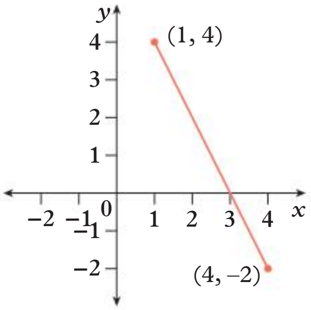
PROBLEM SOLVING AND REASONING

14 Determine the equation of the following graphs.

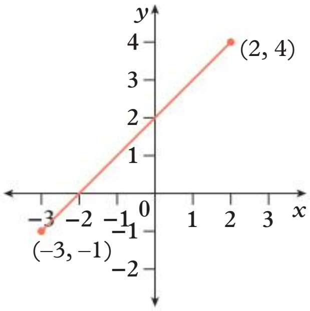
a



b



c



15 a Write an equation for a linear graph to represent each of these situations, where y represents the altitude and x represents the horizontal distance from the point of origin.

- **i** From an altitude of 50 m, a road rises 1 m vertically over a horizontal distance of 200 m.
- **ii** From an altitude of 6 km, a plane descends 2 km vertically over a horizontal distance of 30 km.
- **iii** A road rises 20 m vertically over a horizontal distance of 100 m.

b Sketch the linear graph for each situation. What values of x would be appropriate to use? Rewrite each equation you wrote in your answers to part **a** with the appropriate restriction of x values written as an inequality statement.

CHALLENGE

16 Determine the equation for each linear graph described by the information below. Write your answers without fractions.

a x-intercept of 4, passes through (20, 8)

b gradient of 0, passes through (–15, –23)

c passes through (–25, 10) and (11, –14)

d passes through the origin and (–17, –12)

17 When an equation is written in the form $ax + by = c$, we can determine both of the intercepts simultaneously.

a Determine the coordinates of the x- and y-intercepts of $ax + by = c$ in terms of a, b and c.

b Divide both sides of the equation $ax + by = c$ by c.

c Explain the relationship between the coefficients of x and y in your answer for part **b** and the coordinates in part **a**.

d Use the method in part **b** to help you determine the intercepts of the following lines.

- **i** $2x + 3y = 5$
- **ii** $4x - 7y = 11$
- **iii** $2x - 3y = 12$
- **iv** $9y - 2x = 36$
- **v** $8x - 4y = -32$
- **vi** $20x + 50y = 10$

18 The x-intercept of $ax + by = 1$ is given by the reciprocal of a and the y-intercept is given by the reciprocal of b. Use this fact to:

- **i** write the equations of the following lines in the form $ax + by = 1$, where a and b are real numbers
- **ii** write the equations in the form $px + qy = r$, where p, q and r are integers.

a x-intercept $= \frac{2}{3}$, y-intercept $= \frac{2}{7}$

b x-intercept $= -\frac{3}{5}$, y-intercept $= \frac{3}{2}$

c x-intercept $= \frac{3}{5}$, y-intercept $= -\frac{2}{5}$

d x-intercept $= -\frac{1}{2}$, y-intercept $= -\frac{1}{3}$

e x-intercept = 7, y-intercept = 11

f x-intercept = 1, y-intercept = −1

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching linear relationships using intercepts

Interactive skillsheet
Sketching linear relationships using the gradient-intercept method

Interactive skillsheet
Determining linear equations

Worksheet
Linear graphs

Topic quiz
5A

5B Gradient, midpoint and distance between two points

Learning intentions

By the end of this topic you will be able to …

- ✔ apply the formula to determine the gradient of a line given two points
- ✔ apply the formula to determine the coordinates of the midpoint given two points
- ✔ apply the formula to determine the distance between two points.

Inter-year links

Support	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6I Finding linear equations
Year 9	4H Midpoint and length of a line segment

Determining the gradient

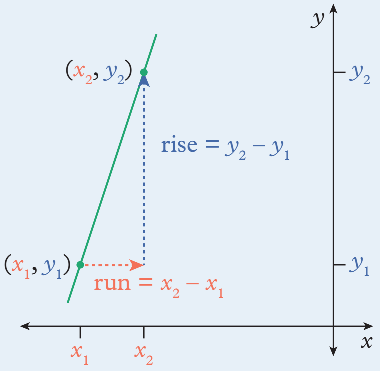
Key content video

- The formula for the gradient, m, between any two points, (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the gradient of a line that passes through (1, 3) and (2, −1) can be calculated as:

$$\begin{aligned} m &= \frac{3 - (-1)}{1 - 2} \\ &= \frac{4}{-1} \\ &= -4 \end{aligned}$$



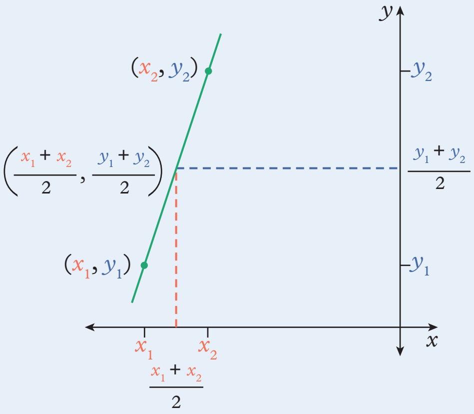
Determining the midpoint of two points

- The formula for the coordinate of the **midpoint** between any two points, (x_1, y_1) and (x_2, y_2), is

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

For example, the coordinates of the midpoint between (1, 3) and (2, −1) can be calculated as

$$\begin{aligned} (x, y) &= \left(\frac{1 + 2}{2}, \frac{3 + (-1)}{2}\right) \\ &= \left(\frac{3}{2}, 1\right) \end{aligned}$$



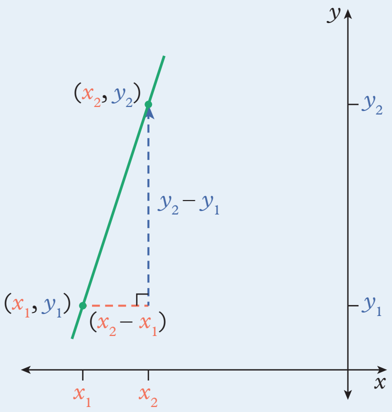
Determining the distance between two points

- The formula for the distance between any two points, (x_1, y_1) and (x_2, y_2), is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, the coordinate of the midpoint between (1, 3) and (2, −1) can be calculated as

$$\begin{aligned} d &= \sqrt{(2 - 1)^2 + (-1 - 3)^2} \\ &= \sqrt{1 + 16} \\ &= \sqrt{17} \end{aligned}$$



Example 5B.1 Determining the gradient, midpoint and length of a line segment

Determine the gradient, midpoint and length of the line segment with endpoints (3, –8) and (–7, –10).

THINK

1 Substitute the given *x*- and *y*-coordinates into the gradient, midpoint and distance formulas. The length of the line segment is equal to the distance between the two points.

2 Calculate and simplify your answers.

WRITE

Let $(x_1, y_1) = (3, -8)$ and $(x_2, y_2) = (-7, -10)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(-10) - (-8)}{(-7) - 3} \\ &= \frac{-10 + 8}{-7 - 3} \\ &= \frac{-2}{-10} \\ &= \frac{1}{5} \end{aligned}$$

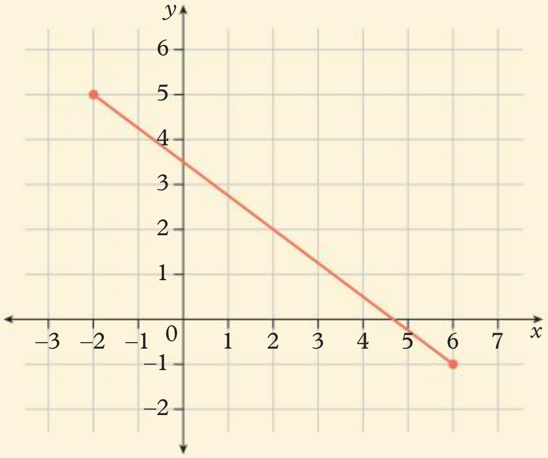
$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + (-7)}{2}, \frac{-8 + (-10)}{2}\right) \\ &= \left(\frac{-4}{2}, \frac{-18}{2}\right) \\ &= (-2, -9) \end{aligned}$$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-7 - 3)^2 + ((-10) - (-8))^2} \\ &= \sqrt{(-10)^2 + (-2)^2} \\ &= \sqrt{100 + 4} \\ &= \sqrt{104} \end{aligned}$$

The gradient is $\frac{1}{5}$, the midpoint is $(-2, -9)$ and the length of the line segment is $\sqrt{104}$ units.

Example 5B.2 Determining the gradient, midpoint and length of a line segment from a graph

Determine the gradient, midpoint and length of the line segment in the following graph.



THINK

1 Identify the x- and y-coordinates of the two points from the graph. Let $(-2, 5)$ be point 1 and $(6, -1)$ be point 2.

2 Substitute the given coordinates into the gradient, midpoint and the distance between two points formulas. The length of the line segment is equal to the distance between the two points.

3 Calculate and simplify your answers.

WRITE

Let $(x_1, y_1) = (-2, 5)$, $(x_2, y_2) = (6, -1)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 5}{6 - (-2)} \\ &= \frac{-6}{8} \\ &= -\frac{3}{4} \end{aligned}$$

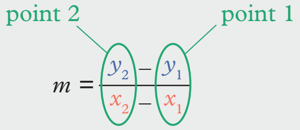
$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-2 + 6}{2}, \frac{5 + (-1)}{2}\right) \\ &= \left(\frac{4}{2}, \frac{4}{2}\right) \\ &= (2, 2) \end{aligned}$$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - (-2))^2 + ((-1) - 5)^2} \\ &= \sqrt{8^2 + (-6)^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

The gradient is $-\frac{3}{4}$, the midpoint is $(2, 2)$ and the length of the line segment is 10 units.

Helpful hints

- ✔ The order in which you substitute points into the formula for the gradient of a line won't affect the final value – you just need to make sure the x- and y-coordinates of a given point match up vertically when written as a fraction:
- ✔ When reading the coordinates of two points from a graph, which point you assign as point 1 or point 2 won't affect the final value of the gradient, the midpoint or the distance between the two points.



ANS p682

Exercise 5B Gradient, midpoint and distance between two points

▲ 1, 2(1st, 2nd columns), 3–5(a–c), 6, 7, 9

■ 2(2nd, 3rd columns), 3(e–h), 4(a, d, e), 5(d–i), 6–9, 11

⬟ 2(e, f), 3(e, h, i), 4(d, e), 5(d–i), 8–12

1 For each line with the given intercepts below:
  - i determine the gradient
  - ii determine the midpoint between the two intercepts
  - iii determine the distance between the two intercepts.

a x-intercept: $(2, 0)$, y-intercept: $(0, 4)$

b x-intercept: $(-3, 0)$, y-intercept: $(0, 3)$

c x-intercept: $(3, 0)$, y-intercept: $(0, -6)$

d x-intercept: $(1, 0)$, y-intercept: $(0, 5)$

ADVANCED

UNDERSTANDING AND FLUENCY

5B.1 **2** For each line passing through a pair of points given below:

i determine the gradient

ii determine the coordinates of the midpoint of the two points

iii determine the distance between the two points.

a (0, 2) and (2, 10) **b** (0, –7) and (3, –4) **c** (0, 5) and (4, 7)

d (–1, –4) and (0, 6) **e** (–2, 1) and (0, –5) **f** (–8, 6) and (0, 4)

3 For each of the following relationships:

i determine the coordinates of the intercepts

ii sketch the graph

iii determine the midpoint between the two intercepts

iv determine the distance between the two intercepts.

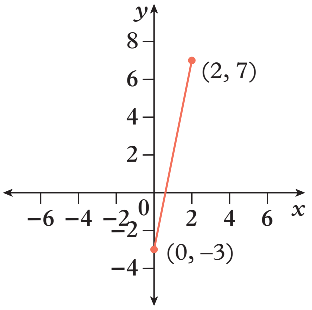
a $y = 2x - 5$ **b** $y = \frac{1}{2}x - 5$ **c** $y = -5x + 3$

d $y = -\frac{1}{5}x + 3$ **e** $y = 4x - 2$ **f** $y = \frac{3}{2}x + \frac{1}{2}$

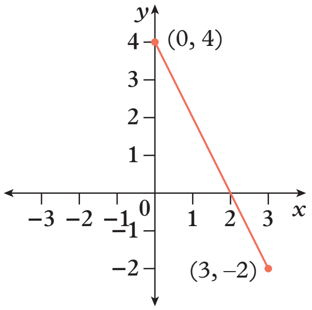
g $3x + y - 1 = 0$ **h** $-6x + y + 12 = 0$ **i** $12y - 5x = 120$

5B.2 **4** Determine the gradient, midpoint and the distance between two points in each of the following graphs.

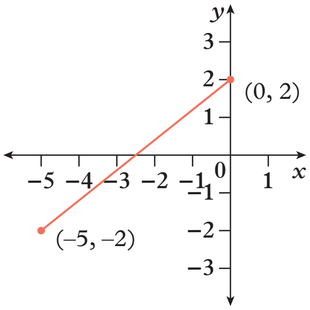
a



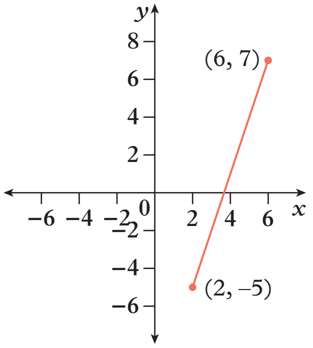
b



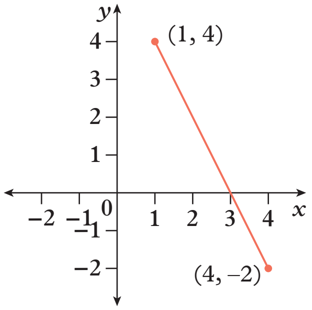
c



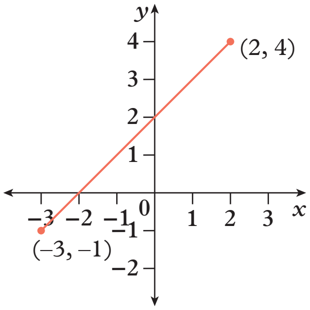
d



e



f



5 Calculate the distance between the two intercepts of the following linear relationships.

a $y = 2x + 1$ **b** $y = 6x - 24$ **c** $y = -3x + 10$

d $y = -2x + 4$ **e** $y = x - 1$ **f** $y = -x - 6$

g $2y + 16 = x$ **h** $7x + 5y + 35 = 0$ **i** $-x + 6y = 24$

6 Liza and Tate are finding the gradient of a line that passes through (–2, 2) and (4, 5). Liza writes $m = \frac{5-2}{4-(-2)}$, while Tate writes $m = \frac{2-5}{-2-4}$.

a Show that Liza and Tate both find the gradient to be $m = \frac{1}{2}$.

b Juan attempts to find the gradient of the same line and writes $m = \frac{5-2}{-2-4}$. Explain what Juan has done wrong and how his answer will compare to the correct gradient found by Liza and Tate.

c Show that $m = \frac{y_1 - y_2}{x_1 - x_2}$ is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

7 Liza and Tate are now finding the equation of the line that passes through $(-2, 2)$ and $(4, 5)$. To find the y-intercept, c, Liza writes the equation $2 = \frac{1}{2}(-2) + c$ and Tate writes $5 = \frac{1}{2}(4) + c$.

a Show that Liza and Tate will both get the same value for c.

b Juan attempts to find the equation of the same line and says it is $y = 4x + 5$. Explain what Juan has done wrong.

8 The length of a line segment between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

a Sketch each of these linear relationships on the same Cartesian plane.

i $x = -6$ for $3 \le y \le 13$ **ii** $y = 3$ for $-6 \le x \le 2$ **iii** $y = -\frac{5}{4}x + \frac{11}{2}$ for $-6 \le x \le 2$

b Calculate the perimeter of the shape formed, correct to two decimal places.

c Calculate the area of the shape formed.

9 a Sketch these linear relationships on the same Cartesian plane:

- $2x + 4y = 8$, for $4 \le x \le 8$
- $y = -2$, for $-2 \le x \le 8$
- $x = -2$, for $-2 \le y \le 2$
- $y = -x$, for $-2 \le x \le 0$
- $y = 0$, for $0 \le x \le 4$.

b Use Pythagoras' theorem to calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

10 a Sketch these linear relationships on the same Cartesian plane.

- $y = -4x + 10$, for $0 \le x \le 2$
- $y = 4x + 10$, for $-2 \le x \le 0$
- $x - 4y = -10$, for $-10 \le x \le -2$
- $4y = x - 10$, for $2 \le x \le 10$
- $y = 4x - 10$, for $0 \le x \le 2$
- $y = -4x - 10$, for $-2 \le x \le 0$
- $x + 4y = -10$, for $-10 \le x \le -2$
- $x + 4y = 10$, for $2 \le x \le 10$

b Calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

d Join the points $(-2, -2)$ and $(2, 2)$ to form a line segment. Write a rule to represent the line segment.

e Join the points $(-2, 2)$ and $(2, -2)$ to form a line segment. Write a rule to represent the line segment.

11 A helicopter rises at a constant speed from a height of 20 m above the ground, covering a vertical distance of 30 m as it flies over a horizontal distance of 100 m. For the next 100 m it travels horizontally. It then descends at a constant speed over a further horizontal distance of 100 m before landing.

a Draw a graph to represent the changing height above the ground of the helicopter over the total 300 m horizontal distance it travelled.

b Identify the rule for each of the three linear sections of the graph you drew for part **a**.

c Use the appropriate rule to calculate the height of the helicopter for each of these horizontal distances on your graph.

i 30 m **ii** 80 m **iii** 150 m **iv** 240 m

d What horizontal distance(s) have been travelled when the helicopter is at a height of 40 m?

12 The point $P(x, y)$ is equidistant from $A(4, 3)$ and $C(-3, 4)$, two vertices of the square $ABCD$.

a Show that P lies on the line $y = 7x$.

b Find the coordinates of the other two vertices of the square $ABCD$.

c What's the perimeter of the square $ABCD$?

Check your Student obook for these digital resources and more:

pro

Interactive skillsheet
Gradients

Interactive skillsheet
Midpoint of a line segment

Interactive skillsheet
Length of a line segment

Topic quiz
5B

ADVANCED

5C Graphing parabolas using intercepts

Learning intentions

By the end of this topic you will be able to ...

- ✔ find the x- and y-intercepts of a quadratic relationship
- ✔ find the coordinates of the turning point for a quadratic relationship with two x-intercepts
- ✔ graph parabolas using the intercepts and turning point.

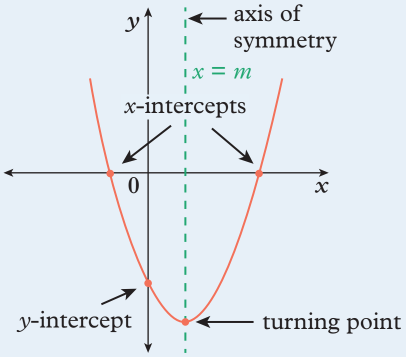
Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	5C Sketching parabolas using intercepts

Parabolas

Key content video

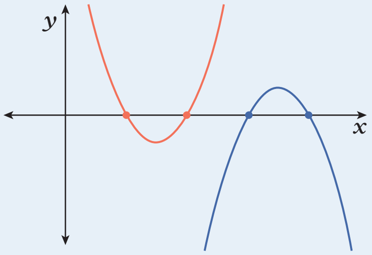
- The graph of a quadratic relationship is called a **parabola**.
- The key features of a parabola include:
  - → x- and y-intercepts (parabolas can have two, one or no x-intercepts)
  - → a **turning point** or **vertex**
  - → an **axis of symmetry**, $x = m$, where m is the x-coordinate of the turning point.



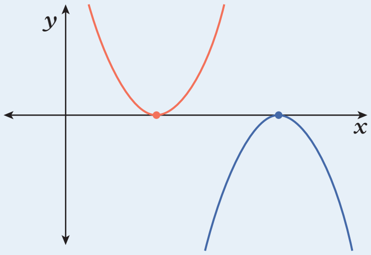
Sketching parabolas

- One way of sketching a parabola is to use the x- and y-intercepts.
  - → The y-intercept is found by substituting $x = 0$ into the equation and simplifying.
  - → The x-intercept(s) are found by substituting $y = 0$ into the equation and solving for x. A parabola can have two, one or no x-intercepts.

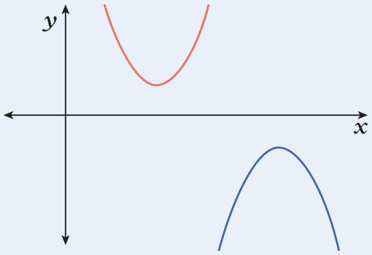
two x-intercepts



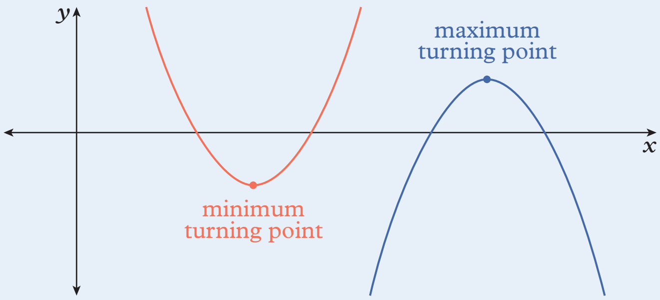
one x-intercept



no x-intercepts



- A **minimum turning point** is the turning point of a graph that opens upwards or is 'upright'.
- A **maximum turning point** is the turning point of a graph that opens downwards or is 'inverted'.
- Turning points can be identified using the axis of symmetry or the halfway point between the x-intercepts (or any two x values with the same y value).



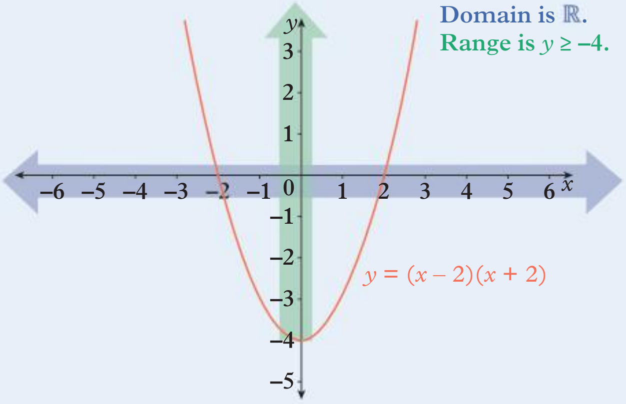
ADVANCED

Domain and range

- The **domain** of a graph is the complete set of possible x values that output real y values.

  For example, the domain of the parabola $y = (x - 2)(x + 2)$ is all real x values, which can be represented as $\mathbb{R}$.
- The **range** is the y values that would result from substituting the complete set of possible x values.

  For example, the range of the parabola $y = (x - 2)(x + 2)$ is all real y values greater than or equal to –4, which can be represented as $y \geq -4$.



Example 5C.1 Finding the x- and y-intercepts of a quadratic relationship

For each quadratic relationship below, find the coordinates of the:

i x-intercepts **ii** y-intercept.

a $y = x^2 - 6x$ **b** $y = x^2 + 7x + 10$

THINK	WRITE
a i 1 To find the x-coordinates of the x-intercepts, substitute $y = 0$ into the quadratic relationship. **2** Factorise the quadratic expression. **3** Solve the equation using the Null Factor Law.	**a i** $y = x^2 - 6x$ For the x-intercepts, $y = 0$: $x^2 - 6x = 0$ $x(x - 6) = 0$ $x = 0$ or $(x - 6) = 0$ $x = 0$ or $x = 6$ The x-intercepts are (0, 0) and (6, 0).
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic relationship and solve for y.	**ii** For the y-intercept, $x = 0$: $y = (0)^2 - 6(0)$ $= 0$ The y-intercept is (0, 0).
b i 1 To find the x-coordinates of the x-intercepts, substitute $y = 0$ into the quadratic relationship. **2** Factorise the quadratic equation. **3** Solve the equation using the Null Factor Law.	**b i** $y = x^2 + 7x + 10$ For the x-intercepts, $y = 0$: $x^2 + 7x + 10 = 0$ $(x + 2)(x + 5) = 0$ $x + 2 = 0$ or $x + 5 = 0$ $x = -2$ or $x = -5$ The x-intercepts are (–2, 0) and (–5, 0).
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic relationship and solve for y.	**ii** For the y-intercept, $x = 0$: $y = (0)^2 + 7(0) + 10$ $= 10$ The y-intercept is (0, 10).

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Example 5C.2 Finding the turning point of a quadratic relationship using the x-intercepts

Find the coordinates of the turning point for $y = x^2 - 6x$.

THINK	**WRITE**
1 Find the x-intercepts by substituting $y = 0$ into the relationship. Solve for x using the Null Factor Law.	$y = x^2 - 6x$ $0 = x(x - 6)$ $x = 0$ or $x = 6$ The x-intercepts are $(0, 0)$ and $(6, 0)$.
2 Calculate the x-coordinate of the turning point by finding the midpoint of the x-intercepts.	For the x-coordinate of the turning point: $x = \frac{0 + 6}{2}$ $= \frac{6}{2}$ $= 3$
3 Calculate the y-coordinate of the turning point by substituting the x-coordinate into the quadratic relationship.	When $x = 3$: $y = (3)^2 - 6(3)$ $= 9 - 18$ $= -9$
4 Write the coordinates of the turning point.	The turning point is $(3, -9)$.

Example 5C.3 Sketching a parabola using the intercepts

Sketch the graph of $y = -x^2 + 4x + 5$ by first finding the x- and y-intercepts. Label the turning point with its coordinates. List the domain and range of the graph.

THINK	**WRITE**
1 Find the x-intercepts by substituting $y = 0$ into the relationship and solving it for x. Factorise so that the Null Factor Law can be used.	For the x-intercepts, $y = 0$: $-x^2 + 4x + 5 = 0$ $-(x^2 - 4x - 5) = 0$ $x^2 - 4x - 5 = 0$ $(x + 1)(x - 5) = 0$ $x + 1 = 0$ or $x - 5 = 0$ $x = -1$ or $x = 5$ The x-intercepts are $(-1, 0)$ and $(5, 0)$.
2 Find the y-intercept by substituting $x = 0$ into the relationship and simplifying.	For the y-intercept, $x = 0$: $y = -(0)^2 + 4(0) + 5$ $= 5$ The y-intercept is $(0, 5)$.

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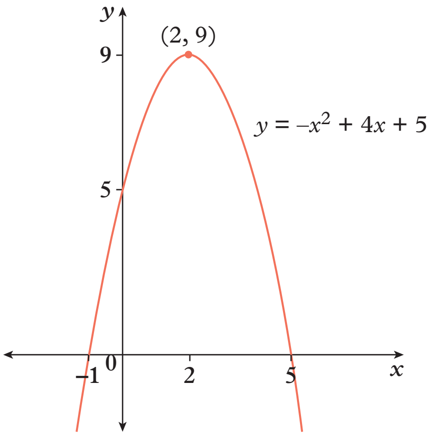
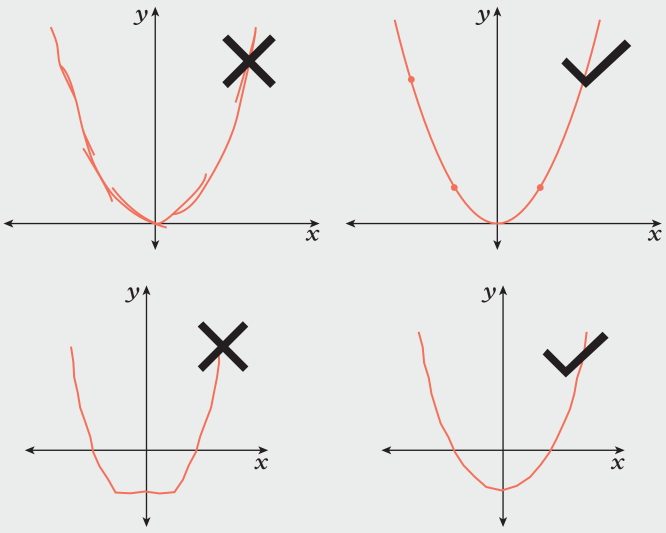
3 Find the coordinates of the turning point. First, calculate the x-coordinate by finding the midpoint of the two x-intercepts, then substitute that value into the equation to find the corresponding y value. Use both values to write down the coordinates of the turning point.	For the x-coordinate of turning point: $x = \frac{-1 + 5}{2}$ $= \frac{4}{2}$ $= 2$ When $x = 2$: $y = -(2)^2 + 4(2) + 5$ $= -4 + 8 + 5$ $= 9$ The turning point is (2, 9).
4 Plot the intercepts and the turning point on the Cartesian plane. **5** Draw an inverted parabola through the points and label the graph with the equation.	
6 List the domain by identifying the x values used to produce the graph. The set of real numbers is represented by ℝ.	Domain is ℝ.
7 List the range by identifying the y values used to produce the graph.	Range is $y \le 9$.

Helpful hints

- ✔ Rather than drawing messy inaccurate parabolas, try starting at the turning point and drawing a smooth curve up or down through the other plotted points. Do not connect the dots with straight lines.
- ✔ Label your working out so that it is clear when you are calculating the x-intercepts, the y-intercept and the turning point. This can make it easier to check your calculations when you are doing a test.
- ✔ Make sure that your parabola has a clearly defined turning point. If your graph has a flat bottom or top where the turning point should be, you may lose marks when doing a test.



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Exercise 5C Graphing parabolas using intercepts

△ 1–6, 7(a–e), 8, 10, 11 □ 3–6, 7(c–g), 8–12, 13(a) , 14(a) ⬠ 7(d–i), 8–11, 13, 14

UNDERSTANDING AND FLUENCY

5C.1 1 For each quadratic relationship below, find the coordinates of:

i the x-intercepts ii the y-intercept.

a $y = x^2 + 6x$ **b** $y = -x^2 + 10x$ **c** $y = x^2 - 6x + 5$

d $y = x^2 - 1$ **e** $y = -x^2 - 4x + 12$ **f** $y = 9 - x^2$

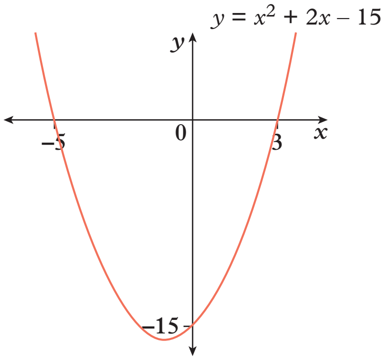
2 For each quadratic relationship in question **1**, state whether its graph is an upright parabola or an inverted parabola.

3 What is the axis of symmetry for each quadratic relationship in question **1**?

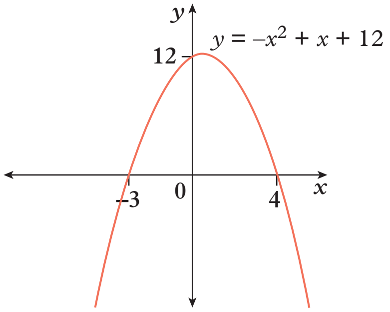
5C.2 4 Find the coordinates of the turning point for each quadratic relationship in question **1**.

5 Find the coordinates of the turning point for each of these parabolas.

a



b



6 Sketch the graph of each quadratic relationship in question **1**, use your answers for questions **1–4** to help you.

5C.3 7 Use intercepts to help you sketch the graph of each quadratic relationship below. Label each turning point with its coordinates. List the domain and range of each graph.

a $y = x^2 - 6x + 8$ **b** $y = -x^2 + 10x - 16$ **c** $y = x^2 + 4x - 12$

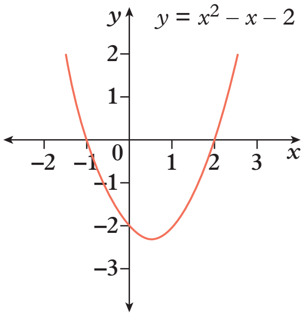
d $y = 4 - x^2$ **e** $y = x^2 + 2x$ **f** $y = -x^2 + 10x - 25$

g $y = x^2 - 4x - 21$ **h** $y = x^2 - x - 6$ **i** $y = x^2 + 7x + 10$

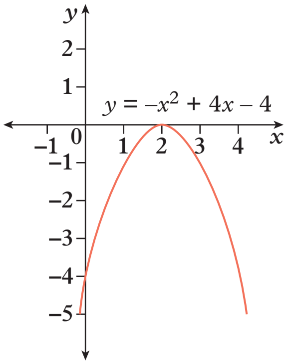
8 Consider the graphs of the three quadratic relationships shown below.

a How many x-intercepts does each parabola have?

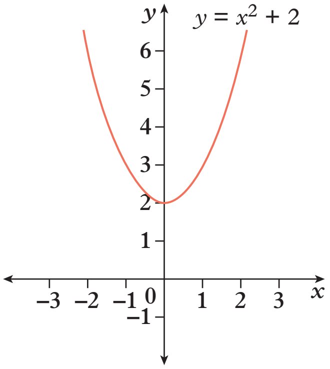
i



ii



iii



b Use the graphs to help you solve the following:

i $x^2 - x - 2 = 0$ **ii** $-x^2 + 4x - 4 = 0$ **iii** $x^2 + 2 = 0$

9 The solutions of a quadratic equation in the general form $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

a Write the coordinates of the x-intercepts of $ax^2 + bx + c = 0$ if $\sqrt{b^2 - 4ac} \geq 0$.

b Use your answer to part **a** to write a formula for the axis of symmetry.

c Substitute the values from the equations in question **8** to verify if your formula for the axis of symmetry applies to quadratic equations with two, one and no x-intercepts.

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PROBLEM SOLVING AND REASONING

10 Consider the graphs of $y = (x + 5)(x - 1)$, $y = 2(x + 5)(x - 1)$ and $y = -(x + 5)(x - 1)$.

a Find the coordinates of the x-intercepts for each graph.

b Explain why the graphs of these equations are different even though all parabolas have the same x-intercepts.

c Find the y-intercept for each parabola.

d Find the coordinates of the turning point for each parabola.

e Sketch all three parabolas on the same Cartesian plane.

11 Thomas throws a discus. The position of the discus can be represented by $h = -0.01(d + 4)(d - 72)$, where h is the height of the discus above the ground in metres and d is the horizontal distance from where the discus was thrown, in metres.

a Sketch the graph of this relationship by finding the intercepts.

b Thomas is standing on a platform. At what height off the ground was the discus thrown?

c What horizontal distance did the discus travel before hitting the ground?

d What is the greatest height that the discus reached?

12 A ball was kicked off the ground. Its path can be represented by $y = -0.05x^2 + 1.6x$, where x is the horizontal distance (in metres) and y is the vertical distance (in metres).

a Sketch the graph of this relationship.

b What was the maximum height the ball reached? What horizontal distance had it travelled when it reached its maximum height?

c What horizontal distance did the ball travel before hitting the ground?

d If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

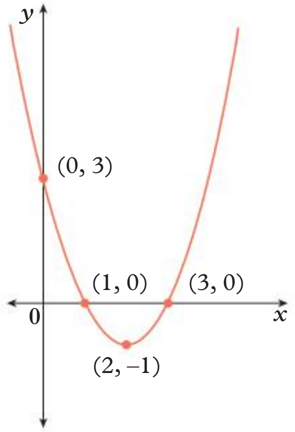
CHALLENGE

13 Use the intercepts to help you sketch the graph of each of these quadratic relationships. Label the turning point with its coordinates.

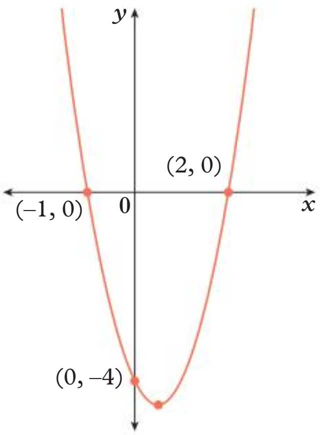
a $y = x^2 + 2x - 7$ **b** $y = -x^2 + 4x + 2$ **c** $y = x^2 - 2x - 1$

14 The following parabolas are all graphs of quadratic relationships in the form $y = a(x - m)(x - n)$. Find the equation for each graph in factorised form.

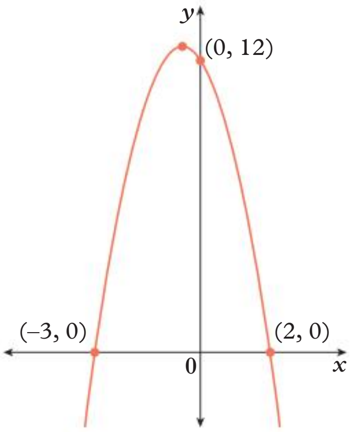
a



b



c



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching parabolas using intercepts

Investigation
Appoximating the location of an intercept

Topic quiz
5C

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5D Graphing parabolas using transformations

Learning intentions

By the end of this topic you will be able to ...

✔ sketch parabolas of the form $y = k(x - b)^2 + c$ using reflections, dilations, vertical translations and horizontal translations.

Inter-year links

Support	Transformations
Year 9	5D Sketching parabolas using transformations

Turning point form

Key content video

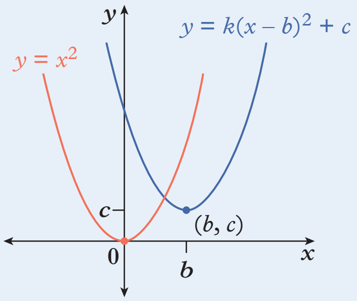
- Quadratic relationships can be written in **turning point form**, as shown below where the coordinates of the turning point are (b, c):

$$y = k(x - b)^2 + c$$

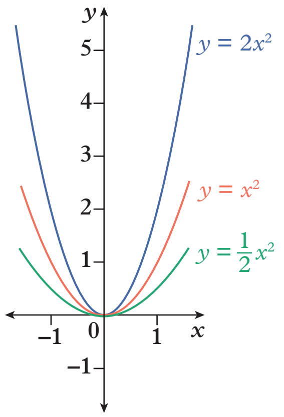
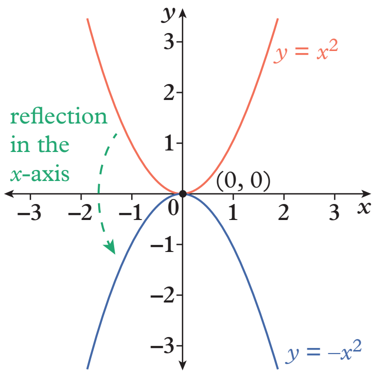
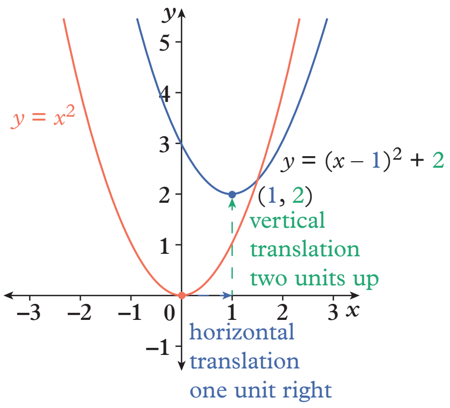
Vertical translation of c units
If $c > 0$, up
If $c < 0$, down

Dilate in the y-direction
If $k > 0$, upright
If $k < 0$, inverted

Horizontal translation of b units
If $b > 0$, right
If $b < 0$, left



- The following **transformations** can be performed on the graph of $y = x^2$ to produce the graph of $y = k(x - b)^2 + c$:
  - → a **dilation** of a factor of k in the y-direction. If k is greater than 1 or less than –1, the parabola becomes thinner. If k is between 1 and –1, the parabola becomes wider.
  - → a **reflection** across the x-axis if $k < 0$
  - → a horizontal **translation** of b units right or left
  - → a vertical translation of c units up or down.

Dilation	Reflection	Translation
		

- A combination of transformations can be performed on the graph of $y = x^2$ to produce the graph of $y = k(x - b)^2 + c$.

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Example 5D.1 Sketching a parabola using vertical and horizontal translations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform a translation to sketch the graph of each quadratic relationship below. Label the turning point and y-intercept on each parabola with their coordinates.

a $y = x^2 + 3$ **b** $y = (x + 4)^2$

THINK

a **1** Identify the transformation. The equation has the form $x^2 + c$ where $c = 3$, so the graph of $y = x^2$ undergoes a vertical translation of 3 units up, resulting in a turning point of $(b, c) = (0, 3)$.

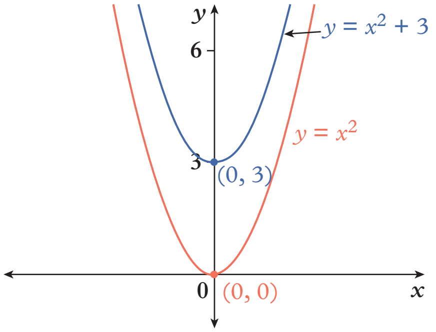
2 Sketch the graphs of $y = x^2$ and $y = x^2 + 3$, and label their turning points. The x-coordinates of both turning points are 0, so the y-intercept and the turning point is the same point for both relationships.

WRITE

a $y = x^2 \boxed{+ 3}$

Vertical translation of three units up

The turning point is (0, 3).



THINK

b **1** Identify the transformation. The equation has the form $(x - b)^2$ where $b = -4$, so the graph of $y = x^2$ undergoes a horizontal translation of 4 units to the left, resulting in a turning point of $(b, c) = (-4, 0)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = (x + 4)^2$, and label their turning points and intercepts.

WRITE

b $y = (x \boxed{+ 4})^2$

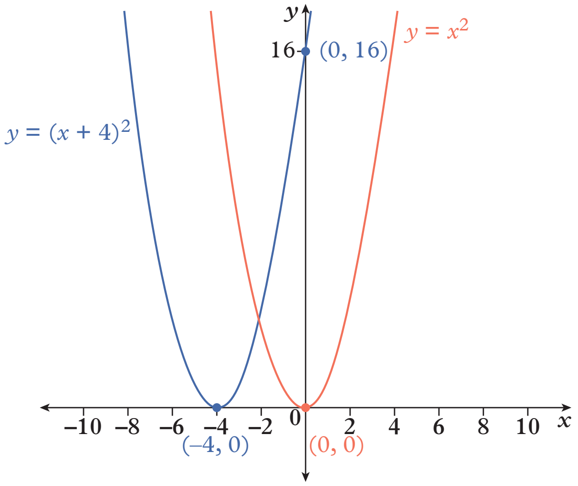
Horizontal translation of four units to the left

The turning point is (–4, 0).

For the y-intercept, $x = 0$:

$$\begin{aligned} y &= (x + 4)^2 \\ &= (0 + 4)^2 \\ &= 16 \end{aligned}$$

The y-intercept is (0, 16).



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Example 5D.2 Sketching a parabola using two transformations

Sketch the graph of each of the following parabolas by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = -(x + 3)^2$ **b** $y = (x - 2)^2 - 1$

THINK

a 1 Identify the transformations. The equation has the form $k(x - b)^2$ where $k = -1$, so the graph of $y = x^2$ is reflected in the x-axis. Because $b = -3$, the graph of $y = x^2$ also undergoes a horizontal translation of three units to the left, resulting in a turning point of $(b, c) = (-3, 0)$.

WRITE

a

$$y = \boxed{-}(x\boxed{+\ 3})^2$$

Reflection in the x-axis

Horizontal translation of three units to the left

The turning point is $(-3, 0)$.

THINK

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

WRITE

For the y-intercept, $x = 0$:

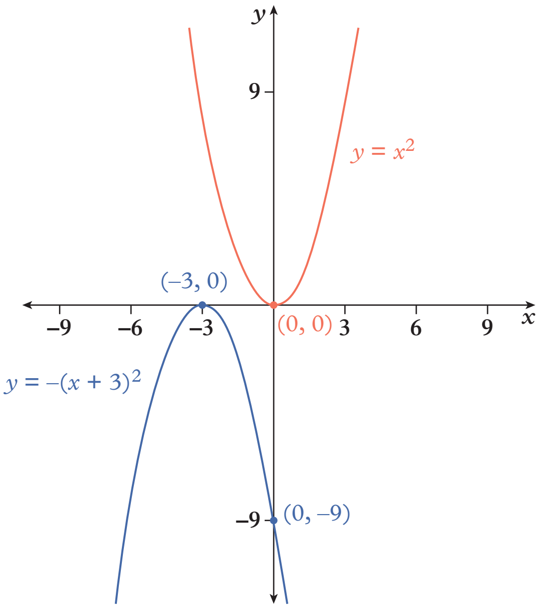
$$\begin{aligned} y &= -(x + 3)^2 \\ &= -(0 + 3)^2 \\ &= -9 \end{aligned}$$

The y-intercept is $(0, -9)$.

THINK

3 Sketch the graphs of $y = x^2$ and $y = -(x + 3)^2$, and label their turning points and intercepts.

WRITE



THINK

b 1 Identify the transformations. The equation has the form $(x - b)^2 + c$ where $b = 2$ and $c = -1$, so the graph of $y = x^2$ undergoes a horizontal translation of two units to the right and a vertical translation of one unit down, giving a turning point of $(b, c) = (2, -1)$.

WRITE

b

$$y = (x\boxed{-\ 2})^2\boxed{-\ 1}$$

Horizontal translation of two units to the right

Vertical translation of one unit down

The turning point is $(2, -1)$.

THINK

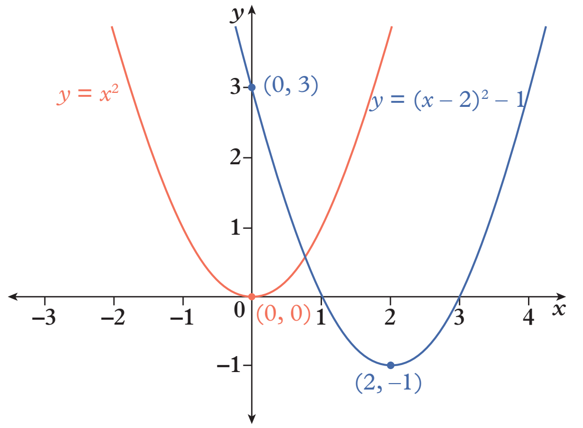
2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

WRITE

For the y-intercept, $x = 0$:

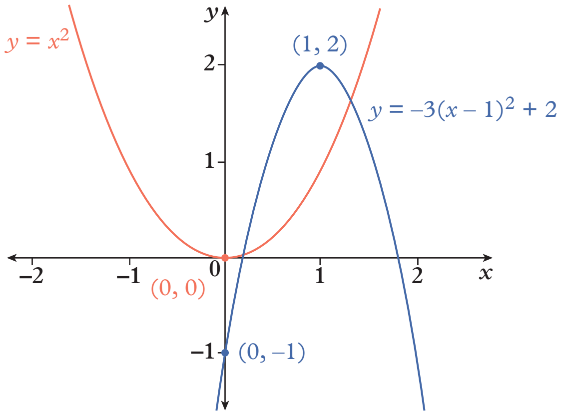
$$\begin{aligned} y &= (x - 2)^2 - 1 \\ &= (0 - 2)^2 - 1 \\ &= 3 \end{aligned}$$

The y-intercept is $(0, 3)$.

3 Sketch the graphs of $y = x^2$ and $y = (x - 2)^2 - 1$, and label their turning points and intercepts.	

Example 5D.3 Sketching a parabola using multiple transformations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform transformations on $y = x^2$ to sketch the graph of $y = -3(x - 1)^2 + 2$. For both graphs, label the y-intercept and the turning point.

THINK	WRITE
1 Identify the transformations. The equation has the form $k(x - b)^2 + c$ where $k = -3$, $b = 1$ and $c = 2$, so the graph of $y = x^2$ is: • dilated in the y-direction by a factor of 3 • reflected in the x-axis • translated horizontally 1 unit to the right • translated vertically 2 units up. resulting in a turning point of $(b, c) = (1, 2)$.	$y = \underbrace{-3}_{\text{Reflection in the } x\text{-axis and dilation in the } y\text{-direction by a factor of 3}}(x \underbrace{- 1}_{\text{Horizontal translation of one unit to the right}})^2 \underbrace{+ 2}_{\text{Vertical translation of two units up}}$ The turning point is (1, 2).
2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.	For the y-intercept, $x = 0$: $y = -3(x - 1)^2 + 2$ $= -3(0 - 1)^2 + 2$ $= -1$ The y-intercept is $(0, -1)$.
3 Sketch the graphs of $y = x^2$ and $y = -3(x - 1)^2 + 2$, and label their turning points and intercepts.	

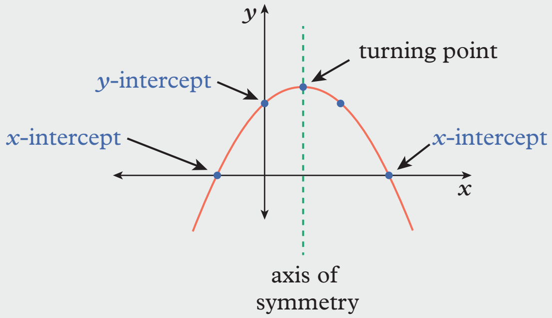
Helpful hints

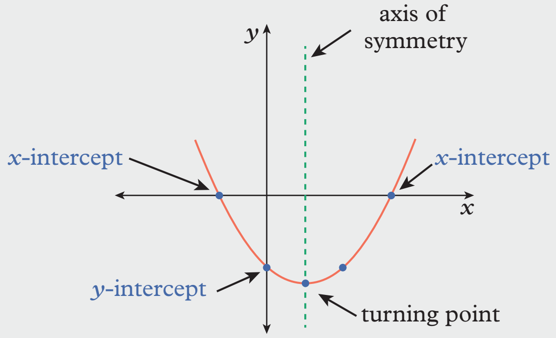
- ✔ In turning point form:
  - → the size and direction of vertical translations are determined by the constant outside the brackets
  - → the size and direction of horizontal translations are determined by the constant that is inside the brackets with x.

$$y = k(x - b)^2 + c$$

Horizontal translation of b units   Vertical translation of c units

- ✔ Always label key features when sketching a graph. These include the intercepts and the turning points.
- ✔ If you need help drawing a symmetrical curve, it can help to draw a dashed line of symmetry.





- ✔ For $y = k(x - b)^2 + c$, if $-1 < k < 1$, the graph of $y = x^2$ has undergone a 'dilation' in the direction of the y-axis, which actually compresses the graph.

  For example, if $y = x^2$ becomes $y = \frac{1}{2}x^2$ then it has been 'dilated' by a factor of $\frac{1}{2}$ in the direction of the y-axis, which really means it has been compressed by a factor of 2 in the direction of the y-axis and has become wider.

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Exercise 5D Graphing parabolas using transformations

▲ 1–7, 8(a, b, d, e), 9–11, 12, 13, 17(a)   ■ 2–7, 8(a, c, d, f), 9–11, 13, 16(a, b)   ⬟ 2–3(b, d, f), 5, 6(d–f), 8, 9, 12, 14, 16, 17

UNDERSTANDING AND FLUENCY

1 For each set of quadratic rules below:

i draw the graphs on the same Cartesian plane

ii compare each parabola with the graph of $y = x^2$

iii identify the transformation (or transformations) performed. Describe the effect of different coefficients for x^2 and any constants that are added or subtracted.

a $y = x^2$
$y = 2x^2$
$y = 4x^2$
$y = \frac{1}{2}x^2$
$y = \frac{1}{4}x^2$

b $y = x^2$
$y = x^2 + 1$
$y = x^2 + 2$
$y = x^2 - 1$
$y = x^2 - 2$

c $y = x^2$
$y = (x - 1)^2$
$y = (x - 2)^2$
$y = (x + 1)^2$
$y = (x + 2)^2$

d $y = x^2$
$y = -x^2$
$y = -x^2 + 2$
$y = -2x^2$
$y = \frac{1}{2}x^2$

5D.1 **2** Perform transformations on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the coordinates of the turning point each time.

a $y = x^2 + 3$ **b** $y = x^2 + 2$ **c** $y = x^2 - 5$

d $y = x^2 + 6$ **e** $y = x^2 - 4$ **f** $y = x^2 - 7.5$

3 Perform a transformation on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the y-intercept and the coordinates of the turning point each time.

a $y = (x - 4)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 5)^2$

d $y = (x + 2)^2$ **e** $y = (x - 8)^2$ **f** $y = (x + 1.5)^2$

4 Match each equation below with a description (**I–VI**) of the transformation/s that must be performed on the graph of $y = x^2$ to produce its graph.

a $y = x^2 - 9$
b $y = (x + 9)^2$
c $y = 9x^2$
d $y = -9x^2$
e $y = -x^2 + 9$
f $y = -(x - 9)^2$

I dilated in the y-direction by a factor of 9
II translated 9 units down
III reflected in the x-axis and then translated 9 units right
IV translated 9 units left
V reflected in the x-axis and then translated 9 units up
VI reflected in the x-axis and dilated by a factor of 9 in the y-direction

5 Write the rule for the parabola that would be produced by performing each transformation (or combination of transformations) below on the graph of $y = x^2$.

a dilated in the y-direction by a factor of 6
b reflected in the x-axis
c translated 5 units right
d translated 2 units down
e dilated in the y-direction by a factor of $\frac{1}{4}$ and reflected in the x-axis
f reflected in the x-axis and translated 3 units up

5D.2 **6** Sketch the graph of each of the following quadratic relationships by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 5)^2 - 3$ **c** $y = -(x - 2)^2 + 7$

d $y = -(x + 4)^2 - 6$ **e** $y = (x + 7)^2 + 4$ **f** $y = -(x - 6)^2 - 1$

7 Identify intervals on the real number line over which the quadratic relationships in question **6** is positive.

5D.3 **8** For each quadratic relationship:

i identify whether the graph is narrower or wider than the graph of $y = x^2$
ii identify whether the parabola is upright or inverted
iii identify the coordinates of the turning point.
iv sketch the graph.

a $y = 2(x - 4)^2 - 8$ **b** $y = -3(x + 1)^2 + 12$ **c** $y = 5(x - 3)^2$

d $y = -4x^2 + 1$ **e** $y = \frac{1}{3}(x + 6)^2 - 3$ **f** $y = -\frac{1}{2}(x - 2)^2 - 9$

9 Write the equation for the parabola produced after performing each set of transformations below on the graph of $y = x^2$.

a Dilated in the y-direction by a factor of 2 then translated 5 units right.
b Reflected in the x-axis then translated 3 units down and 4 units left.
c Dilated in the y-direction by a factor of $\frac{1}{3}$ and reflected in the x-axis.
d Dilated in the y-direction by a factor of 5, reflected in the x-axis then translated 7 units up.

10 What is the range of $y = (x + 2)^2 + 4$? Explain.

11 What is the range of $y = -(x - 6)^2 - 1$? Explain.

ADVANCED

PROBLEM SOLVING AND REASONING

12 Riley tries to kick a goal during a game of football, but accidentally hits the post. The height of the ball above the ground can be represented by the relationship $h = -4(t - 1)^2 + 5$, where h is the height in metres after t seconds.

a What are the coordinates of the turning point of the graph of this relationship?

b Sketch the graph, from $t = 0$ to $t = 1.5$, and list the domain and range.

c Use the graph to find:

i the height at which the ball made contact with Riley's boot

ii the maximum height of the ball during Riley's kick

iii the height of the football when it hits the post, 1.5 seconds after it was kicked.

d If the ball had not hit the post, how long would it have taken for the ball to hit the ground after Riley kicked it? Give your answer correct to one decimal place.

13 A soccer ball is kicked from the ground into the air. The path of the ball follows the relationship $y = -\frac{1}{40}(x - 20)^2 + 10$, where y is the height above the ground of the ball for a horizontal distance x from where the ball was kicked. Both x and y are in metres.

a Sketch the graph of this relationship from when the ball was kicked to when it landed. List its domain and range.

b What was the maximum height of the soccer ball?

c What horizontal distance did the ball travel before it hit the ground?

14 The equation $H = -x^2 + 6x + 1$ models the path of a poorly hit baseball, where H represents the height above the ground of the ball (in metres) and x represents the horizontal distance travelled by the ball (in metres).

a Substitute $x = 0$ into the equation to find the height above the ground of the ball when it was hit by the batter.

b Complete the square for $-x^2 + 6x + 1$.

c Find the maximum height reached by the ball.

d Find the horizontal distance the ball travelled before hitting the ground. Give your answer in metres rounded to two decimal places.

e Use the information found in parts **a–d** to help you sketch the graph of H versus x, labelling all relevant points.

CHALLENGE

15 Sketch each of the following parabolas by first completing the square for the quadratic, labelling all relevant points as exact values.

a $y = x^2 + 6x + 9$ **b** $y = x^2 - 2x + 3$ **c** $y = x^2 - 4x + 1$ **d** $y = 3x^2 - 6x + 5$

16 From the given information, write an equation for each of these quadratic relationships in the turning point form.

a The graph is an upright parabola, with turning point at (2, 6) and y-intercept of 10.

b The graph is an upright parabola, with turning point at (–4, –12) and y-intercept of 20.

c The graph is an inverted parabola, with turning point at (–1, 7) and y-intercept of 4.

d The graph is an inverted parabola, with turning point at (3, –5) and y-intercept of –8.

17 a Show that $(2 - x)^2$ is equivalent to $(x - 2)^2$ by expanding each expression.

b Sketch each of the following graphs, labelling all relevant points.

i $y = (2 - x)^2 + 1$ **ii** $y = (1 - x)^2 - 4$

iii $y = -\frac{1}{3}(3 - x)^2 - 1$ **iv** $y = -(-x - 2)^2 - 1$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Sketching parabolas using transformations

Worksheet Graphing parabolas

Investigation The connection between linear and quadratic graphs

Topic quiz 5D

ADVANCED

5E Cubic and other non-linear relationships

Learning intentions

By the end of this topic you will be able to ...

- ✔ sketch curves in the form $y = x^n$ and describe their features
- ✔ compare polynomial curves and describe their transformations.

Inter-year links

Support	The Cartesian plane
Year 8	4E Roots
Year 9	7B Rotations and translations

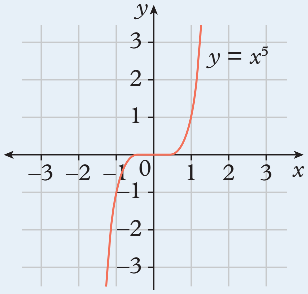
Key content video

Non-linear relationships

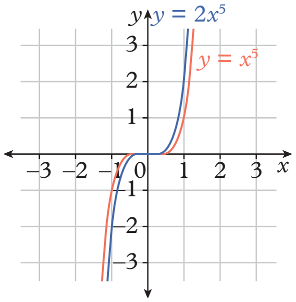
- Linear relationships produce straight-line graphs. **Non-linear relationships** produce curved-line graphs. A non-linear relationship can be sketched using a table of values.
- For all linear and non-linear relationships:
  - → The x-intercept(s) are the point(s) where the graph crosses the x-axis and $y = 0$.
  - → The y-intercept(s) are the point(s) where the graph crosses the y-axis and $x = 0$.

General transformations

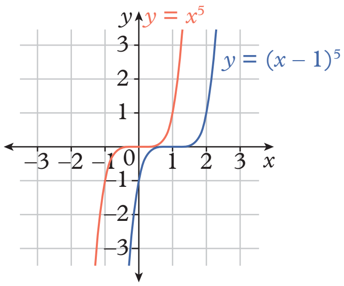
- Transformations can be performed on any non-linear relationship of the form $y = x^n$ where n is an integer and $n \geq -2$, much like the transformations applied to the turning point form of a parabola to create the graph of non-linear relationships in the form $y = k(x - b)^n + c$.



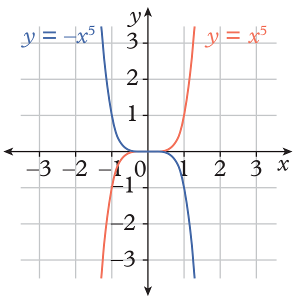
Dilation by a factor of k in the y-direction



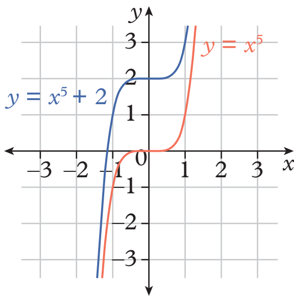
Horizontal translation of b units in the x-direction.



Reflection in the x-axis if $k = -1$.



Vertical translation of c units in the y-direction.



ADVANCED

Example 5E.1 Finding the x- and y-intercepts of a non-linear relationship

For each non-linear relationship, find the coordinates of the:

i x-intercept **ii** y-intercept.

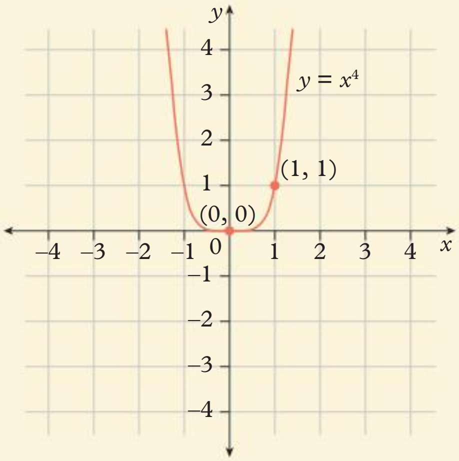
a $y = 2x^3$ **b** $y = 8 + (x - 3)^3$

THINK	**WRITE**
a i To find the x-coordinate of the x-intercept, substitute $y = 0$ into the given equation and solve for x.	**a i** For x-intercept, let $y = 0$: $2x^3 = 0$ $x^3 = 0$ $x = 0$ x-intercept: (0, 0)
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the given equation and solve for y.	**ii** For y-intercept, let $x = 0$: $y = 2(0)^3$ $y = 0$ y-intercept: (0, 0)
b i To find the x-coordinate of the x-intercept, substitute $y = 0$ into the given equation and solve for x.	**b i** For x-intercept, let $y = 0$: $8 + (x - 3)^3$ $(x - 3)^3 = -8$ $x - 3 = -2$ $x = 1$ x-intercept: (1, 0)
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the given equation and solve for y.	**ii** For y-intercept, let $x = 0$: $y = 8 + (0 - 3)^3$ $y = 8 - 27$ $y = -19$ y-intercept: (0, −19)

Example 5E.2 Sketching non-linear relationships using transformations

Sketch the following non-linear relationships by transforming the graph of $y = x^4$.

a $y = 1 + x^4$ **b** $y = -x^4$ **c** $y = (x + 2)^4$



THINK

a **1** Identify the transformations. 1 is added to the right-hand side of the equation, so the graph undergoes a vertical translation of 1 unit up.

2 Identify the corresponding coordinates of the key points. The whole graph has been moved up by 1 unit, so for the same x-coordinate, add 1 unit to the y-coordinate. The corresponding coordinates for (0, 0) and (1, 1) on the graph of $y = 1 + x^4$ are (0, 1) and (1, 2).

3 Sketch the new graph following the shape of $y = x^4$, ensure the graph passes through the points (0, 1) and (1, 2).

b **1** Identify the transformations. The right-hand side of the equation is multiplied by −1, so the graph is reflected in the x-axis.

2 Identify the corresponding coordinates of the key points. The whole graph has been reflected in the x-axis, so for the same x-coordinate, the y-coordinate is multiplied by −1. The corresponding coordinates for (1, 1) on the graph of $y = -x^4$ are (1, −1)

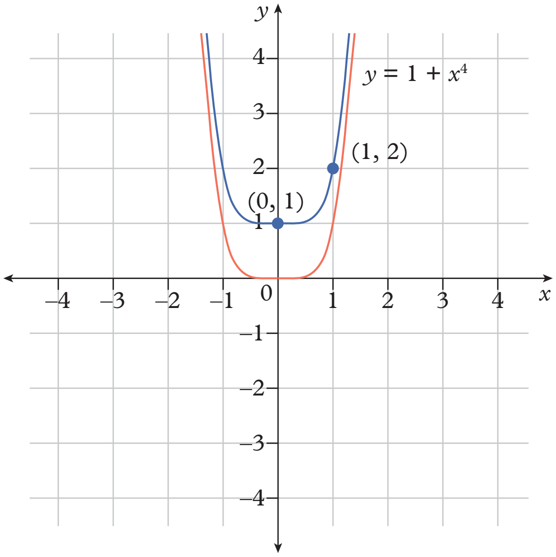
3 Reflect the graph of $y = x^4$ in the x-axis, ensuring the graph passes through the points (1, −1).

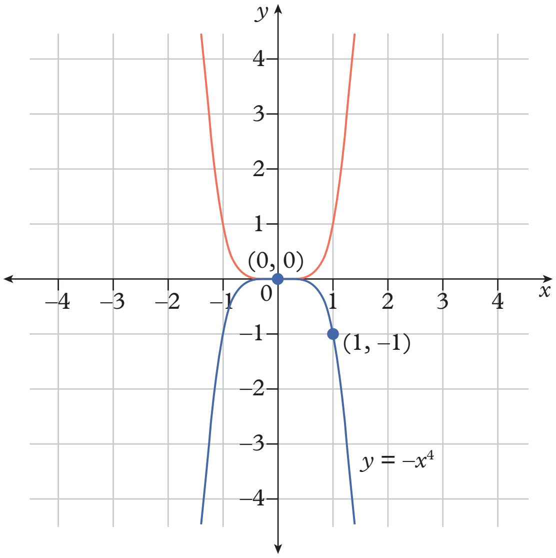
c **1** Identify the transformations. The equation has the form $y = (x - b)^4$, where $b = -2$. The graph of $y = x^4$ undergoes a horizontal translation of 2 units to the left.

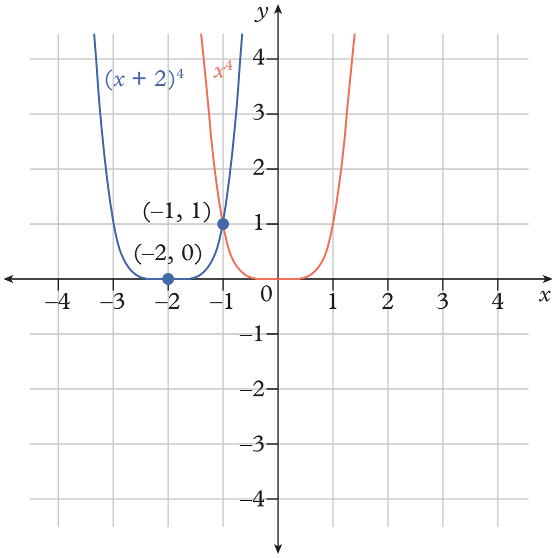
2 Identify the corresponding coordinates of the key points. The whole graph has been moved left by 2 units, so subtract 2 units from the x-coordinate. The corresponding coordinates for (0, 0) and (1, 1) on the graph of $y = (x + 2)^4$ are (−2, 0) and (−1, 1).

3 Sketch the new graph following the shape of $y = (x + 2)^4$, ensure the graph passes through the points (−2, 0) and (−1, 1).

WRITE







Helpful hints

- ✔ Remember that the values of b and c are the opposite sign to how they appear in the equation. For example, in the equation $y = (x + 2)^3 + 4$, $b = -2$ and $c = +4$.
- ✔ Non-linear relationships may have 0, 1 or multiple x-intercepts and y-intercepts.

ANS p693

Exercise 5E Cubic and other non-linear relationships

△ **1–4, 6, 9, 12(a, b), 13(a, b, c)**

■ **3–9, 11, 12, 13(a, b, c)**

⬟ **3–5, 7–9, 11–14**

UNDERSTANDING AND FLUENCY

5E.1 **1** For each non-linear relationship, find the coordinates of the:

i x-intercept **ii** y-intercept.

a $y = 3x^4$ **b** $y = 8 - x^3$ **c** $y = -2x^3$ **d** $y = (x - 2)^5$

2 Write the coordinates of the corresponding points if the following operations are performed on the intercepts you found in question **1**.

i Reflected in the x-axis

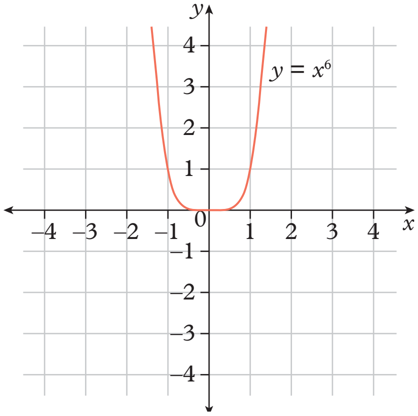
ii Translated 2 units in the x-direction

iii Translated −5 units in the y-direction

iv Translated 3 units in the y-direction first, then 3 units in the x-direction

5E.2 **3** Sketch the following non-linear relationships by transforming the graph of $y = x^6$

a $y = -x^6$ **b** $y = x^6 + 1$ **c** $y = (x + 1)^6$



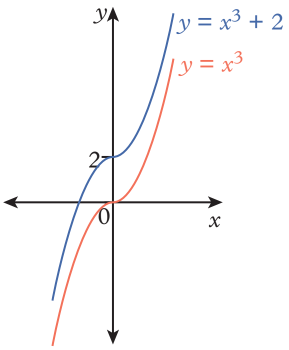
4 Consider the basic cubic relationship, $y = x^3$.

x	−3	−2	−1	0	1	2	3
y	−27						

a Complete the table of values for the relationship $y = x^3$.

b An important feature of the graph of $y = x^3$ is the stationary point at (0, 0), where the slope of the graph is flat (gradient = 0) at a single point. Use the table of values to construct a plot of $y = x^3$. Label the point of inflection with its coordinates on your graph and join the plotted points with a smooth curve.

c Describe the shape of the graphs.



d The graphs of $y = x^3$ and $y = x^3 + 2$ are shown on the right.

i Identify the coordinates of the point of inflection for $y = x^3 + 2$.

ii Describe the translation required to produce the graph of $y = x^3 + 2$ from the graph of $y = x^3$.

e Use your understanding of translations to describe how the graphs of these cubic relationships can be produced from the graph of $y = x^3$.

i $y = x^3 + 1$ **ii** $y = x^3 - 3$ **iii** $y = (x - 2)^3$

iv $y = (x + 4)^3$ **v** $y = (x - 1)^3 + 2$

f Use your answers to part **e** to sketch each relationship. Label the point of inflection with its coordinates on each graph.

5 Sketch the graphs of $y = -x^3$, $y = -x^3 + 4$ and $y = -(x - 3)^3$ on the same Cartesian plane.

6 Complete the descriptions of transformations below.

a To sketch the graph of $y = x^5 + 20$, the graph of $y = x^5$ is translated __________ units __________.

b To sketch the graph of $y = (x - 15)^5$, the graph of $y = x^5$ is translated __________ units __________.

c To sketch the graph of $y = -1 + x^4$, the graph of $y = x^4$ is translated __________ units __________.

d To sketch the graph of $y = (x + 2)^4$, the graph of $y = x^4$ is translated __________ units __________.

7 Write the equation of the new non-linear relationship if the following transformations are performed on $y = x^6$.

a The graph is stretched by a factor of 3 in the y-direction.

b The graph is reflected in the x-axis.

c The graph is translated 3 units in the x-direction.

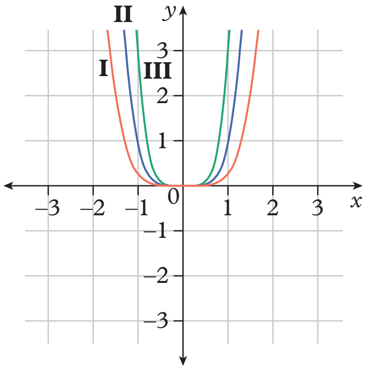
d The graph is translated −2 units in the y-direction.

8 On the same Cartesian plane, sketch the graphs of:

a $y = x^3$ **b** $y = 2x^3$ **c** $y = 3x^3$ **d** $y = \frac{1}{2}x^3$ **e** $y = \frac{1}{3}x^3$

9 Consider the effect of the coefficient you found in question **8**, and match the following equations with their corresponding graphs.

a $y = x^4$ **b** $y = 3x^4$ **c** $y = \frac{1}{3}x^4$

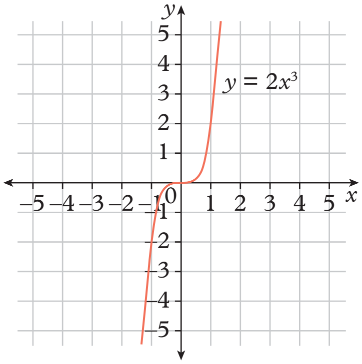


10 Sketch the following relationships by transforming the graph of $y = 2x^3$.

a $y = 2x^3 + 3$ **b** $y = 2(x - 3)^3$

11 Describe the transformations needed to create the graph of the following relationships from $y = 3x^4$.

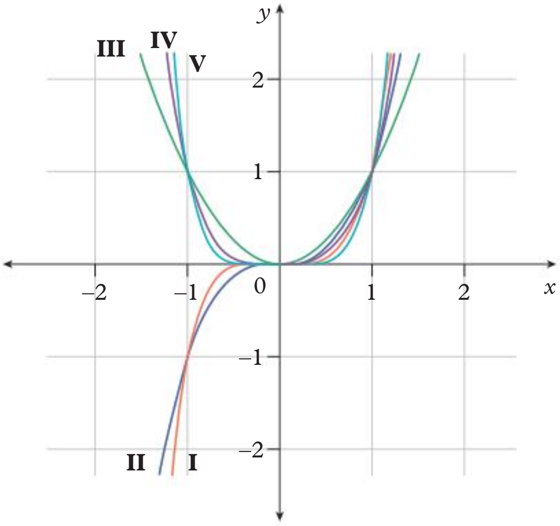
a $y = 6x^4$ **b** $y = x^4$ **c** $y = 3(x - 2)^4$



PROBLEM SOLVING AND REASONING

12 **a** Complete the following table of values for each non-linear relationship.

x	-2	-1	0	1	2
$y = x^2$					
$y = x^3$					
$y = x^4$					
$y = x^5$					
$y = x^6$					



b Use the table of values in part **a** to match the following equations with their graph.

i $y = x^2$ **ii** $y = x^3$ **iii** $y = x^4$
iv $y = x^5$ **v** $y = x^6$

c Find the x-and y-intercept of each relationship in part **b**.

d **i** Which points are always on $y = x^n$ if n is an even number?

ii Which points are always on $y = x^n$ if n is an odd number?

e When n is an even number, explain why for any point on the curve $y = x^n$ with the coordinates (a, b), there is a point with coordinates $(-a, b)$ also on the curve?

f When n is an odd number, explain why for any point on the curve $y = x^n$ with the coordinates (a, b), there is a point with coordinates $(-a, -b)$ also on the curve?

g Summarise what kind of symmetry the curve $y = x^n$ has if n is:

i an even number **ii** an odd number.

13 Consider the non-linear relationship $y = \sqrt{x}$, where only the positive roots of x are taken.

x	0	1	4	9	16	25
y	0					

a Complete the table of values for the relationship $y = \sqrt{x}$.

b Use the table of values to construct a plot of $y = \sqrt{x}$. Join the plotted points with a smooth curve.

c Describe the shape of the graph.

d An important feature of the graph of $y = \sqrt{x}$ is the point on the graph where y is a minimum. Label this point with its coordinates on your graph.

e Sketch $y = -\sqrt{x}$ on the same Cartesian plane by performing the appropriate transformation.

f Write one equation to describe all the points on both graphs. What is the shape of the graph for this equation?

g Sketch a graph of $y = x^2$ on the same Cartesian plane and label the coordinates where the two graphs intersect each other.

h Draw a dotted line that passes through the points where the two graphs intersect each other. Write an equation for this line.

i Describe the transformation required to turn the graph of $y = x^2$ to the graph in part **e**.

14 Use your observations from question **13** to sketch the graph for $x = y^3$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Non-linear relationships

Topic quiz
5E

ANS
p696

Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

ADV **5A** **1** Rearrange each of the following linear relationships into gradient–intercept form.

a $6x + y - 10 = 0$ **b** $x + 2y + 3 = 0$ **c** $-x - y - 1 = 0$ **d** $x - y - 5 = 0$

ADV **5A** **2** Rearrange each of the following linear relationships into general form. Write your answers without fractions.

a $y = 2x - 5$ **b** $y + 3 = x - 2$ **c** $y = \frac{3}{5}x + \frac{7}{10}$ **d** $3y = 2x - \frac{1}{5}$

ADV **5A** **3** For each of the following relationships:

i find the coordinates of the intercepts

ii graph the relationship on the Cartesian plane.

a $2x - 3y = 12$ **b** $4x + 5y = -100$ **c** $18x + 36y = 9$ **d** $14y - 15x = 24$

ADV **5A** **4** Determine the equation of each linear graph described by the information below. Write your answer in gradient–intercept form.

a gradient of 0, passes through $(7, 10)$ **b** gradient of 2, passes through $(-2, 7)$

c gradient of -1, passes through $(2, 7)$ **d** gradient of $\frac{1}{3}$, passes through $(6, 0)$

ADV **5B** **5** For each of the following pairs of points:

i determine the gradient of a line passing through the two points

ii determine the coordinates of the midpoint of the two points

iii determine the distance between the two points.

a $(3, 8)$ and $(7, 10)$ **b** $(13, 2)$ and $(-7, -2)$ **c** $(5, -8)$ and $(-3, 10)$ **d** $(-5, 12)$ and $(3, -2)$

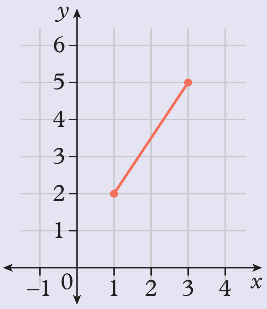
ADV **5B** **6** For each of the following graphs:

i determine the gradient of the line segments

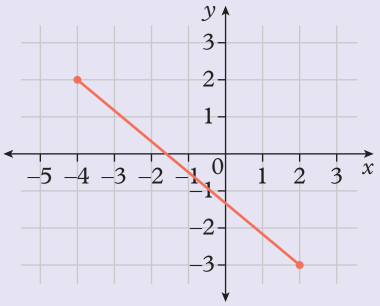
ii determine the coordinates of the midpoint of the two points

iii determine the distance between the two points.

a



b



ADV **5C** **7** Find the x-intercepts and the y-intercept for the parabolas with the following equations. You do not need to sketch the graphs.

a $y = x(x - 4)$ **b** $y = (x + 3)(x - 1)$ **c** $y = x^2 - x - 2$ **d** $y = x^2 + 6x - 16$

ADV **5C** **8** Sketch the graph of $y = (x - 1)(x - 5)$, labelling all relevant points.

ADV **5C** **9** Sketch the graph of $y = -2(x + 1)(x - 2)$, labelling all relevant points.

ADV **5D** **10** Sketch the graphs of each of the following by transformations, labelling all relevant points.

a $y = x^2 + 1$ **b** $y = x^2 - 4$ **c** $y = (x - 2)^2$ **d** $y = (x + 3)^2$

ADV **5D** **11** Sketch the graphs of each of the following by transformations, labelling all relevant points with exact values.

a $y = (x - 3)^2 + 1$ **b** $y = 2(x + 1)^2 + 2$ **c** $y = -x^2 + 1$ **d** $y = -(x + 1)^2 - 1$

ADV **5E** **12** Write the rule for the non-linear relationship resulting from performing a vertical translation of 3 units up and a horizontal translations of 2 units left on the graph of $y = x^4$.

5F Graphing circles

Learning intentions

By the end of this topic you will be able to ...

✔ graph simple circles of the form $x^2 + y^2 = r^2$

✔ graph circles of the form $(x - a)^2 + (y - b)^2 = r^2$ using vertical translations and horizontal translations

✔ determine the equation of a circle from a description or a graph.

Inter-year links

Year 9	7B Rotations and translations

Circles

Key content video

- The graph of $x^2 + y^2 = 1$ is a circle with centre at (0, 0) and a radius of $r = 1$ unit.
- The general equation of a circle graph is: $(x - a)^2 + (y - b)^2 = r^2$

  where:

  → the circle has a radius of r units

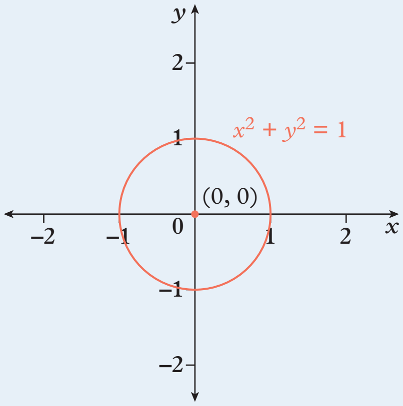
  → the coordinates of the centre of the circle are (a, b).

$$(x - a)^2 + (y - b)^2 = r^2$$

Horizontal translation of a units
If $a > 0$, right
If $a < 0$, left

Vertical translation of b units
If $b > 0$, up
If $b < 0$, down

Radius of r units

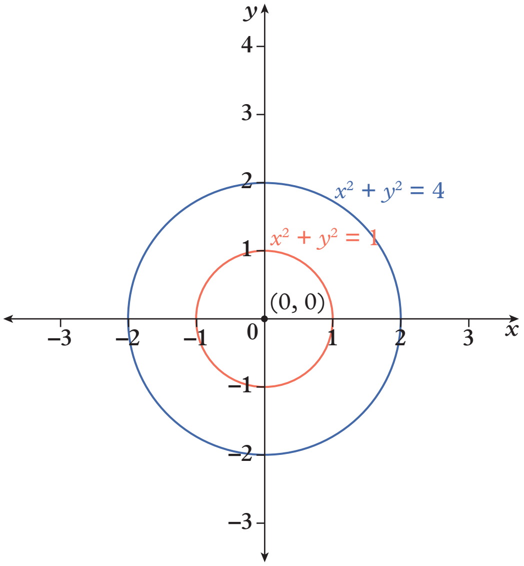
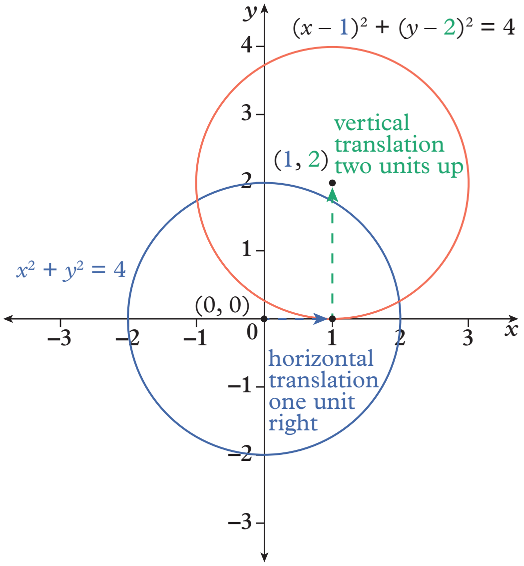


- The following transformations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the graph of $(x - a)^2 + (y - b)^2 = r^2$:

  → a dilation in all directions as r increases

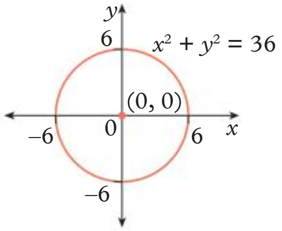
  → a horizontal translation of a units right or left

  → a vertical translation of b units up or down.

Dilation	Translation
	

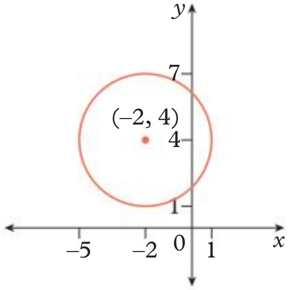
Example 5F.1 Graphing circles with a centre at the origin

Sketch the graph of $x^2 + y^2 = 36$.

THINK	**WRITE**
1 Identify the centre of the circle. The equation has the form $x^2 + y^2 = r^2$, so $h = 0$ and $k = 0$, and the centre of the circle is at the origin (0, 0).	Centre: (0, 0)
2 Identify the radius. The equation has the form $x^2 + y^2 = r^2$ so $r^2 = 36$. A radius cannot be negative, so ignore the negative value.	$r^2 = 36$ $r = \pm\sqrt{36}$ $= \pm 6$ The radius is 6 units.
3 Identifying the horizontal and vertical extremes that are 6 units left, right, below and above the centre (0, 0).	Horizontal extremes: (−6, 0) and (6, 0) Vertical extremes: (0, −6) and (0, 6)
4 Sketch the circle with a centre at the origin and a radius of 6 units. Label the graph with its equation.	

Example 5F.2 Graphing circles in the form $(x - a)^2 + (y - b)^2 = r^2$

Sketch the graph of $(x + 2)^2 + (y - 4)^2 = 9$.

THINK	**WRITE**
1 Identify the translations. The equation has the form $y = (x - a)^2 + (y - b)^2 = r^2$ where $a = -2$ and $b = 4$, so the coordinates of the centre are $(a, b) = (-2, 4)$.	Centre: (−2, 4)
2 Identify the radius. $r^2 = 9$. A radius cannot be negative, so ignore the negative value.	$r^2 = 9$ $r = \pm\sqrt{9}$ $= \pm 3$ The radius is 3 units.
3 Sketch the graph by marking the centre of the circle at (−2, 4), then identifying the horizontal and vertical extremes that are 3 units left, right, below and above the centre: (−5, 4), (1, 4), (−2, 1) and (−2, 7).	

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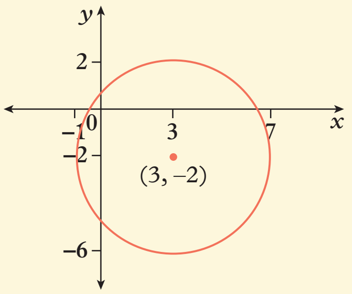
Example 5F.3 Determining the equation of a circle

Determine the equation of the circle in each case below.

a a circle with a radius of 6 units and a centre at (–2, 5)

b



THINK

The general equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$. Identify the values of a, b and r, substitute them into the equation and simplify.

WRITE

a The centre is (–2, 5) = (a, b).

$a = -2$

$b = 5$

Radius: $r = 6$

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - [-2])^2 + (y - 5)^2 = 6^2$$
$$(x + 2)^2 + (y - 5)^2 = 36$$

b The centre is (3, –2) = (a, b)

$a = 3$

$b = -2$

Radius: $r = 4$

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - 3)^2 + (y - [-2])^2 = 4^2$$
$$(x - 3)^2 + (y + 2)^2 = 16$$

Helpful hints

- ✔ Remember that the values of a and b have the opposite signs as in the equation.
  For example, in the equation $(x + 2)^2 + (y - 4)^2 = 9$, $a = -2$ and $b = +4$
- ✔ Make sure you sketch your circles on a set of axes drawn to the same scale, because scales that don't match make your circle look like an oval or an ellipse.
- ✔ Remember that the value of r is the positive square root of the right-hand side of the equation.
  For example, the radius of $(x - 1)^2 + (y - 3)^2 = 4$ is 2, not 4.
- ✔ Make sure that you mark and label the centre of your circles!

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Exercise 5F Graphing circles

△ 1–9, 13, 16, 17

■ 1(e, f), 3–7, 9, 10, 13–18

⬠ 1(e, f), 3(b, d, f, h), 5(b, d, f, h), 6, 7, 10, 11, 14, 15, 18, 19

5F.1 **1** Sketch the graph of each of these equations.

a $x^2 + y^2 = 9$

b $x^2 + y^2 = 1$

c $x^2 + y^2 = 64$

d $4x^2 + 4y^2 = 100$

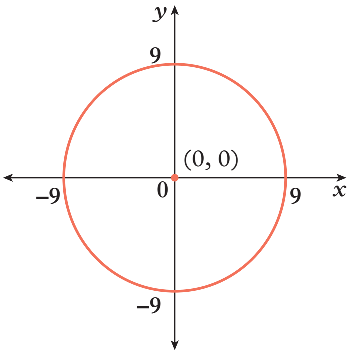
e $15x^2 + 15y^2 = 60$

f $\frac{x^2}{16} + \frac{y^2}{16} = 1$

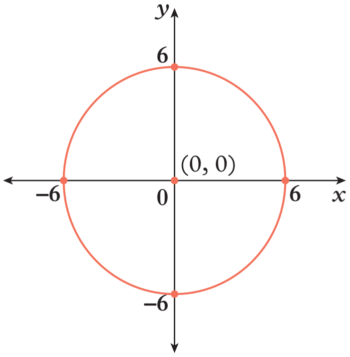
FLUENCY AND UNDERSTANDING

2 Write the equation for each of these circles.

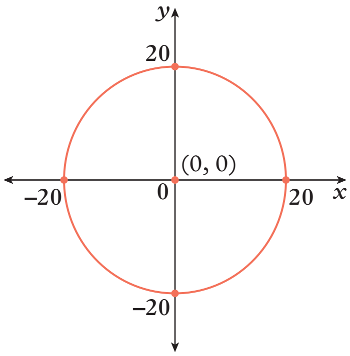
a



b



c



5F.2 **3** Sketch the graph of each of these equations. List the centre and radius.

a $(x - 3)^2 + (y - 2)^2 = 9$

b $(x - 5)^2 + (y - 7)^2 = 25$

c $(x + 1)^2 + (y - 3)^2 = 64$

d $(x - 6)^2 + (y + 6)^2 = 36$

e $(x - 2)^2 + y^2 = 16$

f $x^2 + (y + 5)^2 = 4$

g $(x + 3)^2 + (y + 5)^2 = 49$

h $(x + 4)^2 + (y + 3)^2 = 81$

4 Use digital technology to reproduce the first four graphs from question **3**. Compare those graphs to the ones you sketched.

5F.3 **5** Use the information provided to write the equation for each of these circles.

a a circle with radius 12 units and centre at (0, 0)

b a circle with radius 15 units and centre at (0, 0)

c a circle with radius 1 unit and centre at (4, –2)

d a circle with radius 7 units and centre at (–5, –6)

e a circle with radius 3 units and centre at (4, 4)

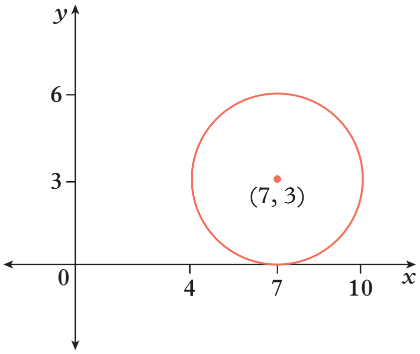
f a circle with radius 6 units and centre at (0, –5)

g a circle with radius 1.5 units and centre at (–1, 2.5)

h a circle with radius $\sqrt{5}$ units and centre at (2, –3)

6 Write the equation for each of these circles.

a



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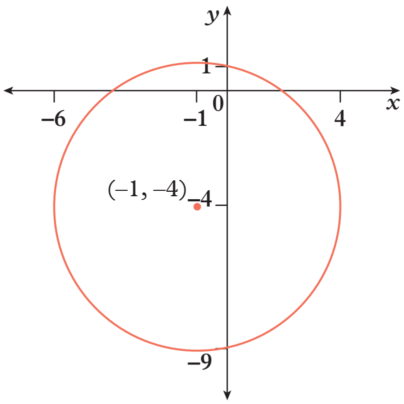
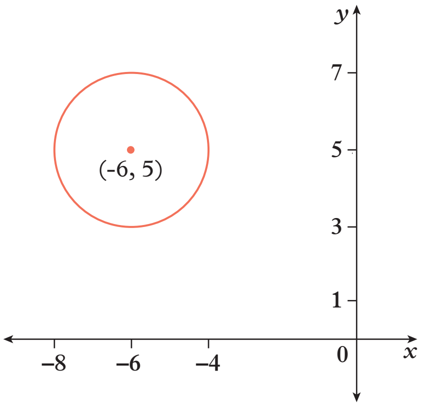
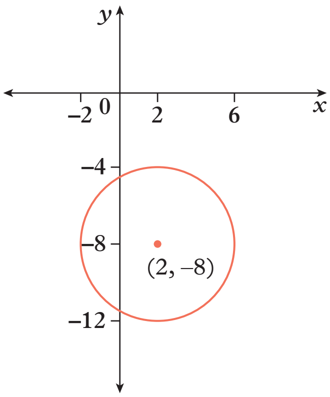


c



d



PROBLEM SOLVING AND REASONING

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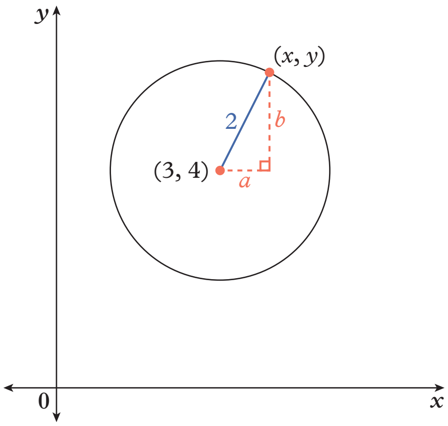
7 **a** The circle with equation $x^2 + y^2 = 1$ is dilated in all directions by a factor of 3, then translated 2 units right and 1 unit down. What is the equation of the transformed circle?

b The circle with equation $x^2 + y^2 = 1$ is transformed to become a circle with equation $(x + 7)^2 + (y - 4)^2 = 25$. Describe the transformations that have occurred. Hint: You must describe the dilation before the translations.

8 A circle is a set of points that are equidistant from a single point. This geometric description of a circle, along with Pythagoras' theorem, explains the equation of a circle.

Consider the diagram on the right, where (x, y) is generalised to be any point on the circle.



a Explain why $a = x - 3$ and $b = y - 4$.

b Use Pythagoras' theorem to explain why the equation of this circle is $(x - 3)^2 + (y - 4)^2 = 2^2$.

9 Consider $x^2 + y^2 = 25$.

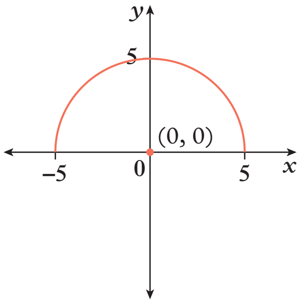
a Draw a graph of the equation.

b Rearrange the equation to make y the subject.

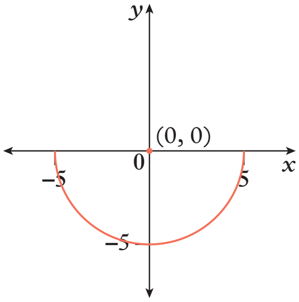
c Match each of these equations to one of the graphs shown below (**A** and **B**). Explain your reasoning.

i $y = -\sqrt{25 - x^2}$ **ii** $y = \sqrt{25 - x^2}$

A



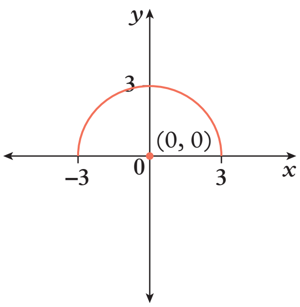
B



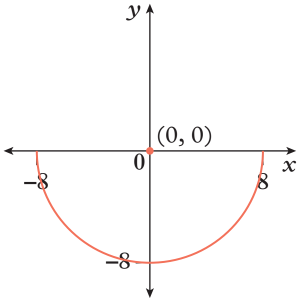
d Explain how the two half circles **A** and **B** are related to the graph of $x^2 + y^2 = 25$.

10 Write the equation for each of these half circles.

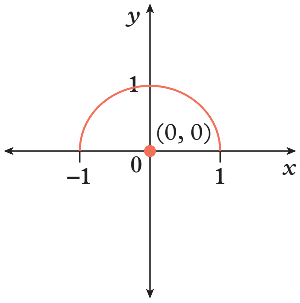
a



b



c



11 Sketch the graph of each of these half circles. List the domain and range of each graph.

a $y = -\sqrt{49 - x^2}$

b $y = \sqrt{16 - x^2}$

c $y = -\sqrt{100 - x^2}$

12 **a** Rearrange $x^2 + y^2 = 25$ to make x the subject of the equation.

b Your answer to part **a** produces two equations. What would the graph of each equation look like?

13 An arch is modelled using the relationship $(x-3)^2 + y^2 = 9$, where x is the horizontal distance from the place the left-hand side of the arch meets the ground and y is the corresponding height at that distance. Both x and y are measured in metres, $x \geq 0$ and $y \geq 0$.

a Sketch the graph of this relationship.

b What is the horizontal distance along the ground from one side of the arch to the other?

c What is the maximum height of the arch?

d Describe the position on the ground below a point where the arch is at half its maximum height.

14 The outline of a circular spa is mapped onto a Cartesian plane representing a 15 m long and 10 m wide outdoor area of a seaside resort, using the relationship $(x-6.2)^2 + (y-4.8)^2 = 2.25$.

a Sketch the graph of this relationship.

b What is the minimum distance from the water to the edge of the outdoor area?

c Calculate the area of the spa to the nearest square metre.

d What percentage of the outdoor area is taken up by the spa?

15 **a** Complete the square for each of these quadratics:

i $x^2 + 6x + 2$

ii $y^2 - 4y + 7$

b Sketch the circle with equation $x^2 + 6x + 2 + y^2 - 4y + 7 = 0$.

16 Use a graphical method to find the points of intersection of the graphs with equations $x^2 + y^2 = 25$ and $y = x + 7$.

17 Use an algebraic method to find the points of intersection of the graphs with equations:

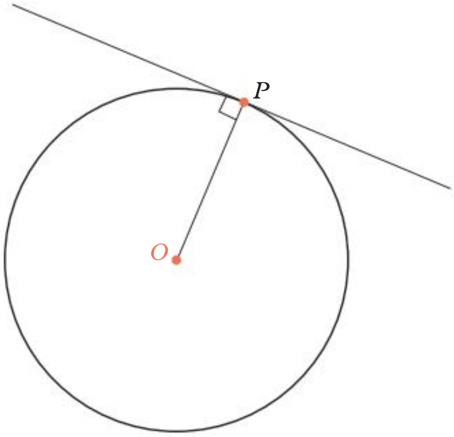
a $x^2 + y^2 = 9$ and $y = x - 3$

b $(x+3)^2 + (y-8)^2 = 4$ and $y = x + 9$.

18 Find the equation for a circle that passes through the four points (–10, 5), (–4, 11), (2, 5) and (–4, –1).

19 A tangent to a circle is a straight line which just touches the circle once, at one point on its circumference. The radius of the circle at the point of tangency is perpendicular to the tangent line.

Consider the circle with equation $(x-1)^2 + (y-2)^2 = 5$.



a What is the centre of this circle?

b Show that (3, 3) is a point on the circle.

c Find the gradient of the radius that connects the centre of the circle to the point (3, 3).

d Find the gradient of the line perpendicular to the radius referred to in part **c**.

e Use the gradient you found in part **d** to find the equation of the tangent to the circle at (3, 3). Sketch the graphs of the circle and your line on a graphing calculator to confirm that the straight line is a tangent to the circle.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circles

Worksheet
Circles on the Cartesian plane

Topic quiz
5F

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5G Graphing exponential and logarithmic relationships

Learning intentions

By the end of this topic you will be able to …

✔ sketch exponential relationships using transformations

✔ compare and contrast exponential and logarithmic graphs.

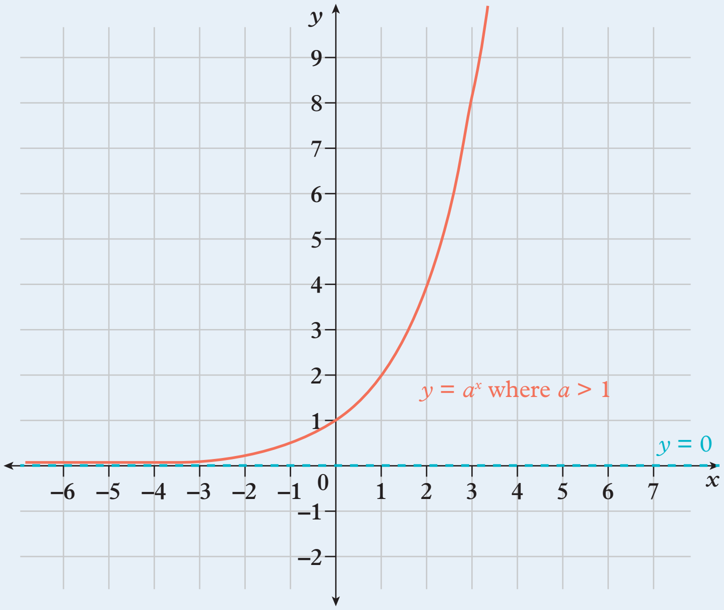
Inter-year links

Year 9	5E Plotting exponential relationships

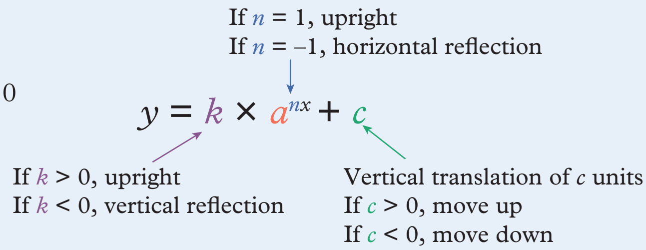
Exponential relationships

Key content video

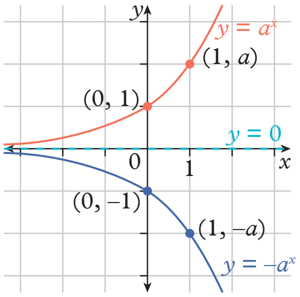
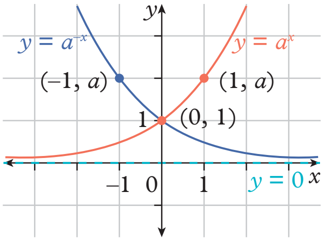
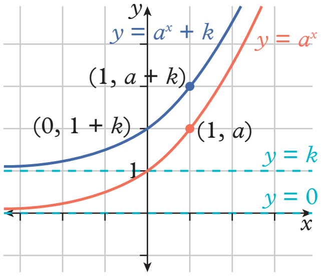
- The basic form of an **exponential relationship** has the equation $y = a^x$ where $a > 0$, $a \neq 1$.



- An **asymptote** is a line whose distance from a curve approaches zero as x or y gets infinitely large or infinitely small.
- The general equation of an exponential relationship is: $y = k \times a^{nx} + c$ where the graph has:
  - → an asymptote: $y = c$
  - → a y-intercept: $(0, k + c)$.
- The following transformations can be performed on the graph of $y = a^x$ to produce a sketch of $y = k \times a^{nx} + c$:
  - → a dilation in the y-direction as k varies
  - → a vertical reflection across the asymptote if $k < 0$
  - → a horizontal reflection across the y-axis if $n = -1$
  - → a vertical translation of c units up or down.

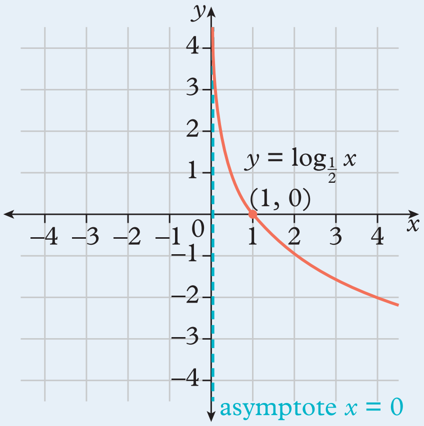


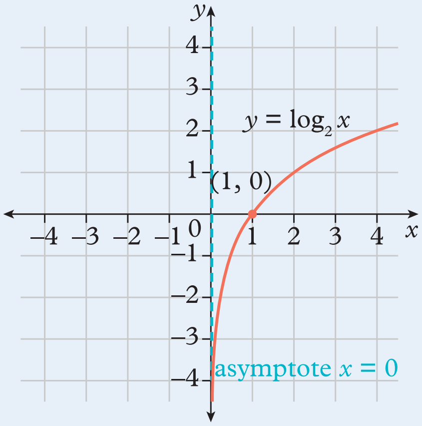
- When $0 < a < 1$, $y = a^x$ can be written as $y = \left(\frac{1}{a}\right)^{-x}$ using the index laws. $y = a^x$ can be graphed by reflecting $y = \left(\frac{1}{a}\right)^x$, where $\frac{1}{a} > 1$, across the y-axis.

Vertical reflection across x-axis	**Horizontal reflection across y-axis**	**Vertical translation**
		

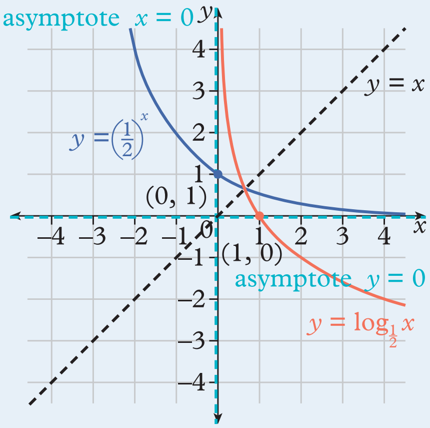
Logarithmic graphs

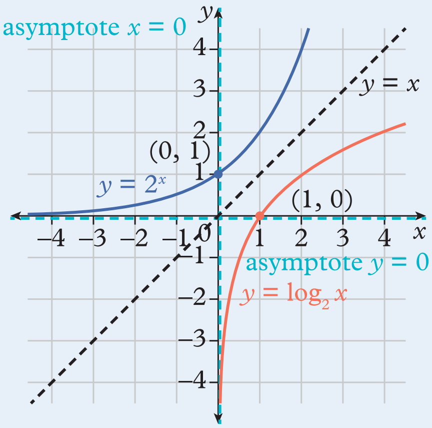
- The shape of the graph of a logarithmic function of the form $y = \log_a x$, where a is a positive number is either an increasing or a decreasing curve.
  - → If the value of a is between 0 and 1 $(0 < a < 1)$, the function is decreasing.
  - → If the value of a is greater than 1 $(a > 1)$, the function is increasing.
  - → The domain of $y = \log_a x$ is $x > 0$, the range of $y = \log_a x$ is all real numbers.
  - → As the value of x increases, the curve starts to flatten.
  - → The x-intercept of $y = \log_a x$ is $(1, 0)$, $y = \log_a x$ does not have a y-intercept.
  - → The asymptote of $y = \log_a x$ is $x = 0$.





- For the same value of a, the shape of the graph $y = \log_a x$ and $y = a^x$ is symmetrical in $y = x$.



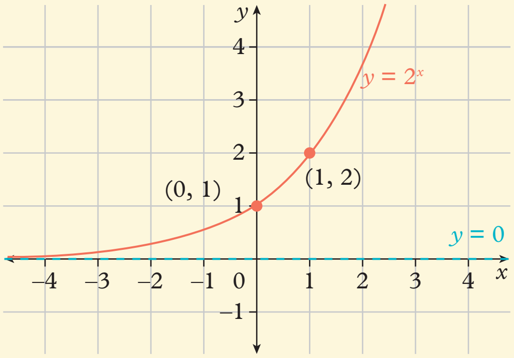


Example 5G.1 Sketching the graph of an exponential relationship by performing reflections

On the same Cartesian plane, sketch the graphs of the following exponential relationships by reflecting the graph of $y = 2^x$. Label the y-intercepts and asymptotes.

a $y = -2^x$

b $y = 2^{-x}$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = k \times 2^{nx}$.

In $y = -2^x$, $k = -1$, $n = 1$.

In $y = 2^{-x}$, $k = 1$, $n = -1$.

When $k = -1$, the graph is being reflected across the x-axis and when $n = -1$, the graph is being reflected across the y-axis. In both graphs, the asymptote remains $y = 0$ because there's no constant term in both equations.

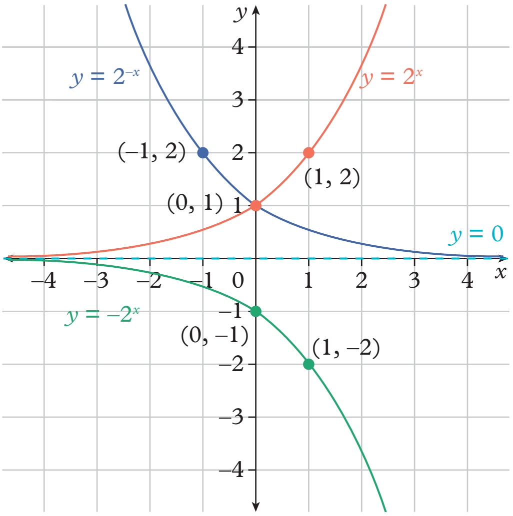
2 Find the corresponding coordinates of (1, 2) and relevant intercepts after the reflection, use these points as reference when sketching the curve.

3 Perform the reflection, sketch the curves through the y-intercepts and reference points, using the horizontal asymptote as a boundary. Label the y-intercepts and asymptotes.

WRITE

a Asymptote $y = 0$, y-intercept from $(0, -1)$, curve passes through $(1, -2)$.

b Asymptote $y = 0$, y-intercept from $(0, 1)$, curve passes through $(-1, 2)$.



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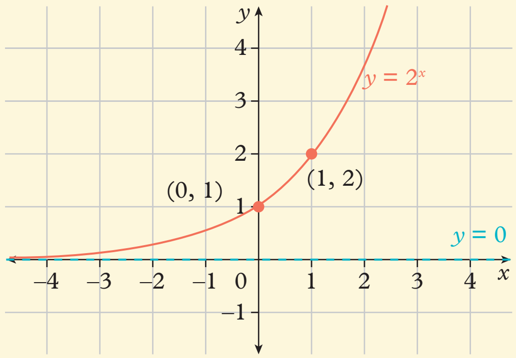
Example 5G.2 Sketching the graph of an exponential relationship by performing vertical translations

On the same Cartesian plane, sketch the graph of the following exponential relationship by transforming the graph of $y = 2^x$. Label the y-intercepts and asymptotes.

a $y = 2^x + 1$

b $y = 2^x - 3$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = 2^x + c$.

In $y = 2^x + 1$, $c = 1$, the asymptote of the graph is $y = 1$, translate the graph of $y = 2^x$ one unit up.

In $y = 2^x - 3$, $c = -3$, the asymptote of the graph is $y = -3$, translate the graph of $y = 2^x$ three units down.

2 A point (x, y) on $y = 2^x$ becomes become $(x, y + c)$ after a vertical translation of c units.

For $y = 2^x + 1$, the y-intercept changes from $(0, 1)$ to $(0, 2)$:

$(0, 1) \rightarrow (0, 1 + c) \rightarrow (0, 1 + 1) \rightarrow (0, 2)$.

For $y = 2^x - 3$, the y-intercept changes from $(0, 1)$ to $(0, -2)$:

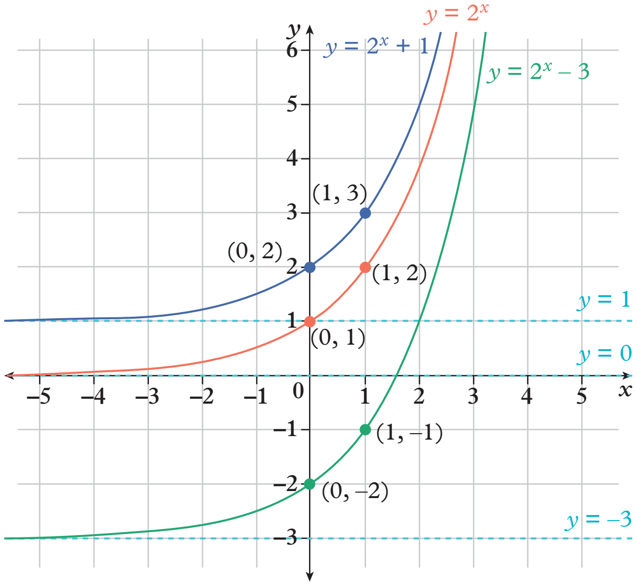
$(0, 1) \rightarrow (0, 1 + c) \rightarrow (0, 1 - 3) \rightarrow (0, -2)$.

3 Perform the translation, sketch the curve through the y-intercept and reference point, using the horizontal asymptote as a boundary. Label the y-intercepts and asymptotes.

WRITE

a Asymptote: $y = 1$, y-intercept $(0, 2)$, curve passes through $(1, 3)$.

b Asymptote: $y = -3$, y-intercept $(0, -2)$, curve passes through $(1, -1)$.

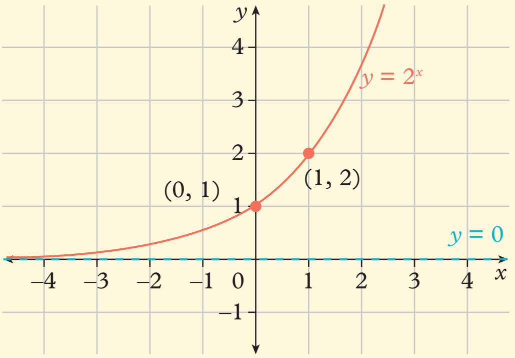


Example 5G.3 Sketching the graph of an exponential relationship using vertical dilations

On the same Cartesian plane, sketch the graph of the following exponential equations by transforming the graph of $y = 2^x$. Label the y-intercept and asymptote.

a $y = 2 \times 2^x$

b $y = \frac{1}{2} \times 2^x$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = k \times 2^{nx}$. The asymptote for both graphs are $y = 0$ because there's no constant term in both equations.

2 When $y = 2^x$ is dilated in the y-direction, a point (x, y) on $y = 2^x$ becomes (x, ky). Since $k = 2$, in $y = 2 \times 2^x$ the y-intercept changes from $(0, 1)$ to $(0, 2)$:

$(0, 1) \rightarrow (0, k \times 1) \rightarrow (0, 2 \times 1) \rightarrow (0, 2)$.

The reference point changes from $(1, 2)$ to $(1, 4)$:

$(1, 2) \rightarrow (1, k \times 2) \rightarrow (1, 2 \times 2) \rightarrow (1, 4)$.

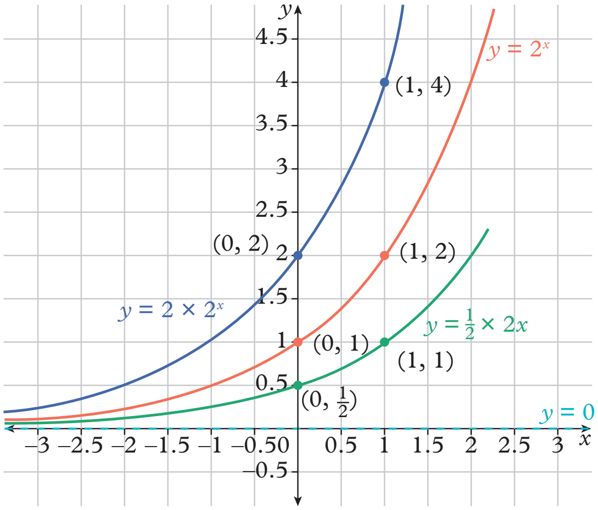
Similarly, in $y = \frac{1}{2} \times 2^x$, $k = \frac{1}{2}$, the y-intercept changes from $(0, 1)$ to $(0, \frac{1}{2})$; the reference point changes from $(1, 2)$ to $(1, 1)$.

3 Sketch the curves for an exponential relationship through the y-intercept and the reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

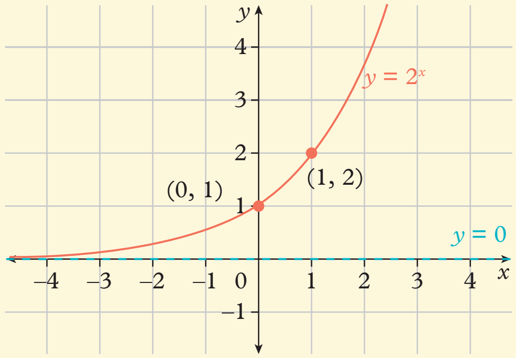
a Asymptote: $y = 0$, y-intercept: $(0, 2)$, curve passes through $(1, 4)$.

b Asymptote: $y = 0$, y-intercept $(0, \frac{1}{2})$, curve passes through $(1, 1)$.



Example 5G.4 Sketching the graph of a simple logarithmic relationship

On the same Cartesian plane, sketch the graph of $y = \log_2 x$ by transforming the graph of $y = 2^x$. Label the intercept(s) and asymptote.



THINK

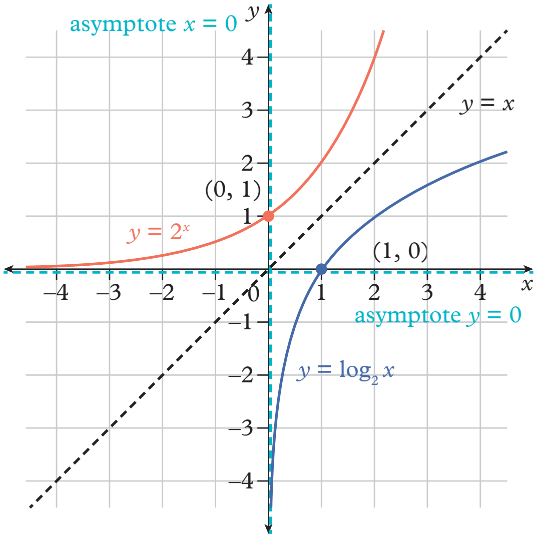
1 The equation $y = 2^x$ and $y = \log_2 x$ have the same base where $a = 2$. So they are symmetric in $y = x$.

2 Reflect the graph $y = 2^x$ across the line $y = x$, using the reference point to sketch the curve. Swap the x- and y- coordinates of the reference point (1, 2) to get the coordinates of the new point (2, 1) after reflecting the curve in $y = x$.

3 After the reflection, the asymptote changes from $y = 0$ to $x = 0$, and the intercept becomes (0, 1).

WRITE

Asymptote $x = 0$ x-intercept: (0, 1)



Helpful hints

- ✔ Using your graphing calculator to alter the exponential function will help you to better understand the transformations. You can start by moving a negative sign to different places in the same relationship to increase or decrease values, or you can compare the graphs for the following equations:
  - $y = -2^x$
  - $y = 2^{-x}$
  - $y = -2^{-x}$
  - $y = 2^x + 1$
  - $y = 2^x - 1$
- ✔ The graph of an exponential or a logarithmic relationship can get increasingly closer to the asymptote but can never cross or touch it.
- ✔ Remember that every exponential or logarithmic graph has an asymptote and you must always include it, even when it is on the axes.
- ✔ The graph of $y = \log_a x$ and $y = a^x$ is symmetrical in $y = x$, that means you can swap the position of the x- and y-coordinate for any point on the given graph to sketch the other graph.

ANS p700

Exercise 5G Graphing exponential and logarithmic relationships

△ 1–11, 13, 15 ■ 2–7, 9–12, 14, 16, 19 ⬟ 3–7(d–f), 9, 10, 15–20

FLUENCY AND UNDERSTANDING

1 For each of the following relationships:

i $y = 3^x$ ii $y = \left(\frac{1}{2}\right)^x$

a plot the graph using a table of values or digital technology

b write the y-intercept for each graph

c write the rule for the asymptote for each graph.

2 The general equation of an exponential relationship is $y = k \times a^{nx} + c$.

i Determine the value of k, n and c in each of the following exponential equations.

ii Describe what transformation needs to be performed on $y = a^x$ to obtain the graph of the following relationships.

iii Determine the coordinates of any intercepts and the equation of the asymptotes.

a $y = 2a^x$ **b** $y = \frac{2}{3}a^x$ **c** $y = -a^x$

d $y = a^{-x}$ **e** $y = a^x + \frac{1}{2}$ **f** $y = a^x - 1$

5G.1 3 Sketch the graph of each exponential relationship by performing a reflection on your graphs in question **1**.

a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

d $y = \left(\frac{1}{2}\right)^{-x}$ **e** $y = -\left(\frac{1}{2}\right)^x$ **f** $y = -\left(\frac{1}{2}\right)^{-x}$

4 Use an index law to show:

a $y = 3^{-x}$ is the same as $y = \left(\frac{1}{3}\right)^x$ **b** $y = \left(\frac{1}{2}\right)^x$ is the same as $y = 2^{-x}$

5 Which exponential equation in question **3** is the same as $y = 2^x$?

5G.2 6 Sketch the graph of each exponential relationship by performing a vertical translation using your graphs from the previous questions.

a $y = 2^x + 2$ **b** $y = 2^x - 2$ **c** $y = 2^x + 1$

d $y = \left(\frac{1}{3}\right)^x + 1$ **e** $y = \left(\frac{1}{3}\right)^x + 2$ **f** $y = \left(\frac{1}{3}\right)^x -2$

5G.3 7 Sketch the graph of each exponential relationship by performing a vertical dilation using your graphs from the previous questions.

a $y = 2 \times 3^x$ **b** $y = 3 \times 3^x$ **c** $y = 0.5 \times 3^x$

d $y = 2 \times \left(\frac{1}{2}\right)^x$ **e** $y = 3 \times \left(\frac{1}{2}\right)^x$ **f** $y = 0.5 \times \left(\frac{1}{2}\right)^x$

ADVANCED

PROBLEM SOLVING AND REASONING

5G.4 **8** Sketch the graph of each logarithmic relationship by performing the appropriate transformations on the graphs in question **1**.

a $y = \log_3 x$ **b** $y = \log_2 x$ **c** $y = \log_{\frac{1}{3}} x$ **d** $y = \log_{\frac{1}{2}} x$

9 What is the rule obtained when the graph of $y = a^x$ is reflected in:

a the x-axis **b** the y-axis **c** the x-axis and the y-axis?

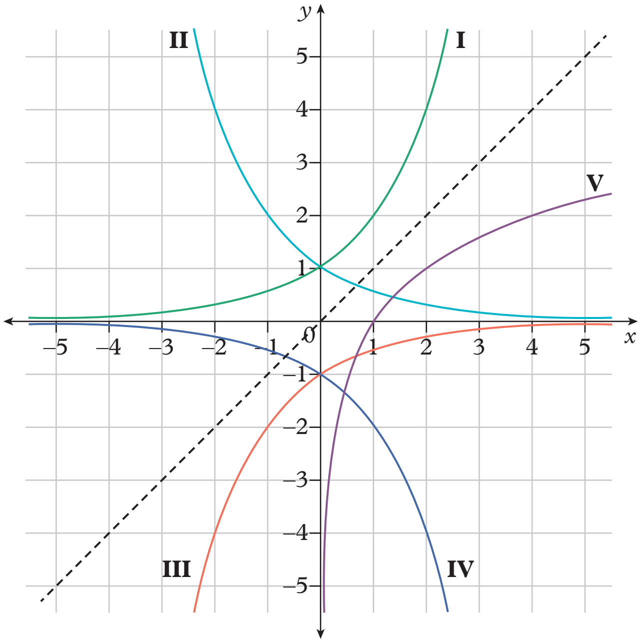
10 What is the domain and range of the following relationships?

a $y = 2^x$ **b** $y = -3^x$ **c** $y = \left(\frac{1}{2}\right)^x$

d $y = 2^x + 2$ **e** $y = \log_2 x$ **f** $y = \log_{\frac{1}{3}} x$

11 Match each of the following graphs with their corresponding equations.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = -2^{-x}$ **d** $y = 2^{-x}$ **e** $y = \log_2 x$

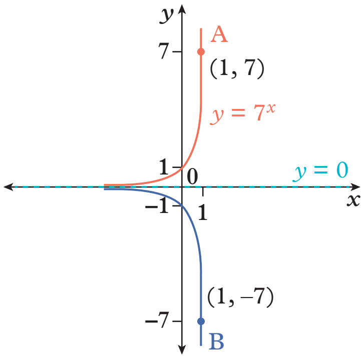


12 For each pair of graphs shown on a Cartesian plane below:

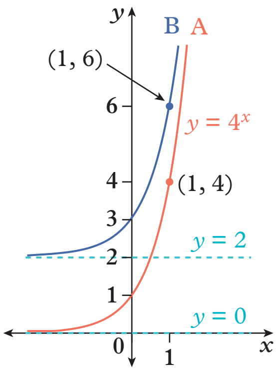
i identify the transformation performed on graph **A** to produce graph **B**

ii write the rule for graph **B**.

a



b



13 Rewrite the following equations in the form $y = k \times 2^x + c$ using the index laws, and then sketch the graphs on the same Cartesian plane using dilations.

a $y = 2^{x+3}$ **b** $y = 2^{x+2}$ **c** $y = 2^{x+1}$

d $y = 2^{x-1}$ **e** $y = 2^{x-2}$ **f** $y = 2^{x-3}$

14 Create the graphs in question **13** using digital technology. Describe the horizontal translation needed to produce the graphs of parts **a–f** from the graph of $y = 2^x$.

PROBLEM SOLVING AND REASONING

ADVANCED

15 a Fill in the following table of values for the exponential and logarithmic relationships.

i

x	−3	−2	−1	0	1	2	3
$y = 5^x$							

ii

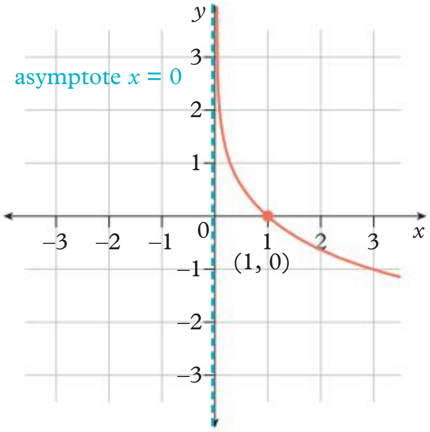
x	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125
$y = \log_5 x$							

b List the coordinates of the points for parts **i** and **ii** in part **a**.

c Describe the pattern you observed in part **b**.

d What is the new coordinate of a point (a, b) after being reflected in the line $y = x$?

16 a Graph the exponential relationship $y = a^x$ on the same Cartesian plane given on the right the following graph for $y = \log_a x$.



b Is base a greater than 1 or less than 1?

17 Write the equation of the new non-linear relationship if the following transformations are performed on $y = 5^x$.

a The graph is stretched by a factor of 3 in the y-direction.

b The graph is reflected in the x-axis.

c The graph is translated 3 units in the x-direction.

d The graph is translated −2 units in the y-direction.

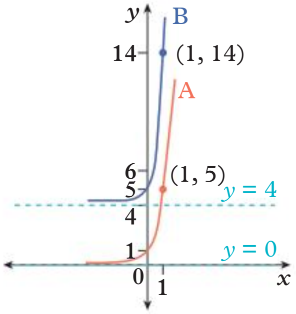
CHALLENGE

18 For each pair of graphs shown on the Cartesian planes below:

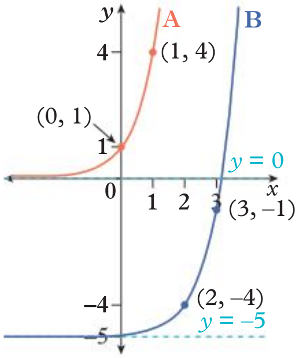
i identify the transformations performed on graph **A** to produce graph **B**

ii write the rules for graphs **A** and **B**.

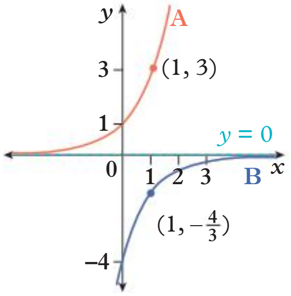
a



b



c



19 a Describe the transformations performed on the graph of $y = 2^x$ to produce the graph of $y = -3 \times 2^{x+1} - 2$.

b Sketch the graph.

c Determine the asymptote and intercept(s) of $y = -3 \times 2^{x+1} - 2$

20 Describe the transformations performed on the graph of $y = 3^{-x} + 1$ to produce the graph of $y = 3^x$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Exponential relationships

Investigation
Intersection of two non-linear relationships

Topic quiz
5G

ADVANCED

5H Graphing hyperbolas

Learning intentions

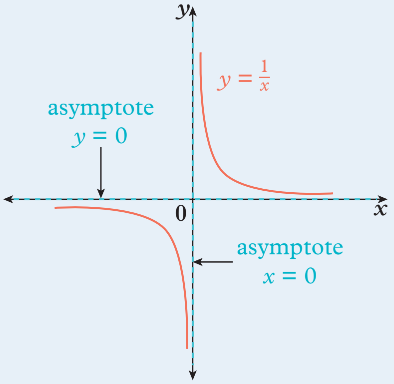
By the end of this topic you will be able to ...

- ✔ sketch simple hyperbolas of the form $y = \frac{k}{x}$
- ✔ sketch hyperbolas using reflections and dilations
- ✔ sketch hyperbolas using vertical translations and horizontal translations.

Inter-year links
Year 9 7B Rotations and translations

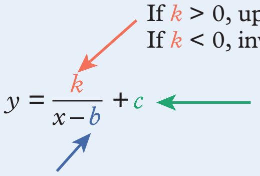
Hyperbolas

Key content video



- The basic form of a simple **rectangular hyperbola** is the graph of $y = \frac{1}{x}$. The graph has two asymptotes and no x- or y-intercepts.
- The asymptotes for $y = \frac{1}{x}$ lie along the x-axis and along the y-axis; that is, they are the lines whose equations are $y = 0$ and $x = 0$.
- The general equation of a hyperbola is: $y = \frac{k}{x-b} + c$ where the graph has asymptotes at $x = b$ and $y = c$.

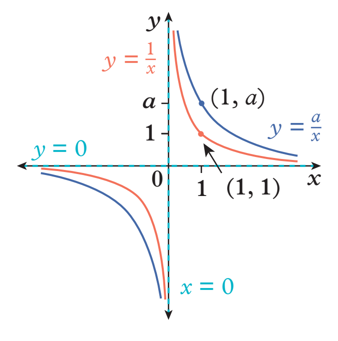
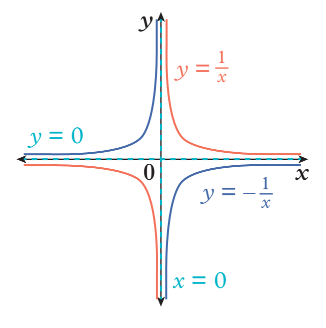
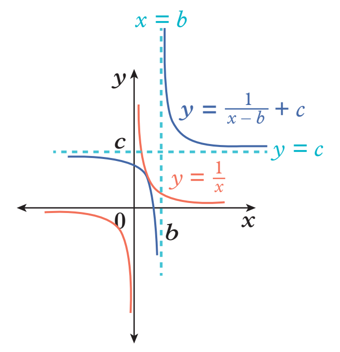
Dilation in the y-direction
If $k > 0$, upright
If $k < 0$, inverted



Vertical translation of c units
If $c > 0$, up
If $c < 0$, down

Horizontal translation of b units
If $b > 0$, right
If $b < 0$, left

- The following transformations can be performed on the graph of $y = \frac{1}{x}$ to produce the graph of $y = \frac{k}{x-b} + c$:
  - → a dilation in the y-direction as k increases
  - → a reflection in the x-axis if $k < 0$
  - → a horizontal translation of b units right or left
  - → a vertical translation of c units up or down.

Dilation	Reflection	Translation
		

Example 5H.1 Sketching the graph of a hyperbola using dilations

Perform transformations to sketch the graph of each hyperbola below on the same Cartesian plane. Label the asymptotes.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{6}{x}$

THINK

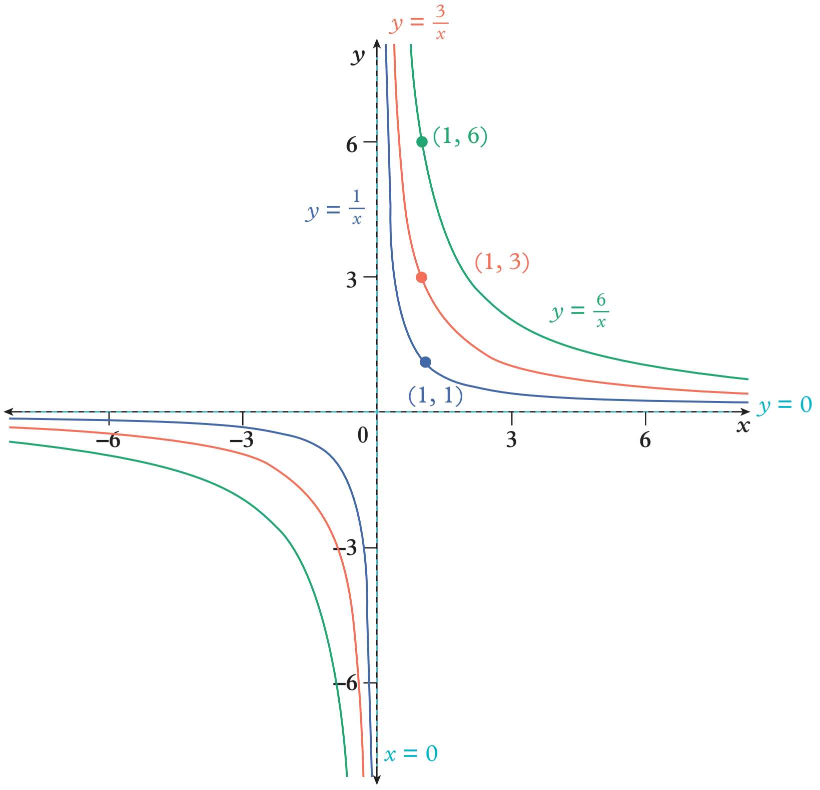
1. Identify the asymptotes. The asymptotes for a hyperbola in the form $y = \frac{k}{x}$ are the horizontal line with equation $y = 0$ and the vertical line with equation $x = 0$.
2. Identify any intercepts.
3. To find the scale of the dilation, label another point on the curve with its coordinates. Find the value of y when $x = 1$.
4. Sketch the basic curve for a rectangular hyperbola using the asymptotes as boundaries. Label a known point on the curve to indicate that a dilation has occurred.

WRITE

a Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 1)$.

b Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 3)$.

c Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 6)$.



ADVANCED

Example 5H.2 Sketching the graph of a hyperbola using translations

Sketch the graph of $y = \frac{1}{x+2} - 4$. Label the asymptotes and intercepts.

THINK

1 Identify the translations that need to be performed on $y = \frac{1}{x}$. Compare to $y = \frac{k}{x-b} + c$ to identify the translations: $b = -2$ (move 2 units left) and $c = -4$ (move 4 units down).

WRITE

$$y = \frac{1}{x \boxed{+ 2}} \boxed{- 4}$$

Horizontal translation of 2 units left

Vertical translation of 4 units down

2 Identify the rule for each asymptote: $x = b$ and $y = c$.

The vertical asymptote is $x = -2$.

The horizontal asymptote is $y = -4$.

3 Find the x-intercept.

When $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{x+2} - 4 \\ \frac{1}{x+2} &= 4 \\ x + 2 &= \frac{1}{4} \\ x &= \frac{1}{4} - 2 \\ &= -1\frac{3}{4} \end{aligned}$$

The x-intercept is $\left(-1\frac{3}{4}, 0\right)$.

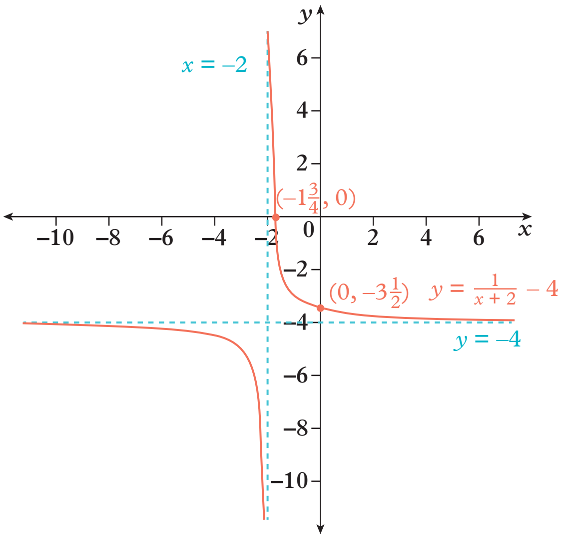
4 Find the y-intercept.

When $x = 0$:

$$\begin{aligned} y &= \frac{1}{0+2} - 4 \\ &= \frac{1}{2} - 4 \\ &= -3\frac{1}{2} \end{aligned}$$

The y-intercept is $\left(0, -3\frac{1}{2}\right)$.

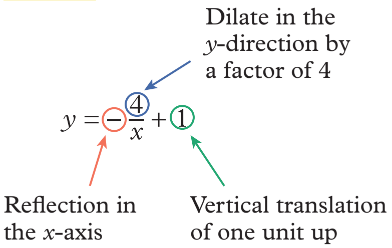
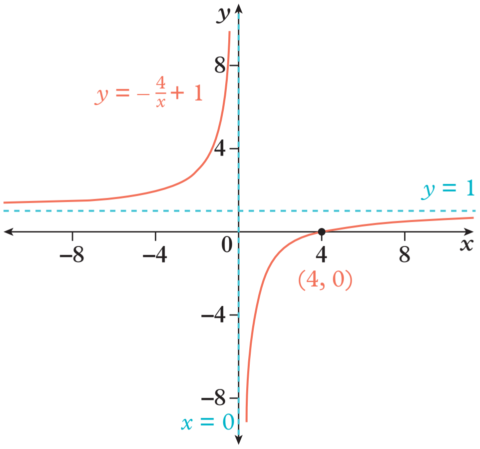
5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.



ADVANCED

Example 5H.3 Sketching the graph of a hyperbola using reflections

Sketch the graph of $y = -\frac{4}{x} + 1$. Label the asymptotes and intercepts.

THINK	WRITE
1 Identify the three translations that need to be performed on $y = \frac{1}{x}$: reflection in the x-axis; $c = 1$ (move 1 unit up); and a dilation in the y-direction by a factor of 4.	
2 Identify the rule for each asymptote: $x = 0$ and $y = 1$.	The vertical asymptote is $x = 0$. The horizontal asymptote is $y = 1$.
3 Find the x-intercept.	For the x-intercept, $y = 0$: $y = -\frac{4}{x} + 1$ $0 = -\frac{4}{x} + 1$ $\frac{4}{x} = 1$ $x = 4$ The x-intercept is $(4, 0)$.
4 Find the y-intercept.	There is no y-intercept.
5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.	

Helpful hints

- ✔ Be careful not to interchange b and c. The h values refers to the horizontal translation, and the c value refers to the vertical translation.
- ✔ Don't forget to sketch the x- and y-asymptotes, even when they are on an axis.
- ✔ The coordinates (b, c) give the point of intersection for the two asymptotes of the hyperbola.
- ✔ When sketching a hyperbola, start by drawing the asymptotes.
- ✔ When finding the domain and range of a hyperbolic relationship, remember that the denominator of a fraction cannot equal zero.

ADVANCED

ANS p703

Exercise 5H Graphing hyperbolas

△ 1–8(a–c), 9, 11, 12 □ 1–8(b–d), 9(a, b), 12–14 ⬠ 1–9(d), 12–16

FLUENCY AND UNDERSTANDING

5H.1 1 Sketch the graph of each of these hyperbolas. Label the asymptotes.

a $y = \frac{4}{x}$ **b** $y = \frac{7}{x}$ **c** $y = \frac{2}{x}$ **d** $y = \frac{5}{x}$ **e** $y = \frac{10}{x}$

2 Sketch the graph of each of these relationships by performing a vertical translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x} + 4$ **b** $y = \frac{1}{x} - 7$ **c** $y = \frac{1}{x} + 5$ **d** $y = \frac{1}{x} - 1$

3 Sketch the graph of each of these relationships by performing a horizontal translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x - 5}$ **b** $y = \frac{1}{x + 4}$ **c** $y = \frac{1}{x - 7}$ **d** $y = \frac{1}{x + 1}$

5H.2 4 Sketch the graph of each of these equations. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x - 3} - 2$ **b** $y = \frac{1}{x + 5} + 3$ **c** $y = \frac{1}{x - 2} + 4$ **d** $y = \frac{1}{x + 4} - 5$ **e** $y = \frac{1}{x - 1} - 8$

5 Use digital technology to produce the graphs from question **4**. Compare them to your sketches.

5H.3 6 Sketch the graph for each equation below, clearly showing the asymptotes and intercepts.

a $y = -\frac{1}{x} + 3$ **b** $y = -\frac{2}{x}$ **c** $y = -\frac{1}{x - 2}$ **d** $y = -\frac{5}{x}$ **e** $y = -\frac{1}{x - 2} + 3$

7 Use transformations to help you sketch the graph of each of these equations. Clearly show the asymptotes and intercepts. List the domain and range for each graph.

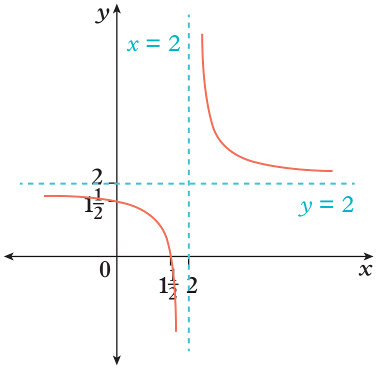
a $y = -\frac{1}{x + 4}$ **b** $y = -\frac{1}{x} - 2$ **c** $y = -\frac{1}{x - 3} + 5$ **d** $y = -\frac{1}{x + 5} - 1$ **e** $y = -\frac{2}{x - 1} - 4$

8 Use digital technology to produce the graphs from question **7**. Compare them to your sketches.

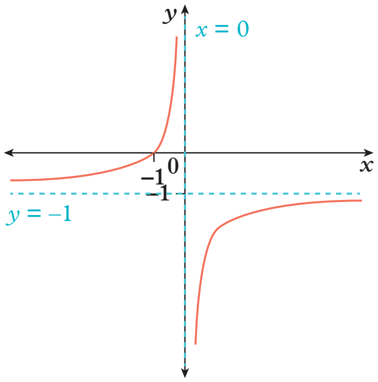
PROBLEM SOLVING AND REASONING

9 Write the equation for each hyperbola shown below.

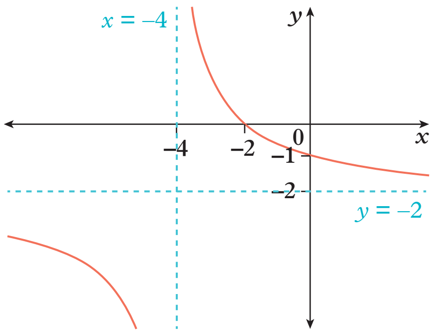
a



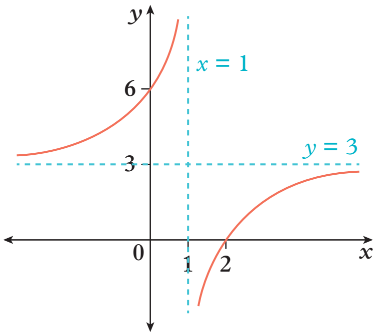
b



c



d



PROBLEM SOLVING AND REASONING

ADVANCED

10 Why does reflecting the graph of $y = \frac{1}{x}$ in the x-axis produce the same graph as reflecting $y = \frac{1}{x}$ in the y-axis?

11 One way to describe the graph of $y = \frac{1}{x}$ is to describe the behaviour of the curve as x approaches very large or very small values. Complete each statement below using a term from this list:

- positive infinity
- zero
- negative infinity.

a When x approaches infinity, y approaches ____________.

b When x approaches zero from the right, y approaches ____________.

c When x approaches zero from the left, y approaches ____________.

d When x approaches negative infinity, y approaches ____________.

12 At a bank, the average waiting time for customers in a queue is related to the number of bank tellers on duty and can be modelled by the relationship $W_n = \frac{12}{n}$, where W_n is the average waiting time in minutes when there are n bank tellers on duty.

a Sketch a graph of this relationship.

b On average, how long does a customer wait when the number of bank tellers on duty is:

i 2? **ii** 3? **iii** 4? **iv** 6? **v** 12? **vi** 24?

c What is the maximum waiting time for a customer? When does this occur?

d According to this model, will there be occasions when the waiting time is zero? Explain.

13 The graph of a hyperbola will always have a horizontal asymptote and a vertical asymptote.

Consider the hyperbola $y = \frac{1}{x-2} + 3$, which has asymptotes at $x = 2$ and $y = 3$.

a Substitute $x = 2$ into the equation and explain why this causes the y-value to be undefined.

b Explain why the error from part **a** will not occur for any other value of x that is not 2.

c Substitute $y = 3$ into the equation and attempt to solve it. What goes wrong when you attempt to solve it for x?

d Explain why the error from part **c** will not occur for any other value of y that is not 3.

CHALLENGE

14 For the graphs of each pair of relationships below:

i identify the number of points of intersection by sketching their graphs

ii use an algebraic method to find the points of intersection.

a $y = \frac{1}{x}$ and $y = x$

b $y = \frac{1}{x}$ and $y = -x$

c $y = \frac{6}{x+2}$ and $y = x + 3$

d $y = \frac{1}{x-4}$ and $y = -\frac{1}{x}$

15 a Find the rule for a hyperbola of the form $y = \frac{8}{x-h}$ that passes through (6, 2).

b Find the rule for a hyperbola of the form $y = \frac{3}{x-h} + 2$ that passes through (–2, 1).

c Find the rule for a hyperbola of the form $y = \frac{a}{x-h}$ that passes through (–7, –1) and (1, 1).

16 Sketch the graphs of each of the following. Remember that the x-asymptote occurs for the value of x that would cause the denominator to equal zero.

a $y = \frac{1}{2x-1}$

b $y = \frac{1}{1-x} + 2$

c $y = \frac{-2}{1-2x} - 1$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Hyperbolas

Investigation
Inverse variation

Topic quiz
5H

5I Direct and inverse variation

Learning intentions

By the end of this topic you will be able to ...

- ✔ identify when two variables are directly proportional to each other
- ✔ identify when two variables are inversely proportional to each other
- ✔ solve modelling problems involving direct and inverse variation.

Inter-year links

Year 9	4G Direct variation

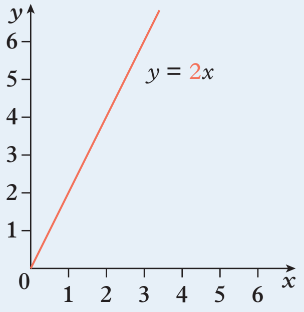
Key content video

Direct variation

- y is said to be **directly proportional** to x if:
  - → $y = 0$ when $x = 0$
  - → y increases as x increases
  - → the rate of change of y with respect to x is constant.
- It is also said that y is proportional to x or x and y vary directly.
- Proportionality is denoted using the symbol '$\propto$'.
- If $y \propto x$:
  - → the equation for the relationship between x and y is $y = kx$ where k, the **constant of variation**, is equal to the rate of change of y with respect to x

$$y = \overset{\swarrow}{k}x \quad \begin{array}{l}\text{constant of variation}\\ = \text{rate of change}\\ = \text{gradient}\end{array}$$

  - → the graph of y against x is a straight line with a gradient of k that passes through the origin, $(0, 0)$.
- If $y \propto x$, then $x \propto y$.

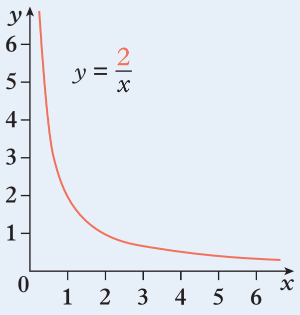


Inverse variation

- y is said to be **inversely proportional** to x if:
  - → y increases as x decreases at a proportional rate; $y \propto \frac{1}{x}$
  - → the product of x and y is constant.
- It is also said that y is proportional to the reciprocal of x or that x and y vary inversely.
- If $y \propto \frac{1}{x}$:
  - → the equation for the inverse relationship between x and y is $y = \frac{k}{x}$

$$y = \frac{k}{x} \longleftarrow \text{constant of variation}$$

  - → the graph of y against x is a hyperbola with asymptotes of $x = 0$ and $y = 0$.
- If $y \propto \frac{1}{x}$, then $x \propto \frac{1}{y}$.
- If y is inversely proportional to x then y is directly proportional to $\frac{1}{x}$.



Example 5I.1 Finding the constant of variation for direct proportion

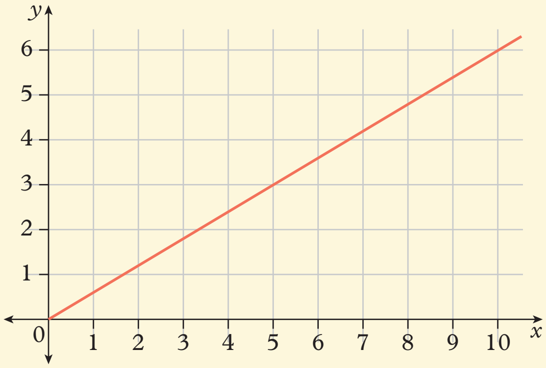
If $y \propto x$, find the constant of variation using the information given about each relationship below.

a When $x = 3$, $y = 18$.

b

x	0	2	4	6	8
y	0	24	48	72	96

c



THINK

a If $y \propto x$, then $y = kx$. Substitute the given values into $y = kx$ and solve the equation for k.

b Select any pair of values in the table and substitute the corresponding values of x and y into $y = kx$, then solve the equation for k.

c To find the constant, k, select any point on the graph, substitute the corresponding values of x and y into $y = kx$, then solve the equation for k.

WRITE

a

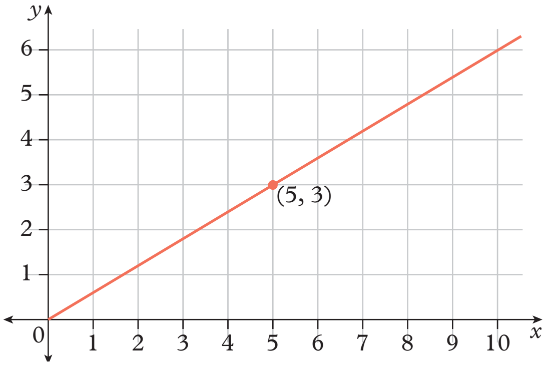
$$y = kx$$
$$18 = k \times 3$$
$$k = 6$$

b

x	0	2	4	6	8
y	0	24	48	72	96

$$y = kx$$
$$24 = k \times 2$$
$$k = 12$$

c



$$y = kx$$
$$3 = k \times 5$$
$$k = \frac{3}{5}$$

Example 5I.2 Finding the constant of variation for inverse proportion

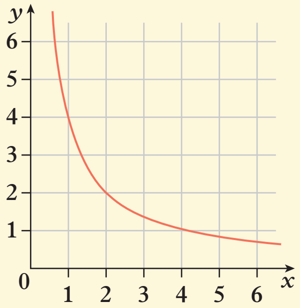
If $y \propto \frac{1}{x}$, use the information given about each relationship below to find the constant of variation.

a When $x = 4$, $y = 3$.

b

x	3	4	5	6	7
y	2.3	1.75	1.4	1.17	1

c



THINK

a If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$. Substitute the given values into $y = \frac{k}{x}$ and solve the equation for k.

b Select any pair of values in the table and substitute the corresponding values of x and y into $y = \frac{k}{x}$, then solve the equation for k.

c To find the constant, k, select any point on the graph, substitute the corresponding values of x and y into $y = \frac{k}{x}$, then solve the equation for k.

WRITE

a
$$y = \frac{k}{x}$$
$$y = \frac{k}{4}$$
$$3 = \frac{k}{4}$$
$$k = 12$$

b

x	3	4	5	6	7
y	2.3	1.75	1.4	1.17	1

$$y = \frac{k}{x}$$
$$1 = \frac{k}{7}$$
$$k = 7$$

c

$$y = \frac{k}{x}$$
$$2 = \frac{k}{2}$$
$$k = 4$$

Helpful hints

- ✔ Direct variation is an example of linear algebra and linear graphs. Remember that for the relationship $y = kx$:
  k = constant of variation = rate of change = gradient
- ✔ For a relationship to be directly or inversely proportional, the constant of variation, k, must be a constant, but it can still be fractional or irrational.

ANS
p706

Exercise 5I Direct and inverse variation

▲ 1(a–c), 2, 3(a–c), 4–7, 8(i), 10, 12 ■ 1–5(a–b), 6–13 ⬟ 1–7(a–b), 11, 13–15

FLUENCY AND UNDERSTANDING

5I.1 **1** If $y \propto x$, find the constant of variation for each of these relationships, using the information given.

a $y = 12$ when $x = 3$

b

x	1	2	3	4
y	9	18	27	36

c $y = 9$ when $x = 6$

d

x	0.5	1	1.5	2
y	2.5	5	7.5	10

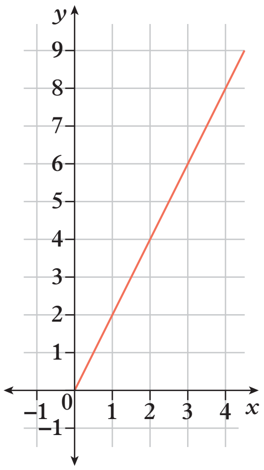
e $y = 10$ when $x = 20$

f

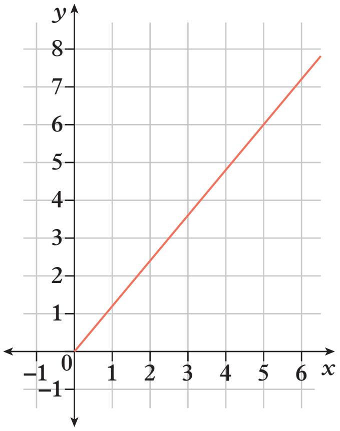
x	5	10	15	20
y	2	4	6	8

2 For each of the following graphs, $y \propto x$. Find the constant of variation.

a



b



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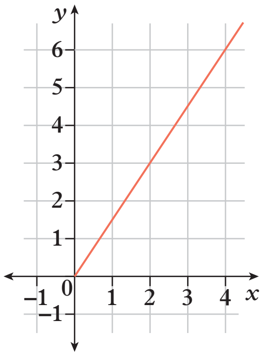

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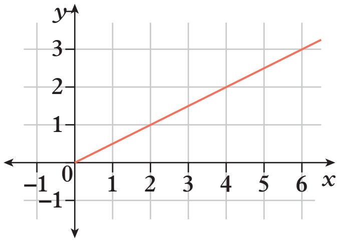


5I.2 **3** If $y \propto \frac{1}{x}$, use the information given to find the constant of variation for each of the following relationships.

a $y = 1$ when $x = 6$

b

x	2	4	6	8
y	6	3	2	1.5

c $y = 0.4$ when $x = 10$

d

x	2	3	4	5
y	3	2	1.5	1.2

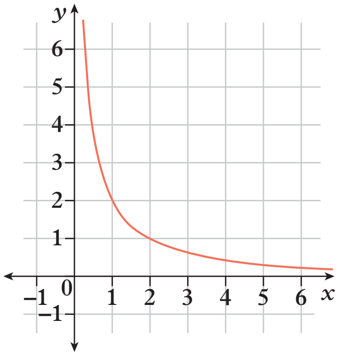
e $y = 3$ when $x = 5$

f

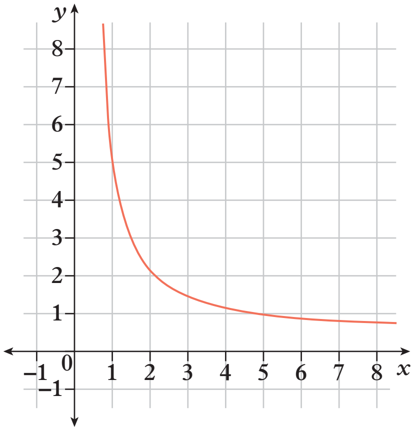
x	2	5	10	20
y	0.5	0.2	0.1	0.05

4 For each of the following graphs, $y \propto \frac{1}{x}$. Find the constant of variation.

a



b



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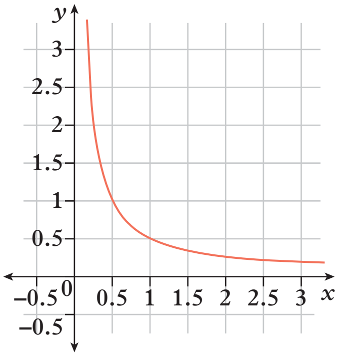

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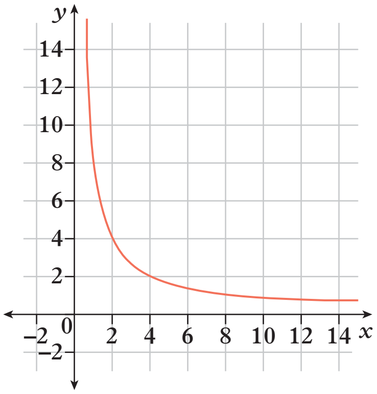


5 Decide whether the variables in each of the following relationships vary directly or inversely, or neither.

a The height of a tree and the length of its shadow

b The cost of hiring a plumber who charges \$70 per hour plus a \$90 call out fee

c The time taken to travel a fixed distance and the speed of travel (assuming the speed is constant)

d The area of a square and its side length

e The height of water in a cylindrical glass and the volume of the water

f The number of people paying a bill and the cost per person

6 Consider the incomplete table on the right, which shows a proportional relationship.

x	0.5	1	2	4	8
y	8	4	2		

a Is y directly proportional or inversely proportional to x?

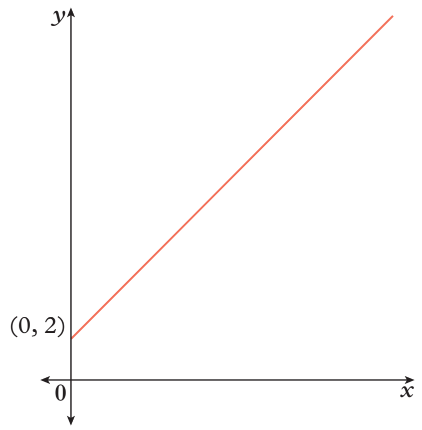
b Find the constant of variation.

c Calculate the two y values missing from the table.

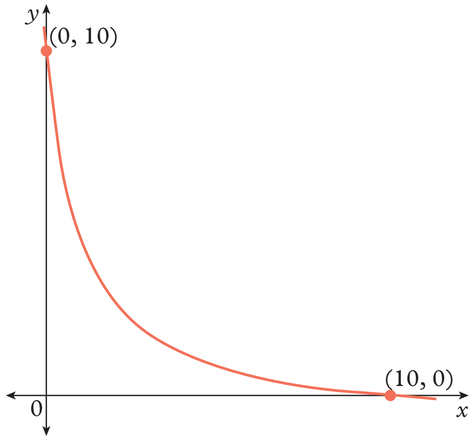
d Plot the points from the table on a graph and draw a smooth curve to join the points.

7 For each of the following graphs, determine if the relationship is directly or inversely proportional, explain why if it's neither.

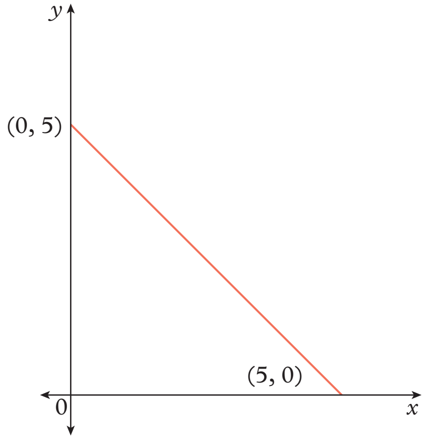
a



b



c



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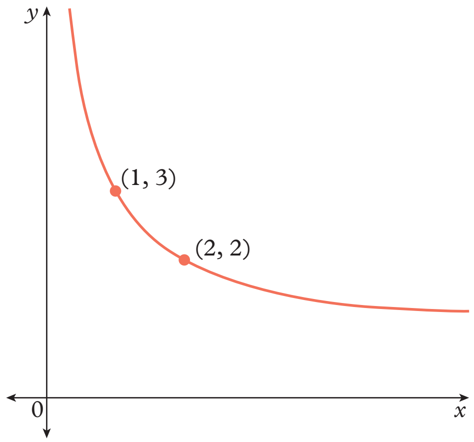


8 **a** Given $y \propto x$, find the change in y if x is:

i doubled **ii** halved **iii** increased by 20% **iv** decreased by 30%.

b Given $x \propto y$, find the change in y if x is:

i doubled **ii** halved **iii** increased by 20% **iv** decreased by 30%.

c Given $y \propto \frac{1}{x}$, find the change in y if x is:

i doubled **ii** halved **iii** increased by 20% **iv** decreased by 30%.

d Given $\frac{1}{y} \propto x$, find the change in y if x is:

i doubled **ii** halved **iii** increased by 20% **iv** decreased by 30%.

9 On Friday, a television was on for 4 hours and used 560 watts of electricity. Let W be the number of watts used and let t be the number of hours the television was on.

a If $W \propto t$, find the constant of variation.

b If the television was on for 6 hours on Saturday, how many watts of electricity did it use?

c If the television used 245 watts on a Sunday, for how many hours was the television on?

10 The number of days it takes to complete a construction job is inversely proportional to the number of people working on the job. If it would take 12 days for 5 workers to complete the job, then how many days would it take for 8 workers to complete the job?

11 Consider the following relationship between x and y.

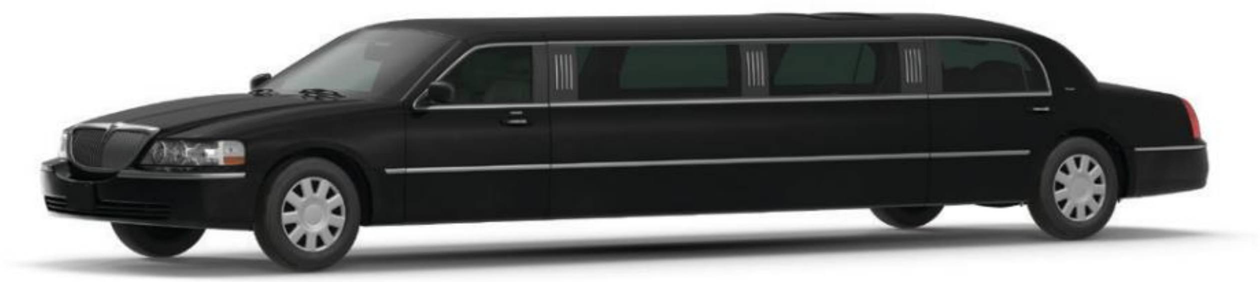
x	0.5	1	1.5	2
y	12	6	4	3

a If $y \propto \frac{1}{x}$, find the constant of variation.

b If $x \propto \frac{1}{y}$, find the constant of variation.

c Your answers for parts **a** and **b** should be the same. Will this always be the case for inversely proportional relationships, or were these answers just a coincidence?

12 To attend a formal, a group of Year 10 students are considering hiring a limousine, which costs a fixed amount regardless of how many passengers ride in the limousine. The rental company tells the students that, if 5 of them hire the limousine, it will cost $34.20 per person.



a What is the total cost of 5 people hiring the limousine for the night?

b What is the constant of variation for $C \propto \frac{1}{n}$, where C is the cost per person and n is the number of people?

c Explain why the total cost and the constant of variation for $C \propto \frac{1}{n}$ will always be the same for situations like this.

13 The circumference of a circle is directly proportional to the diameter of the circle. Malachi tries to find the constant of proportionality for $C \propto d$ by measuring, as best he can, the circumferences and diameters of several circles. His measurements, in centimetres, are shown in the table below.

d	1	2	3	4	5	6	7
C	3.2	6.2	9.3	13	16.2	18.9	21.7

a For each circle, find the constant of variation for $C \propto d$, as a decimal.

b Since there are several different values, Malachi estimates the constant of variation by finding the average of the seven ratios. What is his estimation of the constant? Round your answer to two decimal places.

c What famous constant is Malachi trying to approximate? Is he close?

14 The brightness of light decreases inversely proportionally to the square of the distance from the light source. This means that $I \propto \frac{1}{d^2}$ where I is a measure of light intensity (in watts per square metre) and d is the distance (in metres) from the light source at which the measurement is taken.

Consider a lamp with a light intensity of 1.2 W/m^2 measured at a distance of 2 metres from the lamp.

a Find the constant of variation.

b Find:

  i the light intensity at a distance of 5 metres from the lamp

  ii the distance from the lamp, to the nearest centimetre, if the light intensity is 100 W/m^2.

c By what percentage does the intensity of the light on an object decrease when the object is moved twice as far from the lamp?

15 a Prove that if $x \propto y$ and $y \propto z$, then $x \propto z$.

b Prove that if $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a \propto c$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Direct variation

Interactive skillsheet
Inverse variation

Investigation
Modelling problems involving inverse variation

Topic quiz
5I

ADVANCED

5J Rates of change

Learning intentions

By the end of this topic you will be able to ...

- ✔ describe the rate of change in a graph
- ✔ construct graphical representations of rates of change.

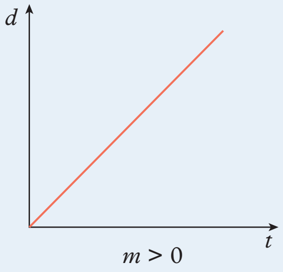
Inter-year links

Year 7	5E Line graphs
Year 8	3I Distance–time graphs
Year 9	4D Gradients and intercepts

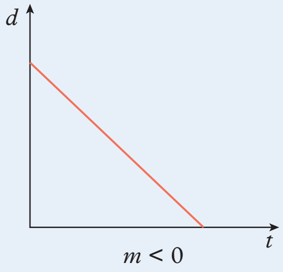
Key content video

Constant rate of change

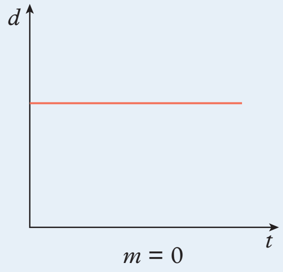
- **Rate of change** indicates how quickly one variable is changing compared to the other. For example, speed is the rate of change of distance over time.
- If a variable is directly proportional to another variable, $y \propto x$ or $y = kx$, the constant of variation k is the rate of change.
- In a linear relationship $y = mx + c$, the rate of change is the gradient of the line m. The rate of change is constant at any point on the line.
  - → If $m > 0$, the linear relationship has a positive constant rate of change. In a distance–time graph, the distance from the origin increases at a constant rate.
  - → If $m < 0$, the linear relationship has a negative constant rate of change. In a distance–time graph, the distance from the origin decreases at a constant rate.
  - → If $m = 0$, the rate of change is 0. In a distance–time graph, the distance from the origin does not change.



$m > 0$



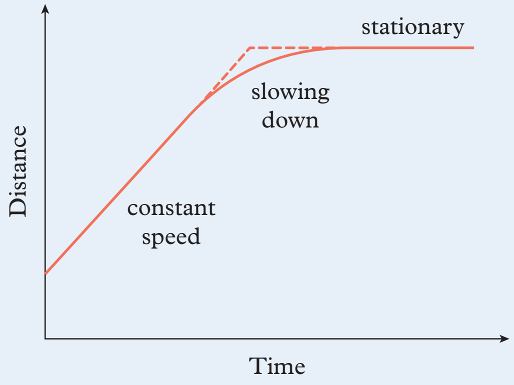
$m < 0$



$m = 0$

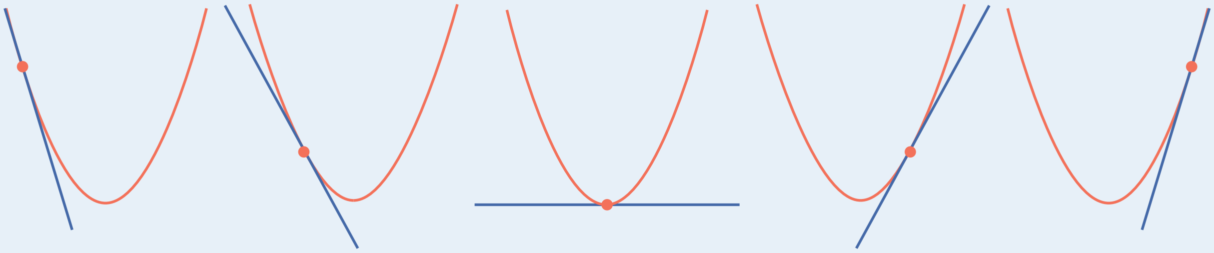
Variable rate of change

- A variable rate of change is represented by curved lines on the Cartesian plane. For example, in the following distance–time graph, the object is moving at a positive constant speed at the beginning, then slowly reduces its speed to zero.



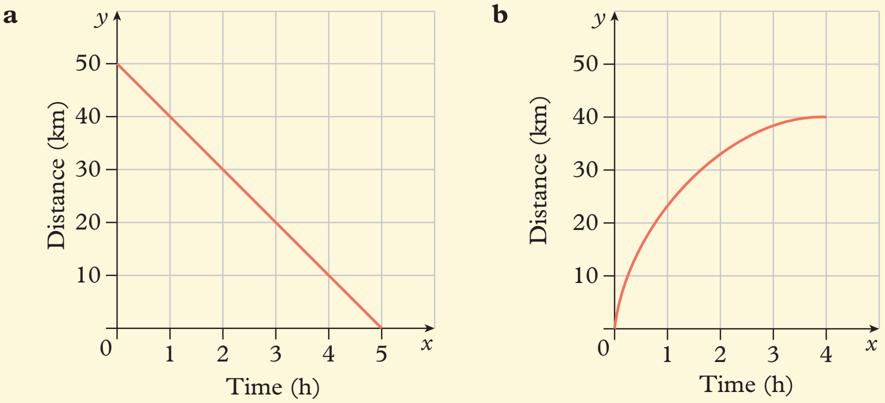
- The **tangent** of a curve at a point is the straight line that just touches, but does not cross, the curve at the point.

- The rate of change of a non-linear relationship at a point is equal to the gradient of the tangent line at the point. The rate of change of a non-linear relationship depends on the value of x.



Example 5J.1 Describing the rate of change of a graph

Determine if the following distance–time graph has a constant speed or variable speed. If constant, calculate the speed. If variable, describe how the speed is changing over time.



THINK

a **1** The graph is linear; the rate of change of all linear graphs is constant.

2 The rate of change of a linear graph is the gradient. Calculate the gradient m by substituting the coordinates of any two points on the graph into the gradient formula.

b **1** The graph is not linear, so it has a variant rate of change.

2 The object travelled approximately 25 km in the first hour, 10 km in the second hour, 5 km in the third hour and 2 km in the fourth hour. So the speed is decreasing.

WRITE

a Constant speed.

$$\begin{aligned} m &= \frac{50 - 0}{0 - 5} \\ &= -10 \end{aligned}$$

b Variable speed.

The speed is decreasing.

Example 5J.2 Constructing graphical representations of rates of change

Sketch a distance–time graph to represent the situation where a car moves at a steady speed, accelerates to pass another car and then decelerates and stops at a set of traffic lights.

THINK

1 Divide the distance–time graph into intervals: steady speed, acceleration to pass, decelerating, and waiting at the traffic lights.

2 Draw a straight-line segment from the origin with a positive gradient to represent the car moving at a steady speed away from the origin.

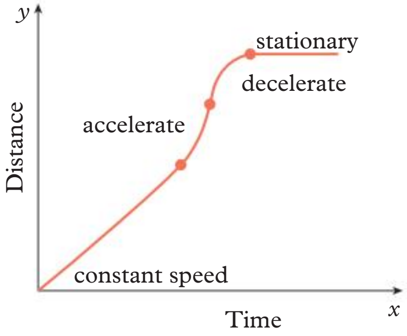
3 Transition smoothly from the straight-line segment to a curved-line segment when the car starts to accelerate. The curved line should have a steeper gradient than the initial straight line, indicating an increasing rate of change.

4 Transition from the steeper curved line segment to a flatter curved line segment as the car decelerates.

5 Gradually reduce the gradient of the curve until it approaches zero, representing the car's decreasing rate of change during deceleration.

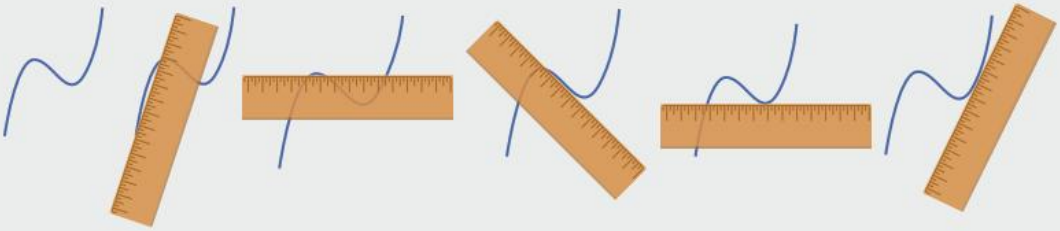
6 Draw a flat horizontal line from the end of the decelerating curve to indicate the car being stationary. This line should be parallel to the x-axis, representing zero speed and no change in position.

WRITE

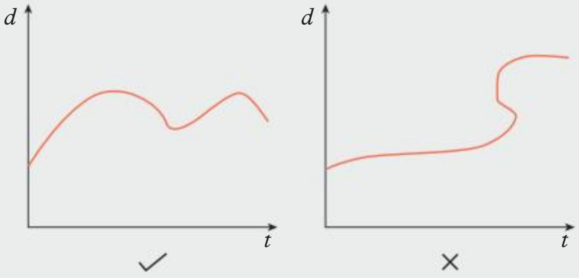


Helpful hints

✔ The rate of change of a non-linear graph can be easily visualized by sliding a ruler along the graph of a non-linear relationship following the graph's shape.



✔ When sketching a distance–time graph, remember time cannot go backwards! If you draw a vertical line from any point on the horizontal axis, it should not intersect with your distance–time graph more than once.



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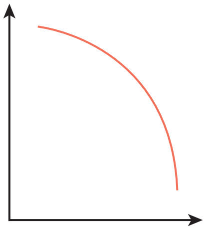
Exercise 5J Rates of change

△ 1–3, 4(a, b), 5, 6(a–c), 8(a, b), 9 □ 3–5, 7–8, 9(a, b), 10 ⬠ 3–4, 7, 10, 11

FLUENCY AND UNDERSTANDING

1 Match each of these descriptions of the quantity on the vertical axis with one of the following graphs.
- **a** Increasing at an increasing rate
- **b** Decreasing at an increasing rate
- **c** Increasing at an decreasing rate
- **d** Decreasing at an decreasing rate

I



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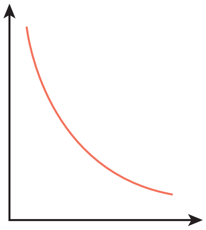
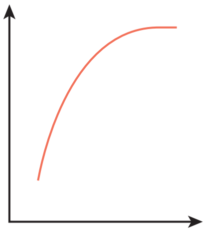


III



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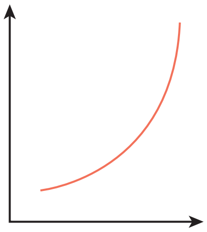
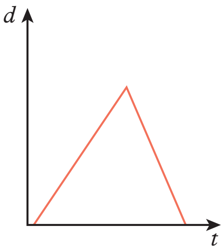


2 Match each of these descriptions to one of the graphs below.
- **a** a tourist walks towards a mailbox at constant speed to drop of a post card and walks back at a constant speed
- **b** a boat accelerates gradually to a higher speed as it moves away from the shore, maintains that speed for a period, and then decelerates to dock at a specific location
- **c** a plane takes off from an airport, accelerates to reach cruising altitude, and maintains a constant speed for the majority of the flight
- **d** a postman walks at a constant speed towards a drop-off address, stops at the address for a while before walking towards the next address at the space constant speed

I



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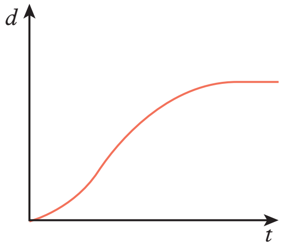

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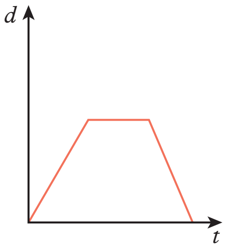

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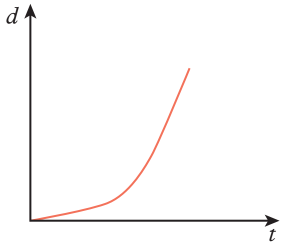
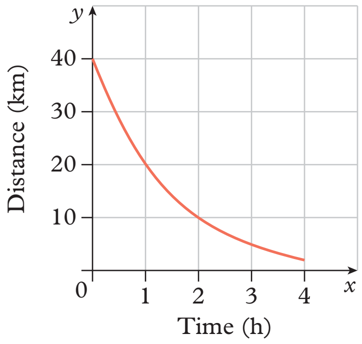
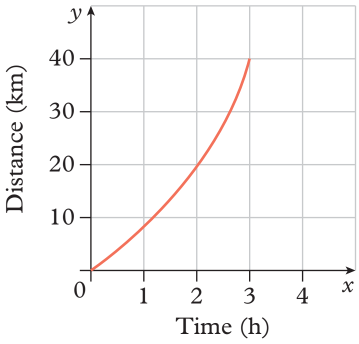


5J.1 **3** Determine if the following distance–time graphs have a constant speed or variable speed. If constant, calculate the speed. If variable, describe how the speed is changing over time.

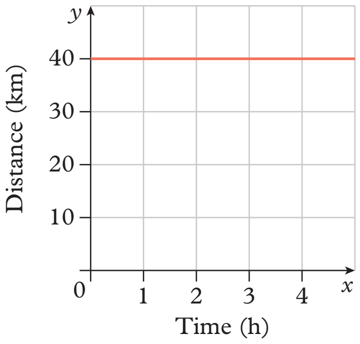
a



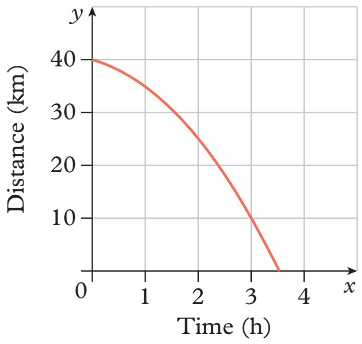
b



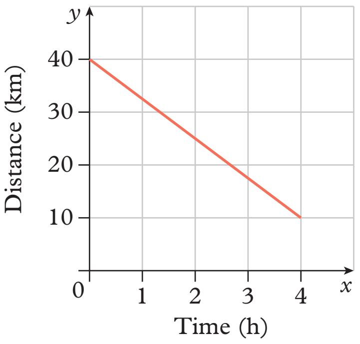
c



d



e



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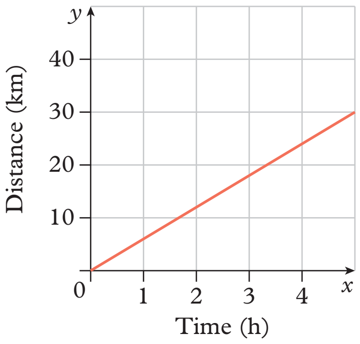


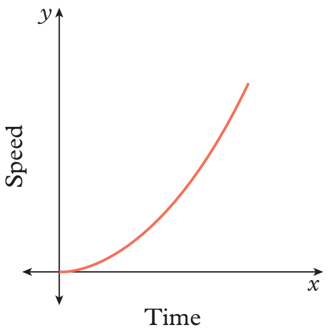
5J.1 **4** Sketch a distance–time graph for the following scenarios.

- **a** a rocket launches into space, initially accelerates rapidly, then maintains a constant speed for a while, and finally decelerates to enter a stable speed to orbit around a celestial body
- **b** a train departs from a station, accelerates steadily to reach its top speed, maintains that speed for some time, and then decelerates to a stop at the next station
- **c** a runner jogs at a consistent pace, sprints for a short distance, and slows to the previous speed to catch their breath
- **d** a skateboarder cruises at a steady pace, swiftly accelerates to avoid an obstacle on the path, and gradually decelerates coming to rest

5 Joseph measured the rate of change of a graph at three places and they were all equal to 5. Is the graph linear? Explain.

6 Describe how the rate of change varies as x increases from 0.

a



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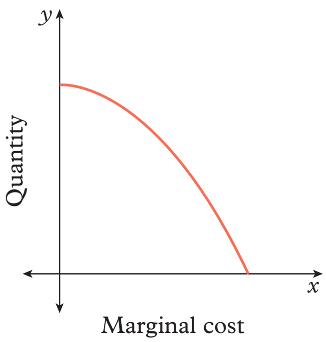

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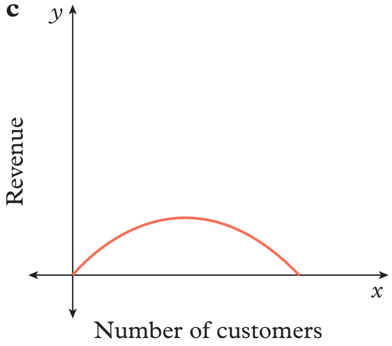

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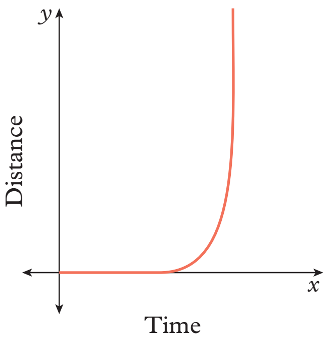

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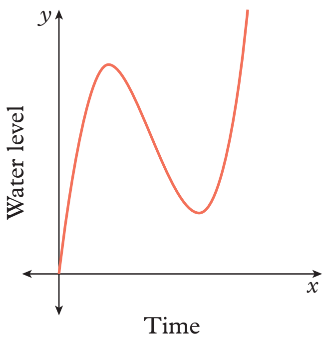

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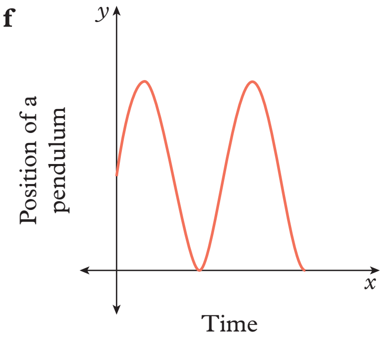
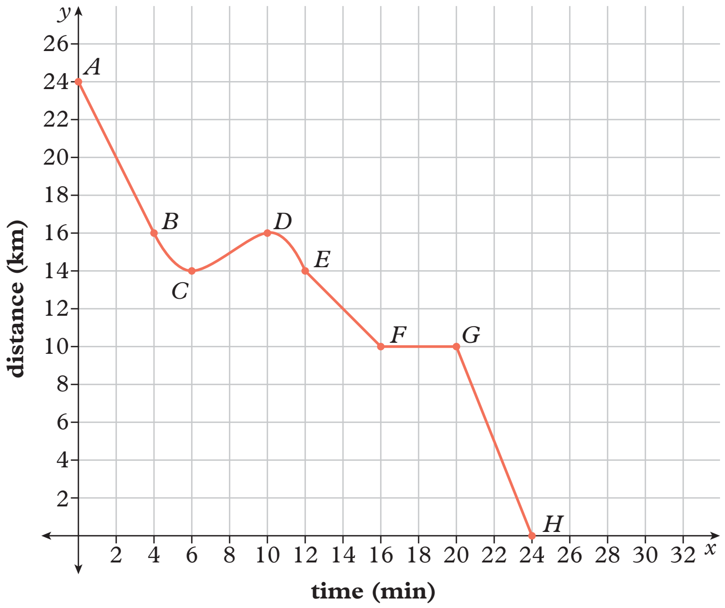


ADVANCED

FLUENCY AND UNDERSTANDING

7 Consider the distance–time graph below. Write a paragraph that describes the distance travelled and motion of the object in each segment. Include information about their initial and final points, and descriptions of the constant and variable speeds.



PROBLEM SOLVING AND REASONING

8 Describe how the y value in the following relationships changes as:

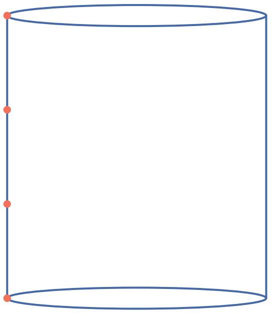
i x increases from from −1 to 0

ii x increase from 0 to 1

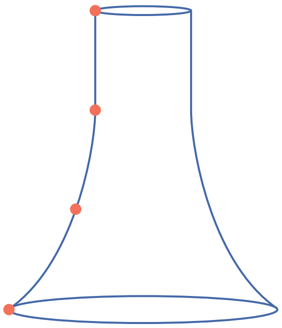
a $y = -x + 1$ **b** $y = x^2 - 1$ **c** $y = \frac{2}{x}$ **d** $y = -2^x$

9 Water is being poured into each of the following vases at a constant rate. Sketch a graph with the height of the water on the vertical axis against the time on the horizontal axis. Note: each red dot on the vase marks an increase of 1 unit in height.

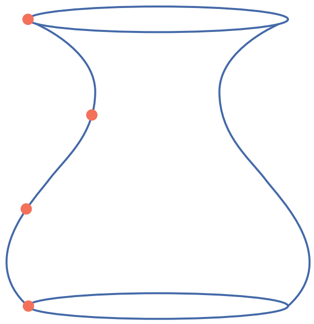
a



b



c



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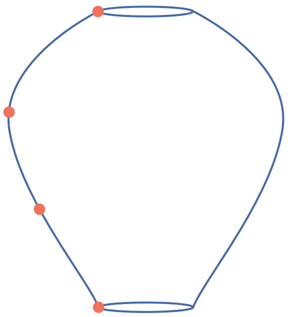

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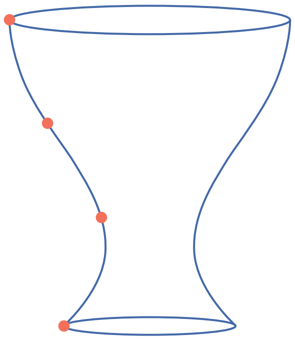

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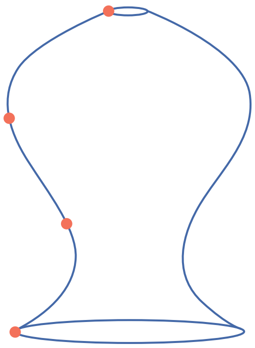
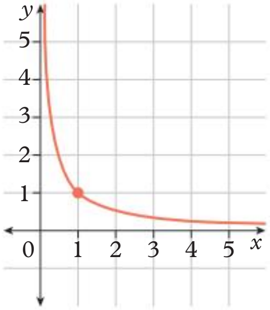


ADVANCED

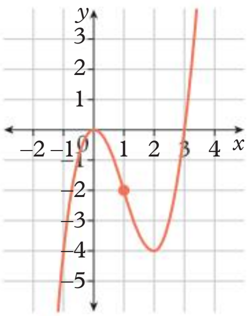
10 For each of the following graphs:

- **i** use a ruler or straight edge to visualise the gradient at the shown point and write down the y-intercept of that tangent, if it exists, to the nearest integer
- **ii** calculate the rate of change using your answer to part **i** at the point shown.

a



b



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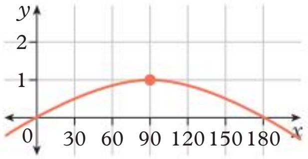
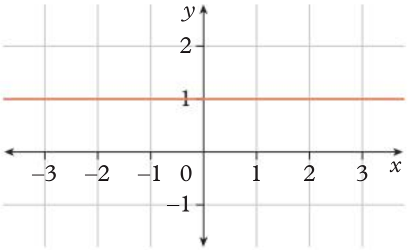
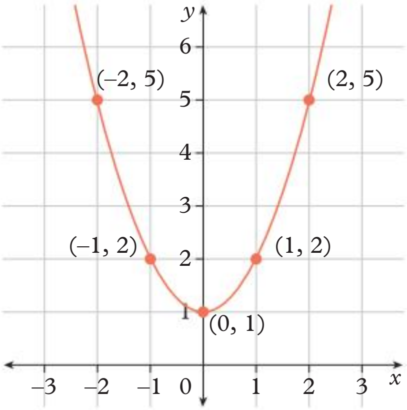


CHALLENGE

11 The graph on the right shows the gradient of $y = x + 1$. The equation of the graph is $y = 1$ because the gradient of $y = x + 1$ is always equal to 1.



- **a** Sketch a graph to show the gradient of $y = x + 2$.
- **b** If the graph of the gradient of an equation is a horizontal line, is the equation necessarily unique? Explain.
- **c** Determine the gradient of line segments connecting adjacent points in the following graph.



- **d** Use your answers to part **c** to approximate the gradient of $y = x^2 + 1$ at the following points

x	−1.5	−0.5	0	0.5	1.5
Gradient					

- **e** Use the table of values from part **d** to sketch the graph of the gradient of $y = x^2 + 1$.
- **f** If the graph of the gradient of an equation is a straight line, is the original equation necessarily unique? Explain.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Rates of change

Topic quiz
5J

Chapter summary

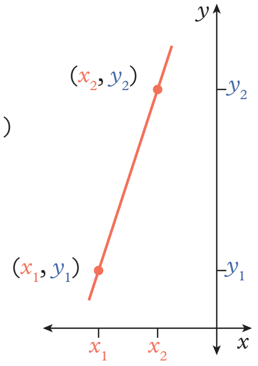
Linear relationships

Gradient-intercept form $y = mx + c$
General form $ax + by + c = 0$
Point-gradient form $y_2 - y_1 = m(x_2 - x_1)$

Gradient = $\dfrac{(y_2 - y_1)}{(x_2 - x_1)}$

Midpoint = $\left(\dfrac{x_2 + x_1}{2}, \dfrac{y_2 + y_1}{2}\right)$

Distance between two points
= $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

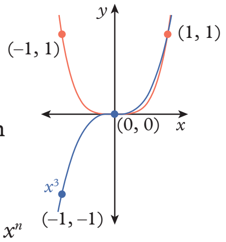


Cubic and other non-linear relationships

For equations in the form $y = x^n$, where $n \geq 2$:

- When n is odd, the curve passes through (1, 1), (0, 0) and (–1, –1), the graph is symmetrical around the origin.
- When n is even, the curve passes through (1, 1), (0, 0) and (–1, 1), the graph is symmetrical across the y-axis.

Transformations can be performed on $y = x^n$ to create the graph of $y = k(x - b)^n + c$



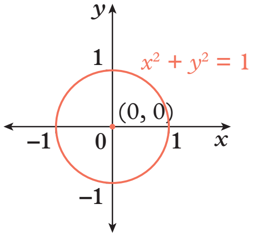
Circles

$(x - a)^2 + (y - b)^2 = r^2$

Radius of r units

Vertical translation of b units
If $b > 0$, up
If $b < 0$, down

Horizontal translation of a units
If $a > 0$, right
If $a < 0$, left



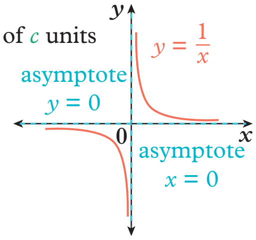
Hyperbolas

Dilation in the y-direction
If $k > 0$, upright
If $k < 0$, inverted

$$y = \frac{k}{x - b} + c$$

Vertical translation of c units
If $c > 0$, up
If $c < 0$, down

Horizontal translation of b units
If $b > 0$, right
If $b < 0$, left



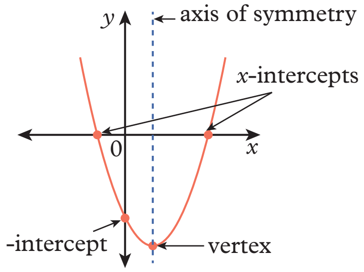
Parabolas

Vertical translation of c units
If $c > 0$, up
If $c < 0$, down

$$y = k(x - b)^2 + c$$

Horizontal translation of b units
If $b > 0$, right
If $b < 0$, left

Stretch in the y-direction
If $k > 0$, vertical stretch up
If $0 < k < 1$, horizontal stretch outward
If $k < 0$, reflect across the x-axis



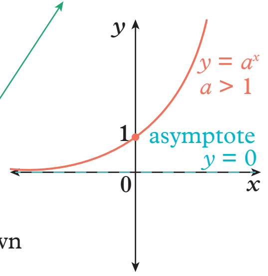
Exponential relationships

If $n = 1$, upright
If $n = -1$, horizontal reflection

$$y = k \times a^{nx} + c$$

If $k > 0$, upright
If $k < 0$, vertical reflection

Vertical translation of c units
If $c > 0$, move up
If $c < 0$, move down



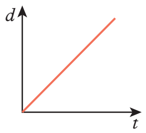
Direct and inverse variation

Direct variation: $y = kx$, $y \propto x$

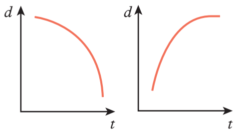
Inverse variation: $y = \dfrac{k}{x}$, $y \propto \dfrac{1}{x}$

Rates of change

Constant rate of change



Variable rate of change



ANS
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Chapter review

Chapter review quiz
Assess your knowledge of this chapter.
✔ Standard
✔ Advanced

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- asymptote
- axis of symmetry
- circle
- constant of variation
- cubic
- dilation
- directly proportional
- distance
- domain
- exponential
- gradient
- gradient-intercept form
- general form
- hyperbola
- intercept
- inversely proportional
- linear graph
- linear relationship
- logarithmic relationship
- midpoint
- minimum turning point
- maximum turning point
- non-linear relationship
- parabola
- point-gradient form
- quadratic
- radius
- range
- reflection
- rate of change
- rise
- run
- tangent
- transformation
- translation
- turning point
- turning point form
- vertex

1 Match the relationships **a–i** with a key term **I–VI**.

a $3x + 4y = 12$ **b** $y = \frac{6}{x}$ **c** $(x - 1)^2 + y^2 = 4$

d $y = x^2 + 5x + 6$ **e** $y = 2^x$ **f** $y = 4x^3$

g $x = -2$ **h** $y \propto x$ **i** $y \propto \frac{1}{x}$

I circle **II** quadratic relationship **III** cubic relationship
IV exponential relationship **V** hyperbolic relationship **VI** linear relationship

2 Which descriptions best explain the transformation of the graph of $y = x^2$ to produce the graph of $y = a(x - b)^2 + c$?

A The dilation of a in the y-direction. This has no impact on any x-intercepts.
B The dilation of a in the y-direction. This has no impact on any y-intercepts.
C If $a < 0$, there will be a reflection across the y-axis.
D If $a < 0$, there will be a reflection across the x-axis.
E There is a horizontal translation c units and a vertical translation of b units.
F There is a horizontal translation b units and a vertical translation of c units.
G Translations affect the shape but not the vertex.
H Translations affect the shape and the vertex.
I Translations affect the vertex but not the shape.
J Translations do not affect the shape nor the vertex.

3 Complete the following sentences using words from the list of key terms.

a The ____________________ is the line that the exponential graph $y = 3^x$ approaches for large negative values of x .

b When two variables are ____________________ proportional, the gradient of the straight line is the same as the ____________________.

c The ____________________ in a parabola can be found by calculating the average or ____________________ of the x-intercepts.

4 State true or false for the following statements about the non-linear relationships in this chapter. If it is false, change the statement to make it true.

a All have changing gradients as x increases.
b All have a horizontal asymptote.
c All have a y-intercept.
d All have parts of the domain that are increasing or decreasing at different rates.
e All have a domain of $\mathbb{R}$ (real numbers).
f For any x-value within the domain, which is not an endpoint, the gradient of the curve can be calculated from the tangent at that point.
g Variables that are inversely proportional are not examples of non-linear relationships.

Multiple choice

ADV 5A **1** The gradient of the line $12x + 4y = 8$ is:
A 3 **B** -3 **C** 13 **D** -12 **E** 4

ADV 5A **2** Which one of the following equations is not equivalent to the others?
A $y = 3x - 6$ **B** $-3x + 6 = y$ **C** $2 - x = -\frac{y}{3}$ **D** $12 + 2y - 6x = 0$ **E** $3x - y - 6 = 0$

ADV 5B **3** Which of the following pairs of points are the closest to each other?
A $(-3, 3), (1, 1)$ **B** $(2, -3), (10, 0)$ **C** $(1, 3), (5, 3)$ **D** $(-1, 2), (7, 2)$ **E** $(0, 6), (2, 0)$

ADV 5B **4** Which of the following pairs of points have a midpoint at $(6, -3)$?
A $(-3, 3), (10, -10)$ **B** $(2, -13), (-10, 3)$ **C** $(19, -7), (-7, 1)$ **D** $(-1, 2), (7, -12)$ **E** $(0, 6), (-3, 0)$

ADV 5C **5** The coordinates of the x-intercepts of the graph of $y = x^2 - 6x + 5$ are:
A $(1, 0), (5, 0)$ **B** $(0, 1), (0, 5)$ **C** $(-3, 0), (2, 0)$ **D** $(-1, 0), (-5, 0)$ **E** $(0, -1), (0, -5)$

ADV 5C **6** Which of the following statements about the graph of $y = x^2 + 2x - 3$ is *incorrect*?
A It is upright. **B** The graph has two x-intercepts.
C The turning point has coordinates $(-4, -1)$. **D** The graph shows a minimum.
E The y-intercept is $(0, -3)$.

ADV 5D **7** Compared with the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is:
A stretched by a factor of 2 in the y-direction. **B** reflected in the x-axis.
C translated 2 units up. **D** translated 2 units right.
E translated 2 units left.

ADV 5D **8** The coordinates of the turning point of the graph of $y = (x - 2)^2 - 5$ are:
A $(-2, -5)$ **B** $(2, -5)$ **C** $(-2, 5)$ **D** $(2, 5)$ **E** $(0, -1)$

ADV 5E **9** Which of the following statements is not true for a relationship in the form $y = x^n$?
A When $n = 1$, the relationship only has one intercept.
B When $n = 2$, the relationship has an axis of symmetry.
C When $n = 3$, the relationship passes through $(0, 0)$.
D When $n = 4$, the relationship passes through $(1, -1)$.
E When $n = 5$, the relationship passes through $(1, 1)$.

ADV 5F **10** The coordinates of the centre of the circle $(x - 2)^2 + (y + 3)^2 = 16$ are:
A $(2, -3)$ **B** $(-2, 3)$ **C** $(-2, -3)$ **D** $(2, 3)$ **E** $(4, 4)$

ADV 5G **11** What is the rule obtained when the graph of $y = 6 \times 2^{3x} + 3$ is reflected across the x-axis?
A $y = 6 \times 2^{-3x} + 3$ **B** $y = -6 \times 2^{3x} + 3$ **C** $y = -6 \times 2^{3x} - 3$
D $y = 6 \times 2^{\frac{x}{3}} + 3$ **E** $y = 6 \times 2^{3x} - 3$

ADV 5G **12** Which of the following relationships has a graph that is a reflection of $y = 2^x$ in the line $y = x$?
A $y = x^2$ **B** $y = \left(\frac{1}{2}\right)^x$ **C** $y = 2^{-x}$ **D** $y = -2^x$ **E** $y = \log_2 x$

ADV 5H **13** Which of these equations represents a hyperbola?
A $y = x^2 - 4$ **B** $y = x^4$ **C** $y = 4^x$ **D** $y = \frac{1}{x - 4}$ **E** $y = \log_4 x$

STN 5I ADV **14** Which of the following represents an example of a directly proportional relationship?

A

x	0	1	2	3
y	2	7	12	17

B

x	0	1	2	3
y	0	1	4	9

C

x	1	2	3	4
y	48	24	16	12

D

x	5	10	15	20
y	12	22	32	42

E

x	0	1	2	3
y	0	6	12	18

ADV 5J **15** If the values in question **14** are plotted on a Cartesian plane and connected with a smooth line, which of the relationships has an decreasing rate of change?

Short answer

ADV 5A **1** Rearrange the following linear relationships into gradient–intercept form.

a $6x + y = 0$ **b** $2y + 3 = x$

ADV 5A **2** Rearrange the following linear relationships into general form. Write your answers without fractions.

a $y + 3x = x - 2$ **b** $y = \frac{3}{2}x + \frac{3}{10}$

ADV 5A **3** For each of the following relationships:

i find the coordinates of the intercepts **ii** graph the relationship on the Cartesian plane.

a $4x + 5y = 20$ **b** $16x + 36y = 4$

ADV 5A **4** Determine the equation of each linear graph described by the information below. Write your answer in gradient–intercept form.

a gradient of 2, passes through $(-2, -1)$ **b** gradient of -3, passes through $(-2, 3)$

ADV 5B **5** For each of the following pairs of points:

i determine the gradient of a line passing through the two points

ii determine the coordinates of the midpoint of the two points

iii determine the distance between the two points.

a $(-3, 4)$ and $(7, 2)$ **b** $(3, -2)$ and $(7, 2)$

ADV 5C **6** For each of these quadratic relationships, find the coordinates of the:

i x-intercept(s) **ii** y-intercept.

a $y = -x^2 + 25$ **b** $y = x^2 - 3x - 4$ **c** $y = 2x^2 - 8x + 8$ **d** $y = -x^2 - x + 6$

ADV 5C **7** Use intercepts to help you sketch the graph of each relationship below. Label the turning point with its coordinates each time.

a $y = x^2 - 4x + 3$ **b** $y = -x^2 - 4x + 5$

ADV 5D **8** For each of the quadratic relationships below:

i describe the translations performed on $y = x^2$ **ii** write the coordinates of the turning point.

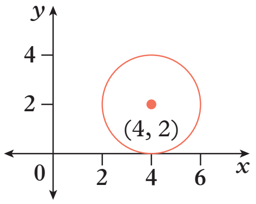
a $y = (x - 5)^2 + 2$ **b** $y = (x + 6)^2 - 4$

c $y = (x - 1)^2 - 1$ **d** $y = (x + 3)^2 + 5$

e $y = (x - 4)^2 + 2$ **f** $y = 3(x + 7)^2 - 3$

ADV 5E **9** The graph of $y = x^4$ is translated 4 units left and 6 units down. What is the equation of the new curve? List the domain and range of the graph.

ADV 5F **10** Write the equation for the circle on the right.



ADV 5G **11** **a** Sketch the graph of $y = 5^x$.

b Describe the transformations performed on the graph of $y = 5^x$ to produce the graph of:

i $y = 5^{x+3}$ **ii** $y = 5^x + 3$ **iii** $y = \log_5 x$

c Sketch all four graphs from parts **a** and **b** on the same Cartesian plane.

ADV 5H **12** Describe the transformations performed on the graph of $y = \frac{1}{x}$ to produce the graph of each of these equations.

a $y = \frac{1}{x} - 5$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{4}{x}$ **d** $y = \frac{1}{x+5} + 6$

STN 5I ADV **13** At a craft store, the cost of fabric is directly proportional to the number of square metres purchased. Mary purchased 7.5 square metres of fabric for $93.75.

a Find the constant of proportionality.

b How much would it cost Mary for 4 square metres of fabric?

c If Mary has $200, how many square metres of fabric can she purchase?

STN 5I ADV **14** Consider the following table which shows an inversely proportional relationship between F and a.

a	2	4	5	10
F	10	5	4	2

a Find the constant of proportionality for $F \propto \frac{1}{a}$. **b** Find the constant of proportionality for $a \propto \frac{1}{F}$.

c Find the value of F if $a = 2.5$.

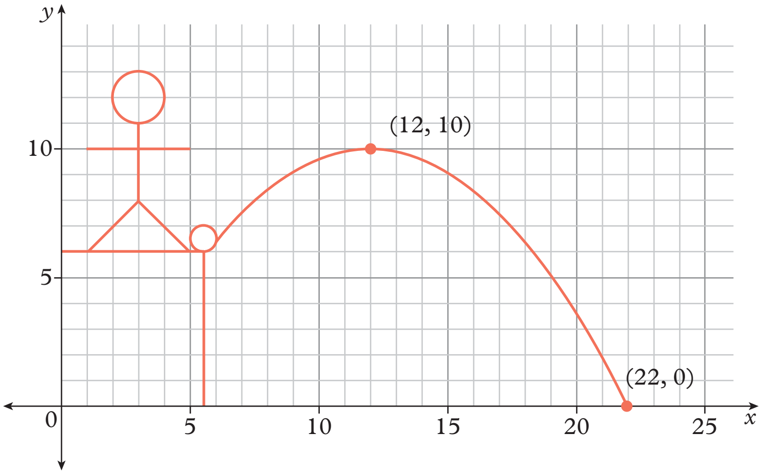
ADV 5J **15** Identify the type of graph that represents each of these equations.

a $y = \frac{1}{x + 4}$ **b** $y = 5^x$ **c** $y = 3x^2 - 5$

ADV 5J **16** Describe how the value of y changes as the value of x increases from −1 to 1 for the relationships in question **15**.

Analysis

ADV **1** Lionel kicks a mini soccer ball from a platform. The following diagram shows Lionel and the path of the ball.



The image can be made from the basic forms of the following equations:

$y = x$, $y = x^2$, $x^2 + y^2 = r^2$, $x = c$, $y = c$

Use transformations, or otherwise, to determine the equations for each line or curve in the diagram. Complete the table, including the relevant domain or range for each equation.

Item	Equation	Domain	Range
Left leg	$y = x + 5$	$1 \le x \le 3$	
Right leg			
Body			
Arms			
Head			
Top of platform			
Side of platform			
Soccer ball			
Path of the kick (in vertex form)			

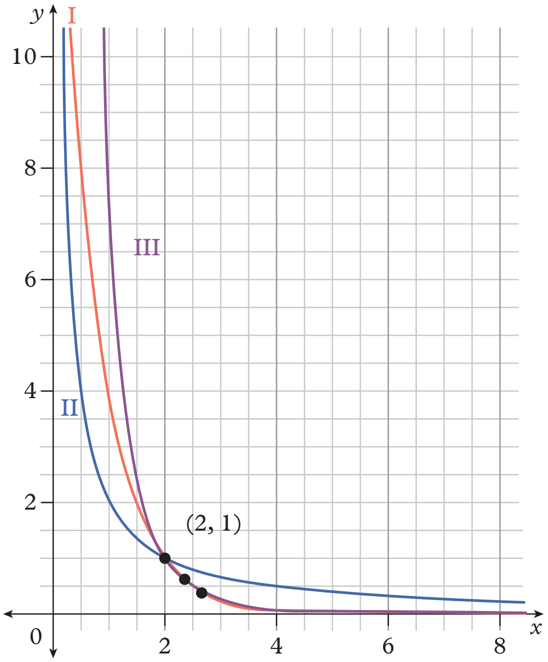
ADV **2** Buzz can hit a golf ball landing at 100 metres with a height of 80 metres. Assume the course is on level ground and ignore any additional bouncing and rolling. It can be modelled with the equation $h = a(d^2 - 100d)$ where h is the height in metres and d is the horizontal distance in metres from where the ball was hit. The value a is a constant.

- **a** Factorise the given equation.
- **b** Determine the horizontal intercepts and describe what they represent.
- **c** Write down the coordinates for the maximum height of the ball.
- **d** Sketch a graph showing the path of the ball. Label the turning point with its coordinates.
- **e** Use the coordinates for the maximum height to determine the value for a in the equation.
- **f** Calculate the height of the ball when it has travelled 20 metres horizontally. Round to one decimal place.
- **g** Where is the other horizontal position when the ball is at the same height as in part **f**?
- **h** Draw lines on your graph to show your answers to parts **f** and **g**.

The flight of a golf ball can also be modelled with an equation of height, h, in metres against time, t, in seconds. It looks similar. The time for Buzz's shot to first land back on the ground is 6 seconds.

- **i** Draw a graph of the ball's flight using appropriate axes labels and scale. Label the maximum height with its coordinates.
- **j** Determine the equation for the ball's path in terms of h and t.
- **k** Calculate the approximate time when the ball has travelled 20 metres horizontally.
- **l** What is the horizontal speed of the ball? Round to one decimal place.

ADV **3** The four graphs shown on the right are all decreasing. They all pass through the point (2, 1).



- **a** Graph I is an exponential graph with an equation $y = k(4)^{-x}$.
  - **i** Find the value of k.
  - **ii** Describe the transformations of the basic graph of $y = 4^x$
- **b** Graph II is part of a hyperbola with equation $y = \frac{b}{x}$.
  - **i** Find the value for b and write down the equation.
  - **ii** Write down the equation for any asymptotes.
- **c** Graph III shows the graph when y is inversely proportional to the cube of x.
  - **i** Determine the constant of proportionality.
  - **ii** Write down the equation.
- **d** Jon, Andy and Rob are training for a half marathon. Their times, in minutes, for the first 2 km are 11.15, 11.75 and 12.0, respectively.
  - **i** Calculate each of their speeds, in km/min, for this first 2 km. Round to three significant figures.
  - **ii** When you plot their speed against time, with time on the x-axis, it will match one of the four graphs above. Which one?
  - **iii** What are the labels and units for each axis?
  - **iv** Explain, in context, why the graph is decreasing.
  - **v** Interpret each asymptote in context.

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Rearrange linear equations from gradient–intercept form to general form and vice versa ☐ Find the x-and y-intercepts and graph the equation of a straight line in any form ☐ Use point–gradient form to find a linear equation	☐ Go back to Topic 5A Linear relationships
☐ Apply the formula to determine the gradient of a line given two points ☐ Apply the formula to determine the coordinates of the midpoint given two points ☐ Apply the formula to determine the distance between two points	☐ Go back to Topic 5B Gradient, midpoint and distance between two points
☐ Find the x- and y-intercepts of a quadratic relationship ☐ Find the coordinates of the turning point for a quadratic relationship with two x-intercepts ☐ Graph parabolas using the intercepts and turning point	☐ Go back to Topic 5C Graphing parabolas using intercepts
☐ Sketch parabolas of the form $y = k(x - b)^2 + c$ using reflections, dilations, vertical translations and horizontal translations	☐ Go back to Topic 5D Graphing parabolas using transformations
☐ Sketch curves in the form $y = x^n$ and describe their features ☐ Compare polynomial curves and describe their transformations	☐ Go back to Topic 5E Cubic and other non-linear relationships
☐ Graph simple circles of the form $x^2 + y^2 = r^2$ ☐ Graph circles of the form $(x - a)^2 + (y - b)^2 = r^2$ using vertical translations and horizontal translations ☐ Determine the equation of a circle from a description or a graph	☐ Go back to Topic 5F Graphing circles
☐ Sketch exponential relationships using transformations ☐ Compare and contrast exponential and logarithmic graphs	☐ Go back to Topic 5G Graphing exponential and logarithmic relationships
☐ Sketch simple hyperbolas of the form $y = \frac{k}{x}$ ☐ Sketch hyperbolas using reflections and dilations ☐ Sketch hyperbolas using vertical translations and horizontal translations	☐ Go back to Topic 5H Graphing hyperbolas
☐ Identify when two variables are directly proportional to each other ☐ Identify when two variables are inversely proportional to each other ☐ Solve modelling problems involving direct and inverse variation	☐ Go back to Topic 5I Direct and inverse variation
☐ Describe the rate of change in a graph ☐ Construct graphical representations of rates of change	☐ Go back to Topic 5J Rates of change

6

Polynomials

Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Expanding binomial products
- ✔ Factorising quadratic expressions
- ✔ Solving quadratic equations
- ✔ Sketching non-linear graphs
- ✔ Sketching parabolas using transformations

Curriculum links

- ADV EXT Defines, operates with and graphs polynomials and applies the factor and remainder theorems to solve problems (MA5-POL-P-01)
  - – Define and operate with polynomials
  - – Divide polynomials
  - – Apply the factor and remainder theorems to solve problems
  - – Graph polynomials
- ADV Uses function notation to describe and graph functions of one variable and graphs inequalities in one and 2 variables (MA5-FNC-P-01)
  - – Define relations and functions, and use function notation
  - – Find the domain and range of a function graph functions

ADVANCED

6A Functions and relations

Learning intentions

By the end of this topic you will be able to ...

- ✔ explain the difference between functions and relations
- ✔ determine whether a graph represents a function
- ✔ use function notation.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	5D Sketching parabolas using transformations

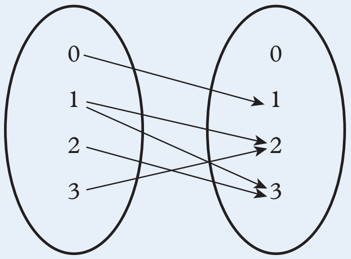
Sets

Key content video

- A **set** is a collection of objects. Objects within a set are called **elements**.
- A set is defined by the elements it contains. For example, the set $\{1, 2, 3, 4\}$ is not the same as the set $\{1, 2, 3\}$ because the first set contains the element '4'.
- Sets can be defined by listing all the elements in a set between brackets, as in $\{1, 2, 3, 4\}$. Sets can also be defined in words, for example 'the set of all prime numbers' or 'the set of all solutions to an equation'.
- Some sets are referred to using symbols. For example, the set of real numbers is written as $\mathbb{R}$, the set of integers as $\mathbb{Z}$ and the set of rational numbers as $\mathbb{Q}$.

Relations

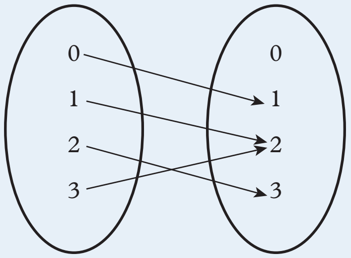
- A **relation** is a correspondence (or map) between the elements of two sets. For example, the relation 'favourite colour of a student' maps from the set of students in a class to the set of colours.
- Relations can be specified in different ways, including in words, using arrow diagrams, as a set of ordered pairs, as a rule or as a graph.



$\{(0, 1), (1, 2), (1, 3), (2, 3), (3, 2)\}$

Functions

- A **function** is a relation which maps every element in one set to exactly one element in another set.
- When a function is graphed on the Cartesian plane, the function is a map from its domain on the x-axis to the set of real numbers $\mathbb{R}$ on the y-axis.
- The **vertical line test** states that if a vertical line drawn through any part of a graph intersects or touches the graph only once, then the graph represents a function. If a vertical line intersects or touches the graph more than once, then the graph doesn't represent a function.
- Functions with a rule can be written using **function notation** in the form $f(x) = [\text{rule}]$. For example, $f(x) = x^2$ (read as 'f of x equals x squared') is the function which squares a number represented by the variable x.
- Function notation can be used to substitute values into an expression. For example, if $f(x) = x + 2$, then $f(3) = 3 + 2 = 5$.
- The **domain** of a function is the set of input values of that function, while the **range** is the set of output values.



$$\rlap{\overbrace{P(x) = x^2 + 2x + 1}^{\text{function}}}P(x) = \underbrace{\underbrace{x^2 + 2x + 1}_{\text{expression}} = 0}_{\text{equation}}$$

Example 6A.1 Determining if a relation is a function

Determine whether the following relations represented by sets of ordered pairs are functions.

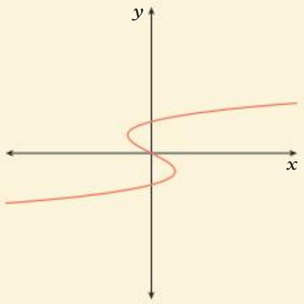
a $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$ **b** $\{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}$

THINK	WRITE
a In this relation, there are four distinct *x*-coordinates. Since each *x*-coordinate maps to exactly one *y*-coordinate, this relation is a function.	**a** This relation is a function.
b In a function, an element in the first set must be mapped to precisely one element in the second set. In this relation, 0 maps to three different *y*-coordinates (0, 1 and 2), so the relation is not a function.	**b** This relation is not a function.

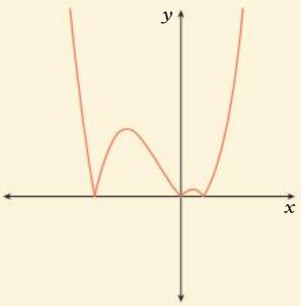
Example 6A.2 Using the vertical line test

Use the vertical line test to determine whether the relations represented by the following graphs are functions.

a



b



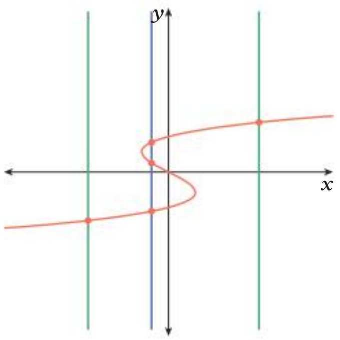
THINK

a Draw vertical lines on the graph and determine whether any of the lines intersect or touch the graph more than once. The blue vertical line intersects the graph three times so the graph fails the vertical line test, which means that it does not represent a function.

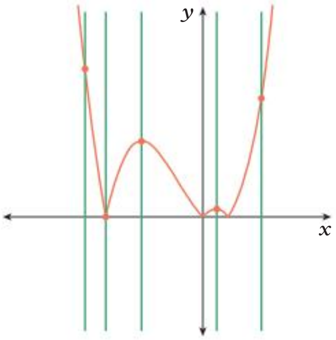
b Draw vertical lines on the graph and determine whether any of the lines intersect or touch the graph more than once. All possible vertical lines drawn on this graph intersect the graph exactly once, so the graph represents a function.

WRITE

a This graph does not represent a function.



b This graph represents a function.



Example 6A.3 Using function notation

For each of the following functions, evaluate $f(0)$, $f(1)$ and $f(-3)$.

a $f(x) = x^3 + 1$ **b** $f(x) = 2^x - x$

THINK	**WRITE**
Substitute the values 0, 1 and –3 into each function in the place of x and simplify.	**a** $f(x) = x^3 + 1$ $f(0) = 0^3 + 1 = 0 + 1 = 1$ $f(1) = 1^3 + 1 = 1 + 1 = 2$ $f(-3) = (-3)^3 + 1 = -27 + 1 = -26$ **b** $f(x) = 2^x - x$ $f(0) = 2^0 - 0 = 1 - 0 = 1$ $f(1) = 2^1 - 1 = 2 - 1 = 1$ $f(-3) = 2^{-3} - (-3) = \frac{1}{8} + 3 = \frac{25}{8}$

Helpful hints

- ✔ All graphs (and their corresponding rules) that you have studied, such as linear graphs, parabolas, exponential graphs, hyperbolas and circles, are examples of relations.
- ✔ Remember that in a function, each element in the first set maps to exactly one element in the second set. In a relation, an element may map to more than one element in the second set.
- ✔ When applying the vertical line test, remember that the graph of a relation fails if any vertical line touches or intersects the graph at more than one point, and passes if no such vertical line can be drawn.
- ✔ In function notation, the variable inside the brackets should match the variable used in the rule of the function.

ANS p711

Exercise 6A Functions and relations

△ **1–6, 7(a–c), 8–11(a,b), 12, 16, 17(a,b), 18(a)**

■ **1(b,d), 2(a,c,e), 3, 5, 6(2nd, 3rd columns), 7(c–f), 8–12, 14, 15, 17, 18(a), 19**

⬠ **2(b,d,f), 4(d–f), 5, 6(3rd, 4th columns), 7(c,f), 11–15, 17(c,d), 18(b), 19–21**

1 Write each of the following sets as a list of elements in brackets.

a the set of whole numbers from 1 to 10

b the set of integers greater than –5 and less than 4

c the set of positive prime numbers less than 20

d the set of the first five letters of the alphabet

6A.1 2 Determine whether the following relations represented by sets of ordered pairs are functions.

a $\{(0, 3), (1, 4), (2, 7), (3, 10)\}$

b $\{(1, 5), (5, 1), (2, 6), (6, 2), (3, 7), (7, 3)\}$

c $\{(1, 9), (5, 8), (-2, 7), (5, 11), (3, 1), (20, 5)\}$

d $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

e $\{(12, -5), (12, 4), (12, 0), (12, 1)\}$

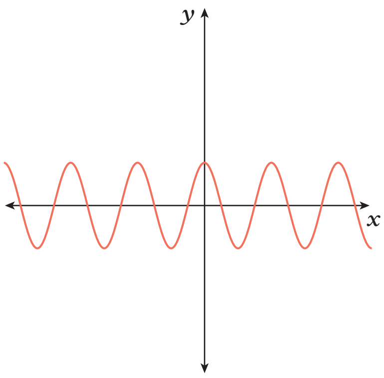
f $\{(-5, 12), (4, 12), (0, 12), (1, 12)\}$

ADVANCED

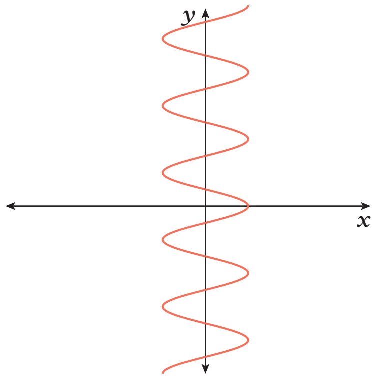
3 For each of the relations in question **2** which are not functions, identify two ordered pairs which can be used to show that the relation is not a function.

6A.2 4 Use the vertical line test to determine whether the relations represented by the following graphs are functions.

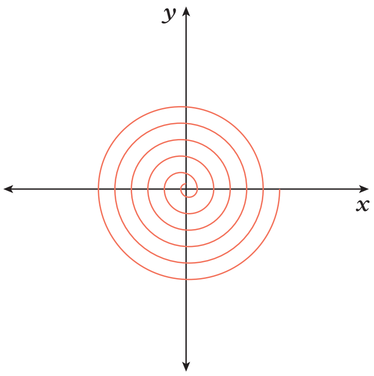
a



b



c



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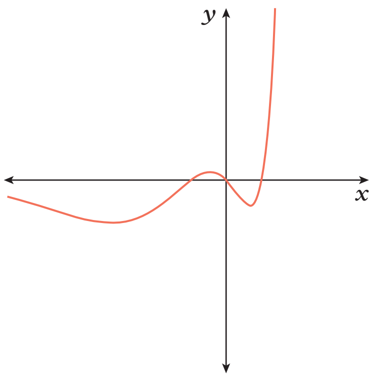

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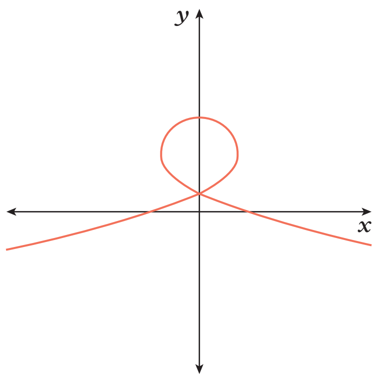

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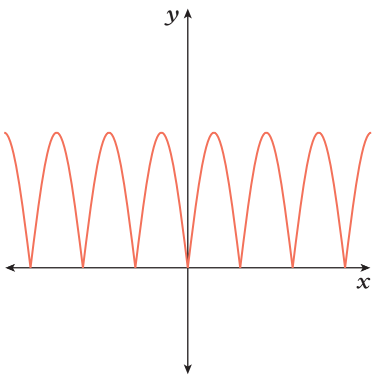
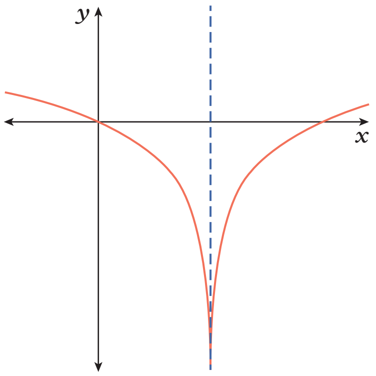


5 Use the vertical line test to determine whether the relations represented by the following graphs are functions. In this question, asymptotes are indicated by dashed blue lines.

a



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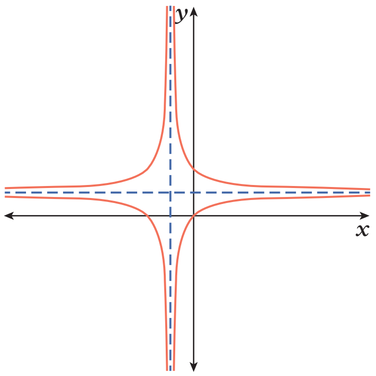

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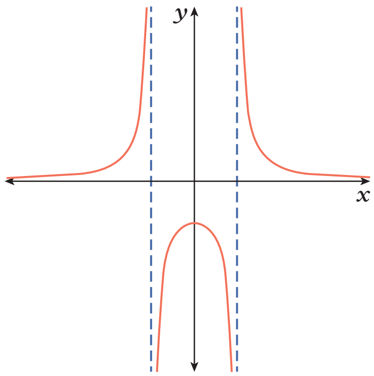

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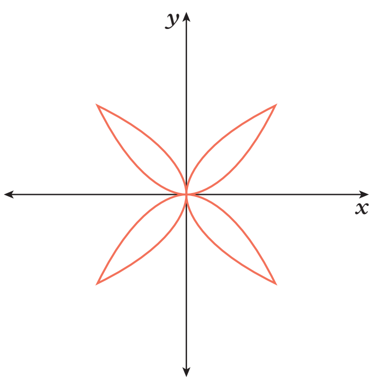

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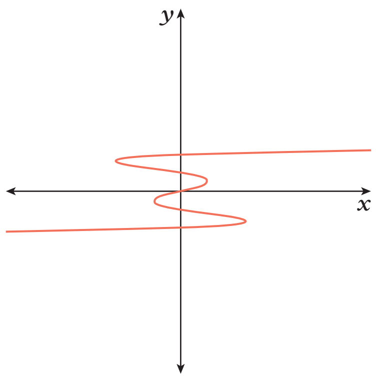
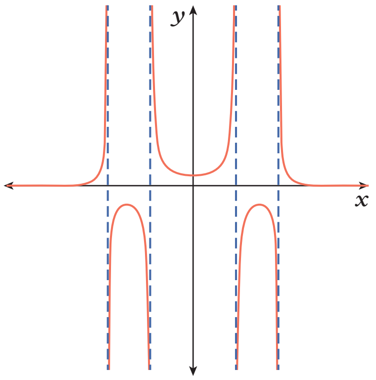


f



PROBLEM SOLVING AND REASONING

6A.3 **6** For each of the functions below, evaluate:

i $f(0)$ **ii** $f(1)$ **iii** $f(-1)$.

a $f(x) = x + 1$ **b** $f(x) = x^2 - 1$ **c** $f(x) = 2x^2 + x$ **d** $f(x) = 3^x + 1$

e $f(x) = 2^x - 1$ **f** $f(x) = 3 \times 2^{-x}$ **g** $f(x) = \frac{3}{x+2}$ **h** $f(x) = \frac{1}{2x-3} + 1$

i $f(x) = \frac{-2}{x-2} - 2$ **j** $f(x) = \log_2(x + 3)$ **k** $f(x) = 2\log_2(2x + 3)$ **l** $f(x) = 3\log_2(5 - 2x)$

7 For each of the functions below, evaluate:

i $f(2)$ **ii** $f(-2)$ **iii** $f(5)$.

a $f(x) = x + 1 + 2^x$ **b** $f(x) = x^2 - 1 + 3^x$ **c** $f(x) = 2x^2 + x + 4 \times 2^{-x}$

d $f(x) = 3^x + \frac{3}{x+5}$ **e** $f(x) = 2^x + \frac{1}{4x-4}$ **f** $f(x) = -3 \times 2^{-x} - \frac{2}{x-3}$

8 **a** Sketch the following linear graphs.

i $y = x$ **ii** $y = 2x + 3$ **iii** $y = -3x - 1$ **iv** $y = 4$ **v** $x = -2$

b Use the vertical line test to determine which of the graphs in part **a** are functions.

c Give a reason why vertical lines are not functions.

d Explain why all linear graphs which are not vertical lines are functions.

9 **a** Sketch the following quadratic relationships.

i $y = x^2$ **ii** $y = 2x^2$ **iii** $y = -(x - 2)^2 + 1$ **iv** $y = (x + 2)^2 + 4$

b Use the vertical line test to determine which of the graphs in part **a** are functions.

c Explain why all quadratic relationships of the form $y = \mathrm{a}x^2 + bx + c$ are functions.

10 **a** Sketch the following exponential relationships.

i $y = 2^x$ **ii** $y = 3 \times 2^{-x}$ **iii** $y = -2^x$ **iv** $y = -2^{-x}$ **v** $y = 2^x + 1$

b Use the vertical line test to determine which of the graphs in part **a** are functions.

c Explain why all exponential relationships of the form $y = A \times b^{nx} + k$ are functions.

11 **a** Sketch the following hyperbolas.

i $y = \frac{1}{x}$ **ii** $y = \frac{-2}{x}$ **iii** $y = \frac{1}{x-2}$ **iv** $y = \frac{-3}{x} + 1$

b Use the vertical line test to determine which of the graphs in part **a** are functions.

c Explain why all hyperbolic relationships of the form $y = \frac{a}{x-h} + k$ are functions.

12 **a** Sketch the following circles.

i $x^2 + y^2 = 1$ **ii** $(x - 1)^2 + y^2 = 4$ **iii** $x^2 + (y + 2)^2 = 9$ **iv** $(x + 1)^2 + (y - 3)^2 = 1$

b Use the vertical line test to determine which of the graphs in part **a** are functions.

c Explain why all circles are not functions.

13 A rule (like $y = x^2$) gives a function if each possible input x gives exactly one output y. Use this condition to explain why the following statements are true.

a The rule $y = x^2$ gives a function. **b** The rule $x = -2$ does not give a function.

14 Two functions are equal if they map the elements of one set X to another set Y in the same way. Consider the functions given by $f(x) = x$ and the infinite set of ordered pairs $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n), \ldots\}$ where n is a positive integer.

a Describe each function in words.

b The first function, $f(x)$, maps from $\mathbb{R}$ to $\mathbb{R}$. Identify two sets that the second function maps between.

c Graph both functions on the same Cartesian plane. Show eight of the ordered pairs from the second function on your graph.

d Use your answers from parts **a** to **c** to determine whether the two functions are equal.

15 Recall that the domain of a function is the set of input values of that function. The *implied domain* of a function is the largest possible domain of the function.

a Determine the implied domain of the following functions.

i $f(x) = x^2$ **ii** $f(x) = 2x + 3$ **iii** $f(x) = 2^x$

ADVANCED

Sometimes it is necessary to exclude elements of a set to write the implied domain. The exclusion of the element a from the set X, is written as $X\backslash\{a\}$.

b Determine the implied domain of the following functions.

i $f(x) = \frac{1}{x}$ **ii** $f(x) = \frac{2}{x-2}$ **iii** $f(x) = \frac{1}{x+3} + 3$

It is possible to consider the domain of relations which are not functions.

c Determine the implied domain of the following circles. Write your answers in the form $a \le x \le b$.
Hint: Refer to your answers to question **12**.

i $x^2 + y^2 = 1$ **ii** $(x-1)^2 + y^2 = 4$ **iii** $x^2 + (y+2)^2 = 9$ **iv** $(x+1)^2 + (y-3)^2 = 1$

16 Recall that the range of a function is the set of output values of the function.

a Sketch each of the following functions.

i $f(x) = x + 1$ **ii** $f(x) = 3x - 1$ **iii** $f(x) = x^2$ **iv** $f(x) = x^2 + 1$

v $f(x) = 3^x$ **vi** $f(x) = 2^x - 1$ **vii** $f(x) = \frac{2}{x+2}$ **viii** $f(x) = \frac{1}{x-3} + 1$

b Determine the range of each of the functions in part **a**. Use appropriate set notation or write an inequality using y where required.

17 Function notation can be used to substitute any expression into a function. For example, if, $f(x) = x^2$, then $f(4x) = (4x)^2 = 16x^2$. For each of the functions below, find:

i $f(3x)$ **ii** $f(x+1)$ **iii** $f(2x-1)$.

a $f(x) = x + 2$ **b** $f(x) = 2x^2$ **c** $f(x) = 2^x$ **d** $f(x) = -x^2 + x$

18 Function notation can be used to describe transformations on a function.

- A dilation by a factor of a in the direction of the y-axis is written in the form $a \times f(x)$.
- A dilation by a factor of $\frac{1}{n}$ in the direction of the x-axis is written in the form $f(nx)$.
- A reflection in the y-axis is written as $f(-x)$.
- A reflection in the x-axis is written as $-f(x)$.
- A translation k units up is written in the form $f(x) + k$.
- A translation h units to the right is written as $f(x-h)$.

a For the functions below, describe the transformation performed on the function $f(x)$ in words.

i $f(x) + 2$ **ii** $f(x) - 3$ **iii** $f(x+5)$ **iv** $f(x-4)$

v $-f(x)$ **vi** $f(-x)$ **vii** $2f(x)$ **viii** $f(2x)$

b Apply each of the transformations in part **a** to the function $f(x) = x^2$ to find the rule of the transformed function in the form $y = a(x-h)^2 + k$.

19 Function notation can be used to describe multiple transformations on a function. Describe the following transformations of a function $f(x)$ in words. Remember that transformations should always be listed in the order dilations, then reflections, then translations.

a $2f(x-1)$ **b** $f(x+1) - 3$ **c** $-2f(x) - 1$ **d** $-4f(-x) + 2$

20 For the functions $f(x) = x + 1$, $g(x) = x^2 + x$ and $h(x) = 2^x$, find:

a $f(g(x))$ **b** $g(f(x))$ **c** $f(h(x))$ **d** $h(f(x))$ **e** $g(h(x))$ **f** $h(g(x))$.

21 A function is called injective (or one-to-one) if no two elements in the domain map to the same element in the range. If a horizontal line drawn through any part of a graph intersects or touches the graph only once, then the graph represents an injective function. This is known as the horizontal line test.

a Sketch the following functions.

i $f(x) = 2x + 1$ **ii** $f(x) = -x^2$ **iii** $f(x) = 2^{-x} + 1$ **iv** $f(x) = \frac{1}{x+2} - 1$

b Use the horizontal line test to determine which of the functions in part **a** are injective.

c Give a reason why no quadratic function $y = ax^2 + bx + c$ on its implied domain is injective.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Functions and relations

Topic quiz
6A

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EXTENSION

6B Polynomials

Learning intentions

By the end of this topic you will be able to …

- ✔ identify the features of a polynomial
- ✔ evaluate a polynomial using substitution
- ✔ expand and simplify a product of polynomials.

Inter-year links

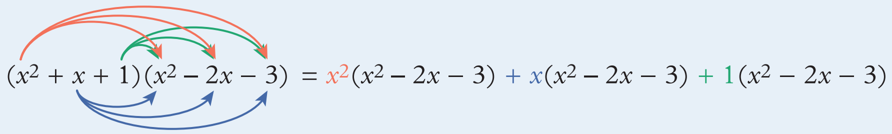
Year 7	6F Simplifying
Year 8	5F Expanding
Year 9	3D Expanding

The distributive law

Key content video

- The distributive law states that multiplication can be 'distributed' by multiplying one factor across another bracketed factor. This means that large numbers and algebraic expressions can be broken down into the sum of a group of smaller numbers or terms, which can then be multiplied separately.

  For example, the distributive law can be applied to binomials and trinomials.



$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

Polynomials

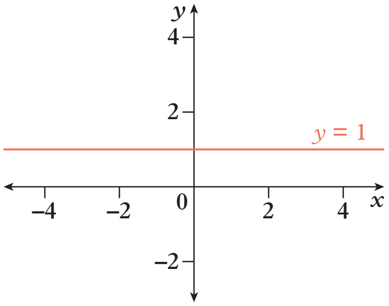
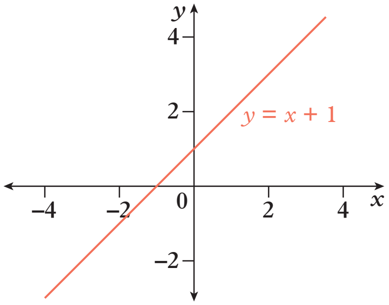
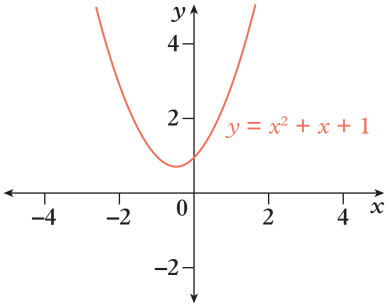
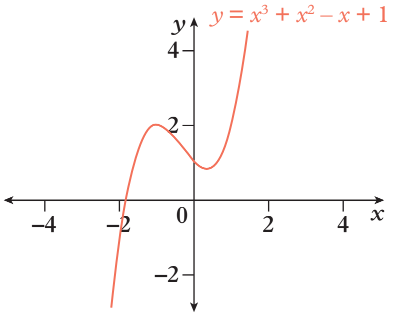
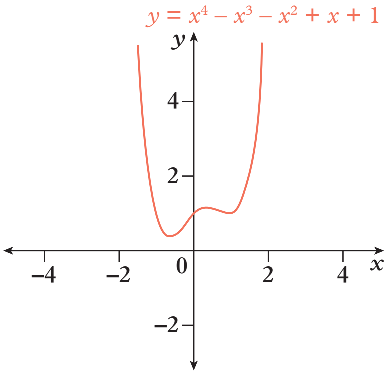
- A **polynomial** is an expression that contains at least one variable, such as x. Each term contains that variable raised to a non-negative integer value (0, 1, 2, 3, …).

  For example, $x^3 + 7x^2 + 3x + 2$ and $3x^2 + 2xy - y^2$ are examples of polynomials, because the index of each variable is a positive integer value.

  The expressions $3x^{\frac{1}{3}} - 9x$ and $\frac{2}{x} + x^2$ are not polynomials because the variables are raised to negative indices or indices that are fractions.
- A single variable polynomial function can be written as:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

  where n is the highest index, and $a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1, a_0$ are coefficients.
- $P(x)$ is read as 'P of x' meaning the polynomial P in terms of the variable x.
- The **degree of a polynomial (n)** is the largest index of the variable. Polynomials are named according to their degree. For example, $P(x) = x^2 - 2x + 1$ is a degree 2 polynomial called a quadratic. Some common polynomials are listed in the table on the next page.
- The **leading term** ($a_n x^n$) includes the variable with the degree of the polynomial and is usually written first. The coefficient of the leading term is called the **leading coefficient**.
- A polynomial is said to be **monic** if its leading coefficient is 1. All other polynomials are **non-monic**.
- The **constant term** (a_0) of a polynomial is the term with no variable.
- Polynomials are generally written with the variables in descending or ascending order of their indices.
- All single variable polynomials are functions. Function notation can be used to indicate the value of a polynomial $P(x)$ at $x = c$ by writing $P(c)$.

Types of polynomials			
Degree	**Name**	**Example**	**Graph**
0	constant	1 (or $1x^0$)	
1	linear	$x + 1$ (or $x^1 + 1$)	
2	quadratic	$x^2 + x + 1$	
3	cubic	$x^3 + x^2 - x + 1$	
4	quartic	$x^4 - x^3 - x^2 + x + 1$	

Example 6B.1 Identifying features of a polynomial

For the polynomial $2x^3 - x^2 + 7x + 4$, identify:

a the number of terms

b the degree of the polynomial

c the constant term

d the leading term

e the leading coefficient

f the coefficient of the x^2 term.

g whether the polyonomial is monic or non-monic

THINK	**WRITE**
a Terms are separated by + and – signs.	**a** There are four terms.
b Look for the highest power of x.	**b** The degree is 3.
c Look for a term without a variable (or a term for which the power of the variable is 0).	**c** The constant term is 4.
d Look for the term that has the highest power of x.	**d** The leading term is $2x^3$.
e Write the coefficient of the leading term.	**e** The leading coefficient is 2.
f Write the coefficient of the term containing x^2. The sign shown between it and the term before belongs to the x^2 term. ($-x^2$ means $-1x^2$.)	**f** The coefficient of x^2 is -1.
g The leading coefficient is not 1 so the polynomial is non-monic.	**g** The polynomial is non-monic.

Example 6B.2 Evaluating a polynomial

If $P(x) = 2x^3 - x^2 + 7x + 4$, evaluate:

a $P(3)$

b $P(0)$

c $P(-1)$.

THINK	**WRITE**
a Substitute $x = 3$ into the expression and evaluate.	**a** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(3) = 2(3)^3 - (3)^2 + 7(3) + 4$ $= 54 - 9 + 21 + 4$ $= 70$
b Substitute $x = 0$ into the expression and evaluate.	**b** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(0) = 2(0)^3 - (0)^2 + 7(0) + 4$ $= 0 - 0 + 0 + 4$ $= 4$
c Substitute $x = -1$ into the expression and evaluate.	**c** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(-1) = 2(-1)^3 - (-1)^2 + 7(-1) + 4$ $= -2 - 1 - 7 + 4$ $= -6$

Example 6B.3 Expanding the product of two polynomials

Expand and simplify each of these products.

a $2x^3(x^2 - 3x + 4)$

b $(x^2 - 2)(x^3 - x + 5)$

c $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$

THINK

a Apply the distributive law. Multiply each term inside the pair of brackets by the term outside the brackets.

b Apply the distributive law. Multiply each term in the second pair of brackets by each term in the first pair of brackets. Simplify like terms.

c Apply the distributive law. Multiply each term in the second pair of brackets by each term in the first pair of brackets. Simplify like terms.

WRITE

a

$$2x^3(x^2 - 3x + 4) = 2x^5 - 6x^4 + 8x^3$$

b

$$\begin{aligned}(x^2 - 2)(x^3 - x + 5) &= x^2(x^3 - x + 5) - 2(x^3 - x + 5)\\ &= x^5 - x^3 + 5x^2 - 2x^3 + 2x - 10\\ &= x^5 - 3x^3 + 5x^2 + 2x - 10\end{aligned}$$

c

$$\begin{aligned}(2x^4 - x^2 + 3)(x^3 + 4x - 1) &= 2x^4(x^3 + 4x - 1) - x^2(x^3 + 4x - 1) + 3(x^3 + 4x - 1)\\ &= 2x^7 + 8x^5 - 2x^4 - x^5 - 4x^3 + x^2 + 3x^3 + 12x - 3\\ &= 2x^7 + 7x^5 - 2x^4 - x^3 + x^2 + 12x - 3\end{aligned}$$

Example 6B.4 Expanding the product of three polynomials

Expand and simplify $(x + 2)(x - 3)(x + 4)$.

THINK	**WRITE**
1 Multiply two of the factors and simplify. It is easier to choose the last two.	$(x + 2)(x - 3)(x + 4) = (x + 2)(x^2 + 4x - 3x - 12)$
2 Multiply the linear factor by the quadratic factor.	$= (x + 2)(x^2 + x - 12)$ $= x(x^2 + x - 12) + 2(x^2 + x - 12)$
3 Simplify like terms.	$= x^3 + x^2 - 12x + 2x^2 + 2x - 24$ $= x^3 + 3x^2 - 10x - 24$

Helpful hints

- ✔ When multiplying through by a negative term, use brackets for clarity. For example, $-7p(3q + 4) = -7p \times 3q + (-7p \times 4)$
- ✔ Show all your working to avoid making errors in arithmetic and errors related to the signs.
- ✔ After expanding expressions containing brackets, always simplify your results by looking for like terms. Simplify, simplify, simplify!

ANS p714

Exercise 6B Polynomials

▲ 1–4, 6–8, 9(a, c, e, g), 10–12, 14

■ 1, 3–8, 9(b, c, f, h), 10, 13, 15, 17(a–d), 19(a, b)

⬟ 1, 5, 6(f, g, j, k), 8, 9(d, f, h), 10, 13, 16–18, 19(c), 20

UNDERSTANDING AND FLUENCY

6B.1 1 For each polynomial below, identify:

- **i** the number of terms
- **ii** the degree of the polynomial
- **iii** the constant term
- **iv** the leading term
- **v** the leading coefficient
- **vi** the coefficient of the x^2 term
- **vii** whether the polynomial is monic or non-monic.

a $2x^3 + 3x^2 + 4x + 5$ **b** $4x^5 + x^4 + 7x^3 - 2x^2 + 9x - 3$ **c** $-5x^4 - 2x^3 + 5x^2 + 1$

d $x^7 + x^6 - x^5 + x^4 + x^3 - x^2 + x$ **e** $9 - 3x - 6x^3 + 2x^2 - 7x^6$ **f** $3 - 11x^{10} + 5x^8$

2 Simplify each of these polynomials by collecting like terms.

a $4x^3 + 2x^2 - 4x - 1 + x^3 - 5x^2 + 3$ **b** $x^5 - 3x^2 + x^3 - 2x^4 - x^3 + 7x^2$

c $x^4 + 2 + 4x^2 + x^4 - 2x^2 - 5 + 6x^3$ **d** $(2x^2 - 5x + 1) + (3x^2 - 6x + 8)$

e $(5x^3 - 2x + 1) - (2x^3 - 7x + 4)$ **f** $(3x^2 + x + x^3) - (4x^4 - 3x^2 + 5x^3)$

6B.2 3 If $P(x) = x^3 - 2x^2 + 5x - 8$, evaluate:

a $P(2)$ **b** $P(0)$ **c** $P(3)$ **d** $P(-3)$.

4 If $P(x) = 3x^4 - 4x^3 + x^2 - 6x + 1$, evaluate:

a $P(3)$ **b** $P(1)$ **c** $P(-1)$ **d** $P(-2)$.

5 If $P(x) = 2x^3 + 3x^2 - 5x + 1$, evaluate:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{2}{3}\right)$ **c** $P(\sqrt{2})$ **d** $P\left(-\frac{1}{\sqrt{3}}\right)$.

6 If $P(x) = 2x^3 - x^2 + 3x - 6$ and $Q(x) = 3x^2 - 7x + 2$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x)$

e $-2Q(x)$ **f** $2P(x) + Q(x)$ **g** $P(x) - 4Q(x)$ **h** $3P(x) - 2Q(x)$

i $P(x) \times Q(x)$ **j** $-3Q(2)$ **k** $P(3) + Q(3)$ **l** $5P(0) - 4Q(0)$.

6B.3 7 Expand and simplify the following expressions.

a $x^3(x^2 + 2x + 7)$ **b** $4x^2(x^2 - 5x + 2)$

c $-6x^3(x^4 - 4x^2 + 1)$ **d** $(x + 3)(x^2 - 2x + 4)$

e $(x^2 - 1)(x^2 + 3x - 7)$ **f** $(x^4 - 3x + 2)(x^3 + 4x^2 - 1)$

6B.4 8 Expand and simplify the following expressions.

a $3x(x + 2)(x + 3)$ **b** $-7x^2(x + 5)(x - 2)$

c $(x + 3)(x + 1)(x + 6)$ **d** $(x - 3)(x - 4)(x + 1)$

e $(3x - 2)(x - 5)(x - 2)$ **f** $(4x + 1)(2x + 7)(x - 4)$

9 Expand and simplify each of these expressions.

a $(x^3 + 4)^2$ **b** $(x^2 + 3x)^2$ **c** $2x(x^2 - 2)^2$ **d** $(x^3 - 5x^2)^2$

e $(x + 2)^3$ **f** $(2x - 3)^3$ **g** $(x + 3)^4$ **h** $(1 - x)^4$

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PROBLEM SOLVING AND REASONING

10 State whether or not each of the following is a polynomial.

a $2x^3 - 3x^2 + 5x - 1$ **b** $2x^{-3} - 3x^{-2} + 5x^{-1} - 1$ **c** $2x^{\frac{3}{7}} - 3x^{\frac{2}{7}} + 5x^{\frac{1}{7}} - 1$

d $\frac{2}{x^3} - \frac{3}{x^2} + \frac{5}{x} - 1$ **e** $\frac{2}{7}x^3 - \frac{3}{7}x^2 + \frac{5}{7}x - \frac{1}{7}$ **f** $2x^3 - 3x^2 + 5x - 1 + \frac{8}{x+1}$

g $\sqrt{2}\,x^3 - \sqrt{3}\,x^2 + \sqrt{5}\,x - 1$ **h** $2\sqrt{x^3} - 3x + 5\sqrt{x} - 1$ **i** $2x^3 - 3x^2 + 5x + 2\sqrt{x} - 1$

11 What is the maximum number of terms in a polynomial of degree 5?

12 What is the minimum number of terms in a polynomial of degree 3?

13 a If $P(x)$ is a polynomial of degree n, what is:

i the maximum number of terms in $P(x)$ **ii** the minimum number of terms in $P(x)$

iii the degree of $2P(x)$ **iv** the degree of $[P(x)]^2$?

b If $Q(x)$ is a polynomial of degree m, where $n > m$, what is:

i the maximum possible degree of $P(x) + Q(x)$ **ii** the maximum possible degree of $P(x) - Q(x)$

iii the maximum possible degree of $P(x)Q(x)$?

14 If $P(2) = 2$, find the value of k in $P(x) = x^3 - 2x^2 + 3x + k$.

15 If $P(-1) = 7$, find the value of k in $P(x) = 3x^4 + 2x^3 - kx - 5$.

16 If $P(3) = 31$ and $P(-2) = -19$, find the value of a and b in $P(x) = x^3 + ax^2 + bx + 1$.

17 If $P(x) = x^3 - 3x^2 + 2x + 1$, write simplified expressions for:

a $P(a)$ **b** $P(-3a)$ **c** $P(a^2)$ **d** $P(-a^2)$ **e** $P(a + 2)$ **f** $P(2a - 1)$.

18 Show that $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$.

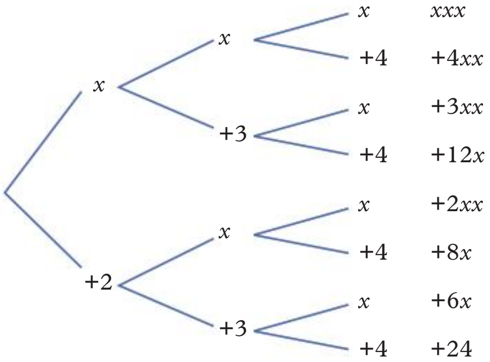
19 Expand each of these expressions.

a $(x^2 - 2)^3$ **b** $(x^3 + x - 1)^2$ **c** $(2x^3 + 1)^4$

CHALLENGE

20 Consider $(x + 2)(x + 3)(x + 4)$. We can expand the expression by adding the products of every combination of one term from each binomial factor.

To ensure all combinations have been multiplied, we can use a tree diagram, as shown below.



The tree diagram shows all the products that need to be added together. The path along each sequence of connected branches on the tree diagram is represented by a set of brackets in this calculation:

$$\begin{aligned}(x + 2)(x + 3)(x + 4) &= (x \times x \times x) + (x \times x \times 4) + (x \times 3 \times x) + (x \times 3 \times 4) + (2 \times x \times x) + (2 \times x \times 4) \\ &\quad + (2 \times 3 \times x) + (2 \times 3 \times 4) \\ &= x^3 + 4x^2 + 3x^2 + 12x + 2x^2 + 8x + 6x + 24 \\ &= x^3 + 9x^2 + 26x + 24\end{aligned}$$

a Use a tree diagram to help you expand $(x + 5)(x - 3)(x - 8)$.

b Use a tree diagram to help you expand $(x - 2)(x + 5)(x - 3)(x - 8)$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Polynomials

Interactive skillsheet
Multiplying polynomials

Worksheet
Working with polynomials

Topic quiz
6B

ADVANCED
EXTENSION

6C Dividing polynomials

Learning intentions

By the end of this topic you will be able to …

- ✔ divide a polynomial by a linear factor
- ✔ divide a polynomial that has coefficients of zero by a linear factor.

Inter-year links

Year 7	1F Dividing whole numbers
Year 8	5E Dividing algebraic terms
Year 9	3C Algebraic fractions with algebraic denominators

Polynomial division

Key content video

$$\begin{array}{rl} & \ x+2 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x+1 & \overline{)\,x^2+3x+4} \leftarrow \text{dividend} \\ & \underline{\ x^2+\ \ x\qquad\ } \\ & \qquad\ \ 2x+4 \\ & \qquad\ \ \underline{2x+2} \\ & \qquad\qquad\ \ 2 \leftarrow \text{remainder} \end{array}$$

- A polynomial can be divided by another polynomial using long division.
- When performing polynomial division, the **dividend** is the expression which is divided by another expression.
- The **divisor** is the expression which divides the dividend.
- The **quotient** is the resulting expression after the division has occurred.
- The **remainder** is the part of the dividend which is left over when the expression is not a multiple of the divisor. The degree of the remainder is always less than the degree of the divisor.
- There are four main steps to remember.
  1. Divide the leading term of the dividend by the leading term of the divisor.
  2. Multiply the resulting term by the divisor and simplify. Write the new expression below the dividend with terms with equal degrees aligned.
  3. Subtract the new expression from the dividend and write the result below.
  4. Repeat steps **1** to **3** with the result from step **3** as the new dividend until an expression with degree less than that of the divisor is reached. This expression is the remainder.
- The quotient and remainder are important elements to identify. In the example above, the quotient is $x + 2$ and the remainder is 2. We can write this as:

$$\underbrace{(x^2+3x+4)}_{\text{dividend}} \div \underbrace{(x+1)}_{\text{divisor}} = \underbrace{x+2}_{\text{quotient}}\ \ \underbrace{\text{remainder } 2}_{\text{remainder}}$$

or as:

$$\frac{\overbrace{x^2+3x+4}^{\text{dividend}}}{\underbrace{x+1}_{\text{divisor}}} = \overbrace{x+2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{2}}{\underbrace{x+1}_{\text{divisor}}}$$

- Alternatively, the polynomial can be written as:

  dividend = divisor × quotient + remainder

  $x^2 + 3x + 4 = (x + 1)(x + 2) + 2$
- In general, if $P(x)$ is divided by $D(x)$ to produce the quotient $Q(x)$ and the remainder $R(x)$, then $P(x) = D(x)Q(x) + R(x)$.
- If the remainder is a negative constant, then it follows that a value must be subtracted from the product of the divisor and the quotient to equal the dividend.

Example 6C.1 Dividing a polynomial by a linear expression

Use long division to find the quotient and remainder for $(x^3 + 2x^2 - 9x - 3) \div (x - 2)$.

THINK

1 Divide the dividend's leading term, x^3, by the divisor's leading term, x, and write the result (x^2) in the quotient line.

2 Expand $x^2(x - 2)$ to give $x^3 - 2x^2$ and write the result aligned under the dividend. Subtract like terms.

3 Divide $4x^2$ by x and write the result ($4x$) in the quotient line.

4 Expand $4x(x - 2)$ to give $4x^2 - 8x$ and write the result aligned under the dividend. Subtract like terms.

5 Divide $-x$ by x and write the result (-1) in the quotient line.

6 Expand $-1(x - 2)$ to give $-x + 2$ and write the result aligned under the dividend. Subtract like terms.

7 Identify the quotient and the remainder.

WRITE

$$\begin{array}{rr}
 & x^2+4x-1 \\
x-2 & \overline{)\,x^3+2x^2-9x-3} \\
x^2(x-2)\rightarrow & -(x^3-2x^2) \\
 & \overline{4x^2-9x-3} \\
4x(x-2)\rightarrow & -(4x^2-8x) \\
 & \overline{-x-3} \\
-1(x-2)\rightarrow & -(-x+2) \\
 & \overline{-5}
\end{array}$$

The quotient is $x^2 + 4x - 1$.
The remainder is -5.

Example 6C.2 Dividing a polynomial that has coefficients of zero by a linear expression

Use long division to find the quotient and remainder for $(3x^4 + 2x^2 - 5) \div (x + 1)$.

THINK

1 To keep like terms aligned, write the 'missing' terms in the dividend, with zero as the coefficient.

2 Divide $3x^4$ by x and write $3x^3$ in the quotient line.

3 Expand $3x^3(x + 1)$ to be $3x^4 + 3x^3$. Subtract like terms.

4 Divide $-3x^3$ by x and write $-3x^2$ in the quotient line.

5 Expand $-3x^2(x + 1)$ to be $-3x^3 - 3x^2$. Subtract like terms.

6 Divide $5x^2$ by x and write $5x$ in the quotient line.

7 Expand $5x(x + 1)$ to be $5x^2 + 5x$. Subtract like terms.

8 Divide $-5x$ by x and write -5 in the quotient line.

9 Expand $-5(x + 1)$ to be $-5x - 5$. Subtract like terms.

10 Identify the quotient and the remainder.

WRITE

$$\begin{array}{rr}
 & 3x^3-3x^2+5x-5 \\
x+1 & \overline{)\,3x^4+0x^3+2x^2+0x-5} \\
3x^3(x+1)\rightarrow & -(3x^4+3x^3) \\
 & \overline{-3x^3+2x^2+0x-5} \\
-3x^2(x+1)\rightarrow & -(-3x^3-3x^2) \\
 & \overline{5x^2+0x-5} \\
5x(x+1)\rightarrow & -(5x^2+5x) \\
 & \overline{-5x-5} \\
-5(x+1)\rightarrow & -(-5x-5) \\
 & \overline{0}
\end{array}$$

The quotient is $3x^3 - 3x^2 + 5x - 5$.
The remainder is 0.

Helpful hints

- ✔ Polynomial expressions need to be written in descending order according to the indices on the variables for long division to work. For example, the expression $3 - 3x + x^2$ needs to be rearranged to $x^2 - 3x + 3$ before it can be divided using long division.
- ✔ Before dividing a polynomial, remember to insert the missing terms with zero as the coefficient. It can be easy to make mistakes if you skip this step.
- ✔ The ultimate purpose of dividing polynomials is so that you can fully factorise cubics, quartics and other polynomials of higher degree. It is best to keep this skill at the front of your mind!

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Exercise 6C Dividing polynomials

△ 1–6, 7(a, c), 8(a, c), 12 □ 1(d–f), 2(a, d, e, g, i, j), 3, 6, 7, 8(b, d), 10, 13 ⬠ 1(e, f), 2(g–j), 3, 7, 8(b, d), 9, 11, 13, 14

UNDERSTANDING AND FLUENCY

1 Use long division to find the quotient and remainder for each of these division problems.

a $(x^2 + 10x + 26) \div (x + 7)$

b $(x^2 - 12x + 26) \div (x - 4)$

c $(5x^2 - 4x - 2) \div (x - 2)$

d $(9x^2 - 37x + 22) \div (x - 4)$

e $(14x^2 + 53x - 6) \div (2x + 7)$

f $(60x^2 - 52x + 79) \div (10x + 3)$

6C.1 **2** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 2x^2 - 5x + 7) \div (x - 3)$

b $(x^3 + 3x^2 + 7x - 1) \div (x + 1)$

c $(x^3 + 6x^2 - 4x - 15) \div (x - 2)$

d $(x^3 - 3x^2 - 20x + 25) \div (x + 4)$

e $(2x^3 + 3x^2 - 5x - 4) \div (x + 2)$

f $(3x^3 - 2x^2 - 7x + 2) \div (x - 1)$

g $(5x^3 - 12x^2 - 6x - 15) \div (x - 3)$

h $(4x^3 + 9x^2 - 8x - 1) \div (x + 1)$

i $(x^4 + 2x^3 - 4x^2 - 2x + 8) \div (x + 1)$

j $(x^4 - 5x^3 + 2x^2 - x - 3) \div (x - 2)$

6C.2 **3** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 + x + 21) \div (x + 3)$ **b** $(2x^3 - 3x^2 - 6) \div (x - 2)$

c $(4x^4 - 3x^2 - 5) \div (x - 1)$ **d** $(3x^4 - 41) \div (x + 2)$

4 For each of the parts **a–f** in question **1**:

i Write the answers for the division problems in the form: quotient + $\frac{\text{remainder}}{\text{divisor}}$.

ii Write the dividends in the form: divisor × quotient + remainder.

5 For each of the parts **a–f** in question **2**:

i Write the answers for the division problems in the form: quotient + $\frac{\text{remainder}}{\text{divisor}}$.

ii Write the dividends in the form: divisor × quotient + remainder.

6 Perform the following divisions and write your answers in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R, and D are polynomial expressions and the degree of R is less than the degree of D.

a $\frac{3x^3 + 6x^2 + 7x + 20}{x + 2}$ **b** $\frac{2x^3 - 11x^2 + 15x + 9}{x - 3}$

c $\frac{x^4 + x^3 - 30x^2 - 8x + 2}{x + 6}$ **d** $\frac{x^4 - 7x^3 + 12x - 34}{x - 7}$

7 Write the following polynomials in the form $(x - 3)Q(x) + R(x)$, where Q and R are polynomial expressions. Hint: divide each polynomial by $(x - 3)$ to find the quotient and remainder.

a $x^2 + 2x - 9$ **b** $x^3 - 7x^2 + 21x - 21$

c $x^3 - 9x^2 + 20x - 17$ **d** $x^4 + 5x^3 - 15x^2 - 23x - 9$

8 Perform the following divisions and write your answer in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R, and D are polynomial expressions and the degree of R is less than the degree of D.

a $(4x + 3) \div (x + 2)$ **b** $(9x - 4) \div (x - 3)$

c $\frac{x - 8}{x - 2}$ **d** $\frac{10x + 4}{2x - 5}$

9 **a** Consider the polynomial division $P(x) \div (x - a)$ where the degree of P is n, and n is a positive integer. Determine the degree of the quotient.

b Consider the polynomial division $P(x) \div D(x)$ where the degree of P is n and the degree of D is m, and m and n are positive integers where $n > m$. Determine the maximum degree of the quotient.

c Explain why when a polynomial $P(x)$ is divided by $x^2 - 3$, the remainder must be of the form $ax + b$ for some constants a and b.

10 Consider the division $\frac{x^3 + 9x^2 + 23x + 17}{x + 4}$.

a Rewrite the division with the numerator written in the form $(x + 4)Q(x) + R(x)$, where Q and R are polynomial expressions.

b Write your answer from part **a** in the form $\frac{(x + 4)Q(x)}{x + 4} + \frac{R(x)}{x + 4}$ and explain how $\frac{x^3 + 9x^2 + 23x + 17}{x + 4} = Q(x) + \frac{R(x)}{x + 4}$ for $x \neq -4$.

11 **a** If $P(x) = x^3 - 3x^2 - 10x + k$, for what value of k does $P(x) \div (x - 2)$ give a remainder of zero?

b From the divisor and the quotient, identify the linear factor and the quadratic factor of $P(x)$. Then show that you can write $P(x)$ as the product of three linear factors.

12 **a** Expand and simplify $(x + 2)x^2 + 5(x + 2)x - 36(x + 2)$.

b Factorise $(x + 2)x^2 + 5(x + 2)x - 36(x + 2)$.

13 Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 3x^2 + 2x - 4) \div (x^2 + 1)$

b $(x^4 + x^3 - 7x^2 + 3x + 5) \div (x^2 - 3)$

c $(2x^4 + 6x^2 - 1) \div (x^2 + 2)$

14 Suppose $f(x) = \frac{P(x)}{D(x)}$ where P and D are polynomial expressions.

a If $P(x) = (x - a)Q(x)$ and $D(x) = x - a + b$, where $b \neq 0$, write an expression for $f(a)$.

b If $P(x) = (x - a)Q(x) + b$ and $D(x) = x - a$, where $b \neq 0$, explain why $f(a)$ is undefined.

c If $P(x) = (x - a)Q(x)$ and $D(x) = x - a$, write an expression for $f(x)$, where $x \neq a$.

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Synthetic division

Topic quiz
6C

6D Remainder and factor theorems

Learning intentions

By the end of this topic you will be able to ...

- ✔ use the remainder theorem to find a linear factor of a polynomial
- ✔ use the factor theorem to find a linear factor of a polynomial
- ✔ factorise cubic polynomials.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	3F Factorising monic quadratic expressions

Key content video

The remainder theorem

- When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.
  For example, when $P(x)$ is divided by $(x - 2)$, the remainder is $P(2)$ and, when $P(x)$ is divided by $(x + 3)$, the remainder is $P(-3)$.

The factor theorem

- A **zero** of a polynomial function is a value a such that $P(a) = 0$.
- When a polynomial $P(x)$ is divided by $(x - a)$ and a is a zero of $P(x)$, then $(x - a)$ is a factor of $P(x)$.
  For example, for $P(x) = x^3 + x^2 - 10x + 8$, since $P(1) = 0$, $(x - 1)$ is a factor of $P(x)$.
- If a linear factor of a cubic polynomial is known, long division can be used to find the quadratic factor.
  For example, the quotient when dividing $P(x) = x^3 + x^2 - 10x + 8$ by $(x - 1)$ is $x^2 + 2x - 8$.
- Factorising the quotient allows us to write the polynomial as a product of its linear factors.
  For example,
  $$\begin{aligned}P(x) &= x^3 + x^2 - 10x + 8\\ &= (x - 1)(x^2 + 2x - 8)\\ &= (x - 1)(x - 2)(x + 4)\end{aligned}$$
- In general, if $P(x)$ is divided by a factor $(x - a)$ to produce the quotient $Q(x)$, then $P(x) = (x - a)Q(x)$.

Example 6D.1 Using the remainder theorem

Find the remainder when $x^3 + 4x^2 + x - 6$ is divided by:

a $x - 2$ **b** $x + 3$.

THINK	WRITE
First, name the polynomial.	$P(x) = x^3 + 4x^2 + x - 6$
a For a divisor of $(x - 2)$, calculate $P(2)$ and write the remainder.	**a** $P(2) = 8 + 16 + 2 - 6$ $= 20$ The remainder is 20.
b For a divisor of $(x + 3)$, calculate $P(-3)$ and write the remainder.	**b** $P(-3) = -27 + 36 - 3 - 6$ $= 0$ The remainder is 0.

Example 6D.2 Deciding whether a divisor is a factor of a polynomial

Decide whether each of these divisors is a factor of $x^3 + 2x^2 - x - 2$.

a $x - 3$ **b** $x + 1$

THINK	WRITE
First, name the polynomial.	$P(x) = x^3 + 2x^2 - x - 2$
a For $(x - 3)$ to be a factor, the remainder should be equal to zero. Check if $P(3) = 0$.	**a** $P(3) = 27 + 18 - 3 - 2$ $= 40$ $(x - 3)$ is not a factor of $P(x)$.
b For $(x + 1)$ to be a factor, the remainder should be equal to zero. Check if $P(-1) = 0$.	**b** $P(-1) = -1 + 2 + 1 - 2$ $= 0$ $(x + 1)$ is a factor of $P(x)$.

Example 6D.3 Using the factor theorem to find a linear factor of a polynomial

Use the factor theorem to find a linear factor for each of the following polynomials.

a $P(x) = x^3 + 5x^2 - 2x - 24$ **b** $P(x) = x^3 - 3x^2 - 9x + 27$

THINK	WRITE
a 1 Write the polynomial.	**a** $P(x) = x^3 + 5x^2 - 2x - 24$
2 Determine a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, etc. That is, substitute x values that are positive or negative factors of the constant term in $P(x)$.	$P(1) = 1 + 5 - 2 - 24$ $= -20$ $\neq 0$ $P(-1) = -1 + 5 + 2 - 24$ $= -18$ $\neq 0$ $P(2) = 8 + 20 - 4 - 24$ $= 0$
3 Use the factor theorem to identify a linear factor of $P(x)$.	So $(x - 2)$ is a factor of $P(x)$.
b 1 Write the polynomial.	**b** $P(x) = x^3 - 3x^2 - 9x + 27$
2 Substitute x values that are positive or negative factors of the constant term in $P(x)$. Try $P(1)$, $P(-1)$, $P(3)$, etc.	$P(1) =1 - 3 - 9 + 27$ $= 16$ $\neq 0$ $P(-1) = -1 - 3 + 9 + 27$ $= 32$ $\neq 0$ $P(3) = 27 -27 - 27 + 27$ $= 0$
3 Use the factor theorem to identify a linear factor of $P(x)$.	So $(x - 3)$ is a factor of $P(x)$.

Example 6D.4 Factorising a cubic polynomial

Factorise $x^3 - 7x^2 + 7x + 15$.

THINK	**WRITE**
1 Name the polynomial.	$P(x) = x^3 - 7x^2 + 7x + 15$
2 Determine a value of x for which $P(x) = 0$. Try $P(1)$ and $P(-1)$.	$P(1) = 1 - 7 + 7 + 15$ $= 16$ $\neq 0$ $P(-1) = -1 - 7 - 7 + 15$ $= 0$
3 Identify a linear factor of $P(x)$.	So $(x + 1)$ is a factor of $P(x)$.
4 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$. The remainder will be zero if the division has been performed correctly.	$\begin{array}{r} x^2 - 8x + 15 \\ x+1\overline{)x^3 - 7x^2 + 7x + 15} \\ \underline{x^3 + x^2 } \\ -8x^2 + 7x + 15 \\ \underline{-8x^2 - 8x } \\ 15x + 15 \\ \underline{15x + 15} \\ 0 \end{array}$
5 Write $P(x)$ as the product of the divisor and quotient.	$P(x) = (x + 1)(x^2 - 8x + 15)$
6 Factorise the quadratic factor.	$= (x + 1)(x - 3)(x - 5)$

Helpful hints

✔ Remember that for a polynomial function $P(x)$, if $P(a) = 0$, then $(x - a)$ is a linear factor of $P(x)$, not $(x + a)$.

✔ The leading coefficient of the expanded polynomial expression is not always the first term of the factorised polynomial expression. But, if we write each linear factor in the form $(x - a)$, the leading coefficient will be the constant factor at the front of the factorised expression.

For example, $6x^2 + 7x + 2$ can be factorised to become $(2x + 1)(3x + 2)$, and can be further factorised to become $6\left(x - \frac{1}{2}\right)\left(x + \frac{2}{3}\right)$.

✔ When factorising, try the factors of the constant value of the polynomial. If $P(a)$ is zero, then a is a factor of the constant term in polynomial $P(x)$.

For example, when determining a linear factor of $x^2 + 2x + 6$, you should try substituting the positive and negative factors of the constant term, 6, which are: 1, –1, 2, –2, 3, –3, 6 and –6.

✔ When a linear factor has been established and you are performing a polynomial division, you will know you have performed the division correctly if the remainder is zero.

ADVANCED

EXTENSION

ANS p716

Exercise 6D Remainder and factor theorems

△ 1–7, 8(a, c, e), 9–11, 14, 15

□ 4–6, 7–8(1st column), 10, 12, 13, 16, 18, 19, 21(a)

⬠ 4, 6(e–h), 7–8(2nd column), 12, 16–18, 20–22

UNDERSTANDING AND FLUENCY

1 Consider the polynomial $P(x) = x^3 + x^2 - 10x + 8$.

a Use long division to find the remainder when $P(x)$ is divided by $(x - 3)$.

b Evaluate $P(3)$. What do you notice?

c Use long division to find the remainder when $P(x)$ is divided by $(x + 2)$.

d Evaluate $P(-2)$. What do you notice?

e Without using long division, use the pattern you noticed in parts **a–d** to show how to work out the remainder when $P(x)$ is divided by $(x - 4)$.

f Use long division to verify your answer to part **e**.

6D.1 **2** Find the remainder when $x^3 - 5x^2 - 8x + 12$ is divided by:

a $x - 1$ **b** $x + 1$ **c** $x - 2$ **d** $x + 2$.

3 Find the remainder when $2x^3 + 3x^2 - 8x + 3$ is divided by:

a $x + 1$ **b** $x - 2$ **c** $x + 2$ **d** $x + 3$.

4 Find the remainder when $x^4 - 7x^3 + 5x^2 + 31x - 30$ is divided by:

a $x - 2$ **b** $x + 1$ **c** $x - 3$ **d** $x + 4$.

6D.2 **5** Decide whether each of these divisors is a factor of $x^3 + 3x^2 - 6x - 8$.

a $x - 1$ **b** $x + 1$ **c** $x - 3$ **d** $x + 3$

e $x - 4$ **f** $x + 4$ **g** $x + 2$ **h** $x - 2$

6 Decide whether each of these divisors is a factor of $x^3 - 4x^2 - 7x + 10$.

a $x - 2$ **b** $x - 3$ **c** $x - 1$ **d** $x + 3$

e $x + 1$ **f** $x + 2$ **g** $x - 5$ **h** $x - 4$

6D.3 **7** Use the factor theorem to find a linear factor for each of the following polynomials.

a $P(x) = x^3 + 8x^2 + 9x - 18$ **b** $P(x) = x^3 - x^2 - 14x + 24$

c $P(x) = x^3 - 4x^2 - 9x + 36$ **d** $P(x) = x^3 - 19x - 30$

e $P(x) = 3x^3 - 10x^2 + x + 6$ **f** $P(x) = 2x^3 - 5x^2 - 14x + 8$

g $P(x) = 8x^3 - 26x^2 + 17x + 6$ **h** $P(x) = 4x^3 + 4x^2 - 21x + 9$

6D.4 **8** Factorise each of these polynomials.

a $x^3 + 6x^2 + 3x - 10$ **b** $x^3 + 3x^2 - 4x - 12$

c $x^3 + 4x^2 - 19x + 14$ **d** $x^3 + 5x^2 - 4x - 20$

e $x^3 - 5x^2 - 8x + 48$ **f** $x^3 - 2x^2 - 9x + 18$

g $2x^3 + x^2 - 8x - 4$ **h** $4x^3 - 9x^2 - 19x + 30$

i $6x^3 - 23x^2 - 6x + 8$ **j** $12x^3 - 17x^2 + 2x + 3$

9 Factorise each of the polynomials in question **7**.

10 a If $P(x) = x^3 + 9x^2 + 23x + 15$, for what three values of k does $P(x) \div (x + k)$ result in a remainder of zero?

b Write $P(x)$ as the product of three linear factors.

11 What is the maximum number of linear factors $P(x)$ can have if $P(x)$ is:

a a quadratic polynomial **b** a cubic polynomial

c a quartic polynomial **d** a polynomial of degree 7

e a polynomial of degree 12 **f** a polynomial of degree n?

ADVANCED
EXTENSION

PROBLEM SOLVING AND REASONING

12 a How many linear factors does the polynomial $P(x)$ have if $P(x) = (x + 2)(x - 3)(x + 4)$?

b Without expanding completely, what is the constant term of $P(x)$? Explain.

c To find linear factors using the factor theorem, we look for values of x for which $P(x) = 0$. What are these x values and how do they relate to the constant term?

13 Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

a Use the factor theorem to find a linear factor of $P(x)$.

b Use the factor theorem to find another linear factor of $P(x)$.

c Use the factor theorem to find a third linear factor of $P(x)$.

d Write $P(x)$ as a product of three factors.

e Check your answer to part **d** by expanding and simplifying the product.

14 a Use your answers to question **5** to identify three factors of the polynomial $P(x) = x^3 + 3x^2 - 6x - 8$.

b Write $P(x)$ as a product of three factors.

c Check your answer to part **b** by expanding and simplifying the product.

15 a Use your answers to question **6** to identify three factors of the polynomial $P(x) = x^3 - 4x^2 - 7x + 10$.

b Write $P(x)$ as a product of three factors.

16 Using only the factor theorem, find three factors of $x^3 + 2x^2 - 11x - 12$. Then write the polynomial as a product of three factors.

17 A polynomial, P, can be written in the form $P(x) = (x - a)Q(x) + r$, where Q is a polynomial.

a Solve $x - a = 0$ for x.

b Substitute the solution to part **a** into $P(x)$.

c Explain why your answer from part **b** is equal to the remainder of the polynomial division $P(x) \div (x - a)$.

d Another polynomial, F, can be written in the form $F(x) = (bx - a)G(x) + h$, where G is a polynomial. Solve $bx - a = 0$ for x.

e Substitute your solution for part **d** into $F(x)$.

f Explain why your answer from part **e** is equal to the remainder of the polynomial division $F(x) \div (bx - a)$.

18 a When $x^3 + 2x^2 + ax + 5$ is divided by $x - 3$ the remainder is 5. Find the value of a.

b When $x^3 + bx^2 - 6x - 7$ is divided by $x + 2$ the remainder is 17. Find the value of b.

c If $x + 4$ is a factor of $x^3 + 13x^2 + cx - 20$, find the value of c.

d When $x^3 + mx^2 + nx + 5$ is divided by $x + 1$ the remainder is 13, and when divided by $x - 2$ the remainder is 31. Find the values of m and n.

e Find the values of a and b if the polynomial $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ is divisible by $x + 1$ and $x - 2$.

CHALLENGE

19 Explain why the strategy used in question **16** is not suitable for factorising each of these polynomials.

a $6x^3 + 13x^2 + 4x - 3$

b $x^3 + 5x^2 + 10x + 8$

20 Find a polynomial of degree 3 with a leading coefficient of 1 that has a remainder of 5 when divided by $x - 2$.

21 Factorise each of these quartic polynomials.

a $x^4 - x^3 - 7x^2 + x + 6$

b $x^4 + 8x^3 + 17x^2 - 2x - 24$

c $2x^4 + 13x^3 + 21x^2 + 2x - 8$

22 a Show that $(x^2 + 4)$ is a factor of $x^4 + 13x^2 + 36$.

b Factorise $x^4 + 13x^2 + 36$.

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Remainder and factor theorems

Worksheet
Dividing and factorising polynomials

Investigation
Rational roots of polynomials

Topic quiz
6D

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Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

ADV 6A EXT **1** Determine which of the following relations are functions.

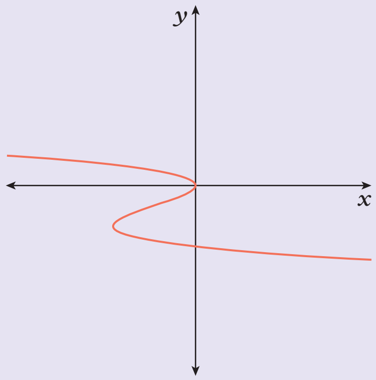
a $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$

b $\{(-5, -5), (-3, 2), (0, 2), (1, 3), (1, 4), (2, 5)\}$

c $y = x^2 + 2x$

d $x^2 + (y - 3)^2 = 9$

e



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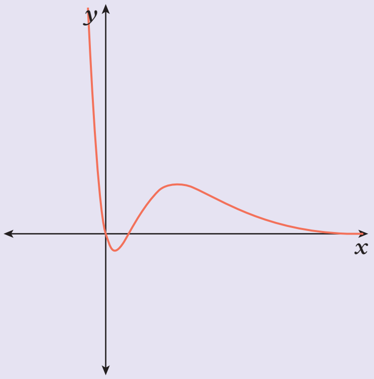


ADV 6B EXT **2** State whether or not each of the following is a polynomial. If it is not a polynomial, state the term(s) in the expression that cannot be included in a polynomial. If it is a polynomial, state:

- **i** its degree
- **ii** its leading term
- **iii** its leading coefficient.

a $5x^2 - 3x^5 + 4x^7$

b $9x^{100} - 9x^{-100}$

c $\frac{5y^{81}}{9} - \sqrt{3}y^{42} + \frac{2}{3}y^{27} - 1.2$

d $81y^5 - 42\sqrt{y^3} + 27y^{\frac{2}{3}} - 1.2$

ADV 6B EXT **3** Suppose $P(x) = 5x^3 - 3x^2 + 2x - 1$ and $Q(x) = 2x^2 - 8x + 7$. Write an expression for each of the following.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $6P(x)$ **d** $P(x) \times Q(x)$

ADV 6C EXT **4** Determine the quotient and remainder for each of the following polynomial divisions.

a $(x^2 + 2x - 29) \div (x - 4)$

b $(x^3 - 3x^2 + 2x + 33) \div (x + 2)$

c $\frac{2x^2 - 8}{x + 3}$

d $\frac{6x^3 + 19x^2 + 4x - 8}{3x + 2}$

ADV 6C EXT **5** Write the following results in the form $Q(x) + \frac{R(x)}{D(x)}$, where $D(x)$, $Q(x)$ and $R(x)$ are polynomial expressions and the degree of R is less than the degree of D.

a $(x^2 + 16x - 9) \div (x + 4)$; quotient: $x + 12$, remainder: -57

b $(5x^3 - 9x^2 + 2x + 1) \div (x - 6)$; quotient: $5x^2 + 21x + 128$, remainder: 769

c $\frac{5x^3 - 6x^2 + 7}{x^2 + 4}$; quotient: $5x - 6$, remainder: $-20x + 31$

ADV 6D EXT **6** Determine the remainder when $P(x) = 3x^3 + 4x^2 - 13x + 6$ is divided by:

a $x + 1$ **b** $x - 1$ **c** $x - 3$ **d** $3x - 2$.

ADV 6D EXT **7** For each of the following, determine whether the linear expression is a factor of the non-linear polynomial.

a $x - 2$ and $x^3 - 5x^2 + 7x + 4$

b $x + 1$ and $5x^3 + 13x^2 + 5x - 3$

c $x^3 - 2$ and $x^2 - 11x + 13$

d $4x - 3$ and $12x^3 + 11x^2 - 23x + 6$

ADV 6D EXT **8** Factorise each of the following polynomials.

a $x^3 - 4x^2 - 25x + 28$

b $x^3 - 7x - 6$

ADV 6D EXT **9** Factorise each of the following polynomials.

a $5x^3 - 15x^2 - 170x + 600$

b $x^4 - x^3 - 43x^2 - 23x + 210$

ADVANCED
EXTENSION

6E Solving polynomial equations

Learning intentions

By the end of this topic you will be able to …

✔ solve polynomial equations using the Null Factor Law.

Inter-year links

Year 7	6C Terms, expressions and equations
Year 8	6A Equations
Year 9	5A Solving quadratic equations

Polynomials and the Null Factor Law

Key content video

- The Null Factor Law states that for real numbers a and b, if $a \times b = 0$, then $a = 0$ or $b = 0$, or both a and b are 0.
- The polynomial equation $P(x) = 0$ can be solved by applying the Null Factor Law when $P(x)$ is in factor form. The solutions of a polynomial equation are also called **roots**.
- If $P(x) = (x - a)(x - b)(x - c)$, then the solutions to $P(x) = 0$ are $x = a$, $x = b$ and $x = c$.

$$\rlap{$\overbrace{}^{\text{function}}$}P(x) = \underbrace{\underbrace{(x-a)(x-b)(x-c)}_{\text{expression}} = 0}_{\text{equation}}$$

$$\begin{aligned} x - a &= 0 \;\Rightarrow\; x = a \\ x - b &= 0 \;\Rightarrow\; x = b \\ x - c &= 0 \;\Rightarrow\; x = c \end{aligned}$$

- A polynomial, $P(x)$, of degree n can have a maximum of n linear factors. Therefore, the number of solutions to the equation $P(x) = 0$ is at most n.
- If a polynomial, $P(x)$, of degree n has repeated factors, then the number of unique solutions to $P(x) = 0$ is less than n.

  For example, the factor $(x + 1)$ is repeated twice in $P(x) = x(x + 1)^2$, so even though the degree of the polynomial is 3, the number of solutions to $P(x) = 0$ is 2.

Factorising polynomials

- To factorise $P(x)$ so that it is a product of linear factors:
  1. apply the factor theorem to find a linear factor
  2. divide $P(x)$ by the linear factor to obtain the quotient
  3. factorise the quotient.
- If $P(x)$ is a cubic polynomial, the quotient is a quadratic factor that may be further factorised into two linear factors (making a total of three linear factors).

  For example, $\begin{aligned}[t] P(x) &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4). \end{aligned}$

Example 6E.1 Solving polynomial equations in factor form

Solve each of these equations.

a $(x-1)(x+3)(x-2)=0$ **b** $(2x+1)(x-5)(x-4)^2=0$

THINK	WRITE
a **1** The left-hand side is in factor form and the right-hand side is equal to 0, so we apply the Null Factor Law.	**a** $(x-1)(x+3)(x-2)=0$
2 Solve each linear equation.	$x-1=0$ or $x+3=0$ or $x-2=0$ $x=1$ or $x=-3$ or $x=2$
3 Write the solutions in ascending order.	$x=-3, 1$ or 2
b **1** Apply the Null Factor Law.	**b** $(2x+1)(x-5)(x-4)^2=0$
2 Solve each linear equation.	$2x+1=0$ or $x-5=0$ or $x-4=0$ $x=-\frac{1}{2}$ or $x=5$ or $x=4$
3 Write the solutions in ascending order.	$x=-\frac{1}{2}$, 4 or 5

Example 6E.2 Solving a cubic polynomial equation

Solve $x^3-7x^2+7x+15=0$.

THINK	WRITE
1 Name the polynomial.	$P(x)=x^3-7x^2+7x+15=0$
2 Use the factor theorem to identify a linear factor of the polynomial.	$P(1)=1-7+7+15$ $=16$ $\neq 0$ $P(-1)=-1-7-7+15$ $=0$ So $(x+1)$ is a factor of $P(x)$.
3 Use long division to find the quotient when $P(x)$ is divided by $(x+1)$.	$\begin{array}{r} x^2-8x+15 \\ x+1\overline{\big)\,x^3-7x^2+7x+15} \\ \underline{x^3+x^2} \\ -8x^2+7x+15 \\ \underline{-8x^2-8x} \\ 15x+15 \\ \underline{15x+15} \\ 0 \end{array}$
4 Write $P(x)$ as the product of the divisor and quotient.	$P(x)=(x+1)(x^2-8x+15)$
5 Factorise the quadratic factor.	$=(x+1)(x-3)(x-5)$
6 Solve $P(x)=0$ using the Null Factor Law.	For $P(x)=0$: $(x+1)(x-3)(x-5)=0$ $x=-1, 3$ or 5

Helpful hints

- ✔ Ensure that the equation shows that something is equal to zero before applying the Null Factor Law. Remember that 'null' means zero. For example, if $(x + 2)(x + 3)(x + 4) = 1$, you do not solve the equations $x + 2 = 1$, $x + 3 = 1$ or $x + 4 = 1$. You need to expand the polynomial, make the equation equal to zero, factorise again, and then apply the Null Factor law.
- ✔ Remember, sometimes a quadratic equation has irrational solutions, or no real solutions at all. If you factorise a polynomial, $P(x)$, into the product of linear and quadratic factors, then you might need to use the quadratic formula $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to determine any irrational solutions, if they exist.

  For example, $P(x) = x^3 - 2x^2 - 6x + 4$
  $$P(x) = (x + 2)(x^2 - 4x + 2)$$
  $$P(x) = (x + 2)(x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$$
- ✔ The discriminant ($\Delta = b^2 - 4ac$) can be used to check whether a quadratic factor can be factorised further. Remember that when $\Delta > 0$ there are two real solutions, when $\Delta = 0$ there is one real solution, and when $\Delta < 0$ there are no real solutions and the expression cannot be further factorised.
- ✔ The zeroes of a polynomial function $P(x)$ are the roots of the equation $P(x) = 0$.

ANS
p718

Exercise 6E Solving polynomial equations

△ 1, 2, 3–4(1st, 2nd columns), 5–8, 9(a, b), 10–12, 14

□ 3–4(2nd, 3rd columns), 5–8, 9(c, d), 12, 13, 15, 17

⬠ 4(2nd, 3rd columns), 5–7, 9(e, f), 13, 15, 16, 18, 19

UNDERSTANDING AND FLUENCY

6E.1 **1** Solve each of these equations.

a $(x + 2)(x + 5)(x - 4) = 0$
b $(x + 1)(x - 3)(x + 4) = 0$
c $(x - 6)(x - 2)(x + 3) = 0$
d $x(x + 2)(x - 9) = 0$
e $(2x - 1)(x + 1)(x - 1) = 0$
f $(3x - 2)(2x + 7)(x + 5) = 0$

2 Solve each of these equations.

a $(x + 3)(x - 4)(x + 7)(x - 1) = 0$
b $(x - 2)(x - 5)(x - 3)(x + 6) = 0$
c $(x + 4)(x + 1)(x + 2)^2 = 0$
d $x^2(x + 5)(x - 6) = 0$
e $5x(3x + 1)(x + 4)(4 - x) = 0$
f $(4x - 3)(2x + 5)(x^2 + 1) = 0$

3 Fully factorise the left-hand side of each of equations and then solve each equation.

a $(x + 1)(x^2 + 8x + 12) = 0$
b $(x - 3)(x^2 - x - 20) = 0$
c $(x - 2)(x^2 - 6x + 8) = 0$
d $(x + 4)(x^2 + 6x - 7) = 0$
e $(x + 3)(x^2 - 16) = 0$
f $x(x - 1)(x^2 - 1) = 0$
g $(x - 5)(x^2 - 10x + 25) = 0$
h $(x + 2)(x^2 + 6x + 9) = 0$
i $(x - 4)(x^2 - x - 6) = 0$
j $(x + 1)(x^2 + 2x + 1) = 0$
k $3(x + 3)(5x^2 + 30x) = 0$
l $(x - 2)(2x^2 + 4x - 6) = 0$

6E.2 **4** Solve each of these equations.

a $x^3 + 6x^2 + 5x - 12 = 0$
b $x^3 - 5x^2 - 4x + 20 = 0$
c $x^3 + x^2 - 36x - 36 = 0$
d $x^3 + 10x^2 + 21x = 0$
e $x^3 - 7x - 6 = 0$
f $x^3 + 2x^2 + 5x + 10 = 0$
g $x^3 + 3x^2 - 9x - 27 = 0$
h $x^3 + 9x^2 + 24x + 16 = 0$
i $x^3 - 3x^2 - 3x - 4 = 0$

5 Solve each of these equations by first taking out a common factor.

a $2x^3 - 2x^2 - 20x - 16 = 0$
b $3x^3 - 15x^2 - 3x + 15 = 0$
c $5x^3 + 10x^2 - 20x - 40 = 0$
d $4x^3 + 24x^2 + 44x + 24 = 0$

6 Solve each of these equations by first taking out a negative common factor.

a $-x^3 - 2x^2 + 9x + 18 = 0$
b $-x^3 + 4x^2 + 17x - 60 = 0$
c $-2x^3 + 8x^2 + 2x - 8 = 0$
d $-3x^3 - 3x^2 + 24x + 36 = 0$

ADVANCED
EXTENSION

UNDERSTANDING AND FLUENCY

7 Solve each of these equations using the quadratic formula after a linear factor has been taken out. Write the solutions as exact values.

a $x^3 + x^2 - 3x + 1 = 0$ **b** $x^3 + 6x^2 + 9x + 2 = 0$

c $x^3 + x^2 - 10x - 12 = 0$ **d** $x^3 + 4x^2 - 27x - 20 = 0$

8 Work through the following steps to solve $P(x) = 0$, where $P(x) = x^4 + x^3 - 7x^2 - x + 6$.

a Identify one of the linear factors of $P(x)$.

b Use long division to find the quotient when $P(x)$ is divided by the linear factor you identified in part **a**. Write $P(x)$ as the product of the divisor and the quotient, $Q(x)$.

c Identify a linear factor of $Q(x)$ and use long division to find its quadratic factor.

d Factorise the quadratic factor you found in part **c**.

e Write $P(x)$ as a product of four linear factors and then solve $P(x) = 0$ using the Null Factor Law.

9 Solve each of these equations.

a $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ **b** $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ **c** $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$

d $x^4 - x^3 - 19x^2 - 11x + 30 = 0$ **e** $x^4 - 5x^3 + 20x - 16 = 0$ **f** $x^4 - 13x^2 + 36 = 0$

PROBLEM SOLVING AND REASONING

10 What is the maximum number of real solutions for $P(x) = 0$ if $P(x)$ is:

a a linear polynomial **b** a quadratic polynomial **c** a cubic polynomial

d a quartic polynomial **e** a polynomial of degree 6 **f** a polynomial of degree n?

11 **a** Using only the factor theorem, find three linear factors of $P(x)$ when $P(x) = x^3 - 2x^2 - 5x + 6$.

b Solve $P(x) = 0$.

12 Explain why $x^3 + x^2 - 2x - 8 = 0$ has only one real solution.

13 Explain why $x^3 - 5x^2 + 3x + 9 = 0$ has only two real solutions.

14 The volume of a lamington is 192 cm^3. Its length is twice its height, and its width is 2 cm more than its height.

a Write a polynomial to represent the volume of the slice.

b Solve a polynomial equation to find the dimensions of the slice.

15 The volume of a Toblerone chocolate box is 450 cm^3. The height of the box's triangular face is 1 cm less than the length of the base of that face. The perpendicular length of the box is five times the length of the base of the triangular face. Find the dimensions of the box.

16 A polynomial equation has the solutions $x = 1$, $x = -1$, $x = 2$ and $x = -3$. Write all possible equations, equal to zero and in factorised form, if the polynomial:

a is of degree 4 and its leading coefficient is 1

b is of degree 5 and its leading coefficient is 1

c is of degree 5 and its leading coefficient is 2

d is of degree 6 and its leading coefficient is 5.

e If the polynomial is of degree n and its leading coefficient is a, where n is a non-zero, positive integer and a is a real number, then write a generalised equation for the factorised form.
(Hint: let p, q and r be non-zero positive integers for three of the four possible indices.)

CHALLENGE

17 $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ has the quadratic factor $(x^2 - x - 2)$. Factorise $P(x)$ and solve $P(x) = 0$.

18 Solve each of these equations.

a $x^5 - x^4 - 17x^3 - 19x^2 + 16x + 20 = 0$ **b** $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48 = 0$

19 Solve the equation $(x + 2)(x - 3)(x + 4) = (x + 2)(x - 10)$.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Solving polynomial equations

Investigation
Solving simultaneous equations numerically using cobweb diagrams

Topic quiz
6E

ADVANCED
EXTENSION

6F Sketching graphs of polynomials using intercepts

Learning intentions

By the end of this topic you will be able to ...

- ✔ sketch the graph of a cubic relationship
- ✔ sketch the graph of a quartic relationship
- ✔ sketch the graph of cubic and quartic relationships with repeated factors.

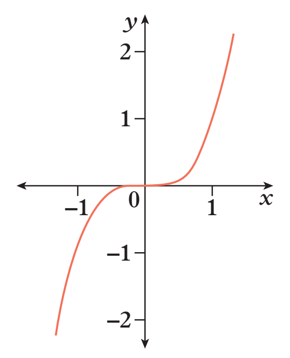
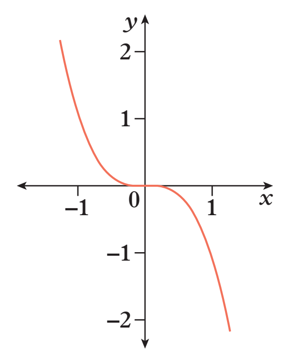
Inter-year links

Year 9 5C Sketching parabolas using intercepts

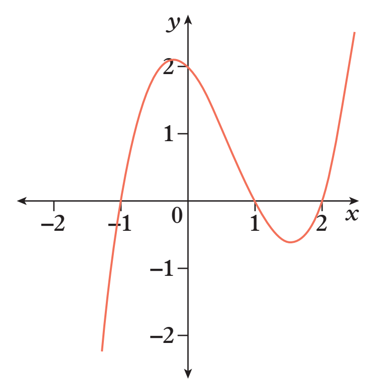
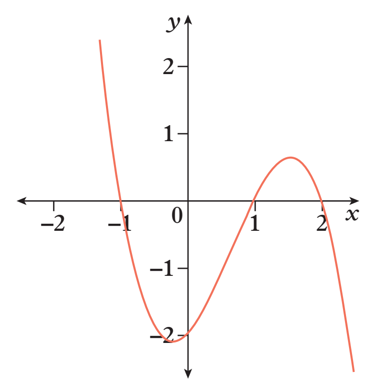
Sketching graphs of cubic relationships

Key content video

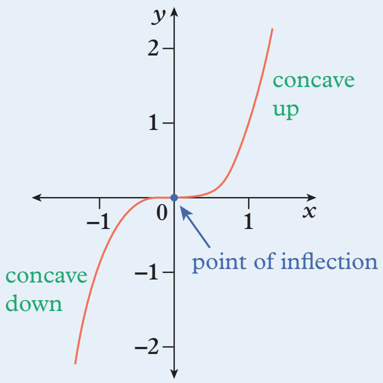
- When the leading coefficient of a cubic function is positive, the curve starts from the bottom left of the Cartesian plane.
- When the leading coefficient of a cubic function is negative, the graph is reflected in the x-axis, and the curve starts from the top left of the Cartesian plane.

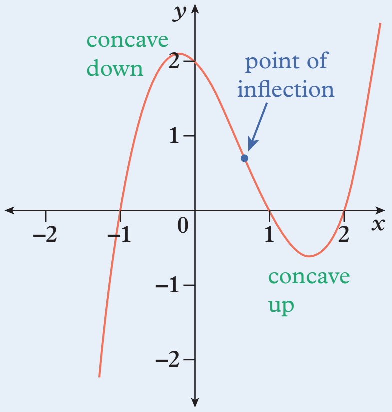
$y = x^3$	$y = -x^3$
	

- The graph of the function $P(x) = (x - a)(x - b)(x - c)$ has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$; and has a y-intercept at $(0, -abc)$.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)$ and has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$; and a y-intercept at $(0, abc)$.

$y = (x - 2)(x - 1)(x + 1)$	$y = -(x - 2)(x - 1)(x + 1)$
	

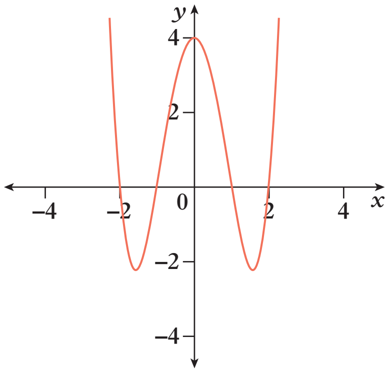
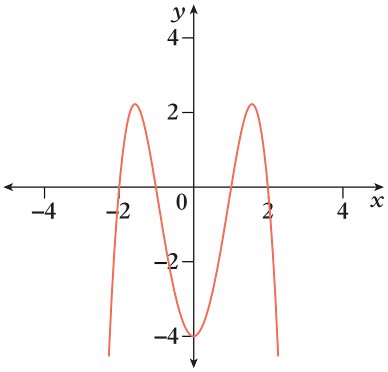
- A **point of inflection** is the point where a curve changes between being 'concave up' and 'concave down'. For example, the point of inflection is shown on each of the curves below.





Sketching graphs of quartic relationships

- The graph of the function $P(x) = (x - a)(x - b)(x - c)(x - d)$ has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$; and has a y-intercept at $(0, abcd)$. The curve starts from the top left of the Cartesian plane.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)(x - d)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)(x - d)$. The curve starts from the bottom left of the Cartesian plane.

$y = (x - 2)(x - 1)(x + 1)(x + 2)$	$y = -(x - 2)(x - 1)(x + 1)(x + 2)$
	

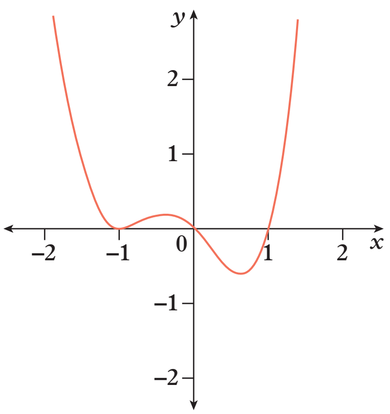
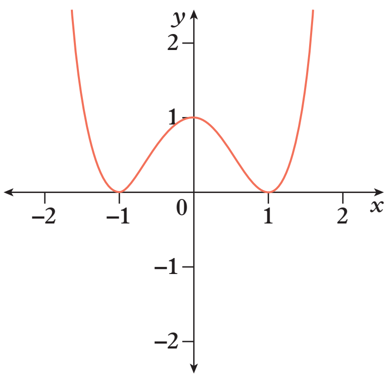
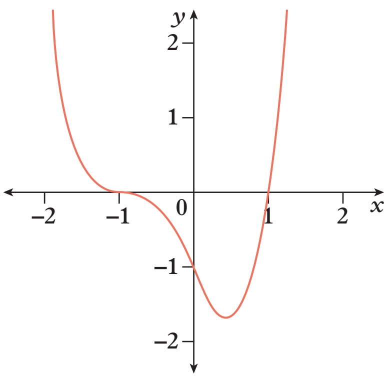
- To sketch the graph of a polynomial relationship:
  1. write the polynomial in factor form
  2. identify the x-intercepts (find x when $y = 0$)
  3. identify the y-intercept (find y when $x = 0$)
  4. draw a smooth curve through the known points
  5. if necessary, find the coordinates of another point to confirm the orientation of the graph.

Repeated factors

- When a linear factor, $(x - a)$, is repeated in a polynomial, the factor is raised to the power of an index greater than 1. For example, in the cubic function $P(x) = (x + 1)(x - 2)^2$, the linear factor $(x - 2)$ is repeated two times. In the quartic function $P(x) = (x - 3)(x + 5)^3$, the linear factor $(x + 5)$ is repeated three times.
- When a linear factor is repeated in a polynomial, $P(x)$, the number of solutions to the equation $P(x) = 0$ is less than the degree of the polynomial. This reduces the number of possible x-intercepts in the graph of $y = P(x)$.
- The **multiplicity** of a root a in a polynomial $P(x)$ is the number of times $(x - a)$ is repeated as a factor of $P(x)$. The sum of the multiplicities of the linear factors of a fully factorised polynomial is equal to the degree of the polynomial.

- When a linear factor, $(x - a)$, is repeated in a polynomial it means that the x-intercept at a has a zero gradient as it passes through the x-axis. For a non-constant polynomial, points with zero gradients correspond to points of inflection and turning points.

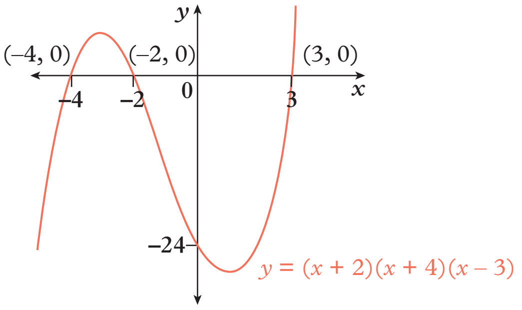
  For example, consider how repeated factors affect the shape of the graphs of the quartic relationships in the table below. Look at the x-intercepts corresponding to each repeated factor.

One factor with multiplicity of 2 (one double root)	**Two factors with multiplicity of 2 (two double roots)**	**One factor with multiplicity of 3 (one triple root)**
$y = x(x - 1)(x + 1)^2$	$y = (x - 1)^2(x + 1)^2$	$y = (x - 1)(x + 1)^3$
		

Example 6F.1 Sketching the graph of a cubic relationship

Sketch the graph of each of these cubic relationships.

a $y = (x + 2)(x + 4)(x - 3)$ **b** $y = -x(x - 1)(x + 2)$

THINK	**WRITE**
a 1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.	When $y = 0$: $(x + 2)(x + 4)(x - 3) = 0$ $x + 2 = 0$ or $x + 4 = 0$ or $x - 3 = 0$ $x = -2, -4$ or 3 The x-intercepts are at (–4, 0), (–2, 0) and (3, 0).
2 Substitute $x = 0$ to find the y-intercept.	When $x = 0$: $y = (2)(4)(-3)$ $= -24$ The y-intercept is at $(0, -24)$.
3 Mark the four intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the bottom left of the Cartesian plane. Label the graph with its equation.	

b **1** Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

When $y = 0$:

$-x(x - 1)(x + 2) = 0$

$-x = 0$ or $x - 1 = 0$ or $x + 2 = 0$

$x = -2, 0$ or 1

The x-intercepts are at $(-2, 0)$, $(0, 0)$ and $(1, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

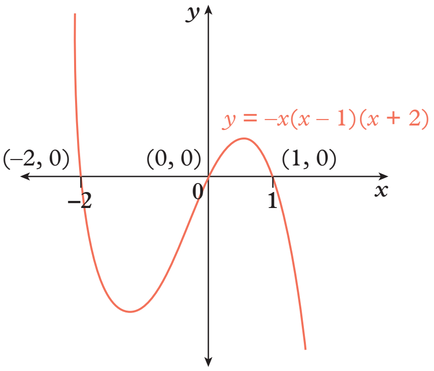
When $x = 0$:

$y = -(0)(-1)(2)$

$= 0$

The y-intercept is at $(0, 0)$.

3 Mark the three intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is negative, so the curve starts from the top left of the Cartesian plane due to the negative sign. Label the graph with its equation.



Example 6F.2 Sketching the graph of a quartic relationship

Sketch the graph of $(x - 2)(x + 3)(x + 1)(x - 4)$.

THINK

WRITE

1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

When $y = 0$:

$(x - 2)(x + 3)(x + 1)(x - 4) = 0$

$x - 2 = 0$ or $x + 3 = 0$ or $x + 1 = 0$ or $x - 4 = 0$

$x = -3, -1, 2$ or 4

The x-intercepts are at $(-3, 0)$, $(-1, 0)$, $(2, 0)$ and $(4, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

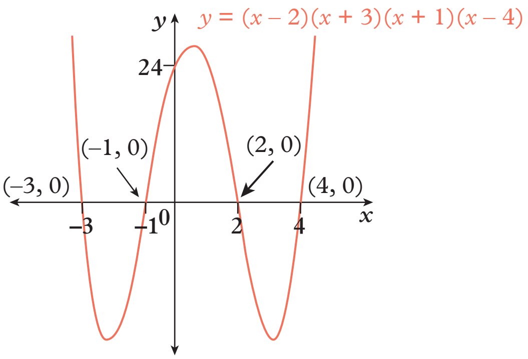
When $x = 0$:

$y = (-2)(3)(1)(-4)$

$= 24$

The y-intercept is at $(0, 24)$.

3 Mark the five intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the top left of the Cartesian plane. Label the graph with its equation.



Helpful hints

- ✔ In this section, you don't need to plot the graph, only to sketch it. The only points to plot are the intercepts. To understand the shape of the graph, ask yourself questions like: 'What happens when x is very large or very small?' and 'Which way does the graph need to turn to pass through all of the intercepts without touching an axis in any other places?'
- ✔ When sketching, remember to use an approximate scale to ensure accuracy. The more accurate your scale, the more accurate the shape of your sketch will be.
- ✔ Plot all the intercepts before attempting to sketch the line.
- ✔ Don't forget to check for both x- and y-intercepts! Some students find the x-intercepts and forget to find the y-intercepts.
- ✔ Make sure your lines are smooth rather than dashed lines. Be careful to create a smooth curve rather than a pointy line when approaching turning points.

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Exercise 6F Sketching graphs of polynomials using intercepts

△ 1–5, 6(a, b, e, g), 7–9, 10(a, b), 13

■ 3–5, 6(c, d, f, h), 8, 9(b, d), 10(c–f), 12, 14

⬟ 4, 5(d–f), 6(e–h), 9(b, d), 10(b, d, f), 11, 12, 15–17

UNDERSTANDING AND FLUENCY

6F.1 **1** Sketch the graph of each of these cubic relationships.

a $y = (x + 1)(x + 3)(x - 1)$ **b** $y = (x - 2)(x - 5)(x + 1)$ **c** $y = (x - 4)(x + 2)(x + 3)$

d $y = (x - 3)(x - 1)(x - 5)$ **e** $y = x(x + 3)(x - 3)$ **f** $y = 2x(x + 4)(x - 1)$

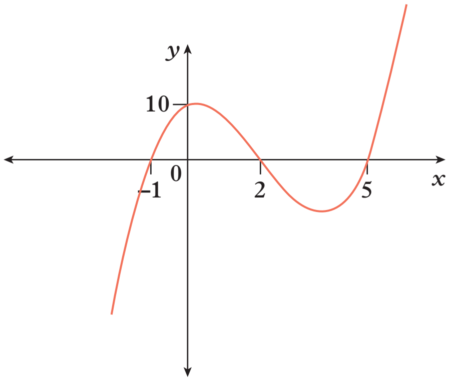
2 For each of the cubic relationships below:

i identify the x-intercepts

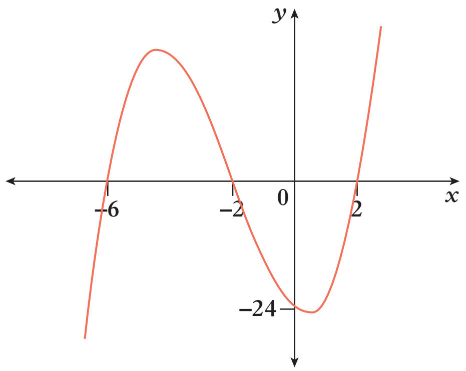
ii identify the y-intercept

iii write the rule in factorised form.

a



b



3 **a** Find the x- and y-intercepts for:

i $y = (x + 7)(x - 2)(x - 3)$

ii $y = -(x + 7)(x - 2)(x - 3)$.

b How is the graph of $y = -(x + 7)(x - 2)(x - 3)$ different from the graph of $y = (x + 7)(x - 2)(x - 3)$?

4 Sketch the graph of each of these cubic relationships.

a $y = -(x + 5)(x - 1)(x - 3)$ **b** $y = -(x - 6)(x + 1)(x - 2)$

c $y = -(x + 2)(x + 3)(x - 5)$ **d** $y = -x(x + 7)(x - 4)$

UNDERSTANDING AND FLUENCY

6F.2 **5** Sketch the graph of each of these quartic relationships.

a $y = (x - 1)(x + 2)(x + 3)(x - 3)$

b $y = (x + 4)(x + 1)(x - 1)(x + 3)$

c $y = x(x - 2)(x - 6)(x + 5)$

d $y = -(x - 3)(x + 3)(x + 1)(x - 1)$

e $y = (2x - 3)(x + 2)(x + 1)(x - 4)$

f $y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 Sketch the graph of each of these polynomial relationships. Hint: You will need to first factorise the polynomial.

a $y = x^3 - 3x^2 - 13x + 15$

b $y = -x^3 - 2x^2 + 16x + 32$

c $y = x^3 - 2x^2 - 3x$

d $y = -2x^3 + 8x^2 - 2x - 12$

e $y = x^4 - 15x^2 - 10x + 24$

f $y = x^4 - 9x^3 + 6x^2 + 56x$

g $y = -x^4 + 10x^2 - 9$

h $y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

7 Use a graphing application to check your graphs from question **6**.

PROBLEM SOLVING AND REASONING

8 Consider the relationship $y = (x + 2)(x - 3)^2$.

a What type of polynomial relationship is this?

b Find the x- and y-intercepts.

c As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?

d Sketch the graph. Use a graphing application to verify your answer.

e What effect does the repeated factor of $(x - 3)$ have on the graph?

9 Sketch the graph of each of these cubic relationships.

a $y = (x + 4)(x - 1)^2$

b $y = (x + 2)^2(x - 5)$

c $y = -(x - 2)(x - 7)(x - 2)$

d $y = -(2x + 1)(x - 4)^2$

10 Sketch the graph of each of these quartic relationships.

a $y = (x - 3)(x + 2)(x + 1)^2$

b $y = -(x - 2)^2(x - 4)(x + 2)$

c $y = -x(x - 2)(x + 3)^2$

d $y = x^2(x + 5)(x - 3)$

e $y = (x - 1)^2(x + 2)^2$

f $y = -x^2(x - 4)^2$

11 A squared factor in a polynomial relationship means that an x-intercept is also a turning point. Investigate the effect of having a cubed factor in a quartic relationship. That is, find what happens at $x = a$ for a graph of the form $y = (x - a)^3(x - b)$. Use a graphing application to try different values for a and b.

12 The motion of a person on a particular water slide can be approximated using the cubic relationship $h = -\frac{1}{5}(t^3 - 11t^2 + 39t - 45)$, where h is the height above the ground in metres and t is the time in seconds since the start of the slide.

a Sketch a graph of this relationship.

b At what height above the ground does a person start the slide?

c How high is a person after 1 second?

d The ride descends to its lowest point. How long does this take?

e How long does it take for the slide to ascend from its lowest point and then descend again to its lowest point?

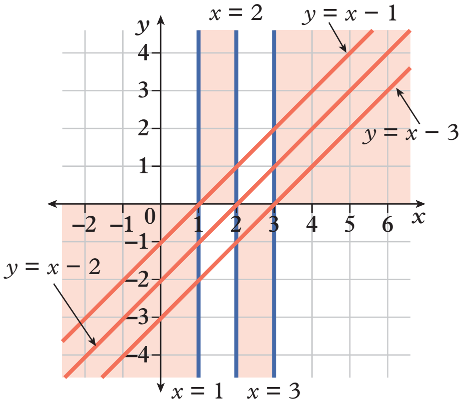
13 Amelia monitors the change in value of shares during the month of June. She finds that the change in value, y (in dollars), after x days can be approximated using a cubic relationship, where y increases from \$0 to \$21 after 5 days and is zero again after 12 days and 20 days.

a Sketch a graph of this cubic relationship.

b Find the equation for this relationship.

c Use this relationship to estimate the change in the value of the shares by the end of June.

14 Shown below are the graphs of the lines $y = x - 1$, $y = x - 2$ and $y = x - 3$ as well as the vertical lines $x = 1$, $x = 2$ and $x = 3$. The shaded regions show where the function of the product of the lines, $y = (x - 1)(x - 2)(x - 3)$, will lie.



a The table below shows if a line will have positive or negative y values for the given x values. Copy the table and complete it by writing 'positive' or 'negative' in each space.

	$x < 1$	$1 < x < 2$	$2 < x < 3$	$x > 3$
$y = x - 1$	negative			
$y = x - 2$				
$y = x - 3$				
$y = (x - 1)(x - 2)(x - 3)$				

b Describe why regions of the graph are shaded.

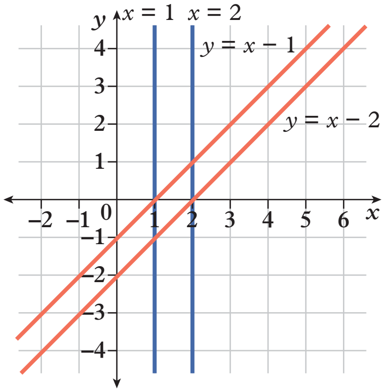
c The table lists inequalities that do not include the values for $x = 1$, $x = 2$, and $x = 3$.

i What is the value of the product $y = (x - 1)(x - 2)(x - 3)$ for these x values?

ii What feature/s of the graph of $y = (x - 1)(x - 2)(x - 3)$ do these x values correspond with?

d Use the information you found in part **c** to help you sketch the graph of $y = (x - 1)(x - 2)(x - 3)$. Label all axis intercepts.

e Shown below are the graphs of the lines $y = x - 1$ and $y = x - 2$ as well as the vertical lines $x = 1$ and $x = 2$.



PROBLEM SOLVING AND REASONING

Copy the table below and complete it by writing ‘positive’ or ‘negative’ in each space.

	$x < 1$	$1 < x < 2$	$x > 2$
$y = x - 1$			
$y = x - 2$			
$y = x - 2$			
$y = (x - 1)(x - 2)^2$			

f Copy the graphs on the previous page in part **e** and shade the regions in which the graph of $y = (x - 1)(x - 2)^2$ will lie.

g Explain why you shaded the regions you did in part **f**.

h Sketch the graph of $y = (x - 1)(x - 2)^2$. Label all axis intercepts.

15 a Sketch $y = x$, $y = x^3$ and $y = x^5$ on the same set of axes.

b Sketch $y = -x$, $y = -x^3$ and $y = -x^5$ on the same set of axes.

c By considering the extreme ends of the graphs from parts **a** and **b**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is odd.

d Sketch $y = x^2$, $y = x^4$ and $y = x^6$ on the same set of axes.

e Sketch $y = -x^2$, $y = -x^4$ and $y = -x^6$ on the same set of axes.

f By considering the extreme ends of the graphs from parts **d** and **e**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is even.

g Use a graphing application to sketch the following graphs. Do *not* find or plot any intercepts.

i $y = x^2 + x$

ii $y = x^3 + x^2 + x$

iii $y = x^4 + x^3 + x^2 + x$

iv $y = x^5 + x^4 + x^3 + x^2 + x$

v $y = x^6 + x^5 + x^4 + x^3 + x^2 + x$

vi $y = x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$

h By considering the extreme ends of the graphs from part **g**, describe the rough shape of polynomials with a leading term of x^n and $-x^n$, when n is odd or even.

CHALLENGE

16 a Zara is using the factor theorem to attempt to factorise $P(x) = x^3 - 10x^2 - 11x + 180$ so she can graph $y = P(x)$. So far, she has found the following: $P(-6) = -330$, $P(-2) = 154$, $P(2) = 126$, $P(6) = -30$ and $P(10) = 70$.

i How many x-intercepts does the graph of $y = P(x)$ have?

ii If Zara knows that the x-intercepts have integer x values, then list a possible set of solutions for $P(x) = 0$.

iii Which number(s) can be ruled out from the set? Explain why.

b Sara is using the factor theorem to attempt to factorise $Q(x) = x^4 - 6x^3 - x^2 + 54x - 72$ so she can graph $y = Q(x)$. So far, she has found the following: $Q(1) = -24$, $Q(-1) = -120$, $Q(2) = 0$, $Q(-2) = -120$, $Q(3) = 0$, $Q(-3) = 0$, $Q(4) = 0$ and $Q(-4) = 336$.

i State the coordinates of the x-intercepts of $y = Q(x)$.

ii Explain how Sara could use the remaining substitutions to sketch the graph of $y = Q(x)$ more accurately.

17 a Sketch a rough graph of $y = x(x - 6)(x - 3)(x + 2)(x + 4)(x + 6)$. You do not need to label any intercepts.

b Sketch a rough graph of $y = (x - 8)(x - 5)^3(x + 1)^2(x + 5)^3(x + 10)^2$. You do not need to label any intercepts.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Sketching graphs of cubic relationships

Interactive skillsheet
Sketching graphs of quartic relationships

Worksheet
Solving polynomial equations and graphing polynomial relationships

Investigation
Maximising the volume of a box

Investigation
Exploring repeated roots

Topic quiz
6F

ADVANCED
EXTENSION

6G Sketching graphs of polynomials using transformations

Learning intentions

By the end of this topic you will be able to ...

✔ perform transformations on the graph of a polynomial relationship.

Inter-year links

Year 9	5D Sketching parabolas using transformations

Key content video

Transformations of a polynomial relationship

- Transformations such as dilations, reflections and translations can be performed on the graph of $y = P(x)$.
- A dilation by a factor of a in the direction of the y-axis produces $y = aP(x)$.
- A reflection in the x-axis produces $y = -P(x)$.
- A reflection in the y-axis produces $y = P(-x)$.
- A vertical translation of k units produces $y = P(x) + k$.
- A horizontal translation of h units produces $y = P(x - h)$.
- A combination of transformations can be performed on the graph of $y = P(x)$ to produce the graph of $y = aP(n(x - h)) + k$.

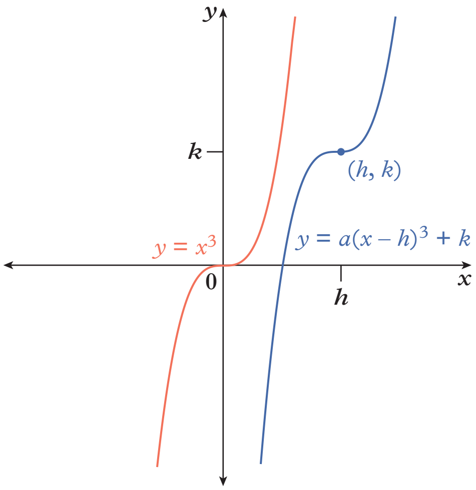
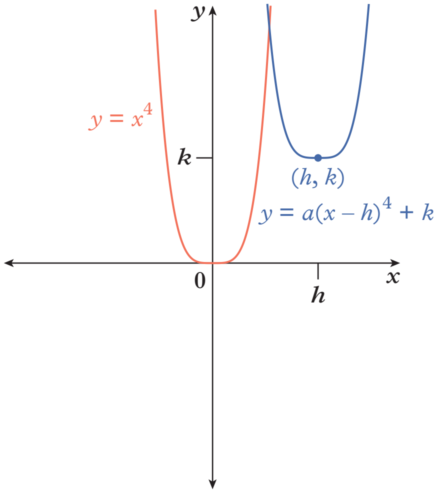
$$y = aP(n(x - h)) + k$$

Vertical translation of k units
For $k > 0$, move up
For $k < 0$, move down

Dilation by factor of a in the y-axis
For $a < 0$, reflect in the x-axis

For $n = -1$, reflect in the y-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left

Cubic relationships	Quartic relationships
	

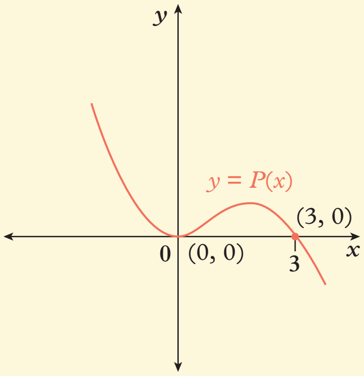
Example 6G.1 Performing transformations on the graph of a polynomial relationship

Perform a transformation on the graph of $y = P(x)$ (shown on the right) to produce the graph of:

a $y = P(x) + 2$

b $y = P(x + 2)$

c $y = -P(x)$.



THINK

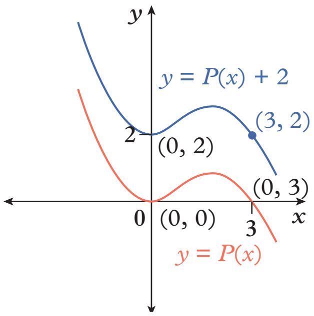
a **1** Identify the transformation. The original graph needs to be translated 2 units up.

2 Determine the impact of the transformation on the intercepts: (0, 0) moves to (0, 2) and (3, 0) moves to (3, 2).

3 Draw $y = P(x)$ and $y = P(x) + 2$ on the same Cartesian plane.

WRITE

a The graph of $y = P(x)$ will be translated 2 units up.

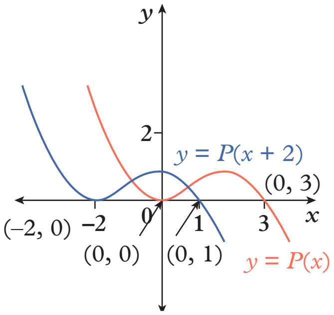


b **1** Identify the transformation. The original graph needs to be translated 2 units left.

2 Determine the impact of the transformation on the intercepts. (0, 0) moves to (–2, 0) and (3, 0) moves to (1, 0).

3 Draw $y = P(x)$ and $y = P(x + 2)$ on the same Cartesian plane.

b The graph of $y = P(x)$ will be translated 2 units left.

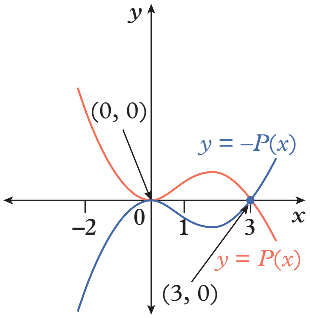


c **1** Identify the transformation. The original graph needs to be reflected in the x-axis.

2 Determine the impact of the transformation on the intercepts: (0, 0) and (3, 0) remain the same.

3 Draw $y = P(x)$ and $y = -P(x)$ on the same Cartesian plane.

c The graph of $y = P(x)$ will be reflected in the x-axis.



Helpful hints

- ✔ You will only become familiar with the shape and turns of a polynomial by sketching them manually. Practice makes permanent!
- ✔ It can be easy to miss the negative sign at the front of a function. Be careful to pay attention, because a negative factor has a large impact on the shape of the graph. It completely flips it!

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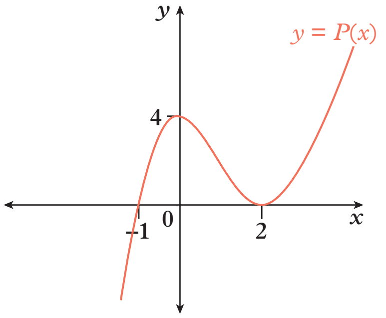
Exercise 6G Sketching graphs of polynomials using transformations

△ 1–7, 9, 10, 12 ■ 2–8, 11, 12, 13(a) ⬟ 2, 5–8, 11, 13–15

UNDERSTANDING AND FLUENCY

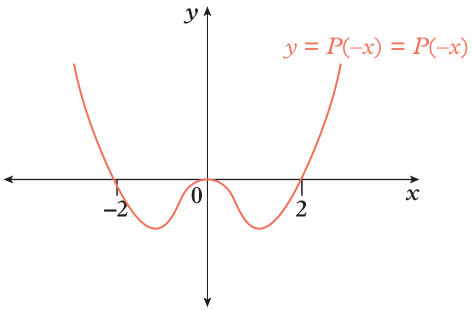
6G.1 **1** Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 1$
b $y = P(x) - 2$
c $y = P(x - 1)$
d $y = P(x + 3)$
e $y = -P(x)$
f $y = 2P(x)$.
g $y = P(-x)$.



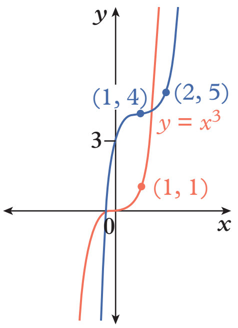
2 Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 3$
b $y = P(x) - 1$
c $y = P(x - 3)$
d $y = P(x + 4)$
e $y = -P(x)$
f $y = 2P(x)$.
g $y = P(-x)$

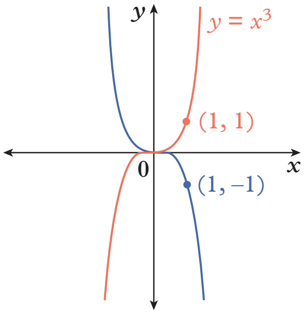


3 Identify the transformations performed on the graph of $y = x^3$ to produce each graph shown in blue below.

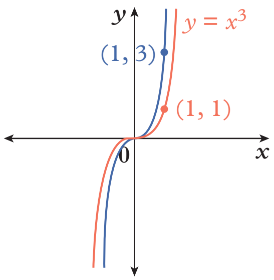
a



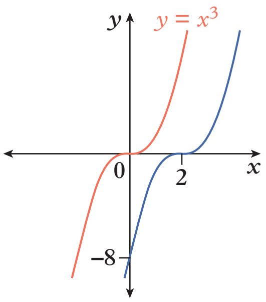
b



c



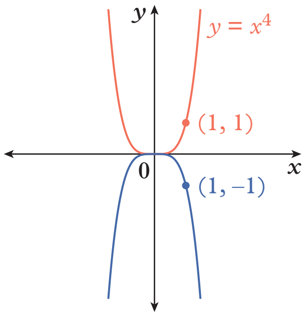
d



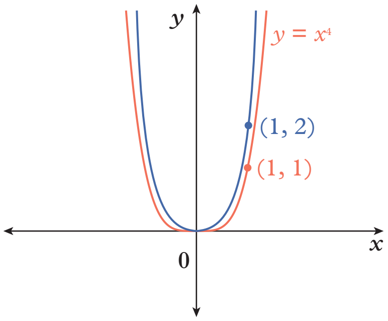
4 Write the equation for each of the graphs in question **3**.

5 Identify the transformations performed on the graph of $y = x^4$ to produce each graph shown in blue below.

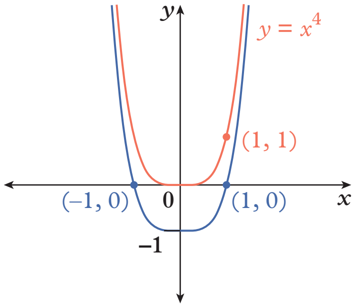
a



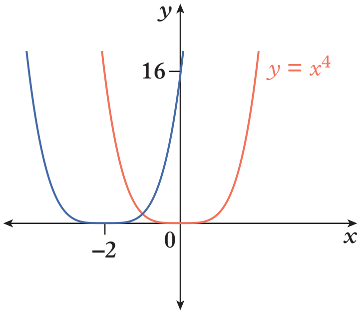
b



c

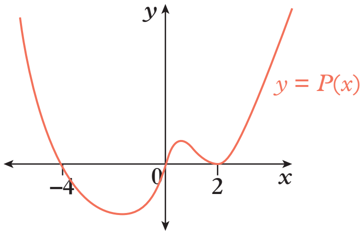


d

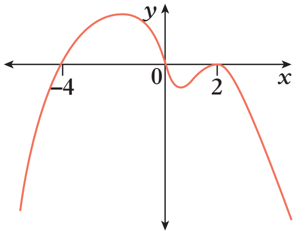


6 Write the equation for each of the transformed graphs in question **5**.

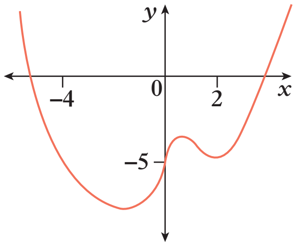
7 Identify the equation for each of these graphs, in terms of $P(x)$, using the graph of $y = P(x)$ shown on the right to help you.



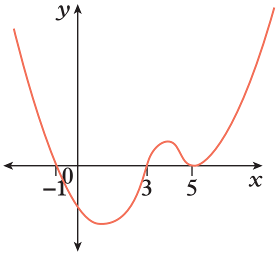
a



b



c



8 Compare the graph produced after reflecting $y = x^3$ in the x-axis with the graph produced after reflecting $y = x^3$ in the y-axis. What is the equation for each of these transformed graphs?

ADVANCED

EXTENSION

9 For each cubic relationship below:

- **i** describe the transformations that need to be performed on $y = x^3$ to produce the graph of the relationship
- **ii** identify the coordinates of the point of inflection
- **iii** find the x- and y-intercepts
- **iv** for parts **c** and **d**, sketch the graph.

a $y = 8x^3 + 1$ **b** $y = \frac{1}{2}x^3 - 4$ **c** $y = \frac{1}{2}(x - 3)^3 + 4$ **d** $y = -2(x + 1)^3 - 2$

10 Identify the x-coordinates of the point of inflection and the x-intercept for cubic relationships of the form $y = a(x - h)^3 + k$.

11 Dilations can also occur in the direction of the x-axis by multiplying the x-coordinates instead of the y-coordinates. Consider the table of values for a polynomial shown on the right.

x	0	1	2	3
$y = P(x)$	5	0	-5	2

When dilating in the x-axis, we replace x with a new expression, then work backwards to find the x values for the transformed graph. This way, the y-coordinates corresponding to the original x values now correspond with the transformed x values.

a Complete each table of values below by performing the appropriate operations to the x-coordinates.

i

$\frac{x}{2}$	0	1	2	3
x				
$y = P\left(\frac{x}{2}\right)$	5	0	-5	2

ii

$2x$	0	1	2	3
x				
$y = P(2x)$	5	0	-5	2

iii

$\frac{x}{5}$	0	1	2	3
x				
$y = P\left(\frac{x}{5}\right)$	5	0	-5	2

iv

$5x$	0	1	2	3
x				
$y = P(5x)$	5	0	-5	2

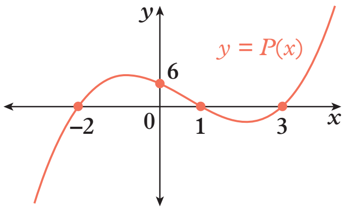
v

$-\frac{x}{2}$	0	1	2	3
x				
$y = P\left(-\frac{x}{2}\right)$	5	0	-5	2

vi

$-2x$	0	1	2	3
x				
$y = P(-2x)$	5	0	-5	2

b Identify the rule for each graph in terms of $P(x)$ using the graph of $y = P(x)$ shown on the right to help you.



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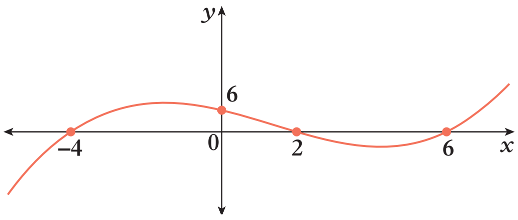

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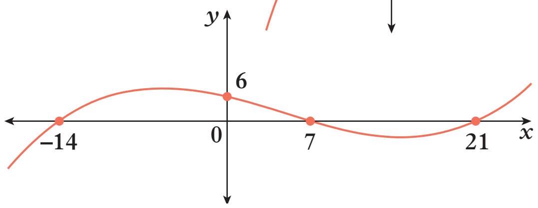

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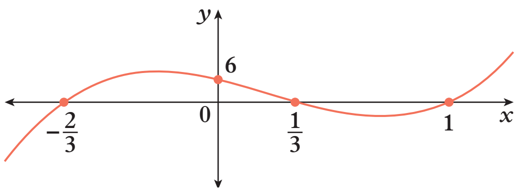

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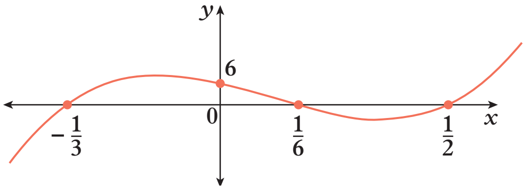


ADVANCED

EXTENSION

PROBLEM SOLVING AND REASONING

12 Dylan is sketching transformations of the graph of the polynomial $y = P(x)$. He has written the transformation he will apply for each of the following. State whether his descriptions are correct or incorrect. Also state the correct transformation for any incorrect description.

a $y = P(x) + 6$; translate 6 units up

b $y = 4P(x)$; dilate by a factor of 4 in the x-axis

c $y = P(x - 5)$; translate 5 units left

d $y = P(3x)$; dilate by a factor of $\frac{1}{3}$ in the x-axis

13 A more complicated transformation of a polynomial $P(x)$ can be written in the form:

$$y = aP(n(x - h)) + k.$$

Here the polynomial $P(x)$ has been:

- dilated by a factor of a in the direction of the y-axis
- dilated by a factor of $\frac{1}{n}$ in the direction of the x-axis
- reflected in the x-axis if $a < 0$
- reflected in the y-axis if $n < 0$
- translated h units to the right and k units up.

a For each of the following polynomials, write a series of transformations of the polynomial $P(x)$ which would result in the given polynomial.

i $y = 2P(3x) + 324$ **ii** $y = \frac{1}{3}P\left(\frac{x}{2}\right)$ **iii** $y = 5P(-x) - 15$ **iv** $y = -2P(-4x) + 12$

b The polynomial $y = P(x)$ is transformed according to the following sequence of transformations. Identify the equation of the transformed polynomial in terms of $P(x)$.

- Dilate by a factor of 3 in the direction of the x-axis.
- Reflect in the x-axis.
- Translate 2 units to the left and 4 units down.

c Consider the transformed polynomial $y = -P\left(\frac{1}{3}x + 2\right) - 4$, which should look similar to your answer to part **b**. Factorise the expression inside the brackets in the form $n(x - h)$ and then write a sequence of transformations that could be applied to $P(x)$ to obtain the transformed polynomial.

d Use parts **b** and **c** to explain why transformations are written in function notation with the expression inside the brackets factorised.

CHALLENGE

14 a Expand $(x - 2)^3 + 27$.

b Using what you found in part **a** to help you, sketch the following graphs.

i $y = 4x^3 - 24x^2 + 48x + 76$ **ii** $y = (x - 2)^3 - 6(x - 2)^2 + 12(x - 2) + 21$

iii $y = -x^3 - 6x^2 - 12x + 19$

15 For polynomials of the form $y = x^n$, a dilation in the x-axis is equivalent to a different dilation in the y-axis. This equivalence can be determined algebraically by expanding or factorising the expression. For example, $y = (2x)^3$ is a dilation of $y = x^3$ by a factor of $\frac{1}{2}$ in the x-axis. When expanded, $y = (2x)^3 = 8x^3$ is a dilation of $y = x^3$ by a factor of 8 in the y-axis.

For each of the following:

a describe the equivalent dilations in both axes of the graph of $y = x^n$ for it to be transformed to the graph of each of the following polynomials

b describe the equivalent dilations in both axes of the graph of each of the polynomials below for it to be transformed to the graph of $y = x^n$.

i $y = (3x)^2$ **ii** $y = \left(\frac{x}{2}\right)^4$ **iii** $y = 25x^2$ **iv** $y = \frac{625}{81}x^4$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching polynomials using transformations

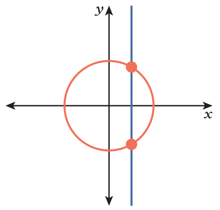
Investigation
Transforming polynomials

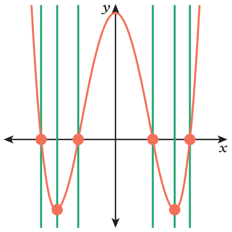
Topic quiz
6G

Chapter summary

Functions and relations

- A relation is a map between two sets.
- A function is a relation where each element in the first set is mapped to exactly one element in the second set.
- Graphs of functions pass the vertical line test.





Polynomials

$$\underbrace{\overbrace{P(x) = \underbrace{x^2 + 2x + 1}_{\text{expression}}}^{\text{function}} = 0}_{\text{equation}}$$

- 1 is a degree 0 polynomial.
- x is a degree 1 polynomial.
- x^2 is a degree 2 polynomial.
- x^3 is a degree 3 polynomial.

Sketching polynomials using intercepts

1. Write the polynomial in factorised form.
2. Identify the x-intercepts (find x when $y = 0$).
3. Identify the y-intercepts (find y when $x = 0$).
4. Draw a smooth curve through the known points.
5. If necessary, find the coordinates of another point to confirm the orientation of the graph.
6. Check the multiplicities of linear factors in the polynomial are correctly represented in the graph.

The distributive law

$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

Division of polynomials

$$\begin{array}{rl} x + 2 & \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x + 1\,\overline{)\,x^2 + 3x + 4} & \leftarrow \text{dividend} \\ \underline{x^2 + x\phantom{{}+4}} & \\ 2x + 4 & \\ \underline{2x + 2} & \\ 2 & \leftarrow \text{remainder} \end{array}$$

$$\frac{\overbrace{x^2 + 3x + 4}^{\text{dividend}}}{\underbrace{x + 1}_{\text{divisor}}} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{2}}{x + 1}$$

The remainder theorem

- When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

The factor theorem

- When a polynomial $P(x)$ is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$.

The Null Factor Law

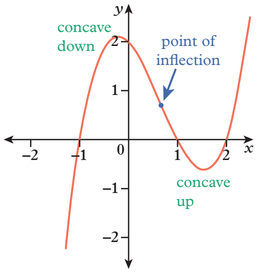
- If $P(x) = (x - a)(x - b)(x - c)$, then $P(x) = 0$ has the solution $x = a, x = b, x = c$.

$$P(x) = (x - a)(x - b)(x - c) = 0$$

$$x - a = 0 \;\Rightarrow\; x = a$$
$$x - b = 0 \;\Rightarrow\; x = b$$
$$x - c = 0 \;\Rightarrow\; x = c$$

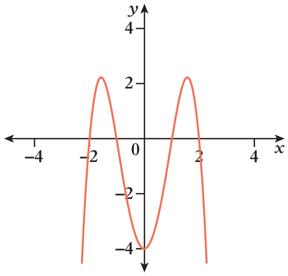
Cubic polynomials

$$P(x) = (x - 2)(x - 1)(x + 1)$$



Quartic polynomials

$$P(x) = -(x - 2)(x - 1)(x + 1)(x + 2)$$



Sketching polynomials using transformations

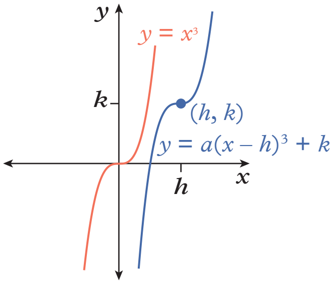
$$y = aP(n(x - h)) + k$$

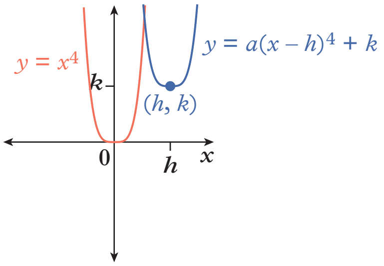
Vertical translation of k units
For $k > 0$, move up
For $k < 0$, move down

Dilation by factor of a in the y-axis
For $a < 0$, reflect in the x-axis

For $n = -1$, reflect in the y-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left





ANS p727

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- coefficient
- constant term
- degree (of a polynomial)
- dividend
- divisor
- element
- function
- function notation
- leading coefficient
- leading term
- monic polynomial
- multiplicity
- non-monic polynomial
- Null Factor Law
- point of inflection
- polynomial
- quotient
- relation
- remainder
- repeated factor
- root
- set
- term
- vertical line test
- zero (of a polynomial)

1 Which of the following expressions are polynomials?

a $x^2 + 2x + 1$ **b** $x^{-2} - 2$ **c** $\dfrac{1}{(x^3 + 3x)}$

d $\log_2(x + 1)$ **e** 2 **f** $y^3 - 2xy + y^2 - x$

2 For the polynomial $3x^6 - 4x^3 + 2x^2 - 1$, identify:

a the number of terms **b** the degree of the polynomial

c the constant term **d** the leading term

e the leading coefficient **f** the coefficient of the x^3 term.

3 **a** Write the definition of a function. **b** Give an example of a relation which is not a function.

4 How does a relation fail the vertical line test?

5 Complete the following sentences using words from the list of key terms.

a When dividing one polynomial by another polynomial, the ________ is the expression being divided by the ________ and the _______ is the resulting expression.

b The __________ of a _______ polynomial is equal to 1 and the _________ of a __________ polynomial is not equal to 1.

c A ___________ of a polynomial $P(x)$ is a _________ of the equation $P(x) = 0$.

d The _________ of a polynomial is the index of the variable in the _______ term.

Multiple choice

ADV EXT **1** Which of these relations is not a function?

A $y = (x - 2)^2 + 2$ **B** $y = \dfrac{2}{x - 2} + 2$ **C** $y = 2^{-x} + x^2$

D $y = x^6 + x^5 + x^4 + x^3$ **E** $x^2 + y^2 = 1$

ADV 6B EXT **2** Which of these expressions is a polynomial?

A $x^2 + \sqrt{x}$ **B** $\dfrac{3x}{x^2 + 1}$ **C** $x^3 - 2x$ **D** $x^2 + x^{-1}$ **E** $\left(\dfrac{1}{x} + 2\right)\left(x + \dfrac{1}{2}\right)$

ADV 6B EXT **3** The degree of the polynomial $4x^2 - 3x^4 + x^3 - 2x$ is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

ADV 6B EXT **4** The coefficient of the leading term in the expression $3 - 2x^2 + 5x^3 + 7x$ is:

A 5 **B** −2 **C** 3 **D** 7 **E** 2

ADV 6B EXT **5** When $x^2 + 6x - 3$ is divided by $x - 2$, the remainder is:

A 13 **B** −19 **C** 5 **D** −11 **E** $x + 8$

ADV 6C EXT **6** Which of these is correct?

A $(5x^2 - x + 2) \div (x - 4) = 5x + 19$, remainder -74

B $(3x^2 - 2x + 1) \div (x - 1) = 3x - 5$, remainder -9

C $(4x^2 - x + 8) \div (x - 2) = 4x - 9$, remainder -10

D $(2x^2 - x + 1) \div (x - 5) = 2x + 9$, remainder 37

E $(6x^2 - 13x + 15) \div (x - 3) = 6x + 5$, remainder 30

ADV 6D EXT **7** When $P(x) = x^3 - 2x^2$ is divided by $(x - 1)$, the remainder is:

A $P(0)$ **B** $P(-2)$ **C** $P(2)$ **D** $P(-1)$ **E** $P(1)$

ADV 6E EXT **8** Which of these expressions is a factor of $x^3 - 4x^2 + x + 6$?

A $x - 1$ **B** $x - 4$ **C** $x + 3$ **D** $x + 6$ **E** $x - 2$

ADV 6E EXT **9** The leading term in a polynomial is raised to the power n. The maximum number of solutions it could have is:

A $n - 1$ **B** $n + 1$ **C** n **D** $\frac{n}{2}$ **E** impossible to tell.

ADV 6E EXT **10** How many different solutions does $(x + 1)(x - 1)(x + 2)(x + 1) = 0$ have?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

ADV 6E EXT **11** The graph of $y = (x - 3)(x + 2)(x - 5)$ has a y-intercept of:

A 3 **B** -2 **C** 30 **D** -30 **E** 5

ADV 6F EXT **12** The graph of $y = (x - 1)^3$ has a point of inflection at:

A $y = -1$ **B** $x = -1$ **C** $x = 0$ **D** $y = 1$ **E** $x = 1$

ADV 6G EXT **13** The transformation that needs to be performed on the graph of $y = x^3$ to produce the graph of $y = (x + 5)^3$ is a translation of:

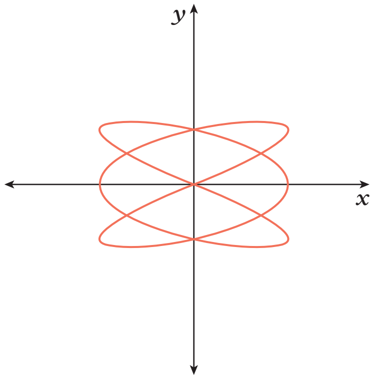
A 5 units right **B** 5 units left **C** 5 units up

D 5 units down **E** 5 units left and 5 units up.

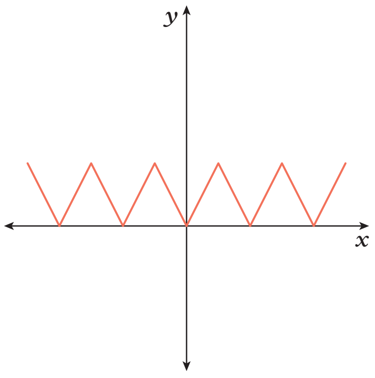
Short answer

ADV 6A **1** Use the vertical line test to determine which of the following relations are functions.

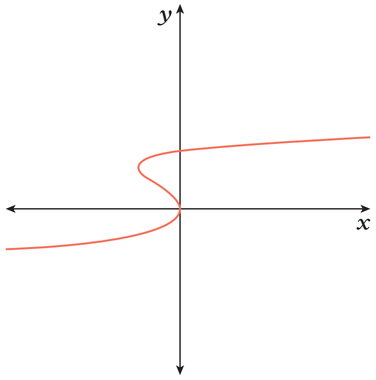
a



b



c



ADV 6A **2** For each of the following functions, evaluate:

i $f(0)$ **ii** $f(5)$ **iii** $f(-2)$.

a $f(x) = x^2 + 1$ **b** $f(x) = x^3 - x + 2^x$ **c** $f(x) = -x^3 + \frac{1}{x + 1}$

ADV 6B EXT **3** Decide whether each of these expressions is a polynomial. For those that are polynomials, correctly identify the type of polynomial using its descriptive name: linear, quadratic, cubic or quartic.

a $5 - 3x^2 + 4x - 6x^3$ **b** $1 - 3x$ **c** $\frac{4x}{5}$

d $\frac{6}{7x}$ **e** $6x^2 - 5x + 12$ **f** $\frac{1}{2}x - 3x^4$

ADV 6C EXT **4** For the polynomial $3x^4 + 2x^6 - 4x^5 + x - 8x^3 - 7x^2 - 5x^8$, identify:

a the number of terms it has **b** its degree **c** the constant term

d the leading term **e** the leading coefficient **f** the coefficient of the x^2 term.

ADV 6C EXT **5** Use long division to find the quotient and remainder for each of these:

a $(x^2 + 5x - 2) \div (x - 2)$ **b** $(3x^2 - x + 4) \div (x + 1)$

c $(x^2 - x + 8) \div (x - 4)$ **d** $(x^3 + 2x^2 - x + 3) \div (x^2 + 1)$

ADV 6C EXT **6** Expand and simplify the right-hand side to verify whether each of these statements is true.

a $2x^2 + 5x - 1 = (2x + 3)(x + 1) - 4$ **b** $x^2 - 3x + 5 = (x - 1)(x - 2) + 7$

ADV 6D EXT **7** Use the factor theorem to find a linear factor of each of the following polynomials, $P(x)$.

a $P(x) = x^3 - x^2 - 5x - 3$ **b** $P(x) = x^3 + 4x^2 + 7x + 12$

c $P(x) = x^3 + 3x^2 - 6x + 2$ **d** $P(x) = x^3 + x^2 + 4$

ADV 6D EXT **8** Fully factorise each of these polynomials.

a $x^3 - 7x^2 + 14x - 8$ **b** $x^3 - 7x - 6$

ADV EXT 6D **9** Two factors of the polynomial $x^3 - 39x - 70$ are $(x - 7)$ and $(x + 5)$. What is the third factor?

ADV 6E EXT **10** Solve each of these equations.

a $(x - 5)(x + 2)(x - 1)(x + 1) = 0$ **b** $x(x + 3)(x - 3)(x - 2) = 0$

c $(x + 3)^2(x - 5)^2 = 0$ **d** $x^2(x + 4)(x - 4) = 0$

ADV 6E EXT **11** Fully factorise the left-hand side of each of these equations, then solve.

a $x(x - 2)(x^2 - 4) = 0$ **b** $(x - 4)(x^2 - 8x + 16) = 0$

ADV 6F EXT **12** Sketch the graph of each of these equations.

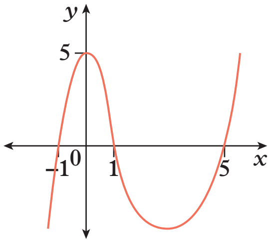
a $y = (x - 3)(x + 1)(x - 2)$ **b** $y = -(x + 2)^2(x - 4)$

ADV 6F EXT **13** For the cubic relationship shown on the right:

a identify the x-intercepts

b identify the y-intercept

c write the equation as a product of factors.



ADV 6G EXT **14** Consider $y = x^4 - 5x^2 + 4$.

a Write this relationship in terms of its factors.

b Identify the x- and y-intercepts.

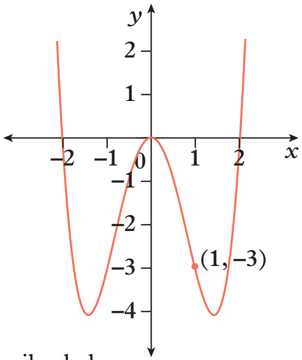
c Sketch the graph.

ADV 6F EXT **15** A quartic relationship is graphed on the right.

a Write its equation in:

i factor form **ii** expanded form.

b Determine the multiplicity of each root of the quartic relationship.



ADV 6F EXT **16** Consider the graph of $y = x(x - 1)^3$.

a Identify the x- and y-intercepts.

b Determine the multiplicity of each root of $y = x(x - 1)^3$.

c Sketch the graph.

ADV 6F EXT **17** For each of the following, write an equation whose graph is in the described shape.

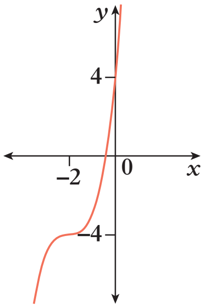
a a cubic polynomial with one double root which passes through the origin

b a quartic polynomial with two double roots whose range is $y \leq 0$

c a cubic polynomial with one triple root which passes through the point (3, 0)

d a quartic polynomial with one double root which passes through the origin and the point (–1, 0).

ADV 6G EXT **18** Describe the transformation/s performed on the graph of $y = x^3$ to produce the graph shown on the right.



ADV 6G EXT **19** Let $P(x)$ be the quartic polynomial relationship in question **15**. Sketch the graphs of the following transformations of $P(x)$.

a $P(x + 1)$

b $P(-x)$

c $-P(x)$

d $-2P(x)$

e $P(x) + 4$

Analysis

ADV EXT **1** You should be familiar with the infinity symbol. It looks like the number 8 on its side. Its shape is similar to the graph of a cubic function with three x-intercepts, combined with its reflection in the x-axis.

a It is possible to model the infinity sign using two cubic relationships graphed for the same set of x values.

i Copy and complete the table below for $y = x^3 - x$.

x	y
–1	
–0.6	
–0.5	
0	
0.5	
0.6	
1	

ii Repeat part **i** for $y = -x^3 + x$.

iii On a Cartesian plane, plot the points from the tables in parts **i** and **ii**. Join the points with a smooth curve.

iv Describe the domain of the two graphs.

v Do the turning points for the two relationships occur at $x = -0.5$ and $x = 0.5$? Explain.

vi Use a graphing application to draw the graphs of these two relationships for the set of x values described in part **iv**. Find the coordinates of the turning points.

b It is also possible to model the infinity symbol by tracing the path a point takes as it moves from an angle of 0° with the x-axis to an angle of 360° with the x-axis.

The coordinates of all points (x, y) on the infinity symbol can be represented by the values $\left(\cos[\theta], \frac{\sin[2\theta]}{2}\right)$, where θ is the angle made with the x-axis by the line joining a point with the origin.

i Construct a table like the one below, for x and y using angles of θ increasing by 15°, between 0° and 360°. (This number of points is necessary for an accurate representation of the graph.)

ii Complete the table, giving your answers correct to two decimal places if necessary. You might find some of the trigonometric values to be negative as the angle increases beyond 90°. You will understand the reason for this when you study the trigonometric ratios for angles greater than 90° in Chapter 9.

Angle (θ)	**x ($\cos[\theta]$)**	**y $\left(\frac{\sin[2\theta]}{2}\right)$**
0°		
15°		
30°		
…		
345°		
360°		

iii Plot the points (x, y) on a Cartesian plane. Join them with a smooth curve. (Alternatively, you could use a spreadsheet or graphing application to plot the graph.)

iv Describe the shape of the graph.

v Describe the domain and range of the graph.

c Compare the two models for infinity symbols from parts **a** and **b**. Do you consider one to be a better model of the infinity symbol than the other? Explain.

ADV EXT **2** For a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ with three (not necessarily distinct) roots, r_1, r_2 and r_3, the following three equations hold:

- $r_1 + r_2 + r_3 = -\frac{b}{a}$
- $r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{c}{a}$
- $r_1 r_2 r_3 = -\frac{d}{a}$

These formulas are known as Vieta's formulas for cubic polynomials.

a Verify Vieta's formulas for the following cubic polynomials.

i $P(x) = 3(x + 1)(x - 3)(x + 4)$ **ii** $P(x) = -2(x - 1)^2\,(x + 10)$

Now you will derive the formulas above. Let $P(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with three roots r_1, r_2 and r_3.

b Write $P(x)$ in factorised form. Hint: Don't forget to consider the leading coefficient of $P(x)$.

c Expand your answer to part **b** and collect like terms.

d By equating the coefficients of the x^2, x and constant terms in your answer to part **c** with those in $P(x) = ax^3 + bx^2 + cx + d$, write three equations which relate r_1, r_2 and r_3 to a, b, c and d.

e Rearrange your answers to part **d** to complete the derivation of Vieta's formulas.

f Use a similar technique to derive Vieta's formulas for quartic polynomials.

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Explain the difference between functions and relations ☐ Determine whether a graph represents a function ☐ Use function notation	☐ Go back to Topic 6A Functions and relations
☐ Identify the features of a polynomial ☐ Evaluate a polynomial using substitution ☐ Expand and simplify a product of polynomials	☐ Go back to Topic 6B Polynomials
☐ Divide a polynomial by a linear factor ☐ Divide a polynomial that has coefficients of zero by a linear factor	☐ Go back to Topic 6C Dividing polynomials
☐ Use the remainder theorem to find a linear factor of a polynomial ☐ Use the factor theorem to find a linear factor of a polynomial ☐ Factorise cubic polynomials	☐ Go back to Topic 6D Remainder and factor theorems
☐ Solve polynomial equations using the Null Factor Law	☐ Go back to Topic 6E Solving polynomial equations
☐ Sketch the graph of a cubic relationship ☐ Sketch the graph of a quartic relationship ☐ Sketch the graph of cubic and quartic relationships with repeated factors	☐ Go back to Topic 6F Sketching graphs of polynomials using intercepts
☐ Perform transformations on the graph of a polynomial relationship	☐ Go back to Topic 6G Sketching graphs of polynomials using transformations

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Semester 1 review

Short answer

1 Simplify each of the following.

a $2x + x^2 + 2 - 5x + 8 - 3x^2 - 5 - 6x$

b $3xy - 4yx^2 + 11xy^2 + 9yx - 5y^2x + 4x^2y + 10xy$

c $\frac{x}{2} + 2x^2 - x + x(5x)$

d $ab + 2a^2 - ab + b(5a) - 2ab$

2 Expand and simplify each of the following.

a $3(2x + 5y)$

b $4x^2y(9xy^4 - 8x^3y)$

c $\frac{-5x}{4}(3x + 6y - 8)$

d $\frac{ab}{4}\left(2a + b - \frac{4}{3}\right)$

3 **a** Ellie receives an annual salary of $71 552. What is her weekly wage?

b David receives a monthly salary of $5123. What is his annual wage?

c Ito receives an annual salary of $120 614. What is his fortnightly wage?

4 Construct a depreciation schedule to show the value of a digital piano at the end of the first three years of the piano's life if it cost $4000 and it depreciates in value by 10% each year.

5 Calculate the interest for each of the following, correct to the nearest cent.

a Abdul invested $3500 at a simple interest rate of 14% per annum for 3 years.

b Nguyen invested $3200 at an interest rate of 7.2% per annum compounding annually for 3 years.

c Iona borrowed $5200 for 40 weeks, at a simple interest rate of 24% per annum.

d Ezra borrowed $8500 at an interest rate of 5.2% per annum compounding monthly for 5 years.

e Louise agrees to pay $120 per fortnight over 2 years for a $5000 loan.

6 Calculate the principal for each of the following.

a Huang earns $500 interest in 5 years at a simple interest rate of 16% per annum.

b Galen borrowed an amount of money for 10 years at an interest rate of 6% per annum compounding annually. At the end of the 10 years, he owed $20 057.50.

c After 4 years with a simple interest rate of 12% per annum, Zala's investment is worth $2516.

7 Calculate the annual interest rate for each of the following, correct to one decimal place.

a Oliver borrowed $850 for 18 months and accrued $50 of simple interest.

b Dani invested $1000 for 6 years, with interest compounding annually, and her investment earned a total of $400 interest.

c Carla invested $10 000 for 20 years, with interest compounding monthly, and her investment earned a total of $15 000 interest.

d For a $900 purchase, Desiree's customers pay instalments of $40 per month for 30 months.

8 Calculate the term of the loan or investment in each of these cases, correct to the nearest year where necessary.

a Isra invested $6200 at a simple interest rate of 18% per annum and the investment grew to $10 664.

b Avoca borrowed $40 000 at a simple interest rate of 23% per annum and now owes $18 400 more than the principal amount.

c Jedda invested $12 000 at an interest rate of 8.1% per annum compounding annually, and that investment is now worth $20 000.

d Christos borrowed $39 990 at an interest rate of 9.6% per annum compounding monthly. He now owes $60 000.

STN ADV **9** Determine the constant of variation, k, for each of the following.

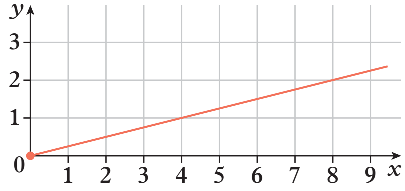
a y is directly proportional to x and, when $x = 4$, $y = 9$.

b y is inversely proportional to x and, when $x = 4$, $y = 9$.

c y is directly proportional to x and, when $x = 10$, $y = -1000$.

d y is inversely proportional to x and, when $x = 5$, $y = 20$.

e



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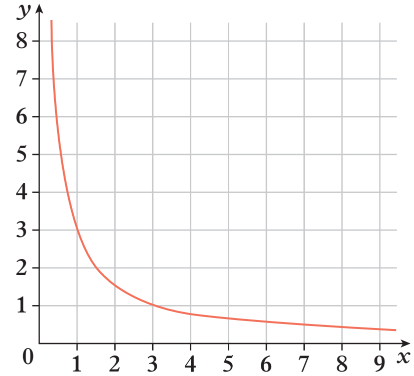


STN ADV **10 i** For each of the relationships in question **9**, determine the value of y when $x = 2$.

ii For each of the relationships in question **9**, determine the value of x when $y = 2$.

ADV **11** Shade the regions indicated by the following inequalities on the Cartesian plane. Mark sure to show if the boundary line is included in each region.

a $y < 2x + 1$

b $y + \frac{x}{3} + 1 \geq 0$

c $\frac{x}{3} \leq 2y$

d $4y + 7x \leq 1$

ADV **12** Solve each of the following equations.

a $(2x - 1)(3x + 5) = 0$

b $4x^2 = 100$

c $(x - 5)^2 - 81 = 0$

d $x^2 - 10x + 24 = 0$

e $x^2 - 15 = 2x$

f $5x^2 = 10x + 40$

g $x^2 + 5x - 10 = 0$

h $\frac{x - 3}{5} = \frac{4}{x - 2}$

ADV **13** Complete the square to write each of the following expressions in turning point form. Do not fully factorise the expressions.

a $x^2 + 10x$

b $x^2 - 6x + 12$

c $x^2 - 5x + 12$

d $2x^2 - 28x - 18$

ADV **14** Sketch the graph of each of the following equations. Write the coordinates of any axis intercepts and turning points and the equations of any asymptotes.

a $y = 2x$

b $y = 2x^2$

c $y = (x + 2)^2$

d $x^2 + y^2 = 2^2$

e $y = 2^x$

f $y = \frac{2}{x}$

ADV **15** Determine the number of x-intercepts the graphs of each of the following equations will have.

a $y = 2x - 10$

b $3x - 6y = 20$

c $x = 12$

d $y = 3$

e $y = -4(x - 1) + 5$

f $y = 3(x - 4)(x + 6)$

g $y = (x - 4)^2$

h $y = -x^2 - 5x + 6$

i $y = (x - 2)^2 + 1$

j $(x - 5)^2 + (y - 3)^2 = 25$

k $y = 2^x + 1$

l $y = 3 - 3^x$

m $y = \frac{1}{x - 3} + 2$

n $y = -\frac{3}{x + 2}$

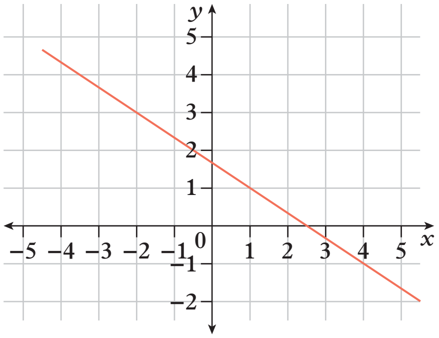
ADV **16** Determine the coordinates of the x- and y-intercepts, where they exist, for the graphs of the equations in question **15**.

ADV **17 i** Sketch the graphs of the equations in question **15**. Write the coordinates of any axis intercepts and turning points and the equations of any asymptotes.

ii State the domain and range for each of the graphs you drew for the equations from question **15**.

ADV **18** From the given information, determine the equation of each of the lines **a–f** below. Write each equation in the form $y = mx + c$.

a



b The gradient of the line is 4 and the line passes through the point (–2, 5).

c The line passes through the points (3, –4) and (8, 12).

d The line is parallel to $3x - 6y = 12$ and passes through the point (4, 3).

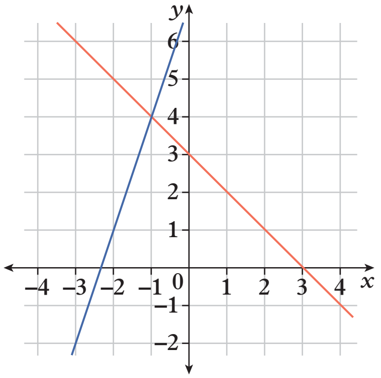
e The line is perpendicular to $y = -\frac{4}{3}x + 5$ and passes through the point (–2, 5).

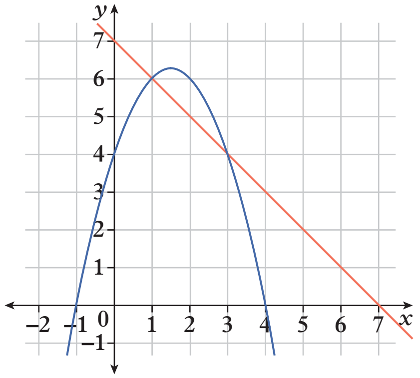
f The line is perpendicular to the line segment between the points (1, 4) and (–2, 5) and passes through the midpoint of the line segment.

ADV **19** Use the graphs provided to solve each of the following pairs of simultaneous equations.

a $y = 3x + 7$
$x + y = 3$

b $y = -x^2 + 3x + 4$
$y = -x + 7$





ADV **20** Find the point(s) of intersection for each of the following pairs of equations.

a $y = 3x - 4$
$y = -7x + 16$

b $y = 2x - 5$
$4x - 5y = 12$

c $2x + 9y = 11$
$2x - 9y = 15$

d $3x - 15y = 10$
$-6x + 5y = 10$

ADV **21** Describe qualitatively how the y value in the following relationships changes as x increases from 0 to 1, using terms such as increasing at a decreasing rate.

a $y = 2^x$

b $y = 1 - 2x$

c $y = \log_2 x$

d $y = \frac{2}{x}$

ADV **22** Sketch a distance–time graph for the following scenarios.

a A sprinter running a race accelerates to a maximum speed and maintains that speed for the remaining distance of the race, then decelerates to a stop after passing the finishing line.

b A bus maintains a constant maximum speed for some time, and then decelerates to a stop for passengers to get on and off, then accelerates to the constant maximum speed again.

ADV **23** Use the substitution method to solve each of these pairs of simultaneous equations.

a $y = x^2 - 4x + 5$
$y = 3x - 1$

b $y = x^2 + 5x - 9$
$y = x^2 - 8x + 43$

c $x^2 + y^2 = 25$
$y = x + 1$

d $y = \frac{6}{x}$
$y = x + 5$

ADV **24 a** Complete the square for $4x^2 - 3x - 7$.

b Using your answer from part **a**, factorise $4x^2 - 3x - 7$.

ADV **25** Evaluate each of the following.

a $8^{\frac{2}{3}}$ **b** $81^{\frac{1}{4}}$ **c** $1000^{-\frac{4}{3}}$

d $\log_2(128)$ **e** $\log_9(3)$ **f** $\log_{10}\left(\frac{1}{100\,000}\right)$

ADV **26** For each of the following:

a write each of the numbers in scientific notation

b state the order of magnitude of each number

c determine the position of values on a base-ten logarithmic scale, relative to 1, correct to three decimal places.

i 1793.85 **ii** 5 105 232.79 **iii** 0.000 001 279 3

ADV **27** Solve each of the following equations.

a $\frac{1-2x}{3} = 5$ **b** $4 + 3(2x - 5) = 11$

c $2(x - 5) - 7(x - 4) = -9$ **d** $10(3x - 4) = 5(3 - 7x)$

e $\frac{4x+5}{5} = \frac{6x-5}{3}$ **f** $\frac{x+1}{2} + \frac{4-x}{6} = \frac{x+4}{4}$

g $\frac{9}{3x-1} = -5$ **h** $\frac{12}{x-5} = \frac{9}{x+7}$

ADV **28** Evaluate and/or simplify each of the following.

a $\frac{a^{\frac{5}{4}} \times a^{\frac{2}{3}}}{a^{\frac{5}{6}}}$ **b** $\left(b^{\frac{2}{3}}c^{\frac{5}{2}}\right)^6$

c $3\log_2(d) + \log_2(2d) - \log_2(d^2)$ **d** $\log_5\left(\frac{1}{e}\right) + \log_5(5) + \log_5(1)$

e $\sqrt{180}$ **f** $\sqrt{2} + \sqrt{72} + \sqrt{27} + \sqrt{50} + \sqrt{3} + \sqrt{12}$

g $\sqrt{75} \times \sqrt{6}$ **h** $\frac{\sqrt{120}}{\sqrt{5}}$

i $3\sqrt{6}\left(4\sqrt{6} + 9\sqrt{2}\right)$ **j** $\left(4\sqrt{3} - 2\sqrt{5}\right)\left(2\sqrt{3} + 3\sqrt{5}\right)$

ADV **29** Write each of the following with a rational denominator.

a $\frac{5}{\sqrt{3}}$ **b** $\frac{4+\sqrt{2}}{\sqrt{6}}$ **c** $\frac{3}{5-3\sqrt{2}}$ **d** $\frac{3+4\sqrt{5}}{2\sqrt{5}+4}$

ADV **30** For (2, 4) and (6, –8):

a determine the distance between the two points.

b determine the coordinate of the midpoint between the two points.

c determine the equation of the linear relationship passing through the two points, write your answer in the general form.

d determine the coordinates of the x- and y-intercepts of the relationship in part **c**.

e graph the relationship from part **c**.

ADV **31 a** Factorise $2x^2 - x - 15$. **b** Factorise $6x^2 - 7x - 20$.

ADV **32** Solve each of the following equations.

a $x^3 = -64$ **b** $(8x)^3 = 27$

c $x^3 - 0.025 = 0.1$ **d** $x^2 - 6x + 4 = 0$

e $4x^2 + 3x - 1 = 0$ **f** $3x^2 - 4x - 12 = 0$

ADV **33** For each of the functions below, evaluate:

i $f(0)$ **ii** $f(2)$ **iii** $f(-2)$.

a $f(x) = x + 3$ **b** $f(x) = 2x^2$

c $f(x) = 2x^2 + x + 1$ **d** $f(x) = 3^x - 1$

ADV **34** Factorise each of the following expressions.

a $16xy^2 - 24x^2y^2 + 40xy$ **b** $20xy - 5x + 16y - 4$ **c** $25x^2 - 49$

d $(x - 3)^2 - 9$ **e** $x^2 - 10x + 25$ **f** $x^2 + 4x + 3$

g $x^2 + 7x - 44$ **h** $3x^2 - 36x - 135$

ADV **35** Describe a sequence of transformations that could have been applied to make the transformed equation from the original each time.

	Original	**Transformed**
a	$y = x^2$	$y = -2x^2 + 8$
b	$y = x$	$y = \frac{3}{8}(x + 2)$
c	$y = 2^x$	$y = \frac{1}{2} \times 2^x - 4$
d	$y = \frac{1}{x}$	$y = \frac{1}{4(x - 2)} + 1$
e	$y = x^3$	$y = -(x + 1)^3 + 8$
f	$x^2 + y^2 = 1$	$(x - 2)^2 + (x - 4)^2 = 1$
g	$y = 3^x$	$y = -\log_3 x$

ADV **36** Sketch the graphs of the transformed equations from question **35**.

ADV **37** Simplify each of the following. Write your answers with positive indices only.

a $\dfrac{x^6 y^{-5} \times x^{-4} y^{-11}}{x^{-3} y^3 \times x y^{-2}}$ **b** $(2x^{-3}y^5)^{-2} \times \left(\dfrac{y^5}{4x^3}\right)^2$ **c** $\dfrac{12^0 x^5 y^{-4}}{4^{-2}(xy)^{-4}} \div \dfrac{6^2 x^{-5} y}{12^2 (x^{-1}y)^3}$

ADV **38** Expand and simplify each of the following.

a $(2x - 7y)(8a - 7b)$ **b** $(x - 4)(x + 7) + (x + 4)(x + 3)$

c $(5x + 2)(3x - 4) + 3(2x + 5)$ **d** $(10x + 3)^2$

e $(x - 5)(3a + 8b + 2c)$ **f** $\left(b - \frac{a}{4}\right)\left(2a - \frac{4}{3} - b\right)$

ADV **39** Simplify each of the following.

a $\dfrac{6x + 3}{2} \times \dfrac{2x}{2x + 1}$ **b** $\dfrac{(x + 10)(x - 4)}{(x - 4)^2} \times \dfrac{(x + 10)(x - 3)}{(x - 2)(x - 3)}$

c $\dfrac{x^2 + 12x + 35}{x^2 + 7x + 10} \times \dfrac{x^2 - 4}{4x + 28}$ **d** $\dfrac{x^2 - 5x + 6}{x^2 + 6x - 16} \div \dfrac{x^2 - 9}{x^2 - 6x + 8}$

ADV **40** Solve the following linear inequalities.

a $2 > 3x + 1$ **b** $23 < -3x + 2$ **c** $2 + x \le 14 - 3x$

d $2x \le 14 - 5x$ **e** $\dfrac{x + 6}{12} > \dfrac{5}{6}$ **f** $\dfrac{x + 6}{3} < \dfrac{3x - 2}{6}$

ADV EXT **41** If $P(x) = x^3 - 3x^2 + 5x - 4$, evaluate each of the following.

a $P(-3)$ **b** $P\left(\frac{1}{2}\right)$ **c** $P(\sqrt{2})$ **d** $P(2\sqrt{3})$

ADV EXT **42** For each of the following polynomials, state the:

i leading term **ii** degree **iii** leading coefficient.

a $P(x) = -6x^5 + 4x^4 - 10x^3 + 5x^3 + 2x^2 + x - 8$

b $Q(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

c $f(x) = \dfrac{3x^2 - 7x^4 - x}{2}$

ADV EXT **43** Perform each of the following polynomial operations. Write the answers to parts **e–h** in the form $Q(x) + \dfrac{R(x)}{D(x)}$.

a $(x^3 + 8x^2 + 9x + 5) + (x^3 - 5x^2 + 4x - 2)$ **b** $(x^3 + 8x^2 + 9x + 5) - (x^3 - 5x^2 + 4x - 2)$

c $(3x^2 + 5x - 7) \times (2x + 3)$ **d** $(x^2 - 4x + 5) \times (x^2 + 5x - 3)$

e $(3x^2 - 6x + 5) \div (x - 3)$ **f** $(x^3 + 5x^2 - 7x - 6) \div (x + 2)$

g $(x^3 + 2x - 1) \div (x + 4)$ **h** $(x^3 + 5x^2 - 7x + 2) \div (x + 5)$

ADV EXT **44** Sketch the graph of each of the following equations.

a $y = 6x^2 - 7x - 10$ **b** $y = (x - 3)(x + 1)(x + 2)$

c $y = (x - 2)^2(x + 2)$ **d** $y = (x - 4)^2(x + 2)^2$

e $y = (x + 1)^3(x - 1)$ **f** $y = x^3 + 7x^2 + 7x - 15$

ADV EXT **45** **a** Given that $x + 3$ is a factor, factorise $x^3 + 9x^2 + 23x + 15$.

b Given that $x - 2$ is a factor, factorise $x^3 - 3x^2 - 28x + 60$.

c Given that $P(5) = 0$, factorise $P(x) = x^3 - 19x - 30$.

ADV EXT **46** Solve each of the following equations.

a $(2x + 5)(3x - 10)(5 - x)(x - 4) = 0$ **b** $x^3 + 6x^2 + 5x - 12 = 0$

Analysis

1 Surjan recently received a $1000 bonus and is considering two investments. They are listed below.

- Investment A: A simple interest rate of 7% per annum
- Investment B: A compound interest rate of 5% per annum, compounding annually

a If Surjan plans to invest his $1000 for 10 years, which investment should he choose?

b Write the rule that gives the value, V_A and V_B, of each investment after n years.

c Complete the table of values on the right to show the performance of each investment. Write each value correct to the nearest cent.

n	0	5	10	15	20
V_A					
V_B					

d Use the values from your completed table to help you sketch graphs on the same set of axes showing the value of each investment against the number of years, for $0 \le n \le 20$.

e Use the graphs you drew for part **d** to help you determine after which year Investment B would be worth more than Investment A.

f After how many years will Investment A be worth more than Investment B is at 5 years?

g Surjan is offered an alternative investment, described below.

Investment C: Simple interest rate of 5% per annum, with $150 added to the account when it is first opened.

i After how many years would Investment A and Investment C be worth the same amount?

ii What would be the value of each investment at that time?

h After how many years would Investment B be worth the same amount as Investment A was worth after 5 years? Write your answer:

i as an exact value **ii** correct to two decimal places.

STN ADV **2** Air resistance in measured in units of force called Newton (N). The air resistance on an object is directly proportional to the object's velocity when wind is moving smoothly in parallel lines (laminar flow). It is also directly proportional to the square of the object's velocity when wind is moving chaotically (turbulent flow).

a During laminar flow, the air resistance on an object travelling at 10 m/s is 25 N.

i Determine the constant of proportionality and write the equation relating the laminar flow air resistance, R_L, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

b During turbulent flow, the air resistance on the object travelling at 10 m/s is 60 N.

i Determine the constant of proportionality and write the equation relating the turbulent flow air resistance, R_T, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

c Sketch the graphs of the two air resistance equations on the same pair of axes.

d Determine the speeds at which the air resistance during laminar and turbulent flow are the same.

e The time it takes for the object to travel a set distance is inversely proportional to its velocity. If an object moves at 8 m/s and takes 12.5 seconds to travel a set distance, then determine the constant of proportionality and write the equation for the velocity, v, of the object in terms of the time, t, it takes to travel the set distance.

f What is the set distance that is used to determine the relationship between velocity and time in part **e**?

g Sketch the graph of the velocity versus the time it takes to travel the set distance.

h Can the time it takes to travel the set distance ever equal zero? Explain.

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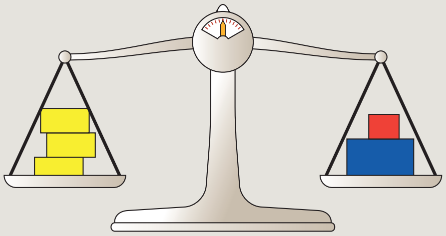
EXPLORATIONS 1

1 Balancing bricks

Indy has a collection of coloured plastic bricks and a balance scale. All bricks of the same colour weigh the same, but bricks of different colours weigh different amounts.

a Indy discovers the following about red, blue, yellow and green bricks:



- 1 blue brick weighs the same as 1 yellow and 1 green brick
- 3 yellow bricks weigh the same as 1 blue and 1 red brick
- 1 blue brick weighs more than 1 green and 3 red bricks
- 14 red bricks weigh more than 1 blue brick
- at least one combination of the other bricks weighs the same as 1 yellow brick.

Find all such combinations.

b Indy discovers the following about black, white, orange and purple bricks:

- 1 orange brick weighs the same as 1 white and 1 black brick
- 3 white bricks weigh the same as 1 orange and 1 purple brick
- 2 white bricks weigh less than 5 purple bricks
- only one combination of the other bricks weighs the same as 1 black brick. Find that combination.

2 Times tables

In each of the following partially completed multiplication tables, the letters A, B, C, … represent different whole numbers, subject to some extra conditions. Find the value of each letter and complete the table.

a Letters A to J represent the numbers 1 to 10, in some order. The sum of numbers A to E is equal to the sum of numbers F to I.

×	*A*	*B*	*C*	*D*	*E*
F	18				
G				20	
H					
I					24
J			10		

b Letters A to H represent the even numbers from 2 to 16, in some order. The sum of E, G and H is twice the sum of A, B, C and D.

×	*A*	*B*	*C*	*D*
E		40		
F			24	
G				112
H				

c Letters A to I represent the numbers 2 to 10, in some order. The sum of the squares of A, B and C is equal to the sum of the squares of D, E, G, H and I.

×	*A*	*B*	*C*	*D*	*E*
F			30		
G		28			
H				27	
I					

3 How irrational!

a Rationalise the denominator to simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}}$.

b Let a and b be positive integers and consider the fraction $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.

- **i** Find, in terms of a and b, an alternative expression for this fraction in which the denominator is an integer.
- **ii** Does your answer always make sense? If not, describe when it fails and find the correct answer in this case.

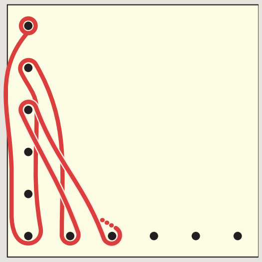
c Rationalise the denominator twice to simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

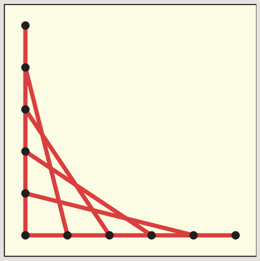
d Let a, b and c be positive integers with $c = a + b$ and consider the fraction $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

- **i** Find, in terms of a, b and c, an alternative expression for this fraction in which the denominator is an integer.
- **ii** Use your rule to simplify $\dfrac{1}{2\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

4 String art

If you have access to a pegboard and string, try the following hands-on activity first. Alternatively, you can use free online geometry software, such as GeoGebra *or* Desmos. *For best results, use as many pegs (or virtual pegs) as possible.*





On a pegboard, vertical and horizontal rows of equally spaced pegs are arranged in an L-shape. A piece of string is looped from the first (top) peg of the vertical row to the first (left-most) peg of the horizontal row, then to the second vertical peg, second horizontal peg, third vertical peg and so on – see the first diagram. When the string is pulled tight, it forms the design in the second diagram. The larger the number of pegs used, the smoother the curve formed by the boundary of the intertwined string. What is this curve?

To model this mathematically, it is best to rotate the board 45° anti-clockwise.

a Follow these instructions to make your own 'string art' on a sheet of grid paper:

- Draw x- and y-axes with the origin in the middle of the page.
- Plot the lines $y = x - 10$ and $y = -x - 10$, extending them to the edge of the page. (There is nothing special about the number 10. Later we will replace it with an arbitrary number, d.)
- Pick any point on one of the lines (for example, $(13, 3)$ on $y = x - 10$). Find the point on the other line whose y-coordinate is the negative of your first point's y-coordinate (for example, $(-7, -3)$ on $y = -x - 10$). Draw a line through your pair of points, extending it to the edge of the page.
- Repeat the previous step several times. As more and more lines are added, a familiar shape should emerge!

Let's prove that the curve produced by the lines in part **a** is the parabola $y = \frac{1}{40}x^2$. To do so, it is enough to show that each such line is a tangent to this parabola. To warm up, we start with a concrete example with two given points.

b **i** Find, in gradient–intercept form, the equation of the line through $(13, 3)$ and $(-7, -3)$.

ii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Now try the same strategy in more generality.

c Let $P(k + 10, k)$ be a point on the line $y = x - 10$, where k is a fixed but arbitrary number.

i Find the point Q on the line $y = - x - 10$ whose y-coordinate is $-k$.

ii Find, in gradient–intercept form, the equation of the line through P and Q.

iii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Finally, let's see how the parabola changes when the original 'lines of pegs' are changed.

d For some arbitrary fixed number $d \neq 0$, consider the lines $y = x - d$ and $y = -x - d$ with points $P(k + d, k)$ and $Q(k - d, -k)$, respectively. There is some parabola of the form $y = ax^2$ which, for all values of k, intersects the line through P and Q exactly once. Verify this claim by finding an expression for a in terms of d.

5 Fifth degree

a Calculate the first five fifth powers: 1^5, 2^5, 3^5, 4^5 and 5^5. What do you notice about their last digits? Does this always happen?

b Fully factorise the expression $n^5 - n$. Hence show that $n^5 - n$ is divisible by 30 for all integers n. Hint: if n, $n - 1$ and $n + 1$ are not divisible by 5, then n is of the form $5k \pm 2$ for some integer k.

c How does the result of part **b** answer the final question of part **a**?

d Show that $n^5 - n$ is divisible by 240 for all odd integers n.

e For what values of n is $n^5 - n$ divisible by 9?

f Deduce from part **b** that every integer has the same final digit as its ninth power. For what other powers is this true?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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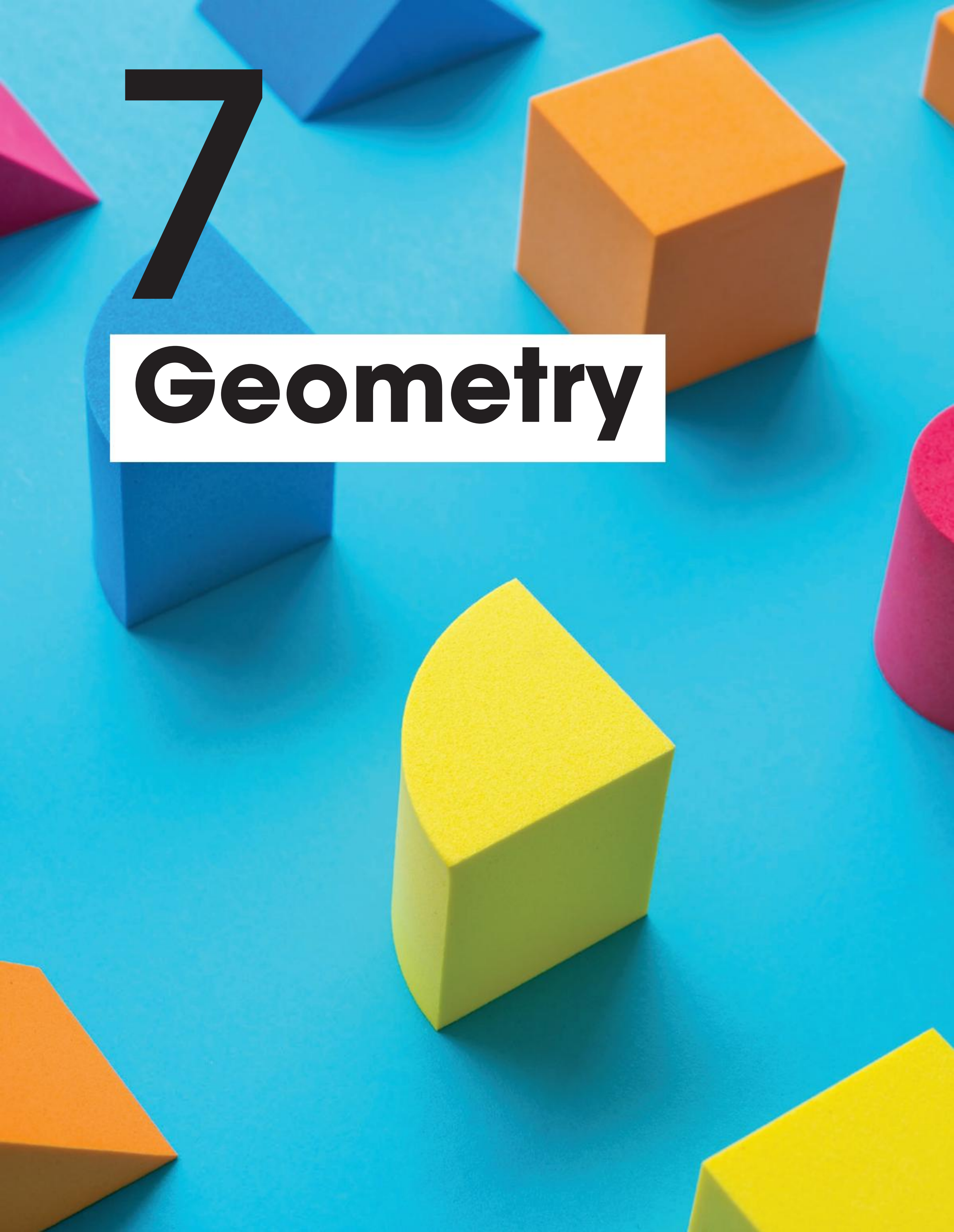
7

Geometry



Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Triangle properties
✔ Quadrilateral properties

Curriculum links

- EXT Establishes conditions for congruent triangles and similar triangles and solves problems relating to properties of similar figures and plane shapes (MA5-GEO-P-01)
  - Apply logical reasoning to numerical problems involving plane shapes
- EXT Constructs proofs involving congruent triangles and similar triangles and proves properties of plane shapes (MA5-GEO-P-02)
  - Construct formal proofs involving congruent and similar triangles
  - Apply logical reasoning to proofs involving plane shapes
- EXT Applies deductive reasoning to prove circle theorems and solve related problems (MA5-CIR-P-01)
  - Prove and apply angle and chord properties of circles
  - Prove and apply tangent and secant properties of circles

Materials

✔ Calculator

7A Geometrical properties

Learning intentions

By the end of this topic you will be able to …

✔ use geometrical properties to solve problems.

Inter-year links

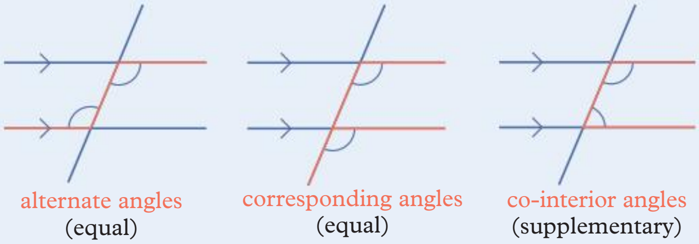
Year 7 7B Angles at a point

Year 8 7A Angles

Key content video

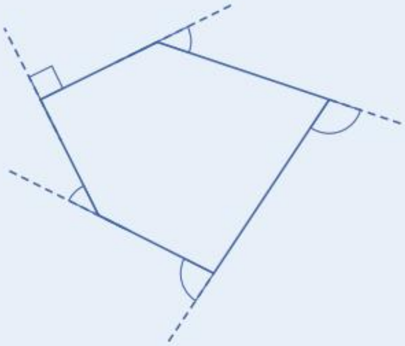
Angles

- **Complementary angles** sum to 90°.
- **Supplementary angles** sum to 180°.
- Angles at a point sum to 360°.
- **Vertically opposite angles** are equal.
- When parallel lines are crossed by a **transversal**, a number of equal and supplementary angles are formed.



Polygons

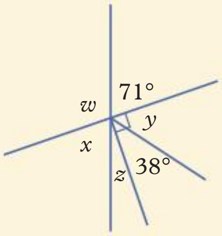
- A **polygon** is a closed shape with straight sides.
  → A **regular polygon** has all sides equal in length and all **interior angles** equal in size.
- A **triangle** is a three-sided polygon with an interior angle sum of 180°.
- The **angle–side relationship** states that the longest side of a triangle is always opposite the largest angle in a triangle.
- The **triangle inequality** states that the sum of the lengths of any two sides of a triangle will always be greater than the length of the third side.
- A **quadrilateral** is a four-sided polygon with an interior angle sum of 360°.
- The interior angle sum of any polygon is $(n - 2) \times 180°$, where n is the number of sides of the polygon.
- A **convex polygon** is a polygon in which all interior angles are less than 180°.
- The **exterior angles** of a convex polygon can be identified by extending the sides of the polygon beyond the vertices of the polygon.



  → The sum of an interior angle and the adjacent exterior angle is 180°.
- The sum of the exterior angles of a convex polygon is 360°.

Example 7A.1 Finding unknown angles using angle properties

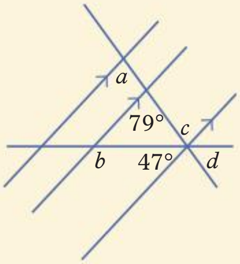
Find the size of the labelled unknown angles in this diagram.



THINK	**WRITE**
1 Angle *w* is supplementary to the 71° angle.	$w = 180° - 71°$ $= 109°$
2 Angle *x* is vertically opposite the 71° angle.	$x = 71°$
3 Angle *y* is complementary to the 38° angle.	$y = 90° - 38°$ $= 52°$
4 Angle *z* is supplementary to the angles 71°, *y* and 38°.	$z = 180° - 71° - 52° - 38°$ $= 19°$

Example 7A.2 Finding sizes of angles using angle relationships with parallel lines

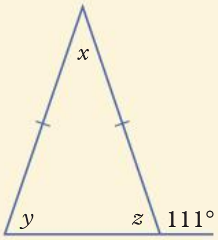
Find the values of the pronumerals in this diagram.



THINK	**WRITE**
1 Angle *a* is a corresponding angle to the 79° angle. Corresponding angles on parallel lines are equal.	$a = 79°$
2 Angle *b* is co-interior with the 47° angle. Co-interior angles on parallel lines are supplementary.	$b = 180° - 47°$ $= 133°$
3 Angle *c* is alternate to the 79° angle. Alternate angles on parallel lines are equal.	$c = 79°$
4 Angle *d* is supplementary to angle *c* and the unlabelled angle beside it, which is vertically opposite and, therefore, equal to 47°.	$d = 180° - 79° - 47°$ $= 54°$

Example 7A.3 Using triangle properties to find the sizes of angles

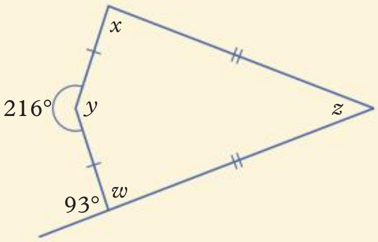
Find the value of each pronumeral in this diagram.



THINK	**WRITE**
1 Angle z is supplementary to the 111° angle.	$z = 180° - 111°$ $= 69°$
2 Angle y is equal to angle z because the triangle is isosceles. Base angles of an isosceles triangle are equal.	$y = z = 69°$
3 Angle x is supplementary to angles y and z because the interior angle sum of a triangle is 180°.	$x = 180° - 69° - 69°$ $= 42°$

Example 7A.4 Using quadrilateral properties to find sizes of angles

Find the value of each pronumeral in the kite on the right.



THINK	**WRITE**
1 Angle w is supplementary to 93°.	$w = 180° - 93°$ $= 87°$
2 Angle x is equal to angle w because the opposite angles formed from the unequal sides of a kite are equal.	$x = w = 87°$
3 Angle y and the angle labelled 216° meet at a point, therefore they add to 360°.	$y = 360° - 216°$ $= 144°$
4 Angles w, x, y and z add to 360° because the interior angle sum of a quadrilateral is 360°.	$z = 360° - 144° - 87° - 87°$ $= 42°$

Helpful hints

✔ When calculating unknown angles, you might not be given the required information to determine all the unknown angles (at first). You may need to use the angle sizes you can calculate to help you find the angle sizes you are unable to determine.

EXTENSION

ANS p738

Exercise 7A Geometrical properties

△ 1–9, 11(a, c), 13, 15, 16

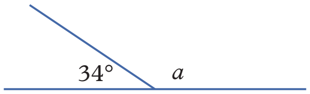
□ 1, 3–5, 8, 10, 11(b, d, e), 12, 14–16, 17(a, b), 18(b, d)

⬠ 3, 5, 6, 10, 11(d–f), 12, 14–16, 17(c), 18, 19

UNDERSTANDING AND FLUENCY

7A.1 **1** Find the sizes of the labelled unknown angles in these diagrams.

a



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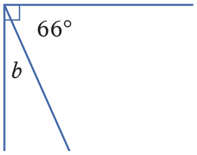

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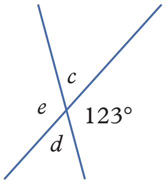
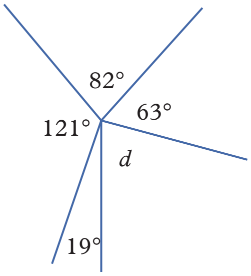
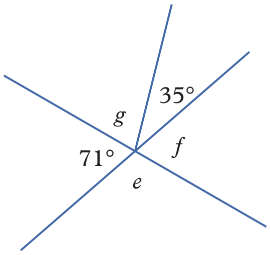


d



e



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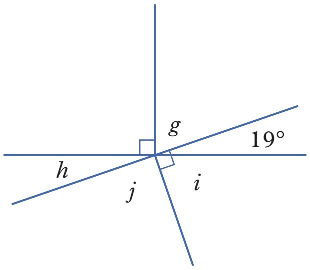
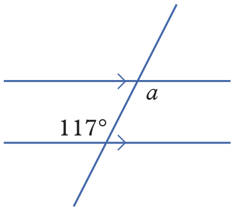


7A.2 **2** Find the value of each pronumeral in these diagrams.

a



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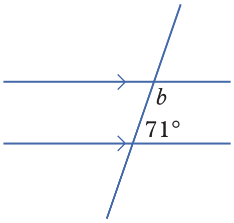

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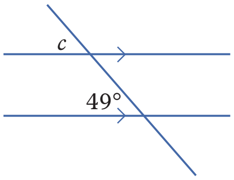

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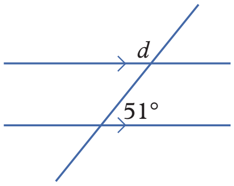

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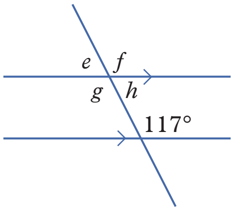

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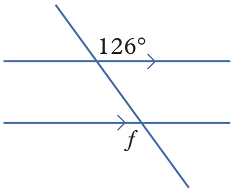
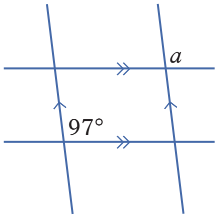


3 Find the value of each pronumeral in these diagrams.

a



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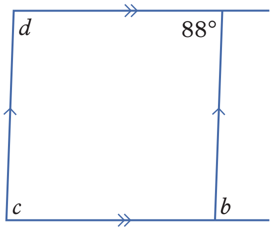

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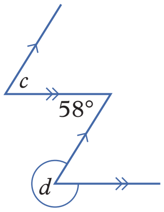

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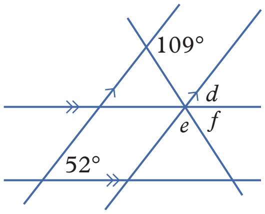

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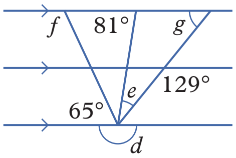

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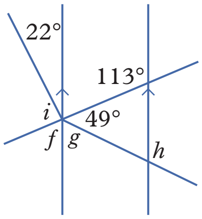
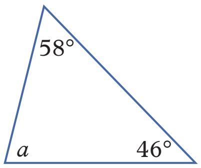


EXTENSION

UNDERSTANDING AND FLUENCY

7A.3 **4** Find the value of each pronumeral in these diagrams.

a



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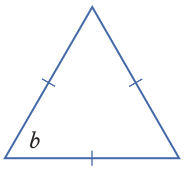

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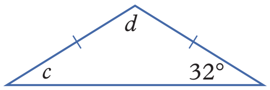

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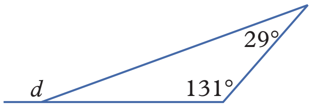

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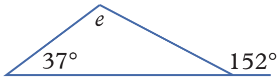

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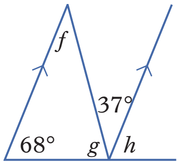
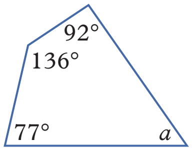


7A.4 **5** Find the value of each pronumeral in these diagrams.

a



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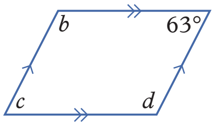

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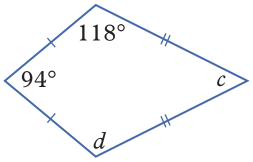

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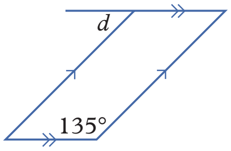

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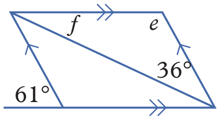

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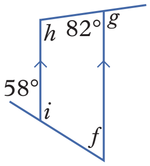
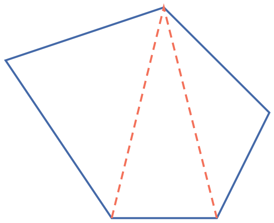
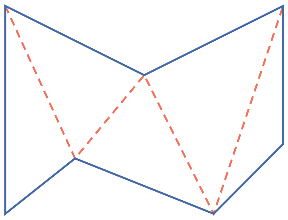


6 Use the expression $(n - 2) \times 180°$ to find the internal angle sum of a polygon with:

a 20 sides **b** 50 sides **c** 100 sides.

7 Every polygon can be dissected into triangles, as shown in the diagrams below.





a How many triangles can a polygon with n sides be dissected into?

b How does the result from part **a** explain the formula for the sum of the interior angles of a polygon?

PROBLEM SOLVING AND REASONING

8 Explain why you can use the formula $x = \frac{(n-2) \times 180°}{n}$ to find the size of an individual internal angle in any regular polygon.

9 Find the size of an individual internal angle in a:

a regular heptagon **b** regular nonagon **c** regular hexagon.

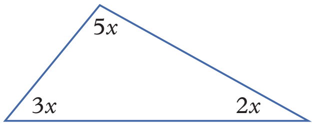
Round your answers to two decimal places where necessary.

10 Find the number of sides of a regular polygon that has each of its internal angles equal to:

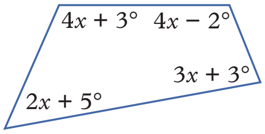
a 160° **b** 144° **c** 170° **d** 179.64°.

11 Find the value of x in each of these diagrams. Round your answers to two decimal places where necessary.

a



b



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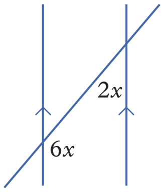

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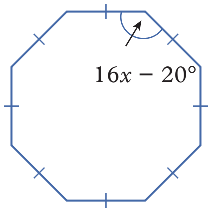

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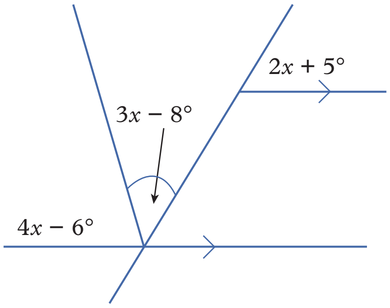

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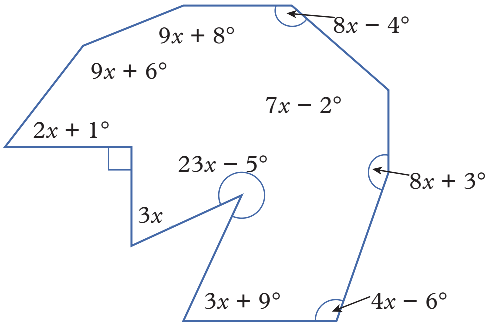
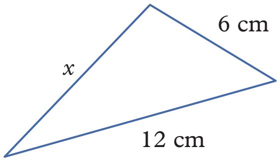


12 Explain why the triangle inequality is true for all triangles.

13 Use the triangle inequality to give a range of possible values for the unknown side lengths in these triangles.

a



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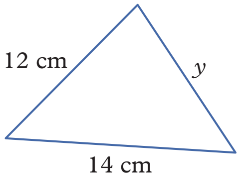

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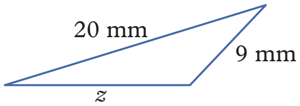
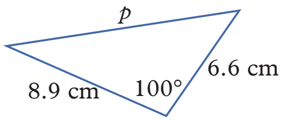
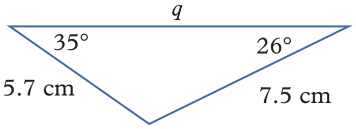


14 Use the angle–side relationship and the triangle inequality to give a range of possible values for the unknown side lengths in these triangles.

a



b



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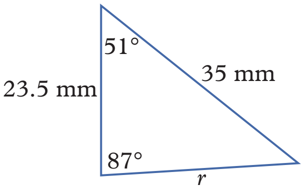


15 a Draw an equilateral triangle and extend each of the sides to create the exterior angles of the triangle. What is the size of each exterior angle of an equilateral triangle?

b What is the sum of the exterior angles of an equilateral triangle?

c Draw a square and extend each of the sides to create the exterior angles of the square. What is the size of the exterior angle of a square?

d What is the sum of the exterior angles of a square?

e Use your diagrams and answers to parts **a–d** to explain why the sum of the exterior angles of a convex polygon is equal to 360°.

16 Decide whether each of these statements is true or false. If false, write an equivalent true statement.

a Angles at a point add to 180°.

b Corresponding angles on parallel lines are supplementary.

c Complementary angles add to 180°.

d A quadrilateral can have a maximum of one concave angle.

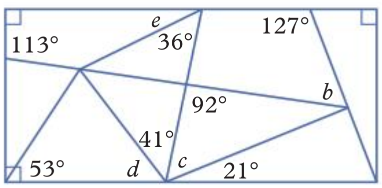
e The sum of an interior angle and the adjacent exterior angle is 360°.

f The number of sides of a polygon determines the interior angle sum of that polygon.

EXTENSION

17 Find the value of each pronumeral in these diagrams.

a



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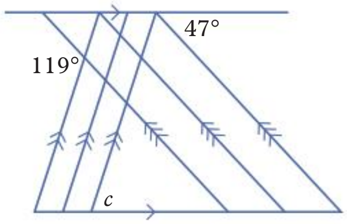

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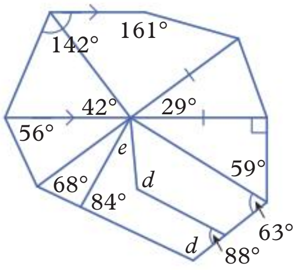
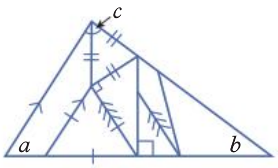


CHALLENGE

18 Find the value of each pronumeral in these diagrams.

a



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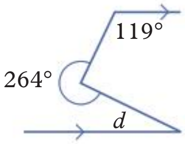

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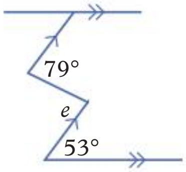

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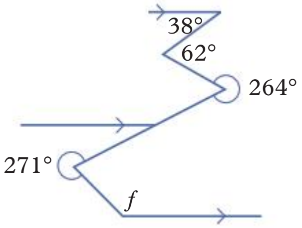

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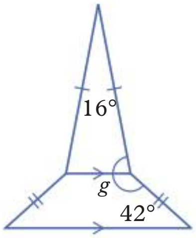

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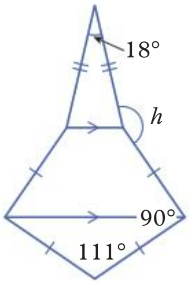


19 Explementary angles are a pair of angles that form a revolution; that is, they add to 360°. Determine the size of each of the following angles:

a the explement of 30°

b the explement of 175°

c the explement of the supplement of 45°

d the explement of the complement of 81°

e the complement of the explement of 324°

f the supplement of the explement of 267°

g the complement of the supplement of the explement of 200°

h the explement of the angle alternate to 64° on parallel lines

i the explement of the angle co-interior to 145° on parallel lines

j the explement of the exterior angle of a regular hexagon.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Angles at a point

Interactive skillsheet
Angles and parallel lines

Topic quiz
7A

7B Congruence and similarity

Learning intentions

By the end of this topic you will be able to ...

- ✔ find unknown side lengths and angles in congruent and similar triangles
- ✔ identify whether a pair of triangles are congruent or similar
- ✔ prove that a pair of triangles are congruent or similar.

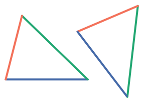
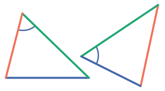
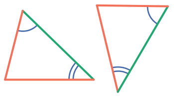
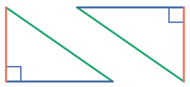
Inter-year links

Year 7	7D Classifying triangles
Year 8	7B Triangles
Year 9	7E Congruence

Key content video

Congruent triangles

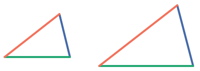
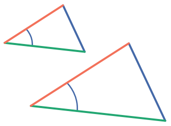
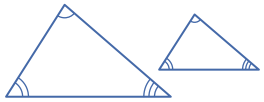
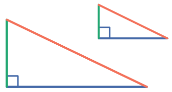
- **Congruent** figures are identical in shape and size but can be in any position or orientation.
  - → The symbol for congruence is ≡ (‘is congruent to’).
- If two triangles meet any of the four conditions shown in the table below, they are congruent.

SSS **(side–side–side)**	**SAS** **(side–angle–side)**	**AAS** **(angle–angle–side)**	**RHS** **(right angle–hypotenuse–side)**
All three side lengths are equal. 	Two of the side lengths are equal and the included angle (between those sides) is equal. 	Two of the angles and a corresponding pair of side lengths are equal. 	For right-angled triangles, the hypotenuses and another pair of side lengths are equal. 

- If a pair of triangles fails any of the four conditions for congruence shown above, they are not congruent.
- If $\triangle ABC \equiv \triangle DEF$, then vertex A corresponds to vertex D, vertex B corresponds to vertex E, and vertex C corresponds to vertex F.

Similar triangles

- **Similar** figures are identical in shape but can be different in size.
  - → The symbol for similarity is ~ (‘is similar to’).
- The scale factor between similar figures can be calculated using the formula: scale factor $= \dfrac{\text{image length}}{\text{original length}}$
- For two triangles to be similar, they must meet one of four conditions of similarity.

SSS **(side–side–side)**	**SAS** **(side–angle–side)**	**AAA** **(angle–angle–angle)**	**RHS** **(right angle–hypotenuse–side)**
The lengths of all three pairs of corresponding sides are in the same ratio. 	The lengths of two pairs of corresponding sides are in the same ratio and the included angles are equal. 	Each of the three angles in one triangle is equal to an angle in the other triangle. 	In two right-angled triangles, the lengths of their hypotenuses are in the same ratio as the lengths of another pair of corresponding sides. 

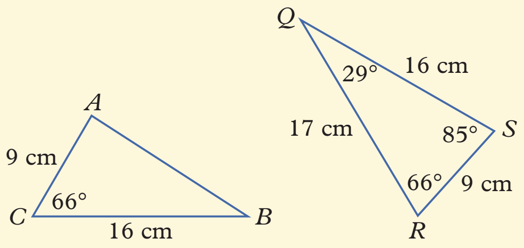
Mathematical proofs

- A mathematical proof explains why something is true through a series of logically connected statements.
- Mathematics proofs use self-evident or given statements. The following list of statements can be used to create the proofs in this section:
  - → Vertically opposite angles are equal.
  - → Alternate angles on parallel lines are equal.
  - → Corresponding angles on parallel lines are equal.
  - → Co-interior angles on parallel lines are supplementary.
  - → The interior angle sum of a triangle is 180°.
  - → If $a = b$ and a = c, then $b = c$ (transitive law).
  - → The same number can be added or subtracted from both sides of an equation to create an equivalent equation (addition and subtraction properties of equality).
- Properties of different types of triangles can also be used.
- The symbol $\therefore$ is shorthand for 'therefore'.
- The letters 'QED' or the 'tombstone', ■ or □, are often written at the end of a proof. The acronym QED stands for the Latin phrase 'quod erat demonstrandum', which means 'which was to be demonstrated'.
- This textbook uses two-column proofs, with statements given in the left-hand column, and reasons why each statement is true given in the right-hand column.

Statements	**Reasons**

Example 7B.1 Identifying whether two triangles are congruent

Decide whether these triangles are congruent, giving a reason for your answer.



THINK

1 Look at the given information to determine which congruence condition to use. We are given the sizes of two sides and an included angle for $\triangle ABC$. This fits the SAS condition for congruence.

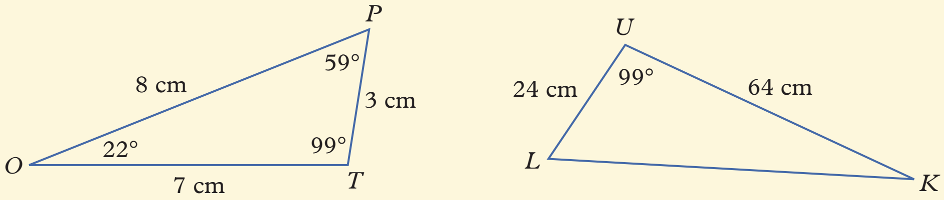
WRITE

$\triangle ABC$ has two sides and an included angle, which fits the SAS condition for congruence.

2 Identify the corresponding sides in the second triangle.	*RS* corresponds to *AC*. *BC* corresponds to *QS*.
3 Compare the sizes of the included angles between the matching sides in each triangle.	$\angle ACB = 66° \neq \angle RSQ$.
4 Write a conclusion.	The two triangles are not congruent because they fail the SAS condition for congruence.

Example 7B.2 Identifying whether two triangles are similar

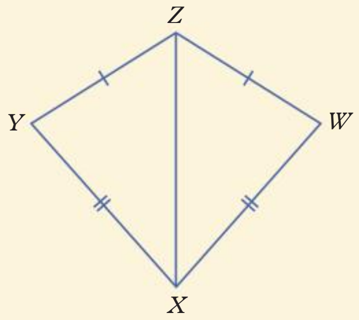
Decide whether these triangles are similar, giving a reason for your answer.



THINK	**WRITE**
1 Look at the given information to determine which similarity condition to use. $\triangle LUK$ has two sides and an included angle, which fits the SAS condition for similarity.	$\triangle LUK$ has two sides and an included angle, which fits the SAS condition for similarity.
2 Use the given angle in $\triangle LUK$ to find the corresponding sides.	$\angle PTO = \angle LUK = 99°$ *PT* corresponds to *LU*. *TO* corresponds to *UK*. *OP* corresponds to *KL*.
3 Determine the scale factor used for one of the pairs of matching sides (*LU* and *PT*).	$\text{scale factor} = \frac{\text{image length}}{\text{original length}}$ $= \frac{LU}{PT}$ $= \frac{24}{3}$ $= 8$
4 Find the scale factor used for the other pair of matching sides (*TO* and *UK*).	$\text{scale factor} = \frac{\text{image length}}{\text{original length}}$ $= \frac{UK}{TO}$ $= \frac{64}{7}$ ≈ 9.14
5 Write a conclusion.	The two triangles are not similar because these two scale factors are not the same, meaning the corresponding side lengths are not in the same ratio. They fail the SAS condition for similarity.

Example 7B.3 Proving that two triangles are congruent

For the following diagram, prove that $\triangle XYZ \equiv \triangle XWZ$ using a congruence condition.



THINK

1 Set up a two-column table for your proof.

2 List the given information.

3 Determine whether the given information meets any of the conditions for congruence.

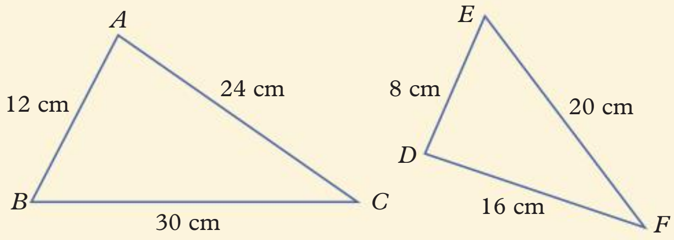
4 Complete the proof.

WRITE

Statements	Reasons
$XY = XW$	Given
$ZY = ZW$	Given
XZ is common to both triangles	Common side
$\therefore \triangle XYZ \equiv \triangle XWZ$ ■	SSS (congruent)

Example 7B.4 Proving that two triangles are similar

Prove that $\triangle ABC$ is similar to $\triangle DEF$ using one of the conditions for similarity.



THINK

1 Identify the corresponding side for the two triangles. For example, AB is the shortest side in $\triangle ABC$ and DE is the shortest side in $\triangle DEF$.

2 Calculate the scale factor for each pair of corresponding sides.

3 Determine whether the given information meets any of the conditions for similarity.

4 Complete the proof.

WRITE

Statements	Reasons
$\frac{DE}{AB} = \frac{8}{12} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{EF}{BC} = \frac{20}{30} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{DF}{AC} = \frac{16}{24} = \frac{2}{3}$	Scale factor between corresponding sides
$\therefore \triangle ABC \sim \triangle DEF$ ■	SSS (similar)

Helpful hints

- ✔ If two triangles meet one of the conditions for congruence, they will meet all the conditions for congruence. Likewise, if two triangles fail one of the conditions for congruence, they will fail all the conditions for congruence.
- ✔ When writing a congruence or similarity statement, the corresponding vertices in each triangle must be written in the same order.

ANS p738

Exercise 7B Congruence and similarity

△ 1–5, 6(a, c, d), 7, 8(a), 9(a), 10(a, b), 12, 15

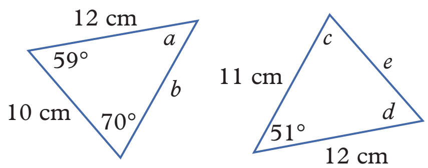
□ 1(c, d), 2(d–f), 3, 4, 5(d–f), 6, 8, 9(b), 10(c,d), 11, 13, 16, 17, 18(a)

⬠ 3, 4, 6(d–f), 7–9, 11, 14, 16, 18(b), 19, 20

UNDERSTANDING AND FLUENCY

1 Find the unknown side lengths and angles in each of these pairs of congruent triangles.

a



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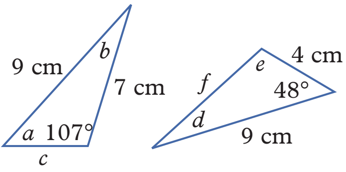

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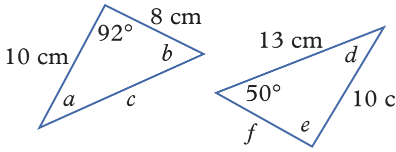
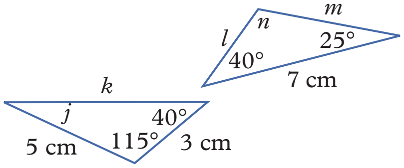
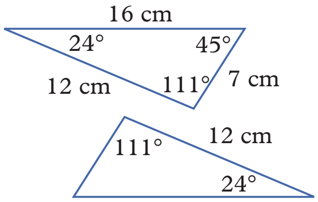


d



7B.1 **2** Decide whether the triangles in each of these pairs are congruent, giving a reason for your answer each time.

a



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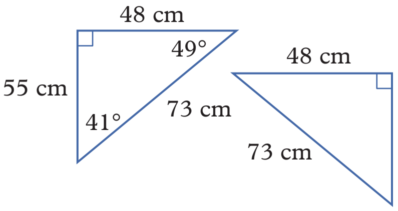

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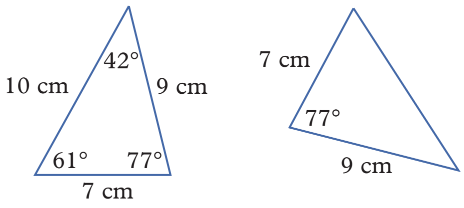

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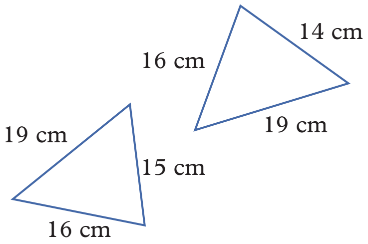

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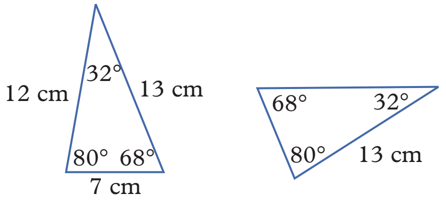

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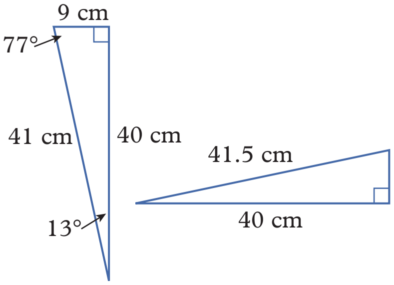
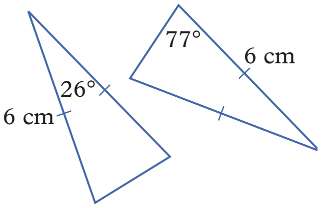


3 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are congruent.

a



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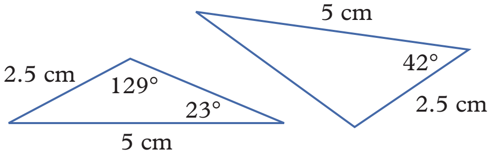

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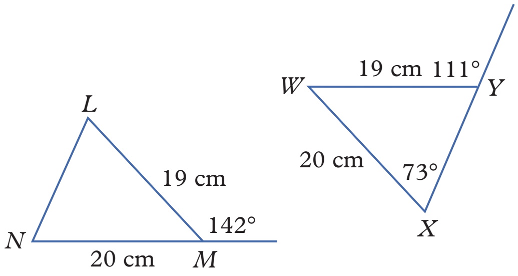

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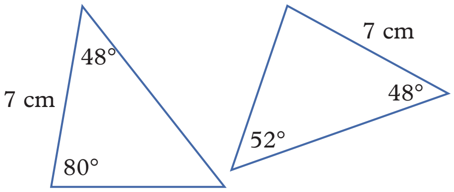
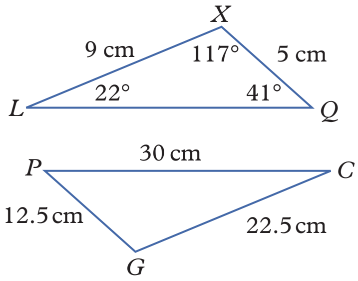


4 Find the scale factor between the triangles in each of these pairs of similar triangles. Assume that the first triangle in each pair is the original.

a



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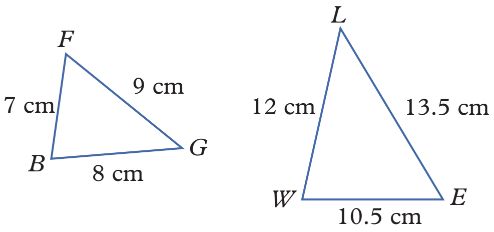

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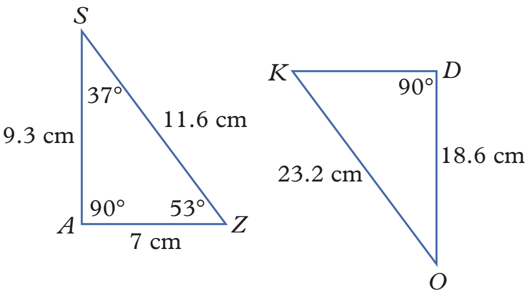

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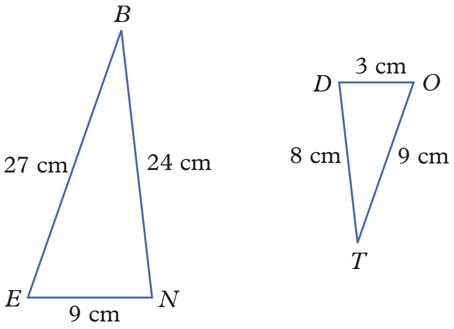
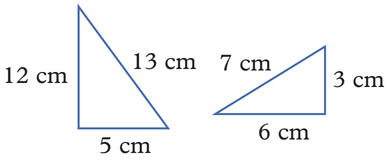


7B.2 **5** Decide whether the triangles in each of these pairs are similar, giving reasons for your answers.

a



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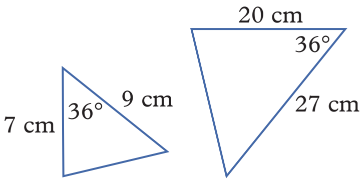

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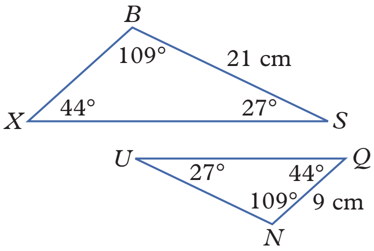

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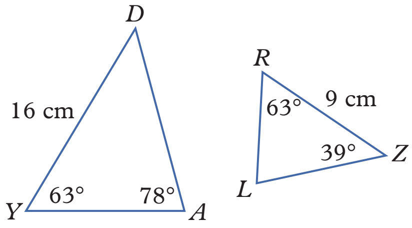

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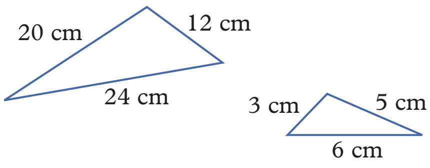

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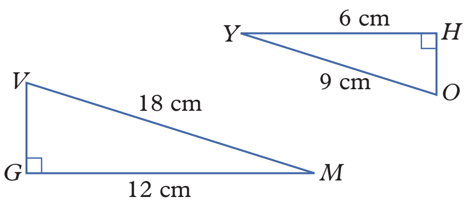
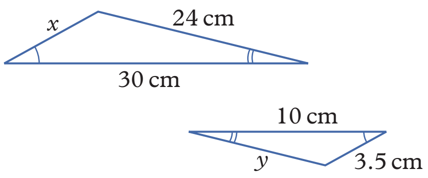


6 Find the unknown side lengths if the triangles in each pair below are similar.

a



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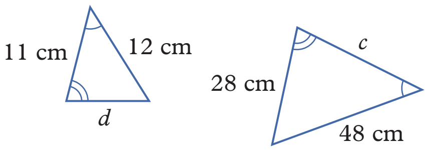

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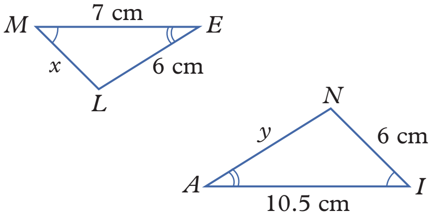

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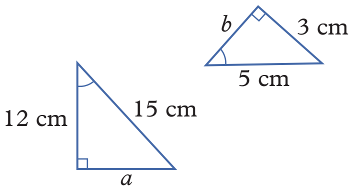

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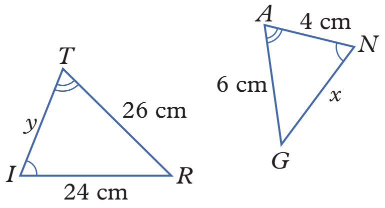

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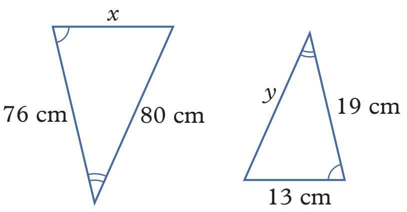
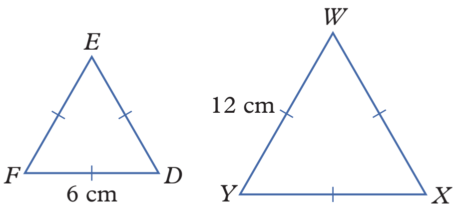


7 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are similar.

a



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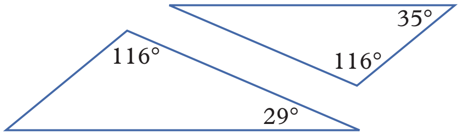

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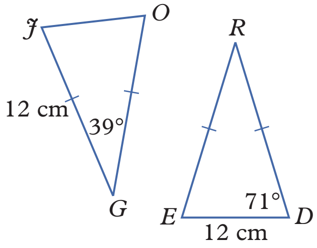

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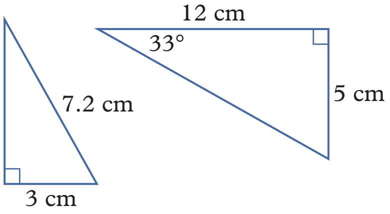
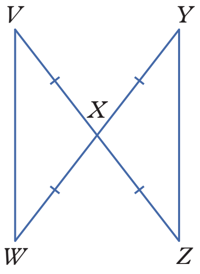
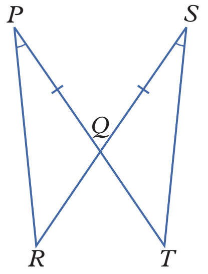


7B.3 **8** **a** For the diagram below, prove that $\triangle VWX \equiv \triangle YZX$.



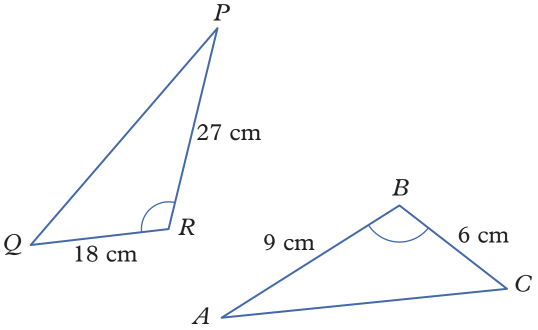
b For the diagram below, given that $\angle RPQ = \angle TSQ$, prove that $\triangle PQR \equiv \triangle SQT$.



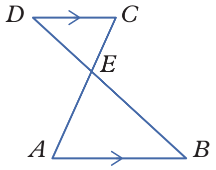
EXTENSION

PROBLEM SOLVING AND REASONING

7B.4 **9** **a** Given that $\angle ABC = \angle PRQ$, prove that $\triangle ABC \sim \triangle PRQ$.

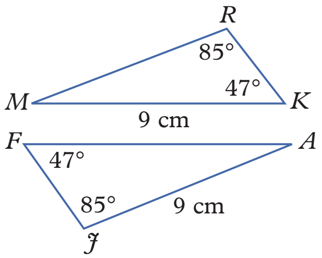


b Given that AB is parallel to CD, prove that $\triangle ABE \sim \triangle CDE$.



10 Lachlan said that the information given for the pairs of triangles below proves that they are congruent triangles. Explain why he is wrong in each case.

a



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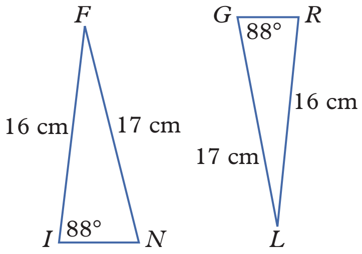

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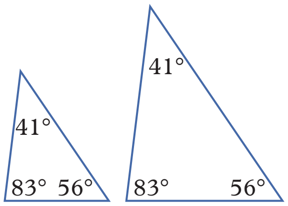

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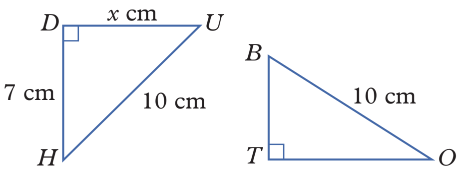


11 Decide whether each of these statements is true or false, giving reasons for your answers.

a If two triangles meet the condition AAA, they are not congruent.

b If two triangles have two corresponding sides of the same length and one corresponding angle of the same size, the triangles are congruent.

c A pair of triangles can meet one congruence condition but fail another.

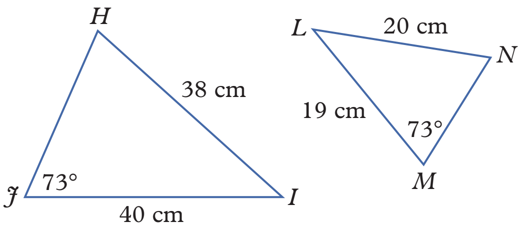
d If $\triangle ABC$ is congruent to $\triangle DEF$, and $\triangle DEF$ is congruent to $\triangle GHI$, then $\triangle GHI$ is congruent to $\triangle ABC$.

e If two quadrilaterals have all matching sides equal in length, then they are congruent.

f All equilateral triangles are congruent.

12 Jessica said that the triangles in each pair below could be proven to be similar. Explain why she is wrong in each case.

a



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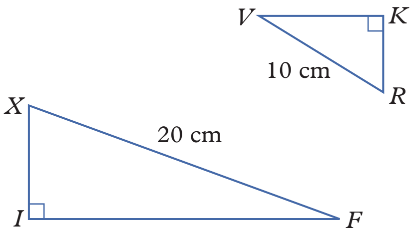
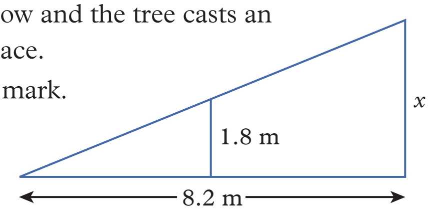


13 A 1 m ruler is placed upright next to a tree. If the ruler casts a 2.3 m shadow and the tree casts an 11.2 m shadow, how tall is the tree? Round your answer to one decimal place.

14 A ski jump is 8.2 m long. It has 1.8 m tall vertical supports at the halfway mark. How tall is the ski jump at its highest point?

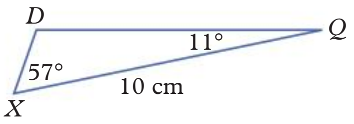


EXTENSION

PROBLEM SOLVING AND REASONING

15 Select two pairs of similar triangles from the options below. Provide reasons for your selection.

A



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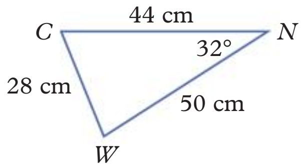
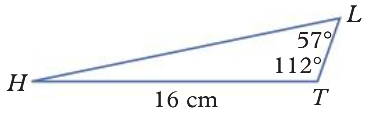
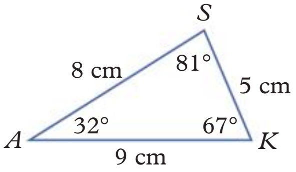


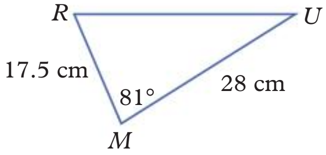
C



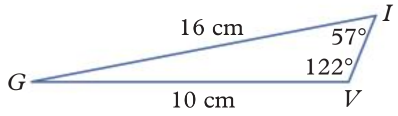
D



E



F



16 Decide whether each of these statements is true or false. Give reasons for your answers.

a If two triangles meet the condition AAA, they are not necessarily similar.

b If $\triangle ABC$ is similar to $\triangle DEF$, and $\triangle DEF$ is similar to $\triangle GHI$, then $\triangle GHI$ is similar to $\triangle ABC$.

c All isosceles triangles are similar.

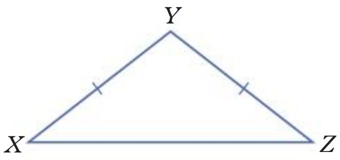
d All squares are similar.

e If two triangles fail the condition SAS, they are not necessarily similar.

f If two quadrilaterals have all sides in the same ratio, they are not necessarily similar.

17 If a triangle has two sides that are equal in length (meaning it is isosceles), then the angles opposite those sides are equal.

You can prove this fact using congruent triangles. To help you, consider $\triangle XYZ$ on the right.



a Draw a line from Y to the midpoint of XZ, and name this point W.

b Name the two triangles you have now formed.

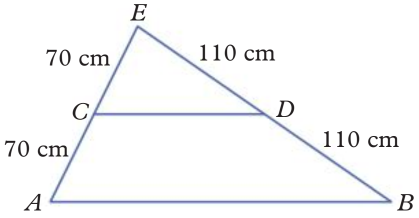
c Which pairs of sides do you know are equal in these two triangles?

d Name the side common to both triangles.

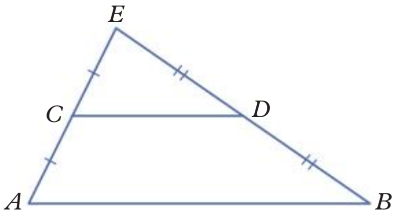
e Use a congruence condition to prove that $\triangle WXY \equiv \triangle WZY$.

f Explain why you now know that $\angle WXY = \angle WZY$.

18 **a** Prove that $\triangle ABE \sim \triangle CDE$.



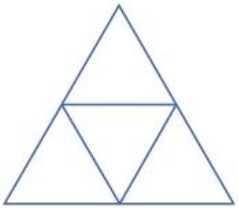
b Prove that $\triangle ABE \sim \triangle CDE$.



CHALLENGE

19 A flagpole has a guide wire attached at its midpoint and tethered to the ground 2.7 m away. A metre ruler, standing vertically, touches the guide wire when placed 75 cm away from where it is tethered to the ground. How tall is the flagpole?

20 Prove that drawing a triangle whose vertices are the midpoints of the sides of an equilateral triangle splits that equilateral triangle into four smaller congruent equilateral triangles. Hint: You will need to make use of your knowledge of isosceles triangles.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Congruent triangles

Interactive skillsheet
Similar triangles

Worksheet
Congruent and similar triangles

Topic quiz
7B

7C Geometric proofs

Learning intentions

By the end of this topic you will be able to ...

✔ prove triangle properties

✔ prove quadrilateral properties.

Inter-year links

Year 7 7E Classifying quadrilaterals

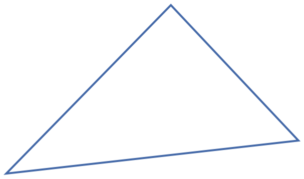
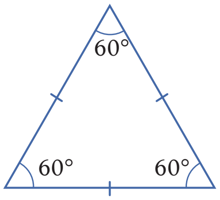
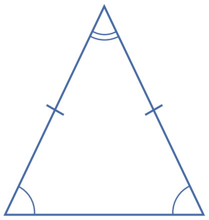
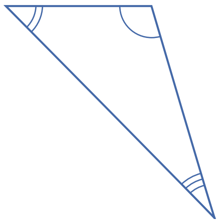
Year 8 7C Quadrilaterals

Key content video

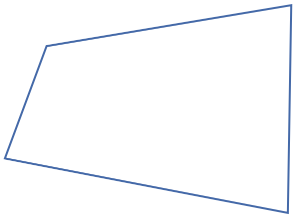
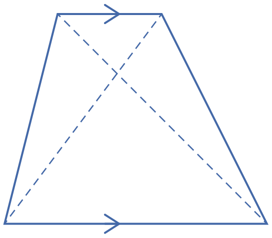
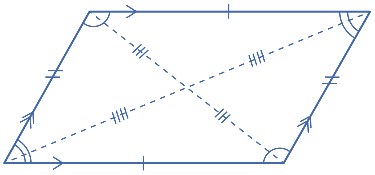
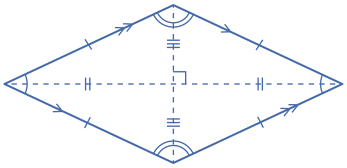
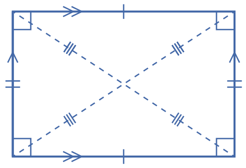
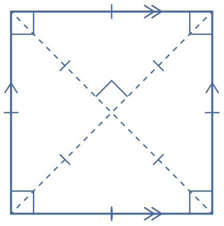
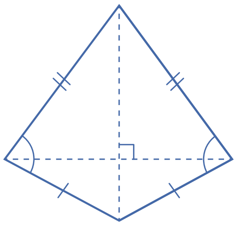
Geometric definitions

- A **definition** is the minimum amount of information required to identify a plane shape.
  → For example, the definition of a triangle is a three-sided polygon.
- A **property** is a characteristic that applies to a plane shape.
  → For example, a property of triangles is that the sum of the interior angles is 180°.
- 'Bisect' means to cut in half.
- The symbol for 'is perpendicular to' is ⊥.
- The symbol for 'is parallel to' is ||.

Triangle properties

Definition	Properties	Example
Triangle A three-sided polygon	The sum of the interior angles is 180° The exterior angle is equal to the sum of the two interior opposite angles	
Equilateral triangle A triangle with all sides equal in length	Each interior angle is 60°	
Isosceles triangle A triangle with exactly two sides equal in length	The angles opposite the equal side lengths are equal	
Scalene triangle A triangle with no two sides equal in length	No angles are equal	

Quadrilateral properties

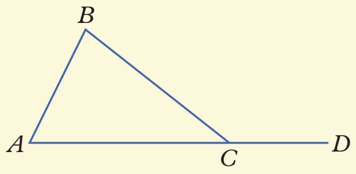
Definition	Properties	Example
Quadrilateral A four-sided polygon	The sum of the interior angles is 360° The exterior angle is equal to the sum of the two interior opposite angles	
Trapezium A quadrilateral with at least one pair of parallel sides	Diagonals are equal in length	
Parallelogram A quadrilateral with two pairs of parallel sides	Opposite sides are equal in length Opposite angles are equal Diagonals bisect each other	
Rhombus A parallelogram with two adjacent sides equal in length	All parallelogram properties All sides are equal in length Diagonals bisect each other at right angles Diagonals bisect the interior angles	
Rectangle A parallelogram with a right angle	All parallelogram properties Diagonals are equal in length All interior angles are right angles	
Square A rectangle with two adjacent sides which are equal in length	All parallelogram properties All rhombus properties All rectangle properties	
Kite A quadrilateral with two pairs of adjacent sides which are equal in length	One pair of opposite angles are equal Diagonals are perpendicular One diagonal bisects the other diagonal One diagonal bisects a pair of interior angles	

Geometric proofs

- A **theorem** is a statement or result that can be demonstrated to be true based on known facts.
- A **test** is a theorem that demonstrates a particular quality.
  - → For example, a test for parallelograms is that if both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- In addition to the given statements in 7B, the following statements can be used to create the proofs in this section:
  - → Angles at a right angle add to 90°.
  - → Angles on a straight line add to 180°.
  - → Angles at a point add to 360°.
  - → If point B lies in the interior of $\angle AOC$, then $\angle AOB + \angle BOC = \angle AOC$ (angle addition postulate).

Example 7C.1 Proving the exterior angle theorem

Use the following diagram to prove the exterior angle theorem, i.e. $\angle BCD = \angle ABC + \angle CAB$.



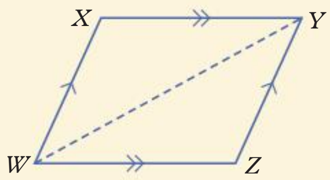
THINK

1. Set up a two-column table for the proof.
2. Use the fact that angles on a straight line sum to 180° to write an equation linking the exterior angle of the triangle to the adjacent interior angle of the triangle.
3. Write an equation linking the interior angles of the triangle.
4. Equate the left-hand sides of the two equations (because the right-hand sides are equal).
5. Simplify by subtracting $\angle BCA$ from both sides.

WRITE

Statements	**Reasons**
$\angle BCD + \angle BCA = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CAB + \angle BCA = 180°$	Interior angle sum of a triangle is 180°
$\angle BCD + \angle BCA = \angle ABC + \angle CAB + \angle BCA$	Transitive law
$\angle BCD = \angle ABC + \angle CAB$ ■	Subtraction property of equality

Example 7C.2 Proving opposite sides of a parallelogram are equal



Use the diagram on the right to prove that the opposite sides of a parallelogram are equal in length.

THINK

1 Set up a two-column table for your proof.

2 Use the fact that the diagonal of the parallelogram is common to both $\triangle WXY$ and $\triangle YZW$.

3 Use your knowledge of angles and parallel lines to identify equal angles.

4 Use a congruence condition to show that the triangles are congruent.

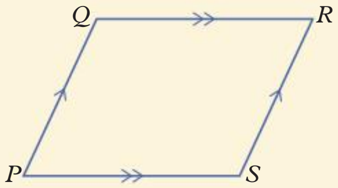
5 Show that the opposite sides of the parallelogram are equal in length (as corresponding sides of the congruent triangles).

6 Complete the proof with a closing statement.

WRITE

Statements	**Reasons**
YW is common to both $\triangle WXY$ and $\triangle YZW$	Common side
$\angle XYW = \angle ZWY$	Alternate angles on parallel lines are equal
$\angle YWX = \angle WYZ$	Alternate angles on parallel lines are equal
$\therefore \triangle WXY \equiv \triangle YZW$	Meets the AAS condition of congruence
$WX = YZ$	Corresponding sides of congruent triangles are equal
$WZ = YX$	Corresponding sides of congruent triangles are equal
∴ Opposite sides of a parallelogram are equal in length ■	

Example 7C.3 Proving opposite angles in a parallelogram are equal



Use the diagram on the right to prove that the opposite angles of a parallelogram are equal in size.

THINK

1 Use your understanding of angles and parallel lines to write two equations about co-interior angles, making sure the same angle appears in both equations.

2 Equate the left-hand sides of the equations you wrote (because the right-hand sides are equal).

3 Simplify by subtracting the same angle from both sides.

4 Repeat steps **1–3** for the other pair of co-interior angles.

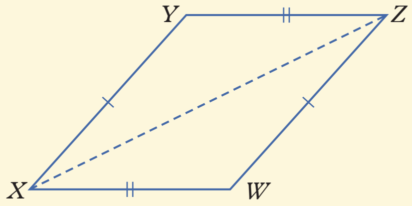
5 Complete the proof with a closing statement.

WRITE

Statements	Reasons
$\angle QPS + \angle PSR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PQR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PSR = \angle QPS + \angle PQR$	Transitive law
$\angle PSR = \angle PQR$	Subtraction property of equality
$\angle PSR + \angle QRS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QPS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QRS = \angle PSR + \angle QPS$	Transitive law
$\angle QRS = \angle QPS$	Subtraction property of equality
∴ Both pairs of opposite angles of the parallelogram are equal in size ■	

Example 7C.4 Proving a quadrilateral is a parallelogram

Use the diagram on the right to prove that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.



THINK

1 Show that the two triangles formed by the diagonal are congruent.

2 Identify the corresponding angles in these congruent triangles.

3 If alternate angles are equal, then the lines must be parallel. Note: || means 'is parallel to'.

4 Use the definition of a parallelogram to prove that *WXYZ* is a parallelogram.

WRITE

Statements	Reasons
$XY = ZW$	Given
$YZ = WX$	Given
XZ is common to both $\triangle XWZ$ and $\triangle ZYX$	Common side
$\therefore \triangle XWZ \equiv \triangle ZYX$	Meets the SSS condition of congruence
$\angle WXZ = \angle YZX$	Corresponding angles of congruent triangles are equal
$\angle XZW = \angle ZXY$	Corresponding angles of congruent triangles are equal
$YZ \parallel XW$	$\angle WXZ$ and $\angle YZX$ are alternate angles. If alternate angles are equal, they lie on parallel lines
$XY \parallel WZ$	$\angle XZW$ and $\angle ZXY$ are alternate angles. If alternate angles are equal, they lie on parallel lines
∴ *WXYZ* is a parallelogram ■	If a quadrilateral has two pairs of parallel sides, then it is a parallelogram

Helpful hints

- ✔ Be patient when constructing mathematical proofs. Start with the information that you are given and build on that information. You may not see the whole proof at first but, by taking it one step at a time, you will work to the required conclusion.
- ✔ If you are unsure where to begin when constructing a mathematical proof, start by writing all the information you are given (using mathematical terminology), as well as what you are trying to prove.
- ✔ Quadrilaterals, and other polygons, are labelled in a cyclic manner around the shape in order. That is, pairs of adjacent letters name the edges of the quadrilateral.

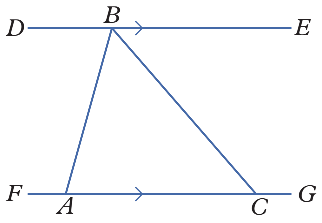
ANS p740

Exercise 7C Geometric proofs

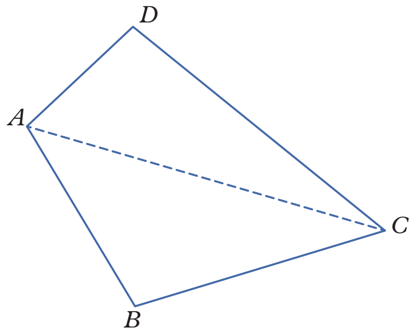
△ 1–6, 8, 9, 12, 15 □ 1–5, 7, 11, 13, 16, 17, 19, 20 ⬠ 1, 5, 6, 10, 14, 15, 17, 18, 19, 21–23

UNDERSTANDING AND FLUENCY

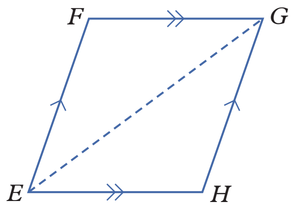
7C.1 1 Use your knowledge of angles and parallel lines, as well as the diagram below, to prove that the sum of the interior angles of a triangle is 180°.



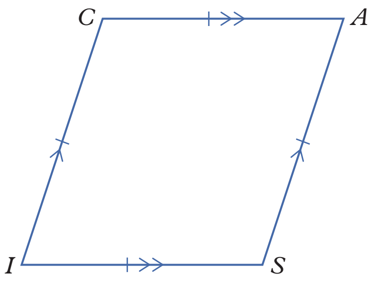
2 Use the property that the sum of the interior angles of a triangle is 180°, as well as the diagram below, to prove that the sum of the interior angles of a quadrilateral is 360°.



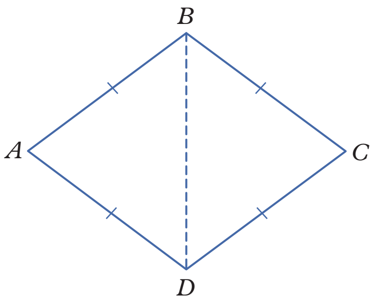
7C.2 3 Given that $FG = EF$, prove that all four sides of the rhombus on the right are equal in length.



7C.3 4 Use the following diagram to prove that a opposite angles of a rhombus are equal.



7C.4 **5** Use the diagram below to prove that the opposite sides of a rhombus are parallel.



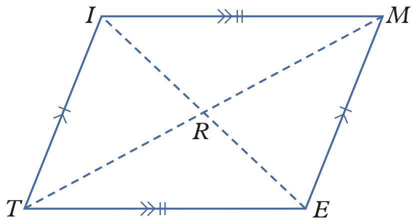
6 Consider the parallelogram on the right.

a What angle is equal to ∠*IMR*?

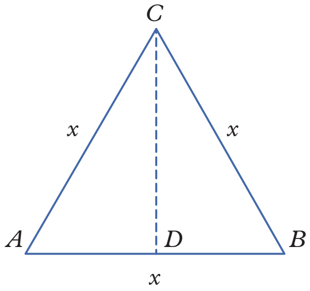
b What angle is equal to ∠*MIR*?

c Prove that $\triangle MIR \equiv \triangle TER$.

d Explain how this shows that the diagonals of a parallelogram bisect each other.



7 Consider the equilateral triangle on the right, where *CD* is a bisector of *AB*. Complete the following parts to show that the interior angles of an equilateral triangle are 60°.

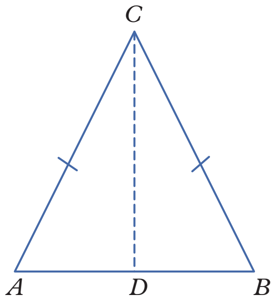


a What are the lengths of *AD* and *BD*?

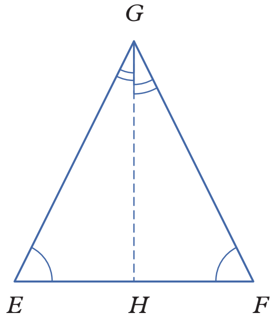
b Use trigonometry to calculate ∠*CAD* and ∠*CBD*.

c Use the property that the sum of the interior angles of a triangle is 180° to calculate ∠*ACB*.

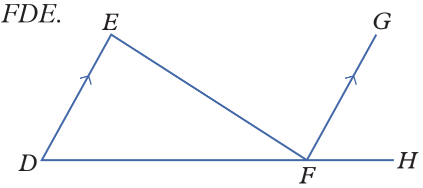
8 Use the diagram on the right, where *CD* is a bisector of *AB*, to prove that if two sides of a triangle are equal, then the angles opposite those sides are equal.



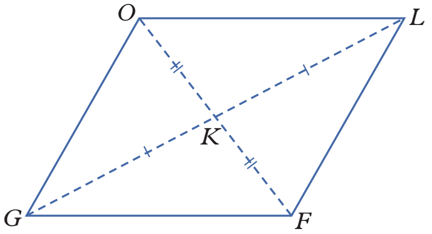
9 Use the diagram below, where *GH* is a bisector of ∠*EGF*, to prove that if two angles of a triangle are equal, then the sides opposite those angles are equal.



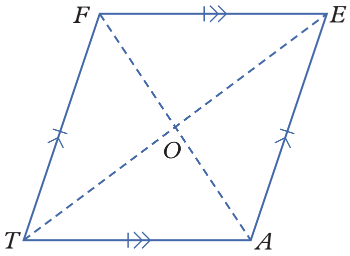
10 For the following diagram, use the fact that *DE* and *FG* are parallel to prove that ∠*EFH* = ∠*DEF* + ∠*FDE*.



11 Prove that the shape below is a parallelogram.



12 Consider the rhombus on the right.



a Follow similar steps to those given in question **6** to prove that the diagonals of a rhombus bisect each other.

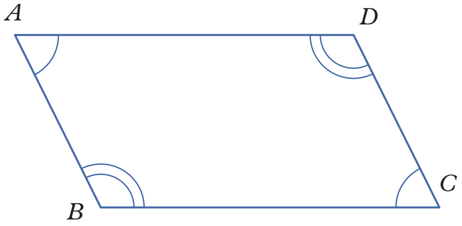
b Show that the four triangles formed by the diagonals are congruent to one another.

c Redraw the rhombus with markings showing all corresponding equal sides.

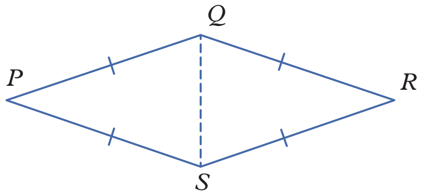
d Explain why $\angle FOE = \angle AOE$.

e Use your answer from part **d** to help you prove that the diagonals of a rhombus bisect each other at right angles.

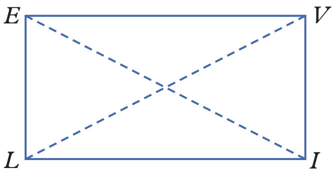
13 Use the diagram on the right, and the property that the interior angles of a quadrilateral sum to 360°, to prove that if both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.



14 Use the diagram below, and the property that the interior angles of a quadrilateral sum to 360°, to prove that if all sides of a quadrilateral are equal, then the quadrilateral is a rhombus.



15 Consider the diagram on the right, where $VE = IL$, $VE \parallel IL$ and $\angle EVI = 90°$.



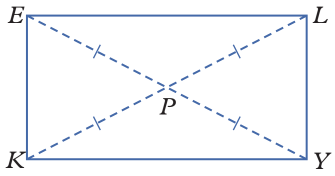
a Prove that $\angle VIL$, $\angle ILE$ and $\angle LEV$ are all 90°.

b Prove that $EL = IV$ and $EL \parallel IV$.

c Use Pythagoras' theorem to show that $LV = EI$.

d Prove that $\triangle LEV \equiv \triangle ELI$.

16 Prove that the shape on the right is a rectangle by following these steps:

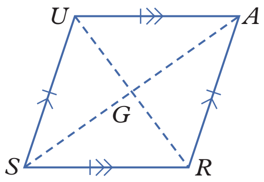


a Use what you know about isosceles triangles and about equal alternate angles to prove that $ELYK$ is a parallelogram.

b Use $\angle LYE + \angle YEL + \angle ELK + \angle KLY$ to prove that $\angle ELY = 90°$.

c Prove that all interior angles are 90°, and therefore that $ELYK$ is a rectangle.

17 Consider the rhombus on the right.



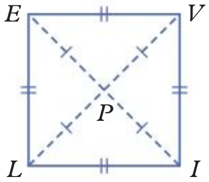
a Prove that $UG = RG$ and that $AG = SG$ (in other words, that the diagonals bisect each other).

b Prove that $\triangle UAG \equiv \triangle RAG$.

c Prove that $\angle UAS = \angle RAS$.

d Explain why this method shows that the diagonals of a rhombus bisect its interior angles.

18 For the diagram below, prove that $\angle EVL = 45°$.

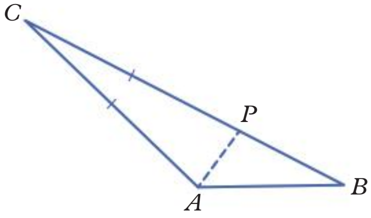


19 The angle–side relationship states that the longest side of a triangle is always opposite the largest angle in a triangle. This can be expanded to state that in $\triangle ABC$, if $BC \geq AC \geq AB$, then $\angle BAC \geq \angle ABC \geq \angle ACB$.

a Explain why the angle–side relationship holds for all equilateral triangles.

b Use the diagram on the right to prove that if $BC > AC$, then $\angle BAC > \angle CBA$. Hint: Start by defining point P.

c For an isosceles triangle with side lengths satisfying $BC \geq AC \geq AB$, explain why $BC > AC = AB$ or $BC = AB > AC$.



The result in part **b** means that if one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

d Use parts **b** and **c** to explain why the angle–side relationship holds for all isosceles triangles.

e Use part **b** to explain why the angle–side relationship holds for all scalene triangles.

20 a Use mathematical reasoning with a square and a rhombus to show that having all corresponding sides the same length is not enough to prove congruence for shapes other than triangles.

b Use mathematical reasoning with a parallelogram and a rhombus to show that meeting the SAS condition of congruence does not prove congruence for shapes other than triangles.

c Use mathematical reasoning with a square and a rectangle to show that meeting the AAS condition of congruence does not prove congruence for shapes other than triangles.

21 a Use mathematical reasoning with a rectangle and a parallelogram to show that having all corresponding sides in the same ratio is not enough to prove similarity for shapes other than triangles.

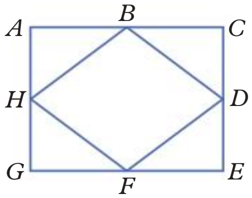
b Use mathematical reasoning with a parallelogram and a rhombus to show that having all corresponding angles the same size is not enough to prove similarity for shapes other than triangles.

22 Use your understanding of the relevant geometric properties to draw each shape described below and show that the triangles within them are congruent.

a A kite is split into two triangles by a line drawn along its line of symmetry.

b A regular hexagon is split into six triangles by drawing three diagonal lines joining opposite vertices.

23 *ACEG* is a rectangle. If the vertices of *HBDF* touch the midpoints of each of the rectangle's sides, prove that *HBDF* is a rhombus and that the opposite angles of *HBDF* are equal.



Check your Student obook pro for these digital resources and more:

Investigation
Finding the shortest path around a rectangle

Topic quiz
7C

EXTENSION

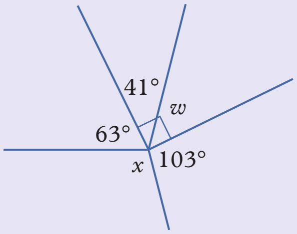
ANS p745

Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

7A **1** Determine the values of the pronumerals.

a



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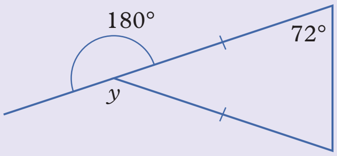
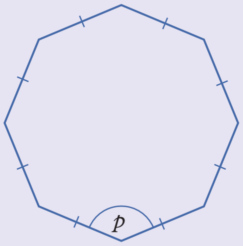


7A **2** Determine the values of the pronumerals.

a



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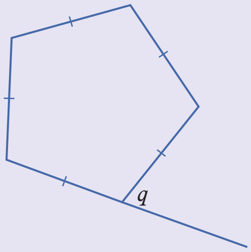
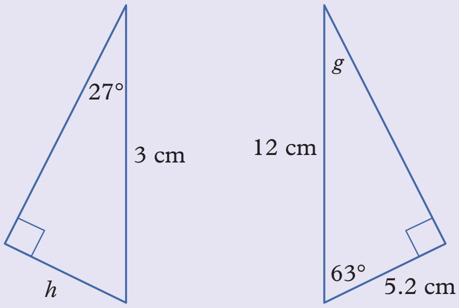


7B **3** Determine the values of the pronumerals.

a



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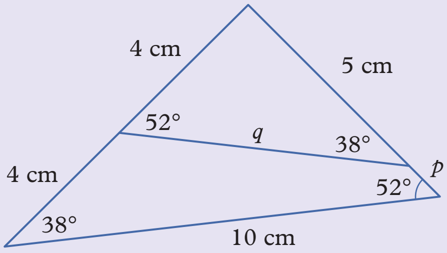
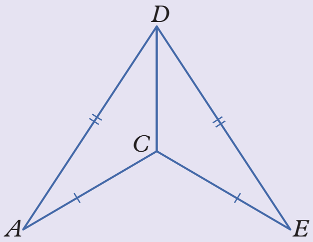
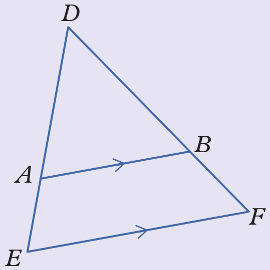


7B **4** **a** Prove that $\triangle ADC \equiv \triangle EDC$ and, therefore, that $\angle ACD = \angle ECD$.

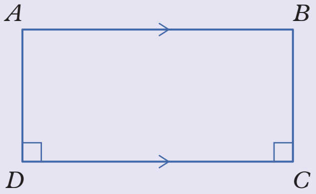


b Prove that $\triangle ADB \sim \triangle EDF$.

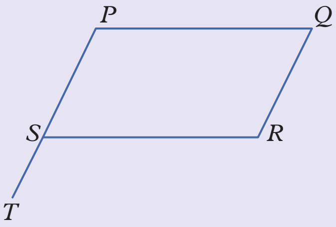


7C **5** Given that $AB \parallel DC$, $BC \perp CD$ and $AD \perp CD$, prove that $ABCD$ is a rectangle and, therefore, that $AC = BD$.

Note: a quadrilateral with four interior angles of 90° is a rectangle.



7C **6** Given that $\angle SPQ = \angle RST = \angle SRQ$, prove that $PQRS$ on the right is a parallelogram.



7D Circle geometry: angles

Learning intentions

By the end of this topic you will be able to ...

✔ understand and apply circle theorems related to angles.

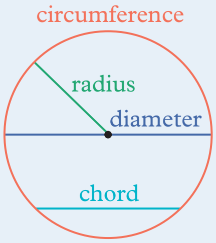
Inter-year links

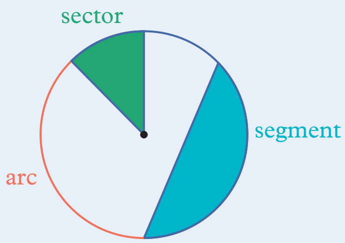
Year 8	7B Triangles

Key content video

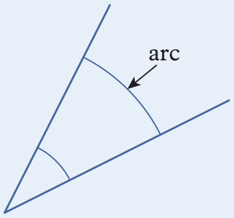
Circle terminology

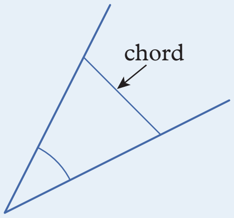
- A circle is a set of points equidistant from a central point.
- **Arc**: A part of the **circumference** of a circle.
  - → A minor arc measures less than a semi-circle (180°). A major arc measures more than a semi-circle.
- **Chord**: A line segment connecting two points on a circle's circumference.
- **Sector**: A portion of a circle enclosed by two radii and an arc on the circumference.
- **Segment**: A region of a circle enclosed by a chord and an arc on the circumference.

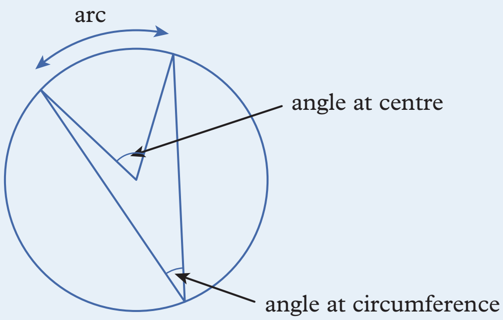




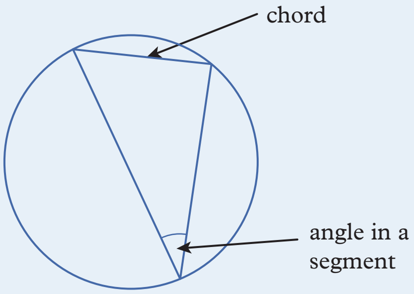
- The angle subtended by an arc or chord at a point is the angle whose rays from that point pass through the endpoints of the arc or chord.







  - → The angle at the centre refers to an angle which is subtended by an arc or chord at the centre of a circle.
  - → The angle at the circumference refers to an angle which is subtended by an arc or chord at the circumference of a circle.
  - → The arc on which an angle at the centre or circumference stands is the arc which subtends that angle.
  - → Two angles which are both subtended by the same arc are said to be standing on the same arc.
  - → An angle in a segment is an angle subtended by the endpoints of the chord that forms that segment to a point on the circumference.
- A cyclic quadrilateral is a quadrilateral whose vertices all lie on the circumference of a single circle.

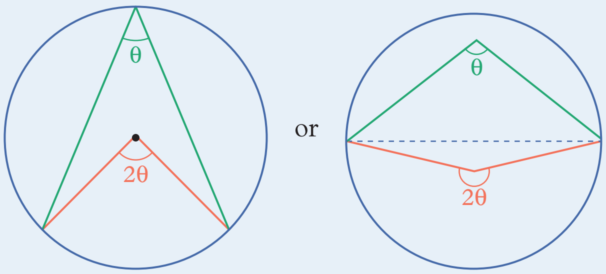


Circle theorems 1–5

- **Circle theorem 1:** The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference.

  This is also known as the 'angle at the centre' theorem.

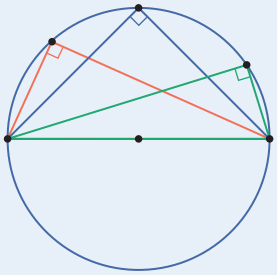
Circle theorem 1



- **Circle theorem 2:** Any angle subtended at the circumference of a circle by the ends of a diameter of the circle is a right angle.

  This is also known as the 'angle in a semi-circle' theorem, or Thales' theorem.

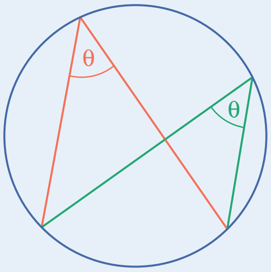
Circle theorem 2



- **Circle theorem 3:** All angles subtended at the circumference by the same arc are equal in size.

  This is also known as the 'angles in the same segment' theorem.

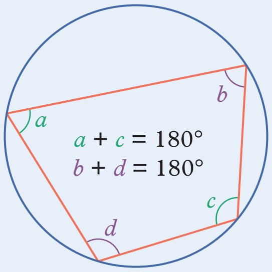
Circle theorem 3



- **Circle theorem 4:** The opposite angles of a cyclic quadrilateral are supplementary.

  This is also known as the 'cyclic quadrilateral' theorem.

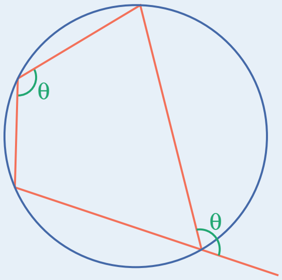
Circle theorem 4



- **Circle theorem 5:** Any exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

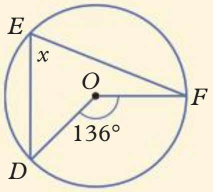
  This is an extension of the 'cyclic quadrilateral' theorem.

Circle theorem 5



Example 7D.1 Using circle theorem 1

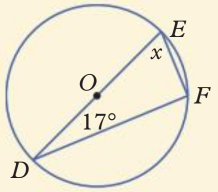
Find the value of x in the circle on the right.



THINK	**WRITE**
1 The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended at the circumference. So $\angle DOF = 2\angle DEF$.	$136^\circ = 2x$
2 Write your answer.	$x = 68^\circ$

Example 7D.2 Using circle theorem 2

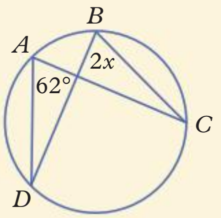
Find the value of x in the circle on the right.



THINK	**WRITE**
1 Any angle subtended at the circumference by the diameter of a circle is a right angle. So $\angle DFE$ is a right angle.	$\angle DFE = 90^\circ$
2 Use the angle sum of a triangle to help you write an equation to find the value of x.	$x + 17^\circ + 90^\circ = 180^\circ$ $x = 180^\circ - 17^\circ - 90^\circ$
3 Solve the equation and write your answer.	$x = 73^\circ$

Example 7D.3 Using circle theorem 3

Find the value of x in the circle on the right.



THINK	**WRITE**
1 All angles subtended by the same arc at the circumference are equal in size, therefore $\angle DBC$ and $\angle DAC$ are equal in size.	$\angle DBC = \angle DAC$
2 Substitute values and solve for x.	$2x = 62^\circ$ $x = 31^\circ$

Helpful hints

✔ If you can't remember the circle theorems at first, carefully inspecting the diagrams can be helpful. The given diagram might help you remember the circle theorem or theorems you need to use!

ANS p746

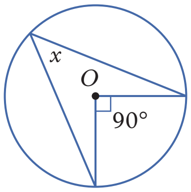
Exercise 7D Circle geometry: angles

△ 1–4, 5(a–c), 6–8 □ 1–5, 9, 10, 12(a) ⬠ 4, 5, 11, 12

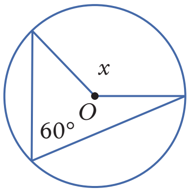
UNDERSTANDING AND FLUENCY

7D.1 1 Find the value of x in each of these circles.

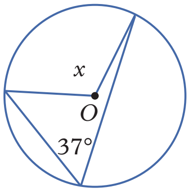
a



b

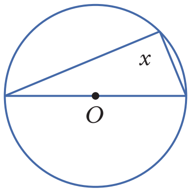


c



7D.2 2 Find the value of x in each of these circles.

a



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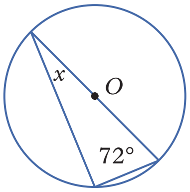

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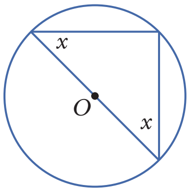
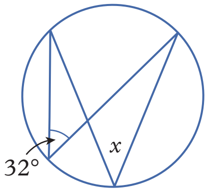


7D.3 3 Find the value of each pronumeral in these circles.

a



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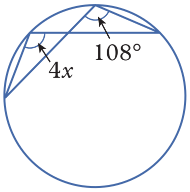

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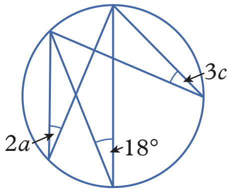
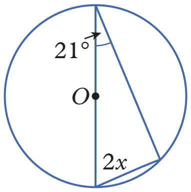


4 Find the value of each pronumeral in these circles, and state which circle theorem is used.

a



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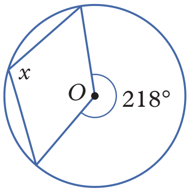

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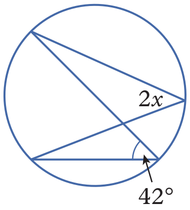

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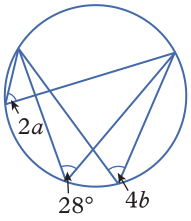

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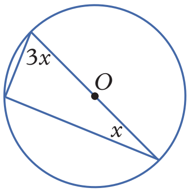

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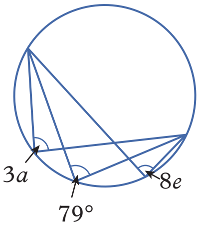
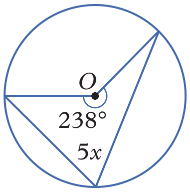


UNDERSTANDING AND FLUENCY

EXTENSION

5 Use your understanding of triangle and circle properties and theorems to find the value of x in each of these circles.

a



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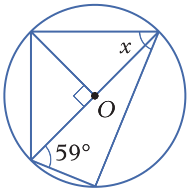

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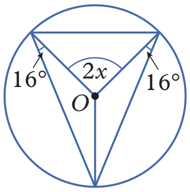

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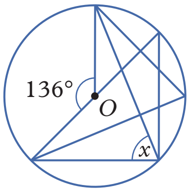

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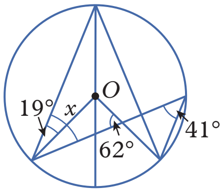

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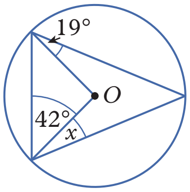
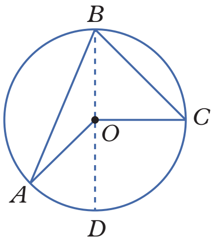


PROBLEM SOLVING AND REASONING

6 This question proves circle theorem 1. Consider the diagram on the right.



a How do you know that $AO = BO = CO$?

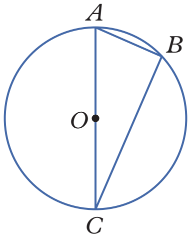
b How do you know that $\angle ABO = \angle OAB$?

c Use the theorem for the exterior angle of a triangle to show that $\angle AOD = 2\angle ABO$.

d Use your answers for parts **b** and **c** to show that $\angle DOC = 2\angle OBC$.

e Explain how this shows that $\angle AOC = 2\angle ABC$.

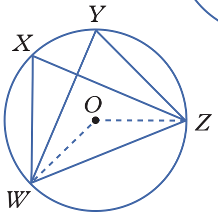
7 This question proves circle theorem 2. Consider the diagram on the right.



a What is the angle at the centre of the circle?

b Use circle theorem 1 to show that $\angle ABC = 90°$.

8 This question proves circle theorem 3. Consider the diagram on the right.

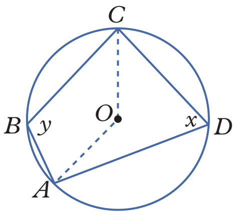


a Use circle theorem 1 to explain why $\angle WOZ = 2\angle WXZ$.

b Similarly, show that $\angle WOZ = 2\angle WYZ$.

c What can you say about $\angle WYZ$ and $\angle WXZ$?

9 This question proves circle theorem 4. Consider the diagram on the right, showing a cyclic quadrilateral.



a Explain why the obtuse angle $\angle AOC = 2x$ and the reflex angle $\angle AOC = 2y$.

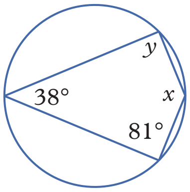
b Use what you know about angles at a point to explain why reflex $\angle AOC = 360° - 2x$.

c Use your answers from parts **a** and **b** to show that $y = 180° - x$.

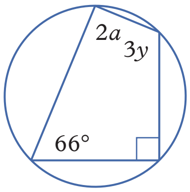
d Copy the diagram, replacing radii OC and OA with OB and OD, then use this to show that $\angle BAD = 180° - \angle BCD$.

e Find the value of each pronumeral in the circles below.

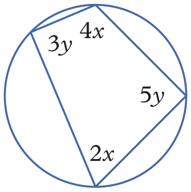
i



ii

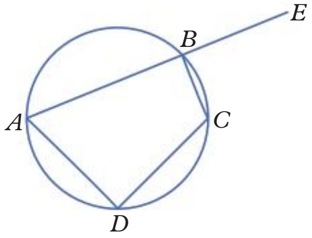


iii



EXTENSION

10 This question proves circle theorem 5. Consider the diagram below, showing another cyclic quadrilateral.



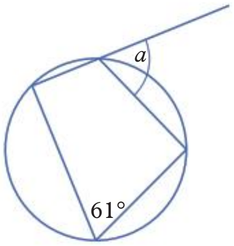
a Use circle theorem 4 to explain why ∠ADC = 180° – ∠ABC.

b Why does ∠EBC = 180° – ∠ABC?

c Explain why this shows that ∠ADC = ∠EBC.

d Find the value of each pronumeral in the diagrams below.

i



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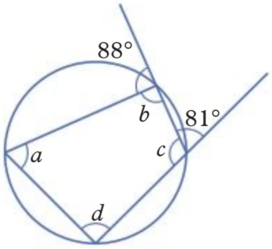
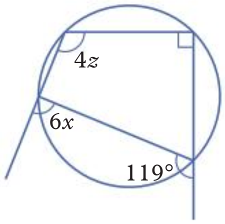


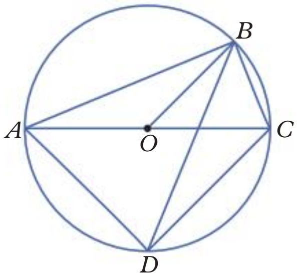
iii



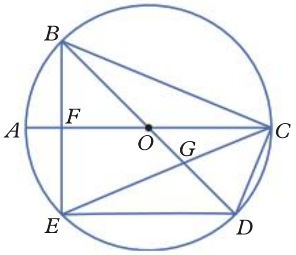
11 Using the diagrams from questions **6–10**, write a complete proof for each of the circle theorems 1–5.

12 Use the given information to answer these questions.

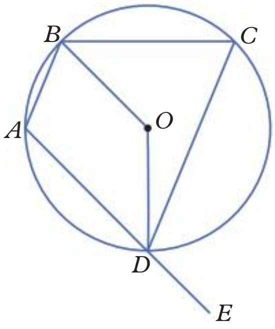
a Find ∠DBC, given that ∠ADB = 68°, OA = OB = OC and ∠BCD = 92°.



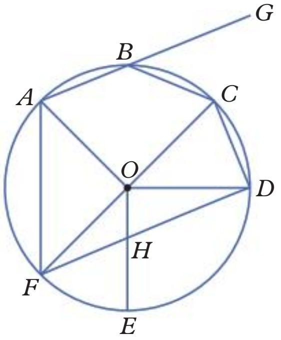
b Find ∠DEG, given that $BE \perp AC$, ∠OCG = 19° and ∠FOG = 142°.



c Find ∠ODC, given that ∠BAD = 96° and ∠CBO = 46°.



d Find ∠OAB, given that ∠GBC = 46°, ∠BCD = 138° and ∠AFD = 70°.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circle geometry: angles

Investigation
Clock angles

Topic quiz
7D

7E Circle geometry: chords

Learning intentions

By the end of this topic you will be able to …

✔ understand and apply circle theorems related to chords.

Inter-year links

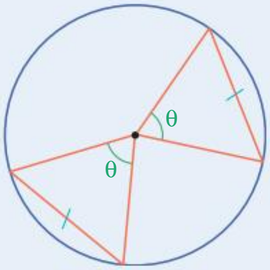
Year 8 7B Triangles

Circle theorems 6–9

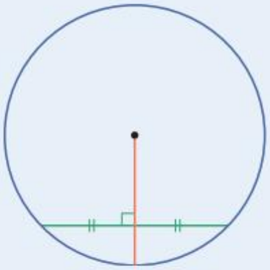
Key content video

- **Circle theorem 6:** Chords of equal length subtend equal angles at the centre of the circle. This is also known as the 'equal chords, equal angles' theorem.
- **Circle theorem 7:** If a radius and a chord intersect perpendicularly, then the radius bisects the chord.
- **Circle theorem 8:** Chords that are equal in length are **equidistant** (at equal distances) from the centre of the circle.
- **Circle theorem 9:** When two chords intersect inside a circle, this divides each chord into two line segments so that the product of the lengths of the line segments for both chords is the same.

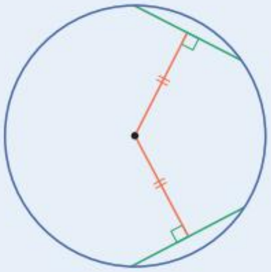
Circle theorem 6



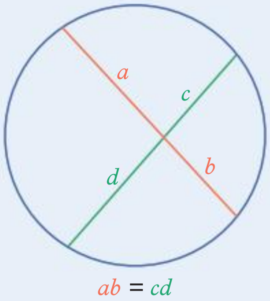
Circle theorem 7



Circle theorem 8

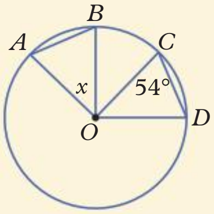


Circle theorem 9



Example 7E.1 Using circle theorem 6

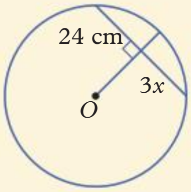
Given $AB = CD$, find the value of x in the circle on the right.



THINK	WRITE
1 Since OC and OD are radii, they are of equal length. This means that $\triangle COD$ is an isosceles triangle.	$OC = OD$
2 Find the two unknown interior angles of $\triangle COD$.	$\triangle COD$ is isosceles, so $\angle OCD = \angle ODC = 54°$. $\angle COD = 180° - 54° - 54°$ $= 72°$
3 Chords that are equal in length subtend equal angles at the centre of the circle. So $\angle AOB = \angle COD$.	$\angle AOB = \angle COD = 72°$
4 Write your answer.	$x = 72°$

Example 7E.2 Using circle theorem 7

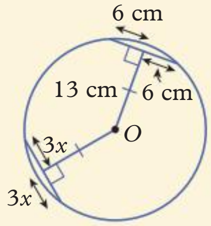
Find the value of x in the circle on the right.



THINK	**WRITE**
1 If a radius and a chord intersect perpendicularly, then the radius bisects the chord. So $3x$ is equal to 24 cm.	$3x = 24$ cm
2 Solve for x.	$x = 8$ cm

Example 7E.3 Using circle theorem 8

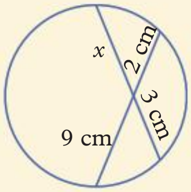
Find the value of x in the circle on the right.



THINK	**WRITE**
1 Chords that are equal in length are equidistant from the centre of the circle. So the two chords must be equal in length.	$3x + 3x = 6 + 6$
2 Solve for x, including units in your answer.	$6x = 12$ $x = 2$ cm

Example 7E.4 Using circle theorem 9

Find the value of x in the circle on the right.



THINK	**WRITE**
1 When two chords intersect, the products of the line segments they are divided into are equal. Write the product of the lengths of the line segments for each chord.	Chord 1: $x \times 3$ cm Chord 2: 2 cm × 9 cm $x \times 3 = 2 \times 9$
2 Simplify the equation.	$3x = 18$
3 Solve for x, including units in your answer.	$x = 6$ cm

Helpful hints

- ✔ All radii of the same circle are equal in size.
- ✔ When chords intersect, each of the equal products results from multiplying the lengths of the two parts of the same line segment.

ANS p747

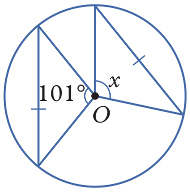
Exercise 7E Circle geometry: chords

△ 1–9 ■ 1–4(b, c), 5–7, 10–12, 14 ⬟ 5–7, 13–17

UNDERSTANDING AND FLUENCY

7E.1 1 Find the value of x in each of these circles.

a



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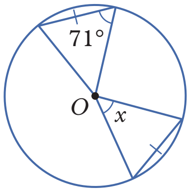

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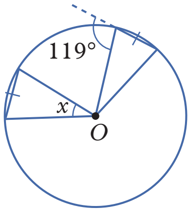
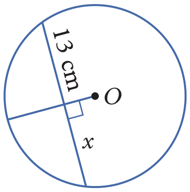


7E.2 2 Find the value of x in each of these circles.

a



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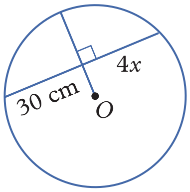

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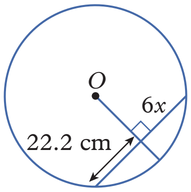
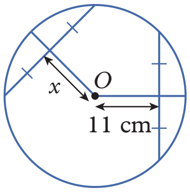


7E.3 3 Find the value of each pronumeral in these circles.

a



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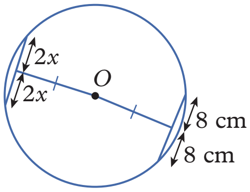

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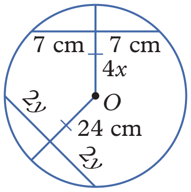
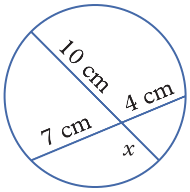


7E.4 4 Find the value of x in each of these circles, correct to one decimal place.

a



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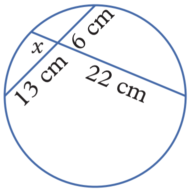

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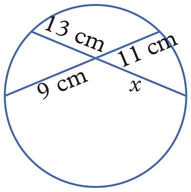
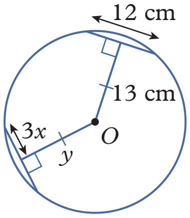


5 Find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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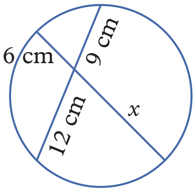

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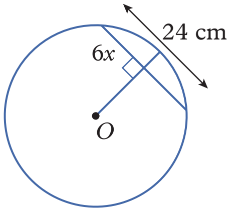

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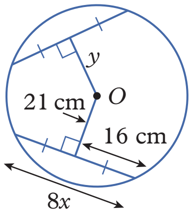

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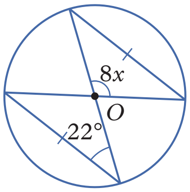

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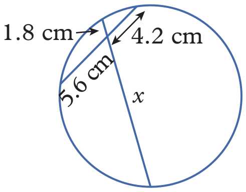

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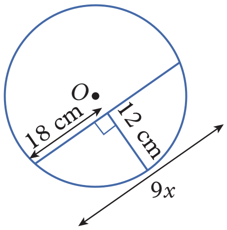

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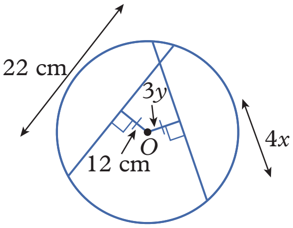

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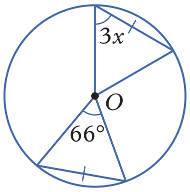
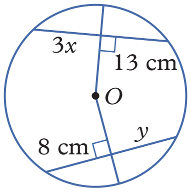


6 Use your understanding of circle properties and theorems, and any other necessary geometric properties, to find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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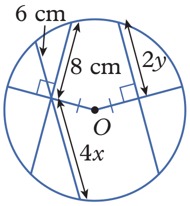

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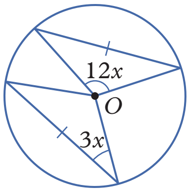

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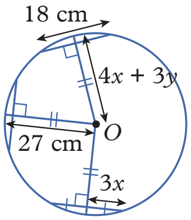

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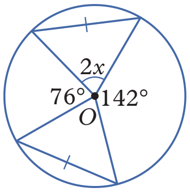

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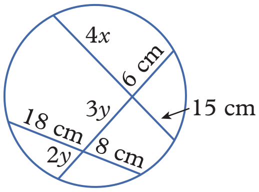
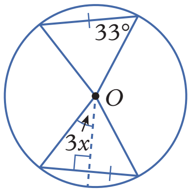


7 Find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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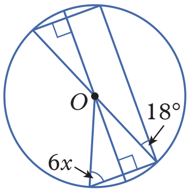

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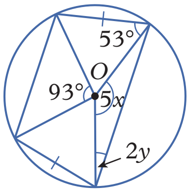

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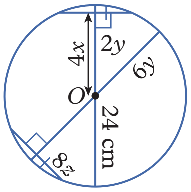

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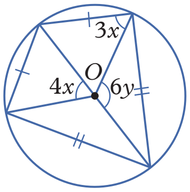

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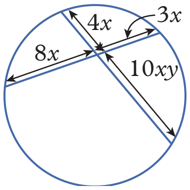
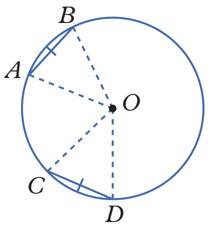
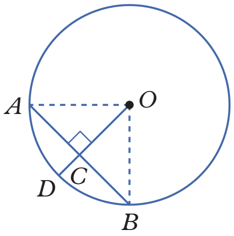


8 This question proves circle theorem 6.
Consider the diagram on the right, in which $AB = CD$.



- **a** Given that OA, OB, OC and OD are radii, what can you say about them?
- **b** What type of triangles does this mean $\triangle OAB$ and $\triangle OCD$ are?
- **c** Explain why $\triangle OAB \equiv \triangle OCD$.
- **d** Explain why $\angle COD = \angle AOB$.

9 This question proves circle theorem 7. Consider the following diagram.



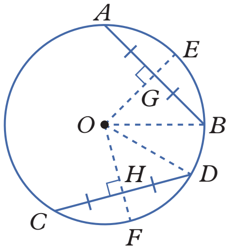
a Explain why $OA = OB$.

b Explain why $\angle OCA = \angle OCB$.

c Use one of the conditions for the congruence of triangles to explain why $\triangle ACO \equiv \triangle BCO$.

d Explain why $AC = BC$.

10 This question proves circle theorem 8. Consider the following diagram, in which $AB = CD$.



a Explain why $OE = OB = OD = OF$.

b Use circle theorem 7 to explain why $AG = GB$ and $CH = HD$.

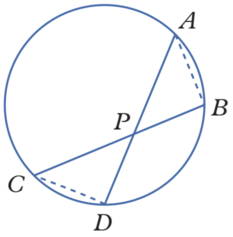
c Likewise, use circle theorem 7 and the fact that $AB = CD$ to explain why $GB = HD$.

d Use one of the conditions for the congruence of triangles to explain why $\triangle OHD \equiv \triangle OGB$.

e Explain why $OH = OG$.

f Use your understanding of Pythagoras' theorem to explain why the shortest distance from the centre of a circle to a chord is perpendicular to that chord.

11 This question proves circle theorem 9. Consider the following diagram.



a Explain why $\angle CPD = \angle APB$.

b Use circle theorem 3 to explain why $\angle DCB = \angle BAD$.

c Explain why $\angle PDC = \angle PBA$.

d Explain how you can tell that $\triangle CPD$ is similar to $\triangle APB$.

e Explain why you can state that $\frac{AP}{BP} = \frac{CP}{DP}$ and, hence, that $AP \times DP = CP \times BP$.

12 Prove that, if a cyclic quadrilateral is a parallelogram, then it must be a rectangle.

13 Using the diagrams from questions **8–11**, write a complete proof for each of the circle theorems 6–9.

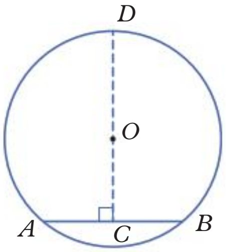
EXTENSION

PROBLEM SOLVING AND REASONING

14 Another form of proof is 'proof by contradiction', where we assume that what we want to prove is not true, and then we demonstrate that this assumption is false.

We can use a proof by contradiction to show that the perpendicular bisector of a chord passes through the centre of a circle.

Consider the following diagram, where *CD* is a perpendicular bisector of *AB*.



Start with the assumption that the perpendicular bisector of a chord does not pass through the centre. That is, that *O* does not lie on the line segment *CD*.

a What is the size of *ACD*?

Now consider the radius of the circle that perpendicularly intersects *AB*.

b Explain how you know that this radius must goes through *C*.

c Explain how this demonstrates that *O* lies on the line segment *CD*.

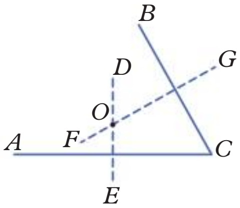
d Explain where the contradiction arises, and what this demonstrates.

CHALLENGE

15 a Non-collinear points are points that do not all lie on the same straight line. Use the diagram below right, where *DE* is a perpendicular bisector of *AC* and *FG* is a perpendicular bisector of *BC*, to prove that there is only one circle that passes through any three non-collinear points.

The following properties can be used in your proof:

- Every point on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.
- A point which is equidistant from two distinct points lies on the perpendicular bisector of the two points.
- Two non-parallel straight lines only intersect at a unique point.
- The perpendicular bisectors of two non-parallel lines are not parallel.

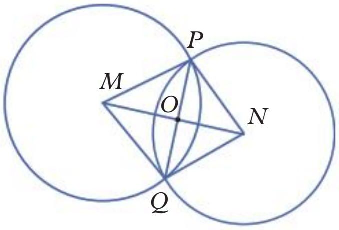


b Explain how the result from part **a** proves that given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle, which passes through all three points.

16 A circle passes through the origin and the points (1, 1), (5, 1) and (1, 6). State the coordinates of the centre of the circle.

17 Consider the diagram on the right, which shows two intersecting circles. Prove that the line joining the centre of the circles (*MN*) bisects the chord common to both circles (*PQ*) at right angles. You may use the following properties in your proof:

- The perpendicular bisector of a chord passes through the centre of a circle.
- There is a unique straight line passing through any two points.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circle geometry: chords

Topic quiz
7E

7F Circle geometry: tangents and secants

Learning intentions

By the end of this topic you will be able to ...

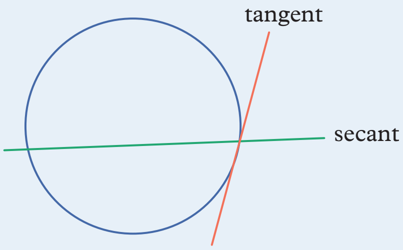
✔ understand and apply circle theorems related to tangents and secants.

Inter-year links

Year 8	7B triangles

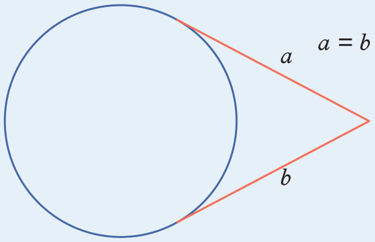
Tangents and secants

Key content video

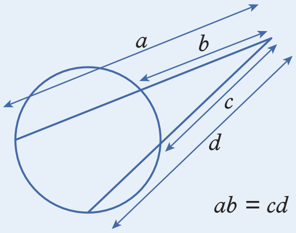


- A **tangent** is a line that touches the circumference of a circle at one point only.
  - → The angle between a radius of a circle and a tangent to a circle is 90°.
- A **secant** is a line that cuts a circle twice.
- **Circle theorem 10:** If two tangents intersect outside a circle, the distances along the tangent from the intersection to the circumference of the circle are equal.
- **Circle theorem 11:** If two secants are drawn from a circle to an intersecting point outside the circle, the product of the entire secant length by the external secant length will be the same for each secant.
- **Circle theorem 12:** If a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length will be equal to the square of the tangent length.
- **Circle theorem 13:** The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.

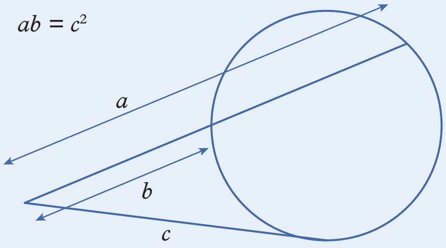
Circle theorem 10



Circle theorem 11

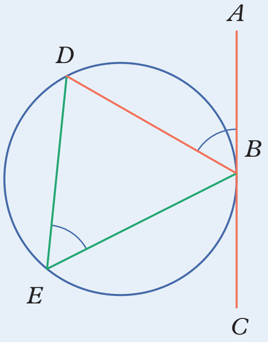


Circle theorem 12



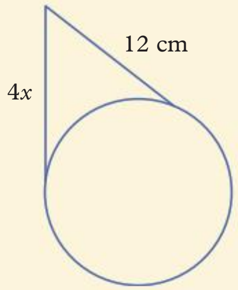
Circle theorem 13

$\angle ABD = \angle DEB$



Example 7F.1 Using circle theorem 10

Find the value of x in the diagram on the right.



THINK

1 The distances from the intersection of two tangents to the circumference of a circle are equal.

2 Solve for x, including units in your answer.

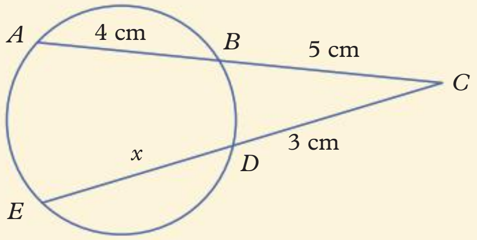
WRITE

$4x = 12$

$x = 3$ cm

Example 7F.2 Using circle theorem 11

Find the value of x in the following circle.



THINK

1 When two secants are drawn from a circle to an intersecting point outside the circle, the product of the entire secant length by the external secant length will be the same for each secant.

2 Write the length of each line segment.

3 Substitute these lengths into the relationship and simplify.

4 Solve for x, including units in your answer.

WRITE

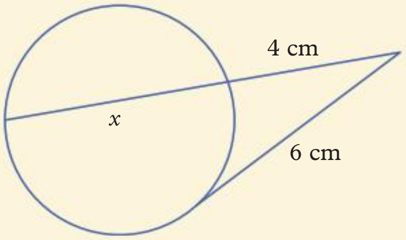
$AC \times BC = CE \times CD$

$AC = 4 + 5$
$= 9$ cm
$BC = 5$ cm
$CE = (3 + x)$ cm
$CD = 3$ cm

$9 \times 5 = (3 + x) \times 3$
$45 = 3(3 + x)$
$15 = 3 + x$
$x = 12$ cm

Example 7F.3 Using circle theorem 12

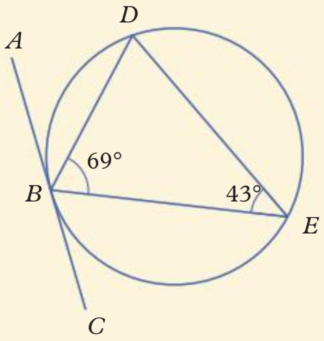
Find the value of x in the circle on the right.



THINK	WRITE
1 When a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length will be equal to the square of the tangent length.	$a \times b = c^2$
2 Write the length for each line segment.	$a = (x + 4)$ cm $b = 4$ cm $c = 6$ cm
3 Substitute into the formula and simplify.	$(x + 4) \times 4 = 6^2$ $4\ (x + 4) = 36$ $x + 4 = 9$
4 Solve for x, including units in your answer.	$x = 5$ cm

Example 7F.4 Using circle theorem 13

Find the value of $\angle CBE$ in the circle on the right.



THINK	WRITE
1 Write the known information.	$\angle BED = 43^\circ$ $\angle DBE = 69^\circ$
2 The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.	$\angle ABD = \angle BED$ $= 43^\circ$
3 Angles on a straight line sum to 180°.	$\angle ABD + \angle BED + \angle CBE = 180^\circ$ $43^\circ + 69^\circ + \angle CBE = 180^\circ$
4 Solve for $\angle CBE$.	$\angle CBE = 60^\circ$

Helpful hints

- ✔ If a straight line touches a circle, it will be either a tangent or a secant.
- ✔ The two points where a secant cuts a circle may not be on separate sides of the circumference – they can be close to each other!

ANS p749

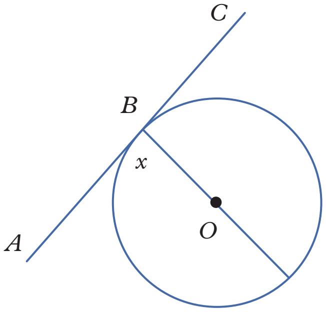
Exercise 7F Circle geometry: tangents and secants

▲ 1–5, 6(a–c), 7–9 ■ 2–5, 6(d–f), 9–12 ⬟ 6, 7, 12–15

UNDERSTANDING AND FLUENCY

1 Use the property that the angle between a radius of a circle and a tangent to a circle is 90° to find the value of x in each of these diagrams.

a



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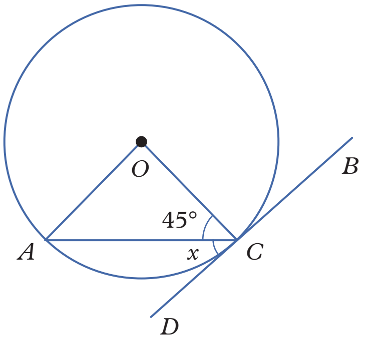

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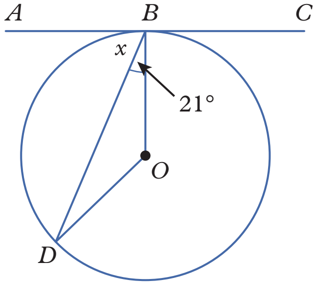
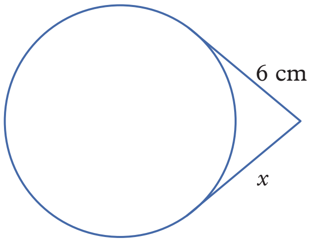


7F.1 **2** Find the value of the unknown pronumerals in each of these diagrams.

a



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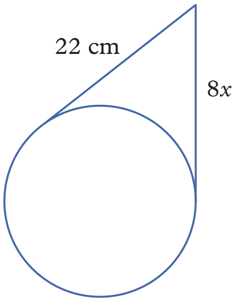

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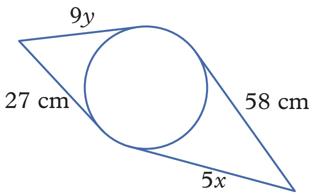
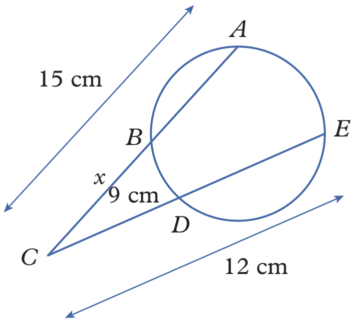


7F.2 **3** Find the value of x in each of these diagrams.

a



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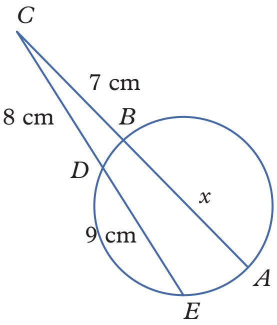

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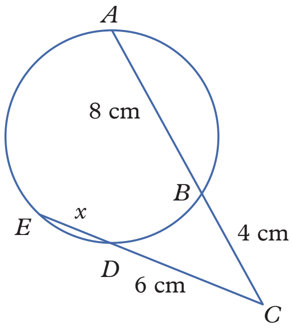
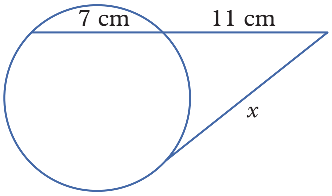


EXTENSION

UNDERSTANDING AND FLUENCY

7F.3 **4** Find the value of x, correct to one decimal place, in each of these diagrams.

a



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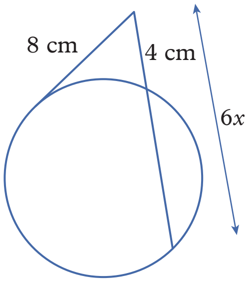

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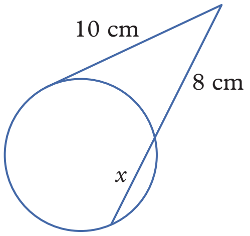
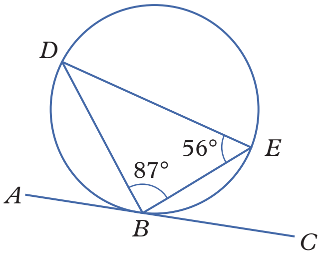


7F.4 **5** Find the value of $\angle CBE$ in each of these diagrams.

a



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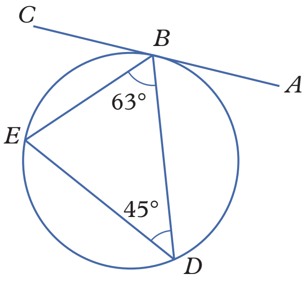

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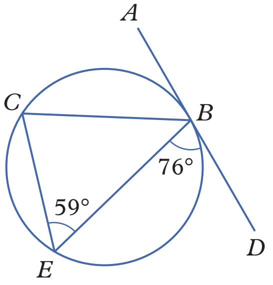
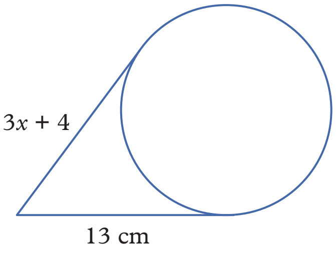


PROBLEM SOLVING AND REASONING

6 Find the value of each pronumeral, giving reasons with your answers.

a



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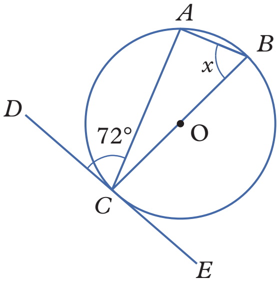

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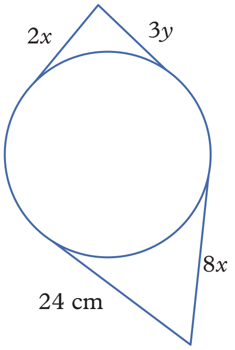

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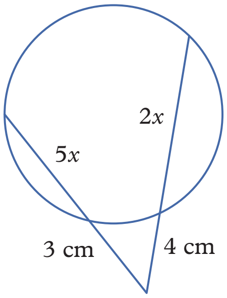

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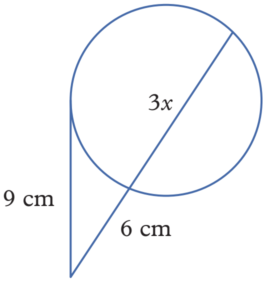

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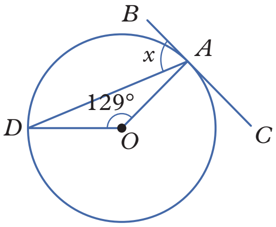
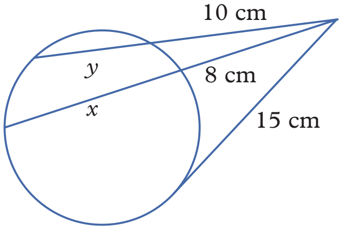


7 Use your understanding of circle theorems and geometric properties to find the value of each pronumeral. Give reasons with your answers.

a



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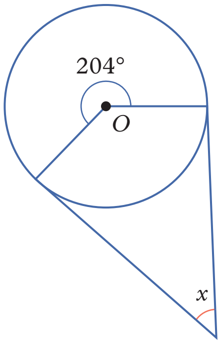

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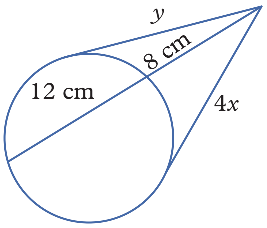
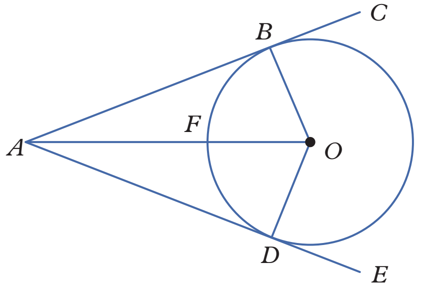
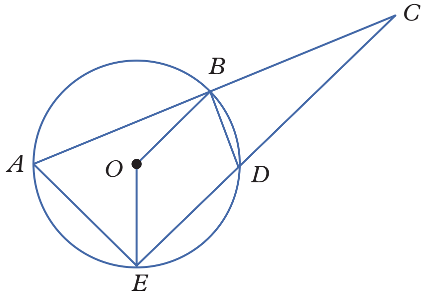


8 This question proves circle theorem 10. Consider the following diagram.



- **a** How do you know that $\angle ABO = \angle ADO = 90^\circ$?
- **b** Explain why $OB = OD$.
- **c** Explain why $\triangle ABO \equiv \triangle ADO$.
- **d** Explain why this means that $AB = AD$.

9 This question proves circle theorem 11. Consider the following diagram.

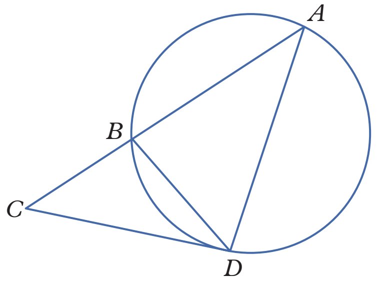


- **a** Explain why $\angle BOE = 2\angle BAE$.
- **b** Explain why the reflex angle $\angle BOE = 360^\circ - 2\angle BAE$.
- **c** Explain how you know $\angle BDE = 180^\circ - \angle BAE$.
- **d** Hence explain why $\angle BDC = \angle BAE$.

Now consider $\triangle DCB$ and $\triangle ACE$.

- **e** What angle do they have in common?
- **f** What pair of angles have you already shown are equal?
- **g** Explain why these triangles must therefore be similar.
- **h** Show that $\frac{CA}{CB} = \frac{EC}{BC}$.
- **i** Explain why this can also be written as $AC \times BC = CE \times CD$.

10 This question proves circle theorem 12. Consider the following diagram.

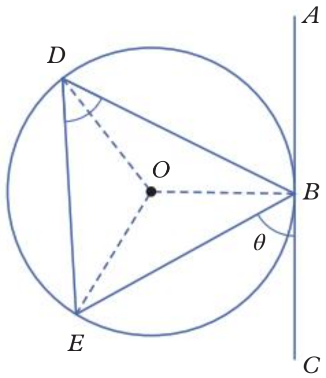


- **a** Which angle do $\triangle BCD$ and $\triangle DAC$ have in common?
- **b** Construct OD and OB, where O is the centre of the circle, and explain why $\angle CAD = \angle CDB$.
- **c** Explain why $\triangle BCD$ and $\triangle DCA$ are similar.
- **d** Explain how you know that $\frac{CB}{CD} = \frac{CD}{CA}$.
- **e** Explain why this can be written as $CA \times CB = (CD)^2$.

PROBLEM SOLVING AND REASONING

EXTENSION

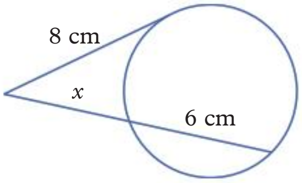
11 This question proves circle theorem 13. Consider the diagram on the right.



- **a** Explain why ∠OBE and ∠OEB are equal.
- **b** Write an equation that links ∠EOB and ∠OBE and explain why this equation is true.
- **c** Explain why ∠OBC is a right angle.
- **d** Write an equation that links ∠OBE and θ, with θ as the subject of the equation.
- **e** Write an equation that links ∠EOB and θ, showing how this equation was formed.
- **f** Explain why ∠$EOB = 2 \times$ ∠EDB.
- **g** Show that ∠$EDB = \theta$.

12 Find the value of each pronumeral, correct to one decimal place, in these diagrams.

a



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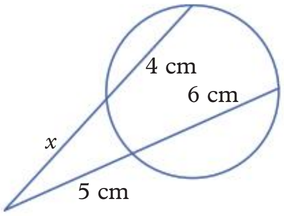
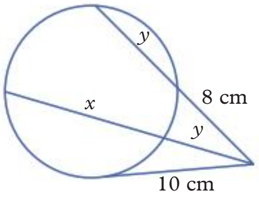


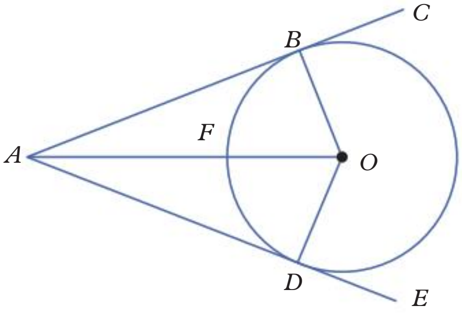
c



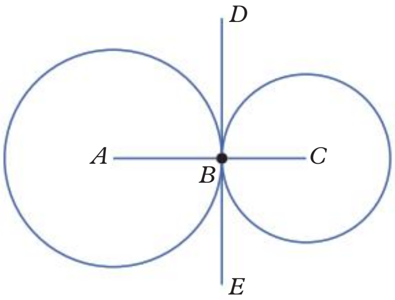
CHALLENGE

13 Using the diagrams from questions **8–11** write a complete proof for each of the circle theorems 10–13.

14 Use the diagram from question **8** (reproduced here) to prove that a straight line drawn from the centre of the circle to the intersection point of two tangents to the circle bisects the angle formed by the two tangents.



15 Use the following diagram, where DE is a common tangent to both circles, to prove that when two circles touch, the point of contact and the centres of the circles are collinear (i.e. they lie on a straight line).



Check your Student obook pro for these digital resources and more:

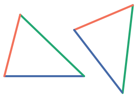
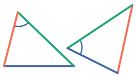
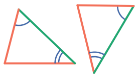
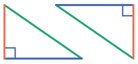
Interactive skillsheet
Circle geometry: tangents and secants

Worksheet
Circle geometry

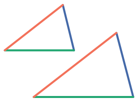
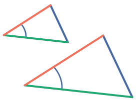
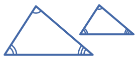
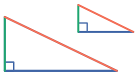
Topic quiz
7F

Chapter summary

Congruent triangles

SSS	SAS	AAS	RHS
			

Similar triangles

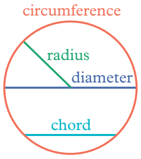
SSS	SAS	AAA	RHS
			

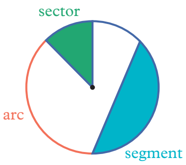
Triangle and quadrilateral properties

Definition	Properties
Triangle A three-sided polygon	The sum of the interior angles is 180° The exterior angle is equal to the sum of the two interior opposite angles
Equilateral triangle A triangle with all sides equal in length	Each interior angle is 60°
Isosceles triangle A triangle with exactly two sides equal in length	The angles opposite the equal side lengths are equal
Scalene triangle A triangle with no two sides equal in length	No angles are equal
Quadrilateral A four-sided polygon	The sum of the interior angles is 360° The exterior angle is equal to the sum of the two interior opposite angles
Trapezium A quadrilateral with at least one pair of parallel sides	Diagonals are equal in length
Parallelogram A quadrilateral with two pairs of parallel sides	Opposite sides are equal in length Opposite angles are equal Diagonals bisect each other
Rhombus A parallelogram with two adjacent sides equal in length	All parallelogram properties All sides are equal in length Diagonals bisect each other at right angles Diagonals bisect the interior angles
Rectangle A parallelogram with a right angle	All parallelogram properties Diagonals are equal in length
Square A rectangle with two adjacent sides which are equal in length	All parallelogram properties All rhombus properties All rectangle properties
Kite A quadrilateral with two pairs of adjacent sides which are equal in length	One pair of opposite angles are equal Diagonals are perpendicular One diagonal bisects the other diagonal One diagonal bisects a pair of interior angles

Terminology

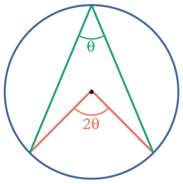
- ‘Bisect’ means to cut in half.
- The symbol for ‘is parallel to’ is ||.
- The symbol for ‘is perpendicular to’ is ⊥.



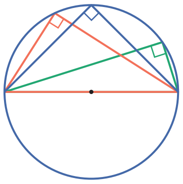


Circle theorems

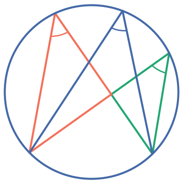
Circle theorem 1



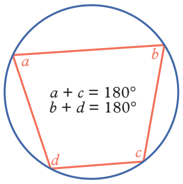
Circle theorem 2



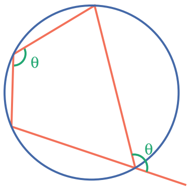
Circle theorem 3



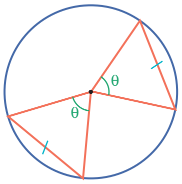
Circle theorem 4



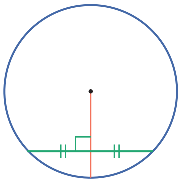
Circle theorem 5



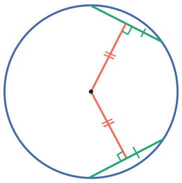
Circle theorem 6



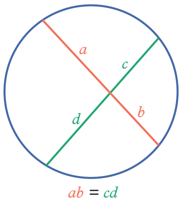
Circle theorem 7



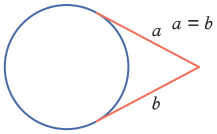
Circle theorem 8



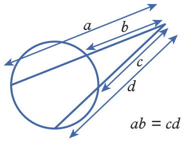
Circle theorem 9



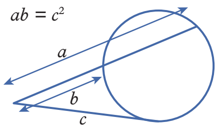
Circle theorem 10



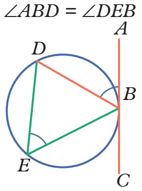
Circle theorem 11



Circle theorem 12



Circle theorem 13



EXTENSION

ANS p752

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- angle–side relationship
- arc
- bisect
- chord
- circumference
- complementary angles
- congruent
- convex polygon
- definition
- equidistant
- equilateral triangle
- exterior angles
- hypotenuse
- interior angles
- isosceles triangle
- kite
- parallel
- parallelogram
- perpendicular
- polygon
- quadrilateral
- rectangle
- regular polygon
- rhombus
- scalene triangle
- secant
- sector
- segment
- similar
- square
- subtend
- supplementary angles
- tangent
- test
- theorem
- transversal
- trapezium
- triangle
- triangle inequality
- vertically opposite angles

1 What makes a polygon a regular polygon?

2 Explain how the angle–side relationship relates to the hypotenuse.

3 Explain the difference between a definition and a property of a plane shape.

4 Identify the key terms being referenced in each of these definitions.

a a polygon in which all interior angles are less than 180°

b a line segment connecting two points on a circle's circumference

5 Using an example, provide a definition in your own words for the following key terms.

a triangle inequality

b tangent

6 Complete the following sentences using words from the key term list.

a The opposite angles of a cyclic quadrilateral are ________.

b A ________ is a statement or result that can be demonstrated to be true based on known facts.

Multiple choice

7A **1** Which of these angles is supplementary to 25°?

A 75° **B** 25° **C** 155° **D** 65° **E** 335°

7A **2** Which is the size of an individual internal angle in a regular decagon?

A 36° **B** 72° **C** 120° **D** 135° **E** 144°

7B **3** Which of these is *not* a test for congruent triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

7B **4** Which of these is *not* a test for similar triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

7B **5** Which of the following is not a property of all rhombuses?

A Opposite sides are equal in length
B Diagonals bisect each other at right angles
C Opposite angles are equal
D All interior angles are right angles
E Diagonals bisect each other

7D **6** For an arc in a circle, the angle subtended at the centre of the circle and the angle subtended at the circumference are complementary angles. What are the values of the two angles?

A 60° and 30° **B** 45° and 45° **C** 90° and 45° **D** 120° and 60° **E** 120° and 240°

7E **7** A circle contains two chords of the same length. Which of these statements is *not* true?

A The arcs on which the two chords stand are the same length.

B The chords are the same perpendicular distance from the centre.

C The chords cannot intersect.

D The chords subtend equal angles at the centre of the circle.

E None of the above.

7F **8** Which of the following best represents the value of x in the following diagram?

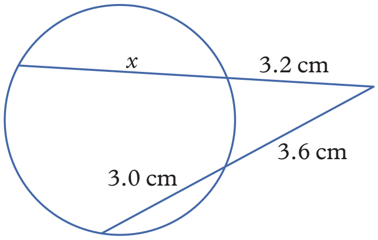
A 2.8 cm

B 3.4 cm

C 4 cm

D 4.2 cm

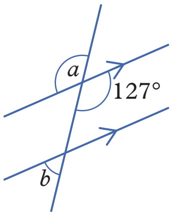
E 4.5 cm



Short answer

7A **1** Determine the values of the pronumerals in these diagrams.

a



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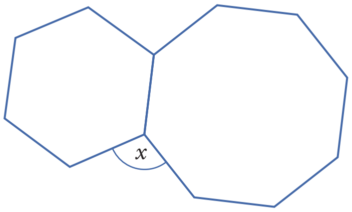


Note: Both polygons are regular.

7B **2** Rectangle $ABCD$ has side lengths of 4 cm and 3 cm. Its diagonals intersect at point P.

a Draw a diagram to represent this rectangle.

b Explain why $\angle ABP$ is equal to $\angle CDP$, and $\angle PAB$ is equal to $\angle PCD$.

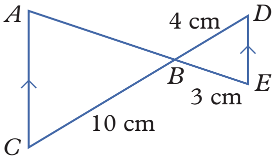
c Explain why $\triangle PAB \equiv \triangle PCD$.

d If $\angle BDC = 37°$, find the size of $\angle CBD$.

7B **3** **a** For the diagram on the right, provide a reason why $\triangle ABC$ is similar to $\triangle EBD$.

b If $\triangle EBD$ is the reduced image of $\triangle ABC$, what is the scale factor?

c What is the length of AB?



7C **4** $\triangle ABC$ has an internal line drawn from vertex B, meeting the opposite side AC at point D.

Also, $\angle ABC$ is a right angle.

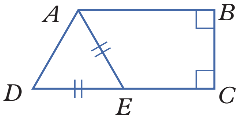
a Draw a sketch of the triangle.

b Use only the given information to prove that $\angle ADB = \angle DBC + \angle BCD$.

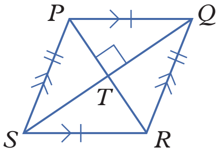
7C **5** Refer to the diagram on the right and write a proof to show that:

a $\angle AEC = 2\angle ADE$

b $2\angle ADE + \angle BAE = 180°$.



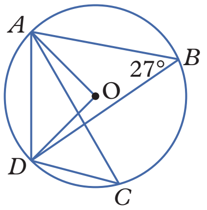
7C **6** Use the information in the following diagram prove that figure $PQRS$ is a rhombus.



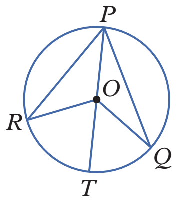
EXTENSION

7D **7** Determine the size of these angles in the diagram on the right.

a $\angle AOD$ **b** $\angle ACD$ **c** $\angle OAD$

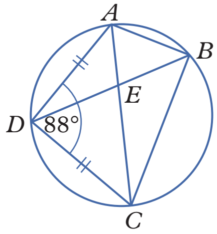


7D **8** For the following diagram, prove that
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$.

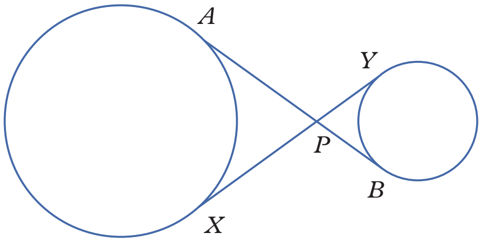


7E **9** **a** Consider the following diagram. Determine the size of $\angle DBC$.

b Prove that $\angle DBC = \angle ABD$.



7F **10** In the following diagram, AB and XY are tangents to the two circles. Prove that $AB = XY$.



Analysis

1 Consider the quadrilateral $FDGC$ on the right. Any facts proven in parts **a–c** can be used as given facts in parts **d–f**.

a Prove that $\triangle CFD \equiv \triangle CGD$.

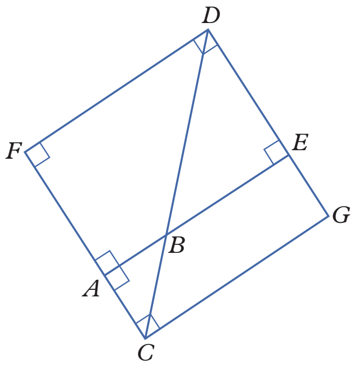
b Prove that $\triangle CAB \sim \triangle CFD$.

c Prove that $\triangle CAB \sim \triangle DEB$.

d If $AC = DE$, prove that B is the midpoint of CD.

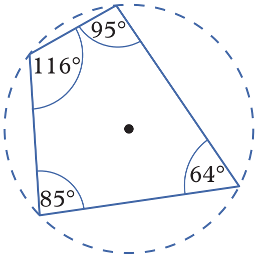
e If $\frac{FC}{AC} = 3$, prove that $\frac{DE}{AC} = 2$.

f If $\angle DBE = 45°$, prove that $FDGC$ is a square.

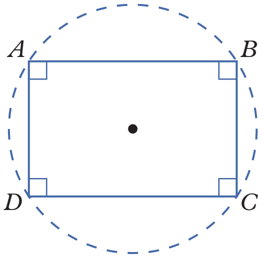


2 All the edges of a cyclic quadrilateral are chords of the same circle. We can consider these chords and the chords that are the diagonals of the cyclic quadrilateral in terms of the circle theorems.

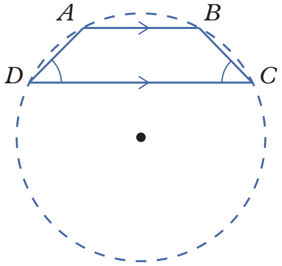
a Explain why this quadrilateral is cyclic.



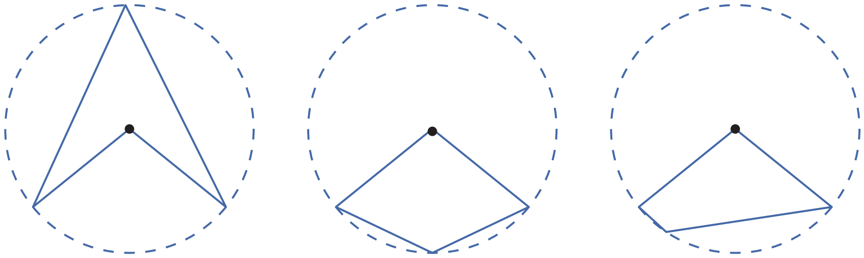
b Prove that all rectangles are cyclic, using $ABCD$ shown below.



c An isosceles trapezium has equal base angles. Prove that all isosceles trapezia are cyclic, using $ABCD$ shown below.

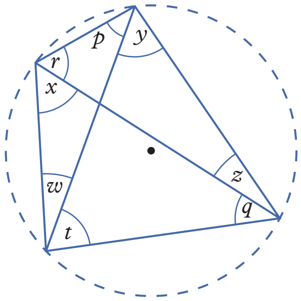


d Explain why quadrilaterals formed using circle theorem 1 (the 'angle at the centre' theorem) are not cyclic.

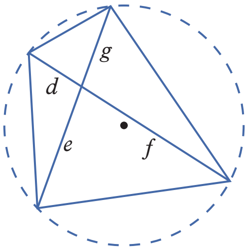


e **i** Consider the the diagram on the right, where diagonals have been added to the cyclic quadrilateral from part **a**.
Explain why $p = q$, $r = t$, $x = y$ and $w = z$.

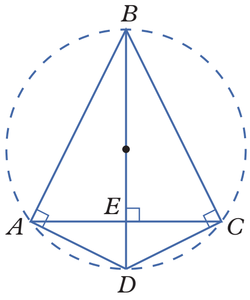
ii If $x = 54°$, determine the values of p, q, r, t, w, y and z.



f Considering only the diagonals of the cyclic quadrilateral shown below, what equation can be written in terms of d, e, f and g?



g Prove that all kites with a pair of right angles, as shown in $ABCD$ below, are cyclic.



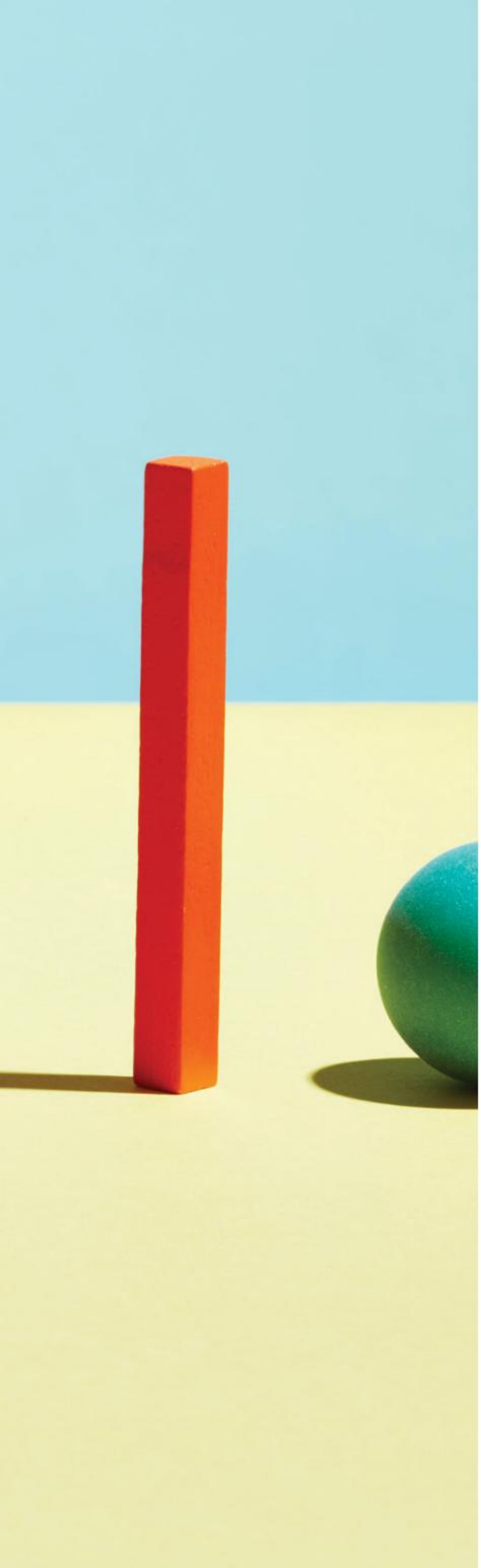
Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Use geometrical properties to solve problems	☐ Go back to Topic 7A Geometrical properties
☐ Find unknown side lengths and angles in congruent and similar triangles ☐ Identify whether a pair of triangles are congruent or similar ☐ Prove that a pair of triangles are congruent or similar	☐ Go back to Topic 7B Congruence and similarity
☐ Prove triangle properties ☐ Prove quadrilateral properties	☐ Go back to Topic 7C Geometric proofs
☐ Understand and apply circle theorems related to angles	☐ Go back to Topic 7D Circle geometry: angles
☐ Understand and apply circle theorems related to chords	☐ Go back to Topic 7E Circle geometry: chords
☐ Understand and apply circle theorems related to tangents and secants	☐ Go back to Topic 7F Circle geometry: tangents and secants

8

Measurement



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Converting between units of length
- ✔ Using Pythagoras' theorem
- ✔ Area of a triangle
- ✔ Area of quadrilaterals
- ✔ Area of a circle

Curriculum links

- Solves problems involving the surface area of right prisms and practical problems involving the area of composite shapes and solids (MA5-ARE-C-01)
- Solves problems involving the volume of composite solids consisting of right prisms and cylinders (MA5-VOL-C-01)
- STN ADV Applies knowledge of the surface area of right pyramids and cones, spheres and composite solids to solve problems (MA5-ARE-P-01)
- STN ADV Applies knowledge of the volume of right pyramids, cones and spheres to solve problems involving related composite solids (MA5-VOL-P-01)

Materials

✔ Calculator

8A Area review

Learning intentions

By the end of this topic you will be able to...

- ✔ calculate the area of composite shapes
- ✔ calculate the area of sectors.

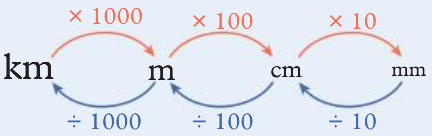
Inter-year links

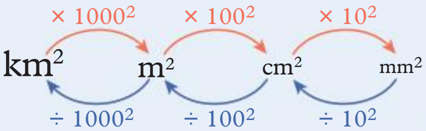
Year 7	8E Area of a triangle
Year 8	8D Area of a circle
Year 9	6A Area of composite shapes

Units of measurement

Key content video

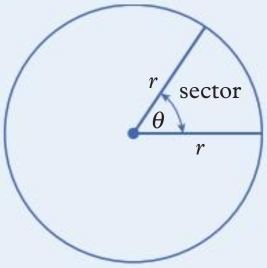
- To convert between units of length, multiply or divide by theappropriate **conversion factor**.
- A unit of **area** can be converted to another unit of area by multiplying or dividing by the appropriate conversion factor.
- 1 ha (hectare) = 10 000 m^2





Composite shapes

- To find the area of a **composite shape**:
  1. split the shape into individual parts
  2. calculate any missing dimensions
  3. calculate the areas of the individual parts
  4. add or subtract the areas to find the total area.
- The area of a sector of a circle is $\frac{\theta}{360°} \times \pi r^2$, where θ is the angle, in degrees, between the two radii forming the sector.



Example 8A.1 Converting units of length and area

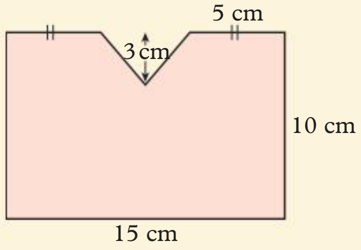
Convert each length or area measurement below to the unit shown in brackets.

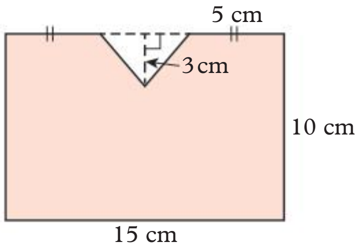
a 5.6 m (cm) **b** 120 mm (m) **c** 1.2 km^2 (m^2) **d** 320 mm^2 (cm^2)

THINK	WRITE
a Identify the correct conversion factor. To convert from metres (m) to centimetres (cm), multiply by 100.	**a** $5.6 \text{ m} = (5.6 \times 100) \text{ cm}$ $= 560 \text{ cm}$
b Identify the correct conversion factor. To convert from millimetres (mm) to metres (m), first divide by 10 (to convert to centimetres), then divide by 100 (to convert to metres).	**b** $120 \text{ mm} = (120 \div 10 \div 100) \text{ cm}$ $= 0.12 \text{ m}$
c Identify the correct conversion factor. To convert from square kilometres (km^2) to square metres (m^2) multiply by 1000^2.	**c** $1.2 \text{ km}^2 = (1.2 \times 1000^2) \text{ m}^2$ $= 1\,200\,000 \text{ m}^2$
d Identify the correct conversion factor. To convert from square millimetres (mm^2) to square centimetres (cm^2), divide by 10^2.	**d** $320 \text{ mm}^2 = (320 \div 10^2) \text{ cm}^2$ $= 3.2 \text{ cm}^2$

Example 8A.2 Calculating the area of a composite shape

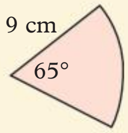
Calculate the area of this composite shape.



THINK	**WRITE**
1 Redraw the shape with dashed lines, splitting it into individual shapes for which you can calculate the area. **2** Label any unlabelled dimensions you know for each individual shape. The total length of the rectangle is 15 cm and the length either side of the triangle is 5 cm. The cut-out triangle must have a base of 5 cm (because 15 – 2 × 5 = 5).	
3 Calculate the area of each individual shape.	rectangle: $A = lb$ $= 10 \times 15$ $= 150\text{ cm}^2$ triangle: $A = \frac{1}{2}bh$ $= \frac{1}{2} \times 5 \times 3$ $= 7.5\text{ cm}^2$
4 Calculate the required area by subtracting the area of the triangle from the area of the rectangle.	total area $= 150 - 7.5$ $= 142.5\text{ cm}^2$

Example 8A.3 Calculating the area of a sector

Calculate the area of this sector correct to one decimal place.



THINK	**WRITE**
1 Identify the values of θ and r.	$\theta = 65°$ $r = 9$ cm
2 Substitute the values into the formula for the area of a sector.	$A = \frac{\theta}{360°} \times \pi r^2$ $= \frac{65°}{360°} \times \pi \times 9^2$
3 Calculate the area, correct to one decimal place, including the appropriate unit.	$= 45.945...$ $\approx 45.9\text{ cm}^2$

Helpful hints

✔ There can be more than one way to split a composite shape into individual component shapes. But the way you split up the shape should not change the total area.

✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, if you are calculating the area of a composite shape, use the full numbers on your calculator when finding the areas of the individual shapes and only round after adding or subtracting.

ANS p754

Exercise 8A Area review

△ 1–3, 4(a, c, e), 5–7, 8(a, c, e), 9, 10 ■ 1, 3(c, e, f), 4(b, d, f), 6, 7(b, d, f), 8, 9, 11, 13, 15 ⬟ 1, 3(c, f), 4(d, f), 6, 7(d, f), 10, 12–16

For questions involving π, use the π key on your calculator.

UNDERSTANDING AND FLUENCY

8A.1 **1** Convert each length or area measurement below to the unit shown in brackets.

a 7 m (cm) **b** 4500 m (km) **c** 8.4 km^2 (m^2)

d 29 mm^2 (cm^2) **e** 24 000 m^2 (ha) **f** 0.88 m (mm)

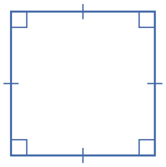
g 34 200 000 cm^2 (km^2) **h** 10.22 km (cm) **i** 3 km^2 (ha)

2 Match each shape below with a formula from this list that could be used to find its area:

a $A = \pi r^2$ **b** $A = l^2$ **c** $A = \frac{1}{2}(a + b)h$

d $A = bh$ **e** $A = \frac{xy}{2}$

I



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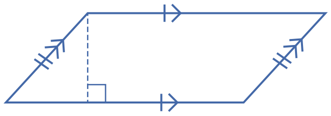

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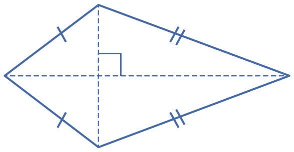

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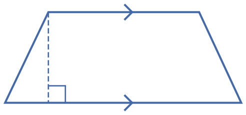

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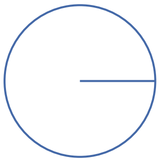
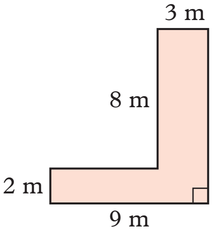
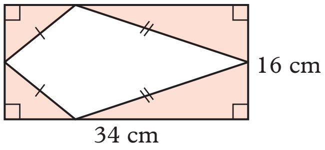


8A.2 **3** Calculate the shaded area of each of these composite shapes. Give your answers to one decimal place where necessary.

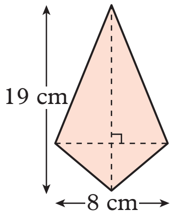
a



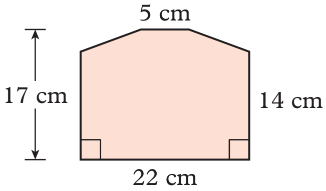
b



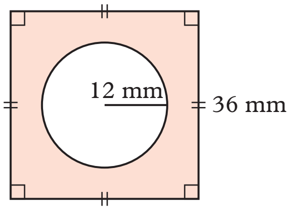
c



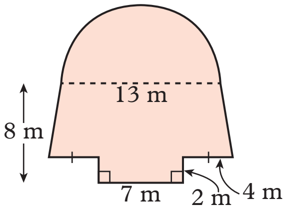
d



e

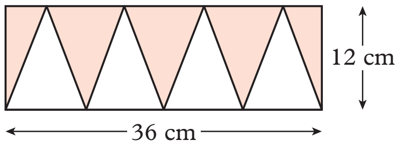


f

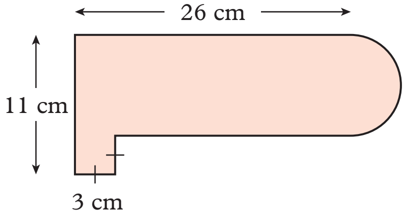


4 Calculate the shaded area of each of these composite shapes. Give your answers in the units given in the brackets, rounded to one decimal place where necessary.

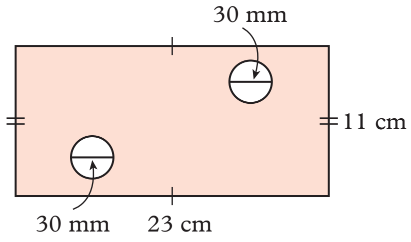
a (cm^2)



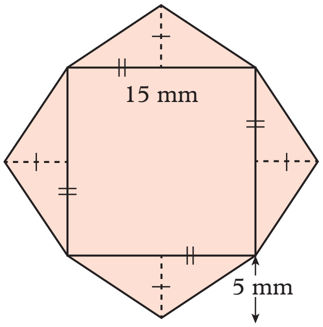
b (cm^2)



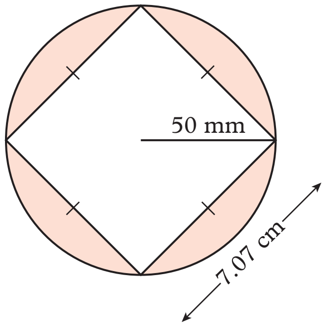
c (cm^2)



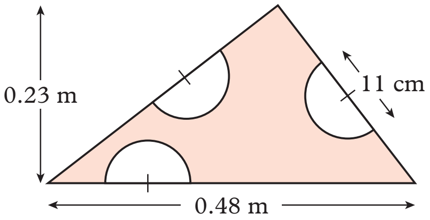
d (mm^2)



e (cm^2)



f (cm^2)



5 The front view of this house is made of a number of basic shapes.

a Identify each shape.

b Calculate the area of each window pane you identified in part **a**.

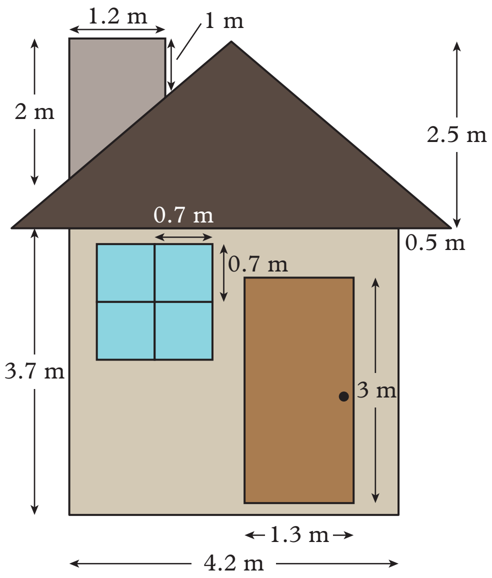
c There are four window panes of equal size. Explain how you can calculate the area of all four window panes if you know the area of only one window pane.

d The owners wish to paint the front of the house (including the door, roof and chimney) and need to find the total area to be painted.

Using your knowledge of how to find the area of a composite shape, explain how to calculate the required area. (Remember that the windows won't need to be painted.)

e Use your answers for parts **c** and **d** to help you find the area of the house to be painted.

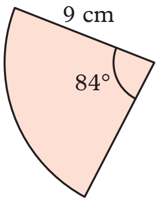
f If 1 L of paint covers 15 m^2, how many litres of paint do the owners need to purchase? Give your answer to the nearest half litre.



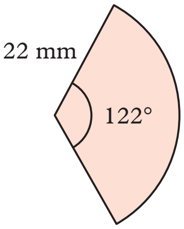
UNDERSTANDING AND FLUENCY

8A.3 **6** Calculate the area of each of these sectors correct to one decimal place.

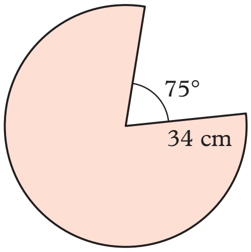
a



b

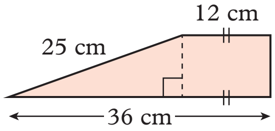


c



7 Use your knowledge of Pythagoras' theorem to find any unknown lengths for each of these composite shapes and then calculate the shaded area of each one. Give your answers to one decimal place where necessary. Hint: Check that all measurements are written in the same units.

a



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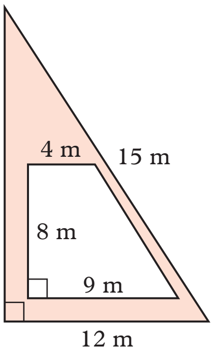
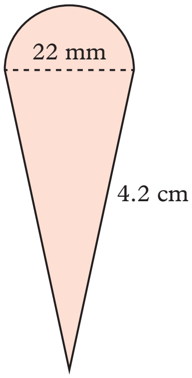


c



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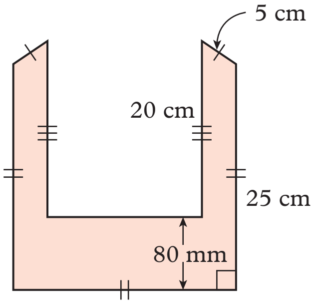

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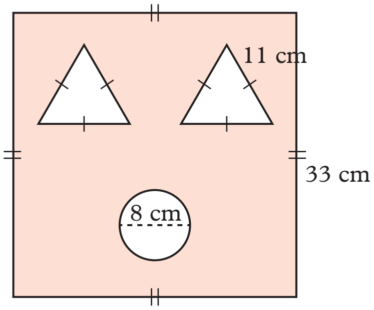

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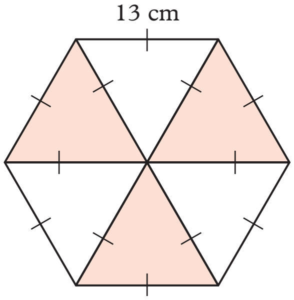
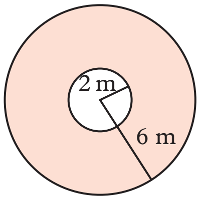


8 An annulus is the region between two circles of different sizes which share the same centre.

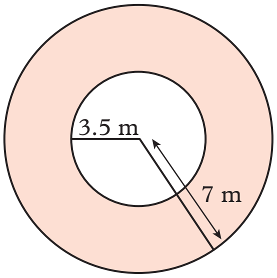
a Explain how you can find the area of the annulus on the right.

b Find the area of the garden bed correct to one decimal place.

c Find the area of the following annuli, correct to one decimal place.



i



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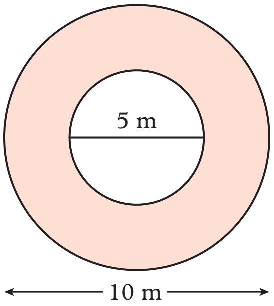

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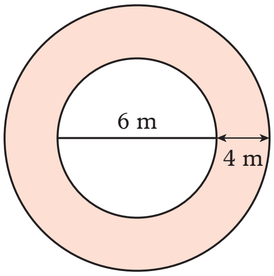


PROBLEM SOLVING AND REASONING

9 A quadrant is a sector with $\theta = 90°$ and a semicircle is a sector with $\theta = 180°$.

a Write a formula for the area of a quadrant with radius *r*.

b Write a formula for the area of a semicircle with radius *r*.

c How do your formulas in parts **a** and **b** relate to the formula for the area of a circle with radius *r*?

d Explain why the area of a sector is $\frac{\theta}{360} \times \pi r^2$, where θ is the angle, in degrees, between the two radii forming the sector.

10 A rectangle has an area of 720 cm^2. One of its sides is five times as long as another.

a If *x* is the length of the smaller side, draw a diagram of the rectangle and label the sides.

b Use this information to find the dimensions of the rectangle.

c If the dimensions are all doubled, what is the area of the new rectangle?

d How does the area of the new rectangle compare to the area of the old rectangle?

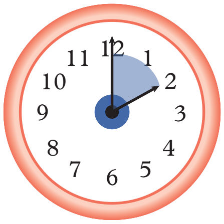
e If the dimensions of the original rectangle are all tripled, what is the area of the new rectangle? Compare this area to the original area of 720 cm^2.

f Explain what happens to the area of the rectangle when the dimensions are all:

i quadrupled

ii multiplied by a factor of *n*.

11 A clock has a minute hand that is 7 cm long and an hour hand that is 5 cm long.



a What is the difference, correct to one decimal place, between the areas covered by the hands when they each complete a full revolution?

b The time shown on the clock face on the right is 2 o'clock. Calculate the angle of the sector between the minute hand and the hour hand.

c Find the difference in the areas that would be covered by the hands if they each moved through the sector from the 12 to the 2 on the clock face. Give your answer correct to one decimal place.

d Find the difference in the area that would be covered by the hands if they each moved through the smaller sector created between them when the time is 7 o'clock. Give your answer correct to one decimal place.

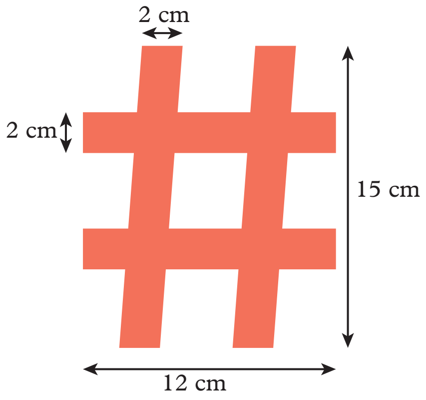
12 A design based on the ancient yin-yang symbol is shown on the right.

a What would be the relationship between the radius of the complete circle and the radius of the semi-circles if they were formed inside it?

b If the radius of the full circle is 24 cm, find the area shaded blue correct to one decimal place (yin).

c Find the area of the circle's white section (yang) without performing any further calculations.

13 The area of a parallelogram is $A = bh$, where *b* is the base length and *h* is the perpendicular height. Find the total shaded area of this hash symbol.



14 Consider the target on the right. The radius of the inner circle is 2 cm and the edge of each ring is a further 2 cm from the centre of the inner circle, as shown.

a Find the area, as an exact value, of the shaded region.

b Find the area, as an exact value, of the unshaded region.

c Find the ratio of the shaded area to the unshaded area.

CHALLENGE

15 A regular hexagon can be thought of as the composite of six congruent equilateral triangles. Give your answers to the following questions as exact values.

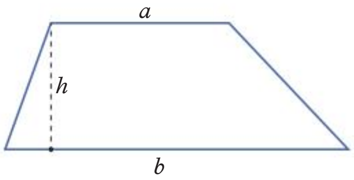
a Use Pythagoras' theorem to find the height, as an exact value, of an equilateral triangle with side lengths 10 cm.

b Hence, find the area of an equilateral triangle with side lengths 10 cm.

c Hence, find the area of a regular hexagon with side lengths 10 cm.

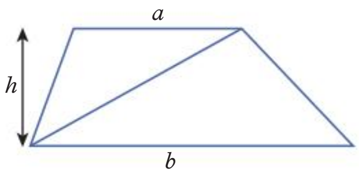
d Find the area of a regular hexagon with side lengths of a.

16 A trapezium is a quadrilateral with a pair of parallel sides. The area of a trapezium is $A = \frac{h}{2}(a + b)$, as shown below.



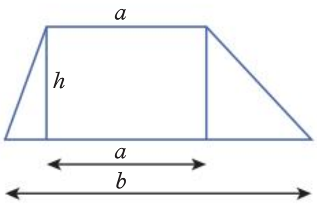
Demonstrate that the trapezium area formula works using the following methods.

a Consider a trapezium as the composite of two triangles, as shown below.

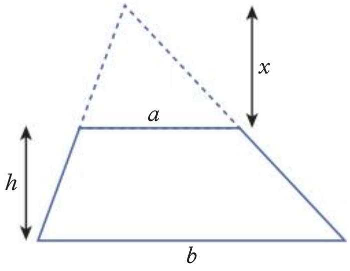


b Consider a trapezium as the composite of one rectangle and two triangles, as shown below.

Hint: you can combine the two triangles into one triangle, since they have the same height.



c Consider a trapezium as a truncated triangle, as shown below.
Hint: The two triangles are similar and so you can substitute out x in the area equation by considering ratios.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Converting between units of area

Interactive skillsheet
Area of composite shapes

Topic quiz
8A

8B Surface area review

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the surface area of prisms and cylinders

✔ calculate the surface area of composite solids consisting of prisms and cylinders.

Inter-year links

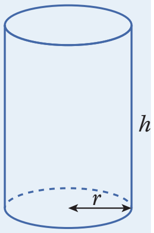
Year 7	7F 3D objects
Year 8	8D Area of a circle
Year 9	6B Surface area

Surface area

Key content video

- The **surface area** (SA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.
- **Prisms** are 3D objects that have a uniform cross-section that is a polygon.
  - → All faces of a prism are flat.
  - → Cubes, rectangular prisms and triangular prisms are all common examples of prisms.
- **Cylinders** are 3D objects with a uniform circular **cross-section** and a curved face. The surface area of the curved face of a cylinder is equal to the circumference ($2\pi r$) of either end multiplied by the height (h):

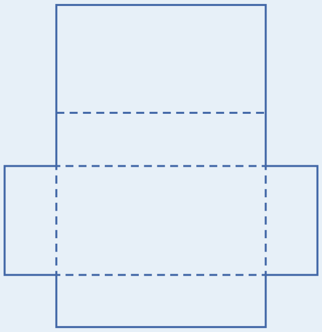
  Area of curved face of a cylinder = $2\pi rh$



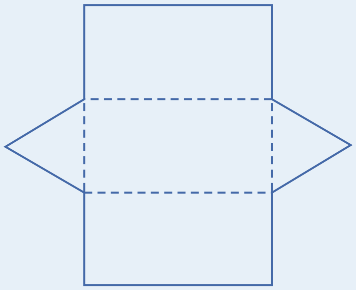
- The surface area of a cylinder (the area of the two circular ends and the curved surface) can be found using the formula:

  $\text{SA} = 2\pi r^2 + 2\pi rh$
- A **net** is a 2D plan that can be folded to form a 3D object. Nets can be used to show all the faces of 3D objects and hence to help calculate the surface areas of those objects.

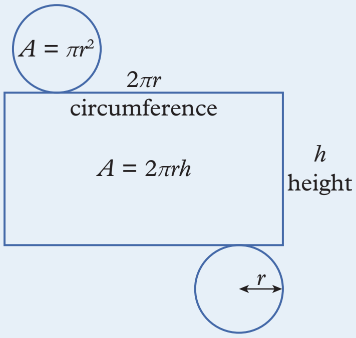
Net of a rectangular prism



Net of a triangular prism

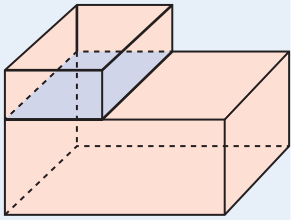


Net of a cylinder



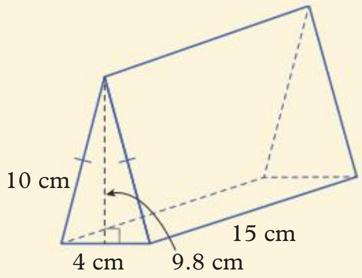
Surface area of composite solids

- A composite solid consists of two or more simple solids.
- To calculate the surface area of a composite solid, identify all faces of the composite solid, add the area of each face to find the surface area of the composite solid.
- The area where two solids join should not be included in the calculation of surface area.



Example 8B.1 Calculating the surface area of a prism

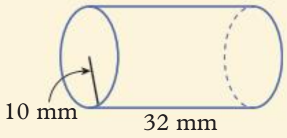
Calculate the surface area of this triangular prism.



THINK	**WRITE**
1 Identify the number of faces and the shapes of the faces of the triangular prism.	The prism has five faces: two identical rectangular faces, two identical triangular faces and a rectangular base.
2 Calculate the area of each face. The triangular faces are isosceles triangles, meaning the two rectangles sharing common sides with them are identical.	rectangular faces: $A = 10 \times 15$ $= 150\ \text{cm}^2$ each rectangular base: $A = 4 \times 15$ $= 60\ \text{cm}^2$ triangular faces: $A = \frac{1}{2} \times 4 \times 9.8$ $= 19.6\ \text{cm}^2$ each
3 Add the areas and include the appropriate unit.	$\text{SA} = 2 \times 150 + 60 + 2 \times 19.6$ $= 399.2\ \text{cm}^2$

Example 8B.2 Calculating the surface area of a cylinder

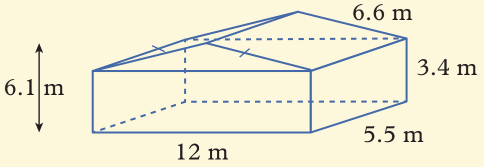
Calculate the surface area of this cylinder, correct to one decimal place.



THINK	**WRITE**
1 Identify the values of r and h for this cylinder.	$r = 10$ mm $\quad h = 32$ mm
2 Substitute the values you identified into the formula for the surface area of a cylinder and simplify.	$\text{SA} = 2\pi r^2 + 2\pi rh$ $= 2 \times \pi \times 10^2 + 2 \times \pi \times 10 \times 32$
3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.	$= 200\pi + 640\pi$ $= 840\pi$ $= 2638.937...$ $\approx 2638.9\ \text{mm}^2$

Example 8B.3 Calculating the surface area of a composite solid

Calculate the surface area of this composite object.

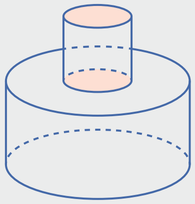


THINK	WRITE
1 Identify the number of faces and the shapes of the faces of the solid.	The solid has nine faces: three pairs of identical rectangular faces (including the pair of sloping faces), a pair of identical triangular faces and the rectangular base.
2 Calculate the area of each face.	rectangular faces (front and back): $A = 3.4 \times 12$ $= 40.8\text{ m}^2$ each rectangular faces (sides): $A = 3.4 \times 5.5$ $= 18.7\text{ m}^2$ each rectangular faces (sloping faces): $A = 6.6 \times 5.5$ $= 36.3\text{ m}^2$ each rectangular base: $A = 12 \times 5.5$ $= 66\text{ m}^2$ triangular faces: $A = \frac{1}{2} \times (6.1 - 3.4) \times 12$ $= \frac{1}{2} \times 2.7 \times 12$ $= 16.2\text{ m}^2$ each
3 Add the areas and include the appropriate unit.	$\text{SA} = 2 \times 40.8 + 2 \times 18.7 + 2 \times 36.3 + 66 + 2 \times 16.2$ $= 81.6 + 37.4 + 72.6 + 66 + 32.4$ $= 290\text{ m}^2$

Helpful hints

✔ If you can't remember the formula to calculate the surface area of a prism or cylinder, consider drawing a net of the solid and calculating the area of the shapes making up the net.

✔ In some instances, the area where the faces of solids join in a composite solid is the same as the area of the top face of the combined solid. Identifying this can help with your calculations. For example, in the diagram on the right the surface area of the composite solid is the surface area of the bottom cylinder plus just the curved surface area of the top cylinder because its exposed circular end is the same as the joined surface calculated as part of the surface area of the bottom cylinder.



✔ The surface area of a prism can also be found by using the formula:

$\text{SA} = 2 \times \text{area of base} + \text{perimeter of base} \times \text{height}$

ANS p755

Exercise 8B Surface area review

△ 1(a, b, f, g, i), 2–7, 9, 11(a)

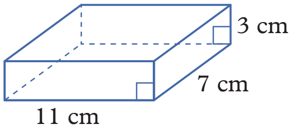
□ 1(d, e, h, i), 3–9, 11, 12(a), 13(a), 14

⬠ 1(f, h, i), 2, 6–8, 11, 12(b), 13–15

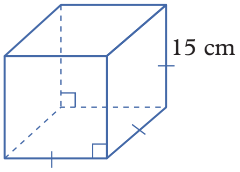
UNDERSTANDING AND FLUENCY

8B.1 **1** Calculate the surface area of each of these prisms.

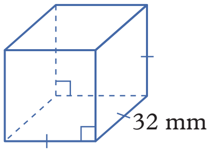
a



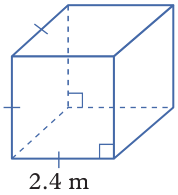
b



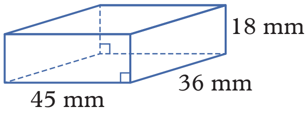
c



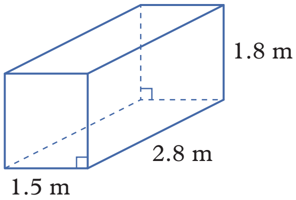
d



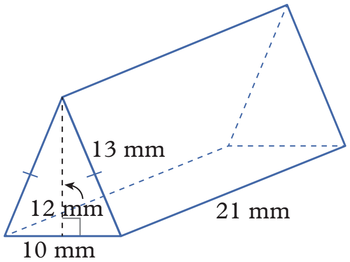
e



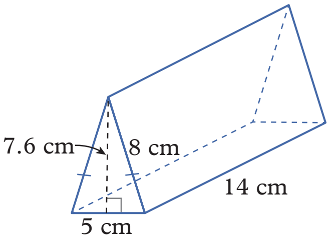
f



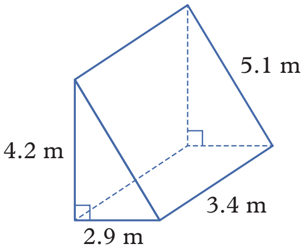
g



h

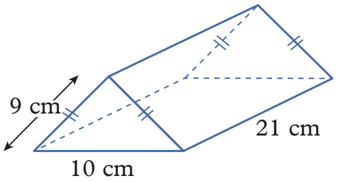


i

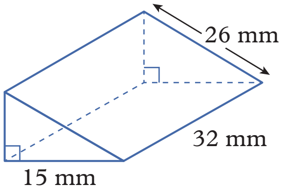


2 Calculate the surface area of each of these triangular prisms correct to one decimal place.
Hint: Use Pythagoras' theorem.

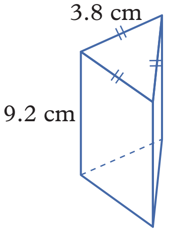
a



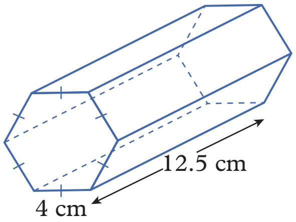
b



c

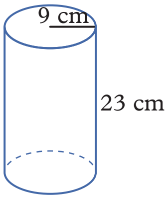


3 If the base of this hexagonal prism has an area of 41.52 cm^2, calculate surface area.

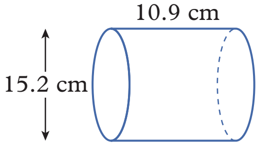


8B.2 **4** Calculate the surface area of each of these cylinders correct to one decimal place.

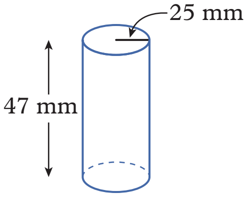
a



b

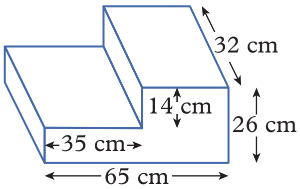


c

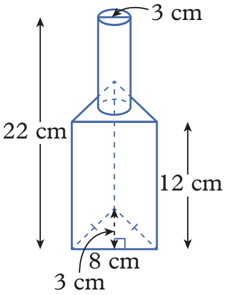


8B.3 **5** Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place where appropriate.

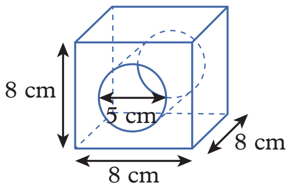
a



b



c



6 Each die in this stack is the shape of a cube with side lengths of 15 mm.

a Calculate the surface area of one die.

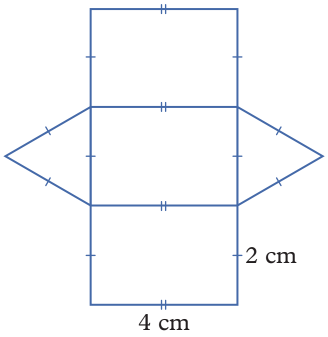
b Use your answer for part **a** to calculate the surface area of nine separate dice.

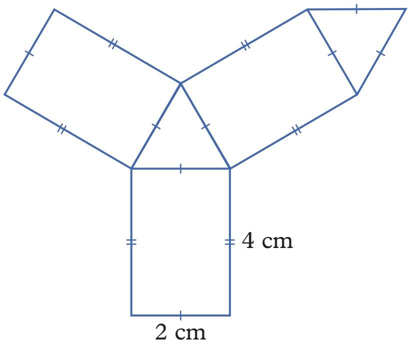
c When stacked, the nine dice form a rectangular prism. Using the information you have been given about each die, draw the rectangular prism in your books and label its dimensions.

d Calculate the surface area of the rectangular prism made of nine dice.

e Find the difference between your answers to parts **b** and **d**. Explain why these answers differ. Support your explanation with calculations.

7 The net of a 3D object is not unique. Here are two nets of the same triangular prism.





a Draw the triangular prism corresponding to the two nets above.

b Draw a third net of the triangular prism, which is different to the two given above.

c Are all diagrams consisting of two equilateral triangles with 2 cm sides and three 4 cm × 2 cm rectangles nets of the same triangular prism?

8 An open pipe has a length of 33 cm and a diameter of 55 mm.

a Calculate the surface area of the outer surface of the pipe correct to the nearest cm^2.

b To find your answer for part **a**, did you need to use the entire formula for the surface area of a cylinder: SA = $2\pi rh + 2\pi r^2$? Explain.

9 Find the surface area of the outside of an open cylinder that has a radius of 4.5 cm and a height of 9.9 cm. Give your answer correct to one decimal place.

10 The teachers at a kindergarten want to add two cylindrical poles to the playground. One pole is to be painted yellow and the other red. The vertical poles have the same diameter of 125 mm, but they vary in height.

a The yellow pole is 1.8 m tall. Calculate the area needing to be painted yellow correct to one decimal place. Assume that the top end and the side of the pole needs painting but the end in the ground does not.

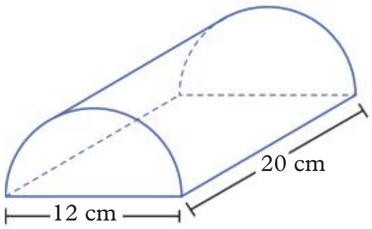
b The red pole is double the height of the yellow pole. Compared to the amount of yellow paint needed, do the teachers need to purchase double the amount of red paint in order to paint this second pole? Why or why not?

c Calculate the area of the pole needing to be painted red, in centimetres correct to one decimal place. Does your result support your answer to part **b**?

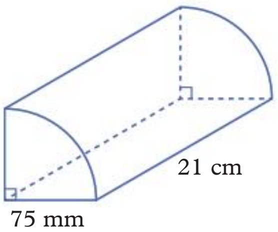
PROBLEM SOLVING AND REASONING

11 Find the surface area of the following cut cylinders. Give your answers in cm^2 correct to one decimal place.

a

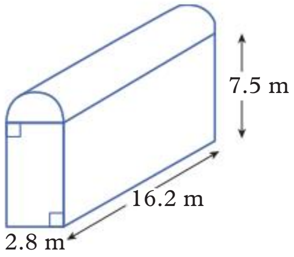


b

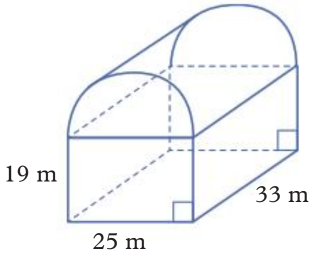


12 Calculate the surface area of each of these composite solids correct to one decimal place.

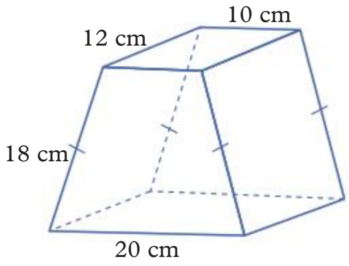
a



b



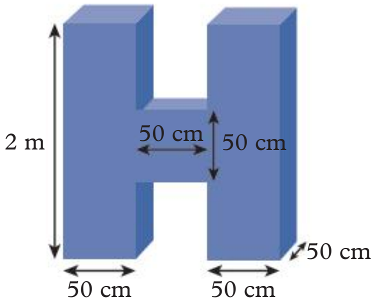
13 Calculate the surface area of this prism.



CHALLENGE

14 Seven cylinders with the same height but different diameters are stacked together as shown on the right. The top cylinder has a diameter of 1 cm and the diameter of each cylinder beneath it increases by 1 cm each time. The bottom cylinder has a diameter of 7 cm. If each cylinder is the same height, 1.5 cm, calculate the surface area when the cylinders are sitting one on top of the other, as shown. Give your answer correct to one decimal place.

15 A company commissions an artist to make a giant version of their logo, an H, for the foyer of their office building. The specifications are in the diagram on the right. The H will be made of wood and then every face of the H will be painted, except the two square faces on the ground.



a Calculate the surface area of the H, not including the two rectangles touching the ground. Answer in square metres.

b The H will be painted blue, as that is the company's colour. The company plans to use a premium blue paint which is sold in 2 L cans. Each litre of this paint is enough to paint one coat on 4 square metres. How many cans of paint must they buy if they paint the H with a triple coat?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Surface area of prisms	**Interactive skillsheet** Surface area of cylinders	**Investigation** Net or not?	**Topic quiz** 8B

8C Volume review

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the volume of prisms and cylinders

✔ calculate the volume of composite solids consisting of prisms and cylinders.

Inter-year links

Year 7 8F Volume and capacity

Year 8 8E Volume and capacity

Year 9 6D Volume of composite solids

Volume

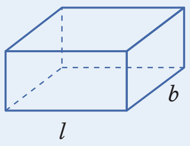
Key content video

- **Volume** is measured in cubic units, such as mm³, cm³ and m³.
- A unit of volume can be converted to another unit of volume by multiplying or dividing by the appropriate conversion factor.

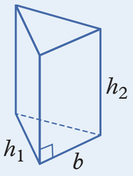
$$\begin{tikzcd}
\mathrm{km}^3 \arrow[r, bend left, "\times 1000^3"] & \mathrm{m}^3 \arrow[l, bend left, "\div 1000^3"] \arrow[r, bend left, "\times 100^3"] & \mathrm{cm}^3 \arrow[l, bend left, "\div 100^3"] \arrow[r, bend left, "\times 10^3"] & \mathrm{mm}^3 \arrow[l, bend left, "\div 10^3"]
\end{tikzcd}$$

- The volume of any prism can be found using the formula $V = Ah$, where A is the area of the base and h is the height of the prism.
  - → The height is perpendicular to the base.
  - → When dealing with a triangular prism, subscripts can be used to represent the two heights, for example h_1 and h_2, as shown in the diagram of a triangular prism on the right, where h_1 is the height of the triangular base.
- The volume of a cylinder can be found by multiplying the area of the base (πr^2) by the height (h).

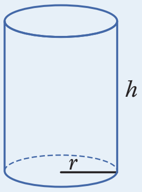
  $V = \pi r^2 h$



$V = lbh$



$V = \frac{1}{2}bh_1h_2$



Volume of composite solids

- The volume of a composite solid can be found by calculating the volumes of the individual component solids and adding them together.

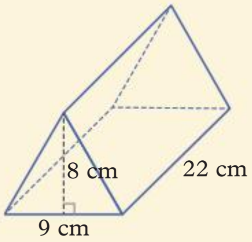
Capacity

- The **capacity** of a 3D object is a measure of how much the object can hold. A container with an inside volume of:
  - → 1 cm³ holds 1 mL of liquid
  - → 1000 cm³ holds 1 L of liquid
  - → 1 m³ holds 1 kL of liquid.

$$\begin{tikzcd}
\mathrm{kL} \arrow[r, bend left, "\times 1000"] & \mathrm{L} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \mathrm{mL} \arrow[l, bend left, "\div 1000"]
\end{tikzcd}$$

Example 8C.1 Calculating the volume of a prism

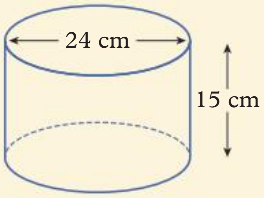
Calculate the volume of this prism.



THINK	**WRITE**
1 Identify the base of the triangular prism and the perpendicular height.	The base is a triangle with base length 9 cm and height 8 cm. The height (h) of the prism is 22 cm.
2 Calculate the area of the base.	$A = \frac{1}{2}bh$ $= \frac{1}{2} \times 9 \times 8$ $= 36 \text{ cm}^2$
3 Substitute the value of the area of the base and the perpendicular height into the formula for the volume of a prism.	$h = 22 \text{ cm}$ $V = bh$ $= 36 \times 22$
4 Include the appropriate unit in your answer.	$= 792 \text{ cm}^3$

Example 8C.2 Calculating the volume of a cylinder

Calculate the volume of this cylinder. Give your answer to one decimal place.



THINK	**WRITE**
a Calculate the radius measurement of the circular base.	$r = D \div 2$ $= 24 \div 2$ $= 12 \text{ cm}$
b Substitute the values for r and h into the formula for the volume of a cylinder.	$h = 15 \text{ cm}$ $V = \pi r^2 h$
c Calculate using the π key on your calculator, round to one decimal place, and include the appropriate unit in your answer.	$= \pi \times 12^2 \times 15$ $= 2160\pi$ $= 6785.840...$ $\approx 6785.8 \text{ cm}^3$

Helpful hints

✔ If the dimensions of a 3D object are given using different units, convert all lengths to the unit specified in the question before you do anything else.

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Exercise 8C Volume review

△ 1, 2(c–i), 3, 4, 5(a, c), 6(a, c, e, g), 7, 8, 9(b), 12, 15

□ 1(b, d, f, h, j), 2(d, e, i, j, k, l), 3(c–e), 5(c), 6(b, d, f, h), 9(a, b), 10, 11, 17

⬠ 1(g–j), 2(i–l), 6(g–i), 9(b, c), 13, 14, 16, 18, 19

UNDERSTANDING AND FLUENCY

1 Convert each of these volume measurements to the unit shown in brackets.

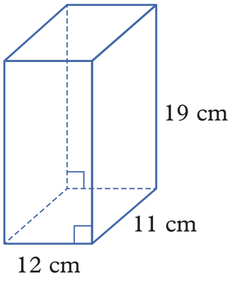
a 12.5 m^3 (cm^3) **b** 240 000 mm^3 (cm^3) **c** 34 200 000 cm^3 (m^3) **d** 0.55 m^3 (mm^3)

e 67 200 mm^3 (m^3) **f** 0.000 009 m^3 (mm^3) **g** 7520 mm^3 (cm^3) **h** 8.74 m^3 (cm^3)

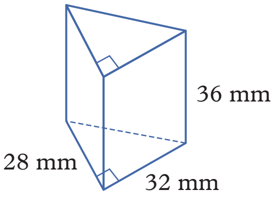
i 142 900 cm^3 (m^3) **j** 73 000 000 mm^3 (m^3)

8C.1 2 Calculate the volume of each of these prisms. Round your answers to one decimal place where necessary.

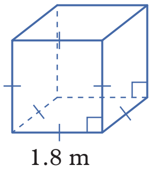
a



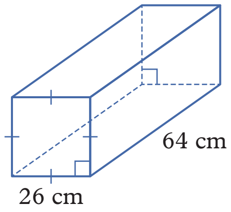
b



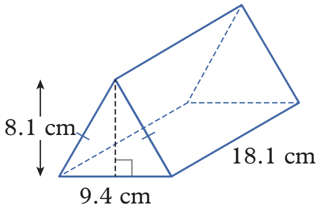
c



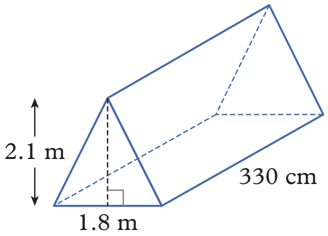
d



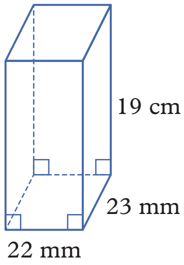
e



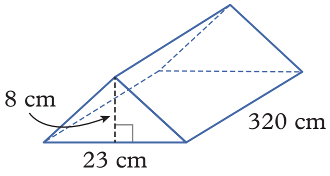
f



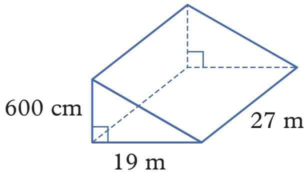
g



h



i



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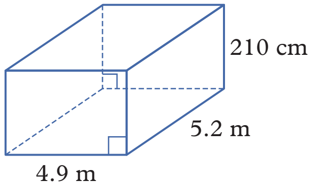
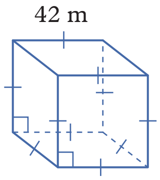
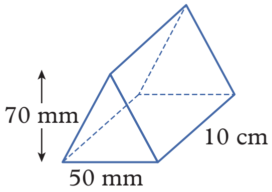


k

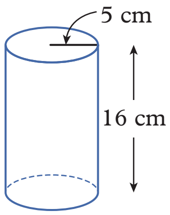


l

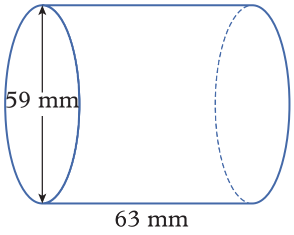


8C.2 3 Calculate the volume of each of these cylinders. Give your answers correct to one decimal place.

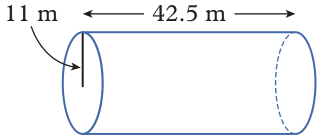
a



b



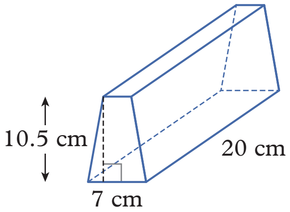
c



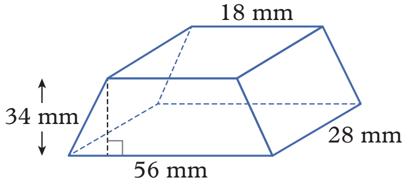
4 Find the capacity in millilitres (mL) or litres (L) of each of the prisms in question **2** correct to one decimal place.

5 Calculate the volume of each of these prisms.

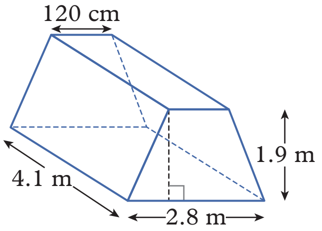
a



b

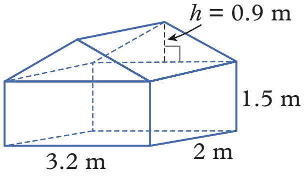


c

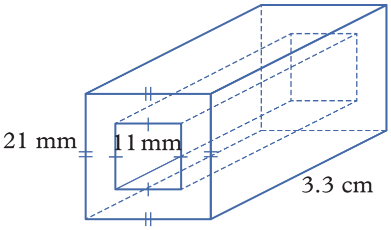


6 Calculate the volume of each of these composite solids. Where appropriate, give your answers correct to one decimal place. Hint: Identify the individual solids first.

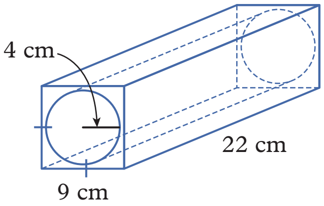
a



b



c



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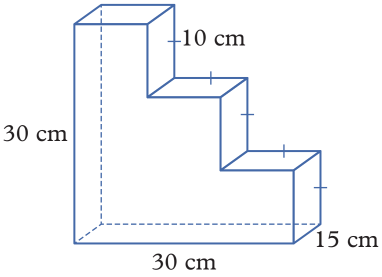
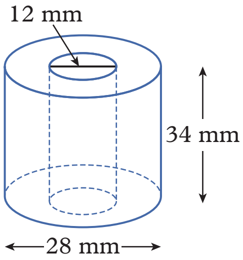


e



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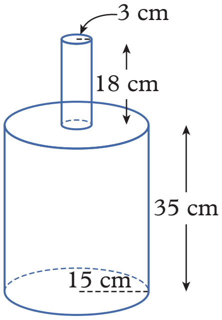

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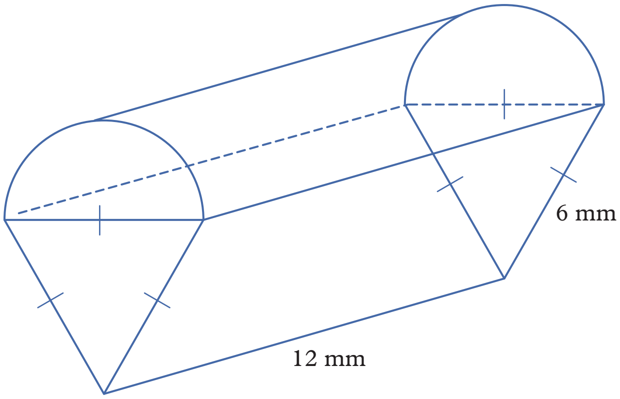

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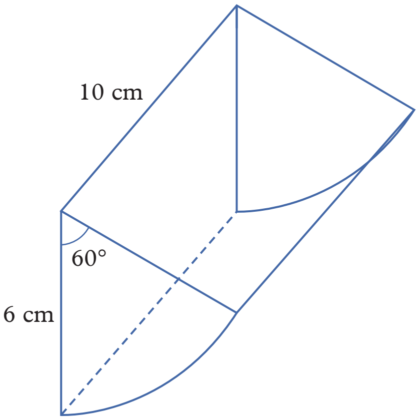
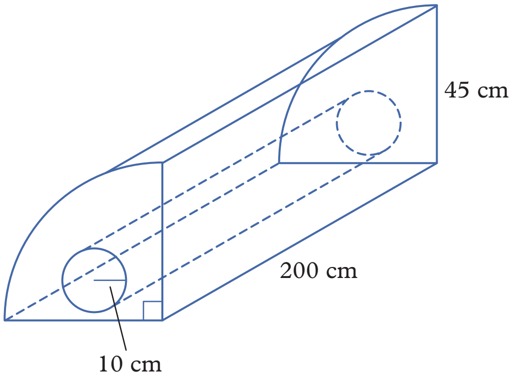


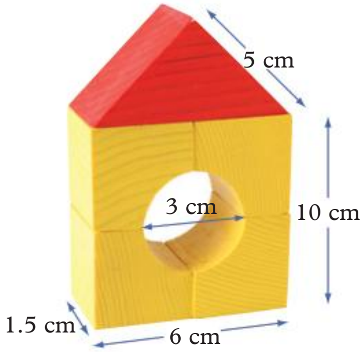
i



7 A round hatbox has a diameter of 23 cm and a height of 12.5 cm. Calculate its volume correct to one decimal place.

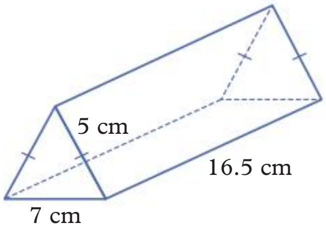
8 In this toy house, constructed from different shaped blocks, there are three common solids that can be identified.

- **a** Identify the three solids.
- **b** Explain how you can calculate the surface area of the toy house.
- **c** Find the surface area of the toy house correct to the nearest cm^2.
- **d** Find the volume of wood used to make this toy house, correct to the nearest cm^3.

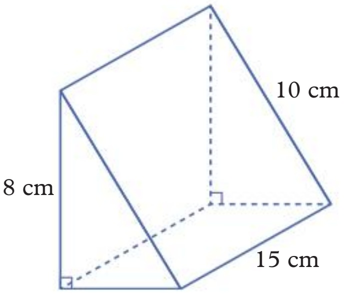


9 Use Pythagoras' theorem to help you calculate the volume of each of these prisms. Give your answers correct to one decimal place where appropriate.

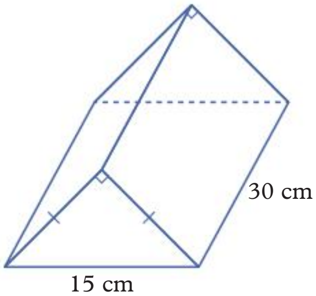
a



b

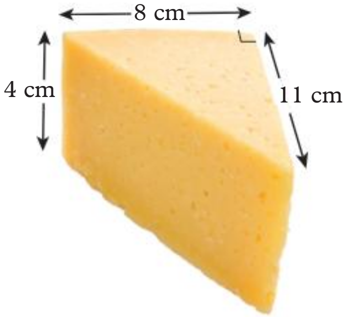


c



10 This piece of cheese is in the shape of a right-angled triangular prism.

- **a** Calculate its volume.
- **b** If 1 cm^3 of cheese has a mass of 0.001 kg, find the mass of the cheese, in kilograms, correct to four decimal places.
- **c** This particular cheese costs \$21 per kilogram. What is the cost of the piece of cheese shown to the nearest cent?



11 A rectangular fish tank is 45 cm long, 20 cm wide and 50 cm tall.

- **a** Calculate the volume of the fish tank.
- **b** What is the capacity of the fish tank if it is filled to the brim? Give your answer in millilitres and litres. (Remember: 1 cm^3 = 1 mL)
- **c** The fish tank should only be filled to a level 6 cm from the top. How many litres of water are required to fill the tank to that height?

12 The volume of a particular prism is 2744 cm^3. If the prism is a cube, what is its side length?

13 **a** If a cylinder has a volume of 1649.34 cm^3 and a radius of 5 cm, what is the height of the cylinder, to the nearest centimetre?

- **b** If a cylinder has a volume of 207 cm^3 and a height of 5.7 cm, find the radius of its circular base, to the nearest millimetre.

14 Jack installs two cylindrical water tanks. The first tank is 2 m tall and the diameter of its base is 1.5 m. The second tank is 1.5 m tall and the diameter of its base is 2 m.

- **a** Which tank has the larger capacity?
- **b** What is the difference in the capacity of the two tanks? Give your answer to the nearest litre.

PROBLEM SOLVING AND REASONING

15 a A particular cylinder has a radius of 8 cm and a height of 15 cm. Calculate its volume correct to one decimal place.

b If the radius of the cylinder from part **a** is halved but the height remains the same:

- **i** calculate the volume by rounding your answer to the nearest cm^3
- **ii** describe how this volume compares to the volume of the original cylinder.

c If the radius of the cylinder from part **a** is doubled but the height remains the same:

- **i** calculate the volume correct to one decimal place
- **ii** describe how this volume compares to the volume of the original cylinder.

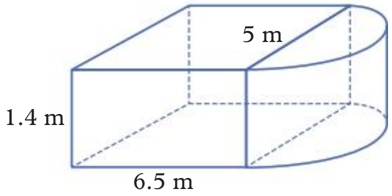
d If the radius of the cylinder from part **a** remains the same but the height is halved:

- **i** calculate the volume by rounding your answer to the nearest cm^3
- **ii** describe how this volume compares to the volume of the original cylinder.

e If the radius of the cylinder from part **a** remains the same but the height is doubled:

- **i** calculate the volume correct to one decimal place
- **ii** describe how this volume compares to the volume of the original cylinder.

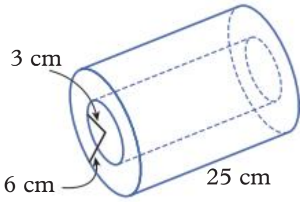
16 A swimming pool is the shape of a rectangular prism. A semi-circular spa is built onto one end of the pool, as shown in the diagram on the right.



a How much water is required if the pool and spa are filled to capacity? Give your answer to the nearest litre.

b To make the pool and spa functional, the water levels must be below the top edge. If the pool is filled to 70% of its depth and the spa is filled to 60% of its depth, how much water is required, to the nearest litre?

17 The plastic pipe on the right is hollow.



a What shape is the base of the pipe? Calculate its area correct to one decimal place.

b Use the area of the base to calculate the volume of plastic used for this pipe by rounding your answer to the nearest cm^3.

c Can you think of a different method for finding the volume? Explain your answer.

CHALLENGE

18 Nicko wishes to make an open box from a 30 cm by 30 cm piece of red cardboard. He decides to cut out a square of length 2 cm from each corner and fold up the sides to form the box. The edges of the box will be joined by sticky tape, so no tabs are needed to create the box.

a Draw a diagram showing the piece of cardboard with the corners cut out. Label the dimensions.

b Determine the length, breadth and height of the box once Nicko has folded up the sides.

c Calculate the volume of the box.

d Would the volume of the box increase or decrease if he cut out 3 cm square corners from the piece of cardboard before forming the rectangular prism? Explain your answer.

19 These nuts have the same dimensions but different shapes. One is circular and the other is in the shape of a regular hexagon. They both have a depth of 6.2 mm.

a How much steel is required to manufacture each nut? Give your answers to one decimal place.

b Which nut requires more steel?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Volume of prisms	**Interactive skillsheet** Volume of cylinders	**Worksheet** Surface area and volume of prisms and cylinders	**Topic quiz** 8C

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Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

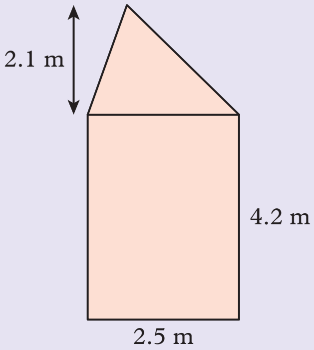
8A **1** Convert each length and area measurement to the unit shown in brackets.

a 45 mm (cm) **b** 400 cm (m)

c 4.53 m (mm) **d** 4000 mm^2 (cm^2)

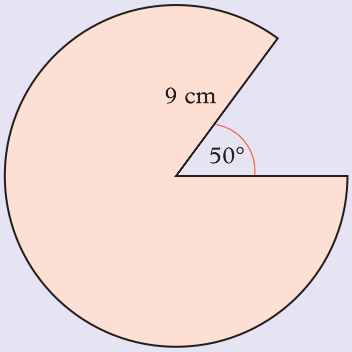
e 5.2 km^2 (m^2) **f** 68 000 cm^2 (m^2)

8A **2** Find the area of the following composite shape.

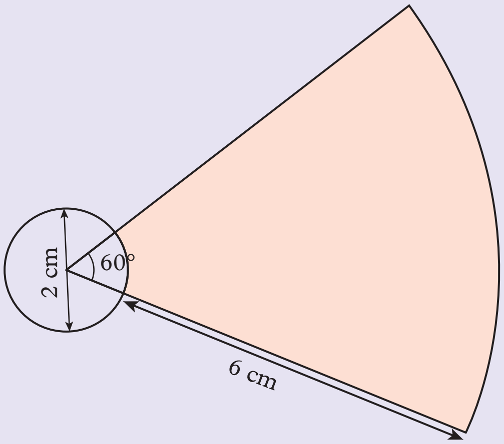


8A **3** Find the area of the following composite shapes. Give your answer correct to the nearest whole number.

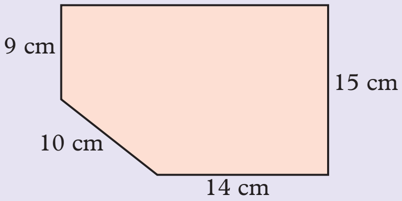
a



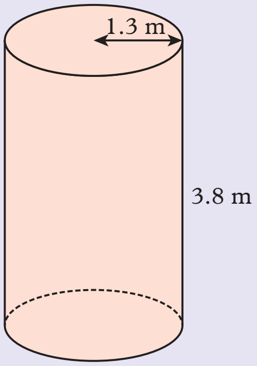
b



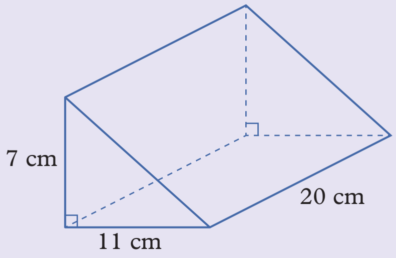
8A **4** Use Pythagoras' theorem to help determine the area of the following shape.



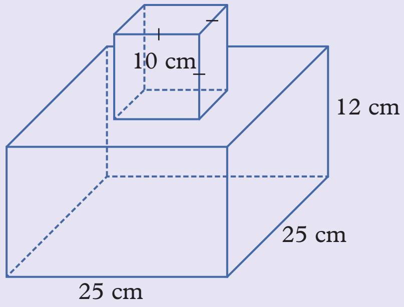
8B **5** Find the surface area of the following cylinder correct to one decimal place.



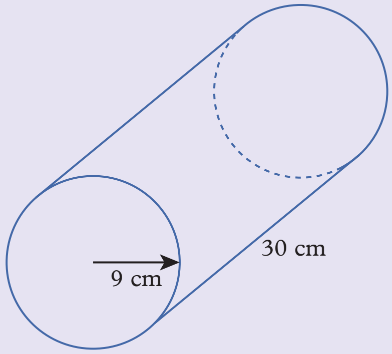
8B **6** Use Pythagoras' theorem to help find the surface area of the following prism correct to one decimal place.



8B **7** A cube with side lengths of 10 cm sits on top of a square-based rectangular prism, as shown below. Find the surface area of the composite solid.

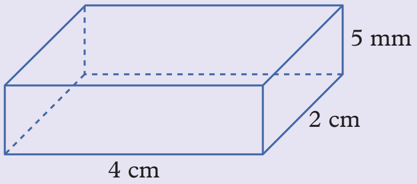


8C **8** Calculate the volume of the following cylinder correct to one decimal place.

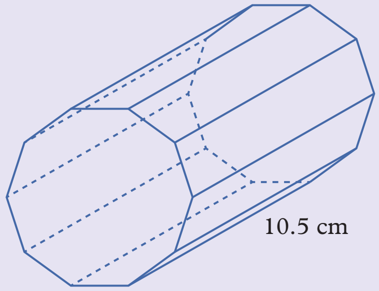


8C **9** Find the volume of this rectangular prism in:

a mm^3 **b** cm^3



8C **10** The following prism has a cross-sectional area of 123.11 cm^2. Find the volume of the prism.



8D Surface area of pyramids and cones

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the surface area of right pyramids and right cones.

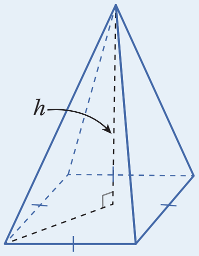
Inter-year links

Year 8	8B Area of triangles and rectangles
Year 9	6C Surface area of cylinders

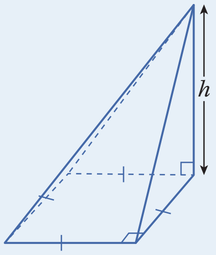
Pyramids

Key content video

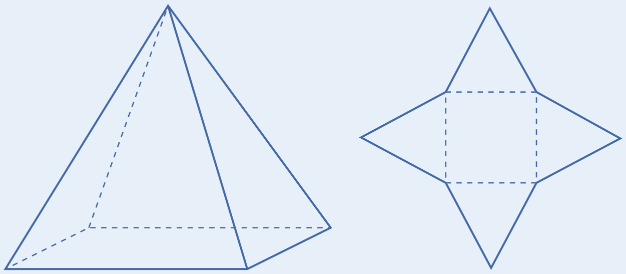
- A **pyramid** is a 3D object with a polygon as its base. All other faces of a pyramid are triangles that meet at the apex (highest point) of the pyramid.
  - → Pyramids are named according to the shape of their base.
  - → A **right pyramid** has an apex directly above the centre of its base.
  - → An **oblique pyramid** does not have its apex over the centre of its base.
- The surface area of a pyramid is the sum of the areas of all of its faces.



right square pyramid



oblique square pyramid

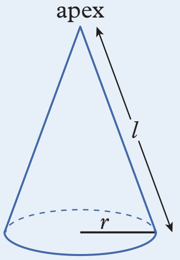


- A **tetrahedron** is a pyramid with a triangular base.
  A **regular tetrahedron** has four triangular faces that are congruent equilateral triangles.
- If the length of the base and the slant edge of a pyramid are known, Pythagoras' theorem can be used to find the height of a triangular face.

Cones

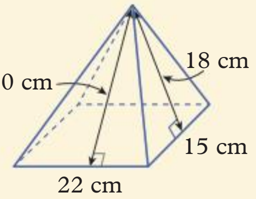
- A **cone** is a 3D object with a circular base and a curved surface that tapers from the circular base to the apex.
- A **right cone** has its apex directly above the centre of its base.
- The surface area of a right cone with radius r and slant height l is given by the formula: SA = $\pi r^2 + \pi rl$.
- The net of a cone consists of a circle and a sector of a circle.

Note: In this topic, only right pyramids and right cones are considered.



Example 8D.1 Calculating the surface area of a pyramid

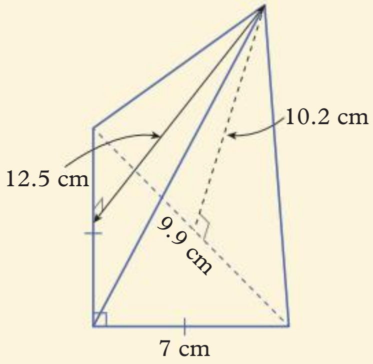
Calculate the surface area of the right rectangular pyramid.



THINK	WRITE
1 Identify the number and shape of the faces of the pyramid. Since it is a right rectangular pyramid, the triangular faces opposite each other are congruent.	The pyramid has five faces: one rectangular face and two pairs of congruent triangular faces.
2 Calculate the area of each face.	rectangular face: $A = 22 \times 15$ $= 330\text{ cm}^2$ front triangular face: $A = \frac{1}{2} \times 22 \times 20$ $= 220\text{ cm}^2$ side triangular face: $A = \frac{1}{2} \times 15 \times 18$ $= 135\text{ cm}^2$
3 Add the areas of the faces together and include the appropriate unit.	$\text{SA} = 330 + 2 \times 220 + 2 \times 135$ $= 1040\text{ cm}^2$

Example 8D.2 Calculating the surface area of a tetrahedron

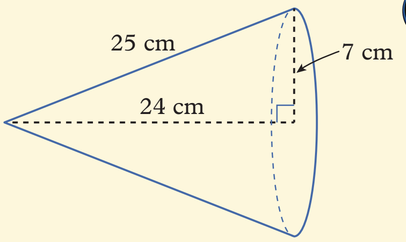
Calculate the surface area of the tetrahedron on the right. The apex of the tetrahedron is directly above the centre of the 9.9 cm edge.



THINK	WRITE
1 Identify the number and shapes of the faces of the tetrahedron.	The base is an isosceles right-angled triangle: $b = 7\text{ cm}, h = 7\text{ cm}$ There are two identical triangular faces: $b = 7\text{ cm}, h = 12.5\text{ cm}$ There is one other triangular face: $b = 9.9\text{ cm}, h = 10.2\text{ cm}$
2 Calculate the area of each face.	base: $A = \frac{1}{2} \times 7 \times 7$ $= 24.5\text{ cm}^2$ triangular face 1: $A = \frac{1}{2} \times 7 \times 12.5$ $= 43.75\text{ cm}^2$ each triangular face 2: $A = \frac{1}{2} \times 9.9 \times 10.2$ $= 50.49\text{ cm}^2$
3 Add the areas of the faces together and include the appropriate unit.	$\text{SA} = 24.5 + 2 \times 43.75 + 50.49$ $= 162.49\text{ cm}^2$

Example 8D.3 Calculating the surface area of a cone

Calculate the surface area of this cone correct to one decimal place.



THINK	**WRITE**
1 Determine the radius and the slant height of the cone.	$r = 7$ cm, $l = 25$ cm
2 Write the formula for the surface area of a cone. Substitute values for r and l and simplify.	$\text{SA} = \pi r^2 + \pi rl$ $= \pi \times 7^2 + \pi \times 7 \times 25$ $= 49\pi + 175\pi$ $= 224\pi$
3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.	$= 703.716...$ $\approx 703.7 \text{ cm}^2$

Helpful hints

- ✔ If a tetrahedron is not regular, you'll need to work out the area of more than one triangular face to calculate the surface area.
- ✔ If the base of a right pyramid is a regular polygon, then the surface area can also be found using the formula:
  $$\text{SA} = \text{base area} + \frac{1}{2} \times \text{perimeter of base} \times \text{slant height}.$$
- ✔ The surface area of a right cone with radius r and slant height l is given by the formula:
  $$\text{SA} = \pi r^2 + \pi rl.$$

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Exercise 8D Surface area of pyramids and cones

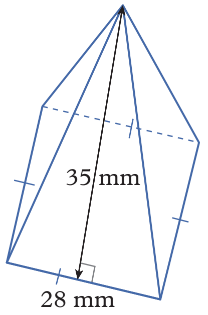
▲ 1–4, 5(a, d, e), 6, 7, 10

■ 1(c, e, f), 2–4, 5(d–f), 6–8, 11, 12(b), 13

⬟ 1(e, f), 2, 5(d–f), 6, 9, 11–15

8D.1 **1** Calculate the surface area of each of these right pyramids.

a



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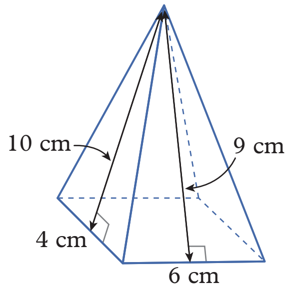
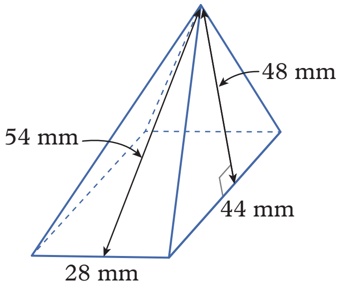
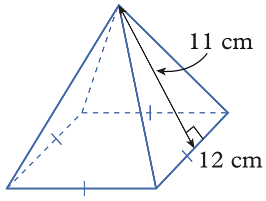


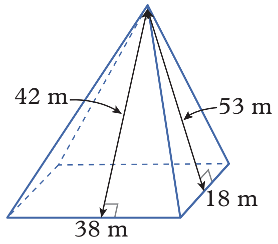
c



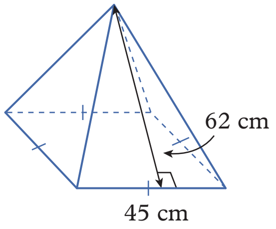
d



e



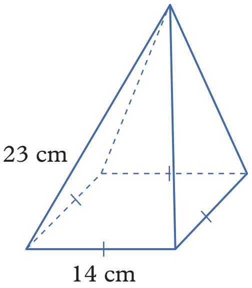
f



UNDERSTANDING AND FLUENCY

2 To calculate the surface area of each of these right pyramids, first find the height of a triangular face. Give your answers to one decimal place where necessary.

a



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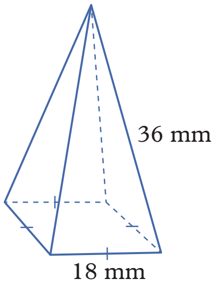

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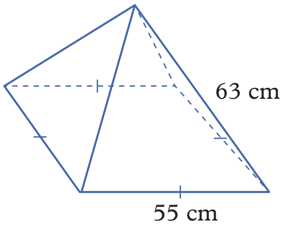


3 At the main entrance to the Louvre Museum in Paris, France, a pyramid has been constructed with a square base of length 35 m. Each identical sloping triangular face has a height of 27 m and is made of glass segments. Find the surface area of the glass walls of the pyramid.

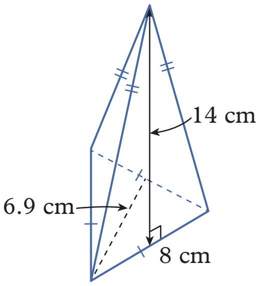
4 The Red Pyramid in Dahshur, Egypt, was originally covered in white limestone. It has a slant edge of length 187 m and a square base of side length 220 m.

a Draw a diagram of this pyramid and label its dimensions.

b Find the surface area of the pyramid that was covered with limestone. Hint: Do not include the base in your calculations.

8D.2 **5** Calculate the surface area of each of these tetrahedrons correct to one decimal place. Hint: Use Pythagoras' theorem to find any missing heights.

a



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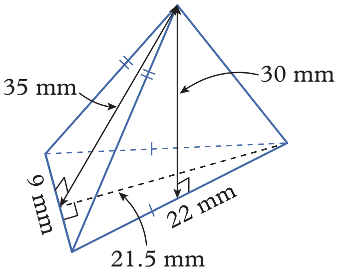

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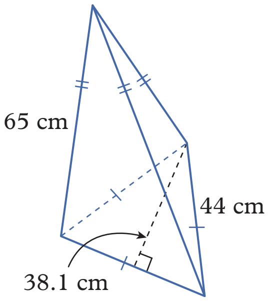

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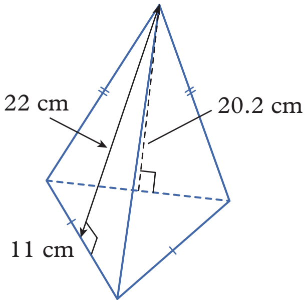

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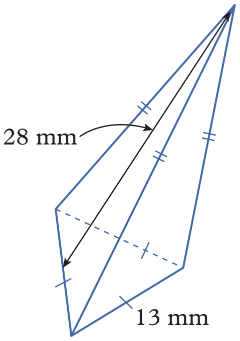

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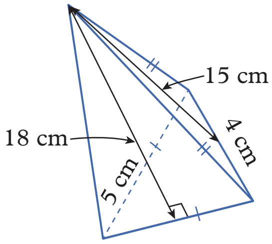
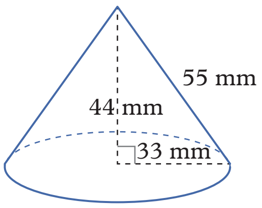


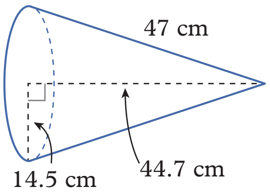
UNDERSTANDING AND FLUENCY

8D.3 **6** Calculate the surface area of each of these cones correct to one decimal place.

a



b



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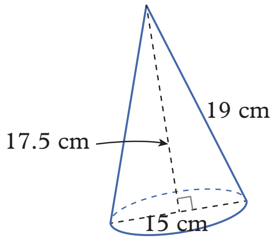

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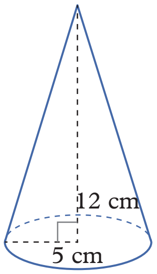

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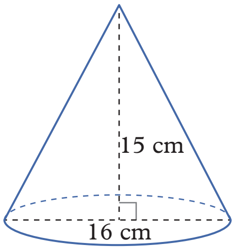

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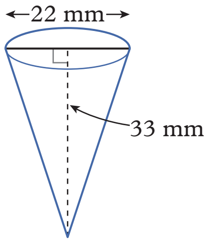


PROBLEM SOLVING AND REASONING

7 A square-based right pyramid has a surface area of 561 cm^2. If the length of the base is 11 cm, find the height of all the triangular faces.

8 A cone has a radius of 15.5 cm and a height of 9.5 cm. Use Pythagoras' theorem to find the slant height of the cone correct to one decimal place and its surface area.

9 **a** What is the surface area of a regular tetrahedron with a side length of 4 cm?

b Develop a formula to determine the surface area of a regular tetrahedron with a side length of x cm.

10 A party hat is in the shape of a cone. The diameter of the base is 12 cm and the slant height is 18 cm. Find how much cardboard is needed to make this party hat assuming there is no overlap. Give your answer correct to one decimal place. Hint: Do you need to use all parts of the formula for the surface area of a cone?

11 An eight-sided die is in the shape of an octahedron.

a How many equilateral triangles form the octahedron?

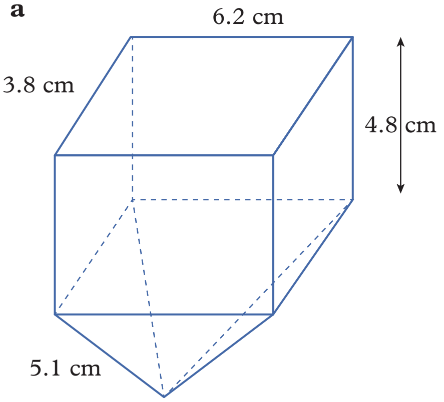
b If the side length of the octahedron is 14 mm, find the surface area of the octahedron correct to one decimal place.

c The octahedron resembles two identical square pyramids sitting base to base. Calculate the surface area of one pyramid and double your answer.

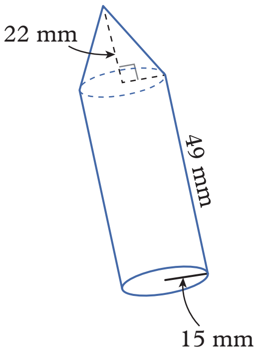
d Are your answers for parts **b** and **c** equal? Why or why not?

12 Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place.

a



b



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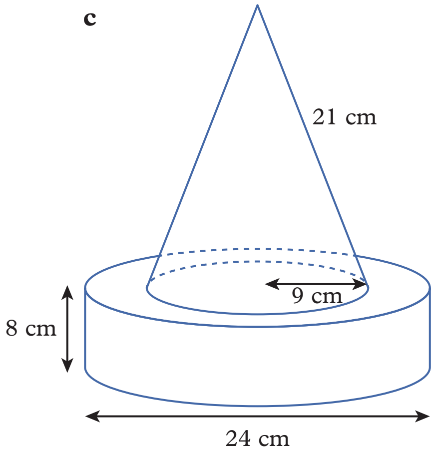
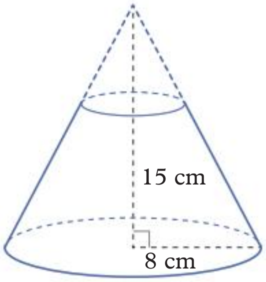


13 The tip of a hollow cone is sliced through and removed. The remaining object is called a truncated cone. If the height of the removed tip was one-third of the height of the whole cone, find the outer surface area of the truncated cone correct to one decimal place. (Assume the cone includes a base.)

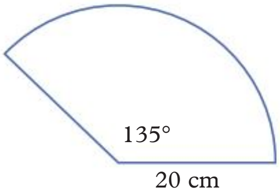


14 An open cone can be made from a circle sector by curving the shape to join the two straight edges. The centre of the circle becomes the apex of the cone and the radius of the circle sector becomes the slant length of the open cone.

a Consider the sector on the right.

i Find the area of this circle sector (and therefore the surface area of the open cone made from this sector). Give your answer to one decimal place.

ii Hence, find the radius of the open cone made from this sector correct to one decimal place.



b A circle sector with radius r and interior angle θ is made into an open cone.

i Find the value of θ if a radius of the cone is half the radius of the sector.

ii Find the value of θ if a radius of the cone is one third the radius of the sector.

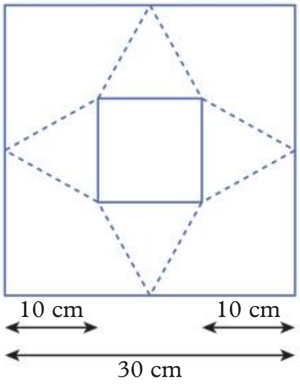
iii Find the radius of the cone in terms of r and θ.

15 Rhonda is creating a square-based right pyramid with a 30 cm by 30 cm square sheet of paper. She draws the square and dotted lines as shown in the image on the right.

a Find the surface area of the pyramid created by Rhonda when she cuts along the dotted lines and folds the triangles up.

b Find the fraction of the square paper that goes to waste.

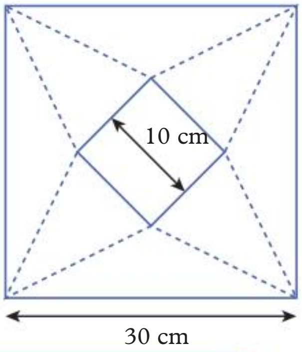
c Rhonda's friend Daniele suggests a different way to cut the paper in order to create a taller right pyramid with the same square base. She draws her design on another piece of paper, as shown below.



i Use Pythagoras' theorem to find the height of the triangles that become the slanted faces of the pyramid. Give your answer correct to one decimal place.

ii Hence, find the surface area of the pyramid that is created by cutting the dotted lines this way. Give your answer correct to one decimal place.

iii Find the percentage of the square paper that goes to waste with this design. Give your answer correct to one decimal place.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Surface area of pyramids and cones

Topic quiz
8D

8E Volume of pyramids and cones

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the volume of pyramids and cones.

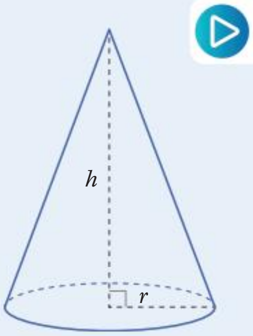
Inter-year links

Year 7	7F 3D objects
Year 9	6D Volume of composite solids

Volume of pyramids and cones

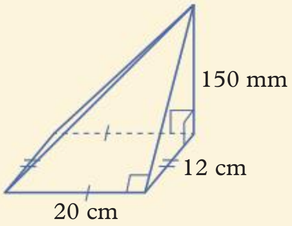
Key content video

- The volume of any pyramid (oblique or right) can be found using the formula $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height of the pyramid.
- The volume of a cone (oblique or right) can be found using $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the perpendicular height of the cone.



Example 8E.1 Calculating the volume of a pyramid

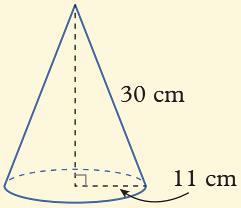
Find the volume of this pyramid.



THINK	**WRITE**
1 Identify the shape of the base of the pyramid and calculate the area of the base.	The pyramid has a rectangular base. $l = 20$ cm $\quad b = 12$ cm $A = lb$ $= 20 \times 12$ $= 240\text{ cm}^2$
2 Convert the measurement given for the perpendicular height to the same units as the sides of the base.	$h = 150$ mm $= 15$ cm
3 Substitute the values of the perpendicular height and the area of the base into the formula for the volume of a pyramid.	$V = \frac{1}{3}Ah$ $= \frac{1}{3} \times 240 \times 15$
4 Calculate the result and include the appropriate unit.	$= 1200\text{ cm}^3$

Example 8E.2 Calculating the volume of a cone

Calculate the volume of this cone, correct to one decimal place.



THINK	**WRITE**
1 Calculate the value of the cone’s perpendicular height by using Pythagoras’ theorem.	$h^2 + 11^2 = 30^2$ $h^2 = 900 - 121$ $h^2 = 779$ $h = \sqrt{779}$
2 Substitute the values of the radius of the base and the perpendicular height into the formula for the volume of a cone.	$r = 11$ cm $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3} \times \pi \times 11^2 \times \sqrt{779}$ $= 3536.573...$
3 Write the answer correct to one decimal place and include the appropriate unit.	$\approx 3536.6 \text{ cm}^3$

Helpful hints

- ✔ If the perpendicular height of a pyramid or cone is not given, you may need to calculate it using Pythagoras’ theorem.
- ✔ The formulas for the volume of a pyramid or a cone are essentially the same, with πr^2 representing the area of the base of a cone.

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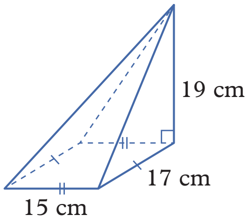
Exercise 8E Volume of pyramids and cones

△ 1–5, 8, 10, 11 □ 1(d–i), 2, 4, 6, 9, 12, 15, 16 ⬠ 1(e, h, i), 2, 4(e, f), 6, 7, 9, 13–18

UNDERSTANDING AND FLUENCY

8E.1 **1** Calculate the volume of each of these pyramids. Give your answers correct to one decimal place where appropriate.

a



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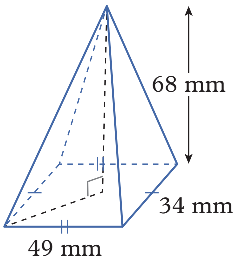
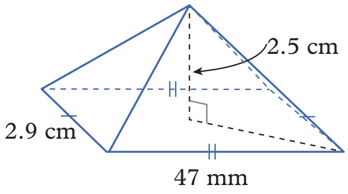


c



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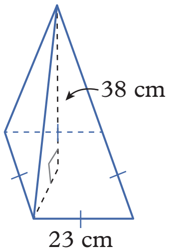

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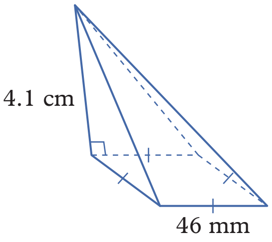

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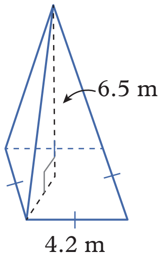

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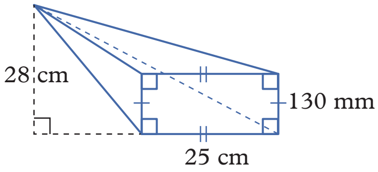

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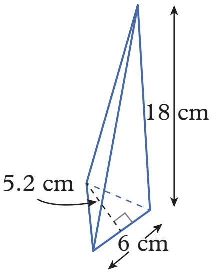

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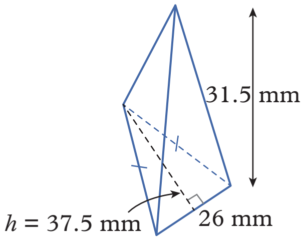
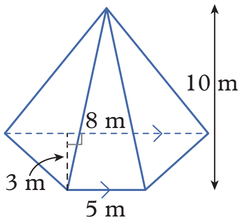


2 Find the volume of each of these pyramids by first calculating the area of the base. Give your answers correct to one decimal place where appropriate.

a



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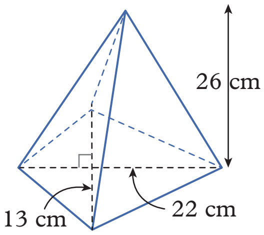

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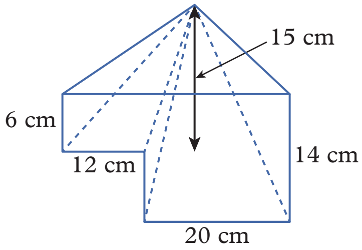
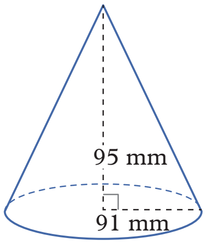


3 The Great Pyramid of Cheops in Giza, Egypt, can be approximated by a square-based pyramid. Its perpendicular height is approximately 147 m and the length of its base is approximately 230 m. Calculate its volume.

8E.2 4 Find the volume of each of these cones. Give your answers correct to one decimal place.

a



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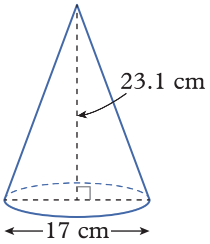

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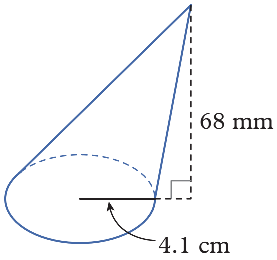

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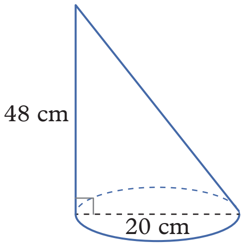

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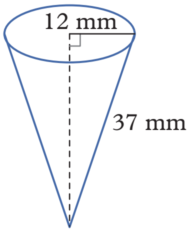

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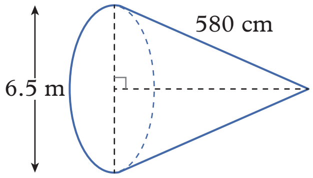
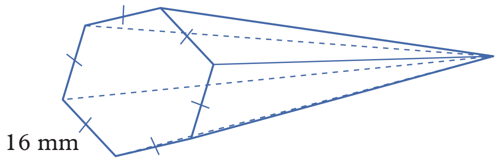


5 A traffic cone has a base with a diameter of 270 mm and it is 850 mm tall. Find the volume of the cone, giving your answer in cubic centimetres correct to one decimal place.

6 Calculate the volume of this hexagonal pyramid of height 70 mm correct to one decimal place.



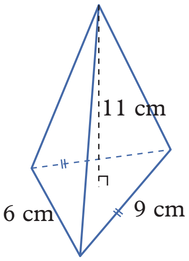
7 **a** Use Pythagoras' theorem to find the height of the triangular base of this tetrahedron. Then find the area of the base. Give your answers correct to one decimal place.

b Use your answer for part **a** to help you find the volume of the tetrahedron. Give your answer in:

i cubic centimetres

ii cubic millimetres.

c Write a formula for finding the volume of a tetrahedron.



8 **a** If a cone has a volume of 89.8 cm^3 and the radius of its base is 3.5 cm, find the height of the cone correct to one decimal place.

b If a second cone has the same volume as the cone from part **a**, but has a height of 10 cm, find the radius of this cone correct to one decimal place.

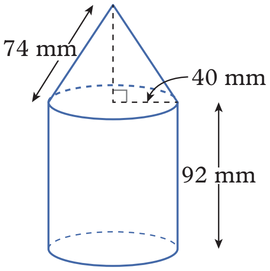
9 A vase has the shape of an open inverted cone sitting in a frame. The circular top of the vase has a radius of 12 cm and the height of the vase is 46 cm.

If the vase is filled to three-quarters of its capacity with water, find how much water, in litres, that the vase contains. Give your answer correct to one decimal place.

10 **a** Calculate the surface area of this composite solid, correct to one decimal place.

b Which unknown dimension is also needed to calculate the volume of this solid?

c Use Pythagoras' theorem to find the unknown length. Calculate the volume of the solid. Give your answers correct to one decimal place.



11 A cone has a base with a radius of 3 cm. The cone is 5 cm tall.

a Calculate the volume of the cone correct to one decimal place.

b Double the radius of the cone.

i What is the new radius?

ii If the height does not change, calculate the volume of the new cone correct to one decimal place.

c Double the radius once again. Using the same height, find the volume of the new cone correct to one decimal place.

d Compare your answers for parts **a**, **b** and **c**. Describe what happens to the volume of a cone if the radius is doubled and the height remains the same.

e This time, double the height of the original cone and leave the radius unchanged. Calculate the volume of the cone if the radius of the base is still 3 cm but the cone is now 10 cm tall. Give your answer correct to one decimal place.

f Double the height of the cone again, but do not change the radius. Calculate the volume of this new cone correct to one decimal place.

g Can you predict the volume of the cone if the height is doubled once again but the radius of the base remains unchanged? Calculate the volume to see if your prediction is correct.

h Write a sentence describing the effect on the volume of a cone when the height is doubled but the radius remains the same.

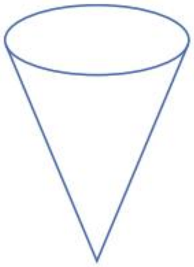
i Why is the volume increased by a different factor when the radius is doubled compared to when the height is doubled?

12 This pyramid framework is 8 cm tall, has a rectangular base with a length of 9 cm and a breadth of 5 cm. A coin of diameter 3 cm and thickness 0.2 cm sits in the centre of the pyramid. What percentage of the space within the pyramid does the coin occupy? Give your answer correct to one decimal place.

13 A cone of height 40 cm and radius 12 cm is placed inside a square-based pyramid. The pyramid has a height of 40 cm and the edges of its square are 24 cm. This means that the pyramid is just big enough to contain the cone.

- **a** Find the volume of the cone correct to one decimal place.
- **b** Find the volume of the pyramid correct to one decimal place.
- **c** Hence, find the volume of empty space between the pyramid and the cone.
- **d** Find the exact ratio of the space in the pyramid taken up by the cone to the space in the pyramid that is empty.

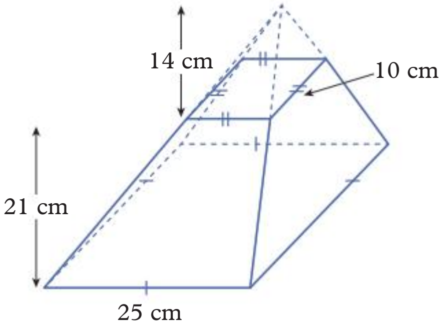
14 A paper cup is in the shape of an open cone. The radius of the cone is 5 cm and the height of the cone is 10 cm.



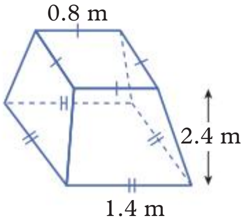
- **a** Find the volume of the cone correct to one decimal place.
- **b** Find the height of the water in the cone when the cup is half full. Give your answer correct to one decimal place.

15 The tip or apex of a square pyramid is cut off parallel to the base and is removed. The object remaining is called a truncated pyramid.

- **a** What shapes are the faces of the truncated pyramid?
- **b** Calculate the volume of the square pyramid before the tip has been removed.
- **c** Calculate the volume of the smaller pyramid that has been cut off correct to one decimal place.
- **d** Use your answers to parts **b** and **c** to calculate the volume of the truncated pyramid on the right.



16 The base of a statue was cut from a concrete block formed as a square-based pyramid. If the height of the original block was 5.6 m before it was truncated, how much concrete was needed to construct this base? Give your answer in cubic metres correct to one decimal place.



17 A drinking glass is 120 mm tall and is in the shape of a truncated cone. The base of the glass has a diameter of 55 mm and the top of the glass has a diameter of 75 mm.

- **a** If the height of the conical piece of glass used to construct the drinking glass was 192 mm, calculate the volume of the drinking glass correct to one decimal place.
- **b** How many millilitres of iced tea can this glass hold when it is filled to its brim? Give your answer correct to the nearest whole number.

18 It is possible to write a formula for finding the volume of a truncated square-based pyramid and a truncated cone. Use your answers to questions **15–17** to assist you to find these formulas.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Volume of pyramids and cones

Investigation
Ice cream cone designs

Topic quiz
8E

8F Surface area and volume of spheres

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the surface area and volume of spheres.

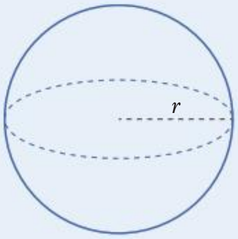
Inter-year links

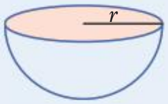
Year 7	8B Circumference of a circle
Year 8	8D Area of a circle
Year 9	6C Surface area of cylinders

Spheres

Key content video

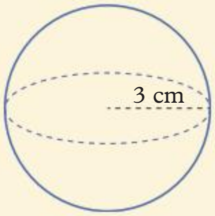
- A **sphere** is a round 3D object, with every point on its surface equidistant from its centre.
  The size of a sphere is determined by its radius (or diameter).
- The formula for the surface area of a sphere is:
  $SA = 4\pi r^2$
- The formula for the volume of a sphere is:
  $V = \frac{4}{3}\pi r^3$
- A hemisphere is half of a sphere.
  In this book, closed hemispheres are drawn with a shaded circular face.





Example 8F.1 Calculating the surface area of a sphere

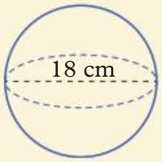
Calculate the surface area of a sphere with a radius of 3 cm, correct to one decimal place.



THINK	WRITE
1 Substitute the length of the radius into the formula for the surface area of a sphere.	$r = 3$ cm $SA = 4\pi r^2$ $= 4 \times \pi \times 3^2$
2 Calculate the result, using the π key on your calculator.	$= 4 \times \pi \times 9$ $= 36\pi$ $= 113.097...$
3 Write your answer correct to one decimal place and include the appropriate unit.	≈ 113.1 cm^2

Example 8F.2 Calculating the volume of a sphere

Calculate the volume of this sphere correct to one decimal place.



THINK	WRITE
1 Calculate the length of the radius of the sphere.	$r = 18 \div 2$ $= 9$ cm
2 Substitute the length of the radius into the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$ $= \frac{4}{3} \times \pi \times 9^3$
3 Calculate the result using the π key on your calculator.	$= 972\pi$ $= 3053.628...$
4 Write the answer correct to one decimal place and include the appropriate unit.	≈ 3053.6 cm^3

Example 8F.3 Calculating the radius of a sphere, given the volume

Calculate the radius of a sphere that has a volume of 500 cm^3. Give your answer correct to one decimal place.

THINK	WRITE
1 Substitute the given volume into the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$ $500 = \frac{4}{3}\pi \times r^3$
2 Rearrange the equation so r^3 is alone on one side of the equation.	$500 \div \frac{4}{3}\pi = r^3$ $119.366... = r^3$
3 Use a calculator to find r.	$r = \sqrt[3]{119.366...}$ $r = 4.923...$
4 Write the answer correct to one decimal place and include the appropriate unit.	$r \approx 4.9$ cm

Helpful hint

✔ When working with surface areas and volumes, always remember to include units with your final answers.
✔ When calculating the surface area of hemispheres, always check whether the circular face is included.

ANS
p758

Exercise 8F Surface area and volume of spheres

△ 1–9, 11, 15, 16(a, b)

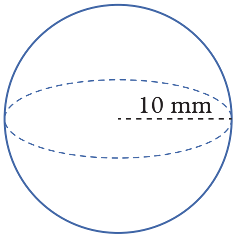
■ 1–4, 6, 8, 12, 14, 16(b, c), 18

⬟ 1(c, d), 2(c, d), 3, 4, 8(b, e, f), 10, 13, 14, 17, 19, 20

UNDERSTANDING AND FLUENCY

8F.1 **1** Calculate the surface area of each of these spheres correct to one decimal place.

a



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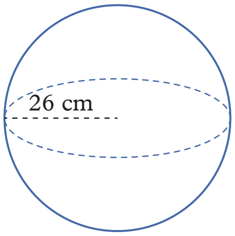

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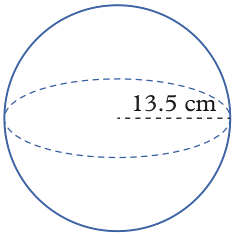

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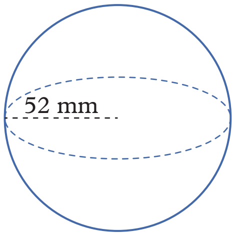
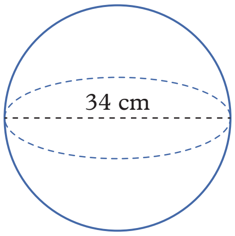


2 Calculate the surface area of each of these spheres by first calculating the radius. Give your answers correct to one decimal place.

a



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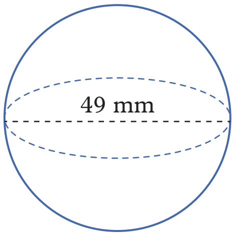

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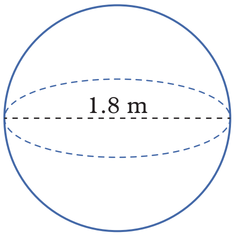

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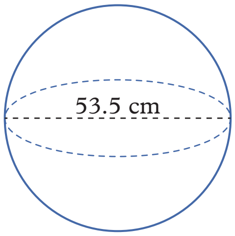


8F.2 **3** Calculate the volume of each sphere in question **1** correct to one decimal place.

4 Calculate the volume of each sphere in question **2** correct to one decimal place.

8F.3 **5** Calculate the radius of a sphere that has a volume of 5575 cm^3. Give your answer correct to the nearest cm.

6 **a** Find the radius of a sphere that has a surface area of 144π cm^2.

b Find the radius of a sphere that has a volume of 972π cm^3.

c Find the radius of a sphere that has a surface area of 2124 cm^2.

d Find the diameter of a hemisphere that has a volume of 1072 cm^3.

7 Consider the solid sphere on the right.

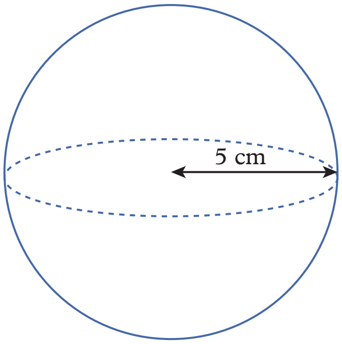
a Find the surface area of the sphere, as an exact value.

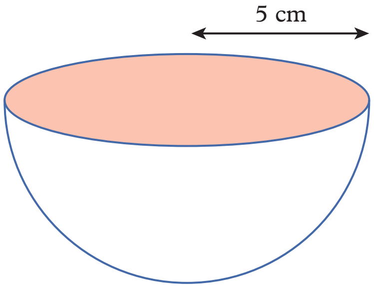
b The sphere is cut exactly in half, as shown below.

i Find the area of the circular face, as an exact value.

ii Find the area of the curved surface, as an exact value.

iii Hence, find the total surface area of the solid hemisphere, as an exact value.



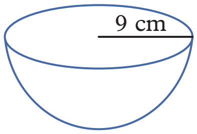


8 For each hemisphere below, calculate:

- **i** its outer surface area correct to one decimal place
- **ii** its volume correct to one decimal place.

Hint: First determine whether each hemisphere is open or closed. In this book closed hemispheres are drawn with a shaded circular face.

a



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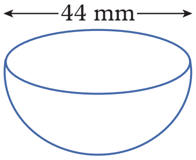

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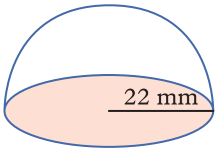

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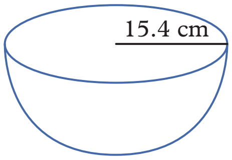

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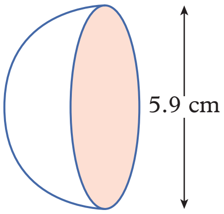

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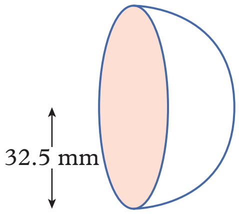


9 When inflated, the beach ball below has a diameter of 76 cm.

a Calculate its surface area correct to one decimal place.

b Find the volume of air, correct to one decimal place, that it can hold.

10 **a** The volume of a fully inflated volleyball is 0.15 m^3. Find the radius of the volleyball, to the nearest centimetre.

b The surface area of a tennis ball is approximately 564 cm^2. Find the radius of the tennis ball correct to one decimal place.

11 The approximate diameters of the planets in our solar system are listed in the table below.

Planet	Diameter (km)	Surface area	Volume
Mercury	4878		
Venus	12 104		
Earth	12 756		
Mars	6787		
Jupiter	142 800		
Saturn	120 000		
Uranus	51 118		
Neptune	49 528		

a Copy and complete the table. Write the values in scientific notation correct to two significant figures ($a \times 10^m$ where $1 \le a < 10$ and m is an integer).

b Approximately how many times greater is Earth's volume compared to the volume of Mercury?

c Approximately how many times greater is Jupiter's surface area compared to Earth's surface area?

d Approximately 71% of Earth's surface is covered by water. To the nearest square kilometre, how much of Earth's surface area is covered by water?

UNDERSTANDING AND FLUENCY

PROBLEM SOLVING AND REASONING

12 The rind of a piece of fruit, like an orange, is the outer layer, which is usually not eaten and can be peeled off to reveal the edible inner part of the fruit. The surface area of the rind of this blood orange is 116.9 cm^2.

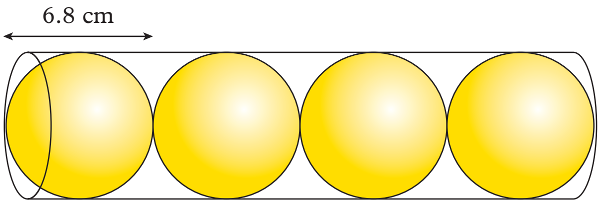
- **a** What is the diameter of the orange correct to one decimal place?
- **b** The thickness of the rind is approximately 1 cm. What volume does the edible part of the orange occupy to the nearest cubic centimetre.

13 A balloon is inflated to become a sphere with a radius of 9.5 cm.

- **a** What is the volume of the balloon correct to one decimal place?
- **b** Two more balloons are inflated to exactly the same size as the first balloon. What is the total volume of the three balloons correct to one decimal place?
- **c** When objects are filled with liquid or gas, you can refer to their capacity rather than volume.

  What is the capacity of the three balloons, in litres? Give your answer correct to one decimal place.
- **d** If the rate that the balloons can be inflated is 2.5 L/s, how long will it take to inflate all three balloons? Give your answer correct to one decimal place.
- **e** What assumptions do you need to make when answering part **d**?

14 Tennis balls are sold in canisters. Each canister holds four tennis balls.

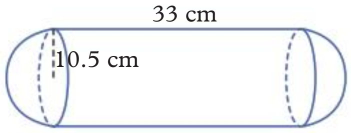


- **a** If each tennis ball has a diameter of 6.8 cm, calculate the volume of one tennis ball correct to one decimal place.
- **b** What is the combined volume of all four tennis balls correct to one decimal place?
- **c** Determine the dimensions of the canister and find its volume correct to one decimal place. Assume that the tennis balls fit perfectly into the canister.
- **d** How much space in the canister is not taken up by tennis balls? Give your answer correct to one decimal place.
- **e** If the tennis balls were to be sold in a container with the same dimensions but in the shape of a rectangular prism instead of a cylinder, without any further calculations, would there be more or less unused space?

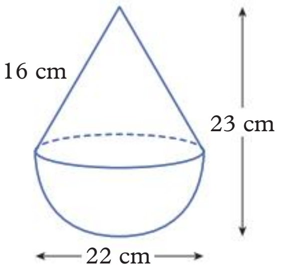
15 Rebecca buys an ice-cream cone. As well as the ice cream on top of the cone, the cone is completely filled with ice cream. Calculate how much ice cream Rebecca has in total by approximating the top as a perfect hemisphere. Give your answer to the nearest millilitre.

16 Calculate the volume of each of these composite solids correct to one decimal place.

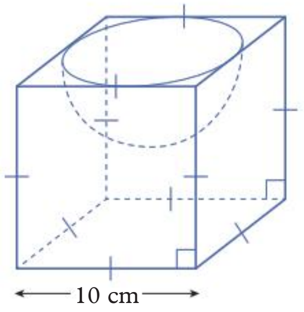
a



b



c



17 The bricks shown are all equal in size and measure 23 cm by 11.5 cm by 7.5 cm. The holes on each brick are hemispheres and the diameter of each hole is 2.5 cm. Calculate the volume of clay required to make the five bricks. Give your answer to the nearest cm^3.

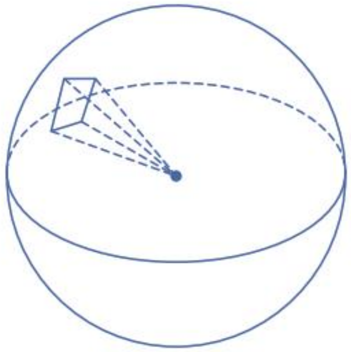
18 a A ping pong ball has a diameter of 40 mm. Find its volume correct to one decimal place.

b Density is defined as the mass per unit volume and can be found using the formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

If the mass of the ping pong ball is 2.7 g, calculate its density in g/cm^3. Give your answer correct to two decimal places.

19 A sphere can be approximated using many right square pyramids, all with equal base lengths, and slant heights equal to the radius of the sphere. The accuracy of the approximation increases as the number of pyramids increases. In this question, you will derive the formula for the surface area of a sphere.



a Suppose your approximation uses n pyramids with base area A. Write an expression for the volume of the sphere in terms of A, n and r, where r is the radius of the sphere.

b Equate your expression from part **a** with the usual formula for the volume of a sphere and simplify to find an expression equal to nA.

c Why does your answer to part **b** give the surface area of the sphere?

d Why doesn't this method work for small values of n (like $n = 6$)?

20 a A solid sphere of radius 10 cm is placed inside the smallest cube that can contain it.

i Find the side length of the cube.

ii Find the volume of empty space in the cube correct to one decimal place.

b A solid cube of side length 10 cm is enclosed in the smallest sphere that can contain it.

i Find the side length of the sphere correct to one decimal place.

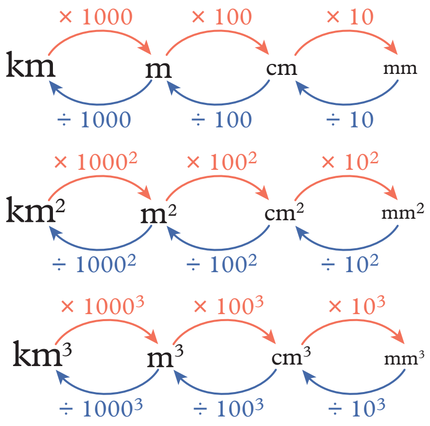
ii Find the volume of empty space in the sphere correct to one decimal place.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Surface area of spheres	**Interactive skillsheet** Volume of spheres	**Worksheet** Surface area and volume of pyramids, cones and spheres	**Topic quiz** 8F

Chapter summary

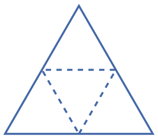
Units



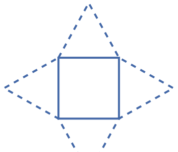
Pyramids

Surface area

The SA is the sum of the areas of all the faces.



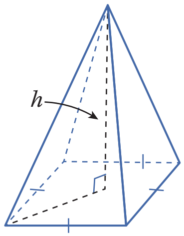
net of a regular tetrahedron



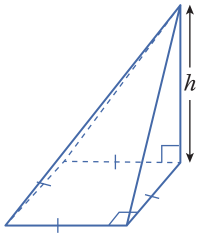
net of a square pyramid

Volume

$V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height of the pyramid.



right square pyramid

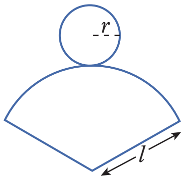


oblique square pyramid

Cones

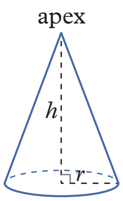
Surface area

SA = $\pi r^2 + \pi rl$



Volume

$V = \frac{1}{3}\pi r^2 h$



Area of a composite shape

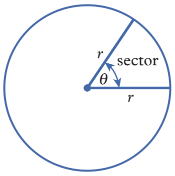
1 Split the shape into individual parts.

2 Calculate any missing dimensions.

3 Calculate the areas of the individual parts.

4 Add or subtract the areas to find the total area required.

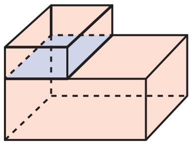
Area of a sector



- Area = $\frac{\theta}{360°} \times \pi r^2$

Surface area

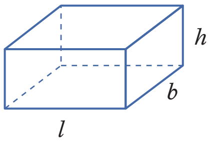
- The surface area (SA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.



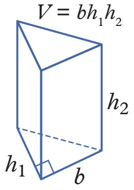
Volume of a prism

- For any prism, $V = Ah$, where A is the area of the base and h is the height of the prism.

$V = lbh$



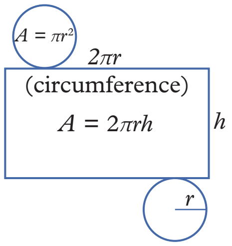
$V = bh_1h_2$



Cylinders

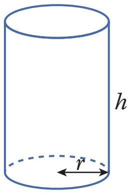
Surface area

SA = $2\pi r^2 + 2\pi rh$



Volume

$V = \pi r^2 h$



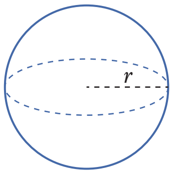
Spheres

Surface area

SA = $4\pi r^2$

Volume

$V = \frac{4}{3}\pi r^3$



ANS
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Chapter review

Chapter review quiz
Assess your knowledge of this chapter.
✔ Core
✔ Standard
✔ Advanced

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- area
- capacity
- circumference
- composite shape
- cone
- conversion factor
- cross-section
- cylinder
- face
- hemisphere
- net
- oblique pyramid
- perpendicular
- prism
- pyramid
- quadrant
- regular tetrahedron
- right cone
- right pyramid
- sector
- sphere
- surface area
- tetrahedron
- uniform
- volume

1 Why is a cylinder not a prism?

2 Draw a diagram to show the conversion factors for converting between mm^2, cm^2, m^2 and km^2.

3 Complete the following sentences using words from the key term list.

a The ________________ of a 3D object is found by summing the areas of each individual ________________.

b The ________________ of an object is how much space it takes up. The ________________ is how much liquid the object can hold.

4 Explain what it means for a 3D object to have a uniform cross-section. Give examples of objects with and without a uniform cross-section.

5 Provide a definition in your own words for the following key terms.

a regular tetrahedron **b** right cone **c** sphere

Multiple choice

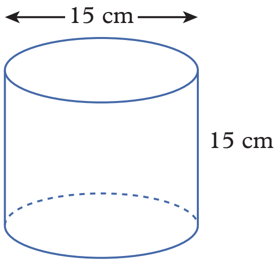
8A **1** Which of the following is equivalent to 300 cm^2?

A 3 m^2 **B** 0.3 m^2 **C** 3000 mm^2 **D** 0.03 m^2 **E** 30 mm^2

8B **2** What is the SA of a cube with side lengths of 8 cm?

A 384 cm^2 **B** 48 cm^2 **C** 64 cm^2 **D** 512 cm^2 **E** 288 cm^2

8B **3** A closed cylinder has a base diameter of 15 cm and a height of 15 cm. Which of the following is closest to the surface area of the cylinder?



A 225 cm^2 **B** 700 cm^2 **C** 750 cm^2 **D** 1060 cm^2 **E** 2800 cm^2

8C **4** A cylinder has a base diameter of 6 cm and a height of 20 cm. Which of the following is closest to the volume of the cylinder?

A 120 cm^3 **B** 188 cm^3 **C** 377 cm^3 **D** 565 cm^3 **E** 2262 cm^3

8C **5** A solid wooden cylinder has a base with radius 10 cm and is 10 cm tall. It has a cylinder of radius 2 cm drilled through its centre, from top to base. What is the volume of timber remaining (in cubic centimetres)?

A 640π cm^3 **B** 1000π cm^3 **C** 960π cm^3 **D** 80π cm^3 **E** 40π cm^3

STN **8D** **ADV** **6** A square-based pyramid has a surface area of 224 cm^2. If the length of the base is 8 cm, what is the height of each triangular face?

A 56 cm **B** 14 cm **C** 10.5 cm **D** 10 cm **E** 3.5 cm

STN **8D** **ADV** **7** How many triangular faces does a tetrahedron have?

A 1 **B** 3 **C** 4 **D** 5 **E** 6

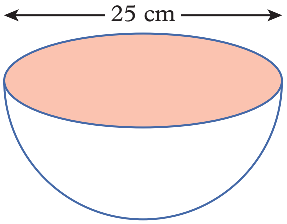
STN **8E** **ADV** **8** A square-based pyramid has the same base length and height as a cube with side lengths of 6 cm. What is the volume of the pyramid?

A 216 cm^3 **B** 72 cm^3 **C** 12 cm^3 **D** 108 cm^3 **E** 144 cm^3

STN **8E** **ADV** **9** Which of the following is closest to the volume of a cone with a diameter of 2.5 cm and a height of 10 cm?

A 16 cm^3 **B** 25 cm^3 **C** 50 cm^3 **D** 65 cm^3 **E** 200 cm^3

STN **8F** **ADV** **10** Which of the following is closest to the total exposed surface area if a solid sphere with a diameter of 25 cm is sliced in half?



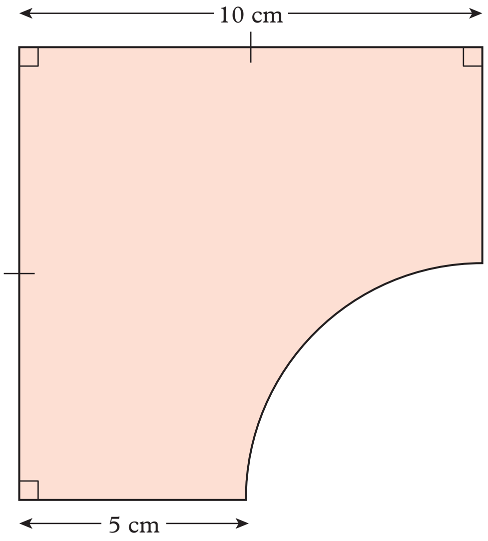
A 500 cm^2 **B** 1500 cm^2 **C** 3000 cm^2 **D** 4000 cm^2 **E** 6000 cm^2

Short answer

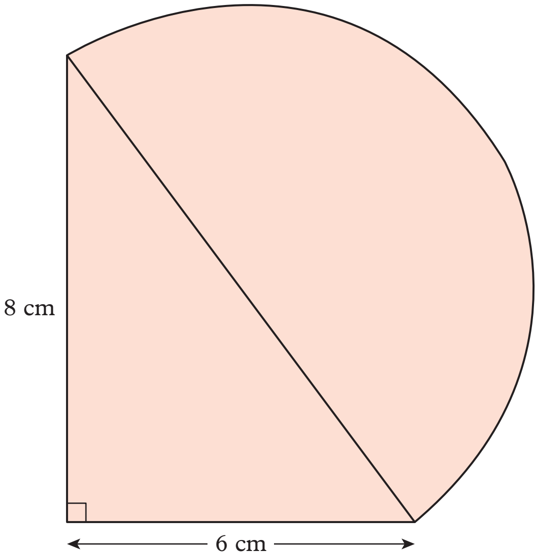
8A **1** A flat metal washer is in the form of an annulus. The radius of the inner ring of the washer is 1 cm, while the outer ring has a radius of 1.5 cm. What is the area of the annulus correct to one decimal place?

8A **2** Determine the area of each of these composite shapes. Give your answers correct to one decimal place.

a



b



8B **3** The base of a triangular prism is a right-angled triangle with a base length of 25 mm and a height of 18 mm. The height of the prism is 30 mm. What is the SA of the prism?

8B **4** A set of three wooden cubes was made for a playground. The smallest cube had a side length of 40 cm. The next sized cube had side lengths twice as long as the smallest cube, while the side lengths of the largest cube were twice as long as those of the middle cube.

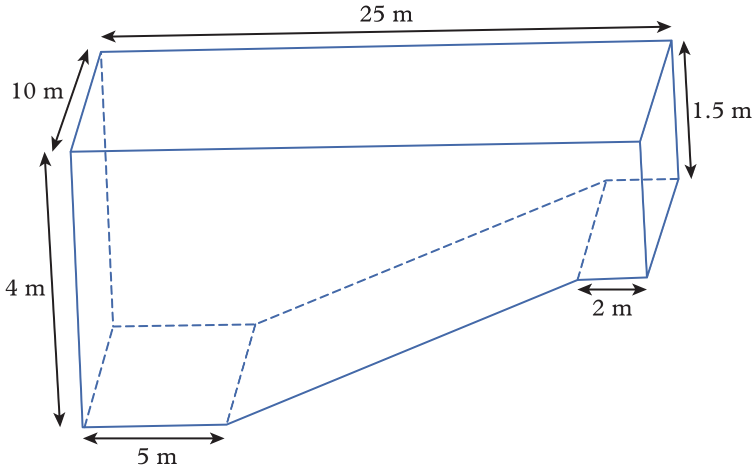
a Calculate the surface area of each cube.

b If 1 L of paint covers an area of 15 m^2, and the paint is only available in 5 L cans, how many cans of paint would be required to paint all three cubes?

8C **5** A can of baked beans has a cylindrical shape with a base diameter of 7 cm and a height of 10 cm. What is the capacity of the can correct to the nearest millilitre?

8C **6** Calculate the volume of each cube in question **4**.

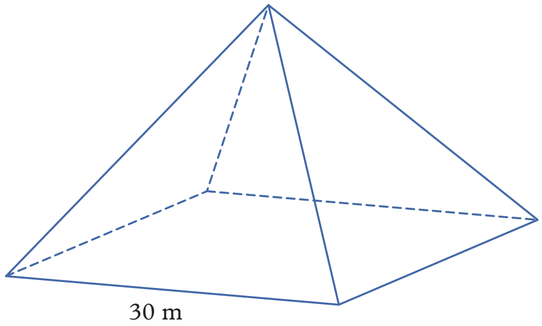
8C **7** An inground pool in the shape of the right prism shown is to be installed. The prism is not drawn to scale. Calculate the volume of soil which needs to be excavated so that the pool fits exactly into the hole.



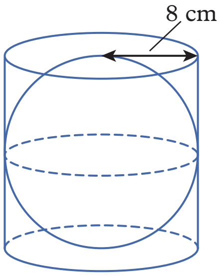
STN **8D** **ADV** **8** A square-based pyramid has a base with sides 5 cm in length. The height of each triangular face is 20 cm. Calculate the surface area of the pyramid.

STN **8D** **ADV** **9** The diameter of a closed cone is 10 cm and the cone's height is 16 cm. Calculate the surface area of the cone correct to one decimal place.

STN **8E** **ADV** **10** A square-based pyramid has a base with sides of length 30 cm. The triangular faces are equilateral. Find the volume of the pyramid correct to two decimal places.



STN **8F** **ADV** **11** A sphere fits neatly inside a cylinder with a base radius of 8 cm. Find the volume of the cylinder and the sphere in terms of π.



STN **8F** **ADV** **12** **a** Calculate the surface area, correct to one decimal place where appropriate, of:

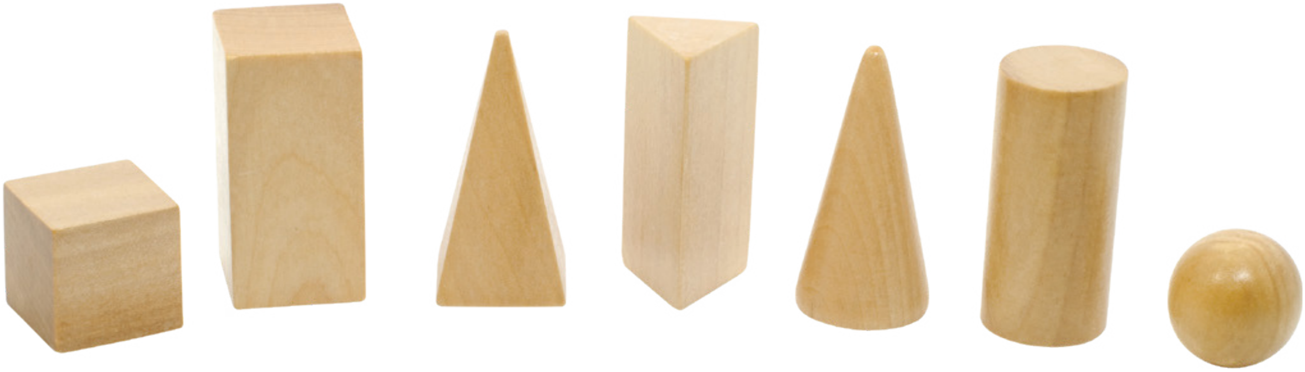
i a solid hemisphere with a diameter of 40 cm

ii a cube with sides 40 cm in length.

b The hemisphere is placed on top of the cube. What is the total surface area of this object?

Analysis

1 Ethan bought this set of wooden blocks for his young cousins.



a Name the solids in order, from left to right.

b The cube has side lengths of 3 cm. Calculate its:

- **i** surface area
- **ii** volume.

c The rectangular prism has a base the same size as the cube, and is twice the height of the cube.

- **i** Compare the surface area of the rectangular prism with that of the cube.
- **ii** Compare the volume of the rectangular prism with that of the cube.

d The triangular prism is the same shape as the rectangular prism cut vertically along the diagonal of its base.

- **i** Draw a labelled diagram showing the shape of the triangular prism's base.
- **ii** Calculate the surface area of the triangular prism correct to one decimal place.
- **iii** Is the surface area of the triangular prism half that of the rectangular prism that you calculated in part **c**? Explain.
- **iv** Compare the volumes of the triangular prism and the rectangular prism.

e The base of the cylinder has a diameter the same length as the side of the cube, and the cylinder is the same height as the rectangular prism. Calculate, correct to one decimal place, the cylinder's:

- **i** surface area
- **ii** volume.

f Create two tables listing the solids in parts **b–e** in increasing order of:

- **i** surface area
- **ii** volume.

STN ADV **g** The square base of the pyramid is the same shape as that of the base of the cube. Its height is the same as that of the rectangular prism. Calculate, correct to one decimal place, the pyramid's:

- **i** surface area
- **ii** volume.

STN ADV **h** The cone has the same base diameter and height as the cylinder. Calculate, correct to one decimal place, the cone's:

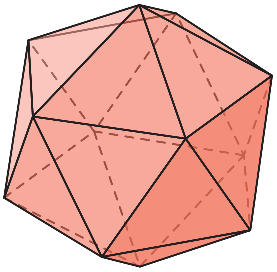
- **i** surface area
- **ii** volume.

STN ADV **i** The sphere has the same diameter as the cone. Calculate, correct to one decimal place, the sphere's:

- **i** surface area
- **ii** volume.

STN ADV **j** Using your answers for parts **g**, **h** and **i**, update your tables from part **f**.

STN ADV **2** A regular icosahedron is a polyhedron (a 3D object with polygonal faces) with 20 equilateral triangle faces.



a Find the area of an equilateral triangle with side length *a*.

b Find the surface area of a regular icosahedron where each equilateral triangle face has side length *a*.

A regular icosahedron can be divided into 20 congruent tetrahedrons. The base of each tetrahedron is one of the faces of the icosahedron, and the height of each tetrahedron is $\frac{\sqrt{3}(3+\sqrt{5})}{12}a$.

c Determine the volume of a regular icosahedron in terms of *a* in exact form and as an approximation correct to three decimal places.

The circumscribed sphere of a regular icosahedron is the sphere which contains the icosahedron and touches all vertices of the icosahedron. The radius of the circumscribed sphere is $r = \frac{a}{4}\sqrt{10 + 2\sqrt{5}}$.

d Determine the surface area and volume of the circumscribed sphere of a regular icosahedron with side length *a* correct to three decimal places.

e For the case $a = 1$, calculate the percentage of the circumscribed sphere which is filled by the icosahedron correct to two decimal places.

A truncated icosahedron is obtained by cutting off each vertex of a regular icosahedron so that each resulting edge is one-third of the length of the edges in the original icosahedron. There are 12 regular pentagonal faces and 20 regular hexagonal faces in a truncated icosahedron. Soccer balls are based on the shape of a truncated icosahedron, where the pentagons are usually coloured black and the hexagons are usually white. The area of a regular pentagon with side length *a* is approximately $A = 1.720a^2$. The area of a regular hexagon with side length a is $A = \frac{3\sqrt{3}}{2}a^2$.

f Determine the surface area of a truncated icosahedron with side length $\frac{a}{3}$ in terms of *a* correct to three decimal places.

The formula for the volume of a truncated icosahedron with side length *a* is $V = \frac{125 + 43\sqrt{5}}{4}a^3$.

g Explain why the volume of a truncated icosahedron is less than the volume of the icosahedron it is constructed from.

h Determine the volume of a truncated icosahedron with side length $\frac{a}{3}$ in terms of *a* correct to three decimal places.

The radius of the circumscribed sphere of a truncated icosahedron with side length *a* is $r = \frac{a}{4}\sqrt{58 + 18\sqrt{5}}$.

i Determine the surface area and volume of the circumscribed sphere of a truncated icosahedron with side length $\frac{a}{3}$ correct to three decimal places.

j For the case $a = 1$, use your answers to parts **h** and **i** to calculate the percentage of the circumscribed sphere which is filled by the truncated icosahedron correct to two decimal places.

k Suggest one reason why soccer balls are in the shape of a truncated isocahedron instead of a regular icosahedron.

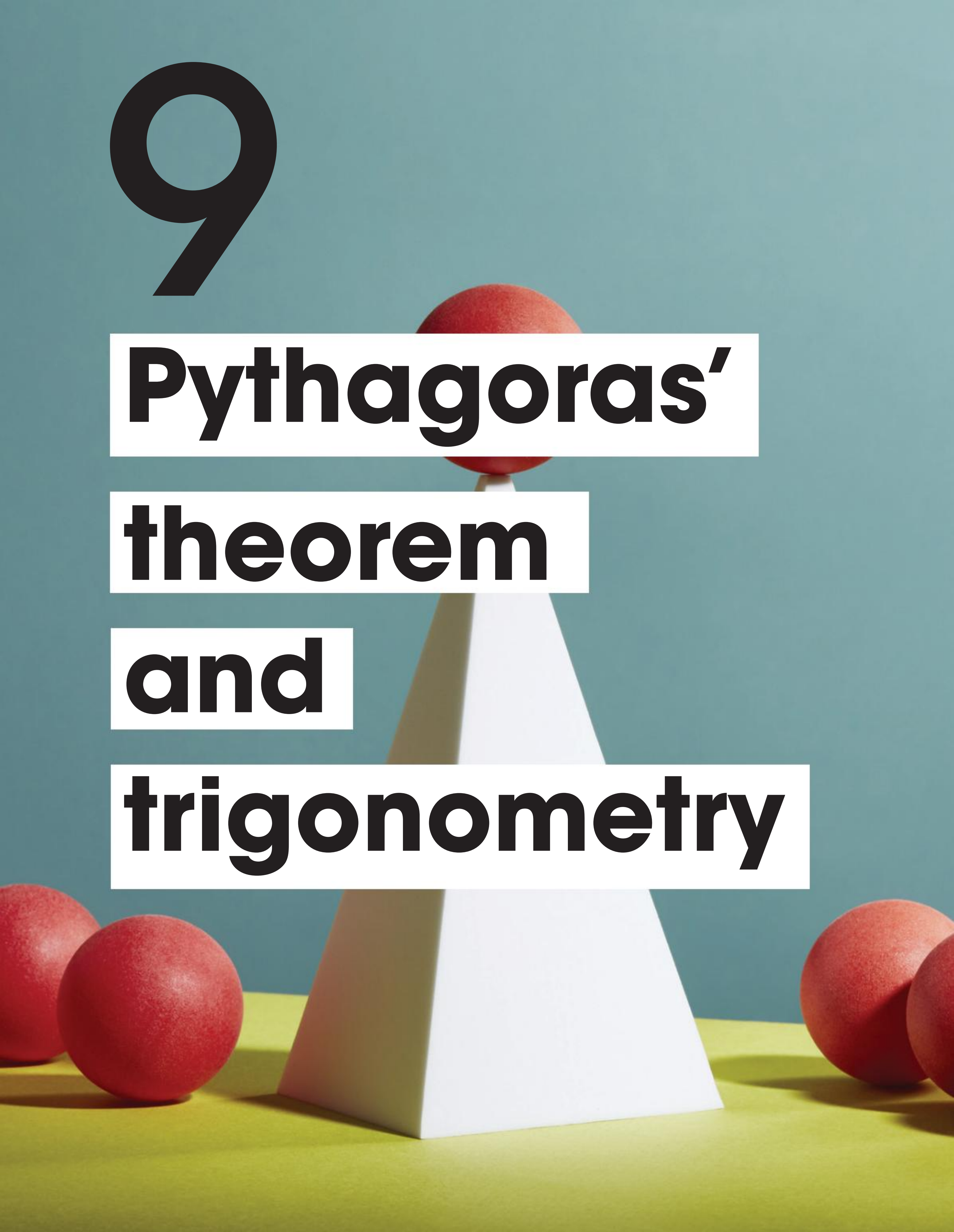
Chapter checklist

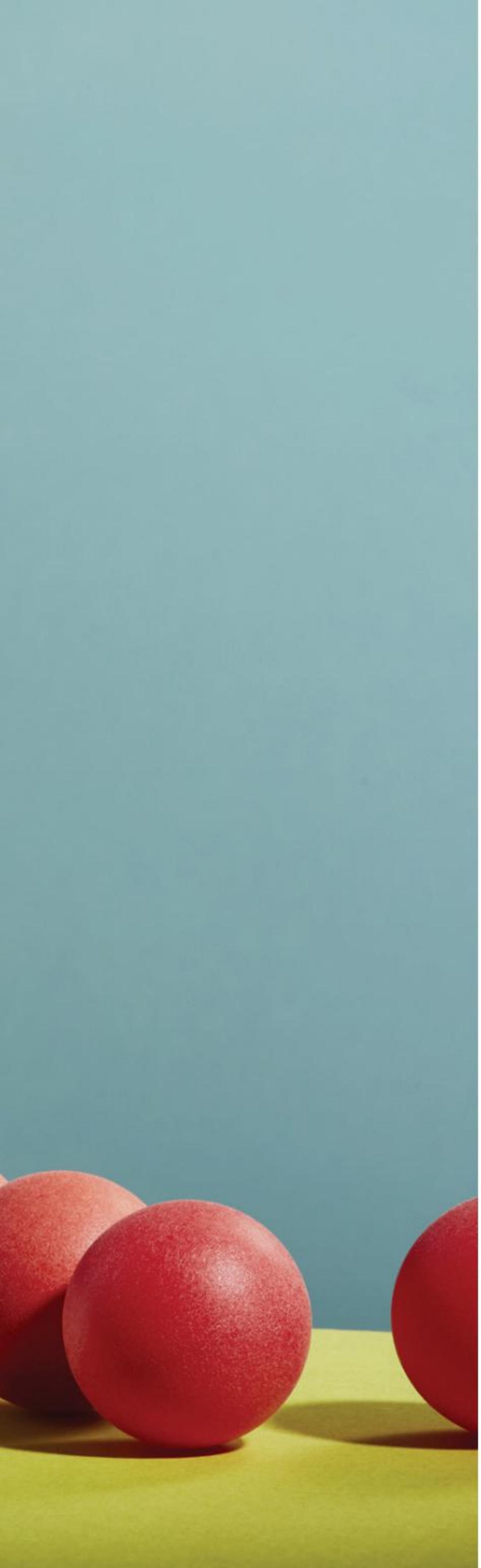
Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Calculate the area of composite shapes ☐ Calculate the area of sectors	☐ Go back to Topic 8A Area review
☐ Calculate the surface area of prisms and cylinders ☐ Calculate the surface area of composite solids consisting of prisms and cylinders	☐ Go back to Topic 8B Surface area review
☐ Calculate the volume of prisms and cylinders ☐ Calculate the volume of composite solids consisting of prisms and cylinders	☐ Go back to Topic 8C Volume review
☐ Calculate the surface area of right pyramids and right cones	☐ Go back to Topic 8D Surface area of pyramids and cones
☐ Calculate the volume of pyramids and cones	☐ Go back to Topic 8E Volume of pyramids and cones
☐ Calculate the surface area and volume of spheres	☐ Go back to Topic 8F Surface area and volume of spheres

9

Pythagoras' theorem and trigonometry





Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Triangle properties
- ✔ Angles and parallel lines
- ✔ Similar triangles
- ✔ Rounding decimals
- ✔ Simplifying surds

Curriculum links

- Applies trigonometric ratios to solve right-angled triangle problems (MA5-TRG-C-01)
- Applies trigonometry to solve problems, including bearings and angles of elevation and depression (MA5-TRG-C-02)
- STN ADV Applies Pythagoras' theorem and trigonometry to solve 3-dimensional problems and applies the sine, cosine and area rules to solve 2-dimensional problems, including bearings (MA5-TRG-P-01)
- ADV Establishes and applies the properties of trigonometric functions and finds solutions to trigonometric equations (MA5-TRG-P-02)

9A Pythagoras' theorem

Learning intentions

By the end of this topic you will be able to ...

- ✔ determine whether a triangle is acute, right-angled or obtuse given the side lengths
- ✔ use Pythagoras' theorem to determine unknown side lengths.

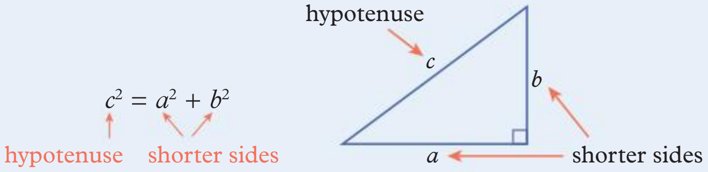
Inter-year links

Year 7	1G Indices and square roots
Year 8	7D Pythagoras' theorem

Pythagoras' theorem

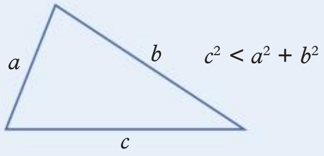
Key content video

- **Pythagoras' theorem** states the relationship between the three side lengths of a right-angled triangle: the sum of the squares of the shorter sides that meet at a right angle is equal to the square of the **hypotenuse**. The relationship for the triangle shown below is:

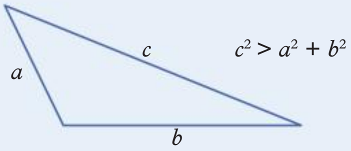


Inequalities for acute and obtuse triangles

- In an acute triangle, the square of the longest side is less than the sum of the square of the other two sides.



- In an obtuse triangle, the square of the longest side is greater than the sum of the square of the other two sides.



Example 9A.1 Determining whether a triangle is acute, right-angled or obtuse

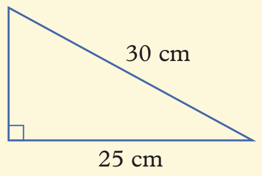
Determine whether a triangle with side lengths of 24 cm, 20 cm and 30 cm is acute, right-angled or obtuse.

THINK	WRITE
1 Identify the length of the longest side and call it c.	If $c = 30$ cm:
2 Calculate the value of c^2.	$c^2 = 30^2$ $= 900$

3 Identify the lengths of the two shorter sides and call them a and b.	If $a = 20$ cm and $b = 24$ cm:
4 Calculate the value of $a^2 + b^2$.	$a^2 + b^2 = 20^2 + 24^2$ $= 400 + 576$ $= 976$
5 Compare c^2 to $a^2 + b^2$. If these two values are equal, the triangle is right-angled. If c^2 is greater than $a^2 + b^2$, the triangle is obtuse. If c^2 is less than $a^2 + b^2$, the triangle is acute.	$c^2 < a^2 + b^2$ The triangle is acute.

Example 9A.2 Using Pythagoras' theorem to calculate an unknown side

Calculate the length of the unknown side of this triangle, give your answer as an exact value.



THINK	WRITE
1 Since the triangle is right-angled, state Pythagoras' theorem and define a, b and c. Remember that c is the hypotenuse and use b for the unknown side.	$c^2 = a^2 + b^2$, where $c = 30$ cm and $a = 25$ cm:
2 Substitute the values for a and c into the equation for Pythagoras' theorem and simplify the equation.	$30^2 = 25^2 + b^2$ $900 = 625 + b^2$
3 Solve the equation for b using inverse operations, include units in your answer.	$b^2 = 900 - 625$ $= 275$ $b = \sqrt{275}$ $= 5\sqrt{11}$ cm

Helpful hints

- ✔ Make sure you correctly identify the hypotenuse. It is always the longest side, and is opposite the right angle on the triangle. It may not be obvious, so always look at the numbers carefully.
- ✔ It doesn't matter which of the shorter sides you assign to be a or b in a right-angled triangle, but the hypotenuse must always be c.
- ✔ Remember to always give your answer with correct units.
- ✔ Any set of three whole numbers that satisfies Pythagoras' theorem is called a Pythagorean triad or Pythagorean triple. For example, 3, 4, 5 and 5, 12, 13 are both Pythagorean triads.

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Exercise 9A Pythagoras' theorem

△ 1–4, 7(a, c, f), 8, 10

□ 1(a, c, e), 2(b, d, f, h), 5–9, 11, 12

⬠ 1(a, c, e), 2(e, g, i), 3(b, d), 7(c, d, f), 11, 13

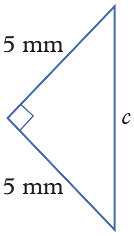
UNDERSTANDING AND FLUENCY

9A.1 **1** Determine whether each set of side lengths form an acute, right-angled or obtuse triangle.

- **a** 1 cm, 2.4 cm, 2.6 cm
- **b** 7.5 cm, 4.2 cm, 8.5 cm
- **c** 5 cm, 7 cm, 6 cm
- **d** 26 cm, 24 cm, 0.1 m
- **e** 1 m, 96 cm, 28 cm
- **f** 3.5 cm, 85 mm, 93 mm

9A.2 **2** Calculate the length of the unknown side in each of these triangles and give your answer as an exact value in simplest form. Leave your answer in centimetres for parts **g–i**.

a



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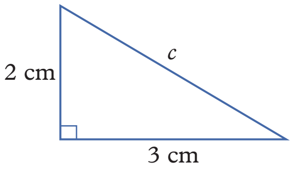
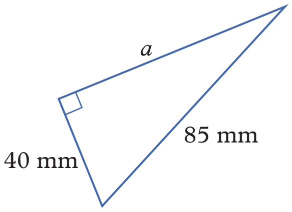


c



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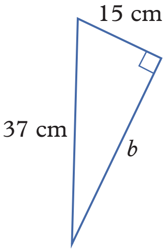
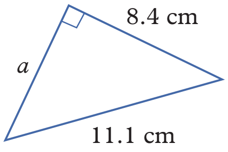


e



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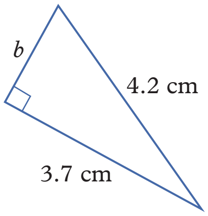
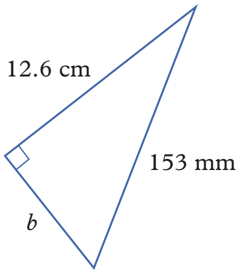
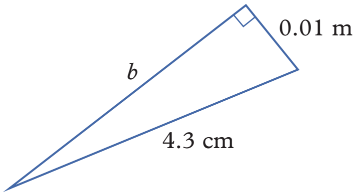


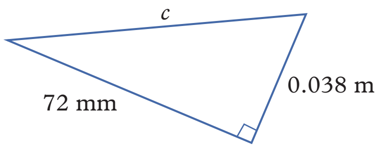
g



h



i



3 Calculate the length of the diagonal of a square with each of these side lengths. Give your answer as an exact value in simplest form.

a 2 cm **b** 30 mm **c** 13 cm **d** 35 cm

4 Calculate the length of the diagonal of a square with side lengths of x cm.

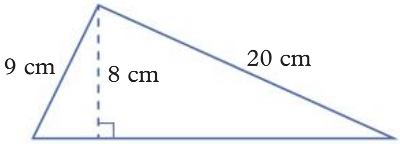
5 An equilateral triangle has side lengths of 10 cm.

- **a** Draw a sketch of the triangle. From the top vertex, draw a line that is perpendicular to the base. Label the lengths of all sides of the figure.
- **b** Use Pythagoras' theorem to calculate the height of the triangle. Give your answer as an exact value.
- **c** Calculate the area of the triangle.

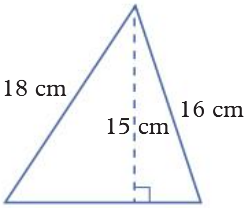
6 Calculate the area of an equilateral triangle with side lengths of x cm; leave your answer as an exact value.

7 Calculate the perimeter of the following triangles by first finding the length of the unknown side, using the given height of each triangle to help you. Correct your answer to two decimal places.

a



b



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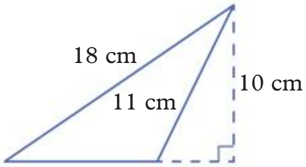
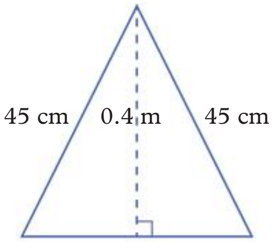


d



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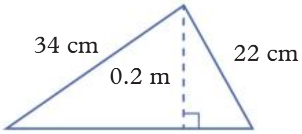

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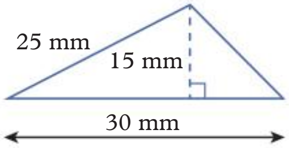


UNDERSTANDING AND FLUENCY

8 Determine whether the triangles in question **7** are acute, right-angled or obtuse.

9 The hypotenuse of a right-angled triangle is 25 cm long; find the length of the other two sides if:

a one of those sides is twice as long as the other

b one of those sides is three times as long as the other

c one of those sides is five times as long as the other.

10 A rectangular running field measures 50 m by 100 m. How much shorter is it to run directly from one corner to the diagonally opposite corner than it is to travel around the outside of the field?

11 The flagpole which sits on Parliament House in Canberra stands 81 m high. The Australian flag flying from the top section of the pole is said to be the largest Australian flag flying in Australia: it's about the size of the side of a double-decker bus! It measures 12.8 m wide, with a diagonal of 14.3 m.

a What is the height of the flag?

b Calculate the perimeter of the flag.

c What is the distance from the base of the flagpole to the lowest point where the flag is attached to the pole?

d Assume the flag is flying flat and directly at right angles to the pole. Calculate the distance from the base of the flagpole to the:

i lower unattached corner of the flag

ii higher unattached corner of the flag.

12 A ladder rests against a vertical wall, with the top of the ladder 7 m above the ground. When the bottom of the ladder is moved 1 m further away from the wall, the ladder is flat on the ground with one end against the wall.

a Draw a diagram to illustrate this situation.

b Calculate the length of the ladder.

PROBLEM SOLVING AND REASONING

13 The orthocentre of a triangle is the intersection point of the perpendicular heights of each side of the triangle. Consider $\triangle ABC$ where $A(-5, 0)$, $B(-1, 8)$ and $C(2, -1)$.

a Determine the equation of the lines of all three perpendicular heights.

b Find the intersection point of the perpendicular heights.

c Find the exact distance each vertex of $\triangle ABC$ is from the orthocentre.

CHALLENGE

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet Pythagoras' theorem

Interactive skillsheet Finding the length of the hypotenuse

Interactive skillsheet Finding the length of a shorter side

Topic quiz 9A

9B Trigonometry

Learning intentions

By the end of this topic you will be able to ...

✔ use trigonometry to find unknown side lengths

✔ use trigonometry to find unknown angles.

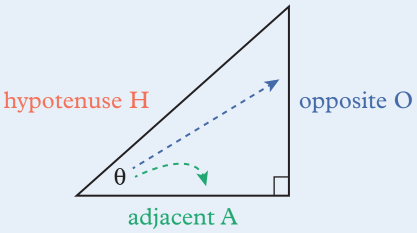
Inter-year links

Year 9 7H Trigonometric ratios

Finding lengths using trigonometry

Key content video

- **Trigonometry** describes the relationship between the angles and side lengths of a triangle.
- The Greek letter θ (theta) is often used as a pronumeral for angles.
- The sides of a right-angled triangle can be labelled as:
  - → the hypotenuse: the longest side of the right-angled triangle opposite the right angle
  - → the opposite side: the side opposite an interior reference angle
  - → the adjacent side: the side adjacent to (or next to) an interior reference angle that is not the hypotenuse.
- The trigonometric ratios are known as the **sine** ratio, the **cosine** ratio and the **tangent** ratio. They are the ratios between two side lengths of a right-angled triangle with respect to a given interior reference angle.
- For right-angled triangles with a reference angle of θ:
  - → $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}$
  - → $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}$
  - → $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$



- A calculator can be used to determine the sine, cosine or tangent values of any angle.
- The equations for the sine, cosine and tangent ratios can be rearranged to determine the side lengths of a right-angled triangle given an angle and one side length.

Using trigonometry to determine unknown angles

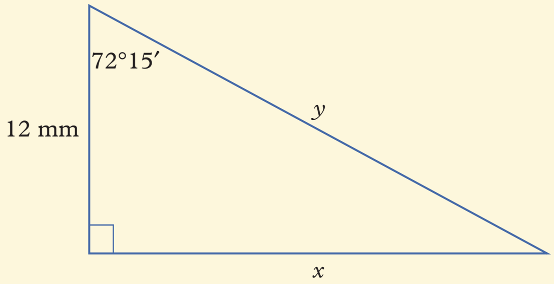
- The non-right angles in a right-angled triangle can be determined using the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a ratio as they are the opposite operations or inverse functions of the sine, cosine and tangent functions.
  - → $\theta = \sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right)$
  - → $\theta = \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right)$
  - → $\theta = \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right)$

Degrees and minutes

- Angles written in degrees can be measured with greater accuracy using parts of degrees, as decimals or using the units of measure minutes (′) and seconds (″).
  - → 1 degree = 60 minutes
  - → 1 minute = 60 seconds
- In this chapter, angles are measured to the nearest minute. For example, the angle 53 degrees 18 minutes is written 53°18″.

Example 9B.1 Using trigonometry to determine unknown side lengths

Find the length of each unknown side in this triangle, correct to one decimal place.



THINK	WRITE
1 Convert the reference angle from degrees and minutes to degrees with decimal places by dividing the minutes by 60.	$\theta = 72°15'$ $15' = \left(\frac{15}{60}\right)° = 0.25°$ $\theta = 72.25°$
2 Identify the hypotenuse (H), opposite side (O) and adjacent side (A) with respect to the reference angle 72°15'.	$\text{H} = y, \text{O} = x, \text{A} = 12 \text{ mm}$
3 Decide which trigonometric ratio to use to find the length of x. Since this involves the sides opposite and adjacent to the reference angle (O and A), use tangent.	$\tan(\theta) = \frac{\text{O}}{\text{A}}$
4 Substitute values for θ, O and A into $\tan(\theta) = \frac{\text{O}}{\text{A}}$.	$\tan(72.25°) = \frac{x}{12}$
5 Solve the equation for x. Use a calculator to multiply 12 by tan(72.25°). Round the value you find for x to one decimal place and include units.	$x = 12 \times \tan(72.25\ °)$ $= 37.4879...$ $\approx 37.5 \text{ mm}$
6 Decide which trigonometric ratio to use to find the length of y. Since this involves the side adjacent to the reference angle and the hypotenuse (A and H), use cosine.	$\cos(\theta) = \frac{\text{A}}{\text{H}}$
7 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.	$\cos(72.25°) = \frac{12}{y}$
8 Solve the equation for y. Use a calculator to divide 12 by cos(72.25°). Round the value you find for y to one decimal place and include units.	$y \times \cos(72.25°) = 12$ $y = \frac{12}{\cos(72.25°)}$ $= 39.3617...$ $\approx 39.4 \text{ mm}$

Example 9B.2 Using the inverse trigonometric functions

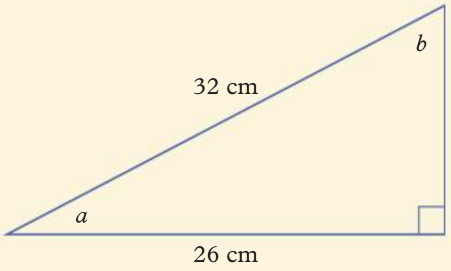
Use a calculator to find the size of θ for these trigonometric ratios to the nearest minute.

a $\sin(\theta) = 0.439$ **b** $\tan(\theta) = \frac{17}{5}$

THINK	WRITE
a 1 Rearrange to make θ the subject of the equation using the inverse sine function.	**a** $\sin(\theta) = 0.439$ $\theta = \sin^{-1}(0.439)$
2 Use the $\sin^{-1}$ key on your calculator to find the inverse sine of 0.439.	$\theta = 26.0400\ldots$
3 Multiply the decimal part of the angle by 60 and round the result to the nearest whole number to convert to minutes.	$0.0400\ldots \times 60 = 2.4$ ≈ 2
4 Write the angle to the nearest minute.	$\theta = 26°2'$
b 1 Rearrange to make θ the subject of the equation using the inverse tangent function.	**b** $\tan(\theta) = \frac{17}{5}$ $\theta = \tan^{-1}\left(\frac{17}{5}\right)$
2 Use the $\tan^{-1}$ key on your calculator to find the inverse tangent of $\frac{17}{5}$.	$\theta = 73.6104\ldots$
3 Multiply the decimal part of the angle by 60 and round the result to the nearest whole number to convert to minutes.	$0.6104\ldots \times 60 = 36.624$ ≈ 37
4 Write the angle to the nearest minute.	$\theta = 73°37'$

Example 9B.3 Using trigonometry to determine unknown angles

Find the size of each of the two unknown angles in this triangle. Give your answers correct to two decimal places.



THINK	WRITE
1 Decide which angle to calculate first, and identify the known side lengths with respect to that chosen angle.	$\theta = a$, H = 32 cm, A = 26 cm
2 Decide which trigonometric ratio to use. Because the adjacent side (A) and hypotenuse (H) are involved, use cosine.	$\cos(\theta) = \frac{\text{A}}{\text{H}}$
3 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.	$\cos(a) = \frac{26}{32}$
4 Rearrange to make a the subject of the equation using the inverse cosine function.	$a = \cos^{-1}\left(\frac{26}{32}\right)$
5 Use the $\cos^{-1}$ key on your calculator to find the inverse cosine of $\frac{26}{32}$.	$= 35.659\ldots$
6 Round your answer correct to two decimal places.	$\approx 35.66°$
7 Use the value of a and the right angle to find the value of b. Remember that the sum of the interior angles of a triangle is equal to 180°.	$b = 180° - 90° - a$ $= 90° - 35.659\ldots°$ $\approx 54.34°$

Helpful hints

- ✔ When finding an unknown side length, label the triangle to identify which side length you know (O, A or H) and which side length you want to find (O, A or H). Then choose the trigonometric ratio whose formula contains both.
- ✔ When finding an unknown angle, label the triangle to identify which two side lengths you know (O, A or H). Then choose the trigonometric ratio whose formula contains both.
- ✔ Remember that $\sin^{-1}$ does *not* mean sine to the power of –1 (the reciprocal of sine).
- ✔ Remember to write $\theta = \sin^{-1}\left(\frac{4}{5}\right)$, not $\sin^{-1}(\theta) = \frac{4}{5}$.
- ✔ It helps to draw diagrams roughly to scale. Remember, the longest side of a triangle is opposite the largest angle and the shortest side is opposite the smallest angle.

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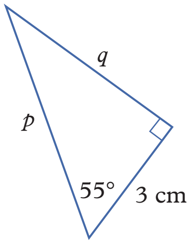
Exercise 9B Trigonometry

△ 1–5, 7–9 ■ 1–4, 6, 10, 12(a, b), 14, 17 ⬟ 1, 3(d–f), 6, 10–13, 15–19

9B.1 **1** Find the lengths of the unknown sides in each of these triangles, correct to one decimal place.

a



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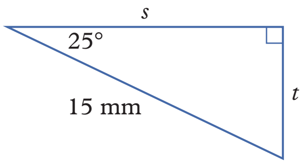

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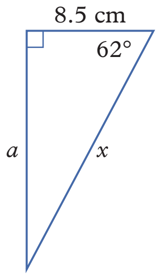

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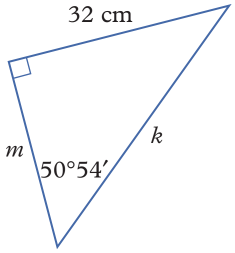

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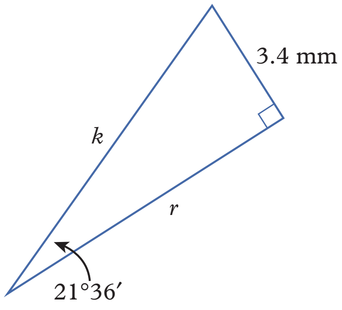

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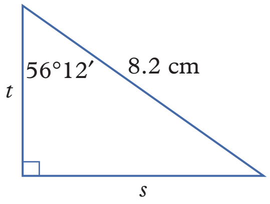


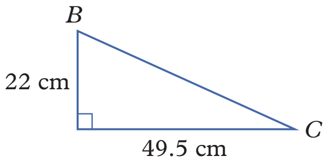
9B.2 **2** Use a calculator to find the size of θ for the trigonometric ratios below. Write your answers to the nearest minute.

a $\sin(\theta) = 0.378$ **b** $\cos(\theta) = 0.845$ **c** $\tan(\theta) = 2.376$

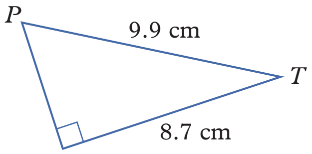
d $\sin(\theta) = \frac{10}{13}$ **e** $\cos(\theta) = \frac{44}{99}$ **f** $\tan(\theta) = \frac{1}{100}$

9B.3 **3** Find the size of each of the two unknown angles in each of these triangles. Give your answers to the nearest degree.

a



b



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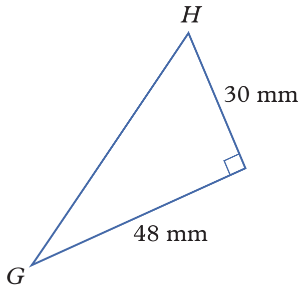

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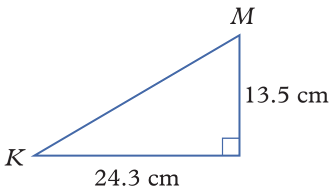

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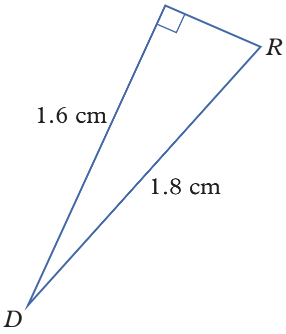

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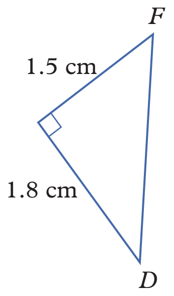
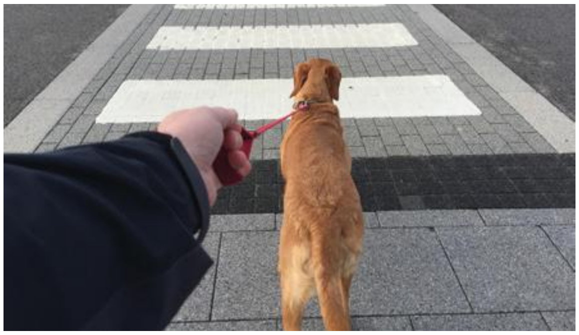


4 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. Shun's dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.



a Draw a diagram to show this information.

b How far in front of Shun is his dog, correct to the nearest centimetre?

c What is the vertical height of the end of the leash in Shun's hand to the nearest centimetre?

5 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick, correct to the nearest millimetre?

6 The Leaning Tower of Pisa is a freestanding bell tower next to the cathedral in the Italian city of Pisa. The top of the tower is displaced 3.9 m horizontally from where it would be if it were standing vertically. The distance from the centre of the base to the centre of the top is 56.28 cm.

a Draw a labelled sketch of the tower using this distance and its horizontal displacement. Mark the angle of 'lean' of the tower.

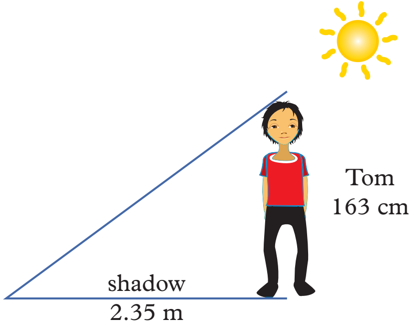
b Describe which trigonometric ratio you would use to calculate the angle of the tower's lean, marked in part **b**.

c Calculate the angle of lean of the tower. Give your answer to the nearest minute.

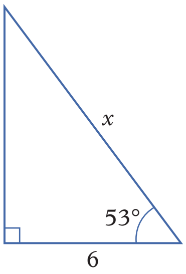
7 Tom and his brother Sam were standing in the sunshine measuring the length of each other's shadow. They each knew their height, so they collated the measurements (see the table on the right).

	Height	**Length of shadow**
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m

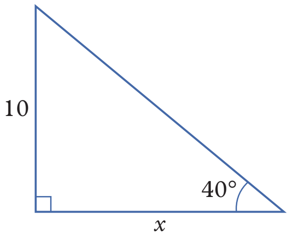
- **a** Tom drew the diagram on the right to represent his shadow and height. Draw a similar diagram to display Sam's measurements.
- **b** Calculate the base acute angle for the triangles in each diagram. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?
- **c** **i** Explain why the two triangles formed by the boys' heights and their shadows must be similar shapes.
  - **ii** Draw a diagram combining the two triangles from the diagram above and your own diagram from part **a**.



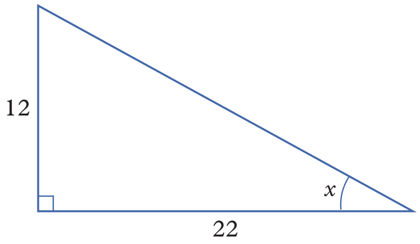
8 **a** Ji-ho has attempted to find the value of x in the following triangles. Explain the mistake Ji-ho made in all three questions.



$$\cos(53°) = \frac{x}{6}$$
$$x = 6\cos(53°)$$
$$x = 3.61089...$$
$$x \approx 3.6$$



$$\tan(40°) = \frac{x}{10}$$
$$x = 10\tan(40°)$$
$$x = 8.390996...$$
$$x \approx 8.4$$



$$\tan(x) = \frac{22}{12}$$
$$x = \tan^{-1}\left(\frac{22}{12}\right)$$
$$x = 61.3895...°$$
$$x \approx 61.4°$$

- **b** Explain how you can ensure you don't make the mistakes in part **a**.

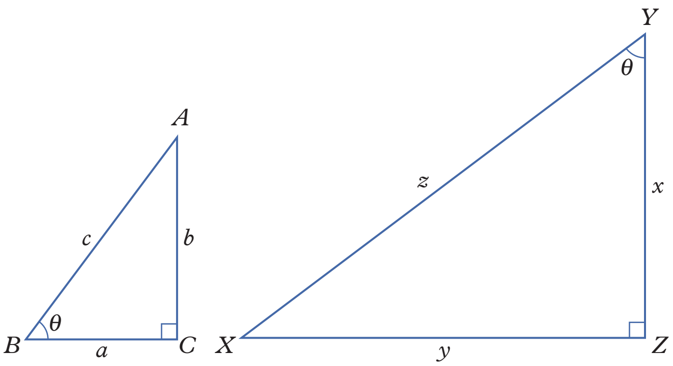
9 Parliament House in Canberra is 300 m long and 300 m wide, being one of the largest buildings in the Southern Hemisphere.

- **a** Draw a labelled diagram to show the shape and size of the building from above.
- **b** Use Pythagoras' theorem to calculate the diagonal length of the building.
- **c** Use trigonometry to calculate the diagonal length of the building. Hint: Identify the size of the angle you can use.
- **d** Comment on your answers for parts **b** and **c**.

10 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide, and is hung so the bottom of the frame is 1.5 m from the floor. If she lowers her eyes 5.7° from the horizontal, the bottom of the frame of the painting is directly in her line of sight. If she raises her eyes 16.7° from the horizontal, the top of the frame is directly in her line of sight.

- **a** At what height are Sandy's eyes?
- **b** What are the dimensions of the painting?

11 Trigonometry and similar triangles are closely related. In this question, we look at why the values of sin(θ), cos(θ) and tan(θ) depend only on θ and not the side lengths of a right-angled triangle.



a In your own words, describe the similarity condition AAA.

b Use the interior angle sum of a triangle and the AAA similarity condition to prove that the following two triangles are similar.

c In your own words, describe the similarity condition SSS.

d For θ in the triangles above, find two equal expressions for each of:

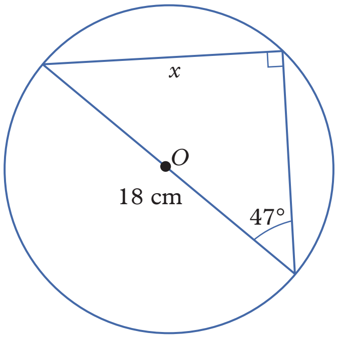
i sin(θ) **ii** cos(θ) **iii** tan(θ).

Recall that if two right-angled triangles meet one of the similarity conditions, they must meet all of them.

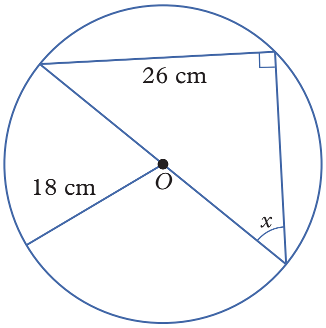
e Use parts **a** to **d** to explain why the trigonometric ratios depend only on the value of θ.

12 Determine the value of x in each of the following graphs, correct your answer to two decimal places.

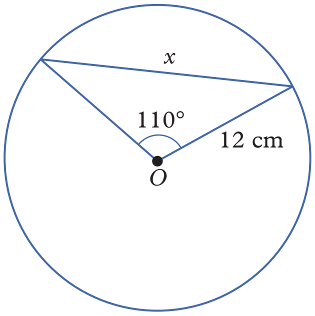
a



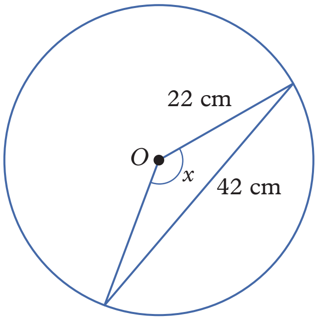
b



c



d

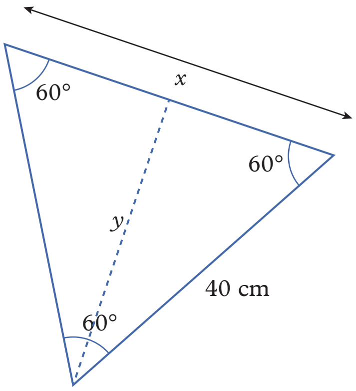


13 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between its parallel sides correct to two decimal places.

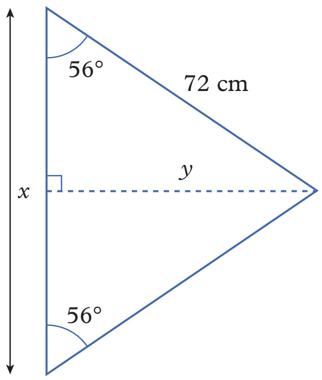
14 a Determine the value of x and y in the following triangles, correct to two decimal places where required.

b Determine the area of each triangle, correct to one decimal place.

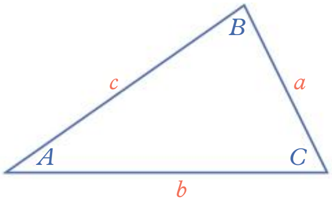
i



ii



CHALLENGE



15 **a** Copy the given triangle into your workbook.

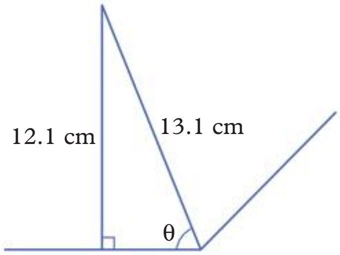
b Draw the height of the triangle from vertex B to base b. Label the point where the height meets the base, D. Label the length of the height h.

c **i** Use the sine ratio for angle A to write a relationship between h and c. Rearrange to make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Rearrange to make h the subject of the formula.

iii Equate your two answers for parts **i** and **ii** and show how this can be written as $\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$.

d Repeat the same reasoning outlined in parts **a**, **b** and **c** to find a similar relationship involving angle B. Show that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.

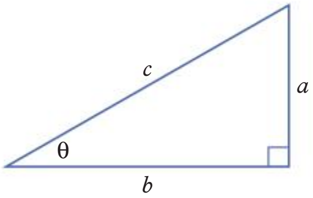


16 The vertex of a regular polygon where line segments have been drawn from the polygon's centre to the vertex and the midpoint of the adjacent edge is shown to the right.

a Determine the value of θ correct to one decimal place.

b Use the angle from part **a** to determine what type of regular polygon the vertex is from.

c Determine the perimeter of the regular polygon correct to the nearest centimetre.



17 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider the triangle in the diagram shown on the right.

a The sine of θ is the ratio of the length of the opposite side to the length of the hypotenuse; that is, $a : c$. When θ is 0°:

i What is the length of a?

ii What does this mean for the value of the ratio $a : c$?

b Now consider the situation when θ becomes almost 90°.

i What is the value of a compared with c?

ii What is the value of the ratio $a : c$?

c Describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.

d Repeat parts **a–c** for the cosine ratio. Consider the relevant line segments for that ratio.

e Once more, repeat parts **a–c** for the tangent ratio, considering the relevant line segments for that ratio.

18 From your investigations in question **17**, you can determine the minimum and maximum values for each of the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of _____ and a maximum value of _____.
The cosine ratio has a minimum value of _____ and a maximum value of _____.
The tangent ratio has a minimum value of _____ and a maximum value of _____.

19 Check your answers for question **18** with a calculator. Provide some values to support your conclusions.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet Trigonometry

Interactive skillsheet Finding lengths by trigonometry

Interactive skillsheet Finding angles by trigonometry

Worksheet Pythagoras' theorem and trigonometry

Topic quiz 9B

9C Applications of trigonometry

Learning intentions

By the end of this topic you will be able to ...

✔ use angles of elevation and depression to calculate distances

✔ use compass bearings and true bearings in trigonometry problems.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	7I Using trigonometry to find side lengths

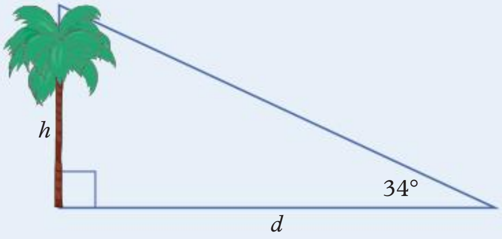
Angles of elevation and depression

Key content video

- The **angle of elevation** is the angle measured upward from the horizontal.

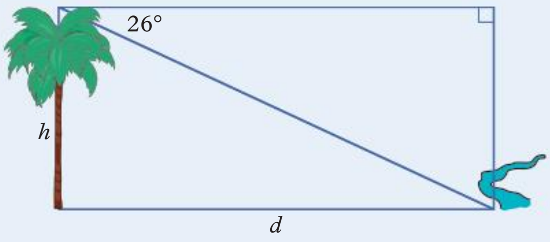
  For example, in the diagram on the right, there is a 34° angle of elevation to the top of a palm tree from a distance d along a flat beach.

  The height of the palm tree is given by $h = d\tan(34^\circ)$.



- The **angle of depression** is the angle measured down from the horizontal.

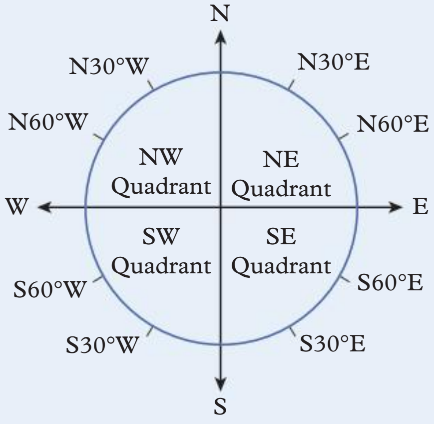
  For example, if someone climbs to the top of the same palm tree and measures a 36° angle of depression from the top of the tree to a nearby ocean inlet, then the distance from the base of the tree to the inlet would be given by $d = \dfrac{h}{\tan(36^\circ)}$.



Bearings

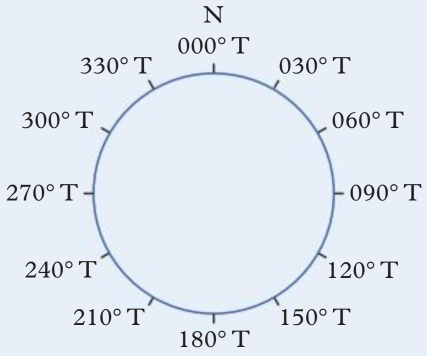
- The four cardinal directions are north, south, east and west.
- **Compass bearings** describe direction using angles between 0° and 90° within one of the four quadrants of a compass (see the diagram on the right). The bearings are written with a capital letter for the north–south direction, followed by the angle, followed by a capital letter for the east–west direction.

  For example, the angle 35° clockwise of north would be written as N35°E, and the opposite direction would be written as S35°W.

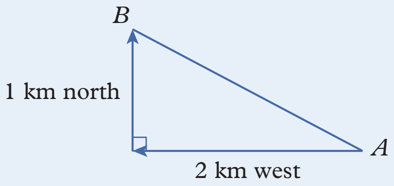


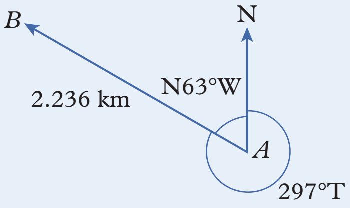
- **True bearings** describe a direction using an angle between 0° and 360° clockwise from north. They are written with three digits for the angle, followed by a capital T.

  For example, the angle 35° clockwise of north would be written as 035°T, and the opposite direction would be written as 215°T.



- It is often useful to know how far a position is in terms of distance along the cardinal directions. For example, if point *B* is 1 km north and 2 km west of point *A*, then Pythagoras' theorem can be used to find the total distance and bearings from point *A* to point *B* to be approximately 2.236 km N63°W or 297°T.



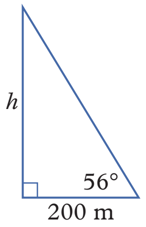
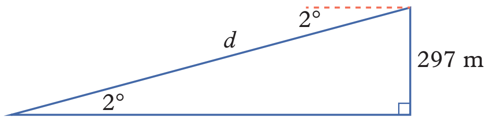


Example 9C.1 Calculating distances using angles of elevation and depression

Cape Pillar in the Tasman National Park is home to the tallest sea cliffs in the Southern Hemisphere.

a If a tourist measures her kayak to be 200 m from the base of the closest cliff face and the angle of elevation to the tip of the cliff is 56°, how tall is the cliff? Round your answer to the nearest metre.

b The next day, the same tourist hikes to the top of the cliff and spots a boat near the horizon at an angle of depression of 2°. How far away is the boat from the tourist? Round your answer to the nearest metre.

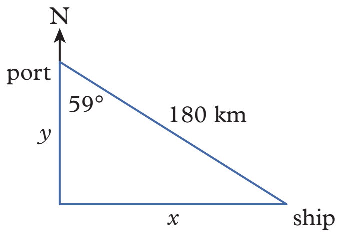
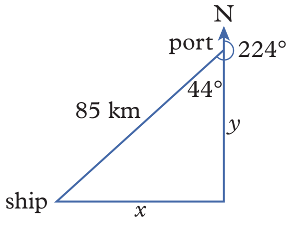
THINK	WRITE
a **1** Draw a diagram to represent the situation. It does not need to be to scale.	**a** 
2 Substitute the known values into the equation for the tangent ratio.	$\theta = 56°$, O $= h$, A $= 200$ m, $\tan(\theta) = \frac{\text{O}}{\text{A}}$ $\tan(56°) = \frac{h}{200}$
3 Solve for the height of the cliff and round to the nearest metre.	$h = 200\tan(56°)$ $= 296.5121\ldots$ ≈ 297 m
b **1** Draw a diagram to represent the situation. It does not need to be to scale. Assign the angle of depression to the corresponding angle of elevation, as they are alternate angles.	**b** 
2 Substitute the known values into the equation for the sine ratio.	$\theta = 2°$, O $= 297$ m, H $= d$ $\sin(\theta) = \frac{\text{O}}{\text{H}}$ $\sin(2°) = \frac{297}{d}$
3 Solve for the distance from the tourist to the boat and round to the nearest metre.	$d = \frac{297}{\sin(2°)}$ $= 8510.1514\ldots$ ≈ 8510 m

Example 9C.2 Calculating distances using bearings

For ships travelling the distances and directions below:

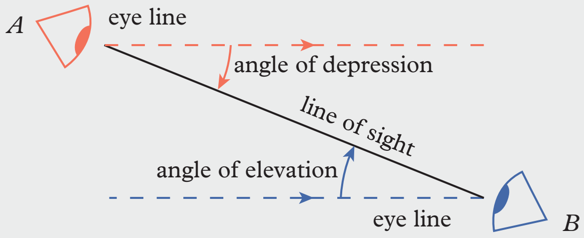
- **i** Draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port.
- **ii** Use trigonometry to calculate the distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 180 km on a bearing of S59°E **b** 85 km on a bearing of 224°T

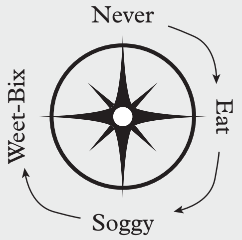
THINK	**WRITE**
a i Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then draw a line in the direction towards the ship, label the distance traveled on the graph. Label the distances along the east–west and north–south axes of your diagram with pronumerals *x* and *y*.	**a i** 
ii 1 Use the sine ratio to calculate the distance east.	**ii** $\sin(\theta) = \frac{O}{H}$ $\sin(59°) = \frac{x}{180}$ $x = 180 \times \sin(59°)$ ≈ 154 km The ship is 154 km east of port.
2 Use the cosine ratio to calculate the distance south.	$\cos(\theta) = \frac{A}{H}$ $\cos(59°) = \frac{y}{180}$ $y = 180 \times \cos(59°)$ ≈ 93 km The ship is 93 km south of port.
b i 1 A true bearing of between 180°T and 270°T is in the SW quadrant; calculate the angle measured clockwise from south.	**b i** 224° – 180° = 44°
2 Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then a line in the correct direction towards the ship, label the distance travelled. Label the distance along the east–west and north–south axes of your diagram with pronumerals *x* and *y*.	
ii 1 Use the sine ratio to calculate the distance west.	**ii** $\sin(\theta) = \frac{O}{H}$ $\sin(44°) = \frac{x}{85}$ $x = 85 \times \sin(44°)$ ≈ 59 km The ship is 59 km west of port.
2 Use the cosine ratio to calculate the distance south.	$\cos(\theta) = \frac{A}{H}$ $\cos(44°) = \frac{y}{180}$ $y = 85 \times \cos(44°)$ ≈ 61 km The ship is 61 km south of port.

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly to scale.
- ✔ The angle of depression from point A to point B is equal to the angle of elevation from point B to point A. This is because they form alternate angles on parallel eye lines.



- ✔ One way of remembering the cardinal directions is the phrase 'Never Eat Soggy Weet-Bix'. The first letter of each word matches the cardinal directions, read clockwise from north.



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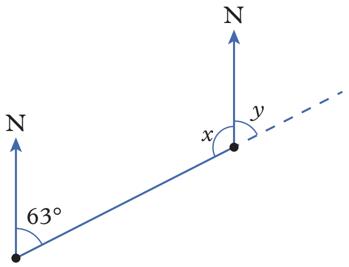
Exercise 9C Applications of trigonometry

△ 1–6, 8, 11, 12, 14, 15 ■ 1, 2, 5–7, 9, 10, 13, 16, 17, 19 ⬟ 1(d–f), 5, 6, 10, 14, 16–18, 20

1 Determine the value of the pronumerals in each of the following.

a



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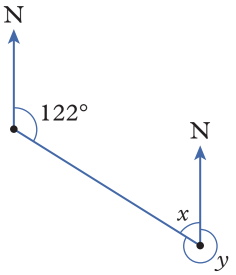
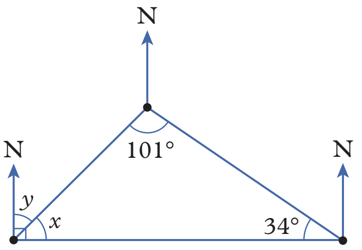


c



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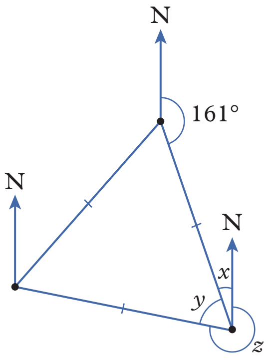

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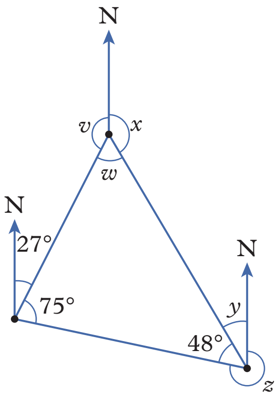
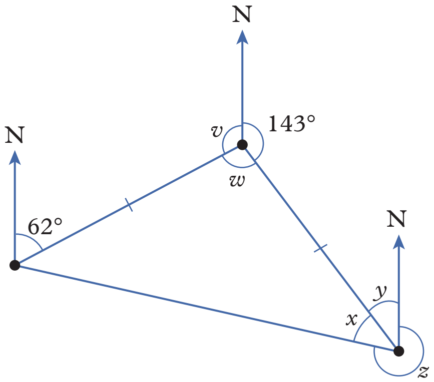


f

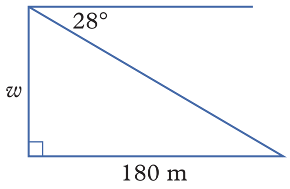


UNDERSTANDING AND FLUENCY

UNDERSTANDING AND FLUENCY

2 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

a



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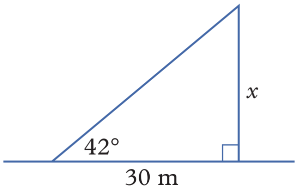

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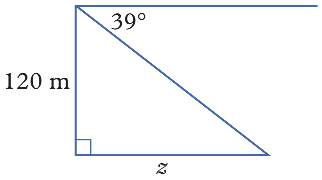

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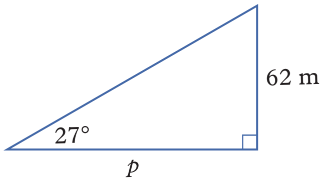

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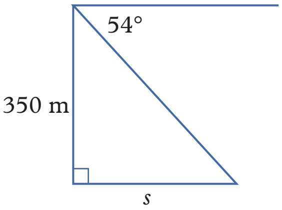

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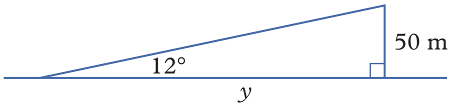


9C.1 3 A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°.

a Find the height of the lighthouse, to the nearest metre.

b The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea, to the nearest metre.

4 Convert the following bearings:

a from compass bearings to true bearings

i S30°E **ii** N52°W **iii** N5°E

b from true bearings to compass bearings.

i 200°T **ii** 100°T **iii** 300°T

9C.2 5 For ships travelling each of the distances and directions below:

i draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port

ii use trigonometry to calculate these distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 250 km on a bearing of N30°E **b** 100 km on a bearing of 300°T

c 75 km on a bearing of S45°W **d** 210 km on a bearing of 090°T

6 If each ship in question **5** turns around and returns to its original port, write a description of the ship's path with respect to its turn-around position using:

i compass bearings and distance in kilometres

ii true bearings and distance in kilometres.

7 Stavros is descending an escalator at an angle of depression of 30°.

a If the escalator is 6 m high at the top, correct to two decimal places, what horizontal distance does it cover?

b If it takes 30 seconds for Stavros to descend the escalator, correct to one decimal place, at what speed, in m/s, is the escalator moving?

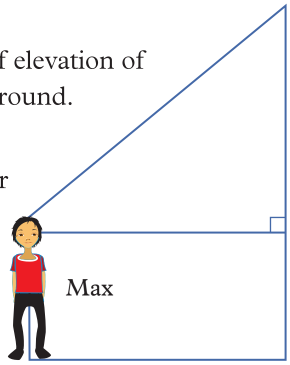
8 Max throws a ball into the air. He views the ball at its highest point to be at an angle of elevation of 40° from him and 3 m horizontally away from him. Max's eye level is 172 cm above ground.

a Copy the diagram on the right and label it with the known values.

b What height does the ball reach above the level of Max's eyes? Correct your answer to the nearest centimetre.

c What is the maximum height the ball reaches above the ground? Correct your answer to the nearest centimetre.

d Why is it important to consider the height of a person's eyes in a situation like this?



9 Can the angle of elevation or angle of depression be an obtuse angle? Explain.

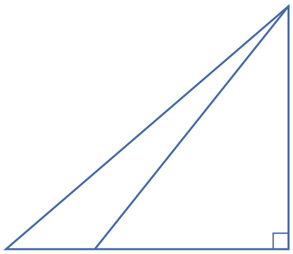
10 Toshiko is standing at the edge of a 10 m cliff and sees a boat near the base of the cliff. Toshiko measures the angle of depression to the point where the bow of the boat (the front of the boat) touches the water as 24°.

a Correct to two decimal places, how far from the base of the cliff is the boat if Toshiko's eye height is 152 cm?

Toshiko spots a shark directly below the boat and measures the angle of depression to the nose of the shark as 25°.

b If the nose of the shark is directly below where the bow of the boat touches the water, correct to two decimal places, how deep is the shark swimming in the water?

11 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)



a Copy the diagram on the right and label it with the known values.

b Find the height of the tree calculated using Ann's observations, correct to one decimal place.

c Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance from the tree, correct to one decimal place.

d What is the distance between Ann and Beth? Correct your answer to one decimal place.

e Suppose Beth had been standing in the same line as Ann, but on the opposite side of the tree.

  i Draw a diagram to show this situation.

  ii What would have been your answer to part **d** in this case?

12 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

a Draw a diagram to display this information.

b Find the height of the tower, correct to one decimal place.

13 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks N22.5°E. They both walk 25 km.

a Display the information on a diagram.

b How far north of his starting point is Ben when he completes his walk? Correct your answer to one decimal place.

c How far east of his starting point is Ben at this stage? Correct your answer to one decimal place.

d If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk? Correct your answer to one decimal place.

14 Jake is riding his bike from home to his friend Luke's house. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's house.

a Draw and label a sketch of Jake's route. Mark the north direction at each significant point on his journey.

b Calculate the direct distance between Jake's house and Luke's house, correct to one decimal place.

c Use trigonometry to find the true bearing from Jake's house to Luke's house. (Remember that bearings are measured from the north in a clockwise direction.)

d What is the true bearing from Luke's house to Jake's house?

e The bearing in part **d** is known as the back-bearing of the bearing in part **c** (and vice versa). What do you notice about the relationship between these two bearings?

f Draw a scale drawing of Jake's route. From your drawing:

  i find the distance between the two houses, correct to one decimal place.

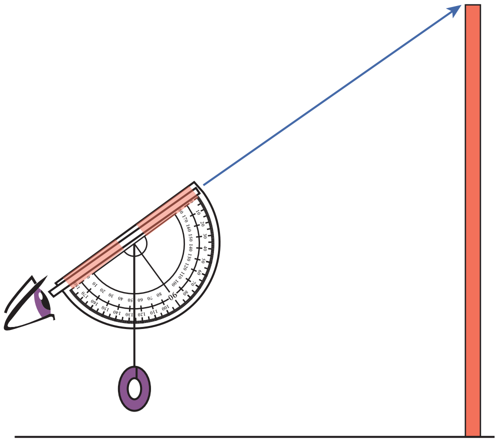
  ii use a protractor to measure the bearings in both directions.

g Comment on any differences between your previous answers and those found using your scale drawing.

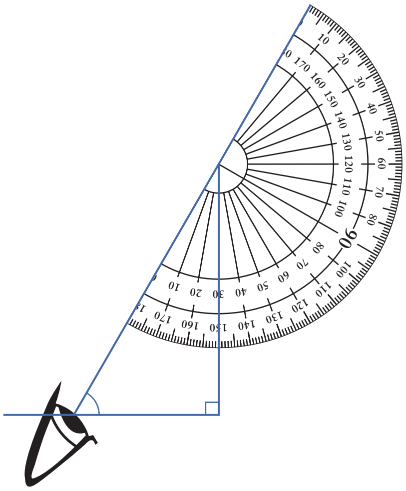
15 A clinometer (or inclinometer) is a tool used to measure angles of inclination. That is, angles of elevation or depression.

A makeshift clinometer can be constructed using a protractor, a straw, some string and a weight. You can then measure angles of elevation and depression by looking at the object through the straw, reading the angle the string makes with the protractor, then using geometry to determine the angle of elevation or depression.

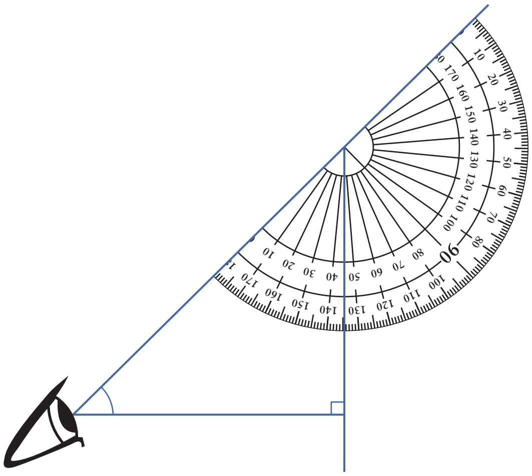
Determine the angle of elevation or depression each of the following clinometers is measuring.



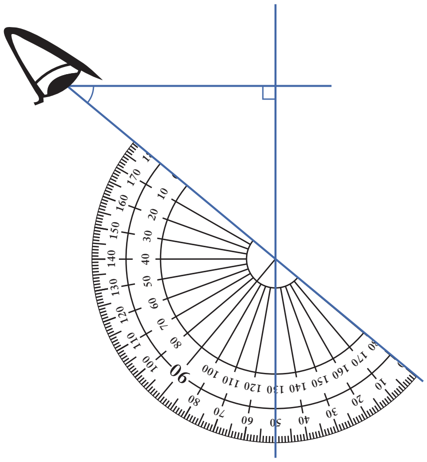
a



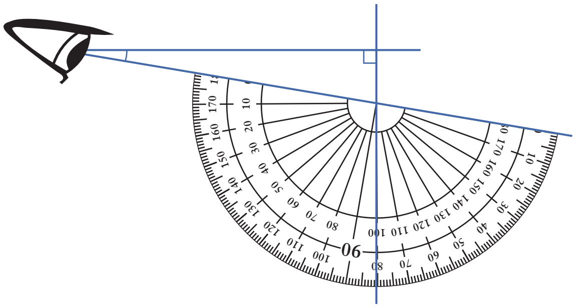
b



c



d



e Describe how you can use the measured angle on the clinometer to determine:

i the angle of elevation

ii the angle of depression.

16 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled *A*, and walk 35 km at a bearing of 055°T to *B*.
- From *B*, walk 25 km at a bearing of 015°T to finish the walk at *C*.

a Draw a sketch of the walk. Try to make it roughly to scale because it can lead to incorrect calculations if the relative position of points is misleading.

b Correct to one decimal place, find how far north:

i *B* is from *A*

ii *C* is from *B*

iii *C* is from *A*.

c Correct to one decimal place, find how far east:

i *B* is from *A*

ii *C* is from *B*

iii *C* is from *A*.

PROBLEM SOLVING AND REASONING

d Consider the right-angled triangle in which *AC* is the hypotenuse. Calculate:
  i the distance from *A* to *C* correct to one decimal place
  ii the true bearing from *A* to *C*
  iii the true bearing from *C* to *A*.

e Draw a scale drawing of the bushwalkers' route.
  i Measure distances and bearings to obtain measured answers for part **d**.
  ii Compare your answers with those you already had for part **d**.

17 A new hike for bushwalkers took them along a different route.

- From the starting point, walk 18 km at a bearing of 300°T.
- Change direction and walk 23 km at a bearing of 200°T to reach your endpoint.

a Find the direct distance from the starting point to the endpoint, correct to one decimal place.

b What is the bearing of the starting point from the endpoint?

18 Jan is standing 2 m in front of a painting on a wall. The bottom of the painting's frame is 1.4 m from the floor, and Jan's eyes are 165 cm from the floor. The painting, including the frame, has a height of 70 cm.

a Draw a diagram to show this information.

b Find the angle of elevation from Jan's eyes to the top of the painting's frame.

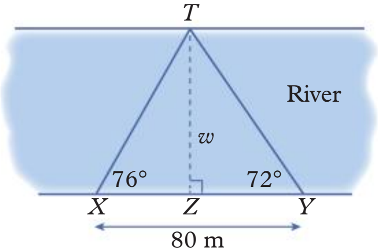
c What is the angle of depression from Jan's eyes to the bottom of the painting's frame?

d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to look directly at the top of its frame, and then directly at the bottom of its frame?

CHALLENGE

19 Council surveyors have been set the task of measuring the width of a river along a straight stretch in preparation for building a bridge across it. They have located a tree on the opposite side of the river and have taken angle measurements from two points *X* and *Y*, which are 80 m apart, as shown in the diagram on the right.

Calculate the width of the river, correct to the nearest metre. Hint: Let the distance *XZ* be represented by a pronumeral.



20 A pilot was flying a routine route from *A*, directly east to *B*, a distance of 3500 km. She didn't know her compass was not operating correctly and she was actually travelling 10° south of her true direction. When she didn't arrive at her destination after travelling 3500 km, the pilot realised she was off course.

a How much further does she need to travel to fly directly to her destination from this point? Give your answer to the nearest kilometre.

b The pilot only has enough fuel for another 60 km, so she decides to land on a nearby runway 29 km away. The pilot measures the angle of depression from the plane to the runway to be 6° at an altitude of 3 km. The runway is a straight line from north to south and so the pilot must approach the runway in that direction. The bearing from the plane to the runway is 115°T. The pilot wants to travel directly east and then make a sharp right-angled turn to land the plane.
  i What is the horizonal distance east before the pilot must make the turn? Give your answer to the nearest kilometre.
  ii What is the vertical distance south from the turn to the runway? Give your answer to the nearest kilometre.

c If the pilot had become aware of the error when she was halfway into the flight and adjusted her direction from that point, would she have made it to her destination?

Check your Student obook pro for these digital resources and more:

Interactive skillsheets
Angles of elevation and depression

Interactive skillsheet
Compass bearings and true bearings

Worksheet
Applying Pythagoras' theorem and trigonometry

Investigation
How do you get to school?

Topic quiz
9C

9D Three-dimensional problems

Learning intentions

By the end of this topic you will be able to …

✔ find lengths and angles within 3D objects.

Inter-year links

Year 7	7F 3D objects
Year 9	7J Using trigonometry to find angles

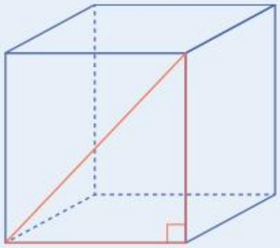
Three-dimensional problems

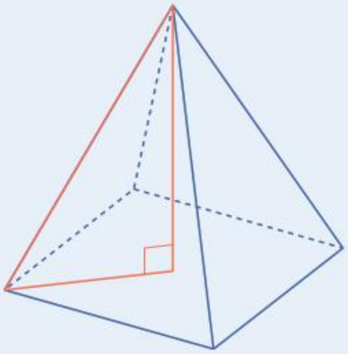
Key content video

- Just as 2D shapes can contain many right-angled triangles, there are also many right-angled triangles to be found in 3D objects.

  Some right-angled triangles can be found on the surface of 3D objects, such as the triangle that has as its hypotenuse the diagonal joining vertex to vertex across the face of a cube.

  Other right-angled triangles can be found within a shape, such as the triangle shown in the diagram of the pyramid on the right. That triangle is formed by a slanting edge of the pyramid (the hypotenuse), a perpendicular line from the base of the square pyramid to the apex of the pyramid, and a line on the base joining the perpendicular to the slanting edge.
- Pythagoras' theorem can be used to find an unknown length, either on the surface of a 3D object or within a 3D object.
- Trigonometry can be used to find an unknown angle or length, either on the surface of a 3D object or within a 3D object.





Example 9D.1 Three-dimensional problems

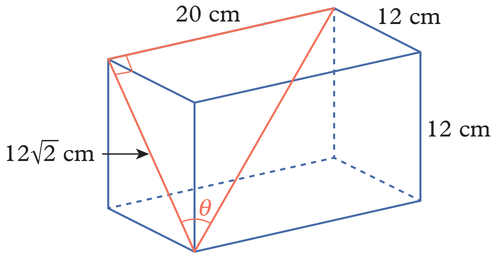
A 20 cm long box has two square faces with 12 cm sides.

a Calculate the length of the diagonal across each of the square faces of the box, to the nearest centimetre.

b Find the length of the diagonal within the box, to the nearest centimetre.

c Find the angle the internal diagonal from part **b** makes with the diagonal across the square face, to the nearest degree.

THINK	WRITE
a Substitute the side lengths of the square into Pythagoras' theorem to find the length of the diagonal across the square face.	**a** $c^2 = a^2 + b^2$ $= 12^2 + 12^2$ $= 12^2 \times 2$ $c = 12\sqrt{2}$ ≈ 17 cm

b	**1** Draw a labelled diagram of the box. Identify the right-angled triangle formed by the two diagonals.	**b** 
	2 Substitute the side lengths of the two short sides of the right-angled triangle into Pythagoras' theorem to find the length of the diagonal within the box (the hypotenuse).	$c^2 = a^2 + b^2$ $= (12\sqrt{2})^2 + (20)^2$ $= 288 + 400$ $= 688$ $c = \sqrt{688}$ ≈ 26 cm
c	**1** Choose a trigonometric ratio to use for calculating the angle. Note that all the sides are now known, so any trigonometric ratio can be used.	**c** $\tan(\theta) = \frac{\text{O}}{\text{A}}$ $\tan(\theta) = \frac{20}{12\sqrt{2}}$
	2 Solve for the angle using the inverse trigonometric function.	$\theta = \tan^{-1}\left(\frac{20}{12\sqrt{2}}\right)$ $\approx 50°$

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly drawn to scale.
- ✔ Remember, the trick is to find right-angled triangles! This can often be done by:
  - → drawing a line across the diagonal of a rectangle
  - → drawing a line through any triangle from one vertex to the opposite side, to meet that side at a right angle.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, rounding the size of angles in the middle of your work can lead to considerable differences in the values of trigonometric ratios.

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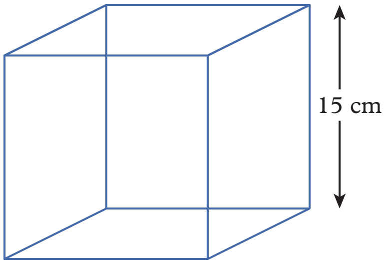
Exercise 9D Three-dimensional problems

△ 1–5, 7, 8, 10, 11 ■ 1–4, 6, 9, 10, 12, 13, 15 ⬟ 3, 6, 9, 10, 12–14, 16, 17

9D.1 **1** The side lengths of a cube are 15 cm.



a Calculate the length of the diagonal of the base of the cube correct to two decimal places.

b Find the length of the diagonal within the cube correct to two decimal places.

c Find the angle the internal diagonal makes with the base diagonal correct to two decimal places.

2 A box has a base 20 cm by 15 cm and is 10 cm tall.

a Provide working to show whether a 30 cm rod will fit in the box.

b If a rod was a snug fit, correct to two decimal places, what angle would it make with the base diagonal of the box?

3 A solid cube of side length 5 cm is sliced along its base diagonal.

a What is the shape of the exposed face? Draw a diagram with its dimensions labelled.

b Compare the size of the exposed face with the size of one of the faces of the cube.

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and the prism's height is 5 cm.

a Draw a labelled diagram of the triangular prism, showing all dimensions.

b Draw a labelled diagram of:

i the front view of the largest side

ii the top view (or base view).

c On your diagram of the front view, draw a diagonal.

i Find the length of the diagonal you drew correct to one decimal place.

ii What angle does the diagonal make with the longer side, correct to two decimal places?

iii Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.

d Calculate, correct to one decimal place:

i the perpendicular height of the base triangle

ii the area of the base.

e The cheese is sliced along the height of the base triangle.

i Draw the shape of one of the exposed faces. Label its dimensions.

ii Find the area exposed on both sides of the slice.

iii What is the diagonal length of the exposed face, correct to the nearest centimetre?

5 Jack was given a tent for Christmas. He wants to make a cover the same size as the tent to protect it when it rains. The front of the tent forms an equilateral triangle, with a height of 1.5 m.

a Sketch the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.

b Use Pythagoras' theorem to find the value of x correct to one decimal place.

c Jack has bought a rectangular tarpaulin sheet with one side long enough to fit the length of the tent from front to back. If the other side length is 3.5 m, explain whether this is sufficient to reach the ground from one side of the tent to the other.

6 The rectangular perimeter of a football field is lined with trees. Ethan is standing 70 m in front of one of the trees. He notices a much taller tree 37° to his left.

a Draw a sketch showing the relative position of the two trees described. Label the known values.

b Find the distance along the ground from Ethan to the base of the taller tree correct to one decimal place.

c He measures the angle of elevation to the top of the taller tree as 13°. Draw a sketch showing the elevation from Ethan to the top of the tall tree. Label the known values.

d Use this diagram to find the height of the tall tree correct to one decimal place.

7 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.

The base of the pyramid is a square with side lengths of 230 m. It stands 140 m tall.

- **a** Draw a labelled sketch of the pyramid.
- **b** Calculate the diagonal length of the base correct to two decimal places.
- **c** The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle that the sloping edge makes with the base diagonal,
correct to two decimal places.
- **d** A different right-angled triangle can be used to find the angle the sloping sides make with the base by measuring from the midpoint of one of the base sides.
  - **i** Draw a labelled diagram to display the dimensions of this triangle.
  - **ii** Calculate the angle the sloping side makes with the base, correct to two decimal places.

8 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of a cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

- **a** Find the diameter of the saucepan correct to one decimal place.
- **b** A stirring spoon is 20 cm in length. Explain whether it could fall below the rim of the saucepan.

9 An ice cream cone has a height of 10 cm. The circumference of the circular top of the cone is 15.5 cm.

- **a** Find the radius of the circular top of the cone correct to one decimal place.
- **b** Draw a labelled two-dimensional sketch of the outline of the cone. Identify any right angles within the shape.
- **c** Find the slant length of the cone correct to one decimal place.

10 A Christmas tree in the shape of a cone is 3 m tall with a base diameter of 2 m. Sandy wants to put streamers from the top to the base of the tree, along the outer surface of the tree. She has one 30 m long roll of streamer paper that she can cut up to do this. How many streamers the full length of the slant of the tree is she able to attach?

11 A regular tetrahedron is a three-dimensional object with four equilateral triangular faces.

a Identify the tetrahedron in the photo below and draw a sketch of it.

b Suppose that the length of each side of the tetrahedron you drew is 10 cm and its height from the base to the upper apex is 8.2 cm. Add these measurements to your sketch from part **a**.

c **i** Draw a diagram of a right-angled triangle with corresponding measurements from the tetrahedron. Label one side of the triangle with the height of the tetrahedron from part **a** and label the hypotenuse with the length of one edge of the tetrahedron.

ii Use your diagram to help you calculate the angle between the edge and the base of the tetrahedron correct to the nearest degree.

d Draw the shape of a face of the tetrahedron. Draw the perpendicular height, and label all known values.

i What is the perpendicular height of the face of the tetrahedron correct to one decimal place?

ii Draw a right-angled triangle showing the perpendicular height of the face and the height of the tetrahedron. Use your diagram to calculate the angle between the base and a face of the tetrahedron correct to the nearest degree.

12 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower, point A is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

a Draw a labelled diagram to display the given information. Label AT, the height of the tower, as h, and the distance XT along the ground as d.

b Use a trigonometric ratio with $\triangle AXT$ to find an equation for h in terms of d.

c Use a trigonometric ratio with $\triangle AYT$ to find an equation for h in terms of d. Note that the length of YT is $(25 + d)$ m.

d Equate your two equations from parts **b** and **c**. Solve to find a value for d correct to one decimal place.

e Use the value you found for d in part **d** with $\triangle AXT$ to find the height of the tower correct to one decimal place.

13 A rectangular prism has width, w, length, l and height, h.

a Write expressions for the lengths of the diagonals along each of the faces.

b Write an expression for the length of the diagonal of the rectangular prism from one vertex to the vertex it is furthest away from.

14 A rectangular prism has a base with length, l, twice its width, w. The height of the prism, h, is three times its width.

a Write the length and height of the prism in terms of its width, w.

b Draw a labelled diagram of the prism.

c Find the length of the base diagonal in terms of w.

d Calculate the length of the diagonal within the prism in terms of w.

e Find the angle that the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

CHALLENGE

15 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

16 Consider a cube of side length x cm.

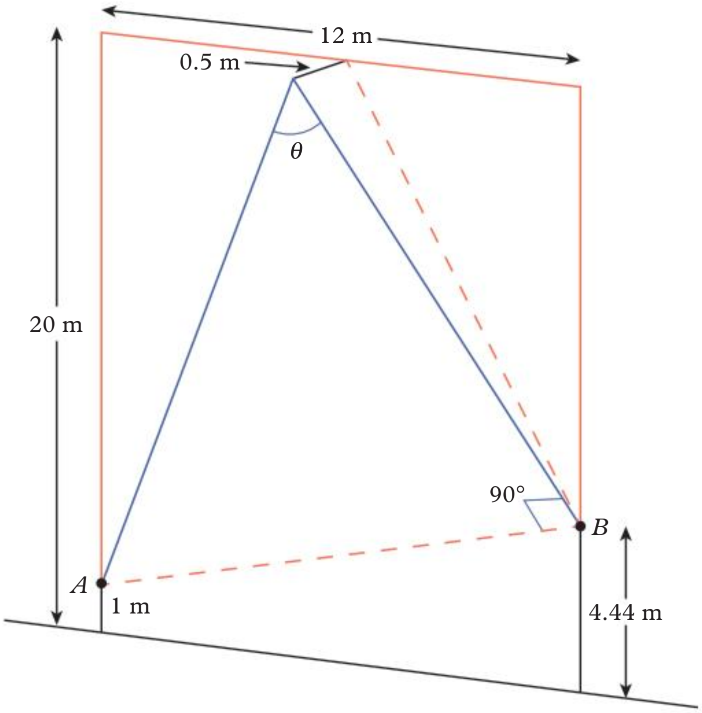
a Draw a diagram of the cube.

b An ant crawls from the top front left-hand corner to the bottom back right-hand corner along the outer surface of the cube.

i Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.

ii Find the length of the shortest path from your answer to part **i**. Give your answer in terms of the side length x.

17 Yolandi goes rock-climbing at an indoor rock-climbing gym. She starts by climbing a rectangular practice wall that is 12 m wide and 20 m high. A belay rope is suspended from the ceiling by a pulley and attached to Yolandi's waist to stop her from falling too far. The pulley is positioned at the centre, 0.5 m away from the wall. Yolandi starts from the left side of the wall with the rope tied to her waist at point A, 1 m above the floor. She then climbs directly to point B, 4.44 m high on the right side of the wall. She then continues directly to the top of the wall at its centre. Given that the angle the rope makes with the line from point A to point B is a right angle, correct to one decimal place, determine:



a the distance Yolandi climbs in total

b the angle, θ, that the rope turns from when Yolandi is on the left side of the wall to when Yolandi is at the right side of the wall.

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Interactive skillsheet
Finding lengths within 3D objects

Interactive skillsheet
Finding angles within 3D objects

Investigation
Bearings and 3D problems

Topic quiz
9D

9E The sine and area rules

Learning intentions

By the end of this topic you will be able to …

- ✔ determine unknown side lengths and angles using the sine rule
- ✔ determine the two possible solutions for an ambiguous case of the sine rule
- ✔ determine the area of a triangle using the included angle area formula.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	7J Using trigonometry to find angles

The sine of angles from 0° to 180°

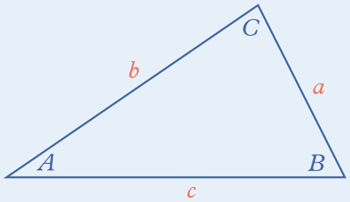
Key content video

- So far, sine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\sin(\theta) = \frac{\text{O}}{\text{H}}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \le \theta \le 90°$.
- The interior angles of a general triangle can be acute and obtuse, so sine must be defined for $0 \le \theta \le 180°$.
- The sine values of supplementary angles are equal, $\sin(180° - \theta) = \sin(\theta)$. For example, $\sin(150°) = \sin(30°)$.

The sine rule

- The sides of a non-right-angled triangle are often labelled using lowercase letters a, b and c. The interior angles opposite those side lengths are then labelled using the capital letters: A, B and C.
- The side lengths and interior angles of any triangle, ABC, are related to each other by the **sine rule**:

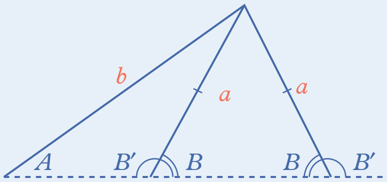
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

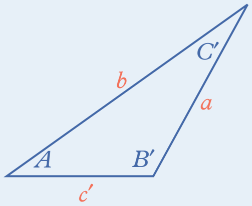


- The sine rule can be used to find missing side lengths or angles in cases where three of the values are already known and two of the known values are an opposite angle-side pair. That is, a pair of $a - A$, $b - B$ or $c - C$.

Ambiguous case of the sine rule

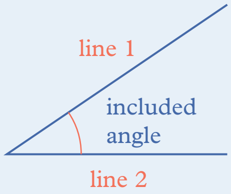
- There are two possible triangles that can be drawn when two side lengths (a and b) of a non-right-angled triangle are known and the angle A opposite the shorter side length is also known.
- The important difference between the two possible triangles is that, for one of them, angle B (opposite side b) is acute while, for the other, angle B is obtuse.
- The acute angle for B can be determined using the sine rule with a calculator.
- The obtuse angle for B (which can be labelled as B') is determined using the fact that the two possible values for B are supplementary: $B + B' = 180°$.
- Once B and B' are both known, they can be used with the sine rule to determine the remaining unknown values for the first triangle (side length c and angle C), as well as for the second triangle (side length c' and angle C').





Included angle area formula

- The included angle is the interior angle made by two lines that meet at a common vertex.
- The area of any triangle can be determined using the included angle of two known side lengths. This relationship is given by the **included angle area formula**:

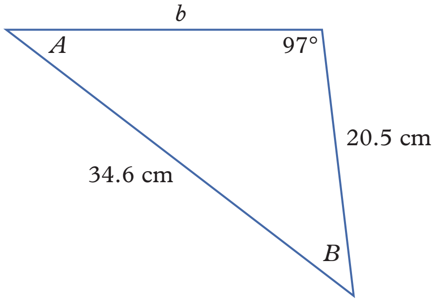


$$\text{area} = \frac{1}{2}\underbrace{bc}_{\text{adjacent side lengths}}\sin(\underbrace{A}_{\text{included angle}})$$

Example 9E.1 The sine rule

For $\triangle ABC$, find the unknown angles and side lengths correct to one decimal place using this information:

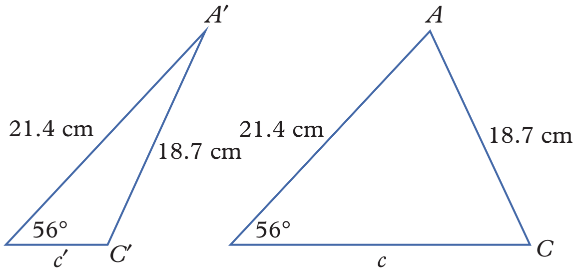
$a = 20.5$ cm, $c = 34.6$ cm, $C = 97°$

THINK	**WRITE**
1 Draw a labelled diagram, roughly to scale.	
2 To solve for A, substitute the known values into an equation for the sine rule containing a and c.	$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$ $\frac{20.5}{\sin(A)} = \frac{34.6}{\sin(97°)}$ $\frac{\sin(A)}{20.5} = \frac{\sin(97°)}{34.6}$ $\sin(A) = \frac{20.5 \times \sin(97°)}{34.6}$ $= 0.5881\ldots$ $A = \sin^{-1}(0.5881\ldots)$ $\approx 36.0°$
3 The size of angle B can now be calculated, because the interior angles of a triangle add to 180°.	$A + B + C = 180°$ $B \approx 180° - 97° - 36.0°$ $\approx 47.0°$
4 To solve for b to find the unknown side length, substitute the known values into any equation for the sine rule containing b.	$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ $\frac{b}{\sin(47°)} = \frac{34.6}{\sin(97°)}$ $b = \frac{34.6 \times \sin(47°)}{\sin(97°)}$ $b \approx 25.5$ cm

Example 9E.2 Ambiguous case of the sine rule

For △*ABC*, find the two possible sets of unknown angles and side lengths correct to one decimal place using this information:

$b = 18.7$ m, $c = 21.4$ m, $B = 56°$

THINK	**WRITE**
1 Draw a labelled diagram roughly to scale. Because this is an ambiguous case of the sine rule, draw both possible triangles and use the convention for differentiating both sets of unknown values from each other (C and C').	
2 To solve for *C*, substitute the known values into the equation for the sine rule containing *b* and *c*.	$\frac{c}{\sin(C)} = \frac{b}{\sin(B)}$ $\frac{21.4}{\sin(C)} = \frac{18.7}{\sin(56°)}$ $\sin(C) = \frac{21.4 \times \sin(56°)}{18.7}$ $= 0.9487\ldots$ $C = \sin^{-1}(0.9487\ldots)$ $\approx 71.6°$
3 The size of angle *A* can now be calculated, because the interior angles of a triangle add to 180°.	$A + B + C = 180°$ $A \approx 180° - 56° - 71.6°$ $\approx 52.4°$
4 To solve for *a*, substitute the known values into any equation for the sine rule containing *a*.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ $\frac{a}{\sin(52.4°)} = \frac{18.7}{\sin(56°)}$ $a = \frac{18.7 \times \sin(52.4°)}{\sin(56°)}$ $a \approx 17.9$ m
5 Calculate C' using the fact that C' is supplementary to *C*.	$C + C' = 180°$ $C' \approx 180° - 71.6°$ $C' \approx 108.4°$
6 A' can now be calculated by subtracting the known angles from 180°.	$A' + B' + C' = 180°$ $A' \approx 180° - 56° - 108.4°$ $\approx 15.6°$
7 To solve for a', substitute the known values into any equation for the sine rule containing a'.	$\frac{a'}{\sin(A')} = \frac{b}{\sin(B)}$ $\frac{a'}{\sin(15.6°)} = \frac{18.7}{\sin(56°)}$ $a' = \frac{18.7 \times \sin(15.6°)}{\sin(56°)}$ $a' \approx 6.1$ m

Example 9E.3 The area rule

Find the area of $\triangle ABC$ correct to one decimal place given the following information:

$b = 7.6$ cmm, $A = 104°$, $B = 28°$

THINK

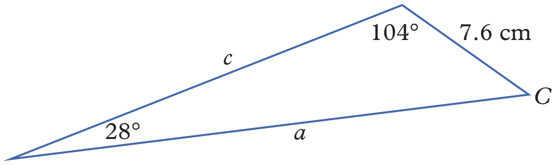
1 Draw a labelled diagram. Note that at least two side lengths must be known to calculate the area using the area rule.

2 To solve for a, substitute the known values into an equation for the sine rule containing a and b.

3 Calculate the included angle between a and b using the fact that the interior angles of a triangle add to 180°.

4 Substitute the relevant known values into the area rule.

WRITE



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

$$\frac{a}{\sin(104°)} = \frac{7.6}{\sin(28°)}$$

$$a = \frac{7.6 \times \sin(104°)}{\sin(28°)}$$

$$a \approx 15.7 \text{ cm}$$

$$C + A + B = 180°$$

$$C = 180° - 104° - 28°$$

$$= 48°$$

$$\text{area} = \frac{1}{2}ab \sin(C)$$

$$= \frac{1}{2} \times 15.7 \times 7.6 \times \sin(48°)$$

$$\approx 44.4 \text{ cm}^2$$

Helpful hints

- ✔ When using the sine rule, it helps to label the diagram with side lengths a, b and c, and their opposite angles A, B and C, respectively.
- ✔ To determine whether the given information for any triangle represents an ambiguous case of the sine rule, run through this short checklist:
  - → Do I only know the values for two side lengths and one angle?
  - → Is the known angle opposite the shorter known side?

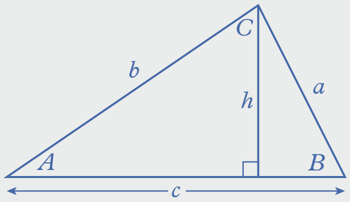
  If the answer is yes to both these questions, it is an ambiguous case.
- ✔ One way to remember area $= \frac{1}{2}bc \sin(A)$ is using the following diagram, where $\sin(A) = \frac{h}{b}$ multiply both sides by b, we have $h = b \times \sin(A)$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times c \times h$$

$$= \frac{1}{2} \times c \times b \times \sin(A)$$

$$= \frac{1}{2} bc \sin(A)$$



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Exercise 9E The sine and area rules

△ 1(a–d), 2–7, 8(a–c), 12, 13, 15, 16, 19 □ 3–7, 8(b, d, f), 9, 11, 12, 14, 17, 19, 20, 22 ⬟ 4, 5, 7, 8(b, f), 10, 14, 16–19, 21–23

UNDERSTANDING AND FLUENCY

1 Use a calculator to evaluate each of the following correct to four decimal places where necessary.

a $\sin(30°)$ **b** $\sin(150°)$ **c** $\sin(40°)$ **d** $\sin(140°)$

e $\sin(80°)$ **f** $\sin(100°)$ **g** $\sin(22°)$ **h** $\sin(158°)$

2 Use the sine rule to find the unknown value for each of the following triangles, ABC.

a If $A = 45°$, $B = 72°$ and $a = 10$ mm, calculate b correct to one decimal place.

b If $C = 65°$, $a = 12$ cm and $c = 13.4$ cm, calculate A correct to one decimal place.

c If $B = 37°$, $c = 8$ cm and $b = 15$ cm, calculate A correct to one decimal place.

9E.1 **3** For each of these triangles, ABC, use the given information to find the unknown angles and side lengths, correct to one decimal place. Draw a labelled sketch of the triangle.

a $a = 8.5$ cm, $B = 81°$ and $A = 37°$ **b** $c = 37.3$ mm, $b = 33.7$ mm, $C = 85°$

c $a = 22.5$ cm, $c = 28.3$ cm, $A = 49°$ **d** $b = 5.8$ cm, $B = 26°$, $A = 67°$

9E.2 **4** Use the given information to find the two possible sets of unknown angles and side lengths for each of these, correct to one decimal place. Draw a labelled sketch of both possible triangles.

a $c = 33.7$ mm, $b = 37.3$ mm, $C = 40°$ **b** $a = 14.9$ mm, $b = 26$ mm, $A = 32°$

c $b = 12.3$ cm, $c = 32.8$ cm, $B = 16°$ **d** $c = 22.1$ mm, $a = 23.8$ mm, $C = 67°$

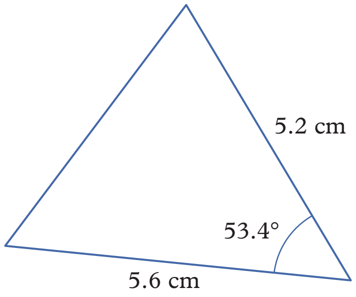
5 Find the area of each of these triangles, ABC, correct to one decimal place.

a $a = 9.5$ cm, $b = 7.6$ cm, $C = 28°$ **b** $b = 2.8$ mm, $c = 8.9$ mm, $A = 87°$

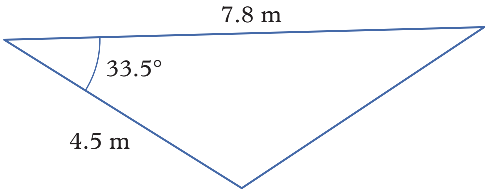
c $a = 4.7$ cm, $c = 9.2$ cm, $B = 57°$ **d** $c = 12.5$ cm, $b = 9.2$ cm, $A = 43°$

9E.3 **6** Find the area of each of these triangles, correct to one decimal place.

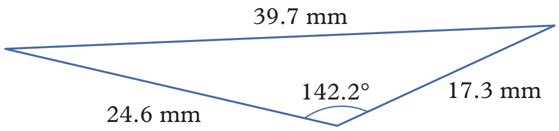
a



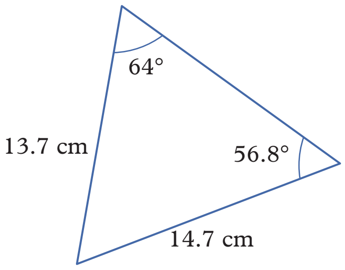
b



c



d



7 For each set of known values for the triangle ABC, explain why or why not:

i the sine rule can be used

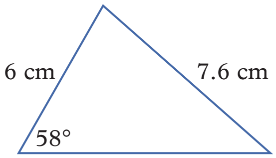
ii the set implies an ambiguous case of the sine rule.

a $c = 65$ cm, $b = 70$ cm, $C = 64°$ **b** $a = 60$ cm, $b = 70$ cm, $C = 64°$

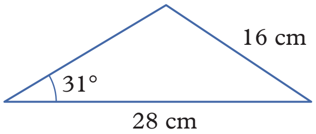
c $b = 60$ cm, $B = 70°$, $A = 64°$ **d** $c = 65$ cm, $b = 70$ cm, $B = 64°$

8 Find the perimeter of each triangle. (Hint: check which triangles result in an ambiguous case of the sine rule and decide which of the two possible triangles match the diagram.)

a



b



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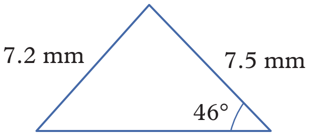
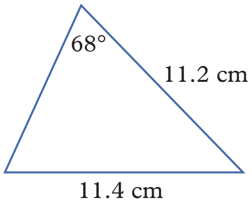
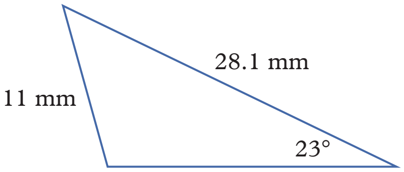


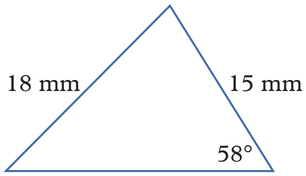
d



e



f



9 **a** Copy the given triangle into your workbook.

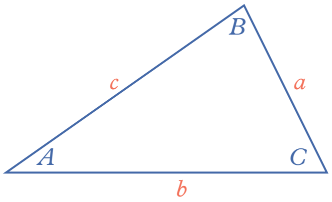
b Draw a perpendicular from vertex B to base b. Label the point where the perpendicular meets the base, D. Label the length of the perpendicular h.

c **i** Use the sine ratio for angle A to write a relationship between h and c. Make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Make h the subject of the formula.

iii Equate your two answers for parts **i** and **ii** and show how this can be written as $\frac{a}{\sin(A)} = \frac{a}{\sin(C)}$.

d Repeat the same reasoning outlined in parts **a**, **b**, and **c** to find a similar relationship involving angle B. Show that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.



10 Consider the right-angled triangle, $\triangle ABC$.

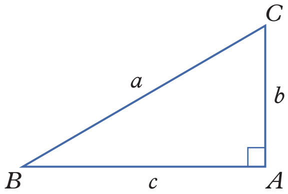
a Evaluate $\sin(A)$.

b Use the triangle to help you write expressions for $\sin(B)$ and $\sin(C)$.

c Substitute your expressions from parts **a** and **b** into the sine rule. Simplify.

d Explain your answer. Does the sine rule hold for right-angled triangles?

e What would have been your answer for part **c** if the triangle had been right-angled at $\angle ABC$ or $\angle BCA$?



11 Refer to the diagram in question **10**. The area of the triangle can be calculated using the formula:

area $= \frac{1}{2}ab\sin(C)$

a Use the diagram to help you obtain an expression for $\sin(C)$.

b Substitute the expression you found in part **a** into the area formula and simplify.

c Explain the resulting formula.

12 Chloe measures the angle of elevation to the top of a building as 27°. When she walks 15 m closer to the building, she measures the angle of elevation to be 47°.

a Find the size of:

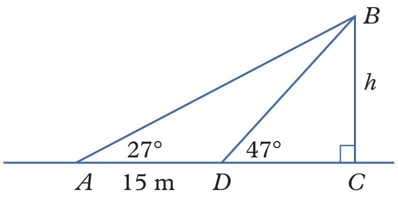
i $\angle ADB$ **ii** $\angle ABD$.

b Use the sine rule in $\triangle ABD$ to calculate the length of BD, correct to one decimal place.

c Use the sine ratio in $\triangle BCD$ to find the height of the building.

d Use the cosine ratio in $\triangle BCD$ to find how far Chloe was from the building when she took her second reading.

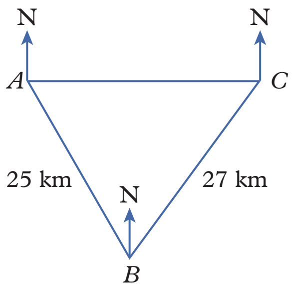
e How far was Chloe from the building initially?



UNDERSTANDING AND FLUENCY

PROBLEM SOLVING AND REASONING

13 A group of bushwalkers started from point A, travelled 25 km to point B at a bearing of S30°E, then continued for 27 km to point C, which lies due east of their starting point.



a What is the size of $\angle BAC$?

b Use the sine rule in $\triangle ABC$ to find the size of $\angle ACB$ to the nearest degree.

c What is the size of $\angle ABC$?

d Use the sine rule again to find the distance the bushwalkers would need to travel from their finishing point to return directly to their starting point. Give your answer to one decimal place.

e Give the bearing:

i of C from B

ii of B from C

iii the bushwalkers would need to take to return directly to their starting point.

f Calculate the area enclosed by the bushwalkers' hike.

14 The banks of a creek run directly east–west. From where Kim stands on the banks of the creek, the bearing to a tree on the opposite side of the creek is 032°T. If she moves 20 m due east on the bank of the creek, the bearing to the same tree is 310°T.

a Draw a labelled diagram displaying as much information as possible.

b Find the distance (in metres, to one decimal place) of the tree from:

i Kim's first viewing point

ii Kim's second viewing point.

c Calculate the width of the creek.

15 A give-way sign is the shape of an equilateral triangle, with side lengths of 40 cm. Calculate the area of the sign, to the nearest square centimetre.

16 A lighthouse stands 50 m tall on top of a cliff. From a ship at sea, the angle of elevation to the base of the lighthouse is 3°, while the angle of elevation to the top of the lighthouse is 4.2°.

a Draw a labelled diagram to display this information.

b Use the sine rule to find the straight-line distance from the ship to:

i the base of the lighthouse **ii** the top of the lighthouse.

c Calculate the distance in kilometres (to one decimal place) from the ship to the base of the cliff.

d Find the height of the cliff, to the nearest metre.

17 A pirate ship departs from Ashy Bay on a bearing of 295°T to Basalt Cove, 23 km away. From Basalt Cove, the pirate ship needs to travel 12.4 km to Coral Quay, but the inexperienced crew have forgotten which bearing they need to travel along. The crew knows that Coral Quay is due west from Ashy Bay but so is Deserted Depths, which is also 12.4 km from Basalt Cove.

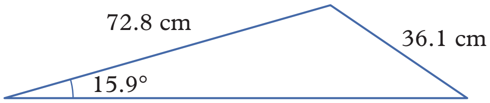
a Determine the bearings of Coral Quay and Deserted Depths from Basalt Cove, correct to the nearest degree.

b Determine the distances from Ashy Bay to Coral Quay and Deserted Depths, correct to one decimal place.

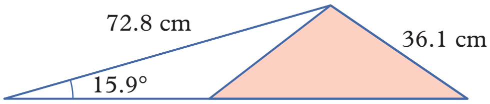
One of the crew members incorrectly recalls that Coral Quay is closer to Ashy Bay than Deserted Depths is. The pirate ship travels to the location closer to Ashy Bay but arrives at Deserted Depths instead. The pirate ship now must travel from Deserted Depths to Coral Quay.

c Calculate the minimum total distance the pirate ship will have travelled when it finally reaches Coral Quay, correct to one decimal place.

18 Consider the triangle below. Note that it is not drawn to scale.

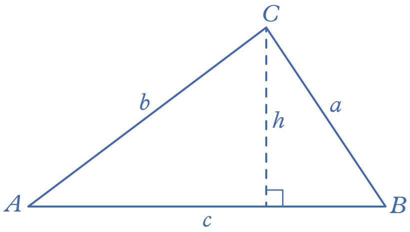


- **a** Explain why the triangle is ambiguous.
- **b** Correct to one decimal place, determine the two possible lengths of the third side.
- **c** Correct to one decimal place, calculate the area of the two possible triangles.
- **d** When the two triangles are overlapped at the 15.9° angle, the excess is an isosceles triangle.



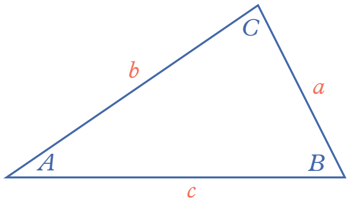
  - **i** Determine the area of the isosceles triangle using your answers to part **c** correct to one decimal place.
  - **ii** Write an expression for the area of the isosceles triangle using the angle between the equal side lengths.

19 Consider the $\triangle ABC$ on the right with height h and base c.



- **a** Write a formula for the area of $\triangle ABC$ in terms of h and c.
- **b** Write h in terms of A and b, and use your answer to write a formula for the area of $\triangle ABC$ in terms of A, b and c.
- **c** Write h in terms of B and a, and use your answer to write a formula for the area of $\triangle ABC$ in terms of B, a and c.
- **d** Compare your answers to parts **b** and **c** to the included angle area formula.

20 Consider the triangle ABC.



- **a** Write three different expressions for the area of the triangle using the included angle area formula.
- **b** Equate the expressions from part **a** and show that the resulting equation is equivalent to the sine rule.

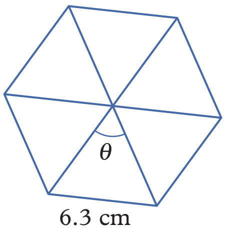
All interior angles of an equilateral triangle are 60°.

- **c** Use a calculator to determine the value of $(\sin(60°))^2$ as a fraction.
- **d** Hence, write the exact value of $\sin(60°)$.
- **e** Hence, write a rule for the area of an equilateral triangle with a side length of l.

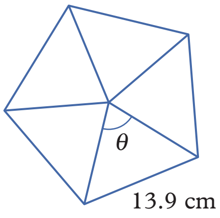
21 Regular n-sided polygons can be broken up into n congruent, isosceles triangles by joining the centre of the polygon to each vertex. For each of the following regular polygons shown, determine:

- **i** the size of the angle θ at the centre
- **ii** the distance from the centre to each vertex, correct to one decimal place
- **iii** the area of each isosceles triangle, correct to one decimal place
- **iv** the area of the regular polygon, correct to one decimal place.

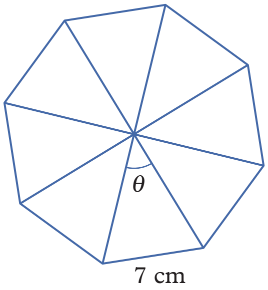
a



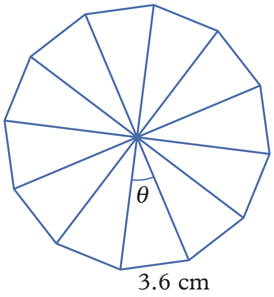
b



c



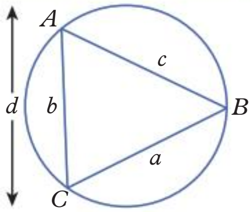
d



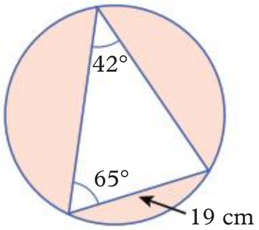
- **e** Write a rule for the area of an n-sided regular polygon with side length l.

22 The circumcircle of a triangle is the circle that passes through the vertices of the triangle. The ratio of a side to the sine of the opposite angle is equal to the diameter of the circumcircle.

That is, $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = d$.



a Determine the diameter of the circumcircle of the triangle in the circle below correct to two decimal places.



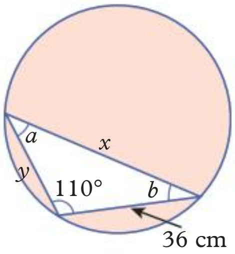
b Correct to two decimal places, determine the area of:

i the triangle **ii** the circumcircle **iii** the shaded area.

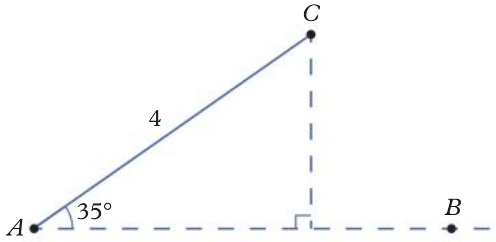
c The diameter of the circle on the right is 56 cm.

i Determine the value of each of the pronumerals correct to two decimal places.

ii Determine the shaded area correct to two decimal places.



23 So far, when dealing with the ambiguous case of the sine rule, we have assumed a triangle (or triangles) can be formed. However, it is possible for the side opposite the known angle to be too short to form a triangle. Consider the diagram below.

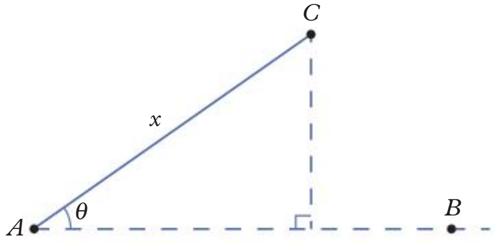


a Determine the perpendicular distance from C to the ray AB, correct to two decimal places.

b What is the minimum length of the line segment BC such that ABC is a triangle? Explain.

c What is the maximum length of the line segment BC such that there are two possible triangles ABC? Explain.

d For the diagram below, write an inequality for the length of line segment BC such that ABC is a triangle and also ambiguous.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet The sine rule

Interactive skillsheet The area rule

Investigation Proof of Heron's formula

Topic quiz 9E

ANS
p767

Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.
✔ Core
✔ Standard
✔ Advanced

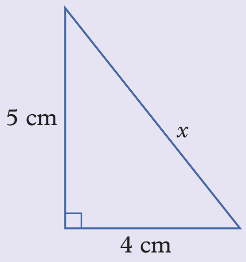
9A 1 Determine whether the following lengths can form a right-angled triangle.

a 9 cm, 40 cm, 41 cm
b 39 cm, 89 cm, 80 cm
c 137 cm, 104 cm, 89 cm
d 21 cm, 221 cm, 220 cm

9A 2 Determine the value of x in each of the following:

i as a simplified exact value
ii correct to two decimal places.

a



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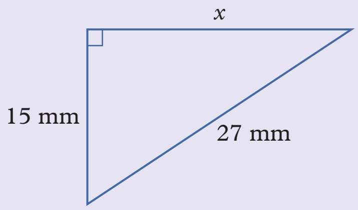
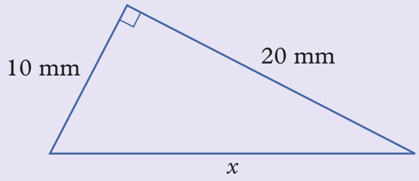
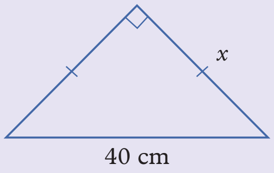


c

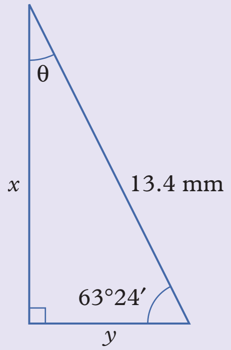


d



9B 3 Determine the value of each of the pronumerals correct to one decimal place.

a



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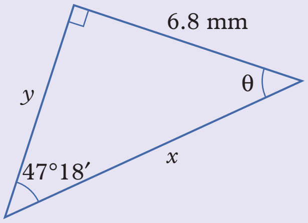

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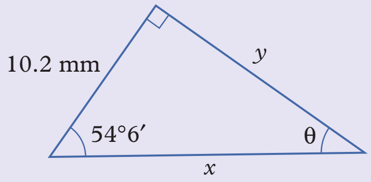
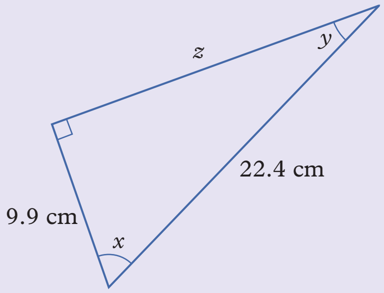


9B 4 Determine the value of each of the pronumerals correct to one decimal place.

a



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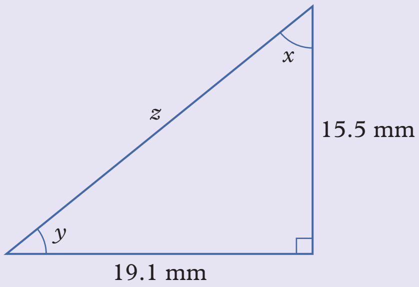
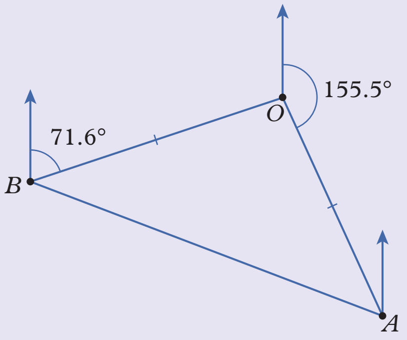


9C **5** Determine the true bearing of B from A in each of the following.

a The bearing of B from A is S85°E.

b The bearing of A from B is N35°W.

c



9C **6** Calculate each of the following correct to one decimal place.

a The angle of elevation of the top of a large tree from a point on the ground 6 m away is 78°. Determine the height of the tree.

b A building is 120 m tall. Another building is 105 m tall. Determine the angle of depression from the top of the taller building to the top of the shorter building if the tops of the buildings are 30 m apart.

c The angle of depression of an overboard passenger from the deck of a ship is 24°. A 12 m rope pulled taut just reaches from the deck of the ship to the overboard passenger in the water. Determine how far, in total, the overboard passenger needs to be pulled towards the boat through the water and then up the side of the ship to get them back onto the deck of the boat.

d A child holds the 8 m string of a kite 130 cm from the ground. If the angle of elevation of the kite is 65° how high above the ground is the kite flying?

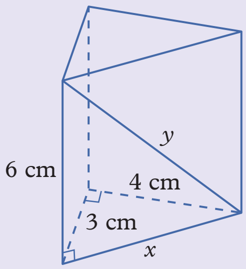
9C **7** Calculate each of the following distances correct to one decimal place and bearings correct to the nearest degree.

a A snorkeller swims to a point 230 m at a bearing of 160° from their boat. Determine how far south and east they are from the boat.

b A farmer walks 1.2 km east and then 800 m north along two sides of a paddock. Determine the true bearing of the location the farmer ended at from their starting point.

STN 9D ADV **8** Determine the value of each of the pronumerals correct to one decimal place.

a



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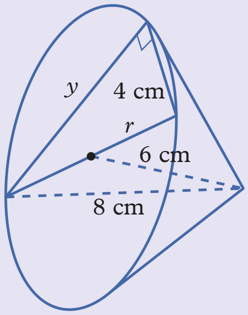
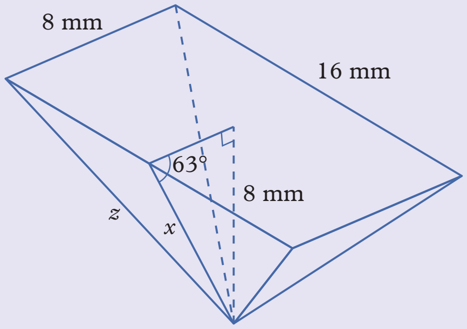


STN 9D ADV **9** Determine the value of each of the pronumerals correct to one decimal place.

a



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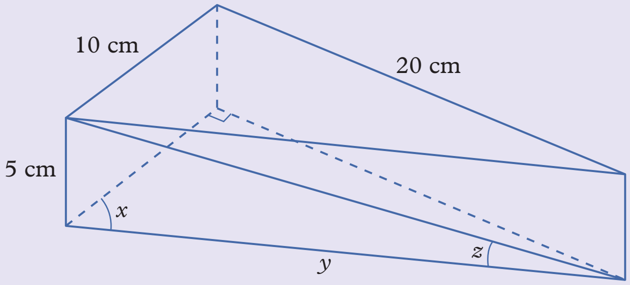
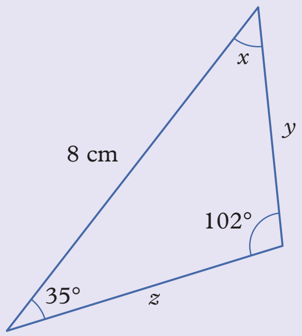


STN 9E ADV

10 Determine the value of each of the pronumerals correct to one decimal place.

a



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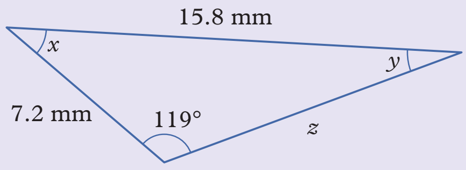

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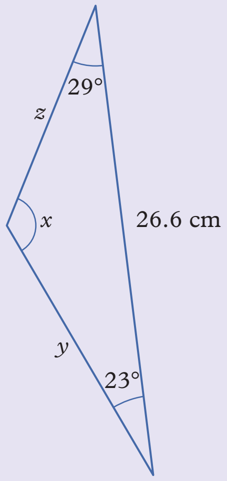

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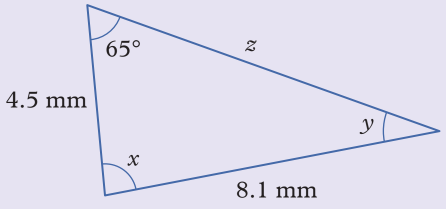
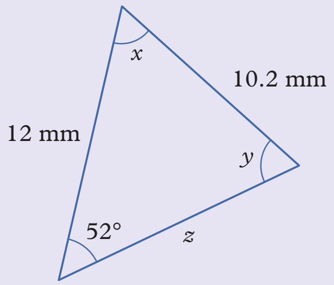


STN 9E ADV

11 Determine all possible values of each of the pronumerals correct to one decimal place.

a



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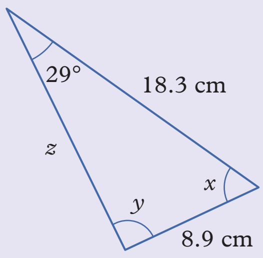
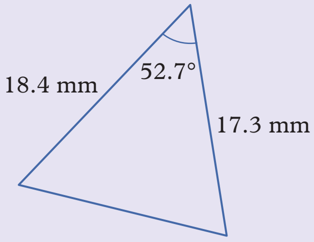


STN 9E ADV

12 Determine the area of the following triangles correct to two decimal places.

a



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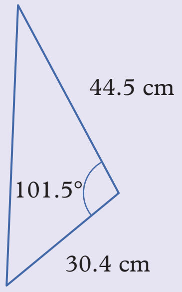

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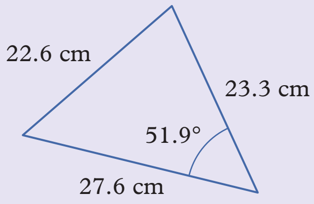

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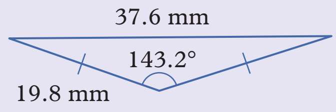


9F The cosine rule

Learning intentions

By the end of this topic you will be able to …

- ✔ determine unknown side lengths of a triangle using the cosine rule
- ✔ determine unknown angles of a triangle using the cosine rule.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving linear equations
Year 9	7J Using trigonometry to find angles

The cosine of angles from 0° to 180°

Key content video

- So far, cosine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\cos(\theta) = \frac{A}{H}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \leq \theta \leq 90°$.
- The interior angles of a general triangle can be acute and obtuse, so cosine must be defined for $0 \leq \theta \leq 180°$. The cosine values of supplementary angles are the negative of each other, $\cos(180° - \theta) = -\cos(\theta)$. For example, $\cos(120°) = -\cos(60°)$.
- In addition, $\cos(0°) = 1$ and $\cos(90°) = 0$.

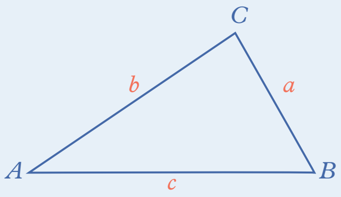
Using the cosine rule

- The side lengths and interior angles of any triangle, $\triangle ABC$, are related to each other by the **cosine rule**:

  $$a^2 = b^2 + c^2 - 2bc\cos(A).$$

  → *a*, *b and c* are the three side lengths of a triangle.

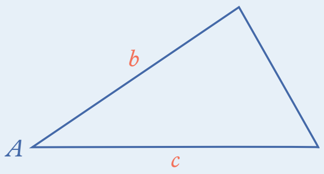
  → *A* is the included angle between *b* and *c*.

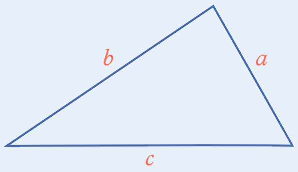


- The cosine rule is particularly useful for determining angles and side lengths in two situations when the sine rule is not suitable for finding the unknowns:

  Situation 1: The only known angle is the included angle between the only known side lengths.

  Situation 2: All the side lengths are known, and all the angles are unknown.

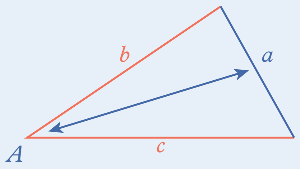




- To help remember the cosine rule, remember that the side length and the angle at either end of the equation form an opposite side-angle pair, and the side lengths in the middle are the adjacent side lengths to the included angle.

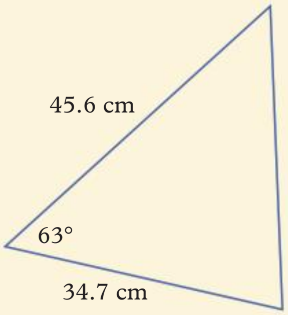
opposite side-angle pair

$$a^2 = \underbrace{b^2 + c^2 - 2bc}_{\text{adjacent side lengths where } A \text{ is included}}\cos(A)$$



Example 9F.1 Using the cosine rule to find a side length

Find the unknown side length in this triangle, correct to one decimal place.



THINK	**WRITE**
1 Identify the known values.	The included angle is 63° between the adjacent side lengths 45.6 cm and 34.7 cm.
2 Substitute the known values into the cosine rule.	$a^2 = b^2 + c^2 - 2bc\cos(A)$ $= (45.6)^2 + (34.7)^2 - 2 \times (45.6) \times (34.7) \times \cos(63°)$ $= 1846.7335\ldots$
3 Find the square root and round the answer to one decimal place. Write the units.	$a = \sqrt{1846.7335\ldots}$ ≈ 43.0 cm

Example 9F.2 Using the cosine rule to find an unknown angle

The following lengths are given for the sides of $\triangle ABC$:

$a = 2.8$ m, $b = 5.3$ m, $c = 7.1$ m

Determine the size of the included angle between a and b correct to one decimal place.

THINK	**WRITE**
1 Substitute the side lengths into the cosine rule.	$c^2 = a^2 + b^2 - 2ab\cos(C)$ $(7.1)^2 = (2.8)^2 + (5.3)^2 - 2 \times (2.8) \times (5.3) \times \cos(C)$
2 Solve for $\cos(C)$.	$50.41 = 35.93 - 29.68\cos(C)$ $14.48 = -29.68\cos(C)$ $\cos(C) = -0.4878\ldots$
3 Use the inverse cosine to calculate C.	$C = \cos^{-1}(-0.4879\ldots)$ $\approx 119.2°$

Helpful hints

- ✔ When you are calculating an angle using the cosine rule, always check that the value you have calculated for the cosine value is between −1 and 1.
- ✔ If you are calculating all the interior angles of a triangle using the cosine rule, check your calculations by making sure the three angles add to 180°.

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Exercise 9F The cosine rule

△ 1(a–d), 2(a, c, e, g, i), 3, 4, 5(a–c), 6, 8, 9, 10, 13, 15

□ 2(g–i), 3, 5–8, 11–13, 17, 18, 20

⬠ 3(c, d), 5, 8, 11, 12, 14, 16, 17, 19, 21–23

UNDERSTANDING AND FLUENCY

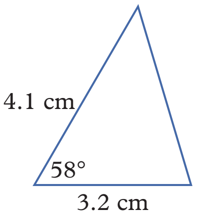
1 Use a calculator to evaluate each of the following correct to four decimal places where necessary.

a cos (60°) **b** cos (120°) **c** cos (40°) **d** cos (140°)

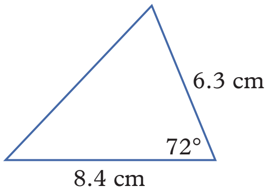
e cos (80°) **f** cos (100°) **g** cos (22°) **h** cos (158°)

9F.1 2 Find the unknown side length in each of these triangles, correct to one decimal place.

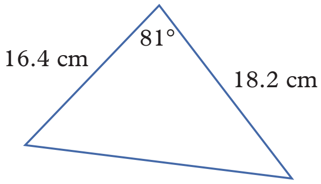
a



b



c



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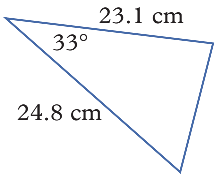

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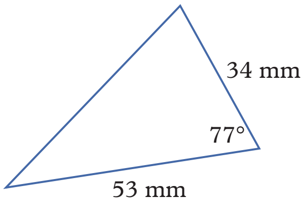

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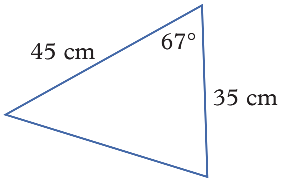
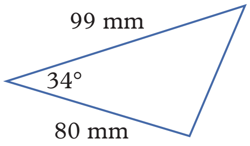


g



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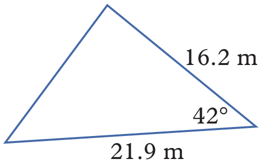
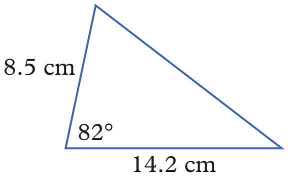


i



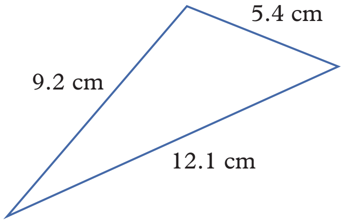
9F.2 3 Each of the following sets of side lengths are for a different △*ABC*. Find the sizes of all the angles of each triangle, correct to one decimal place.

a $a = 4.5$ cm, $b = 6$ cm and $c = 5$ cm **b** $a = 13$ mm, $b = 12$ mm and $c = 7$ mm

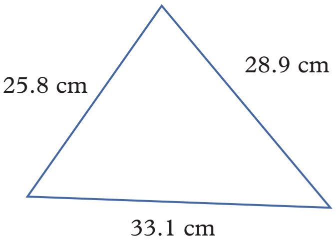
c $a = 1.5$ m, $b = 2.3$ m and $c = 1.8$ m **d** $a = 16$ cm, $b = 19$ cm and $c = 12$ cm

4 Find the sizes of the angles of each of these triangles, correct to one decimal place.

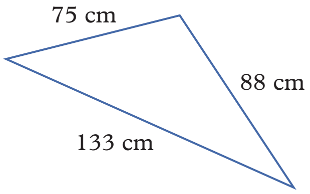
a Find the size of the largest interior angle.



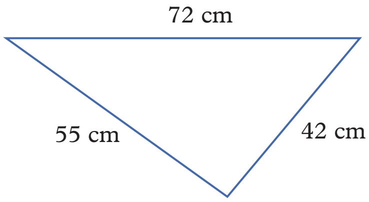
b Find the size of the smallest interior angle.



c Find the size of the median interior angle.



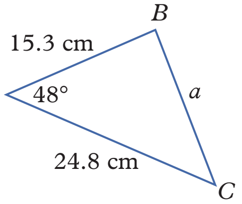
d Find the size of the largest exterior angle.



UNDERSTANDING AND FLUENCY

5 Find the unknown side lengths and angles of each of these triangles.

a



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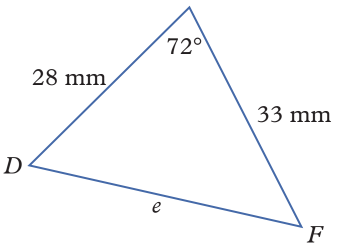

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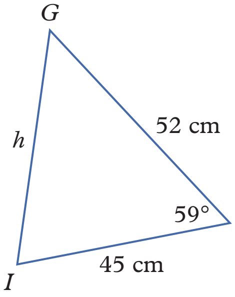

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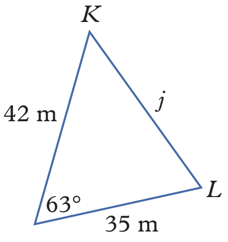
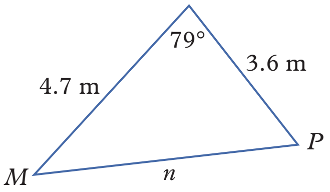


e



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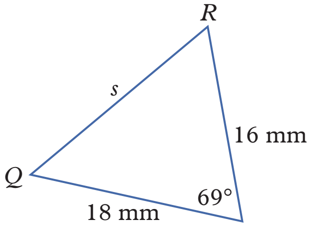
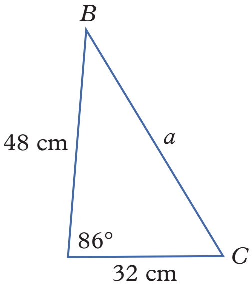


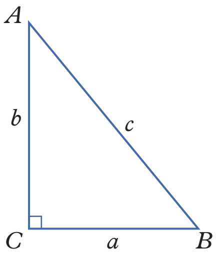
PROBLEM SOLVING AND REASONING

6 Sometimes there is a choice as to whether the cosine rule or the sine rule can be used to solve a problem. Consider the triangle on the right.



- **a** Explain why it is not possible to use the sine rule to calculate the length of *a*.
- **b** Use the cosine rule to find the length of *a*.
- **c** **i** Explain how you could now use the sine rule to find the sizes of angles *B* and *C*.
  - **ii** Calculate the sizes of those two angles.
- **d** Angles *B* and *C* from part **c** could also have been calculated using the cosine rule. Find their sizes using this rule.
- **e** Comment on your answers to parts **c ii** and **d**.

7 Consider the right-angled triangle *ABC* with the right angle at *C*.

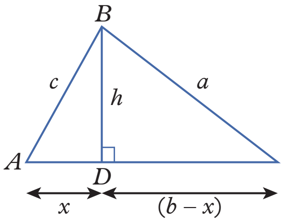


- **a** Copy and complete the cosine rule for triangle *ABC*: $c^2 =$ ________________.
- **b** Evaluate $\cos(90°)$.
- **c** Substitute the value from part **b** into your equation in part **a** and simplify. Describe the result.

The cosine rule has the adjustment term $-2ab\cos(C)$ compared to Pythagoras' theorem.

- **d** Suppose *C* is an acute angle. Use the adjustment term to explain why the side opposite angle *C* is shorter than the hypotenuse in the case where *C* is a right angle.
- **e** Suppose *C* is an obtuse angle. Use the adjustment term to explain why the side opposite angle *C* is longer than the hypotenuse in the case where *C* is a right angle.

8 Consider $\triangle ABC$, with the perpendicular h from B dividing the base b into two sections of length x and $(b - x)$.



a Use $\triangle CDB$ and Pythagoras' theorem to complete the relationship $a^2 = h^2 +$ __________.

b Use $\triangle ADB$ and Pythagoras' theorem to complete the relationship $h^2 =$ ____________.

c Substitute the value of h^2 you wrote for part **b** into your equation in part **a**.
Simplify the resulting equation and confirm that it can be written as $a^2 = b^2 + c^2 - 2bx$.

d Use $\triangle ADB$ and the cosine ratio to write a relationship with x as the subject of the equation.

e Substitute your value for x into the equation you wrote for part **c**. Your resulting equation should be:
$a^2 = b^2 + c^2 - 2bc\cos(A)$.

f Use a similar method to that used for writing an equation for a^2 in part **e** to write equations for b^2 and c^2.

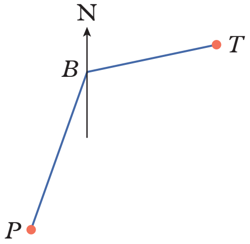
g The cosine rule can be rearranged to find the size of an angle of a triangle. Write three versions of the cosine rule with the following cosine values as the subject.

i $\cos(A)$ **ii** $\cos(B)$ **iii** $\cos(C)$

h Using the cosine rule, what information would you need to calculate:

i a side length **ii** the size of an angle?

9 From a buoy at sea, Todd rows 2.5 km at a bearing of N80°E, while Pete rows 3.5 km at a bearing of S20°W.



a Copy the diagram and label all known sides and angles.

b What is the size of $\angle TBP$?

c Join TP. This represents the distance between the two rowers. Use the cosine rule with $\triangle BTP$ to calculate this distance.

d Use the cosine rule with $\triangle BTP$ to calculate the size of $\angle BPT$.

e Use your answer for part **d** to find the bearing of Todd from Pete.

f Calculate the size of $\angle BTP$.

g Use your answer for part **f** to find the bearing of Pete from Todd.

10 A triangular garden bed has side lengths 5.3 m, 4.8 m and 4.5 m.

a Draw a labelled sketch of the garden bed.

b Identify the vertex with the largest angle. (Hint: The largest angle lies opposite the longest side.)

c Use the cosine rule to find the size of the angle you identified in part **b** to the nearest degree.

d Explain how, using the angle you identified, you can now calculate the area of the bed. What is its area?

11 A deep-sea diver is attached to two safety ropes from two boats on the surface of the ocean. The boats are 50 m apart. One rope is 110 m long and the other is 100 m long.

a Draw a labelled diagram to describe the situation when the diver is at the maximum depth that is possible while attached to the two ropes.

b Use the cosine rule to find the angles the two ropes make with the surface of the water.

c On your diagram from part **a** construct a right-angled triangle that will enable you to find the maximum depth to which the diver can descend. Explain a technique you could use to find this depth.

d Calculate the maximum depth to which the diver can descend while attached to the two safety ropes.

12 Emily is practising shooting hockey goals. The hockey goal is 3 m wide. She stands 5 m directly in front of one of the posts.

a Draw a diagram to display the situation.

b Use Pythagoras' theorem to calculate the distance from Emily to the other goalpost.

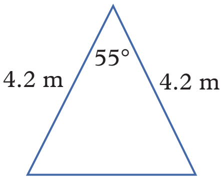
c Use the cosine rule to find the angle within which Emily must shoot to score a goal.

d Emily moves her position, so she is 5 m from each of the goal posts. Explain whether she has a greater chance of scoring a goal from this position.

13 The frame formed by the end support poles of a swing forms a triangular shape with the ground. The poles are each 4.2 m long with an angle of 55° between them. Write your answers correct to one decimal place.

a How far apart are the ends of the poles on the ground?

b Find the vertical height of the top of the frame above the ground.



14 The frame for a bridge consists of a series of triangular shapes connected together as shown in the photo on the right.

The sloping steel parts are 3.9 m long and the horizontal steel parts are 2.8 m long.

a Find the angle between any two sloping pieces of steel.

b Draw a sketch of several sections of the bridge, showing the angles and lengths of the connecting pieces.

c Find the height of the frame of the bridge in metres, correct to one decimal point.

15 The diagonal length of an octagon, between vertices that are four edges apart, measures 4 cm.

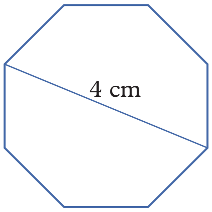
a Draw a sketch of the octagon. Add the three remaining diagonals that join vertices that are four edges apart.

b Consider one of the triangles within your sketch.

i What is the size of the angle at the centre of the octagon?

ii Find the sizes of the other two angles of the triangle.

c Calculate the length of the outer edge (the perimeter) of the whole octagon.



16 A light aircraft is set to fly on a course N78°E for a distance of 350 km. A strong wind moving from east to west blows the plane off course by 6°.

a Draw a labelled diagram to show this situation.

b If the pilot flies the planned distance on this incorrect path, how far off course will he be at the end of his journey?

17 A cube has a side length of 25 cm.

a Draw a labelled sketch of the cube.

b Use the cosine rule to find the length of the diagonal of the base. Leave your answer in a simplified exact form.

c Use the cosine rule again to find the length of the diagonal of the cube. Leave your answer in a simplified exact form.

18 The angles of a triangle are in the ratio 3 : 5 : 7.

a Find the size of all the angles.

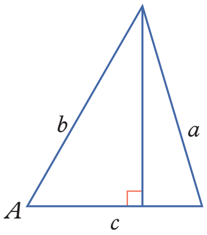
The two sides adjacent to the largest angle measure 2.58 cm and 4.30 cm.

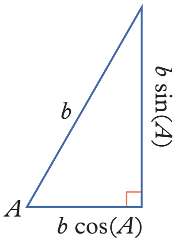
b Correct to two decimal places, calculate the length of the third side.

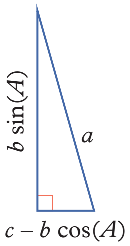
c Draw a labelled diagram to display the measurements of all the angles and sides.

PROBLEM SOLVING AND REASONING

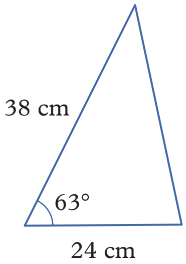
19 Consider the diagrams on the right. Using Pythagoras' theorem on the rightmost triangle, we can write the following equation: $(b\sin(A))^2 + (c - b\cos(A))^2 = a^2$.



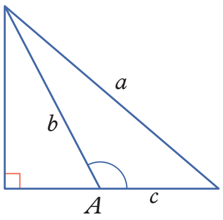


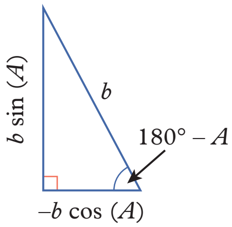


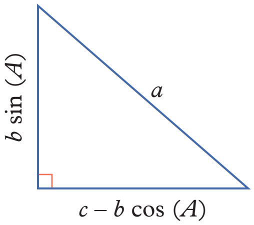
a Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length correct to one decimal place.



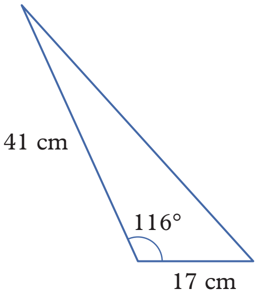
The equation also holds true if A is an obtuse angle as $\sin(180° - A) = \sin(A)$, and $\cos(A)$ will be negative and so adds the extra length on.







b Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length correct to one decimal place.



c Use the fact that $(b\sin(A))^2 + (b\cos(A))^2 = b^2$ to show that $(b\sin(A))^2 + (c - b\cos(A))^2 = a^2$ is equivalent to $b^2 + c^2 - 2bc\cos(A) = a^2$.

CHALLENGE

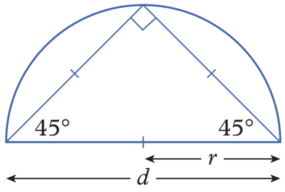
20 A right-angled isosceles triangle is inscribed in a semi-circle of radius r.

a Find the lengths of the triangle's two equal sides, in terms of r, using:

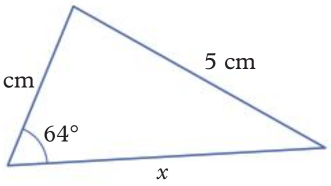
  i Pythagoras' theorem

  ii trigonometry.

b Comment on your two answers for part **b**.



21 Azar has written the equation $5^2 = 3^2 + x^2 - 2(3)x\cos(64°)$ for the triangle on the right.

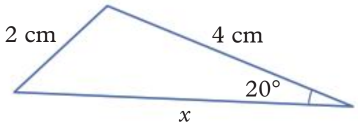


a Write the equation in the form $x^2 + bx + c = 0$, where b and c are real numbers.

b Solve the equation in part **a**. Give your answers correct to one decimal place.

c Are both solutions to the equation in part **b** possible side lengths of the triangle? Explain, stating the length of the unknown side length.

Azar has written the equation $2^2 = 4^2 + x^2 - 2(4)x\cos(20°)$ for the triangle below, which is not drawn to scale.

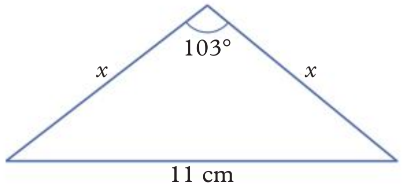


d Solve this equation. Give your answers correct to one decimal place.

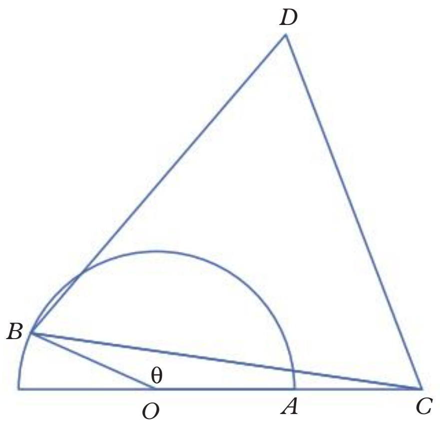
e Are both solutions to the equation in part **d** possible side lengths of the triangle? Explain by referring to ambiguous triangles.

f Compare using the sine and cosine rules to find the third side of a possibly ambiguous triangle.

g Consider the isosceles triangle shown below. Write and solve an equation for the value of the side lengths, x, correct to one decimal place.



22 Point B moves along the arc of semi-circle O. The angle formed between OB and OA is θ. The radius of the semi-circle is 1. C is on the same line as OA and $OA = AC$, connecting BC and creating an equilateral triangle BCD. As B moves along the arc of the semi-circle, the area of $\triangle BCD$ also changes. Find the area of $\triangle BCD$ in terms of $\cos(\theta)$.



23 The triangle inequality states that a triangle with side lengths $a \geq b \geq c > 0$ exists precisely when $a < b + c$.

a Explain how the inequality $a \geq b + c$ relates to the triangle inequality.

b Use part **a**, the cosine rule and the fact that $-1 \leq \cos(\theta) \leq 1$ for all $\theta \in \mathbb{R}$ to prove the triangle inequality by contradiction.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
The cosine rule

Topic quiz
9F

9G The unit circle

Learning intentions

By the end of this topic you will be able to ...

- ✔ determine the quadrant in which an angle lies
- ✔ determine the sign of a trigonometric expression given an angle
- ✔ find the reference angle in the first quadrant of any angle using symmetry properties.

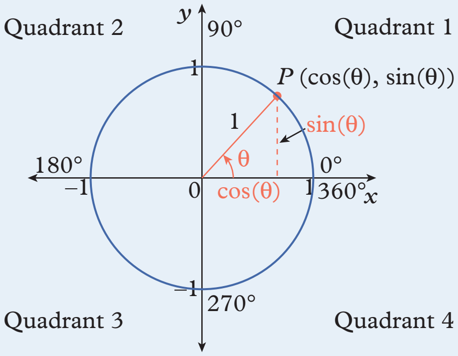
Inter-year links

Year 9 7H Trigonometric ratios

The unit circle

Key content video

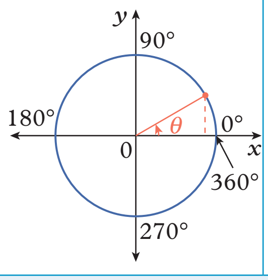
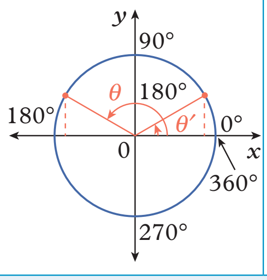
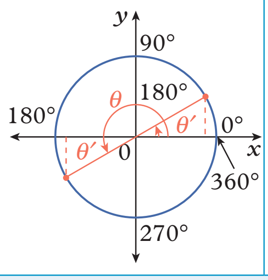
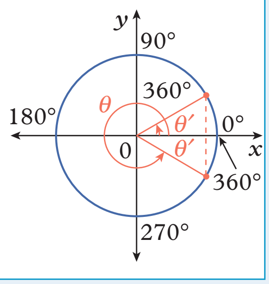
- The **unit circle** is a circle on the Cartesian plane, having its centre at the origin and a radius of 1 unit. The **quadrants** on the Cartesian plane are labelled in an anti-clockwise direction from the positive *x*-axis.



- If θ is the angle measured anti-clockwise from the positive *x*-axis to the radius that meets the unit circle at point *P*, then:
  - → The *x*-coordinate of the point on the unit circle is the cosine of θ.
  - → The *y*-coordinate of the point on the unit circle is the sine of θ.
  - → The gradient of the line from the origin to the point on the unit circle is the tangent of θ.

The reference angle

- The sine and cosine values for any angle θ can be related by symmetry to an acute reference angle θ' in quadrant 1.
- The **reference angle** is the acute angle between the *x*-axis and the line connecting the origin to the point on the unit circle.

	Quadrant 1	**Quadrant 2**	**Quadrant 3**	**Quadrant 4**
Angle (θ)	$0° < \theta < 90°$	$90° < \theta < 180°$	$180° < \theta < 270°$	$270° < \theta < 360°$
Reference angle (θ')	$\theta' = \theta$ 	$\theta' = 180° - \theta$ 	$\theta' = \theta - 180°$ 	$\theta' = 360° - \theta$ 

Example 9G.1 Determining the quadrant in which an angle lies

Identify the quadrant in which each of these angles lies.

a 34° **b** 122° **c** 230° **d** 325°

THINK

a Sketch the unit circle and mark the angle 34°. Because the size of the angle is between 0° and 90°, the angle is in quadrant 1.

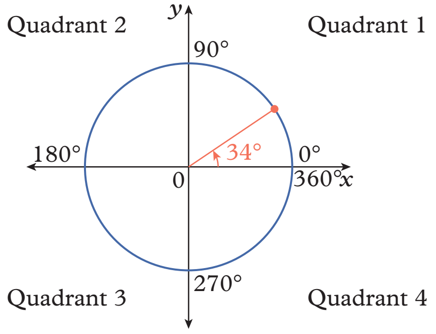
b Sketch the unit circle and mark the angle 122°. Because the size of the angle is between 90° and 180°, the angle is in quadrant 2.

c Sketch the unit circle and mark the angle 230°. Because the size of the angle is between 180° and 270°, the angle is in quadrant 3.

d Sketch the unit circle and mark the angle 325°. Because the size of the angle is between 270° and 360°, the angle is in quadrant 4.

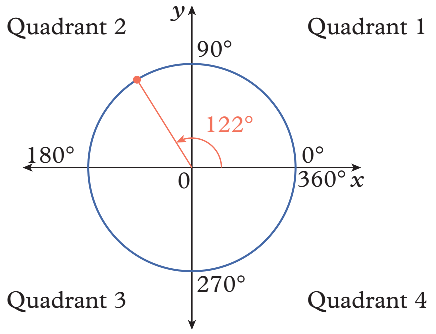
WRITE

a



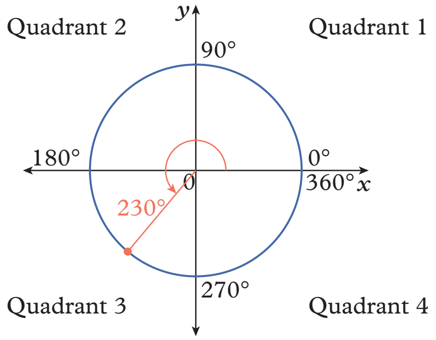
34° is in quadrant 1.

b



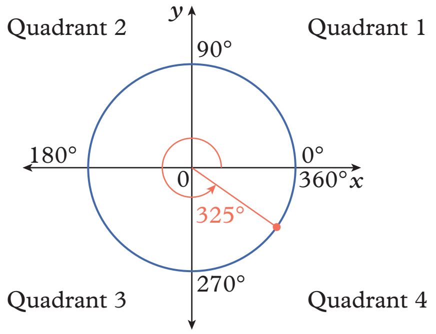
122° is in quadrant 2.

c



230° is in quadrant 3.

d



325° is in quadrant 4.

ADVANCED

Example 9G.2 Determining the sign of a trigonometric expression

Identify the sign (positive or negative) of each of these trigonometric expressions.

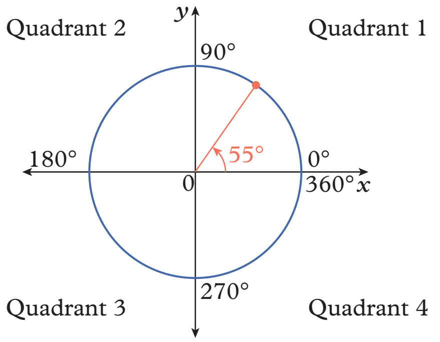
a cos(55°) **b** tan(103°) **c** sin(230°) **d** tan(349°)

THINK

a Sketch the unit circle and mark the angle 55°. The angle is in quadrant 1, and the cosine value is positive in quadrant 1 because the *x*-coordinate of a point in quadrant 1 is positive.

WRITE

a



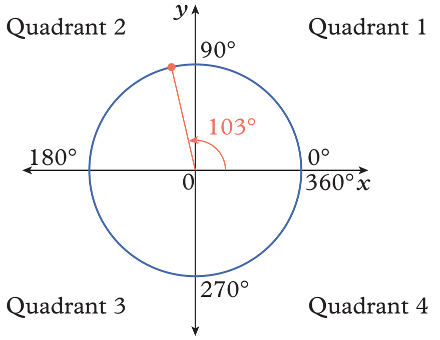
cos(55°) is positive.

THINK

b Sketch the unit circle and mark the angle 103°. The angle is in quadrant 2, and the tangent value is negative in quadrant 2 because the gradient of the line from the origin to a point in quadrant 2 is negative.

WRITE

b



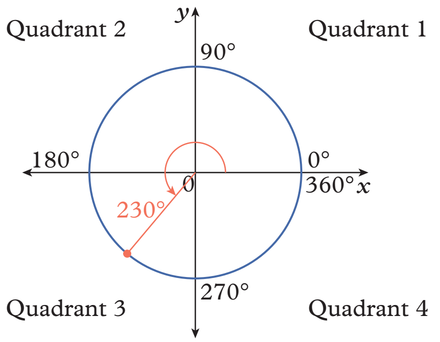
tan(103°) is negative.

THINK

c Sketch the unit circle and mark the angle 230°. The angle is in quadrant 3, and the sine value is negative in quadrant 3 because the *y*-coordinate of a point in quadrant 3 is negative.

WRITE

c



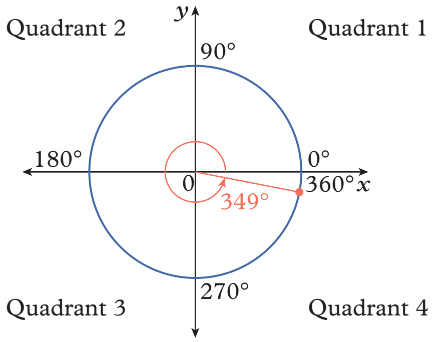
sin(230°) is negative.

THINK

d Sketch the unit circle and mark the angle 349°. The angle is in quadrant 4, and the tangent value is negative in quadrant 4 because the gradient of a line from the origin to a point in quadrant 4 is negative.

WRITE

d



tan(349°) is negative.

Example 9G.3 Determining the reference angle

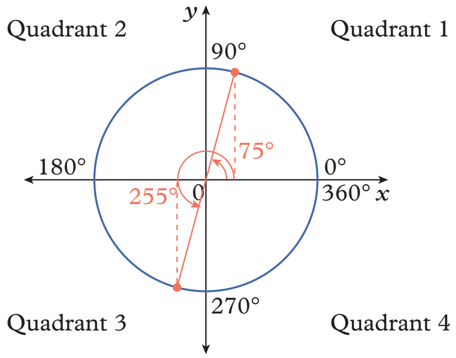
Write each trigonometric expression in terms of the reference angle using the correct sign.

a sin(255°) **b** cos(317°) **c** tan(145°)

THINK

WRITE

a **1** Sketch the unit circle and mark the angle 255°.



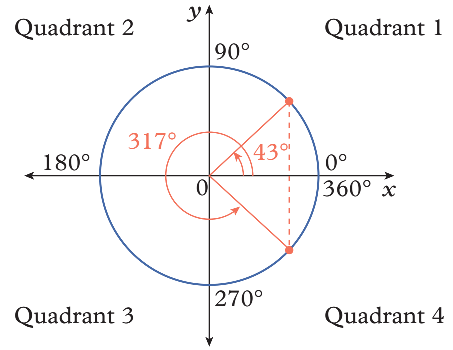
2 The angle is in quadrant 3, so the reference angle is $\theta - 180°$.

$255° - 180° = 75°$

3 sin(255°) is negative because the y-coordinate of a point in quadrant 3 is negative.

$\sin(255°) = -\sin(75°)$

b **1** Sketch the unit circle and mark the angle 317°.



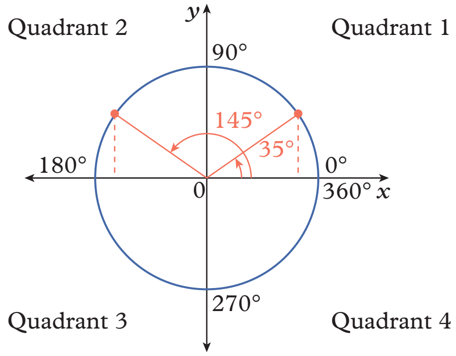
2 The angle is in quadrant 4, so the reference angle is $360° - \theta$.

$360° - 317° = 43°$

3 cos(317°) is positive because the x-coordinate of a point in quadrant 4 is positive.

$\cos(317°) = \cos(43°)$

c **1** Sketch the unit circle and mark the angle 145°.



2 The angle is in quadrant 2, so the reference angle is $180° - \theta$.

$180° - 145° = 35°$

3 tan(145°) is negative because the gradient of a line from the origin to any point in quadrant 2 is negative.

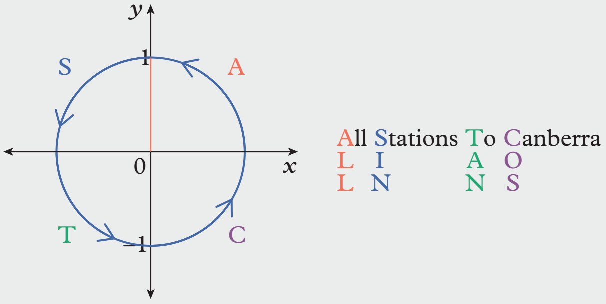
$\tan(145°) = -\tan(35°)$

Helpful hints

✔ The sine, cosine and tangent values have different signs (positive or negative) depending on the quadrant in which they lie.

- Sine is positive where the y-coordinate of the point on the unit circle is positive (quadrants 1 and 2), and negative where the y-coordinate is negative (quadrants 3 and 4).
- Cosine is positive where the x-coordinate of the point on the unit circle is positive (quadrants 1 and 4), and negative where the x-coordinate is negative (quadrants 2 and 3).
- Tangent is positive where the gradient from the origin to the point on the unit circle is positive (quadrants 1 and 3), and negative where the gradient is negative (quadrants 2 and 4).

✔ An easy way to remember the sign of the trigonometric functions in the four quadrants is with a mnemonic. Something like **A**ll **S**tations **T**o **C**anberra is good:



ANS p770

Exercise 9G The unit circle

▲ 1–3, 4(a–j), 5, 6, 8, 10 ■ 3, 4(f–o), 5, 6, 8, 9, 11, 13 ⬟ 3, 5–9, 12–14

UNDERSTANDING AND FLUENCY

1 For each of the following, shade the appropriate regions of a unit circle diagram.

a Shade the regions where the sine of angles in that region is positive.

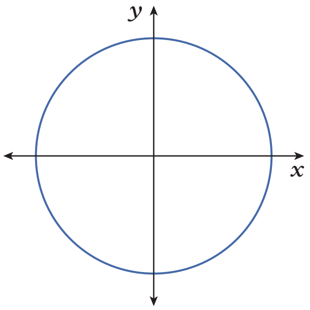
b Shade the regions where the cosine of angles in that region is positive.

c Shade the regions where the tangent of angles in that region is positive.

d Shade the regions where the sine of angles in that region is negative.

e Shade the regions where the cosine of angles in that region is negative.

f Shade the regions where the tangent of angles in that region is negative.



9G.1 2 Identify the quadrant in which each of these angles lies.

a 345° **b** 103° **c** 204°

d 265° **e** 74° **f** 139°

9G.2 3 Identify the sign (positive or negative) of each of these trigonometric expressions.

a sin(100°) **b** cos(200°) **c** tan(140°) **d** cos(295°) **e** tan(260°)

f sin(340°) **g** cos(94°) **h** sin(130°) **i** cos(190°) **j** tan(350°)

4 Write the reference angle for each of the following.

a 140° **b** 92° **c** 200° **d** 190° **e** 302°

f 315° **g** 260° **h** 100° **i** 280° **j** 150°

k 345° **l** 170° **m** 220° **n** 255° **o** 299°

ADVANCED

9G.3 **5** Express each of the following trigonometric expressions in terms of an appropriate reference angle. Hint: remember to consider whether the sign of each expression is positive or negative.

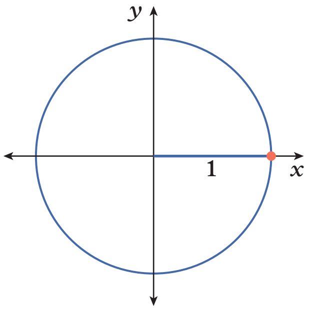
a $\sin(241^\circ)$ **b** $\cos(304^\circ)$ **c** $\tan(95^\circ)$ **d** $\sin(164^\circ)$ **e** $\cos(110^\circ)$

f $\cos(200^\circ)$ **g** $\tan(200^\circ)$ **h** $\tan(358^\circ)$ **i** $\sin(285^\circ)$ **j** $\sin(175^\circ)$

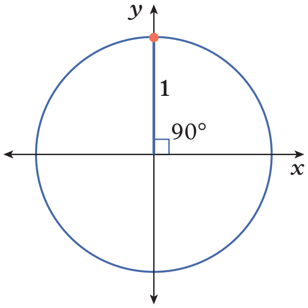
6 The unit circle can be used to evaluate the sine and cosine of angles that are multiples of 90° (including 0°) by considering the coordinates of points on the unit circle. For each of the following identify:

- **i** the cosine of the marked angle (the x-coordinate of the point shown)
- **ii** the sine of the marked angle (the y-coordinate of the point shown).

a



b



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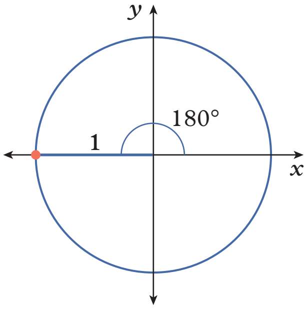

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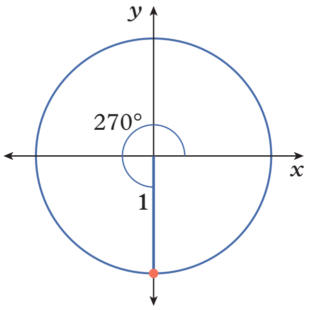
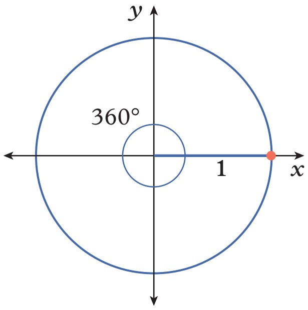


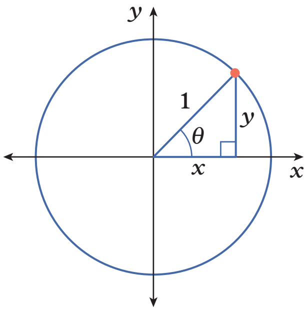
e



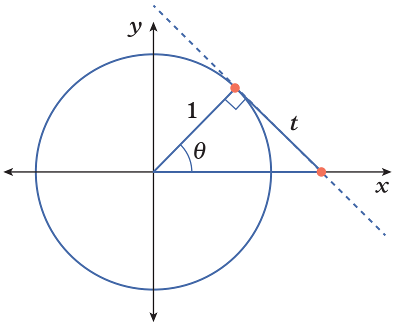
7 The tangent of an angle using the unit circle can be defined in different ways.

a State the tangent of θ in the right-angled triangles shown.

i



ii



b Explain how the definition from part **a i** can be described as 'the tangent of the angle is equal to the gradient of the line from the origin to the point on the unit circle'.

c The lengths x and y also refer to the coordinate of the point where the radius touches the circle. Write the expression of the tangent of θ from part **a i** in terms of sine and cosine.

d Use your expression from part **c** to write the following in terms of tangent.

i $\dfrac{\sin(20^\circ)}{\cos(20^\circ)}$ **ii** $\dfrac{\sin(152^\circ)}{\cos(152^\circ)}$ **iii** $\dfrac{\cos(218^\circ)}{\sin(218^\circ)}$ **iv** $\dfrac{\sin(347^\circ)}{\cos(347^\circ)}$

e The unit circle can also be used to evaluate the tangent of angles that are multiples of 90° (including 0°) by considering the gradients of a line from the origin to the point on the unit circle. Since the gradient is either horizontal or vertical at these angles, the tangent value is either zero or undefined, respectively. For each of the following state the tangent of the marked angle using the gradient of the radius shown in question **6**.

i $\tan(0^\circ)$ **ii** $\tan(90^\circ)$ **iii** $\tan(180^\circ)$ **iv** $\tan(270^\circ)$ **v** $\tan(360^\circ)$

PROBLEM SOLVING AND REASONING

8 Angles greater than or equal to 360° are coterminal with angles between 0° and 360°, as they terminate or end at the same point on the unit circle. That is, every angle greater than or equal to 360° corresponds to an angle between 0° and 360° as they point in the same direction. To determine a coterminal angle, add or subtract 360° from the given angle. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign, or its value if it is a multiple of 90°.

a $\sin(400°)$ **b** $\cos(530°)$ **c** $\tan(800°)$ **d** $\sin(630°)$

e $\cos(777°)$ **f** $\cos(1024°)$ **g** $\tan(2000°)$ **h** $\cos(900°)$

9 Using the unit circle, we can also consider negative angles as angles measured clockwise from the positive x-axis. Negative angles are also coterminal with angles between 0° and 360°. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign or its value if it is a multiple of 90°.

a $\sin(-45°)$

b $\cos(-225°)$

c $\tan(-135°)$

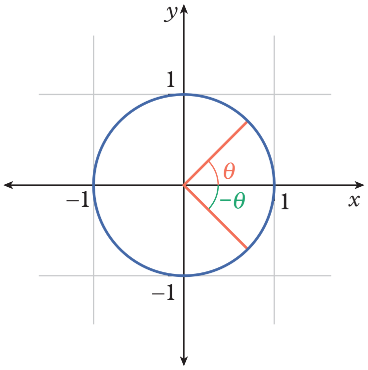
d $\sin(-315°)$

e $\cos(-508°)$

f $\tan(-436°)$

g $\cos(-630°)$

h $\sin(-1000°)$



10 Complete the following with different angles between 0° and 360°.

a $\sin(158°) = \sin(\_\_\_°) = -\sin(\_\_\_°) = -\sin(\_\_\_°)$

b $\cos(213°) = \cos(\_\_\_°) = -\cos(\_\_\_°) = -\cos(\_\_\_°)$

c $\tan(285°) = \tan(\_\_\_°) = -\tan(\_\_\_°) = -\tan(\_\_\_°)$

11 Consider the following three equalities.

$\sin(234°) = -\sin(126°)$ $\cos(234°) = \cos(126°)$ $\tan(234°) = -\tan(126°)$

a Use a calculator to confirm that the pairs of trigonometric expressions are equal.

b Evaluate $234° + 126°$.

c Explain how these equalities relate to the fourth quadrant.

12 In this question, you will verify that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ for all values of θ.

a Use the right-angled triangle on the right to verify that, when θ is an acute angle, $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

b Suppose θ is in the second quadrant.

i Write θ', the corresponding reference angle of θ, and whether $\tan(\theta)$ is positive or negative.

ii Use part **i** to write an equation of the form $\tan(\theta)=\_\tan(\theta')$

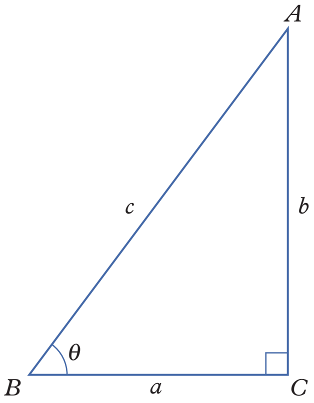
ii Verify that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

c Repeat part **b** for θ in the third quadrant.

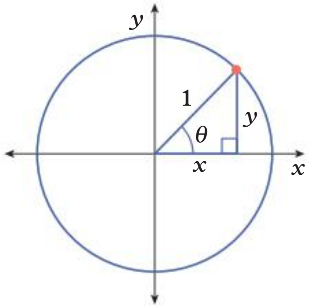
d Repeat part **b** for θ in the fourth quadrant.

e Verify that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ for $\theta = 0°, 90°, 180°$ and $270°$.

f Use the idea of a coterminal angle in question **8** to justify why $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ for values of θ outside $0° \le \theta < 360°$.



13 Using the diagram shown, we can write the equation $x^2 + y^2 = 1$ using Pythagoras' theorem.



a Write the equation in terms of the sine and cosine of θ.

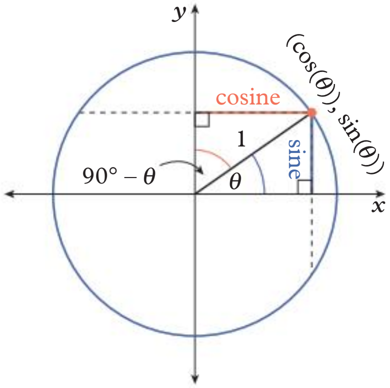
b Rearrange the equation in part **a** to make:

  i $\sin(\theta)$ the subject
  ii $\cos(\theta)$ the subject.

c We can use the equation in part **a** (and its rearrangements in part **b**) to determine the cosine value of an angle if we know the sine of the angle and vice versa. The sign of the solution can be determined by considering which quadrant the angle lies in. Use the sine and cosine values provided to determine the value of the trigonometric expression in brackets correct to four decimal places.

  i $\sin(50°) = 0.7660$ $(\cos(50°))$
  ii $\cos(116°) = -0.4384$ $(\sin(116°))$
  iii $\sin(177°) = 0.0523$ $(\cos(177°))$
  iv $\cos(240°) = -0.5$ $(\sin(240°))$

14 The name 'cosine' comes from an abbreviation of the phrase 'the sine of the complementary angle'. That is, $\sin(90° - \theta) = \cos(\theta)$. We can use this definition to turn sine expressions into cosine expressions of the complementary angle and vice versa. Write the following sine expressions using cosine, and the following cosine expressions using sine.



a $\sin(30°)$
b $\sin(52°)$
c $\cos(24°)$
d $\cos(45°)$

This is true even when θ is not an acute angle. Write the following sine expressions using cosine, and cosine expressions using sine, using:

  i the angle from the subtraction $90° - \theta$
  ii the coterminal angle between 0° and 360° if applicable
  iii the reference angle using the correct sign or its value if it is a multiple of 90° if applicable.

e $\sin(158°)$
f $\cos(351°)$
g $\sin(211°)$
h $\sin(-15°)$
i $\cos(482°)$
j $\cos(-555°)$

9H Exact values

Learning intentions

By the end of this topic you will be able to …

- ✔ determine the exact values of the sine, cosine and tangent functions
- ✔ apply trigonometric identities involving complementary, supplementary and negative angles.

Inter-year links
Year 9 7H Trigonometric ratios

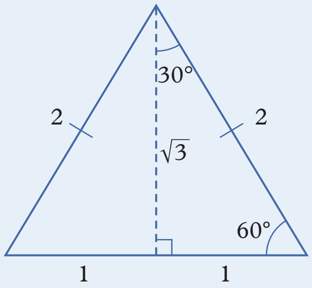
Exact values of trigonometric ratios

Key content video

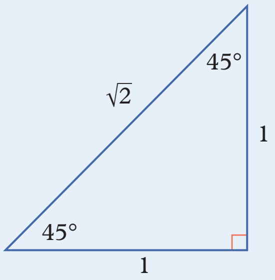
- The table on the right can be used to find exact values of the trigonometric ratios for some special angles.
- There are two special triangles whose side lengths provide the exact values of the trigonometric ratios for the three angles specified in the table above. We can use geometry and Pythagoras' theorem to determine all angles and lengths of the two triangles.

θ	**sin(θ)**	**cos(θ)**	**tan(θ)**
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

half-equilateral triangle



half-square triangle



Trigonometric identities

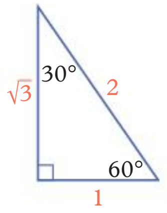
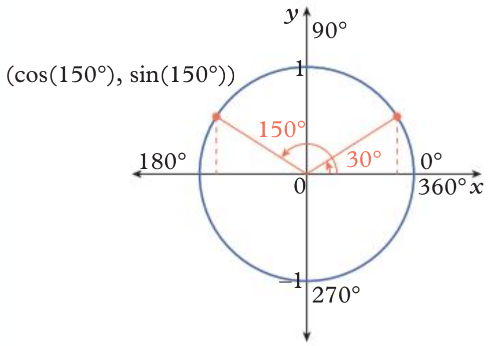
- Negative angles are angles measured clockwise from the positive *x*-axis. The sine, cosine and tangent of negative angles can be identified using the unit circle and reference angles.
- The table below gives some useful trigonometric identities, which hold for all values of θ.

Complementary angles	**Supplementary angles**	**Negative angles**
$\sin(90° - \theta) = \cos(\theta)$	$\sin(180° - \theta) = \sin(\theta)$	$\sin(-\theta) = -\sin(\theta)$
$\cos(90° - \theta) = \sin(\theta)$	$\cos(180° - \theta) = -\cos(\theta)$	$\cos(-\theta) = \cos(\theta)$
$\tan(90° - \theta) = \frac{1}{\tan(\theta)}$	$\tan(180° - \theta) = -\tan(\theta)$	$\tan(-\theta) = -\tan(\theta)$

- **Coterminal angles** are angles that differ by a multiple of 360°. Two coterminal angles have the same sine, cosine and tangent ratios.

Example 9H.1 Finding the exact values of a trigonometric ratio

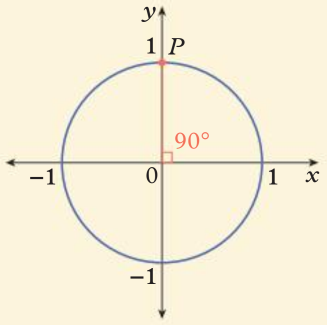
Use the unit circle and exact values for the trigonometric ratios to evaluate: sin(150°), cos(150°), tan(150°)

THINK	**WRITE**
1 Sketch the unit circle and mark the angle 150°. It is in quadrant 2, so the reference angle is 180° – 150° = 30°. 	
2 Calculate sin(30°), cos(30°) and tan(30°) using the side lengths of the special triangle above.	$\sin(30°) = \frac{1}{2}, \cos(30°) = \frac{\sqrt{3}}{2}, \tan(30°) = \frac{\sqrt{3}}{3}$
3 Apply the correct signs to the trigonometric ratios for quadrant 2.	In quadrant 2, cosine and tangent are both negative, but sine is positive. $\sin(150°) = \frac{1}{2}, \cos(150°) = -\frac{\sqrt{3}}{2}, \tan(150°) = -\frac{\sqrt{3}}{3}$

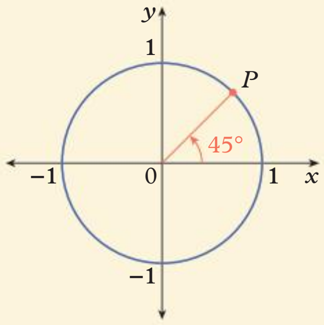
Example 9H.2 Identifying points on the unit circle

Identify the coordinates of the point *P* in each case below.

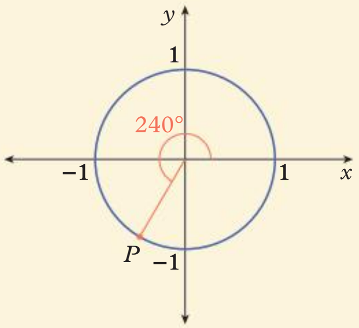
a



b



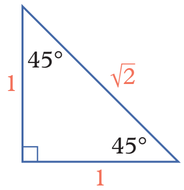
c



THINK	**WRITE**
a **1** Any point on the unit circle has an *x*-coordinate of cos(θ) and a *y*-coordinate of sin(θ). Write *P* in coordinate form using $\theta = 90°$.	**a** $P = (\cos(90°), \sin(90°))$
2 Identify the exact values of the trigonometric ratios. Remember that the unit circle has a radius of 1.	$= (0, 1)$

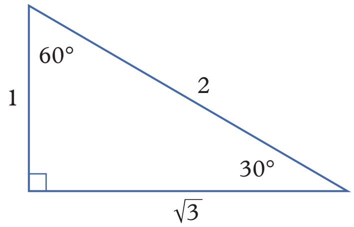
b 1 Write P in coordinate form using $\theta = 45°$.

2 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values.



c 1 Write P in coordinate form using $\theta = 240°$.

2 θ is in quadrant 3, so the reference angle is $240° - 180° = 60°$ Use the half-equilateral triangle to determine $\sin(60°)$ and $\cos(60°)$



Apply the correct signs to the trigonometric ratios for quadrant 3.

b $P = (\cos(45°), \sin(45°))$

$\cos(45°) = \frac{\sqrt{2}}{2}$

$\sin(45°) = \frac{\sqrt{2}}{2}$

$P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c $P = (\cos(240°), \sin(240°))$

$\sin(60°) = \frac{\sqrt{3}}{2}, \cos(60°) = \frac{1}{2}$

$\sin(240°) = -\frac{\sqrt{3}}{2}, \cos(240°) = -\frac{1}{2}$

$P = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

Example 9H.3 Using trigonometric identities

Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

a $\cos(-\theta)$ **b** $\tan(360° - \theta)$ **c** $\sin(90° + \theta)$

THINK

a Apply the negative angle trigonometric identity for cosine.

b 1 Identify that on the unit circle, 360° and 0° are coterminal angles, so 360° can be replaced by 0°. Subtracting from 0° is equivalent to taking the negative angle.

2 Apply the negative angle trigonometric identity for tangent.

c 1 Replace 90° with (180° – 90°). Group the resulting expression to be in the form of the supplementary angle trigonometric identity for sine.

2 Apply the supplementary angle trigonometric identity for sine. The expression in brackets takes the place of θ in the identity.

3 Apply the complementary angle trigonometric identity for sine.

WRITE

a $\cos(-\theta) = \cos(\theta)$

b $\tan(360° - \theta) = \tan(0° - \theta)$

$= \tan(-\theta)$

$= -\tan(\theta)$

c $\sin(90° + \theta) = \sin((180° - 90°) + \theta)$

$= \sin(180° - (90° - \theta))$

$= \sin(90° - \theta)$

$= \cos(\theta)$

Helpful hints

✔ It's helpful to memorize the special trigonometric ratios for 30°, 60° and 45°, and remember $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ and $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

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Exercise 9H Exact values

△ 1–9, 11, 12 □ 2, 4–7, 9–14, 16 ⬠ 2–3(d–f), 5–7, 10(d–f), 11–15, 17, 18

FLUENCY AND UNDERSTANDING

1 Write a list of all angles from 0° to 360° which have the following reference angles.

a 30° **b** 45° **c** 60°

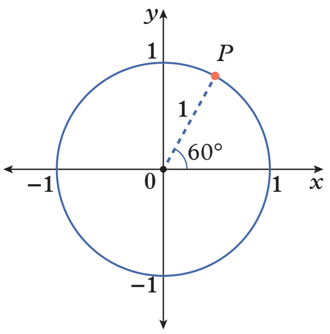
9H.1 2 Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each of these angles.

a 240° **b** 300° **c** 120°

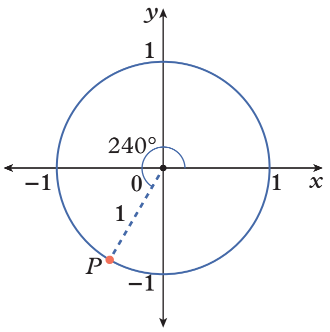
d 315° **e** 135° **f** 150°

9H.2 3 Identify the coordinates of the point *P* in each of these cases.

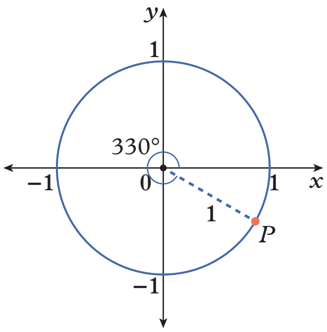
a



b



c



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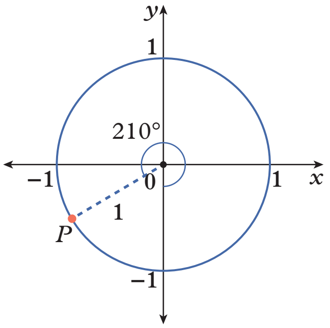

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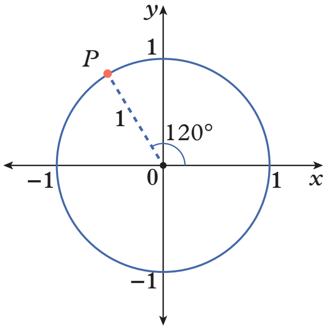
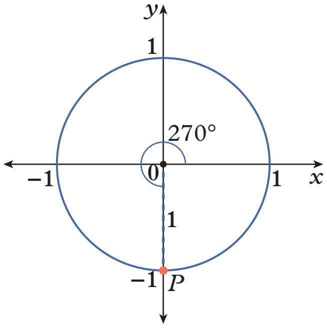


f



4 Give the exact coordinates of the point on the unit circle when the radial arm is rotated by each of these angles from the positive *x*-axis.

a 225° **b** 300° **c** 150°

d 135° **e** 315° **f** 180°

9H.3 5 Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

a $\sin(-\theta)$ **b** $\tan(-\theta)$ **c** $\cos(360° - \theta)$

d $\sin(360° - \theta)$ **e** $\cos(90° + \theta)$ **f** $\tan(90° + \theta)$

6 Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$

a $\sin(360° + \theta)$ **b** $\cos(540° - \theta)$ **c** $\tan(-90° + \theta)$

d $\cos(-90° - \theta)$ **e** $\sin(270° + \theta)$ **f** $\tan(270° - \theta)$

g $\sin(180° + \theta)$ **h** $\tan(540° + \theta)$ **i** $\cos(270° - \theta)$

ADVANCED

PROBLEM SOLVING AND REASONING

7 The constructions of the half-square and half-equilateral triangles are helpful to learn in case you forget any of the side lengths or angles, as you can then reconstruct the triangle from the name.

a Draw a square, mark the side lengths as 1 unit, and add one of the diagonals.

b Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

c Draw an equilateral triangle with a horizontal base, mark the side lengths 2 units, and add the perpendicular height to the base such that it bisects the top angle and the base length.

d Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

8 The exact values of sine and cosine for the angles 30°, 45° and 60°, together with their multiples, can be recorded on the unit circle.

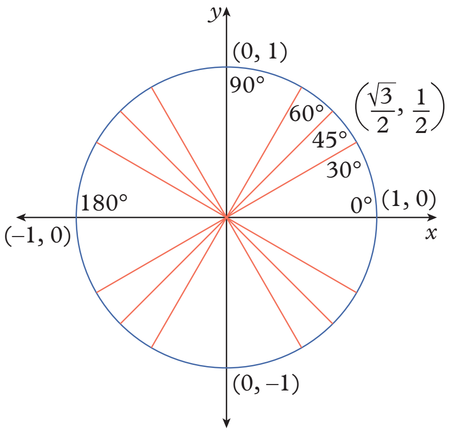
a Draw a unit circle. Mark the coordinates of the points where the circle cuts the horizontal and vertical axes. Also, mark points on the circle representing angles of 30°, 45° and 60°, together with their multiples up to 360°.

b Label each of the points shown on the unit circle with its exact cosine and sine value as coordinates.

For example, the point representing 30° would be labelled as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Take care to use the correct signs.

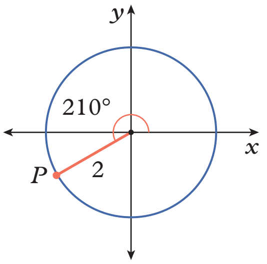
c Give the exact coordinates of each of these angles on the unit circle.

i 225° **ii** 300° **iii** 150° **iv** 135° **v** 315° **vi** 90°

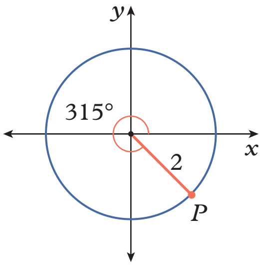


9 Identify the coordinates of the point P in each of the following diagrams.

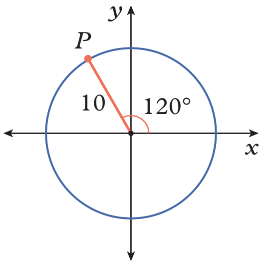
a



b



c



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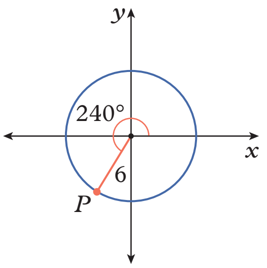

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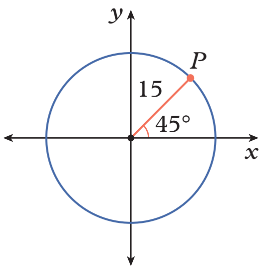

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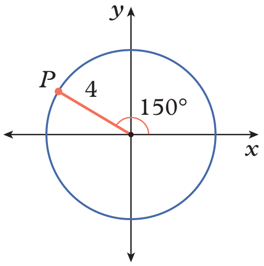


10 A trigonometric ratio involving an angle in quadrants 3 or 4 is often easier to evaluate using negative angles. For each of the trigonometric ratios below:

- **i** rewrite the expression using a negative angle
- **ii** simplify the expression to one with an acute or right angle using an appropriate trigonometric identity
- **iii** evaluate the trigonometric ratio.

a sin(300°) **b** cos(315°) **c** tan(330°)

d cos(270°) **e** sin(330°) **f** tan(300°)

11 There are many more trigonometric identities. The double angle trigonometric identity for sine is $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Verify this identity for the following trigonometric ratios by:

- **i** evaluating the left-hand side using reference angles and the unit circle
- **ii** applying the identity to the right-hand side and simplifying the result.

a $\sin(60°) = \sin(2 \times 30°)$ **b** $\sin(90°) = \sin(2 \times 45°)$ **c** $\sin(120°) = \sin(2 \times 60°)$

d $\sin(180°) = \sin(2 \times 90°)$ **e** $\sin(240°) = \sin(2 \times 120°)$ **f** $\sin(360°) = \sin(2 \times 180°)$

12 The angle addition trigonometric identity for sine is:

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

Use this identity to verify:

a $\sin(60°) = \sin(30° + 30°)$ **b** $\sin(90°) = \sin(45° + 45°)$ **c** $\sin(90°) = \sin(60° + 30°)$

d $\sin(150°) = \sin(90° + 60°)$ **e** $\sin(120°) = \sin(90° + 30°)$ **f** $\sin(210°) = \sin(150° + 60°)$

g $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

13 Some of the exact values of other angles can be determined using the identity from question **12**.

a Find the exact value of:

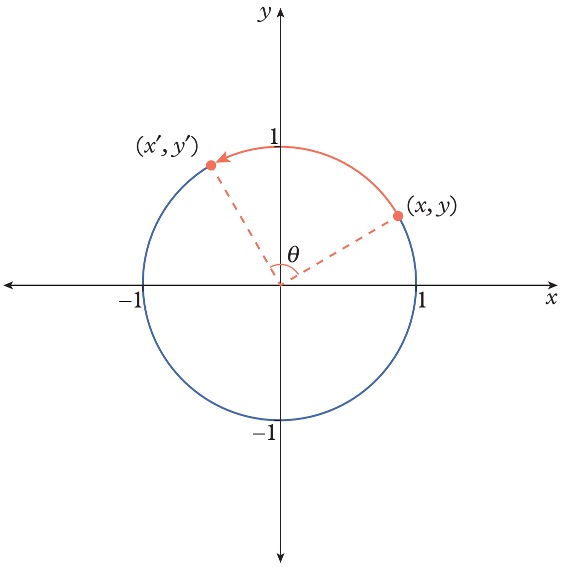
- **i** $\sin(105°) = \sin(60° + 45°)$
- **ii** $\sin(75°) = \sin(30° + 45°)$
- **iii** Explain why your answers to parts **i** and **ii** are equal.

b Write an angle subtraction trigonometric identity for sine. Hint: Write $\sin(x - y) = \sin(x + (-y))$ and apply the angle addition identity.

c Use your answer to part **b** to find the exact value of:

- **i** $\sin(15°) = \sin(45° - 30°)$ **ii** $\sin(75°) = \sin(90° - 15°)$

14 The formula $(x', y') = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$ can be used to rotate a point (x, y) by θ anti-clockwise about the origin. The rotated point (x', y') is called the image.



a Use the formula to verify that when the point (0, 1) is rotated 90° anti-clockwise about the origin, the image is (1, 0).

b Find the image of each of the following points after they are rotated 60° about the origin.

- **i** (1, 0)
- **ii** (0, 1)
- **iii** (1, 1)
- **iv** (–2, 3)
- **v** (–3, –1)
- **vi** (4, –2)

Negative angles are used to rotate a point clockwise about the origin.

c Substitute $-\theta$ into the formula to find a new formula for rotating points clockwise about the origin.

d Use your answer to part **c** to rotate each of the points in part **b** 30° clockwise about the origin.

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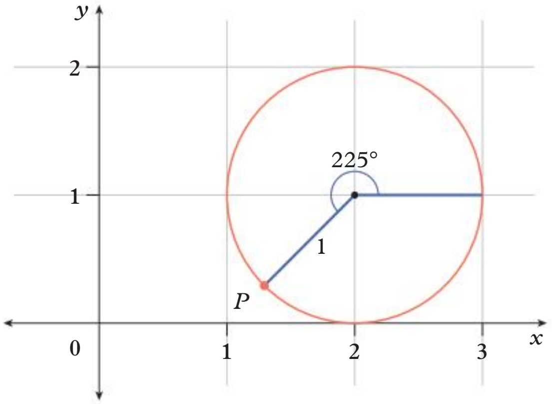
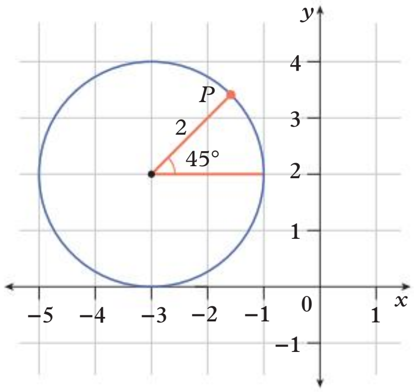
CHALLENGE

15 Identify the coordinates of the point P in each case.

a



b



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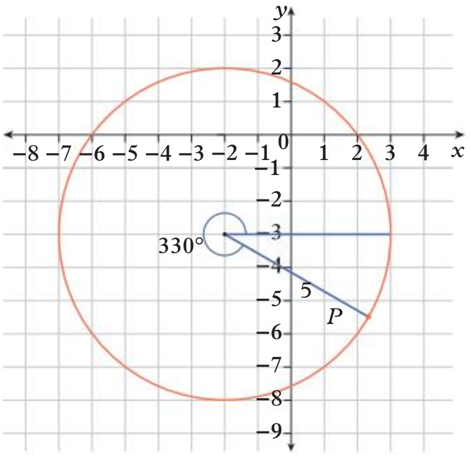

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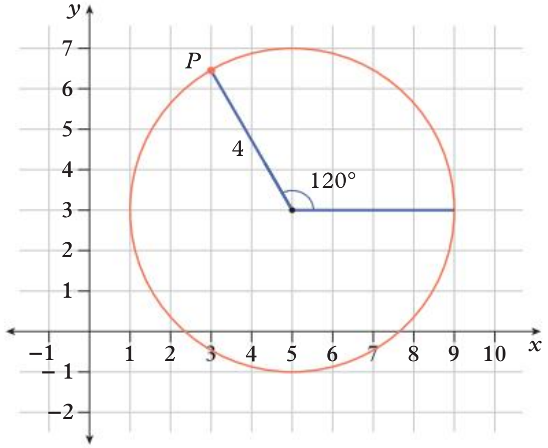
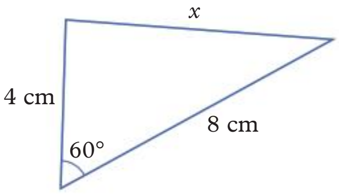


16 Determine the exact value of x for both triangles.

a



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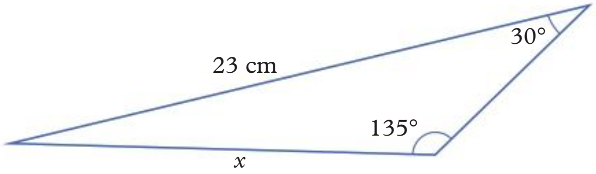

17 The formula to rotate a point anti-clockwise about the origin in question **14** can be used to derive identities for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. Recall that any point on the unit circle can be written in the form $(\cos(\theta), \sin(\theta))$.

a Show that the point $(0, 1)$ rotated by α anti-clockwise about the origin is $(\cos(\alpha), \sin(\alpha))$.

b Use the formula to rotate the point $(\cos(\alpha), \sin(\alpha))$ by β anti-clockwise about the origin.

c Write the image of the point $(0, 1)$ rotated by $\alpha + \beta$ anti-clockwise about the origin in the form $(\cos(\theta), \sin(\theta))$.

Rotating a point by $\alpha + \beta$ anti-clockwise about the origin is the same as first rotating by α anti-clockwise about the origin and then rotating the image by β anti-clockwise about the origin.

d Use your answers to parts **a** and **b** to find formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. Hint: Your answer for $\sin(\alpha + \beta)$ should be the same as the formula in question **13**.

e Derive a formula for $\cos(\alpha - \beta)$.

18 Use your answers to questions **13** and **17** to derive identities for the following in terms of $\tan(\alpha)$ and $\tan(\beta)$. Hint: Note that $\sin(x) = \tan(x)\cos(x)$ (when $\cos(x) \neq 0$).

a $\tan(\alpha + \beta)$

b $\tan(\alpha - \beta)$

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Coordinates of points on the unit circle

Interactive skillsheet
Exact values of trigonometric ratios

Interactive skillsheet
Trigonometric identities

Investigation
Remembering the exact values

Topic quiz
9H

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9I Trigonometric graphs

Learning intentions

By the end of this topic you will be able to ...

- ✔ read a point on a trigonometric graph
- ✔ determine the period and amplitude of a trigonometric function
- ✔ sketch graphs of simple trigonometric functions.

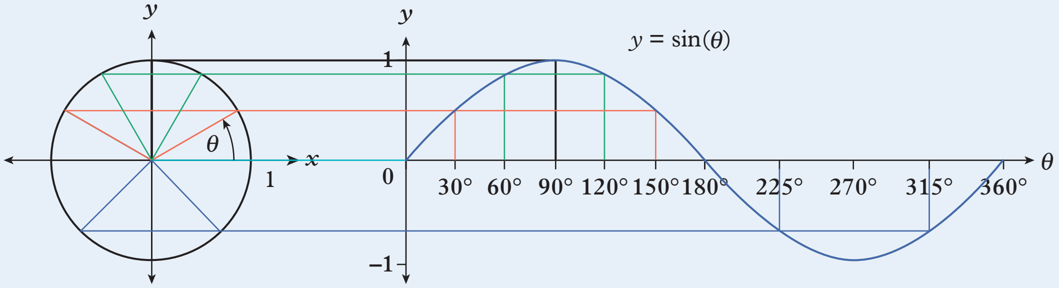
Inter-year links

Year 9 5B Plotting quadratic relationships

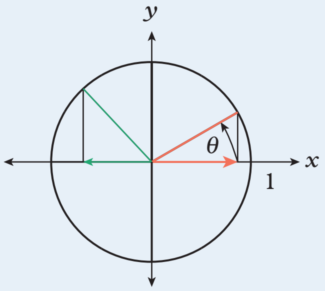
Trigonometric functions

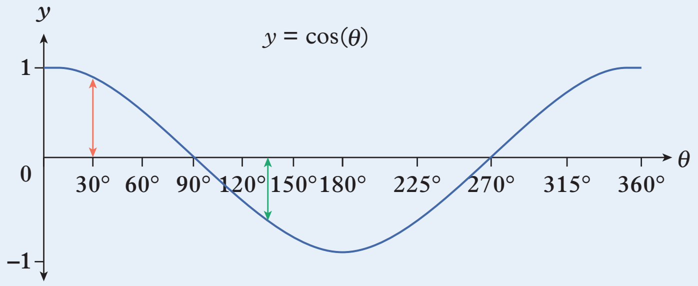
Key content video

- A trigonometric function can be represented as a graph on the Cartesian plane.
- The horizontal axis represents the angle and the vertical axis represents the value for the trigonometric ratio.
- The graph of $y = \sin(\theta)$ has y values corresponding to the values of the y-coordinates on the unit circle for a given angle θ.

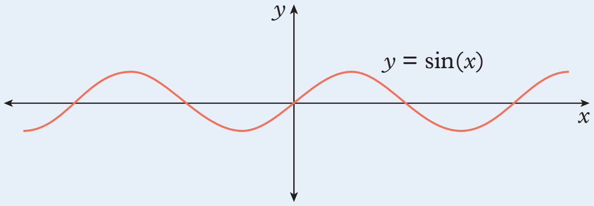


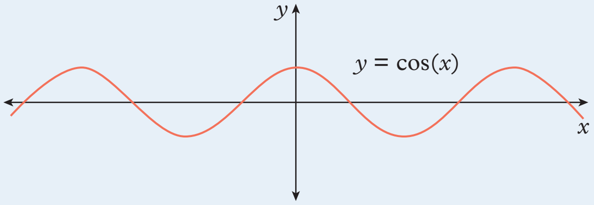
- The graph of $y = \cos(\theta)$ has y values corresponding to the values of the x-coordinates on the unit circle for a given angle θ.





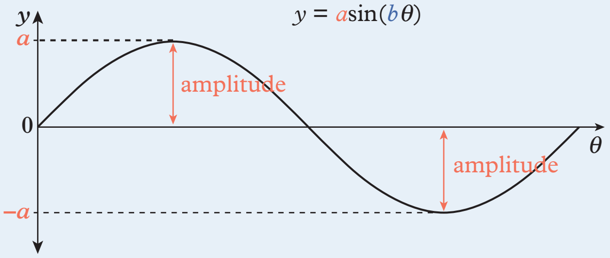
- For sine and cosine functions, the shape of the graph is called a sinusoid. They both look like waves with repeating peaks and troughs going along the horizontal axis.

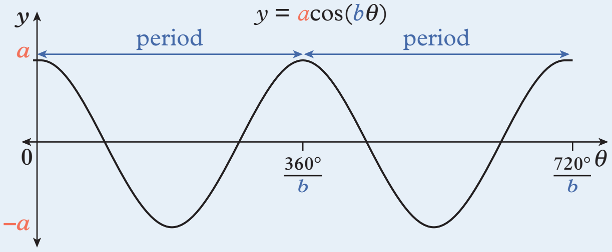




Amplitude and period

- The **amplitude** of a trigonometric function is the distance from the centre of the curve to the maximum/minimum y value.
- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$, the amplitude is equal to the magnitude of a. That means, if $a = 2$ or $a = -2$, then the amplitude is 2.
- Trigonometric functions are **periodic functions**, because their output is repeated as the angle moves around the circle.
- The **period** of a periodic function is the length of the interval along the horizontal axis that is required for the function to repeat itself.
- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$, the period is given by: period $= \dfrac{360°}{b}$





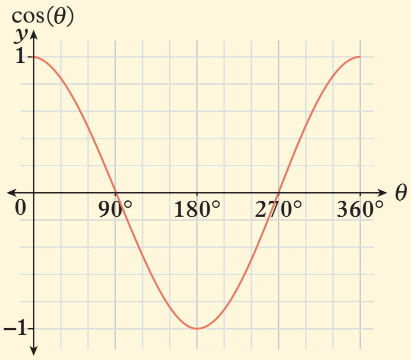
Sketching trigonometric graphs

- For trigonometric functions of the form $y = a\sin(b\theta)$ or $y = a\cos(b\theta)$:
  - → as the size of a increases, the amplitude increases, so the sinusoid is stretched on the y-axis.
  - → as b increases, the period decreases, so the sinusoid is compressed on the x-axis.
- For functions of the form $y = a\sin(b\theta)$, the y-intercept is at $y = 0$.
- For functions of the form $y = a\cos(b\theta)$, the y-intercept is at $y = a$.

Example 9I.1 Reading a trigonometric ratio from a graph

Consider the graph of the cosine function on the right. Use the graph to read the cosine value for each of these angles. Give your answers as correct to one decimal place, if necessary.

a 360° **b** 80°

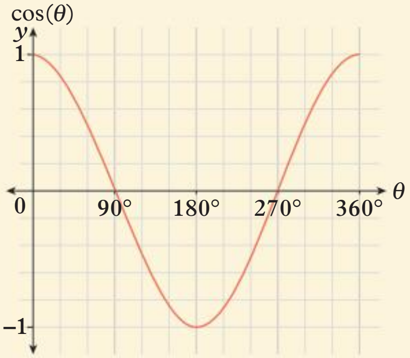


THINK	**WRITE**
a There is a maximum on the graph at $\theta = 360°$, so the vertical coordinate is exactly 1.	**a** $(360°, \cos(360°)) = (360°, 1)$ $\cos(360°) = 1$
b Identify an approximate coordinate for the graph just to the left of $(90°, 0)$.	**b** $(80°, \cos(80°)) \approx (80°, 0.2)$ $\cos(80°) \approx 0.2$

Example 9I.2 Reading an angle from a trigonometric graph

Consider this graph of the cosine function on the right. Use the graph to find the angles from 0° to 360° that have the following cosine values. Give your answers to the nearest 10°.

a 0.6 **b** -0.2



THINK	WRITE
a **1** There are two coordinates on the graph where $\cos(\theta) = 0.6$: one near 50° and one near 310°.	**a** $(\theta, 0.6) \approx (50°, 0.6)$ or $(310°, 0.6)$
2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.	1st quadrant: $\theta' = \theta$ $= 50°$ 4th quadrant: $\theta' = 360° - \theta$ $= 360° - 310°$ $= 50°$ Cosine is positive in the first and fourth quadrant, so $\cos(50°) = \cos(310°)$. $\theta \approx 50°$ or $310°$
b **1** There are two coordinates on the graph where $\cos(\theta) = -0.2$: one near 100° and one near 260°.	**b** $(\theta, 0.2) \approx (100°, -0.2)$ or $(260°, -0.2)$
2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.	2nd quadrant: $\theta' = 180° - \theta$ $= 180° - 100°$ $= 80°$ 3rd quadrant: $\theta' = \theta - 180°$ $= 260° - 180°$ $= 80°$ Cosine is negative in the second and third quadrant, so $\cos(100°) = \cos(260°)$. $\theta \approx 100°$ or $260°$

Example 9I.3 Determining the amplitude and period of a trigonometric function

Determine the amplitude and period of $y = -\frac{1}{2}\sin(6\theta)$.

THINK	WRITE
1 Identify the amplitude as the magnitude of the coefficient of the sine function.	amplitude $= \frac{1}{2}$
2 Write the period as 360° divided by the coefficient in front of θ and simplify.	period $= \frac{360°}{6}$ $= 60°$

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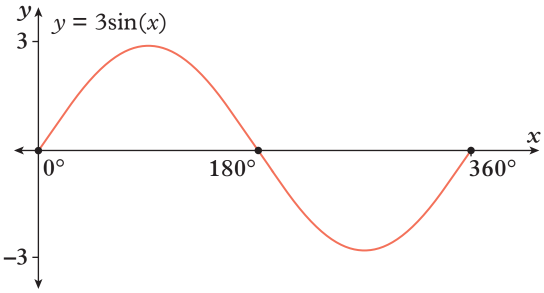
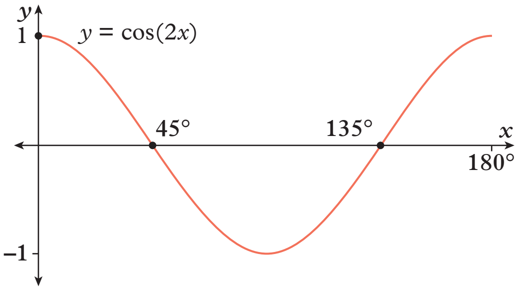
Example 9I.4 Sketching trigonometric graphs

Sketch the graph of each of these functions over a one-period interval.

a $y = 3\sin(\theta)$ **b** $y = \cos(2\theta)$

THINK	**WRITE**
a **1** Identify the amplitude as the magnitude of the coefficient in front of the sine function.	**a** amplitude = 3
2 Identify the period.	period = 360°
3 This is a sine function, so sketch a sinusoid starting at the point (0, 0) and extending for one period. **4** Use the amplitude to label the y-axis with the maximum and minimum values. For a sine function the x-intercepts are every half period starting from 0.	
b **1** Identify the amplitude as the magnitude of the coefficient in front of the cosine function.	**b** amplitude = 1
2 Calculate the period.	period $= \dfrac{360°}{2}$ $= 180°$
3 This is a cosine function, so sketch a sinusoid starting with a maximum at $\theta = 0$ and extending for one period. **4** Use the amplitude to label the y-axis with the maximum and minimum values. For a cosine function the x-intercepts are every half period starting from the first quarter of a period.	

Helpful hints

- ✔ Remember to consider all 4 quadrants of the unit circle and convert to the correct reference angle when trying to determine the angles for a given y value on a trigonometric graph. There are usually 2 angles with the same y value per period.
- ✔ Keep in mind that $\sin(30°) = \frac{1}{2}$ and $\cos(60°) = \frac{1}{2}$. That is, the graphs do not equal half the amplitude at the halfway point between the x-axis and the maximum/minimum.
- ✔ Sine and cosine graphs have the same basic shape. You only need to consider the period, amplitude and the starting point of the graph.

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Exercise 9I Trigonometric graphs

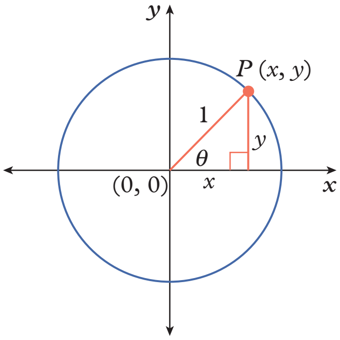
△ 1–8, 9(b–e), 10(b–e), 11, 14

□ 3–6, 8, 9–10(c–f), 13, 15–17

⬠ 5, 6, 8, 9–10(d–f), 12, 13, 15–18

UNDERSTANDING AND FLUENCY

1 Trace the point P around the unit circle in an anti-clockwise direction from its starting point on the positive x-axis back to its starting point. As you trace, consider the size of angle θ as P moves. Also consider what happens to the x-coordinate for $P(x, y)$ as it moves along the x-axis.



- **a** As you traced anti-clockwise around the unit circle, you will have noticed that the size of angle θ increased from 0° to 360°, and that the x-coordinate of point P decreased as it moved from 0° to 90°. Describe what happened to the x-coordinate of P when the size of the angle moved between:

  i 90° and 180° **ii** 180° and 270° **iii** 270° and 360°.

- **b** The change in x value from part **a** is the change in the cosine of angle θ as it increases from 0° to 360°. Complete this table with the aid of your calculator to obtain a clearer picture of the values for cosine as the angle increases from 0° to 360°. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos(θ)													

- **c** Plot the points from the table on grid or graph paper, with the size of angle θ along the horizontal axis scale and the cosine value, $\cos(\theta)$, up the vertical axis scale. Join your points with a smooth curve.
- **d** Sketch on graph paper what would happen if you extended the angle values to 720°.
- **e** What is the value of $\cos(\theta)$ at its first and second maximum. What is the period of the cosine graph?
- **f** Note the difference between the maximums and the minimums of the graph. What is the amplitude of the graph?
- **g** What is the y-intercept of the cosine graph?

2 Consider the change in the y-coordinate as point $P(x, y)$ moves and angle θ increases in size from 0° to 360°. This represents the change in the sine value of the angle.

- **a** Repeat the procedure from question **1b**, constructing a table for the sine values, to obtain a clearer picture of the values for sine as the angle sweeps from 0° to 360°.
- **b** Sketch the sine graph from 0° to 360°.
- **c** What is the period of the graph?
- **d** What is the amplitude of the graph?
- **e** What is the y-intercept of the sine graph?

9I.1 **3** Consider the graph on the right of the cosine function.

a From the graph, write the cosine value for each of these angles.

i 0° **ii** 90°

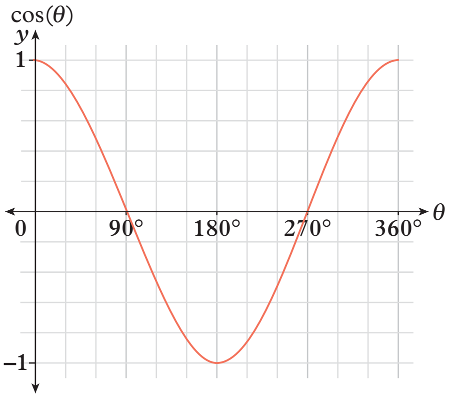
iii 270° **iv** 180°

b Use the graph on the right to help you write the approximate cosine value (correct to one decimal place) for each of these angles.

i 40° **ii** 100°

iii 340° **iv** 200°

v 280° **vi** 140°



9I.2 **4** Use the graph from question **3** to help you find the angles from 0° to 360° with the following cosine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

5 Consider the graph of the sine function on the right.

a From the graph, write the sine value for each of these angles.

i 360° **ii** 90°

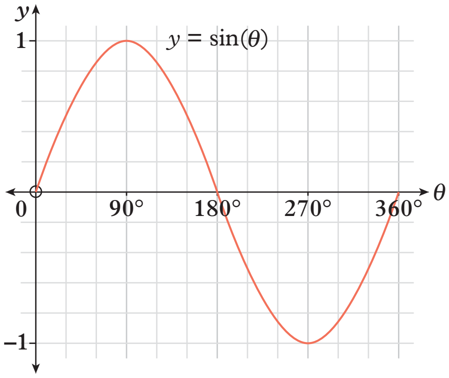
iii 270° **iv** 180°

b Use the graph to help you write the approximate sine value (correct to one decimal place) for each of these angles.

i 40° **ii** 130°

iii 350° **iv** 190°

v 230° **vi** 140°



6 Use the graph from question **5** to help you find the angles from 0° to 360° with the following sine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

7 Consider the function $y = 2\sin(3\theta)$. Calculate the y value for each of the following θ values and plot each pair of coordinates on a Cartesian plane. Draw a line to sketch the graph.

a Complete the table of values below. The first two columns have been completed for you.

b Plot each point (θ, y) on a Cartesian plane.

c Sketch the graph of $y = 2\sin(3\theta)$ for $0° \le \theta \le 240°$.

3θ	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ	0°	30°							
$\sin(3\theta)$	0	1							
$y = 2\sin(3\theta)$	0	2							

9I.3 **8** Determine the amplitude and period in degrees for each of the following.

a $y = 3\sin(2\theta)$ **b** $y = 5\cos(3\theta)$ **c** $y = -4\sin\left(\frac{\theta}{2}\right)$ **d** $y = -6\cos\left(\frac{\theta}{6}\right)$

e $y = \frac{3}{4}\sin\left(\frac{3\theta}{4}\right)$ **f** $y = \frac{5}{3}\cos\left(\frac{3\theta}{5}\right)$ **g** $y = -\frac{7}{8}\sin\left(\frac{4\theta}{7}\right)$ **h** $y = \frac{5}{12}\cos\left(\frac{24\theta}{5}\right)$

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PROBLEM SOLVING AND REASONING

9I.4 **9** Sketch the graphs of the following equations for 0°≤ x ≤ 360°.

a $y = 4\sin(\theta)$
b $y = \sin(4\theta)$
c $y = \sin\left(\frac{\theta}{4}\right)$
d $y = -4\sin(\theta)$
e $y = 5\sin(2\theta)$
f $y = -3\sin\left(\frac{\theta}{2}\right)$

10 Sketch the graphs of the following equations for 0°≤ x ≤ 360°.

a $y = 3\cos(\theta)$
b $y = \cos(3\theta)$
c $y = -3\cos(\theta)$
d $y = \cos\left(\frac{\theta}{3}\right)$
e $y = -\frac{1}{2}\cos(4\theta)$
f $y = 7\cos\left(\frac{\theta}{2}\right)$

11 The graphs of the sine function and the cosine function are quite similar. If you were presented with a graph of $y = \sin(\theta)$ and $y = \cos(\theta)$, and the functions weren't labelled, how could you tell which graph was which?

12 A grandfather clock has a pendulum that swings back and forth in a regular periodic motion. This motion is so regular that it has been the same method used, for hundreds of years, for keeping time. The weight of the bob on the end of the pendulum does not affect the period, but the period is affected by the length of the pendulum.

a Explain why the motion of a pendulum is periodic.

b The equation used to find the period of the motion of a pendulum is $T = \pi\sqrt{\frac{l}{g}}$, where T is the time (in seconds), l is the length of the pendulum (in metres) and g is the acceleration due to gravity (9.8 m/s²).

Find the period of a pendulum with each of the following lengths:

i 1 m
ii 1.5 m
iii 2 m.

c Explain how the length of a pendulum affects the period of its motion.

13 Periodic functions occur in many everyday activities. Consider this table of the sunrise and sunset times recorded for the east coast of Australia on the first day of the month throughout a year. The times are recorded as Eastern Standard Time.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sunrise (am)	4:55	5:20	5:40	5:57	6:12	6:29	6:38	6:29	6:02	5:28	4:57	4:44
Sunset (pm)	6:46	6:42	6:20	5:46	5:17	5:01	5:04	5:19	5:34	5:47	6:05	6:28

a Plot the sunrise and sunset figures from the table on separate graphs, placing the sunrise graph directly below the sunset graph. Use the horizontal axis scale for the months and the vertical axis scale for the times.

b Describe the shapes of the two curves you graphed.

c Are these two curves similar to those for the sine and cosine functions? Explain any similarities or differences.

d Are the curves periodic? Explain why or why not.

e Because of the way you positioned these two graphs, the trend in daylight hours throughout the year is visually apparent. Explain how this is so.

f If you were not told that these were figures for the east coast of Australia, how could you reason that they must be figures for somewhere in the Southern Hemisphere?

14 State the number of cycles (a repetition of a sinusoid) the graphs of each of the following equations will have in the domain 0°≤ x ≤ 360°.

a $y = \sin(3\theta)$
b $y = \frac{1}{2}\sin(11\theta)$
c $y = 5\sin\left(\frac{\theta}{2}\right)$
d $y = a\sin(b\theta)$
e $y = \cos(4\theta)$
f $y = -4\cos(90\theta)$
g $y = \frac{5}{2}\cos\left(\frac{3\theta}{2}\right)$
h $y = a\cos(b\theta)$

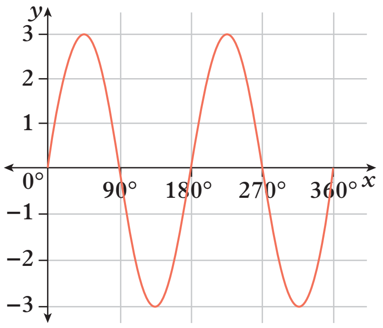
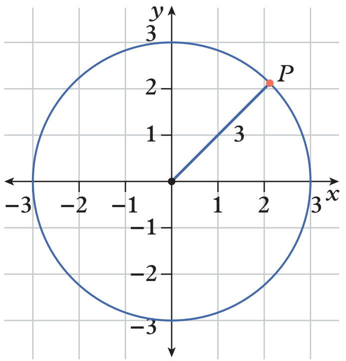
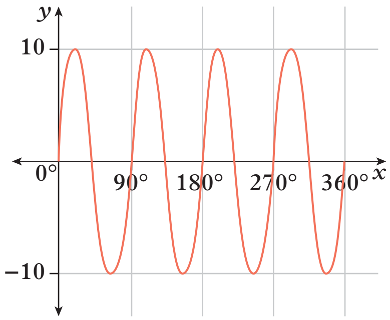
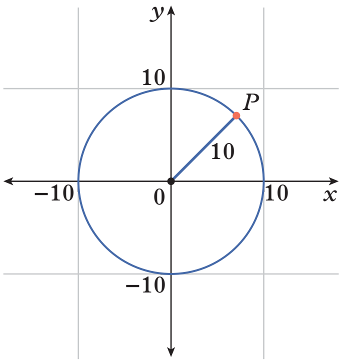
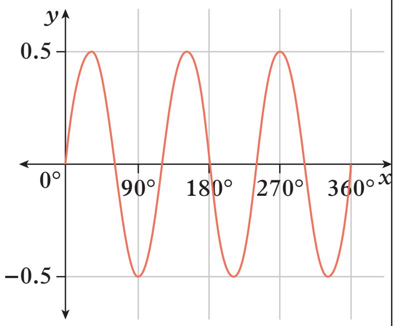
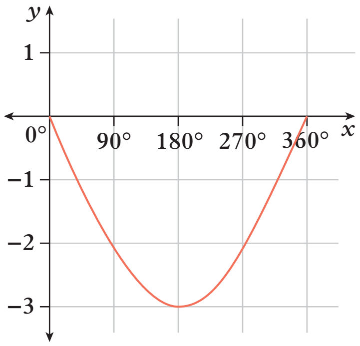
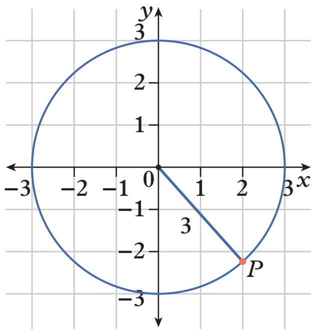
ADVANCED

15 The graphs for sine and cosine can be considered a point moving at a constant speed anti-clockwise around a circle. The amplitude and period, then, have analogues on the circle that they are related to.

a Complete the table below.

b Determine what feature of the circle the amplitude is related to and how.

c Determine what feature of the circle the period is related to and how.

	Graph	Circle	Radius	Revolutions per 360°
			3	2
i				
ii				
iii				

ADVANCED

CHALLENGE

16 The tangent ratio is also a periodic function.

a Complete a table of values, correct to one decimal place, for θ values from 0° to 360° as shown.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)																	

b Plot these points on a Cartesian plane and join them with a smooth curve.

c **i** Describe the shape of the graph.

ii It is different from the sine curve and the cosine curve. Why is this so?

iii At $\theta = 90°$, a vertical asymptote occurs. Explain what you understand this to mean. Where does the next asymptote occur?

iv Explain why the graph is periodic. What is its period?

v What is the amplitude of the graph?

17 Since the sine, cosine and tangent functions are periodic, they can be graphed on any domain by first graphing one period, and then copying the graph to the left or right as required.

a To graph $y = \sin(\theta)$ on the domain $-360° \le \theta \le 360°$, complete the following steps.

i Graph $y = \sin(\theta)$ on the domain $0° \le \theta \le 360$ °.

ii Extend your graph to the left, keeping the same shape, amplitude and period, until the graph extends from $\theta = -360°$ to $\theta = 360°$.

iii Verify that the points at $\theta = -360°$ and $\theta = 360°$ are correctly located on your graph.

b To graph $y = \cos(\theta)$ on the domain $-450° \le \theta \le -90°$, complete the following steps.

i In pencil, graph $x = \cos(\theta)$ on the domain $0° \le \theta \le 360°$.

ii In pencil, extend your graph to the left, keeping the same shape, amplitude and period, until the graph extends from $\theta = -450°$ to $\theta = 360°$.

iii Erase the parts of your graph outside the domain $-450° \le \theta \le -90°$.

iv Verify that the points at $\theta = -450°$ and $\theta = -90°$ are correctly located on your graph.

c Graph $y = \sin(\theta)$ and $y = \cos(\theta)$ on the same set of axes on the following domains.

i $-270° \le \theta \le 90°$

ii $-180° \le \theta \le 180°$

iii $-720° \le \theta \le 0°$

18 When graphing one period of a trigonometric function, it is common to choose a complete period, which includes the point $\theta = 0°$. As you saw in question **16**, the tangent function passes through the origin, has a vertical asymptote at 90° and has a period of 180°.

a Determine the equation of the first vertical asymptote to the left of the one at $\theta = 90°$.

b Draw the asymptote from part **a** and the one at $\theta = 90°$ on the same set of axes.

c Use your answer to question **16** to graph one period of $y = \tan(\theta)$ between the two asymptotes.

d What is the equation of the next vertical asymptote to the left of the ones already drawn on your graph?

e Find a rule you can use to locate all asymptotes of $y = \tan(\theta)$ on any domain.

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Interactive skillsheet
Period of a trigonometric function

Interactive skillsheet
Amplitude of a trigonometric function

Interactive skillsheet
Sketching trigonometric graphs

Topic quiz
9I

9J Solving trigonometric equations

Learning intentions

By the end of this topic you will be able to ...

✔ solve simple trigonometric equations.

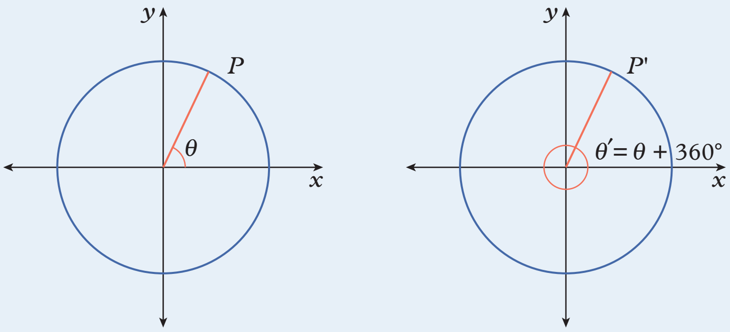
Inter-year links

Year 9 7J Using trigonometry to find angles

Trigonometric equations

Key content video

- The trigonometric functions $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are all periodic, as their values repeat with every revolution of the angle, θ.
- If θ is the angle measured anti-clockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → the x-coordinate of the point on the unit circle is $\cos(\theta)$
  - → the y-coordinate of the point on the unit circle is $\sin(\theta)$
  - → the gradient from the origin to the point P on the unit circle is $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.
- Adding or subtracting 360° to θ, the intersection of the new angle θ' meets the unit circle at point P' on the unit circle, then:
  - → the x-coordinate of P' on the unit circle is $\cos(\theta \pm 360°)$
  - → the y-coordinate of P' on the unit circle is $\sin(\theta \pm 360°)$
  - → the gradient from the origin to the point P' on the unit circle is $\tan(\theta \pm 360°)$.
- Because P and P' would have the same coordinates on the unit circle:
  - → $\sin(\theta) = \sin(\theta \pm 360°)$
  - → $\cos(\theta) = \cos(\theta \pm 360°)$
  - → $\tan(\theta) = \tan(\theta \pm 360°)$.

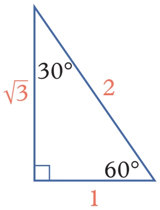


- Equations involving trigonometric functions have infinitely many solutions, as trigonometric functions are periodic.

  For example, $\sin(\theta) = 0$ for $\theta = 0, 180°, 360°, 540°, 720°, \ldots$
- The solutions to a trigonometric equation are often restricted to a domain given by an interval, such as $0° \le \theta \le 360°$.
- Most calculators only give one of the solutions for $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ as the domains are restricted to only half a revolution of the unit circle.
- Once one solution for a trigonometric equation is identified, the rest can be determined using the symmetry of the unit circle within a specified domain.

Example 9J.1 Solving a trigonometric equation within a given domain

Solve the equation $\sin(\theta) = -\frac{1}{2}$, $0° \leq \theta \leq 360°$.

THINK	WRITE
1 Determine the first solution to the equation. Use the special triangles to determine the angle in the first quadrant that has a sine of $\frac{1}{2}$.	 $\sin(\theta) = \frac{1}{2}$ $\theta = \sin^{-1}\left(\frac{1}{2}\right)$ $= 30°$
2 Identify the quadrants in which sine is negative.	We were given that $\sin(\theta) = -\frac{1}{2}$ and sine is negative in quadrants 3 and 4.
3 Write possible values for θ using symmetry properties of trigonometric ratios in the given domain.	So: $\theta = 180° + 30°$ $= 210°$ or: $\theta = 360° - 30°$ $= 330°$
4 Write the solution to the equation.	There are two solutions: $\theta = 210°$ or $\theta = 330°$

Helpful hints

- ✔ The trigonometric functions repeat themselves every 360°, because these are always coterminal angles. But don't forget that most values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ repeat themselves in another quadrant of the unit circle as well.

ANS p777

Exercise 9J Solving trigonometric equations

△ 1–5, 7, 9, 11, 16(a, b), 18(a, e)

■ 1–3, 6, 8, 10(a, e, i), 11, 12, 15, 16–18(a, c, e), 19

⬠ 1, 2, 6, 10(g–i), 11–14, 17–18(a, c, e), 20–22

9J.1 1 Solve each equation for the domain $0° \leq \theta \leq 360°$.

a $\cos(\theta) = \frac{\sqrt{3}}{2}$ **b** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{\sqrt{3}}{2}$ **e** $\cos(\theta) = \frac{1}{\sqrt{2}}$ **f** $\tan(\theta) = 1$

2 Using a calculator to determine the reference angle, solve each equation for the domain $0° \leq \theta \leq 360°$ correct to the nearest degree.

a $\cos(\theta) = -0.4$ **b** $\sin(\theta) = -0.3$ **c** $\tan(\theta) = 0.9$

d $\sin(\theta) = 0.6$ **e** $\cos(\theta) = 0.6$ **f** $\tan(\theta) = -7$

UNDERSTANDING AND FLUENCY

3 Consider the equation $\sin(\theta) = 1, 0° \leq \theta \leq 360°$.

a What angle has a sine value of 1? Hint: Refer to the unit circle.

b Are there any other angles within the domain with a sine value of 1? If so, which one(s)?

c What is the solution to the equation?

d Write a list of all the solutions for the domain $0° \leq \theta \leq 720°$.

4 Consider the equation $\cos(\theta) = -1$.

a What solution(s) can be found in the domain $0° \leq \theta \leq 360°$?

b What are the solutions in the domain $0° \leq \theta \leq 720°$?

5 Consider the equation $\sin(\theta) = 0, 0° \leq \theta \leq 360°$.

a What angle has a sine value of 0? Hint: Refer to the unit circle.

b Are there any other angles within the domain with a sine value of 0? If so, which one(s)?

c What is the solution to the equation?

d Write a list of all the solutions for the domain $0° \leq \theta \leq 720°$.

6 Solve each equation for the domain $0° \leq \theta \leq 360°$.

a $\cos(\theta) = 0$ **b** $\tan(\theta) = 0$ **c** $\sin(\theta) = -1$ **d** $\cos(\theta) = 1$

PROBLEM SOLVING AND REASONING

7 Give the smallest domain within $0° \leq \theta \leq 360°$ for there to be two solutions for each equation.

a $\sin(\theta) = -1$ **b** $\cos(\theta) = 0$ **c** $\cos(\theta) = \frac{1}{2}$

d $\sin(\theta) = \frac{\sqrt{3}}{2}$ **e** $\sin(\theta) = \frac{1}{\sqrt{2}}$ **f** $\cos(\theta) = -\frac{\sqrt{3}}{2}$

8 Find the smallest domain for each equation to give:

i two solutions within $0° \leq \theta \leq 360°$

ii four solutions within $0° \leq \theta \leq 720°$.

a $\sin(\theta) = -\frac{1}{2}$ **b** $\cos(\theta) = -\frac{1}{2}$ **c** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **d** $\cos(\theta) = -\frac{1}{\sqrt{2}}$

9 It is possible that a trigonometric equation has no real solution. Explain why each of these equations has no solution within any real domain.

a $3 - \sin(\theta) = 1$ **b** $2\cos(\theta) = 3$ **c** $\frac{1}{4}\sin(\theta) = 1$

10 Indicate whether each equation has a real solution within the domain $0° \leq \theta \leq 360°$. For those without a real solution, explain why this is so. There is no need to solve the equation.

a $\tan(\theta) = 4$ **b** $1 - \sin(\theta) = 1$ **c** $2\cos(\theta) = 2$

d $\frac{1}{2}\sin(\theta) = -\frac{1}{2}$ **e** $1 + \cos(\theta) = -1$ **f** $4\sin(\theta) = -\frac{1}{4}$

g $3 - \cos(\theta) = \frac{1}{3}$ **h** $\frac{1}{2}\cos(\theta) = -\frac{1}{4}$ **i** $\frac{1}{4} + \sin(\theta) = -1$

11 Take care when you consider the domain of a trigonometric equation. Don't always assume that a trigonometric equation has two or more solutions. Consider the following equations with the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

a $\sin(\theta) = \frac{1}{2}, 0° \leq \theta \leq 90°$ **b** $\sin(\theta) = -\frac{1}{2}, 0° \leq \theta \leq 90°$ **c** $\cos(\theta) = -\frac{1}{2}, 0° \leq \theta \leq 180°$

d $\tan(\theta) = 0.3, 0° \leq \theta \leq 180°$ **e** $\sin(\theta) = -0.4, 0° \leq \theta \leq 270°$ **f** $\cos(\theta) = 0.8, 0° \leq \theta \leq 270°$

12 If the pronumeral inside the brackets of the trigonometric function is multiplied by a coefficient, then start by redefining the specified domain in terms of the expression inside the brackets. For example, follow these steps to solve $\cos(3\theta) = \frac{1}{2}, 0° \leq \theta \leq 180°$.

1 Write the equation and the domain of θ.	$\cos(3\theta) = \frac{1}{2}, 0° \leq \theta \leq 180°$
2 Multiply the domain by the coefficient of θ.	$\cos(3\theta) = \frac{1}{2}, 0° \leq 3\theta \leq 540°$
3 Solve the equation normally for all solutions in the new domain.	$3\theta = 60°, 300°, 420°$
4 Divide all solutions by the coefficient of θ to find the solutions to the original equation.	$\theta = 20°, 100°, 140°$

Solve each of the following equations for the domain $0° \leq \theta \leq 180°$.

a $\sin(2\theta) = \frac{1}{\sqrt{2}}$ **b** $\cos(4\theta) = -1$ **c** $\tan(3\theta) = -1$

PROBLEM SOLVING AND REASONING

13 Solve each equation for the domain indicated.

- **a** $2\sin(\theta) - 3 = \sin(\theta) - 2,\ 0° \le \theta \le 180°$
- **b** $4\cos(\theta) + 5 = 4 + 5\cos(\theta),\ 0° \le \theta \le 180°$
- **c** $\sqrt{2}\cos(\theta) + 1 = 0,\ 0° \le \theta \le 270°$
- **d** $2\sin(\theta) + \sqrt{3} = 0,\ 0° \le \theta \le 270°$
- **e** $4 - 4\tan(\theta) = 0,\ 90° \le \theta \le 270°$
- **f** $\sqrt{48}\tan(\theta) + 2 = -3 - \sqrt{3}\tan(\theta),\ 90° \le \theta \le 270°$

14 Use the method shown in question **11** to solve each equation for the domain $0° \le \theta \le 1080°$.

- **a** $\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2}$
- **b** $\cos\left(\frac{\theta}{3}\right) = \frac{1}{\sqrt{2}}$
- **c** $\tan\left(\frac{2\theta}{5}\right) = -\sqrt{3}$

15 Explain why solving $\tan(\theta) = 1$ is different to solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$.

16 Determine the sum of the solutions of the following equations for the domain $0° \le \theta \le 360°$.

- **a** $\sin(\theta) = \frac{1}{2}$
- **b** $\cos(\theta) = \frac{1}{2}$
- **c** $\tan(\theta) = \frac{1}{\sqrt{3}}$
- **d** $\sin(\theta) = -\frac{1}{2}$
- **e** $\cos(\theta) = -\frac{1}{2}$
- **f** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

17 Solve each equation for the domain $-360° \le \theta \le 0°$.

- **a** $\sin(\theta) = \frac{\sqrt{3}}{2}$
- **b** $\cos(\theta) = -\frac{1}{\sqrt{2}}$
- **c** $\tan(\theta) = \sqrt{3}$
- **d** $\sin(\theta) = -1$
- **e** $\cos(\theta) = 0$
- **f** $\tan(\theta) = 0$

18 Solve each equation for the domain $-180° \le \theta \le 180°$.

- **a** $\sin(\theta) = 0$
- **b** $\cos(\theta) = \frac{\sqrt{3}}{2}$
- **c** $\tan(\theta) = -1$
- **d** $\sin(\theta) = \frac{1}{\sqrt{2}}$
- **e** $\cos(\theta) = -1$
- **f** $\tan(\theta) = \frac{1}{\sqrt{3}}$

19 The equation $\sin(\theta) = \cos(\theta)$ can be solved by using the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

- **a** Show that $\sin(\theta) = \cos(\theta)$ is equivalent to $\tan(\theta) = 1$.
- **b** Solve the following equations for $0° \le \theta \le 360°$.
  - **i** $\sin(\theta) = \cos(\theta)$
  - **ii** $\sin(\theta) = \sqrt{3}\cos(\theta)$
  - **iii** $\sqrt{3}\sin(\theta) + \cos(\theta) = 0$
  - **iv** $3\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

CHALLENGE

20 Solve the following equations.

- **a** $2\sin(2\theta) - 1 = 0,\ 0° \le \theta \le 360°$
- **b** $4\cos\left(\frac{\theta}{3}\right) + 2 = 0,\ 180° \le \theta \le 540°$
- **c** $10\sin(2\theta - 30°) + 5 = 0,\ 180° \le \theta \le 540°$
- **d** $12\cos\left(\frac{\theta}{4} + 180°\right) - \sqrt{72} = 0,\ -900° \le \theta \le 2160°$

21
- **a** Solve the equation $2x^2 - 3x - 2 = 0$.
- **b** Hence, solve the equation $2(\sin(\theta))^2 - 3\sin(\theta) - 2 = 0,\ 0° \le \theta \le 360°$.

22
- **a** Solve the equation $6x^2 - 17x + 12 = 0$.
- **b** Why are there no solutions to the equation $6(\sin(\theta))^2 - 17\sin(\theta) + 12 = 0$?
- **c** Why are there no solutions to the equation $6(\cos(\theta))^2 - 17\cos(\theta) + 12 = 0$?
- **d** How many solutions are there to the equation $6(\tan(\theta))^2 - 17\tan(\theta) + 12 = 0$?

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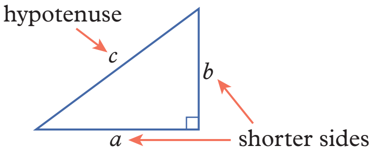
Interactive skillsheet Solving trigonometric equations	**Worksheet** Trigonometric rules and equations	**Investigation** Trigonometry and linear graphs	**Topic quiz** 9J

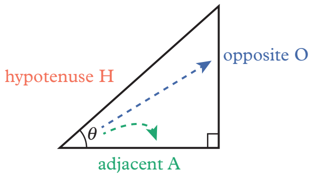
Chapter summary

Pythagoras' theorem and trigonometric ratios

$$\underset{\text{hypotenuse}}{c^2} = \underbrace{a^2 + b^2}_{\text{shorter sides}}$$

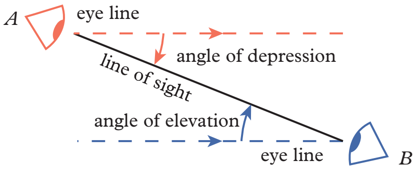


- $\sin(\theta) = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{\text{O}}{\text{H}}$
- $\cos(\theta) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{\text{A}}{\text{H}}$
- $\tan(\theta) = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{\text{O}}{\text{A}}$



Angles of elevation and depression

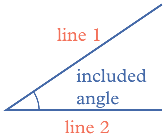
- The angle of depression from A to B is equal to the angle of elevation from B to A.



Exact values

θ	30°	45°	60°
sin(θ)	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos(θ)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan(θ)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Included angle area formula

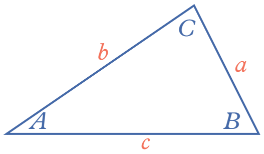


$$\text{area} = \frac{1}{2}\underbrace{bc}_{\text{adjacent side lengths}}\sin(\underbrace{A}_{\text{included angle}})$$

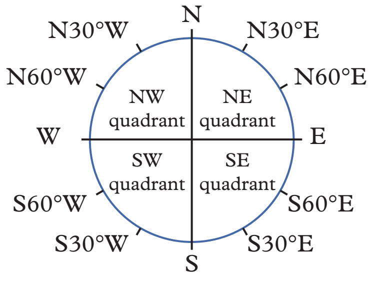
The sine and cosine rules

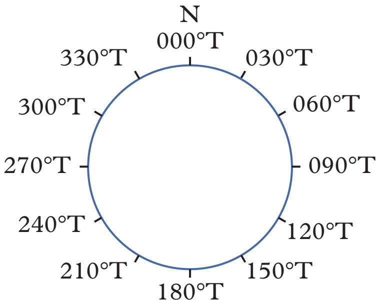
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

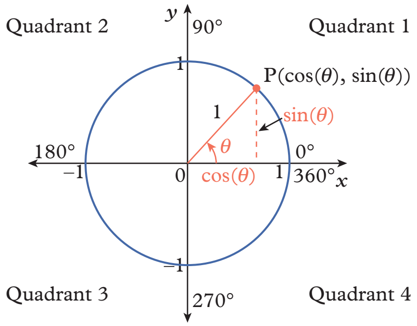


Compass bearings and true bearings



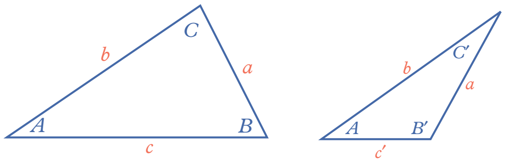


The unit circle

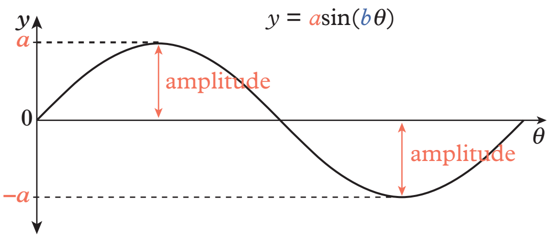


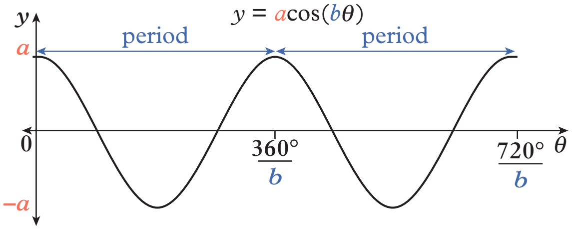
Ambiguous case of the sine rule

- If two side lengths (a and b) are known and the angle (A) opposite the shorter side length is also known, then there are two possible triangles.



Trigonometric graphs





ANS
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Chapter review

Chapter review quiz
Assess your knowledge of this chapter.
✔ Core
✔ Standard
✔ Advanced

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- amplitude
- angle of depression
- angle of elevation
- compass bearing
- cosine
- cosine rule
- coterminal angles
- hypotenuse
- included angle area formula
- period
- periodic function
- Pythagoras' theorem
- quadrant
- reference angle
- sine
- sine rule
- tangent
- trigonometry
- true bearing
- unit circle

1 Write Pythagoras' theorem in words using at least one word from the key term list.

2 Use a diagram to explain how to calculate the sine, cosine and tangent ratios for an acute angle θ in a right-angled triangle.

3 Use an example to explain the difference between compass bearings and true bearings.

4 Complete the following sentences using words from the key term list.

a The angle of ________ is the angle measured down from the horizontal.

b The angle of ________ is the angle measured upwards from the horizontal.

5 Draw and label the triangle with sides a, b and c and angles A, B and C, which is assumed in the sine and cosine rules.

6 Identify which rule(s) could be used given the following pieces of information.

a two sides and an included angle

b an opposite angle–side pair and another angle or side

7 Using an example, provide a definition in your own words for the following key terms.

a coterminal angles

b periodic function

8 Complete the following sentences using words from the key term list.

a The ______________ of a trigonometric function is the distance from the centre of the curve to the maximum or minimum y-value.

b For a given angle, the ______________ is the acute angle between the x-axis and the line connecting the origin to the point on the ___________.

Multiple choice

9A **1** Which of these describes a triangle with side lengths measuring 4 cm, 5 cm and 6 cm?

A right-angled **B** obtuse-angled **C** acute-angled **D** isosceles **E** equilateral

9A **2** The ratio $\dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

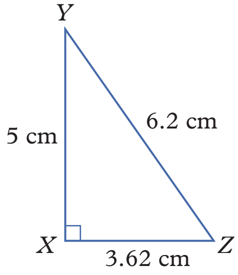
A sine **B** cosine **C** tangent **D** π **E** none of these

9B **3** For the triangle on the right, what is the size of $\angle YZX$, to the nearest degree?

A 54° **B** 55° **C** 35° **D** 36° **E** 90°

9B **4** For the triangle on the right, which of these ratios represents the sine of $\angle ZYX$?

A $\dfrac{5}{6.2}$ **B** $\dfrac{3.62}{6.2}$ **C** $\dfrac{3.62}{5}$ **D** $\dfrac{5}{3.62}$ **E** $\dfrac{6.2}{3.62}$



9C **5** What is the equivalent true bearing of N12°W?

A 348°T **B** 012°T **C** 192°T **D** 168°T **E** 078°T

STN 9D ADV **6** Which of these values is closest to the diagonal length within a cube of side length 5 cm?

A 5 cm **B** 8 cm **C** 9 cm **D** 10 cm **E** 7 cm

STN 9E ADV **7** What is the size of acute angle A in $\frac{10}{\sin(80°)} = \frac{9}{\sin(A)}$ closest to?

A 55° **B** 60° **C** 65° **D** 115° **E** 130°

STN 9F ADV **8** In $\triangle ABC$, if you know the size of angle B, what do you also need to know to apply the cosine rule to find b?

A a and b **B** a and c **C** c and b **D** angle A **E** angle A and a

ADV 9G **9** If a point P on the unit circle in quadrant 2 is given in terms of an acute reference angle, θ, then which of these coordinates for P are correct?

A $(-\cos(\theta), \sin(\theta))$ **B** $(\cos(\theta), -\sin(\theta))$ **C** $(-\cos(\theta), -\sin(\theta))$ **D** $(\cos(\theta), \sin(\theta))$ **E** $(-\sin(\theta), \cos(\theta))$

ADV 9H **10** Which of these is the exact sine value for 30°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$ **E** $\sqrt{3}$

ADV 9I **11** What is the period of $y = \sin(\theta)$?

A 90° **B** 180° **C** 270° **D** 360° **E** 720°

ADV 9I **12** What is the amplitude of $y = 3\cos(4\theta)$?

A 3 **B** −3 **C** 4 **D** $\frac{1}{3}$ **E** $\frac{1}{4}$

ADV 9J **13** Which of the following is not a solution to $2\cos(2\theta) = 1$?

A 30° **B** 60° **C** 150° **D** 210° **E** 330°

Short answer

For these questions, calculate each angle correct to the nearest degree and each length correct to two decimal places (unless stated otherwise).

9A **1** An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:

a an exact value **b** an approximate value, correct to one decimal place.

9B **2** A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

9C **3** Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

STN 9D ADV **4** Find the length of the longest diagonal within each 3D object:

a a cylinder with base circumference of 20 cm and height of 15 cm

b a box 10 cm long, 8 cm wide and 15 cm tall

c a cube of side length 25 cm.

STN 9D ADV **5** Calculate the angle each diagonal you calculated in question **4** makes with the height of the object correct to the nearest degree.

STN 9E ADV **6** Standing in a park, Anabel takes a bearing of 065°T to a tree. She walks 15 m due east and finds the bearing to the same tree to be 295°T.

a Draw a labelled diagram to represent this situation.

b Find the distance from the tree to Anabel, in both positions.

STN 9F ADV **7** In $\triangle ABC$, $a = 5$ cm, $b = 6.1$ cm and $A = 39°$.

a Explain why two distinct triangles can be drawn with these measurements. Draw them.

b Find the dimensions of all the angles and sides of the two triangles you drew in part **a**.

STN 9F ADV **8** A triangular garden bed in a park has sides of lengths 50 m, 40 m and 20 m.

a Draw a sketch of the shape of the garden bed.

b Find the size of the angle at each vertex of the triangular shape and the area of the triangle.

c What is the shortest distance from the vertex opposite the longest side to the longest side?

ADV 9G **9** Express each of the following trigonometric expressions in terms of an appropriate reference angle.

a $\sin(120°)$ **b** $\cos(210°)$ **c** $\tan(300°)$ **d** $\cos(315°)$

ADV 9H **10** Use the unit circle to find each of the following values.

a $\sin(300°)$ **b** $\cos(225°)$ **c** $\sin(135°)$ **d** $\cos(315°)$

ADV 9H **11** Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

a $\sin(90° - \theta)$ **b** $\cos(180° - \theta)$ **c** $\tan(-\theta)$ **d** $\cos(360° - \theta)$

ADV 9I **12** Sketch the graph of the following functions for the domain $0° \le \theta \le 360°$.

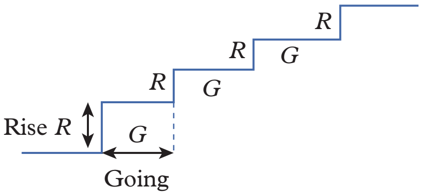
a $y = -3\sin(\theta)$ **b** $y = \cos(2\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$ **d** $y = -2\cos(3\theta)$

ADV 9J **13** Solve each of these equations for the domain $0° \le \theta \le 360°$.

a $\cos(\theta) = -1$ **b** $\sin(\theta) = 0$ **c** $\cos(\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin(\theta) = -\frac{1}{2}$

Analysis

1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as listed and shown in the diagram below.



- The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm.
- The going (G) must have a length within the range of 250–355 mm.
- The regulation also states that $2R + G$ must lie within the range of 550–700 mm.

a Consider a staircase with the minimum rise measurement of 115 mm and the minimum going measurement of 250 mm.

  i Explain whether this staircase would be within the building regulation guidelines.

  ii What is the slope of the staircase described? Give your answer to the nearest degree.

b Repeat part **a** using the maximum values allowed for the rise and going measurements.

c If the going measurement is the maximum value allowed, 355 mm, and the value of $2R + G$ falls within the required limits:

  i find a range of acceptable values for the rise **ii** calculate the range of acceptable slope values.

d The number of steps is counted by the number of goings from the bottom to the top, starting with a going and ending with a going. Draw a set of five stairs with a going measurement of 300 mm and a rise of 150 mm.

  i What is the total horizontal distance of the five stairs?

  ii Find the total vertical height of the five stairs.

  iii Calculate the direct distance between the base of the first step and the total vertical height calculated in part **ii**.

  iv What is the slope of the staircase? Correct your answer to the nearest degree.

e What do you consider would be ideal measurements for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and provide its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, because a wheelchair may tip backwards or roll down if the ramp is too steep. The code specifies that an internal ramp should be no steeper than 1 to 14, while an external ramp should have a gentler slope of 1 to 20.

  i Draw diagrams of 20 m long internal and external ramps with these slopes.

  ii For each ramp you drew in part **i**, find the going and rise measurements, rounding to three decimal places. Mark these measurements on your diagrams.

ADV **2** Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel.

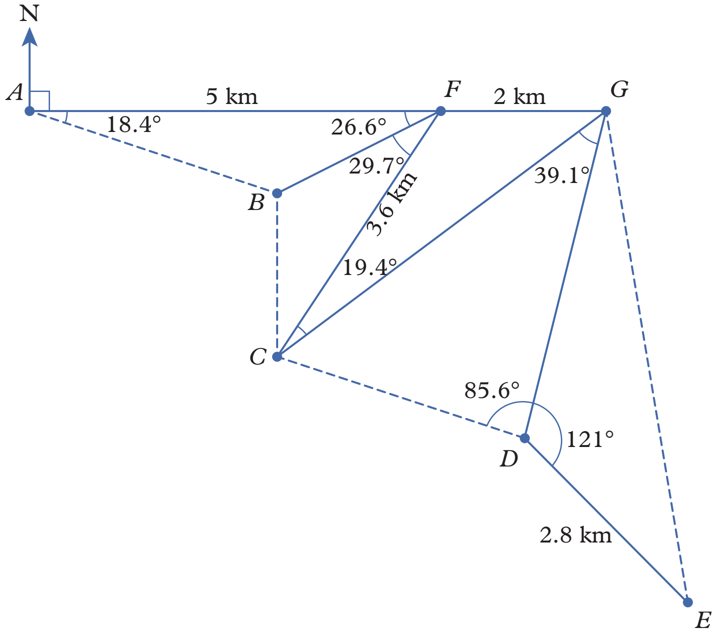
Tom used a video to record the height of the tag with respect to its starting position. He also made a record of the results in the table below.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	–20	–30	–20	0	20	30

- **a** Graph the data from the table, showing time on the *x*-axis.
- **b** Suggest whether the graph models a sine curve or a cosine curve.
- **c** How long does it take for the wheel to rotate once?
- **d** What was the starting position of the tag on the wheel? Explain how you deduced this.
- **e** What is the radius of the wheel?
- **f** Calculate the circumference of the wheel.
- **g** At what speed does Emma ride? Give your answer in metres per second.
- **h** What is the position of the tag after 10 seconds of riding at this constant speed? Explain how you arrived at this answer.

STN ADV **3** Triangulation is a method to determine unknown distances and angles using trigonometry. In this question, we consider a simplified example using the diagram below, which shows some known distances and angles between seven points. A dashed line between two points means that a direct measurement could not be taken, resulting in the need to use triangles and trigonometry.

- **a** Use the sine and cosine rules and your knowledge of triangles to find the following angles and side lengths correct to one decimal place. In your working, use the values correct to one decimal place.
  - **i** $\angle ABF$
  - **ii** AB
  - **iii** BF
  - **iv** BC
  - **v** $\angle CBF$
  - **vi** $\angle BCF$
  - **vii** $\angle CGF$
  - **viii** $\angle CFG$
  - **ix** CG
  - **x** $\angle DCG$
  - **xi** CD
  - **xii** DG
  - **xiii** EG
  - **xiv** $\angle DEG$
  - **xv** $\angle DGE$
- **b** Use your answers to part **a** to determine the distance between points *A* and *E* correct to one decimal place, and the true bearing from *A* to *E* correct to the nearest minute.
- **c** How could you make your answers to part **b** more accurate?



Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Determine whether a triangle is acute, right-angled or obtuse given the side lengths ☐ Use Pythagoras' theorem to determine unknown side lengths	☐ Go back to Topic 9A Pythagoras' theorem
☐ Use trigonometry to find unknown side lengths ☐ Use trigonometry to find unknown angles	☐ Go back to Topic 9B Trigonometry
☐ Use angles of elevation and depression to calculate distances ☐ Use compass bearings and true bearings in trigonometry problems	☐ Go back to Topic 9C Applications of trigonometry
☐ Find lengths and angles within 3D objects	☐ Go back to Topic 9D Three-dimensional problems
☐ Determine unknown side lengths and angles using the sine rule ☐ Determine the two possible solutions for an ambiguous case of the sine rule ☐ Determine the area of a triangle using the included angle area formula	☐ Go back to Topic 9E The sine and area rules
☐ Determine unknown side lengths of a triangle using the cosine rule ☐ Determine unknown angles of a triangle using the cosine rule	☐ Go back to Topic 9F The cosine rule
☐ Determine the quadrant in which an angle lies ☐ Determine the sign of a trigonometric expression given an angle ☐ Find the reference angle in the first quadrant of any angle using symmetry properties	☐ Go back to Topic 9G The unit circle
☐ Determine the exact values of the sine, cosine and tangent functions ☐ Apply trigonometric identities involving complementary, supplementary and negative angles	☐ Go back to Topic 9H Exact values
☐ Read a point on a trigonometric graph ☐ Determine the period and amplitude of a trigonometric function ☐ Sketch graphs of simple trigonometric functions	☐ Go back to Topic 9I Trigonometric graphs
☐ Solve simple trigonometric equations	☐ Go back to Topic 9J Solving trigonometric equations

10

Networks

Index

Prerequisite skills

Before starting work on this chapter, complete the Colouring in the plane investigation.

Investigation
Colouring in the plane

Curriculum links

STN
- Solves problems involving the characteristics of graphs/networks, planar graphs and Eulerian trails and circuits (MA5-NET-P-01)
  - Examine and describe a graph/network
  - Define a planar graph and apply Euler's formula for planar graphs
  - Explain the concept of Eulerian trails and circuits in the context of the Königsberg bridges problem

10A Introduction to networks

Learning intentions

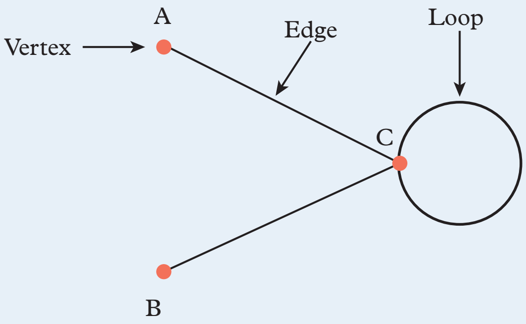
By the end of this topic, you will be able to …

- ✔ draw networks to represent real-world situations
- ✔ use terminology relating to networks/graphs
- ✔ reason with a network/graph diagram.

Networks

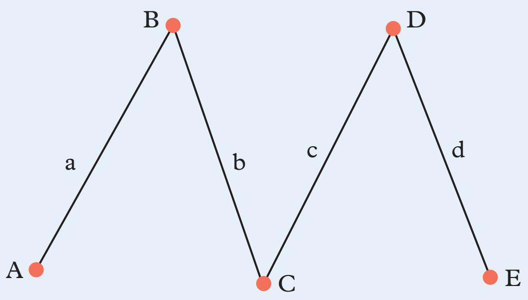
Key content video

- A **network** (or **graph**) is a collection of objects and the relations between them. The objects in a network are called **vertices** (or nodes) and a relation between two vertices is called an **edge** (or line).
  - → Vertices and edges often have physical meanings. For example, vertices may correspond to things such as people, cities or electronic devices. Edges may correspond to connections such as friendships, highways or power cables.
  - → For convenience, assume that all networks have at least one vertex.
  - → The terms 'network' and 'graph' are often used interchangeably; however, network should be used when relating to a real-world application and graph should be used when referring to an abstract entity.
- Edges connect exactly two vertices. Vertices on either side of an edge are said to be **neighbours**.
- The **degree** of a vertex is the number of edges connected to that vertex.
- A **loop** is an edge with the same vertex at both ends.
  - → A loop contributes two to the degree of that vertex.
- Graphs are defined by their vertices and edges, not by how they are drawn. Any given graph can be drawn in multiple different ways.
- The **handshaking lemma** (edges version) states that in every graph the sum of vertex degrees is twice the total number of edges.
  - → Intuitively, both ends of an edge are counted in the overall sum of vertex degrees.

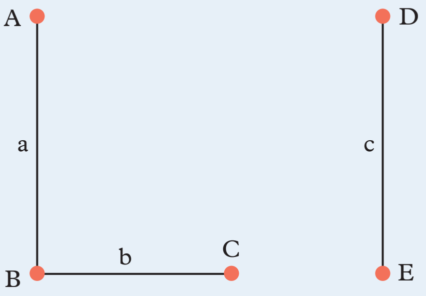


Walks

- A **walk** is a sequence of vertices and the edges between them, starting and ending with a vertex.
  - → A graph is said to be **connected** if there is a walk between every pair of vertices. If there is not a walk between every pair of vertices, a graph is said to be disconnected.



Connected graph



Disconnected graph

Example 10A.1 Determining the degree of vertices in a graph

State the degree for each vertex in this graph.

THINK

1 Count the number of edges connected to each vertex.

2 Remember that a loop adds two to the degree of a vertex.

WRITE

Vertex A has degree 2.

Vertex B has degree 4.

Example 10A.2 Drawing a network

Annabel, Benjamin, Cecilia and Derek are students. Their friendships are given in the table below. Display this information in a network.

Student	Annabel	Benjamin	Cecilia	Derek
Friends	Benjamin, Cecilia	Annabel, Cecilia	Annabel, Benjamin, Derek	Cecilia

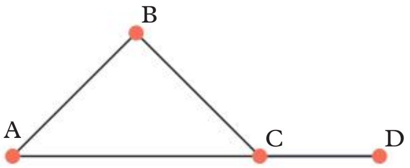
THINK

1 Draw a vertex for each student.

2 Label the vertices with the students' initials.

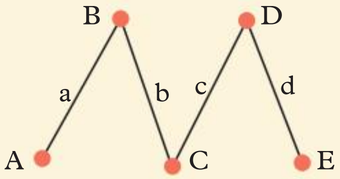
3 Draw an edge between students who are friends with each other.

WRITE



Example 10A.3 Finding a walk in a graph

Find a walk from A to E in this graph.



THINK

1 Start at vertex A.

2 Move along the edges and through the vertices of the graph until you reach vertex E.

3 Write all the vertices and edges that were touched, in order, during your walk. Include the start and finish vertex in your list.

WRITE

A, a, B, b, C, c, D, d, E

Helpful hints

- ✔ Two graphs may look quite different even if they represent the same information.
- ✔ When drawing a graph, make sure that the vertices and edges are legible and easy to interpret.

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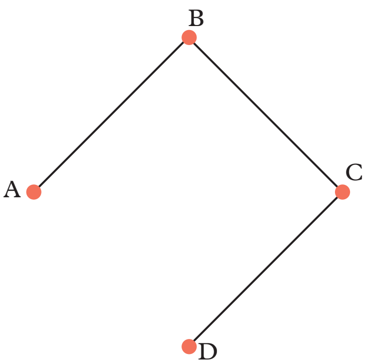
Exercise 10A Introduction to networks

△ 1–4, 6, 7, 9, 11, 13, 15–17 □ 4–16 ⬠ 4–12, 15, 18, 19

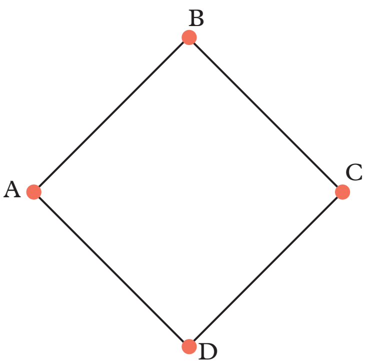
UNDERSTANDING AND FLUENCY

1 State the number of **i** vertices and **ii** edges in each of the following graphs.

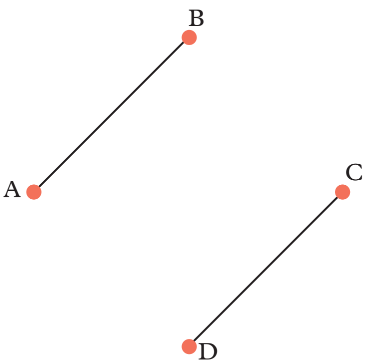
a



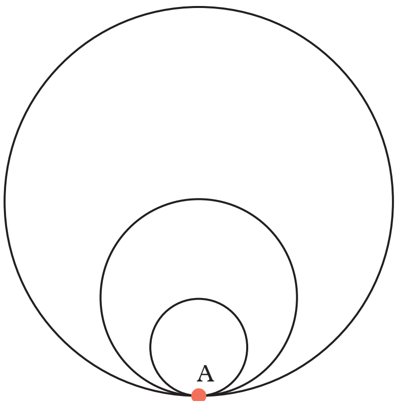
b



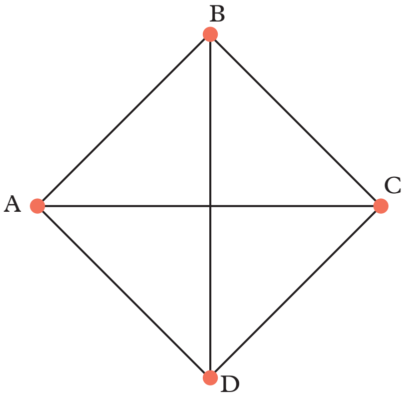
c



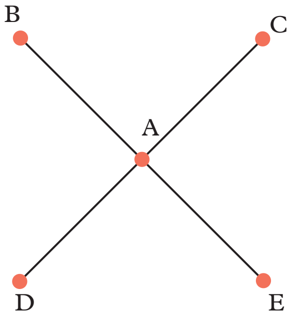
d



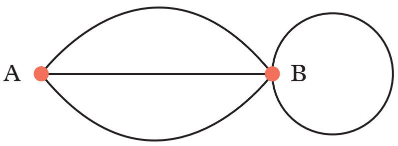
e



f



g



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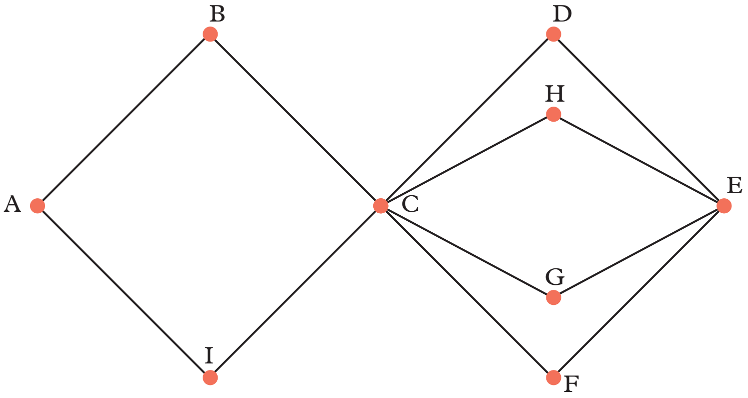


UNDERSTANDING AND FLUENCY

10A.1 2 Make a table of vertex degrees for each graph in question **1**.

3 Which graph in question **1** is disconnected?

10A.2 4 The following table represents regional NSW airports and the destinations serviced by those airports. Display this information in a network.

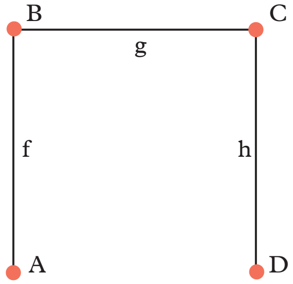
Airport	Albury (ABX)	Bathurst (BHS)	Coffs Harbour (CFS)	Dubbo (DBO)	Evans Head (EVH)
Destinations	BHS, DBO	ABX, CFS, DBO	BHS, EVH	ABX, BHS	CFS

5 A house contains a printer, computer, tv and gaming console. All these devices are connected to a single router via ethernet cables.

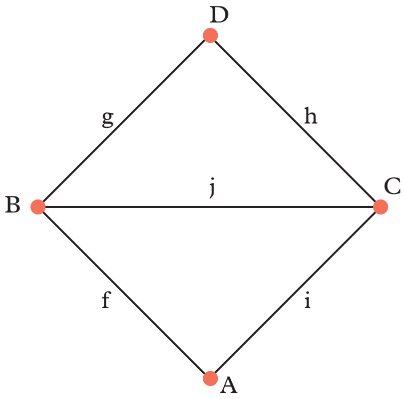
- **a** Draw a network to represent this information. Use vertex names P, C, T, G and R for the printer, computer, tv, gaming console and router, respectively, and use edges to represent ethernet connections between the devices.
- **b** How many ethernet connections are required at R?

10A.3 6 Which graphs contain a walk from A to D? State the vertices and edges in that walk (if it exists).

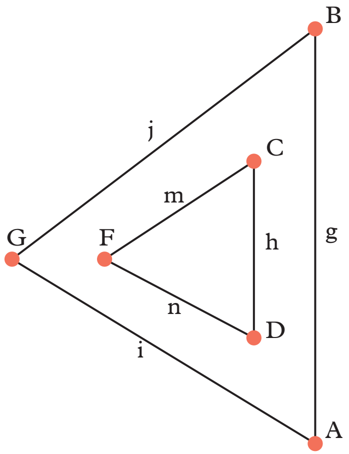
a



b

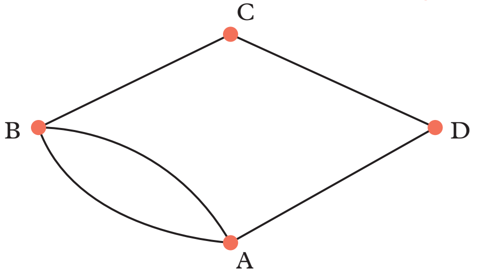


c



7 Consider the network on the right, where the vertices represent 4 people at a party and the edges represent handshakes.

- **a** How many handshakes did each person make?
- **b** What is the total number of handshakes?

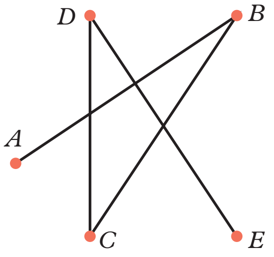


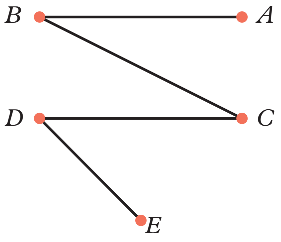
8 What is the sum of the vertex degrees in the network from question 7?

9 A graph has five vertices and each vertex has degree 4. How many edges does it have?

10 Why is it impossible to have a graph with three vertices, where the vertex degrees are 3, 2 and 2? Hint: If there were such a graph, how many edges must it have?

11 Explain why these two diagrams represent the same graph.





12 a Draw the network which represents the following list of people and their friends.

Person	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Friends	Vanessa, Zhang	Wasim, Uma	Vanessa, Zhang, Yvonne, Xavier	Wasim, Yvonne	Wasim, Xavier	Uma, Wasim

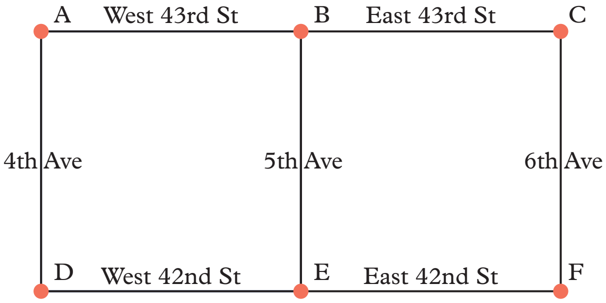
b Fill out the table of vertex degrees.

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree						

c Uma is telling Vanessa a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is about.

d Yvonne is telling Wasim a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is *not* about.

13 The streets of Manhattan are divided into East and West by 5th Avenue. The following network is a map of Manhattan between 4th and 6th Avenues and 43rd and 42nd streets.



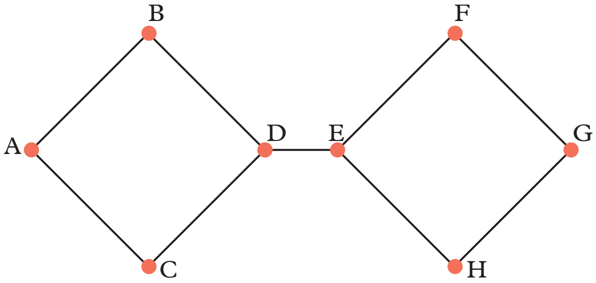
You are given some directions: Start at corner A. Follow West 43rd St to corner B. Follow 5th Avenue to corner E. Follow East 42nd St to corner F. Then follow 6th Avenue to corner C.

a Is this a valid walk?

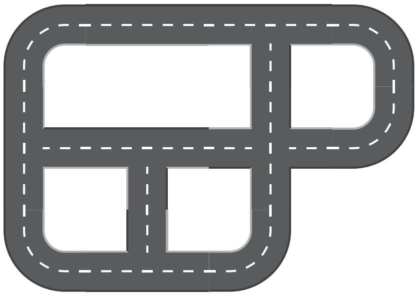
b If so, where does this walk lead to?

14 You now have a different set of directions for the network in question **12**: Start at corner A. Follow East 42nd St to corner F. Explain why this walk is not valid.

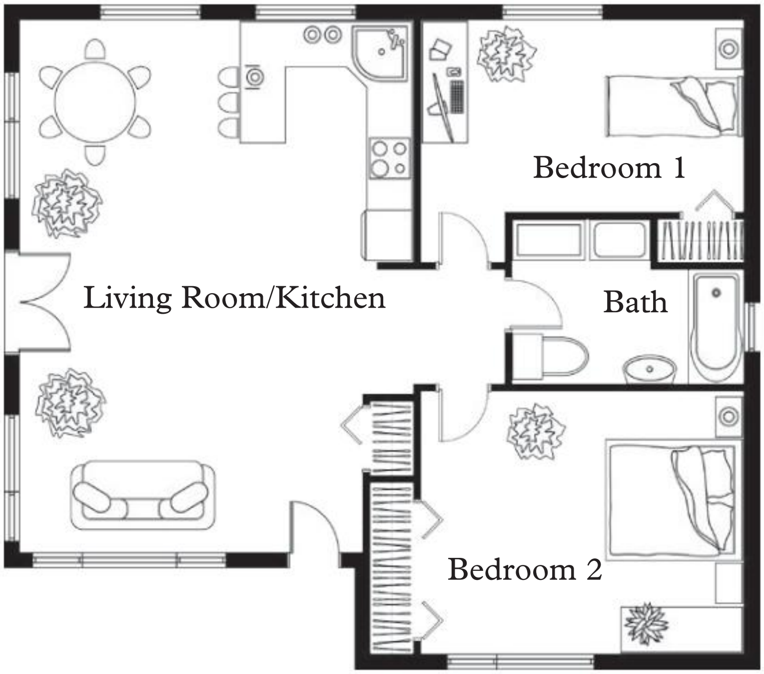
15 Disconnect this graph by removing exactly one edge.



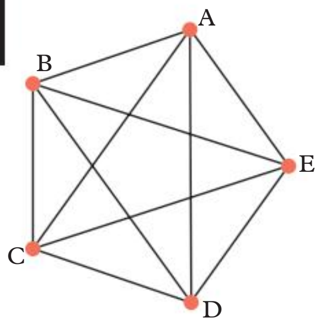
16 Redraw the roads below as a network using vertices to represent intersections, and edges to represent a span of road connecting two intersections.



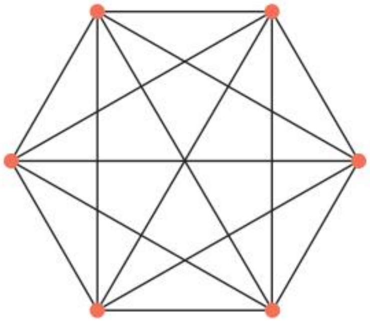
17 Draw the following house plan as a network. Use vertices to represent the Living Room/Kitchen, Bathroom, Bedroom 1 and Bedroom 2, and use edges to represent the doorways that connect the rooms.



18 Copy the graph on the right and use a red and blue pen to colour all the edges. Try to do this without drawing a red triangle or a blue triangle (i.e. three vertices where the edges between them are all the same colour). Hint: First colour as many edges in red as you can without drawing a triangle, then do the same for the blue edges.



19 **a** Copy the graph below and colour all the edges blue or red.



b Find a red or blue triangle in your coloured graph.

c Suppose you use a graph with coloured edges to represent whether dinner guests have met before or not: blue edges represent people who have met before, and red edges represent people who have not met before. If you are hosting a dinner party and want to ensure that either three people will all have met before, or three people will have never met before, what is the smallest number of people you should invite? Hint: Compare your answers to questions **18** and **19b**.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Networks

Topic quiz
10A

10B Planar graphs

Learning intentions

By the end of this topic you will be able to ...

- ✔ identify planar graphs
- ✔ identify non-planar graphs
- ✔ use Euler's formula for connected planar graphs.

Planar graphs

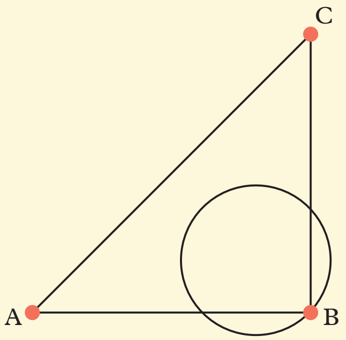
Key content video

- A **planar graph** is a graph that can be drawn in two dimensions so that no two edges of the graph cross.
  - → If a graph is drawn with edges crossing but can be redrawn without edges crossing then it is a planar graph.
  - → A **non-planar graph** is a graph that cannot be drawn in two dimensions without some edges crossing.
- A drawing of a planar graph without edges crossing divides the graph into regions, called **faces**, bordered by edges. There is always one infinite face, surrounding the graph.
- The number of vertices (v), the number of edges (e) and the number of faces (f) in a connected planar graph are related by **Euler's formula**:

  $v - e + f = 2.$
- The **degree** of a face is the number of edges along its border.
- The handshaking lemma (faces version) states that in every planar graph the sum of face degrees is twice the number of edges.
  - → Intuitively, both sides of an edge are counted in the overall sum of face degrees.
- Two edges with the same endpoints are **parallel**.
- A **simple graph** is a graph without loops or parallel edges.
- For a simple, connected graph with $v \geq 3$ vertices and e edges, if $e > 3v - 6$ then the graph is non-planar.
  - → If $e \leq 3v - 6$ then the graph may or may not be planar.

Example 10B.1 Determining whether a graph is planar

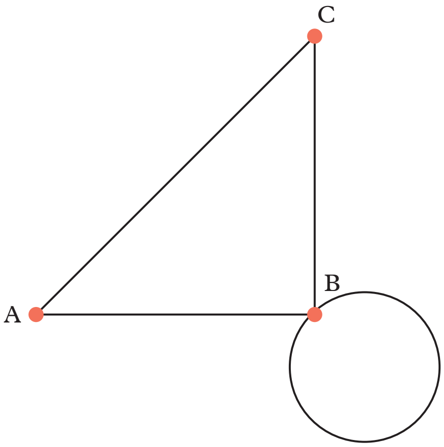
Try to redraw the following graph without crossing edges, and hence determine whether it is planar or non-planar.



THINK

If the graph can be redrawn without crossing edges, it is a planar graph.

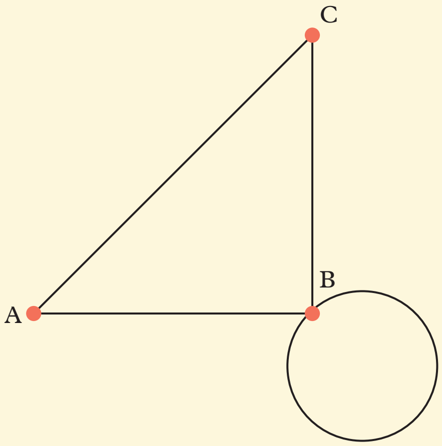
WRITE



The graph can be redrawn without crossing edges, so it is a planar graph.

Example 10B.2 Using Euler's formula to find the number of edges

Consider the connected planar graph from Example 10B.1 (reproduced here).



a Count the number of vertices in the graph.

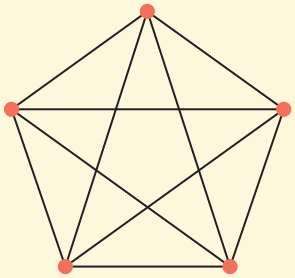
b Count the number of faces in the graph.

c Use Euler's formula (instead of counting) to find the number of edges in the graph.

THINK	**WRITE**
a There are three vertices: A, B and C.	**a** $v = 3$
b There are three faces: inside the triangle, inside the circle, and outside both shapes.	**b** $f = 3$
c Euler's formula states $v - e + f = 2$. Substitute in the values for v and f and solve for e.	**c** $3 - e + 3 = 2$ $e = 4$

Example 10B.3 Using $e > 3v - 6$ to prove a graph is non-planar

Consider this simple and connected graph.



Use the number of edges and vertices to prove this graph is non-planar.

THINK	WRITE
1 Count the number of vertices in the graph.	$v = 5$
2 Count the number of edges in the graph.	$e = 10$
3 If the number of edges in the graph exceeds $3v - 6$, then the graph has too many edges and cannot be planar.	$3v - 6 = 9$ As $e > 3v - 6$ $(10 > 9)$, the graph is non-planar.

Helpful hints

- ✔ $3v - 6$ is an upper limit on the number of edges a simple, connected, planar graph can have. If the number of edges exceeds this limit, then any drawing of the graph must have some edges crossing and the graph cannot be planar. If the number of edges does not exceed this limit, we still don't know if the graph is planar or not.
- ✔ The handshaking lemma (faces version) is true for connected and disconnected planar graphs, whereas Euler's formula only applies to connected planar graphs.

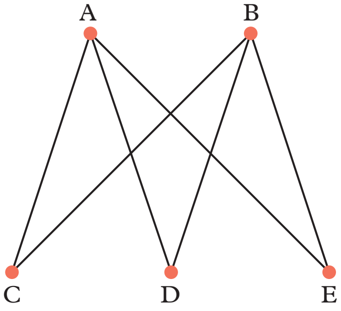
ANS p781

Exercise 10B Planar graphs

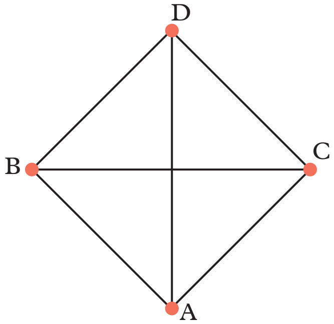
△ 1–5, 9, 11, 14, 17 ■ 1–11, 13, 15, 17 ⬠ 1–8, 10–13, 15–17

10B.1 **1** Try to redraw the following graphs without edges crossing, and hence determine whether they are planar or non-planar.

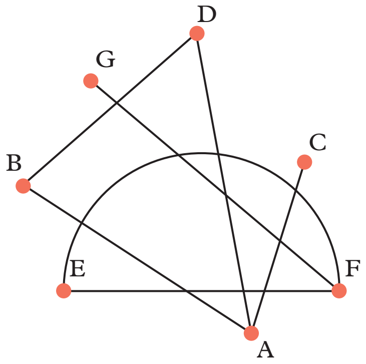
a



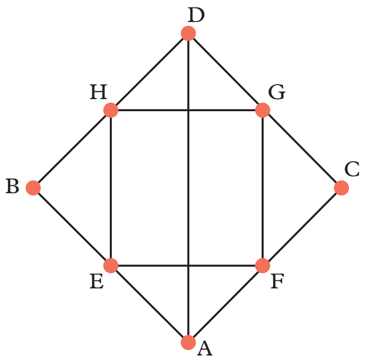
b



c



d

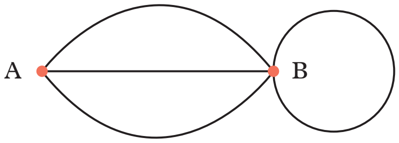


10B.2 2 Consider the connected graph on the right.

a Count the number of vertices in the graph.

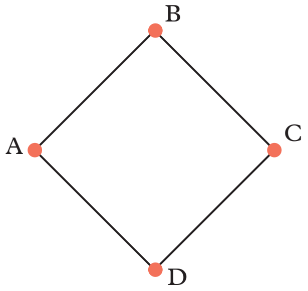
b Count the number of faces in the graph.

c Use Euler's formula (instead of counting) to find the number of edges in the graph.

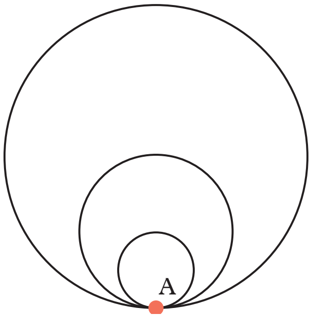


3 Count the number of edges, faces and vertices in the graphs below and show Euler's formula is satisfied.

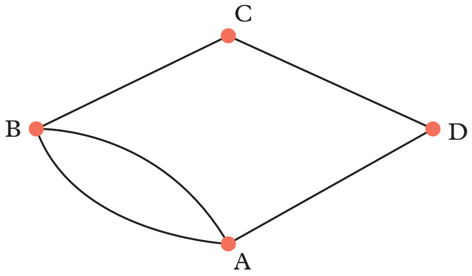
a



b



c



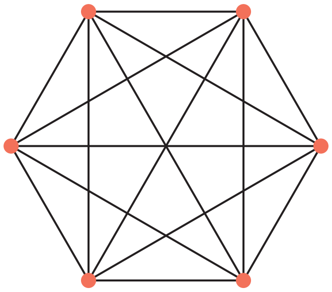
4 For the graphs in question **3**, verify that the sum of face degrees is twice the number of edges.

5 Which graphs from question **3** are simple?

6 A connected planar graph has 2 vertices and 4 faces. How many edges does it have?

7 Why does the graph from 10A question **1c** not satisfy Euler's formula?

10B.3 8 Consider this simple and connected graph.



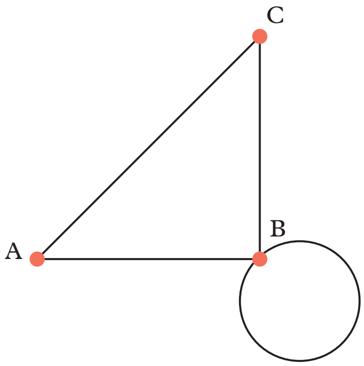
Use the number of edges and vertices to prove this graph is non-planar.

9 Consider this graph from Example 10B.2, which has 4 edges.

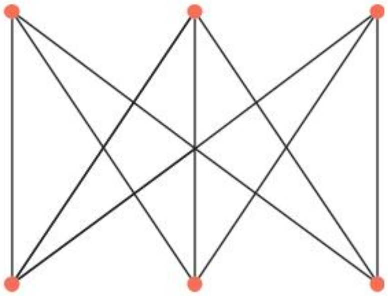
a State the degree of each face.

b Find the sum of the face degrees.

c How is the sum of the face degrees related to the number of edges?



10 If there are *fewer* than $3v - 6$ edges, a graph might be planar or non-planar.
For example, consider the following non-planar graph.



a Count the number of edges and vertices.

b Is this graph simple?

c Is this graph connected?

d Show that this non-planar graph has fewer than $3v - 6$ edges.

11 Two houses need to be connected to three underground utilities: power, water and internet. Is it possible to make the connections without utility lines crossing?

12 Repeat question **11** with three houses instead of two.

13 A simple, connected, planar graph has 3 vertices.

a What is the largest number of possible edges?

b Draw this graph.

14 The vertex degrees of a connected planar graph are 5, 3 and 2. How many faces does the graph have?
Hint: determine the number of edges and then use Euler's formula or draw the graph.

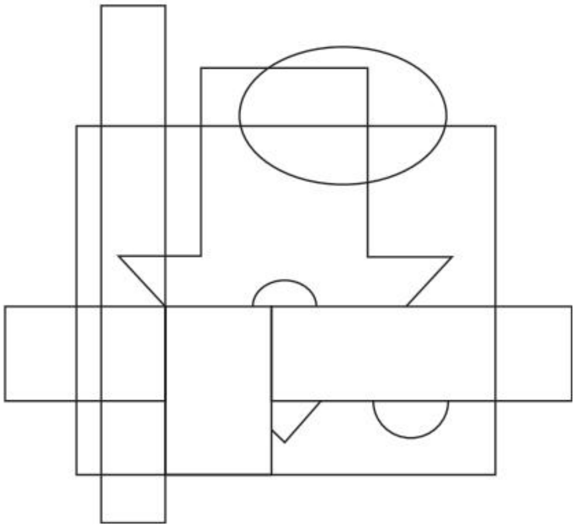
15 Can a connected, simple, planar graph have 5 faces and 4 vertices? Explain your answer.

16 a What does a face of degree 1 look like?

b What does a face of degree 2 look like?

c Given that a simple graph has no loops or parallel edges, what is the smallest face degree that a simple graph can have?

17 The four-colour theorem states only four colours are needed to colour a planar graph without loops so that neighbouring region have different colours. Colour in this planar graph using only four colours. Remember to colour the outside face.



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Interactive skillsheet
Planar graphs

BLM
Four-colour theorem

Topic quiz
10B

10C Platonic solids

Learning intentions

By the end of this topic you will be able to ...

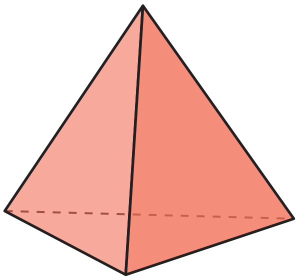
✔ identify the five Platonic solids

✔ think of the Platonic solids as planar graphs.

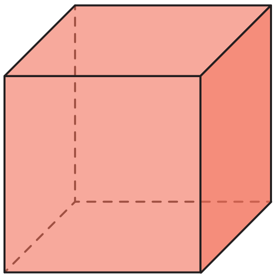
Platonic solids

Key content video

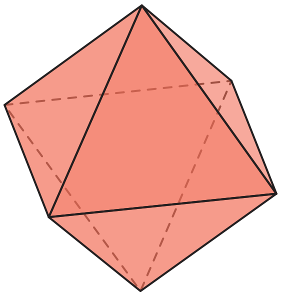
- The **Platonic solids** get their name from the famous philosopher Plato, who associated them with the four elements: fire, earth, air, water. A fifth Platonic solid was associated with the heavens and constellations. The Platonic solids are:



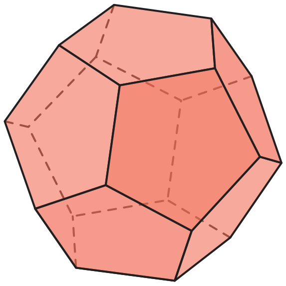
Tetrahedron (fire)



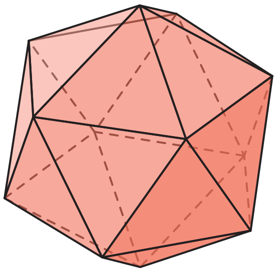
Cube (earth)



Octahedron (air)



Dodecahedron (heavens)

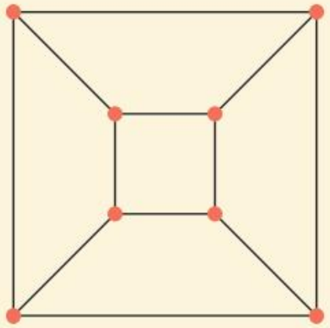


Icosahedron (water)

- A Platonic solid is a convex three-dimensional shape where each face is a polygon with n equal sides, and where m faces meet at each corner.
  - → Each face is identical in shape and size.
- There are only five Platonic solids. This was postulated by Euclid in approximately 300 BCE.
- Each Platonic solid can be viewed as a planar graph.
  - → Imagine a Platonic solid with vertices at each corner and edges around each face. Now puncture a hole in one of the faces and peer through that hole. You will see a simple, connected, planar graph.

Example 10C.1 Viewing Platonic solids as planar graphs

Imagine that we puncture a hole in one of the faces of a Platonic solid. Peer through that hole into the shape. Each Platonic solid will appear as a simple, connected, planar graph. We represent each side of the face with an edge, and the point at which faces meet with a vertex. If you peer inside a cube you will see something like this:



Use the handshaking lemma (faces version) to calculate the number of edges on the cube.

THINK	**WRITE**
1 The graph is planar, so the handshaking lemma (faces version) gives the number of edges.	
2 A cube has six faces, and each face is bordered by 4 edges and therefore has a degree of 4.	The sum of face degrees is $4 + 4 + 4 + 4 + 4 + 4 = 24$.
3 Half of the sum of face degrees is 12, which is the number of edges.	Therefore, there are $\frac{24}{2} = 12$ edges.

Example 10C.2 Using Euler's formula to find the number of vertices

Use Euler's formula to find the number of vertices in the graph from Example 10C.1.

THINK	**WRITE**
1 From the previous example, we know the number of faces and edges.	$f = 6, e = 12$
2 The graph is connected and planar, so Euler's formula holds.	$v - e + f = 2$ $v - 12 + 6 = 2$
3 Solve Euler's formula for v.	$v = 2 + 12 - 6$ $v = 8$

Helpful hints

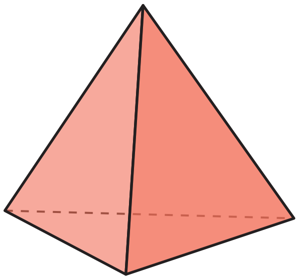
✔ Each face of the cube is a square, so each face has degree 4. This makes it easy to calculate the sum of face degrees as 4 times the number of faces. This is useful when applying the handshaking lemma (faces version).

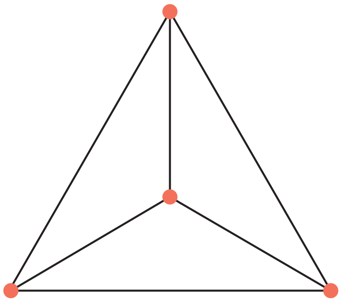
ANS p782

Exercise 10C Platonic solids

△ 1–9, 13 □ 1, 2, 5–12, 14–16 ⬠ 1, 3, 5, 9–12, 14, 16, 17

10C.1 **1** Peer inside the tetrahedron and you will see a simple, connected, planar graph with four faces.





Use the handshaking lemma (faces version) to calculate the number of edges on the tetrahedron.

2 Label the following statements as true or false.

a Every Platonic solid corresponds to a planar graph.

b Every planar graph corresponds to a Platonic solid.

3 **a** Which Platonic solids have triangular (3-sided) faces?

b Which have square (4-sided) faces?

c Which have pentagonal (5-sided) faces?

4 **a** Which Platonic solids have 3 faces meeting at each corner?

b Which have 4 faces meeting at each corner?

c Which have 5 faces meeting at each corner?

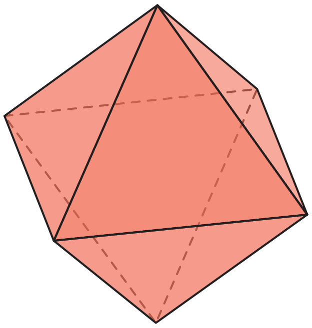
5 The game of Dungeons & Dragons has five kinds of dice known as d4, d6, d8, d12 and d20. The number after 'd' indicates the number of faces the die has. Match these dice with the Platonic solids.

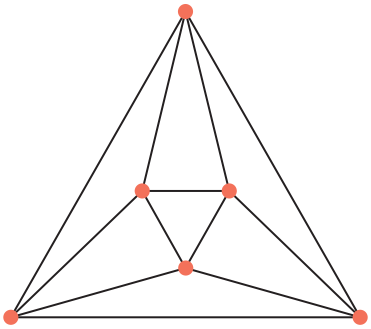
6 If the faces of a Platonic solid have degree 3, what solid could it be?

7 A connected planar graph has three faces. Each face has degree 4. How many edges does it have?

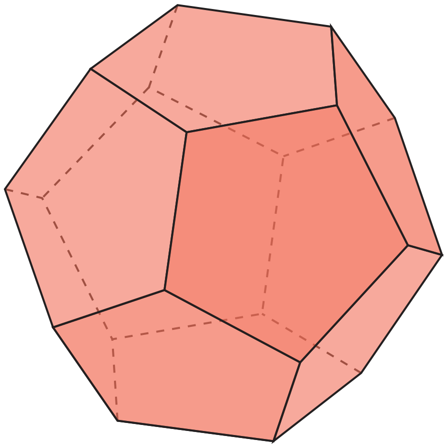
10C.2 **8** Use Euler's formula to find the number of vertices in the graph from question **7**.

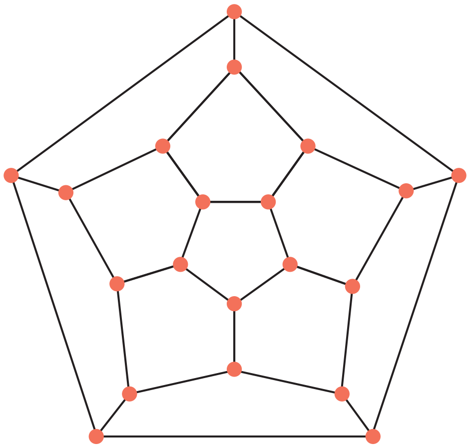
9 Repeat question **1** for the octahedron, given that it has 8 faces.



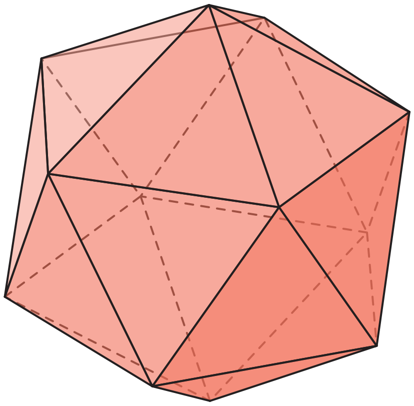


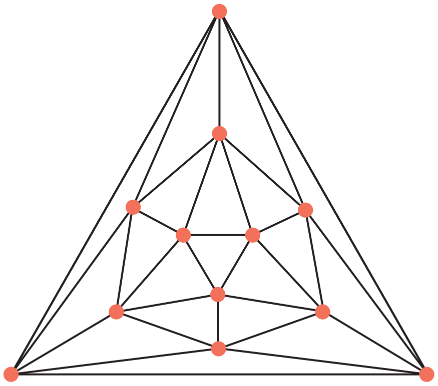
10 Repeat question **1** for the dodecahedron, given that it has 12 faces.





11 Repeat question **1** for the icosahedron, given that it has 20 faces.





12 Use Euler's formula and your answers to questions **1**, **9**, **10** and **11** to complete the following table.

Platonic solid	**Tetrahedron**	**Cube**	**Octahedron**	**Dodecahedron**	**Icosahedron**
Vertices		8			
Faces	4	6	8	12	20
Edges		12			

13 For the cube:

a state the sequence of vertex degrees

b state the sequence of face degrees

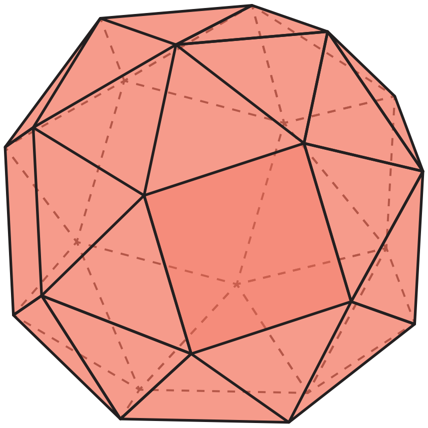
c show both sequences sum to 24.

14 One unique feature of the Platonic solids is that each face is identical. Other solids can also be created, but the faces will be different. For example, the snub cube is a non-Platonic solid with two kinds of faces. It has 6 square faces and 32 triangular faces.

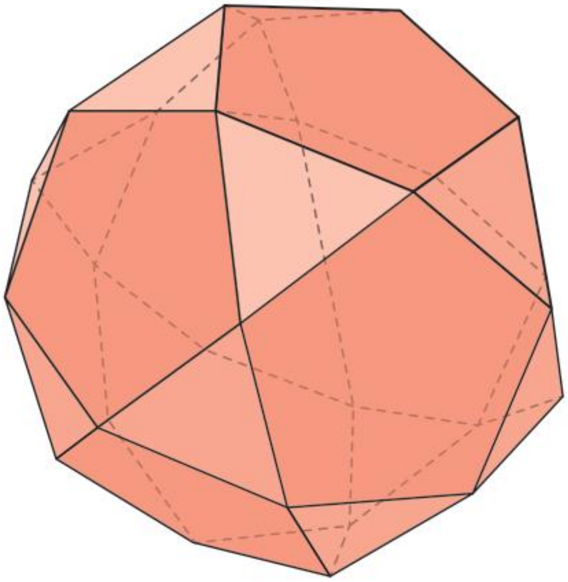
a What is the sum of the face degrees?

b How many edges does it have?

c How many vertices does it have?



15 Repeat question **14** for the icosidodecahedron, given it has 20 triangular faces and 12 pentagonal faces.



16 Each Platonic solid can be imagined as a planar graph with v vertices, e edges and f faces. Earlier in this section we used the handshaking lemma to show if each vertex of a Platonic solid has degree $m \geq 3$ then:

$2 \times e$ = sum of vertex degrees $= m \times v$.

Similarly, if each face of the Platonic solid is a polygon with $n \geq 3$ equal sides then:

$2 \times e$ = sum of face degrees $= n \times f$.

Putting this into Euler's formula, and rearranging, gives:

$$\frac{1}{e} = \frac{1}{n} + \frac{1}{m} - \frac{1}{2}$$

a Fill out the missing entries in the following table with the value of $\frac{1}{n} + \frac{1}{m} - \frac{1}{2}$.

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$		
$m = 4$			
$m = 5$			$\frac{1}{5} + \frac{1}{5} - \frac{1}{2} = -\frac{1}{10}$

b Which entries in this table have a positive value for $\frac{1}{e}$? How do these entries relate to the Platonic solids?

c Is it possible to have a graph where $\frac{1}{e}$ is negative?

d Does this means Euclid was right and our five shapes are the only Platonic solids?

17 Why don't we need to consider $m \geq 6$ in the previous question?
Hint: This would force $n < 3$; compare with 10B question **16**.

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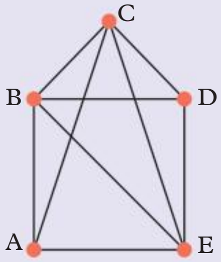
Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.

10A **1** Fill out the table of vertex degrees for the network below

Vertex	A	B	C	D	E
Degree					



10A **2** Is the network in question **1** connected?

10A **3** Explain why there is no network with a vertex degree sequence 4, 3, 2?

10B **4** Is the network in question **1** simple?

10B **5** Is the network in question **1** planar?

10B **6** Consider a simple, connected network of 6 vertices where each vertex has degree 5.

- **a** How many edges does this graph have?
- **b** Is this graph planar?

10B **7** Draw a network with 6 vertices, where each vertex has degree 3 and the graph is:

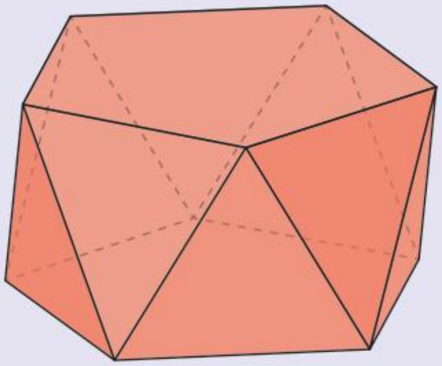
- **a** planar
- **b** non-planar.

10B **8** Suppose the network in question **1** is planar. Using Euler's formula, how many faces should it have?

10C **9** In Dungeons & Dragons d4 represents a die with 4 sides and d20 represents a die with 20 sides. A chaotic goblin wizard taunts you and you take two d4 emotional damage.

- **a** What is the maximum possible total of the numbers shown on two d4 dice?
- **b** What is the smallest amount of emotional damage you can take?
- **c** You attempt a witty retort to the chaotic goblin wizard; however goblins are thick-skinned creatures. You roll a d20 and if the die shows a score of 13 or more your retort is successful. Are you more likely to succeed or to fail?

10C **10** A pentagonal antiprism has two pentagonal faces and ten triangular faces.



- **a** What is the sum of the face degrees?
- **b** How many edges does it have?
- **c** How many vertices does it have?

10D Special types of walks

Learning intentions

By the end of this topic, you will be able to ...

✔ classify walks in a graph/network

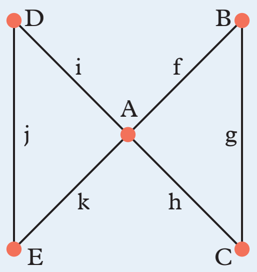
✔ know when a trail or circuit is Eulerian

✔ determine when Eulerian trails or circuits exist.

Trails, circuits, paths and cycles

Key content video

- Recall from section 10A that a walk is a sequence of vertices and the edges between them.
  - → A **trail** is a walk where every edge is different. For example, A, k, E, j, D, i, A, h, C is a trail which contains the vertex A twice.
  - → A **path** is a trail where every vertex is different. For example, A, k, E, j, D is a path, since every edge and vertex are distinct.
- There are also special names for walks that contain at least one edge and which return to where they started.
  - → A **closed walk** is a walk with at least one edge, that starts and ends at the same vertex.
  - → A **circuit** is a closed walk where every edge is different. For example, A, k, E, j, D, i, A, h, C, g, B, f, A is a circuit.
  - → A **cycle** is a circuit where every vertex is different, except for the start/end vertex. For example, A, k, E, j, D, i, A is a cycle since only the start/end vertex is repeated.



Type of walk	Repeated edges allowed	Repeated vertices allowed	Must start and finish at same vertex
Walk	Yes	Yes	No
Closed walk	Yes	Yes	Yes
Trail	No	Yes	No
Circuit	No	Yes	Yes
Path	No	No	No
Cycle	No	No*	Yes

* Excluding the start/end vertex.

Eulerian trails and circuits

- **Eulerian trails** and **Eulerian circuits** fill a graph exactly, in the sense that they contain every edge exactly once.
- There is an easy way to know when an Eulerian trial or circuit exists: just count the number of vertices with odd degree. Specifically, suppose your graph is connected and has n vertices with odd degree.
  - → If $n = 0$ then there must be an Eulerian circuit and the graph is **Eulerian**.
  - → If $n = 2$ then there must be an Eulerian trail and the graph is **semi-Eulerian**.
  - → Otherwise there is no Eulerian circuit or Eulerian trail, and the graph is not Eulerian.

Example 10D.1 Classifying walks

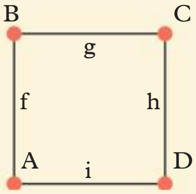
For the graph below, classify the following sequences as accurately as possible.

a A, i, D, i, A

b A, i, D, h, C

c A, i, D, i, A, f, B

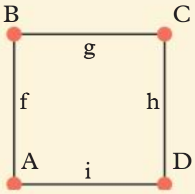
d A, i, C



THINK	WRITE
a This sequence re-uses edge i, so it must be a walk. As it starts and finishes with the same vertex (A), it is a closed walk.	**a** This is a closed walk.
b This sequence has no repeated edges or vertices. As it starts and finishes at different vertices, it is a path.	**b** This is a path.
c This sequence re-uses edge i, so it must be a walk. It starts and finishes at different vertices, so it is not closed.	**c** This is a walk.
d Edge i does not connect vertices A and C, so this walk is not possible.	**d** This is an invalid walk.

Example 10D.2 Finding an Eulerian circuit in a graph

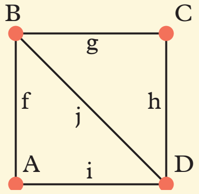
Find an Eulerian circuit, if one exists, in the network from Example 10D.1 (reproduced here).



THINK	WRITE
1 Check the number of vertices with odd degree to see if an Eulerian circuit exists. There are zero vertices with an odd degree, so there must be an Eulerian circuit.	
2 Start at A, then move clockwise around the network until you return to A. This is a circuit because it starts and ends at the A without re-using edges. This circuit is Eulerian because it contains every edge exactly once.	An Eulerian circuit for this network is A, f, B, g, C, h, D, i, A.

Example 10D.3 Finding an Eulerian trail in a graph

Find an Eulerian trail, if one exists, in the graph below.



THINK

1 Check the number of vertices with odd degree to see if an Eulerian trail exists. There are two vertices with an odd degree, so there must be an Eulerian trail.

2 Start at B, then move clockwise around the network until you return to B. Then use the diagonal edge to finish at D. This is a trail because edges are not re-used. This trail is Eulerian because it contains every edge exactly once.

WRITE

An Eulerian trail for this graph is B, g, C, h, D, i, A, f, B, j, D.

Helpful hints

- ✔ A visual way to understand walks is to place your finger on a vertex, then move your finger along the graph and finish at any vertex. Everything that your finger touches (i.e. both vertices and edges) is a walk. Trails, circuits, paths and cycles are just special kinds of walks.
- ✔ If a trail or circuit contains every edge exactly once, then it said to be Eulerian.
- ✔ If a graph has exactly 2 vertices of odd degree then these are the start and endpoints of the Eulerian trail.
- ✔ Any walk can be trimmed into a path by removing cycles.

ANS
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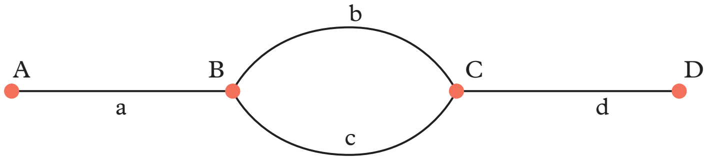
Exercise 10D Special types of walks

△ 1, 2, 4–11, 13, 15, 16 □ 2, 4–12, 14, 15 ⬠ 2–4, 6, 7, 9–12, 14, 17

1 Consider the following graph.



Remember from Section 10A that vertices are neighbours if they are connected by an edge.

a List the neighbours of B.

b Is A a neighbour of C?

10D.1 **2** For the graph in question **1**, classify the following sequences as accurately as possible.

- **a** A, a, B, c, C
- **b** A, a, B, c, C, b, B
- **c** A, b, B, a
- **d** B, c, C, b, B, c, C

3 Consider two different walks from A to D in the graph from question **1**.

One walk is: A, a, B, b, C, d, D.

The other walk is: A, a, B, c, C, d, D.

- **a** Write the walks as sequences of vertices only.
- **b** In this example, is it a good idea to abbreviate walks to just a sequence of vertices?

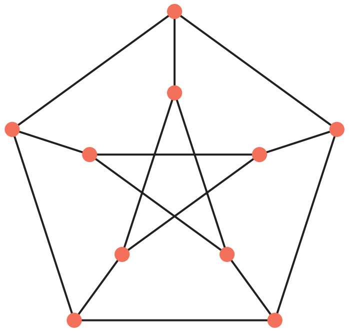
4 For the graph in question **1**, consider the walk A, a, B, b, C, c, B, a, A.

What kind of walk is this?

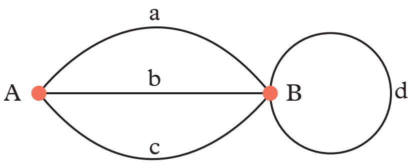
5 Find a circuit in the graph from question **1**.

10D.2 **6** Find a different Eulerian circuit for the graph in Example 10D.2.

7 Explain why the following graph is not Eulerian by counting the number of vertices with odd degree.

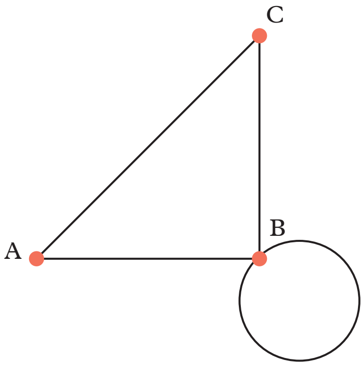


8 Explain why the following graph is semi-Eulerian by counting the number of vertices with odd degree.

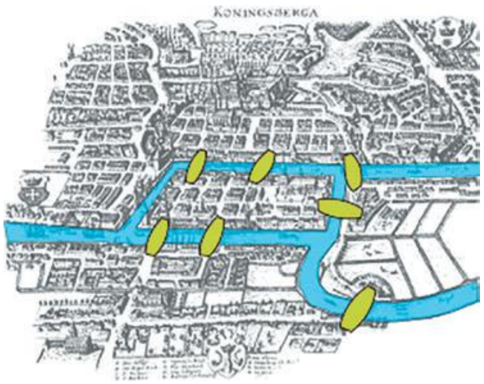


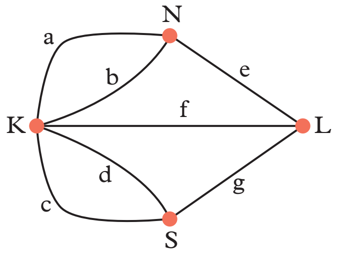
10D.3 **9** Find an Eulerian trail in the graph from question **8**.

10 Explain why the following graph is Eulerian by counting the number of vertices with odd degree.



11 The old city of Königsberg lies on the river Pregel. The city has seven bridges, as shown below. Work through this problem to see if there is there a walk through Königsberg which crosses every bridge exactly once.





We represent city regions with vertices: N for the North side of the river, S for the south side, K for Kneiphof Island and L for Lomse Island. The bridges are represented by edges, and we label them a, b, c, d, e, f and g. The network above shows the city regions as vertices and the bridges between them as edges. The question can be rephrased: does this graph have an Eulerian trail?

The answer to the seven bridges of Königsberg problem is far from obvious. In 1736 Leonhard Euler said the problem has nothing to do with how the edges are connected. The only thing that matters is the number of vertices with odd degree.

a Fill out the table of vertex degrees.

Vertex	N	K	S	L
Degree				

b How many vertices have odd degree?

c Is the network Eulerian, semi-Eulerian or not Eulerian?

d Is there a walk through Königsberg which crosses every bridge exactly once? Explain your answer.

12 The mayor of Königsberg wants to encourage tourism with a grand tour of the bridges. Tour groups must start and end outside the mayor's offices in the northern part of the city and traverse every bridge exactly once.

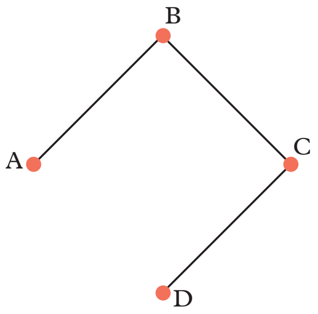
a The mayor will allow you to build one additional bridge. How can you explain to the mayor that the task is still impossible?

b The mayor really wants this to happen and has secured funding for two additional bridges, on the condition that both of them start from the northern side of the river. Is the task now possible? Explain your answer.

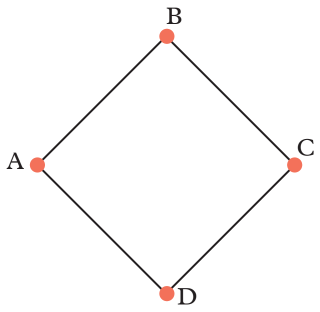
c The mayor is willing to budge a little and as well as the two additional bridges will allow the tour to finish in any location, provided it starts from the mayor's offices. Is the task now possible? Explain your answer.

13 Determine whether the following graphs are Eulerian, semi-Eulerian or not Eulerian by counting the number of vertices with odd degrees.

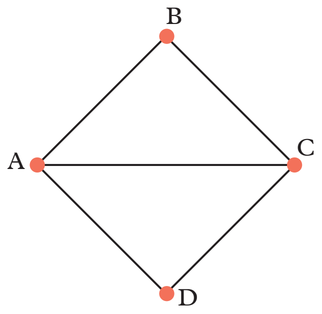
a



b



c



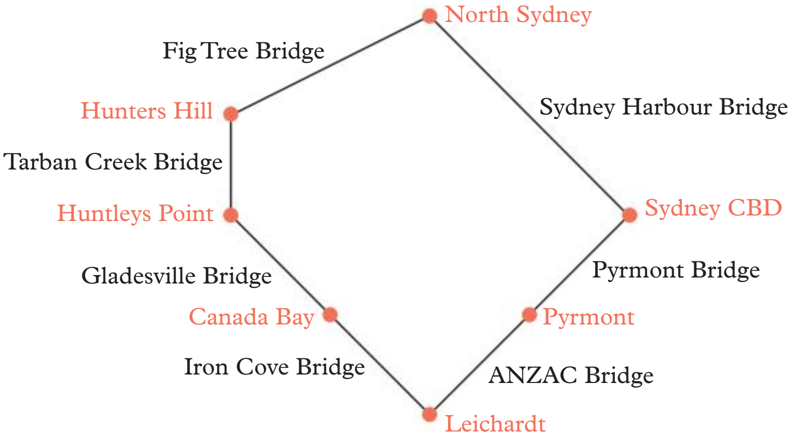
14 Which Platonic solid has an Eulerian circuit along its edges?

15 Design a garbage truck route that visits every street in these maps exactly once or explain why this task is impossible.

a

b

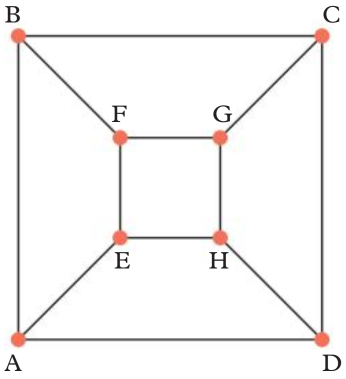
16 In 2017, the 'Seven Bridges Walk' toured some of the scenic bridges of Sydney.



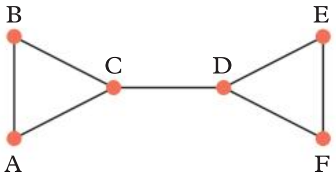
Count the vertex degrees, then state why it is possible to cross the seven bridges of Sydney without visiting the same bridge more than once.

17 A Hamiltonian cycle is a cycle that visits every vertex exactly once. Unlike Eulerian circuits, a simple way to determine if a graph has a Hamiltonian cycle has never been found.

a Find a Hamiltonian cycle for the cube.



b Is there a Hamiltonian cycle in the graph below? Explain your answer.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Eulerian trails and circuits

Topic quiz
10D

10E Applications of networks

Learning intentions

By the end of this topic, you will be able to …

✔ use directed edges in networks

✔ model authentic real-world situations using networks.

Directed networks

Key content video

- A **directed edge** is an edge that has a direction indicated by an arrow.
  - → For example, suppose person B is the mother of person A. We can represent this directional relationship with a directed edge. The arrow on the tip of the edge represents the direction of the 'mother of' relationship

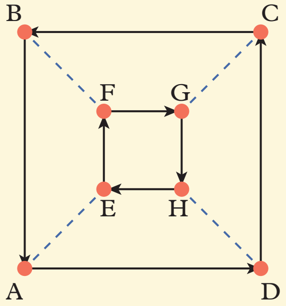
- A **directed network** is a network with directed edges.

Example 10E.1 Using networks to represent kin systems

The Warlpiri are a people whose country is found in Central Australia around the Tanami Desert. The following kin system is inspired by the Warlpiri people and has been simplified for the purposes of this example. Suppose everyone belongs to one of eight kin groups A, B, C, D, E, F, G or H, and that:

- Marriage is recommended between kin groups A and E, B and F, C and G, or D and H. Recommended marriages are represented by the dashed blue edges in the network below. Here and throughout we will simplify things by assuming all children are born from recommended marriages.
- The kin group of the mother determines the group of all her children.

	Kin group							
Mother	A	B	C	D	E	F	G	H
Children	D	A	B	C	F	G	H	E



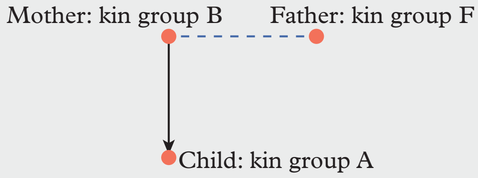
The directed edges in the network on the right represent the 'mother of' relation.

What is the kin group of the parents for children in kin group A?

THINK	WRITE
1 Follow the 'mother of' arrow backwards from A to find the kin group of their mother.	Their mother is in kin group B.
2 Once you know the kin group of their mother, follow the dashed line to find the recommended kin group of her partner.	Their father is in kin group F.

Helpful hints

✔ Draw a family tree, and then decorate that tree with kin groups.



ANS p783

Exercise 10E Applications of networks

△ **1–3, 6–9** □ **1, 3, 5, 10–13** ⬠ **1, 3–5, 10–12, 14**

UNDERSTANDING AND FLUENCY

Use the kin diagram in Example 10E.1 to answer questions **1** to **5**.

10E.1 **1** Suppose you are in kin group D.

a What is the kin group of your siblings?

b What is the recommended kin group of your spouse?

c What is the kin group of your sister's children?

d What is the kin group of your brother's children?

2 Repeat question **1**, supposing that you are in kin group H.

3 Complete this paternity table. Assume that children are only born from recommended marriages.

	Kin group							
Father	A	B	C	D	E	F	G	H
Children	F					A		

4 Suppose you belong to kin group A. What is the kin group of:

a your mother's sister's children

b your mother's brother's children

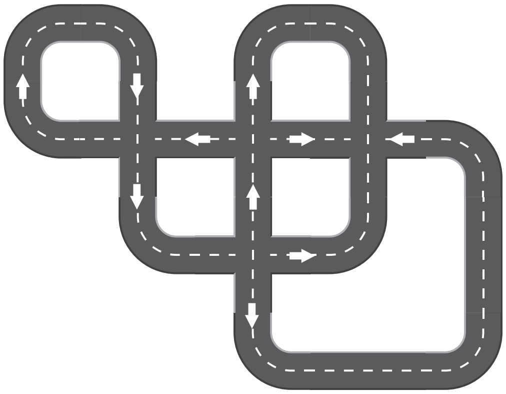
c your father's sister's children

d your father's brother's children?

5 Complete the following table.

	Kin group							
You	A	B	C	D	E	F	G	H
Cousins	A or G						G or A	

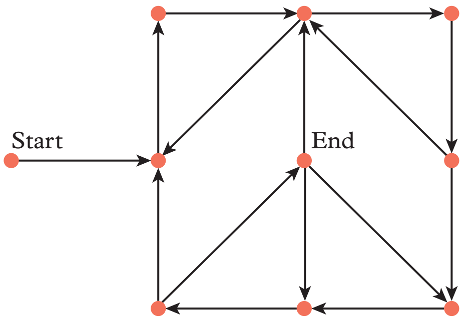
6 Draw this system of one-way streets as a directed network using labelled vertices to represent intersections and directed edges to represent one-way streets.



STANDARD

UNDERSTANDING AND FLUENCY

7 Find your way to the end of the labyrinth. You can only travel along edges in the direction of the arrows.



a How many vertices did you pass on your way?

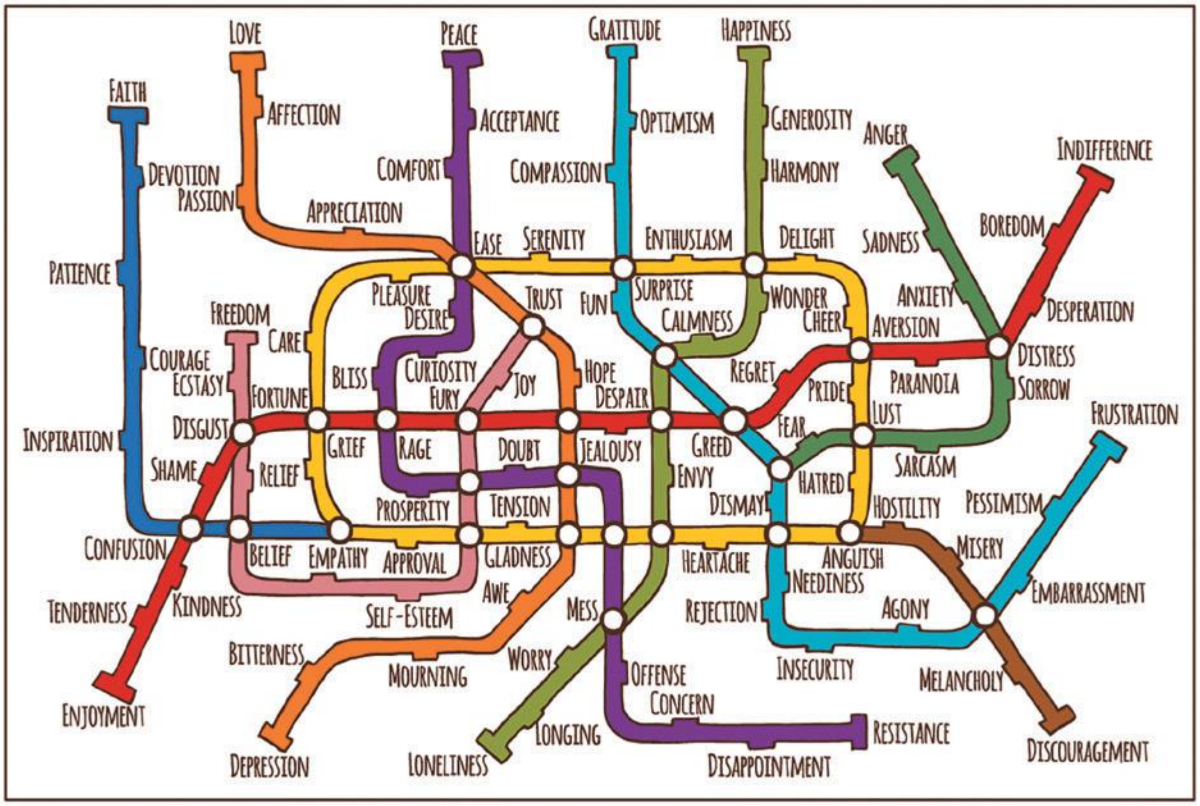
b How many edges did you travel through?

8 In the underground map of feelings, how many stops does it take to get from:

a Comfort (purple line) to Concern (purple line).

b Heartache (yellow line) to Serenity (yellow line). Is it faster to pass through Pride or Tension?

c Indifference (red line), changing at Grief, to Empathy (yellow line).



Use the kin diagram in Example 10E.1 to answer questions **9** to **12**.

9 A child is born into kin group A. What is the kin group of the:

a maternal grandmother

b maternal grandfather

c paternal grandmother

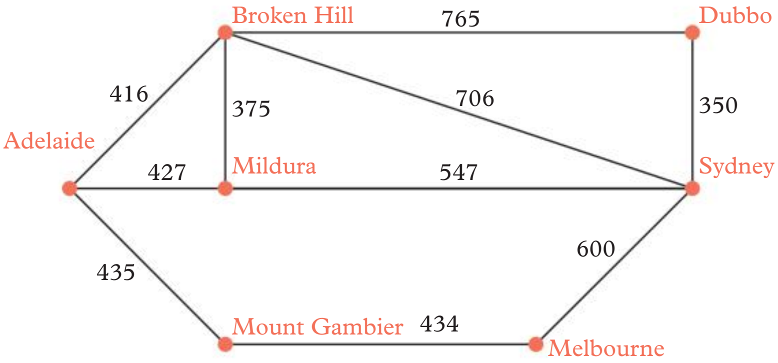
d paternal grandfather?

10 Are you always in the same kin group as your paternal grandfather?
Hint: Use the table from question **3**.

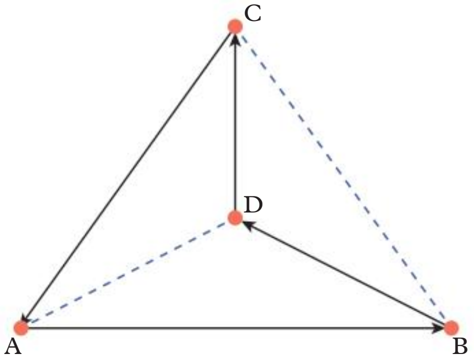
11 Are you ever recommended to marry your cousin?
Hint: Use the table from question **5**.

12 You are in kin group A and your mother re-marries. What is the kin group of your step-sisters and step-brothers?

13 Consider the network of regional flights. Each edge has a number that represents the cost (in \$) of each route. What is the cheapest route from Dubbo to Mount Gambier?



14 Suppose we define a new kin system based on Example 10E.1. Instead of using a cube we use a tetrahedron with just four kin groups.



a Fill out the maternity table.

	Kin group			
Mother	A	B	C	D
Children	B			

b Fill out the paternity table.

	Kin group			
Father	A	B	C	D
Children	C			

c Can any other Platonic solids be used to define kin group relations?

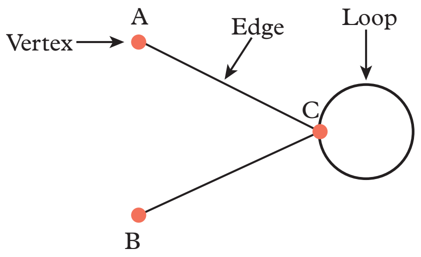
Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Directed networks

Topic quiz
10E

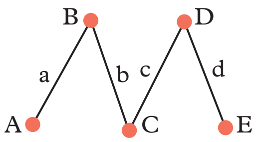
Chapter summary

Networks

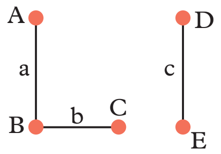


Walks

- Walk: sequence of vertices and the edges between them.
- Connected graph: a walk exists between every pair of vertices.



Connected graph



Disconnected graph

Planar graphs

- A graph that can be drawn without edges crossing.
- There is always one infinite 'outside' face.
- Euler's formula for connected planar graphs:
  $v - e + f = 2$

Handshaking lemma

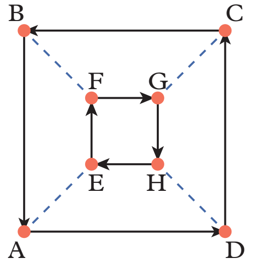
- Sum of vertex degrees = 2 × number of edges
- Sum of face degrees = 2 × number of edges

Simple graphs

- No loops or parallel edges.
- Simple, connected graph with $v \geq 3$ vertices and e edges:
  → If $e > 3v - 6$ graph is non-planar.
  → If $e \leq 3v - 6$ graph may or may not be planar.

Directed networks

- A directed edge has a direction indicated by an arrow.
- A directed network is a network with directed edges.

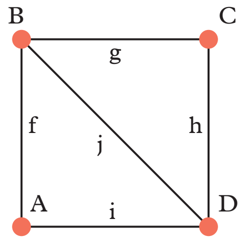


Trials and circuits

Type of walk	Repeated edges allowed	Repeated vertices allowed	Must start and finish at same vertex
Walk	Yes	Yes	No
Closed walk	Yes	Yes	Yes
Trail	No	Yes	No
Circuit	No	Yes	Yes
Path	No	No	No
Cycle	No	No*	Yes

* Excluding the start/end vertex.

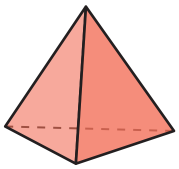
- Eulerian trails and Eulerian circuits contain every edge exactly once.
- If a connected graph has n vertices with odd degree:
  → if $n = 0$, there is an Eulerian circuit and graph is Eulerian
  → if $n = 2$, there is an Eulerian trail and graph is semi-Eulerian
  → otherwise no Eulerian circuit or trail and graph is not Eulerian.

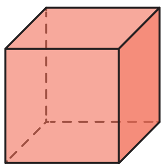


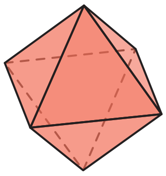
There are two vertices of odd degree, so an Eulerian trail exists.

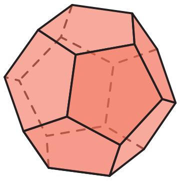
Platonic solids

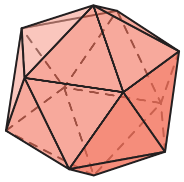
- Convex, three-dimensional shapes where each face is a regular polygon.


Tetrahedron


Cube


Octahedron


Dodecahedron


Icosahedron

ANS p783

Chapter review

Chapter review quiz
Assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- circuit
- closed walk
- connected
- cycle
- degree
- directed edge
- directed network
- edge
- Euler's formula
- Eulerian
- Eulerian circuit
- Eulerian trail
- faces
- graph
- handshaking lemma
- loop
- neighbours
- network
- non-planar graph
- path
- planar graph
- Platonic solids
- semi-Eulerian
- simple graph
- trail
- vertices
- walk

1 Explain what makes a graph connected.

2 Name the five Platonic solids.

3 How can you determine the degree of a face?

4 What makes a circuit an Eulerian circuit?

5 Identify the key terms being referenced in each of these definitions.

a a walk without repeated edges or repeated vertices

b a graph that can be drawn without crossing edges

6 Using an example, provide a definition in your own words for the following key terms.

a cycle **b** simple graph

7 Complete the following sentences using words from the key term list.

a A graph is said to be ________ if there is a ________ between every pair of vertices.

b The ________ (faces version) states that in every planar graph the sum of face degrees is twice the number of ________.

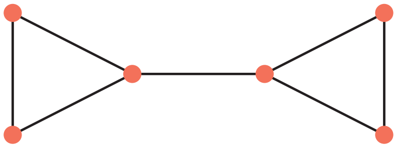
Multiple choice

10A **1** A graph has a vertex degree sequence of 3, 2, 1. How many edges does it have?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

10A **2** What is the largest vertex degree in this graph?

A 1
B 2
C 3
D 4
E 5

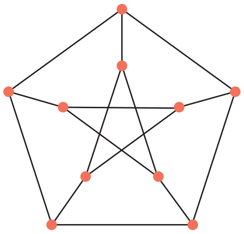


10B **3** What is the largest face degree in the graph from question **2**?

A 4 **B** 5 **C** 6 **D** 7 **E** 8

10B **4** What is the largest number of edges that you can remove from this network without disconnecting the graph?

A 3
B 4
C 5
D 6
E 7



10B **5** What is the smallest number of edges that must be removed from the graph in question **4** to disconnect the graph?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

10C **6** Which platonic solid has 12 faces?

A Tetrahedron **B** Cube **C** Octahedron **D** Dodecahedron **E** Icosahedron

10C **7** Which platonic solid has 12 vertices?

A Tetrahedron **B** Cube **C** Octahedron **D** Dodecahedron **E** Icosahedron

10C **8** Every vertex in a connected graph has an even degree. This means the graph is:

A not Eulerian **B** half-Eulerian **C** semi-Eulerian **D** Eulerian **E** Directed

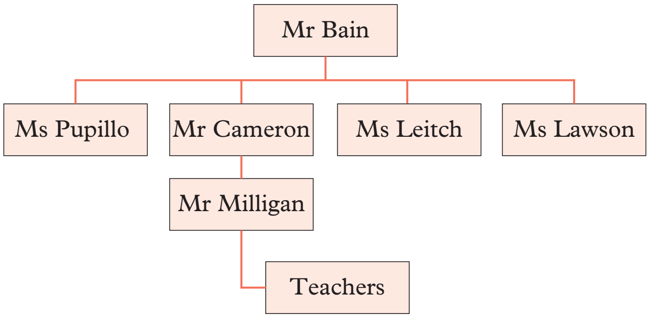
10D **9** Suppose three vertices in a graph have an odd degree, and the rest have an even degree. This graph is:

A not Eulerian **B** half-Eulerian **C** semi-Eulerian **D** Eulerian **E** Impossible

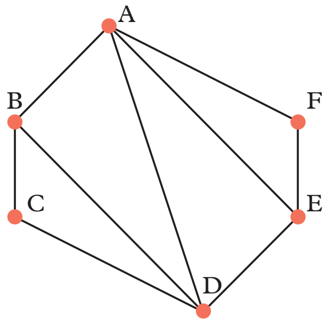
10E **10** The following organisational chart shows the hierarchy within Oban High School. A teacher reports a serious incident to Mr Milligan. How many steps will it take for this report to be handed up from Mr Milligan to Mr Bain?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Oban High School Organisation Chart



Short answer



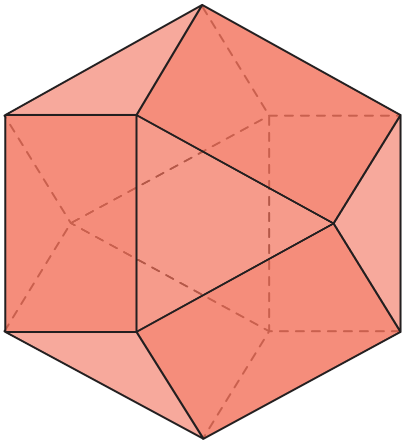
10A **1** Fill out the table of vertex degrees for the graph on the right.

Vertex	A	B	C	D	E	F
Degree						

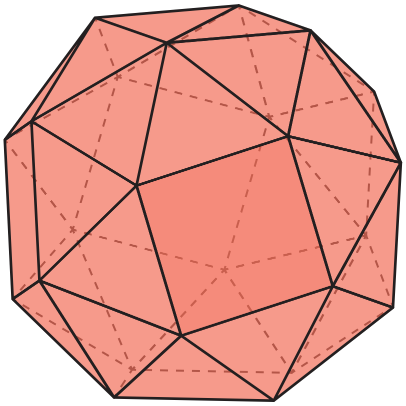
10B **2** The graph from question **1** is planar. How many faces does it have?

10B **3** Use Euler's formula to find the number of edges in the graph from question **1**.

10C **4** The cuboctahedron is related to the cube and octahedron. According to Archimedes, it may have been known to Plato. It has 6 square faces and 8 triangular faces. What is the sum of the face degrees?

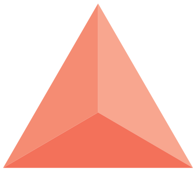
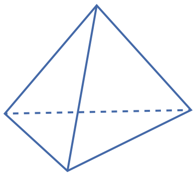
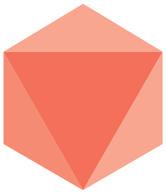
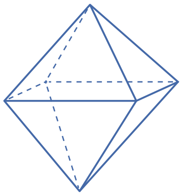
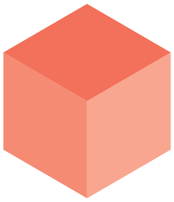
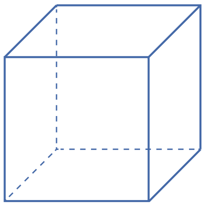
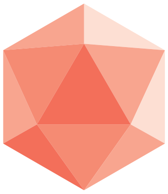
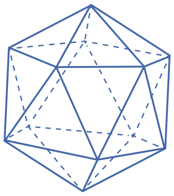
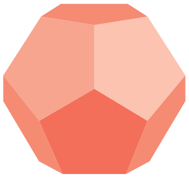
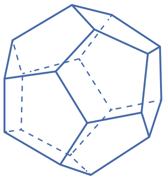


10C **5** The snub cube is a convex three-dimensional shape with 6 square faces and 32 triangular faces. Why is it not considered a Platonic solid?



10C **6** Why is it impossible for a planar graph to have exactly five faces, where each face has a degree of three?

10C **7** Fill out the table for the number of faces, vertices and edges of the Platonic solids.

Solid	Faces	Vertices	Edges
TETRAHEDRON  			
OCTAHEDRON  			
CUBE  			
ICOSAHEDRON  			
DODECAHEDRON  			

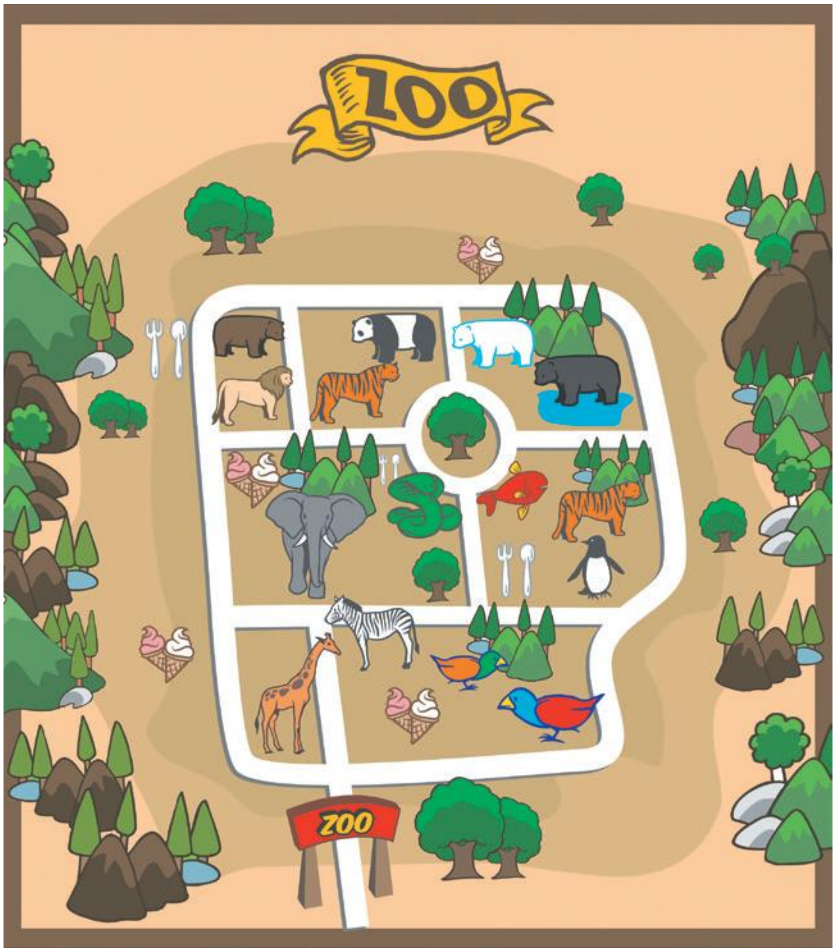
10D **8** Consider the following town map as a network of intersections (vertices) connected by roads (edges).

a How many vertices are in the network?

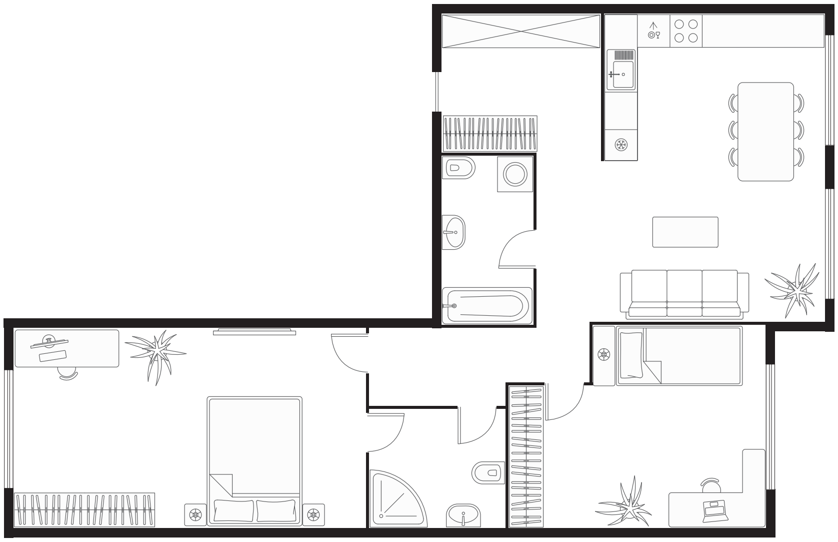
b What is the highest vertex degree?

c Can you design a bus route that visits every street exactly once?

10D **9** Why is it impossible to pass through every footpath in the zoo below exactly once?



10D **10** You have an eccentric uncle who can only go to sleep once they have passed through and closed every internal door in their apartment, shown below. If they start in the bathroom by the entrance, in which bedroom do they sleep at night?



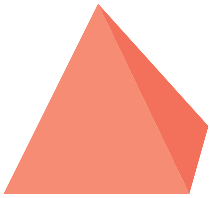
10E **11** By referring to the kin system in Example 10E.1, complete the table to show that you are always in the same kin group as your maternal great-great-grandmother (i.e. your mother's mother's mother's mother).

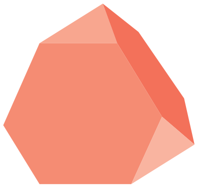
	Kin group							
You	A	B	C	D	E	F	G	H
Mother	B							
Grandmother	C							
Great-grandmother	D							
Great-great-grandmother	A							

Analysis

1 **a** Platonic solids can be truncated into different shapes by flattening the pointy bits. For example, the tetrahedron has faces of degree 3, but the truncated tetrahedron has faces of degree 3 or 6.

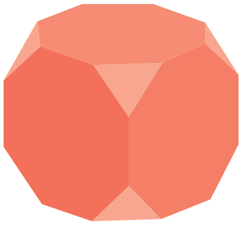
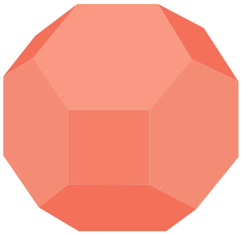
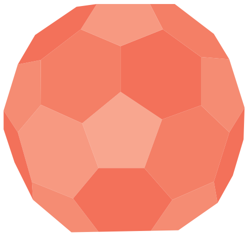
TETRAHEDRON





TRUNCATED TETRAHEDRON

Fill out the table of face degrees.

Shape	 TRUNCATED CUBE	 TRUNCATED OCTAHEDRON	 TRUNCATED ICOSAHEDRON
Face degrees			

b For the truncated tetrahedron:

  i How many faces have a degree of 3?

  ii How many faces have a degree of 6?

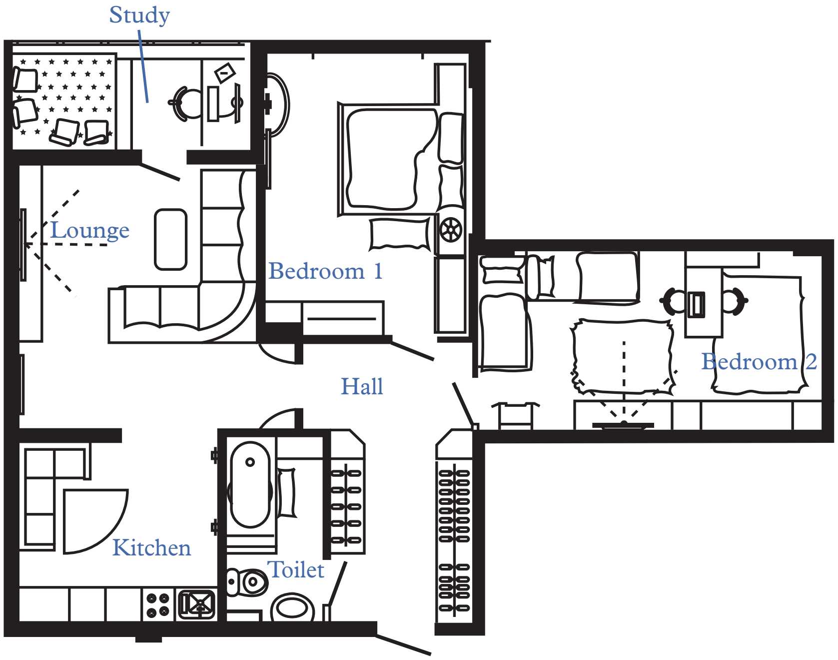
  iii What is the sum of all the face degrees?

  iv How many edges does it have?

  v Using Euler's formula or otherwise, state the number of vertices.

c Repeat **iii**, **iv** and **v** from part **b** for the truncated cube.

2 **a** Draw a network which represents the following floor plan. Use vertices to represent rooms, and represent rooms connected by a door with an edge.



b Can you walk through the house passing through every doorway exactly once? Why or why not?

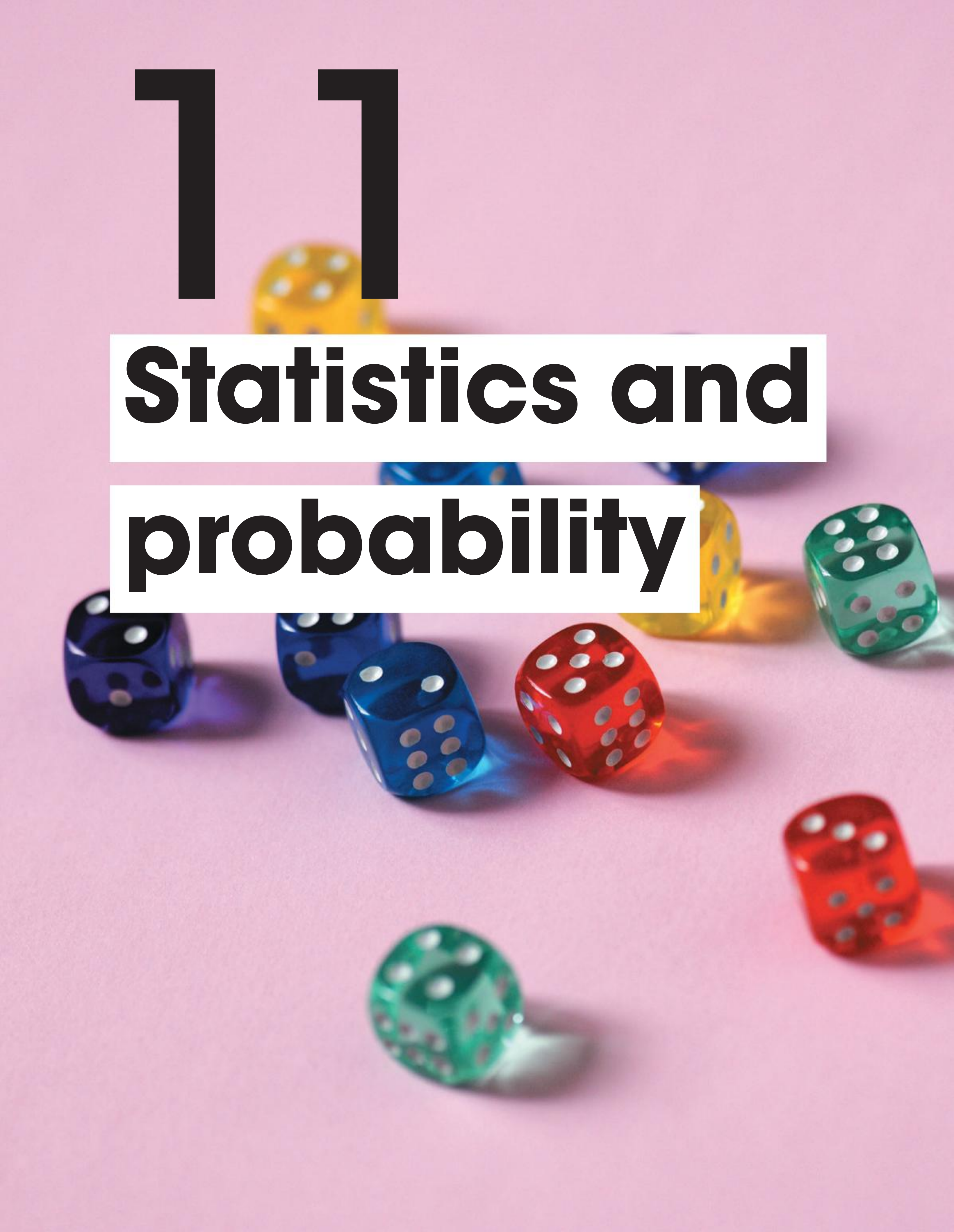
Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Draw networks to represent real-world situations ☐ Use terminology relating to networks/graphs ☐ Reason with a network/graph diagram	☐ Go back to Topic 10A Introduction to networks
☐ Identify planar graphs ☐ Identify non-planar graphs ☐ Use Euler's formula for connected planar graphs	☐ Go back to Topic 10B Planar graphs
☐ Identify the five Platonic solids ☐ Think of the Platonic solids as planar graphs	☐ Go back to Topic 10C Platonic solids
☐ Classify walks in a graph/network ☐ Know when a trail or circuit is Eulerian ☐ Determine when Eulerian trails or circuits exist	☐ Go back to Topic 10D Special types of walks
☐ Use directed edges in networks ☐ Model authentic real-world situations using networks	☐ Go back to Topic 10E Applications of networks

11

Statistics and probability



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Calculating theoretical probability
- ✔ Fractions, decimals and percentages
- ✔ Adding and subtracting fractions
- ✔ Multiplying and dividing fractions

Curriculum links

- Displays and interprets datasets involving bivariate data (MA5-DAT-C-02)
- Solves problems involving probabilities in multistage chance experiments and simulations (MA5-PRO-C-01)
- STN ADV Plans, conducts and reviews a statistical inquiry into a question of interest (MA5-DAT-P-01)
- ADV Solves problems involving Venn diagrams, 2-way tables and conditional probability (MA5-PRO-P-01)

11A Bivariate data and scatter plots

Learning intentions

By the end of this topic you will be able to ...

- ✔ interpret and draw scatter plots
- ✔ determine the strength and direction of a linear relationship from a scatter plot
- ✔ draw a line of best fit by eye.

Inter-year links
Year 9 8C Distributions of data

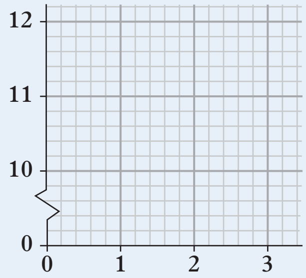
Bivariate data

Key content video

- **Bivariate data** contains two variables, where each data value in one variable is paired with a data value in the other variable. The purpose of gathering bivariate data is to investigate the relationship between the two variables.
  - → An association between two variables exists if as one variable changes in value, the other variable also changes in value.
  - → When investigating numerical bivariate data, we usually look at how changing one variable (the **independent variable**) affects the value of the other variable (the **dependent variable**).
- A **causal relationship** between two variables exists when a change in the independent variable directly causes a change in the dependent variable. For example, a change in the number of people in a household directly causes a change in the electricity usage of the household.
  - → Some pairs of variables may have an association without a causal relationship.

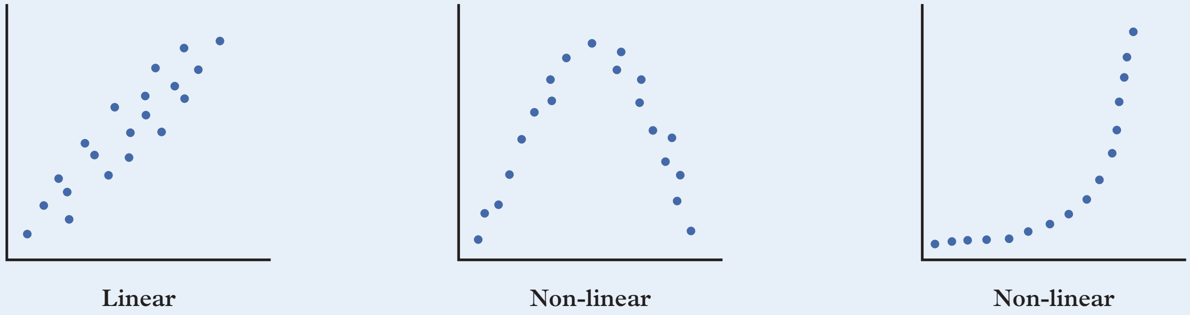
Scatter plots

- A **scatter plot** can be used to display numerical bivariate data.
  - → When plotting numerical bivariate data on a scatter plot the independent variable should be plotted on the horizontal axis and the dependent variable should be plotted on the vertical axis.
  - → If there is no obvious independent and dependent variables, either variable can be plotted on either axis.
- Scatter plots are usually drawn focusing on the area of the axes between which the data points lie. A break in the scale on the axes, denoted by a zigzag, indicates that the focus is on the axes a distance away from the origin.



Form

- When considering bivariate data, we need to determine whether the pattern of the data is either linear or non-linear. For a linear relationship the points can be approximated by a straight line.

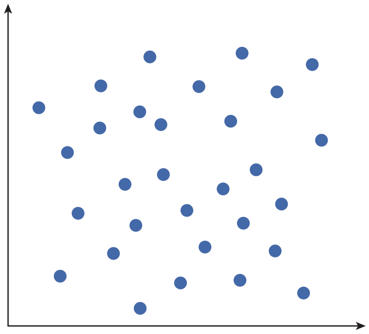
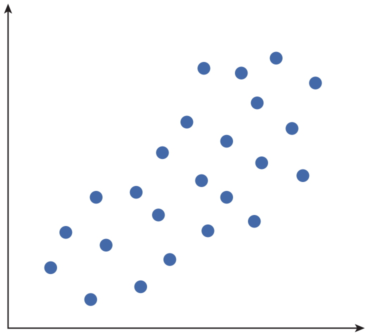
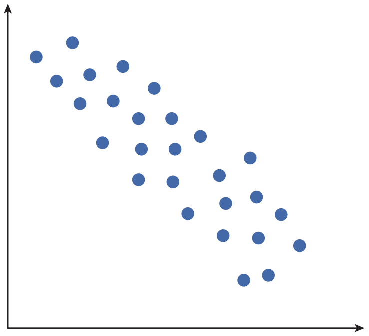
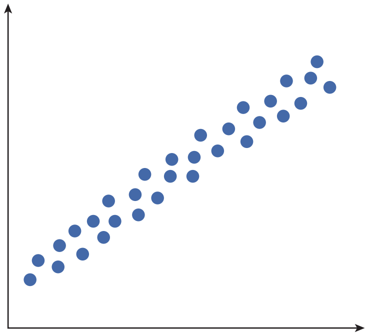


Direction

- If there is a **positive relationship** between two numerical variables, as one variable increases in value the other variable also increases in value.
- If there is a **negative relationship** between two numerical variables, as one variable increases in value the other variable decreases in value.

Strength

- The **correlation** between two numerical variables is a measure of the strength of the linear relationship between those variables. If the relationship is strong, the points will lie close to a straight line.

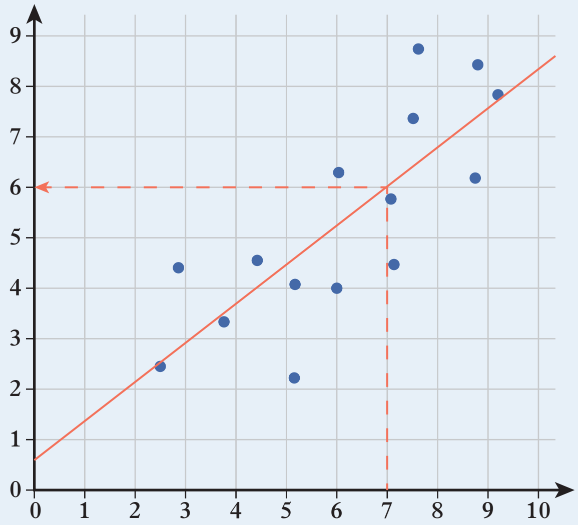
Sample scatter plot	**Description**	**Strength of relationship**	**Direction of relationship**
	no obvious pattern in the data	no correlation	not applicable
	a weak linear pattern in the data	weak	positive
	a clear linear pattern in the data	moderate	negative
	a strong linear pattern in the data	strong	positive

Lines of best fit by eye

- A **line of best fit** is a line that best represents the relationship between the two variables on a scatter plot.
- To draw a line of best fit by eye, draw a straight line on your scatter plot so that it lies as close to as many points as possible.
  - → There should be approximately the same number of points on each side of a line of best fit.

Using a line of best fit to make predictions

- A line of best fit can help predict the value of one variable when the other variable is known.
  - → As the strength of the correlation between two plotted variables increases, the predictions made using a line of best fit become more reliable.
  - → Predictions made within the original **range** of the data plotted (interpolation) are more reliable than predictions made outside the original range of data (extrapolation).
- To make a prediction using a line of best fit, locate the value on the axis you are predicting for, and draw a straight line (horizontal or vertical) to the line of best fit. From this point, draw another straight line (horizontal or vertical) to the other axis to get your predicted value.

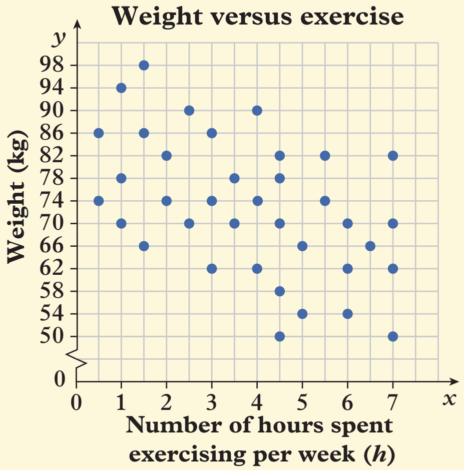


Example 11A.1 Describing the relationship in a scatter plot

Describe the relationship shown in the scatter plot on the right:

a in terms of its strength and direction

b in the context of the data it represents.



THINK	WRITE
a While there is only a weak linear correlation in the data the dependent variable decreases as the independent variable increases.	**a** The scatter plot shows a weak, negative linear relationship between the two variables.
b Describe the relationship shown in the context of what it tells you about the data.	**b** As the number of hours spent exercising increases, weight generally decreases.

Example 11A.2 Drawing a scatter plot

Draw a scatter plot for this data set.

Hours of sleep per night	8	6	7	9	10	11	8	7	6	5	4	8	7	6	10
Exam marks	72	61	82	92	68	72	49	68	61	56	32	62	41	45	55

THINK

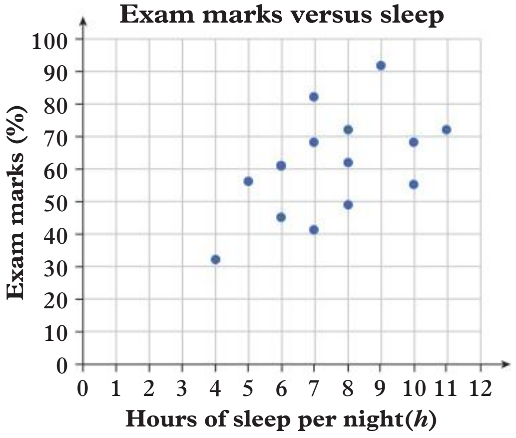
1 Determine which of the two variables is the independent variable and which variable is the dependent variable. A change in the number of hours of sleep per night could affect the exam marks, so the number of hours of sleep per night is the independent variable.

2 Use the horizontal axis for the independent variable (hours of sleep per night) and the vertical axis for the dependent variable (exam mark).

3 Identify the smallest and largest value for each variable and mark scales on the axes to cover these values. (Show breaks in the axes if the values on either axis do not begin at zero.)

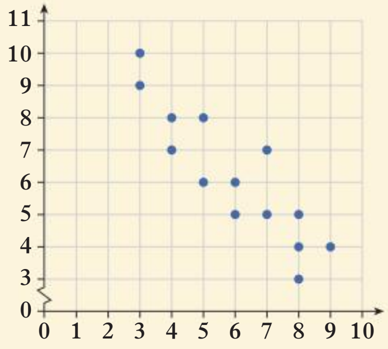
4 Create a scatter plot by plotting each pair of variables from the table on the axes.

WRITE



Example 11A.3 Drawing a line of best fit by eye

Draw a line of best fit by eye for the data shown in the scatter plot on the right.

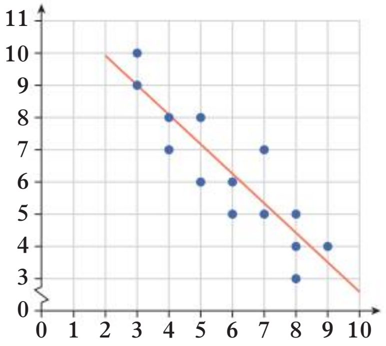


THINK

1 Look at the direction of the plotting of the points. There is a clear downward trend, so the line of best fit will need to be drawn in this direction.

2 Draw a straight line on the scatter plot in the direction of the data, with an approximately equal number of points below and above the line.

WRITE



Helpful hints

- ✔ Two variables may show strong positive or negative correlation in a scatter plot, but they do not necessarily have a causal relationship.
- ✔ When creating a scatter plot by hand, use graph paper to ensure it is appropriately scaled and the points are accurately plotted.
- ✔ A line of best fit may, but does not need to, pass through any of the points on the scatter plot.
- ✔ A line of best fit drawn by eye is not expected to be perfect. As long as it looks like a good fit for the data it will be sufficient.

ANS p784

Exercise 11A Bivariate data and scatter plots

△ 1–4, 6, 8–10 □ 1–8, 13, 15 ⬠ 1–4, 7, 9–15

UNDERSTANDING AND FLUENCY

1 Identify whether the following situations involve one variable (univariate data) or two variables (bivariate data).

a measuring the heights of plants over the first 8 weeks of their lives

b researching how distance from work affects the time taken to travel to work

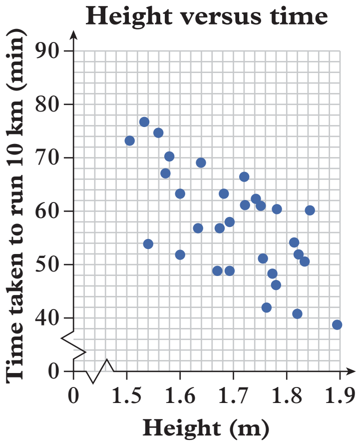
c measuring the time it takes for a group of students to run 400 metres

d researching the cost of staying in different accommodation in Jindabyne for 1 week

11A.1 **2** Describe the relationship shown in each of the scatter plots below:

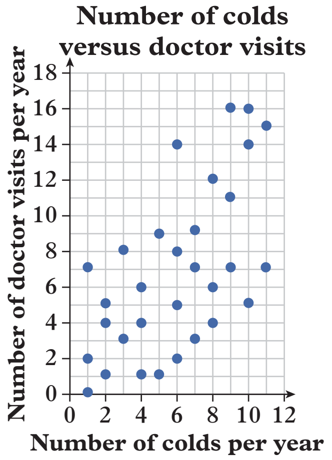
i in terms of strength and direction **ii** in the context of the data it represents.

a



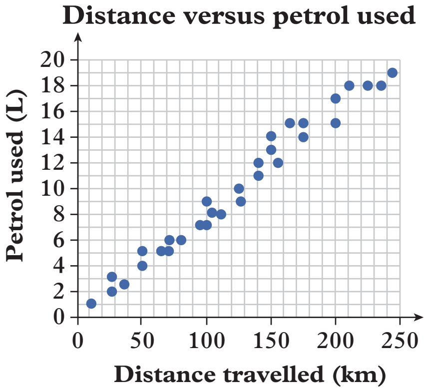
Height versus time

b



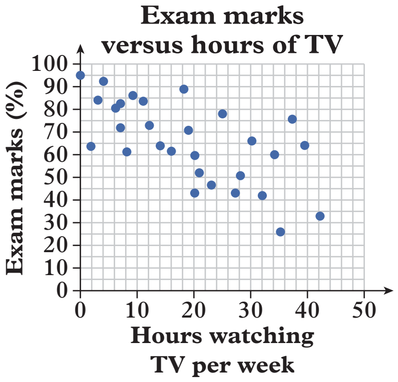
Number of colds versus doctor visits

c



Distance versus petrol used

d



Exam marks versus hours of TV

UNDERSTANDING AND FLUENCY

11A.2 3 Draw a scatter plot for each of these data sets.

a

Height (cm)	150	181	166	168	175	178	172	155	161	168	179	184	188	159	167
Shoe size	4	9	9	7	9	8	7	4	5	7	7	10	12	7	6

b

Temperature (°C)	33	15	16	25	27	36	41	11	8	32	20	25	31	18	28
Number of pies sold	5	25	33	16	19	2	3	42	35	12	21	11	10	51	15

c

Number of customers	45	56	16	32	78	98	26	66	45	76	86	42	59	31	67
Daily profit ($100s)	12	16	8	19	21	48	7	32	22	29	36	13	18	16	12

4 Describe the relationship shown in each scatter plot you drew for question **3** in the context of the data it represents.

11A.3 5 For each of these data sets, construct a scatter plot and draw a line of best fit by eye.

a

x	3	2	7	5	6	3	1	4	10	9	7	4	8	10	1	5	7	3	8	6
y	5	4	7	6	8	2	2	3	10	9	9	8	6	8	5	4	4	6	10	5

b

x	4	7	1	2	10	5	8	6	2	9	4	3	10	7	1	9	8	5
y	5	4	9	9	2	6	4	5	7	4	8	8	1	6	8	1	5	7

c

x	6	8	3	9	11	14	5	1	15	13	8	5	11	12	7	4
y	5	10	5	6	11	9	3	2	11	13	8	7	8	10	3	1

PROBLEM SOLVING AND REASONING

6 Kieran and Marie want to investigate the relationship between the number of different types of animals in zoos and the number of visitors that the zoos receive per day.

a For this investigation, predict the correlation you expect to see between the two variables in terms of form, strength and direction.

b Kieran and Marie collected the data in the table below. Draw a scatter plot to represent this data.

Number of animal types	9	26	11	37	19	42	15	14	25	32	47	34	31	37	22	18
Visitors per day	42	261	150	541	247	387	226	291	317	401	761	396	309	311	177	151

c Describe the trend shown by the scatter plot you drew for part **b**. Compare the trend to the prediction you made in your answer to part **a**.

7 A strong positive linear correlation was found between the number of hours studied and the result achieved on a test.

a Does this mean that if you study for many hours you will get high marks? Explain.

b To help prevent assumptions like the one made in part **a**, there is a saying: 'correlation does not mean causation'. What does this mean?

c Explain why the saying quoted in part **b** is useful to remember, particularly when dealing with strong correlations.

8 When examining the causal relationship between two variables in real life, we think about how one variable changes depending on how the other changes. Height and weight are a good example of this.

a Would you say that the height of a person depends on how much they weigh? Or does the weight of a person depend on how tall the person is?

Another way to look at this is to think about what effect changes in one variable would have.

b Would you say that, as a person gets taller, this generally makes them heavier? Or, as a person gets heavier, this generally makes them taller?

c If we consider height and weight, which is:

i the dependent variable

ii the independent variable?

9 For each of the pairs of variables **a–d** below, determine:

- **i** if there might be a correlation between them
- **ii** if there might be a causal relationship between them
- **iii** which is the independent variable and which is the dependent variable, if a causal relationship does exist between them
- **iv** the strength and direction of any linear correlation that exists.

a time (years) and height of a tree (metres)

b value of a car (dollars) and age (years)

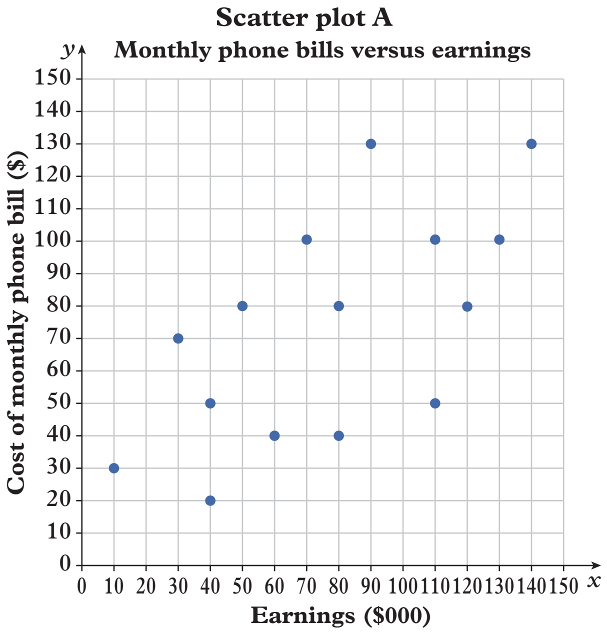
c weight of a person (kilograms) and number of languages he or she speaks

d shoe size and height of a person (metres)

e the number of girls in a park and the number of boys in a park

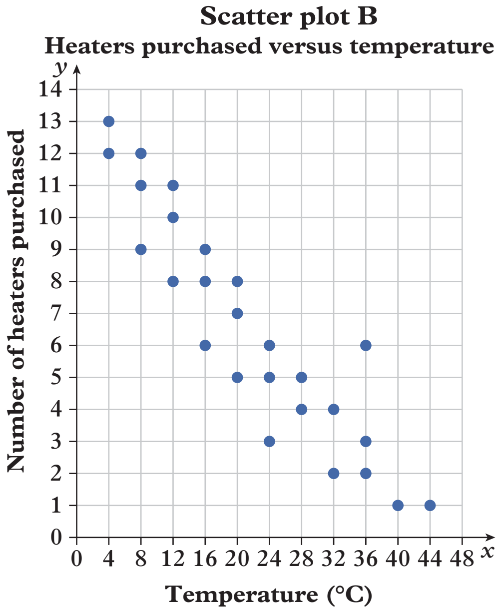
f the number of fans sold and the number of heaters sold

10 Consider the two scatter plots below.



Scatter plot A

Monthly phone bills versus earnings



Scatter plot B

Heaters purchased versus temperature

a For each of the scatter plots, describe the strength and direction of the linear relationship shown in the context of the data.

b If the temperature on a day was 26°C, how many heaters do you predict would be bought?

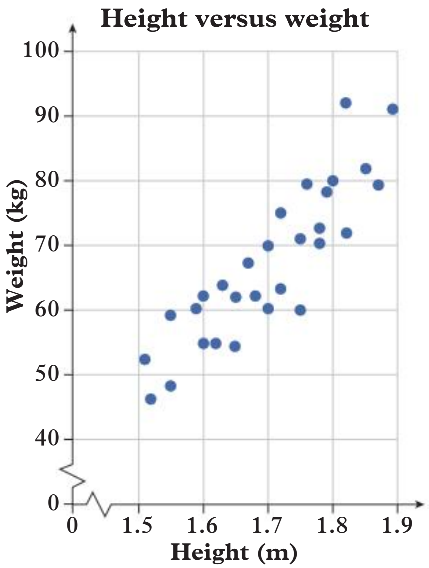
c If somebody had earnings of $100 000, what would you predict their monthly phone bill to be?

d How confident are you of your predictions in parts **b** and **c**? Explain why you would not be as confident predicting a value from scatter plot A as you would be predicting a value from scatter plot B.

e If somebody had earnings of $300 000, what would you predict their monthly phone bill to be?

f How confident are you of your prediction in part **e**? Explain why you would not be as confident predicting the value in part **e** as you would be predicting the value in part **c**.

11 Consider the scatter plot on the right, showing the relationship between the heights and weights of the students in a Year 10 class.



a Describe the trend shown.

b Draw a line of best fit by eye for this scatter plot.

c Use your line of best fit to predict the weight of a person who is 185 cm tall.

d How confident are you of your prediction in part **c**? Explain your answer.

e Predicting a value within a data set, as you did in part **c**, is called interpolation. Sometimes you may want to predict a value outside a data set. This is called extrapolation.

Identify whether each of these predictions uses interpolation or extrapolation.

i The predicted weight of a person who is 160 cm tall is 55 kg.

ii The predicted weight of a person who is 100 cm tall is 16 kg.

iii The predicted weight of a person who is 128 cm tall is 25 kg.

iv The predicted weight of a person who is 226 cm tall is 141 kg.

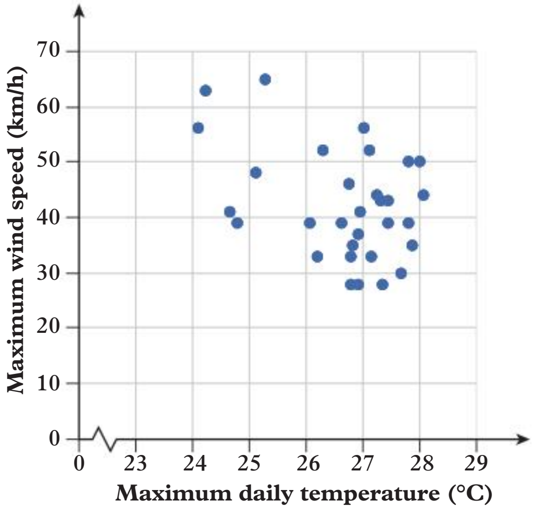
f Use your line of best fit to predict the height of a person who has a weight of 150 kg.

g Comment on your answer to part **f**. How confident are you about making a prediction outside the range of the given data set?

PROBLEM SOLVING AND REASONING

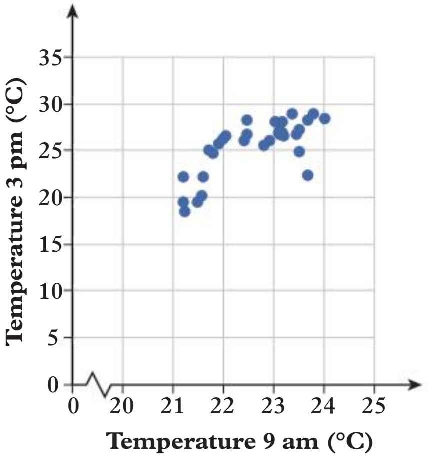
12 For each of the following, state whether there is a correlation between the two variables and, if there is, describe it in terms of form, strength and direction.

a The scatter plot below plots the maximum wind speed, in kilometres per hour, against the maximum temperature, in °C, for each day in January 2023 in Sydney, New South Wales.



(Source: BOM)

b The scatter plot below plots the temperature at 3 pm, in °C, against the temperature at 9 am, in °C, for each day in January 2023 in Sydney, New South Wales.



(Source: BOM)

13 Look up the weather data for the previous month in your local area. Determine any correlations between the numerical variables given by constructing a variety of scatter plots with the data.

14 Can a scatter plot be drawn with a negative section of the x- and/or y-axis? Explain.

15 Explain why it is important to plot a scatter plot with the original data rather than with grouped data.

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Scatter plots

Worksheet
Scatter plots and lines of best fit

Topic quiz
11A

11B Sampling methods

Learning intentions

By the end of this topic you will be able to ...

✔ evaluate the appropriateness of sampling methods and sample size.

Inter-year links

Year 7	9A Collecting and classifying data

Key content video

Statistical inquiries and surveys

- When conducting a statistical inquiry, there must be a question of interest.
  - → A statistical inquiry should have an **aim**, which is the purpose of the inquiry, and a **hypothesis**, which is a proposed explanation that is being further investigated.
- In statistics, a **population** is every potential piece of data under consideration.
- A **survey** is a list of questions designed to extract information about a population.
- Surveys usually take a **sample** of a population and use the sample data to make **inferences** (conclusions drawn from evidence and reasoning) about the population.
- Population means and medians can be estimated by using the information obtained in surveys.

Sampling methods and sample size

- The goal of selecting a sample is to get an accurate representation of the population, which means it is important to select the sample randomly to avoid possible **bias**.
  - → Bias is an inclination towards one person (group of people, thing or idea) over another.
- **Stratified sampling** divides the population into small groups known as strata (for example, based on age) and takes a random sample from each stratum. The size of each sample is proportional to how large the category is.
- **Systematic sampling** selects data at fixed intervals (for example, every fifth person). The starting point should be random.
- **Self-selected sampling** means respondents choose whether to take part in a survey when asked. Self-selected sampling often leads to a biased sample, as people tend not to respond to surveys unless they have a strong interest in the topic or a motivation to take part in the survey.
- Other non-random sampling methods include convenience sampling and quota sampling.
  - → Convenience sampling is when a sample that is easily accessible is selected.
  - → Quota sampling is similar to stratified sampling, but the researcher selects the size of each group.
- When collecting a random sample from a population, natural variation means the sample statistics are unlikely to exactly match the population parameters.
- The size of a sample should depend on the size of the population. A larger population will require a larger sample size to accurately reflect the population parameters.
  - → Large samples are also more accurate than small samples, being less susceptible to the impact of outliers.

Ethical considerations

- **Ethics** is the system of moral principles that distinguish between right and wrong behaviour.
- Ethics need to be considered throughout the statistical investigation process, from data collection through to data interpretation and reporting.
  - → When collecting data, you should consider whether the data was collected in a fair manner with the aim of finding the truth as opposed to validating pre-existing beliefs.

Example 11B.1 Identifying the sampling method

Jordan is conducting a survey to find out people's favourite foods. She calls every 100th phone number from a register to complete the survey. Determine the sampling method Jordan used.

THINK	**WRITE**
This sample was taken from people who were randomly selected from the population at a fixed interval of every 100th phone number, so it is an example of systematic sampling.	This is systematic sampling.

Example 11B.2 Evaluating surveys

Decide whether each of these surveys will provide fair or biased results, giving a reason for your answer.

a Asking every 20th person in the local phone book if they enjoy running, to determine how many people in the local community exercise.

b Surveying 1000 random households in Sydney in order to find the average number of people living in residential homes in NSW.

THINK	**WRITE**
a 1 Is the sample large enough and gathered in an unbiased way so that it theoretically represents the population (the local community)? 2 Consider the question asked. Does it answer what the questioner is trying to discover (how many people exercise)?	**a** The sample is randomly selected from the community and is of sufficient size. However, this survey is likely to provide biased results, because the question does not ask if people exercise, just if they enjoy running.
b Is the sample large enough and gathered in an unbiased way so that it theoretically represents the population (people in NSW)?	**b** This survey is likely to provide biased results because, although the sample is of a reasonable size, it only samples from one city in NSW and is therefore unlikely to be representative of the whole population.

Example 11B.3 Analysing surveys

Caleb wanted to know the average age of people at a skate park. He surveyed every 10th person he saw there over one week, surveying a total of 20 people.

a Decide whether Caleb's data collection method is fair.

b If appropriate, provide a suggestion to improve the survey.

THINK	**WRITE**
a Is the sampling method random, with an equal chance of selecting every person in the population?	**a** The method appears fair, although it could be biased depending on the times that Caleb visited the skate park. The sample could be larger.
b Are there any ways to improve this survey?	**b** Caleb could take a larger sample and ensure that he visits the skate park at different times of the day.

Helpful hints

✔ Make sure samples are taken from the whole population, not from a subset of the population. A sample will only be random if it is randomly selected from the whole population.

ANS p787

Exercise 11B Sampling methods

△ 1–7 □ 1, 3, 5, 7–9, 11 ⬠ 1, 3, 5–11

UNDERSTANDING AND FLUENCY

11B.1 **1** Determine the sampling method used in the following situations.
- **a** Xavier surveys 30 people on the main street to complete his survey about their occupation.
- **b** Mia surveys 20 students from public and private schools in proportions that reflect the town's population.
- **c** Ali surveys the first 50 people who will answer his questionnaire about school dinners.
- **d** Tina surveys 10 adults from each age group (20–29, 30–39, etc.) about their voting intentions.

11B.2 **2** Decide whether each survey below will provide fair or biased results, giving reasons for your answers.
- **a** Asking 500 random people at an AFL match their favourite sport, in order to determine the country's favourite sport
- **b** Asking everybody at your school their opinion on a school issue
- **c** Asking every 10th person on the electoral roll if they like dogs, to determine the most popular pet
- **d** Asking a random sample of 100 people (stratified by age) from around the country who they think should be prime minister at the next election
- **e** Asking 10 000 people randomly selected around the country their opinion on an issue in your community
- **f** Asking everybody you see at a local shopping mall their opinion on a global issue

11B.3 **3** For each scenario below:
- **i** decide whether the data collection method is fair
- **ii** if appropriate, provide a suggestion to improve the survey.
- **a** Tess wanted to know which is the most popular beach in NSW. She asked every 10th person on the electoral roll in her town, 'What is your favourite beach in NSW?'
- **b** Stuart wanted to know the average height of basketball players in his town. He surveyed 10 basketball players at random.
- **c** Isaac wanted to know the most popular computer brands. He asked a random sample of 1000 people, 'Do you like Apple Macs?'
- **d** Chelsea wanted to know the most popular brand of soft drink in her town. She asked all the local supermarkets which brand they sold the most of.
- **e** Sienna wanted to know the average number of hours people spent sitting down each day. She asked over 100 people, chosen at random from people she saw walking through a mall.
- **f** Kane wanted to know the most popular party foods to serve at an upcoming school dance. He surveyed every second person in his neighbourhood about their favourite party foods.

UNDERSTANDING AND FLUENCY

4 Consider the proportions of the Australian population in 2019 shown in the tables below.

Age group (years)	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54
Percentage	6.179	6.382	6.133	5.923	6.935	7.525	7.458	7.021	6.291	6.616	6.051
Number to survey											

Age group (years)	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99	100 and over
Percentage	6.091	5.473	4.829	4.171	2.895	1.993	1.234	0.605	0.174	0.02
Number to survey										

(Source: ABS, Australian Demographic Statistics Tables)

Determine the number of people in each stratum that should be surveyed for a stratified sample of 10 000 people.

5 Complete the table for a stratified sample of 1500 red kangaroos in the Western Plains region in 2020.

Zone	**Frequency**	**No. to survey**
Tibooburra	200 465	
Broken Hill	1 197 474	
Lower Darling	583 802	
Cobar	102 480	
Bourke	117 975	

Zone	**Frequency**	**No. to survey**
Narrabri	190 468	
Coonabarabran	152 052	
Griffith North	126 160	
Griffith South	253 492	

(Source: NSW Government)

PROBLEM SOLVING AND REASONING

6 Online surveys are very common. However, for an online survey to have any credibility, many issues need to be considered. For example:

- How can we ensure that a random sample has been surveyed that is representative of the target audience?
- How can we ensure that people completing the survey are who they claim to be?
- How do we deal with the non-response rate?

Write some guidelines you feel would be important to follow when choosing a sample for an online survey.

7 After answering an online survey, you frequently receive unwanted advertising emails from companies unknown to you, 'Great holiday deals on Hayman Island', 'Take advantage of this special deal', etc. Some groups make money by selling email addresses to interested companies. Is this simply part of the digital age, or can you somehow monitor this behaviour? What are your thoughts?

8 A television channel invites its viewers to vote on a topic that has just been discussed on air. Later in the program, the results of the votes are revealed, such as, '98% of viewers agree that …' Discuss whether the reported figures in this type of survey are valid.

9 Over the past 20 years, there has been a steady decrease in the use of landline phones, and an increase in the screening of phone calls from unknown numbers. Discuss the impact that both of these factors may have had on collecting data from surveys.

10 Confirmation bias is the tendency to favour information that confirms already existing beliefs. Confirmation bias can occur throughout the statistical investigation process.

Ellie is running to be the class president at her school. She conducts a survey of students in her school asking for their opinions on a range of topics and reports the results as favourable to her campaign.

Discuss where confirmation bias may have occurred in the collection of Ellie's data.

11 Conduct your own statistical inquiry by either administering a survey or using data taken from the ABS (Australian Bureau of Statistics). Clearly state both the aim and the hypothesis of your inquiry, and write a report of your findings, including appropriate data visualisations.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sampling methods

Worksheet
Conducting your own statistical inquiry

Topic quiz
11B

11C Analysing reports

Learning intentions

By the end of this topic you will be able to …

✔ evaluate where graphs or claims may be misleading.

Inter-year links

Year 8 9B Constructing graphs

Summary statistics in claims and reports

Key content video

- When evaluating a report or claim, you need to consider how the information was collected and how the information has been interpreted.
- Data should be fairly represented, so as not to skew the outcome. Presenting only some of the information or selecting inaccurate statistics to describe the data is misleading. Consider why the data has been presented as it has, and what may have been omitted. For example, if only summary statistics are provided, this may not provide adequate information about the distribution of the data.
  - → Where only 'averages' are provided, the distribution of the data is not known, so the average may not be very informative. It is also important to know if the stated average is the mean or the median.
  - → Look at the range of data being presented and consider whether any data has been purposefully omitted. For example, if you are looking at temperature data over a year, all months should be considered, as temperature varies due to the seasons in a year.
- Be prepared to question the claims made in reports. Some reports are motivated by an agenda that can result in an unwarranted claim being pushed to further that agenda.

Misleading graphs

- Reports may contain graphs or data displays that are misleading. Look at the scales used on the axes of graphs to be certain they are evenly spaced with consistent numbering.
  - → Look at the value the vertical axis starts at – if it does not start at 0, vertical differences will be enlarged and the differences in heights will be exaggerated.
- Graphs and axes on graphs should be properly labelled. By not labelling the graph correctly, incorrect assumptions can be made about the data being presented.

Example 11C.1 Identifying how a statement might misrepresent data

Explain how the statement 'Cricket is deemed the world's most popular sport' might be a misrepresentation of the data that 67% of people surveyed at cricket matches in 15 countries answered that cricket was their favourite sport.

THINK	WRITE
1 Look at the sample chosen. Does it reflect the population (the world)? 2 Look at the question or result given. Is the conclusion fair?	This statement misrepresents the data by using a biased sample (people at cricket matches), who are much more likely to answer that cricket is their favourite sport than people not attending a cricket match. The question was also only asked in 15 countries, not every country in the world.

Example 11C.2 Reviewing a published report

A new toll tunnel has just been opened to allow cars to travel from one side of the river to the other. The alternative route is via a no-toll bridge. Shortly after the tunnel opened, a newspaper reported:

'The tunnel is now so popular that the probability of a car using the tunnel rather than the bridge is 85%'.

Public perception was that this is not the case. Some residents decided to do a car count over a long time and found 967 cars from a total of 2472 went over the bridge. Comment on the newspaper report.

THINK	**WRITE**
1 From the collected data, calculate the probability of a car using the tunnel.	Using the collected data: $\Pr(\text{car uses bridge}) = \frac{967}{2472} \times 100\%$ $\approx 39\%$ So: $\Pr(\text{car uses tunnel}) \approx 61\%$
2 Compare the collected data with the reported data, and comment on the comparison.	The collected data shows that 61% of cars use the tunnel, while the newspaper reports this figure to be 85%. It seems likely that the newspaper report is exaggerated as this gap is unlikely to occur from natural variation.

Example 11C.3 Identifying misleading features in graphs

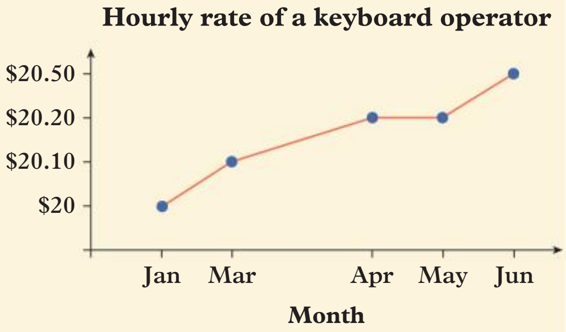
The graph on the right displays the hourly pay rate for a keyboard operator employed by a large company. The time period covered is the first 6 months of the year.

a This graph is misleading. Give reasons why.

b The company manager claims that the hourly rate of the keyboard operator has increased dramatically over the last 6 months. The graph appears to indicate this.

  i Is this true?

  ii How does the graph seem to support the manager's interpretation of the data?



THINK	**WRITE**
a Look for anything that might make the graph misleading.	**a** This graph is misleading for the following reasons: • The scale on the *x*-axis is not uniform. • The month of February is missing. • The scale on the *y*-axis does not start at zero, and no break is indicated. • The scale on the *y*-axis is not uniform.
b **i** Look at the values on the *y*-axis.	**b** **i** The keyboard operator's hourly pay rate has only increased from $20 to $20.50 over the 6-month period, an increase of 50c per hour. This is not a 'dramatic' increase.
ii Consider the scale on the *y*-axis.	**ii** Because the scale on the *y*-axis does not start at zero, the range of $20 to $20.50 has been stretched out to appear larger than it would on a complete scale. This gives the impression that the increase is greater than it really is.

Helpful hints

✔ When reviewing graphs, look at where the numbering on the axes starts. Heights of columns in a column graph may not be proportional to their frequency unless the scale is accurate.

ANS p788

Exercise 11C Analysing reports

△ 1–5, 7, 10–12, 14 □ 1–3, 5–8, 12–14, 16 ⬠ 1, 2, 5–8, 10, 13–18

UNDERSTANDING AND FLUENCY

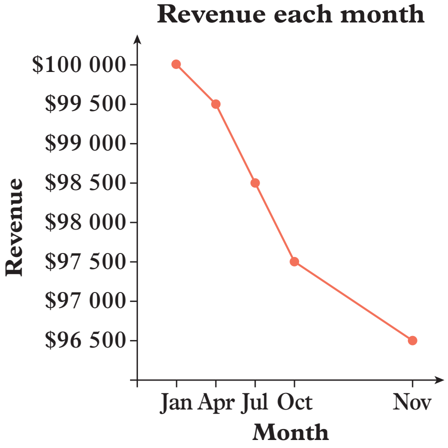
11C.1 1 Explain how each of the following headlines might be misrepresenting the data described in brackets.

- **a** 'Soft rock is the most popular music genre.' (70% of people surveyed said they enjoy listening to soft rock.)
- **b** 'New Apple tablet tops popularity stakes.' (48 people out of 50 surveyed in an Apple store said the Apple tablet was their favourite.)
- **c** 'Women now earn more than men.' (10 people were randomly surveyed on a street.)
- **d** 'New anti-ageing cream is unbeatable.' (100% of the anti-ageing cream company's shareholders were surveyed and agreed.)

11C.2 2 Each of these media statements is followed by a 'fine print' statement in brackets qualifying the original claim. Explain why the claims are not so 'amazing' when you read the fine print.

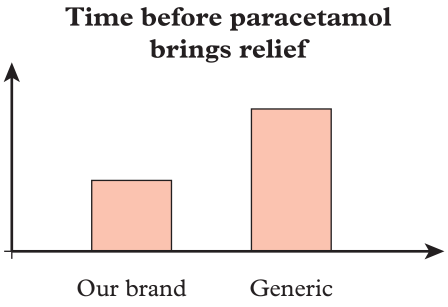
- **a** An advertisement for a phone's battery life claims it 'lasts up to 7 days' (based on standby power consumption).
- **b** An advertisement for toilet paper claims it is 'Voted Australia's favourite!' (a total of 42 people were surveyed).
- **c** An advertisement for a moisturising lotion states '67% of women notice a difference after just one week!' (compared to the use of regular soap).

11C.3 3 The graph on the right displays the revenue a company made each month over the course of a year.



- **a** This graph is misleading. Give reasons why.
- **b** The chief finance officer claims that the company's revenue has declined significantly this year, but declined at a slower rate at the end of the year. The graph appears to indicate this.
  - **i** Is this true?
  - **ii** How was this effect achieved?
  - **iii** By what percentage has the revenue declined from January to November?

4 The graph on the right displays the average time before a brand of paracetamol takes brings relief compared to a generic competitor.



- **a** This graph is misleading. Give reasons why.
- **b** The company claims their brand of paracetamol is twice as effective as their generic competitor's. The graph appears to indicate this.
  - **i** Is this true?
  - **ii** How was this effect achieved?

UNDERSTANDING AND FLUENCY

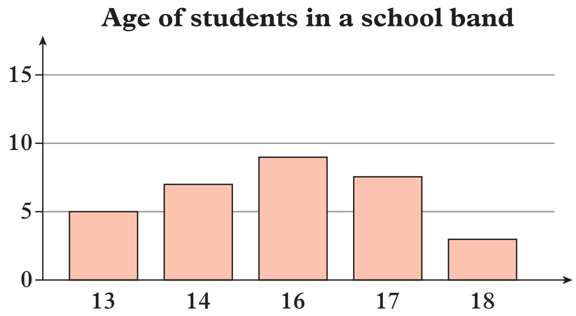
5 The graph on the right displays the number of secondary students of different ages that joined a school band.

a This graph is misleading. Give reasons why.

b The band conductor claims that the number of students that joined the band is approximately symmetric and students of all ages joined. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

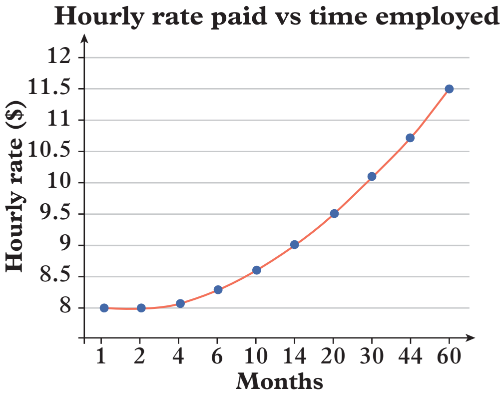


6 Fatima's employer constructed the graph on the right.

a What is the graph supposed to represent?

b Explain how the graph misrepresents the data.

c Redraw the graph so that it is fair.



PROBLEM SOLVING AND REASONING

7 The marketing manager of a drug company claims that the company's new pain relief drug blocks pain for up to 18 hours. This is based on a sample of 15 people recording the number of hours they were pain-free. Here is the raw data:

2	18	3	2	2
1	2.5	2	1.5	2.5
2	1.5	2	1.5	1.5

a Why is the company's marketing manager making this claim? Explain why it is misleading.

b Calculate the three measures of centre (mean, median and mode) for the given data.

c The marketing manager changes tactics and decides to claim that the pain relief drug blocks pain for an average of 3 hours. Comment on this claim. Is the claim now fair? Explain.

8 What questions should you ask yourself when you are presented with a claim that uses statistics?

9 Some advertisements claim that eating a good breakfast leads to success at school. Consider an alternative argument:

> 'Students who don't eat breakfast are more likely to be absent from school or experience lethargy.'

Does this statement mean the same thing as eating a good breakfast leading to success at school? Why do the advertisements use one statement rather than the other?

10 Use the statistics shown in this table to comment on the number of refugees granted asylum by Australia to 2018, in relation to the population as a whole, compared with other countries. Hint: Find the number of refugees per million of the country's population.

Country	Refugee population*	Estimated total population in 2018
Australia	56 934	24 992 370
Germany	1 063 835	82 927 920
Jordan	715 298	9 956 011
Pakistan	1 438 955	212 215 000
Turkey	3 652 362	82 319 730

*Refers to the number of people residing in the country of asylum. For most industrialised countries, this refers to the number of people recognised as refugees after arriving as asylum seekers over the past 10 years. Excludes resettled refugees.

(Source: UNHCR)

11 We are often unaware of companies sponsoring studies.

Consider this headline: 'A hotdog a day increases risk of cancer by 21%.'

Whether this statement is true or not is open to debate. However, if the study was later revealed to have been conducted by an animal-rights group whose purpose is to convince people to be vegan, would this affect the credibility of the study? Suggest how this type of reporting of information from a biased source could be monitored.

12 Influencers often promote products on social media. Do these influencers really believe in the products, or are they simply paid to advertise them? Are they given these products free for their own use?

What are your thoughts when you see advertisements such as these? Do the advertisements prompt you to purchase a product?

13 Consider the following table and text, which were published online.

> 'Australia has been ranked among the top 10 countries for its handling of the coronavirus pandemic, with New Zealand taking out the top spot in a poll conducted by a prominent Australian think tank.
>
> The Lowy Institute assessed the response of 98 countries in how they managed the pandemic in the 36 weeks following their hundredth confirmed case of the virus.
>
> It judged countries that had fewer reported cases and deaths (both in aggregate and per capita basis), as well as nations where testing rates were high.
>
> China was excluded from the ranking because of a lack of publicly available data on testing.
>
> The United States, Brazil, Iran and Mexico were ranked the worst handlers of the pandemic.
>
> Countries that proved more successful in containing the virus were largely in the Asia-Pacific region.'

Rank	Country
1	New Zealand
2	Vietnam
3	Taiwan
4	Thailand
5	Cyprus
6	Rwanda
7	Iceland
8	Australia
9	Latvia
10	Sri Lanka

(Source: SBS News)

a From the table, where is Australia ranked in terms of their response to handling the coronavirus pandemic?

b Why would it be misleading to say that Australia ranked 8th out of 10 countries?

c How does the data missing from the report make it difficult to determine where Australia sits among all countries in the world.

d The article does not directly provide the data to support its claims that 'Australia has been ranked among the top 10 countries'. What could you do to research this?

14 The CEO of a company constructed the following table showing the composition of her company.

	Total employees		Full-time		Part-time	
	25 and over	Under 25	25 and over	Under 25	25 and over	Under 25
Managers	18	4	14	4	4	0
Non-managers	42	61	40	31	2	30

She made these claims about her employees.

a 82% of 25-and-over employees are managers.

b 6% of under-25s in the company are managers.

c 29% of employees work part-time.

d 49% of under-25 non-managers work full-time.

Discuss whether each statement is correct. Provide mathematical evidence to support your answers. If a particular claim is incorrect, provide the correct figure correct to the nearest percentage.

15 The top 10 Facebook users in 2020 came from the countries listed in the following table. The table shows the number of users (in millions) and the population (in millions) of these countries.

Number	Country	Users (millions)	Population (millions)	% users
1	India	320	1380	
2	USA	190	331.0	
3	Indonesia	140	273.5	
4	Brazil	130	212.6	
5	Mexico	93	128.9	
6	Philippines	83	109.6	
7	Vietnam	68	97.34	
8	Thailand	51	69.80	
9	Egypt	45	102.3	
10	Bangladesh	41	164.7	

a Complete the last column of the table (correct to one decimal place).

b Which of these countries has the:

i highest percentage of users **ii** lowest percentage of users?

c China has the largest population of any country (1.4 billion), yet it does not make the top 10 list of Facebook users. Suggest a reason for this.

d Using the data from the table, draw different graphs to support the following statements.

i Far more people in the USA and India use Facebook than in other countries.

ii People in India don't use Facebook as much as those in other countries.

e Explain why you chose to draw your graphs the way you did in part **d**.

16 Use the statistics shown in the following table to comment on how the operating profit before tax of different Australian industries varied during the 2010s.

	Industry				
Year	**Mining**	**Manufacturing**	**Construction**	**Retail**	**Media**
2010–11	$85 156 m	$27 986 m	$30 282 m	$19 860 m	$10 430 m
2012–13	$54 374 m	$16 353 m	$26 340 m	$19 417 m	$8116 m
2014–15	$16 295 m	$20 652 m	$36 027 m	$18 724 m	$9400 m
2016–17	$33 258 m	$24 639 m	$36 043 m	$18 847 m	$7054 m
2018–19	$98 551 m	$33 822 m	$43 485 m	$17 582 m	$2935 m

(Source: ABS)

Note: figures provided for every other year of the decade.

17 Find a media article that uses statistics to support its claims (including at least one visual display). Write a paragraph analysing these claims and commenting on how trustworthy you think the article is.

18 Write an informative guide to teach somebody how and what to look for when reading statistics and the claims that they are supposedly supporting.

Check your Student obook pro for these digital resources and more:

Worksheet
Analysing reports

Topic quiz
11C

11D Theoretical probability

Learning intentions

By the end of this topic you will be able to ...

✔ calculate the probability of experiments with and without equally likely outcomes

✔ calculate theoretical probabilities using tree diagrams and arrays.

Inter-year links

Year 7	9G Theoretical probability
Year 8	9F Theoretical probability
Year 9	8E Two-step chance experiments

Theoretical probability

Key content video

- The **sample space** of an **experiment** is a list of all the different possible **outcomes** of that experiment.
- The different outcomes of an experiment may not be equally likely to occur.
- An **event** is a set of one or more outcomes from the sample space of an experiment.
- The probabilities of all events within a sample space sum to 1.
- If all the outcomes of an experiment are equally likely to occur, the probability of an event occurring can be calculated using the formula:

$$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

- The **complement** of an event A is the event that A does not occur.
  - → The complement of A is written as A′, and is sometimes referred to as 'not A'.
  - → The sum of the values of the probabilities of complementary events is 1.

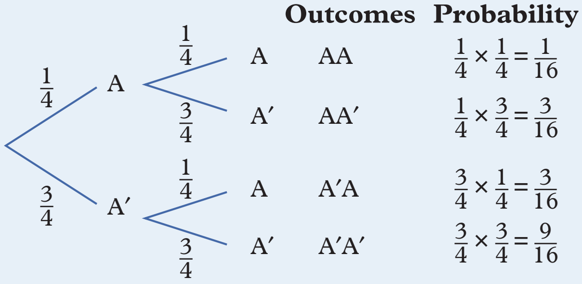
    $\Pr(A) + \Pr(A') = 1$
- Events can be described using a variety of different terminology. For example:
  - → 'at least' and 'at most' include the number being referenced
  - → 'not' means everything that is not being referenced
  - → 'and' means both conditions must be met.

Mutually exclusive events

- If two events are **mutually exclusive** then they cannot both occur at the same time. For example, when rolling a standard six-sided die, the events 'rolling an odd number' and 'rolling an even number' are mutually exclusive.

Tree diagrams

- **Tree diagrams** can be used to display the outcomes of multi-step experiments.
  - → The possible outcomes for each step of the experiment are represented by the branches.
  - → The branches might not all have the same probability.
  - → The final outcomes are listed at the ends of the branches.
  - → The probability of the final outcomes can be calculated by multiplying together the individual probabilities along the branches.



- If there are two potential outcomes at any step of a multi-step experiment, then the outcomes will be complementary and their probabilities will sum to 1.

Independent events

- Two events are **independent** if one event occurring has no effect on the probability of the other event occurring.
  - → If the events in multi-step experiments are independent, then the probabilities at one step will not be affected by what happens in the previous step.
- The probabilities of outcomes for multi-step independent events can be calculated by multiplying the probabilities for each individual step together.

  Pr(A and B) = Pr(A) × Pr(B)

Arrays

- The sample space of a two-step experiment can be displayed in an **array** or table.
- Arrays can be used to calculate the probability of an experiment. To do this, all the cells in the array must be equally likely to occur. Divide the number of cells that satisfy the given event by the total number of cells in the array.
- The table below shows the outcomes for the two-step experiment of flipping a coin and then rolling a fair six-sided die.
  - → The outcomes of the first step are the headings of the rows and the outcomes of the second step are the headings of the columns. Reading across the columns and then down the rows to each cell will help you list all the possible combinations of outcomes.

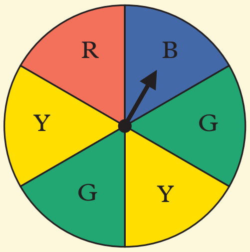
		Rolling a fair six-sided die					
		1	**2**	**3**	**4**	**5**	**6**
Flipping a coin	**H**	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)
	T	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)

Example 11D.1 Identifying the sample space and calculating theoretical probability

The spinner on the right divided into equally sized sectors (two yellow, two green, one red, one blue) is spun and the colour that the pointer lands on is recorded.

a List the sample space for this experiment.

b Calculate the probability of the spinner landing on a green sector.



THINK	**WRITE**
a List all of the possible outcomes for this experiment inside curly brackets (also called braces).	**a** {red, blue, green, yellow}
b **1** The spinner has six equally sized sections, each of which has an equally likely chance of the pointer landing on it.	**b** $\Pr(\text{green}) = \frac{2}{6}$ $= \frac{1}{3}$
2 Divide the number of equally sized sections that are green by the total number of equally sized sections and simplify.	

Example 11D.2 Calculating the probability of multi-step experiments with equally likely outcomes

A coin is flipped twice. Calculate the probability that the outcome will be:

a exactly one head **b** no heads **c** at least one head.

THINK

1 Draw a tree diagram to represent all the different outcomes of this experiment.

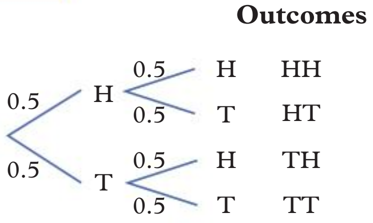
2 Write the individual probabilities on the branches of the tree diagram. For each flip of the coin there are two different possible outcomes which are equally likely to occur, therefore they each have a $\frac{1}{2}$ or 0.5 chance of occurring.

3 The probabilities of the final outcomes can be determined by multiplying the probabilities along the related branches together.

4 Identify the required final outcomes.

5 All final outcomes have the same probability of occurring, so the required probability can be determined by multiplying the number of final outcomes by their probability.

WRITE



All final outcomes have a probability of:

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

a Favourable outcomes are HT and TH.

$$\begin{aligned}\text{Pr(exactly one head)} &= 2 \times \frac{1}{4} \\ &= \frac{2}{4} \\ &= \frac{1}{2}\end{aligned}$$

b The favourable outcome is TT.

$\text{Pr(no heads)} = \frac{1}{4}$

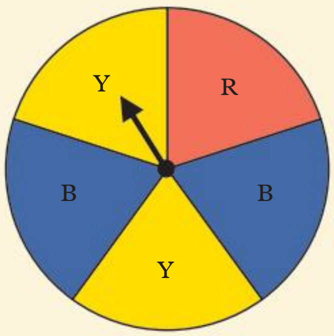
c Favourable outcomes are HH, HT and TH.

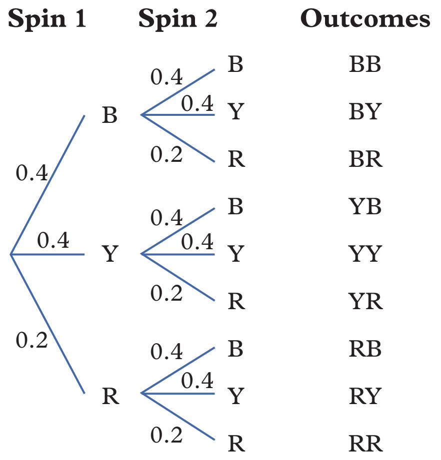
$$\begin{aligned}\text{Pr(at least one head)} &= 3 \times \frac{1}{4} \\ &= \frac{3}{4}\end{aligned}$$

Note: The probability of this event could also be found by subtracting the probability of the complementary event (‘no heads’) from 1.

Example 11D.3 Calculating the probability of multi-step experiments without equally likely outcomes

The spinner on the right divided into equally sized sectors (two blue, two yellow, one red) is spun twice, with the colour the pointer lands on being recorded both times. Calculate the probability of landing on a blue sector both times.



THINK	WRITE
1 Calculate the probabilities of landing on each colour. There are 5 equally sized sectors, so the probability of landing on a blue sector is $\frac{2}{5}$, and so on.	$\Pr(\text{blue}) = \Pr(B)$ $= \frac{2}{5}$ $= 0.4$ $\Pr(\text{yellow}) = \Pr(Y)$ $= \frac{2}{5}$ $= 0.4$ $\Pr(\text{red}) = \Pr(R)$ $= \frac{1}{5}$ $= 0.2$
2 Draw a tree diagram. Mark the individual probabilities on the branches and show all outcomes.	
3 Identify the outcome with two blue sectors.	
4 Calculate the probability of this outcome by multiplying the probabilities on each branch together.	$\Pr(\text{two blues}) = 0.4 \times 0.4$ $= 0.16$

Helpful hints

- ✔ The word ‘event’ has a specific meaning in probability that is separate to its meaning in everyday English. Read the definition carefully and make sure you do not get confused.
- ✔ Remember that you can only use the theoretical probability formula,

  $$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}},$$

  if the outcomes you are dealing with are equally likely to occur.
- ✔ Addition of decimals is often easier than addition of fractions, and multiplication of fractions is often easier than multiplication of decimals. Therefore, depending on the context, it may be useful to write your probabilities as decimals or fractions.

  For example, evaluating $0.25 \times 0.7 \times 0.8$ is more difficult than $\frac{1}{4} \times \frac{7}{10} \times \frac{4}{5}$ whereas $0.15 + 0.125$ is easier to evaluate than $\frac{3}{20} + \frac{1}{8}$.
- ✔ Be careful not to represent decimals incorrectly. For example, $\frac{1}{3}$ is not equal to 0.33.

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Exercise 11D Theoretical probability

△ 1–8, 10, 12, 13, 17, 18 ■ 1, 2, 6, 7, 9, 10, 14, 17–19 ⬠ 1, 2, 8, 10, 11, 14, 15–18, 20, 21

UNDERSTANDING AND FLUENCY

11D.1 **1** List the sample space for each of these experiments.

- **a** rolling an eight-sided die (with sides numbered 1–8) and recording the uppermost number
- **b** randomly selecting a letter from the word AUSTRALIA
- **c** spinning a spinner with two red, three white and four blue sections (all sections being equal in size) and recording the colour

2 State whether the outcomes in the sample spaces for the experiments in question **1** are equally likely to occur or not.

3 A letter is selected at random from the word MATHEMATICS.

- **a** List the sample space for this experiment.
- **b** Calculate the probability of selecting an M.

11D.2 **4** A multiple-choice question consists of two parts. Each part has an equally likely chance of being true or false. What is the probability that:

- **a** the answers are both true
- **b** the answers are both false
- **c** one answer is true, and one answer is false?

5 In a room there are two lights, each with its own on/off switch.

- **a** Draw a tree diagram to show all the possible combinations of on or off the two lights could be.
- **b** If it is equally likely that each light is either on or off, what is the probability that the lights will:
  - **i** both be on
  - **ii** both be off
  - **iii** not be in the same mode?

11D.3 **6** Two fair six-sided dice are rolled. Each time a 6 is rolled on either of the dice, it is recorded.

- **a** If A represents the event ‘rolling a 6’, what does A′ represent?
- **b** Draw a tree diagram showing the probabilities of the different outcomes of the experiment. Write the probabilities for each step on the branches.
- **c** Use your tree diagram to find the probability of rolling:
  - **i** a double 6
  - **ii** at least one 6.

7 The experiment in question **6** is repeated. This time, all the numbers rolled on both dice are recorded.

- **a** Explain how this experiment differs from the experiment in question **6**.
- **b** How many outcomes are possible in this case?
- **c** Draw an array to represent the experiment, and use it to determine the probability that the two numbers rolled are:
  - **i** the same
  - **ii** both prime numbers.

8 Three eight-sided dice (numbered 1–8) are thrown and the number rolled on each die is noted. It is recorded whether each number is even or odd.

Use a tree diagram to find the probability of rolling:

a no odd numbers

b one or two odd numbers

c more odd numbers than even numbers.

9 'Unders and overs' is a game that is played by rolling two six-sided dice. The numbers rolled are added together to give a total. Players predict whether the total will be under 7, equal to 7 or over 7.

a Draw an array to show all the possible outcomes when rolling two dice.

b Colour your array from part **a** to show which outcomes are:

i under 7 **ii** equal to 7 **iii** greater than 7.

c Calculate the probability of the prediction being correct if you predicted:

i under 7 **ii** equal to 7 **iii** greater than 7.

d Describe what your strategy would be if you played this game.

10 A ten-sided die (numbered 1–10) is rolled and the number facing up is recorded. Let A represent the event 'rolling an even number', B represent the event 'rolling a multiple of 4', and C represent the event 'rolling a prime number'.

a List the outcomes in each event: A, B and C.

b Which two events are mutually exclusive?

c Find each of the following:

i Pr(A) **ii** Pr(B) **iii** Pr(C).

11 When rolling two unbiased dice of the same kind, the probability of rolling a double depends on the number of sides on each die.

a Consider rolling a pair of standard six-sided dice.

i How many outcomes are possible when the two dice are rolled?

ii How many of these outcomes are doubles? List them.

iii What is the probability of rolling a double?

b Consider rolling a pair of eight-sided dice. Repeat the questions from part **a**.

c Write a general statement for the probability of rolling a double using a pair of dice with n sides.

12 A marble collection is made up of clear and opaque marbles. There are 15 clear marbles, and the probability of choosing an opaque marble from the collection is $\frac{2}{5}$. How many opaque marbles are in the collection?

13 Two fair six-sided dice are thrown and the product of the two numbers rolled is recorded.

a What is the probability that this number is:

i odd **ii** prime **iii** a square number?

b From the three outcomes given in part **a**, what is:

i the most likely outcome **ii** the least likely outcome?

14 A game involves a regular six-sided die and a fair coin. The die is rolled, and the number recorded. The coin is then flipped. If the coin lands on heads, then the die score is doubled, whereas if the coin lands on tails, then the die score remains the same.

a List the sample space of possible scores for this game.

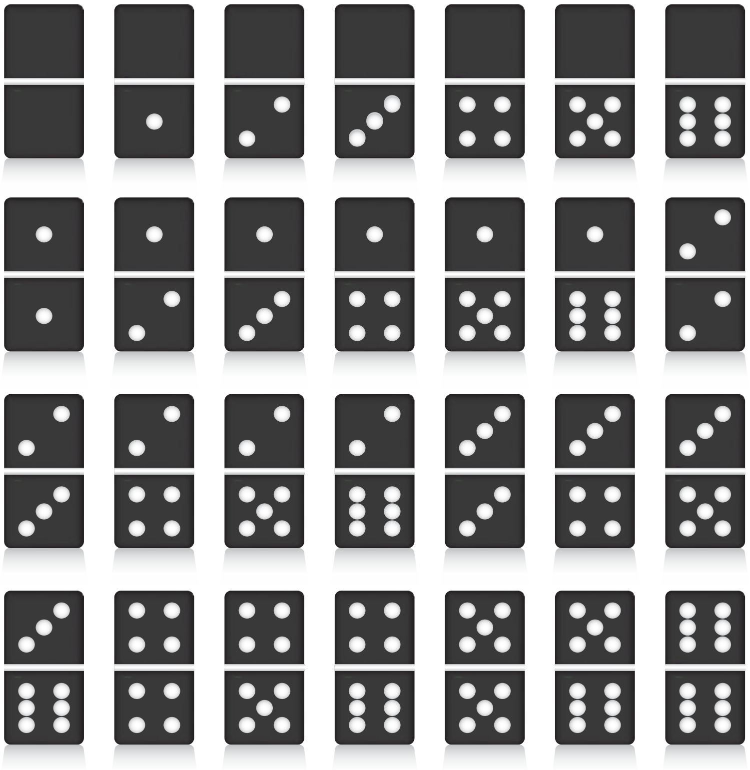
b Explain why each outcome in the sample space is not equally likely to occur.

c Find the probability that a player achieves a score of:

i 10 **ii** 4 **iii** at least 8.

15 Two standard six-sided dice are rolled and the two numbers facing up are recorded. What is the probability that the sum of the numbers is greater than the product of the numbers?

16 A set of dominoes contains tiles which have been divided in two with numbers represented by dots (similar to dice). A blank space on a tile represents 0.



a How many tiles are in the complete set of dominoes (shown above)?

b Explain the order of the tiles in the photo.

c For each of the values (0, 1, 2, 3, 4, 5 and 6), count the number of tiles on which the value appears. If the value appears on both halves of a tile, count the tile once. You should get the same number of tiles for each of the values from 0 to 6. What is this number?

d One tile is chosen at random. What is the probability that it shows:

- **i** double 6
- **ii** the same number twice
- **iii** two different numbers
- **iv** one even number and one odd number
- **v** at least one prime number
- **vi** two numbers, one of which is a multiple of the other (not including zero as a multiple)?

17 A coin is biased so that the chance of it showing a tail when flipped is 0.7.

a Draw a tree diagram to show the outcomes and their probabilities if the coin is flipped three times.

b From your tree diagram, what is:

- **i** the most likely outcome
- **ii** the least likely outcome?

c What is the probability of flipping:

- **i** only one head
- **ii** only one tail
- **iii** more heads than tails?

18 Explain why the events in the multi-step experiment in question **17** are independent.

19 Snakes and ladders is played on a board with numbered grid squares. Ladders connect certain pairs of squares together, and snakes connect other pairs of squares together. The aim of the game is to progress from square 1 to square 100 by rolling a fair six-sided die and moving the number of squares rolled. You must move up a ladder or down a snake if you land on the squares that show the base of a ladder or the head of a snake.

Imagine a grid on which the first ladder base is on square 7.

a What is the minimum number of die rolls which would take you to square 7?

b What is the maximum number of die rolls which could take you to square 7?

c Consider the minimum number of rolls required.

  i List the die rolls which would take you to square 7.

  ii How many different options are there?

20 A bag contains Christmas decorations of four different colours: red, green, gold and silver. There are four silver decorations and eight gold decorations. One decoration is drawn at random from the bag. The probability it is silver or gold is 0.6. If there are three green decorations, how many red decorations are in the bag?

21 Consider the target on the right.

a Discuss whether the chance of landing on a red section is the same as that of landing on a white section.

b The diameter of the innermost red circle (the bullseye) is 2 cm, and the widths of each of the subsequent rings is 2 cm. What is the radius of each of the rings?

c Calculate the area of each of the rings in terms of π.

d Assuming the arrow is equally likely to hit any point within the target, calculate the probability of the arrow landing:

  i in the bullseye

  ii in the outermost red ring.

Round your answers to three decimal points.

Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** Complementary events
- **Interactive skillsheet** Arrays
- **Interactive skillsheet** Tree diagrams
- **Worksheet** Probability and tree diagrams
- **Investigation** Sicherman dice
- **Topic quiz** 11D

ANS
p791

Checkpoint

Checkpoint quiz
Check your knowledge of the first part of this chapter.
✔ Core
✔ Standard
✔ Advanced

11A **1** A record of all the winning times for the men's 100 m sprint at the Olympic Games between 1952 and 2016 has been recorded in the following tables.

Year	1952	1956	1960	1964	1968	1972	1976	1980	1984
Time (s)	10.79	10.62	10.32	10.06	9.95	10.14	10.06	10.25	9.99

Year	1988	1992	1996	2000	2004	2008	2012	2016
Time (s)	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81

a Draw a scatter plot of the data.

b Draw a line of best fit by eye on your scatter plot.

c Use your line of best fit from part **b** to make a prediction for when the 9.5 second barrier might be broken. (Remember that the Olympic Games are only held every four years.)

STN **11B** **2** Describe the relationship in the scatter plot you drew in question **1** in terms of:

a strength

b direction.

STN **11B** **3** For each of the following surveys:

i determine the sampling method

ii decide whether each of these surveys will provide fair or biased results, giving reasons for your answers.

a Arya asked every fifth competitor of the 128 competitors in a swimming tournament what their favourite sport is to determine the most popular sport in her high school.

b Billy asked a sample of 200 people at his local train station whether they think New South Wales should implement a carbon tax.

STN **11C** **4** Explain how each of the following headlines might be misrepresenting the given statistic.

a 'The majority of people prefer name brands to generic brands.' (51% of people surveyed said they preferred name brands to generic brands.)

b '9 out of 10 dentists recommend our toothpaste.' (Ten dentists were surveyed, nine of whom are sponsored by the toothpaste company.)

11D **5** List the sample space for each of the following experiments.

a A fair coin is tossed.

b A standard six-sided die is rolled.

c A card is picked from a standard 52-card deck and its suit (clubs, diamonds, hearts, spades) noted.

d A letter is selected at random from the word LOLLIPOP.

11D **6** A letter from the word CHANCE is selected at random. Find the probability that the selected letter is:

a 'H' **b** 'C' **c** a vowel **d** not a vowel.

11D **7** A biased coin, which has a 60% chance of landing on heads, is tossed twice.

a Draw a tree diagram to represent this situation.

b Find the probability the coin lands on:

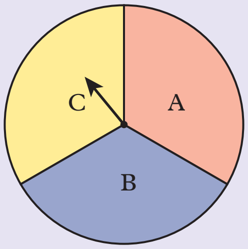
i heads both times

ii heads once and tails once.

11D **8** The spinner on the right is spun twice.

a Draw a tree diagram to represent the possible outcomes of the two spins.

b Find the probability the spinner lands on 'A' both times.



11E Experiments with and without replacement

Learning intentions

By the end of this topic you will be able to ...

✔ calculate probabilities of multi-step experiments with and without replacement.

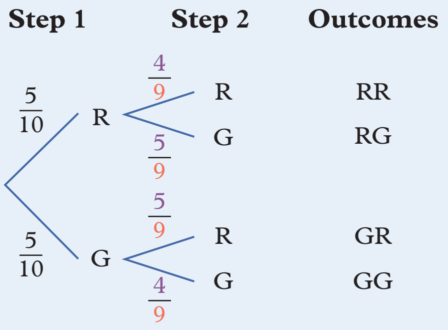
Inter-year links

Year 7	9G Theoretical probability
Year 8	9F Theoretical probability
Year 9	8F Experiments with replacement

Experiments with and without replacement

Key content video

- When multi-step experiments involve items being selected from a group, the item can either be replaced or not replaced after each selection.
- If the items in a multi-step experiment involving selection are replaced, the probabilities for each step of the experiment remain the same.
- If the items in a multi-step experiment involving selection are not replaced, the probabilities for each step of the experiment change.
- The outcomes for each step of a multi-step experiment with replacement are independent, as the probabilities for the second step are not dependent on the outcome of the first step.
- The outcomes for each step of a multi-step experiment without replacement are dependent (not independent), as the probabilities for the second step depend on the outcome of the first step.
- For experiments without replacement, the number of items to select from decreases by 1 after each selection.



The number of outcomes in the sample space is reduced for the second selection.

Depending on the first selection, the number of successful outcomes may also be reduced.

- Arrays can be used to show the reduced sample space for experiments without replacement. For example, if two letters are picked at random from the word 'SUN' without replacement, the sample space would be:

		Second letter		
		S	**U**	**N**
First letter	**S**	–	(S, U)	(S, N)
	U	(U, S)	–	(U, N)
	N	(N, S)	(N, U)	–

Example 11E.1 Calculating probability for experiments with replacement

A barrel contains coloured balls: two red, one blue and three green. One ball is drawn from the barrel, its colour is noted, then it is replaced. A second ball is then chosen and its colour is also noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two balls selected are the same colour.

c Determine the probability that a red and green ball are selected in any order.

THINK

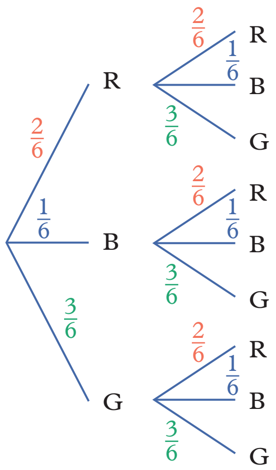
a Draw a tree diagram, writing all possible outcomes along with their probabilities. Because the selected ball is replaced after the first selection, the probabilities for the second selection remain the same.

b On your tree diagram, identify the final outcomes in which the two balls are the same colour. Add the probabilities for those outcomes to find the required probability, and then simplify.

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{2}{6} = \frac{4}{36}$
RB	$\frac{2}{6} \times \frac{1}{6} = \frac{2}{36}$
RG	$\frac{2}{6} \times \frac{3}{6} = \frac{6}{36}$
BR	$\frac{1}{6} \times \frac{2}{6} = \frac{2}{36}$
BB	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
BG	$\frac{1}{6} \times \frac{3}{6} = \frac{3}{36}$
GR	$\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$
GB	$\frac{3}{6} \times \frac{1}{6} = \frac{3}{36}$
GG	$\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

b Favourable outcomes are RR, BB and GG.

$$\Pr(RR) = \frac{4}{36}$$

$$\Pr(BB) = \frac{1}{36}$$

$$\Pr(GG) = \frac{9}{36}$$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{4}{36} + \frac{1}{36} + \frac{9}{36}\\ &= \frac{14}{36}\\ &= \frac{7}{18}\end{aligned}$$

c Favourable outcomes are RG and GR.

$$\Pr(RG) = \frac{6}{36}$$

$$\Pr(GR) = \frac{6}{36}$$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{36} + \frac{6}{36}\\ &= \frac{12}{36}\\ &= \frac{1}{3}\end{aligned}$$

Example 11E.2 Calculating probability for experiments without replacement

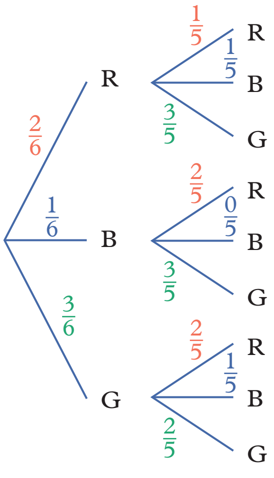
Repeat the experiment from Example 11E.1, but this time assume the first ball is not replaced before the second ball is chosen.

THINK

a Draw a tree diagram. The number of balls from which the second ball is chosen decreases by 1, so the possible outcomes at the second step are different from the possible outcomes for the first step.

WRITE

a



Outcomes	**Probability**
RR	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RB	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RG	$\frac{2}{6} \times \frac{3}{5} = \frac{6}{30}$
BR	$\frac{1}{6} \times \frac{2}{5} = \frac{2}{30}$
BB	$\frac{1}{6} \times \frac{0}{5} = 0$
BG	$\frac{1}{6} \times \frac{3}{5} = \frac{3}{30}$
GR	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$
GB	$\frac{3}{6} \times \frac{1}{5} = \frac{3}{30}$
GG	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

b On your tree diagram, identify the final outcomes in which the two colours are the same. Add the probabilities for those outcomes to find the required probability and simplify. (Note the outcome BB is not possible, because there is only one blue ball.)

b Favourable outcomes are RR and GG.

$$\Pr(RR) = \frac{2}{30}$$

$$\Pr(GG) = \frac{6}{30}$$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{2}{30} + \frac{6}{30}\\ &= \frac{8}{30}\\ &= \frac{4}{15}\end{aligned}$$

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

c Favourable outcomes are RG and GR.

$$\Pr(RG) = \frac{6}{30}$$

$$\Pr(GR) = \frac{6}{30}$$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{30} + \frac{6}{30}\\ &= \frac{12}{30}\\ &= \frac{2}{5}\end{aligned}$$

Helpful hints

✔ The sum of the probabilities at each step of a multi-step experiment is always equal to 1.

✔ For multi-step experiments, it is often helpful to leave fractions unsimplified until the final answer. In Example 11E.1, the probability that the balls are the same colour was calculated as $\frac{4}{36} + \frac{1}{36} + \frac{9}{36}$, which is easy to add. However, if each fraction were simplified then the denominators would all be different, which would make addition more difficult.

ANS p791

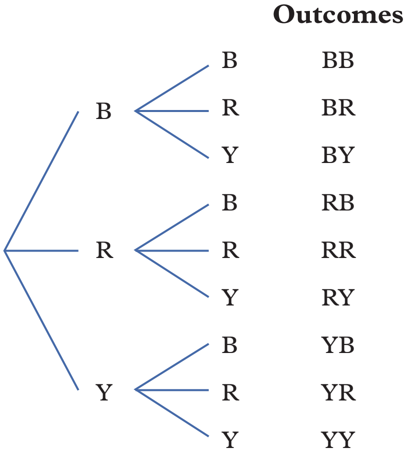
Exercise 11E Experiments with and without replacement

△ 1–3, 5–7, 10, 12 ◻ 3, 4, 8, 9, 11, 13–15, 17, 18 ⬠ 4, 7, 9, 11, 13–19

UNDERSTANDING AND FLUENCY

1 A bag contains 22 coloured marbles: 6 blue, 8 red and 8 yellow. One marble is drawn from the bag, its colour is noted, then it is replaced. A second marble is then chosen and its colour is also noted.

a Complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

11E.1 2 A standard deck of playing cards consists of four suits (hearts, diamonds, clubs and spades) with 13 cards in each suit (52 cards in total). One card is chosen from a deck and its suit is noted. It is replaced, then a second card is chosen and the suit noted.

a Draw a tree diagram to represent this experiment.

b Determine the probability that two hearts are selected.

c Determine the probability that a club and a diamond are selected in any order.

d Determine the probability that two cards of the same suit are drawn.

e Determine the probability that two cards of different suits are drawn.

3 John removes the loose coins from his pockets each night and throws them into a tin. There are 75 silver coins and 25 gold coins in the tin. He picks out a coin from the tin and notes its colour. After replacing it in the tin, he repeats this process another two times.

a Determine the probability of John drawing out:

i a silver coin

ii a gold coin.

b Draw a tree diagram to display this experiment. Write the individual probabilities on the branches of the tree diagram.

c Use your tree diagram to calculate the probability of John drawing out:

i all silver coins

ii no silver coins

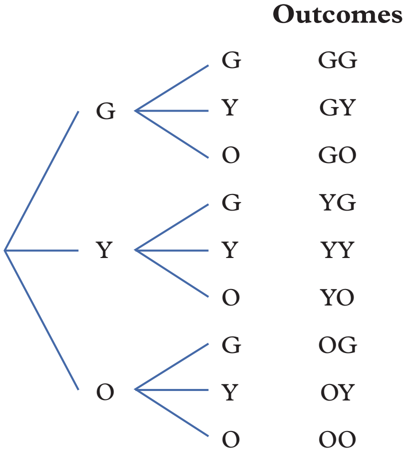
iii at least one coin of each colour.

4 Consider the digits 1, 2, 3, 4 and 5. You are asked to write a two-digit number using these digits. The digits may be used more than once.

- **a** Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.
- **b** How many outcomes are there?
- **c** How many of these outcomes have two digits which are the same?
- **d** Calculate the probability that the number has:
  - **i** two different digits
  - **ii** two digits the same
  - **iii** the first digit less than the second.

5 A bag contains 16 coloured marbles: 6 green, 5 yellow and 5 orange. One marble is drawn from the bag, its colour is noted and it is not replaced. A second marble is then chosen and its colour is also noted.

- **a** Complete the tree diagram representing this situation by adding probabilities to the branches.



- **b** Determine the probability that the two chosen marbles are the same colour.

11E.2 6 A bag contains 8 cubes: two black, three white, and three grey. One cube is drawn and its colour is noted. A second cube is drawn without the first cube being replaced and its colour is noted.

- **a** Draw a tree diagram to represent the situation.
- **b** Determine the probability that the two cubes selected are the same colour.
- **c** Determine the probability that a black and a white cube are selected in any order.

7 The letters of the word PROBABILITY are written on separate pieces of paper and placed in a box. One letter is drawn from the box and it is recorded whether the letter is a consonant or a vowel. A second letter is then drawn without replacing the first letter. Determine the probability that:

- **a** the letters are both vowels
- **b** one letter is a vowel and the other is a consonant.

8 From the set of all 26 letters in the English alphabet, two letters are drawn:

- **a** with replacement
- **b** without replacement.

For each of these situations, what is the probability that the letters are both vowels or both consonants?

9 Consider again the situation described in question **4** but, this time, you are only able to form a two-digit number without repeated digits.

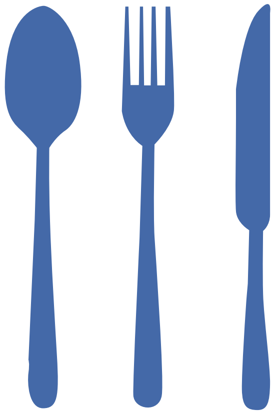
- **a** Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.
- **b** How many outcomes are there?
- **c** Calculate the probability that the number has:
  - **i** two digits the same
  - **ii** two different digits
  - **iii** the first digit smaller than the second.

UNDERSTANDING AND FLUENCY

10 Sam has a small bag of lollies. The bag contains 10 red, 8 blue, 6 orange, 4 yellow and 12 green lollies.

a Sam prefers the red lollies. What is the probability that, if he chooses one lolly at random, it will be red?

b Sam eats all the red lollies. He then chooses another lolly at random, eats it, then chooses another one at random and eats it. What is the probability that the last two lollies Sam ate were the same colour?



11 You are setting the table for you and your friend to have dinner. You each need a knife, fork and spoon. The cutlery drawer in the kitchen contains 15 knives, 10 forks and 15 spoons.

Without looking in the drawer, you choose three pieces of cutlery at random.

a What is the chance that your first choice was a knife?

b If your first choice is a knife, what is the probability that your next choice will be a fork, and the next choice a spoon?

c What is the probability of choosing a knife, a fork and a spoon in any order?

d Assume that you have already chosen a knife, fork and spoon for yourself. What is the probability that you will now choose these three items for your friend in your next three selections?

12 Consider a pencil case containing 30 pencils: 10 black, 2 red, 4 blue, 3 green and 11 yellow.

Two pencils are taken out one after the other without replacement. The chance that the first pencil chosen is black, then the next one is red is calculated as:

$$\begin{aligned}\Pr(\text{black then red}) &= \Pr(\text{black}) \times \Pr(\text{red})\\ &= \frac{10}{30} \times \frac{2}{29}\\ &= \frac{2}{87}\end{aligned}$$

a Use this information to help you determine the probability that:

- **i** the first pencil chosen is black, then the next one is not red
- **ii** the first pencil chosen is black, then the next one is red or blue
- **iii** the first pencil chosen is black, then the next one is red, blue or green
- **iv** both pencils are black
- **v** both pencils are red
- **vi** the first pencil chosen is red, then the next one is blue.

b Investigate to see whether undertaking the two events in the reverse order affects the probability by determining the probability that:

- **i** the first pencil chosen is red, then the next one is black
- **ii** the first pencil chosen is blue, then the next one is red.

c Comment on your answers to part **b**.

13 A bag contains 5 balls: three white and two black. Three balls are drawn, and each ball is placed back in the bag immediately after it has been drawn.

a Find the probability that all three balls drawn are black.

b Find the probability that all three balls drawn are white.

14 Explain why the events in multi-step experiments with replacement are independent, and the events in multi-step experiments without replacement are dependent.

PROBLEM SOLVING AND REASONING

15 As with question **13**, a bag contains 5 balls: three white and two black. This time, three balls are drawn without replacement.

- **a** Explain why at least one white ball will be drawn.
- **b** Find the probability that all three balls drawn are white.

16 Five balls labelled with numbers 1–5 are placed in a bag. Two balls are selected, at random, one at a time.

- **a** Write the sample space for the sum of the two numbers if:
  - **i** the first ball is replaced after being selected
  - **ii** the first ball is not replaced after being selected.
- **b** Find the probability that the sum of the two balls chosen is 4 if:
  - **i** the first ball is replaced after being selected
  - **ii** the first ball is not replaced after being selected.
- **c** Find the probability that the sum of the two balls chosen is 6 if:
  - **i** the first ball is replaced after being selected
  - **ii** the first ball is not replaced after being selected.

17 It's not always necessary to draw tree diagrams to determine the number of outcomes of an experiment.

First, consider forming a two-digit number from the digits 1 to 5 with replacement (as was described in question **4**). There are five digits to choose from when writing the first digit. You can use all the digits again for the second digit, so there are five from which to choose again. This means that there are 5×5 possible outcomes.

Next, consider forming a two-digit number from the digits 1 to 5 *without* replacement. There are five digits to choose from when writing the first digit. But, since you are not able to repeat any of the digits, there are only four digits from which to choose the second digit. This means that there are 5×4 possible outcomes.

- **a** Use this method to determine the number of possible outcomes when forming a three-digit number from the digits 1 to 5:
  - **i** with replacement
  - **ii** without replacement.
- **b** Repeat part **a** using the digits 1 to 9 to form a three-digit number.

CHALLENGE

18 A restaurant offers a menu of six entrées, six main meals and six desserts.

- **a** How many different combinations of courses does the restaurant offer?
- **b** During the evening, the kitchen runs out of one of the entrées and two of the desserts. What is the decrease in the number of combinations available?

19
- **a** Suppose that number plates for cars contain six digits and digits can be repeated. Remember that a number plate can start with the digit 0.
  - **i** How many different digits can be used in each place of the six-digit number?
  - **ii** How many different number plates could be produced using this system?
- **b** A new system was introduced whereby each number plate contains three digits followed by three letters.
  - **i** How many number plates would be possible using this system?
  - **ii** Explain why the system with a mixture of digits and letters for number plates is better than the system with six digits.

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Interactive skillsheet
Experiments without replacement

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Replacement and sample sizes

Investigation
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Topic quiz
11E

ADVANCED

11F Two-way tables

Learning intentions

By the end of this topic you will be able to ...

✔ create two-way relative frequency tables

✔ use two-way tables to solve problems.

Inter-year links

Year 9 8E Two-step chance experiments

Two-way tables

Key content video

- **Two-way tables** represent the relationships between two attributes. The results can be displayed as raw numbers or as relative frequencies. For example, both of these two-way tables display the same data.

	Dark hair	**Light hair**	**Total**
Dark eyes	16	4	20
Pale eyes	8	12	20
Total	24	16	40

	Dark hair	**Light hair**	**Total**
Dark eyes	0.4	0.1	0.5
Pale eyes	0.2	0.3	0.5
Total	0.6	0.4	1

Identifying the required outcome

- A **compound event** is an event with two or more favourable outcomes.
- The language used to describe an outcome can help you to identify which cells of the two-way table are being referred to.
  - → 'and' means both outcomes must be met. For example, 16 people have dark eyes and dark hair.
  - → 'or' can be inclusive, meaning if both outcomes are met the outcome is favourable. For example, $4 + 8 + 12 = 24$ have pale eyes or light hair.
  - → 'or' can also be exclusive, meaning if both outcomes are met the outcome is not favourable. For example, $4 + 8 = 12$ people have pale eyes or light hair but not both.

Example 11F.1 Understanding a two-way table

Consider this two-way table, which shows whether Year 9 and Year 10 students prefer jam or Vegemite.

a How many students were surveyed?

b How many students are in Year 10 and prefer Vegemite?

	Year 9	**Year 10**	**Total**
Jam	28	38	66
Vegemite	25	34	59
Total	53	72	125

THINK	WRITE
a The total number of people surveyed is displayed in the bottom right-hand corner.	**a** 125 students were surveyed.
b Identify the cell that is in the 'Year 10' column and the 'Vegemite' row.	**b** 34 students are in Year 10 and prefer Vegemite.

Example 11F.2 Creating a two-way relative frequency table

This two-way table shows whether or not a group of 50 people from Sydney and Perth own a pet. Convert it into a two-way relative frequency table.

	Sydney	**Perth**	**Total**
Owns pet	13	19	32
Does not own pet	11	7	18
Total	24	26	50

THINK

Divide the values in each cell by the value in the total cell in the bottom right-hand corner.

The totals should add to 1 in the bottom right-hand corner.

WRITE

	Sydney	**Perth**	**Total**
Owns pet	$\frac{13}{50} = 0.26$	$\frac{19}{50} = 0.38$	$\frac{32}{50} = 0.64$
Does not own pet	$\frac{11}{50} = 0.22$	$\frac{7}{50} = 0.14$	$\frac{18}{50} = 0.36$
Total	$\frac{24}{50} = 0.48$	$\frac{26}{50} = 0.52$	1

Example 11F.3 Creating and completing a two-way relative frequency table

A survey collected information on whether motorists from the country and city drive automatic or manual cars. Of the respondents, 45% are from the city and drive automatic cars, 43% are from the country and 34% drive manual cars.

Use this information to complete a two-way relative frequency table.

THINK

1 Draw the table with one of the categories (car type) placed in the rows and the other category (location) placed in the columns.

2 Convert the percentages to relative frequencies by dividing by 100, e.g. 43% = 0.43.

3 Insert the given information in the relevant cells. Remember that the total in the bottom right-hand corner is always 1.

4 The numbers in each row and column add to give the value in the total cells, so wherever there are two out of three cells in a row or column, the unknown value can be found by using addition or subtraction.

WRITE

	City	**Country**	**Total**
Automatic	0.45	0.66 – 0.45 = 0.21	1 – 0.34 = 0.66
Manual	0.57 – 0.45 = 0.12	0.43 – 0.21 = 0.22	0.34
Total	1 – 0.43 = 0.57	0.43	1

Example 11F.4 Calculating probability using a two-way table

Consider the two-way table in example 11F.1, showing whether Year 9 and 10 students prefer jam or Vegemite. What is the probability of a student chosen randomly from the group being in Year 9 or preferring jam (or both)?

	Year 9	Year 10	Total
Jam	28	38	66
Vegemite	25	34	59
Total	53	72	125

THINK

1 Identify the relevant cells in the table that represent this outcome (any cell in the Year 9 column or Jam row).

2 Divide the total of these cells by the total in the table. Simplify if possible.

WRITE

$$\frac{28 + 38 + 25}{125} = \frac{91}{125}$$

Helpful hints

✔ When adding totals in rows and columns, be careful not to double count the cells where the rows and columns intersect.

✔ In some problems it may appear that there is insufficient information to answer the question. Start by writing down what you know and then use this information to find the unknown values.

ANS p793

Exercise 11F Two-way tables

△ 1, 3–10, 12 ■ 2, 4–9, 11, 14, 15(a, b), 16(a–d) ⬠ 4, 6–9, 11, 13, 15(b, c), 16, 17

UNDERSTANDING AND FLUENCY

11F.1 11F.4 **1** Consider this two-way table that shows whether students in primary school (PS) and high school (HS) prefer to watch or play sport.

a How many students were surveyed in total?

b How many students in high school prefer to watch sport?

c What is the probability of a student chosen randomly from the group being in high school and preferring to watch sport?

	PS	HS	Total
Watch sport	8	7	15
Play sport	22	18	40
Total	30	25	55

2 Consider this two-way table that shows the hair colours and eye colours of a group of people.

a How many people were surveyed in total?

b How many people with dark hair have blue/green eyes?

c What is the probability of a person chosen randomly from the group having dark hair or blue/green eyes but not both?

	Fair	Dark	Total
Blue/green	23	11	34
Brown	6	35	41
Total	29	46	75

3 The two-way table on the right shows whether Year 9 and Year 10 students prefer alternative or mainstream music. Use the two-way table to find the probability, as a simplified fraction, of a person chosen randomly from the group:

	Year 9	**Year 10**	**Total**
Alternative	14	23	37
Mainstream	29	19	48
Total	43	42	85

a being a Year 9 student who prefers mainstream music

b preferring alternative music

c being a Year 10 student

d being a Year 10 student who prefers alternative music.

4 The following two-way table shows whether Year 7, 8 and 9 students prefer tennis, basketball or hockey. Use this table to find the probability, as a simplified fraction, that a person chosen randomly from this group:

	Year 7	**Year 8**	**Year 9**	**Total**
Tennis	23	26	37	86
Basketball	19	42	34	95
Hockey	31	13	25	69
Total	73	81	96	250

a is in Year 9

b plays basketball

c is in Year 8 and plays tennis

d plays hockey or tennis

e does not play hockey

f is in Year 7 and doesn't play basketball.

11F.2 **5** The following two-way table shows whether students from two classes prefer sweet or savoury food. Convert the two-way table into a two-way relative frequency table.

	Class A	**Class B**	**Total**
Prefer sweet food	23	32	55
Prefer savoury food	29	16	45
Total	52	48	100

6 The following two-way table shows whether a group of introverts and extroverts prefer to read books, play games or play sports. Convert the two-way table into a two-way relative frequency table. Write your answers as fractions in their simplest form.

	Reads books	**Plays games**	**Plays sports**	**Total**
Introvert	56	44	32	132
Extrovert	12	44	68	124
Total	68	88	100	256

11F.3 **7** A survey collected information on whether people with light or dark hair preferred pop music or rock music. Of the respondents, 22% had light hair and preferred pop music, 65% had dark hair and 40% preferred rock music.

Use this information to complete a two-way relative frequency table.

8 A survey collected information on whether people who subscribe to Spotify also subscribe to Netflix. Of the respondents, $\frac{1}{6}$ had a subscription to both Spotify and Netflix, $\frac{1}{4}$ had a subscription to Spotify, and $\frac{1}{3}$ did not have a subscription to Netflix. Use this information to complete a two-way relative frequency table. Write the values in the table as fractions in their simplest form.

9 **a** Copy and complete the two-way table on the right, which shows whether people prefer to watch films at the cinema or at home and whether they prefer action or comedy films.

b Use it to calculate the probability, as a simplified fraction, that a person chosen randomly from the group is someone who prefers to:

i watch films at the cinema

ii watch comedy films at home

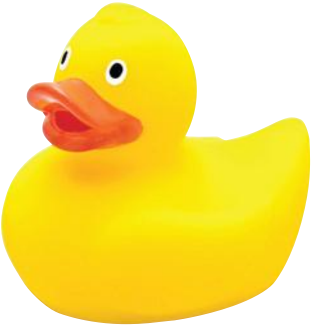
iii watch action films.

	Cinema	**Home**	**Total**
Action	22		
Comedy		14	33
Total			85

ADVANCED

10 A group of 200 people, each with a single pet, were surveyed on their pets. Of the 113 who owned cats, 29 had a purebred. This gives a total of 104 pet owners who owned a purebred. Create a two-way table showing this information and use it to calculate the probability, as a simplified fraction, that a person chosen at random is an owner of a purebred that is not a cat.

11 In a group of 500 men and women who were surveyed on their bathing habits, 60% of women preferred a bath to a shower, whereas 80% of men said they preferred a shower rather than a bath; 55% of the group was female.



a Create a relative frequency two-way table to represent this information using decimals. Hint: The statement '60% of women prefer a bath' refers to 60% of the proportion of women, not the total.

b If a person was randomly selected from the group of 500 people, find the probability that they are:

i a male who prefers to shower

ii someone who prefers a bath

iii a female who prefers to shower.

c Of the group of 500 people, find the number of people who:

i prefer a bath to a shower **ii** are female **iii** are male and prefer a bath.

12 Elsa recorded the make and colour of cars that went past her house over a week and put her results into a two-way table as shown below. Add totals to her results and then find the probability, as a simplified fraction, that a car going past her house is:

a white **b** a Holden **c** a white Holden

d blue or red **e** not a Toyota **f** not black

g neither red nor a Ford **h** blue but not a Hyundai **i** a Mitsubishi but not silver.

	Ford	**Hyundai**	**Holden**	**Mitsubishi**	**Toyota**	**Other**
Silver	11	9	17	8	6	12
White	15	12	16	11	8	10
Red	12	10	8	6	13	14
Blue	8	15	12	7	9	11
Black	13	8	10	10	7	8
Other	4	3	4	2	1	2

13 $n(\text{A})$ means the number of elements in category A. For the relative frequency two-way table shown below, determine the population size if:

a $n(\text{A}) = 30$ **b** $n(\text{B}) = 72$ **c** $n(\text{A and B}) = 98$ **d** $n(\text{A or B}) = 102$

	A	**A'**	**Total**
B	0.1	0.35	0.45
B'	0.4	0.15	0.55
Total	0.5	0.5	1

14 Consider the following two-way relative frequency table. If the total frequency is 120, construct the two-way frequency table.

	A	**A'**	**Total**
B	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{7}{12}$
B'	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{5}{12}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

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15 Complete each pair of two-way frequency and two-way relative frequency tables.

a

	A	**A'**	**Total**
B	40		75
B'			
Total			

	A	**A'**	**Total**
B	0.2		
B'			
Total	0.4		1

b

	C	**C'**	**Total**
D			9
D'		9	
Total			

	C	**C'**	**Total**
D		0.25	
D'			
Total	0.3		1

c

	E	**E'**	**Total**
F	18		
F'		12	
Total			

	E	**E'**	**Total**
F		$\frac{1}{9}$	$\frac{11}{18}$
F'			
Total			1

16 Consider the two-way table on the right, which shows whether a group of Year 9 and Year 10 students prefer Maths or English.

	Year 9	**Year 10**	**Total**
Maths		11	
English	10		
Total	22		50

a Copy and complete the table.

b Use it to find the probability that a student selected randomly from the group:

i prefers Maths **ii** is in Year 10 **iii** is in Year 10 and prefers Maths.

c A student is selected randomly from the group. You know that they are in Year 10.

i How many students are in Year 10?

ii How many students in Year 10 prefer Maths?

iii What is the probability that a student in Year 10 prefers Maths?

d What is the difference between parts **b iii** and **c iii**?

The problem represented in part **c iii** is an example of conditional probability. It looks at the probability of an outcome given certain conditions. In part **c iii**, you are looking for the probability that a student likes Maths, given that they are in Year 10. This means that you consider only the limited group of the condition rather than the entire population.

e Use these steps to calculate the probability of a student being in Year 9, given that they prefer English.

i What is the condition? How many students are in this group?

ii What is the specific group you are after? Hint: It is not just somebody in Year 9.

iii How many students are in this specific group?

iv Use your answers to parts **i** and **iii** to calculate the probability of selecting somebody in Year 9, given that they prefer English.

17 Use the two-way table in question **4** to calculate the probability of randomly selecting a person from the group that:

a plays hockey, given that they are in Year 8

b plays tennis, given that they are in Year 9

c is in Year 7, given that they play basketball

d is in Year 8, given that they play tennis

e is in Year 9, given that they *don't* play hockey

f is not in Year 7, given that they play basketball.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Two-way tables

Topic quiz
11F

11G Venn diagrams

Learning intentions

By the end of this topic you will be able to …

✔ use set notation to represent sets of data

✔ create Venn diagrams to represent two or more sets of data

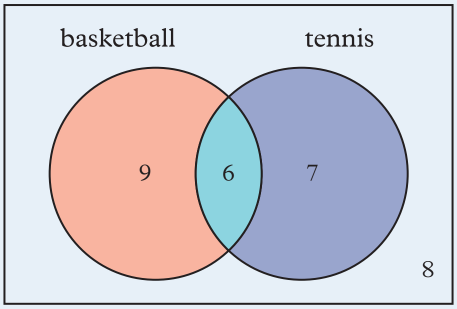
✔ calculate the probabilities of outcomes using data displayed in Venn diagrams.

Inter-year links

Year 9 8E Two-step chance experiments

Venn diagrams

Key content video

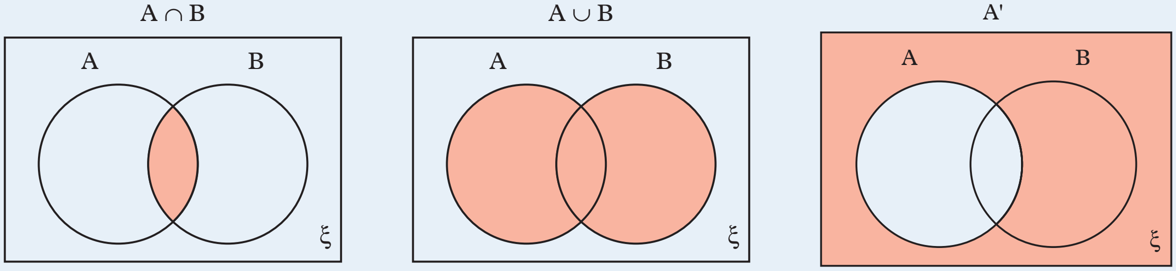


- In mathematics, a set is a collection of distinct objects.
- A **Venn diagram** is used to display the relationship between different sets of data.
- In a Venn diagram, each circle represents a different set and the rectangle containing the circles represents the entire data set.
- Frequencies are placed within Venn diagrams to represent the number of data points for each region of the diagram.

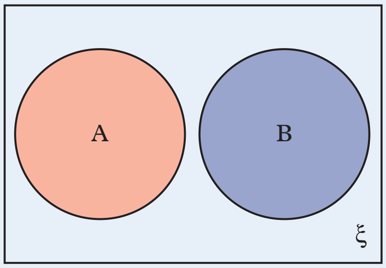
  For example, the Venn diagram on the right shows whether Year 10 students play either basketball or tennis. 6 students play both basketball and tennis (the region where these sets overlap).

Set notation

- All elements contained in a rectangle of a Venn diagram belong to the **universal set**, which is represented by the Greek letter xi (ξ).
- $A \cap B$ means the **intersection** of sets A and B, which includes only the elements that are in A and B. This can be read as 'A and B'.
- $A \cup B$ means the **union** of sets A and B, which includes all elements in either A or B or both. This can be read as 'A or B'.
  - → The intersection ($A \cap B$) and union ($A \cup B$) of sets A and B are also sets themselves.
- A' (or A^c) means the complement of A, which includes all elements not in A.



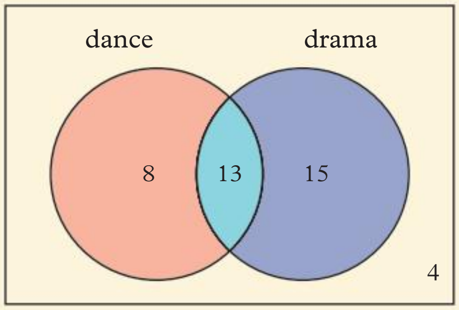
- If set A and set B do not have any common elements, there's no element in $A \cap B$ and the circles representing sets A and B can be drawn with no overlap.



Example 11G.1 Understanding Venn diagrams

The following Venn diagram shows the results of a survey where people were asked if they attend optional dance and drama classes.

a How many people attend both the dance and drama classes?
b How many people attend the drama class?
c How many people do not attend the dance class?
d How many people were surveyed in total?



THINK	WRITE
a Identify the required region, which is the intersection of both sets. This is the middle section where the two circles overlap.	**a** 13 people attend both classes.
b Identify the required region, which is the 'drama' circle. Add the numbers in this region.	**b** 13 + 15 = 28 attend the drama class.
c Identify the required region, which is everything outside the 'dance' circle. Add the numbers in this region.	**c** 15 + 4 = 19 people do not attend the dance class.
d Find the total of all numbers in the Venn diagram.	**d** 8 + 13 + 15 + 4 = 40 people were surveyed in total.

Example 11G.2 Using Venn diagrams to calculate relative frequencies

Use the Venn diagram from Example 11G.1 to find the relative frequency that a person chosen randomly from the survey group attends the dance class.

THINK	WRITE
1 Identify the required region, which is the 'dance' circle. Add the numbers in this region.	8 + 13 = 21 people attended the dance class.
2 Find the total number of people who were surveyed by adding all the numbers in the Venn diagram (don't forget the number outside the circles).	8 + 13 + 15 + 4 = 40 were surveyed.
3 To determine the relative frequency, divide the number in the required region by the total number.	$\text{Relative frequency} = \frac{21}{40}$ $= 0.525$

Example 11G.3 Creating a Venn diagram

In a group of 40 people, 15 have a cat and 22 have a dog. There are 11 people who do not have either a cat or a dog. Draw a Venn diagram to represent this situation.

THINK

1 Draw two circles, one representing cats and the other representing dogs. Draw a rectangle around the two circles.

2 Place the number of people who do not have either a cat or a dog (11) in the rectangle outside the circles.

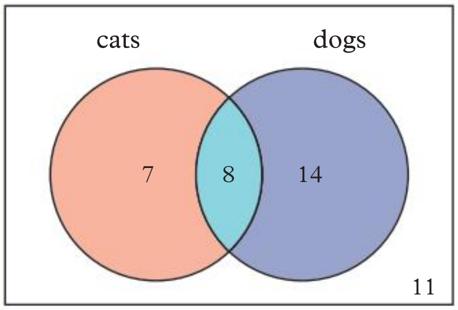
3 Determine the number of people who have a cat only by subtracting the number of people who have a dog (22) and the number of people who do not have either a cat or a dog (11) from the total (40).

4 Repeat step 3 to determine the number of people who have a dog only.

5 Determine the number of people who have a cat and a dog by subtracting the number of people who have a dog only (14) from the number of people who have a dog (22).

WRITE

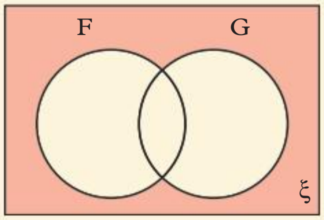
Number of people who have a cat only
$= 40 - 22 - 11$
$= 7$
Number of people with a dog only
$= 40 - 15 - 11$
$= 14$
Number of people with a cat and a dog
$= 22 - 14$
$= 8$



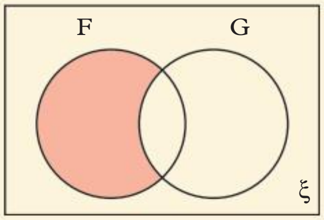
Example 11G.4 Using set notation

Use set notation to describe the following regions in these Venn diagrams.

a



b



THINK

a 1 Look at the shaded region. This represents the elements not in F and not in G.

2 Use set notation to represent the region. Use the complement symbol F′ for 'not in F' and the symbol ∩ for 'and'.

b 1 Look at the shaded region. This represents the elements in F and not in G.

2 Use set notation to represent the region.

WRITE

a $F' \cap G'$

b $F \cap G'$

Note: There is sometimes more than one way to describe a region. For example, the region in part **a** could also be described as $(F \cup G)'$.

Example 11G.5 Converting a two-way table to a Venn diagram

The two-way table on the right details information about the cards found in a standard deck of playing cards. Convert the two-way table into a Venn diagram.

	Ace (A)	Not Ace (A')	Total
Heart (H)	1	12	13
Not Heart (H′)	3	36	39
Total	4	48	52

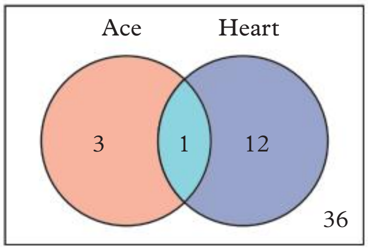
THINK

1 Draw two circles on your Venn diagram to represent the two sets. Label each circle with one of the possible options from each set.

2 Fill in the sections of the Venn diagram with the information from the two-way table.

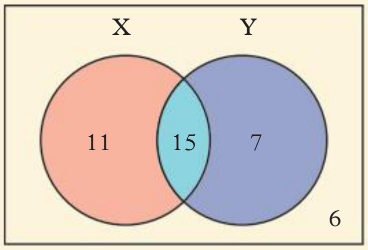
3 Check that the total of the numbers in the Venn diagram equals the total in the bottom right-hand corner of the two-way table.

WRITE



Example 11G.6 Converting a Venn diagram to a two-way table

Convert the following Venn diagram into a two-way table.



THINK

1 Create a table with columns and rows for the sets in the Venn diagram and their complements.

2 Add a row and a column for the totals.

3 Fill in the inner cells of the two-way table with the information from the Venn diagram.

4 Calculate the values in the total row and column, and check the bottom right-hand corner equals the total of the values in the Venn diagram.

WRITE

	X	X'	Total
Y	15	7	22
Y'	11	6	17
Total	26	13	39

Helpful hints

✔ When creating Venn diagrams, check that the total in all the individual regions equals the number in the entire data set.

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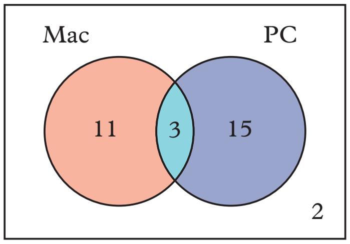
Exercise 11G Venn diagrams

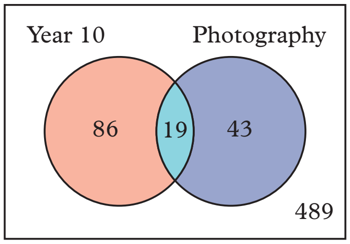
△ 1, 3, 4, 6–9, 11, 16, 17 ☐ 2, 4–8, 10, 12–14, 17, 18, 24 ⬠ 5, 7, 8, 10, 13–15, 17–23, 25

UNDERSTANDING AND FLUENCY



11G.1 1 Consider the Venn diagram on the right that shows whether people own a Mac or a PC.

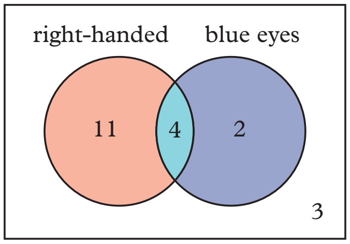
- **a** How many people own a Mac?
- **b** How many people own a PC but not a Mac?
- **c** How many people don't own either a Mac or a PC?
- **d** How many people were surveyed in total?



2 Consider the Venn diagram on the right that shows whether people are in Year 10 and whether they take a Photography class.

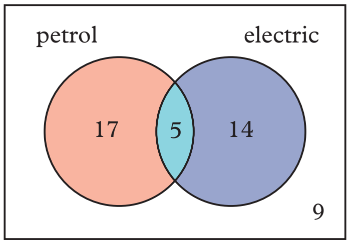
- **a** How many people take Photography?
- **b** How many people are in Year 10 and take Photography?
- **c** How many people are not in Year 10?
- **d** How many people were surveyed in total?

11G.2 3 The following Venn diagram shows whether people are right-handed and whether they have blue eyes.



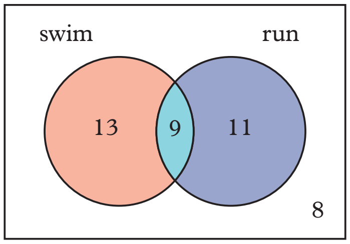
Use this diagram to find the relative frequency of a person:

- **a** being right-handed with blue eyes
- **b** having blue eyes
- **c** being right-handed with eyes not blue
- **d** not having blue eyes
- **e** being left-handed with blue eyes
- **f** being left-handed.



4 The Venn diagram on the right shows whether people drive petrol or electric cars. Use this diagram to find the relative frequency of a person who:

- **a** drives petrol cars only
- **b** drives both petrol and electric cars
- **c** drives electric cars
- **d** drives neither petrol nor electric cars
- **e** does not drive petrol cars
- **f** drives petrol or electric cars (but not both).



5 The Venn diagram on the right shows whether people like to swim or run for exercise. Use this diagram to find the relative frequency of a person who:

- **a** swims and runs
- **b** does not run
- **c** runs but does not swim
- **d** does not swim or run
- **e** swims
- **f** swims or runs.

11G.3 6 In a group of 50 students, 24 are in Year 10, 19 walk to school, and 16 are not in Year 10 and do not walk to school. Draw a Venn diagram to represent this situation.

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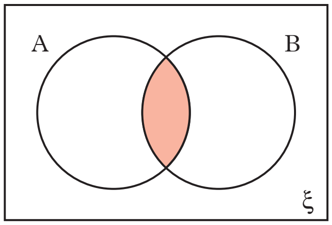
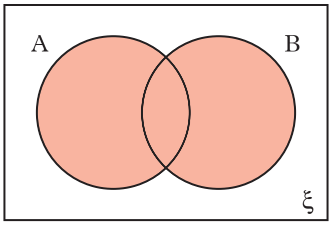
7 In a group of 160 students, 78 take music, 95 take drama, and 46 take music and drama. Draw a Venn diagram to represent this situation.

11G.4 8 Use set notation to describe the following shaded regions in the following Venn diagrams.

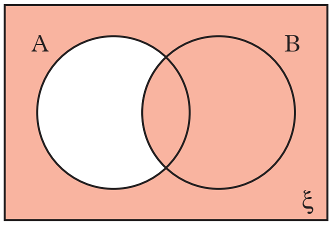
a



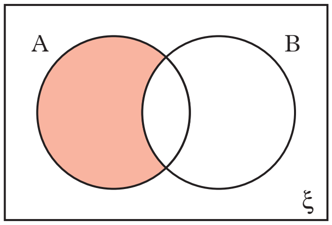
b



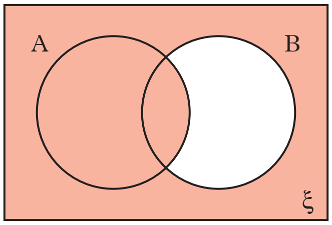
c



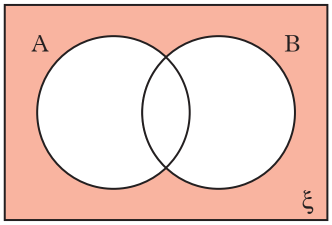
d



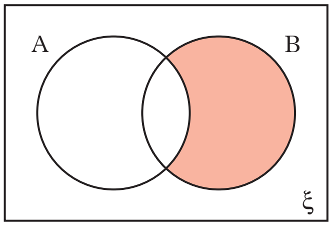
e



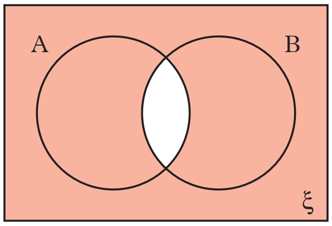
f



g



h



9 The following two-way table details information about the sports that 140 students play. Convert the two-way table into a Venn diagram.

	Netball (N)	**Not netball (N′)**	**Total**
Field hockey (F)	22	10	32
Not field hockey (F′)	45	63	108
Total	67	73	140

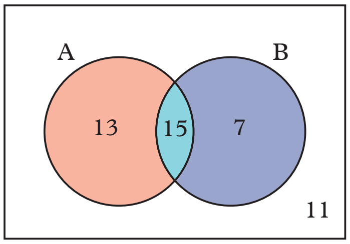
10 A group of 200 Year 10 students were asked if they have subscriptions to Netflix and Spotify. The results are partially displayed in the two-way table below.

	Netflix (N)	**Not Netflix (N′)**	**Total**
Spotify (S)	85	62	147
Not Spotify (S′)			53
Total	126		200

a Fill in the missing numbers in the two-way table.

b Convert the completed two-way table into a Venn diagram.

11 Convert the following Venn diagram into a two-way table.



12 Explain why in a Venn diagram mutually exclusive events can be represented by circles which do not overlap, whereas non-mutually exclusive events need to be represented by circles which overlap.

UNDERSTANDING AND FLUENCY

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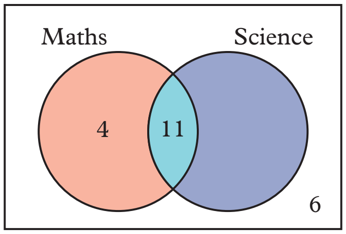
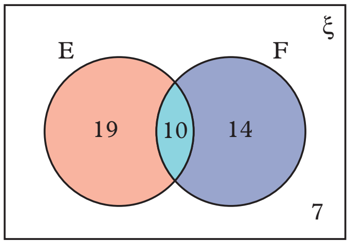


13 A class of 28 students were each asked if they enjoyed studying mathematics and science. The results are partially displayed in the Venn diagram on the right.

a The number of students who enjoy studying science, but not maths, is missing. What is this number?

b Convert the completed Venn diagram into a two-way table.



14 Consider the Venn diagram on the right.

a Given that $n(\text{F})$ means the number of elements in set F, find:

i $n(\text{E})$ **ii** $n(\text{E} \cup \text{F})$ **iii** $n(\text{F}')$

iv $n(\text{E} \cap \text{F})$ **v** $n(\xi)$ **vi** $n(\text{E}' \cap \text{F})$.

b Given that Pr(E) means the probability of selecting an element from set E, find:

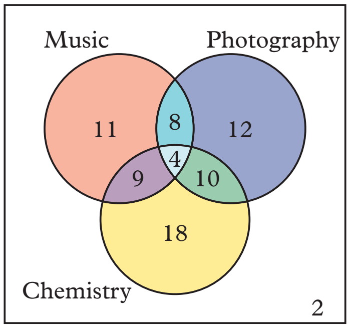
i $\Pr(\text{F})$ **ii** $\Pr(\text{E}')$ **iii** $\Pr(\text{E} \cap \text{F})$

iv $\Pr(\text{E} \cup \text{F})'$ **v** $\Pr(\text{E}' \cap \text{F}')$ **vi** $\Pr(\text{F} \cup \text{E}')$.

15 Consider the Venn diagrams in question **8**.

a State which pairs are complementary by writing both set equalities using the symbol for the complementary event, ′. For example, $(\text{A} \cap \text{B})' = \text{A}' \cup \text{B}'$ and $(\text{A} \cup \text{B})' = \text{A}' \cap \text{B}'$.

b Explain the relationship between the complements of intersections and unions. Note: These relationships are known as De Morgan's laws.



16 Venn diagrams can be used to show the relationship between three or more sets. Consider the Venn diagram on the right which shows whether students take any of three elective subjects at school.

Copy this Venn diagram five times and shade the section that represents students:

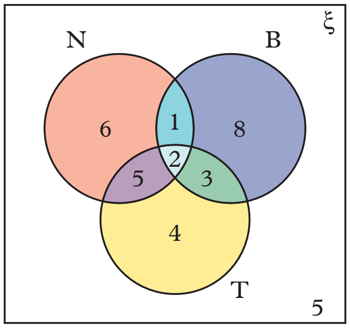
a taking all three elective subjects

b taking only Photography

c taking Music or Chemistry

d taking Music

e not taking any of these three electives.



17 The Venn diagram on the right shows the number of students who play netball, basketball and tennis. Find the probability that a student chosen randomly:

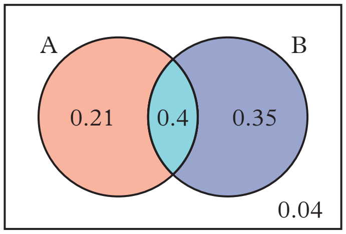
a plays netball but not tennis **b** plays basketball only

c plays all three sports **d** doesn’t play any of these sports

e plays basketball or tennis **f** plays tennis and netball.

18 Use the following information to draw a Venn diagram that shows the relationship between three different sets of data. In a group of 100 people surveyed:

- 35 liked action films
- 46 liked horror films
- 5 people liked all three genres
- 16 only liked horror
- 45 liked romance
- 55 did not like romance films
- 15 people liked both action and romance
- 18 liked both horror and romance
- 19 people did not like films in any of these three genres.



19 The Venn diagram on the right shows the relative frequency of each intersection between the events A, B and their complements. If the sample size was 400, redraw the Venn diagram using frequencies.

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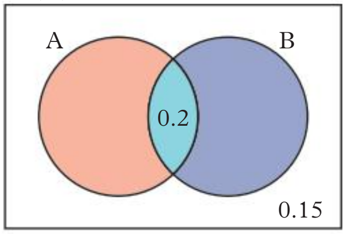
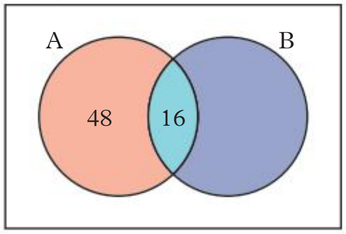
20 The following Venn diagrams represent the same sample of people for the events A and B.

a Complete the Venn diagrams.

b Calculate:

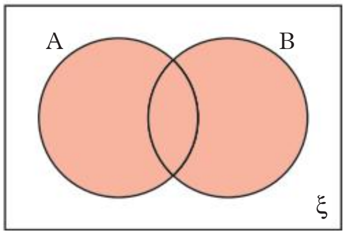
i Pr(A) **ii** Pr(B)



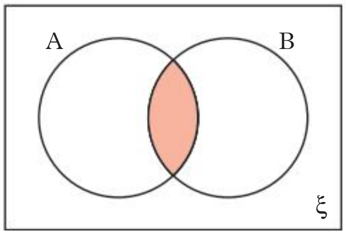


21 a Use set notation to describe the shaded regions in the following diagrams.

i



ii



b Use the diagrams from part **a** to write the probability of an element appearing in either set A or set B: $\Pr(A \cup B)$.

22 The rule to calculate $\Pr(A \cup B)$ (from question **21b**) is known as the addition rule. Explain why this can be simplified to $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ when sets A and B are mutually exclusive.

23 Use the addition rule to solve the following problems.

a Given that $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

b Find Pr(A), given that $\Pr(B) = \frac{1}{3}$, $\Pr(A \cup B) = \frac{4}{9}$ and $\Pr(A \cap B) = \frac{1}{6}$.

CHALLENGE

24 The probability of an event can change when new information is obtained. For example, the probability of selecting a diamond from a deck of cards is $\frac{1}{4}$.

However, if you are told the card is red, then the probability it is a diamond is now $\frac{1}{2}$, based on this new information. This would usually be asked with wording such as: "What is the probability that a randomly selected card from a deck is a diamond, given that it is red?"

The Venn diagram on the right shows whether people are from the country and if they like cricket or not.

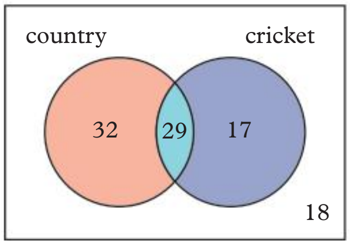
Use this diagram to calculate the probability that a person chosen randomly:

a likes cricket, given that they are from the country

b is from the country, given that they like cricket

c does not like cricket, given that they are from the country

d is not from the country, given that they do not like cricket.



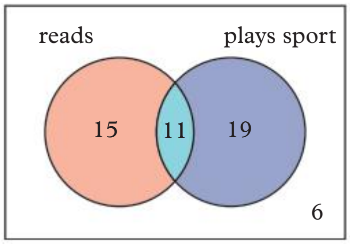
25 The Venn diagram on the right shows whether people like to read and/or play sport. Use this diagram to calculate the probability that a person chosen randomly:

a reads given that they play sport

b plays sport given that they read

c does not play sport given that they read

d does not read given that they do not play sport.



Check your Student obook pro for these digital resources and more:

pro

- Interactive skillsheet Venn diagrams
- Interactive skillsheet Using 'and' and 'or' in probability
- Interactive skillsheet Set notation
- Investigation Venn diagram applications
- Topic quiz 11G

11H Conditional probability

Learning intentions

By the end of this topic you will be able to …

- ✔ understand conditional statements
- ✔ calculate conditional probability using tree diagrams
- ✔ calculate conditional probability using two-way tables and Venn diagrams.

Inter-year links

Year 9 8G Experiments without replacement

Conditional probability

Key content video

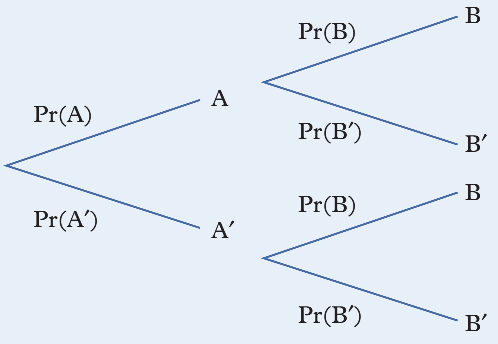
- **Conditional probability** is the probability of an event occurring, given the knowledge that another event has occurred.
  - → Conditional probability is often about a single trial of an experiment. For example, if a standard six-sided die is rolled, what is the probability the outcome is odd, given that the outcome is prime?
  - → Conditional probability restricts the sample space to a subset of the universal set. In the above example, the information 'given that the outcome is prime' restricts the sample space to $\{2, 3, 5\}$ from $\{1, 2, 3, 4, 5, 6\}$. This gives $\Pr(\text{odd given prime}) = \frac{2}{3}$, since two of the three primes are odd.
- $\Pr(A|B)$ is the probability that event A occurs given that event B has occurred. This can be read as 'the probability of A given B'.
- $\Pr(B|A)$, which represents the probability that event B occurs given that event A has occurred, is different from $\Pr(A|B)$.
- Conditional probability can be calculated by using the following formulas:

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \quad \text{and} \quad \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- The above formulas can be rearranged to give:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) \quad \text{and} \quad \Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

  - → This explains why we multiply the probabilities on branches in tree diagrams:



$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$$

$$\Pr(A \cap B') = \Pr(A) \times \Pr(B'|A)$$

$$\Pr(A' \cap B) = \Pr(A') \times \Pr(B|A')$$

$$\Pr(A' \cap B') = \Pr(A') \times \Pr(B'|A')$$

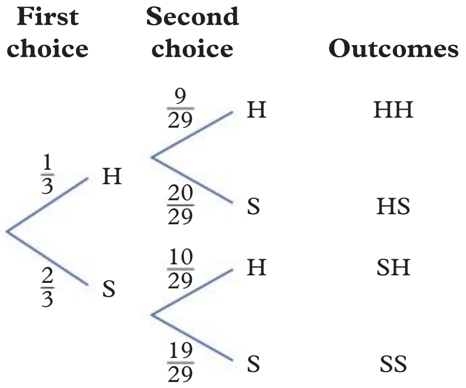
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$ and $\Pr(B|A) = \Pr(B)$, since the fact that A or B occurred does not change the probability of the other event happening.
  - → If $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, then A and B are independent events.
- Conditional probability can be identified when any of the following terms are used in a probability question: 'if … then …', 'given …' or 'knowing that …'
- When calculating conditional probability using two-way tables or Venn diagrams, the words 'of' and 'from' indicate the set from which the choice is being made. For example, 'from the group who …'

Example 11H.1 Calculating conditional probability using a tree diagram

A sandwich tray contains 30 sandwiches, 10 of which are ham (H) and 20 of which are salad (S). Jack takes two sandwiches at random, one after the other.

a Draw a tree diagram to display the possible outcomes of Jack's two choices.

b If Jack's first sandwich is ham, what is the probability that his second sandwich will also be ham?

THINK	WRITE
a Draw a tree diagram to display all the outcomes. The probabilities for the second choice depend on the outcome of the first choice. Write the probabilities on the branches of your tree diagram.	**a** H represents ham and S represents salad. 
b In this case, you are looking for the probability that the second choice is H, given that the first choice is H. Follow the branches H then H on the tree diagram. The probability on the second branch is the probability of the first and second sandwiches both being ham.	**b** $\Pr(H \mid H) = \frac{9}{29}$

Example 11H.2 Testing for independence

For two events, A and B, it is known that $\Pr(A) = 0.6$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.5$.

a Evaluate:

- **i** $\Pr(A \mid B)$
- **ii** $\Pr(B \mid A)$.

b Are A and B independent or dependent events?

THINK	WRITE
a Use the conditional probability formula to evaluate each probability.	**a i** $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.5}{0.75} = \frac{2}{3}$ **ii** $\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.5}{0.6} = \frac{5}{6}$
b Use your answers to part **a** to check if $\Pr(A \mid B)$ is equal to $\Pr(A)$, or if $\Pr(B \mid A)$ is equal to $\Pr(B)$. If yes, then the events are independent. If not, the events are dependent.	**b** A and B are dependent events, since $\Pr(A \mid B)$ is not equal to $\Pr(A)$. Equivalently, $\Pr(B \mid A)$ is not equal to $\Pr(B)$.

ADVANCED

Example 11H.3 Calculating conditional probability using a two-way table

The table below shows the results of a survey taken at a library to find whether people under 18 and people 18 and over prefer using Facebook or Twitter (or neither).

	Under 18	**18 or over**	**Total**
Facebook	20	30	50
Twitter	5	10	15
Neither	10	5	15
Total	35	45	80

a Find the relative frequency that a person prefers Facebook, given they are 18 or over

b Find the relative frequency that a person is 18 and over, given they prefer Facebook.

c Explain why your answers to parts **a** and **b** are not equal.

THINK	**WRITE**
a **1** Identify the total number of people surveyed who are 18 or over by looking at the total number in the '18 or over' column.	**a** total number of people 18 or over = 45
2 Identify the number of people who prefer Facebook in the '18 or over' column.	number of people 18 or over who prefer Facebook = 30
3 Divide the number of people who are 18 or over who prefer Facebook by the total number of people who are 18 or over and give the answer as a fraction, simplifying if possible.	$\begin{aligned}\Pr(\text{prefer Facebook} \mid \text{18 or over}) &= \frac{30}{45}\\ &= \frac{2}{3}\end{aligned}$
b **1** Identify the total number of people who prefer Facebook by looking at the Facebook row.	**b** total number of people who prefer Facebook = 50
2 Identify the number of people who are 18 or over in the 'Facebook' row.	number of the people who prefer Facebook who are 18 or over = 30
3 Divide the number of people who prefer Facebook who are 18 or over by the total number of people who prefer Facebook and give the answer as a fraction, simplifying if possible.	$\begin{aligned}\Pr(\text{18 and over} \mid \text{prefer Facebook}) &= \frac{30}{50}\\ &= \frac{3}{5}\end{aligned}$
c Identify that parts **a** and **b** are asking for information about different groups of people who were surveyed. Recall that Pr(A|B) is usually not equal to Pr(B|A).	**c** The selections are made from different groups of people. In part **a**, the selection is made from those aged 18 or over while, in part **b**, the selection is made from those who prefer Facebook.

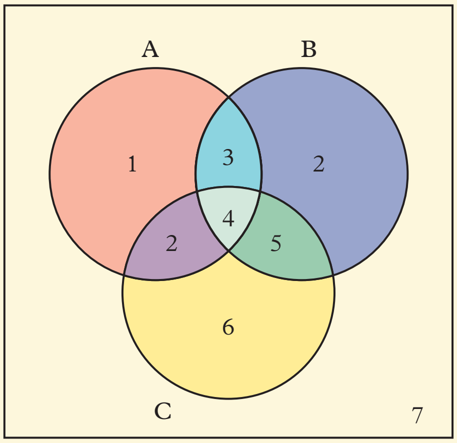
Example 11H.4 Calculating conditional probability using a Venn diagram

The Venn diagram on the right shows the distribution of a group of students who study Art (A), Business Studies (B) and Computing (C).

Use the diagram to determine the relative frequency that a student studies:

a all three of these subjects, given that they study two of the subjects

b Art and Computing, given that they do not study Business Studies.



THINK

a **1** Identify the number of students in the 'given' set. This is the total in all the overlapping sections of the diagram.

2 Identify the number of students within this group who study all three subjects.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

b **1** Identify the number who do not study Business Studies. This is all those outside the circle B.

2 Identify the number of students within this group who study Art and Computing.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

WRITE

a number of students studying two or more subjects $= 3 + 4 + 5 + 2$

$= 14$

number of students studying all three subjects $= 4$

$$\Pr(\text{all three subjects} \mid \text{two subjects}) = \frac{4}{14}$$

$$= \frac{2}{7}$$

b number of students not studying Business Studies $= 1 + 2 + 6 + 7$

$= 16$

number of students in this group studying Art and Computing $= 2$

Pr(Art and Computing | not Business Studies)

$$= \frac{2}{16}$$

$$= \frac{1}{8}$$

Helpful hints

✔ When working with conditional probability, remember to determine the number of elements in the conditional (or given) set first.

✔ When calculating conditional probability, the given number always goes on the denominator.

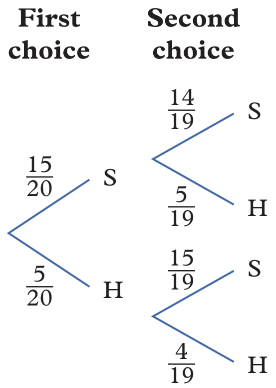
ANS
p796

Exercise 11H Conditional probability

△ 1, 3–9, 11, 12, 14, 16, 18, 19 ■ 2–5, 7, 8, 10, 12, 13, 15–17, 19, 21 ⬠ 3, 5, 7, 8, 10, 12, 13, 17, 19–24

UNDERSTANDING AND FLUENCY

1 A box of 20 chocolates contains 15 chocolates with soft centres and 5 chocolates with hard centres. Two chocolates are taken at random, one after the other. The tree diagram on the right represents the outcomes of the two choices.



- **a** Explain the probabilities of $\frac{15}{20}$ and $\frac{5}{20}$ on the first two branches of the tree diagram.
- **b** Explain the probabilities on the four branches for the four possible second choices.
- **c** If the first chocolate is hard, what is the probability that the second chocolate will also be hard?
- **d** Knowing that the first chocolate is a soft one, what is the probability the second chocolate will be hard?
- **e** What is the probability of the second chocolate being soft, given the first chocolate is hard?

11H.1 **2** A bag contains 6 balls: two black and four white. A ball is selected at random, its colour noted, and then another ball is selected without the first ball being replaced.

- **a** Draw a tree diagram to display this situation.
- **b** If the first ball is white, what is the probability the second ball will also be white?
- **c** Knowing that the first ball is black, what is the probability the second ball is white?
- **d** What is the probability that the second ball is black, given that the first ball is white?

3 A box contains 7 balls: four green and three red. Three balls are selected without replacement.

- **a** Draw a tree diagram to display this situation.
- **b** If the first ball is green, what is the probability the second ball will also be green?
- **c** Knowing that the first two balls are both green, what is the probability that the third ball will also be green?
- **d** What is the probability that the third ball is green, given that a green ball and then a red ball were drawn in the first two selections?

11H.2 **4** For two events, A and B, it is known that $\Pr(A) = 0.4$, $\Pr(B) = 0.5$ and $\Pr(A \cap B) = 0.2$.

- **a** Evaluate:
  - **i** $\Pr(A|B)$
  - **ii** $\Pr(B|A)$.
- **b** Are A and B independent or dependent events?

5 For two independent events, M and N, it is known that $\Pr(M) = \frac{2}{3}$ and $\Pr(N) = \frac{1}{4}$. Find each of the following.

- **a** $\Pr(M \cap N)$
- **b** $\Pr(M|N)$
- **c** $\Pr(N|M)$

11H.3 **6** Consider the following two-way table showing cards from a standard deck of cards.

	Ace (A)	**Not ace (A′)**	**Total**
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

Use this two-way table to determine the relative frequency of a card being:

- **a** not an Ace, given it is a heart
- **b** not an Ace, given it is not a heart.

7 The two-way table on the right shows the modes of transport a group of students use to get to school. State the number of:

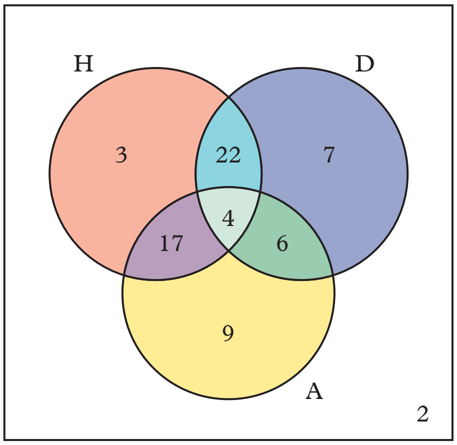
- **a** primary school students who catch public transport to school
- **b** high school students who are driven to school
- **c** students who walk or ride to school
- **d** students who are in high school.

	Primary school	**High school**	**Total**
Catch public transport	15	27	42
Are driven	22	18	40
Walk/ride	4	14	18
Total	41	59	100

8 Using the two-way table from question **7**, calculate the relative frequency of a student:

- **a** catching public transport to school, given they are at primary school
- **b** being at high school, given they are driven to school
- **c** walking or riding to school
- **d** being at high school
- **e** being driven to school, given they are at high school.

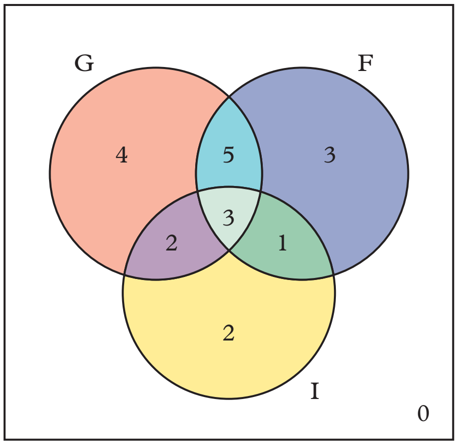
11H.4 9 The following Venn diagram shows the distribution of a group of 70 students who were asked if they enjoy watching horror films (H), drama films (D) and action films (A).



Use the Venn diagram to determine the relative frequency that a student enjoys:

- **a** all three genres of film, given that they enjoy two of the film genres
- **b** horror films, given that they enjoy action films.

10 Twenty members of the Swiss national soccer team were asked which languages they speak fluently, out of German (G), French (F) and Italian (I). Their responses are shown in the Venn diagram on the right.



- **a** One of the twenty players is selected at random. Find the probability that they speak:
  - **i** German
  - **ii** German, but not French
  - **iii** German, given that they speak French
  - **iv** French, given that they speak German
  - **v** all three languages, given that they speak at least two of the languages.
- **b** An Italian-speaking player is selected at random. Find the probability that this player can also speak French.

11 For two events, A and B, it is known that $\Pr(A|B) = 0.4$, $\Pr(A \cap B) = 0.1$ and $\Pr(A) = 0.3$.

a Find $\Pr(B|A)$. Give your answer as a fraction. **b** Find $\Pr(B)$. Give your answer as a decimal.

12 In each case, decide if A and B are independent events or dependent events.

a $\Pr(A) = 0.4, \Pr(B) = 0.3, \Pr(A \cap B) = 0.2$ **b** $\Pr(A) = \frac{3}{5}, \Pr(B) = \frac{2}{3}, \Pr(A \cap B) = \frac{2}{5}$

c $\Pr(A) = 0.5, \Pr(B) = 0.8, \Pr(A \cap B') = 0.1$ **d** $\Pr(A) = 0.6, \Pr(B) = 0.5, \Pr(A|B) = 0.3$

13 A computer generates random questions for students to use for practice. From the collected data, the chance that a student gets the first question correct is 0.8. If the student gets this question correct, the computer generates a harder question, with the probability of the student getting this one correct being reduced by 0.1. If the student gets the first question wrong, the computer gives the student a question of the same level of difficulty as the first question.

a Draw a tree diagram to display the possible outcomes of the two practice questions.

b Find the probability, expressed as a decimal, that the student gets the second question:

i correct, given they got the first question correct

ii correct, given they got the first question wrong

iii wrong, given they got the first question correct.

14 At a small firm with 50 employees, some work full-time and others work part-time. There are 15 employees who work from home (WFH) in total and 30 of the 34 full-time employees are hybrid workers who split their time between the office and at home.

a Draw a two-way table to record this data. Complete all the entries in the table.

b How many employees are:

i full-time employees who work from home

ii part-time

iii hybrid workers?

c Find the relative frequency of an employee:

i working part-time, given they are a hybrid worker

ii being a hybrid worker, given they work part-time

iii being a WFH employee given they work full-time

iv working full-time, given they are a WFH employee.

d Explain the differences in your answers for the following parts of part **c**:

i parts **i** and **ii** **ii** parts **iii** and **iv**.

15 The two-way table on the right shows the number and types of paper clips in a pile on a desk.

	Small	**Large**	**Total**
Silver	27	35	
Coloured	18	10	
Total			

a Complete the totals for the rows and columns.

b Determine whether each of these statements is true, showing the calculations you used. (Percentages are rounded to the nearest whole number.) For those statements which are not true, show working to provide a correction.

i There is an equal number of small and large paper clips.

ii There are fewer coloured paper clips than silver ones.

iii The percentage of small paper clips which are silver is 44%.

iv The percentage of coloured paper clips which are large is 22%.

v 69% of the paper clips are silver.

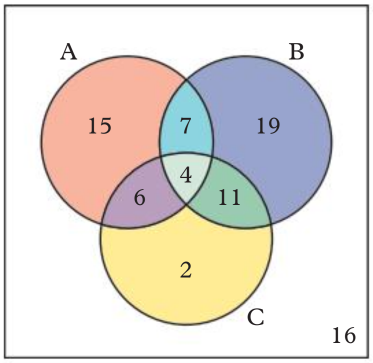
16 A standard six-sided die is rolled twice, and both times a 4 is rolled. Explain why, if the die is rolled again, the probability of rolling a 4 again is unchanged.

17 Alisha believes that as conditional probability restricts the sample space, it will always result in a probability that is greater than at the previous step in a multi-step experiment. Kaitlyn thinks this is not the case, and the probability could be either greater than or less than at the previous step. Explain who is correct and why.

ADVANCED

18 A jar contains black and white balls. Two balls are drawn without replacement. The probability of selecting a black ball on the first draw is 0.47, and the probability of selecting a second black ball, given that the first ball is black, is 0.22. What is the probability of selecting a white ball on the second draw, given that the first ball is black? Give your answer as a decimal.

19 Consider the Venn diagram on the right, which shows survey data gathered from 80 people.



a Find each of the following.

i $n(\mathrm{A})$ **ii** $n(\mathrm{B})$ **iii** $n(\mathrm{C}')$

b One person from the 80 is selected at random. Find:

i $\Pr(\mathrm{B}|\mathrm{A})$ **ii** $\Pr(\mathrm{A}|\mathrm{B})$ **iii** $\Pr(\mathrm{A}'|\mathrm{C}')$.

c Explain how the answers to part **a** are helpful in answering part **b**.

20 For two events, X and Y, it is known that $\Pr(\mathrm{X}|\mathrm{Y}) = 0.2$ and $\Pr(\mathrm{Y}|\mathrm{X}) = 0.5$.

a Find:

i $\Pr(\mathrm{X}'|\mathrm{Y})$ **ii** $\Pr(\mathrm{Y}'|\mathrm{X})$.

b If $\Pr(\mathrm{X} \cap \mathrm{Y}) = 0.05$, then find $\Pr(\mathrm{X})$ and $\Pr(\mathrm{Y})$.

21 There are b blue marbles and r red marbles in a bag and there are at least two marbles of each colour. Two marbles are drawn without replacement.

a Draw a tree diagram to represent this situation. Include all probabilities in terms of b and r on the branches.

b Find the probability that the second ball drawn is blue. Comment on this value.

22 If an event, A, relates to the second step of a multi-step experiment, then it may be necessary to split A into cases in order to determine its probability. That is, $\Pr(\mathrm{A}) = \Pr(\mathrm{A} \cap \mathrm{B}) + \Pr(\mathrm{A} \cap \mathrm{B}')$, meaning that A can either occur with B or without B, where B relates to the first step of the experiment. Then, using the conditional probability formula, $\Pr(\mathrm{A}) = \Pr(\mathrm{B}) \times \Pr(\mathrm{A}|\mathrm{B}) + \Pr(\mathrm{B}') \times \Pr(\mathrm{A}|\mathrm{B}')$.

Imagine that a soccer team has a 40% chance of winning their upcoming game if it rains and a 70% chance of winning their upcoming game if it does not rain.

a Find the probability that the team wins their upcoming game if there is a 20% chance of rain. Give your answer as a decimal.

b If the probability of the team winning their upcoming game is 50%, what is the probability that it rains on that day? Give your answer as a fraction.

23 Bayes' theorem is a useful conditional probability formula, which states that $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{B}|\mathrm{A}) \times \Pr(\mathrm{A})}{\Pr(\mathrm{B})}$.

Bayes' theorem can be proved using the conditional probability formulas for $\Pr(\mathrm{A}|\mathrm{B})$ and $\Pr(\mathrm{B}|\mathrm{A})$.

a Prove Bayes' theorem using $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{B})}$ and $\Pr(\mathrm{B}|\mathrm{A}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{A})}$.

b Two boxes each contain white balls and black balls. Box X contains 7 white and 3 black balls, whereas Box Y contains 2 white and 3 black balls. One box is chosen at random and then a ball is selected from it at random. Let X be the event 'selecting box X' and W be the event 'selecting a white ball.'

i Find $\Pr(\mathrm{W})$. Give your answer as a decimal.

ii Use Bayes' theorem to find $\Pr(\mathrm{X}|\mathrm{W})$, which is the probability that Box X was chosen, given that the selected ball is white. Give your answer as a fraction.

24 a Use Venn diagrams to show to that $\Pr(\mathrm{A}) = \Pr(\mathrm{A} \cap \mathrm{B}) + \Pr(\mathrm{A} \cap \mathrm{B}')$.

b Prove that if A and B are independent events, then A and B′ are also independent.

PROBLEM SOLVING AND REASONING

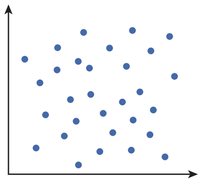
CHALLENGE

Check your Student obook pro for these digital resources and more:

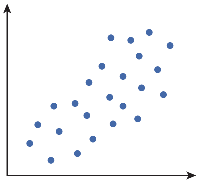
Interactive skillsheet Conditional probability	**Worksheet** Conditional probability	**Investigation** Bertrand's box	**Topic quiz** 11H

Chapter summary

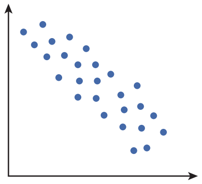
Scatter plots



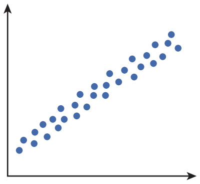
- No obvious pattern
- No correlation



- Weak linear pattern
- Weak correlation
- Positive direction



- Clear linear pattern
- Moderate correlation
- Negative direction



- Strong linear pattern
- Strong correlation
- Positive direction

Analysing reports

- Graph axes should be evenly spaced with consistent numbering
- Graphs should be properly labelled
- Check that no data has been purposefully omitted
- Question the claims made in reports
- Ask whether the data is presented in a fair manner

Line of best fit

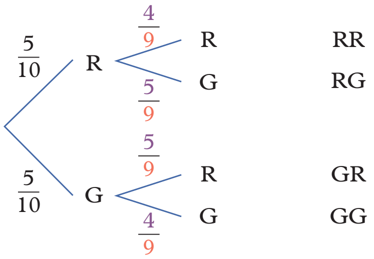
- A straight line that lies as close to as many points as possible
- Approximately the same number of points on each side of the line
- Can be used to make predictions

Sampling methods

- Stratified sampling divides the population into strata and takes a random sample from each strata.
- Systematic sampling selects data at fixed intervals.
- Convenience sampling is when a sample that is easily accessible is selected.
- Quota sampling is when the researcher selects the size of each group to sample.

Experiments without replacement

- When multi-step experiments involve items being selected from a group, if the items are not replaced after each selection, the probabilities will change.



Independent events

- Two events are independent if the outcome of one event has no effect on the outcome of the other event.
- If A and B are independent events, then $Pr(A|B) = Pr(A)$.
- If $Pr(A \cap B) = Pr(A) \times Pr(B)$, then A and B are independent events.
- The outcomes for each step of a chance experiment with replacement are independent.
- The outcomes for each step of a chance experiment without replacement are not independent.

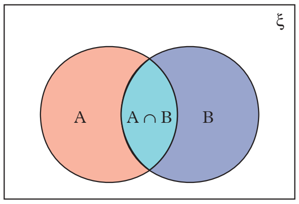
Two-way tables

	School A	School B	Total
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Set notation

- A set is a collection of distinct objects.
- ξ = the universal set containing all the elements under consideration
- $A \cap B$ = the intersection of two sets, A and B
- $A \cup B$ = the union of two sets, A and B
- A' = the complement of set A, everything from ξ that is not in A
- $n(A)$ = the numbers of elements in set A

Venn diagrams



Conditional probability

- $Pr(A|B)$ = the probability that event A occurs given that event B has occured.

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \qquad Pr(A|B) \neq Pr(B|A)$$

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Chapter review

Chapter review quiz
Assess your knowledge of this chapter.
✔ Core
✔ Standard
✔ Advanced

Quizlet
Test your knowledge of this topic by working individually or in teams.

Mathematical literacy review

The following key terms are used in this chapter:

- aim
- array
- bivariate data
- causal relationship
- complement
- compound event
- conditional probability
- correlation
- dependent variable
- ethics
- event
- experiment
- hypothesis
- independent events
- independent variable
- inferences
- intersection
- line of best fit
- mutually exclusive events
- negative relationship
- outcomes
- population
- positive relationship
- sample
- sample space
- scatter plot
- self-selected sampling
- set
- stratified sampling
- survey
- systematic sampling
- tree diagram
- two-way table
- union
- universal set
- Venn diagram

1 How many variables does bivariate data contain?

2 Explain the difference between the intersection and the union of two sets.

3 Explain the difference between a positive and a negative relationship between two numerical variables.

4 What is meant by mutually exclusive events?

5 Identify the key terms being referenced in each of these definitions.

a a sampling method where data is selected at fixed intervals with a random starting point

b the set containing all elements, represented by the Greek letter xi (ξ)

6 Using an example, provide a definition in your own words for the following key terms.

a causal relationship

b independent events

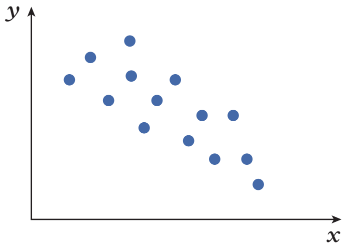
7 Complete the following sentences using words from the key term list.

a The ________ of an event A is the event that A does not occur.

b ________ is the probability of an event occurring, given the knowledge that another event has occurred.

Multiple choice

Questions **1–2** refer to the scatter plot on the right, which shows a linear relationship between two variables, *x* and *y*.



11A **1** How would you describe the linear relationship between the two variables?

A strong
B weak
C moderate
D no correlation
E impossible to determine

11A **2** Which of these terms also describes the linear relationship between the two variables displayed on the graph?

A positive
B negative
C horizontal
D vertical
E random

STN 11B ADV **3** If a stratified sample of 20 people is to be taken from a group of 280 children and 120 adults, how many adults should be chosen?

A 6
B 10
C 14
D 20
E 60

11C **4** The graph on the right displays the number of penguins in a nature reserve over a period of 6 months.

Why is the graph misleading?

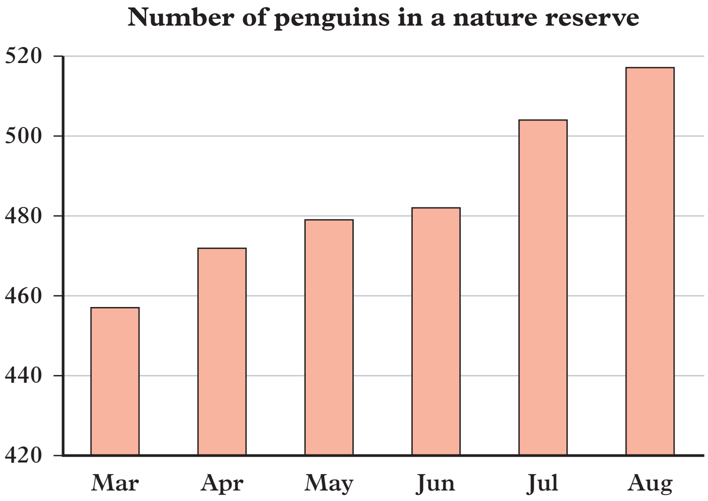
A The *x*-axis doesn't start at 0.

B The *y*-axis doesn't start at 0.

C Not all values are present on the *x*-axis.

D The scale on the *y*-axis is inconsistent.

E The maximum value on the *y*-axis is too high.



11D **5** A coin is flipped three times. How many outcomes would appear on a tree diagram representing this experiment?

A 2 **B** 4 **C** 6 **D** 8 **E** 10

11D **6** A letter is randomly selected from a word. The sample space for the experiment is {A, E, L, P}. Which of the following could be the word?

A HELP **B** PLEASE **C** PLEAD **D** APPLE **E** ELLA

11E **7** A standard deck of cards contains 52 cards, with 13 cards of each of four suits (clubs, diamonds, hearts and spades). One card is drawn from a deck of playing cards and its suit is noted. It is replaced, then a second card is drawn. What is the probability two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

11E **8** Repeat the experiment from question **7** but, this time, assume the first card is not replaced before the second card is drawn. What is the probability that two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

ADV **11F** **9** Consider the following two-way table.

	Under 30s	**30 and over**	**Total**
Prefer coffee	176	98	
Prefer tea	101	25	126
Total	277		400

The relative frequency of a person preferring coffee is:

A $\frac{137}{200}$ **B** $\frac{49}{200}$ **C** $\frac{11}{25}$ **D** $\frac{101}{400}$ **E** $\frac{176}{277}$

ADV **11F** **10** If this two-way table is converted to a two-way relative frequency table, which relative frequency will not appear?

	A	**A'**	**Total**
B	5	5	10
B'	7	8	15
Total	12	13	25

A 0.2 **B** 0.28 **C** 0.52 **D** 0.5 **E** 0.4

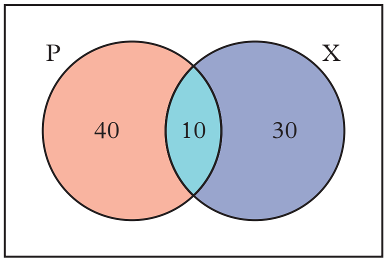
ADV **11F** **11** Based on the following incomplete two-way table, what is Pr(A)?

	A	**A'**	**Total**
B	12		24
B'		16	
Total			50

A 0.44 **B** 0.5 **C** $\frac{6}{11}$ **D** $\frac{14}{25}$ **E** 0.48

ADV **11G** **12** A group of 80 students were asked if they owned a PlayStation (P) or an Xbox (X). Of the students, 40 students owned a PlayStation, 30 students owned an Xbox, and 10 said they owned both. Which of the following Venn diagrams correctly displays this information?

A



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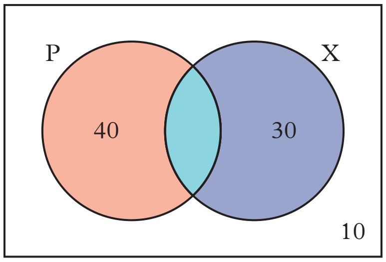

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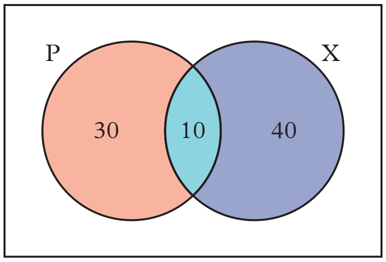

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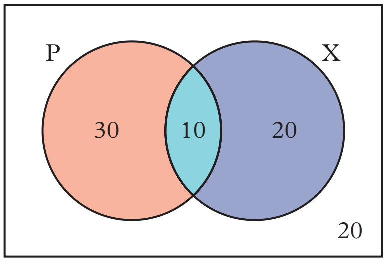
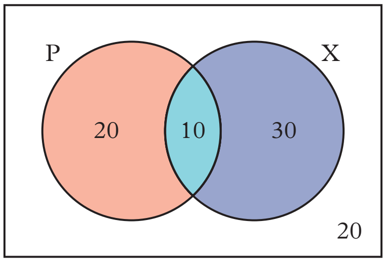
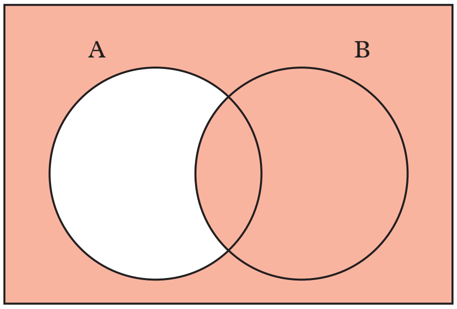


E

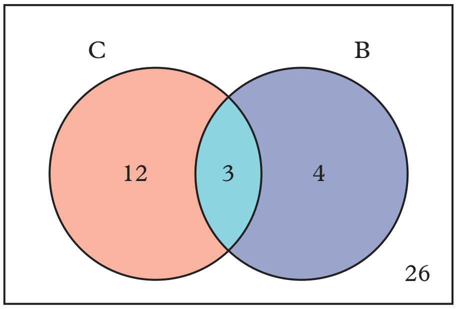


ADV **11G** **13** Which of the following is represented by the shaded section in the Venn diagram?



A $A \cap B$ **B** $A \cup B$ **C** $A' \cup B$ **D** $A' \cap B$ **E** $A \cap B'$

ADV **11G** **14** The following Venn diagram shows the number of students who play chess and badminton.



One of these students is selected at random. What is the probability that they play chess?

A $\frac{1}{15}$ **B** $\frac{12}{19}$ **C** $\frac{1}{3}$ **D** $\frac{4}{15}$ **E** $\frac{7}{45}$

ADV **11H** **15** Based on the Venn diagram in question **14**, what is the probability that a student plays badminton given that they also play chess?

A $\frac{1}{3}$ **B** $\frac{7}{45}$ **C** $\frac{1}{15}$ **D** $\frac{3}{7}$ **E** $\frac{1}{5}$

ADV **11H** **16** For two events, A and B, it is known that $\Pr(B) = 0.6$ and $\Pr(A \cap B) = 0.2$. What is $\Pr(A|B)$?

A $\frac{1}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{5}$ **D** $\frac{2}{3}$ **E** $\frac{2}{5}$

Short answer

11A **1** This table shows the number of children enrolled at a day-care centre over the 12 years since its opening.

Years since opening	1	2	3	4	5	6	7	8	9	10	11	12
Number of children	25	27	26	33	32	35	37	39	40	39	40	45

a Create a scatter plot for the data.

b Describe the relationship between the two variables.

c Draw a line of best fit by eye on the scatter plot you have created.

d Use your line of best fit from part **c** to predict the number of children the day-care centre could expect to have 20 years after opening.

STN 11B ADV **2** **a** A survey was conducted by selecting the first person in the top left-hand column of every page of the local telephone book. What type of sampling would this be?

b A survey was conducted by randomly selecting a proportionate number of students in each year level. What type of sampling would this be?

STN 11C ADV **3** Critically analyse each of these reports.

a A survey conducted at a yoga studio revealed that 90% of the studio's clients felt they were getting enough exercise each week. It was later quoted that: 'Australians are getting fitter, as 90% of them say they exercise regularly each week.'

b Four teenage boys were still left on a bus when it reached the end of the line. A reporter asked them if they had part-time jobs. Two of them said they did. It was later reported that, '50% of teenagers today have part-time jobs.'

c A survey showed that four out of five children believe they do their fair share of chores at home. A report later appeared in a newspaper with the heading, 'Families lucky: 80% of children do fair share of chores.'

d The children in the stands watching a school swimming carnival, held indoors, were asked whether they were wearing sunscreen to prevent them getting sunburnt. They all said that they had not worn sunscreen. It was later reported that 'School children ignore warnings about wearing sunscreen to prevent skin cancer.'

11D **4** A ten-sided die (with sides numbered 1–10) is rolled and the number facing up is recorded.

a List the sample space.

b If A is the event that the number facing up is odd and B is the event that the number facing up is a multiple of 3, then list the elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$ **v** $A \cap B'$.

11D **5** A bag contains 26 tiles with the letters of the alphabet. One letter is randomly chosen, whether it's a vowel or consonant is recorded, and then the tile is replaced. This is repeated another two times. Draw a tree diagram to find the probability of choosing:

a no vowels **b** at least one vowel

c three consonants **d** no more than two consonants.

11E **6** A three-digit number is made using the digits 0 to 9. The digits can be repeated.

a Describe any restrictions on the formation of such a three-digit number.

b How many three-digit numbers are possible?

c What is the probability the number will be 111?

d What is the probability that all the digits in the number will be the same?

11E **7** The experiment in question **6** is repeated, this time not allowing any repetition of the digits.

a Describe any restrictions on the formation of the number.

b How many numbers are possible in this case?

c What is the probability the number will be 107?

d How many numbers with a value greater than 500 are possible?

ADV 11F **8** For the following questions, refer to this two-way table for Years 8, 9 and 10 students and their preferred brands of make-up.

	Year 8	**Year 9**	**Year 10**	**Total**
Napoleon Perdis	32	25	45	
Rimmel	22		22	54
Covergirl	17	31	13	
Maybelline		29		
Total	120		95	310

a Copy and complete the table.

b Calculate the probability that a student chosen at random from the group:

- **i** prefers Maybelline
- **ii** is in Year 8 and prefers Rimmel
- **iii** is in Year 9 and prefers Napoleon Perdis.

c Calculate the probability that a student chosen from the Year 9 students:

- **i** prefers Napoleon Perdis
- **ii** prefers Covergirl or Maybelline.

ADV 11F **9** The table on the right shows the number of iPhone users in a group of students.

State the number of:

a Year 8 students who do not use an iPhone

b students who do not use an iPhone

c Year 10 students

d students who use an iPhone.

	Year 8	**Year 10**	**Total**
iPhone	20		
Other type of phone		12	
Total	35		82

ADV 11G **10** A survey of 120 secondary school students was conducted about whether they regularly play video games and board games. Of the 20 Year 9 students surveyed, 8 play video games, 10 play board games, and 3 play both. Of the students in other year levels who were surveyed, 55 play video games, 60 play board games and 35 play both.

Let A be the set of Year 9 students and let V be the set of students who play video games.

a Use a Venn diagram with circles for the sets A and V to display this information.

b Calculate:

- **i** $\Pr(A \cap V)$
- **ii** $\Pr(A \cup V)$
- **iii** $\Pr(A' \cap V')$
- **iv** $\Pr(V)$.

Let B be the set of students who play board games.

c Use a Venn diagram with circles for the sets B and V to display the information.

d Calculate:

- **i** $\Pr(B \cap V)$
- **ii** $\Pr(B' \cup V)$
- **iii** $\Pr(B \cap V')$
- **iv** $\Pr(B)$.

ADV 11G **11** Sets A, B and C overlap, and can be described as follows:

- $n(A) = 14$
- $n(B) = 13$
- $n(C) = 15$
- $n(A \cap B) = 6$
- $n(B \cap C) = 5$
- $n(A \cap C) = 7$
- $n(A \cap B \cap C) = 4$

Draw a Venn diagram to display this information.

ADV 11H **12** Two pens are taken in turn from a box containing four blue pens, five black pens and two red pens. The pens are not replaced after they have been selected. Calculate the probability that:

a the second pen is red, given the first pen is blue

b the second pen is red, given the first pen is black

c the two pens are the same colour

d the two pens are different colours.

ADV 11H **13** A letter is chosen at random from the word CONDITIONAL. Find the probability that the chosen letter is:

a C

b N

c O, given that it is a vowel

d A vowel, given that it is O

e A vowel, given that it is adjacent to an N.

Analysis

1 A student must select two from these four electives in Year 10: Drama, Art, Italian and Food Technology. The student writes each subject on a card and then chooses two at random from a hat.

  a Would this selection be with or without replacement? Explain your reasoning.

  b **i** Draw a tree diagram to represent the possible subject combinations.

  ii How many combinations are possible?

  c Find the probability of selecting:

  i Art and Food Technology **ii** Italian and Drama.

ADV **d** The senior students at the school were surveyed about the Food Technology and Art electives offered. 12% of the respondents were Year 12 students who preferred Art to Food Technology; 28% of the respondents were Year 11 students who preferred Food Technology to Art. Overall, 52% of the respondents were in Year 11.

  i Construct a two-way table showing these percentages as relative frequencies.

  Based on these survey results, if a person was randomly selected from the 200 students who participated in the survey, find the probability that they are:

  ii a Year 12 student who prefers Food Technology

  iii a Year 11 student who prefers Art

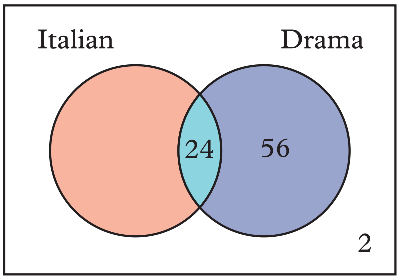
  iv a student who prefers Art.

  Of these 200 students, find the number of people who:

  v are in Year 12 and prefer Food Technology

  vi prefer Food Technology.

ADV **e** An analysis of the numbers of students studying Italian and/or Drama was also undertaken and the results are collated into the following Venn diagram.



  i How many of the 150 students study both Italian and Drama?

  ii How many students study neither subject?

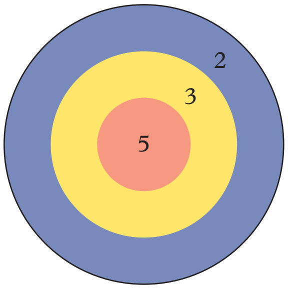
  iii How many students study Italian only?

  iv If selecting a student at random from the cohort, what is the probability that they study Italian only?

  v If selecting a student at random from the cohort, what is the probability that they study Drama only?

2 An archery target consists of an inner circle of radius 10 cm, encircled by two rings, each with width 10 cm. An arrow which lands in the outermost ring is worth 2 points, an arrow which lands in the middle ring is worth 3 points, and an arrow which lands in the inner circle is worth 5 points, as shown in the diagram on the right.

An archer shoots two arrows, both of which hit this target.



  a Find the fraction of the target area that is taken up by the region worth:

  i 5 points **ii** 3 points **iii** 2 points.

  b Write the sample space for the sum of the two scores from the archer's two shots.

Assume that each arrow has an equally likely chance of hitting anywhere on the target.

  c Find the probability that, from their two shots, the archer scores:

  i 10 points **ii** exactly 5 points **iii** at least 8 points.

It is not realistic to assume that the archer has an equally likely chance to hit any point on the target. Instead, assume that, for each shot, the archer has a 20% chance of hitting the outermost ring, a 50% chance of hitting the middle ring, and a 30% chance of hitting the inner circle.

d Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 6 points **iii** no more than 5 points.

It is reasonable to think that the number of points the archer scores on the second shot is dependent on their score from the first shot. Assume that the archer learns from their first shot and will therefore score at least as many points on the second shot as they did on their first shot.

If the archer scores 2 points on their first shot, then the probabilities for their second shot remain 20%, 50%, and 30% for scoring 2, 3, or 5 points, respectively. However, if the archer scores 3 points on their first shot, then they have an even chance of scoring either 3 or 5 points on their next shot. If the archer scores 5 points on their first shot, then they will score 5 points again on their second shot.

ADV **e** Under these conditions, find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 8 points **iii** at least 6 points.

Chapter checklist

Now that you have completed this chapter, reflect on your ability to do the following.

I can do this	I need to review this
☐ Interpret and draw scatter plots ☐ Determine the strength and direction of a linear relationship from a scatter plot ☐ Draw a line of best fit by eye	☐ Go back to Topic 11A Bivariate data and scatter plots
☐ Evaluate the appropriateness of sampling methods and sample size	☐ Go back to Topic 11B Sampling methods
☐ Evaluate where graphs or claims may be misleading	☐ Go back to Topic 11C Analysing reports
☐ Calculate the probability of experiments with and without equally likely outcomes ☐ Calculate theoretical probabilities using tree diagrams and arrays	☐ Go back to Topic 11D Theoretical probability
☐ Calculate probabilities of multi-step experiments with and without replacement	☐ Go back to Topic 11E Experiments with and without replacement
☐ Create two-way relative frequency tables ☐ Use two-way tables to solve problems	☐ Go back to Topic 11F Two-way tables
☐ Use set notation to represent sets of data ☐ Create Venn diagrams to represent two or more sets of data ☐ Calculate the probabilities of outcomes using data displayed in Venn diagrams	☐ Go back to Topic 11G Venn diagrams
☐ Understand conditional statements ☐ Calculate conditional probability using tree diagrams ☐ Calculate conditional probability using two-way tables and Venn diagrams	☐ Go back to Topic 11H Conditional probability

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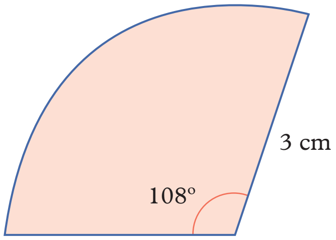
Semester 2 review

Short answer

1 Determine the area of each of the following shapes, correct to two decimal places.

a



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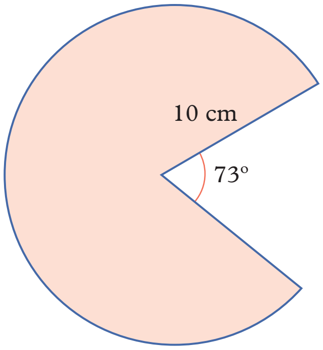

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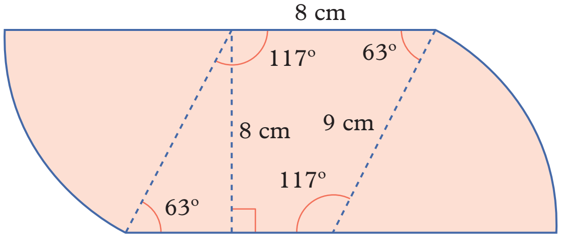

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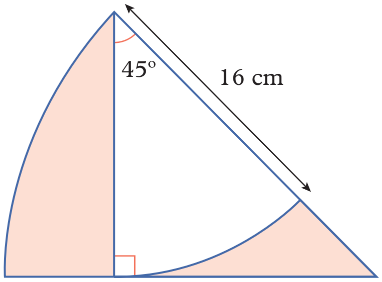

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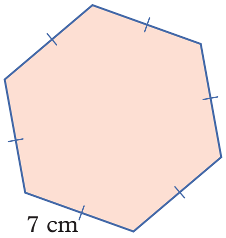

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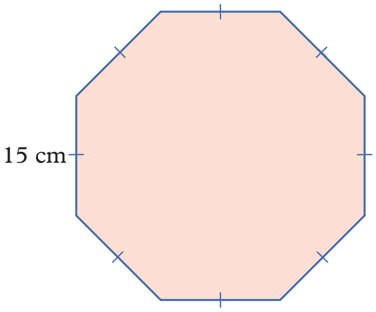


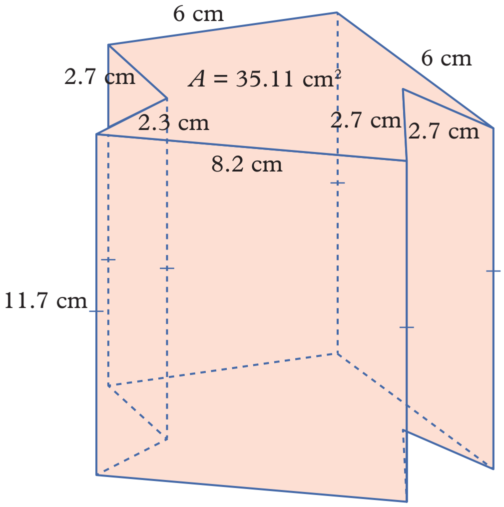
2 For each of the following objects, determine:

i the surface area

ii the volume.

Write your answers correct to two decimal places where required.

a



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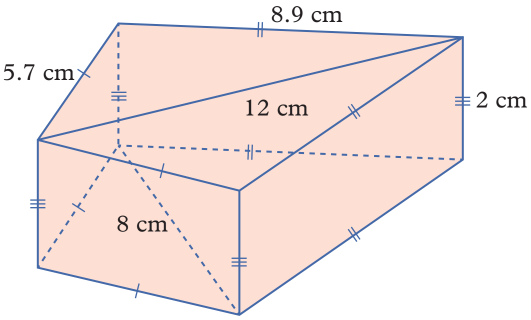

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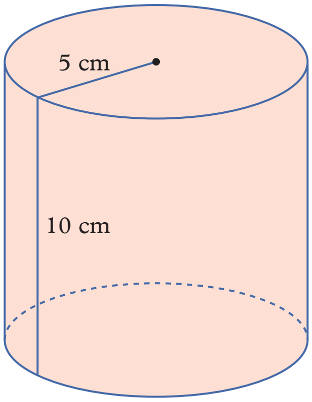

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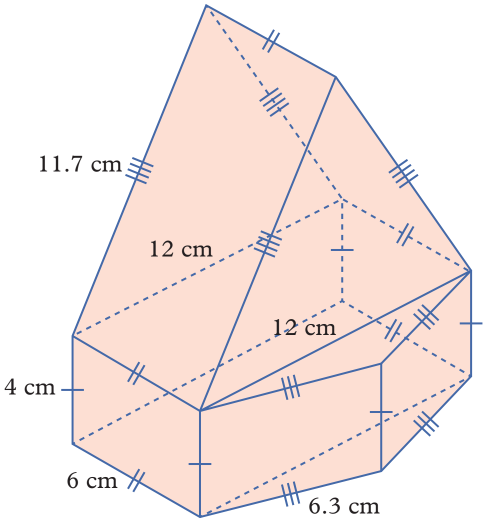


3 Determine if triangles with the following side lengths are acute, right-angled or obtuse.

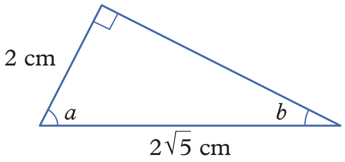
a 6 cm, 8 cm, 10 cm **b** 12 cm, 22 cm, 25 cm **c** 15 cm, 36 cm, 39 cm

4 For each of the following shapes, determine:

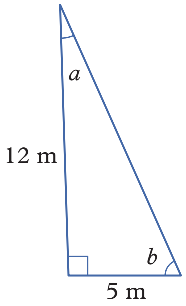
i the unknown length

ii the size of the unknown angles; correct your answers to the nearest degree

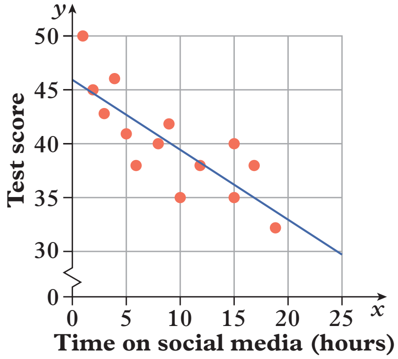
a



b



5 The test scores out of 50 of a group of students have been plotted against the time, in hours, those students spent on social media during the two days prior to the test.



a Use the line of best fit above to predict:

i a student's test score if the student spent 20 hours on social media in the two days before the test

ii a student's test score if the time the student spent on social media is 11 hours

iii the time spent on social media if the student's test score is 44

iv the time spent on social media if the student's test score is 30.

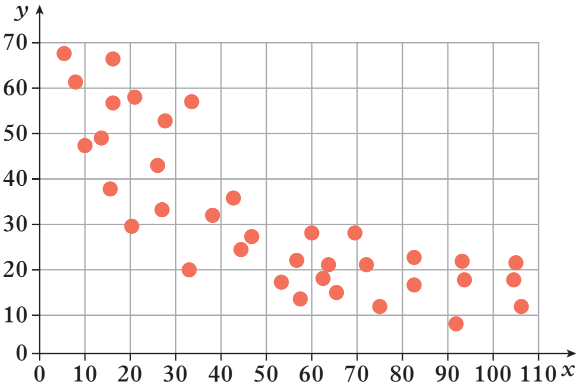
b For each answer in part **a**, state whether the prediction is reliable or not.

6 Construct a scatter plot for the following data set.

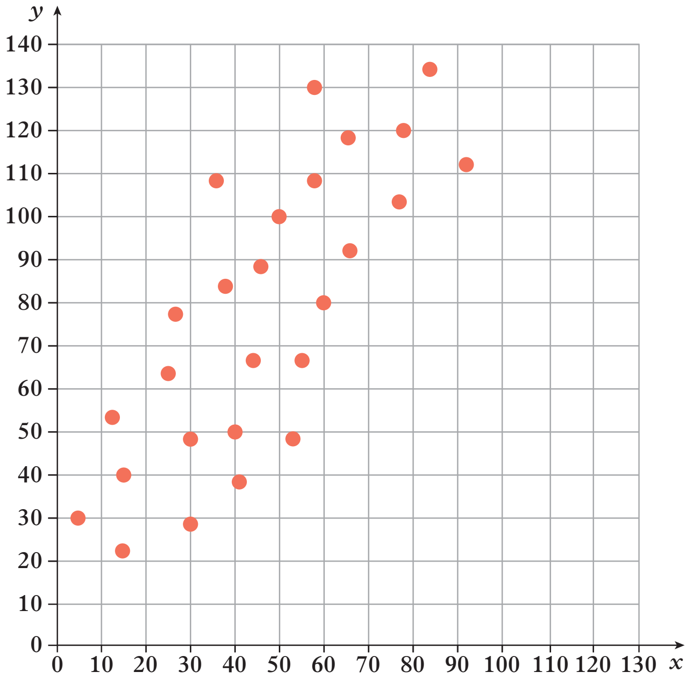
x	2	5	4	7	3	8	0	6	6	4
y	7	9	6	10	8	7	6	7	9	5

7 For each of the following scatter plots, describe the correlation between the variables in terms of strength and direction.

a



b



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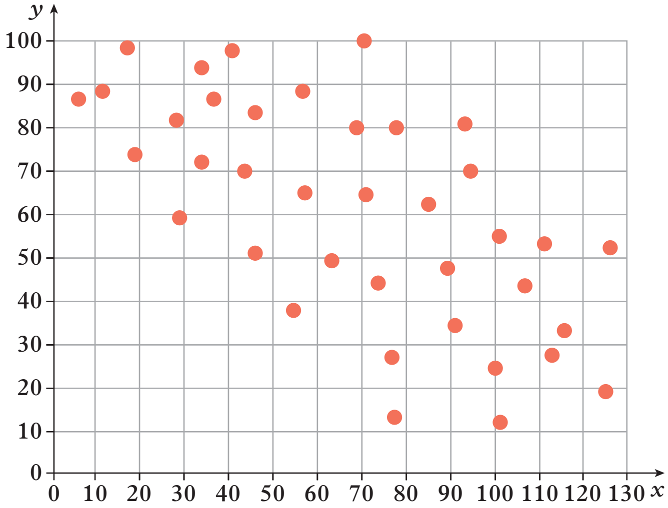

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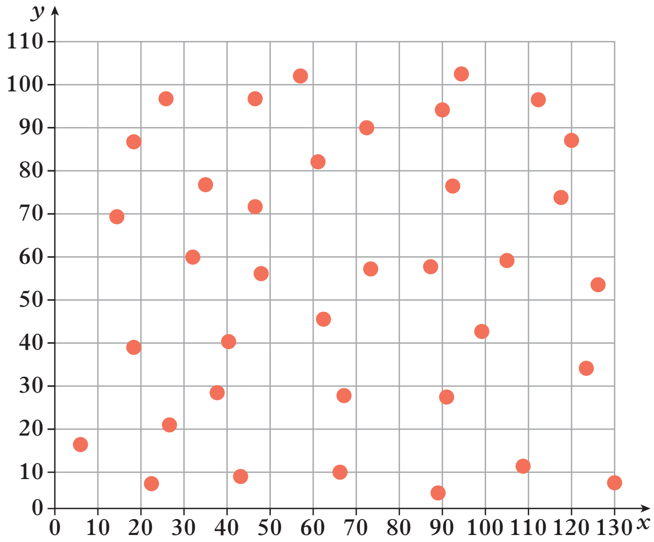


8 **a** The Forrestfield-Airport Link escalator in Perth, Western Australia, is the longest uninterrupted escalator in the Southern Hemisphere. The escalator is 15 m high and covers a horizontal distance of 35 m.

Determine the angle of elevation of the Forrestfield-Airport Link escalator correct to one decimal place.

b The Twelve Apostles are giant, natural limestone structures that are approximately 45 m tall, standing in the water off the Great Ocean Road in Victoria. If you were standing at the lookout on the adjacent 70 m high cliff, how far from the nearest of the Twelve Apostles would you be (measuring horizontally) if the angle of depression from you to the top of the nearest of the Twelve Apostles was 18°? Give your answer correct to the nearest metre.

9 **a** A hiker walks 2.5 km due north and then walks 1.2 km due west of that point.

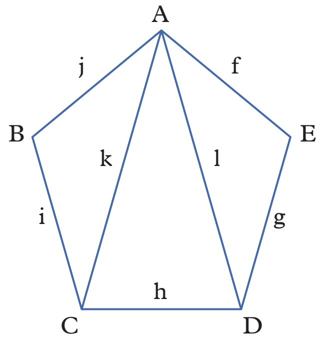
- **i** Determine the true bearing of the hiker's final position from the starting point, correct to the nearest degree.
- **ii** Determine the true bearing of the starting point from the hiker's final position, correct to the nearest degree.
- **iii** Determine the straight-line distance the hiker is from the starting point correct to one decimal place.

b A snorkeller swims, from a boat at point *B*, 35 m on a bearing of 120° T to point *C*, then continues for 20 m on a bearing of 200°T to point *D*.

- **i** Draw a diagram showing the snorkeller's swimming path. Include all the bearings and distances provided.
- **ii** How far south of the boat is the snorkeller at point *C*, correct to one decimal place?
- **iii** How far east of the boat is the snorkeller at point *C*, correct to one decimal place?
- **iv** How far south of point *C* is the snorkeller at point *D*, correct to one decimal place?
- **v** How far west of point *C* is the snorkeller at point *D*, correct to one decimal place?
- **vi** What is the true bearing from point *D* on which the snorkeller must swim in a straight line to return to the boat, correct to the nearest degree?
- **vii** To return to the boat from point *D*, how far does the snorkeller need to swim along the bearing you wrote for part **vi**, correct to one decimal place?

STN **10** Fill out the table of vertex degrees for the network on the right.

Vertex	A	B	C	D	E
Degree					



STN **11** For the network in question **10**, classify the sequences as walks, trails, circuits, closed walks or invalid walks.

a E, g, D, l, A, f, E

b A, k, C, h, D, h, C

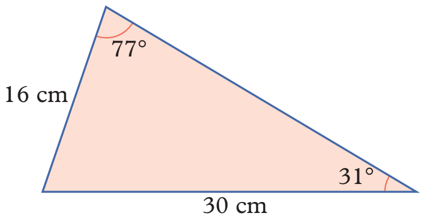
c B, i, C, h, D, l, A

d E, g, C, k, A

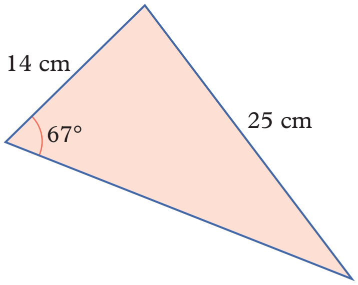
STN **12** Plan a return trip through North and South America, using every flight path on the map exactly once.

STN ADV **13** Determine the area of each of the following. Write your answers correct to two decimal places.

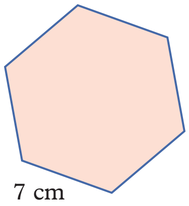
a



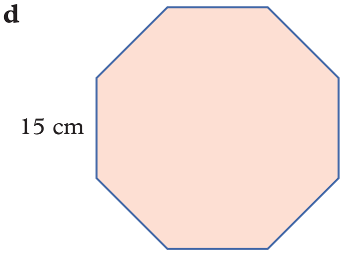
b



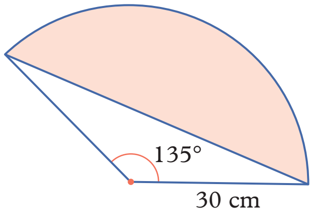
c



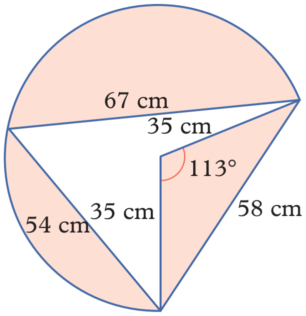
d



e

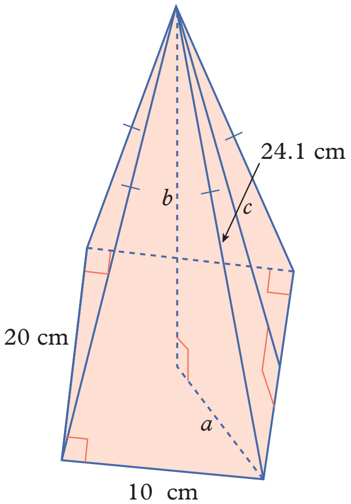


f



STN ADV **14** Determine the value of the each of the pronumerals in these diagrams. Write your answers correct to two decimal places where required.

a



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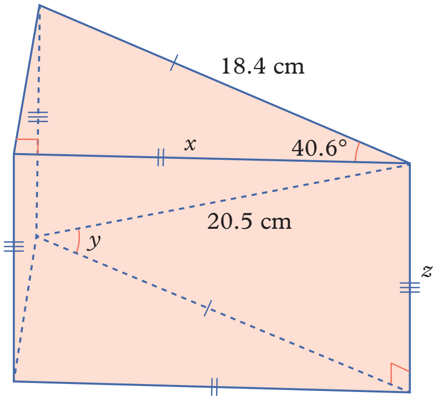
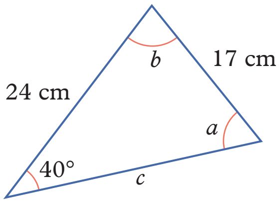
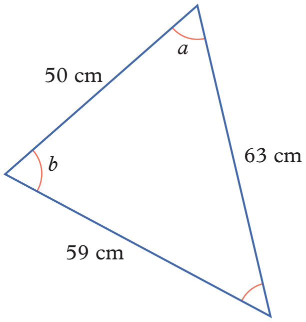


STN ADV **15** Determine all possible values for the pronumerals in each of the following diagrams.

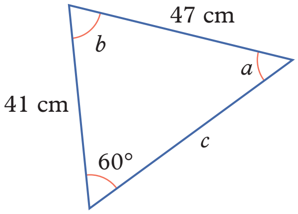
a



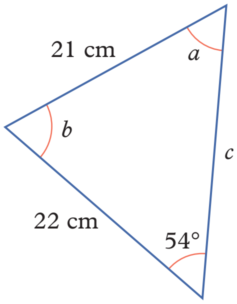
b



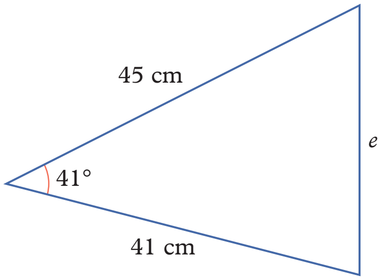
c



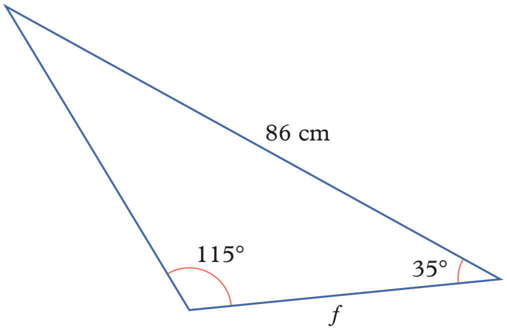
d



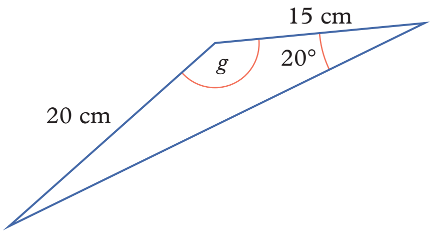
e



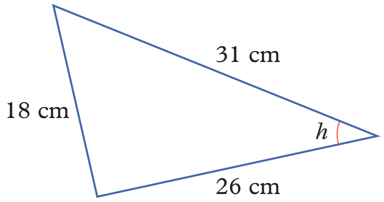
f



g



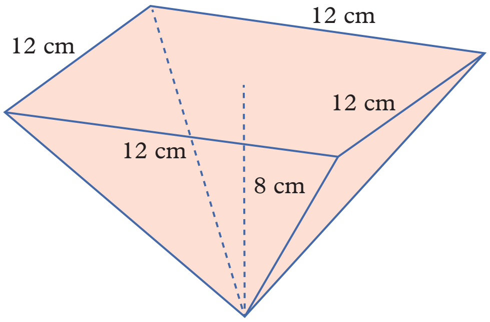
h



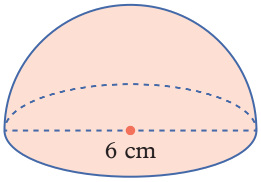
STN ADV **16** For each of the following, determine the:

i surface area **ii** volume.

a



b



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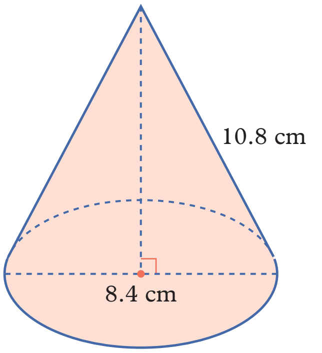
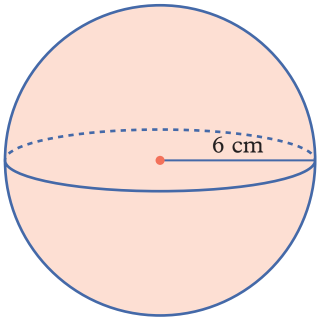
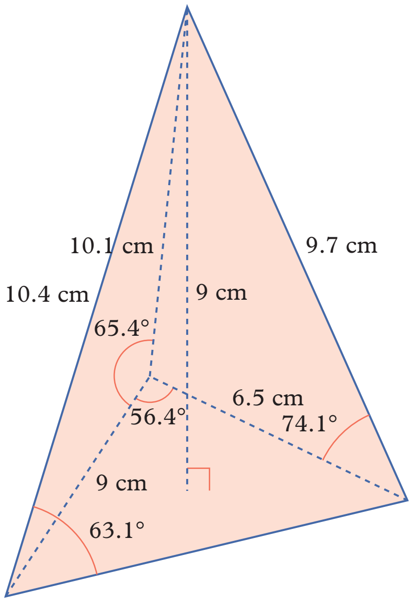


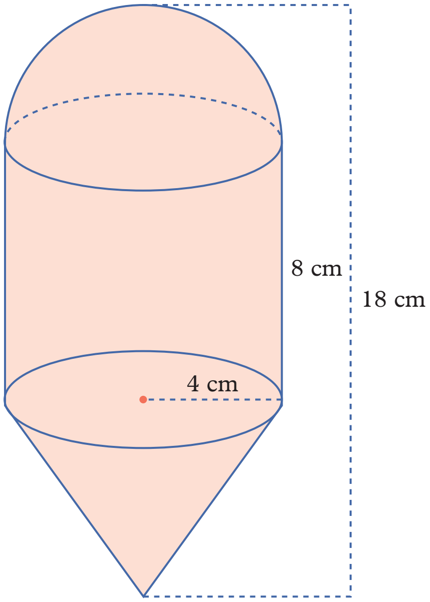
d



e



f



STN ADV **17** Decide whether each of these surveys will provide fair or biased results, giving reasons for your answers.

a Determining the most popular video game by asking every fifth competitor of the 128 competitors in a video game tournament (in which they are all playing the same game) what their favourite game is.

b Asking a random sample of 50 students from a school with 2000 students whether a tie should be required to be worn with their school uniform, in order to determine student opinion about whether ties should be included in the school uniform policy or not.

c Asking a random sample of 1000 people in Queensland (population 5 million) whether they think Queensland should adopt Daylight Saving Time.

d Asking everyone in Australia what languages they speak in order to determine the most spoken language in the Southern Hemisphere.

STN ADV **18** For each of the following:

i decide whether the data collection method is fair

ii decide whether the interpretation of the data is fair

iii if appropriate, provide a suggestion to improve the survey.

a Peta wanted to know what would make more people travel on public transport. She asked every fifth person entering her local shopping centre in a five-hour period on a busy Saturday, 'What would make you more likely to take public transport?' Of the people surveyed, 60% responded that they would take public transport if it was more reliable and ran more frequently. Peta concluded that more people would take public transport if it was more reliable and ran more frequently.

b Rob wanted to know the average height of professional basketball players. He researched the heights of the players of all 30 teams in the NBA and calculated an average height of 2.01 m. Rob concluded that the average height of a basketball player is 2.01 m.

ADV **19** Evaluate each of the following by first writing it in terms of a positive acute angle (the reference angle). Write your answer as an exact value.

a $\sin(120°)$ **b** $\tan(315°)$ **c** $\cos(300°)$

ADV **20** Solve each of the following equations.

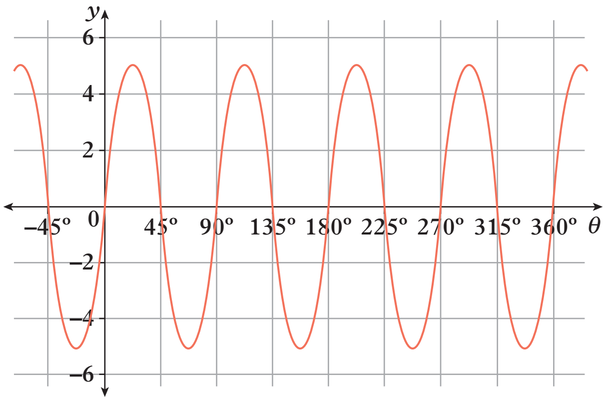
a $\sin(\theta) = -\frac{1}{2}, 0° \le \theta \le 360°$ **b** $\cos(\theta) = \frac{\sqrt{2}}{2}, 0 \le \theta \le 360°$

c $\tan(\theta) = \sqrt{3}, 0° \le \theta \le 360°$ **d** $\sin(\theta) = -1, 0 \le \theta \le 360°$

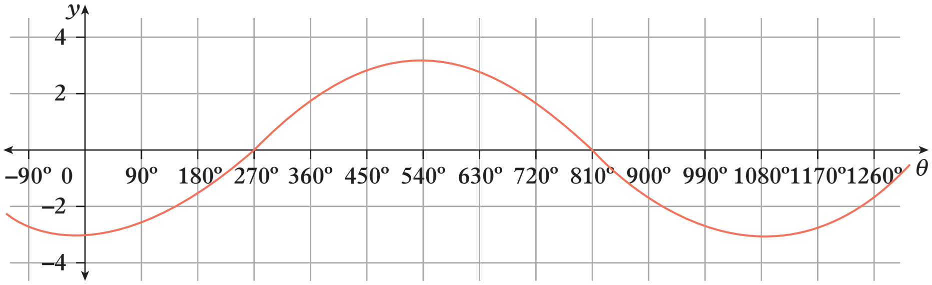
e $\cos(\theta) = 1, 0° \le \theta \le 360°$ **f** $\tan(\theta) = -1, 0 \le \theta \le 360°$

ADV **21** State the amplitude and period of each of the following graphs.

a



b



ADV **22** A large box contains fluorescent and LED light globes from two companies: Phillips and Osram. A sample of 90 light globes is taken and 46 are Phillips LED light globes, 39 are fluorescent and 21 are Osram LED light globes.

a Show this information on a Venn diagram.

b From the sample of 90 light globes, calculate the relative frequency of:

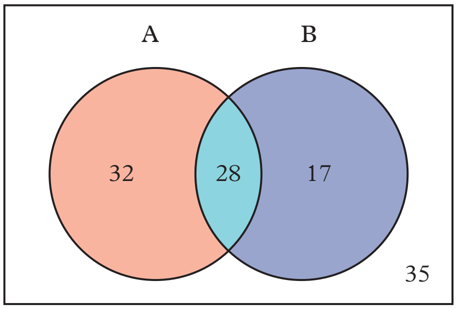
i Phillips fluorescent light globes

ii LED light globes

iii Osram light globes

iv fluorescent or not Phillips or both.

ADV **23** Consider this Venn diagram.



a Given that $n(A)$ means the number of elements in set A, find:

i $n(A)$ **ii** $n(A \cup B)$

iii $n(A')$ **iv** $n(A \cap B)$

v $n(\xi)$ **vi** $n(A' \cap B)$

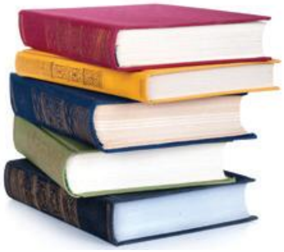
b Given that $\Pr(A)$ means the probability of selecting an element from set A, find:

i $\Pr(A)$ **ii** $\Pr(A')$

iii $\Pr(A \cap B)$ **iv** $\Pr((A \cup B)')$

v $\Pr(A' \cap B')$ **vi** $\Pr(A \cup B')$

ADV **24** Simone was organising her book collection, which contained 65 books. She found there were 40 fiction books, and 24 books that were part of a series of which 14 were fiction.



a Construct a two-way table to represent this information.

b Convert the two-way table into a Venn diagram.

c Determine the probability of each of the following.

- **i** Simone randomly selects a fiction book that is not part of a series.
- **ii** Simone randomly selects a non-fiction book that is part of a series.
- **iii** Given that Simone has selected a fiction book, it is part of a series.
- **iv** Given that Simone has selected a book that is not part of a series, it is non-fiction.

ADV **25** A printing company is hired by two other companies to print trading cards. Company A orders twice as many cards to be printed as Company B. Due to differences in the trading card designs, there is an 8% chance that any one card for company A is misprinted compared to the 4% chance of a misprinted card for company B. A printed trading card is selected at random from a recent print run.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following.

- **i** The card is for company A and is misprinted.
- **ii** The card is for company B and is not misprinted.
- **iii** Given that the card is for company A, it is misprinted.
- **iv** Given that the card is misprinted, it is for company A.

ADV **26** While driving, Fergus is listening to his playlist. It contains 52 tracks and Fergus knows that three songs will play at random without repeats before he reaches his destination. The playlist contains songs from four albums:

- album A has 13 tracks
- album B has 11 tracks
- album C has 12 tracks
- album D has 16 tracks.

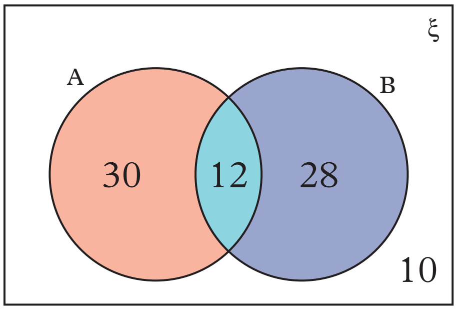
Fergus is thinking about the probability of tracks from album A playing or not playing.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following.

- **i** All three songs that play are from album A.
- **ii** Two songs that play are from album A.
- **iii** Given that the first song that plays is from album A, the next two songs are not from album A.
- **iv** Given that the first two songs that play are not from album A, the third song is from album A.

ADV **27** Consider the Venn diagram shown on the right. Determine:



a $\Pr(A)$

b $\Pr(B')$

c $\Pr(A \cap B)$

d $\Pr(A \cup B')$

e $\Pr(A|B)$

f $\Pr(B|A')$

ADV **28** Determine if events A and B in each of the following situations are independent or dependent.

a $\Pr(A) = 0.16, \Pr(B) = 0.25, \Pr(A \cap B) = 0.4$

b $\Pr(A) = \frac{2}{3}, \Pr(B) = \frac{5}{6}, \Pr(A \cap B) = \frac{5}{9}$

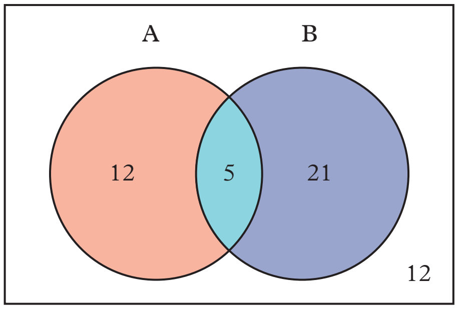
c $\Pr(A) = \frac{7}{15}, \Pr(A|B) = \frac{7}{15}, \Pr(A \cap B) = \frac{1}{3}$

d $\Pr(B) = 0.625, \Pr(B|A) = 0.35, \Pr(A \cap B) = 0.35$

ADV **29** **a** Display the information in this two-way table in a Venn diagram.

	A	A'	Total
B	11	57	68
B'	47	25	72
Total	58	82	140

b Display the information in this Venn diagram in a two-way table.



ADV **30** A school has 212 Year 10 students. Of the 98 students with a smartphone, 56 have a gaming console and 80 students do not have a gaming console.

a Show this information in a two-way table.

b Calculate the relative frequency of students that:

i do not have a smartphone

ii have a gaming console

iii have neither a smartphone nor a gaming console

iv have a smartphone or a gaming console but not both.

ADV **31** **a** Convert this two-way table to a relative frequency two-way table.

	A	A'	Total
B	75	5	80
B'	35	10	45
Total	110	15	125

b Convert this relative frequency two-way table to a two-way table given that the sample size is 120.

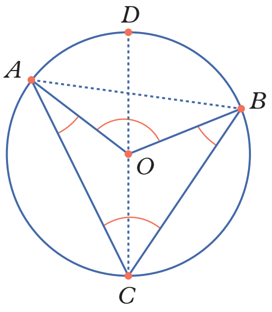
	A	A'	Total
B	$\frac{17}{40}$	$\frac{29}{120}$	$\frac{2}{3}$
B'	$\frac{1}{8}$	$\frac{5}{24}$	$\frac{1}{3}$
Total	$\frac{11}{20}$	$\frac{9}{20}$	1

EXT **32** Calculate the area of the image if the perimeter of the shapes in question **1** parts **e** and **f** have been dilated by the following scale factors.

i 3

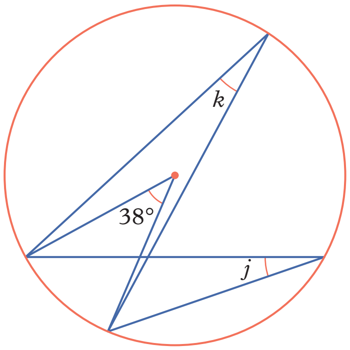
ii $\frac{1}{2}$

EXT **33** Given that *CD* is a diameter of the circle below and *O* is the centre point, prove that $\angle AOB = 2\angle ACB$ (without using any circle theorems).

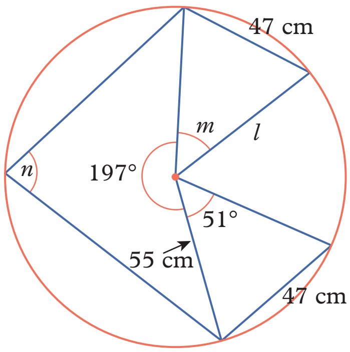


EXT **34** Determine the value of each of the pronumerals in these diagrams. Write your answers correct to two decimal places where required.

a



b



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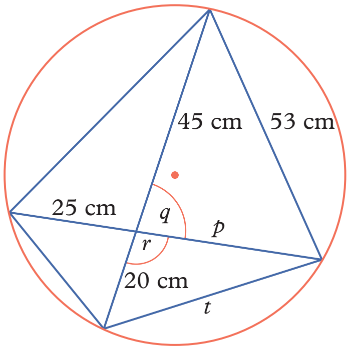

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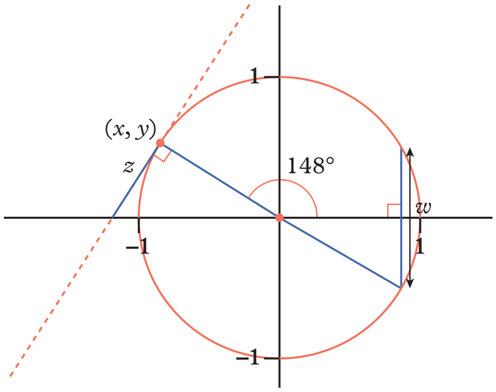


Analysis

1 A teacher decided to retest the class two weeks after a recent test to see how much the students were able to remember. The scores the 24 students scored on the two tests are given in the table below.
The teacher hoped to be able to predict the retest scores from the test scores.

Test score (%)	95	90	90	85	85	80	80	80	75	75	75	70	70	70	65	65	65	60	60	60	55	50	40
Retest score (%)	80	85	70	50	65	65	50	90	65	60	40	80	60	75	60	65	35	35	70	50	50	30	30

a State the independent variable.

b Construct a scatter plot of the scores from the retest (vertical axis) against the scores from the original test (horizontal axis).

c Describe the correlation between the scores for the retest and the scores for the original test.

d Add a line of best fit to your scatter plot from part **b**.

e **i** If a student scored 70% for the original test, use the line of best fit to predict the student's score for the retest.

ii State whether or not each prediction is reliable.

f The teacher claims that students who scored 75% or more originally did better on the retest than students who scored less than 75%. Evaluate the teacher's claim.

2 Erica is planning a hike through a local national park. She plans to:

- drive to the carpark, at point P
- then walk on a bearing of 036.9°T for 10 km towards the lower bridge, at point L

- then walk on a bearing of 108.4°T for 6.3 km towards the upper bridge, at point U
- then walk on a bearing of 233.1°T for 10 km towards the new bridge, at point N.

From the new bridge there is then a 4 km track due west back to the carpark.

a Draw a diagram to show Erica's hike plan.

b Determine the length of the planned hike.

c What is the average minimum speed Erica must walk to make it back to her car in 6 hours?

d Erica reaches the upper bridge and sees a warning sign that says it is a 50 m drop to the ground below. From the upper bridge, Erica can see a kangaroo on the ground at an angle of depression of 12°. If Erica's eye height is 172 cm, determine the horizontal distance to the kangaroo, correct to the nearest metre.

STN ADV **e** For emergency purposes, the park has a straight-line emergency route to point U from the carpark. Determine the straight-line distance from point P to point U, in kilometres correct to one decimal place.

STN ADV **f** Determine the true bearing of point U from point P, correct to the nearest degree.

STN ADV **g** Determine the area of the region Erica hiked around, correct to two decimal places.

h The park has multiple tracks that can be taken from the carpark. The tracks vary in difficulty. Hikers often mark which way they went on a signpost and this has allowed the park rangers to determine the following:

- From point P, 20% of hikers walk to point R and 80% walk to point L.
- From point L, 5% of hikers walk to point U and 95% walk to point A.
- From point R, 10% of hikers walk to point U and 90% walk to point A.
- They all then follow tracks that lead to point N.

Determine the probability that:

i a randomly selected hiker took the path through points P, R, U and N

ii a randomly selected hiker took the path through points P, L, A and N

iii a randomly selected hiker took the path through points P, R, A and N

iv given that they walked to point U, a randomly selected hiker took the path through points P, L, U and N.

STN **i** Draw a network that represents the paths between the points P, R, L, U, A and N.

STN **j** **i** Is the network from part **f** planar?

ii Why is there no Eulerian circuit in the network? Is there an Eulerian trail?

iii Suppose there is a path from all points to N. Is the corresponding network planar?

ADV **3** A special Ferris wheel with enclosed, air-tight carriages has been designed so that half of the wheel is submerged underwater. Passengers board the carriage at sea level, the Ferris wheel rotates upwards to its highest point, then back to sea level, then underwater to the deepest point, then back to sea level. A full rotation of the Ferris wheel takes one hour to complete. The deepest point the Ferris wheel can reach is 20 m.

a State the amplitude and period of the Ferris wheel's height in relation to time.

b Write the equation of the height, h m, of the carriage t minutes after boarding using an appropriate function.

c Sketch the graph of the height of the carriage against the time it takes the Ferris wheel to complete one full rotation.

d Determine the height the carriage is above sea level after 20 minutes:

i as an exact value **ii** correct to two decimal places.

e Determine, as an exact value, the depth the carriage is below sea level after 35 minutes.

f Determine the time it takes after boarding for the carriage to be $10\sqrt{2}$ m below sea level.

g People on a nearby boat watch the Ferris wheel go around. Describe the vertical and horizontal location of a carriage relative to the centre of the Ferris wheel 22.5 minutes after it was boarded:

i using exact values **ii** correct to two decimal places.

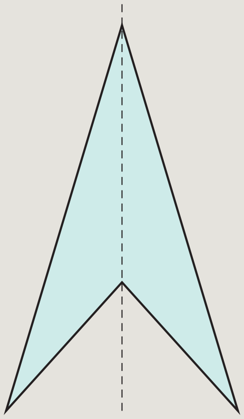
h Determine the distance the carriage has travelled during the time described in part **g**:

i as an exact value **ii** correct to two decimal places.

ANS p803

EXPLORATIONS 2

1 Thorn polygons



A *thorn polygon* is a shape that has at least one axis of symmetry and the maximum possible number of interior reflex angles. The symmetrical dart shown is an example of a thorn quadrilateral.

a How many reflex angles can a triangle have? What is another name for a thorn triangle?

b There are two different types of thorn pentagon: one with the reflex angles adjacent and one with them separated. Draw an example of each type.

c Explain why every thorn heptagon has at least one pair of adjacent reflex angles.

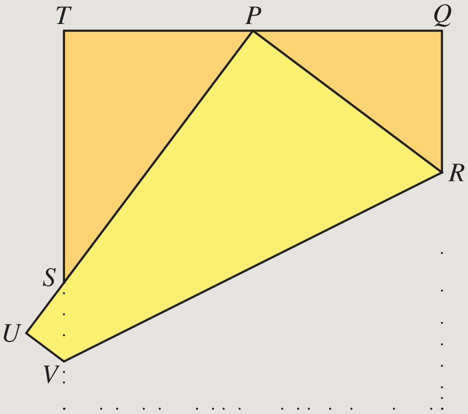
d Explain why no thorn octagon can have two obtuse angles.

e Show that an n-sided thorn polygon has exactly $n - 3$ reflex angles. Hint: This requires two steps: 1) show that a symmetrical n-sided polygon with $n - 3$ reflex angles always exists, and 2) show that more than $n - 3$ reflex angles is impossible.

f Find the maximum number of sides in a thorn polygon in which no three consecutive angles are all reflex angles.

g Show that, for every $n \geq 3$, there exists an n-sided thorn polygon that can be bisected along its line of symmetry to form two congruent thorn polygons.

2 Know when to fold 'em



A square piece of origami paper has side a length of 1 unit. The bottom-right corner is folded to a point P on the top edge, as shown, creating triangles PQR, PTS and SUV.

a Suppose P is at the midpoint of the top edge of the square.

  i Set up an equation to find the length of QR.

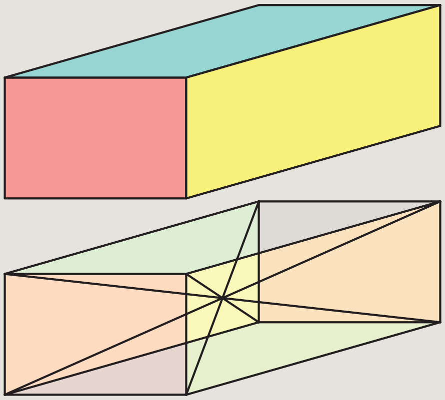
  ii Use similarity to find the side lengths of triangle SUV.

  iii Calculate the area of the folded yellow region of the paper.

b Find all possibilities for the length of PQ given that triangle SUV has side lengths in the ratio 7 : 24 : 25.

c Show that, for any Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$, there are two different positions for P such that triangle SUV has sides in the ratio $a : b : c$. What is the second position for the ratio in part **a**?

3 Six pyramids



a A 4 cm × 6 cm × 12 cm rectangular prism is divided into six right rectangular-based pyramids, as shown. Each face of the prism is the base of one pyramid and the centre of the prism is the apex of each pyramid.

  i Find the volume of each pyramid.

  ii For each pyramid, find the sum of the lengths of its eight edges.

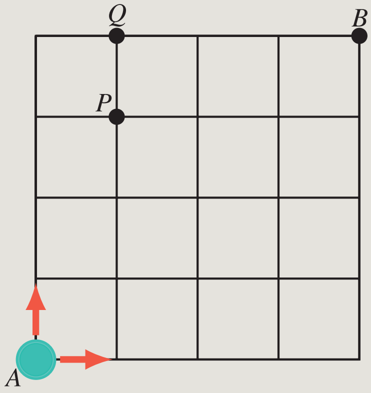
  iii Find the surface area of each pyramid, to 2 decimal places.

b Given an 85 cm × 132 cm × 720 cm rectangular prism, show that its six pyramids each have integer surface area, in square centimetres.

c Let (a, b, c) be any Pythagorean triple satisfying $a^2 + b^2 = c^2$ and define $u = 2a(4b^2 - c^2)$, $v = 2b(4a^2 - c^2)$ and $w = 8abc$.

Show that the six pyramids of a u cm × v cm × w cm rectangular prism each have integer surface area, in square centimetres.

4 Chance encounters



In the 4 × 4 grid shown, a counter is placed at *A*. Each second, the counter is moved along a grid line to an adjacent grid point, always increasing its distance from *A*. Where there are two possible moves, they are equally likely. After 8 seconds, the counter reaches point *B*.

a Explain why there is a one-in-four chance that the counter starting at *A* will be at *P* after 4 seconds.

b What is the probability that the counter starting at *A* will be at *Q* after 5 seconds?

A second counter is placed at *B*. Each second, it is moved along a grid line, always increasing its distance from *B* and with each move equally likely. The two counters start moving from *A* and *B* at the same time.

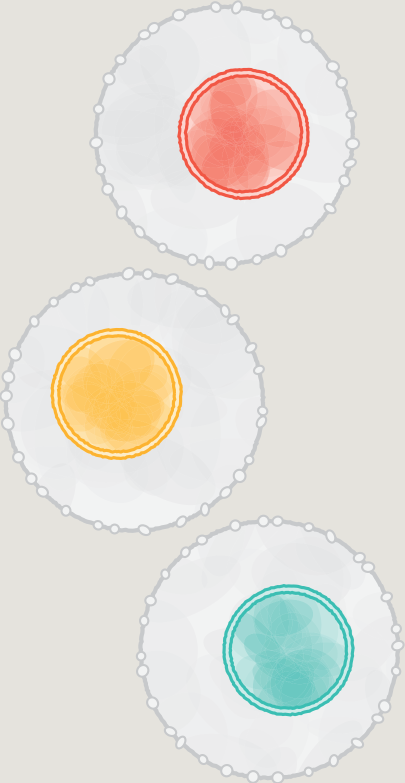
c Show on a diagram all the grid points at which the counters can meet. Explain why there are no others.

d What is the probability that the counters meet?

e The game is changed so that each counter has probability $\frac{1}{3}$ of moving vertically and probability $\frac{2}{3}$ of moving horizontally. Is it more or less likely that the counters will meet?

5 Algal bloom

Dr Al Goldblum discovered that a certain species of poisonous singled-celled algae has three different forms, called kirons, lirons and mirons. A kiron lives for one week and then changes into a liron. A liron also lives for one week and then changes into a miron. One week later, a miron divides into one kiron and one liron.



A spreadsheet is a useful way to track the numbers of kirons, lirons, mirons and total algae over time. Dr Al always takes measurements at the same time of the week: 12 pm on Wednesday.

a In one experiment, Dr Al started in week 1 with a single kiron in a flask of water.

  i What will be the total number of algae in the flask in week 10?

  ii In which week will the total number of algae first exceed 100?

b Show that, from week 6 onwards, the number of kirons is equal to the sum of the numbers of kirons one week earlier and five weeks earlier. Is this also true for lirons and mirons?

c In another experiment, Dr Al started in week 1 with an equal number of kirons and mirons, but no lirons. Some weeks later, there were exactly 100 algae in total. What is the smallest possible number of algae at the beginning of the experiment?

d Dr Al's goal is to develop a herbicide to kill the poisonous algae. Batch X31 destroys all lirons within one day, but it does not affect the other two forms. It remains active for several weeks.

  i On Wednesday in week 11, the X31 herbicide is added to the flask that started with a single kiron in week 1. During which week will all forms of algae be completely eradicated?

  ii If X31 is added to a lake infested with all three forms, what is the maximum number of days it will take to eradicate the algae?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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[STEAM project 1]

How can we use technology so that we improve the lives of people in the world's poorest nations?

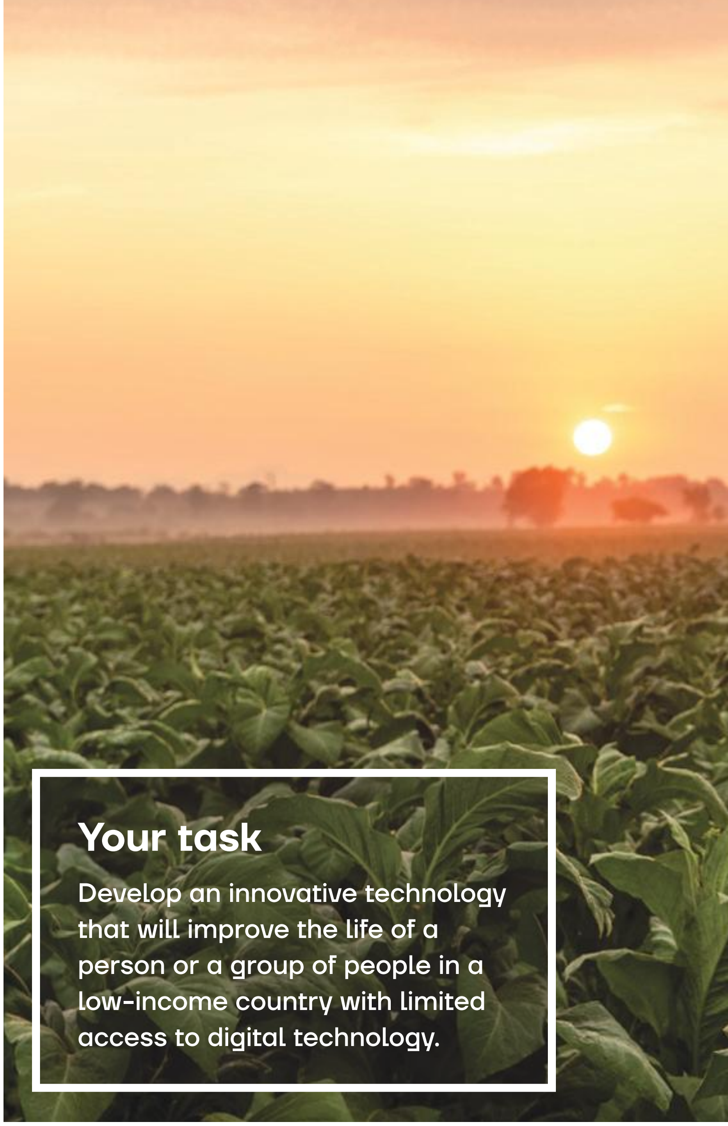
In Australia we are surrounded by technology every day. It is in the phones we use, the televisions we watch and the cars we drive. The term 'technology' is used for any machinery or equipment that applies the scientific knowledge we have discovered. Wheels and computers are both examples of technology.

In high-income countries, emergency response teams often rely on technological data supplied by electronic sensors to respond to natural disasters such as storms, fires and plagues. Drones might be used to conduct search and rescue operations. Doctors can use technology to remotely diagnose people who are sick and to perform operations that save lives.

At the end of 2017, the number of high-speed mobile subscriptions in member countries of the Organisation for Economic Co-operation and Development (OECD) reached a milestone: more subscriptions than the number of people. These mobile phones have been used to alert people to natural disasters, or to call for help in the event of floods and fires.

Technology is not just used for communication during natural disasters. It is also used to create medicine, improve farming practices and for education.

However, not everyone has access to technology.



Your task

Develop an innovative technology that will improve the life of a person or a group of people in a low-income country with limited access to digital technology.

Figure 1 Technology can be used to enhance and improve agricultural practices.

The digital divide

The term 'digital divide' is used to describe the gap between those who have access to digital technology – such as mobiles, computers and the internet – and those who do not. The Australian Bureau of Statistics has identified that almost 2.6 million Australians do not use the internet and cannot access technology in an emergency. Access is even lower in lower-income countries throughout Africa and Asia.

The OECD has identified that targeted innovation that uses technology can boost productivity, increase economic growth and help solve problems in society.

Figure 2 Technology has made attending a doctor's appointment easier and more accessible for people who may have difficulty attending in person.

HSIE

In Geography this year, you will be learning about the spatial variations in human wellbeing globally. You will need to explore a range of factors that lead to inequalities, such as social, political, economic and technological differences. In Commerce, you will explore variations in living standards between countries, and how living standards can be improved.

To complete this task successfully, you will need to research the initiatives of international governments and non-government organisations (NGOs) aimed at improving human wellbeing, particularly regarding health. You should consider how technology could be effectively accessed, resourced and used by a group of people to address their health concerns.

MATHS

In Maths this year, you will extend your skills in representing, comparing and interpreting data. You will use digital technology to work with data, but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to use skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Chapter 11 ‘Statistics and probability’ of *Oxford Maths 10 NSW Curriculum*.

SCIENCE

In Science this year, you will learn how an understanding of evolution can contribute to the selection of desired traits (such as drought resistance) in plants and animals. You will also learn how genetic engineering can be used to develop medicines that will cure cancers and prevent disease.

To complete this task successfully, you will need to consider how the values and needs of different societies can influence the focus of scientific research. You will also need to consider the ethics of the technology that you will be offering to your selected individual or group of people.

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping and what they need. The people you are helping, who will use your design, are called your end-users.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?
- What does it feel like to face these problems? What words would you use to describe these feelings?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your innovative technology, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1 Describe the limitations in energy, communications, transport and support personnel that will need to be considered as part of the solution.

2 Describe how many copies of the solution prototype will need to be made to make a difference in the country you have chosen.

3 Identify who could pay for the construction of the solution prototypes.

4 Describe the social culture that is experienced by the individuals and groups who are affected by the problem. Why might some technologies be viewed as unwanted or even dangerous?

Ideate

Once you know who you're designing for, and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your technological design must fulfil (i.e. cost, size and weight for transportation, and cultural appropriateness).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each group member should select one design to draw. Label each part of the design. Include the material that will be used for its construction.

Include in the individual designs:

a a detailed diagram of the design

b a description of how it will change the life of your selected individual or group

c an outline of any similar designs that are already available to buy

d an outline of why your idea or design is better than others that are already available.

Present your design to your group.

Build the prototype

Choose one solution and build two or three prototypes. The prototype may be full size, or it may be a scale model (10 cm = 1 m).

Use the following questions as a guideline for your prototype solution.

- What materials or technology will you need to build or represent your prototype solution?
- What skills will you need to construct your prototype design? Does your group have these skills, or who can teach you those skills?
- How will you make sure your prototype design is able to be used by your selected individual or group? Will they need training?
- How will you display or describe the way the prototype design will work?

Test

Prototype 1

Use the scientific method to design an experiment that will test the effectiveness and strength of your first prototype solution. You will test the prototype more than once to compare results, so you will need to control your variables between tests.

What criteria will you use to determine the success of your prototype? Conduct your tests and record your results in an appropriate table.

Prototype 2

If your prototype will be used to help an individual, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use it? Would it make their work easier or harder? Would they consider buying it?)

Prototype 3

Use the information you have obtained from testing the first two versions to adapt your last prototype to be more effective and usable for the group you are helping. You may want to use the first two prototypes to demonstrate how the design has been improved over time.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your designed solution.

In your presentation, you will need to:

- outline the situation of the country in which your selected individual or group lives
- outline the challenges faced by your selected individual or group
- include a working model or a detailed series of diagrams with a description of how the prototype of the solution will be used
- include a description of how you changed your design prototype as a result of testing or feedback
- include a description of how the design prototype will improve the lives of your selected individual or group.

Check your Student obook pro for the following digital resources to help you with this STEAM project:

Student booklet
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage your project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to pitch your idea
This 'how-to' video will help you with the 'Communicate' phase of your project.

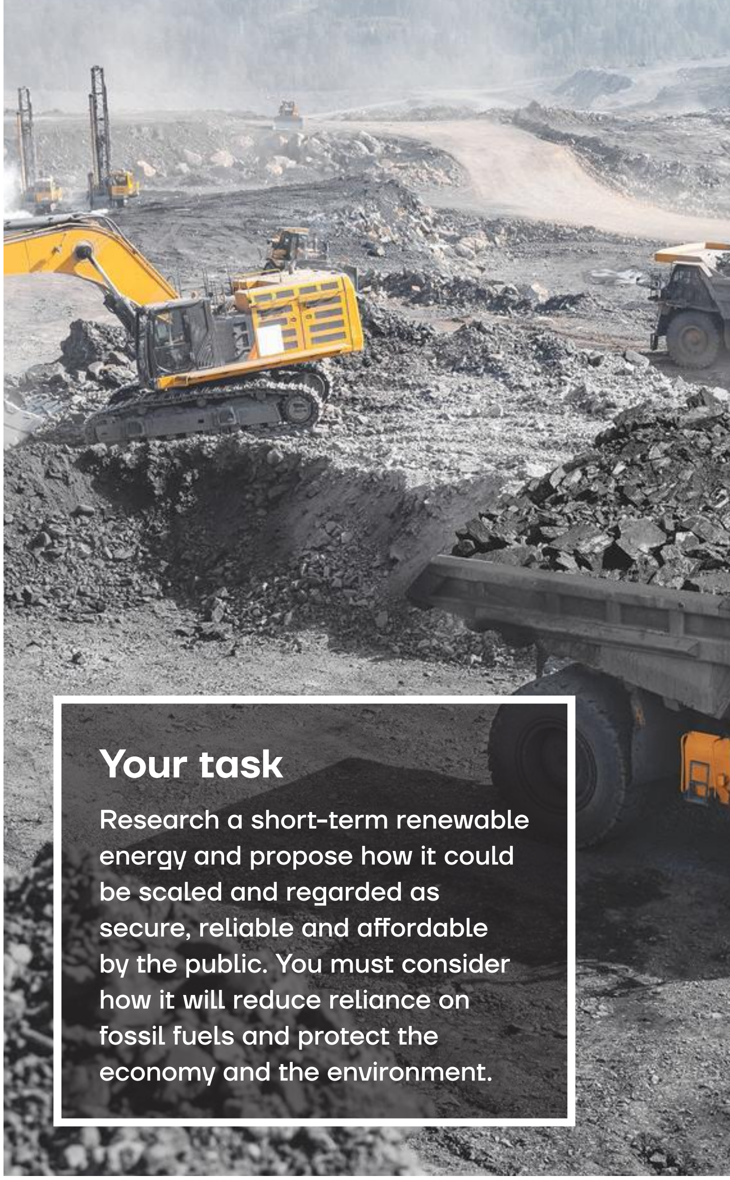
[STEAM project 2]

How can Australia reduce its reliance on fossil fuels so that we protect the environment and the economy?

Fossil fuels such as coal, oil and gas are made from fossilised, decomposed organisms that aged over millions of years in the Earth's crust. They contain carbon, which can be burned for energy. Due to the length of time it takes for these fuels to form as part of the carbon cycle, they are classified as long-term renewable sources of energy, sometimes called non-renewable.

Australia is a major user, producer and exporter of fossil fuels. Nearly 80 per cent of Australia's electricity is generated from coal and gas. Seventy-five per cent of coal mined in Australia is exported, making Australia the largest net exporter of this fuel in the world. In 2019, it was reported that Australia was the world's third-largest exporter of fossil fuels. Economically, the production and export of fossil fuels contributes hugely to Australia's GDP, and the mining industry is an important employer.

Coal emits higher amounts of CO_2 than oil or gas when used to produce energy. Measuring fossil fuel exports according to their potential to emit CO_2 makes Australia's carbon footprint per capita one of the largest in the world. This is contentious globally, particularly for nations most affected by a changing climate.



Your task

Research a short-term renewable energy and propose how it could be scaled and regarded as secure, reliable and affordable by the public. You must consider how it will reduce reliance on fossil fuels and protect the economy and the environment.

Figure 1 Most of Australia's energy is generated from coal. Almost 80 per cent of the coal produced in Australia is from open-cut mines.

Renewable alternatives

To protect both the environment and the economy, Australia needs to focus more on short-term renewable energy. When deciding on energy alternatives, it is important that energy supply be secure, reliable and affordable.

Short-term renewable energy sources include hydropower, solar power, wind power, bioenergy and ocean energy. Australia's landscape is suitable for many of these alternatives, but large investments in technology are required. As we invest more in the technology that makes short-term renewable energy possible, the better we get at making it, the more affordable it becomes and the more demand for renewable energy grows (a cyclical process).

Figure 2 A solar farm in Canberra. Solar energy is a source of renewable energy.

HSIE

In Geography this year, you will learn about environmental change and management. You will explore the environmental, technological and economic factors that have influenced the change and the consequences of human actions on the sustainability of the environment.

In Commerce, you will investigate how the performance of Australia's economy is measured and how Australia's economic growth has depended on natural resources. You will explore the impact that environmental policies can have on Australia's economy and living standards.

To complete this task successfully, you will need to understand how stakeholders such as governments, communities and businesses can work together to initiate environmental change and management plans that protect both the environment and the economy.

MATHS

In Maths this year, you will extend your skills in representing and interpreting data, including univariate, bivariate and multivariate data sets. This will include critical consideration of media reports that use statistics and present graphs. You will use digital technology to work with data but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to have skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Chapter 11 'Statistics and probability' of *Oxford Maths 10 NSW Curriculum*.

SCIENCE

In Science this year, you will learn about the impacts of fossil fuel combustion reactions in the production of carbon dioxide and carbon monoxide. You will also examine how increased reliance on this form of energy has affected the way carbon cycles through Earth's spheres, and how the resulting increase in greenhouse gases (including carbon dioxide) has led to enhanced global warming, which is contributing to melting sea ice and permafrost, rising sea levels and an increased number of extreme weather events.

To complete this task successfully, you will need to consider how energy that is generated can be used efficiently.

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping (your end-users) and what they need.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for? Will I be helping the government or members of the public?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your solution for the potential replacement of fossil fuels, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1 What type of energy source are you trying to replace? How much of it is currently used and how is it used?
2 How will the renewable energy be used? Will it be easy for the user to access?
3 Will the renewable energy require many changes in the vehicles or equipment being used? Who will pay for this change in infrastructure? How much will it cost?
4 How long will it take to generate the resources needed to make this renewable energy resource accessible and affordable for most people?

Ideate

Once you know who you're designing for, and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your design must fulfil (i.e. type of equipment, number and amount of materials, area that needs to be covered).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each team member should draw one individual design. Label each part of the design. Include the material that will be used to construct a model of the design.

Include in the individual designs:

a a description of what you see as the biggest problem with the energy source that you are replacing
b a description of the renewable energy that you propose could be used instead

c a description of how this renewable energy could be scaled up so that it can be used more effectively.

Present your design to your group.

Build the prototype

As a group, choose one design and plan how to model or build it. You may need to produce two or three to-scale prototypes for your group's design. Keep each iteration so that you can show the progress of your ideas.

Use the following questions as a guideline for your prototype.

- How will you replicate or model the renewable energy source?
- How will you model how the renewable energy will be used?
- What are the limitations of the renewable energy source? Will it produce enough energy for the equipment that currently uses fossil fuels?
- Calculate the number, density or requirements of energy sources in your area. How will your model provide for these demands?

Test

Use the scientific method to design an experiment that will test the limitations of your renewable energy prototype idea. You will need to model and test more than one prototype to compare results, so you will need to consider all variables between tests.

What criteria will you use to determine the success of your renewable energy prototype?

If your prototype will be used by a particular group of individuals, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use the alternative energy source? Would it make their life easier or harder? Would they consider buying it? How much would they be prepared to pay to access this form of energy?)

Conduct your tests and record your results in an appropriate table.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your alternative energy design.

In your presentation, you will need to:

- outline the energy needs of the selected individual or group you are supporting
- outline the energy challenges faced by your selected individual or group
- create a working model or a detailed series of diagrams, with a description of how the design prototype will be used to replace the current energy demands
- describe how you changed your design prototype as a result of testing or feedback
- describe how the renewable energy prototype will improve the life of your selected individual or group
- estimate the cost of production for each element of your energy design
- estimate the number of each element of your energy design required in your local government area
- estimate the total implementation cost to individuals, or to local, state or national government bodies
- compare how this energy system could be implemented in developed and developing countries.

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How to define a problem
This 'how-to' video will help you to narrow your ideas down and define a specific problem.

Answers

CHAPTER 1 Financial mathematics

EX p6

1A Earning money

1 **a** **i** $76 **ii** $57

b **i** $58 **ii** $43.50

c **i** $63.20 **ii** $47.40

d **i** $39.60 **ii** $29.70

e **i** $84.20 **ii** $63.15

f **i** $72.60 **ii** $54.45

g **i** $51 **ii** $38.25

h **i** $57.80 **ii** $43.35

2 **a** $1080.30 **b** $56 175.60 **c** $4681.30

3 **a** $59 540 **b** $88 504 **c** $51 974

d $82 524 **e** $65 568 **f** $98 544

4 **a** $1748 per week **b** $2215 per fortnight

c $10 422 per month

5 **a** $3450 **b** $9750 **c** $4770

6 **a** $358.40 **b** $607.50 **c** $249.20

7 $6032.45

8 $61 867

9 **a** $13 932 **b** $6701 **c** $17 299

d $9047 **e** $12 978 **f** $44 903.40

10 **a** $83 906 **b** $17 736.45 **c** $1678.12

d $64 491.43

11 $73 817.98

12 No, Rose's sister is not correct. To calculate Rose's weekly salary, you need to multiply her monthly salary by 12 and divide by 52.

To the nearest dollar, Rose's weekly salary is $1242

13 **a** $6635/month ($79 620/year vs $77 230/year)

b $3995/fortnight ($103 870/year vs $103 448/year)

c $1645/week ($85 540/year vs $78 480/year)

14 **a** $1119/week **b** $58 971 **c** $51 500

15 $25.60

16 $424 000

17 **a** Tyreek **b** Tyreek **c** $75 000

18 7.4%

19 **a** $22 620 **b** $21 046 **c** Refund of $1574

20 Option A: less than $80 000

Option B: between $80 000 and $135 000

Option C: above $135 000

EX p12

1B Simple interest

1 **a** 2 **b** 4 **c** 2.25

d 3.75 **e** $\frac{7}{12}$ **f** 1

g $1\frac{1}{4}$ **h** $\frac{1}{13}$ **i** 6

j 1.5 **k** 2 **l** $\frac{1}{5}$

2 **a** $1200 **b** $204 **c** $31 500

d $11 520 **e** $2387.50 **f** $8250

3 **a** **i** $640 **ii** $8640 **iii** $360

b **i** $6000 **ii** $21 000 **iii** $437.50

c **i** $202.50 **ii** $1452.50 **iii** $40.35

d **i** $2870 **ii** $16 870 **iii** $562.33

e **i** $2550 **ii** $27 550 **iii** $1530.56

f **i** $589.50 **ii** $7139.50 **iii** $297.48

4 **a** **i** $1080 **ii** $13 080

b **i** $6600 **ii** $72 600

c **i** $18 750 **ii** $143 750

d **i** $231 **ii** $5731

e **i** $876.56 **ii** $9126.56

f **i** $494.64 **ii** $5074.64

5 **a** 4 years **b** 2.5 years **c** 3 years

6 **a** $15 000 **b** $6500 **c** $20 000

7 **a** 4.4% **b** 10.5% **c** 0.4%

8 **a** $2160 **b** 4 years **c** $18 000

d 8.4% p.a. **e** $2592 **f** 5 years

g $20 000 **h** 2.4% p.a.

9 **a** $10 160 **b** $6000 **c** $22 680

d $72 064 **e** $14 592 **f** $13 064

g $24 320 **h** $23 175

10 **a** $624.75 **b** $3564.75 **c** $178.24

d $2866.50 **e** $698.25

11 **a** $2000 **b** $2400 **c** 5%

d $2700 **e** 30 months

12 **a** **i** 7.5% **ii** $8250

b **i** 7% **ii** $6750

c **i** 4.5% **ii** $612.50

d **i** 12% **ii** $400

13 **a** $1615.50, $1546, $2421.55 **b** 8, 2, 9, 2, 3, 7

c 0.30, 0.07, 0.62, 0.12, 0.18, 0.65

d $1.94 **e** $2423.49

14 **a** **i** $5000 **ii** no

iii between 2 months and less than 6 months

b 5.4%

c $2430

d $2812.50; $382.50 more

e It increases by $59.00.

15 **a** Yes. there were no withdrawals and over $250 has been deposited this month.

b $42.02

c $9994.52

d No bonus interest would have been calculated; final balance is $9861.27.

16 a The interest rate won't change over that time.
b $204 **c** $29 411.76 **d** $12 107.32
17 a 2.6 years **b** 12.5% **c** 28%
d She will have trouble finding a financial institution offering such high interest rates.
18 a $120.00 for 4 days
$476.00 for 3 days
$515.90 for 7 days
$700.90 for 6 days
$850.90 for 5 days
$916.75 for 1 day
b $925.69 **c** 21.9% p.a.
19 a i $60 **ii** $60.09
b i $315 **ii** $317.21
c i $1487.50 **ii** $1511.38
d i $451.50 **ii** $459.64
20 a 3.61% **b** 8.46%
c 12.95% **d** 21.89%

EX
p20

1C Compound interest

1 a i 1 year **ii** 2
b i 1 year **ii** 3
c i 6 months **ii** 8
d i 1 month **ii** 24

2

	Years	Simple interest	Future value for simple interest	Compound interest	Future value for compound interest
a	3	$10	$130	$I = \$121 \times 10\% = \$12.10$	$A = \$121 + \$12.10 = \$133.10$
b	4	$10	$140	$13.31	$146.41
c	5	$10	$150	$14.64	$161.05
d	6	$10	$160	$16.11	$177.16

3 a i $42 000 **ii** $44 100 **iii** $6305
b i $6120 **ii** $6242.40 **iii** $367.25
c i $1040 **ii** $1081.60 **iii** $124.86
d i $20 700 **ii** $21 424.50 **iii** $2174.36
e i $6955 **ii** $7441.85 **iii** $1462.78
f i $16 179.20 **ii** $16 567.50 **iii** $1165.12
4 a i 1 year **ii** 3
b $442 **c** $8942 **d** $464.98
e $9406.98 **f** $489.16 **g** $9896.15
h Final value with simple interest is $9826, which is $70.15 less.
5 a $3401.40 **b** $2809.61 **c** $2919.29
d $2008.86 **e** $19 335.94 **f** $38 421.56
6 a $12 155.06 **b** $5955.08 **c** $816.29
d $53 019.97 **e** $112 649.26 **f** $19 993.02
7 a i $8682.19 **ii** $1182.19
b i $10 270.50 **ii** $1270.50
c i $31 263.59 **ii** $8763.59
d i $53 022.34 **ii** $13 022.34
8 a $4101.25 **b** $7325.28 **c** $16 925
d $34 945.12 **e** $127 567.39 **f** $507 382.93
9 a $901.25 **b** $2735.28 **c** $925
d $9045.12 **e** $27 567.39 **f** $252 382.93
10 a i 1.2% **ii** $r = 0.012, n = 12$
b i 2.95% **ii** $r = 0.0295, n = 8$
c i 0.7% **ii** $r = 0.007, n = 24$
d i 0.8% **ii** $r = 0.008, n = 60$
11 a $6071.85 **b** $13 406.88 **c** $22 895.02
d $68 216.57 **e** $61 373.74 **f** $162 619.66
g $551 913.02 **h** $1 017 670.16
12 a Multiplying by 1 keeps the amount and multiplying by 0.05 evaluates 5% of the amount. Therefore, multiplying by 1.05 increases the amount by 5%.
b $10\,000 \times (1.05)^n$ **c** $10\,000 \times (1 + r)^n$
d $PV(1 + r)^n$
13 a 3 **b** 1 year
c $PV = \$16\,000, r = 0.0575, n = 3$
d $18 921.74 **e** $2921.74 **f** Yes
14 a $63 404.70 **b** $18 404.70
15 a Option 1: $10 444.50, Option 2: $10 403.75
b Option 1, since it yields greater interest
c Option 1: $11407.50, Option 2: $11459.21. Option 2 is better since it yields greater interest.
16 a 2.15% **b** $r = 0.0215$ **c** 20
d $12 854.25 **e** $4454.25
17 a $13 360
b i $13 897.66 **ii** $13 944.79
iii $13 960.82 **iv** $13 977.02
c It increases. Yes.
18 fewer
19 a i 8 **ii** 0.017
b FV
c $6500 = PV(1 + 0.017)^8$; $PV = \$5679.97$
d $5680
20 a $6000 **b** 8%
c (3, 7558.27) and (4, 8162.93) **d** $11 105.58
21 a option 1: $53 450; option 2: $53 296; option 3: $53 251
b option 1: $56 900; option 2: $56 808; option 3: $56 713
c option 1: $60 350; option 2: $60 553; option 3: $60 401
d If Thao intends to keep the money in the account for a reasonable period of time (more than 2 years), option 2 is the best option. For a shorter period of time, option 1 is the best option.

22 **a** **i** 6.03% **ii** 6.07% **iii** 6.08%
b **i** 6.06% **ii** 6.03% **iii** 6.02%

EX p26

Checkpoint

1 **a** \$1483.50 **b** \$77 142 **c** \$6428.50
2 **a** \$4340 **b** \$3374
3 **a** \$5222 **b** \$13 649 **c** \$34 014
d \$40 959
4 **a** \$58 383 **b** \$9441 **c** \$1168
d \$47 774
5 **a** \$600 **b** \$2100 **c** \$1615
6 **a** **i** \$600 **ii** \$5600
b **i** \$840 **ii** \$12 840
c **i** \$29 216 **ii** \$162 016
7 **a** \$900 **b** \$5400 **c** \$112.50
8 **a** \$13 857 **b** \$2148 **c** \$6491
9 **a** **i** 1 month **ii** 24
b **i** 3 months **ii** 12
10 **a** \$4181 **b** \$15 765
11 **a** \$39 668 **b** \$13 668

EX p30

1D Compound interest calculations

1 **a** \$20 122.05 **b** \$8522.70
c \$4003.95 **d** \$6930.08
2 **a** \$4274.64 **b** \$7775.87 **c** \$13 545.38
d \$10 550.71 **e** \$145 348.17
3 **a** \$925.36 **b** \$2224.13 **c** \$8454.62
d \$6099.29 **e** \$79 651.83
4 **a** r
b

$$\begin{aligned}15\,000 &= 12\,000(1+r)^3\\ \frac{15\ 000}{12\ 000} &= (1+r)^3\\ \frac{15}{12} &= (1+r)^3\\ \frac{5}{4} &= (1+r)^3\end{aligned}$$

c $\sqrt[3]{1.25}$, $\sqrt[3]{1.25}$, 1.077 217...
d 7.7% p.a.
5 **a** **i** 0.0817 **ii** 16.3%
b **i** 0.0583 **ii** 5.8%
c **i** 0.0392 **ii** 15.7%
6 **a** **i** 0.1086 **ii** 10.9%
b **i** 0.0226 **ii** 4.5%
c **i** 0.0323 **ii** 6.5%
7 **a** $FV = \$8000$, $PV = \$5500$, $n = 4$
b 0.098 **c** 9.8%
8 **a** **i** 0.145 **ii** 14.5%
b **i** 0.107 **ii** 10.7%
c **i** 0.044 **ii** 8.8%
d **i** 0.034 **ii** 13.6%
e **i** 0.055 **ii** 10.9%
f **i** 0.021 **ii** 8.4%
9 **a** 36 **b** 0.006
c PV **d** \$12 093.83
10 **a** n
b

$$\begin{aligned}30\,000 &= 25\,000(1+0.05)^n\\ 30\,000 \div 25\,000 &= (1.05)^n\\ 1.2 &= (1.05)^n\end{aligned}$$

c 4
d \$25 000 must be invested for 4 years at 5.0% compounded annually to increase to just over \$30 000.
11 **a** $FV = \$1100$, $PV = \$850$, $r = 0.042$
b $\frac{22}{17} = (1.042)^n$ **c** 7
d 7
12 **a** 6 **b** 5 **c** 9
d 13 **e** 6 **f** 6
13 **a** 6 **b** 5 **c** 4.5
d 3.25 **e** 3 **f** 1.5
14 **a** \$8548.04 **b** \$8523.71
c **i** 5.74% **ii** 5.62%
iii 5.59% **iv** 5.58%
15 **a** 3.6% **b** 10.8% **c** 46.8%
d Quarters. 3.6% is a reasonable interest rate for a savings account.
e Months. 3.6% is too low because credit card interest rates are usually quite high. However, they are not as high as 46.8%.
16 **a** $FV = PV(1 + Cr)^{\frac{T}{C}}$
b As C decreases, r decreases, because $r = RC$.
c As C decreases, n increases, because $n = \frac{T}{C}$.
d As C decreases, FV increases. While $r = RC$ decreases, $n = \frac{T}{C}$ increases, and because $\frac{T}{C}$ is the exponent, the exponential increase makes up for the decrease in r with the net effect that FV increases.
e When $C = T$, $FV = PV(1 + TR)^{\frac{T}{T}} = PV(1 + TR)^1 = PV + PVRT$, which is the formula for the value of a simple interest investment. Therefore, if the time period equals the term of investment, then it is just simple interest.
17 **a** **i** 6 **ii** 25
b **i** 5.41 **ii** 24.12
c The n value must be rounded up to the next interest payment for the account balance to be above the required minimum amount.
d **i** 13 **ii** 34 **iii** 43

EX p37

1E Depreciation

1

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$36 000	\$3600	\$32 400
2	\$32 400	\$3600	\$28 800
3	\$28 800	\$3600	\$25 200

2 $18 000

3

Year	Value at start of year	Depreciation amount	Value at end of year
1	$8000	$800	$7200
2	$7200	$720	$6480
3	$6480	$648	$5832

4 $8192

5 **a** $57 600 **b** 7 years

6 $2600

7 **a** $1200 **b** 6 years

8 $1443

9 $569

10 20%

11 5%

12 The 25% depreciation in the second year is calculated on a smaller quantity than the 25% depreciation is calculated on in the first year, therefore it is a smaller value. This means the total depreciation is less than 50% of the initial value.

13 Straight-line depreciation is similar to simple interest, as it is dealing with a fixed quantity for each time period, which is calculated using the original value.

Diminishing-value depreciation is similar to compound interest, as it is dealing with a changing quantity for each time period, which is calculated using the value from the previous step.

14 **a** $14 000 **b** $2000 **c** $6000
d 7 years

15 **a** $20 000 **b** $16 000 **c** 20%

16 **a** straight-line depreciation: $2400;
diminishing-value depreciation: $4800

b Straight-line depreciation:

Year	Value at start of year	Depreciation amount	Value at end of year
1	$40 000	$2400	$37 600
2	$37 600	$2400	$35 200
3	$35 200	$2400	$32 800

Diminishing-value depreciation:

Year	Value at start of year	Depreciation amount	Value at end of year
1	$40 000	$4800	$35 200
2	$35 200	$4224	$30 976
3	$30 976	$3717	$27 259

c In the 7th year

d The diminishing-value method of depreciation gives a larger tax deduction in the early years of the machinery's life, giving Kumar more money in the short term.

17 As the percentage of diminishing-value depreciation increases, the amount of time for it to result in a lower amount of depreciation decreases. For example, if the percentage is 12% the depreciation amount would be less in the 7th year, whereas if the percentage was 20% the depreciation amount would be less in the 5th year. For example, if the percentage was 15% each year, then the deduction would be less than 2400 after 4 years.

18 **a** 29.29% **b** 20.63% **c** 15.91%
d 12.94%

EX p40

CHAPTER 1 review

Mathematical literacy review

1 The interest rate will be expressed per annum when the time period is one year.

2 An individual's net income can be calculated by subtracting any tax deductions from their total income.

3 Straight-line depreciation produces a constant amount of depreciation each year, whereas diminishing-value depreciation calculates the depreciation as a percentage of the value at the start of each year.

4 **a** leaving loading
b depreciating asset

5 **a** the initial amount of money borrowed or invested in a loan or investment
b the value of an asset after it has been depreciated

6 **a** Penalty rates
b time period, interest

Multiple choice

1 D 2 E 3 C 4 C 5 A
6 D 7 C 8 B 9 B 10 C

Short answer

1 $4647

2 **a** $1646 **b** $3958.65 **c** $4885

3 **a** $9227 **b** $19 759 **c** $3551
d $24 059

4 **a** $1695 **b** $89 327 **c** $69 647

5 **a** $450 **b** $5950

6 **a** **i** $25 200 **ii** $175 200
b **i** $116.16 **ii** $9566.16

7 **a** 1.25 **b** $18 000 **c** 11.25%

8 **a** Interest: $675.00, $705.38, $737.12
Investment: $15 675, $16 380.38, $17 117.49
b Interest: $438.40, $466.46, $496.31, $528.07
Investment: $7288.40, $7754.86, $8251.17, $8779.24

9 **a** $2600.16 **b** $26 978.10 **c** $191 793.77

10 **a** $6691.22 **b** $14 700.60 **c** 0.04

11 **a** 0.003
b **i** 0.3% **ii** 1.2%
iii 3.6% **iv** 15.6%
c 1.2% or 3.6%

12

Year	Value at start of year	Depreciation amount	Value at end of year
1	$4450	$371	$4079
2	$4079	$371	$3708
3	$3708	$371	$3337

13

Year	Value at start of year	Depreciation amount	Value at end of year
1	\$12 000	\$1200	\$10 800
2	\$10 800	\$1080	\$9720
3	\$9720	\$972	\$8748

14 12.5%

Analysis

1 **a** 16
b **i** 0.013 75 **ii** 0.014 25 **iii** 0.0155
c \$12 673.74
d PV = \$12 673.74, $r = 0.014\,25$, $n = 8$
e \$14 192.70 **f** \$15 093.32 **g** \$3093.32
h 25.78% **i** 6.45% **j** 6.42%

2 **a** \$92 560 **b** \$1950 **c** \$120
d \$72 **e** \$95 995 **f** \$21 665
g \$1920 **h** \$72 410

CHAPTER 2 Algebra

EX p48

2A Indices

1 **a** a^4 **b** b^{20} **c** c^{15} **d** d
e y^{10} **f** m^{14} **g** n^{14} **h** h^7
i x^6 **j** p^{18} **k** w^3 **l** a^{10}
2 **a** 1
b **i** 1 **ii** 1 **iii** 1 **iv** 1
3 **a** 3 **b** 1 **c** 8 **d** 1
e 1 **f** 2 **g** 0 **h** 2
i 6 **j** 1 **k** 1 **l** 1
4 **a** a^5b^5 **b** k^9p^9 **c** $64c^3$
d $7^{13}y^{13}$ **e** x^4y^4 **f** c^7d^7
g $9w^2$ **h** 6^6g^6 **i** 2^8p^8
5 **a** $\frac{a^4}{b^4}$ **b** $\frac{f^7}{g^7}$ **c** $\frac{k^6}{64}$
d $\frac{5^6}{x^6}$ **e** $\frac{a^2}{b^2c^2}$ **f** $\frac{m^5}{3^5n^5}$
6 **a** $5x^5$ **b** $6a^7$ **c** $9x^2$
d $8x^{12}$ **e** $4a^3$ **f** $2x^5$
g 2 **h** $30x^{12}$ **i** $3a^3$
7 **a** $8y^4$ **b** $24x^9$ **c** $72k^5$
d $45b^{12}$ **e** $60a^{20}$ **f** $10c^5$
g $2n^2$ **h** v^6 **i** $25m^{13}$
8 **a** p^9 **b** $18t^{20}$ **c** $25m^{39}n^{48}$ **d** $\frac{1}{j^6}$
9 **a** false; $x^7 \times x \times x^7 = x^{7+1+7} = x^{15}$
b false; $(5a)^3 = 5^3 \times a^3 = 125 \times a^3$
c true; $-k^0 = -(1) = -1$
d false; $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4} = a^5b^7$
e true; $x^7 \times y^4 \div y^6 = \frac{x^7y^4}{y^6} = \frac{x^7\overset{1}{\cancel{y^4}}}{\underset{y^2}{\cancel{y^6}}} = \frac{x^7}{y^2}$
f false; $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$
g false; $\frac{w^6 \times w^6}{w^{12}} = \frac{w^{6+6}}{w^{12}} = \frac{w^{12}}{w^{12}} = 1$
h true; $\frac{(b^5)^4 \times b^2}{(b^3)^7} = \frac{b^{20} \times b^2}{b^{21}} = \frac{b^{22}}{b^{21}} = b$
i false; $\frac{a^8b^5}{a^4b^3} \times \frac{a^2b^2}{a^9b^4} = \frac{a^{8+2}b^{5+2}}{a^{4+9}b^{3+4}} = \frac{a^{10}b^7}{a^{13}b^7} = \frac{1}{a^3}$
10 **a** 9 **b** 0 **c** 10 **d** 4
e 7 **f** 9 **g** 6 **h** 5
i 8
11 2^5
12 3^{2x}
13 $\frac{5^{4x}}{3^{3x}}$
14 **a** $x = 4$ and $y = 8$
b $x = -2$, $y = 12$
15 **a**

$$\begin{aligned}\frac{2^{13} \times 3^{11} \times 5^9}{30^7} &= \frac{2^{13} \times 3^{11} \times 5^9}{(2 \times 3 \times 5)^7}\\ &= \frac{2^{13} \times 3^{11} \times 5^9}{2^7 \times 3^7 \times 5^7}\\ &= 2^6 \times 3^4 \times 5^2\\ &= \left(2^3 \times 3^2 \times 5\right)^2\end{aligned}$$

b $a = 5 \times 7$ or $a = 5 \times 7^3$ or $a = 5 \times 7^5$ or $a = 5 \times 7^7$ or $a = 5^3 \times 7$ or $a = 5^3 \times 7^3$ or $a = 5^3 \times 7^5$ or $a = 5^3 \times 7^7$ or $a = 5^5 \times 7$ or $a = 5^5 \times 7^3$ or $a = 5^5 \times 7^5$ or $a = 5^5 \times 7^7$
c The prime factorisation of any square number can be written as a product of squares of prime numbers.
16 50
17 **a** $(x + y)^2 = x^2 + y^2 + 2xy$. Since $x, y \neq 0$, $2xy \neq 0$.
b **i** $(2 + 3)^3 = 125 \neq 35 = 2^3 + 3^3$
ii $(4 + 5)^4 = 6561 \neq 881 = 4^4 + 5^4$
iii $(1 + 5)^5 = 7776 \neq 3126 = 1^5 + 5^5$
c **i** $125 = 35 + 3 \times 90$, $m = 30$
iii $7776 = 3126 + 5 \times 930$, $m = 930$
d $(x + y)^2 = x^2 + y^2 + 2 \times xy$
e $(x + y)^3 = x^3 + y^3 + 3 \times (x^2y + xy^2)$
f (For example) $x = 1$, $y = 3$, $n = 4$
$(1 + 3)^4 = 1^4 + 3^4 + 4 \times \frac{87}{2}$, but $m = \frac{87}{2}$ is not an integer

EX p54

2B Negative indices

1 **a** 1 **b** $-\frac{1}{2}$ **c** 3 **d** $-\frac{1}{5d}$
e $\frac{1}{2w}$ **f** $2n$ **g** $\frac{xy}{z}$ **h** $\frac{a^2}{b}$
2 **a** x^3 **b** a^{-5} **c** 1 **d** y^4
e a^5 **f** m^7 **g** n^{-1} **h** n^2
i k^{12} **j** d^{-15} **k** y **l** n^2
3 **a** $\frac{1}{x^3}$ **b** a^5 **c** 1 **d** $\frac{1}{y^4}$
e $\frac{1}{a^5}$ **f** $\frac{1}{m^7}$ **g** n **h** $\frac{1}{n^2}$
i $\frac{1}{k^{12}}$ **j** d^{15} **k** $\frac{1}{y}$ **l** $\frac{1}{n^2}$
4 **a** b^{-1} **b** n^2m^{-1} **c** 1 **d** $-x^5y^{-5}$
e $b^{-1}c^2$ **f** $ab^{-3}c^3$ **g** $-y^{-1}$ **h** yz^{-1}
5 **a** $\frac{1}{x^5}$ **b** $\frac{d^4}{c^9}$ **c** $\frac{j^5}{k^3}$ **d** $\frac{y^8}{x}$
e $\frac{2a^3}{b^6}$ **f** $\frac{3}{p^4q^5}$ **g** $\frac{w^4y^7}{x^5}$ **h** $\frac{m^5}{k^3n^8}$
i $\frac{6d^2f}{e^7}$ **j** $\frac{11}{a^3b^5c^9}$ **k** x^3 **l** b^4c^6
m $4m^3p^7$ **n** $\frac{b^2}{a^4}$ **o** $\frac{4y^5}{x^2}$ **p** $\frac{d^8e^4}{c}$

6 **a** $\frac{1}{x^4y^4}$ **b** $\frac{1}{4^6a^6}$ **c** $\frac{1}{9n}$
d $\frac{1}{5^3p^3}=\frac{1}{125p^3}$ **e** $\frac{1}{9p^2}$ **f** $\frac{1}{k^5m^5}$
g $\frac{64}{x^3}$ **h** $\frac{d}{7}$

7 **a** $\frac{1}{x^6}$ **b** $\frac{4}{y^6}$ **c** w^{12} **d** $4x^{14}$
e $\frac{9}{a^8}$ **f** $2m$ **g** $8h^{12}$ **h** $\frac{30}{c^9}$
i $2d^3$ **j** $\frac{2}{t^{16}}$ **k** x **l** $\frac{12}{a^8}$

8 **a** $\frac{1}{x^2}$ **b** $\frac{2}{x^5}$ **c** $42x$ **d** $\frac{20}{x^4}$
e $\frac{1}{x^2}$ **f** $2x^9$ **g** $\frac{1}{4x^9}$ **h** $\frac{3}{5x^4}$
i $\frac{1}{x^{11}}$ **j** $\frac{10}{x^6}$ **k** $\frac{3}{x^7}$ **l** $\frac{1}{x^{21}}$

9 **a** x^{-2} **b** y^2 **c** $a^{-4}b^2$ **d** $m^{-2}n^{-6}$
e $\frac{1}{64}x^{-6}$ **f** p^2q^2 **g** $123g^{234}$ **h** $4t^3$
i $10^{10}u^{-10}$ **j** $\frac{1}{3}pq^{-2}$ **k** $\frac{1}{9}x^2$ **l** $\frac{81}{8}n^3m^{-3}$

10 **a** $a^{11}b^7$ **b** $2304p^5$ **c** $32y^{15}$ **d** $3m^{12}n^6$
e $\frac{d^{12}}{h^{11}}$ **f** $\frac{125w^3}{y^3}$ **g** $a^{30}b^{12}$ **h** $\frac{b^8}{a^{16}c^{28}}$
i u^3wx^{17} **j** $125c^4n$ **k** $m^{34}n$ **l** $\frac{1}{y^7}$

11 **a** false, $\frac{a^5}{a^9}=\frac{1}{a^4}$ **b** false, $\frac{x^3}{x^6}=\frac{1}{x^3}$
c true **d** false, $\left(\frac{3a^2b^{-1}}{c^3}\right)^2=\frac{9a^4}{b^2c^6}$
e true **f** true
g false, $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z}=\frac{3y^5}{x^9z}$ **h** true

12 **a** 3 **b** –4 **c** –2
d 3 **e** 1 **f** 5

13 **a** $x^{2a+5}y^{4b+1}$ **b** $m^{2x+4}n^3$

14 **a** x^{3a-4} **b** y^{3m+5} **c** a^{x+3} **d** b^{x+4}

15 **a** $\frac{3a^2b^2}{2c}$ **b** $\frac{3}{2b^8c^2}$ **c** $\frac{1}{y}$

16 **a** $\frac{8c^{12}}{27a^6b^9}$ **b** answer may vary

17 B

18 **a** $a^m \div a^n = a^m \times a^{-n}$
$= a^{(m-n)}$

b $a^m \div a^n = \frac{1}{a^{-m}\times a^n}$
$= \frac{1}{a^{(-m+n)}}$
$= a^{-((-m)+n)}$

EX 2C Simplifying

p59

1 **a** **i** 3 **ii** Sample answer: $-4abcd$
b **i** –4 **ii** Sample answer: $2mn$
c **i** 1 **ii** Sample answer: $5xy^2$
d **i** 9 **ii** Sample answer: $-3k^2m^4p$

2 **a** –6 **b** –3 **c** 10 **d** 9
e –52 **f** –24 **g** –15 **h** –9
i $\frac{21}{4}$ **j** $-\frac{1}{6}$ **k** $-\frac{41}{4}$ **l** $-\frac{1}{2}$

3 **a** $9x$ **b** $-10ab$ **c** $10y^2$
d $8m^2n$ **e** $12a+9d$ **f** $8k+2m$
g $2-3y$ **h** $7a-4a^2$ **i** $3x^2+2xy$
j $\frac{19a}{24}-\frac{ab}{4}$ **k** $\frac{3m^2n}{4}-\frac{8mn^2}{15}$ **l** $\frac{7xy}{10}-\frac{x}{4}+\frac{y}{3}$

4 **a** $11a$ **b** $-4n+2n^2+m-mn$
c $3x^2-2x+1$ **d** $-3x^2y+3xy-5$
e $2p^3+15p^2-9$ **f** $-a^2-b^2+3ab^2+9a^2b$
g $7ab$ **h** $7abc$
i $5x^2y+3xy$ **j** 0

5 **a** $10abcd$ **b** $-21mnxy$ **c** $4k^2p$

6 **a** ac **b** $6n$ **c** $\frac{2x}{y}$

7 **a** $4x^5y^3$ **b** $6a^6b^3$ **c** $\frac{24m^7}{n^2}$
d $\frac{25c}{2}$ **e** $\frac{5x^4y^2}{4}$ **f** $\frac{m^7n^4}{3}$
g $6x^4y^3$ **h** $\frac{2x^2y^2}{5}$ **i** $\frac{27a^3}{160}$

8 **a** 2 **b** –69 **c** 246
d –20 736 **e** –18 **f** 4

9 **a** 1550 cm^2 **b** 0.62 m^2

10 **a** $7x+6y$ **b** $6x+4y-6$ **c** $20y-12$

11 **a** 61 cm **b** 44 cm **c** 28 cm

12 The answers are square units:
a $42ab$ **b** $4a^2$ **c** $10ab$
d $9\pi a^2$ **e** $24ab+\frac{9}{8}\pi b^2$ **f** $37ab$

13 **a** 504 m^2 **b** 64 m^2 **c** 120 m^2
d 144π m^2 **e** $288+\frac{81\pi}{8}$ **f** 444 m^2

14 **a** $3\pi x^2$
b Any value larger than about 6.5147 mm and smaller than 7.2836 mm. Some possible values are: 6.7 mm, 7 mm and 7.1 mm.
c $6\pi x$
d 126.3 mm, 131.9 mm, 133.8 mm

15 **a** $12a$ **b** $32ab$
c $-3a+15ab$ **d** $24xy^2-x$
e $-6a^2+25a^2b^6$ **f** $16x^2y^2+13y^2$
g $23a^2b^2$ **h** $\frac{(5y^2)}{2}+7xz$

16 **a** $abc-6c$ m^3 **b** $2ab+2ac+2bc-12$ m^2

EX 2D Expanding

p63

1 **a** $3a+21$ **b** $5b-20$ **c** $12-2c$
d $4+4d$ **e** $-6e-18$ **f** $-7f+14$
g $-3g+3$ **h** $-45+9h$ **i** x^2+2x
j $2y^2-8y$ **k** $-k^2-5k$ **l** $-3p^2+3p$
m $3mn+km$ **n** $12ab+24ac$ **o** $-w^2+8wx$
p $-4p-10p^2$

2 **a** $3a-12$ **b** $-5b+15$
c $8x+12$ **d** $-14y-28$
e $10a-5a^2$ **f** $2m^3+6m$
g $24a^2b+8ab$ **h** $8wx-4w^2$
i $15k^2-10k^4$ **j** $12x^3y-6x^2y^4$
k $15c^2d-10cd^2$ **l** $-8a^3c+6a^2bc$

3 **a** $9x+10$ **b** $4y-13$ **c** $mn+3m$
d $8x^2-5x$ **e** $2p^3-3p$ **f** x^7-3x^5

4 **a** $6x + 2$ **b** $m^2 + 14m - 18$

c $6a^2 + 14a - 6$ **d** $k^5 + 7k^2 + 5$

e $3p^6 - 4p^4 + 12p$ **f** $10y^8 + y^3 + 8y$

5 **a** $ab + 2a + 4b + 8$ **b** $cd + 6c + 3d + 18$

c $mn + 5m + 7n + 35$ **d** $xy + 8x + 6y + 48$

e $kp - k + 5p - 5$ **f** $fg + 3f - 2g - 6$

g $xy - 4x - 9y + 36$ **h** $2mn + 2m + 3n + 3$

i $12xy - 28x + 15y - 35$

j $15ac - 12ad - 10bc + 8bd$

k $x^2 + 7x + 10$ **l** $y^2 + 8y - 9$

m $m^2 - 14m + 33$ **n** $p^2 - 6p + 8$

o $30 - 11k + k^2$ **p** $4x^2 + 19x + 21$

q $3a^2 - 13a - 10$ **r** $8w^2 - 23w + 14$

s $12p^2 + 52p + 35$ **t** $8x^2 + 10x - 3$

u $15y^2 - 26y + 8$

6 **a** $2a^2 + 6a - ab - 3b$ **b** $4t^2 - 3tu - u^2$

c $2x^2 + 11xy + 12y^2$ **d** $-x^2 + x + 20$

e $6x^3 + 18x^2y + xy + 3y^2$ **f** $ab^4 - 2a^2b^2 + b^2 - 2a$

g $ab + ac + b^2 + bc$ **h** $b^2 + bd + 3bc + 3cd$

i $3a - ac + 6b - 2bc$ **j** $3a^4b + 2a^3 + 3ab^3 + 2b^2$

k $5x^5 - 2x^4y - 15xz + 6yz$ **l** $c^4 - 16a^6b^2$

7 **a** **i** $x^2 - 4$ **ii** $k^2 - 49$ **iii** $m^2 - 64$

iv $w^2 - 36$ **v** $y^2 - 1$ **vi** $a^2 - b^2$

b The product of the two binomial factors produces an expression that is the difference of two squared numbers or pronumerals. The first term in each binomial factor is the same, and the second term in one factor is the negative of the second term in the other factor.

c In the expression $a^2 - b^2$, the terms a^2 and b^2 are both squares, and there is a minus sign between them indicating 'find the difference'.

d No; multiplication is commutative.

e **i** $a^2 - 9$ **ii** $x^2 - 100$ **iii** $m^2 - n^2$

iv $9 - x^2$ **v** $1 - d^2$ **vi** $4x^2 - 25$

vii $4 - 9k^2$ **viii** $16g^2 - h^2$ **ix** $25y^2 - 4w^2$

8 **a** **i** $x^2 + 6x + 9$ **ii** $y^2 + 16y + 64$

iii $a^2 + 2ab + b^2$ **iv** $k^2 - 10k + 25$

v $p^2 - 12p + 36$ **vi** $x^2 - 2xy + y^2$

b The two binomial factors are the same. The product of factors like these produces an expression made up of the squares of the two terms in the factor, and twice the product of the two terms in the factor.

c The expression $(a + b)^2$ is the square of $(a + b)$.

d **i** $a^2 + 4a + 4$ **ii** $x^2 + 10x + 25$

iii $p^2 + 8p + 16$ **iv** $y^2 + 20y + 100$

v $k^2 + 18k + 81$ **vi** $m^2 + 2mn + n^2$

vii $25 + 10x + x^2$ **viii** $1 + 4d + 4d^2$

ix $9w^2 + 42w + 49$

e The expression $(a - b)^2$ is the square of $(a - b)$.

f **i** $a^2 - 6a + 9$ **ii** $b^2 - 8b + 16$

iii $x^2 - 18x + 81$ **iv** $n^2 - 22n + 121$

v $4 - 4x + x^2$ **vi** $g^2 - 2gh + h^2$

vii $16w^2 - 8w + 1$ **viii** $25 - 20y + 4y^2$

ix $9m^2 - 12mp + 4p^2$

9 **a** $x^4 - 4$ **b** $x^4 - 25$ **c** $x^4 - y^4$

d $x^4 + 8x^2 + 16$ **e** $x^4 - 6x^2 + 9$ **f** $x^6 - 2x^3 + 1$

10 **a** $2x^3 + 9x^2 + 7x - 3$

b $2x^3 + 5x^2 + x - 2$

c $x^3 - 4x^2 + 3x + 2$

d $2x^4 + x^3 + 3x^2 + 2x - 2$

e $2x^4 - 6x^3 - x^2 + 9x - 3$

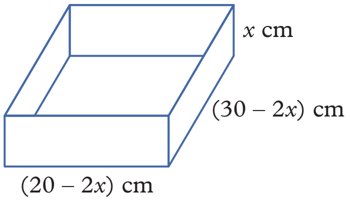
f $3x^4 + 5x^3 - 5x^2 - 6x$

11 **a** **i** $(30 - 2x)$ cm

ii $(20 - 2x)$ cm

iii x cm

b



c **i** $x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$

ii $30 \times 20 - 4x^2 = 600 - 4x^2$ (using the net)

d **i** 1008 cm³ **ii** 564 cm²

12 **a** $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

b $x^2 + 4xy + 6x + 4y^2 + 12y + 9$

c $4a^2 - 9b^2 - c^2 + 6bc$

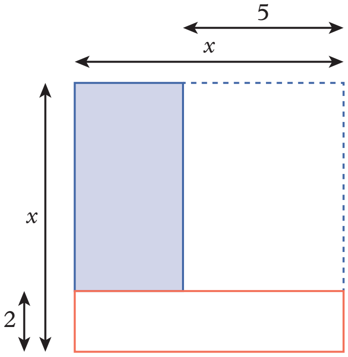
13 **a** $x^3 + 3x^2 + 3x + 1$

b $x^3 - 9x^2 + 27x - 27$

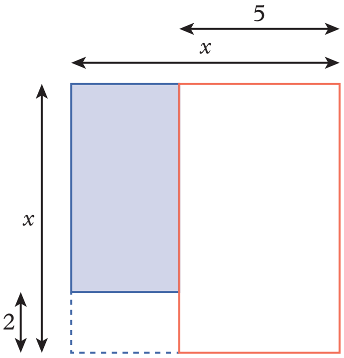
c $x^4 + 8x^3 + 24x^2 + 32x + 16$

d $x^4 - 2x^2y^2 + y^4$

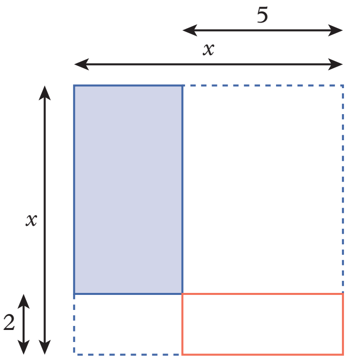
14 **a** **i**



ii



iii



b The rectangle with area 10 is subtracted as part of the $2x$ rectangle and as part of the $5x$ rectangle. Since the rectangle with area 10 is subtracted twice when it should only be subtracted once, it must be added back once.

EX p69

2E Algebraic fractions

1 **a** $\frac{x}{2}$ **b** $\frac{1}{y}$ **c** 4 **d** $\frac{2x}{3}$

e $\frac{x+3}{5}$ **f** $2(x-5)$ **g** $x-4$ **h** $x+1$

2 **a** $\frac{x+5}{x+1}$ **b** $\frac{x-6}{x-2}$ **c** $\frac{x-1}{x+1}$ **d** $\frac{x+2}{x-8}$

e 1 **f** 1

3 **a** $\frac{2}{y}$ **b** $2b$ **c** $\frac{5}{a}$ **d** $\frac{x}{2}$

e $\frac{1}{3m}$ **f** $\frac{n+7}{2}$ **g** $3y$ **h** $\frac{x}{x+1}$

i $\frac{5(y-3)}{y-4}$ **j** $\frac{t+2}{r+2}$ **k** $\frac{x-1}{x-4}$ **l** $\frac{p-2}{3(p+2)}$

4 **a** $\frac{1}{2}$ **b** 5 **c** 1 **d** 3

e 5 **f** x

5 **a** $\frac{5c+3}{12}$ **b** $\frac{7m-5}{12}$ **c** $\frac{3x-10}{10}$ **d** $\frac{12k+5}{20}$

e $\frac{29x-7}{40}$ **f** $\frac{-x+2}{6}$ **g** $\frac{b+10}{24}$ **h** $\frac{-5}{12}$

6 **a** $\frac{5x+13}{6}$ **b** $\frac{11x-8}{30}$ **c** $\frac{3x-7}{8}$

d $\frac{x+10}{12}$ **e** $\frac{x-12}{24}$ **f** $\frac{4x+1}{10}$

g $\frac{13x-6}{15}$ **h** $\frac{7x+17}{18}$ **i** $\frac{23x+5}{6}$

j $\frac{7x+15}{20}$ **k** $\frac{40-10x}{21}$ **l** $\frac{3x^2+1}{40}$

7 **a** $\frac{2a+1}{a-4}$ **b** $\frac{x+1}{x-3}$

c $\frac{2}{t}$ **d** $\frac{m+2}{(m-3)(m+7)}$

e $\frac{(x+3)(x+5)}{(x-3)}$ **f** $\frac{1}{x+5}$

g $\frac{3(2y+3)}{y+3}$ **h** $\frac{m(m+1)}{(m-3)(m-1)}$

i $\frac{c^2+6}{c+2}$ **j** $\frac{n-p}{n}$

k $\frac{y^2}{x^2(x+1)}$ **l** $\frac{10(a^2+b^2)}{a-b}$

8 **a** $\frac{2(x-5)}{x-2}$ **b** $\frac{(x+8)(x-6)}{(x-3)(x+6)}$

c $\frac{(3x+2)(5x+1)}{(x+1)(x+2)}$ **d** $\frac{x}{2(x+8)}$

9 **a** $\frac{5m+3}{m(m+1)}$ **b** $\frac{4k^2+3}{k(k^2+1)}$

c $\frac{4y-7}{(y+2)(y-3)}$ **d** $\frac{x^2+2x+6}{x(x+3)}$

e $\frac{t^2+t+10}{(t+5)(t-1)}$ **f** $\frac{8x+5}{x(2x+1)}$

g $\frac{2x-1}{(x+2)(x+1)}$ **h** $\frac{-2h+11}{(2h-1)(h+2)}$

i $\frac{2m^2+3}{(m-3)(2m+1)}$ **j** $\frac{a^3+12a+4}{a(3a+1)}$

k $\frac{x^2-3x-6}{2(x+1)}$ **l** $\frac{2x+1}{x(x+1)}$

10 **a** $\frac{x+1}{(x+2)(x-2)}$ **b** $\frac{2x^2-4x+4}{(x+1)(x-1)}$

c $\frac{x+4}{(x+2)(x+1)}$ **d** $\frac{4x^3-8x^2+3}{x(x-1)(x-2)}$

e $\frac{x^2+10x+4}{x(x+2)(x-2)}$ **f** $\frac{x^3+7x^2+15x}{(x+1)(x+2)(x+3)}$

g $\frac{-2x^2+5x-8}{x(x+2)(x-2)}$ **h** $\frac{-3x^2-5x+10}{x(x+2)}$

11 $a = x - 3$

12 $b = 3x + 1$

13 **a** $\frac{(1-x^4)}{x^2}$ **b** $\frac{(x^4+2x^2+1)}{x^2}$

c $\frac{1}{x-1}$ **d** $\frac{1}{2x^2}$

14 **a** $\frac{1}{x(x+1)}$

b **i** $\frac{1}{42}$ **ii** $\frac{1}{90}$

iii $\frac{1}{9900}$ **iv** $\frac{1}{1\,001\,000}$

15 **a** $\frac{x}{y(y+1)}$

b **i** $\frac{5}{42}$ **ii** $\frac{7}{110}$

iii $\frac{67}{10\,100}$ **iv** $\frac{11}{999\,000}$

16 **a** -1 **b** -4 **c** $\frac{x-3}{x+4}$

d $-x-1$ **e** $\frac{(5-x)}{(x+5)}$ **f** $\frac{2}{(2-x)}$

17 ±1

EX p72

Checkpoint

1 **a** x^9 **b** y^4 **c** $2m$

d a^{10} **e** $9x^2y^4$ **f** $\frac{8a^3}{b^6}$

2 **a** 1 **b** 2 **c** 1 **d** 3

3 **a** $\frac{1}{y^3}$ **b** $\frac{4}{x^2}$ **c** $\frac{1}{6a^7}$

d $2b^3$ **e** $\frac{1}{2a^2b^5}$ **f** $\frac{5d^4}{c^3}$

4 $\frac{12}{a^7b}$

5 **a** $6x$ **b** $8m^2$

c $7a+5b$ **d** $-3c+5d$

e $5a^2-2a$ **f** $-2x^2y+3xy+xy^2$

g $\frac{3x}{20}+\frac{y}{4}$ **h** $\frac{5m^2}{12}+\frac{m}{2}$

6 **a** $3ab$ **b** $8xy$
c $3ab^2$ **d** $-3x + 12$

7 **a** $3a + 21$ **b** $6y + 7$
c $3x - 8$ **d** $-7x + 10$

8 **a** $a^2 + 3a + 2$ **b** $x^2 + 2x - 24$
c $k^2 - 25$ **d** $2x^2 + x - 6$

9 **a** $x^3 + 2x^2y - xy - 2y^2$
b $\frac{1}{2}x^3 + \frac{27}{4}x^2 + 5x - 6$

10 **a** $\frac{1-x}{6}$ **b** $\frac{17y+2}{6}$
c $\frac{5m^2-4m+5}{20}$ **d** $\frac{11x+1}{21}$

11 **a** $\frac{a}{3}$ **b** $\frac{x+4}{5}$
c $\frac{1}{x+7}$ **d** $\frac{m-4}{m}$

12 **a** $\frac{x^2-2x+6}{(x-2)(x+1)}$ **b** $\frac{x^2-48}{4x(x+2)}$

EX
p77

2F Factorising

1 **a** $5(a + 7)$ **b** $6(5b + 1)$
c $3(8 + 11c)$ **d** $4(3d + 2)$
e $2(4e - 9)$ **f** $7(5f - 3)$
g $8(3 + 2g)$ **h** $10(1 - 4h)$
i $9(j + k)$ **j** $4(4w - 7x)$
k $m(m + 2)$ **l** $n(n - 6)$
m $a(5 + a)$ **n** $p(9 - p)$
o $3q(q + 1)$ **p** $2r(r - 4)$

2 **a** $3(x - 4)$ **b** $3(2y - 5)$
c $8x(x - 2)$ **d** $2k(2k + 3)$
e $c(5 + 11c)$ **f** $4u(2 - u)$
g $3m(2m - 1)$ **h** $y^2(1 + 5y)$
i $2t(7 - 4t)$ **j** $3p^2(3p + 1)$
k $a(4b + 3)$ **l** $tu(5t - 3)$
m $2mn(2m + n)$ **n** $3xy(2x + 1)$
o $3c(5c + 7d)$ **p** $6a^2bc(3b^2 - 2a^2c)$

3 **a** b, b
b $-5x, -5x, -5x$

4 **a** $-6(mn + 3)$ **b** $-3b(a + c)$
c $-y(y + 10)$ **d** $-9x(2x + 1)$
e $-4k(1 - 2k)$ **f** $-10a(2a^2 + 3)$
g $-2xy(6x - 7)$ **h** $-w^5(1 - w^2)$
i $-2b^2(2b - 1 + 3b^2)$

5 **a** $y(x - 5) + 2(x - 5) = (x - 5) \times y + (x - 5) \times 2$
$= (x - 5)(y + 2)$
b $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2) \times 4a + (3 + k^2) \times (-9)$
$= (3 + k^2)(4a - 9)$

6 **a** $(x + 3)(a + 5)$ **b** $(k - 2)(m + 4)$
c $(y + 5)(y + 2)$ **d** $(x - 1)(x + 9)$
e $(k + 6)(k - 3)$ **f** $(p - 9)(p - 6)$
g $(4 - a)(7 + a)$ **h** $(2n - 5)(3n + 4)$
i $(y^2 + 2)(x + 8)$ **j** $(a - 4)(a^2 + 3)$
k $(y^2 + 1)(2x - 5)$ **l** $(7 - x^3)(5x + 2)$

7 **a** $(a + 4)(b + 3)$ **b** $(x - 7)(x + 2y)$
c $(n + 5)(m - 6)$ **d** $(a + 2)(a + 5)$
e $(x - 1)(x + 4)$ **f** $(c + 8)(c - 3)$
g $(2 - y)(3y + 4)$ **h** $(p + 2)(p^2 + 5)$
i $(x^2 + 1)(x^2 + 3)$ **j** $(m^2 + 2)(2m - 3)$
k $(1 + k^2)(4 + k^3)$ **l** $(y - 3)(x + 5)$

8 **a** $x^2 - 25$
b Write each term as a square; that is, $x^2 - 25 = x^2 - 5^2$. Write the product of the sum and difference of the two values that are squared, x and 5. The factorised form is $(x + 5)(x - 5)$.

9 **a** $(x + 2)(x - 2)$ **b** $(a + 6)(a - 6)$
c $(10 + y)(10 - y)$ **d** $(8b + 3)(8b - 3)$
e $(5 + 7p)(5 - 7p)$ **f** $(2a + 9b)(2a - 9b)$
g $3(m + 1)(m - 1)$ **h** $2(2k + 3)(2k - 3)$
i $(xy + 4w)(xy - 4w)$ **j** $(m^3 + n)(m^3 - n)$
k $4(a^5 + 3b^3)(a^5 - 3b^3)$ **l** $(x^2y + w^2)(x^2y - w^2)$
m $(h + 8)(h - 2)$ **n** $(c - 1)(c - 7)$
o $4(1 - x)$ **p** $(1 + a + b)(1 - a - b)$
q $(y - 3)(7 - y)$ **r** $9(2x - 3)$

10 **a** $(x + \sqrt{7})(x - \sqrt{7})$ **b** $(a + \sqrt{13})(a - \sqrt{13})$
c $(\sqrt{19} + y)(\sqrt{19} - y)$ **d** $(4k + \sqrt{5})(4k - \sqrt{5})$
e $(p + 8 + \sqrt{2})(p + 8 - \sqrt{2})$
f $(m - 4 + \sqrt{6})(m - 4 - \sqrt{6})$
g $(x + 7 + \sqrt{3})(x + 7 - \sqrt{3})$
h $(y - 1 + \sqrt{19})(y - 1 - \sqrt{19})$
i $(d - 17 + \sqrt{17})(d - 17 - \sqrt{17})$

11 **a** $x^2 + 10x + 25$
b Find the square root of the first and last terms in the expansion ($\sqrt{x^2} = x$ and $\sqrt{25} = 5$). Note that the middle term is twice the product of these two values, and is positive because it is preceded by a '+' sign ($+10x$). Write the factorised form as the square of the sum of x and 5; that is, $(x + 5)^2$.

12 **a** $y^2 - 16y + 64$
b Find the square root of the first and last terms in the expansion ($\sqrt{y^2} = y$ and $\sqrt{64} = 8$). Note that the middle term is twice the product of these two values, and is negative because it is preceded by a '–' sign ($-16y$). Write the factorised form as the square of the difference between y and 8; that is, $(y - 8)^2$.

13 **a** $(x + 6)^2$ **b** $(m + 4)^2$ **c** $(p + 10)^2$
d $(y - 3)^2$ **e** $(x - 7)^2$ **f** $(g - 2)^2$
g $(1 + a)^2$ **h** $(5 - x)^2$ **i** $(9 - b)^2$
j $(w + x)^2$ **k** $(k - m)^2$ **l** $(3x + 2)^2$

14 **a** **i** $(x + 10)$ cm **ii** $(2x + 3)$ cm
b **i** $4x$ cm **ii** $6(x + 1)$ cm

15 **a** $\frac{a}{b}$ **b** $\frac{x}{3y}$ **c** $c^2 - 4b$
d mn **e** $\frac{x-y}{2}$ **f** $\frac{3}{x}$
g $\frac{3x}{4}$ **h** $\frac{m+7}{5m}$ **i** $\frac{a+b}{a-b}$
j $\frac{x+3}{x-3}$ **k** $\frac{4a+5}{2a}$ **l** $\frac{2x+7}{7-2x}$

16 a $\frac{6}{x-7}$ **b** 1

17 a $(n+2)$ and $(n+4)$ **b** $3n+6$

c $3(n+2)$; this expression is 3 times the middle odd number.

d $425 \times 3 = 1275$

e $n+(n+2)+(n+4)+(n+6)+(n+8) = 5n+20$
$= 5(n+4)$

The expression describes a value that is 5 times the middle number.

f $2675 \times 5 = 13\,375$

18 a The sum of k odd numbers (where k is an even number)
$= k \times$ (the average of the two middle odd numbers)
or $k \times (n+k-1)$
or $k \times$ (the average of the first and last of the odd numbers)

b 10 (the average of 81 and 83) $= 10 \times 82 = 820$
or $10(73+10-1) = 820$ or $10([73+91] \div 2) = 820$

19 15

20 a 399 **b** 896 **c** 6391 **d** 2484

21 a 101 **b** 50

c $50 \times 101 = 5050$ **d** $\frac{n}{2}(n+1)$

EX
p82

2G Factorising quadratic expressions

1 a 7 and 1 **b** 8 and 2 **c** 9 and 5
d 10 and 6 **e** 9 and 4 **f** 8 and 3

2 a $(x+7)(x+1)$ **b** $(x+8)(x+2)$
c $(x+9)(x+5)$ **d** $(x+10)(x+6)$
e $(x+9)(x+4)$ **f** $(x+8)(x+3)$

3 a $(x+2)(x+1)$ **b** $(x+3)(x+2)$
c $(x+10)(x+1)$ **d** $(x+9)(x+3)$
e $(x+5)(x+2)$ **f** $(x+7)(x+2)$
g $(x+3)(x+3) = (x+3)^2$ **h** $(x+11)(x+1)$
i $(x+9)(x+2)$ **j** $(x+8)(x+4)$
k $(x+9)(x+7)$ **l** $(x+10)(x+4)$
m $(x+6)(x+5)$ **n** $(x+11)(x+2)$
o $(x+12)(x+3)$

4 a 8 and –1 **b** –6 and –1 **c** –7 and 5
d 5 and –3 **e** –4 and 2 **f** –6 and –5

5 a $(x+8)(x-1)$ **b** $(x-6)(x-1)$
c $(x-7)(x+5)$ **d** $(x+5)(x-3)$
e $(x-4)(x+2)$ **f** $(x-6)(x-5)$

6 a $(x+6)(x-2)$ **b** $(x-10)(x+2)$
c $(x-3)(x-2)$ **d** $(x+9)(x-3)$
e $(x-8)(x-5)$ **f** $(x-5)(x+4)$
g $(x+7)(x-3)$ **h** $(x-11)(x-3)$
i $(x-7)(x+4)$ **j** $(x+4)(x-3)$
k $(x-9)(x+2)$ **l** $(x-8)(x-7)$

7 a $2(x+8)(x+1)$ **b** $3(x+3)(x+2)$
c $7(x+4)(x-2)$ **d** $-5(x+3)(x+4)$
e $-4(x-5)(x-2)$ **f** $-(x+6)(x-1)$
g $2(x-2)(x+9)$ **h** $19(x-3)(x+1)$
i $-3(x-13)(x+2)$ **j** $-2(x-1)(x-2)$
k $\frac{1}{2}(x-1)(x+4)$ **l** $\frac{1}{3}(x+6)(x-3)$

8 a $x+4$ **b** $\frac{1}{x-2}$ **c** $\frac{x}{x-2}$
d $\frac{x-5}{x+3}$ **e** $\frac{2(x-5)}{x+2}$ **f** $\frac{x}{x+2}$
g $\frac{x+1}{1-x}$ **h** $\frac{2(x-3)}{x+4}$ **i** $\frac{x-5}{2(x+5)}$

9 a 2 **b** $\frac{x-6}{3}$ **c** $\frac{5x-2}{6}$ **d** $\frac{29x-19}{24}$

10 a $(x+11)(x+4)$ **b** $(x-5)(x+3)$
c $(x+2)^2$ **d** $3(x-4)(x-2)$
e $5(x+3)(x-3)$ **f** $2x(x-4)$

11 a x^2+3^2 is the sum of two squares.

b $(x-3)^2+4^2$ is the sum of two squares.

c $(x+1)^2+2^2$ is the sum of two squares.

d There are no numbers that add to –4 and multiply to 5.

e The expression simplifies to x^2+2^2, which is the sum of two squares.

f There are no numbers that add to 12 and multiply to 40.

12 a $2x^2$ square metres and $(12x+18)$ square metres

b $(2x^2+12x+18)$ square metres

c $(x+3)$ m

13 $(x+6)$ m

14 a i $x^2+2xy+y^2$ **ii** $x^2+5xy+6y^2$
b i $(x+2)(x+8)$ **ii** $(x+2y)(x+8y)$
c i $(x+5)(x+2)$ **ii** $(x+5y)(x+2y)$
d i $(x-4y)(x-2y)$
ii $(y-7x)(y-2x)$ or $(7x-y)(2x-y)$
iii $(4x+y)(x+y)$

15 a $x(x+6)$ **b** not possible
c not possible **d** not possible
e not possible **f** $(x+1)(x+5)$
g not possible **h** not possible
i $(x+2)(x+4)$ **j** $(x+3)^2$

16 a $k = 5, 8, 9$ (positive) or $k = 0$ or
$k = -7, -16, -27, -40, \ldots$ (negative)

The negative case can be expressed as any number of the form $n(-n-6) = -n(n+6)$, where n is any positive integer.

b $k = 19, 36, 51, 64, 75, 84, 91, 96, 99, 100$ (positive) or $k = 0$ or $k = -21, -44, -69, -96, \ldots$ (negative)

The negative case can be expressed as any number of the form $-n(n+20)$, where n is any positive integer.

The positive case can be expressed as any number of the form $n(20-n)$, where n is positive and less than 20.

17 a $(x^2+4)(x+2)(x-2)$ **b** $4(4a+b)(4a-b)$
c $(a-x+y)(a-x-y)$

EX p88

2H Completing the square

1 **a** **i** $x^2 + 6x + 9$ **ii** $(x + 3)(x + 3) = (x + 3)^2$

b **i** $x^2 - 18x + 81$ **ii** $(x - 9)(x - 9) = (x - 9)^2$

c **i** $(x + 4)^2$ **ii** $(x + 2)^2$ **iii** $(x - 6)^2$

iv $(x + 8)^2$ **v** $(x - 5)^2$ **vi** $(x - 10)^2$

2 **a** 9 **b** 25 **c** 16

d 4 **e** 1 **f** 64

3 **a** $x^2 + 14x + 49$ **b** $x^2 - 6x + 9$

c $x^2 + 18x + 81$ **d** $x^2 + 20x + 100$

e $x^2 - 12x + 36$ **f** $x^2 + 22x + 121$

4 **a** $(x + 7)^2$ **b** $(x - 3)^2$ **c** $(x + 9)^2$

d $(x + 10)^2$ **e** $(x - 6)^2$ **f** $(x + 11)^2$

5 $(x + 3)^2 - (\sqrt{2})^2 = (x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$

6 **a** $(x + 4 + \sqrt{7})(x + 4 - \sqrt{7})$

b $(x - 5 + \sqrt{11})(x - 5 - \sqrt{11})$

c $(x + 2 + \sqrt{10})(x + 2 - \sqrt{10})$

d $(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

e $(x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$

f $(x - 6 + \sqrt{23})(x - 6 - \sqrt{23})$

g $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$

h $(x - 7 + \sqrt{3})(x - 7 - \sqrt{3})$

i $(x + 7)(x + 1)$

j $\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$

k $\left(x - \frac{5}{2} - \frac{\sqrt{13}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{13}}{2}\right)$

l $\left(x + \frac{11}{2} - \frac{\sqrt{85}}{2}\right)\left(x + \frac{11}{2} + \frac{\sqrt{85}}{2}\right)$

7 **a** $2\left(x + 2 - \sqrt{3}\right)\left(x + 2 + \sqrt{3}\right)$

b $-\left(x + 3 - \sqrt{11}\right)\left(x + 3 + \sqrt{11}\right)$

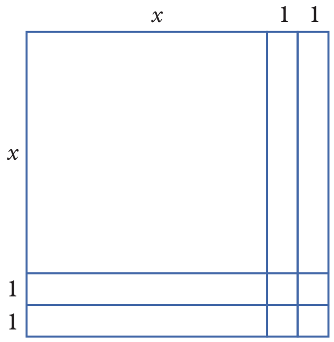
c $-5\left(x - 1 - \sqrt{2}\right)\left(x - 1 + \sqrt{2}\right)$

d $3\left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)$

8 **a** x by $y - x$ **b** x^2

c y by $y - x$ **d** y^2

9 **a**



b $x^2 + 4x + 4$ **c** $(x + 2)^2$

10 **a** $x(x + 6)$

b $(x + 3 + \sqrt{8})(x + 3 - \sqrt{8})$

c $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$

d $(x + 3 + \sqrt{6})(x + 3 - \sqrt{6})$

e $(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$

f $(x + 1)(x + 5)$

g $(x + 3 + \sqrt{3})(x + 3 - \sqrt{3})$

h $(x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$

i $(x + 2)(x + 4)$

j $(x + 3)^2$

11 **a** $x^2 + 7x + 6 = \left(x + \frac{7}{2}\right)^2 - \left(\frac{5}{2}\right)^2$

$= \left(x + \frac{7}{2} + \frac{5}{2}\right)\left(x + \frac{7}{2} - \frac{5}{2}\right)$

$= (x + 6)(x + 1)$

b $6 \times 1 = 6$ and $6 + 1 = 7$, so the factors are $(x + 6)$ and $(x + 1)$.

12 **a** $c > 16$ **b** $c > 25$ **c** $-2 < b < 2$

13 **a** Can be written as $(x + 3)^2 + 1$, which is the sum of two squares.

b Can be written as $(x - 4)^2 + 5$, which is the sum of two squares.

c The leading coefficient is not a highest common factor.

d The leading coefficient is not a highest common factor.

e It can be written as $2(x + 1)^2 + 2$, which is the sum of two squares.

14 **a** $2(x + 7)(x - 4)$

b not possible as it is the sum of two squares: $\left(x - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2$

c $x(x + 6)(x - 3)$

d $-(x - 4)(x + 3)$

e $-2(x + 7)(x - 7)$

f not possible as it is the sum of two squares: $(2x)^2 + 5^2$

15 **a** $2\left(x + 3 - \sqrt{\frac{13}{2}}\right)\left(x + 3 + \sqrt{\frac{13}{2}}\right)$

b $2\left(x + \frac{7}{4} - \frac{\sqrt{41}}{4}\right)\left(x + \frac{7}{4} + \frac{\sqrt{41}}{4}\right)$

c $3\left(x + \frac{11}{6} - \frac{\sqrt{133}}{6}\right)\left(x + \frac{11}{6} + \frac{\sqrt{133}}{6}\right)$

16 **a** $\left(x + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c}\right)\left(x + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c}\right)$

b $\frac{b^2 - 4c}{4}$

EX p93

2I Factorising non-monic quadratic expressions

1 **a** **i** $ac = 4, b = 5$ **ii** 1, 4 **iii** $x + 4x$

b **i** $ac = 18, b = 11$ **ii** 2, 9 **iii** $2x + 9x$

c **i** $ac = 16, b = 17$ **ii** 1, 16 **iii** $x + 16x$

d **i** $ac = -24, b = 10$ **ii** -2, 12 **iii** $-2x + 12x$

e **i** $ac = -18, b = -7$ **ii** -9, 2 **iii** $-9x + 2x$

f **i** $ac = 30, b = -13$ **ii** $-3, -10$ **iii** $-3x - 10x$

2 **a** $(3x + 2)(x + 3)$ **b** $(4x + 1)(x + 4)$

c $(2x + 1)(x + 2)$ **d** $(2x - 3)(3x + 1)$

e $(3x - 2)(x + 4)$ **f** $(5x - 3)(x - 2)$

3 **a** $(2x + 3)(x + 2)$ **b** $(3x + 2)(x + 1)$

c $(2x + 1)(2x + 5)$ **d** $(7x + 2)(x + 1)$

e $(5x + 4)(x + 2)$ **f** $(2x + 3)(x + 3)$

g $(3x + 2)(2x + 1)$ **h** $(5x + 1)(2x + 3)$

i $(4x + 5)(2x + 1)$

4 **a** $(3x + 1)(x - 3)$ **b** $(4x - 3)(x + 2)$
c $(2x - 5)(x - 2)$ **d** $(5x + 4)(x - 1)$
e $(4x - 1)(2x + 3)$ **f** $(2x - 3)(2x + 5)$
g $(7x - 4)(x - 2)$ **h** $(4x - 1)(3x + 1)$
i $(3x + 4)(x - 5)$

5 **a** $2(2x + 3)(x + 1)$ **b** $3(4x + 1)(x + 2)$
c $2(5x - 1)(x + 4)$ **d** $(4x + 5)(2x + 3)$
e $3(2x + 5)(x - 2)$ **f** $4(3x + 1)(x - 1)$
g $-2(2x + 1)(x + 1)$ **h** $-3(4x - 1)(x + 2)$
i $-(3x + 4)(2x + 3)$

6 **a** $2x + 3$ **b** $\frac{1}{x + 3}$ **c** $2x + 7$
d $\frac{2x - 1}{x + 5}$ **e** $\frac{5x + 2}{2x - 3}$ **f** -4

7 **a** $\frac{5x - 3}{x - 3}$ **b** $\frac{2x + 7}{2x + 1}$ **c** 1
d $\frac{(2x + 7)(x + 4)}{(2x - 7)(x - 4)}$ **e** $\frac{5x + 2}{2}$ **f** $\frac{x + 1}{x - 3}$

8 **a** $3(x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$
b $2(x - 5 + \sqrt{5})(x - 5 - \sqrt{5})$
c $5(x + 1 + \sqrt{7})(x + 1 - \sqrt{7})$
d $3(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$
e $2\left(x - 3 + \sqrt{\frac{21}{2}}\right)\left(x - 3 - \sqrt{\frac{21}{2}}\right)$
f $4\left(x + 1 + \frac{\sqrt{5}}{2}\right)\left(x + 1 - \frac{\sqrt{5}}{2}\right)$
g $6\left(x - 2 + \sqrt{\frac{29}{6}}\right)\left(x - 2 - \sqrt{\frac{29}{6}}\right)$
h $3\left(x + 4 + \sqrt{\frac{47}{3}}\right)\left(x + 4 - \sqrt{\frac{47}{3}}\right)$
i $9\left(x - 1 + \frac{\sqrt{7}}{3}\right)\left(x - 1 - \frac{\sqrt{7}}{3}\right)$

9 $\frac{-14}{3x + 4}$

10 **a** $(4x - 1)(x + 2)$
b $-4x^2 - 7x + 2 = -(4x^2 + 7x - 2) = -(4x - 1)(x + 2)$
$= (-4x + 1)(x + 2)$
$= (1 - 4x)(x + 2)$
c **i** $(3 - 2x)(x + 1)$
ii $(1 - 3x)(x - 5) = (3x - 1)(5 - x)$
iii $(7 - 5x)(x + 2)$
iv $(2 - x)(4x + 3)$
v $(1 - 2x)(3x + 2)$
vi $(1 - 5x)(5x - 1)$

11 **a** $\sqrt{9x^2 - 4x - 5}$ metres **b** $(9x + 5)(x - 1)$

12 **a** $(x + 4 - \sqrt{11})(x + 4 + \sqrt{11})$
b $2\left(x + 2 - \sqrt{\frac{3}{2}}\right)\left(x + 2 + \sqrt{\frac{3}{2}}\right)$
c $3(x + 1)\left(x + \frac{5}{3}\right)$ or $(x + 1)(3x + 5)$

13 Parts **a**, **d** and **e** cannot be factorised.
b $(5x - 4)(3x + 2)$
c $3(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$
f $-3(2x - 9)(x - 7)$

14 **a** Room A: $2x + 3$ by $4x - 5$ metres
Room B: $4x - 5$ by $4x - 5$ metres
b $(24x^2 - 38x + 10)$ m²
c $6x - 2$ by $6x - 2$ metres
d $(6x - 2)^2 - (4x - 5)^2$ m²
e $(10x - 7)(2x + 3)$ m²
f $36 - \frac{24}{x} + \frac{4}{x^2}$
g 28

15 **a** **i** $(x^2 + 5)(x^2 + 2)$
ii $(x^2 + 12)(x^2 + 2)$
iii $(x^2 + 3)(x^2 + 1)$
b **i** $(x^2 + 1)(x - 1)(x + 1)$
ii $(x^2 + 4)(x - 2)(x + 2)$
iii $(x^2 + 9)(x - 3)(x + 3)$
c **i** $(x - 2)(x + 2)(x - 3)(x + 3)$
ii $(x - 1)(x + 1)(x - 5)(x + 5)$
iii $(x - 1)(x + 1)(x - \sqrt{10})(x + \sqrt{10})$

16 **a** $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$
b $a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)$
c $b^2 - 4ac$

EX
p97

CHAPTER 2 review

Mathematical literacy review

1 **a** $x + 5$ or $\frac{4}{x + 1} - 7x$ **b** $6x^3 - 2x + 9$
c 9 or 5 **d** $\frac{4}{x + 1}$ **e** -2 or -7
f 6 **g** 3

2 **a** V **b** VII **c** VIII **d** VI
e I **f** III **g** IV **h** II

3 **a** negative index, index
b expanding, highest common factor

4 E

Multiple choice

1 D **2** A **3** E **4** B **5** E
6 C **7** D **8** C **9** B **10** B
11 A **12** E **13** D **14** E **15** C

Short answer

1 **a** a^7 **b** $24x^7y^5$ **c** 3 **d** $3ac^3$

2 **a** $\frac{x^3}{y^4}$ **b** $\frac{a^4}{3b^5}$ **c** $\frac{6p^4r^7}{5q^5}$

3 **a** x^8 **b** $\frac{1}{pq^2}$ **c** $\frac{1}{864a^8b^3}$

4 **a** $-6xy - 3x^2 + 2x^2y - 5xy^2$
b $-3ab^2 - 2a^2b^2 + 2a^2b$
c $-4pq - 8p^2 + 3q^2$

5 **a** $a^2b^2c^2$ and $(2abc)^2$
b $5xyz^2$, $-2z^2xy$ and $3yz^2x$
c $2p^2q^2$, $-(pq)^2$ and $p^2 \times q \times -q$

6 **a** 14 **b** 39 **c** -16

7 **a** $x^2 + 4x - 21$ **b** $x^2 + 4x + 4$

c $x^2 - 8x + 16$ **d** $2x^2 - 7x - 4$

e $a^2 - b^2$ **f** $a^3 + a^2b^2 + ab + b^3$

8 **a** $6x^4 - 2x^2 - 20$

b $4x^2 - 9y^2$

c $9x^4 - 24x^2 + 16$

9 **a** $2x^2 - 5x - 6$ **b** $4x - 13x^2$ **c** $x^2 - x$

10 **a** $\frac{25x}{12}$ **b** $\frac{9x-22}{20}$ **c** $\frac{14x-11}{24}$ **d** $\frac{31}{12x}$

11 **a** $(x-3)(x-4)$ **b** $(x+1)^2(x-1)$

c $2(8+3x)(x-2)$

12 **a** $2(3x+2y)(3x-2y)$

b $-3(2x-3)$

c $8x$

13 **a** $(xy + \sqrt{13})(xy - \sqrt{13})$

b $(x + 1 + \sqrt{3})(x + 1 - \sqrt{3})$

c $(\sqrt{7} + a - 1)(\sqrt{7} - a + 1)$

14 **a** $(x+2)(x+6)$ **b** $(x-9)(x+7)$

c $-(x-4)(x-2)$ **d** $-6(x+3)(x-2)$

15 **a** $\frac{x+4}{x+3}$ **b** $\frac{(x-3)(x+1)}{(x+4)(x+3)}$

16 **a** $(x + 5 + \sqrt{5})(x + 5 - \sqrt{5})$

b $(x - 4 + \sqrt{17})(x - 4 - \sqrt{17})$

17 **a** $(3x+2)(x+9)$ **b** $(4x-1)(2x+3)$

c $(2x-11)(x+2)$ **d** $(4x-3)(3x-2)$

18 **a** $2(x + 3 + \sqrt{11})(x + 3 - \sqrt{11})$

b cannot be factorised

c $3(2x-5)(x+6)$

d $-4\left(x - \frac{3}{2} + \frac{\sqrt{7}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{7}}{2}\right)$

Analysis

1 **a** $x^2 + 12$ is not in the correct form to apply the difference of two squares rule.

b No. Since b is an integer, the constant term would need to be b^2, but 12 is not a perfect square.

c $\left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4} + 12$

$-\frac{b^2}{4} + 12 \le 0$

$b^2 \ge 48$

Since b is an integer, its square is a square number. The smallest square number greater than 48 is 49.

$\therefore b^2 \ge 49$

d The possible values of b are ±7, ±8 and ±13.

Factor pairs of 12	b = Sum of factors	Quadratic trinomial	Factorised form
1, 12	13	$x^2 + 13x + 12$	$(x+1)(x+12)$
–1, –12	–13	$x^2 - 13x + 12$	$(x-1)(x-12)$
2, 6	8	$x^2 + 8x + 12$	$(x+2)(x+6)$
–2, –6	–8	$x^2 - 8x + 12$	$(x-2)(x-6)$
3, 4	7	$x^2 + 7x + 12$	$(x+3)(x+4)$
–3, –4	–7	$x^2 - 7x + 12$	$(x-3)(x-4)$

e The possible values of b are ±10, ±11, ±14 and ±25.

2 **a** $a = 2, b = -15$

b $a = 3, b = 1, c = 5$

c $a = 1, b = 12, c = 15$

d $a = -6, b = -15, c = 88$

e $c = 5$, $d = 5a - 3b$. So the 12 options for a, b are

a	1	–1	2	–2	3	–3	4	–4	6	– 6	12	– 12
b	12	–12	6	–6	4	–4	3	–3	2	– 2	1	– 1

f **i** $f = ad + bc$

ii 2 values

iii 4 values. For example: $6x^2 + dx + 5 = 6x^2 + 11x + 5$ or $6x^2 + 31x + 5$ or $6x^2 + 13x + 5$ or $6x^2 + 17x + 5$

iv 12 values

v $\frac{mn}{2}$

vi mn possible values

3 **a** $A = (3x+1)^2 - x^2 = (4x+1)(2x+1)$

b $A = (3x+1)^2 - 2x^2 = (3x + 1 + x\sqrt{2})(3x + 1 - x\sqrt{2})$

c $A = (3x+1)^2 - 4x^2 = (5x+1)(x+1)$

d $A = \frac{1}{2}[(3x+1)^2 - x^2] = \frac{1}{2}(4x+1)(2x+1)$

e $A = \frac{1}{2}[(3x+1)^2 - 4x^2] = \frac{1}{2}(5x+1)(x+1)$

f side length of big square: $5x + 3$;
side length of small square: $4x$

g side length of big square: $\left(\frac{m+n}{2}\right)x + a$;
side length of small square: $\left(\frac{m-n}{2}\right)x$

CHAPTER 3 Real numbers

3A Rational and irrational numbers

EX p106

1 **a** IV **b** VI **c** II **d** V **e** I **f** III

2 **a** rational **b** rational **c** irrational

d rational **e** irrational **f** rational

g rational **h** irrational **i** rational

j rational **k** rational **l** irrational

m rational **n** irrational **o** irrational

3 **a**, **c**, **d**, **f**, **g**, **i**, **j**, **l**

4 **a** 4.5826 **c** 2.8284 **d** –2.4495

f 11.1803 **g** –15.5885 **i** 1.4422

j 5.1962 **l** 0.4899

5 $\sqrt{3}, -3\sqrt{5}, \sqrt{8}, \sqrt[4]{32}$

6 **a** 11 **b** 30 **c** –32 **d** –1

e 24 **f** 14 **g** 10 **h** 5

i 13 **j** –6 **k** 11 **l** 9

7 rational numbers: **a**, **b**, **c**, **f**, **g**, **j**, **k**, **l**

irrational numbers: **d**, **e**, **h**, **i**

8 **d**, **h**, **i**

9 **a** rational **b** irrational **c** irrational **d** rational

e irrational **f** irrational **g** irrational **h** irrational

10 Irrational, since it cannot be written exactly as a terminating or recurring decimal.

11 **a** 4.6856 **b** 6.7807 **c** 1.6042
d 4.9892 **e** 5.9161 **f** 54.9909
g 3.4641 **h** 2.6788 **i** 4.6291
j 0.3769 **k** 1.5157 **l** 0.1139

12 **a** $\sqrt{\frac{5}{8}} \approx 0.790\,569, \frac{\sqrt{5}}{\sqrt{8}} \approx 0.790\,569$
b yes
c $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
d Irrational, because it cannot be written exactly as a terminating or recurring decimal.
e **i** $\sqrt{\frac{32}{2}} = \sqrt{16} = 4$, rational
ii $\sqrt{\frac{15}{3}} = \sqrt{5} = 2.236\ldots$, irrational
iii $\sqrt{\frac{7}{28}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, rational

13 **a** **i** 2 **ii** 3
b 2, 3
c By choosing between which two perfect squares the number lies.
d **i** 3, 4 **ii** –4, –5 **iii** 11, 12
iv –6, –7 **v** 13, 14 **vi** 17, 18

14 **a** **i** 8, 9 **ii** –5, –4 **iii** 4, 5
iv 15, 18 **v** 5, 8 **vi** –15, –12
b **i** $\frac{3}{5}, \frac{4}{5}$ **ii** $\frac{5}{4}, \frac{5}{3}$ **iii** $\frac{13}{2}, 7$
iv $-1, -\frac{1}{2}$ **v** $-\frac{7}{2}, -\frac{10}{3}$ **vi** $-\frac{13}{3}, -\frac{22}{5}$

15 **a** 2, 3 **b** 1, 2 **c** 3, 4 **d** 13, 14
e 14, 21 **f** 2, 3 **g** –27, –22 **h** $6, \frac{20}{3}$

16 **a** $\sqrt{2^2 + 1^2} = \sqrt{5}$ **b** 2.24

17 **a** 8 cm **b** **i** $8\sqrt{2}$ cm **ii** 11.31 cm

18 **a** **i** $\sqrt{35}$ cm **ii** 5.92 cm **iii** 5.91 cm
b **i** $\sqrt{51}$ mm **ii** 7.14 mm **iii** 7.14 mm
c **i** $\sqrt{95}$ cm **ii** 9.75 cm **iii** 9.74 cm

19 **a** **i** 23.66 cm **ii** 23.68 cm **iii** 23.64 cm
b **i** 28.57 cm **ii** 28.56 cm **iii** 28.56 cm
c **i** 38.99 cm **ii** 39.00 cm **iii** 38.96 cm

20 The answers are only the same if the digit in the third decimal place is smaller or equal to four, that is when the rounded value is the same as the truncated value.

21 **a** $\frac{2}{5}$ **b** $\frac{13}{20}$ **c** $\frac{49}{100}$
d $\frac{8}{5}$ **e** $\frac{257}{100}$ **f** $\frac{851}{1000}$

22 There are no positive real numbers that when raised to an even power will result in a negative solution; however, a negative number raised to an odd power will result in a negative solution.

23 **a** 1–IV, 2–III, 3–V, 4–VI, 5–II, 6–I
b To find another number with the same recurring decimal part, so that we could subtract to eliminate the decimal part.

24 **a** $\frac{2}{9}$ **b** $\frac{7}{9}$ **c** $\frac{7}{3}$ **d** $\frac{17}{3}$

25 **a**
$$\begin{aligned} \text{Let } x &= 0.363\,636\ldots \\ 100x &= 36.363\,636\ldots \\ 100x - x &= 36.363\,636\ldots - 0.363\,636\ldots \\ 99x &= 36 \\ x &= \frac{36}{99} \\ &= \frac{4}{11} \\ \text{So } 0.\overline{36} &= \frac{4}{11} \end{aligned}$$
b Because the recurring part of the decimal contained two digits, it was necessary to shift the decimal point two places to obtain the same recurring decimal again.

26 **a** $\frac{14}{99}$ **b** $\frac{7}{33}$ **c** $\frac{16}{33}$ **d** $\frac{172}{99}$

27 **a** $\frac{1}{3}$ **b** $\frac{8}{33}$ **c** $\frac{785}{999}$ **d** $\frac{41}{99}$
e $\frac{17}{30}$ **f** $\frac{35}{11}$ **g** $\frac{383}{90}$ **h** $\frac{3899}{1110}$

EX
p111

3B Simplifying surds

1 **a** 4 **b** 9 **c** 7 **d** 2
e 10 **f** 5 **g** 18 **h** 44
i 1 **j** 2

2 **a** $9\sqrt{7}$ **b** $4\sqrt{3}$ **c** $2\sqrt{6}$ **d** $5\sqrt{10}$
e $3\sqrt{2}$ **f** $6\sqrt{5}$ **g** $8\sqrt{11}$ **h** $30\sqrt{5}$
i $6\sqrt{10}$

3 **a** $2\sqrt{7}$ **b** $5\sqrt{2}$ **c** $2\sqrt{6}$ **d** $3\sqrt{3}$
e $4\sqrt{3}$ **f** $5\sqrt{3}$ **g** $3\sqrt{7}$ **h** $2\sqrt{15}$
i $2\sqrt{14}$ **j** $3\sqrt{5}$ **k** $8\sqrt{2}$ **l** $6\sqrt{3}$
m $6\sqrt{5}$ **n** $7\sqrt{3}$ **o** $9\sqrt{2}$ **p** $5\sqrt{5}$
q $15\sqrt{2}$ **r** $12\sqrt{2}$ **s** $3\sqrt{30}$ **t** $9\sqrt{7}$

4 **a** $10\sqrt{5}$ **b** $6\sqrt{2}$ **c** $12\sqrt{3}$ **d** $27\sqrt{2}$
e $16\sqrt{5}$ **f** $9\sqrt{10}$ **g** $60\sqrt{6}$ **h** $-12\sqrt{3}$
i $28\sqrt{6}$ **j** $-14\sqrt{5}$ **k** $64\sqrt{3}$ **l** $60\sqrt{7}$
m $-20\sqrt{7}$ **n** $21\sqrt{11}$ **o** $-24\sqrt{17}$ **p** $1000\sqrt{3}$
q $4\sqrt{5}$ **r** $3\sqrt{6}$ **s** $16\sqrt{62}$ **t** $3\sqrt{11}$

5
$$\begin{aligned} 5\sqrt{27} &= 5 \times \sqrt{27} \\ &= 5 \times \sqrt{9 \times 3} \\ &= 5 \times \sqrt{9} \times \sqrt{3} \\ &= 5 \times 3 \times \sqrt{3} \\ &= 15\sqrt{3} \end{aligned}$$

6 **a** $2\sqrt{11}$ **b** $5\sqrt{5}$ **c** $2\sqrt{30}$ **d** $6\sqrt{2}$
e $5\sqrt{6}$ **f** $-6\sqrt{6}$ **g** $3\sqrt{14}$ **h** $4\sqrt{15}$
i $-9\sqrt{2}$ **j** $7\sqrt{5}$ **k** $4\sqrt{3}$ **l** $2\sqrt{3}$
m $-8\sqrt{3}$ **n** $60\sqrt{5}$ **o** $6\sqrt{21}$ **p** $9\sqrt{2}$
q $-160\sqrt{2}$ **r** –20 **s** $-10\sqrt{10}$ **t** $3\sqrt{2}$

7 **a** $4\sqrt{5}$ cm **b** $6\sqrt{5}$ cm **c** $2\sqrt{13}$ cm **d** $12\sqrt{11}$ m

8 **a**
$$\begin{aligned} 3\sqrt{6} &= \sqrt{3^2} \times \sqrt{6} \\ &= \sqrt{9} \times \sqrt{6} \\ &= \sqrt{9 \times 6} \\ &= \sqrt{54} \end{aligned}$$
b To be simplified to a whole number coefficient it must have first been expressed as the square root of a perfect square.
c They are equal.

d Put the entire surd and the original surd into your calculator and press equal. You should get the same answer.

e $\sqrt{48}$

9 a $\sqrt{45}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{252}$
e $\sqrt{160}$ **f** $\sqrt{275}$ **g** $\sqrt{384}$ **h** $\sqrt{125}$
i $\sqrt{240}$ **j** $-\sqrt{128}$ **k** $\sqrt{1008}$ **l** $-\sqrt{75}$

10 a $\sqrt{a^2 b}$

c Square the whole part then multiply it by the surd part to find the number that will be under the root sign in the entire surd.

11 a $\sqrt{m^2 n}$ **b** $\sqrt{49a}$ **c** $\sqrt{5x^2}$ **d** $\sqrt{100xy}$
e $\sqrt{2cd^2}$ **f** $\sqrt{36abc}$ **g** $\sqrt{a^3b^2}$ **h** $\sqrt{x^4 y^3 z^2}$

12 a $x\sqrt{y}$ **b** xy **c** $b\sqrt{a}$ **d** $xz\sqrt{y}$
e x^2 **f** $x^2\sqrt{y}$ **g** x^2y **h** $a^2b\sqrt{c}$

13 a $2^2 \times 3$ **b** $\sqrt{2^2 \times 3} = \sqrt{2^2} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$

14 a 2×3^3 **b** $\sqrt{3^3 \times 2} = \sqrt{3^2 \times 3 \times 2}$ **c** $3\sqrt{6}$

15 a $3\sqrt{2}$ **b** $2\sqrt{7}$ **c** $4\sqrt{2}$ **d** $3\sqrt{3}$
e $2\sqrt{6}$ **f** $5\sqrt{2}$ **g** $6\sqrt{5}$ **h** $16\sqrt{2}$
i $15\sqrt{7}$ **j** $16\sqrt{10}$ **k** $28\sqrt{3}$ **l** $18\sqrt{5}$
m $6\sqrt{5}$ **n** $35\sqrt{3}$ **o** $54\sqrt{2}$ **p** $5\sqrt{5}$
q $30\sqrt{2}$ **r** $12\sqrt{2}$

16 a $\sqrt{18}\text{ m} = 3\sqrt{2}\text{ m}$ **b** $\sqrt{2}\text{ m}$

c The side length of the 18 m² square is three times the side length of the 2 m² square. The area a square divided into nine smaller squares has a total area of 3 × 3 = 9 smaller squares, which is the specified amount.

d i $\sqrt{27}\text{ m} = 3\sqrt{3}\text{ m}$ **ii** $\sqrt{48}\text{ m} = 4\sqrt{3}\text{ m}$
iii $\sqrt{50}\text{ m} = 5\sqrt{2}\text{ m}$ **iv** $\sqrt{75}\text{ m} = 5\sqrt{3}\text{ m}$

17 a i $2\sqrt{2}$ **ii** $3\sqrt{3}$ **iii** $5\sqrt{5}$ **iv** $a\sqrt{a}$
b i $4\sqrt{2}$ **ii** $9\sqrt{3}$ **iii** $25\sqrt{5}$ **iv** $a^2\sqrt{a}$
c i $2\sqrt[3]{2}$ **ii** $3\sqrt[3]{3}$ **iii** $5\sqrt[3]{5}$ **iv** $a\sqrt[3]{a}$
d i $2\sqrt[4]{2}$ **ii** $3\sqrt[4]{3}$ **iii** $5\sqrt[4]{5}$ **iv** $a\sqrt[4]{a}$

18 a $\sqrt{5}, \sqrt{10}, \sqrt{21}, \sqrt{22}, \sqrt{27}, \sqrt{30}, \sqrt{42}, \sqrt{46}$
b $\sqrt{6}, 3, \sqrt{11}, 2\sqrt{3}, 2\sqrt{6}, 6, \sqrt{39}, 4\sqrt{3}$
c $4, 2\sqrt{5}, \sqrt{22}, 3\sqrt{3}, \sqrt{43}, 2\sqrt{11}, 3\sqrt{6}, \sqrt{83}$
d $2\sqrt{6}, 4\sqrt{5}, 3\sqrt{14}, 2\sqrt{57}, 12\sqrt{2}, 3\sqrt{43}, 20, 7\sqrt{10}$

EX
p116

3C Multiplying and dividing surds

1 a $\sqrt{35}$ **b** $\sqrt{33}$ **c** $-\sqrt{14}$ **d** $-\sqrt{105}$
e $\sqrt{65}$ **f** $-\sqrt{22}$ **g** $\sqrt{21}$ **h** $\sqrt{95}$
i $3\sqrt{77}$ **j** $20\sqrt{6}$ **k** $-28\sqrt{22}$ **l** $-36\sqrt{30}$
m $66\sqrt{34}$ **n** $64\sqrt{42}$ **o** $-24\sqrt{33}$ **p** $132\sqrt{15}$

2 a 7 **b** 3 **c** 2 **d** 6
e 5 **f** 11 **g** 10 **h** 9

3 a The answer is rational. **b** a

4 a 9 **b** -77 **c** 40 **d** 60
e $3\sqrt{10}$ **f** $30\sqrt{30}$ **g** $-32\sqrt{26}$ **h** $3\sqrt{33}$
i 84 **j** 15 **k** $-6\sqrt{26}$ **l** $\sqrt{105}$
m $72\sqrt{42}$ **n** $\frac{3\sqrt{195}}{8}$ **o** $-5\sqrt{70}$ **p** $-11\sqrt{85}$

5 a $\sqrt{17}$ **b** $\sqrt{3}$ **c** 1 **d** $-\sqrt{5}$
e $\sqrt{6}$ **f** $\sqrt{7}$ **g** $4\sqrt{6}$ **h** $2\sqrt{5}$
i $-3\sqrt{5}$ **j** $2\sqrt{5}$ **k** $-4\sqrt{10}$ **l** $\frac{\sqrt{5}}{2}$

6 a 4 **b** 2 **c** 3 **d** 2

7 a $-12\sqrt{5}$ **b** $5\sqrt{3}$ **c** $-3\sqrt{2}$ **d** $2\sqrt{5}$
e $4\sqrt{7}$ **f** $-\frac{7\sqrt{5}}{3}$ **g** $\frac{3}{\sqrt{3}}$ **h** $\frac{\sqrt{5}}{5}$

8
$$\begin{aligned} 6\sqrt{3} \times 3\sqrt{2} &= 6 \times \sqrt{3} \times 3 \times \sqrt{2} \\ &= 6 \times 3 \times \sqrt{3} \times \sqrt{2} \\ &= 18 \times \sqrt{3 \times 2} \\ &= 18\sqrt{6} \end{aligned}$$

9
$$\begin{aligned} \frac{24\sqrt{40}}{8\sqrt{20}} &= \frac{24}{8} \times \frac{\sqrt{40}}{\sqrt{20}} \\ &= 3 \times \sqrt{\frac{40}{20}} \\ &= 3\sqrt{2} \end{aligned}$$

10 a 6 **b** $\frac{3\sqrt{5}}{4}$ **c** $\frac{11\sqrt{17}}{7\sqrt{21}}$
d $-\frac{4\sqrt{3}}{3\sqrt{5}}$ or $-\frac{4\sqrt{5}}{5\sqrt{3}}$ **e** $\frac{9\sqrt{5}}{4}$
f $-\frac{2}{3}$ **g** -48 **h** $\frac{\sqrt{15}}{16\sqrt{2}}$ **i** 1

11 a 3 **b** $\frac{9}{10}$ **c** 24 **d** $\frac{1}{3}$
e -60 **f** 2 **g** 84 **h** 576
i $\frac{24}{5}$ **j** $\frac{1}{8}$ **k** $-\frac{4}{3}$ **l** $-\frac{7}{3}$

12 a $2\sqrt{10}$ **b** $\frac{\sqrt{3}}{5}$ **c** $\frac{8}{5}$
d $-\frac{11\sqrt{2}}{8}$ **e** $\frac{8\sqrt{2}}{3}$ **f** $-\frac{2\sqrt{6}}{5}$
g $\frac{\sqrt{5}}{9}$ **h** $-\frac{9\sqrt{7}}{5}$ **i** $-\frac{10\sqrt{11}}{9}$

13 a 5 **b** 8 **c** 2 **d** 7
e 9 **f** 80 **g** 54 **h** 192
i 891

14 a rational
b $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$
c $\sqrt{80}, \sqrt{54}, \sqrt{192}, \sqrt{891}$
d The number beneath the square root of an entire surd is equal to the square of the surd.

15 a x **b** ab **c** $6y$ **d** $x + y$

16 a $\frac{25}{7}$ **b** $\frac{1}{5}$ **c** $\frac{3}{2}$ **d** $\frac{10}{13}$

17 a Method 1: $1680\sqrt{6}$ Method 2: $42\sqrt{2} \times 40\sqrt{3}$
b yes
c Use Method 2 only when the surd part can be simplified. Method 2 has the added advantage that the numbers are smaller.

18 a $6\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{3\sqrt{13}}{5}$ **d** $\frac{6\sqrt{5}}{7}$
e $-7\sqrt{3}$ **f** $\frac{8\sqrt{3}}{3}$ **g** $-16\sqrt{5}$ **h** $\frac{\sqrt{6}}{5}$
i $\frac{-2\sqrt{21}}{3}$ or $\frac{-2\sqrt{7}}{\sqrt{3}}$

19 a $\frac{11\sqrt{2}}{8}$ **b** $8\sqrt{6}$ **c** $\frac{-7\sqrt{2}}{16}$ **d** $8\sqrt{3}$
e $-\frac{15\sqrt{3}}{2}$ **f** $\frac{6\sqrt{2}}{5}$ **g** -8 **h** $-\frac{32\sqrt{6}}{11}$

20 $ab\sqrt{bc}$

21 a i 50.00 cm^2 **ii** 50 cm^2

b i 137.48 cm^2 **ii** $30\sqrt{21}$ cm^2

c i 19.44 cm^2 **ii** $3\sqrt{42}$ cm^2

d i 169.71 mm^2 **ii** $120\sqrt{2}$ mm^2

e i 6.00 cm^2 **ii** 6 cm^2

f i 23.24 m^2 **ii** $6\sqrt{15}$ m^2

g i 2.51 cm^2 **ii** $\frac{4\pi}{5}$ cm^2

h i 37.70 cm^2 **ii** 12π cm^2

i i 0.39 m^2 **ii** $\frac{\pi}{8}$ m^2

22 The perimeter of the square is $4\sqrt{a}$ and the area is a. Equating the perimeter and area to form the equation $4\sqrt{a} = a$, which has the solutions $a = 0$, $a = 16$. Since the square roots of 0 and 16 are not surds, there is no value of a such that the perimeter and area are equal if the side length is a surd.

23 a $4\sqrt{5}$ m **b** $4\sqrt{17}$ m **c** $12\sqrt{2}$ m

d $8\sqrt{5}$ m **e** $2\sqrt{3}$ m **f** $\frac{10\sqrt{7}}{3}$ m

24 a $\sqrt{5}$ **b** $\sqrt{6}$ **c** $\sqrt{35}$

d $\sqrt{14}$ **e** $\sqrt{39}$ **f** $\sqrt{66}$

EX
p121

3D Adding and subtracting surds

1 a yes **b** no **c** no **d** yes

e no **f** no **g** yes **h** yes

2 a $7\sqrt{3} + \sqrt{2}$ **b** $4\sqrt{3} - \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $8\sqrt{13} + 4\sqrt{39} - \sqrt{3}$

e $13\sqrt{2} - 6\sqrt{5}$ **f** $4\sqrt{6} + 4\sqrt{5} + 1$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $\sqrt{21} - \sqrt{3}$

i $7\sqrt{6} + 3\sqrt{5} + 5$ **j** $-7\sqrt{15} - 3\sqrt{7}$

3 a $\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}$ **b** none

c $3\sqrt{12}, \sqrt{3}, 2\sqrt{27}$ **d** $9\sqrt{32}, 4\sqrt{8}, \sqrt{128}, 5\sqrt{200}$

4 a $13\sqrt{2}$ **b** $24 - 4\sqrt{3}$ **c** $6\sqrt{6} + 3\sqrt{5}$

d $28\sqrt{7}$ **e** $59\sqrt{2}$ **f** $28 - 4\sqrt{2}$

5 a $-15\sqrt{3}$ **b** $-6\sqrt{5}$ **c** $-4\sqrt{7}$

d $5\sqrt{3}$ **e** $23\sqrt{2}$ **f** 0

6 a $4\sqrt{5}$ **b** 0

c $4\sqrt{6}$ **d** $\sqrt{2}$

e $-19\sqrt{6} + 8\sqrt{5}$ **f** $-25\sqrt{3} + 2\sqrt{2}$

g $16\sqrt{2} - 30\sqrt{3}$ **h** 0

i $6\sqrt{3} - 2\sqrt{7} + 10\sqrt{2} - 4\sqrt{5}$ **j** $3\sqrt{5} - 2\sqrt{6} - 6\sqrt{3}$

7 a $5\sqrt{5}$ **b** $42\sqrt{3}$

c $29\sqrt{2} - 80\sqrt{3}$ **d** $18\sqrt{6} - 5\sqrt{5} - 14\sqrt{3}$

e $66\sqrt{7} - 8\sqrt{6} + 112\sqrt{3}$

f $10\sqrt{2} - 24\sqrt{5} + 60\sqrt{6} - 12\sqrt{10}$

g $3\sqrt{3} - 25\sqrt{5} - 61$

h $88\sqrt{2} - 113\sqrt{3}$

8 a 0 **b** $-22\sqrt{x}$

c $7\sqrt{x} - 6\sqrt{3x}$ **d** $-5\sqrt{2xy}$

e $20\sqrt{6xy}$ **f** $-x\sqrt{6y}$

g $12y\sqrt{3x}$ **h** $34x\sqrt{y} - 5x\sqrt{3y}$

i $8x\sqrt{5xy} + 10\sqrt{3xy} - 5x\sqrt{5y} - 24x\sqrt{3y}$

9 a $12 + 6\sqrt{3}$ **b** $8\sqrt{5} - 32$

c $20 - 30\sqrt{2}$ **d** $12\sqrt{2} + 8\sqrt{3}$

e $10\sqrt{6} - 6\sqrt{5}$ **f** $-10\sqrt{42} + 15\sqrt{70} - 10\sqrt{7}$

g $45\sqrt{2} + 12\sqrt{15}$ **h** $70 - 60\sqrt{7}$

i $12\sqrt{33} - 30\sqrt{77} + 66$ **j** $3\sqrt{5} - 10 + 25\sqrt{3}$

k $-36\sqrt{2} + 6\sqrt{30} - 81\sqrt{5}$

10 a $10 + 20\sqrt{6} + 6\sqrt{3} + 36\sqrt{2}$

b $15\sqrt{10} - 24\sqrt{2} + 30\sqrt{5} - 48$

c $28\sqrt{10} + 56\sqrt{30} - 24 - 48\sqrt{3}$

d $12\sqrt{3} + 15\sqrt{2} + 12\sqrt{6} + 30$

e $24\sqrt{6} - 288 - 15\sqrt{15} + 90\sqrt{10}$

f $56\sqrt{5} - 48\sqrt{30} + 21\sqrt{10} - 36\sqrt{15}$

11 a $31 + 10\sqrt{6}$ **b** $16 - 6\sqrt{7}$

c $57 + 12\sqrt{15}$ **d** $159 - 24\sqrt{42}$

12 a
$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

b
$$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

13 a 10 **b** 116 **c** 10 **d** –67

14 a The terms in the second bracket are exactly the same as the terms in the first bracket, but the operation sign between the terms in the second bracket is the inverse of the sign between the terms in the first bracket.

b Rational, because they can be written as the difference of two squares, and the surds become rational numbers after being squared each time.

c
$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$$
$$\begin{aligned}(a - b)(a + b) &= a^2 + ab - ba - b^2\\ &= a^2 - b^2\end{aligned}$$

15 Sample answer:
$$\begin{aligned}\text{LHS} &= \sqrt{1} + \sqrt{4}\\ &= 1 + 2\\ &= 3\\ \text{RHS} &= \sqrt{1 + 4}\\ &= \sqrt{5}\\ \text{LHS} &\neq \text{RHS}\end{aligned}$$

16 a $\left(32\sqrt{10} + 15\sqrt{30} - 12\sqrt{6}\right)$ m^2

b $\left(72\sqrt{5} - 40 + 10\pi\right)$ cm^2

c $\left(15 - 3\sqrt{3} - 2.5\sqrt{2} + 0.5\sqrt{6}\right)$ m^2

17 a $180\sqrt{2} - 18\sqrt{5}$ cm^2 **b** $409 + 80\sqrt{6}$ m^2

c $\left(151 - 24\sqrt{7}\right)\pi$ cm^2 **d** $\left(7 + 4\sqrt{2}\right)\pi$ cm^2

18 a $\left(10\sqrt{10} + 16\sqrt{5}\right)$ m **b** $\left(36 + 2\pi\sqrt{5}\right)$ cm

19 a $8\sqrt{6}$ **b** $3\sqrt{15}$ **c** $\frac{73\sqrt{3}}{18}$

d $2\sqrt{10} + 6\sqrt{5}$ **e** $\sqrt{11} - 3\sqrt{7}$ **f** $6\sqrt{6} - 6$

20 $25 + 3\sqrt{2} + 3\sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{10} + \sqrt{11} + \sqrt{13} + \sqrt{14} + \sqrt{15}$

21 a i add $5\sqrt{5}$ **ii** add $\sqrt{3} + 7$

iii multiply $\sqrt{2}$ **iv** multiply $2\sqrt{5}$

b **i** $23\sqrt{5}, 28\sqrt{5}, 33\sqrt{5}, 38\sqrt{5}, 43\sqrt{5}, 48\sqrt{5}$

ii $6\sqrt{3} + 28, 7\sqrt{3} + 35, \sqrt{3} + 42, 9\sqrt{3} + 49, 10\sqrt{3} + 56, 11\sqrt{3} + 63$

iii $12\sqrt{2}, 24, 24\sqrt{2}, 48, 48\sqrt{2}, 96$

iv $400\sqrt{5}, 4000, 8000\sqrt{5}, 80\,000, 160\,000\sqrt{5}, 1\,600\,000$

c **i** $255\sqrt{5}$ **ii** $65\sqrt{3} + 315$

iii $93\sqrt{2} + 186$ **iv** $168\,421\sqrt{5} + 1\,684\,210$

22 **a** **i** $\frac{15\sqrt{2} + 30}{8} = \frac{15\sqrt{2}}{8} + \frac{15}{4}$ **ii** $2 + 2\sqrt{2}$

b **i** $\frac{3\sqrt{5} + 4\sqrt{3} + 6\sqrt{2}}{6} = \frac{\sqrt{5}}{2} + \frac{2\sqrt{3}}{3} + \sqrt{2}$ **ii** $\frac{3\sqrt{5}}{2}$

EX p126

3E Rationalising the denominator

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$ **iii** $\frac{3}{4}$ **iv** $\frac{2}{5}$

b The fraction appearing first in the multiplication remained unchanged.

c It is the same as 1, so each fraction has been multiplied by 1.

2 **a** **i** $\frac{2}{\sqrt{3}}$ **ii** $\frac{6}{3\sqrt{3}}$ **iii** $\frac{2\sqrt{3}}{3}$ **iv** $\frac{4}{2\sqrt{2}}$

b part **iii**

c Multiplication of both the numerator and the denominator by the surd in the denominator.

3 **a** $\frac{\sqrt{6}}{2}$ **b** $\frac{2\sqrt{7}}{7}$ **c** $\sqrt{3}$ **d** $\frac{4\sqrt{35}}{7}$

e $\frac{5\sqrt{2}}{6}$ **f** $\frac{7\sqrt{5}}{20}$ **g** $\frac{2\sqrt{15}}{5}$ **h** $-\frac{\sqrt{5}}{2}$

i $\frac{\sqrt{7}}{49}$ **j** $\frac{2\sqrt{110}}{5}$ **k** $\frac{4\sqrt{10}}{3}$ **l** $\frac{9\sqrt{30}}{20}$

4 **a** $-\sqrt{3}$ **b** $-\frac{\sqrt{10}}{2}$ **c** $\frac{4\sqrt{14}}{21}$ **d** $\sqrt{3}$

e $\frac{6\sqrt{5}}{7}$ **f** -6 **g** $\frac{5\sqrt{x}}{x}$ **h** $\frac{\sqrt{y}}{y}$

i $\frac{2x\sqrt{2y}}{y}$ **j** $\frac{\sqrt{xy}}{xy}$ **k** $\frac{\sqrt{2xy}}{2xy}$ **l** $-\frac{\sqrt{xyz}}{2xyz}$

5 **a** $3 - \sqrt{2}$ **b** $2 + \sqrt{3}$ **c** $\sqrt{6} + 5$

d $-4 - 3\sqrt{5}$ **e** $-3\sqrt{5} - 2$ **f** $4\sqrt{7} - 1$

g $1 - 2\sqrt{3}$ **h** $\sqrt{2} - \sqrt{3}$ **i** $5 + \sqrt{3}$

j $5\sqrt{6} + 2$ **k** $-3\sqrt{7} - 2$ **l** $-5 + 2\sqrt{5}$

6 **a** $2 - \sqrt{3}$ **b** $\frac{9 + 3\sqrt{2}}{7}$ **c** $\frac{30 - 5\sqrt{5}}{31}$

d $\frac{-9 - 12\sqrt{5}}{71}$ **e** $\frac{5\sqrt{2} - 6}{7}$ **f** $\frac{12 - 2\sqrt{3}}{11}$

g $\frac{3\sqrt{5} + 3}{4}$ **h** $-6\sqrt{42} + 15\sqrt{7}$ **i** $4\sqrt{6} + 8\sqrt{2}$

j $\sqrt{15} - 2\sqrt{3}$ **k** $\frac{10\sqrt{5} - 3\sqrt{30}}{23}$ **l** $\frac{6\sqrt{10} + 12\sqrt{5}}{5}$

7 **a** $\frac{2\sqrt{6} - 3\sqrt{2} + 2\sqrt{3} - 3}{3}$

b $\frac{20 - 5\sqrt{2} + 8\sqrt{3} - 2\sqrt{6}}{14}$

c $\frac{3 + \sqrt{5} + 3\sqrt{3} + \sqrt{15}}{4}$

d $\frac{3\sqrt{6} - 3 + \sqrt{30} - \sqrt{5}}{5}$

e $\frac{6\sqrt{15} - 4\sqrt{3} - 6\sqrt{5} + 4}{41}$

f $\frac{10\sqrt{10} - 15\sqrt{2} - 2\sqrt{5} + 3}{11}$

8 **a** $\frac{x^2 + x\sqrt{y}}{x^2 - y}$ **b** $\frac{x + y\sqrt{x}}{x - y^2}$

c $\frac{x^2\sqrt{x} + xy\sqrt{x} - xy - y^2}{x^3 - y^2}$ **d** $\frac{x^2 - 2x\sqrt{y} + y}{x^2 - y}$

9 **a** $\frac{3\sqrt{2}}{2}$ **b** $\frac{3\sqrt{2} + 1}{2}$

10 **a** $\frac{3\sqrt{5} + 1}{5}$ **b** $\frac{4\sqrt{3} - 1}{3}$ **c** $\frac{9\sqrt{2} + 4\sqrt{3}}{6}$

d $\frac{\sqrt{3} + 9\sqrt{2}}{12}$ **e** $\frac{9\sqrt{5} - \sqrt{6}}{9}$ **f** $\frac{12\sqrt{6} + 5\sqrt{15}}{60}$

11 **a** $\frac{\sqrt{5} + 1}{4} + \frac{3\sqrt{5} - 3}{4}$ **b** $\frac{2\sqrt{5} - 1}{2}$

12 **a** $2\sqrt{6}$ **b** -8

c $\frac{2\sqrt{6} + 5\sqrt{3} - 3}{5}$ **d** $5\sqrt{2} - 4\sqrt{5} - 13$

e $\frac{106\sqrt{3} - 5\sqrt{6}}{238}$ **f** $\frac{16\sqrt{3} - 25}{13}$

13 **a** $\frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{3}}{15}$ **c** $\frac{\sqrt{7}}{2}$

d $\frac{4\sqrt{3}}{9}$ **e** $\frac{\sqrt{6} - 2}{2}$ **f** $\frac{7 - 3\sqrt{5}}{2}$

14 **a** 2 cm^2 **b** 1 cm^2 **c** $\frac{1}{\sqrt{2}}$ cm

d $\frac{1}{2}$ **e** $\frac{1}{2}\sqrt{2}$ cm

f The area of the shaded rectangle is given by both $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}\sqrt{2}$. Therefore, they are equal because rationalising $\frac{1}{\sqrt{2}}$ gives $\frac{1}{2}\sqrt{2}$.

g $\frac{7}{10}\sqrt{10}$

15 **a** **i** $\sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}$

ii $\sqrt{5} + \sqrt{3}, -\sqrt{5} - \sqrt{3}$

iii $2\sqrt{5} - 5\sqrt{6}, -2\sqrt{5} + 5\sqrt{6}$

iv $2\sqrt{5} + 3\sqrt{7}, -2\sqrt{5} - 3\sqrt{7}$

b **i** -1 **ii** 1 **iii** $-2\sqrt{6} - 5$

iv $2\sqrt{6} - 5$ **v** 1

c **i** $\sqrt{5} + \sqrt{3}$ or $-\sqrt{5} - \sqrt{3}$

ii $2\sqrt{2} - 5\sqrt{5}$ or $-2\sqrt{2} + 5\sqrt{5}$

iii $\sqrt{11} + 3\sqrt{2}$ or $-\sqrt{11} - 3\sqrt{2}$

iv $5\sqrt{13} - 4\sqrt{3}$ or $-5\sqrt{13} + 4\sqrt{3}$

d **i** $\frac{\sqrt{5} + \sqrt{3}}{2}$ **ii** $\frac{5\sqrt{5} - 2\sqrt{2}}{117}$

iii $\frac{\sqrt{11} + 3\sqrt{2}}{7}$ **iv** $\frac{4\sqrt{3} - 5\sqrt{13}}{277}$

e **i** Lucy multiplied by the conjugate that is the negative of the denominator, so it did not rationalise. The product is not the difference of two squares, it is the negative of the square of the denominator.

ii Lucy could use the conjugate of her new denominator $-2\sqrt{6} - 5$ and multiply her final line by $\frac{-2\sqrt{6} - 5}{-2\sqrt{6} - 5}$.

f **i** $\sqrt{5} - \sqrt{3}$ **ii** $-4\sqrt{6} - 8$

iii $\frac{\sqrt{15} - \sqrt{6}}{3}$ **iv** $\frac{3\sqrt{30} - 10\sqrt{2}}{7}$

v $2\sqrt{5} + 2\sqrt{2}$

vi $\dfrac{15\sqrt{6} - 10\sqrt{10} - 3\sqrt{3} + 2\sqrt{5}}{7}$

vii -1

viii $\dfrac{x^2\sqrt{x} - x\sqrt{y} + xy\sqrt{x} - y\sqrt{y}}{x^3 - y}$

g $2\sqrt{6}$

h $\dfrac{\sqrt{6}\left(\sqrt{2} + \sqrt{3} - \sqrt{5}\right)}{3} = \dfrac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{3}$

Checkpoint

EX p129

1 **a** rational **b** rational **c** irrational
d rational **e** irrational **f** irrational

2 **a** surd **b** not surd **c** not surd
d not surd **e** surd **f** not surd

3 **a** $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $5\sqrt{7}$ **d** $10\sqrt{6}$

4 **a** $10\sqrt{3}$ **b** $21\sqrt{6}$ **c** $150\sqrt{2}$ **d** $80\sqrt{5}$

5 **a** $\sqrt{35}$ **b** $-2\sqrt{77}$ **c** $-90\sqrt{2}$ **d** $12\sqrt{14}$

6 **a** $\sqrt{6}$ **b** $-\dfrac{2\sqrt{17}}{3}$ **c** $-\dfrac{3}{2}$ **d** $\dfrac{5\sqrt{3}}{2}$
e $-\dfrac{\sqrt{30}}{3}$ **f** $\dfrac{6\sqrt{74}}{37}$

7 **a** 6 **b** $\dfrac{2\sqrt{6}}{9}$

8 **a** $12\sqrt{2}$ **b** $-6\sqrt{2}$
c $12\sqrt{3} + 18\sqrt{5}$ **d** $17\sqrt{11} - 2\sqrt{7}$

9 **a** $13\sqrt{2}$ **b** $2\sqrt{3}$
c $8\sqrt{6} + 14\sqrt{3}$ **d** $17\sqrt{10} - 17\sqrt{2}$

10 **a** $6\sqrt{3}$
b $18\sqrt{6} - 18\sqrt{2}$
c $2\sqrt{7} + 3\sqrt{5} - \sqrt{35} - 6$
d $15\sqrt{15} + 10\sqrt{10} + 8\sqrt{6} + 36$
e $30\sqrt{2} - 54\sqrt{30}$
f $10\sqrt{154} - 12\sqrt{14} + 160\sqrt{11} - 192$

11 **a** $\dfrac{5\sqrt{2}}{2}$ **b** $4\sqrt{3}$ **c** $\dfrac{2\sqrt{35}}{7}$ **d** $\dfrac{\sqrt{66}}{6}$

12 **a** $5\sqrt{2} - 5$ **b** $\dfrac{\sqrt{3} + 1}{2}$
c $\dfrac{4\sqrt{5} + 30}{41}$ **d** $\dfrac{-13\sqrt{7} - 35}{14}$
e $\dfrac{9\sqrt{5} - 47}{164}$ **f** $\dfrac{-\sqrt{10} - 8\sqrt{5} - 2\sqrt{3} - 8\sqrt{6}}{62}$

3F Fractional indices

EX p132

1 **a** $\sqrt{5}$ **b** $\sqrt[3]{12^2}$ **c** $\sqrt[4]{x^3}$
d $\sqrt[3]{32^2}$ **e** $\sqrt[4]{100}$ **f** $\sqrt[5]{x^4}$

2 **a** $6^{\frac{1}{2}}$ **b** $x^{\frac{2}{3}}$ **c** $5^{\frac{3}{2}}$
d $x^{\frac{3}{2}}$ **e** $x^{\frac{5}{2}}$ **f** $x^{\frac{3}{4}}$

3 **a** $x^{\frac{3}{5}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{7}{8}}$ **d** $x^{\frac{17}{30}}$
e x **f** $x^{\frac{11}{12}}$ **g** $x^{\frac{1}{2}}$ **h** $x^{\frac{1}{2}}$

4 **a** $x^{\frac{4}{11}}$ **b** $x^{\frac{1}{4}}$ **c** $x^{\frac{1}{14}}$ **d** $x^{\frac{1}{20}}$
e $x^{\frac{1}{6}}$ **f** $x^{\frac{1}{10}}$ **g** $x^{\frac{7}{4}}$ **h** 1

5 **a** $x^{\frac{15}{28}}$ **b** $x^{\frac{1}{5}}$ **c** 1 **d** $x^{\frac{3}{10}}$
e $x^{\frac{1}{18}}$ **f** $x^{\frac{1}{2}}$ **g** x^{12} **h** $x^{\frac{1}{35}}$

6 **a** $10x^{\frac{7}{12}}$ **b** $30x^{\frac{31}{40}}$ **c** $-36x^{\frac{41}{63}}$ **d** $4x^{\frac{1}{12}}$
e $2x^{\frac{9}{40}}$ **f** $4x^{\frac{1}{30}}$ **g** $6x^{\frac{5}{6}}$ **h** $\dfrac{1}{4}x^{\frac{1}{10}}$

7 **a** $5^{\frac{3}{5}}x^{\frac{1}{5}}$ **b** $16^{\frac{1}{4}}x^{\frac{1}{10}}$ **c** $27^{\frac{2}{3}}x^{\frac{1}{4}}$ **d** $2x^{\frac{1}{5}}y^{\frac{3}{8}}$
e $4x^{\frac{2}{15}}y^{\frac{2}{7}}$ **f** $5x^{\frac{1}{4}}y^{\frac{3}{10}}$ **g** $5xy^{\frac{3}{10}}$ **h** $7x^{\frac{1}{3}}y^{\frac{20}{33}}$

8 **a** $28x^{\frac{7}{12}}y^{\frac{19}{20}}$ **b** $-72x^{\frac{3}{2}}y$ **c** $45x^{\frac{19}{24}}y^{\frac{7}{5}}$
d $5x^{\frac{2}{5}}y^{\frac{1}{6}}$ **e** $2x^{\frac{7}{40}}y^{\frac{4}{33}}$ **f** $x^{\frac{1}{14}}y^{\frac{1}{35}}$

9 **a** $a^m \times a^n = a^{m+n}$ **b** 3
c 3 **d** The answers are the same.
e **i** 2 **ii** 5 **iii** 10 **iv** x
f The fractional power of $\frac{1}{2}$ performs the same operation as finding the square root ($\sqrt{}$).

10 **a** 3
b 3
c The answers are the same.
d **i** 2 **ii** 5 **iii** 10 **iv** x
e $\sqrt[3]{}$ is a cube root and $\sqrt{}$ is a square root.
f The fractional power of $\frac{1}{3}$ performs the same operation as finding the cube root ($\sqrt[3]{}$).

11 **a** $(x^2)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$
b Since $(x^2)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$ are both equal to $x^{\frac{2}{3}}$, we only need to show that $x^{\frac{2}{3}}$ is equal to $\sqrt[3]{x^2}$ and $\left(\sqrt[3]{x}\right)^2$. So $x^{\frac{2}{3}} = x^{\frac{1}{3}\times 2} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2$ and $x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$.

12 $x^{\frac{m}{n}} = x^{m\times\frac{1}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$ and $x^{\frac{m}{n}} = x^{\frac{1}{n}\times m} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

13 **a** 5 **b** 6 **c** 1 **d** 12
e 10 **f** 14 **g** 2 **h** 3
i 5 **j** 1 **k** 1 **l** 2

14 **i** The answers would be the reciprocal of the original answers.
ii **a** $\frac{1}{5}$ **b** $\frac{1}{6}$ **c** 1 **d** $\frac{1}{12}$
e $\frac{1}{10}$ **f** $\frac{1}{14}$ **g** $\frac{1}{2}$ **h** $\frac{1}{3}$
i $\frac{1}{5}$ **j** 1 **k** 1 **l** $\frac{1}{2}$
iii Parts **g**, **h**, **i**, **j**, **k** and **l** can have a negative base as the root is odd (3 or 5). The cube root and 5th root of a negative number are real numbers as $(-a)^3 = -a$ so $\sqrt[3]{-a} = -a$, but the square root of a negative number is not a real number.
iv **g** $-\frac{1}{2}$ **h** $-\frac{1}{3}$ **i** $-\frac{1}{5}$
j -1 **k** -1 **l** $-\frac{1}{2}$

15 **a** $3\sqrt{2}$ **b** $2\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{3}$
e $2\sqrt{38}$ **f** $3\sqrt[3]{2}$ **g** $4\sqrt[3]{2}$ **h** $8\sqrt{5}$
i $9\sqrt{3}$ **j** $12\sqrt{3}$ **k** $5\sqrt{5}$ **l** $6\sqrt{6}$

16 **a** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **b** $x^{\frac{3}{2}}y^{\frac{1}{3}}$ **c** $x^{\frac{3}{2}}y^{\frac{5}{2}}$
d $3^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{2}}$ **e** $4x^{-\frac{1}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{4}{3}}y^{\frac{2}{3}}$

17 **a** $108x^{\frac{9}{20}}y^{\frac{13}{15}}$ **b** $\dfrac{1}{5}x^{\frac{2}{9}}y^{\frac{7}{20}}$ **c** $2x^{\frac{1}{5}}y^{\frac{1}{4}}$
d $6x^4y^2$ **e** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{1}{2}}y^{\frac{5}{6}}$
g $96^{\frac{1}{2}}x^2y^{\frac{1}{2}}z^4$ **h** $x^4y^{\frac{2}{9}}z^{\frac{2}{3}}$

18 **a** $4x\sqrt{y}$ **b** $\dfrac{2y^3\sqrt{3y}}{x\sqrt{x}}$ **c** $xy\sqrt{3x}$
d $3\sqrt[3]{xy^2}$ **e** $2\sqrt[4]{x^3y^2z}$ **f** $\dfrac{7y^2\sqrt{7y}}{\sqrt{x}}$

19 a $18x^{\frac{5}{12}}y^{\frac{7}{15}}$ **b** $\frac{72y^{\frac{1}{12}}}{x^{\frac{5}{12}}}$ **c** $\frac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}$

d $\frac{1}{4x^{\frac{1}{15}}y^{\frac{7}{20}}}$ **e** $\frac{1}{x^{\frac{1}{12}}y^{\frac{3}{4}}}$ **f** $\frac{16}{x^{\frac{8}{5}}y^5}$

g $10x^{\frac{7}{30}}$ **h** $\frac{y^{\frac{13}{30}}}{3^{\frac{1}{2}}x^{\frac{1}{24}}}$

20 a $\frac{3^{\frac{2}{3}}}{9}$ or $3^{-\frac{4}{3}}$ **b** $x = -\frac{2}{3}$

21 a $x = -\frac{5}{2}$ **b** $x = 11$ **c** $x = -\frac{1}{2}$

22 a Both denominators specify how many repeated parts the base is broken into. When multiplying by a fraction, the parts are added; when raising to the power, the parts are multiplied. Both numerators describe the number of those repeated parts to operate together.

b **i** 3 **ii** 27 **iii** 243

iv 823 543 **v** 59 049 **vi** 8

c **i** $81^{\frac{1}{4}} = \sqrt[4]{81}$ **ii** $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3$

iii $81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5$ **iv** $343^{\frac{7}{3}} = \left(\sqrt[3]{343}\right)^7$

v $81^{\frac{5}{2}} = \left(\sqrt[2]{81}\right)^5 = \left(\sqrt{81}\right)^5$ **vi** $256^{\frac{3}{8}} = \left(\sqrt[8]{256}\right)^3$

d The denominator of the exponent and the root of a surd both specify how many repeated parts the base is broken into. The numerator of the exponent and the index of a surd describe the number of those repeated factors of the base to multiply together.

23 a
$$\begin{aligned}\sqrt{ab} &= (ab)^{\frac{1}{2}}\\ &= a^{\frac{1}{2}} \times b^{\frac{1}{2}}\\ &= \sqrt{a} \times \sqrt{b}\end{aligned}$$

b
$$\begin{aligned}\sqrt{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{2}}\\ &= \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\\ &= \frac{\sqrt{a}}{\sqrt{b}}\end{aligned}$$

24 a $\sqrt{10^5}$ **b** $\sqrt[4]{2^{17}}$ **c** $\sqrt[25]{3^4}$ **d** $\sqrt[200]{5^{1111}}$

EX p137

3G Logarithms

1 a $\log_3(27) = 3$ **b** $\log_4(4) = 1$ **c** $\log_5(625) = 4$

d $\log_{10}(1000) = 3$ **e** $\log_6(36) = 2$ **f** $\log_9(729) = 3$

g $\log_3(81) = 4$ **h** $\log_2(128) = 7$

2 a $2^5 = 32$ **b** $3^2 = 9$ **c** $5^2 = 25$

d $7^2 = 49$ **e** $4^2 = 16$ **f** $2^{10} = 1024$

g $6^3 = 216$ **h** $10^2 = 100$

3 a 4 **b** 4 **c** 3 **d** 3 **e** 2 **f** 3

g 2 **h** 5 **i** 4 **j** 4 **k** 2 **l** 4

m 3 **n** 3 **o** 2 **p** 5

4 a i $\frac{1}{6}$ **ii** $\log_6\left(\frac{1}{6}\right) = -1$

b i $\frac{1}{4}$ **ii** $\log_4\left(\frac{1}{4}\right) = -1$

c i $\frac{1}{4}$ **ii** $\log_2\left(\frac{1}{4}\right) = -2$

d i $\frac{1}{32}$ **ii** $\log_2\left(\frac{1}{32}\right) = -5$

e i $\frac{1}{25}$ **ii** $\log_5\left(\frac{1}{25}\right) = -2$

f i $\frac{1}{10}$ **ii** $\log_{10}\left(\frac{1}{10}\right) = -1$

g i $\frac{1}{1000}$ **ii** $\log_{10}\left(\frac{1}{1000}\right) = -3$

h i $\frac{1}{27}$ **ii** $\log_3\left(\frac{1}{27}\right) = -3$

i i 2 **ii** $\log_{\frac{1}{2}}(2) = -1$

j i 9 **ii** $\log_{\frac{1}{3}}(9) = -2$

k i 125 **ii** $\log_{\frac{1}{5}}(125) = -3$

l i 100 **ii** $\log_{0.01}(100) = -1$

5 a −3 **b** −3 **c** −2 **d** −1

e −4 **f** −6 **g** −3 **h** −2

6 a i 16 **ii** $\frac{1}{16} = 0.0625$

b no real number

c **i** no real number **ii** no real number

iii no real number **iv** no real number

v no real number **vi** no real number

d There is no real logarithm of a negative number to any positive integer base.

7 a no

b **i** no **ii** no **iii** no

iv no **v** no

c There is no power of base 1 that gives an answer other than 1, so there is no logarithm of any other number to base 1.

8 a Raise 2 to the power of 7. Then, take the logarithm with base 2 of 2^7. The result is 7.

b **i** 7 **ii** 5 **iii** 9

iv 140 **v** −6 **vi** 123

c The logarithm with base a is the exponent to raise a to get a^x. x is the exponent to raise a to get a^x. Therefore, $\log_a(a^x) = x$.

9 a Take the logarithm to base 2 of 7. Then, raise 2 to the power of that. The result is 7.

b **i** 7 **ii** 5 **iii** 11

iv 531 **v** 8 **vi** 147

c The logarithm to base a is the exponent, y, to raise a to get x. We raise a to that exponent, y. Therefore, $a^{\log_a(x)} = x$.

10 a
$$\begin{aligned}\log_x(64) &= 2\\ x^{\log_x(64)} &= x^2\\ 64 &= x^2\\ x &= 8\end{aligned}$$

b 8, as $x > 0$ so $x = -8$, while a solution to $x^2 = 64$ is not a solution for $\log_x(64) = 2$.

c $x^2 = 8^2$ **d** It is the same.

11 a 2 **b** 5 **c** 4 **d** 2

e 2 **f** 3 **g** 4 **h** 2

12 a
$$\begin{aligned}\log_4(x) &= 3\\ 4^{\log_4(x)} &= 4^3\\ x &= 4^3\end{aligned}$$

b 64

c It can be easily solved by multiplying the base by itself the number of times indicated by the exponent.

13 a 27 **b** 25 **c** 6 **d** 10 000

e 32 **f** 64 **g** $\frac{1}{27}$ **h** $\frac{1}{4}$

14 a $(a^2)^{\frac{1}{2}} = a^{2\times\frac{1}{2}} = a^1 = a$

$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\times 2} = a^1 = a$

b $12^2 = 144$

$12 = 144^{\frac{1}{2}}$

$\log_{144}(12) = \log_{144}(144^{\frac{1}{2}})$

$\log_{144}(12) = \frac{1}{2}$

c $\frac{1}{x}$

15 a $\frac{1}{3}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$ **d** $\frac{1}{9}$

16 a 1 **b** 1 **c** 1 **d** 1

17 a $\frac{1}{2}$ **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{7}{3}$

e $\frac{3}{2}$ **f** $\frac{3}{8}$ **g** 2 **h** $-\frac{1}{3}$

18 a $x = -\frac{1}{2}$ **b** $x = -\frac{7}{2}$ **c** $x = -\frac{7}{4}$ **b** $x = -\frac{11}{4}$

e $x = -\frac{3}{2}$ **f** $x = -\frac{3}{4}$ **g** $x = \frac{9}{4}$ **h** $x = \frac{5}{8}$

EX p142

3H Logarithm laws

1 a 1 **b** 1 **c** 1 **d** 4 **e** 3 **f** 0

g 0 **h** 0

2 a $\log_{10}(3) + \log_{10}(5)$ **b** $\log_2(6) + \log_2(3)$

c $\log_5(2.5) + \log_5(10)$ **d** $\log_2(x) + \log_2(y)$

3 a $\log_3(12)$ **b** $\log_{10}(15)$ **c** $\log_2(50)$

d $\log_4(72)$ **e** $\log_a(15)$ **f** $\log_2(xy)$

4 a $\log_{10}(5) - \log_{10}(7)$ **b** $\log_2(5) - \log_2(3)$

c $\log_5(11) - \log_5(20)$ **d** $\log_2(a) - \log_2(b)$

5 a $\log_3\left(\frac{12}{5}\right)$ **b** $\log_{10}\left(\frac{50}{15}\right)$ **c** $\log_2\left(\frac{25}{10}\right)$

d $\log_4\left(\frac{36}{9}\right)$ **e** $\log_a\left(\frac{3}{15}\right)$ **f** $\log_2\left(\frac{x}{y}\right)$

6 a 4 **b** 2 **c** 5 **d** 4 **e** 2 **f** 3

g 2 **h** 2 **i** 9 **j** 2 **k** 2 **l** 4

7 a 2 **b** 2 **c** 3

8 a $2 \times \log_3(7)$

b $9 \times \log_5(2)$

c $3 \times \log_4(8)$

d $4 \times \log_9(11)$

9 a 5 **b** 2 **c** 5

d 8 **e** 9 **f** 12

g 10 **h** 2

10 a −2 **b** −1 **c** −2

d −14 **e** −30 **f** −21

g $-\frac{1}{3}$ **h** −5

11 a 1 **b** 4 **c** 4

d $\frac{1}{2}$ **e** 1 **f** 1

12 a 3 **b** 2 **c** 2

d −1 **e** −2 **f** 3

13 a 3 **b** 2 **c** 6 **d** 4

e 1 **f** 7 **g** 5 **h** 3

i 3 **j** 3

14 a 3 **b** 3 **c** 9 **d** 4

e 216 **f** 64 **g** −3 **h** 36

i 1 **j** $\frac{1}{2}$ **k** $\frac{1}{2}$ **l** 4

15 a 0.5

b $\log_{10}(9) = \log_{10}(3^2) = 2\log_{10}(3)$

c $\frac{\log_{10}(3)}{\log_{10}(9)} = \frac{\log_{10}(3)}{2\log_{10}(3)} = \frac{1}{2} = 0.5$

16 Hailey, because $\frac{\log_3(8)}{\log_3(2)} = \frac{3\log_3(2)}{\log_3(2)} = 3$

17 a 4 **b** $\frac{3}{2}$ **c** 3

d $\frac{3}{4}$ **e** $\frac{2}{3}$ **f** 3

18 a i $\log_5(2x) + \log_5(3) - \log_5(12) = 0$

$\log_5(6x) - \log_5(12) = 0$

$\log_5\left(\frac{6x}{12}\right) = 0$

$\log_5\left(\frac{x}{2}\right) = 0$

ii $5^0 = \frac{x}{2}, \frac{x}{2} = 1, x = 2$

b i $\log_5(2x) + \log_5(3) = \log_5(6x) = \log_5(12)$

ii $6x = 12, x = 2$

19 a 3 **b** 4 **c** 20 **d** $\frac{2}{3}$

e $\frac{1}{14}$ **f** $\frac{25\,000}{3}$ or $8333\frac{1}{3}$

20 a 1.545 **b** 0.341 **c** 1.024

21 $x = 4$ or $x = 5$

22 $\log_{10}(2^{2x+1}) = \log_{10}(4)$

$(2x + 1)\log_{10}(2) = \log_{10}(2^2)$

$(2x + 1)\log_{10}(2) = 2\log_{10}(2)$

$2x + 1 = 2$

$2x = 1$

$x = \frac{1}{2}$

23 $y = 77$

EX p147

3I Logarithmic scales

1 a 12 **b** −5 **c** 3

d 8 **e** −7 **f** 3

2 a 3 units to the right of 1

b 6 units to the left of 1

c 1 unit to the right of 1

d 2.30 units to the right of 1

e 1.70 units to the left of 1

f 3.10 units to the left of 1.

3 a 100 000 **b** 0.1 **c** 2510

d 0.002 00 **e** 1.26 **f** 0.000 025 1

4 a 4.215×10^4 **b** 4.215×10^5

c 4.215×10^7 **d** 0.729

e 7.29 **f** 729

5 **a** 4.215×10^2 **b** 4.215×10^1
c 4.215×10^{-2} **d** 0.007 29
e 0.000 729 **f** 0.000 000 729

6 **a** 6 orders higher **b** 3 orders lower
c 15 orders lower **d** 2 orders higher
e 3 orders lower **f** 12 orders higher

7 **a** divide by 1000
b multiply by 10 000
c multiply by 100 000 000 000 (or 10^{11})
d multiply by 1 995 262
e divide by 126
f divide by 125 892 541 179 (or $10^{11.1}$)

8 **a** **i** $7.293\,85 \times 10^3$ **ii** $1.052\,357\,9 \times 10^5$
iii 2.135×10^{-3} **iv** 5.0123×10^4
v 6.1290×10^1 **vi** 3.2793×10^{-6}
b **i** 3 **ii** 5 **iii** −3
iv 4 **v** 1 **vi** −6
c **i** 3 and 4 **ii** 5 and 6 **iii** −3 and −2
iv 4 and 5 **v** 1 and 2 **vi** −6 and −5
d **i** 3.863 **ii** 5.022 **iii** −2.671
iv 4.700 **v** 1.787 **vi** −5.484

9 **a** Increasing a value by 0.5 orders of magnitude corresponds to the value becoming 3.16 times greater.
b Increasing a value by 1.5 orders of magnitude corresponds to the value becoming 31.6 times greater.
c Increasing a value by 2.5 orders of magnitude corresponds to the value becoming 316 times greater.
d Increasing a value by 3.5 orders of magnitude corresponds to the value becoming 3160 times greater.
e Multiplying a value by 2 increases its order of magnitude by 0.301.
f Multiplying a value by 4 increases its order of magnitude by 0.602.
g Multiplying a value by 8 increases its order of magnitude by 0.903.
h Multiplying a value by 16 increases its order of magnitude by 1.20.

10 **a** 4.724 units to the right of 1
b 6.276 units to the left of 1
c 6.676 unit to the left of 1
d 8.990 units to the right of 1
e 5.154 units to the left of 1
f 0.979 units to the left of 1

11 **a** **i** 5.5 **ii** 3.1 **iii** 4.9
iv 2.8 **v** 7.3 **vi** 8.2
b **i** 3 162 278 μm **ii** 1 000 000 μm
iii 3 162 000 μm **iv** 32 μm
v 63 095 734 μm **vi** 19 953 μm
c **i** 10 **ii** 3.162 **iii** 31.62
iv 63.10 **v** 1 259 000 **vi** 1.259

12 **a** The number of cases becomes 10 times greater every 100 days.
b **i** 50 days **ii** 200 days **iii** 500 days
iv 31 623 cases **v** 1000 cases
c The growth in cases has slowed down below the point where the cumulative number of cases in country B is the same as if the number of cases had been increasing at a rate of 10 times the amount of cases every 100 days.

13 **a** linear **b** logarithmic
c logarithmic **d** linear
e logarithmic **f** linear

14 **a** The logarithm of a negative number is not a real number.
b The logarithm of zero is undefined.

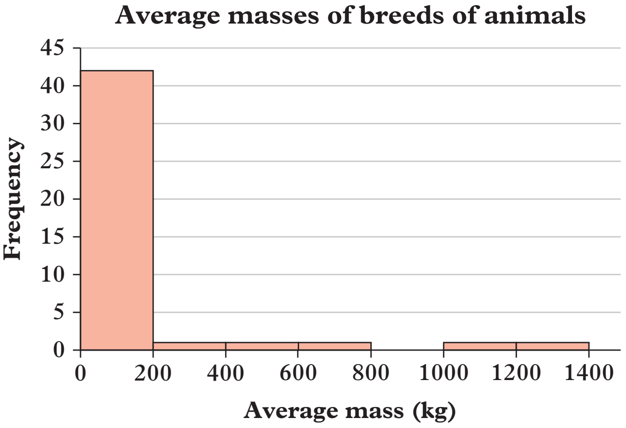
15 **a** **i**

Class interval	Frequency
0–<200	42
200–<400	1
400–<600	1
600–<800	1
800–<1000	0
1000–<1200	1
1200–<1400	1

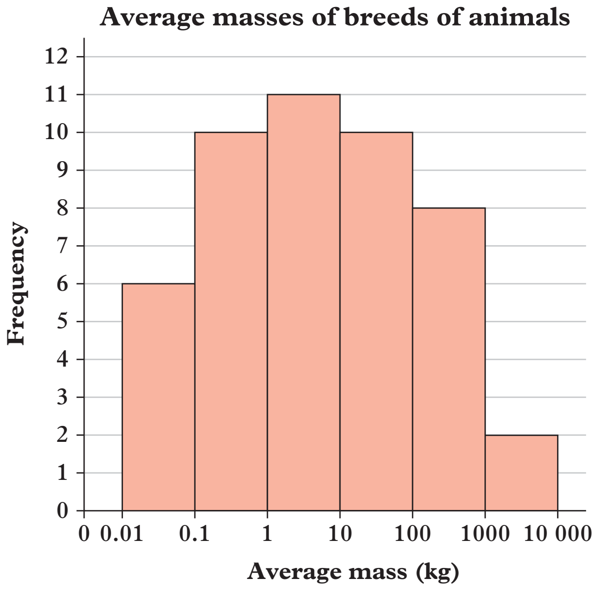
ii

Class interval	Frequency
0.01–<0.1	6
0.1–<1	10
1–<10	11
10–<100	10
100–<1000	8
1000–<10 000	2

b **i**



ii



c Linear scale: positively skewed
Logarithmic scale: approximately symmetric

d The logarithmic scale spreads out the average masses in the 0–<200 column (42 out of 47 masses) into five different columns and groups the data in the remaining columns (with only 5 masses) into two columns.

16 a 1 day **b** 1 day **c** constant
d decreasing **e** increasing

17 a 2 hours

b With the base-two logarithmic scale you can more easily identify the amount of the active substance remaining after a high number of hours, however it is more difficult to identify the amount of active substance remaining for periods between each half-life interval (every two hours).

18 a i 10 **ii** 10 **iii** 10 000
iv 100 000 **v** 1 000 000 000 **vi** 1.259

b i –10 dB **ii** 10 dB **iii** –20 dB
iv –6 dB **v** 0 dB **vi** 20 dB

c i 5000 W **ii** 10.0 W **iii** 99.8 W
iv 500 000 W **v** 240 000 000 W
vi 24 000 W

EX
p153

CHAPTER 3 review

Mathematical literacy review

1 a True

b False. All recurring decimals can be expressed as fractions with the denominator and numerator being integers and are therefore rational numbers. For example, $0.\dot{3} = \frac{1}{3}$.

c False. All terminating decimals can be written as an integer divided by a power of ten and are therefore rational numbers. For example, $0.25 = \frac{25}{100} = \frac{1}{4}$.

d True

e False. Numbers with fractional indices can be either rational or irrational. For example, $32^{\frac{2}{5}} = 4$ (rational) and $5^{\frac{1}{8}} = 1.709\,975\,9\ldots$ (irrational).

f False. A square root can be either rational or irrational. For example, $\sqrt{25} = 5$ (rational) and $\sqrt{26} = 5.099\,019\,5\ldots$ (irrational).

g True

2 E

3 a base-ten

b product

4 a a number that can be written as a fraction with both the numerator and denominator being integers. Rational numbers include whole numbers, fractions, terminating decimals and recurring decimals. For example, $\frac{2}{3}$ is a rational number.

b two or more surd terms that in simplest form have the same number or pronumeral under the root sign. For example, $\sqrt{3}$, $3\sqrt{3}$ and $-2\sqrt{3}$ are like surds.

5 a $9 = 3^2$ is written in exponential form, and the corresponding $\log_3 9 = 2$ is written in logarithmic form.

b $2 + \sqrt{3}$ is the conjugate of $2 - \sqrt{3}$ and is used to rationalise the denominator of $\frac{5}{2 - \sqrt{3}}$.

Multiple choice

1	A	**2**	D	**3**	C	**4**	D	**5**	B
6	B	**7**	E	**8**	A	**9**	B	**10**	B

Short answer

1 a rational **b** irrational **c** rational
d irrational **e** irrational **f** rational

2 a, b, d, f

3 a 4.899 **b** 0.059

4 a $0.65 = \frac{65}{100}$, which is a fraction and therefore rational.

b $0.2175 = \frac{2175}{10\,000}$, which is a fraction and therefore rational.

c $0.5\dot{2} = \frac{47}{90}$, which is a fraction and therefore rational.

d $1.\overline{23} = 1\frac{23}{99}$, which is a fraction and therefore rational.

5 a $4\sqrt{6}$ **b** $15\sqrt{5}$ **c** $3\sqrt{3}$

6 a $72\sqrt{2}$ **b** -150

7 a $24\sqrt{3}$ m^2 **b** $90\sqrt{2}$ cm^2

8 a $\sqrt{77}$ **b** $27\sqrt{30}$

9 a $\sqrt{3}$ **b** $4\sqrt{3}$

10 a $-2\sqrt{2}$ **b** $2\sqrt{7} + 5\sqrt{5}$
c $2\sqrt{10} + 4\sqrt{5}$ **d** $3\sqrt{6} + 2 + 3\sqrt{5}$

11 a $2\sqrt{5} - 22\sqrt{6}$ **b** $4\sqrt{3} + 6\sqrt{2}$
c $15\sqrt{5}$ **d** $6\sqrt{2} + 24\sqrt{5} + 4\sqrt{10}$

12 a $6 - 8\sqrt{2} + 12\sqrt{3} - 16\sqrt{6}$ **b** $92 + 16\sqrt{15}$

13 a $\frac{5\sqrt{6}}{12}$ **b** $\frac{3\sqrt{3}}{2}$

c $\frac{8\sqrt{30} - 4\sqrt{15}}{21}$

d $\frac{-6\sqrt{7} - 10\sqrt{3} - 3\sqrt{42} - 15\sqrt{2}}{12}$

14 a i $x^{\frac{3}{2}}$ **ii** $\sqrt{x^3}$

b i $x^{\frac{5}{6}}$ **ii** $\sqrt[6]{x^5}$

c i $\frac{1}{9}x^{\frac{5}{12}}$ **ii** $\frac{1}{9}\sqrt[12]{x^5}$

d i $24^{\frac{1}{2}}x^2y^{\frac{3}{2}}$ **ii** $2x^2y\sqrt{6y}$

e i $3x^{\frac{1}{8}}y^{\frac{1}{5}}$ **ii** $3\sqrt[8]{x}\sqrt[5]{y}$

f i $\left(\frac{1}{5}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}}$

ii $\frac{x}{10}\sqrt{\frac{3xy}{2z}}$ or $\frac{x\sqrt{6xyz}}{20z}$ (rationalising the denominator)

15 a $\log_3 81 = 4$ **b** $\log_6 216 = 3$

c $\log_2 32 = 5$ **d** $\log_x w = y$

16 a $2^3 = 8$ **b** $4^3 = 64$

c $10^{-4} = \frac{1}{10\,000}$ **d** $x^b = z$

17 a 3 **b** 2 **c** 3 **d** 3

18 a 2 **b** $\frac{1}{125}$ **c** 4

d $\frac{1}{2}$ **e** $\frac{12}{5}$ **f** $\frac{625}{4}$

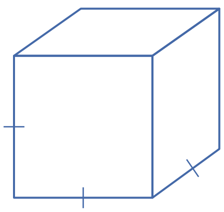
19 a 26 500 **b** 840

c 7.51 **d** 0.090 91

20 a 0.0316 **b** 178

Analysis

1 a



$7\sqrt{10} + 11\sqrt{7}$ cm

b $(1337 + 154\sqrt{70})$ cm²

c irrational

d $(8022 + 924\sqrt{70})$ cm²

e $20.16, correct to the nearest cent

f You need to show that a 1 L can of the protective paint contains enough paint to cover $(8022 + 924\sqrt{70})$ cm² three times, an area of $1.575\ldots \times 3 \approx 4.725\ldots < 5$; it does.

g 94.5% correct to one decimal place

h The diagonal of the base is

$$\begin{aligned}\sqrt{\left(7\sqrt{10} + 11\sqrt{7}\right)^2 + \left(7\sqrt{10} + 11\sqrt{7}\right)^2} &= \sqrt{2} \times \left(7\sqrt{10} + 11\sqrt{7}\right)\\ &= 7\sqrt{20} + 11\sqrt{14}\\ &= 14\sqrt{5} + 11\sqrt{14}\\ &= 72.463...\end{aligned}$$

72.463 cm < 80 cm, so it won't fit.

i Yes, diagonally from bottom corner to opposite top corner.

j You need to assume that the interior measurements are the same as exterior measurements.

2 a $\frac{a+b}{a} \approx 1.7$, $\frac{a}{b} \approx 1.4$; the fractions are not equal to each other

b Sample answer: When $a = 2.8$ and $b = 1.7$, the fractions are almost equal to each other.

c Answers will vary, but both fractions should be close to the golden ratio (≈ 1.62).

d GF corresponds to $a + b$ and BG corresponds to a.

e $EB = \frac{\sqrt{5}}{2}$, $\mathrm{AG} = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $BG = \frac{\sqrt{5}}{2} - \frac{1}{2}$

f $\varphi = \frac{\sqrt{5} + 1}{2}$

3 a 0 dB

b 3 dB

c 20 dB

d 100 times more intense

e 1000 times less intense

f 84.7 dB

g A decrease of 15 dB is a reduction of approximately 32 times the noise level. A decrease of 30 dB is a reduction of 1000 times the noise level.

CHAPTER 4 Equations

EX
p163

4A Solving linear equations

1 a $x = \frac{19}{3}$ **b** $x = -40$ **c** $x = \frac{2}{7}$

d $x = \frac{24}{5}$ **e** $x = -53$ **f** $x = -3$

g $x = -\frac{5}{3}$ **h** $x = -8$ **i** $x = \frac{29}{12}$

j $x = -\frac{19}{2}$ **k** $x = -\frac{43}{6}$ **l** $x = \frac{62}{5}$

m $x = -\frac{32}{9}$ **n** $x = \frac{15}{2}$ **o** $x = -\frac{7}{3}$

2 a $x = -8\frac{1}{2}$ **b** $x = 1.7$ **c** $x = 4$

d $x = -2.464$ **e** $x = 2.4$ **f** $x = \frac{15}{77}$

g $x = \frac{1}{2}$ **h** $x = -\frac{1}{3}$ **i** $x = -\frac{20}{21}$

3 a $x = -5$ **b** $x = 3$ **c** $x = -2$

d $x = 6$ **e** $x = -\frac{1}{3}$ **f** $x = 7$

g $x = 3$ **h** $x = \frac{1}{9}$ **i** $x = 6$

4 a $x = 2$ **b** $x = 5$ **c** $x = -3$

d $x = 1$ **e** $x = -7$ **f** $x = -2$

g $x = 3$ **h** $x = 4$ **i** $x = \frac{3}{7}$

j $x = -4$ **k** $x = 2$ **l** $x = -5$

5 a $x = 5$ **b** $x = -2$ **c** $x = 3$

d $x = -1$ **e** $x = 6$ **f** $x = 2$

g $x = -4$ **h** $x = 8$ **i** $x = 4$

j $x = -1$ **k** $x = 5$ **l** $x = 9$

6 a $x = -8$ **b** $x = 4$ **c** $x = -2$

d $x = 2$ **e** $x = -1$ **f** $x = 4$

7 a $x = -8$ **b** $x = 31$ **c** $x = 8$

d $x = 1$ **e** $x = 17$ **f** $x = 13$

g $x = -4\frac{2}{3}$ **h** $x = 6\frac{1}{7}$ **i** $x = -2\frac{1}{8}$

8 **a** $x = 2$ **b** $x = -5$ **c** $x = -6$

d $x = -4$ **e** $x = 4$ **f** $x = \frac{15}{2}$

g $x = \frac{10}{3}$ **h** $x = \frac{2}{35}$ **i** $x = 2$

j $x = -1$ **k** $x = \frac{1}{4}$ **l** $x = -\frac{12}{5}$

m $x = -\frac{11}{8}$ **n** $x = \frac{35}{2}$ **o** $x = \frac{5}{6}$

9 **a** $x = 3$ **b** $x = -3$ **c** $x = 12$

d $x = -\frac{3}{7}$ **e** $x = \frac{14}{15}$ **f** $x = \frac{1}{11}$

10 **a** $x = \frac{8}{17}$ **b** $x = -33$ **c** $x = -1$

d $x = 16$ **e** $x = \frac{4}{25}$ **f** $x = 1$

11 **a** $x = -16$ **b** $x = -\frac{1}{2}$ **c** $x = -\frac{37}{40}$

d $x = \frac{11}{5}$ **e** $x = \frac{31}{37}$ **f** $x = -\frac{17}{5}$

12 **a** $x = b - a$ **b** $x = k + p$ **c** $x = \frac{d}{c}$

d $x = \frac{h-g}{3}$ **e** $x = \frac{y-5}{4}$ **f** $x = mn - a$

g $x = \frac{5+2w}{7}$ **h** $x = a(b-c)$ **i** $x = \frac{h}{k}$

j $x = \frac{e+f}{2}$ **k** $x = \frac{v-2y}{2}$ **l** $x = p(w-q)$

m $x = \frac{a}{b+c}$ **n** $x = \frac{2ny+m}{k}$ **o** $x = \frac{5c-3d}{8}$

13 **a** Decide that n = the number of weeks of saving. Any letter will do.

b $24n + 105 = 219$

c $n = 4.75$

d It will take Liam 5 weeks to save for his new tennis racquet.

14 **a** Let m be the cost of a makeup kit.

$4m + 29 = 387$

$m = 89.5$

Cost: \$89.50

b Let the length be l m and the breadth b m.

$l = b + 12$

$2l + 2b = 88$

$2(b + 12) + 2b = 88$

$4b + 24 = 88$

$b = 16, l = 28$

length: 28 m, breadth: 16 m

c Let their scores be l, s and b, respectively.

$l = s + 5, b = s - 3$

$l + s + b = (s + 5) + s + (s - 3)$

$3s + 2 = 29$

$s = 9, l = 14, b = 6$

Lisa: 14 goals, Sara: 9 goals, Bindi: 6 goals

d Let n = the number of people

$27n + 400 = 1800; n = 51.851\ldots$

51 people

15 **a** $6d + 3.8 = 4d + 9.2$, where d is the cost of a dumpling

b \$2.70

16 **a** $A = 50\text{ cm}^2$ **b** $h = 7$ mm

c $a = 7$ m **d** $a = \frac{2A}{h} - b$

e **i** $a = 52$ cm **ii** $a = 7.6$ m

17 **a**

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{c} = \frac{b}{ab} + \frac{a}{ab}$$

$$\frac{1}{c} = \frac{a+b}{ab}$$

$$c = \frac{ab}{a+b}$$

b **i** 3 cm **ii** 400 ohms

c

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

$$\frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{1}{a} = \frac{b}{bc} - \frac{c}{bc}$$

$$\frac{1}{a} = \frac{b-c}{bc}$$

$$a = \frac{bc}{b-c}$$

d **i** 40 cm **ii** 16 ohms

18 $5x + 20 = 3x + 50; x = 15$; 15, 17, 19, 21, 23

19 **a** $x = \frac{33}{5}$ **b** $x = -\frac{41}{34}$ **c** $x = \frac{33}{52}$

20 **a** $x = \frac{p-qr}{nr-m} = \frac{qr-p}{m-nr}$

b $x = \frac{b^2 - 2aby - a^2}{a^2y - 2ab - b^2y} = \frac{a^2 + 2aby - b^2}{b^2y + 2ab - a^2y}$

c $x = -\frac{1}{y}$

EX p172

4B Solving linear inequalities

1 **c**, **d**, **e**, **h**, **j**, **l**

2 **a** $x < 4$ **b** $x \le \frac{1}{2}$ **c** $x > 8$

d $x < -10$ **e** $x \ge -12$ **f** $x > 22$

g $x \le 10$ **h** $x > 147$ **i** $x \le 5$

j $x \ge \frac{500}{3}$ **k** $x \ge 25$ **l** $x > 98$

3 **a** $x > -6$ **b** $x \ge -\frac{1}{2}$ **c** $x > -10$

d $x > 10$ **e** $x \le 6$ **f** $x < -22$

g $x \ge -9$ **h** $x < -141$ **i** $x \ge -5$

j $x \le -\frac{10}{3}$ **k** $x \le 10$ **l** $x < -98$

4 **a** $x > \frac{5}{8}$ **b** $x \le -\frac{1}{3}$ **c** $x < 12$

d $x > \frac{22}{3}$ **e** $x \le \frac{12}{5}$ **f** $x < -11$

g $x \ge \frac{2}{7}$ **h** $x < -11$ **i** $x \ge \frac{19}{26}$

j $x \le \frac{50}{63}$ **k** $x \le \frac{35}{17}$ **l** $x < \frac{56}{15}$

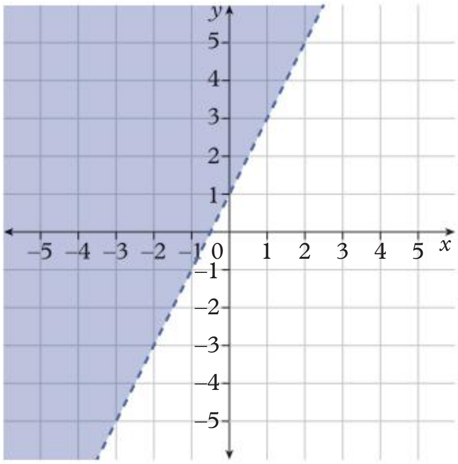
5 **a** $x < 6$ **b** $x < 1$ **c** $x > \frac{9}{5}$

d $x < \frac{4}{3}$ **e** $x < \frac{15}{2}$ **f** $x \le 2$

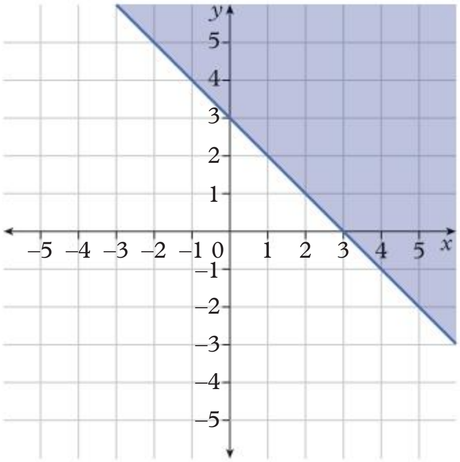
6	**a**	**i** no	**ii** no	**iii** no
	b	**i** no	**ii** no	**iii** yes
	c	**i** yes	**ii** yes	**iii** yes
	d	**i** yes	**ii** yes	**iii** yes
7	**a**	**i** yes	**ii** yes	**iii** yes
	b	**i** no	**ii** yes	**iii** yes
	c	**i** yes	**ii** yes	**iii** no
	d	**i** yes	**ii** yes	**iii** no

8 **a–I**, **b–IV**, **c–III**, **d–II**

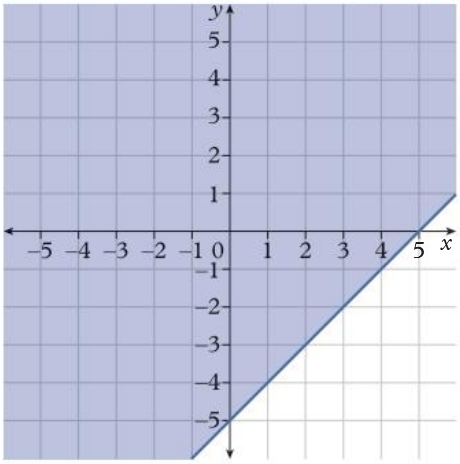
9 **a**



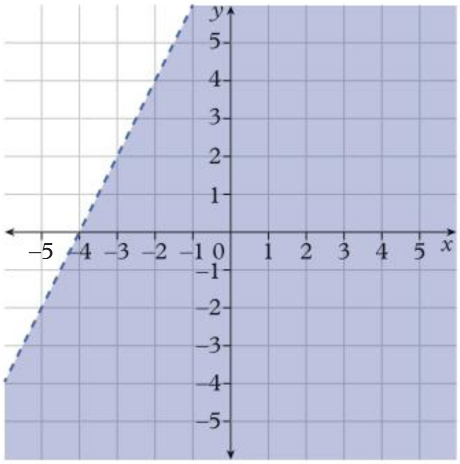
b



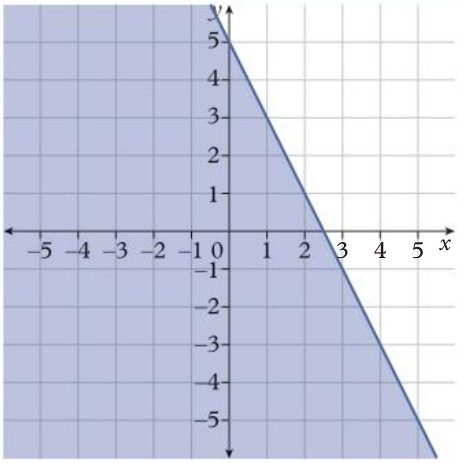
c



d



10 **a**



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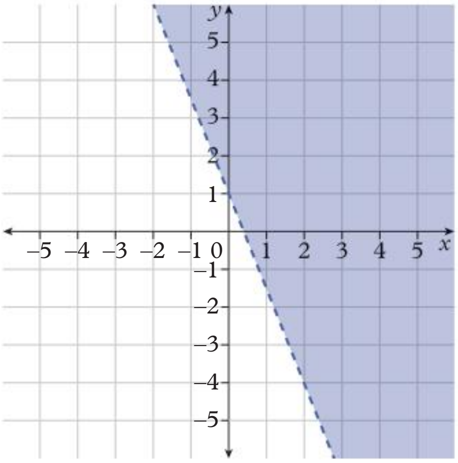
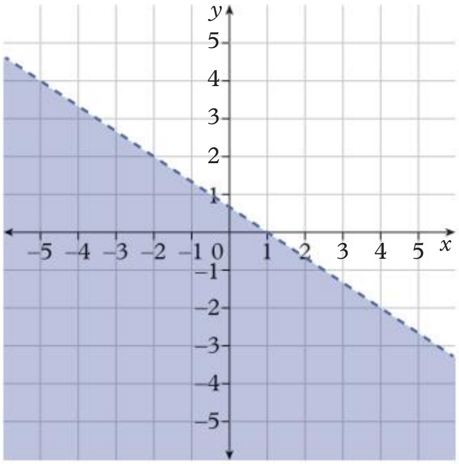
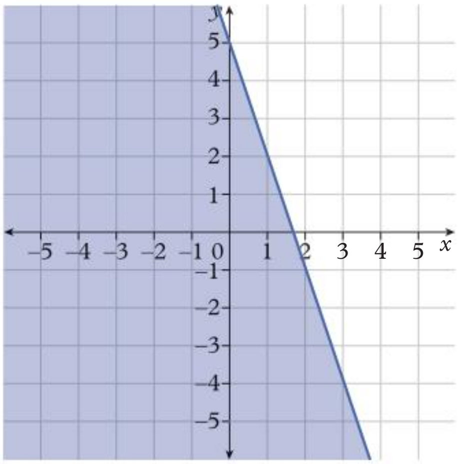


c



11 a



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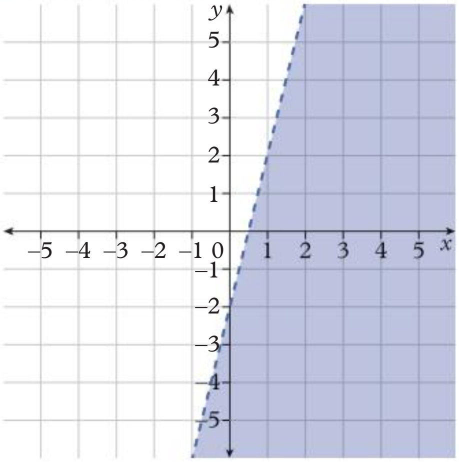

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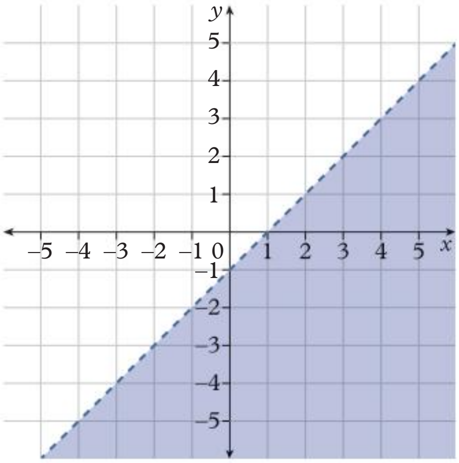

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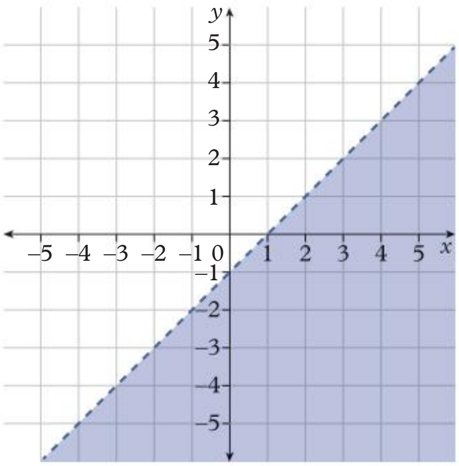

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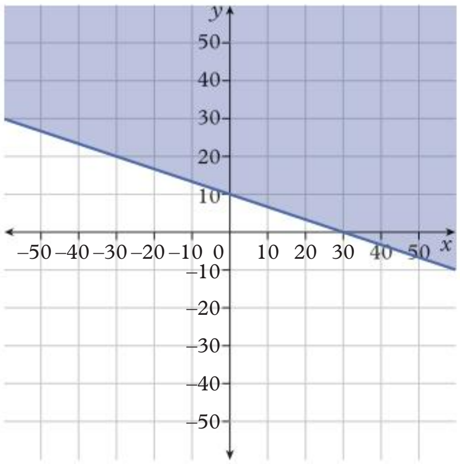
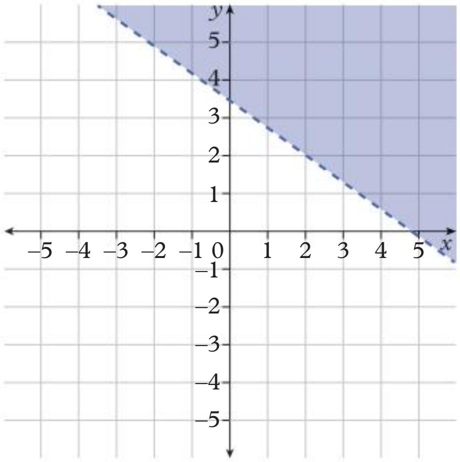


f



12 a $y > x + 3$ **b** $y \le \frac{x}{2} - 1$

c $y \le -x - 1$ **d** $y > -x + 1$

13 a $2(b + l) > 20$ **b** $22f + 6p \le 100$

c $55s + 80b \le 2145$ **d** $8p \ge 4 + 3g$

e $s + b + 3.75 \le 12$

14 a $10x + 20y \le 400$

b The right-hand side of the inequality should be $10x + 20y + 100$, because 400 is the maximum of how much Ken can spend, not the actual spending.

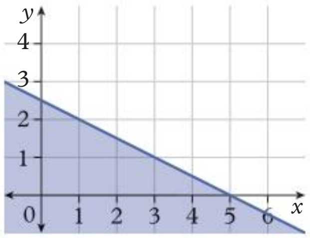
c $5x + 5y \ge 100$,

d $100

e Ken should stock up 40 product A. Because the profit for selling each product is $5 for both product A and B, as A is cheaper to acquire, Ken can make more profit by selling product A.

15 a $10x + 20y \le 50$

b



c Because the quantity of snow globes and hoodies can't be negative.

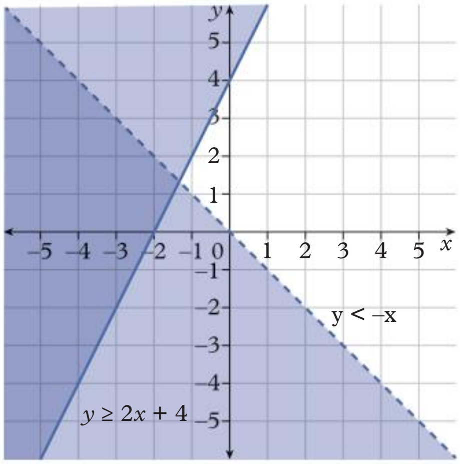
d No

e 0 snow globes, 1 hoodie
0 snow globes, 2 hoodies
1 snow globe, 0 hoodies
1 snow globe, 1 hoodie
1 snow globe, 2 hoodies
2 snow globes, 0 hoodies
2 snow globes, 1 hoodie
3 snow globes, 0 hoodies
3 snow globes, 1 hoodie
4 snow globes, 0 hoodies
5 snow globes, 0 hoodies

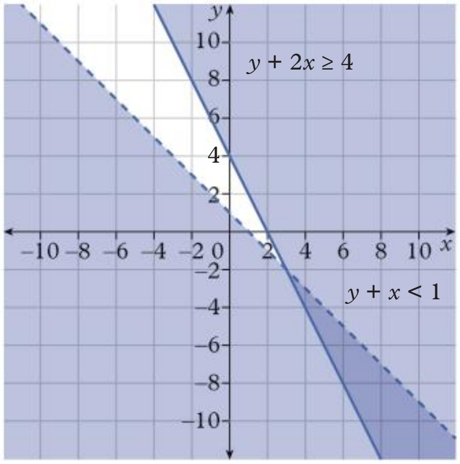
f $x + 0.7y + 3 \leq 7$

g all combinations with three or less snow globes

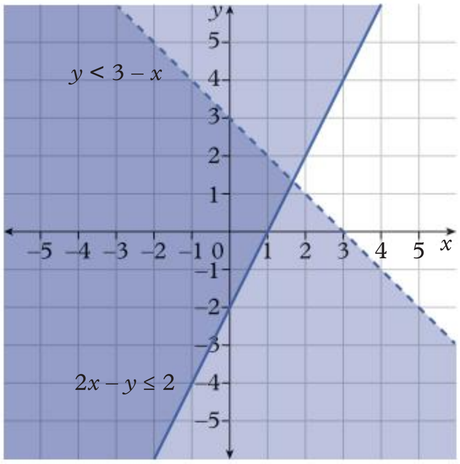
16 a i



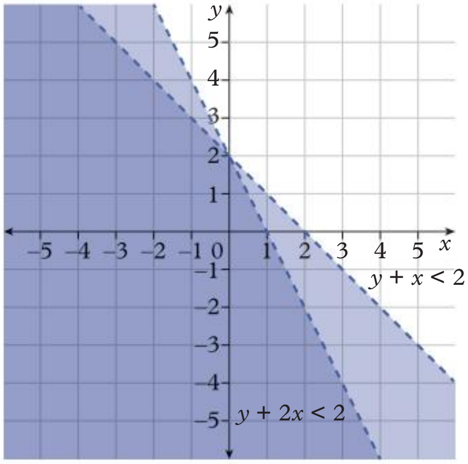
ii



iii



iv



b **i** yes **ii** no **iii** no
iv yes **v** no **vi** no

EX p181

4C Simultaneous linear equations

1 **a** is a solution **b** is a solution
c is not a solution **d** is a solution
e is not a solution **f** is not a solution

2 **a** **i** (4, 3) **ii** $x = 4, y = 3$
b **i** (2, –4) **ii** $x = 2, y = -4$
c **i** (–5, –2) **ii** $x = -5, y = -2$
d **i** (–2, 3) **ii** $x = -2, y = 3$

3 **a** $x = 5, y = 4$ **b** $x = 2, y = 6$
c $x = 6, y = 4$ **d** $x = -2, y = 5$
e $x = -1, y = 3$ **f** $x = -3, y = 1$
g $x = 3, y = 2$ **h** $x = 4, y = -2$
i $x = -2, y = -3$ **j** $x = -5, y = 6$
k $x = 2, y = -2$ **l** $x = 3, y = 5$
m $x = -4, y = -1$ **n** $x = 1, y = -4$
o $x = -6, y = -5$ **p** $x = -1, y = 2$

4 **a** $x = 2, y = 7$ **b** $x = 4, y = 1$
c $x = -1, y = 3$ **d** $x = 6, y = 5$
e $x = -4, y = -2$ **f** $x = 2, y = 12$
g $x = 1, y = 3$ **h** $x = -3, y = 11$

i $x = 5, y = -5$ **j** $x = 5, y = 3$

k $x = -4, y = -1$ **l** $x = 5, y = 6$

5 **a** $x = 1, y = 8$ **b** $x = 6, y = 4$

c $x = -3, y = -5$ **d** $x = 5, y = -1$

e $x = -1, y = 4$ **f** $x = 4, y = -7$

g $x = 2, y = 1$ **h** $x = -2, y = 6$

6 **a** $x = 2, y = 5$ **b** $x = -3, y = 1$

c $x = 1, y = -5$

7 **a** $x = 5, y = 3$ **b** $x = -2, y = 9$

c $x = 3, y = -7$

8 **a** $x = -4, y = -2$ **b** $x = 6, y = 2$

c $x = -1, y = 2$ **d** $x = -1, y = 5$

e $x = 5, y = 3$ **f** $x = -3, y = -4$

9 **a** $x = 3, y = 2$ **b** $x = 2, y = -1$

c $x = -4, y = 7$ **d** $x = 5, y = 3$

e $x = -3, y = -4$ **f** $x = -2, y = 6$

g $x = 1, y = 4$ **h** $x = -3, y = 1$

i $x = 2, y = -1$ **j** $x = -4, y = 2$

k $x = 5, y = 5$ **l** $x = -3, y = -2$

10 **a** **i** The graphical method is most appropriate when the equations of the lines are easy to sketch accurately on the same set of axes and you have a grid on which to read the intersection point.

ii The substitution method is most appropriate when one or both of the equations have a variable as the subject of the equation.

iii The elimination method is most appropriate when the terms with the same variable in each equation have the same coefficient or the negative value of the other coefficient.

b **i** The graphical method is least appropriate when the equations of the lines are not easy to sketch accurately on the same set of axes or the intersection point cannot be read off accurately.

ii The substitution method is least appropriate when neither equation has a variable as the subject of the equation.

iii The elimination method is least appropriate when the coefficients of the same variable cannot be simply multiplied by a number to make them the same or the negative value of each other.

c **i** The graphical method can be made more appropriate by using digital technology to sketch and locate the intersection point.

ii The substitution method can be made more appropriate by writing one or both equations with a variable as the subject.

iii The elimination method can be made more appropriate by multiplying the equations by a number to make the coefficients of the same variable the same or the negative value of each other.

11 **a** $x = 2, y = 0$

b $x = \frac{1}{3}, y = -3$

c $x = \frac{3}{2}, y = \frac{2}{5}$ or $x = 1.5, y = 0.4$

d $x = -9, y = -7$

e $x = 4, y = 2$

f $x = -2, y = -3$

g $x = 3, y = 10$

h $x = 5, y = -2$

i $x = 2, y = -1$

j $x = -2, y = 2$

k $x = 7, y = 4$

l $x = -4, y = 1$

m $x = -0.8, y = 2.3$

n $x = -\frac{1}{4}, y = 5$ or $x = -0.25, y = 5$

o $x = 28, y = 41$

12 **a** $2l + 2b = 810$ or $2(l + b) = 810$

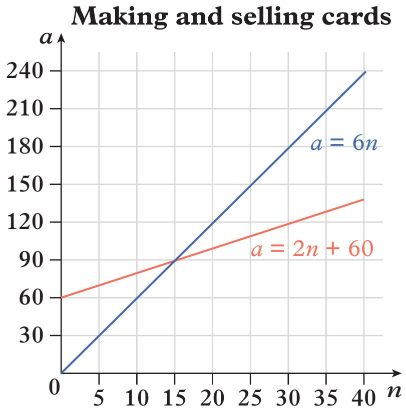
b $l = 2b$

c 270 cm long and 135 cm wide

13 **a** To make n cards, the cost of plain white cards is $2 \times n$ or $2n$. So, the amount of money spent (in \$) is the cost of the plain white cards plus the start-up cost of \$60.

b Each card sells for \$6, so for selling n cards, the amount of money received (in \$) is $6 \times n$ or $6n$.

c

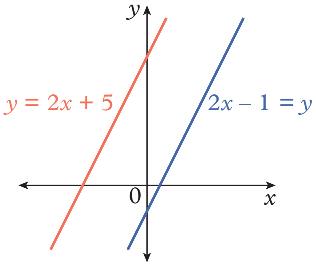


d (15, 90); this shows the number of cards that need to be sold (15) so that the amount of money received is the same as the amount of money spent (\$90).

e 15 cards

f 16 to 40 cards

14 Both graphs have a gradient of 2. This means the lines are parallel, so there is no point of intersection, thus no solution to the equation.



15 They are equivalent equations. When graphed, the position of the lines is identical; that is, one line lies on top of the other. There are an infinite number of points where the lines intersect so there are infinite solutions.

16 a Strategy A: Equation ③ is $10x - 6y = -2$.

① – ③: $-3x = -12$, $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy B: Equation ④ is $-10x + 6y = 2$.

① + ④: $-3x = -12$, so $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy C: Equation ⑤ is $35x - 30y = -70$ and equation ⑥ is $-35x - 21y = -7$.

⑤ – ⑥: $-9y = -63$, so $y = 7$

Substituting $y = 7$ into ②: $5x - 21 = -1$, so $x = 4$

The solution is $x = 4$, $y = 7$.

The strategies all provide the same solution of $x = 4$ and $y = 7$.

b Some possible answers are:

Multiply equation ① by 5 to form equation ⑦ and multiply equation ② by –7 to form equation ⑧. Add equations ⑦ and ⑧.

Graph equations ① and ② and identify the coordinates of the point of intersection.

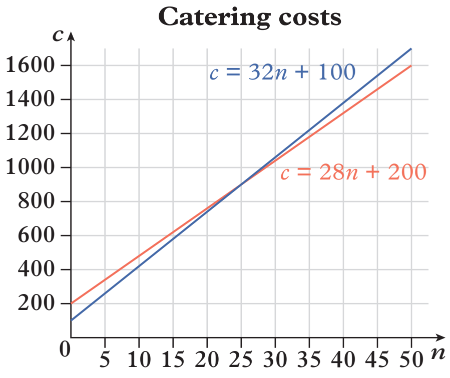
Rearrange equation ② to obtain $3y = 5x - 1$ and substitute that into equation ①. In other words, substitute $5x - 1$ for $3y$ in $7x - 2(3y) = -14$.

17 n = number of people and c = catering cost

a $c = 28n + 200$

b $c = 32n + 100$

c



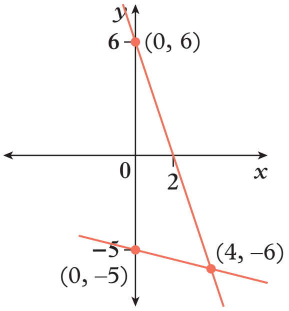
d Cool Food Club is cheaper for 18 people; at $n = 18$, the line for Cool Food Club is lower than the line for Angie's Catering. This represents a lower cost for 18 people.

e Angie's Catering

f 25 people at a cost of $900

g When catering for fewer than 25 people, Cool Food Club is cheaper. When catering for more than 25 people, Angie's Catering is cheaper. The cost is the same when catering for 25 people.

18 a



b $y = -\frac{1}{4}x - 5$, $y = -3x + 6$

c $x = 4$, $y = -6$

d If $x = 4$ and $y = -6$:

For $y = -\frac{1}{4}x - 5$, RHS $= -\frac{1}{4} \times 4 - 5 = -6 =$ LHS.

For $y = -3x + 6$, RHS $= -3 \times 4 + 6 = -6 =$ LHS.

19 a $15 + (-5) = 10$ and $15 + 5 = 20$

b $22 + (-12) = 10$ and $22 + (-2) = 20$

c Since there are three variables but only two equations, if we add, subtract or substitute, the resulting equation will still have two variables in it when we need it to have only one.

20 a

$$\begin{aligned}
ax + by &= c \\
ax + by - (ax + qy) &= c - (r) \\
by - qy &= c - r \\
(b - q)y &= c - r \\
y &= \frac{c - r}{b - q} \\
ax + b\left(\frac{c - r}{b - q}\right) &= c \\
ax + \frac{bc - br}{b - q} &= \frac{c(b - q)}{b - q} \\
ax &= \frac{cb - cq}{b - q} - \frac{bc + br}{b - q} \\
ax &= \frac{cb - cq - bc - br}{b - q} \\
ax &= \frac{br - cq}{b - q} \\
x &= \frac{br - cq}{ab - aq}
\end{aligned}$$

b $x = \frac{4 \times 7 - 9 \times 6}{3 \times 4 - 3 \times 6} = \frac{13}{3}$ and $y = \frac{9 - 7}{4 - 6} = -1$

c

$$\begin{aligned}
ax + by &= c \ (\times\, p) \\
apx + bpy &= cp \\
px + qy &= r \ (\times\, a) \\
apx + aqy &= ar
\end{aligned}$$

$$\begin{aligned}
apx + bpy - (apx + aqy) &= cp - (ar)\\
bpy - aqy &= cp - ar\\
(bp - aq)y &= cp - ar\\
y &= \frac{cp - ar}{bp - aq}\\
ax + b\left(\frac{cp - ar}{bp - aq}\right) &= c\\
ax + \frac{bcp - abr}{bp - aq} &= \frac{c(bp - aq)}{bp - aq}\\
ax &= \frac{bcp - acq}{bp - aq} - \frac{bcp - abr}{bp - aq}\\
ax &= \frac{bcp - acq - bcp + abr}{bp - aq}\\
ax &= \frac{abr - acq}{bp - aq}\\
x &= \frac{br - cq}{bp - aq}
\end{aligned}$$

d $x = -13$ and $y = 12$

EX p185

Checkpoint

1 **a** $x = -\frac{2}{7}$ **b** $x = \frac{17}{11}$

c $x = -7$ **d** $x = 2$

2 **a** $x = 19$ **b** $x = \frac{5}{7}$

c $x = \frac{1}{30}$ **d** $x = -1$

3 **a** $x = -\frac{7}{2}$ **b** $x = \frac{23}{37}$

c $x = \frac{9}{13}$ **d** $x = \frac{14}{19}$

4 **a** $x = \frac{c - by}{a}$ **b** $x = \frac{ef + f}{gp}$

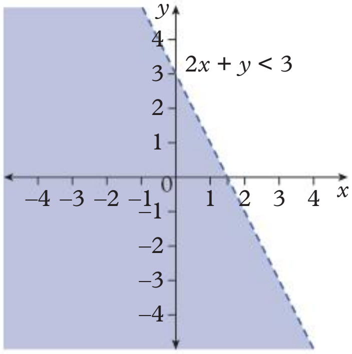
c $x = \frac{jk + mn}{m - j}$ **d** $x = \frac{rs + q}{p + rt}$

5 **a** $x \le -4$ **b** $x \ge \frac{3}{2}$

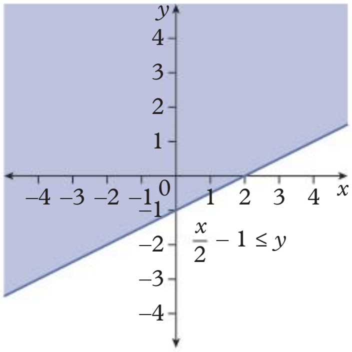
c $x > -20$ **d** $x \ge \frac{3}{29}$

6 **a** yes **b** no **c** yes **d** yes

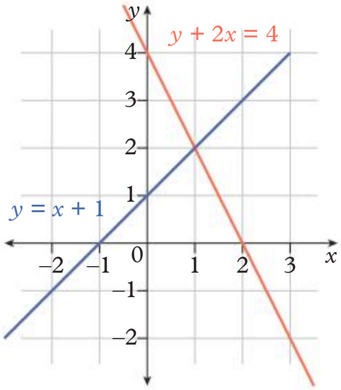
7 **a**



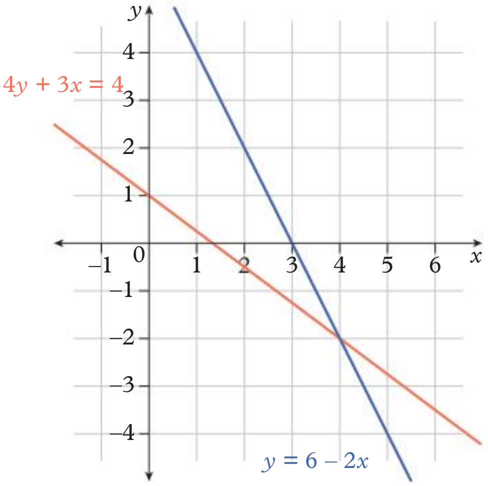
b



8 **a** $x = 1, y = 2$



b $x = 4, y = -2$



9 **a** $x = 5, y = 13$ **b** $x = 1, y = 4$

10 **a** $x = 0, y = \frac{1}{2}$ **b** $x = 1, y = 7$

EX p189

4D Solving quadratic equations

1 **a** $x = -2$ or $x = 4$ **b** $x = -7$ or $x = -1$

c $x = 3$ or $x = 6$ **d** $x = \pm 9$

e $x = 0$ or $x = 5$ **f** $x = 0$ or $x = \frac{1}{2}$

g $x = 0$ or $x = -1$ **h** $x = 0$ or $x = -2$

i $x = 0$ or $x = \frac{3}{4}$ **j** $x = \pm\sqrt{2}$

k $x = \pm\sqrt{5}$ **l** $x = \pm 4\sqrt{3}$

m $x = -1\frac{1}{2}$ or $x = 4$ **n** $x = \pm\frac{1}{3}$

o $x = -\frac{2}{7}$ or $x = 1\frac{1}{4}$

2 **a** $x = -3$ or $x = -4$ **b** $x = -5$ or $x = 2$

c $x = 1$ or $x = 8$ **d** $x = \pm 5$

e $x = \pm 9$ **f** $x = \pm 2$

g $x = 0$ or $x = 3$ **h** $x = 0$ or $x = -1$

i $x = 0$ or $x = -4$ **j** $x = 0$ or $x = \frac{2}{5}$

k $x = 0$ or $x = 6$ **l** $x = 0$ or $x = -10$

3 **a** $x = -6$ or $x = 2$ **b** $x = 1$ or $x = 4$

c $x = \pm 3$ **d** $x = -5$ or $x = 8$

e $x = 0$ or $x = 9$ **f** $x = 2 \pm \sqrt{3}$

g $x = 0$ or $x = 4$ **h** $x = 2$ or $x = 9$

i $x = -3$ or $x = 6$ **j** $x = -7$ or $x = 3$

k $x = 4$ or $x = 6$ **l** $x = -3$

4 **a** Both sides of the original equation have been divided by 3, so the equation remained in balance and the values of the variables are unchanged.

b $x = -6$ or $x = 8$

5 **a** $x = -5$ or $x = 6$ **b** $x = 2$ or $x = 4$
c $x = -8$ or $x = -1$ **d** $x = 3$ or $x = -9$
e $x = \pm 4$ **f** $x = 0$ or $x = -7$
g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -1\frac{2}{3}$
i $x = 1\frac{1}{2}$ or $x = -10$

6 **a** $x = -2$ or $x = -1$ **b** $x = -5$ or $x = 3$
c $x = 1$ or $x = 4$ **d** $x = -3$ or $x = 2$
e $x = -2$ or $x = 6$ **f** $x = -8$ or $x = 7$
g $x = \pm 2$ **h** $x = \pm 3$
i $x = 0$ or $x = \frac{1}{2}$

7 **a** $x = 2$ **b** $x = 3 \pm \sqrt{3}$ **c** $x = -1, x = 3$
d $x = 6 \pm 2\sqrt{10}$ **e** $x = 1 \pm \sqrt{2}$ **f** $x = 6 \pm \sqrt{33}$

8 **a** $x = -1 \pm \sqrt{7}$ **b** $x = 2 \pm \sqrt{2}$
c $x = -4 \pm \sqrt{5}$ **d** $x = 6 \pm 2\sqrt{2}$
e $x = -10 \pm 5\sqrt{3}$ **f** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$
g $x = -3 \pm \sqrt{2}$ **h** $x = -1 \pm \sqrt{5}$
i $x = 2 \pm \sqrt{3}$ **j** $x = -4 \pm \sqrt{10}$
k $x = 3 \pm \sqrt{\frac{21}{2}}$ or $x = 3 \pm \frac{\sqrt{42}}{2}$
l $x = -\frac{1}{2} \pm \sqrt{\frac{13}{20}}$ or $x = -\frac{1}{2} \pm \frac{\sqrt{65}}{10}$

9 **a** $x = -5$ or $x = -\frac{3}{2}$ **b** $x = \frac{2}{5}$ or $x = 4$
c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ **d** $x = -7$ or $x = \frac{3}{4}$
e $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$ **f** $x = -\frac{3}{7}$ or $x = 2$

10 **a** **i** yes **ii** no
b **i** yes **ii** yes
c **i** no **ii** yes
d **i** no **ii** no

11 **a** **i** yes **ii** no **iii** no
b **i** no **ii** yes **iii** no
c **i** no **ii** no **iii** yes
d **i** no **ii** yes **iii** no

12 **a** $x^2 + 2x - 15 = 0$
b $x^2 - 8x + 15 = 0$
c $x^2 - 5 = 0$
d $5x^2 - 40x + 75 = 0$

13 **a** **i** yes **ii** no **iii** no
b **i** no **ii** yes **iii** no
c **i** no **ii** no **iii** yes
d **i** no **ii** yes **iii** no

14 Dividing both sides of the first equation by 3 gives the equivalent equation $x(x - 2) = 0$ and this does not affect the solutions (with $x = 2$ and $x = 0$ being a solution for both equations). Dividing both sides of the equation by $3x$ loses the solution $x = 0$. In addition, since the value of x is not fixed, dividing by $3x$ includes dividing by 0, which is undefined.

15 **a** two **b** one **c** two
d one **e** two **f** none

16 **a** $x = \pm 3$ **b** $x = -11$ or $x = 7$
c $x = 4$ **d** no solutions
e $x = 0$ or $x = -4$ **f** $x = 3 \pm \sqrt{10}$

17 **a** **i** 8 m **ii** 6 m
b 6 m
c 3 s
d It is not possible to have negative time values.

18 **a** 25 kg **b** 9 kg **c** 5 weeks
d No. If the amounts were the same each week, the relationship would be linear.

19 **a** $x(x + 3) = 70$ or $x^2 + 3x - 70 = 0$
b $x = -10$ or $x = 7$. The feasible solution is $x = 7$, while $x = -10$ is not feasible because it is not possible to have a negative length.
c Width 7 m, length 10 m

20 $x(x - 5) = 66$, where x is the length in centimetres. The dimensions are length 11 cm, width 6 cm.

21 **a** $x = \pm\sqrt{6}$ **b** $x = -\frac{1}{2}$

22 11

EX p195

4E The quadratic formula

1 **a** none **b** two **c** two
d none **e** one **f** two

2 **a** **i** 64 **ii** two **iii** $x = -3$ or $x = 5$
b **i** 0 **ii** one **iii** $x = -2$
c **i** 196 **ii** two **iii** $x = -7$ or $x = 7$
d **i** 324 **ii** two **iii** $x = 0$ or $x = 6$
e **i** 8 **ii** two **iii** $x = 3 - \sqrt{2}$ or $x = 3 + \sqrt{2}$
f **i** -8 **ii** none **iii** no solutions

3 **a** **i** $x = \frac{7 \pm \sqrt{29}}{2}$ **ii** $x \approx 0.81$ or $x \approx 6.19$
b **i** $x = \frac{1 \pm \sqrt{13}}{6}$ **ii** $x \approx -0.43$ or $x \approx 0.77$
c **i** $x = \frac{-1 \pm \sqrt{33}}{4}$ **ii** $x \approx -1.69$ or $x \approx 1.19$
d **i** $x = \frac{-1 \pm \sqrt{109}}{6}$ **ii** $x \approx -1.91$ or $x \approx 1.57$
e **i** $x = \frac{-5 \pm \sqrt{97}}{12}$ **ii** $x \approx -1.24$ or $x \approx 0.40$
f **i** $x = 1 \pm \sqrt{2}$ **ii** $x \approx -0.41$ or $x \approx 2.41$
g **i** $x = -4 \pm 2\sqrt{3}$ **ii** $x \approx -7.46$ or $x \approx -0.54$
h **i** $x = \frac{3 \pm 2\sqrt{2}}{2}$ **ii** $x \approx 0.09$ or $x \approx 2.91$
i **i** $x = 2 \pm \sqrt{6}$ **ii** $x \approx -0.45$ or $x \approx 4.45$
j **i** $x = \frac{-5 \pm 2\sqrt{10}}{5}$ **ii** $x \approx -2.26$ or $x \approx 0.26$
k **i** $x = \frac{-3 \pm \sqrt{97}}{4}$ **ii** $x \approx -3.21$ or $x \approx 1.71$
l **i** $x = -1 \pm 2\sqrt{2}$ **ii** $x \approx -3.83$ or $x \approx 1.83$

4 **a** $x = \frac{-3 \pm \sqrt{13}}{2}$ **b** no solutions **c** $x = -3 \pm \sqrt{5}$
d no solutions **e** $x = \frac{-5 \pm \sqrt{17}}{4}$ **f** $x = 4$

g $x = -1 \pm \sqrt{5}$ **h** $x = \frac{7 \pm \sqrt{61}}{2}$ **i** $x = -4 \pm \sqrt{26}$

j $x = \frac{1 \pm \sqrt{21}}{2}$ **k** no solutions **l** $x = \frac{-\sqrt{5} \pm 3}{2}$

5 a $m = \frac{-3 \pm \sqrt{37}}{2}$ **b** $u = -2 \pm \sqrt{5}$

c $r = \frac{-9 \pm \sqrt{77}}{2}$ **d** $y = -1$ or $y = 3$

e $h = \frac{-7 \pm \sqrt{61}}{6}$ **f** $w = 3 \pm \sqrt{14}$

6 $x = \frac{3 \pm \sqrt{13}}{2}$

7 a, b $x = 2$ or $x = 3$

c In cases such as this, the method described in part **a** is often found to be quicker and easier.

d If integer solutions exist, the method described in part **a** is usually quicker.

8 a There is no real number that is the square root of a negative number, so we cannot obtain a result using the quadratic formula. There are no solutions.

b The square root of zero is zero, so there will be only one distinct value obtained using the quadratic formula: $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$

c The square root of a positive number gives a positive result so there will be two distinct solutions:

$$x = \frac{-b - \text{positive}}{2a} \text{ or } x = \frac{-b + \text{positive}}{2a}.$$

9 Step 3: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 4: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

Step 5: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Step 7: $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)^2 = 0$

Step 10: $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$

Step 11: $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

Step 12: $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

10 a

$$\begin{aligned} a(x - h)^2 + k &= 0 \\ a(x - h)^2 &= -k \\ (x - h)^2 &= -\frac{k}{a} \\ x - h &= \pm\sqrt{-\frac{k}{a}} \\ x &= h \pm \sqrt{-\frac{k}{a}} \end{aligned}$$

b The quadratic equation would have:

two real solutions if k and a have different signs,

one real solution if $k = 0$ and

no real solutions if k and a have the same signs.

c i two **ii** one **iii** zero

d i $x = 5 \pm \frac{1}{2}$ **ii** $x = -4$

11 a $620 **b** 59

12 length 81.4 m, width 61.4 m

13 length 369 mm, width 131 mm

14 a i Δ = 33. Solutions will be irrational.

ii Δ = 25. Solutions will be rational.

iii Δ = 9. Solutions will be rational.

iv Δ = 41. Solutions will be irrational.

v Δ = 49. Solutions will be rational.

b The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $\sqrt{b^2 - 4ac}$ is an integer and a, b and c are integers then the solutions are rational. $\sqrt{b^2 - 4ac}$ is an integer if $b^2 - 4ac$, the discriminant, is a square number, since the square root of a square number is an integer.

15 a i (−2, 0) and (−5, 0) **ii** (2, 0) and (4, 0)

iii no x-intercepts **iv** (4, 0)

v no x-intercepts

b The discriminant relates to the number of solutions to a quadratic when it equals zero, and therefore the number of x-intercepts of its graph.

16 a $k = 3$

b i $k < 3$ **ii** $k > 3$

17 a $k = 1$ **b** $k < 2$

c $k < -9$ **d** $k = \frac{9}{4}$

e $k < -6$ and $k > 6$ **f** $-10 < k < 10$

18 a $b^2 - 20a$ **b** $b^2 = 20a$ **c** $a = 5, b = 10$

EX
p200

4F Solving cubic equations

1 a $x^3 = 5$ **b** $x^3 = -12$ **c** $x^3 = 100$

d $x^3 = 15$ **e** $x^3 = 2$ **f** $x^3 = \frac{1}{100}$

2 a $a = 10$ **b** $x = 3$ **c** $y = -4$

d $m = \frac{3}{5}$ **e** $t = \frac{8}{7}$ **f** $z = -\frac{5}{11}$

g $n = 0$ **h** $x = -\frac{1}{3}$ **i** $p = -\frac{1}{2}$

j $s = 4$ **k** $x = -2$ **l** $m = 10$

m $q = 1.2$ **n** $x = 1.1$ **o** $n = 0.6$

3 a $a = \sqrt[3]{10}$ **b** $b = \sqrt[3]{9}$ **c** $c = -\sqrt[3]{3}$

d $d = \sqrt[3]{12}$ **e** $e = -\sqrt[3]{15}$ **f** $f = -\sqrt[3]{36}$

4 a $a \approx 2.15$ **b** $b \approx 2.08$ **c** $c \approx -1.44$

d $d \approx 2.29$ **e** $e \approx -2.47$ **f** $f \approx -3.30$

5 a $a = -\sqrt[3]{3}$ **b** $b = \frac{\sqrt[3]{81}}{4}$ **c** $c = \sqrt[3]{\frac{1}{3}}$

d $d = \sqrt[3]{9}$ **e** $e = -\sqrt[3]{36}$ **f** $a = -\sqrt[3]{24}$

g $n = \sqrt[3]{1.2}$ **h** $x = \sqrt[3]{0\ .1}$ **i** $m = \sqrt[3]{2.2}$

j $x = \sqrt[3]{\frac{1}{2}}$ **k** $y = \sqrt[3]{16}$ **l** $s = \sqrt[3]{2}$

6 a $s \approx 2.15$ cm **b** $h \approx 2.67$ mm **c** $r \approx 2.88$ m

7 a i $V = 9s^3$ **ii** 2 cm

b i $V = 6s^3$ **ii** 3 cm

c i $V = 5s^3$ **ii** 4 mm

d i $V = 6s^3$ **ii** 10 m

8 5.60 cm

9 a 3.13 m **b** 3.16 cm

10 a i $x = 0.01$

ii $x^3 = 0.1$

iii $x = 10$

iv $x^3 = 100$

b **i** rational, $x = 30$
iv irrational
iii rational, $x = 0.04$
iv irrational
v rational, $x = -200$
vi irrational

11 **a** **i** $x = 1.1$
ii $x = 11$
iii $x = 110$
iv $x = 0.11$

b **i** $x = -11$
ii $x = 0.011$
iii $x = 1100$
iv $x = -0.0011$

c **i** $x = -2.3$
ii $x = 0.023$
iii $x = 2300$
iv $x = 230$

12 **a**
$$\begin{aligned}\sqrt[3]{ab} &= \sqrt[3]{m^3 n^3}\\ &= \sqrt[3]{m \times m \times m \times n \times n \times n}\\ &= \sqrt[3]{m \times n \times m \times n \times m \times n}\\ &= \sqrt[3]{(m \times n)^3}\\ &= m \times n\\ &= \sqrt[3]{a} \times \sqrt[3]{b}\end{aligned}$$

b **i** $x = 2\sqrt[3]{3}$ **ii** $x = 3\sqrt[3]{2}$
iii $x = 5\sqrt[3]{2}$ **iv** $x = 2\sqrt[3]{2}$

13
$$\begin{aligned}\sqrt[3]{-x} &= \sqrt[3]{-1 \times x}\\ &= \sqrt[3]{-1 \times 1 \times x}\\ &= \sqrt[3]{-1 \times (-1 \times -1) \times x}\\ &= \sqrt[3]{(-1)^3 \times x}\\ &= \sqrt[3]{(-1)^3} \times \sqrt[3]{x}\\ &= -1 \times \sqrt[3]{x}\\ &= -\sqrt[3]{x}\end{aligned}$$

14 **a** $x = -4$ **b** $x = 4$ **c** $x = 0.3$
d $x = -\frac{1}{2}$ **e** $x = -0.2$ **f** $x = \frac{2}{3}$

EX
p204

CHAPTER 4 review

Mathematical literacy review

1 **a** I, II **b** VI **c** III, IV, V
d III, V **e** IV **f** II, V

2 quadratic formula

3 discriminant

4 **a** simultaneous equations
b substitution method, subject of the equation
c elimination method
d Null Factor Law
e linear equation, linear inequality, boundary line

Multiple choice

1 B **2** A **3** E **4** C
5 D **6** E **7** B **8** A
9 A **10** C **11** A **12** E

Short answer

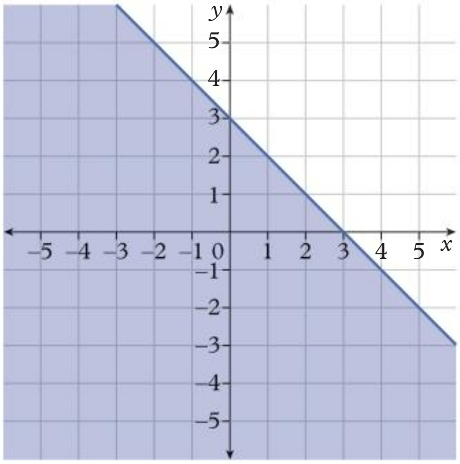
1 **a** $x = -1\frac{1}{2}$ **b** $x = 9\frac{2}{3}$

2 **a** $x = 9$ **b** $x = \frac{5}{13}$

3 **a** $b = \frac{2A}{h} - a$ **b** $h = \frac{A}{2(l+b)}$

4 **a** $x > -1$ **b** $x < -2\frac{2}{3}$

5 **a**



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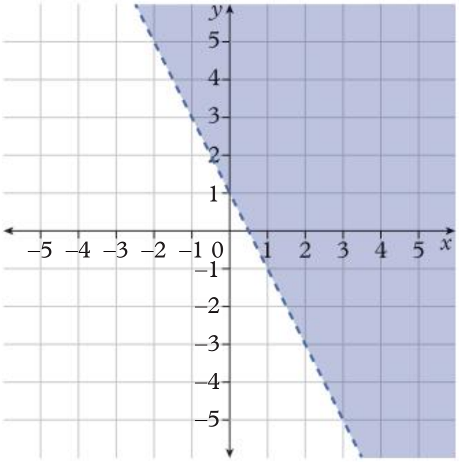

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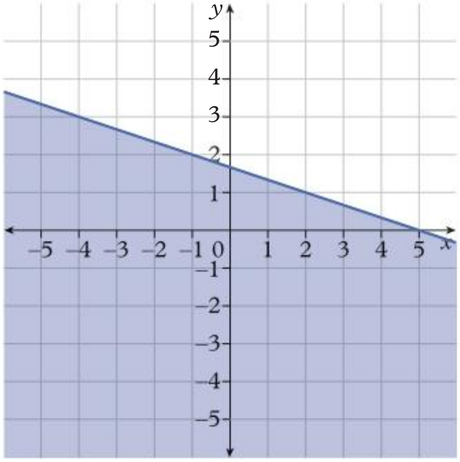

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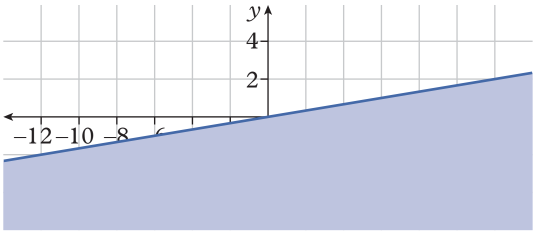


6 **a** $x = 1, y = 2$ **b** $x = 6, y = 17$

7 **a** $x = 1, y = -1$ **b** $x = -2, y = -5$

c $x = 2, y = 10$

8 **a** $x = 2, y = 4$ **b** $x = -2, y = 3$

c $x = 3, y = -1$

9 **a** $x = 2$ **b** $x = \frac{3}{4}$ or $x = -\frac{4}{3}$

c $x = \pm\sqrt{2}$ **d** $x = -2$ or $x = 4$

e $x = \pm\sqrt{\frac{7}{5}}$ **f** $x = 3$

10 **a** $x = -3 \pm \sqrt{5}$ **b** $x = \frac{3 \pm \sqrt{21}}{2}$

11 **a** 1 **b** 0 **c** 0

d 2 **e** 2 **f** 0

12 **a** **i** $x = \frac{-3 \pm \sqrt{53}}{2}$

ii $x \approx -5.14$ or $x \approx 2.14$

b **i** $x = \frac{3 \pm \sqrt{29}}{10}$

ii $x \approx -0.24$ or $x \approx 0.84$

c **i** $x = \frac{5 \pm \sqrt{17}}{2}$

ii $x \approx 0.44$ or $x \approx 4.56$

d **i** $x = \frac{-5 \pm \sqrt{17}}{2}$

ii $x \approx -4.56$ or $x \approx -0.44$

e **i** $x = \frac{1 \pm \sqrt{17}}{4}$

ii $x \approx -0.78$ or $x \approx 1.28$

f **i** $x = \frac{-7 \pm \sqrt{13}}{6}$

ii $x \approx -1.77$ or $x \approx -0.57$

13 **a** **i** $x = \sqrt[3]{10}$ **ii** $x \approx 2.15$

b **i** $x = \sqrt[3]{0.25}$ **ii** $x \approx 0.63$

c **i** $x = \sqrt[3]{25}$ **ii** $x \approx 2.92$

d **i** $x = \sqrt[3]{1.5}$ **ii** $x \approx 1.14$

e **i** $x = \sqrt[3]{4}$ **ii** $x \approx 1.59$

f **i** $x = -\sqrt[3]{3}$ **ii** $x \approx -1.44$

Analysis

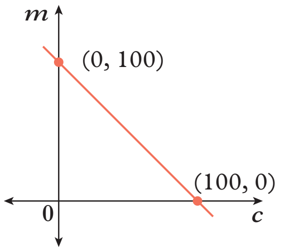
1 **a** $1.6m + 0.8c \le 100$

b $c \le 85$. Answer must be an integer.

c $m \le 47.5$. Answer must be an integer.

d $P = 1.7m + 1.7c$

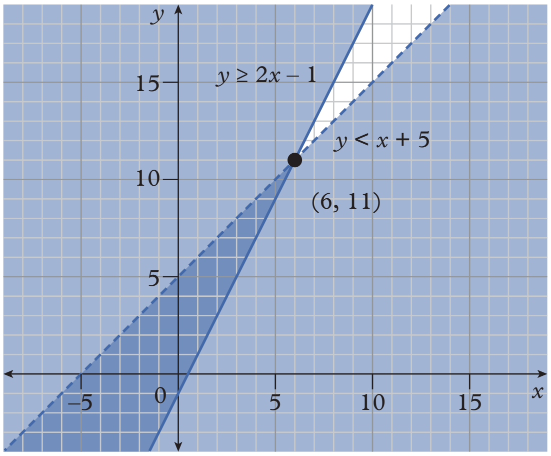
e



f 24 muffins, 76 croissants

2 **a** $x = 6, y = 11$

b



c Yes. It is the y-intercept for $y \ge 2x - 1$ and the boundary line is included.

d No. It is the y-intercept for $y < x + 5$ and the boundary line is not included.

e No. The point of intersection (6, 11) only satisfies one of the inequalities and is not included in the region.

f Yes

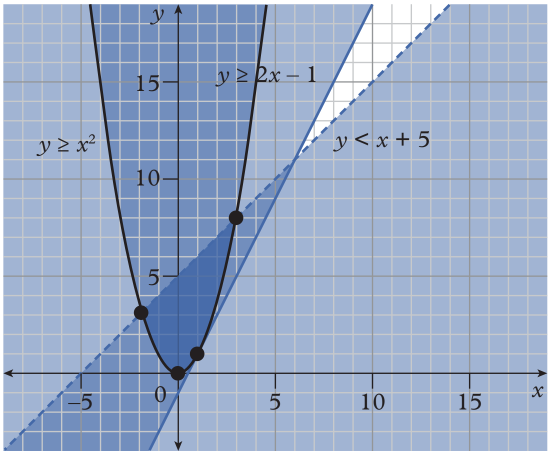
g $a \ge 0, b > 0$

h $a \le 0, b \le 0$

i $a \ge -7, b > 5$

j (1, 1)

k



l $a \ge -7, b > 5, c \le -1$

3 **a** **i** $a = 2, b = 7, c = 3$

ii $a = 3, b = 1$

iii $a = 3, b = 0, c = -4$

iv $a = b = \pm 2, c = -1$

v $a = 4, b = 3, c = 5$

b **i** $f = \frac{c}{d} + \frac{d}{c}$

ii $x = -\frac{c}{d}, -\frac{d}{c}$

iii $x = \frac{-f \pm \sqrt{f^2 - 4}}{2}$

c $f \ge 2$ or $f \le -2$

CHAPTER 5 Linear and non-linear relationships

EX
p214

5A Linear relationships

1 **a** **i** $y = 2x + 1$ **ii** $-2x + y - 1 = 0$

b **i** $y = -3x + 4$ **ii** $3x + y - 4 = 0$

c **i** $y = -\frac{3}{4}x - 3$

ii $\frac{3}{4}x + y + 3 = 0$ or $3x + 4y + 12 = 0$

d **i** $y = 3x - 6$ **ii** $-3x + y + 6 = 0$

e **i** $y = -2x + 6$ **ii** $2x + y - 6 = 0$

f **i** $y = -5x - 2$ **ii** $5x + y + 2 = 0$

g **i** $y = -3x + 2$ **ii** $3x + y - 2 = 0$

h **i** $y = 4x - 5$ **ii** $-4x + y + 5 = 0$

i **i** $y = -2x + 3$ **ii** $2x + y - 3 = 0$

j **i** $y = \frac{4}{5}x + 3$ **ii** $-4x + 5y - 15 = 0$

k **i** $y = \frac{3}{2}x + 2$ **ii** $3x - 2y + 4 = 0$

l **i** $y = -\frac{1}{3}x - 2$ **ii** $x + 3y + 6 = 0$

2 **a** **i** gradient = 2

ii x-intercept $\left(-\frac{1}{2}, 0\right)$, y-intercept $(0, 1)$

b **i** gradient = −3

ii x-intercept $\left(\frac{4}{3}, 0\right)$, y-intercept $(0, 4)$

c **i** gradient = $-\frac{3}{4}$

ii x-intercept $(-4, 0)$, y-intercept $(0, -3)$

d **i** gradient = 3

ii x-intercept $(2, 0)$, y-intercept $(0, -6)$

e **i** gradient = −2

ii x-intercept $(3, 0)$, y-intercept $(0, 6)$

f **i** gradient = −5

ii x-intercept $\left(-\frac{2}{5}, 0\right)$, y-intercept $(0, -2)$

g **i** gradient = −3

ii x-intercept $\left(\frac{2}{3}, 0\right)$, y-intercept $(0, 2)$

h **i** gradient = 4

ii x-intercept $\left(\frac{5}{4}, 0\right)$, y-intercept $(0, -5)$

i **i** gradient = −2

ii x-intercept $\left(\frac{3}{2}, 0\right)$, y-intercept $(0, 3)$

j **i** gradient = $\frac{4}{5}$

ii x-intercept $\left(\frac{15}{4}, 0\right)$, y-intercept $(0, 3)$

k **i** gradient = $\frac{3}{2}$

ii x-intercept $\left(-\frac{4}{3}, 0\right)$, y-intercept $(0, 2)$

l **i** gradient = $-\frac{1}{3}$

ii x-intercept $(-6, 0)$, y-intercept $(0, -2)$

3 **a** **i** x-intercept: $(0, 0)$

ii y-intercept: $(0, 0)$

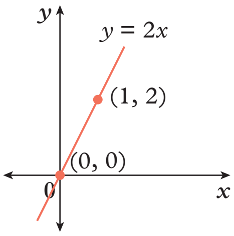
b No; you only have one point.

c You need the coordinates of one other point on the line.

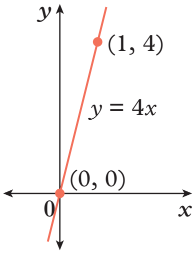
d $y = 2$

e $(0, 0)$ and $(1, 2)$

f



4 **a**



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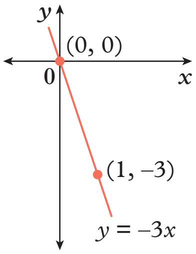

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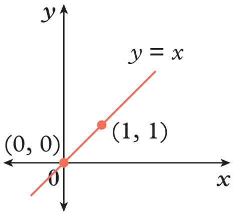

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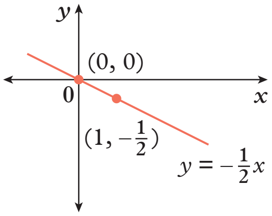
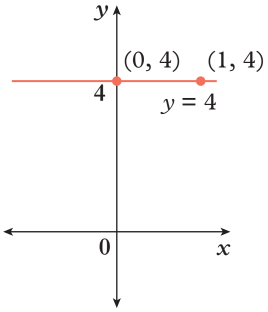
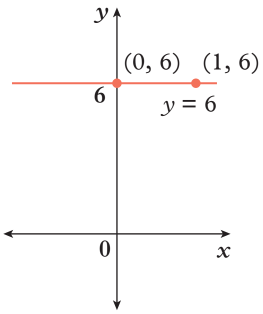


5 **a** **i** $m = 0$ **ii** $c = 4$

b



6 a



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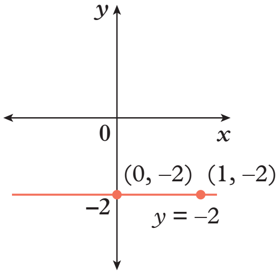

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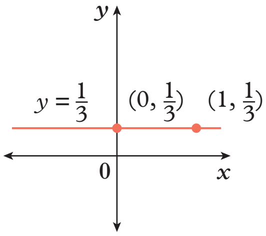

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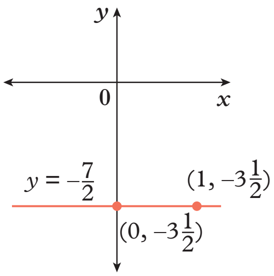
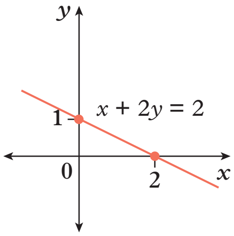


7 a



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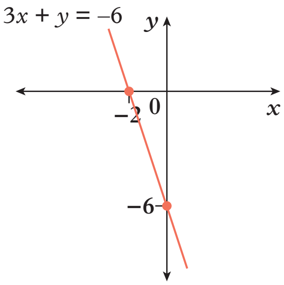

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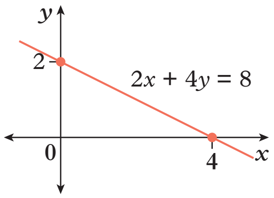

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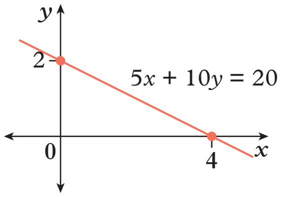

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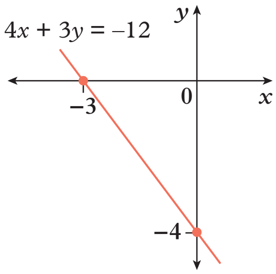

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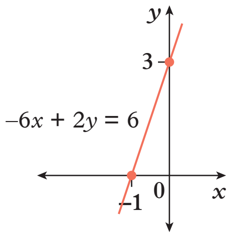

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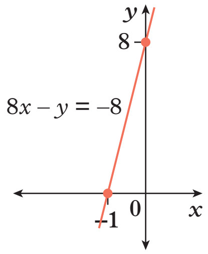

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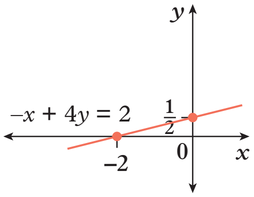

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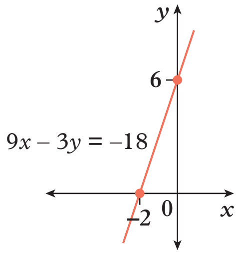

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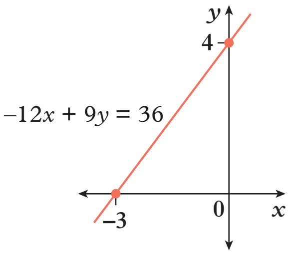

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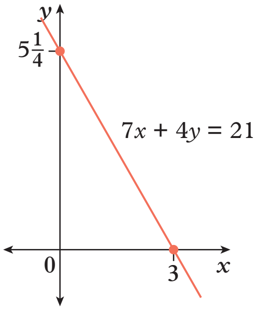

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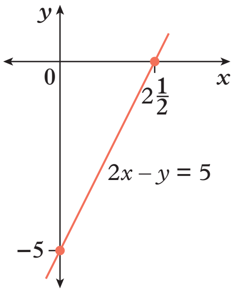
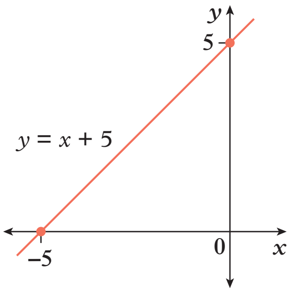


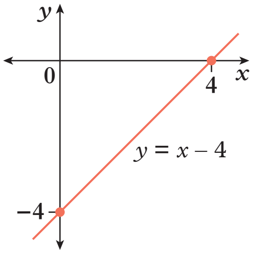
8 **a** **i** x-intercept: (–5, 0) **ii** y-intercept: (0, 5)

iii



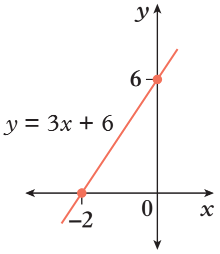
b **i** x-intercept: (4, 0) **ii** y-intercept: (0, –4)

iii



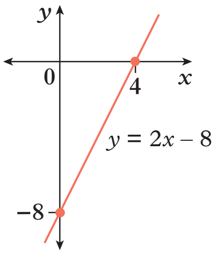
c **i** x-intercept: (–2, 0) **ii** y-intercept: (0, 6)

iii



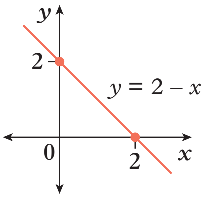
d **i** x-intercept: (4, 0) **ii** y-intercept: (0, –8)

iii



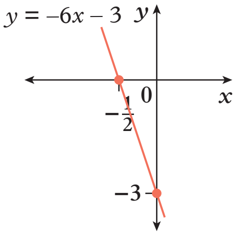
e **i** x-intercept: (2, 0) **ii** y-intercept: (0, 2)

iii



f **i** x-intercept: $\left(-\frac{1}{2}, 0\right)$ **ii** y-intercept: (0, –3)

iii



9 **a** gradient–intercept method

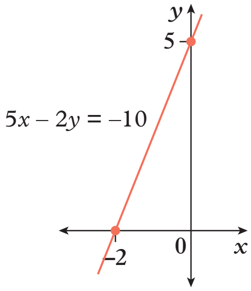
b x-and-y-intercept method

c gradient–intercept method

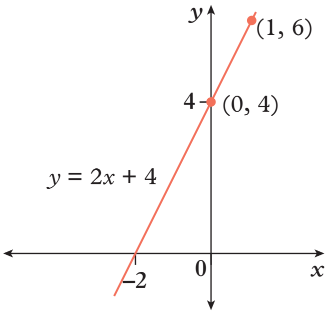
d Locate the y-intercept, and draw a line through that point parallel to the x-axis.

e Locate the x-intercept, and draw a line through that point parallel to the y-axis.

10 **a**



b



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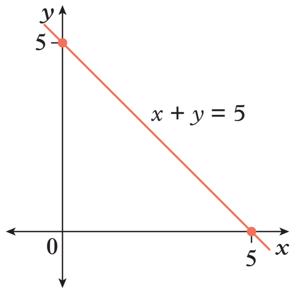

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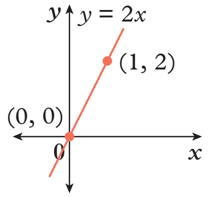

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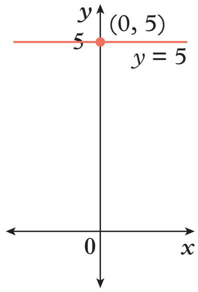

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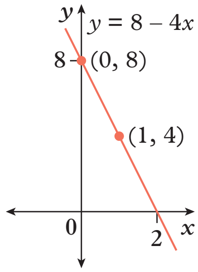

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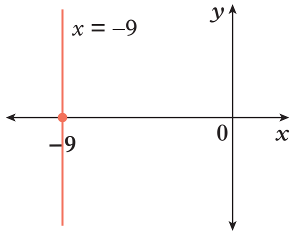

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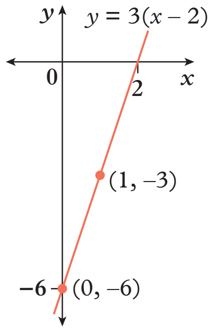

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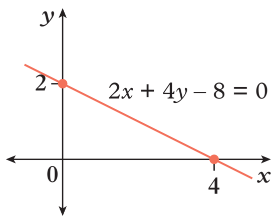

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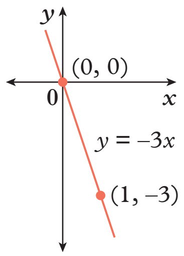

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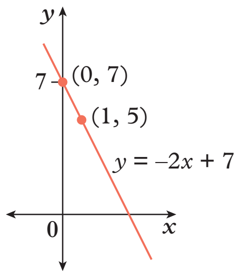

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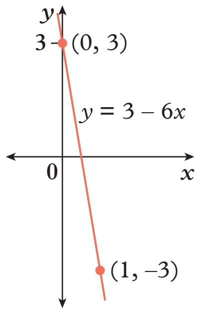


11 a $c = y_1 - mx_1$

b $y = mx + y_1 - mx_1$

c $y - y_1 = mx - mx_1$

$y - y_1 = m(x - x_1)$

12 a i $y - 4 = 3(x - 1)$ **ii** $y = 3x + 1$

b i $y - 9 = 2(x - 3)$ **ii** $y = 2x + 3$

c i $y - 2 = -4(x + 1)$ **ii** $y = -4x - 2$

d i $y - 7 = -1(x - 2)$ **ii** $y = -x + 9$

e i $y + 6 = 5(x + 3)$ **ii** $y = 5x + 9$

f i $y - 11 = -3(x + 4)$ **ii** $y = -3x - 1$

g i $y - 4 = \frac{6}{5}(x - 15)$ **ii** $y = \frac{6}{5}x - 14$

h i $y + 5 = -\frac{3}{4}(x - 8)$ **ii** $y = -\frac{3}{4}x + 1$

i i $y + 6 = \frac{5}{2}(x + 3)$ **ii** $y = \frac{5}{2}x + \frac{3}{2}$

j i $y - 4 = -\frac{7}{9}(x + 6)$ **ii** $y = -\frac{7}{9}x - \frac{2}{3}$

13 a i translation of 6 units up

ii translation of 3 units left

iii translation of 2 units left and 2 units up

b i $y - 6 = 2x$

ii $y = 2(x + 3)$

iii $y - 2 = 2(x + 2)$

c **i** translation of 1 unit left and 2 units down

ii translation of 1 unit right and 2 units up

iii translation of 1 unit right and 2 units up

iv translation of 4 units left and 6 units up

v translation of $\frac{3}{2}$ units right and $\frac{4}{3}$ units up

vi translation of 4 units right and 2 units up

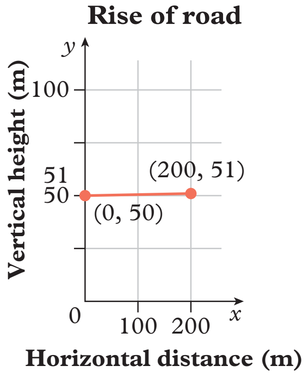
14 **a** $y = 3x - 11$ **b** $y = -2x + 6$ **c** $y = x + 2$

15 **a** **i** $y = \frac{1}{200}x + 50$

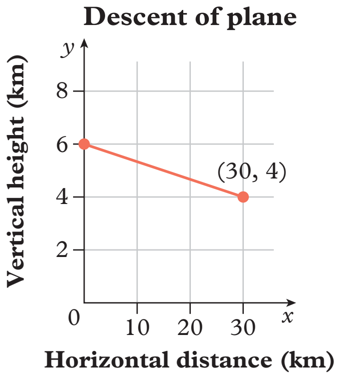
ii $y = -\frac{1}{15}x + 6$

iii $y = \frac{1}{5}x$

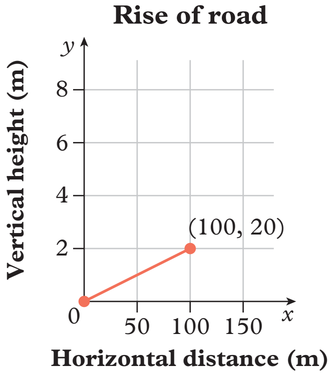
b **i** $y = \frac{1}{200}x + 50$, for $0 \le x \le 200$



ii $y = -\frac{1}{15}x + 6$, for $0 \le x \le 30$



iii $y = \frac{1}{5}x$, for $0 \le x \le 100$



16 **a** $x - 2y = 4$

b $y = -23$

c $2x + 3y + 20 = 0$

d $12x - 17y = 0$ or $17y = 12x$

17 **a** $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right)$ **b** $\frac{a}{c}x + \frac{b}{c}y = 1$

c The reciprocals of the coefficients of x and y $\left(\frac{a}{c} \text{ and } \frac{b}{c}\right)$ are the corresponding x and y values for the x- and y-intercept of the line. This is because when the equation is equal to 1, we multiply both sides of the equation by the reciprocal of the coefficient of either term when solving for the intercept.

d **i** $\frac{2}{5}x + \frac{3}{5}y = 1, \left(\frac{5}{2}, 0\right), \left(0, \frac{5}{3}\right)$

ii $\frac{4}{11}x - \frac{7}{11}y = 1, \left(\frac{11}{4}, 0\right), \left(0, -\frac{7}{11}\right)$

iii $\frac{1}{6}x - \frac{1}{4}y = 1, (6, 0), (0, -4)$

iv $\frac{1}{4}y - \frac{1}{18}x = 1, (-18, 0), (0, 4)$

v $-\frac{1}{4}x + \frac{1}{8}y = 1, (-4, 0), (0, 8)$

vi $2x + 5y = 1, \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{5}\right)$

18 **a** **i** $\frac{3}{2}x + \frac{7}{2}y = 1$ **ii** $3x + 7y = 2$

b **i** $-\frac{5}{3}x + \frac{2}{3}y = 1$ **ii** $-5x + 2y = 3$

c **i** $\frac{5}{3}x - \frac{5}{2}y = 1$ **ii** $10x - 15y = 6$

d **i** $-2x - 3y = 1$ **ii** $2x + 3y = -1$

e **i** $\frac{1}{7}x + \frac{1}{11}y = 1$ **ii** $11x + 7y = 77$

f **i** $x - y = 1$ **ii** $x - y = 1$

EX
p219

5B Gradient, midpoint and distance between two points

1 **a** **i** -2 **ii** $(1, 2)$ **iii** $2\sqrt{5}$

b **i** 1 **ii** $\left(-\frac{3}{2}, \frac{3}{2}\right)$ **iii** $3\sqrt{2}$

c **i** 2 **ii** $\left(\frac{3}{2}, -3\right)$ **iii** $3\sqrt{5}$

d **i** -5 **ii** $\left(\frac{1}{2}, \frac{5}{2}\right)$ **iii** $\sqrt{26}$

2 **a** **i** 4 **ii** $(1, 6)$ **iii** $2\sqrt{17}$

b **i** 1 **ii** $\left(\frac{3}{2}, -\frac{11}{2}\right)$ **iii** $3\sqrt{2}$

c **i** $\frac{1}{2}$ **ii** $(2, 6)$ **iii** $2\sqrt{5}$

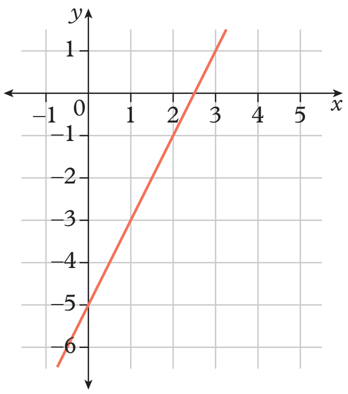
d **i** 10 **ii** $\left(-\frac{1}{2}, 1\right)$ **iii** $\sqrt{101}$

e **i** -3 **ii** $(-1, -2)$ **iii** $2\sqrt{10}$

f **i** $-\frac{1}{4}$ **ii** $(-4, 5)$ **iii** $2\sqrt{17}$

3 **a** **i** $(0, -5), \left(\frac{5}{2}, 0\right)$

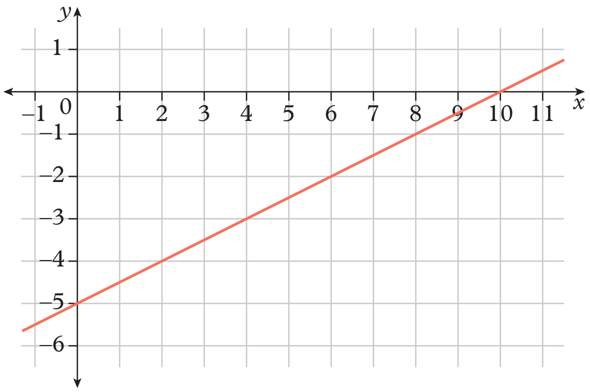
ii



iii $\left(\frac{5}{4}, -\frac{5}{2}\right)$ **iv** $\frac{5\sqrt{5}}{2}$

b **i** $(0, -5), (10, 0)$

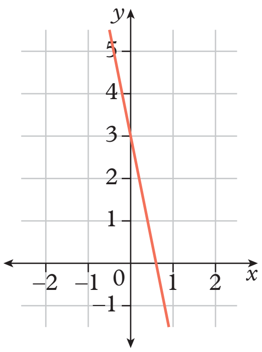
ii



iii $\left(5, -\frac{5}{2}\right)$ **iv** $5\sqrt{5}$

c **i** $(0, 3), \left(\frac{5}{3}, 0\right)$

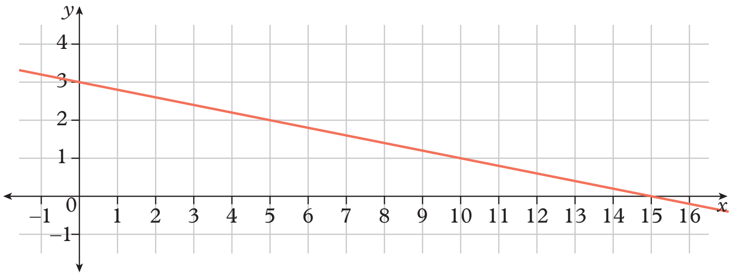
ii



iii $\left(\frac{5}{6}, \frac{3}{2}\right)$ **iv** $\frac{\sqrt{106}}{3}$

d **i** $(0, 3), (15, 0)$

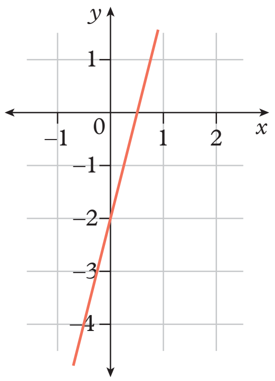
ii



iii $\left(\frac{7}{2}, \frac{3}{2}\right)$ **iv** $3\sqrt{26}$

e **i** $(0, -2), \left(\frac{1}{2}, 0\right)$

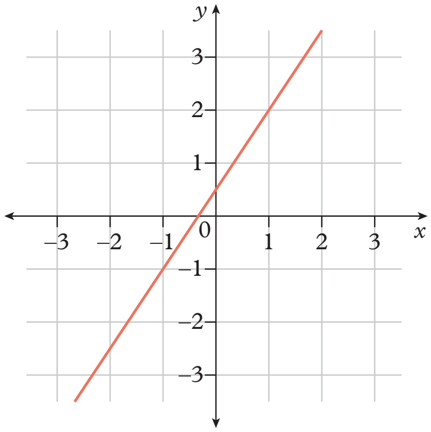
ii



iii $\left(\frac{1}{4}, -1\right)$ **iv** $\frac{\sqrt{17}}{2}$

f **i** $\left(0, \frac{1}{2}\right), \left(-\frac{1}{3}, 0\right)$

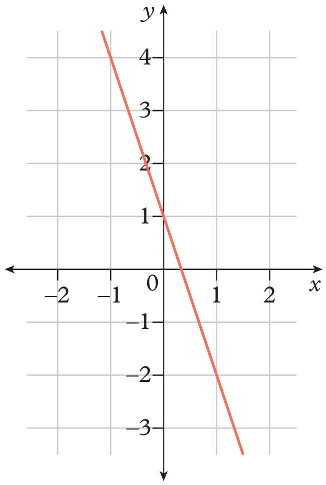
ii



iii $\left(-\frac{1}{6}, \frac{1}{4}\right)$ **iv** $\frac{\sqrt{13}}{12}$

g **i** $(0, 1), \left(\frac{1}{3}, 0\right)$

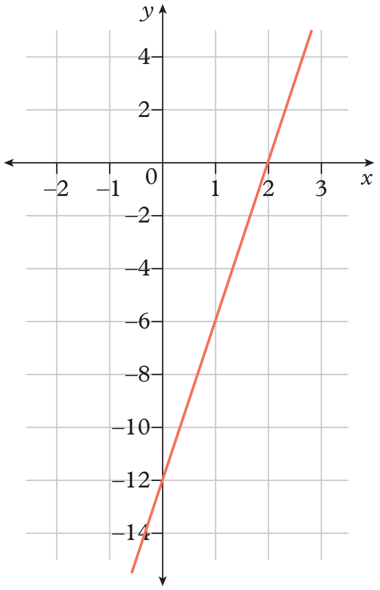
ii



iii $\left(\frac{1}{6}, \frac{1}{2}\right)$ **iv** $\frac{\sqrt{10}}{3}$

h **i** (0, 12), (2, 0)

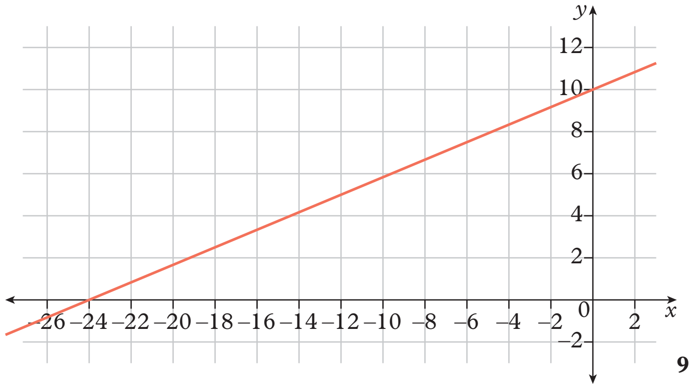
ii



iii (1, 6) **iv** $2\sqrt{37}$

i **i** (0, 10), (−24, 0)

ii



iii (−12, 5) **iv** 26

4 **a** gradient = 5, midpoint (1, 2), distance $2\sqrt{26}$

b gradient = −2, midpoint $\left(\frac{3}{2}, 1\right)$, distance $3\sqrt{5}$

c gradient = $\frac{4}{5}$, midpoint $\left(-\frac{5}{2}, 0\right)$, distance $\sqrt{41}$

d gradient = 3, midpoint (4, 1), distance $4\sqrt{10}$

e gradient = −2, midpoint $\left(\frac{5}{2}, 1\right)$, distance $3\sqrt{5}$

f gradient = 1, midpoint $\left(-\frac{1}{2}, \frac{3}{2}\right)$, distance $\sqrt{50}$

5 **a** $\frac{\sqrt{5}}{2}$ **b** $4\sqrt{37}$ **c** $\frac{10\sqrt{10}}{3}$

d $2\sqrt{5}$ **e** $\sqrt{2}$ **f** $6\sqrt{2}$

g $8\sqrt{5}$ **h** $\sqrt{74}$ **i** $4\sqrt{37}$

6 **a**
$$\begin{aligned} m &= \frac{5-2}{4-(-2)} & \text{and } m &= \frac{2-5}{-2-4} \\ &= \frac{3}{6} & &= \frac{-3}{-6} \\ &= \frac{1}{2} & &= \frac{1}{2} \end{aligned}$$

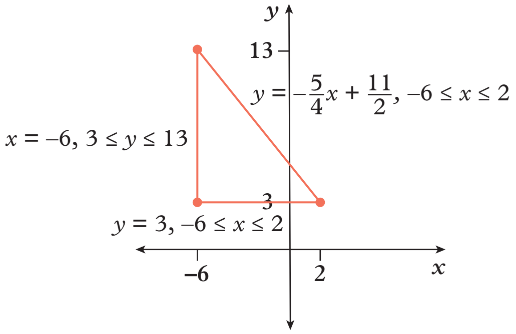
b Juan has not vertically aligned the corresponding x- and y-coordinates of the same point, so he will end up with the negative of the actual gradient, $-m = -\frac{1}{2}$.

c
$$\begin{aligned} m &= \frac{y_1 - y_2}{x_1 - x_2} \\ &= \frac{y_1 - y_2}{x_1 - x_2} \times \frac{-1}{-1} \\ &= \frac{-(y_1 - y_2)}{-(x_1 - x_2)} \\ &= \frac{-y_1 + y_2}{-x_1 + x_2} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

7 **a**
$$\begin{aligned} 2 &= \tfrac{1}{2}(-2) + c & \text{and} \quad 5 &= \tfrac{1}{2}(4) + c \\ 2 &= -1 + c & 5 &= 2 + c \\ c &= 3 & c &= 3 \end{aligned}$$

b Juan has substituted the x-coordinate of one of the points for the gradient and the corresponding y-coordinate for the y-intercept.

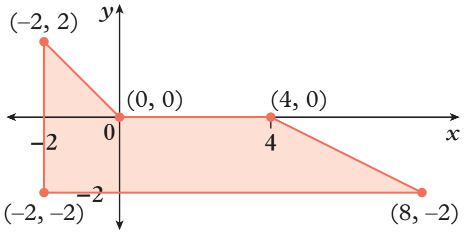
8 **a** **i, ii, iii**



b perimeter = 30.81 units

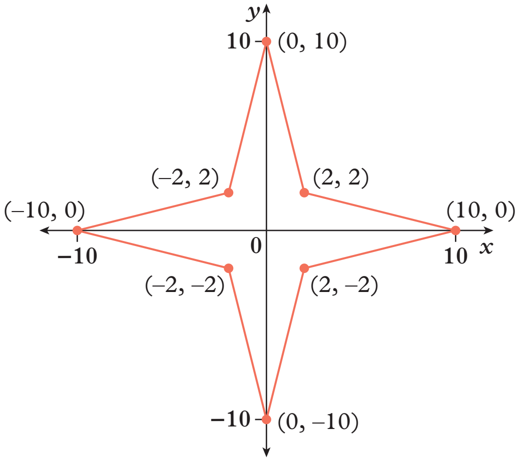
c area = 40 units^2

9 **a**



b 25.3 units **c** 18 square units

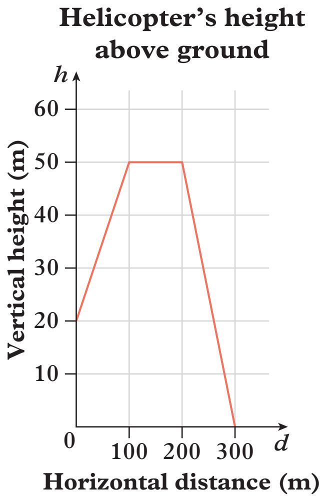
10 **a**



b 66 units **c** 80 square units

d $y = x$ **e** $y = -x$

11 a



b In the order of the flight: $h = 0.3d + 20$, $h = 50$, $h = -0.5d + 150$

c **i** 29 m **ii** 44 m

iii 50 m **iv** 30 m

d $66\frac{2}{3}$ m and 220 m

12 a $PA^2 = (3 - y)^2 + (4 - x)^2$,

$PC^2 = (4 - y)^2 + (-3 - x)^2$

Because $PA = PB$, $PA^2 = PB^2$

$$(3 - y)^2 + (4 - x)^2 = (4 - y)^2 + (-3 - x)^2$$
$$9 - 6y + y^2 + 16 - 8x + x^2 = 16 - 8y + y^2 + 9 + 6x + x^2$$
$$-6y - 8x = -8y + 6x$$
$$2y = 14x$$
$$y = 7x$$

b (0, 0), (1, 7)

c 20

EX p226

5C Graphing parabolas using intercepts

1 a **i** (–6, 0), (0, 0) **ii** (0, 0)

b **i** (0, 0), (10, 0) **ii** (0, 0)

c **i** (1, 0), (5, 0) **ii** (0, 5)

d **i** (–1, 0), (1, 0) **ii** (0, –1)

e **i** (–6, 0), (2, 0) **ii** (0, 12)

f **i** (–3, 0), (3, 0) **ii** (0, 9)

2 a upright **b** inverted **c** upright

d upright **e** inverted **f** inverted

3 a $x = -3$ **b** $x = 5$ **c** $x = -3$

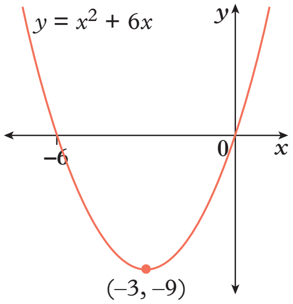
d $x = 0$ **e** $x = -2$ **f** $x = 0$

4 a (–3, –9) **b** (5, 25) **c** (3, –4)

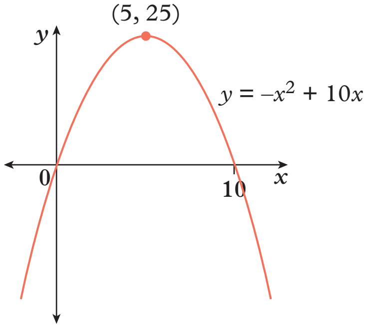
d (0, –1) **e** (–2, 16) **f** (0, 9)

5 a (–1, –16) **b** $\left(\frac{1}{2}, 12\frac{1}{4}\right)$

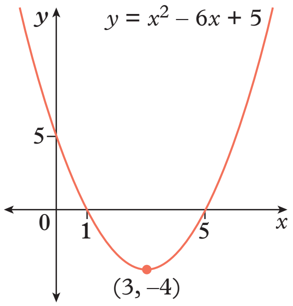
6 a



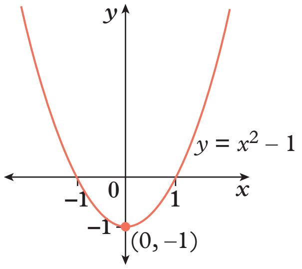
b



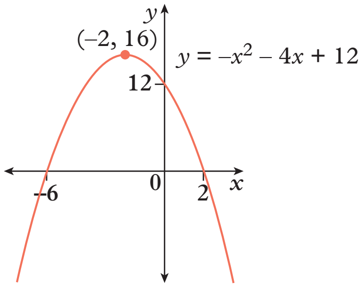
c



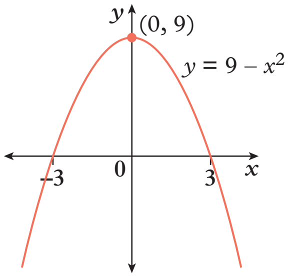
d



e

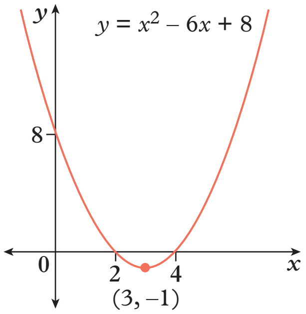


f



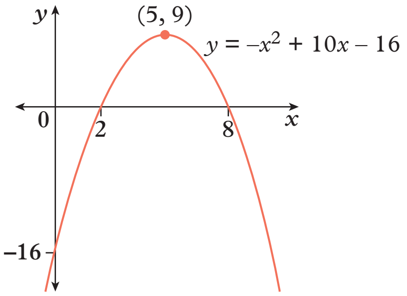
7 **a** Domain is $\mathbb{R}$.

Range is $y \geq -1$.



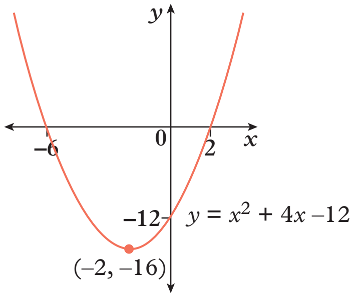
b Domain is $\mathbb{R}$.

Range is $y \leq 9$.



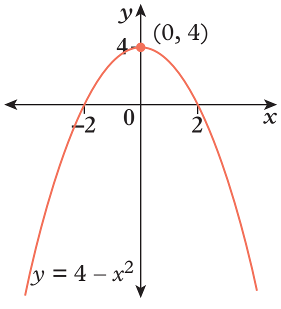
c Domain is $\mathbb{R}$.

Range is $y \geq -16$.



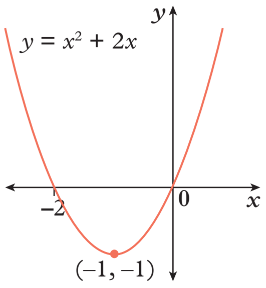
d Domain is $\mathbb{R}$.

Range is $y \leq 4$.



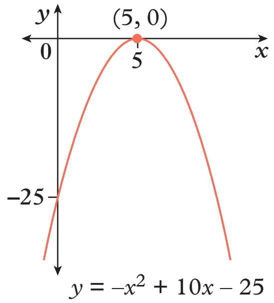
e Domain is $\mathbb{R}$.

Range is $y \geq -1$.



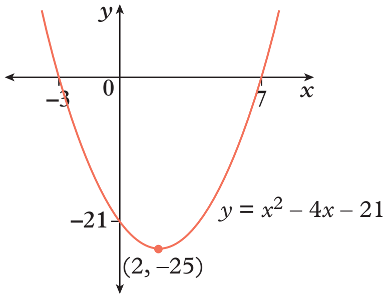
f Domain is $\mathbb{R}$.

Range is $y \leq 0$.



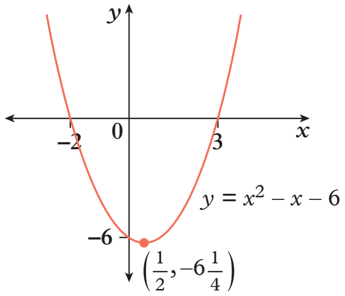
g Domain is $\mathbb{R}$.

Range is $y \geq -25$.



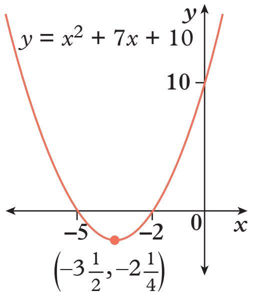
h Domain is $\mathbb{R}$.

Range is $y \geq -6\frac{1}{4}$.



i Domain is $\mathbb{R}$.

Range is $y \geq -2\frac{1}{4}$.



8 **a** **i** two **ii** one **iii** none

b **i** $x = -1$ or 2 **ii** $x = 2$ **iii** no solution

9 **a** $\left(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0\right), \left(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0\right)$

b $x = -\frac{b}{2a}$

c The formula $x = -\frac{b}{2a}$ for the axis of symmetry applies for all parabolas.

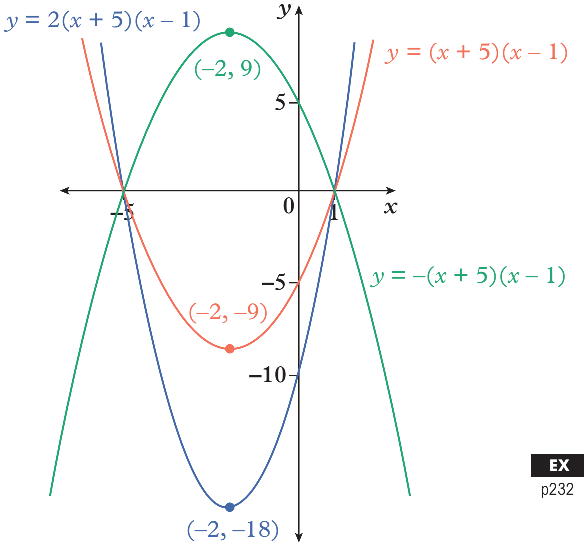
10 **a** For all these graphs, the x-intercepts are at $(-5, 0)$ and $(1, 0)$.

b The x-intercepts are the points when $y = 0$. For these equations, when $y = 0$ the equations all have the same solution for x. But, for other y values, the x values differ.

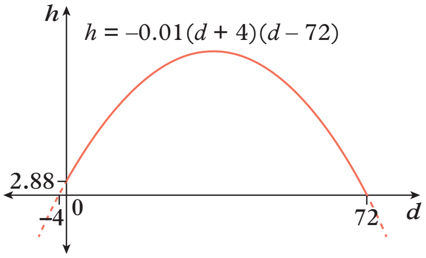
c For $y = (x + 5)(x - 1)$, the y-intercept is at $(0, -5)$.
For $y = 2(x + 5)(x - 1)$, the y-intercept is at $(0, -10)$.
For $y = -(x + 5)(x - 1)$, the y-intercept is at $(0, 5)$.

d For $y = (x + 5)(x - 1)$, the turning point is $(-2, -9)$.
For $y = 2(x + 5)(x - 1)$, the turning point is $(-2, -18)$.
For $y = -(x + 5)(x - 1)$, the turning point is $(-2, 9)$.

e



11 **a**

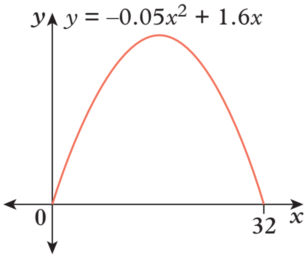


b 2.88 m

c 72 m

d 14.44 m

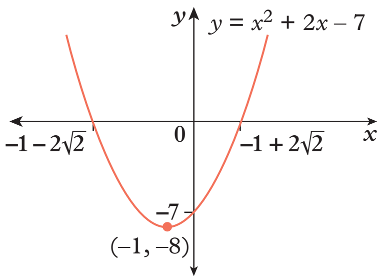
12 **a**



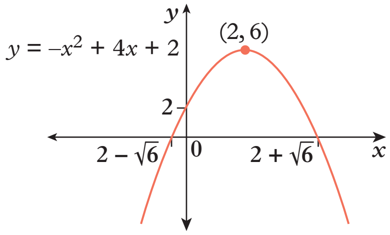
b height of 12.8 m at horizontal distance of 16 m

c 32 m **d** 30.8 m

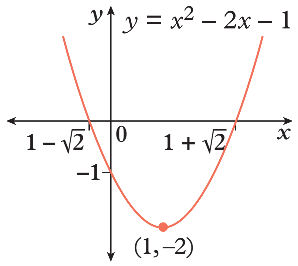
13 **a**



b



c

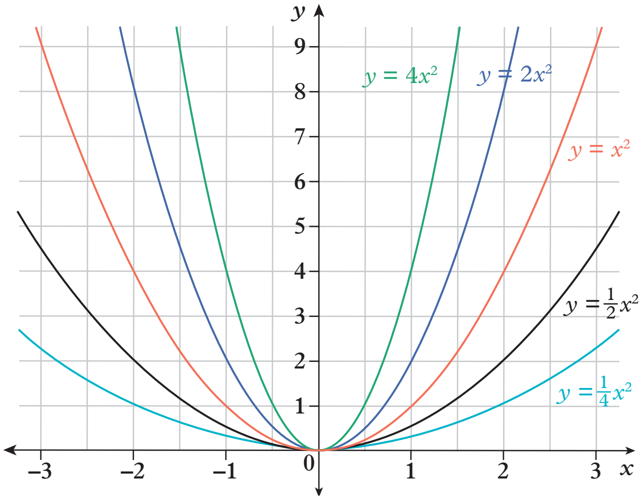


14 **a** $y = (x - 1)(x - 3)$ **b** $y = 2(x + 1)(x - 2)$

c $y = -2(x - 2)(x + 3)$

EX p232 **5D** Graphing parabolas using transformations

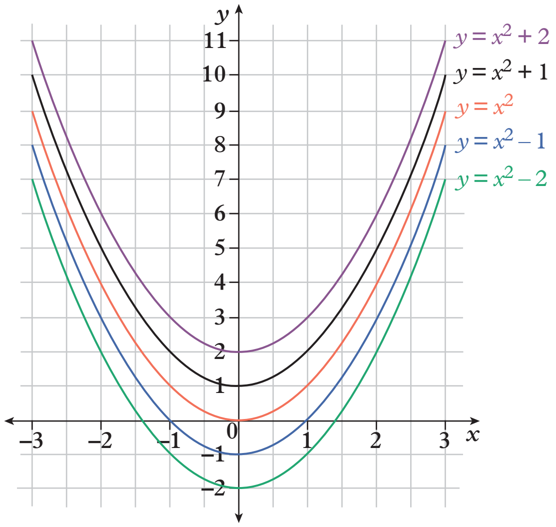
1 **a** **i**



ii All the graphs have the same minimum turning point (0, 0), the same axis of symmetry and the same x- and y-intercepts (same point as the turning point). The graphs of $y = 2x^2$ and $y = 4x^2$ are narrower than the graph of $y = x^2$; and the graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ are wider than the graph of $y = x^2$.

iii stretching; a graph is narrower when the coefficient of x^2 is greater than 1; a graph is wider when the coefficient of x^2 is between 0 and 1.

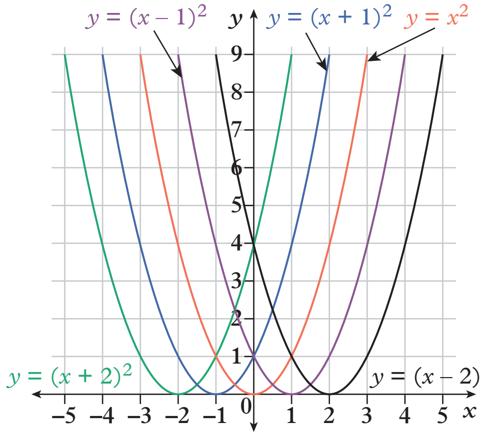
b **i**



ii All graphs have the same axis of symmetry and the same shape but they have different turning points.

iii vertical translation; the graph of $y = x^2$ moves up when a positive constant is added; the graph of $y = x^2$ moves down when a negative constant is added (or a positive constant is subtracted)

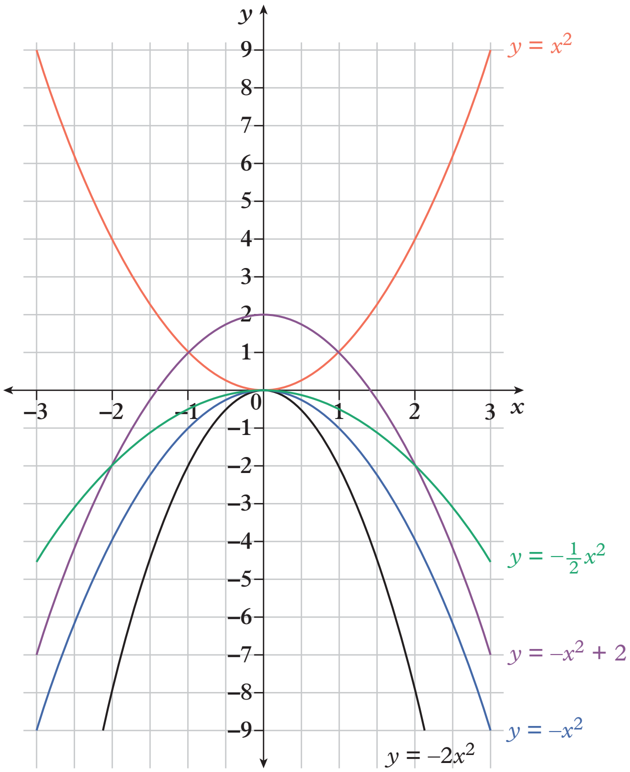
c **i**



ii All the graphs have the same shape but different turning points.

iii horizontal translation; the graph of $y = x^2$ moves to the right when a positive constant is subtracted from x before squaring; the graph of $y = x^2$ moves to the left when a negative constant is subtracted from x (or a positive constant is added to x) before squaring

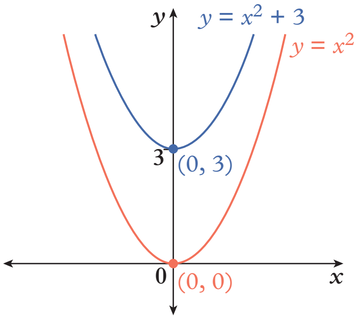
d **i**



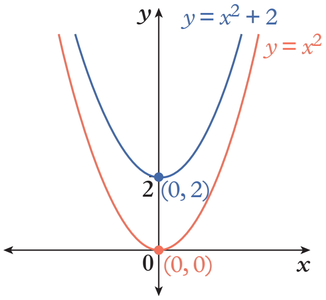
ii The graph of $y = -x^2$ is a reflection in the x-axis of the graph of $y = x^2$, but it has the same shape, the same axis of symmetry and the coordinates of its turning point (0, 0) are the same as for the graph of $y = x^2$. The graph of $y = -x^2 + 2$ results from the graph of $y = x^2$ being reflected in the x-axis and moved 2 units up. The graph of $y = -2x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is narrower. The graph of $y = -\frac{1}{2}x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is wider.

iii All the graphs have been reflected in the x-axis when compared to the graph of $y = x^2$ because each has a negative coefficient of x^2. There is also a vertical translation for $y = -x^2 + 2$ because a constant is added to x^2; there is a dilation for $y = -2x^2$ (it is narrower than $y = x^2$ because its coefficient of x^2 is greater than 1); and there is a dilation for $y = -\frac{1}{2}x^2$ (it is wider than $y = x^2$ because its coefficient of x^2 is between 0 and 1).

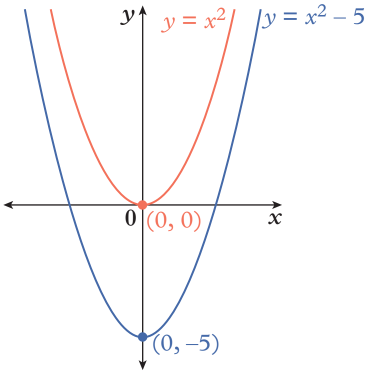
2 **a**



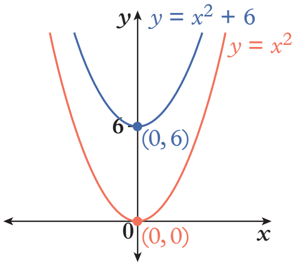
b



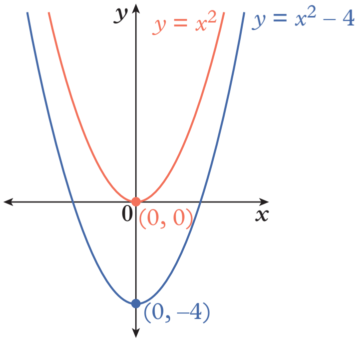
c



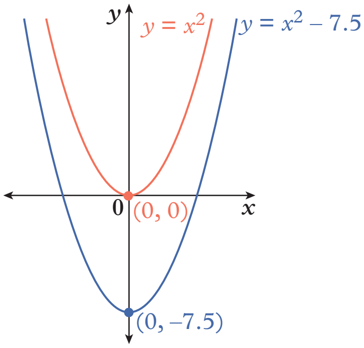
d



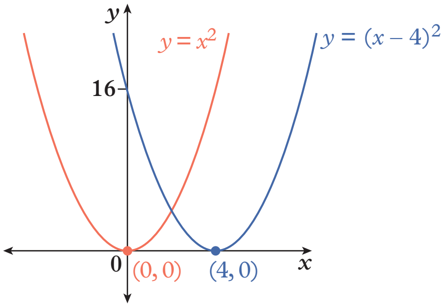
e



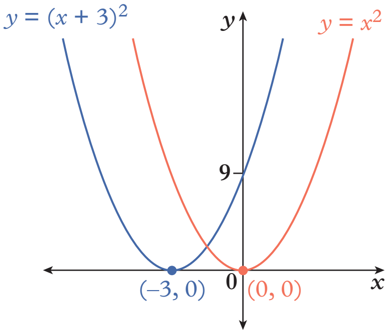
f



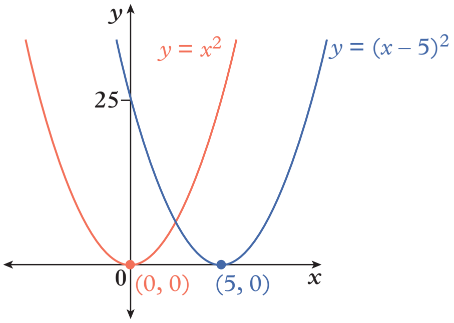
3 **a**



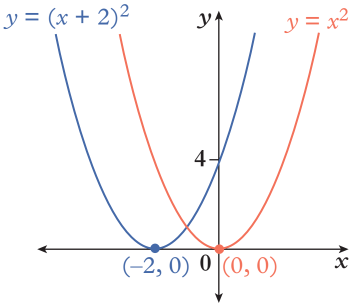
b



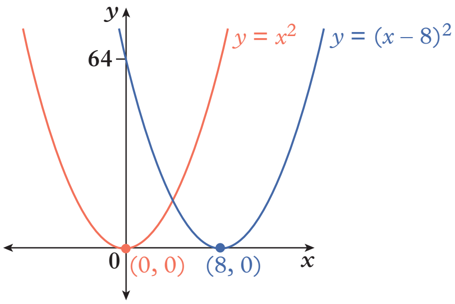
c



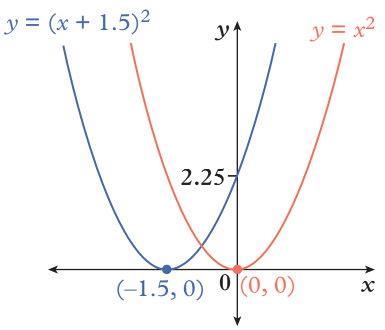
d



e



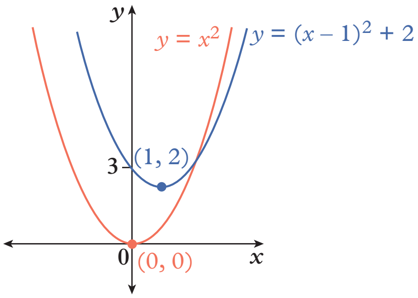
f



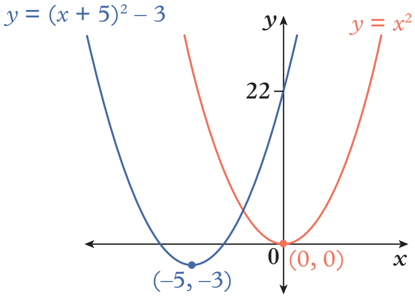
4 **a** II **b** IV **c** I
d VI **e** V **f** III

5 **a** $y = 6x^2$ **b** $y = -x^2$ **c** $y = (x - 5)^2$
d $y = x^2 - 2$ **e** $y = -\frac{1}{4}x^2$ **f** $y = -x^2 + 3$

6 **a**



b



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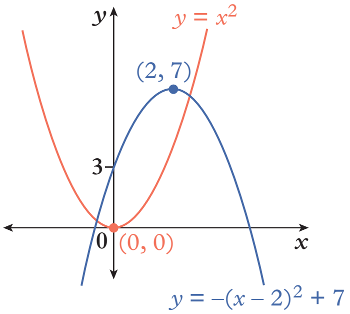
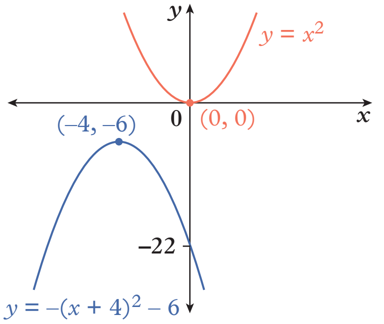


d



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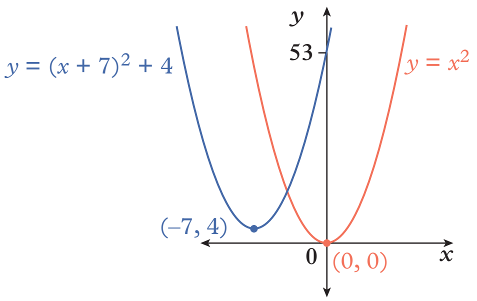
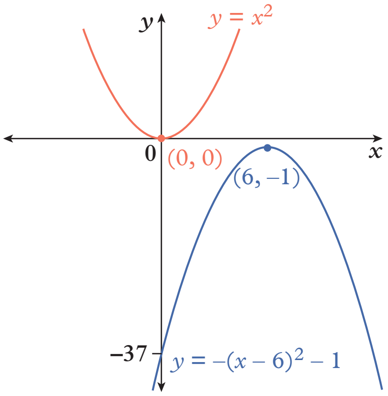
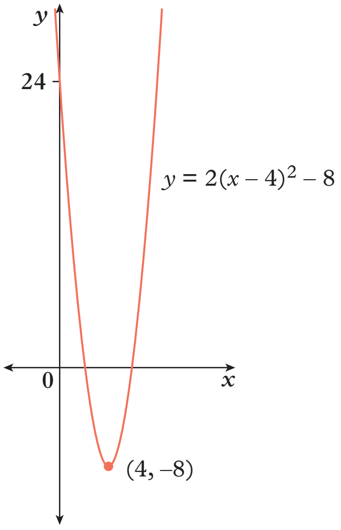


f



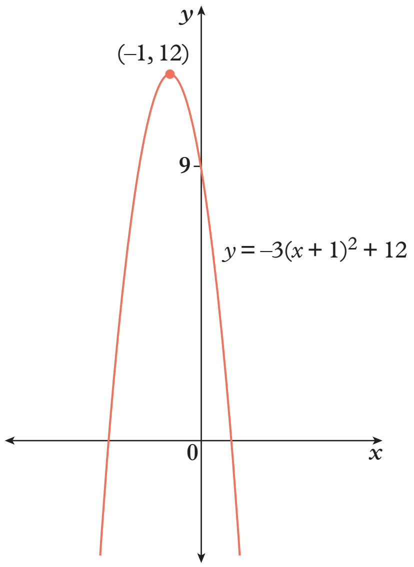
7 **a** The quadratic relationship is always positive
b $x < -5 - \sqrt{3}$ and $x > -5 + \sqrt{3}$
c $2 - \sqrt{7} < x < 2 + \sqrt{7}$
d The quadratic relationship is never positive
e The quadratic relationship is always positive
f The quadratic relationship is never positive

8 **a** **i** narrower **ii** upright **iii** $(4, -8)$
iv



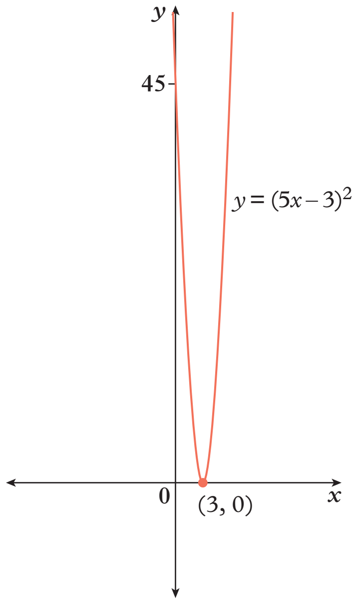
b **i** narrower **ii** inverted **iii** (–1, 12)

iv



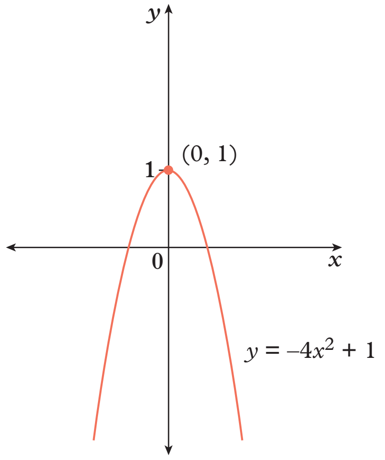
c **i** narrower **ii** upright **iii** (3, 0)

iv



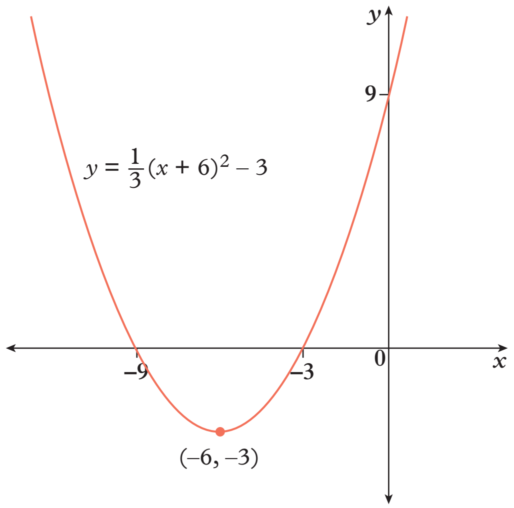
d **i** narrower **ii** inverted **iii** (0, 1)

iv



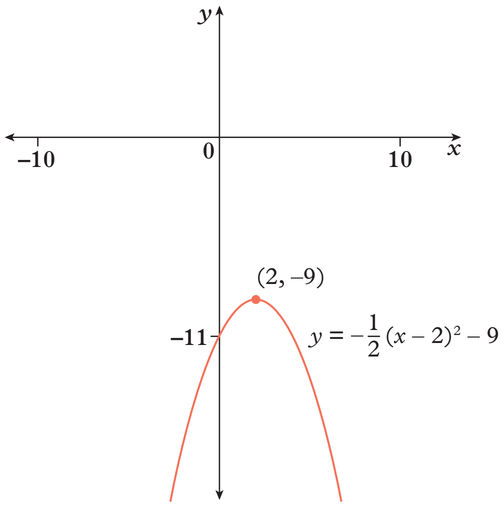
e **i** wider **ii** upright **iii** (–6, –3)

iv



f **i** wider **ii** inverted **iii** (2, –9)

iv



9 **a** $y = 2(x - 5)^2$ **b** $y = -(x + 4)^2 - 3$

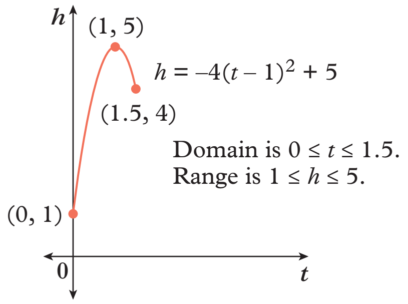
c $y = -\frac{1}{3}x^2$ **d** $y = -5x^2 + 7$

10 4 is the smallest, because the graph is upright and has a minimum turning point at (–2, 4). There is no largest value because the y values increase as the x values move away from –2. Thus, the range is $y \geq 4$.

11 –1 is the largest, because the graph is inverted and has a maximum turning point at (6, –1). There is no smallest value because the y values decrease as the x values move away from 6. Thus, the range is $y \leq -1$.

12 **a** (1, 5)

b

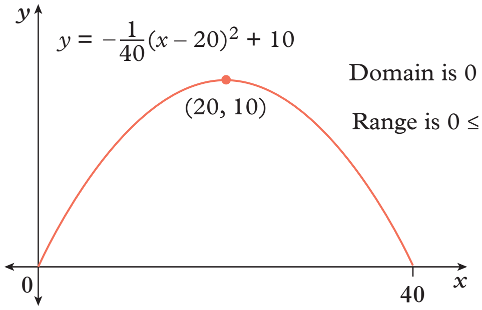


Domain is $0 \leq t \leq 1.5$.
Range is $1 \leq h \leq 5$.

c **i** 1 m **ii** 5 m **iii** 4 m

d 2.1 seconds

13 **a**



Domain is $0 \leq x \leq 40$.

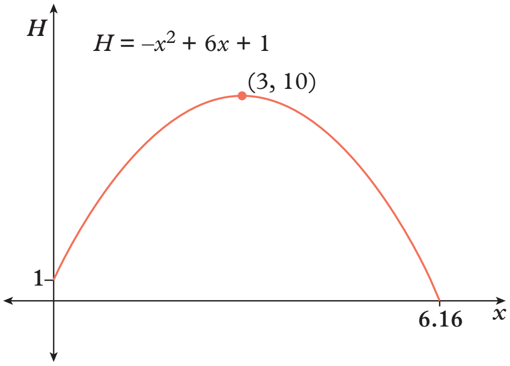
Range is $0 \leq y \leq 10$.

b 10 m **c** 40 m

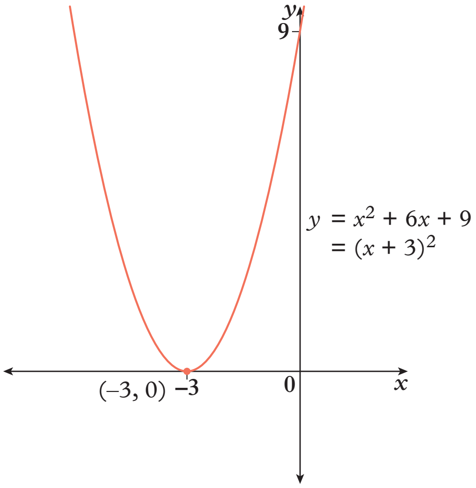
14 **a** 1 metre **b** $-(x - 3)^2 + 10$

c 10 metres **d** 6.16 metres

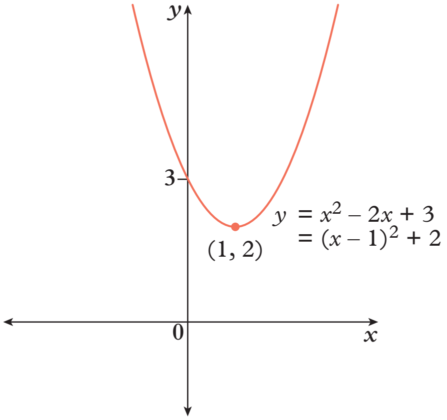
e



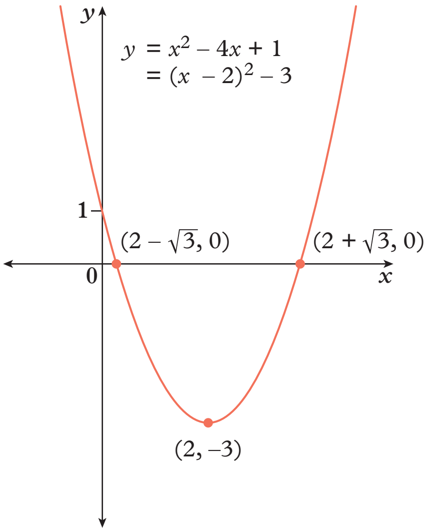
15 **a**



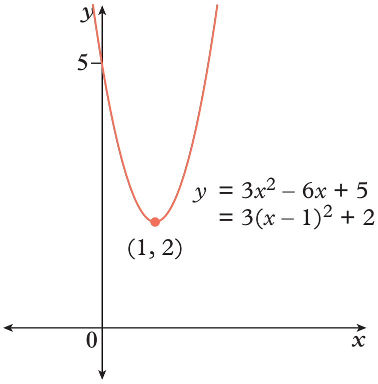
b



c



d



16 **a** $y = (x - 2)^2 + 6$ **b** $y = 2(x + 4)^2 - 12$

c $y = -3(x + 1)^2 + 7$ **d** $y = -\frac{1}{3}(x - 3)^2 - 5$

17 **a** $(2 - x)^2 = (2 - x)(2 - x)$

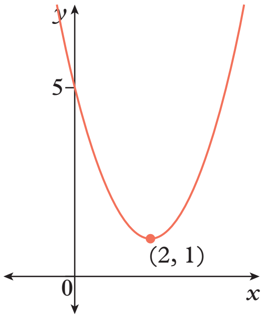
$= 4 - 2x - 2x + x^2$

$= x^2 - 4x + 4$

$= (x - 2)(x - 2)$

$= (x - 2)^2$

b **i**



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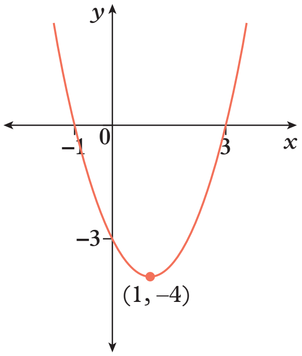

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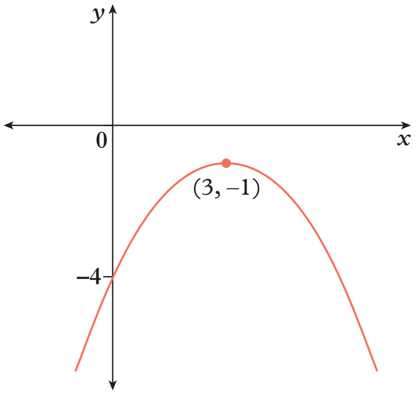
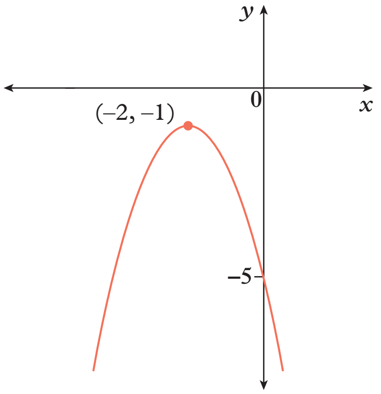


iv



EX p238

5E Cubic and other non-linear relationships

1 **a** **i** (0, 0) **ii** (0, 0)

b **i** (2, 0) **ii** (0, 8)

c **i** (0, 0) **ii** (0, 0)

d **i** (2, 0) **ii** (0, –32)

2 **a** **i** (0, 0), (0, 0) **ii** (2, 0), (2, 0)

iii (0, –5), (0, –5) **iv** (3, 3), (3, 3)

b **i** (2, 0), (0, –8) **ii** (4, 0), (2, 8)

iii (2, –5), (0, 3) **iv** (5, 3), (3, 11)

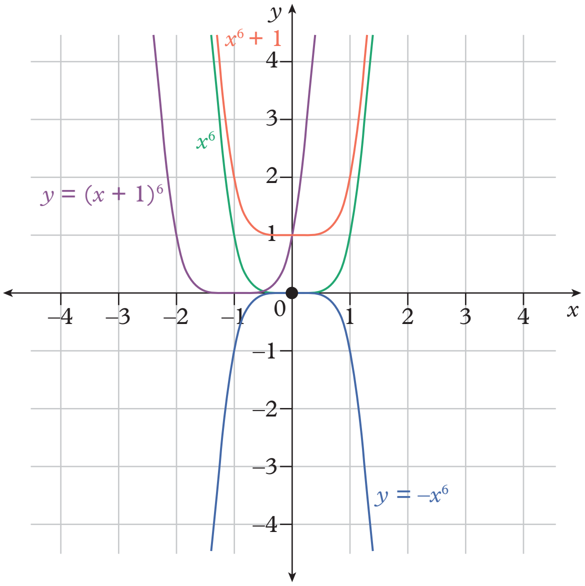
c **i** (0, 0), (0, 0) **ii** (2, 0), (2, 0)

iii (0, –5), (0, –5) **iv** (3, 3), (3, 3)

d **i** (2, 0), (0, 32) **ii** (4, 0), (2, –32)

iii (2, –5), (0, –37) **iv** (5, 3), (3, –29)

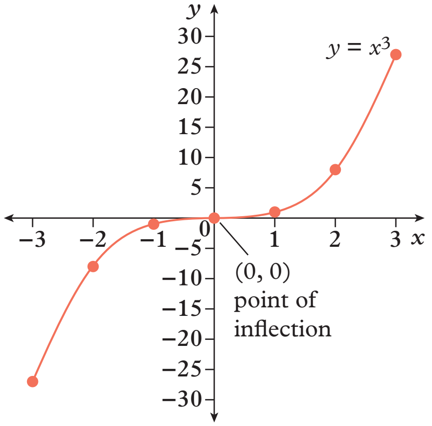
3



4 **a**

x	–3	–2	–1	0	1	2	3
y	–27	–8	–1	0	1	8	27

b



c As x increases, y increases. Graph appears to level out at $x = 0$ before rising more steeply again.

d **i** (0, 2) **ii** translate graph 2 units up

e **i** translate 1 unit up

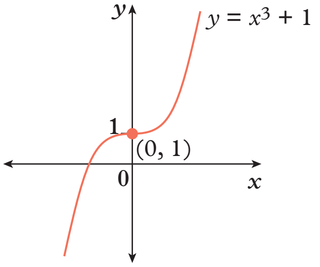
ii translate 3 units down

iii translate 2 units right

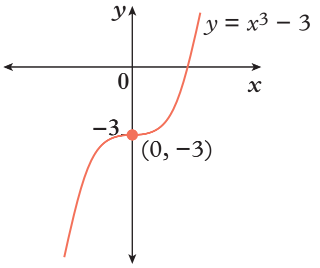
iv translate 4 units left

v translate 1 unit right and 2 units up

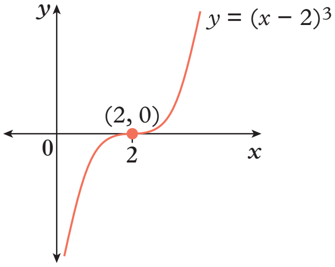
f **i**



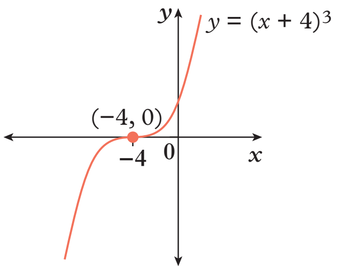
ii



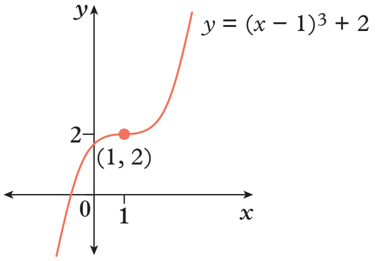
iii



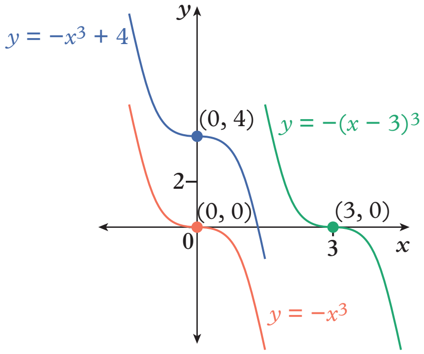
iv



v



5



6 **a** To sketch the graph of $y = x^5 + 20$, the graph of $y = x^5$ is translated 20 units up.

b To sketch the graph of $(x - 15)^5$ the graph of $y = x^5$ is translated 15 units right.

c To sketch the graph of $y = -1 + x^4$, the graph of $y = x^4$ is translated 1 units down.

d To sketch the graph of $y = (x + 2)^4$, the graph of $y = x^4$ is translated 2 units left.

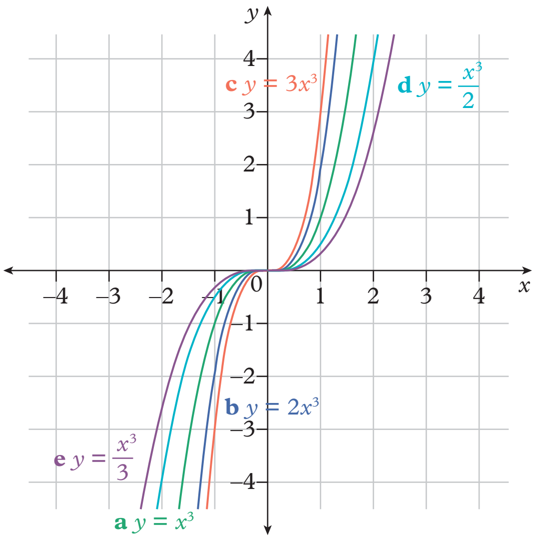
7 **a** $y = 3 \times x^6$

b $y = -x^6$

c $y = (x - 3)^6$

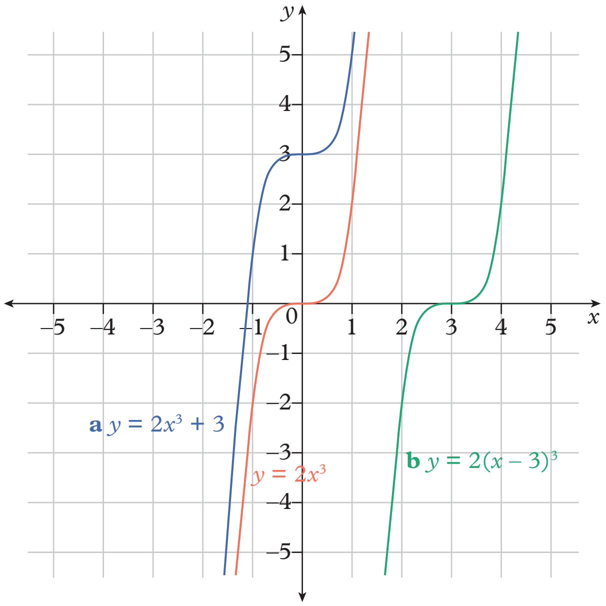
d $y = x^6 - 2$

8 **a-e**



9 **a** II **b** III **c** I

10



11 **a** Dilation in the y-direction by a factor of 2.

b Dilation in the y-direction by a factor of $\frac{1}{3}$.

c Horizontal translation of 2 units to the right.

12 **a**

x	−2	−1	0	1	2
$y = x^2$	4	1	0	1	4
$y = x^3$	−8	−1	0	1	8
$y = x^4$	16	1	0	1	16
$y = x^5$	−32	−1	0	1	32
$y = x^6$	64	1	0	1	64

b **i** III **ii** II **iii** IV

iv I **v** V

c The x-and y-intercepts of all curves are (0, 0).

d **i** (0, 0), (1, 1) (−1, 1)

ii (0, 0), (1, 1) (−1, −1)

e When n is an even number, x^n can be written as x^{2m} where m is an integer. x^{2m} can be rewritten as $(x^2)^m$ using the index law. Substitute (a, b) into $(x^2)^m$, $b = (a^2)^m$. Substitute $(-a)$ into $(x^2)^m$:

$$\begin{aligned}(x^2)^m &= ((-a)^2)^m\\ &= (a^2)^m\\ &= b\end{aligned}$$

So if (a, b) is on the curve, $(-a, b)$ is also on the curve.

f When n is an odd number, x^n can be written as x^{2m+1} where m is an integer. x^{2m+1} can be rewritten as $x \times (x^2)^m$ using the index laws. Substitute (a, b) into $x \times (x^2)^m$, $b = a \times (a^2)^m$. Substitute $(-a)$ into $x \times (x^2)^m$:

$$\begin{aligned}x \times (x^2)^m &= -a \times (-a)^2)^m\\ &= -a \times (a^2)^m\\ &= -(a \times (a^2)^m)\\ &= -b\end{aligned}$$

So if (a, b) is on the curve, $(-a, -b)$ is also on the curve.

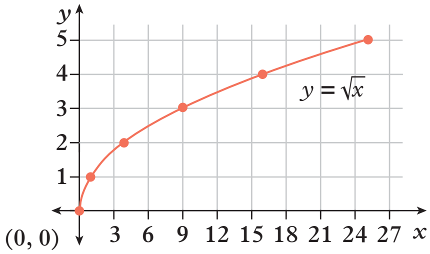
g **i** When n is even, the graph is symmetrical about the y-axis (reflect onto itself).

ii When n is odd, the graph has point symmetry about the origin (rotate 180 degrees about the origin onto itself).

13 **a**

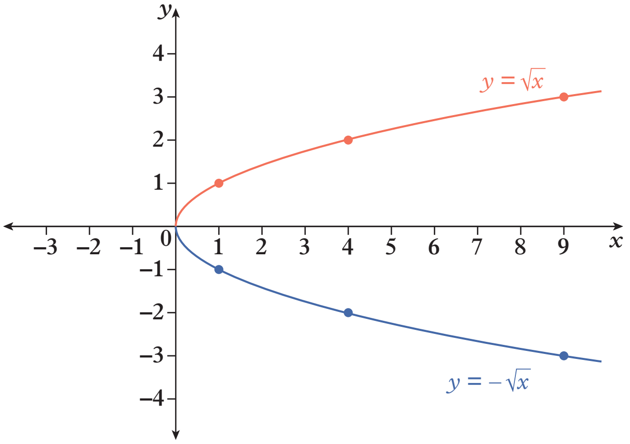
x	0	1	4	9	16	25
y	0	1	2	3	4	5

b, d



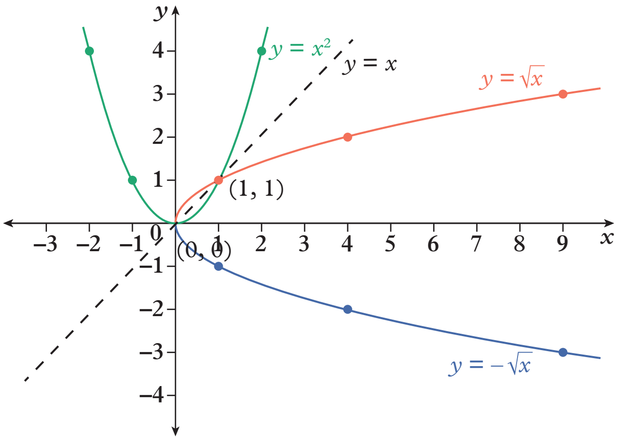
c It is a smooth curve with a minimum value at (0, 0); the curve is half a parabola on its side. No points can be graphed for negative values of x because it is not possible to find the square root of a negative number.

e



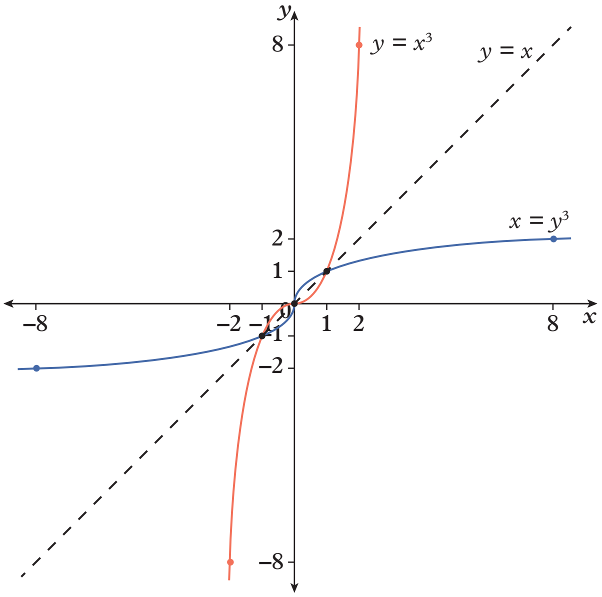
f $y = \pm\sqrt{x}$ or $y^2 = x$, the graph is a parabola on its side.

g, h



i The graph of $y^2 = x$ and $y = x^2$ are reflected in the line $y = x$, or the graph of $y = x^2$ is rotated 90 degrees clockwise to make $y^2 = x$.

14



EX p241

Checkpoint

1 **a** $y = -6x + 10$ **b** $y = -\frac{1}{2}x - \frac{3}{2}$

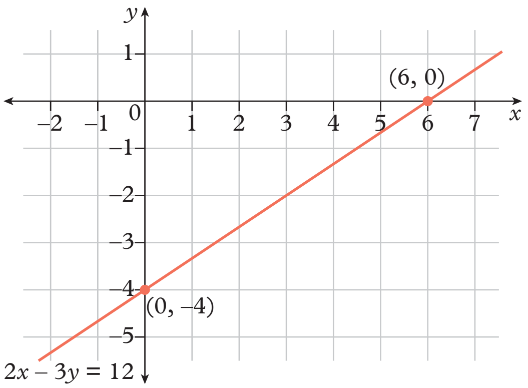
c $y = -x - 1$ **d** $y = x - 5$

2 **a** $-2x + y + 5 = 0$ **b** $-x + y + 5 = 0$

c $-6x + 10y - 7 = 0$ **d** $-10x + 15y + 1 = 0$

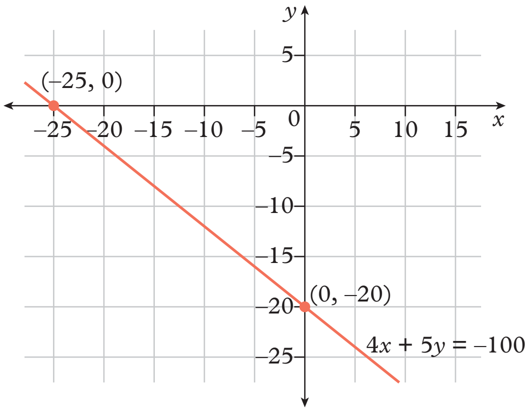
3 **a** **i** $(0, -4), (6, 0)$

ii



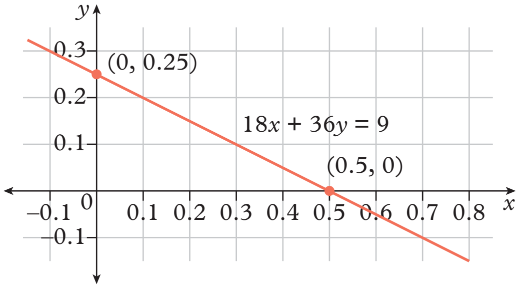
b **i** $(0, -20), (-25, 0)$

ii



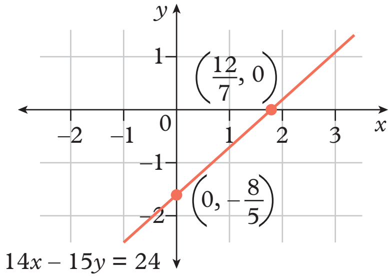
c **i** $\left(0, \frac{1}{4}\right), \left(\frac{1}{2}, 0\right)$

ii



d **i** $\left(0, \frac{12}{7}\right), \left(-\frac{8}{5}, 0\right)$

ii



4 **a** $y = 10$

b $y = 2x + 11$

c $y = -x + 9$

d $y = \frac{x}{3} - 2$

5 **a** **i** $\frac{1}{2}$ **ii** $(5, 9)$ **iii** $2\sqrt{5}$

b **i** $\frac{1}{5}$ **ii** $(3, 0)$ **iii** $2\sqrt{101}$

c **i** $-\frac{9}{4}$ **ii** $(1, 1)$ **iii** $2\sqrt{97}$

d **i** $-\frac{7}{4}$ **ii** $(-1, 5)$ **iii** $2\sqrt{65}$

6 **a** **i** $\frac{3}{2}$ **ii** $\left(2, \frac{7}{2}\right)$ **iii** $\sqrt{13}$

b **i** $-\frac{5}{6}$ **ii** $\left(-1, -\frac{1}{2}\right)$ **iii** $\sqrt{61}$

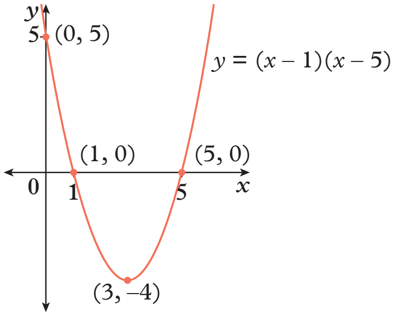
7 **a** x-intercepts: $(0, 0)$ and $(4, 0)$, y-intercept: $(0, 0)$

b x-intercepts: $(-3, 0)$ and $(1, 0)$, y-intercept: $(0, -3)$

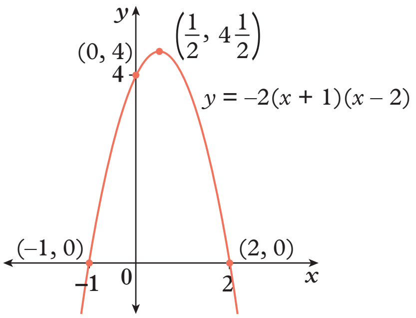
c x-intercepts: $(-1, 0)$ and $(2, 0)$, y-intercept: $(0, -2)$

d x-intercepts: $(-8, 0)$ and $(2, 0)$, y-intercept: $(0, -16)$

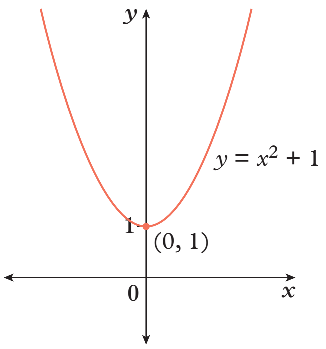
8



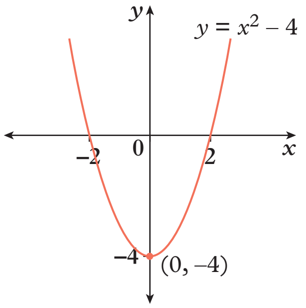
9



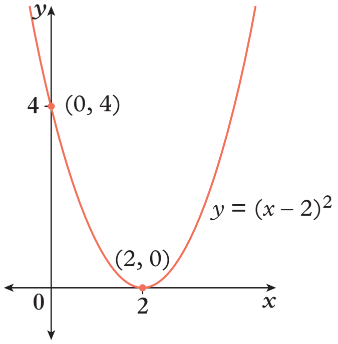
10 **a**



b



c



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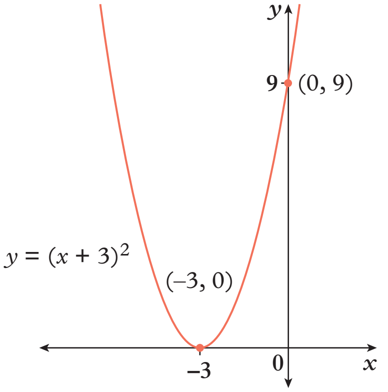
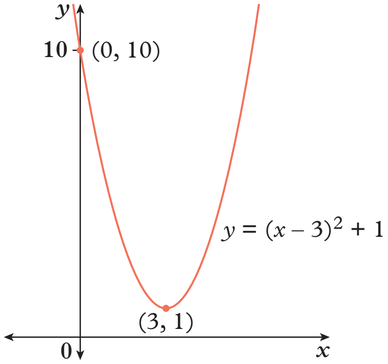


11 a



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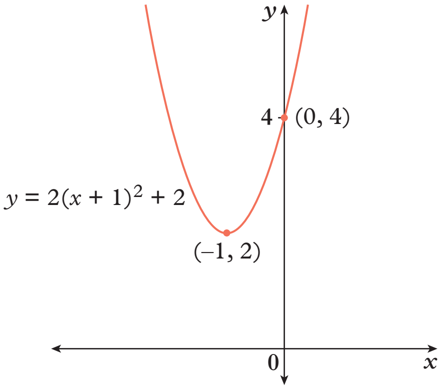

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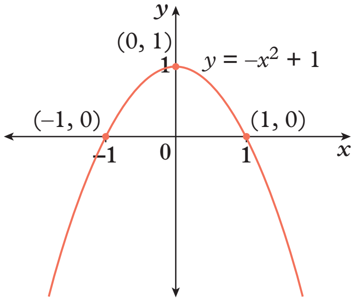

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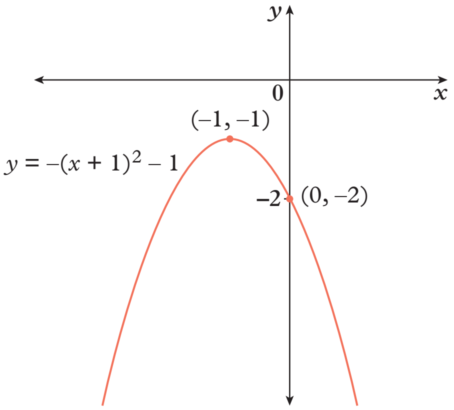
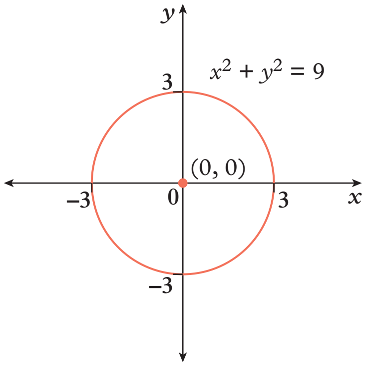


12 $y = (x + 2)^4 + 3$

EX
p244

5F Graphing circles

1 a



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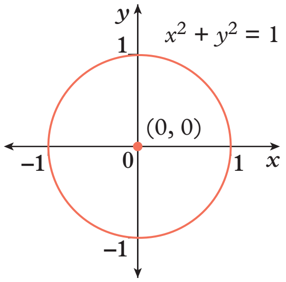

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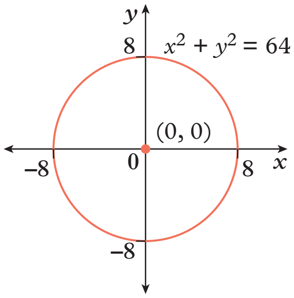

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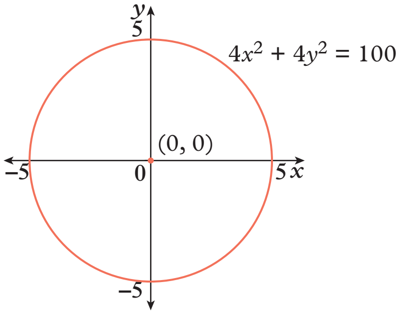
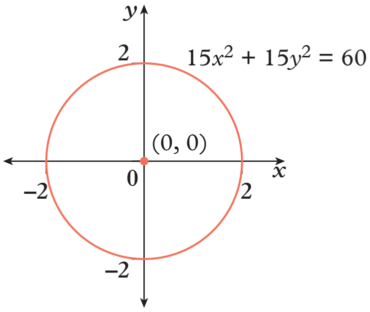
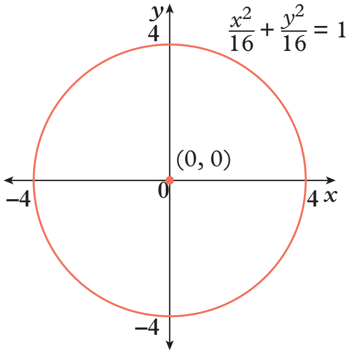


e



f

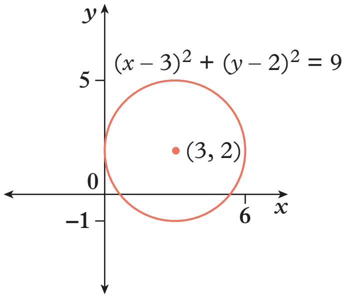


2 **a** $x^2 + y^2 = 9^2$ or $x^2 + y^2 = 81$

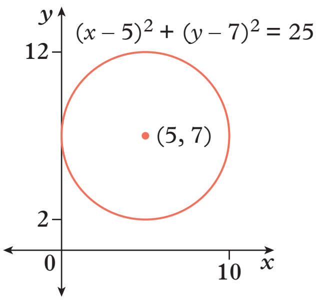
b $x^2 + y^2 = 36$

c $x^2 + y^2 = 400$

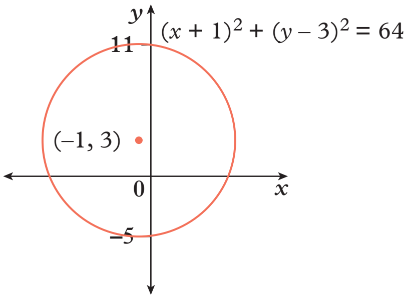
3 **a** centre (3, 2), radius 3 units



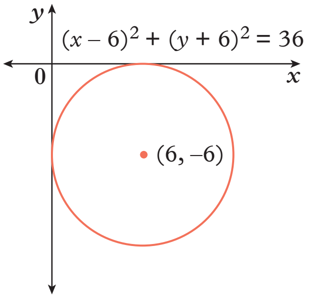
b centre (5, 7), radius 5 units



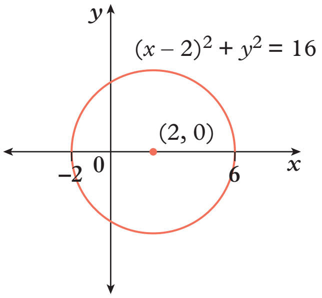
c centre (−1, 3), radius 8 units



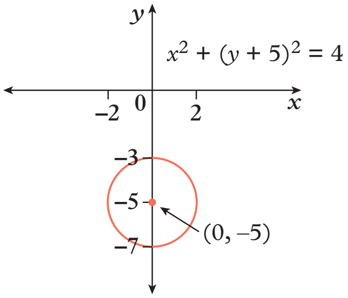
d centre (6, −6), radius 6 units



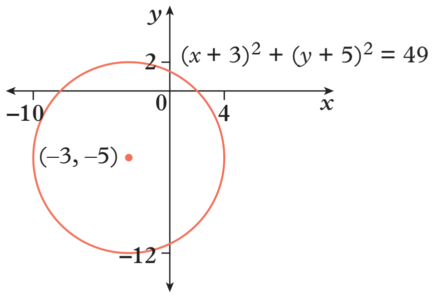
e centre (2, 0), radius 4 units



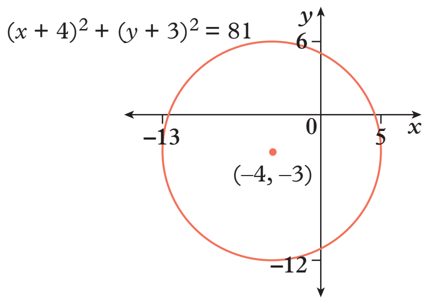
f centre (0, −5), radius 2 units



g centre $(-3, -5)$, radius 7 units



h centre $(-4, -3)$, radius 9 units



5 **a** $x^2 + y^2 = 144$

b $x^2 + y^2 = 225$

c $(x-4)^2 + (y+2)^2 = 1$

d $(x+5)^2 + (y+6)^2 = 49$

e $(x-4)^2 + (y-4)^2 = 9$

f $x^2 + (y+5)^2 = 36$

g $(x+1)^2 + (y-2.5)^2 = 2.25$

h $(x-2)^2 + (y+3)^2 = 5$

6 **a** $(x-7)^2 + (y-3)^2 = 9$ **b** $(x+1)^2 + (y+4)^2 = 25$

c $(x+6)^2 + (y-5)^2 = 4$ **d** $(x-2)^2 + (y+8)^2 = 16$

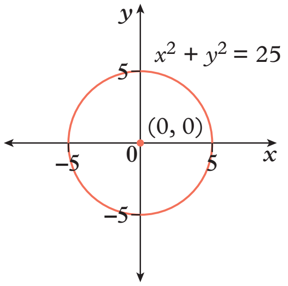
7 **a** $(x-2)^2 + (y+1)^2 = 9$

b A dilation in all directions by a factor of 5, then a translation of 7 units left and 4 units up

8 **a** a is the horizontal distance between the coordinates $(3, 4)$ and (x, y). This distance is measured as $x - 3$, hence $a = x - 3$. b is the vertical distance between the coordinates $(3, 4)$ and (x, y). This distance is measured as $y - 4$, hence $b = y - 4$.

b Pythagoras' theorem, $a^2 + b^2 = c^2$, is useful for determining the side lengths of a right-angled triangle. The right-angled triangle in the diagram can be explained by $a^2 + b^2 = 2^2$. Substituting a and b into the equation gives $(x-3)^2 + (y-4)^2 = 2^2$. Since every point (x, y) on the circle can be found by this process, this is the equation of the circle.

9 **a**



b $y = \pm\sqrt{25 - x^2}$

c **i** B; because $y \le 0$ for $-5 \le x \le 5$

ii A; because $y \ge 0$ for $-5 \le x \le 5$

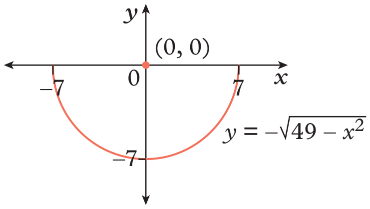
d $y = \sqrt{25 - x^2}$ gives the upper half of the circle and $y = -\sqrt{25 - x^2}$ gives the lower half of circle. Together, $y = \pm\sqrt{25 - x^2}$, or $x^2 + y^2 = 25$, gives the whole circle.

10 **a** $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{64 - x^2}$

c $y = \sqrt{1 - x^2}$

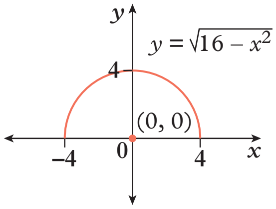
11 **a** Domain is $-7 \le x \le 7$.

Range is $-7 \le y \le 0$.



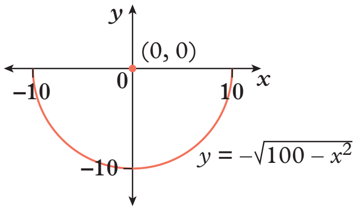
b Domain is $-4 \le x \le 4$.

Range is $0 \le y \le 4$.



c Domain is $-10 \le x \le 10$.

Range is $-10 \le y \le 0$.

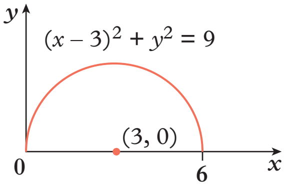


12 **a** $x = \pm\sqrt{25 - y^2}$

b For $x = \sqrt{25 - y^2}$, the graph is the right-hand half of the circle with equation $x^2 + y^2 = 25$.

For $x = -\sqrt{25 - y^2}$, the graph is the left-hand half of the circle with equation $x^2 + y^2 = 25$.

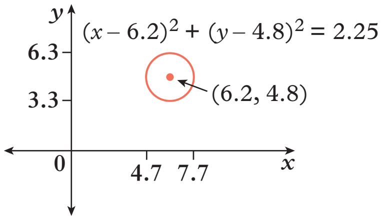
13 **a**



b 6 m **c** 3 m

d When $y = 1.5$, $x \approx 0.4$ and 5.6, so the arch is at half its maximum height at a distance of 0.4 m from either end.

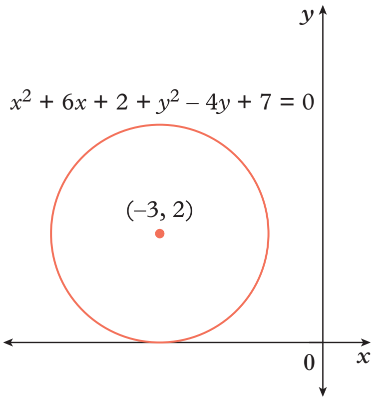
14 **a**



b 3.3 m **c** 7 m² **d** 4.7%

15 **a** **i** $(x+3)^2-7$ **ii** $(y-2)^2+3$

b



16 (–4, 3), (–3, 4)

17 **a** (0, –3), (3, 0) **b** (–3, 6), (–1, 8)

18 $(x+4)^2 + (y-5)^2 = 36$

19 **a** (1, 2)

b Substituting in (3, 3) gives $(3-1)^2 + (3-2)^2 = 5$, which is true. Therefore, since (3, 3) satisfies the equation, it is a point on the circle.

c $\frac{1}{2}$

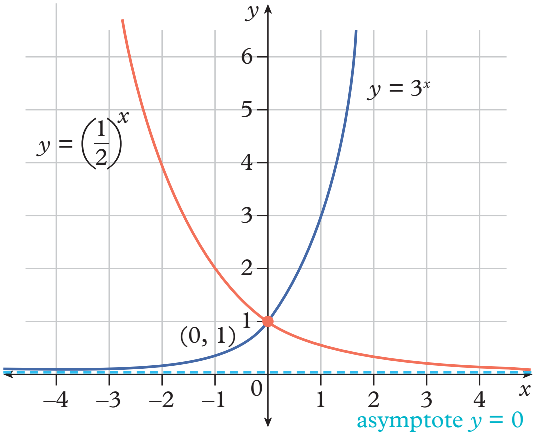
d –2

e $y = -2x + 9$

EX p254

5G Graphing exponential and logarithmic relationships

1 **a**



b **i** (0, 1) **ii** (0, 1)

c **i** $y = 0$ **ii** $y = 0$

2 **a** **i** $k = 2, n = 1, c = 0$

ii Dilation in the y-direction by a factor of 2.

iii (0, 2), $y = 0$

b **i** $k = \frac{2}{3}, n = 1, c = 0$

ii Dilation in the y-direction by a factor of $\frac{2}{3}$.

iii $\left(0, \frac{2}{3}\right)$, $y = 0$

c **i** $k = -1, n = 1, c = 0$

ii Reflection in the x-axis.

iii (0, – 1), $y = 0$

d **i** $k = 1, n = -1, c = 0$

ii Reflection in the y-axis.

iii (0, 1), $y = 0$

e **i** $k = 1, n = 1, c = \frac{1}{2}$

ii Translate $\frac{1}{2}$ units up.

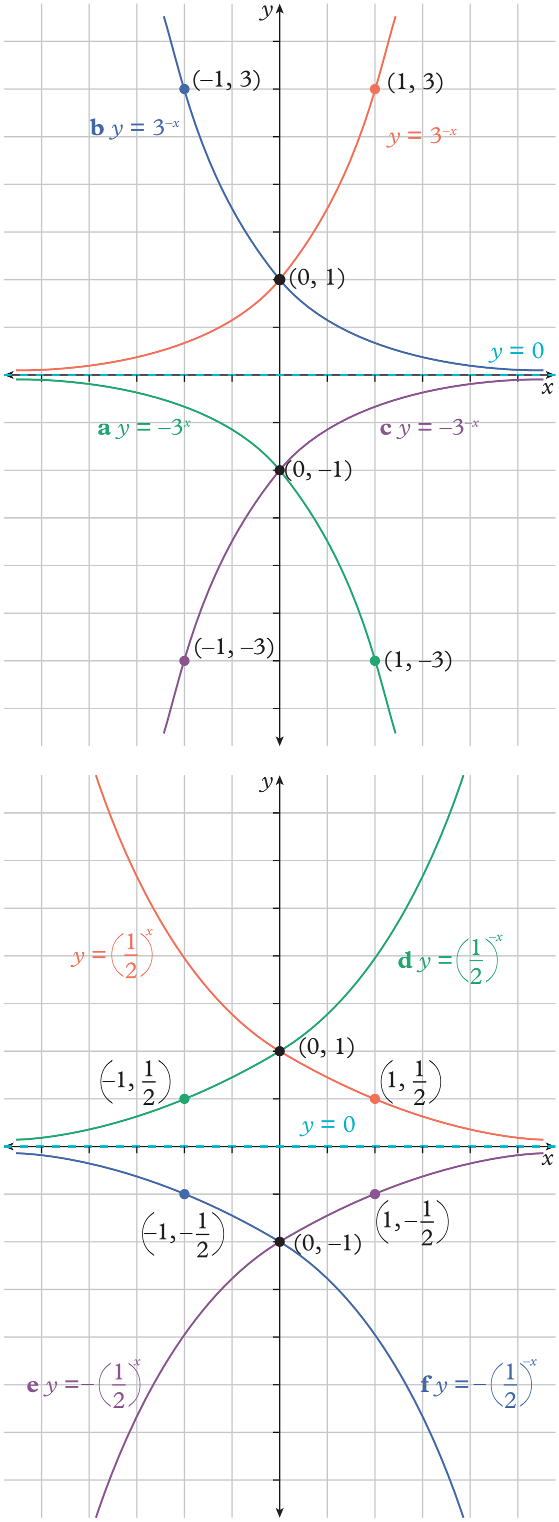
iii $\left(0, \frac{3}{2}\right)$, $y = \frac{1}{2}$

f **i** $k = 1, n = 1, c = -1$

ii Translate 1 unit down.

iii (0, 0), $y = -1$

3



4 **a** $y = 3^{-x}$

$y = 3^{-1 \times x}$

$y = (3^{-1})^{x}$

$y = \left(\frac{1}{3}\right)^{x}$

b $y = \left(\frac{1}{2}\right)^{x}$

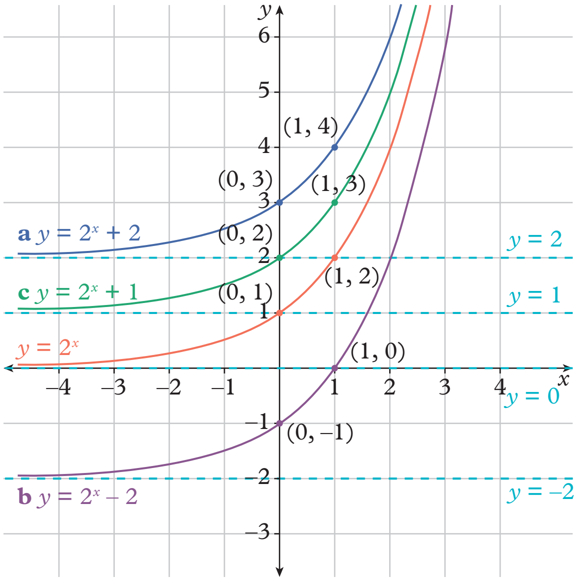
$y = (2^{-1})^{x}$

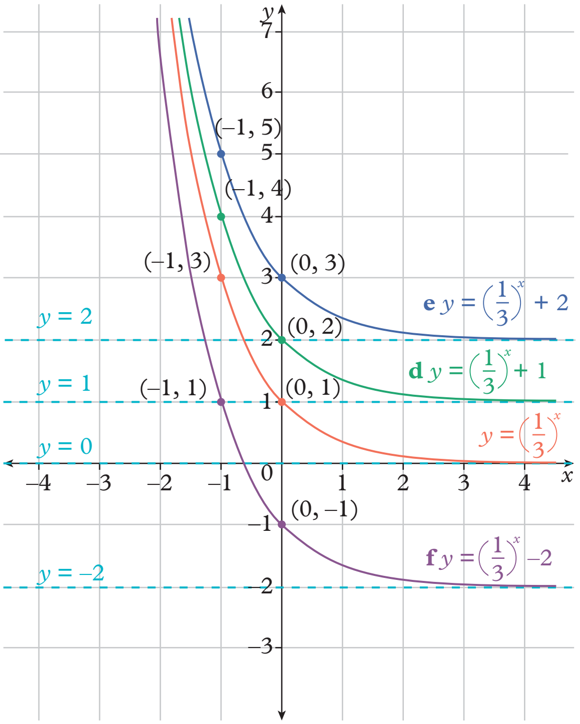
$y = 2^{-1 \times x}$

$y = 2^{-x}$

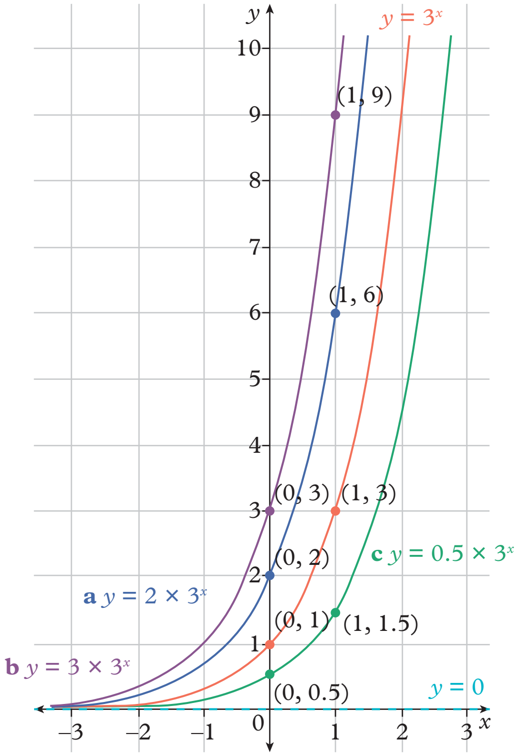
5 $y = \left(\frac{1}{2}\right)^{-x}$

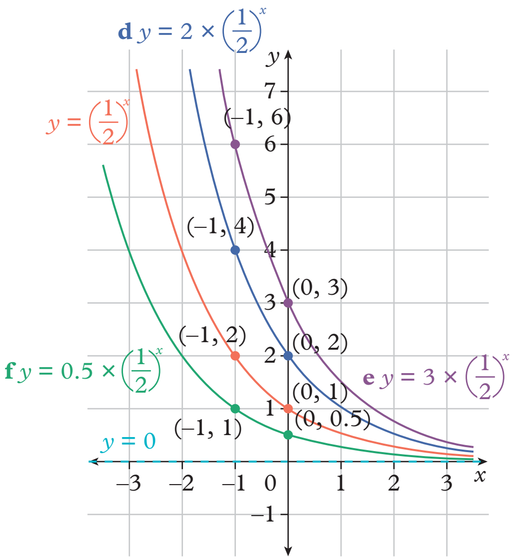
6



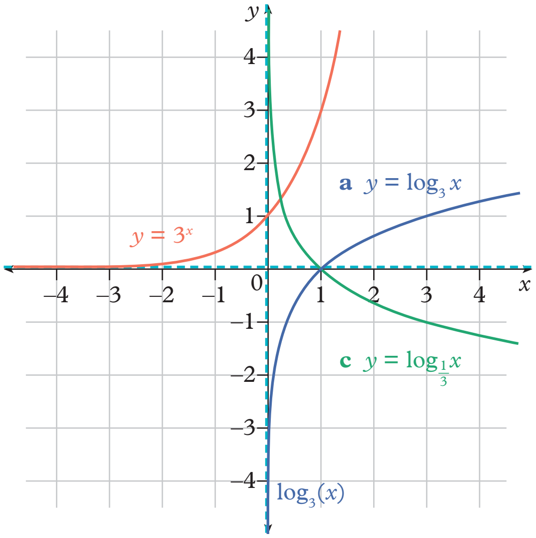


7

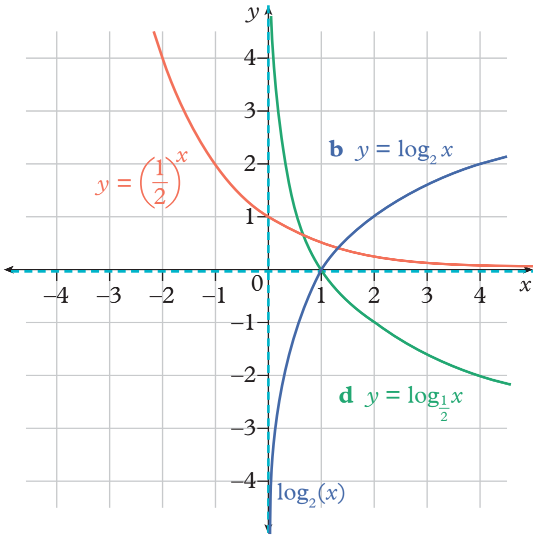




8 **a, c**



b, d



9 **a** $y = -a^x$ **b** $y = a^{-x}$ **c** $y = -a^{-x}$

10 **a** Domain: $\mathbb{R}$, range: $y > 0$

b Domain: $\mathbb{R}$, range: $y < 0$

c Domain: $\mathbb{R}$, range: $y > 0$

d Domain: $\mathbb{R}$, range: $y > 2$

e Domain: $x > 0$, range: $\mathbb{R}$

f Domain: $x > 0$, range: $\mathbb{R}$

11 **a** I **b** IV **c** III

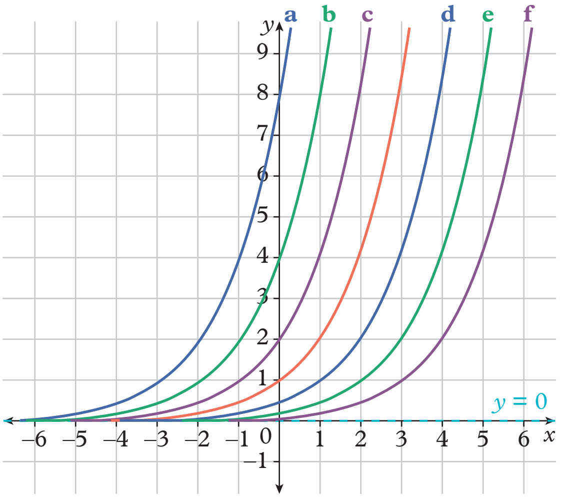
d II **e** V

12 **a** **i** reflection in the x-axis **ii** $y = -7^x$

b **i** translation 2 units up **ii** $y = 4^x + 2$

13 **a** $y = 8 \times 2^x$ **b** $y = 4 \times 2^x$ **c** $y = 2 \times 2^x$

d $y = \frac{1}{2} \times 2^x$ **e** $y = \frac{1}{4} \times 2^x$ **f** $y = \frac{1}{8} \times 2^x$



14 **a** 3 units left **b** 2 units left **c** 1 unit left

d 1 unit right **e** 2 units right **f** 3 units right

15 **a** **i**

x	-3	-2	-1	0	1	2	3
$y = 5^x$	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125

ii

x	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125
$y = \log_5 x$	-3	-2	-1	0	1	2	3

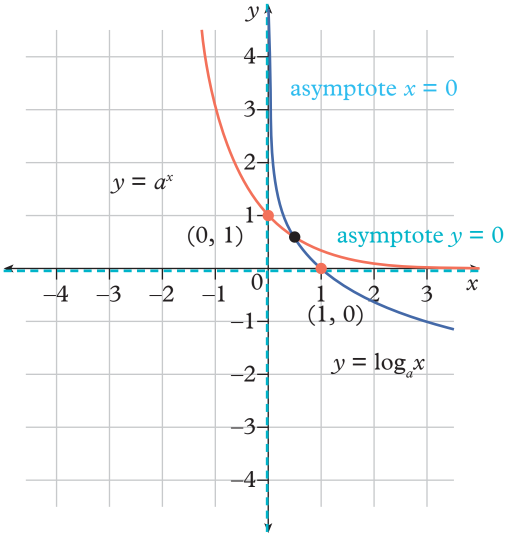
b **i** $\left(-3, \frac{1}{125}\right), \left(-2, \frac{1}{25}\right), \left(-1, \frac{1}{5}\right),$
$(0, 1), (1, 5), (2, 25), (3, 125)$

ii $\left(\frac{1}{125}, -3\right), \left(\frac{1}{25}, -2\right), \left(\frac{1}{5}, -1\right),$
$(1, 0), (5, 1), (25, 2), (125, 3)$

c The x-coordinate in part **i** is the y-coordinate in part **ii** and vice versa.

d (b, a)

16 **a**



b $a < 1$

17 **a** $y = 3 \times 5^x$ **b** $y = -5^x$

c $y = 5^{x-3}$ **d** $y = 5^x - 2$

18 **a** **i** dilation in the y-direction by factor of 2, translation 4 units up

ii The equation for A is $y = 5^x$. The equation for B is $y = 2 \times 5^x + 4$.

b **i** translation of 2 units right and 5 units down

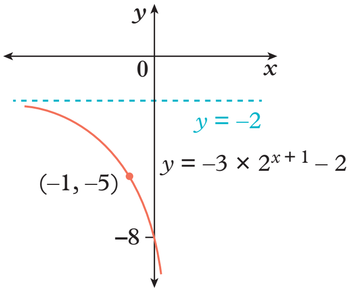
ii The equation for A is $y = 4^x$. The equation for B is $y = 4^{x-2} - 5$.

c **i** dilation in the y-direction by factor of 4, reflection in x-axis and y-axis

ii The equation for A is $y = 3^x$. The equation for B is $y = -4 \times 3^{-x}$.

19 **a** dilation in the y-direction by factor of 3, reflection in the x-axis, translation of 1 unit left and 2 units down

b



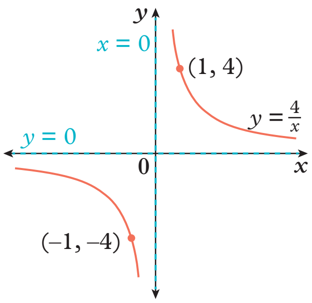
c y-intercept $(0, -8)$, asymptote: $y = -2$

20 Translation of 1 unit down, reflection in the y-axis.

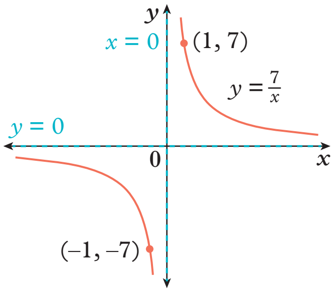
EX 5H Graphing hyperbolas

p261

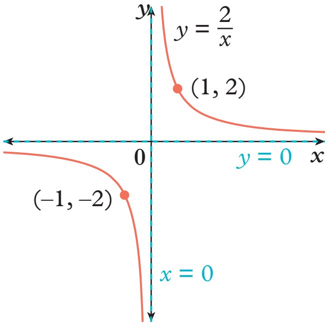
1 **a**



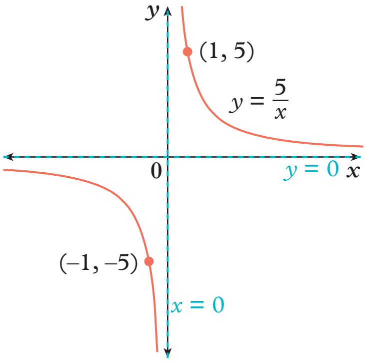
b



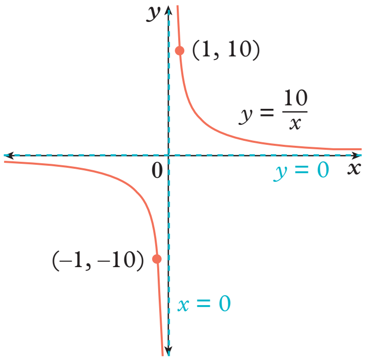
c



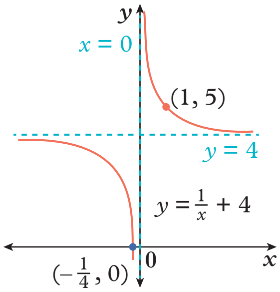
d



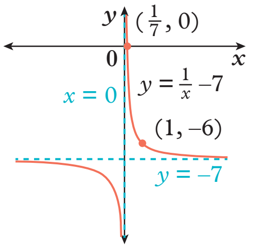
e



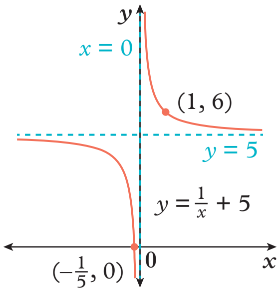
2 **a**



b



c



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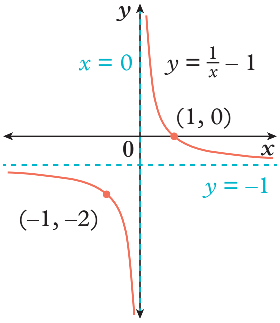
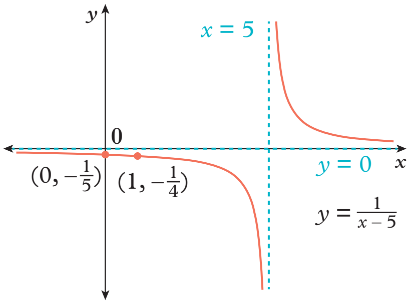


3 **a**



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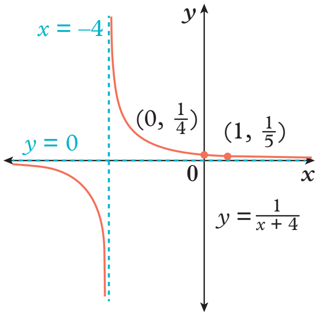

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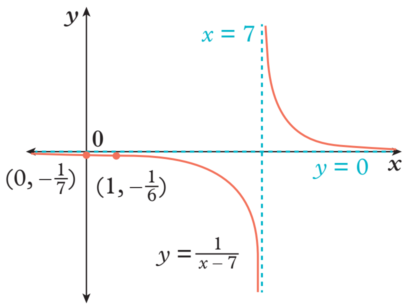

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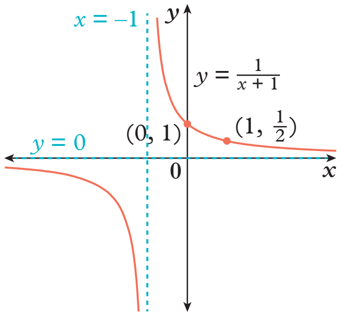
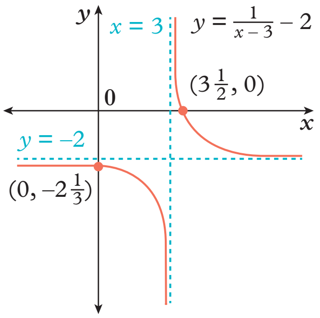


4 a



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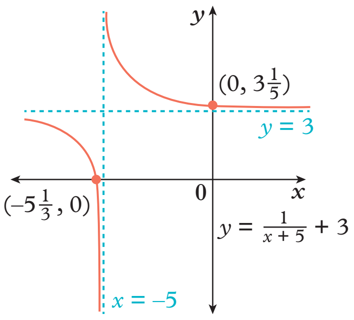

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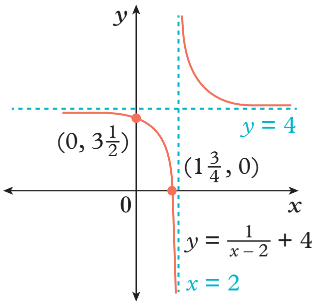

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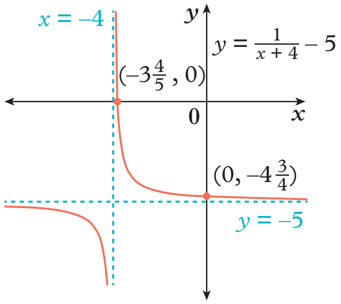
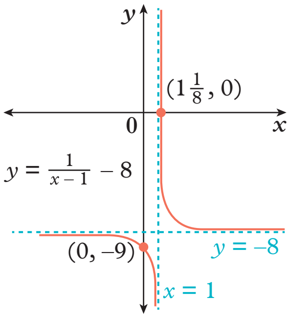
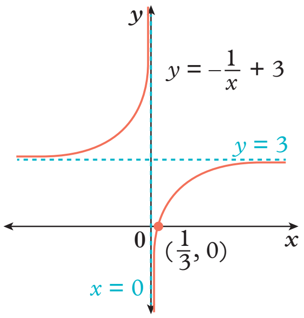


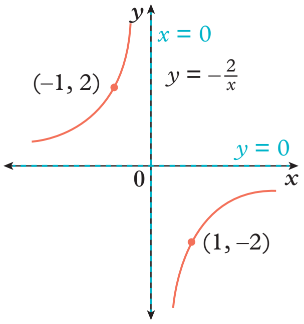
e



6 a



b



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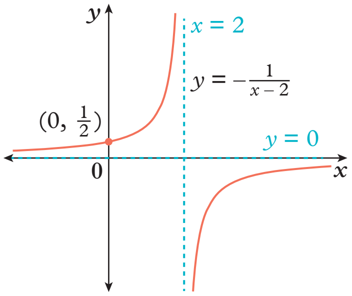
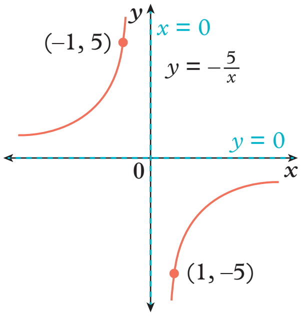
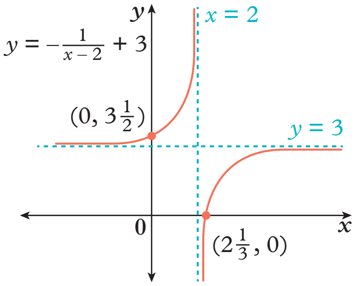


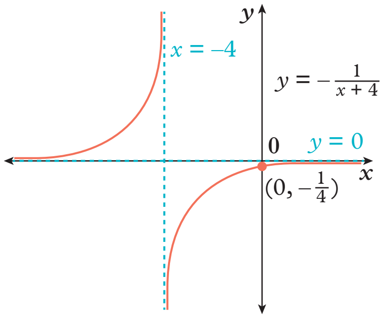
d



e



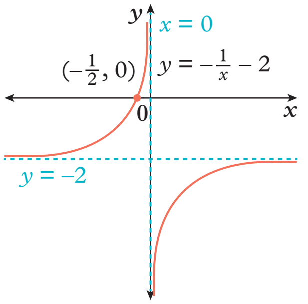
7 a



The domain is $x < -4$ and $x > -4$.

The range is $y < 0$ and $y > 0$.

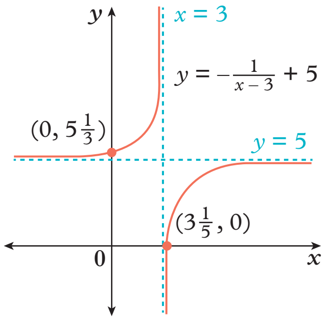
b



The domain is $x < 0$ and $x > 0$.

The range is $y < -2$ and $y > -2$.

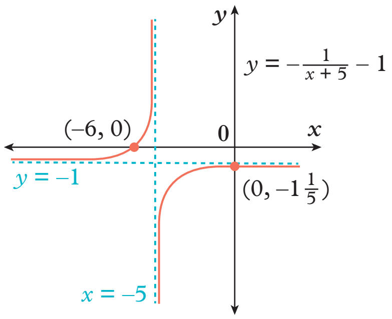
c



The domain is $x < 3$ and $x > 3$.

The range is $y < 5$ and y > 5.

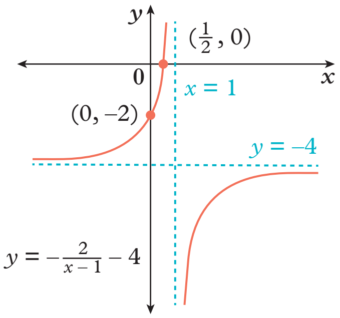
d



The domain is $x < -5$ and $x > -5$.

The range is $y < -1$ and $y > -1$.

e



The domain is $x < 1$ and $x > 1$.

The range is $y < -4$ and $y > -4$.

8 a i The y value decrease at a constant rate.

ii The y value decrease at a constant rate.

b i The y value decreases at an decreasing rate

ii The y value increases at an increasing rate

c i The y value decreases at an increasing rate

ii The y value decreases at an decreasing rate

d i The y value decreases at an increasing rate

ii The y value decreases at an increasing rate

9 a $y = \frac{1}{x-2} + 2$ **b** $y = -\frac{1}{x} - 1$

c $y = \frac{4}{x+4} - 2$ **d** $y = -\frac{3}{x-1} + 3$

10 Reflecting in the x-axis: the curve in the first quadrant is reflected to the fourth quadrant; the curve in the third quadrant is reflected to the second quadrant.

The equation becomes $y = -\left(\frac{1}{x}\right)$ or $y = -\frac{1}{x}$.

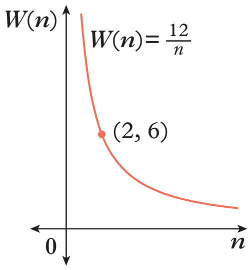
Reflecting in the y-axis: the curve in the first quadrant is reflected to the second quadrant; the curve in the third quadrant is reflected to the fourth quadrant.

The equation becomes $y = \frac{1}{(-x)}$ or $y = -\frac{1}{x}$.

11 a zero **b** positive infinity

c negative infinity **d** zero

12 **a**



b **i** 6 min **ii** 4 min **iii** 3 min

iv 2 min **v** 1 min **vi** 30 s

c 12 minutes when there is one bank teller on duty

d No, because $\frac{12}{n} \neq 0$; however, W approaches 0 as n increases.

13 **a** $y = \frac{1}{0} + 3$, the y value is undefined since it is not possible to divide by 0.

b If x equals any other value than 2 then the denominator will not be 0.

c Substituting $y = 3$ into the equation gives $0 = \frac{1}{x-2}$, which is impossible since a fraction cannot equal 0 unless its numerator is 0 (and its denominator is not 0).

d If y equals any other value than 3 then the fraction $\frac{1}{x-2}$ will equal a non-zero number, which can be solved for x.

14 **a** **i** two **ii** $(-1, -1)$ and $(1, 1)$

b **i** none

c **i** two **ii** $(0, 3)$ and $(-5, -2)$

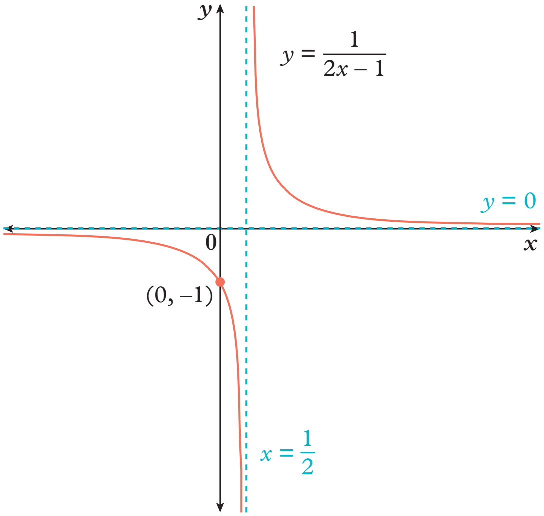
d **i** one **ii** $(2, -\frac{1}{2})$

15 **a** $y = \frac{8}{x-2}$

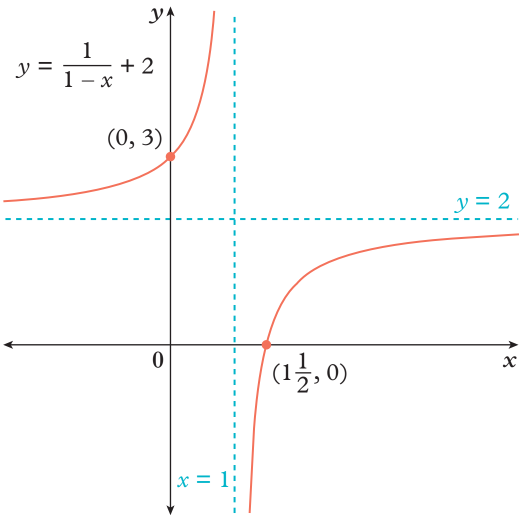
b $y = \frac{3}{x-1} + 2$

c $y = \frac{4}{x+3}$

16 **a**



b



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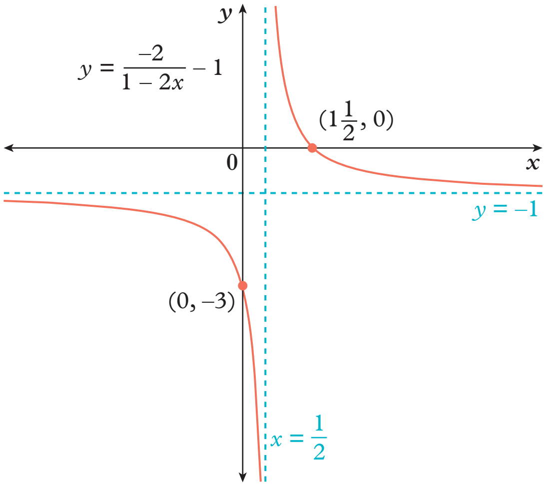


EX p266 **5I** Direct and inverse variation

1 **a** 4 **b** 9 **c** 1.5

d 5 **e** 0.5 **f** 0.4

2 **a** 2 **b** 1.2 **c** 1.5

d 0.5

3 **a** 6 **b** 12 **c** 4

d 6 **e** 15 **f** 1

4 **a** 2 **b** 5 **c** 0.5

d 8

5 **a** directly **b** neither

c inversely **d** neither

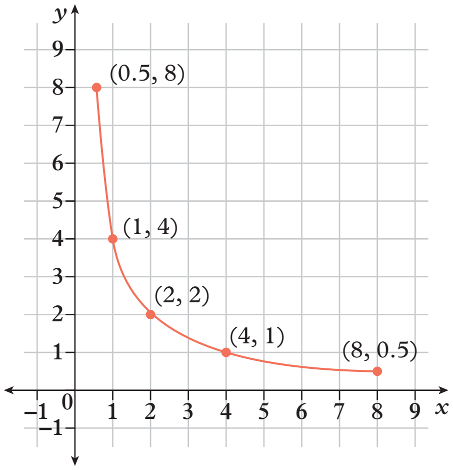
e directly **f** inversely

6 **a** inversely proportional

b 4

c 1, 0.5

d



7 **a** The graph does not pass through (0, 0) therefore not direct variation. The graph is a straight line, therefore not inverse variation.

b The graph is not a straight line, therefore not direct variation. The graph intersects the vertical and horizontal axes therefore not inverse variation.

c The graph does not pass through (0, 0) or y decreases as x increases, therefore not direct variation. The graph is a straight line, therefore not inverse variation.

d The graph is not a straight line, therefore not direct variation. The x-and y-coordinates of the points on the graph do not multiply to give the same number, therefore not inverse variation.

8 **a** **i** doubled **ii** halved
iii increased by 20% **iv** decreased by 30%
b **i** doubled **ii** halved
iii increased by 20% **iv** decreased by 30%
c **i** halved **ii** doubled
iii decreased by $\frac{1}{6}$ **iv** increased by $\frac{3}{7}$
d **i** halved **ii** doubled
iii decreased by $\frac{1}{6}$ **iv** increased by $\frac{3}{7}$

9 **a** 140 **b** 840 watts **c** 1.75 hours

10 7.5 days

11 **a** 6 **b** 6

c It will always be the same. If y is inversely proportional to x then $y = \frac{k}{x}$ for some real number, k. Rearranging to make x the subject gives $x = \frac{k}{y}$, which is the same constant of variation.

12 **a** $171 **b** 171

c The total cost, T, is fixed. Therefore, the cost per person, C, will be $\frac{T}{n}$, where n is the number of people. This gives the equation $C = \frac{T}{n}$, which shows that C is inversely proportional to n and that T is the constant of variation.

13 **a** 3.2, 3.1, 3.1, 3.25, 3.24, 3.15, 3.1

b 3.16

c He is trying to approximate π, which is roughly 3.14. Therefore, he is relatively close.

14 **a** 4.8

b **i** 0.192 W/m^2 **ii** 22 cm

c 75% (i.e. the light intensity is one quarter of its previous value)

15 **a** If $x \propto y$ and $y \propto z$, then $x = k_1 y$ and $y = k_2 z$. Therefore, $x = k_1 k_2 z$, which shows that $x \propto z$.

b If $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a = \frac{k_1}{b}$ and $b = \frac{k_2}{c}$. Therefore, $a = \frac{k_1}{k_2/c} = \frac{k_1 c}{k_2} = \frac{k_1}{k_2}c$, which shows that $a \propto c$.

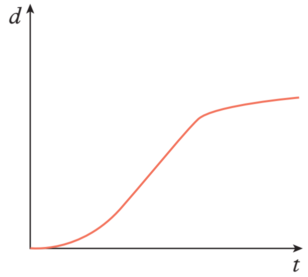
EX p273

5J Rates of change

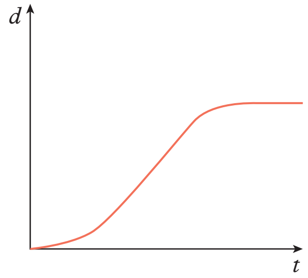
1 **a** IV **b** I **c** III **d** II
2 **a** I **b** II **c** IV **d** III

3 **a** Variable speed, the speed decreases over time.
b Variable speed, the speed increases over time.
c constant speed = 0
d Variable speed, the speed increases over time.
e constant speed = –7.5
f constant speed = 6

4 **a**



b



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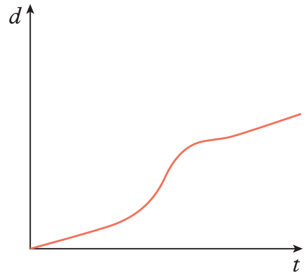
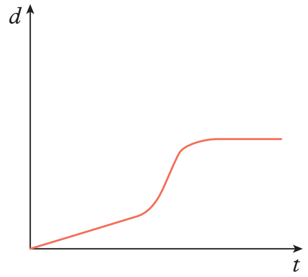
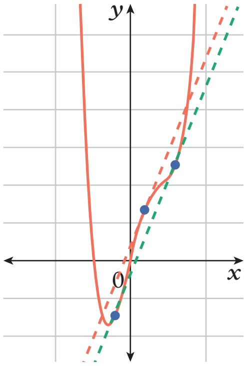


d



5 Not necessarily, the graph of a function could have a gradient of 5 at three different x values without being linear, such as the one below.



6 **a** The rate starts at zero and steadily increases (becomes more positive).

b The rate starts at zero and steadily decreases (becomes more negative).

c The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative).

d The rate starts at nearly zero and increases at an increasing rate.

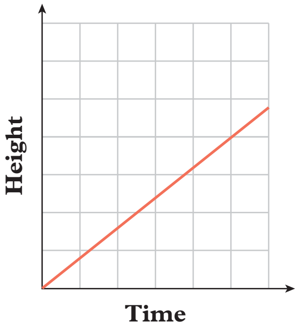
e The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative), then begins increasing (becomes less negative) until it is zero again then continues increasing (becomes more positive).

f The rate alternates between decreasing and increasing repeatedly.

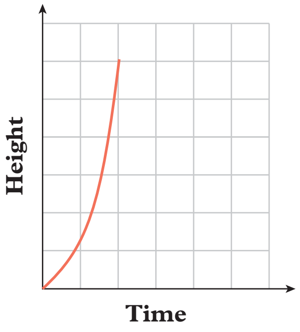
7 The object travelled 8 km from A towards B at a constant speed. Then travelled 2 km in the same direction towards C at an decreasing speed. From C, the object travelled in the opposite direction at a constant speed for 2 km. From D, the object changed direction again and travelled at an increasing speed for 2 km. The object travelled in the same direction at a constant speed for 4 km until reaching F. The object didn't move for 2 mins between F and G. From G, the object travelled at a constant speed for 10 km to arrive at H.

8 **a** **i** The y-value decreases at a constant rate.
ii The y-value decreases at a constant rate.
b **i** The y-value decreases at an decreasing rate
ii The y-value increases at an increasing rate
c **i** The y-value decreases at an increasing rate
ii The y-value decreases at an decreasing rate
d **i** The y-value decreases at an increasing rate
ii The y-value decreases at an increasing rate

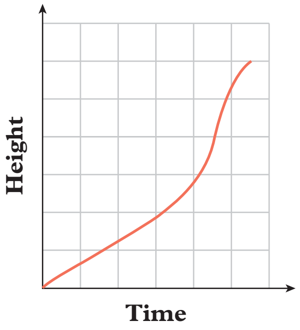
9 **a**



b



c



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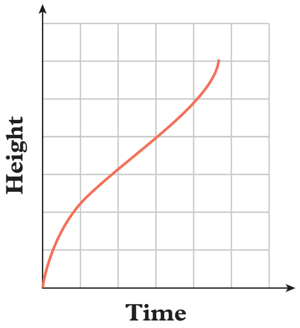
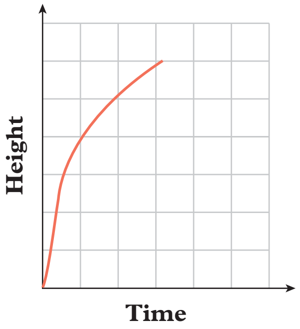
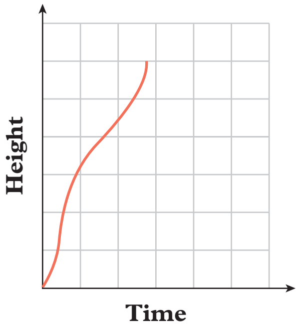


e

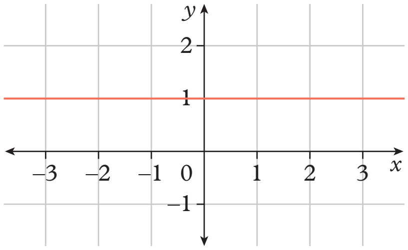


f



10 **a** **i** y-intercept = 2 **ii** −1
b **i** y-intercept = 1 **ii** −3
c **i** y-intercept = 1 **ii** 0

11 a



b No, the gradient of a linear relationship in the form $y = x + b$ is 1, the graph of the gradient is always a horizontal line passing through 1 on the vertical axis. The graph of the gradient is the same for different values of b. Similarly, the gradient of linear relationship in the form $y = ax + b$ is a, the graph of the gradient is always a horizontal line passing through a on the vertical axis regardless of the value of b.

c Gradient between (–2, 5) and (–1, 2) is –3

Gradient between (–1, 2) and (0, 0) is –1

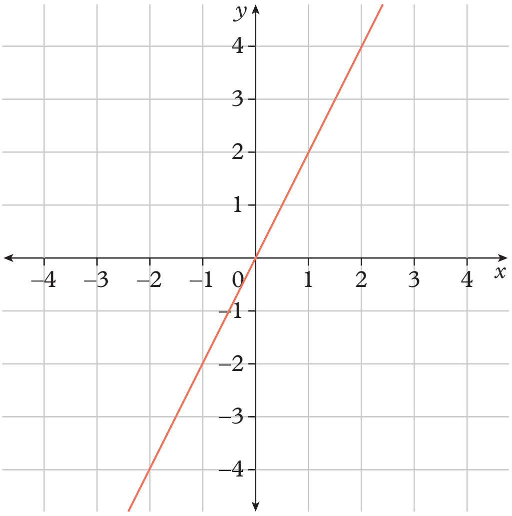
Gradient between (0, 0) and (1, 2) is 1

Gradient between (1, 2) and (2, 5) is 3

d

x	–1.5	–0.5	0	0.5	1.5
Gradient	–3	–1	0	1	3

e



f No. If a quadratic relationship $y = a(x - h)^2$ is translated k units up or down, the rates of change are still the same for a given x value, the graph of the gradient remains the same regardless the value of k.

EX p278

Chapter 5 review

Mathematical literacy review

1 **a** VI **b** V **c** I **d** II **e** IV

f III **g** VI **h** VI **i** V

2 A, D, F, I

3 **a** asymptote

b directly, constant of variation.

c axis of symmetry, midpoint

4 **a** True

b False. Exponential and hyperbolic relationships have a horizontal asymptote.

c False. Circles, square roots and hyperbolic relationships may not have a y-intercept.

d True

e False. Circles, square roots and hyperbolic relationships have a limited domain.

f True

g False. They are examples of non-linear relationships.

Multiple choice

1	B	**2**	B	**3**	C	**4**	C	**5**	A
6	C	**7**	D	**8**	B	**9**	D	**10**	A
11	C	**12**	E	**13**	D	**14**	E	**15**	C

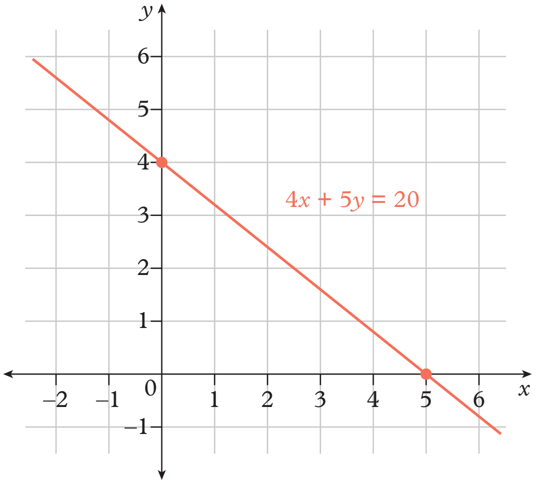
Short answer

1 **a** $y = -6x$ **b** $y = \frac{1}{2}x - \frac{3}{2}$

2 **a** $2x + y + 2 = 0$ **b** $-15x + 10y - 3 = 0$

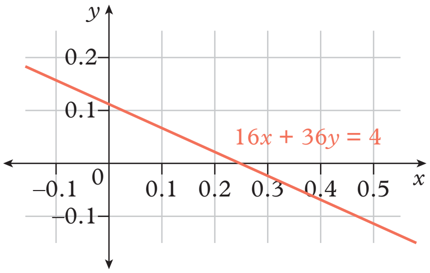
3 **a** **i** (0, 4), (5, 0)

ii



b **i** $\left(0, \frac{1}{9}\right), \left(\frac{1}{4}, 0\right)$

ii



4 **a** $y = 2x + 3$ **b** $y = -3x - 3$

5 **a** **i** $-\frac{1}{5}$ **ii** (2, 3) **iii** $2\sqrt{26}$

b **i** 1 **ii** (5, 0) **iii** $4\sqrt{2}$

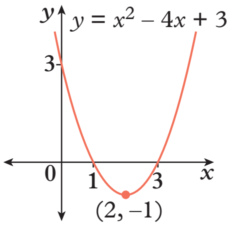
6 **a** **i** (–5, 0) and (5, 0) **ii** (0, 25)

b **i** (–1, 0) and (4, 0) **ii** (0, –4)

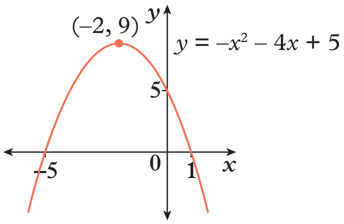
c **i** (2, 0) **ii** (0, 8)

d **i** (–3, 0) and (2, 0) **ii** (0, 6)

7 **a**



b



8 **a** **i** 5 units right, 2 units up **ii** (5, 2)

b **i** 6 units left, 4 units down **ii** (–6, –4)

c **i** 1 unit right, 1 unit down **ii** (1, –1)

d **i** 3 units left, 5 units up **ii** (–3, 5)

e **i** 4 units right, 2 units up **ii** (4, 2)

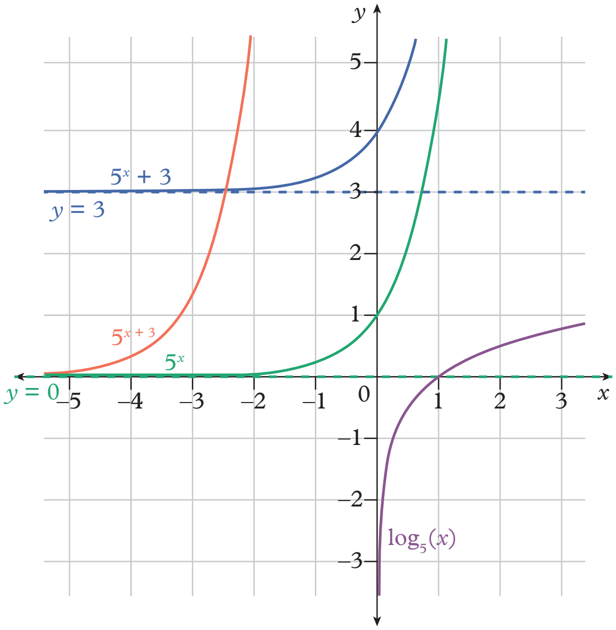
f **i** dilaton in the *y*-direction with a dilation factor of 3, then moved 7 units left, 3 units down

ii (–7, –3)

9 $y = (x + 4)^4 - 6$; the domain is $\mathbb{R}$; the range is $y > -6$

10 $(x - 4)^2 + (y - 2)^2 = 4$

11 **a, c**



b **i** translate 3 units left **ii** translate 3 units up

iii reflect across the line $y = x$

12 **a** translate 5 units down

b reflect in the *x*-axis

c a dilation in the *y*-direction by a factor of 4

d translate 5 units left and 6 units up

13 **a** 12.5 **b** \$50 **c** 16 square metres

14 **a** 20 **b** 20 **c** 8

15 **a** hyperbola

b exponential relationship

c parabola

16 **a** The *y* value is decreasing at a decreasing rate.

b The *y* value is increasing at an increasing rate.

c The *y* value is decreasing at an decreasing rate from –1 to 0, then increasing at an increasing rate from 0 to 1.

Analysis

1

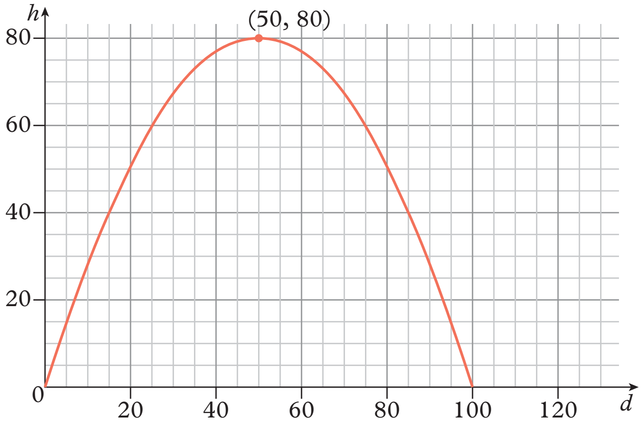
Item	**Equation**	**Domain**	**Range**
Left leg	$y = x + 5$	$1 \le x \le 3$	$6 \le y \le 8$
Right leg	$y = -x + 11$	$3 \le x \le 5$	$6 \le y \le 8$
Body	$x = 3$	$x = 3$	$8 \le y \le 11$
Arms	$y = 10$	$1 \le x \le 5$	$y = 10$
Head	$(x - 3)^2 + (y - 12)^2 = 1$	$2 \le x \le 4$	$11 \le y \le 13$
Top of platform	$y = 6$	$0 \le x \le 5.5$	$y = 6$
Side of platform	$x = 5.5$	$x = 5.5$	$0 \le y \le 6$
Soccer ball	$(x - 5.5)^2 + (y - 6.5)^2 = 0.25$	$5 \le x \le 6$	$6 \le y \le 7$
Path of the kick	$y = -0.1(x - 12)^2 + 10$	$6 \le x \le 22$	$0 \le y \le 10$

2 **a** $h = ad(d - 100)$

b (0, 0), (100, 0); $d = 0$ is where the ball starts, $d = 100$ is where the ball lands.

c (50, 80)

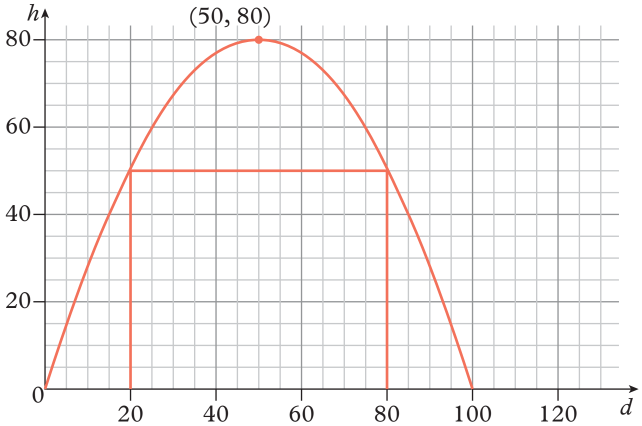
d



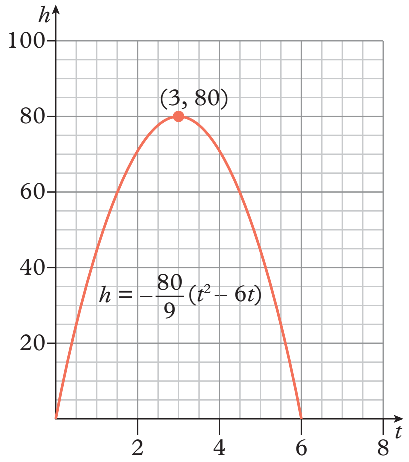
e $a = -0.032$ **f** $h = 51.2$ m

g 80 m

h



i



j $h = -\frac{80}{9}(t^2 - 6t)$ **k** 0.4 seconds

l 16.7 m/s

3 **a** **i** $k = 16$

ii reflection across the y-axis and a vertical dilation of 16

b **i** $b = 2, y = \frac{2}{x}$ **ii** $x = 0, y = 0$

c **i** $k = 8$ **ii** $y = \frac{8}{x^3}$

d **i** Jon: 0.179 km/min; Andy: 0.170 km/min; Rob: 0.167 km/min

ii Graph II

iii The x-axis has the label time, t, with units minutes. The y-axis has the label speed, s, with units km/min.

iv The longer their time to complete the 2 km, the slower their speed; that is, the speed is decreasing as time is increasing.

v For the horizontal asymptote $s = 0$, you can't have a zero speed and run 2 km. For the vertical asymptote $t = 0$, there is no speed that will work to run 2 km in zero time.

CHAPTER 6 Polynomials

EX
p288

6A Functions and relations

1 **a** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b {−4, −3, −2, −1, 0, 1, 2, 3}

c {2, 3, 5, 7, 11, 13, 17, 19}

d {a, b, c, d, e}

2 **a, b, d, f** function

c, e not a function

3 **c** (5, 8) and (5, 11)

e (12, −5) and (12, 4), for example

4 **a, d, f** function

b, c, e not a function

5 **a, c, f** function

b, d, e not a function

6 **a** **i** 1 **ii** 2 **iii** 0

b **i** −1 **ii** 0 **iii** 0

c **i** 0 **ii** 3 **iii** 1

d **i** 2 **ii** 4 **iii** $\frac{4}{3}$

e **i** 0 **ii** 1 **iii** $-\frac{3}{2}$

f **i** 3 **ii** $\frac{3}{2}$ **iii** 6

g **i** $\frac{3}{2}$ **ii** 1 **iii** 3

h **i** $\frac{2}{3}$ **ii** 0 **iii** $\frac{4}{5}$

i **i** −1 **ii** 0 **iii** $-\frac{4}{3}$

j **i** $\log_2(3)$ **ii** $\log_2(4)$ **iii** 1

k **i** $2\log_2(3)$ **ii** $2\log_2(5)$ **iii** 0

l **i** $3\log_2(5)$ **ii** $3\log_2(3)$ **iii** $3\log_2(7)$

7 **a** **i** 7 **ii** $-\frac{3}{4}$ **iii** 38

b **i** 12 **ii** $\frac{28}{9}$ **iii** 267

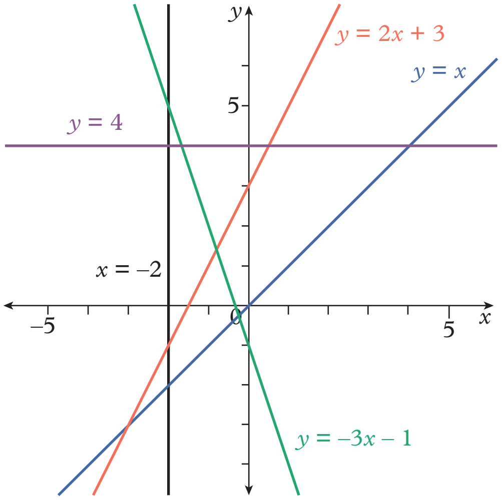
c **i** 11 **ii** 22 **iii** $\frac{441}{8}$

d **i** $\frac{66}{7}$ **ii** $\frac{10}{9}$ **iii** $\frac{2433}{10}$

e **i** 0 **ii** 1 **iii** $-\frac{3}{2}$

f **i** $\frac{17}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{513}{16}$

8 **a**

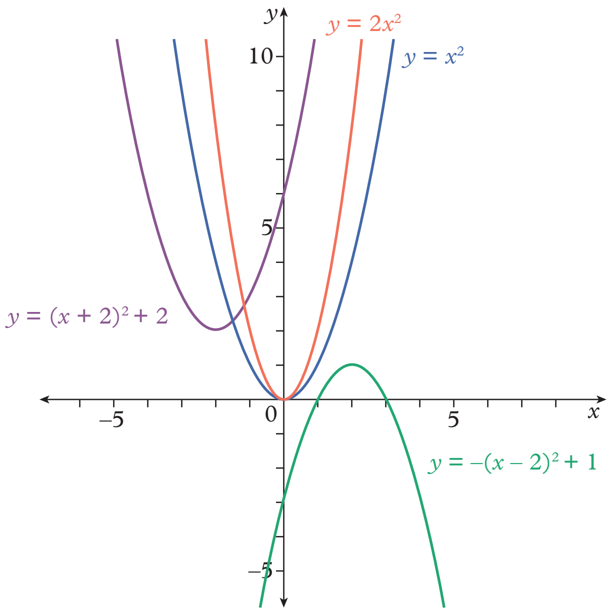


b $y = x$, $y = 2x + 3$, $y = -3x - 1$ and $y = 4$ are functions.

c One x-value maps to infinitely many y-values.

d If $y = mx + b$, then each x-value maps to the unique y-value $mx + b$ (for example).

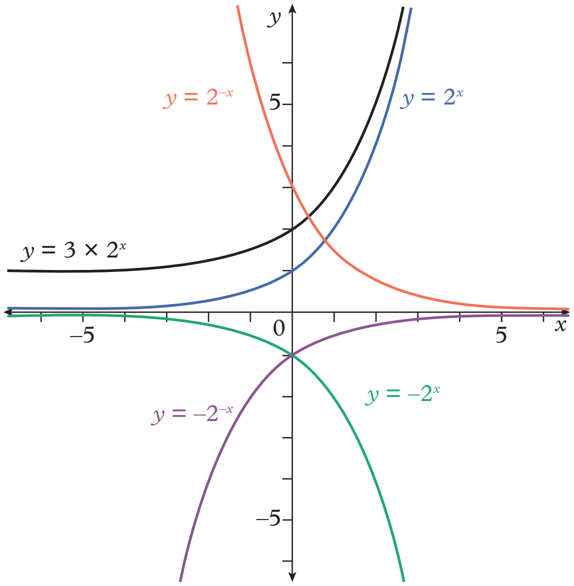
9 a



b They are all functions.

c If $y = ax^2 + bx + c$, then each x-value maps to the unique y-value $ax^2 + bx + c$ (for example).

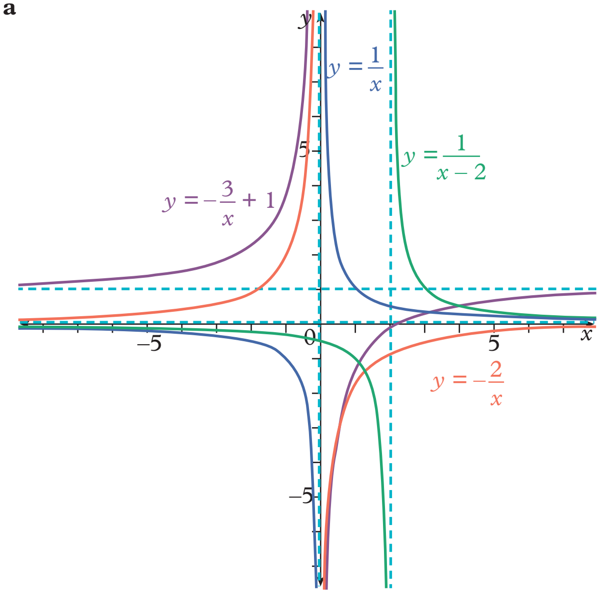
10 a



b They are all functions.

c If $y = A \times b^{nx} + k$, then each x-value maps to the unique y-value $A \times b^{nx} + k$ (for example).

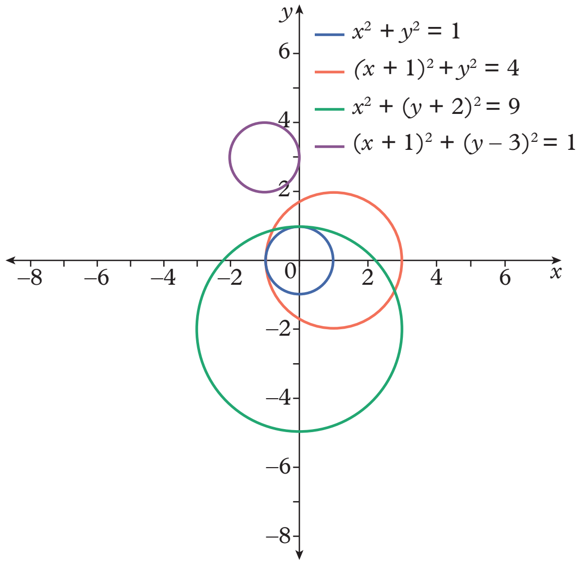
11 a



b They are all functions

c If $y = \frac{a}{x-h} + k$, then each x-value maps to the unique y-value $\frac{a}{x-h} + k$ (for example).

12 a



b None are functions.

c For any circle, you can draw a vertical line which passes through the highest and lowest points on the circle, so all circles fail the vertical line test (for example).

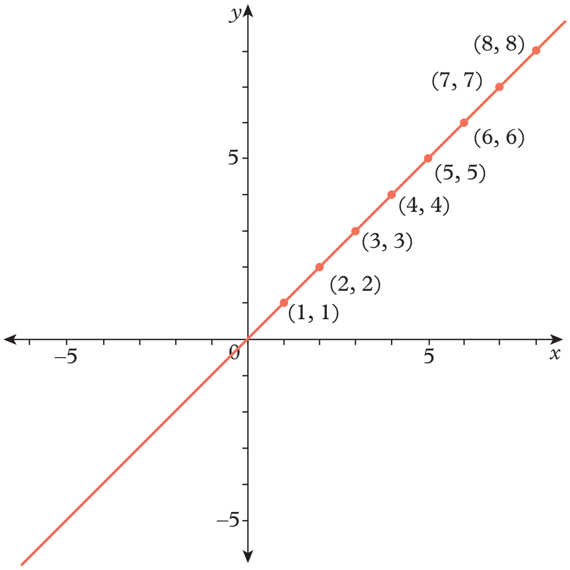
13 a If $y = x^2$, then each x-value maps to the unique y-value x^2, which is unique because the square of a number is unique.

b If $x = -2$, then y can be infinitely many values, so there is more than one output.

14 a For each function, the output is the same as the input.

b From the positive integers ($\mathbb{Z}^+$) to the real numbers ($\mathbb{R}$) or the positive integers ($\mathbb{Z}^+$)

c



d The two functions are not equal because even though they act the same way, the domains of the functions are different.

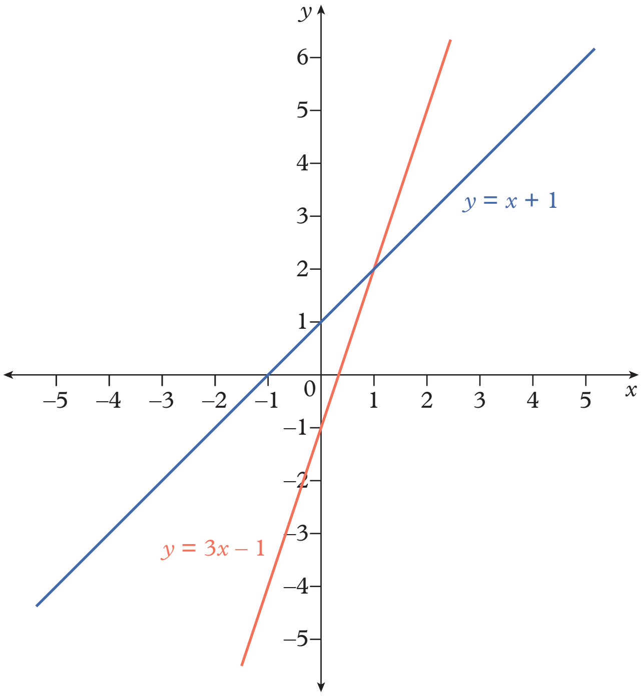
15 **a** **i** $\mathbb{R}$ **ii** $\mathbb{R}$ **iii** $\mathbb{R}$

b **i** $\mathbb{R}\backslash\{0\}$ **ii** $\mathbb{R}\backslash\{2\}$ **iii** $\mathbb{R}\backslash\{-3\}$

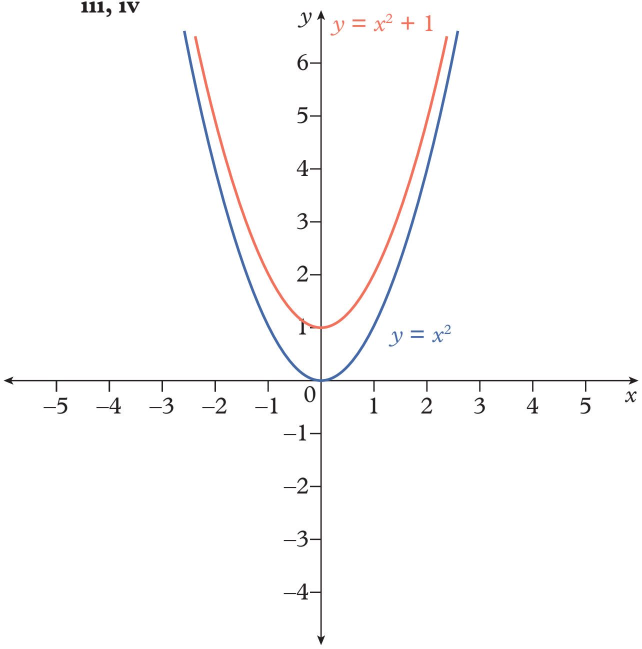
c **i** $-1 \leq x \leq 1$ **ii** $-1 \leq x \leq 3$

iii $-3 \leq x \leq 3$ **iv** $-2 \leq x \leq 0$

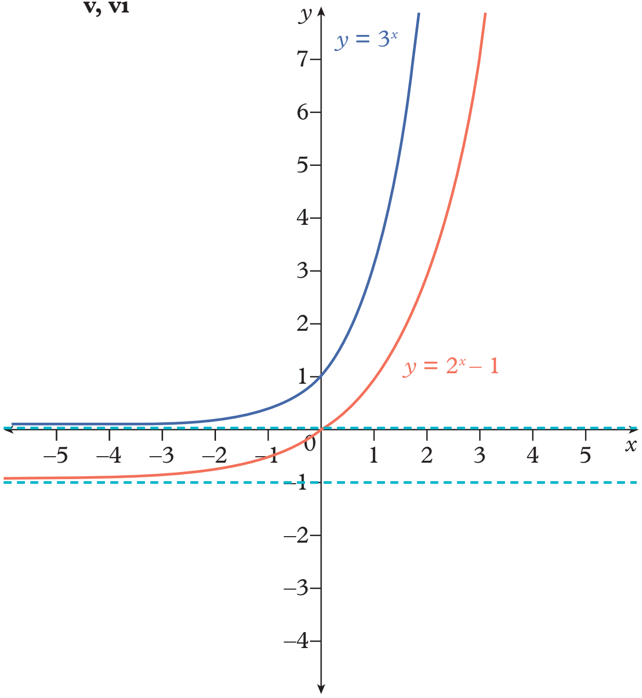
16 **a** **i, ii**



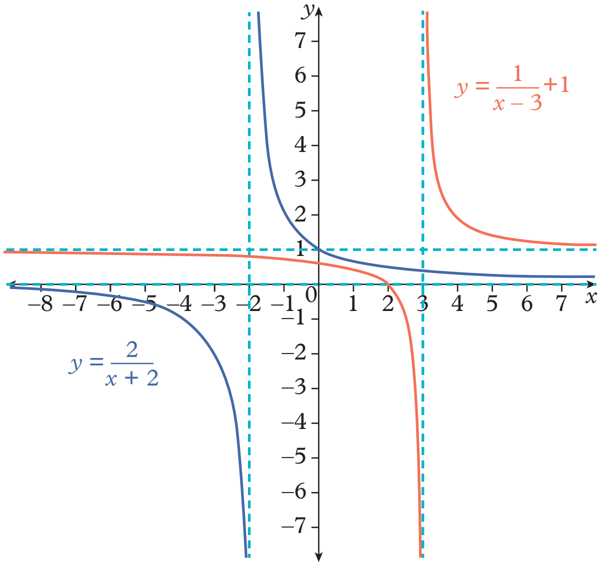
iii, iv



v, vi



vii, viii



b **i** $\mathbb{R}$ **ii** $\mathbb{R}$ **iii** $y \geq 0$

iv $y \geq 1$ **v** $y \geq 0$ **vi** $y \geq -1$

vii $\mathbb{R}\backslash\{0\}$ **viii** $\mathbb{R}\backslash\{1\}$

17 **a** **i** $f(3x) = 3x + 2$

ii $f(x + 1) = x + 3$

iii $f(2x - 1) = 2x + 1$

b **i** $f(3x) = 18x^2$

ii $f(x + 1) = 2(x + 1)^2 = 2x^2 + 4x + 2$

iii $f(2x - 1) = 2(2x - 1)^2 = 8x^2 - 8x + 2$

c **i** $f(3x) = 2^{3x} = 8^x$

ii $f(x + 1) = 2^{x+1} = 2 \times 2^x$

iii $f(2x - 1) = 2^{2x-1} = \frac{2^{2x}}{2} = \frac{4^x}{2}$

d **i** $f(3x) = 3x - 9x^2$

ii $f(x + 1) = -x(x + 1) = -x^2 - x$

iii $f(2x - 1) = -2(x - 1)(2x - 1) = -4x^2 + 6x - 2$

18 **a** **i** translate 2 units up

ii translate 3 units down

iii translate 5 units left

iv translate 4 units right

v reflect in the x-axis

vi reflect in the y-axis

vii dilate by a factor of 2 in the direction of the y-axis

viii dilate by a factor of $\frac{1}{2}$ in the direction of the x-axis

b **i** $f(x) + 2 = x^2 + 2$ **ii** $f(x) - 3 = x^2 - 3$

iii $f(x + 5) = (x + 5)^2$ **iv** $f(x - 4) = (x - 4)^2$

v $-f(x) = -x^2$ **vi** $f(-x) = x^2$

vii $2f(x) = 2x^2$ **viii** $f(2x) = 4x^2$

19 **a** dilate by a factor of 2 in the direction of the y-axis and then translate 1 unit to the right

b translate 1 unit to the left and 3 units down

c dilate by a factor of 2 in the direction of the y-axis, reflect in the x-axis and then translate 1 unit down

d dilate by a factor of 4 in the direction of the y-axis, reflect in the x-axis and in the y-axis, and then translate 2 units up

20 **a** $f(g(x)) = x^2 + x + 1$

b $g(f(x)) = (x + 1)(x + 2) = x^2 + 3x + 2$

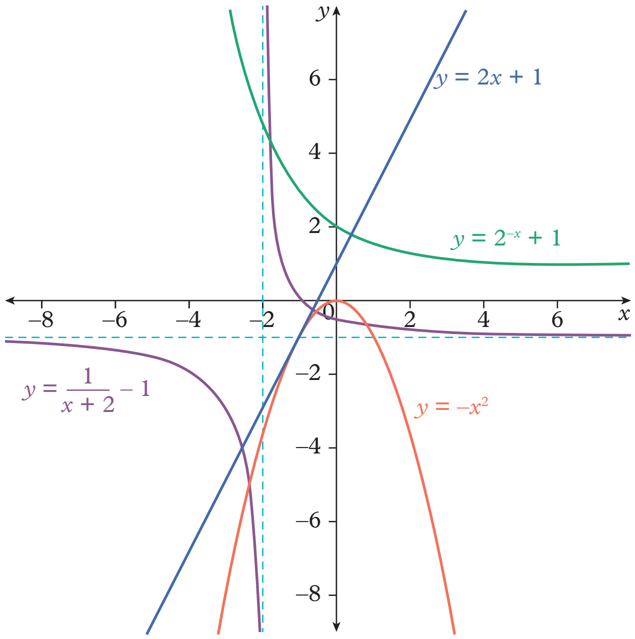
c $f(h(x)) = 2^x + 1$

d $h(f(x)) = 2^{x+1} = 2 \times 2^x$

e $g(h(x)) = 2^{2x} + 2^x = 4^x + 2^x$

f $h(g(x)) = 2^{x^2+x}$

21 **a**



b $y = 2x + 1, y = 2^{-x} + 1$ and $y = \frac{1}{x+2} - 1$ are injective

c Such quadratics have the shape of a parabola, which always fails the horizontal line test by choosing any y-value in the range other than that of the turning point.

EX
p296

6B Polynomials

1 **a** **i** 4 **ii** 3 **iii** 5

iv $2x^3$ **v** 2 **vi** 3

vii non-monic

b **i** 6 **ii** 5 **iii** −3

iv $4x^5$ **v** 4 **vi** −2

vii non-monic

c **i** 4 **ii** 4 **iii** 1

iv $-5x^4$ **v** −5 **vi** 5

vii non-monic

d **i** 7 **ii** 7 **iii** 0

iv x^7 **v** 1 **vi** −1

vii monic

e **i** 5 **ii** 6 **iii** 9

iv $-7x^6$ **v** −7 **vi** 2

vii non-monic

f **i** 3 **ii** 10 **iii** 3

iv $-11x^{10}$ **v** −11 **vi** 0

vii non-monic

2 **a** $5x^3 - 3x^2 - 4x + 2$ **b** $x^5 - 2x^4 + 4x^2$

c $2x^4 + 6x^3 + 2x^2 - 3$ **d** $5x^2 - 11x + 9$

e $3x^3 + 5x - 3$ **f** $-4x^4 - 4x^3 + 6x^2 + x$

3 **a** 2 **b** −8 **c** 16 **d** −68

4 **a** 127 **b** −5 **c** 15 **d** 97

5 **a** $-\frac{1}{2}$

b $\frac{137}{27} = 5\frac{2}{27}$

c $7 - \sqrt{2}$

d $\frac{13\sqrt{3} + 18}{9} = \frac{13\sqrt{3}}{9} + 2$

6 **a** $2x^3 + 2x^2 - 4x - 4$ **b** $2x^3 - 4x^2 + 10x - 8$

c $-2x^3 + 4x^2 - 10x + 8$ **d** $6x^3 - 3x^2 + 9x - 18$

e $-6x^2 + 14x - 4$ **f** $4x^3 + x^2 - x - 10$

g $2x^3 - 13x^2 + 31x - 14$ **h** $6x^3 - 9x^2 + 23x - 22$

i $6x^5 - 17x^4 + 20x^3 - 41x^2 + 48x - 12$

j 0 **k** 56 **l** −38

7 **a** $x^5 + 2x^4 + 7x^3$

b $4x^4 - 20x^3 + 8x^2$

c $-6x^7 + 24x^5 - 6x^3$

d $x^3 + x^2 - 2x + 12$

e $x^4 + 3x^3 - 8x^2 - 3x + 7$

f $x^7 + 4x^6 - 4x^4 - 10x^3 + 8x^2 + 3x - 2$

8 **a** $3x^3 + 15x^2 + 18x$ **b** $-7x^4 - 21x^3 + 70x^2$

c $x^3 + 10x^2 + 27x + 18$ **d** $x^3 - 6x^2 + 5x + 12$

e $3x^3 - 23x^2 + 44x - 20$ **f** $8x^3 - 2x^2 - 113x - 28$

9 **a** $x^6 + 8x^3 + 16$

b $x^4 + 6x^3 + 9x^2$

c $2x^5 - 8x^3 + 8x$

d $x^6 - 10x^5 + 25x^4$

e $x^3 + 6x^2 + 12x + 8$

f $8x^3 - 36x^2 + 54x - 27$

g $x^4 + 12x^3 + 54x^2 + 108x + 81$

h $x^4 - 4x^3 + 6x^2 - 4x + 1$

10 **a** polynomial **b** not a polynomial

c not a polynomial **d** not a polynomial

e polynomial **f** not a polynomial

g polynomial **h** not a polynomial

i not a polynomial

11 6

12 1

13 **a** **i** $n + 1$ **ii** 1 **iii** n **iv** $2n$

b **i** n **ii** n **iii** $n + m$

14 $k = -4$

15 $k = 11$

16 $a = -1, b = 4$

17 **a** $a^3 - 3a^2 + 2a + 1$

b $-27a^3 - 27a^2 - 6a + 1$

c $a^6 - 3a^4 + 2a^2 + 1$

d $-a^6 - 3a^4 - 2a^2 + 1$

e $a^3 + 3a^2 + 2a + 1$

f $8a^3 - 24a^2 + 22a - 5$

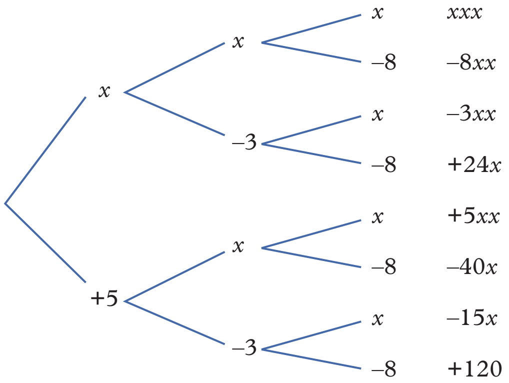
18
$$\begin{aligned}(2x-1)^4 &= (2x-1)(2x-1)(2x-1)(2x-1)\\ &= (2x-1)(2x-1)(4x^2-4x+1)\\ &= (2x-1)(8x^3-8x^2+2x-4x^2+4x-1)\\ &= (2x-1)(8x^3-12x^2+6x-1)\\ &= 16x^4-24x^3+12x^2-2x-8x^3+12x^2-6x+1\\ &= 16x^4-32x^3+24x^2-8x+1\end{aligned}$$

19 **a** $x^6 - 6x^4 + 12x^2 - 8$

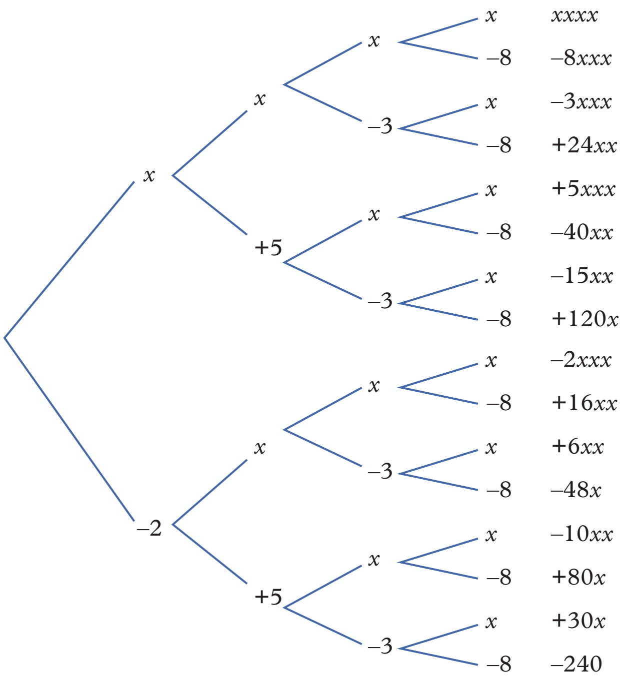
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$

c $16x^{12} + 32x^9 + 24x^6 + 8x^3 + 1$

20 **a** $x^3 - 6x^2 - 31x + 120$



b $x^4 - 8x^3 - 19x^2 + 182x - 240$



EX p300

6C Dividing polynomials

1 **a** quotient $x + 3$, remainder 5

b quotient $x - 8$, remainder −6

c quotient $5x + 6$, remainder 10

d quotient $9x - 1$, remainder 18

e quotient $7x + 2$, remainder −20

f quotient $6x - 7$, remainder 100

2 **a** quotient $x^2 + x - 2$, remainder 1

b quotient $x^2 + 2x + 5$, remainder -6

c quotient $x^2 + 8x + 12$, remainder 9

d quotient $x^2 - 7x + 8$, remainder -7

e quotient $2x^2 - x - 3$, remainder 2

f quotient $3x^2 + x - 6$, remainder -4

g quotient $5x^2 + 3x + 3$, remainder -6

h quotient $4x^2 + 5x - 13$, remainder 12

i quotient $x^3 + x^2 - 5x + 3$, remainder 5

j quotient $x^3 - 3x^2 - 4x - 9$, remainder -21

3 **a** quotient $x^2 - 3x + 10$, remainder -9

b quotient $2x^2 + x + 2$, remainder -2

c quotient $4x^3 + 4x^2 + x + 1$, remainder -4

d quotient $3x^3 - 6x^2 + 12x - 24$, remainder 7

4 **a** **i** $x + 3 + \frac{5}{x + 7}$

ii $(x + 7)(x + 3) + 5$

b **i** $x - 8 - \frac{6}{x - 4}$

ii $(x - 4)(x - 8) - 6$

c **i** $5x + 6 + \frac{10}{x - 2}$

ii $(x - 2)(5x + 6) + 10$

d **i** $9x - 1 + \frac{18}{x - 4}$

ii $(x - 4)(9x - 1) + 18$

e **i** $7x + 2 - \frac{20}{2x + 7}$

ii $(2x + 7)(7x + 2) - 20$

f **i** $6x - 7 + \frac{100}{10x + 3}$

ii $(10x + 3)(6x - 7) + 100$

5 **a** **i** $x^2 + x - 2 + \frac{1}{x - 3}$

ii $(x - 3)(x^2 + x - 2) + 1$

b **i** $x^2 + 2x + 5 - \frac{6}{x + 1}$

ii $(x + 1)(x^2 + 2x + 5) - 6$

c **i** $x^2 + 8x + 12 + \frac{9}{x - 2}$

ii $(x - 2)(x^2 + 8x + 12) + 9$

d **i** $x^2 - 7x + 8 - \frac{7}{x + 4}$

ii $(x + 4)(x^2 - 7x + 8) - 7$

e **i** $2x^2 - x - 3 + \frac{2}{x + 2}$

ii $(x + 2)(2x^2 - x - 3) + 2$

f **i** $3x^2 + x - 6 - \frac{4}{x - 1}$

ii $(x - 1)(3x^2 + x - 6) - 4$

6 **a** $3x^2 + 7 + \frac{6}{x + 2}$

b $2x^2 - 5x + \frac{9}{x - 3}$

c $x^3 - 5x^2 - 8 + \frac{50}{x + 6}$

d $x^3 + 12 + \frac{50}{x - 7}$

7 **a** $(x - 3)(x + 5) + 6$

b $(x - 3)(x^2 - 4x + 9) + 6$

c $(x - 3)(x^2 - 6x + 2) - 11$

d $(x - 3)(x^3 + 8x^2 + 9x + 4) + 3$

8 **a** $4 - \frac{5}{x + 2}$ **b** $9 + \frac{23}{x - 3}$

c $1 - \frac{6}{x - 2}$ **d** $5 + \frac{29}{2x - 5}$

9 **a** $n - 1$

b $n - m$

c The degree of the remainder is less than the degree of the divisor $x^2 - 3$, so it has at most a linear term and a constant term and is of the form $ax + b$.

10 **a** $\frac{(x + 4)(x^2 + 5x + 3) + 5}{x + 4}$

b $\frac{(x + 4)(x^2 + 5x + 3)}{x + 4} + \frac{5}{x + 4} = x^2 + 5x + 3 + \frac{5}{x + 4}$

The fraction $\frac{(x + 4)(x^2 + 5x + 3)}{x + 4}$ can be simplified by dividing the numerator and denominator by their common factor $(x + 4)$.

11 **a** $k = 24$

b linear factor $(x - 2)$; quadratic factor $(x^2 - x - 12)$

$$\begin{aligned} P(x) &= (x - 2)(x^2 - x - 12) \\ &= (x - 2)(x - 4)(x + 3) \end{aligned}$$

12 **a** $x^3 + 7x^2 - 26x - 72$

b $(x + 2)(x^2 + 5x - 36) = (x + 2)(x - 4)(x + 9)$

13 **a** quotient $x - 3$, remainder $x - 1$

b quotient $x^2 + x - 4$, remainder $6x - 7$

c quotient $2x^2 + 2$, remainder -5

14 **a** $f(a) = 0$

b $f(a) = \frac{P(a)}{D(a)} = \frac{b}{0}$, which is undefined as the denominator is zero.

c $f(x) = Q(x)$

EX p305

6D Remainder and factor theorems

1 **a** 14

b 14; same answer

c 24

d 24; same answer

e $P(4) = 64 + 16 - 40 + 8 = 48$; the remainder is 48.

f

$$\begin{array}{r} x^2+5x+10 \\ x-4\overline{\big)\,x^3+x^2-10x+8} \\ \underline{x^3-4x^2} \\ 5x^2-10x+8 \\ \underline{5x^2-20x} \\ 10x+8 \\ \underline{10x-40} \\ 48 \end{array}$$

Long division produces a quotient of $x^2 + 5x + 10$ and remainder of 48.

2 **a** 0 **b** 14 **c** –16 **d** 0

3 **a** 12 **b** 15 **c** 15 **d** 0

4 **a** 12 **b** –48 **c** 0 **d** 630

5 **a** no **b** yes **c** no **d** no
e no **f** yes **g** no **h** yes

6 **a** no **b** no **c** yes **d** no
e no **f** yes **g** yes **h** no

7 **a** $(x-1)$ or $(x+3)$ or $(x+6)$
b $(x-2)$ or $(x-3)$ or $(x+4)$
c $(x-3)$ or $(x+3)$ or $(x-4)$
d $(x+2)$ or $(x+3)$ or $(x-5)$
e $(x-1)$ or $(x-3)$
f $(x+2)$ or $(x-4)$
g $(x-2)$
h $(x+3)$

8 **a** $(x-1)(x+2)(x+5)$
b $(x-2)(x+2)(x+3)$
c $(x-1)(x-2)(x+7)$
d $(x-2)(x+2)(x+5)$
e $(x+3)(x-4)(x-4)$ or $(x+3)(x-4)^2$
f $(x-2)(x-3)(x+3)$
g $(x-2)(2x+1)(x+2)$
h $(x+2)(4x-5)(x-3)$
i $(x-4)(3x+2)(2x-1)$
j $(x-1)(4x-3)(3x+1)$

9 **a** $(x-1)(x+3)(x+6)$
b $(x-2)(x-3)(x+4)$
c $(x-3)(x+3)(x-4)$
d $(x+2)(x+3)(x-5)$
e $(x-1)(3x+2)(x-3)$
f $(x+2)(2x-1)(x-4)$
g $(x-2)(4x+1)(2x-3)$
h $(x+3)(2x-1)(2x-3)$

10 **a** $k = 1, 3$ or 5
b $P(x) = (x+1)(x+3)(x+5)$

11 **a** 2 **b** 3 **c** 4
d 7 **e** 12 **f** n

12 **a** 3
b –24; the constant term of $P(x)$ will be the product of the constant terms of the three linear factors, i.e. $2 \times -3 \times 4 = -24$
c –2, 3, –4; they are factors of –24

13 **a** $P(-1) = 0$, so $(x+1)$ is a factor
b $P(2) = 0$ so $(x-2)$ is a factor
c $P(-3) = 0$ so $(x+3)$ is a factor
d $P(x) = (x+1)(x-2)(x+3)$
e $(x+1)(x-2)(x+3)$
$= (x+1)(x^2+x-6)$
$= x^3+x^2-6x+x^2+x-6$
$= x^3+2x^2-5x-6$

14 **a** $(x+1)$, $(x+4)$ and $(x-2)$
b $P(x) = (x+1)(x+4)(x-2)$
c $(x+1)(x+4)(x-2)$
$= (x+1)(x^2+2x-8)$
$= x^3+2x^2-8x+x^2+2x-8$
$= x^3+3x^2-6x-8$

15 **a** $(x-1)$, $(x+2)$ and $(x-5)$
b $P(x) = (x-1)(x+2)(x-5)$

16 $(x+1)(x-3)(x+4)$

17 **a** $x = a$
b $P(a) = r$
c The product $(a-a)Q(a)$ in the expression for $P(a)$ is equal to zero, so the value of $P(a)$ is equal to the remaining term, r, which is the remainder of the division $P(x) \div (x-a) = \dfrac{(x-a)Q(x)+r}{x-a}$
$= Q(x) + \dfrac{r}{x-a}$
d $x = \dfrac{a}{b}$
e $F\left(\dfrac{a}{b}\right) = h$
f The product $\left(b \times \dfrac{a}{b} - a\right)G\left(\dfrac{a}{b}\right)$ in the expression for $F\left(\dfrac{a}{b}\right)$ is equal to zero, so the value of $F\left(\dfrac{a}{b}\right)$ is equal to the remaining term, h, which is the remainder of the division $F(x) \div (bx-a) = \dfrac{(bx-a)G(x)+h}{bx-a}$
$= G(x) + \dfrac{h}{bx-a}$

18 **a** $a = -15$ **b** $b = 5$
c $c = 31$ **d** $m = 6$ and $n = -3$
e $a = -33$, $b = -15$

19 **a** You need to substitute x values that are fractions $\left(\dfrac{1}{3} \text{ and } -\dfrac{3}{2}\right)$; the polynomial factorises to $(3x-1)(2x+3)(x+1)$.
b There is only one linear factor; the polynomial factorises to $(x+2)(x^2+3x+4)$.

20 Sample answer: $x^3+x^2+2x-11$

21 **a** $(x-1)(x+1)(x+2)(x-3)$
b $(x-1)(x+2)(x+3)(x+4)$
c $(x+1)(x+2)(2x-1)(x+4)$

22 **a**

$$\begin{array}{lrl} & & x^2+9 \\ & x^2+4\big) & \overline{x^4+13x^2+36} \\ x^2(x^2+4) & & \underline{-(x^4+4x^2)} \\ & & +9x^2+36 \\ 9(x^2+4) & & \underline{-(9x^2+36)} \\ & & 0 \end{array}$$

b $(x^2+4)(x^2+9)$

EX
p307

Checkpoint

1 **a**, **c** and **f** are functions

2 **a** polynomial
i 7 **ii** $4x^7$ **iii** 4
b not a polynomial; $-9x^{-100}$

c polynomial

i 81 **ii** $\frac{5y^{81}}{9}$ **iii** $\frac{5}{9}$

d not a polynomial; $-42\sqrt{y^3}$, $27y^{\frac{2}{3}}$

3 **a** $5x^3 - x^2 - 6x + 6$

b $5x^3 - 5x^2 + 10x - 8$

c $30x^3 - 18x^2 + 12x - 6$

d $10x^5 - 46x^4 + 63x^3 - 39x^2 + 22x - 7$

4 **a** quotient: $x + 6$, remainder: -5

b quotient: $x^2 - 5x + 12$, remainder: 9

c quotient: $2x - 6$, remainder: 10

d quotient: $2x^2 + 5x - 2$, remainder: -4

5 **a** $x + 12 - \frac{57}{x + 4}$

b $5x^2 + 21x + 128 + \frac{769}{x - 6}$

c $5x - 6 + \frac{-20x + 31}{x^2 + 4}$

6 **a** 20 **b** 0 **c** 84 **d** 0

7 **a** is not a factor **b** is a factor

c is not a factor **d** is a factor

8 **a** $(x - 1)(x + 4)(x - 7)$

b $(x + 1)(x + 2)(x - 3)$

9 **a** $5(x - 4)(x - 5)(x + 6)$

b $(x - 2)(x + 3)(x + 5)(x - 7)$

EX p310

6E Solving polynomial equations

1 **a** $x = -5, -2$ or 4 **b** $x = -4, -1$ or 3

c $x = -3, 2$ or 6 **d** $x = -2, 0$ or 9

e $x = -1, \frac{1}{2}$ or 1 **f** $x = -5, -3\frac{1}{2}$ or $\frac{2}{3}$

2 **a** $x = -7, -3, 1$ or 4 **b** $x = -6, 2, 3$ or 5

c $x = -4, -2$ or -1 **d** $x = -5, 0$ or 6

e $x = -4, -\frac{1}{3}, 0$ or 4 **f** $x = -2\frac{1}{2}$ or $\frac{3}{4}$

3 **a** $(x + 1)(x + 2)(x + 6) = 0; x = -6, -2$ or -1

b $(x - 3)(x - 5)(x + 4) = 0; x = -4, 3$ or 5

c $(x - 2)(x - 4)(x - 2) = 0$ or $(x - 2)^2(x - 4) = 0$;
$x = 2$ or 4

d $(x + 4)(x + 7)(x - 1) = 0; x = -7, -4$ or 1

e $(x + 3)(x + 4)(x - 4) = 0; x = -4, -3$ or 4

f $x(x - 1)(x + 1)(x - 1) = 0$ or $x(x - 1)^2(x + 1) = 0$;
$x = -1, 0$ or 1

g $(x - 5)(x - 5)(x - 5) = 0$ or $(x - 5)^3 = 0; x = 5$

h $(x + 2)(x + 3)(x + 3) = 0$ or $(x + 2)(x + 3)^2 = 0$;
$x = -3$ or -2

i $(x - 4)(x + 2)(x - 3) = 0; x = -2, 3$ or 4

j $(x + 1)(x + 1)(x + 1) = 0$ or $(x + 1)^3; x = -1$

k $15x(x + 3)(x + 6) = 0; x = -6, -3$ or 0

l $2(x - 2)(x - 1)(x + 3) = 0; x = -3, 1$ or 2

4 **a** $x = -4, -3$ or 1 **b** $x = -2, 2$ or 5

c $x = -6, -1$ or 6 **d** $x = -7, -3$ or 0

e $x = -2, -1$ or 3 **f** $x = -2$

g $x = -3$ or 3 **h** $x = -4$ or -1

i $x = 4$

5 **a** $x = -2, -1$ or 4 **b** $x = -1, 1$ or 5

c $x = -2$ or 2 **d** $x = -3, -2$ or -1

6 **a** $x = -3, -2$ or 3 **b** $x = -4, 3$ or 5

c $x = -1, 1$ or 4 **d** $x = -2$ or 3

7 **a** $x = 1, -1 - \sqrt{2}$ or $-1 + \sqrt{2}$

b $x = -2, -2 - \sqrt{3}$ or $-2 + \sqrt{3}$

c $x = -3, 1 - \sqrt{5}$ or $1 + \sqrt{5}$

d $x = 4, -4 - \sqrt{11}$ or $-4 + \sqrt{11}$

8 **a** $(x - 1)$ or $(x + 1)$ or $(x + 3)$ or $(x - 2)$.

b Sample answer: $P(x) = (x - 1)(x^3 + 2x^2 - 5x - 6)$

c Sample answer: The linear factor is $(x + 1)$, and the quadratic factor is $(x^2 + x - 6)$.

d Sample answer: $(x + 3)(x - 2)$

e $P(x) = (x - 1)(x + 1)(x + 3)(x - 2); x = -3, -1, 1$ or 2

9 **a** $x = -1, 1, 2$ or 3 **b** $x = -2, -1, 3$ or 4

c $x = -4, -3, -2$ or 2 **d** $x = -3, -2, 1$ or 5

e $x = -2, 1, 2$ or 4 **f** $x = -3, -2, 2$ or 3

10 **a** 1 **b** 2 **c** 3

d 4 **e** 6 **f** n

11 **a** $P(x) = (x - 1)(x + 2)(x - 3)$

b $x = -2, 1$ or 3

12 $x^3 + x^2 - 2x - 8 = 0$ has factorised form
$(x - 2)(x^2 + 3x + 4) = 0$, because the quadratic factor
$x^2 + 3x + 4$ cannot be factorised.
So, $(x - 2)(x^2 + 3x + 4) = 0$ has one solution $(x = 2)$
because there is only one linear factor.

13 $x^3 - 5x^2 + 3x + 9 = 0$ has factorised form
$(x + 1)(x - 3)(x - 3) = 0$, which has two solutions
$(x = -1, 3)$ because two of the linear factors are the same.

14 **a** $x(2x)(x + 2) = 2x^3 + 4x^2$

b $2x^3 + 4x^2 = 192; x^3 + 2x^2 - 96 = 0$;
$(x - 4)(x^2 + 6x + 24) = 0; x = 4$;
height 4 cm, length 8 cm, width 6 cm

15 $\frac{1}{2}x(5x)(x - 1) = 450; x = 6$;

base 6 cm, height 5 cm, length 30 cm

16 **a** $(x - 1)(x + 1)(x - 2)(x + 3) = 0$

b $(x - 1)^2(x + 1)(x - 2)(x + 3) = 0$ or
$(x - 1)(x + 1)^2(x - 2)(x + 3) = 0$ or
$(x - 1)(x + 1)(x - 2)^2(x + 3) = 0$ or
$(x - 1)(x + 1)(x - 2)(x + 3)^2 = 0$

c $2(x - 1)^2(x + 1)(x - 2)(x + 3) = 0$ or
$2(x - 1)(x + 1)^2(x - 2)(x + 3) = 0$ or
$2(x - 1)(x + 1)(x - 2)^2(x + 3) = 0$ or
$2(x - 1)(x + 1)(x - 2)(x + 3)^2 = 0$

d $5(x - 1)^3(x + 1)(x - 2)(x + 3) = 0$ or
$5(x - 1)(x + 1)^3(x - 2)(x + 3) = 0$ or
$5(x - 1)(x + 1)(x - 2)^3(x + 3) = 0$ or
$5(x - 1)(x + 1)(x - 2)(x + 3)^3 = 0$ or
$5(x - 1)^2(x + 1)^2(x - 2)(x + 3) = 0$ or
$5(x - 1)^2(x + 1)(x - 2)^2(x + 3) = 0$ or
$5(x - 1)^2(x + 1)(x - 2)(x + 3)^2 = 0$ or
$5(x - 1)(x + 1)^2(x - 2)^2(x + 3) = 0$ or
$5(x - 1)(x + 1)^2(x - 2)(x + 3)^2 = 0$ or

$5(x - 1)(x + 1)(x - 2)^2(x + 3)^2 = 0$

e $a(x - 1)^p(x + 1)^q(x - 2)^r(x + 3)^{n-p-q-r} = 0$, where p, q, r are non-zero, positive integers such that $p + q + r \leq n$.

17 $P(x) = (x + 1)(x - 2)(x - 4)(x + 3)$; $x = -3, -1, 2$ or 4

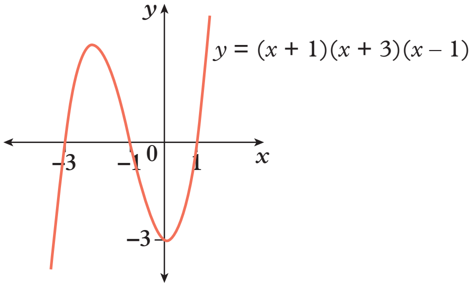
18 **a** $x = -2, -1, 1$ or 5 **b** $x = -3, -2, 1, 2$ or 4

19 $x = \sqrt{2}, -\sqrt{2}$, or -2

EX p316

6F Sketching graphs of polynomials using intercepts

1 **a**



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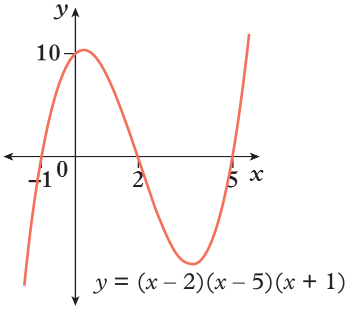

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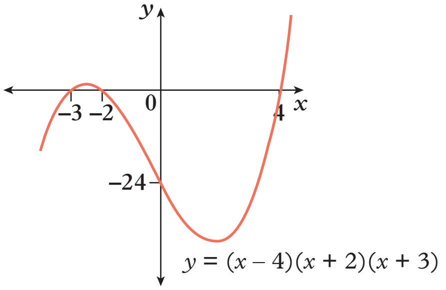

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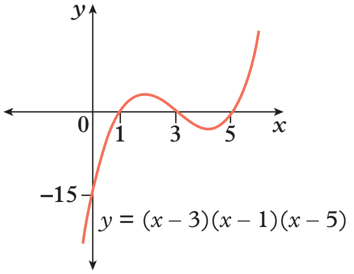

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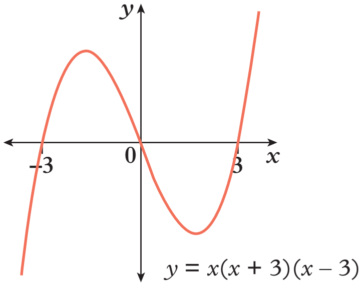

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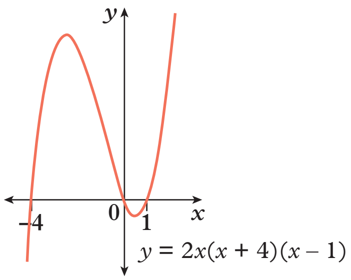


2 **a** **i** $(-1, 0)$, $(2, 0)$ and $(5, 0)$ **ii** $(0, 10)$

iii $y = (x + 1)(x - 2)(x - 5)$

b **i** $(-6, 0)$ $(-2, 0)$ and $(2, 0)$ **ii** $(0, -24)$

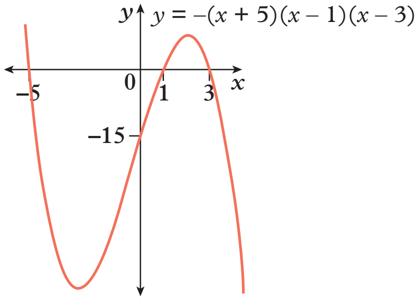
iii $y = (x + 6)(x + 2)(x - 2)$

3 **a** **i** $(-7, 0)$, $(2, 0)$ and $(3, 0)$, and $(0, 42)$

ii $(-7, 0)$, $(2, 0)$ and $(3, 0)$, and $(0, -42)$

b One graph is the reflection of the other graph in the x-axis.

4 **a**



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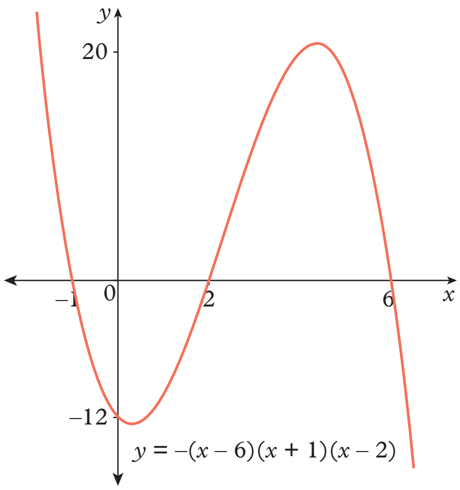

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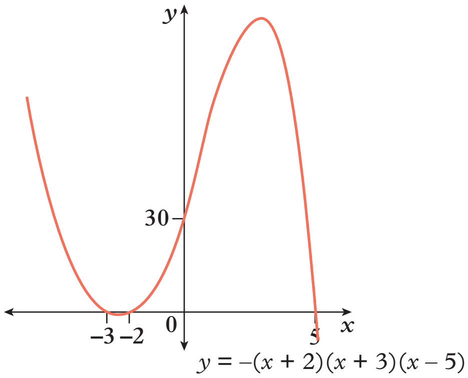

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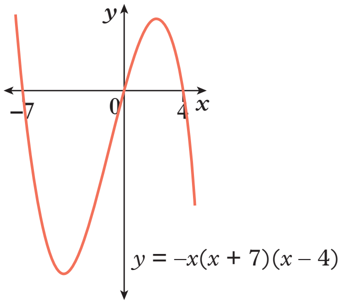
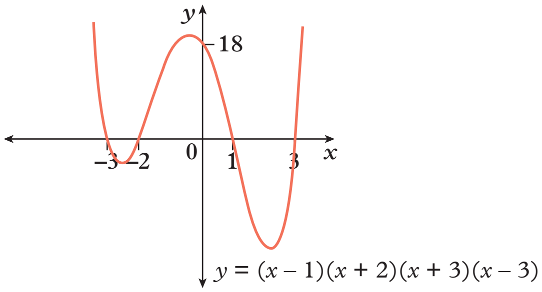


5 a



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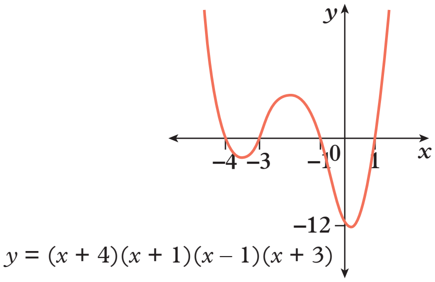

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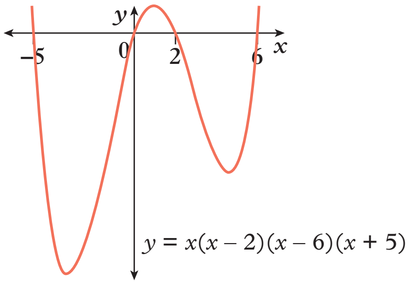

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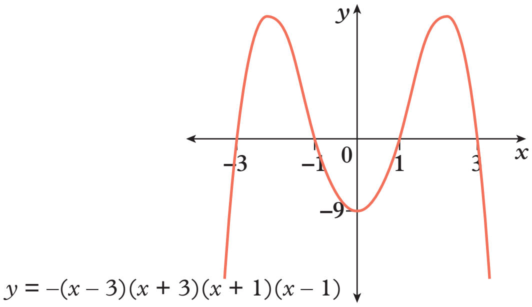

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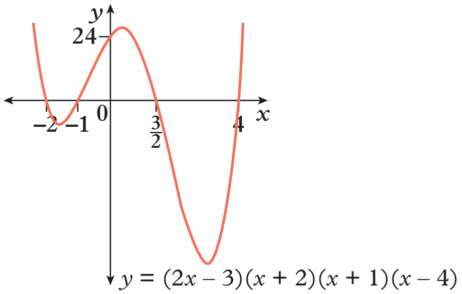

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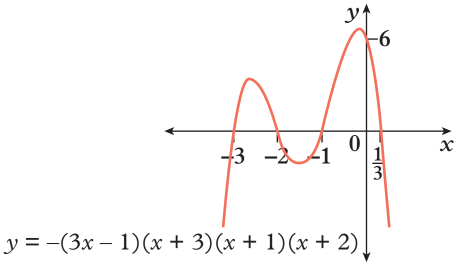
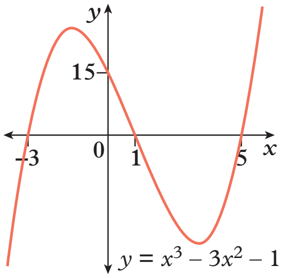


6 a



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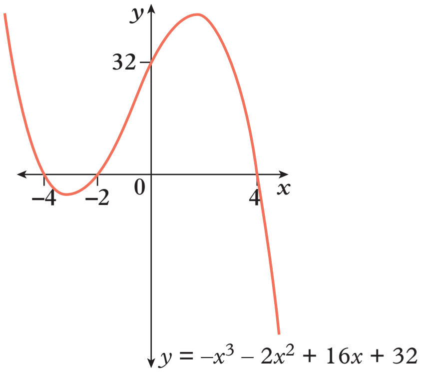

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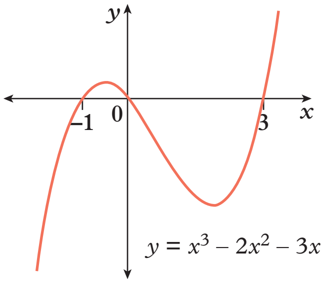

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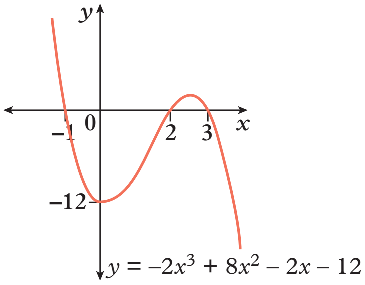

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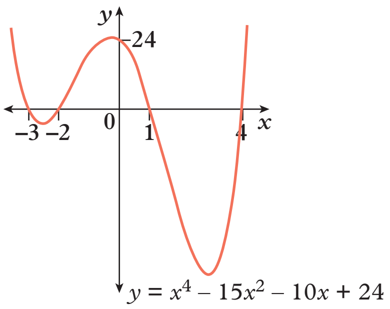

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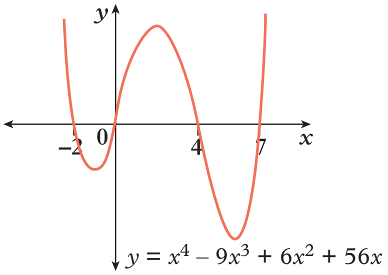

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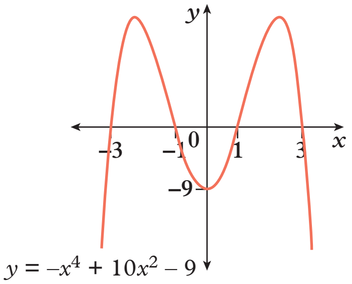
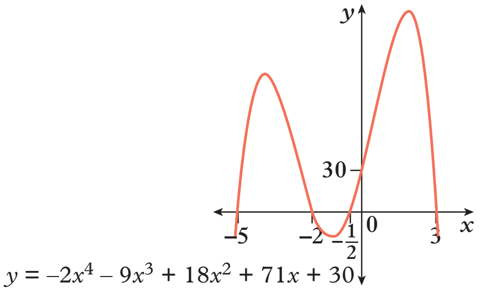


h

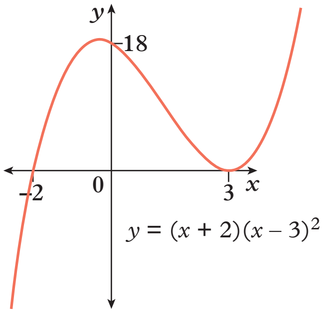


8 **a** cubic

b The x-intercepts are (–2, 0) and (3, 0); the y-intercept is (0, 18).

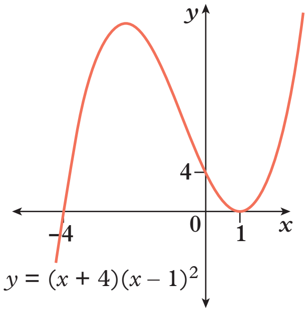
c bottom left

d

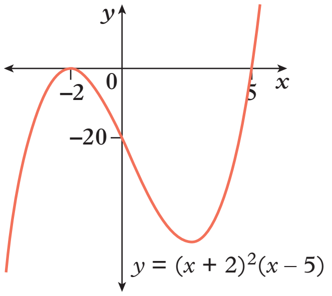


e The x-intercept at (3, 0) is also a turning point.

9 **a**



b



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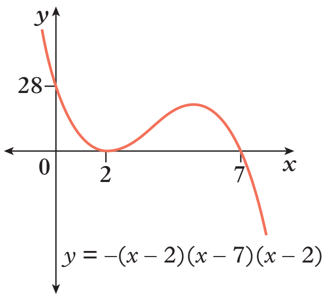

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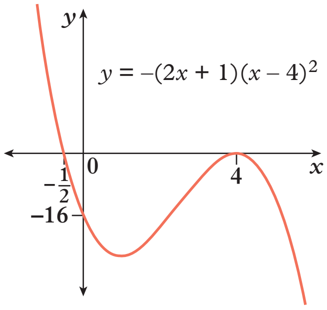
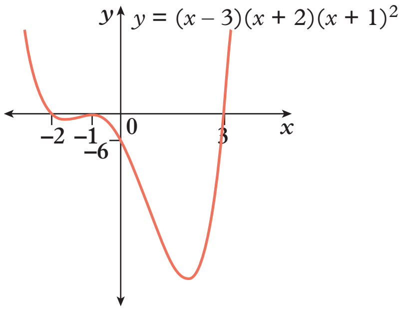
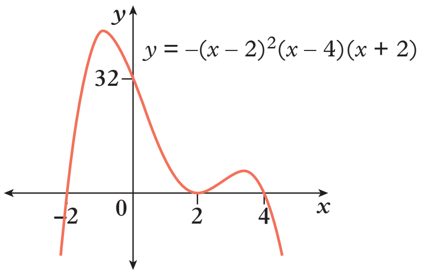


10 **a**



b



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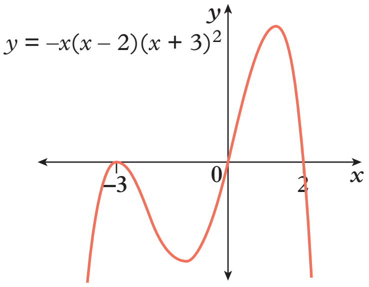

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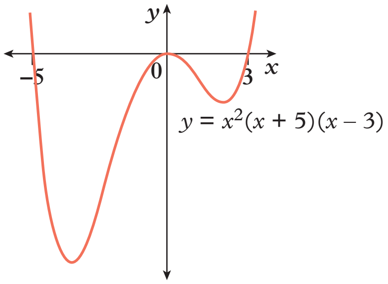
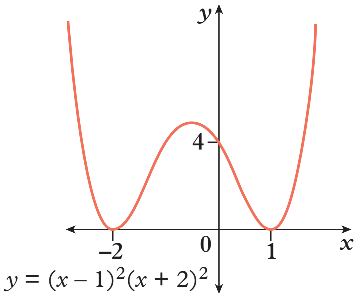
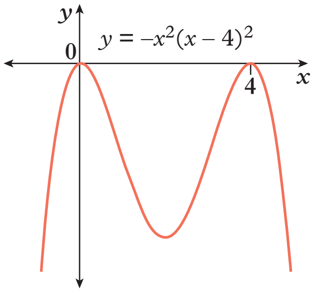


e

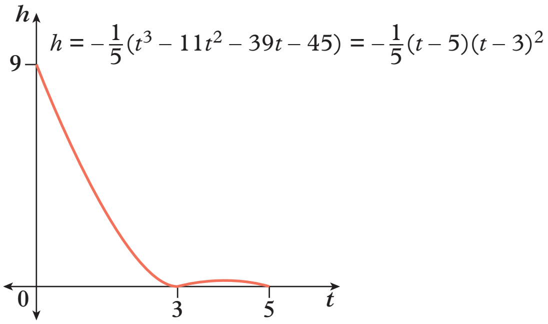


f



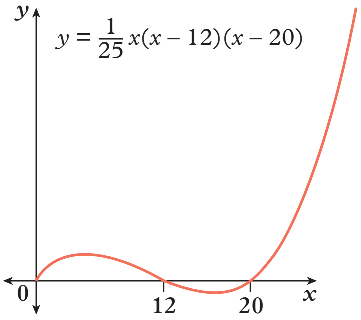
11 The x-intercept at $(a, 0)$ is also a point of inflection.

12 a



b 9 m **c** 3.2 m **d** 3 s **e** 2 s

13 a



b $y = \frac{1}{25}x(x-12)(x-20)$ or

$y = 0.04x^3 - 1.28x^2 + 9.6x$

c \$216

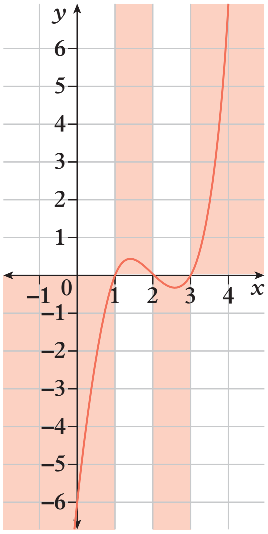
14 a

	$x < 1$	$1 < x < 2$	$2 < x < 3$	$x > 3$
$y = x - 1$	negative	positive	positive	positive
$y = x - 2$	negative	negative	positive	positive
$y = x - 3$	negative	negative	negative	positive
$y = (x - 1)(x - 2)(x - 3)$	negative	positive	negative	positive

b The shaded regions are where the product $y = (x-1)(x-2)(x-3)$ is positive (above the y-axis) and negative (below the y-axis).

c **i** 0 **ii** x-intercepts

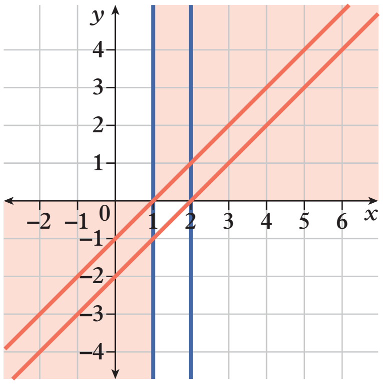
d



e

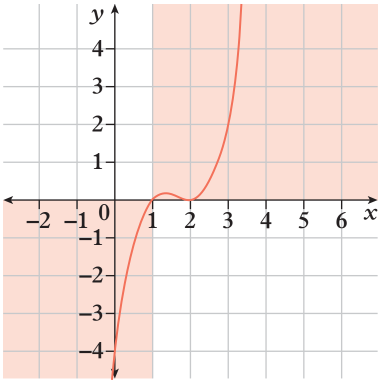
	$x < 1$	$1 < x < 2$	$x > 2$
$y = x - 1$	negative	positive	positive
$y = x - 2$	negative	negative	positive
$y = x - 2$	negative	negative	positive
$y = (x - 1)(x - 2)^2$	negative	positive	positive

f

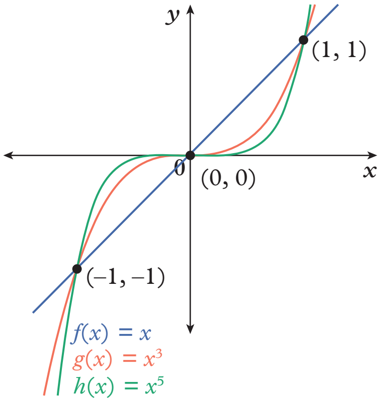


g The product, $y = (x - 1)(x - 2)^2$, is negative for $x < 1$ so below the y-axis was shaded. The product, $y = (x - 1)(x - 2)^2$, is positive for $1 < x < 2$ and $x > 2$ so above the y-axis was shaded.

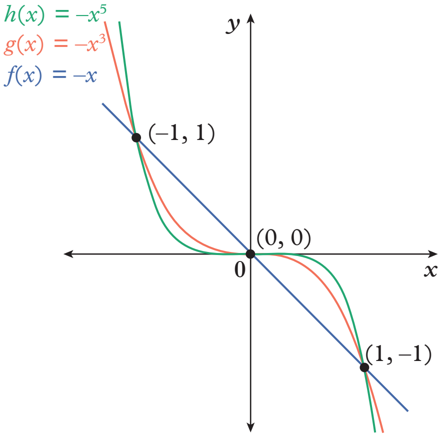
h



15 a

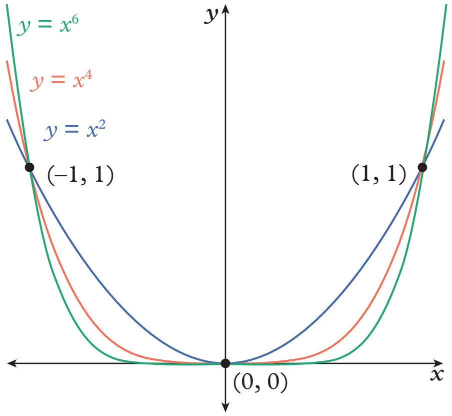


b



c For $y = x^n$ where n is odd, the graphs have roughly the same shape as $y = x$. For $y = -x^n$ where n is odd, the graphs have roughly the same shape as $y = -x$.

d



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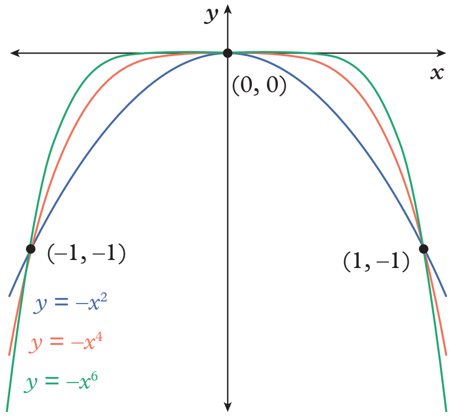
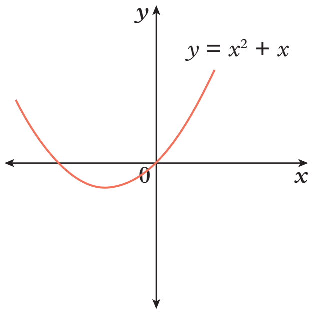


f For $y = x^n$ where n is even, the graphs have roughly the same shape as $y = x^2$. For $y = -x^n$ where n is even, the graphs have roughly the same shape as $y = -x^2$.

g i



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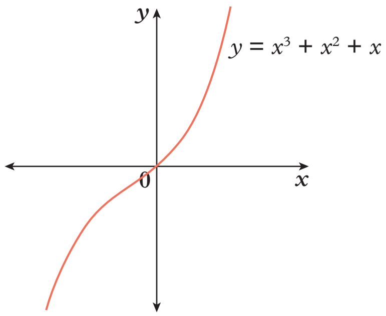

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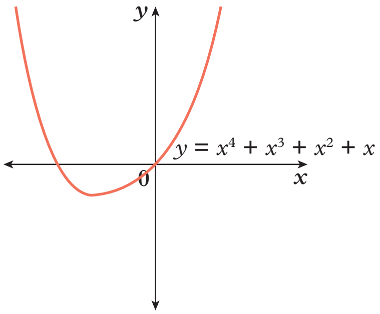

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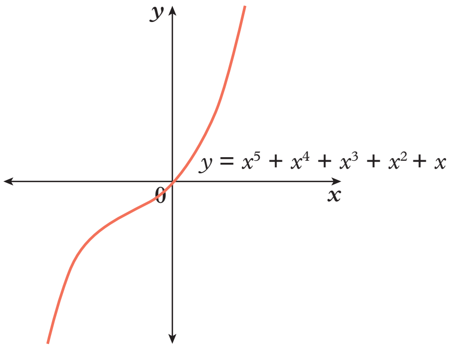

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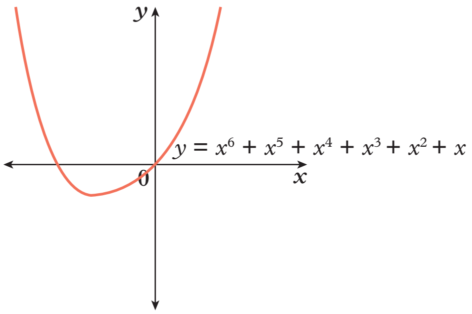

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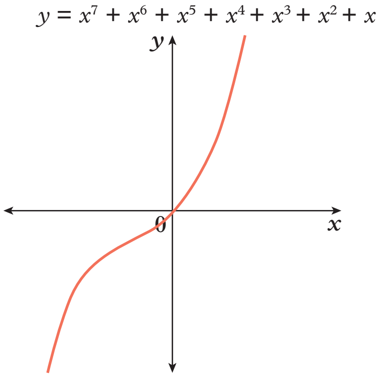


h For a leading term of x^n where n is odd, the graphs have roughly the same shape as $y = x$. For $y = -x^n$ where n is odd, the graphs have roughly the same shape as $y = -x$. For x^n where n is even, the graphs have roughly the same shape as $y = x^2$. For $y = -x^n$ where n is even, the graphs have roughly the same shape as $y = -x^2$.

16 a i 3

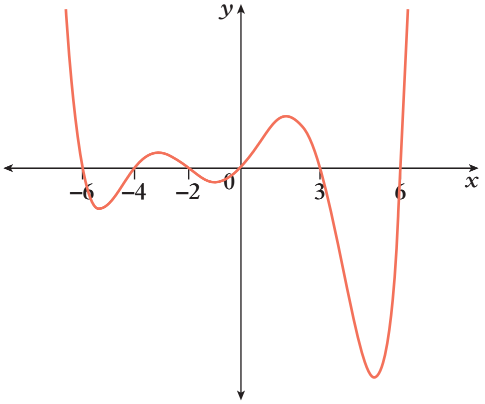
ii $\{-5, -4, -3, 3, 4, 5, 7, 8, 9\}$

iii 7, as it is not a factor of 180

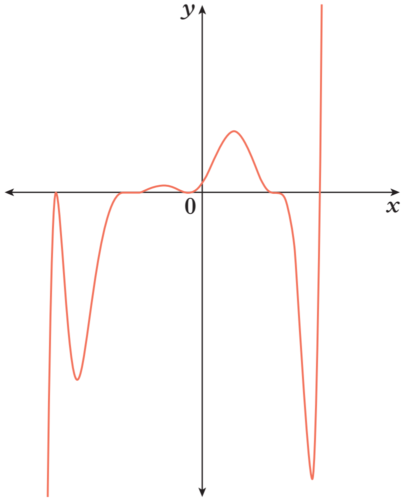
b i $(-3,0)$, $(2,0)$, $(3,0)$, $(4,0)$

ii The remaining substitutions correspond with the coordinates $(1, -24)$, $(-1, -120)$, $(-2, -120)$, $(-4, 336)$. Sara can plot these to sketch the graph of $y = Q(x)$ more accurately.

17 a $y = x(x - 6)(x - 3)(x + 2)(x + 4)(x + 6)$



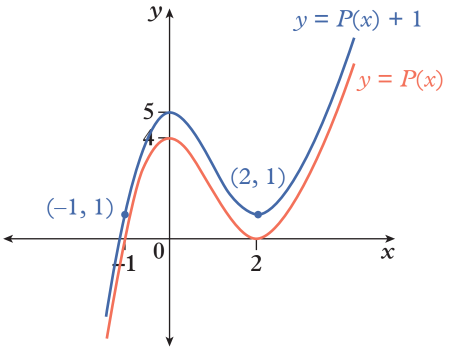
b $y = (x - 8)(x - 5)^3(x + 1)^2(x + 5)^3(x + 10)^2$



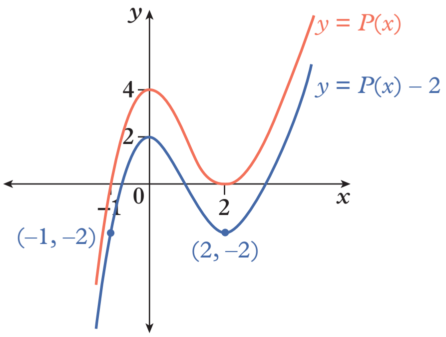
EX p322

6G Sketching graphs of polynomials using transformations

1 a



b



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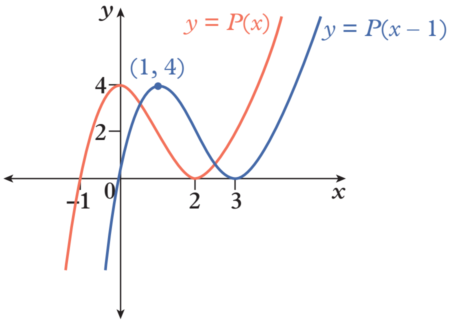

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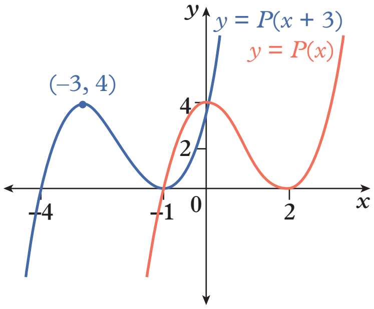

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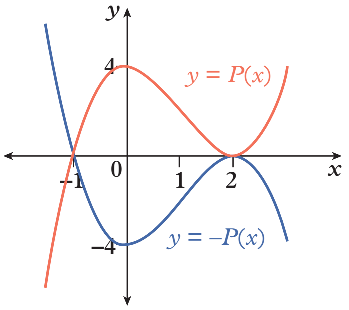

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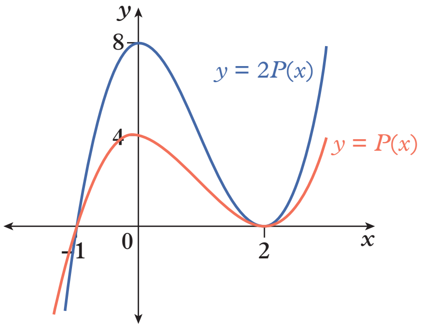

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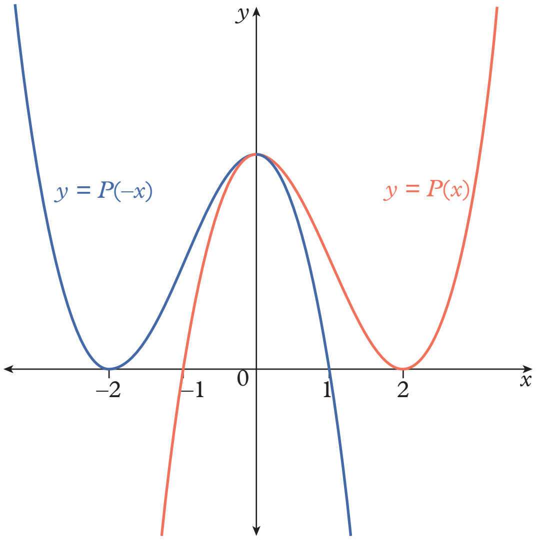
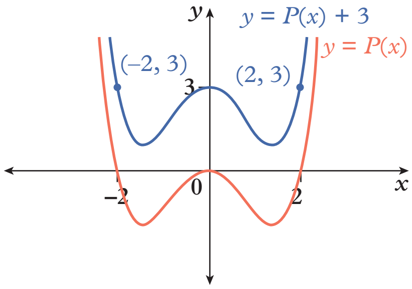


2 **a**



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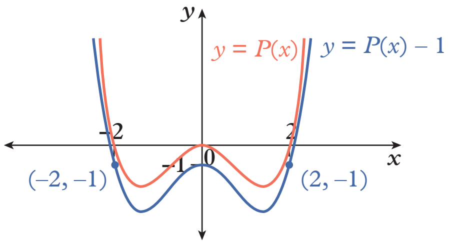

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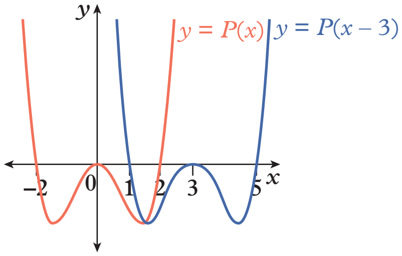

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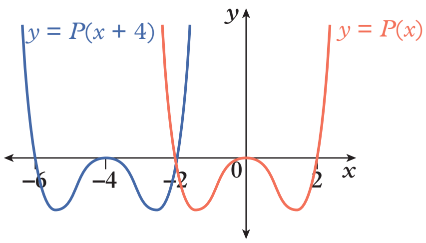

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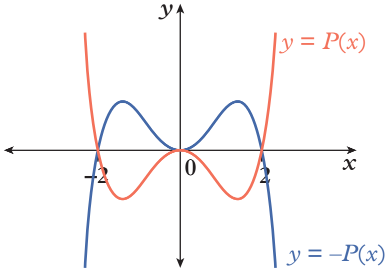

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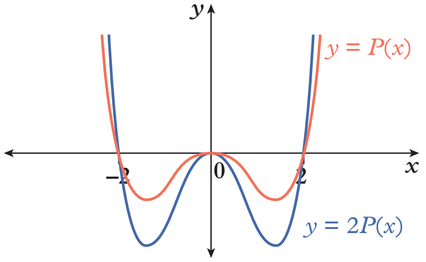
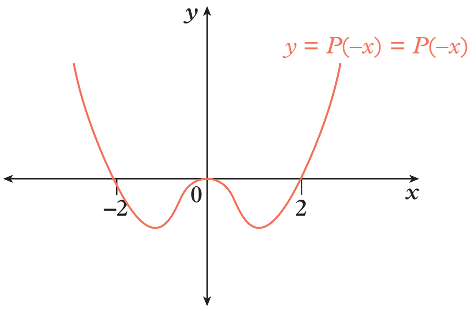


g



3 **a** translation of 4 units up and 1 unit right

b reflection in the x-axis

c vertical dilation by a factor of 3

d translation of 2 units right

4 **a** $y = (x - 1)^3 + 4$

b $y = -x^3$

c $y = 3x^3$

d $y = (x - 2)^3$

5 **a** reflection in the x-axis

b vertical dilation by a factor of 2

c translation of 1 unit down

d translation of 2 units left

6 **a** $y = -x^4$

b $y = 2x^4$

c $y = x^4 - 1$

d $y = (x + 2)^4$

7 **a** $y = -P(x)$

b $y = P(x) - 5$

c $y = P(x - 3)$

8 Both are the same graph, with the same equation, $y = -x^3$.

9 **a** **i** a dilation in the y-axis by a factor of 8 and a translation of 1 unit up

ii (0, 1)

iii The x-intercept is $\left(-\frac{1}{2}, 0\right)$ and the y-intercept is (0, 1).

b **i** a dilation in the y-axis by a factor of $\frac{1}{2}$ and a translation of 4 units down

ii (0, –4)

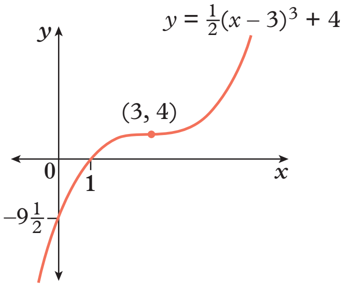
iii The x-intercept is (2, 0) and the y-intercept is (0, –4).

c **i** a dilation in the y-axis by a factor of $\frac{1}{2}$ and a translation of 3 units right and 4 units up

ii (3, 4)

iii The x-intercept is (1, 0); the y-intercept is $\left(0, -9\frac{1}{2}\right)$.

iv

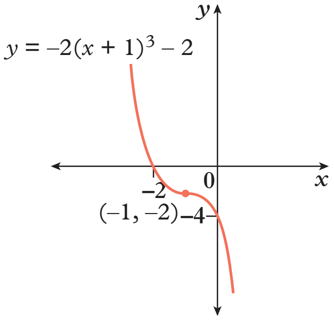


d **i** a dilation in the y-axis by a factor of 2, a reflection in the x-axis and a translation of 1 unit left and 2 units down

ii (–1, –2)

iii The x-intercept is (–2, 0); the y-intercept is (0, –4).

iv



10 Point of inflection: (h, k) x-intercept: $\left(\sqrt[3]{\frac{-k}{a}} + h, 0\right)$

11 **a** **i**

$\frac{x}{2}$	0	1	2	3
x	0	2	4	6
$y = P\left(\frac{x}{2}\right)$	5	0	–5	2

ii

$2x$	0	1	2	3
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$
$y = P(2x)$	5	0	–5	2

iii

$\frac{x}{5}$	0	1	2	3
x	0	5	10	15
$y = P\left(\frac{x}{5}\right)$	5	0	–5	2

iv

$5x$	0	1	2	3
x	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$y = P(5x)$	5	0	–5	2

v

$-\frac{x}{2}$	0	1	2	3
x	0	–2	–4	–6
$y = P\left(-\frac{x}{2}\right)$	5	0	–5	2

vi

$-2x$	0	1	2	3
x	0	$-\frac{1}{2}$	–1	$-\frac{3}{2}$
$y = P(-2x)$	5	0	–5	2

b **i** $y = P\left(\frac{x}{2}\right)$ **ii** $y = P\left(\frac{x}{7}\right)$

iii $y = P(3x)$ **iv** $y = P(6x)$

12 **a** correct

b incorrect; dilation by a factor of 4 in the y-axis

c incorrect; translate right 5 units

d correct

13 **a** **i** dilate by a factor of 2 in the direction of the y-axis and by a factor of $\frac{1}{3}$ in the direction of the x-axis, and then translate 324 units up

ii dilate by a factor of $\frac{1}{3}$ in the direction of the y-axis and by a factor of 2 in the direction of the x-axis

iii dilate by a factor of 5 in the direction of the y-axis, reflect in the y-axis and translate 15 units down

iv dilate by a factor of 2 in the direction of the y-axis and by a factor of $\frac{1}{4}$ in the direction of the x-axis, reflect in both the x- and y-axes and then translate 12 units up

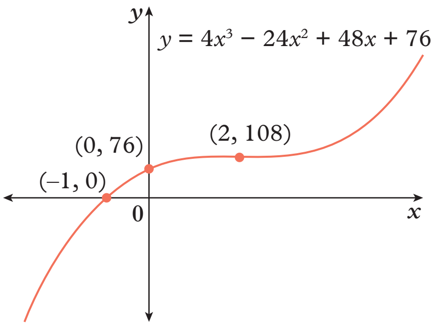
b $y = -P\left(\frac{1}{3}(x + 2)\right) - 4$

c $y = -P\left(\frac{1}{3}(x + 6)\right) - 4$, dilate by a factor of 3 in the direction of the x-axis, reflect in the x-axis and then translate 6 units to the left and 4 units down

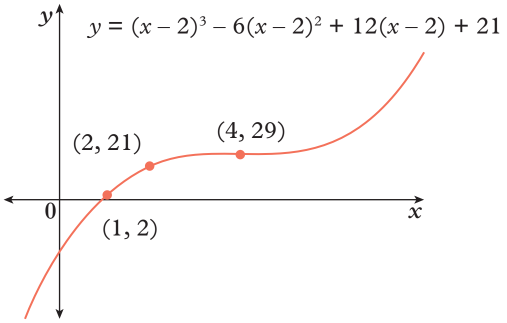
d If the expression in the brackets isn't factorised, the horizontal translation is unclear.

14 a $y = x^3 - 6x^2 + 12x + 19$

b i



ii



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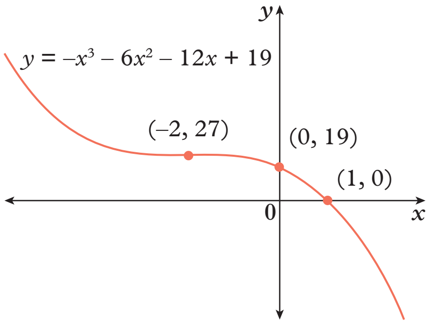


15 a i a dilation by a factor of $\frac{1}{3}$ in the x-axis

a dilation by a factor of 9 in the y-axis

ii a dilation by a factor of 2 in the x-axis

a dilation by a factor of $\frac{1}{16}$ in the y-axis

iii a dilation by a factor of $\frac{1}{5}$ in the x-axis

a dilation by a factor of 25 in the y-axis

iv a dilation by a factor of $\frac{3}{5}$ in the x-axis

a dilation by a factor of $\frac{625}{81}$ in the y-axis

b i a dilation by a factor of 3 in the x-axis

a dilation by a factor of $\frac{1}{9}$ in the y-axis

ii a dilation by a factor of $\frac{1}{2}$ in the x-axis

a dilation by a factor of 16 in the y-axis

iii a dilation by a factor of 5 in the x-axis

a dilation by a factor of $\frac{1}{25}$ in the y-axis

iv a dilation by a factor of $\frac{5}{3}$ in the x-axis

a dilation by a factor of $\frac{81}{625}$ in the y-axis

EX p327

CHAPTER 6 review

Mathematical literacy review

1 a, e, f polynomial

b, c, d not a polynomial

2 a 4 **b** 6 **c** −1

d $3x^6$ **e** 3 **f** −4

3 a A function is a relation where every element in the first set is mapped to exactly one element in the second set.

b $x^2 + y^2 = 1$ (for example)

4 A relation fails the vertical line test if a vertical line can be drawn through the graph of the relation and the line passes through more than one point on the graph.

5 a When dividing one polynomial by another polynomial, the dividend is the expression being divided by the divisor and the quotient is the resulting expression.

b The leading coefficient of a monic polynomial is equal to 1 and the leading coefficient of a non-monic polynomial is not equal to 1.

c A zero of a polynomial $P(x)$ is a root of the equation $P(x) = 0$.

d The degree of a polynomial is the index of the variable in the leading term.

Multiple choice

1 E **2** C **3** E **4** A

5 A **6** E **7** E **8** E

9 C **10** D **11** C **12** E

13 B

Short answer

1 a, c not a function

b function

2 a i 1 **ii** 26 **iii** 5

b i 1 **ii** 152 **iii** $-\frac{23}{4}$

c i 1 **ii** $-\frac{749}{6}$ **iii** −9

3 **a** yes, cubic **b** yes, linear
c yes, linear **d** not a polynomial
e yes, quadratic **f** yes, quartic

4 **a** 7 **b** 8 **c** 0
d $-5x^8$ **e** -5 **f** -7

5 **a** $x + 7$, remainder 12
b $3x - 4$, remainder 8
c $x + 3$, remainder 20
d $x + 2$, remainder $1 - 2x$

6 **a** true **b** false

7 **a** $x + 1$ or $x - 3$ **b** $x + 3$
c $x - 1$ **d** $x + 2$

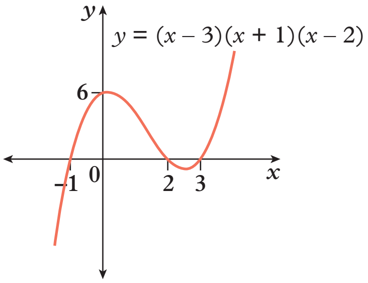
8 **a** $(x - 1)(x - 2)(x - 4)$
b $(x - 3)(x + 1)(x + 2)$

9 $x + 2$

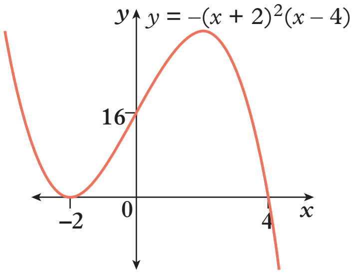
10 **a** $x = -2, -1, 1$ or 5 **b** $x = -3, 0, 2$ or 3
c $x = -3$ or 5 **d** $x = -4, 0$ or 4

11 **a** $x(x - 2)(x + 2)(x - 2)$ or $x(x - 2)^2(x + 2)$; $x = -2, 0$ or 2
b $(x - 4)(x - 4)(x - 4)$ or $(x - 4)^3$; $x = 4$

12 **a**

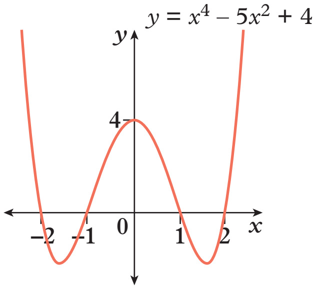


b



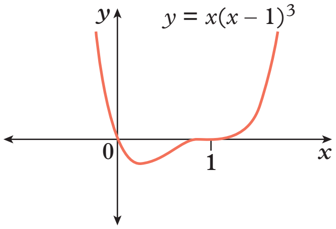
13 **a** $(-1, 0)$, $(1, 0)$ and $(5, 0)$
b $(0, 5)$
c $y = (x - 1)(x + 1)(x - 5)$

14 **a** $y = (x - 1)(x + 2)(x + 1)(x - 2)$
b The x-intercepts are $(-2, 0)$, $(-1, 0)$, $(1, 0)$ and $(2, 0)$; the y-intercept is $(0, 4)$.
c



15 **a** $y = x^2(x - 2)(x + 2)$ **b** $y = x^4 - 4x^2$

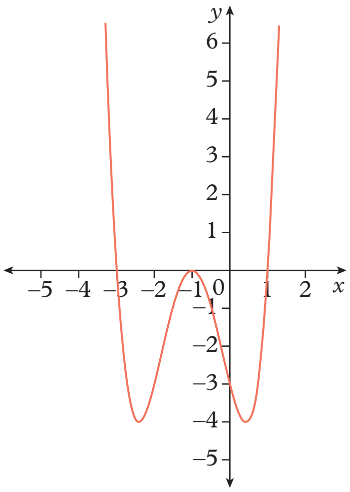
16 **a** The x-intercepts are $(0, 0)$ and $(1, 0)$; the y-intercept is $(0, 0)$.
b $x = 0$ has a multiplicity of 1
$x = 1$ has a multiplicity of 3
c



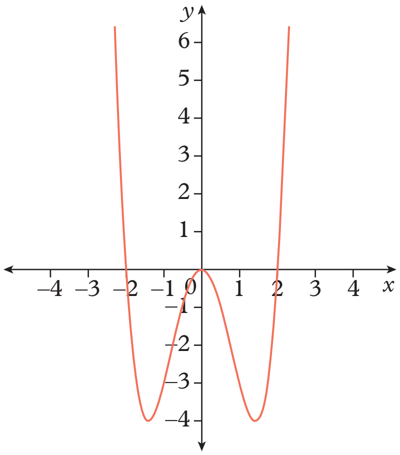
17 **a** $y = x(x - 1)^2$ (for example)
b $y = (x - 1)^2\ (x + 1)^2$ (for example)
c $y = (x - 3)^3$ (for example)
d $y = x(x + 1)\ (x - 1)^2$ (for example)

18 The graph of $y = x^3$ is translated 2 units left and 4 units down.

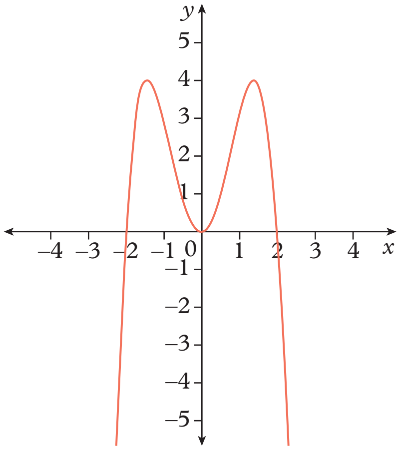
19 **a**



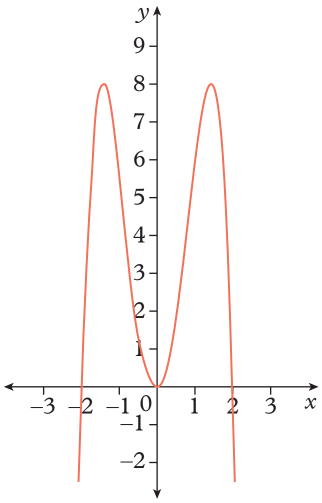
b



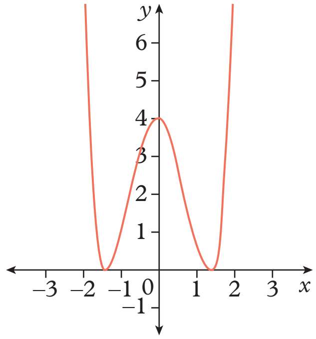
c



d



e



Analysis

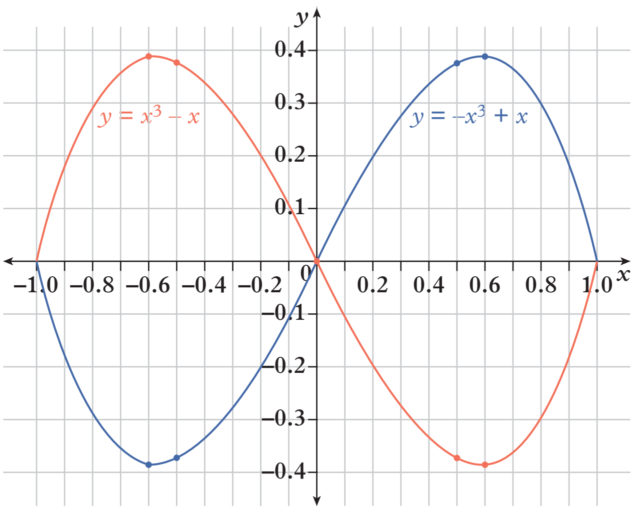
1 a i $y = x^3 - x$

x	–1	–0.6	–0.5	0	0.5	0.6	1
y	0	0.384	0.375	0	–0.375	–0.384	0

ii $y = -x^3 + x$

x	–1	–0.6	–0.5	0	0.5	0.6	1
y	0	–0.384	–0.375	0	0.375	0.384	0

iii



iv $-1 \le x \le 1$

v No, the turning points do not occur midway between the x-intercepts. They occur at an x value between –0.6 and –0.5 (close to –0.6), and between 0.5 and 0.6 (close to 0.6).

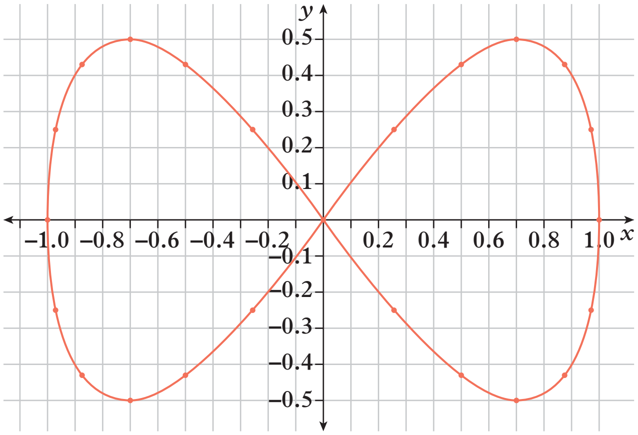
vi The turning points are at $\left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$, $\left(-\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$, $\left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$ or, to three decimal places, (–0.577, 0.385), (–0.577, –0.385), (0.577, 0.385) and (0.577, –0.385).

b i, ii

Angle (θ)	x **cos [θ]**	y $\left(\frac{\sin[2\theta]}{2}\right)$
0°	1	0
15°	0.97	0.25
30°	0.87	0.43
45°	0.71	0.5
60°	0.5	0.43
75°	0.26	0.25
90°	0	0
105°	–0.26	–0.25
120°	–0.5	–0.43
135°	–0.71	–0.5
150°	–0.87	–0.43
165°	–0.97	–0.25
180°	–1	0
195°	–0.97	0.25
210°	–0.87	0.43
225°	–0.71	0.5
240°	–0.5	0.43
255°	–0.26	0.25
270°	0	0
285°	0.26	–0.25
300°	0.5	–0.43
315°	0.71	–0.5
330°	0.87	–0.43
345°	0.97	–0.25

Angle (θ)	x **cos** [θ]	y $\left(\frac{\sin[2\theta]}{2}\right)$
360°	1	0

iii



iv It looks like the infinity symbol.

v $-1 \le x \le 1, -0.5 \le y \le 0.5$

c The model in part **b** is a better model for the infinity symbol, because the curve at $x = -1$ and $x = 1$ is less pointy.

2 **a** **i** $a = 3, b = 6, c = -33, d = -36$

$r_1 = -1, r_2 = 3, r_3 = -4$

$r_1 + r_2 + r_3 = -1 + 3 - 4 = -2 = -\frac{6}{3} = -\frac{b}{a}$

$r_1r_2 + r_1r_3 + r_2r_3 = (-1) \times 3 + (-1) \times (-4) + 3 \times (-4)$

$= -11 = -\frac{33}{3} = \frac{c}{a}$

$r_1r_2r_3 = (-1) \times 3 \times (-4) = 12 = \frac{36}{3} = -\frac{d}{a}$

ii $a = -2, b = -16, c = 38, d = -20$

$r_1 = 1, r_2 = 1, r_3 = -10$

$r_1 + r_2 + r_3 = 1 + 1 - 10 = -8 = -\frac{16}{2} = -\frac{b}{a}$

$r_1r_2 + r_1r_3 + r_2r_3 = 1 - 10 - 10 = -19 = -\frac{38}{2} = \frac{c}{a}$

$r_1r_2r_3 = 1 \times 1 \times (-10) = -10 = -\frac{20}{2} = -\frac{d}{a}$

b $P(x) = a(x - r_1)(x - r_2)(x - r_3)$

c $P(x) = ax^3 - a(r_1 + r_2 + r_3)x^2 + a(r_1r_2 + r_1r_3 + r_2r_3)x - ar_1r_2r_3$

d $b = -a(r_1 + r_2 + r_3), c = a(r_1r_2 + r_1r_3 + r_2r_3),$
$d = -ar_1r_2r_3$

e $r_1 + r_2 + r_3 = -\frac{b}{a}, r_1r_2 + r_1r_3 + r_2r_3 = \frac{c}{a}, r_1r_2r_3 = -\frac{d}{a}$

f $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ with roots r_1, r_2, r_3, r_4

$r_1 + r_2 + r_3 + r_4 = -\frac{b}{a}$

$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = \frac{c}{a}$

$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -\frac{d}{a}$

$r_1r_2r_3r_4 = \frac{e}{a}$

EX p332

SEMESTER 1 review

Short answer

1 **a** $-2x^2 - 9x + 5$ **b** $6xy^2 + 22xy$
c $-\frac{x}{2} + 7x^2$ **d** $2a^2 + 3ab$

2 **a** $6x + 15y$
b $36x^3y^5 - 32x^5y^2$
c $-\frac{15x^2}{4} - \frac{15xy}{2} + 10x$
d $\frac{b}{2} + \frac{ab^2}{4} - \frac{ab}{3}$

3 **a** \$1376 **b** \$61 476 **c** \$4639

4

Year	Value at start of year	Depreciation amount	Value at end of the year
1	\$4000	\$400	\$3600
2	\$3600	\$360	\$3240
3	\$3240	\$324	\$2916

5 **a** \$1470 **b** \$742.16 **c** \$960
d \$2517.72 **e** \$1240

6 **a** \$625 **b** \$11 200 **c** \$1700

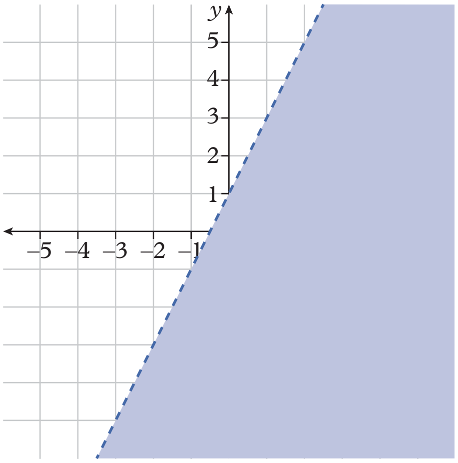
7 **a** 3.9% **b** 5.8%
c 4.6% **d** 13.3%

8 **a** 4 years **b** 2 years
c 7 years **d** 4 years

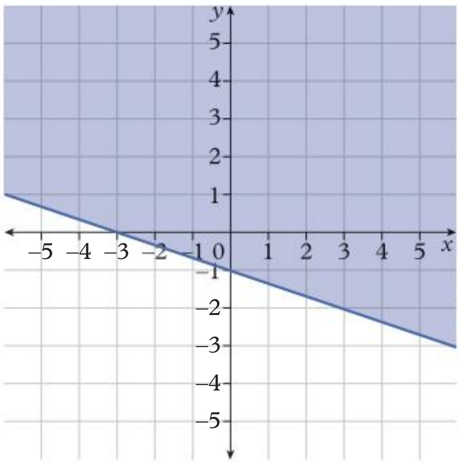
9 **a** $k = \frac{9}{4}$ **b** $k = 36$
c $k = -100$ **d** $k = 100$
e $k = \frac{1}{4}$ **f** $k = 3$

10 **a** **i** $y = \frac{9}{2}$ **ii** $x = \frac{8}{9}$
b **i** $y = 18$ **ii** $x = 18$
c **i** $y = -200$ **ii** $x = -\frac{1}{50}$
d **i** $y = 50$ **ii** $x = 50$
e **i** $y = \frac{1}{2}$ **ii** $x = 8$
f **i** $y = \frac{3}{2}$ **ii** $x = \frac{3}{2}$

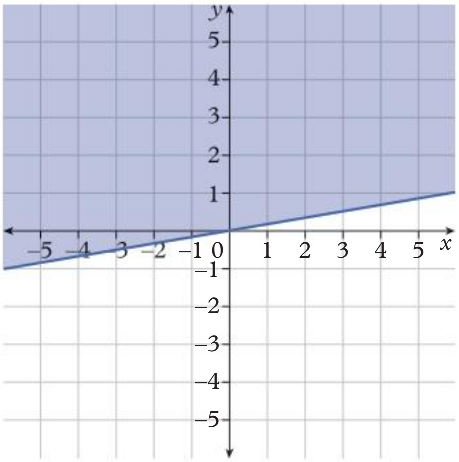
11 **a**



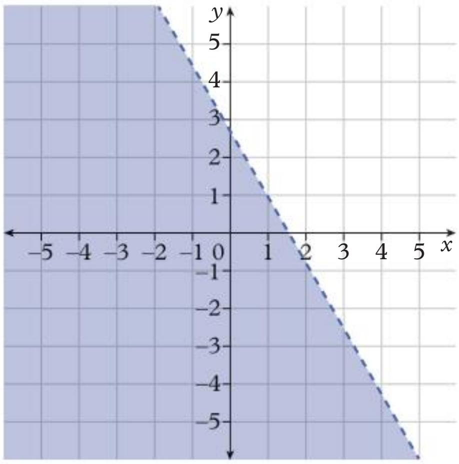
b



c



d



12 a $x = \frac{1}{2}, x = -\frac{5}{3}$

b $x = 5, x = -5$

c $x = -4, x = 14$

d $x = 4, x = 6$

e $x = -3, x = 5$

f $x = -2, x = 4$

g $x = \frac{-5 \pm \sqrt{65}}{2}$

h $x = -2, x = 7$

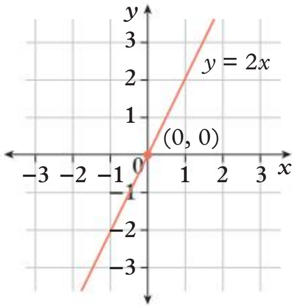
13 a $(x + 5)^2 - 25$

b $(x - 3)^2 + 3$

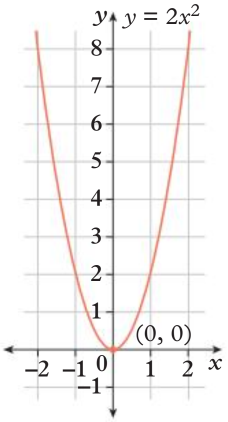
c $\left(x - \frac{5}{2}\right)^2 + \frac{23}{4}$

d $2(x - 7)^2 - 116$

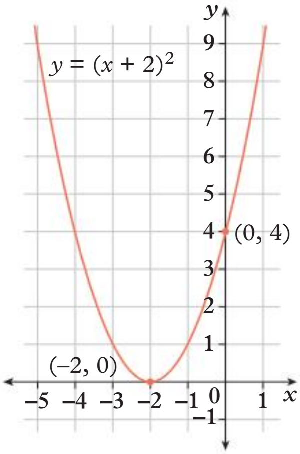
14 a



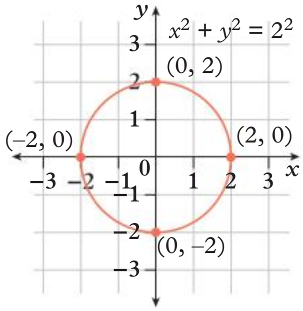
b



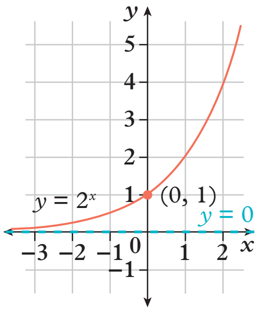
c



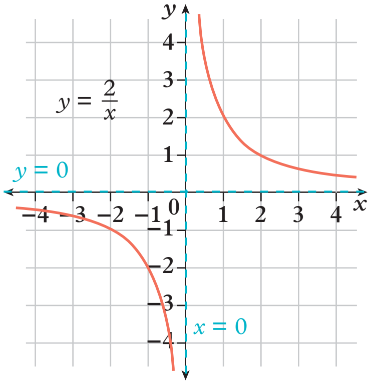
d



e



f



15 **a** 1 **b** 1 **c** 1 **d** 0 **e** 1
f 2 **g** 1 **h** 2 **i** 0 **j** 2
k 0 **l** 1 **m** 1 **n** 0

16 **a** *x*-intercept (5, 0); *y*-intercept (0, −10)

b *x*-intercept $\left(\frac{20}{3}, 0\right)$; *y*-intercept $\left(0, -\frac{10}{3}\right)$

c *x*-intercept (12, 0)

d *y*-intercept (0, 3)

e *x*-intercept $\left(\frac{9}{4}, 0\right)$; *y*-intercept (0, 9)

f *x*-intercepts (4, 0), (−6, 0); *y*-intercept (0, −72)

g *x*-intercept (4, 0); *y*-intercept (0, 16)

h *x*-intercepts (−6, 0), (1, 0); *y*-intercept (0, 6)

i *y*-intercept (0, 5)

j *x*-intercepts (9, 0), (1, 0); *y*-intercept (0, 3)

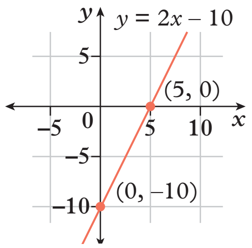
k *y*-intercept (0, 2)

l *x*-intercept (1, 0); *y*-intercept (0, 2)

m *x*-intercept $\left(\frac{5}{2}, 0\right)$; *y*-intercept $\left(0, \frac{5}{3}\right)$

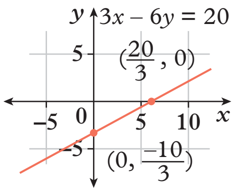
n *y*-intercept $\left(0, -\frac{3}{2}\right)$

17 **a** **i**



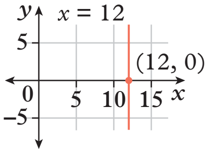
ii Domain is ℝ; range is ℝ.

b **i**



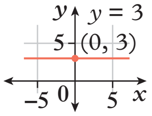
ii Domain is ℝ; range is ℝ.

c **i**



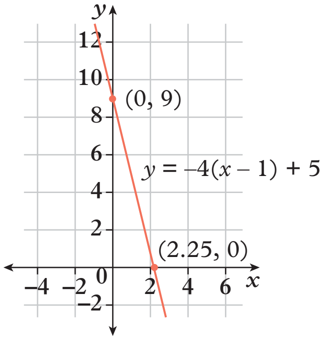
ii Domain is {12}; range is ℝ.

d **i**



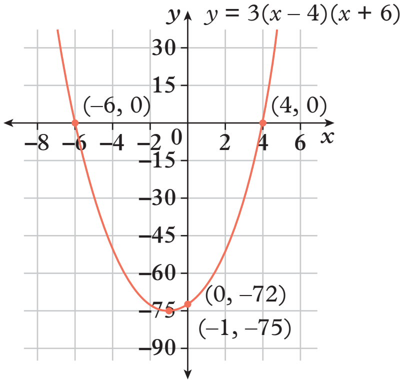
ii Domain is ℝ; range is {3}.

e **i**



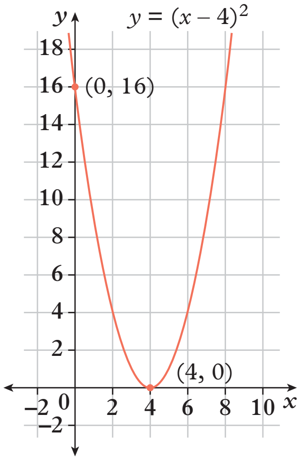
ii Domain is ℝ; range is ℝ.

f **i**



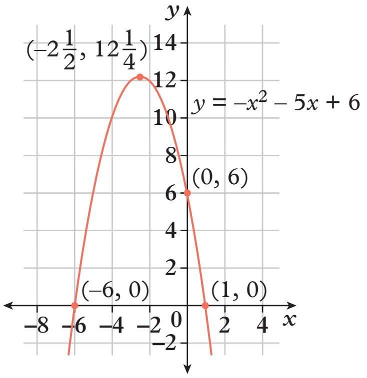
ii Domain is ℝ; range is $y \geq -75$.

g **i**



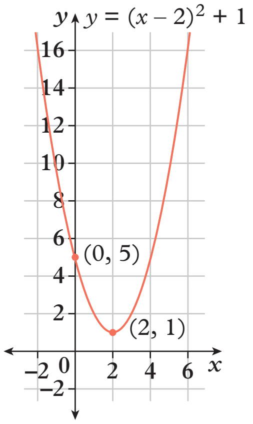
ii Domain is $\mathbb{R}$; range is $y \geq 0$.

h **i**



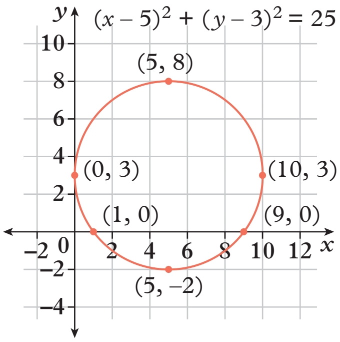
ii Domain is $\mathbb{R}$; range is $y \leq \frac{49}{4}$.

i **i**



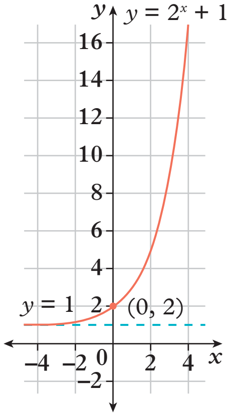
ii Domain is $\mathbb{R}$; range is $y \geq 1$.

j **i**



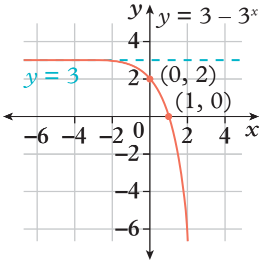
ii Domain is $0 \leq x \leq 10$; range is $-2 \leq y \leq 8$.

k **i**



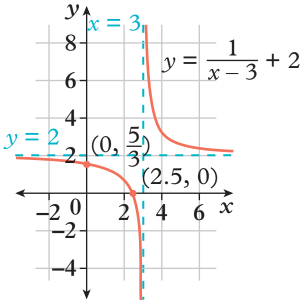
ii Domain is $\mathbb{R}$; range is $y > 1$.

l **i**



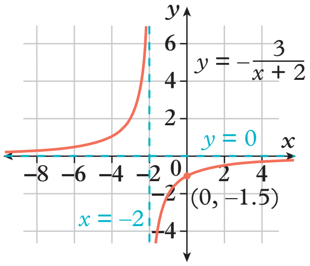
ii Domain is $\mathbb{R}$; range is $y < 3$.

m **i**



ii Domain is $x < 3$ and $x > 3$;
range is $y < 2$ and $y > 2$.

n **i**



ii Domain is $x < -2$ and $x > -2$;
range is $y < 0$ and $y > 0$.

18 **a** $y = -\frac{2}{3}x + \frac{5}{3}$ **b** $y = 4x + 13$

c $y = \frac{16}{5}x - \frac{68}{5}$ **d** $y = \frac{1}{2}x + 1$

e $y = \frac{3}{4}x + \frac{13}{2}$ **f** $y = 3x + 6$

19 **a** $x = -1, y = 4$

b $x = 1, y = 6$ or $x = 3, y = 4$

20 **a** $x = 2, y = 2$ **b** $x = \frac{13}{6}, y = -\frac{2}{3}$

c $x = \frac{13}{2}, y = -\frac{2}{9}$ **d** $x = -\frac{8}{3}, y = -\frac{6}{5}$

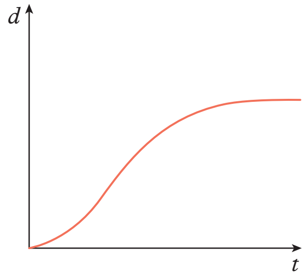
21 **a** The y-value increases at an increasing rate as x increases from 0 to 1.

b The y-value decreases at a constant rate as x increases from 0 to 1.

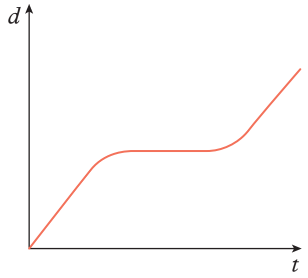
c The y-value increases at a decreasing rate as x increases from 0 to 1.

d The y-value decreases at a decreasing rate as x increases from 0 to 1.

22 a



d



23 a $x = 1, y = 2$ or $x = 6, y = 17$

b $x = 4, y = 27$

c $x = -4, y = -3$ or $x = 3, y = 4$

d $x = 1, y = 6$ or $x = -6, y = -1$

24 a $4\left(x - \frac{3}{8}\right)^2 - \frac{121}{16}$

b $4(x + 1)\left(x - \frac{7}{4}\right) = (x + 1)(4x - 7)$

25 a 4 **b** 3 **c** $\frac{1}{10\,000}$

d 7 **e** $\frac{1}{2}$ **f** -5

26 a i $1.793\,85 \times 10^3$

ii $5.105\,232\,79 \times 10^6$

iii 1.2793×10^{-6}

b i 3 **ii** 6 **iii** -6

c i 3.254 **ii** 6.708 **iii** -5.893

27 a $x = -7$ **b** $x = \frac{11}{3}$

c $x = \frac{27}{5}$ **d** $x = \frac{11}{13}$

e $x = \frac{20}{9}$ **f** $x = -2$

g $x = -\frac{4}{15}$ **h** $x = -43$

28 a $a^{\frac{13}{12}}$

b b^4c^{15}

c $\log_2(2d^2) = 2\log_2(d) + 1$

d $1 - \log_5(e)$

e $6\sqrt{5}$

f $6\sqrt{3} + 12\sqrt{2}$

g $15\sqrt{2}$

h $2\sqrt{6}$

i $54\sqrt{3} + 72$

j $8\sqrt{15} - 6$

29 a $\frac{5\sqrt{3}}{3}$ **b** $\frac{2\sqrt{6} + \sqrt{3}}{3}$

c $\frac{9\sqrt{2} + 15}{7}$ **d** $\frac{14 - 5\sqrt{5}}{2}$

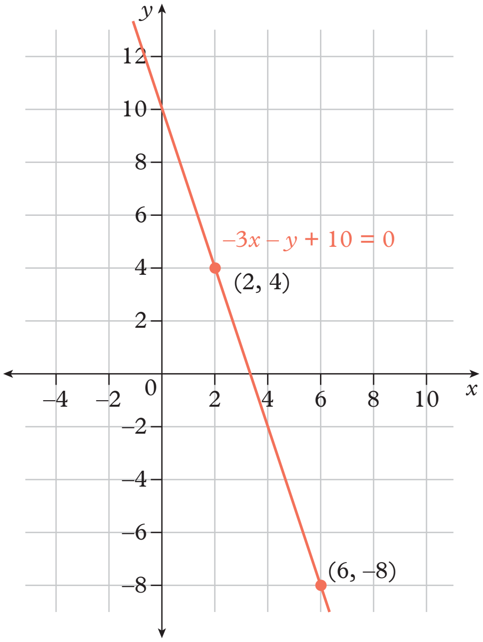
30 a $4\sqrt{10}$

b $(4, -2)$

c $-3x - y + 10 = 0$

d $(0, 10)$ and $\left(\frac{10}{3}, 0\right)$

e



31 a $(2x + 5)(x - 3)$ **b** $(3x + 4)(2x - 5)$

32 a $x = -4$ **b** $x = \frac{3}{8}$

c $x = 0.5$ **d** $x = 3 \pm \sqrt{5}$

e $x = -1, x = \frac{1}{4}$ **f** $x = \frac{2 \pm 2\sqrt{10}}{3}$

33 a i 3 **ii** 5 **iii** 1

b i 0 **ii** 8 **iii** 8

c i 1 **ii** 11 **iii** 7

d i 0 **ii** 8 **iii** $-\frac{8}{9}$

34 a $-8xy(3xy - 2y - 5) = 8xy(2y - 3xy + 5)$

b $(5x + 4)(4y - 1)$

c $(5x - 7)(5x + 7)$

d $x(x - 6)$

e $(x - 5)^2$

f $(x + 1)(x + 3)$

g $(x + 11)(x - 4)$

h $3(x - 15)(x + 3)$

35 a dilated by a factor of 2 from the x-axis
reflected in the x-axis
translated 8 units up

b dilated by a factor of $\frac{3}{8}$ from the x-axis
translated 2 units left

c dilated by a factor of $\frac{1}{2}$ from the x-axis, or
translated 1 unit right
translated 4 units down

d dilated by a factor of $\frac{1}{4}$ from the x-axis
translated 2 units right and 1 unit up

e reflected in the x-axis

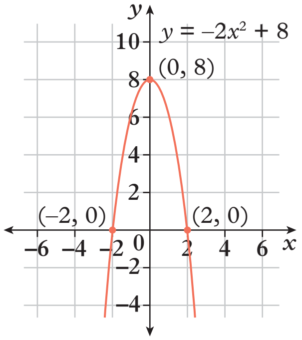
translated 1 unit left and 8 units up

f translated 2 units right and 4 units up

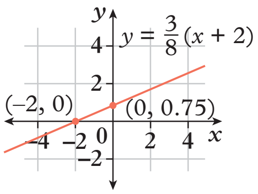
g reflected across $y = x$

reflected in the x-axis

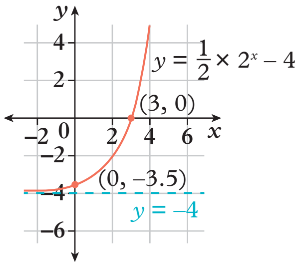
36 a



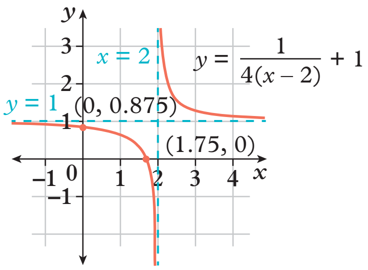
b



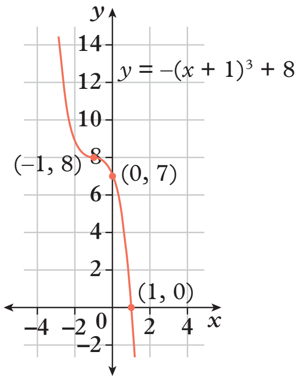
c



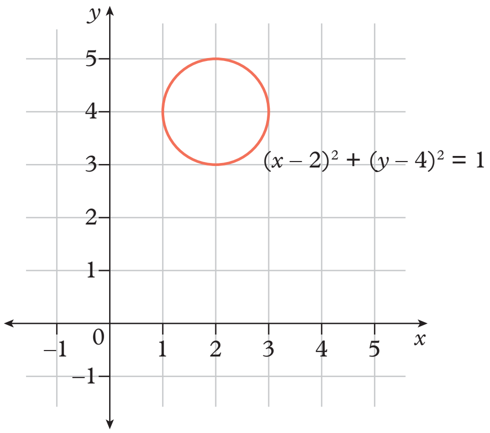
d



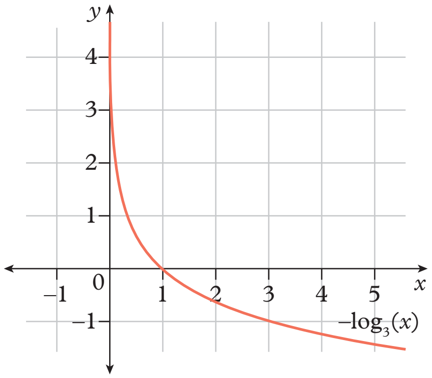
e



f



g



37 a $\frac{x^4}{y^{17}}$ **b** $\frac{1}{64}$ **c** $64x^{11}y^2$

38 a $16ax - 14bx - 56ay + 49by$

b $2x^2 + 10x - 16$

c $15x^2 - 8x + 7$

d $100x^2 + 60x + 9$

e $3ax + 8bx + 2cx - 15a - 40b - 10c$

f $\frac{a}{3} - \frac{a^2}{2} + \frac{9ab}{4} - \frac{4b}{3} - b^2$

39 a $3x$

b $\frac{(x + 10)^2}{(x - 4)(x - 2)}$

c $\frac{x - 2}{4}$

d $\frac{(x - 4)(x - 2)}{(x + 3)(x + 8)}$

40 a $x < \frac{1}{3}$ **b** $x < -7$ **c** $x \le 3$

d $x \le 2$ **e** $x > 4$ **f** $x > 14$

41 a -73 **b** $-\frac{17}{8}$

c $7\sqrt{2} - 10$ **d** $34\sqrt{3} - 40$

42 a i $-6x^5$ **ii** 5 **iii** -6

b i $5x^4$ **ii** 4 **iii** 5

c i $-\frac{7}{2}x^4$ **ii** 4 **iii** $-\frac{7}{2}$

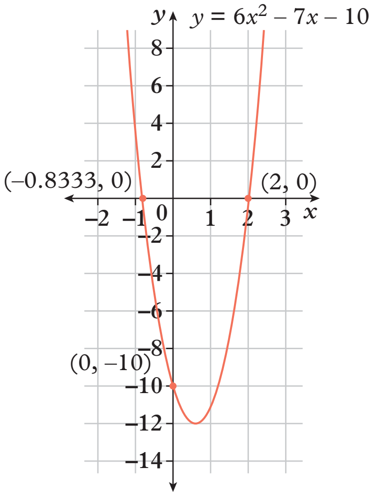
43 a $2x^3 + 3x^2 + 13x + 3$ **b** $13x^2 + 5x + 7$

c $6x^3 + 19x^2 + x - 21$ **d** $x^4 + x^3 - 18x^2 + 37x - 15$

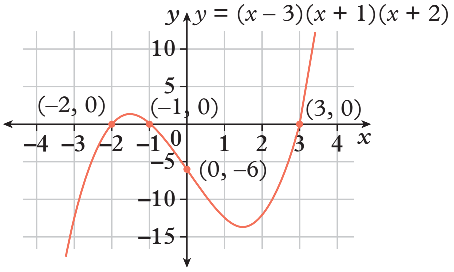
e $3x + 3 + \frac{14}{x - 3}$ **f** $x^2 + 3x - 13 + \frac{20}{x + 2}$

g $x^2 - 4x + 18 - \frac{73}{x + 4}$ **h** $x^2 - 7 + \frac{37}{x + 5}$

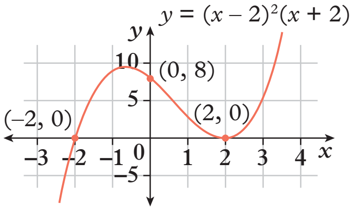
44 a



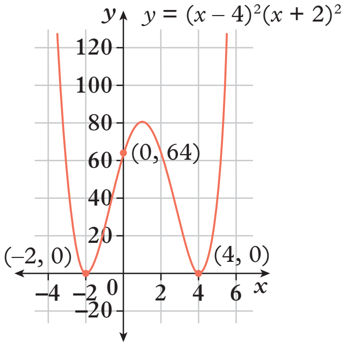
b



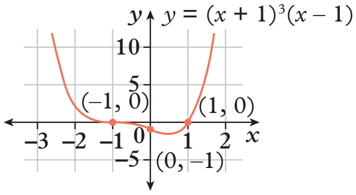
c



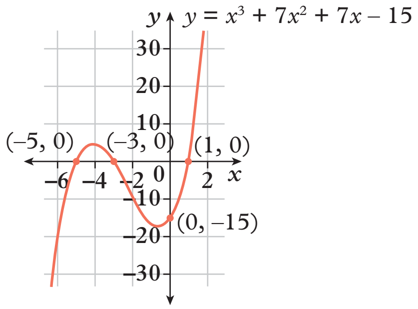
d



e



f



45 a $(x + 3)(x + 1)(x + 5)$

b $(x - 2)(x + 5)(x - 6)$

c $(x - 5)(x + 2)(x + 3)$

46 a $x = -\frac{5}{2}, x = \frac{10}{3}, x = 5, x = 4$

b $x = 1, x = -3, x = -4$

Analysis

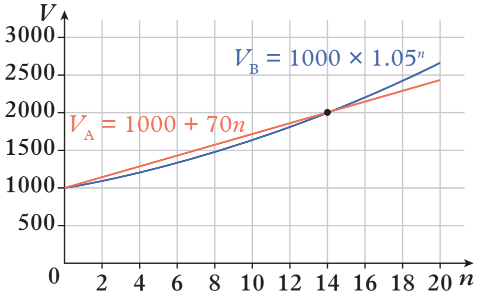
1 a Investment A

b $V_A = 1000 + 70n$ and $V_B = 1000 \times 1.05^n$

c

n	0	5	10	15	20
V_A	1000	1350	1700	2050	2400
V_B	1000	1276.28	1628.89	2078.93	2653.30

d



e After the 14th year

f After 4 years

g i After 12 years

ii $1840

h i $\log_{1.05}\left(\frac{27}{20}\right)$ years or $\frac{\log(1.05)}{\log 1.35}$

ii 6.15 years

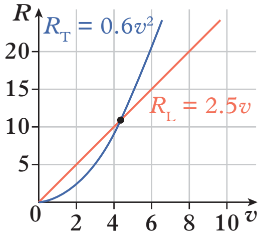
2 a i $k = 2.5$; $R_L = 2.5v$

ii $R_L = 50$ N

b i $k = 0.6$; $R_T = 0.6v^2$

ii $R_T = 240$ N

c

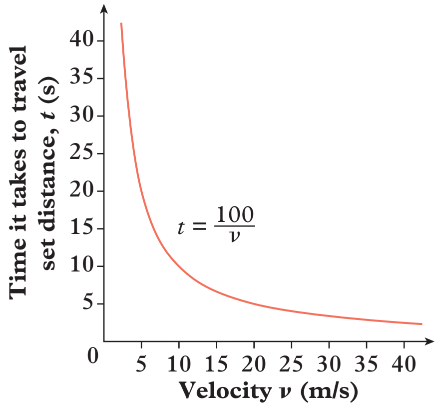


d $v = 0$ m/s; $v = \frac{25}{6} = 4\frac{1}{6}$ m/s

e $k = 100$; $v = \frac{100}{t}$

f 100 m

g



h As the velocity increases the time it takes to travel 100 m approaches zero, but it never reaches zero as $v = \frac{100}{t}$ and $\frac{100}{0}$ is undefined.

EX p338

EXPLORATIONS 1

1 **a** 4 red bricks

b 2 purple bricks

2 **a**

×	2	7	10	5	3
9	18	63	90	45	27
4	8	28	40	20	12
6	12	42	60	30	18
8	16	56	80	40	24
1	2	7	10	5	3

b

×	6	4	2	8
10	60	40	20	80
12	72	48	24	96
14	84	56	28	112
16	96	64	32	128

c Four tables fit the given criteria:

×	10	7	5	3	2
6	60	42	30	18	12
4	40	28	20	12	8
9	90	63	45	27	18
8	80	56	40	24	16

×	10	7	5	3	8
6	60	42	30	18	48
4	40	28	20	12	32
9	90	63	45	27	72
2	20	14	10	6	16

×	10	7	5	9	2
6	60	42	30	54	12
4	40	28	20	36	8
3	30	21	15	27	6
8	80	56	40	72	16

×	10	7	5	3	2
6	60	42	30	54	48
4	40	28	20	36	32
9	30	21	15	27	24
8	20	14	10	18	16

3 **a** $\sqrt{3} - \sqrt{2}$

b **i** $\frac{\sqrt{a} - \sqrt{b}}{a - b}$

ii It fails when $a = b$. Then $\frac{\sqrt{a}}{2a}$ is the correct simplified fraction.

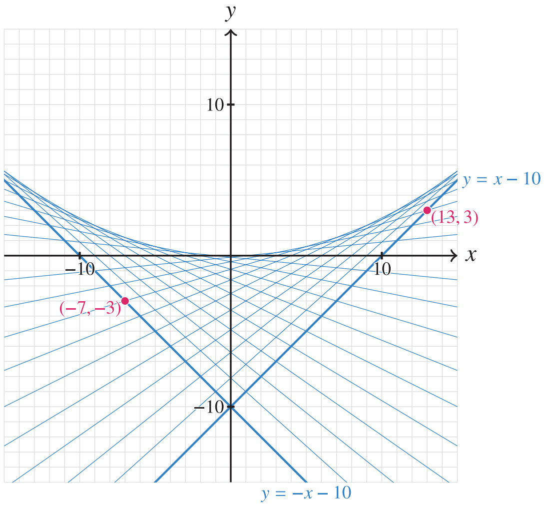
c $\frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$

d **i** $\frac{a\sqrt{b} + b\sqrt{a} - \sqrt{abc}}{2ab}$

ii It works for all a and b.

iii $\frac{1}{\sqrt{3} + \sqrt{5} + \sqrt{8}} = 3\frac{\sqrt{5} + 5\sqrt{3} - 2\sqrt{30}}{30}$

4 **a**



b **i** $y = \frac{3}{10}x - \frac{9}{10}$

ii Show that $\frac{1}{40}x^2 = \frac{3}{10}x - \frac{9}{10}$ simplifies to $(x - 6)^2 = 0$, which has exactly one solution.

c **i** $Q(k - 10, -k)$

ii $y = \frac{k}{10}x - \frac{k^2}{10}$

iii Simplify to $(x - 2k)^2 = 0$ or show that the discriminant is zero.

d Show that $ax^2 = \frac{k}{d}x - \frac{k^2}{d}$ has discriminant equal to zero when $a = \frac{1}{4d}$.

5 **a** 1, 32, 243, 1024, 3125. The last digit of the power is equal to the base. Yes, this also works for other fifth powers.

b $n(n-1)(n+1)(n^2+1)$. Show that 2, 3 and 5 each divide at least one of the four factors. If n is of the form $5k \pm 2$, show that 5 divides $n^2 + 1$.

c Since $n^5 - n$ ends in 0, n^5 and n must have the same last digit.

d Explain why the second and third factors are divisible by 2 and 4, or vice versa.

e Any n that is equal to, one more than, or one less than a multiple of 9.

f $n^9 - n = (n^5 - n)(n^4 + 1)$ is a multiple of 30. It is true for all 1st, 9th, 13th, …, $(4m+1)$th, … powers.

CHAPTER 7 Geometry

EX p345

7A Geometrical properties

1 **a** $a = 146^\circ$
b $b = 24^\circ$
c $c = 57^\circ, d = 57^\circ, e = 123^\circ$
d $d = 75^\circ$
e $e = 109^\circ, f = 71^\circ, g = 74^\circ$
f $g = 71^\circ, h = 19^\circ, i = 71^\circ, j = 90^\circ$

2 **a** $a = 117^\circ$ **b** $b = 109^\circ$
c $c = 49^\circ$ **d** $d = 129^\circ$
e $e = 63^\circ, f = 117^\circ, g = 117^\circ, h = 63^\circ$
f $f = 126^\circ$

3 **a** $a = 97^\circ$
b $b = 88^\circ, c = 88^\circ, d = 92^\circ$
c $c = 58^\circ, d = 302^\circ$
d $d = 52^\circ, e = 71^\circ, f = 57^\circ$
e $d = 180^\circ, e = 30^\circ, f = 115^\circ, g = 51^\circ$
f $f = 67^\circ, g = 64^\circ, h = 116^\circ, i = 91^\circ$

4 **a** $a = 76^\circ$ **b** $b = 60^\circ$
c $c = 32^\circ, d = 116^\circ$ **d** $d = 160^\circ$
e $e = 115^\circ$ **f** $f = 37^\circ, g = 75^\circ, h = 68^\circ$

5 **a** $a = 55^\circ$
b $b = 117^\circ, c = 63^\circ, d = 117^\circ$
c $c = 30^\circ, d = 118^\circ$
d $d = 45^\circ$
e $e = 119^\circ, f = 25^\circ$
f $f = 58^\circ, g = 98^\circ, h = 98^\circ, i = 122^\circ$

6 **a** 3240° **b** 8640° **c** 17 640°

7 **a** $n - 2$
b The sum of the interior angles of the triangles that a polygon can be dissected into is equal to the sum of the interior angles of the polygon. As the interior angle sum of a triangle is 180°, the interior angle sum of an n-sided polygon is $(n-2) \times 180^\circ$.

8 The internal angle sum of a polygon $= (n-2) \times 180^\circ$
A regular polygon with n equal sides has n equal angles.
So the size of each angle $= \dfrac{(n-2) \times 180^\circ}{n}$

9 **a** 128.57° **b** 140° **c** 120°

10 **a** 18 **b** 10 **c** 36 **d** 1000

11 **a** 18° **b** 27° **c** 22.5°
d 9.69° **e** 21° **f** 17.63°

12 If a side length in a triangle was greater than the sum of the side lengths of the other two sides, then the other two side lengths would not be able to meet at a point if each one was attached to a different end of the longer side length. Therefore, a triangle could not be formed with these side lengths.

13 **a** $6 \text{ cm} < x < 18 \text{ cm}$
b $2 \text{ cm} < y < 26 \text{ cm}$
c $11 \text{ mm} < z < 29 \text{ mm}$

14 **a** $8.9 \text{ cm} < p < 15.5 \text{ cm}$
b $7.5 \text{ cm} < q < 13.2 \text{ cm}$
c $23.5 \text{ mm} < r < 35 \text{ mm}$

15 **a** 120° **b** 360°
c 90° **d** 360°
e The exterior angles of a convex polygon can be joined together to form a full revolution, which is 360°.

16 **a** false; angles at a point add to 360°.
b false; corresponding angles on parallel lines are equal.
c false; complementary angles add to 90°.
d true
e false; the sum of an interior angle and the adjacent exterior angle is 180°
f true

17 **a** $b = 30^\circ, c = 44^\circ, d = 74^\circ, e = 29^\circ$
b $c = 72^\circ$
c $d = 109^\circ, e = 8^\circ$

18 **a** $a = 60^\circ, b = 30^\circ, c = 90^\circ$ **b** $d = 35^\circ$
c $e = 79^\circ$ **d** $f = 163^\circ$
e $g = 220^\circ$ **f** $h = 154.5^\circ$

19 **a** 330° **b** 185° **c** 225° **d** 351°
e 54° **f** 87° **g** 70° **h** 296°
i 325° **j** 300°

EX p353

7B Congruence and similarity

1 **a** $a = 51^\circ, b = 11 \text{ cm}, c = 70^\circ, d = 59^\circ, e = 10 \text{ cm}$
b $a = 48^\circ, b = 25^\circ, c = 4 \text{ cm}, d = 25^\circ, e = 107^\circ, f = 7 \text{ cm}$
c $a = 38^\circ, b = 50^\circ, c = 13 \text{ cm}, d = 38^\circ, e = 92^\circ, f = 8 \text{ cm}$
d $j = 25^\circ, k = 7 \text{ cm}, l = 3 \text{ cm}, m = 5 \text{ cm}, n = 115^\circ$

2 **a** congruent; AAS
b congruent; RHS
c congruent; SAS
d not congruent; SSS does not apply
e not congruent; AAS does not apply
f not congruent; RHS does not apply

3 **a** congruent; SAS or AAS **b** not congruent; SAS does not apply
c congruent; SAS **d** congruent; AAS

4 **a** $\frac{5}{2}$ **b** $\frac{3}{2}$ **c** 2 **d** $\frac{1}{3}$

5 **a** not similar; does not meet the requirements of the SSS condition for similarity

b not similar; does not meet the requirements of the SAS condition for similarity

c similar; meets the requirements of the AAA condition for similarity

d similar; meets the requirements of the AAA condition for similarity

e similar; meets the requirements of the SSS condition for similarity

f similar; meets the requirements of the RHS condition for similarity

6 **a** $x = 10.5$ cm, $y = 8$ cm

b $c = 44$ cm, $d = 7$ cm

c $x = 4$ cm, $y = 9$ cm

d $a = 9$ cm, $b = 4$ cm

e $x = 5\frac{7}{13}$ cm, $y = 17\frac{1}{3}$ cm

f $x = 52$ cm, $y = 20$ cm

7 **a** similar; passes SSS condition

b similar; meets the requirements of the AAA condition for similarity

c not similar; does not meet the requirements of the AAA condition for similarity

d not similar; does not meet the requirements of the RHS condition for similarity

8 **a**

Statements	**Reasons**
$VX = YX$	Given
$WX = ZX$	Given
$\angle VXW = \angle YXZ$	Vertically opposite angles are equal
$\therefore \triangle VWX \equiv \triangle YZX$ □	SAS (congruent)

b

Statements	**Reasons**
$QP = QS$	Given
$\angle PQR = \angle SQT$	Vertically opposite angles are equal
$\angle RPQ = \angle TSQ$	Given
$\therefore \triangle PQR \equiv \triangle SQT$ □	AAS (congruent)

9 **a**

Statements	**Reasons**
$\angle ABC = \angle PRQ$	Given
$\frac{PR}{AB} = \frac{27}{9} = 3$	Scale factor between corresponding sides
$\frac{RQ}{BC} = \frac{18}{6} = 3$	Scale factor between corresponding sides
$\therefore \triangle ABC \sim \triangle PRQ$ □	SAS (similar)

b

Statements	**Reasons**
$\angle AEB = \angle CED$	Vertically opposite angles are equal
$\angle EAB = \angle ECD$	Alternate angles on parallel lines are equal
$\angle EBA = \angle EDC$	Alternate angles on parallel lines are equal
$\therefore \triangle ABE \sim \triangle CDE$ □	AAA (similar)

10 **a** The 9 cm side length is opposite the 85° angle in one figure, and adjacent to the 85° angle in the other figure. None of the angles are equal, so the case of an isosceles triangle with two equal sides is ruled out.

b The 88° angle is opposite the 17 cm side in one figure and opposite the 16 cm side in the other figure.

c Three pairs of equal angles is not a condition for congruence. One triangle could be a dilation of the other.

d The congruence condition RHS does not apply because the only side measurement given is the hypotenuse.

11 **a** False; the triangles may or may not be congruent, depending on their side lengths.

b False; the corresponding angle must be between the two corresponding sides for these triangles to be congruent.

c False; if a pair of triangles meets one condition of congruence, they will meet all conditions of congruence (but, perhaps, further calculations are necessary to show the other conditions of congruence are met).

d True; figures which are congruent to each other all have the same shape and size.

e False; a quadrilateral with all sides of equal length could be a square or a rhombus. A square and a rhombus are only ever congruent if all the interior angles are right angles.

f False; all equilateral triangles are similar, but they are only congruent if their side lengths are the same.

12 **a** It appears at first glance that the scale factor is $\frac{1}{2}$. However, the sides compared are not corresponding sides.

b The RHS test requires a right angle, the hypotenuse and one of the shorter sides (which is not provided). This means that the pair of triangles fails the RHS test.

13 4.9 m

14 3.6 m

15 Pairs of similar triangles: **A** and **C** (AAA), **D** and **E** (SAS)

16 **a** False; they must be similar if their corresponding angles are equal

b True; the corresponding angles in these three triangles must be equal

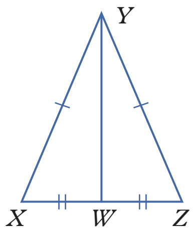
c False; the angles in isosceles triangles are not fixed

d True; all squares have all angles equal to 90° and all four lengths equal to each other, which makes them all the same shape

e False; if they fail SAS they cannot be similar

f True; consider a rhombus and a square with the same side lengths.

17 **a**



b $\triangle YWX$ and $\triangle YWZ$

c $YX = YZ$ and $WX = WZ$

d YW is common to both triangles

e

Statements	**Reasons**
$YX = YZ$	Equal sides in an isosceles triangle
$WX = WZ$	By construction
YW is common to both triangles	Common side
$\therefore \triangle WXY \equiv \triangle WZY$ □	SSS (congruent)

f $\angle WXY = \angle WZY$ as they are corresponding angles in congruent triangles.

18 **a**

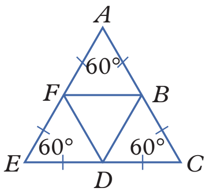
Statements	**Reasons**
$\angle AEB = \angle CED$	Common angle to both triangles
$AE = 70 + 70 = 140$	Substitution
$BE = 110 + 110 = 220$	Substitution
$\frac{DE}{BE} = \frac{110}{220} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{70}{140} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \triangle ABE \sim \triangle CDE$ □	SAS (similar)

b

Statements	**Reasons**
$\angle AEB = \angle CED$	Common angle
$AC = CE$	Given
$AE = AC + CE$	Given
$AE = CE + CE = 2AE$	Substitution
$BD = DE$	Given
$BE = BD + DE$	Given
$BE = DE + DE = 2DE$	Substitution
$\frac{DE}{BE} = \frac{DE}{2DE} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{CE}{2CE} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \triangle ABE \sim \triangle CDE$ □	SAS (similar)

19 7.2 m

20



Statements	**Reasons**
$AC = AE = CE$	Side lengths of an equilateral triangle are equal
$AB = BC = CD = DE = EF = FA$	Points B, D and F lie at the midpoints of equal side lengths
$\angle ACE = \angle CEA = EAC = 60°$	Interior angles of an equilateral triangle are 60°
$\triangle ABF \equiv \triangle CDB \equiv \triangle EDF$	SAS (congruent)
$FB = BD = DF$	Corresponding sides on congruent triangles
$\therefore \triangle FBD$ is equilateral	A triangle with equal side lengths is equilateral
$\angle FBD = \angle BDF = \angle DFB = 60°$	Interior angles of an equilateral triangle are 60°
$\triangle ABF$ is isosceles	A triangle with two equal side lengths is isosceles
$\angle ABF = \angle AFB$	Angles opposite equal sides of an isosceles triangle are equal
$\angle ABF = \angle AFB = 60°$	Interior angle sum of a triangle is 180°
$\therefore \triangle ABF$ is equilateral	A triangle with equal angles is equilateral
$AB = FA = FB$	Side lengths on equilateral triangles are equal
$AB = BC = CD = DE = EF = FA = FB = BD = DF$	Transitive law
$\therefore \triangle CDB$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \triangle EDF$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \triangle ABF \equiv \triangle CDB \equiv \triangle EDF \equiv FBD$ □	SSS (congruent)

EX
p363

7C Geometric proofs

1

Statements	**Reasons**
$\angle DBA + \angle ABC + \angle CBE = 180°$	Angles on a straight line sum to 180°
$\angle DBA = \angle BAC$	Alternate angles on parallel lines are equal
$\angle CBE = \angle BCA$	Alternate angles on parallel lines are equal
$\therefore \angle BAC + \angle ABC + \angle BCA = 180°$ □	Transitive law

2

Statement	Reasons
$\angle ADC + \angle DCA + \angle DAC = 180^\circ$	Interior angle sum of a triangle is 180°
$\angle ABC + \angle BCA + \angle BAC = 180^\circ$	Interior angle sum of a triangle is 180°
$\therefore \angle ADC + \angle DCA + \angle CAD + \angle ABC + \angle BCA + \angle CAB = 360^\circ$	Substitution
$\angle DAC + \angle BAC = \angle DAB$	Angle addition postulate
$\angle DCA + \angle BCA = \angle DCB$	Angle addition postulate
$\therefore \angle ADC + \angle DCB + \angle CBA + \angle DAB = 360^\circ$	Transitive law
$\therefore$ sum of interior angles of $ADCB$ is 360° □	

3

Statements	Reasons
$\angle FGE = \angle HEG$	Alternate angles on parallel lines are equal
$\angle FEG = \angle HGE$	Alternate angles on parallel lines are equal
EG is common to both $\triangle FGE$ and $\triangle HEG$	Common side
$\therefore \triangle FGE \equiv \triangle HEG$	AAS (congruence)
$FG = HE$	Corresponding sides in congruent triangles are equal
$EF = GH$	Corresponding sides in congruent triangles are equal
$FG = EF$	Given
$\therefore FG = EF = HE = GH$	Transitive law

4

Statements	Reasons
$\angle ACI + \angle CIS = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle ASI + \angle CIS = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle ACI + \angle CIS = \angle ASI + \angle CIS$	Transitive law
$\angle ACI = \angle ASI$	Subtraction property of equality
$\angle ICA + \angle CAS = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CIS = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CAS = \angle ICA + \angle CIS$	Transitive law
$\angle CAS = \angle CIS$	Subtraction property of equality
$\therefore$ the opposite angles of a rhombus are equal □	

5

Statements	Reasons
$AB = CD$	Given
$AD = CB$	Given
BD is common to both $\triangle ABD$ and $\triangle CDB$	Common side
$\triangle ABD \equiv \triangle CDB$	SSS (congruence)
$\angle ABD = \angle CDB$	Corresponding angles in congruent triangles are equal
$\angle ADB = \angle CBD$	Corresponding angles in congruent triangles are equal
$AB \parallel CD$	If alternate angles are equal, they lie on parallel lines
$AD \parallel CB$	If alternate angles are equal, they lie on parallel lines

6 **a** $\angle ETR$ **b** $\angle TER$

c

Statements	Reasons
$\angle MIR = \angle TER$	Alternate angles on parallel lines are equal
$\angle IMR = \angle ETR$	Alternate angles on parallel lines are equal
$IM = ET$	Given
$\therefore \triangle MIR \equiv \triangle TER$ □	AAS (congruent)

d IR and ER together form the parallelogram's diagonal IE, while MR and TR form the diagonal TM. The diagonals bisect each other because we know that $IR = ER$ and $MR = TR$ as they are corresponding sides of congruent triangles.

7 **a** $AD = BD = 0.5x$ **b** $\angle CAD = \angle CBD = 60^\circ$

c $\angle ACB = 60^\circ$

8

Statement	Reasons
$AC = BC$	Given
$AD = BD$	Given
CD is common to both $\triangle ACD$ and $\triangle BCD$	Common side
$\therefore \triangle ACD \equiv \triangle BCD$	SSS (congruence)
$\angle CAD = \angle CBD$ □	Corresponding angles in congruent triangles are equal

9

Statement	Reasons
$\angle GEH = \angle GFH$	Given
$\angle EGH = \angle FGH$	Given
GH is common to both $\triangle EGH$ and $\triangle FGH$	Common side
$\therefore \triangle EGH \equiv \triangle FGH$	AAS (congruence)
$EG = FG$ □	Corresponding sides in congruent triangles are equal

10

Statements	**Reasons**
$\angle EFH + \angle DFE = 180°$	Angles on a straight line sum to 180°
$\angle DFE + \angle DEF + \angle FDE = 180°$	Interior angle sum of a triangle is 180°
$\angle EFH + \angle DFE = \angle DFE + \angle DEF + \angle FDE$	Transitive law
$\angle EFH = \angle DEF + \angle FDE$ □	Subtraction property of equality

11

Statements	**Reasons**
$OK = FK$	Given
$LK = GK$	Given
$\angle OKL = \angle FKG$	Vertically opposite angles are equal
$\therefore \triangle OKL \equiv \triangle FKG$	SAS (congruence)
$\angle LOK = \angle GFK$	Corresponding angles in congruent triangles are equal
$\angle OLK = \angle FGK$	Corresponding angles in congruent triangles are equal
$OL \parallel FG$	If alternate angles are equal, they lie on parallel lines
$\angle OKG = \angle FKL$	Vertically opposite angles are equal
$\therefore \triangle OKG \equiv \triangle FKL$	SAS (congruence)
$\angle GOK = \angle LFK$	Corresponding angles in congruent triangles are equal
$\angle FLK = \angle OGK$	Corresponding angles in congruent triangles are equal
$OG \parallel FL$	If alternate angles are equal, they lie on parallel lines
$\therefore$ $GOLF$ is a parallelogram □	A quadrilateral with parallel opposite sides is a parallelogram

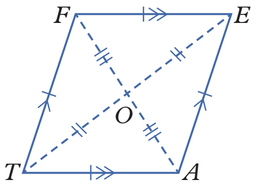
12 **a**

Statements	**Reasons**
$\angle EFO = \angle TAO$	Alternate angles on parallel lines are equal
$\angle FEO = \angle ATO$	Alternate angles on parallel lines are equal
$FE = AT$	Given
$\therefore \triangle FEO \equiv \triangle ATO$	AAS (congruence)
$FO = AO$	Corresponding sides in congruent triangles are equal
$EO = TO$	Corresponding sides in congruent triangles are equal
$\therefore$ the diagonals bisect each other □	

b

Statements	**Reasons**
$\triangle FEO \equiv \triangle ATO$	Proven in part **a**
$\angle AEO = \angle FTO$	Alternate angles on parallel lines are equal
$\angle EAO = \angle TFO$	Alternate angles on parallel lines are equal
$AE = FT$	Given
$\therefore \triangle AEO \equiv \triangle FTO$	AAS (congruence)
$FE = AE$	Given
$FO = AO$	Proven in part **a**
EO is common to both $\triangle FEO$ and $\triangle AEO$	Common
$\triangle FEO \equiv \triangle AEO$	SSS (congruence)
$\therefore \triangle FEO \equiv \triangle AEO \equiv \triangle FTO \equiv \triangle ATO$ □	Transitive law

c



d $\angle FOE$ and $\angle AOE$ are corresponding angles in congruent triangles, therefore they are equal.

e

Statements	**Reasons**
$\angle FOE + \angle AOE = 180°$	Angles on a straight line sum to 180°
$\angle FOE = \angle AOE$	As shown in part **d**
$\angle FOE + \angle FOE = 180°$	Substitution
$\angle FOE = 90°$	Solving equation
$\angle AOE = 90°$	Transitive law
$\therefore$ the diagonals of a rhombus bisect each other at right angles □	

13

Statement	**Reasons**
$\angle ADC = \angle CBA$	Given
$\angle DCB = \angle BAD$	Given
$\angle ADC + \angle DCB + \angle CBA + \angle BAD = 360°$	Interior angle sum of a quadrilateral is 360°
$\angle ADC + \angle DCB + \angle ADC + \angle DCB = 360°$	Transitive law
$2\angle ADC + 2\angle DCB = 360°$	Simplifying equation
$\angle ADC + \angle DCB = 180°$	Simplifying equation
$\therefore AD \parallel BC$	If co-interior angles are supplementary, they lie on parallel lines
$\angle CBA + \angle DCB + \angle CBA + \angle DCB = 360°$	Transitive law
$2\angle CBA + 2\angle DCB = 360°$	Simplifying equation
$\angle CBA + \angle DCB = 180°$	Simplifying equation
$\therefore AB \parallel DC$	If co-interior angles are supplementary, they lie on parallel lines
$\therefore ABCD$ is a parallelogram □	A quadrilateral with two pairs of parallel sides is a parallelogram

14

Statement	**Reasons**
$PQ = PS = RQ = RS$	Given
QS is common to both $\triangle PQS$ and $\triangle RSQ$	Common side
$\therefore \triangle PQS \equiv \triangle RSQ$	SSS (congruence)
$\angle QPS = \angle SRQ$	Corresponding angles in congruent triangles are equal
$\angle PQS = \angle RSQ$	Corresponding angles in congruent triangles are equal

Statement	Reasons
$\angle PSQ = \angle RQS$	Corresponding angles in congruent triangles are equal
$\angle PQR = \angle PQS + \angle RQS$	Angle addition postulate
$\angle PSR = \angle PSQ + \angle RSQ$	Angle addition postulate
$\angle PSR = \angle RQS + \angle PQS$	Transitive law
$\angle PQR = \angle PSR$	Transitive law
$\angle PQR + \angle SRQ + \angle PSR + \angle QPS = 360°$	Interior angle sum of a quadrilateral is 360°
$\angle PQR + \angle SRQ + \angle PQR + \angle SRQ = 360°$	Transitive law
$2\angle PQR + 2\angle SRQ = 360°$	Simplifying equation
$\angle PQR + \angle SRQ = 180°$	Simplifying equation
$\therefore PQ \parallel SR$	If co-interior angles are supplementary, they lie on parallel lines
$\angle PSR + \angle SRQ + \angle PSR + \angle SRQ = 360°$	Transitive law
$2\angle PSR + 2\angle SRQ = 360°$	Simplifying equation
$\angle PSR + \angle SRQ = 180°$	Simplifying equation
$\therefore PS \parallel QR$	If co-interior angles are supplementary, they lie on parallel lines
$\therefore$ *PQRS* is a parallelogram	A quadrilateral with two pairs of parallel sides is a parallelogram
$\therefore$ *PQRS* is a rhombus □	A parallelogram with two adjacent sides equal in length is a rhombus

15 a

Statements	Reasons
$EV = IL$	Given
EI is common to both $\triangle EVI$ and $\triangle ILE$	Common side
$\angle VEI = \angle LIE$	Alternate angles on parallel lines are equal
$\therefore \triangle EVI \equiv \triangle ILE$	SAS (congruence)
$\angle EVI = 90°$	Given
$\therefore \angle ILE = 90°$	Corresponding angles in congruent triangle are equal
$VE \parallel IL$	Given
$\angle EVI + \angle VIL = 180°$	Co-interior angles on parallel lines are supplementary
$\angle VIL = 90°$	Substitution and solving equation
$\angle ILE + \angle LEV = 180°$	Co-interior angles on parallel lines are supplementary
$\angle LEV = 90°$ □	Substitution and solving equation

b

Statements	Reasons
$EL = IV$	Corresponding sides in congruent triangles are equal
$\angle LEV = \angle EVI = 90°$	Proved in part **a**
$\angle LEV + \angle EVI = 180°$	Substitution
$\therefore EL \parallel IV$ □	If co-interior angles are supplementary, they are on parallel lines

c

Statements	Reasons
$EI^2 = EL^2 + IL^2$	Pythagoras' theorem
$LV^2 = EL^2 + EV^2$	Pythagoras' theorem
$EV = IL$	Given
$LV^2 = EL^2 + IL^2$	Substitution
$LV^2 = EI^2$	Transitive law
$\therefore LV = EI$ □	Transitive law

d

Statements	Reasons
$EV = IL$	Given
$LV = EI$	Proved in part **c**
EL is common to both $\triangle LEV$ and $\triangle ELI$	Common side
$\therefore \triangle LEV \equiv \triangle ELI$ □	SSS (congruence)

16 a

Statements	Reasons
$PE = PY$	Given
$PL = PK$	Given
$\angle EPL = \angle YPK$	Vertically opposite angles are equal
$\therefore \triangle ELP \equiv \triangle YKP$	SAS (congruence)
$\angle PEL = \angle PYK$	Corresponding angles in congruent triangles are equal
$EL \parallel YK$	If alternate angles are equal, they lie on parallel lines
$\angle EPK = \angle YPL$	Vertically opposite angles are equal
$\therefore \triangle EPK \equiv \triangle YPL$	SAS (congruence)
$\angle PEK = \angle PYL$	Corresponding angles in congruent triangles are equal
$EK \parallel YL$	If alternate angles are equal, they lie on parallel lines
$\therefore$ *ELYK* is a parallelogram □	If a quadrilateral has two pairs of parallel sides, it is a parallelogram

b

Statements	Reasons
$\angle LYE + \angle YEL + \angle ELK + \angle KLY = 180°$	Interior angle sum of a triangle is 180°
$\angle YEL = \angle ELK$	Angles opposite equal sides in an isosceles triangle are equal
$\angle LYE = \angle KLY$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle KLY + \angle ELK + \angle ELK + \angle KLY = 180°$	Substitution
$2\angle ELK + 2\angle KLY = 180°$	Solving equation
$\angle ELK + \angle KLY = 90°$	Solving equation
$\angle ELY = 90°$ □	Transitive law

c Since $ELYK$ is a parallelogram,
$\angle ELY + \angle LYK = 180°$ (co-interior angles)
Since $\angle ELY = 90°$, $\angle LYK = 90°$.
So $\angle YKE = 90°$ and $\angle KEL = 90°$, since opposite angles in a parallelogram are equal. Since $ELYK$ is a parallelogram, and its interior angles are 90°, $ELYK$ is a rectangle.

17 a

Statements	**Reasons**
$UA = RS$	Sides of a rhombus are equal
$\angle AUG = \angle SRG$	Alternate angles on parallel lines are equal
$\angle UAG = \angle RSG$	Alternate angles on parallel lines are equal
$\therefore \triangle GUA \equiv \triangle GRS$	AAS (congruence)
$UG = RG$	Corresponding sides in congruent triangles are equal
$AG = SG$ □	Corresponding sides in congruent triangles are equal

b

Statements	**Reasons**
$UA = RA$	Given
$UG = RG$	Proved in part **a**
GA is common to both $\triangle UAG$ and $\triangle RAG$	Common side
$\therefore \triangle UAG \equiv \triangle RAG$ □	SSS (congruence)

c

Statements	**Reasons**
$\angle UAG = \angle RAG$	Corresponding angles in congruent triangles are equal
$\angle UAS = \angle RAS$ □	Transitive law

d The two equal angles are the angles at vertex A. The angle at A has been bisected by the diagonal SA to produce them. The same argument from part **b** can be applied to the other interior triangles to show, as was shown in question **12b**, that all four interior triangles of a rhombus are congruent to each other. The same argument from part **c** can then be applied at each vertex of the rhombus to show that the diagonals of a rhombus bisect the interior angles of a rhombus.

18

Statements	**Reasons**
$EP = IP$	Given
$EV = IV$	Given
PV is common to both $\triangle EVP$ and $\triangle IVP$	Common side
$\therefore \triangle EVP \equiv \triangle IVP$	SSS (congruence)
$\angle VPE = \angle VPI$	Corresponding angles in congruent triangles are equal
$\angle VPE + \angle VPI = 180°$	Angles on a straight line sum to 180°
$\therefore \angle VPE = \angle VPI = 90°$	Solving equation
$\angle EVP = \angle VEP$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle EVP = \angle VEP = 45°$	Angles in a triangle sum to 180° and solving equation
$\therefore \angle EVL = 45°$ □	Transitive law

19 a In an equilateral triangle, and $BC = AC = AB$ and $\angle BAC = \angle ABC = \angle ACB$, which satisfies the required inequalities.

b

Statements	**Reasons**
Let P be the point on BC such that $PC = AC$	Since $BC > AC$, this point must exist and can be found using a ruler
$\angle BAP > 0$	Since $BC > AC$, $P \neq B$
$\triangle ACP$ is an isosceles triangle	Definition of isosceles triangle
$\angle PAC = \angle CPA$	Angles opposite equal sides in an isosceles triangle are equal
$\angle CPA = \angle BAP + \angle \mathrm{PBA}$	Exterior angle of a triangle is equal to the sum of the opposite two interior angles
$\angle PAC = \angle BAP + \angle PBA$	Substitution
$\angle BAC = \angle BAP + \angle PAC$	Angle addition postulate
$\angle BAC = \angle BAP + \angle BAP + \angle PBA$	Substitution
$\angle BAC > \angle PBA$	Since $\angle BAP > 0$, the RHS is greater than $\angle PBA$
$\angle PBA = \angle CBA$	Same angle
$\angle BAC > \angle CBA$ □	Substitution

c In an isosceles triangle, there are exactly two equal sides. Given that the side lengths satisfy $BC \geq AC \geq AB$, then either $BC = AC$ or $AC = AB$, which means that $BC > AC = AB$ or $BC = AB > AC$.

d In an isosceles triangle, equal sides are opposite equal angles. Consider the two cases from part **c**.
If $BC = AC > AB$, then $\angle BAC = \angle ABC$. Apply the result from part **b** to sides AC and AB (or BC and AB) to show $\angle ABC > \angle ACB$ (or $\angle BAC > \angle ACB$). By transitivity, $\angle BAC = \angle ABC > \angle ACB$, which means $\angle BAC \geq \angle ABC \geq \angle ACB$ is satisfied.

If $BC > AC = AB$ then $\angle ABC = \angle ACB$. Apply the result from part **b** to sides BC and AC (or BC and AB) to show $\angle BAC > \angle ABC$ (or $\angle BAC > \angle ACB$). By transitivity, this gives $\angle BAC > \angle ABC = \angle ACB$, which means $\angle BAC \geq \angle ABC \geq \angle ACB$ is satisfied.

e In a scalene triangle, all sides are different lengths, so $BC > AC > AB$. Use part **b** to compare sides BC and AC to get $\angle BAC > \angle ABC$, and then again to compare sides AC and AB to get $\angle ABC > \angle BCA$. By transitivity, $\angle BAC > \angle ABC > \angle ACB$, which means $\angle BAC \geq \angle ABC \geq \angle ACB$ is satisfied.

20 a Consider a square and a rhombus which have equal side lengths. The interior angles of the square are all 90°, but those in the rhombus are not. This means the two figures do not have the same shape, so they are not congruent.

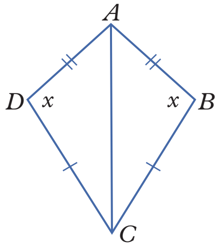
b A rhombus has four sides of equal length. If the SAS condition for congruence held for these shapes, the adjacent sides of the parallelogram would be equal in length. This would make it a rhombus.

c The AAS condition for congruence considers two angles (in this case, right angles) and one side. This could hold true for the square and rectangle for two angles and the included side, but the other sides do not need to be equal lengths unless the rectangle is also square.

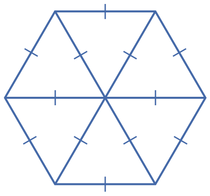
21 **a** A rectangle and a parallelogram can have all corresponding sides in the same ratio without being similar. They will not be the same shape unless all the interior angles of the parallelogram are right angles, which would make the shape a rectangle.

b A parallelogram and a rhombus might have their corresponding angles all the same size, but they only have the same shape if the sides of the parallelogram are all equal. This would make the parallelogram a rhombus.

22 **a** A kite has two pairs of adjacent sides equal, and one side in common to both triangles. The SSS condition for congruence applies.



b The six equilateral triangles formed all have the same side length. The SSS condition for congruence applies.



23

Statements	Reasons
$\angle HAB = \angle DCB = \angle DEF = \angle HGF = 90°$	Interior angles of a rectangle are 90°
$AB = CB = EF = GF$	Opposite sides of a rectangle are equal and B and F lie at the midpoints of these sides
$AH = CD = ED = GH$	Opposite sides of a rectangle are equal and H and D lie at the midpoints of these sides
$\therefore \triangle ABH \equiv \triangle CBD \equiv \triangle EFD \equiv \triangle FGH$	SAS (congruence)
$BH = BD = FD = FH$	Corresponding sides in congruent triangles are equal
$\therefore$ $HBDF$ is a rhombus	A quadrilateral with four equal sides is a rhombus
$\angle ABH = \angle CBD = \angle EFD = \angle GFH$	Corresponding angles in congruent triangles are equal
$\angle ABH + \angle HBD + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle GFH + \angle HFD + \angle DFE = 180°$	Angles on a straight line sum to 180°
$\therefore \angle HBD = \angle HFD$	Substitution and transitive law
$\angle BHA = \angle BDC = \angle FDE = \angle FHG$	Corresponding angles in congruent triangles are equal
$\angle AHB + \angle BHF + \angle GHF = 180°$	Angles on a straight line sum to 180°
$\angle CDB + \angle BDF + \angle EDF = 180°$	Angles on a straight line sum to 180°
$\therefore \angle BHF = \angle BDF$ □	Substitution and transitive law

EX
p367

Checkpoint

1 **a** $w = 49°$, $x = 104°$

b $y = 144°$

2 **a** $p = 135°$

b $q = 72°$

3 **a** $g = 27°$, $h = 1.3$ cm

b $p = 1.4$ cm, $q = 6.25$ cm

4 **a**

Statements	Reasons
$AD = DE$	Given
$AC = CE$	Given
CD is common to both $\triangle ADC$ and $\triangle EDC$	Common side
$\triangle ADC \equiv \triangle EDC$	SSS (congruence)
$\angle ACD = \angle ECD$ □	Corresponding angles in congruent triangles are equal

b

Statements	Reasons
$AB \parallel EF$	Given
$\angle DAB = \angle DEF$	Corresponding angles on parallel lines are equal
$\angle DBA = \angle DFE$	Corresponding angles on parallel lines are equal
$\angle ADB = \angle EDF$	Common angle
$\therefore \triangle ADB \sim \triangle EDF$ □	AAA (similarity)

5

Statements	Reasons
$\angle ADC = 90°$	Given $AD \perp CD$
$\angle BCD = 90°$	Given $BC \perp CD$
$\angle DAB + \angle ADC = 180°$	Co-interior angles on parallel lines are supplementary
$\angle DAB = 90°$	Substituting and solving equation
$\angle ABC + \angle BCD = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ABC = 90°$	Substituting and solving equation
$\therefore$ $ABCD$ is a rectangle	A quadrilateral with all interior angles of 90° is a rectangle
$AC = BD$ □	Diagonals of a rectangle are equal

6

Statements	Reasons
$\angle SPQ = \angle RST$	Given
$PQ \parallel SR$	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel
$PS \parallel QR$, $\angle RST = \angle SRQ$	If alternate angles are congruent, they lie on parallel lines
$PQRS$ is a parallelogram □	If a quadrilateral has two pairs of opposite sides that are parallel, it is a parallelogram

EX
p371

7D Circle geometry: angles

1 **a** 45° **b** 120° **c** 74°

2 **a** 90° **b** 18° **c** 45°

3 **a** $x = 32°$ **b** $x = 27°$

c $a = 9°, c = 6°$

4 In the following answers, CT stands for circle theorem.

a $x = 34.5°$, CT2 and angle sum of a triangle

b $x = 109°$, CT1

c $x = 21°$, CT3

d $a = 14°, b = 7°$, CT3

e $x = 22.5°$, CT2 and triangle angle sum

f $a = 26\frac{1}{3}°, e = 9\frac{7}{8}°$, CT3

5 **a** 12.2° **b** 76° **c** 32°

d 68° **e** 55° **f** 29°

6 **a** Each one is a radius of the circle.

b $\triangle AOB$ is isosceles.

c $\angle AOD = \angle ABO + \angle OAB = 2\angle ABO$

d $\triangle AOB$ is isosceles, so $\angle OBC = \angle OCB$.

$\angle DOC = \angle OBC + \angle OCB = 2\angle OBC$

e $\angle AOC = \angle AOD + \angle DOC = 2\angle ABO + 2\angle OBC = 2(\angle ABO + \angle OBC) = 2\angle ABC$

7 **a** 180°

b Since the angle at the centre is twice the angle at the circumference subtended by the same arc, the angle at the circumference is equal to $\frac{1}{2}$ of 180°, or 90°.

8 **a** $\angle WOZ$ is an angle at the centre of the circle, while $\angle WXZ$ is an angle at the circumference. Both are subtended by the same arc, and so circle theorem 1 applies so that $\angle WOZ = 2\angle WXZ$.

b Similarly, $\angle WOZ$ and $\angle WYZ$ are both subtended by the same arc, so circle theorem 1 applies and $\angle WOZ = 2\angle WYZ$.

c It follows that $\angle WYZ = \angle WXZ$.

9 **a** Obtuse $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle x at the circumference.

Similarly, the reflex angle $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle y at the circumference.

In both cases, circle theorem 1 applies, so that $\angle AOC = 2x$ and $\angle AOC = 2y$.

b obtuse $\angle AOC$ + reflex $\angle AOC = 360°$

That is: $2x$ + reflex $\angle AOC = 360°$

or: reflex $\angle AOC = 360° - 2x$

c Since: reflex $\angle AOC = 2y$

$2y = 360° - 2x$

Simplifying: $y = 180° - x$

d obtuse $\angle BOD = 2\angle BCD$

reflex $\angle BOD = 2\angle BAD$

obtuse $\angle BOD$ + reflex $\angle BOD = 360°$

(angles at a point)

So: $2\angle BAD + 2\angle BCD = 360°$

Rearranging and simplifying: $\angle BAD = 180° - \angle BCD$

e **i** $x = 142°, y = 99°$

ii $a = 45°, y = 38°$

iii $x = 30°, y = 22.5°$

10 **a** $\angle ADC + \angle ABC = 180°$ (opposite angles of a cyclic quadrilateral)

So: $\angle ADC = 180° - \angle ABC$

b supplementary angles on a straight line

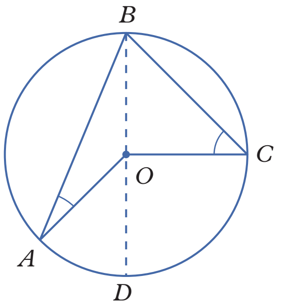
c both are equal to the same expression $(180° - \angle ABC)$

d **i** $a = 61°$

ii $a = 81°, b = 92°, c = 99°, d = 88°$

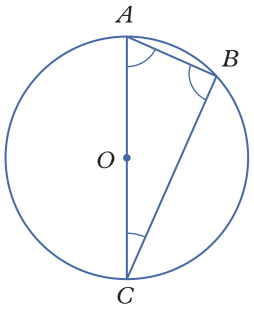
iii $x = 15°, z = 29.75°$

11 Circle theorem 1



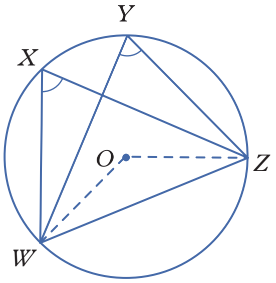
Statements	Reasons
$OA = OB = OC$	Radii of the circle
$\triangle AOB$ and $\triangle COB$ are isosceles	If two sides in a triangle are equal, the triangle is isosceles
$\angle ABO = \angle OAB$ $\angle CBO = \angle OCB$	Angles opposite equal sides in an isosceles triangle are equal
$\angle AOD = \angle ABO + \angle OAB$ $\angle DOC = \angle CBO + \angle OCB$	The exterior angle of a triangle is equal to the sum of the two opposite interior angles
$\angle AOD = 2\angle ABO$ $\angle DOC = 2\angle CBO$	Transitive law, substitution and simplifying
$\angle AOC = \angle AOD + \angle DOC$	Angle addition postulate
$\angle AOC = 2\angle ABO + 2\angle CBO$ $= 2(\angle ABO + \angle CBO)$	Transitive law and factorising
$\angle ABC = \angle ABO + \angle CBO$	Angle addition postulate
$\angle AOC = 2\angle ABC$ □	Transitive law

Circle theorem 2



Statements	Reasons
$\angle AOC = 180^\circ$	A straight angle equals 180°
$\angle AOC = 2\angle ABC$	Circle theorem 1
$2\angle ABC = 180^\circ$	Transitive law
$\angle ABC = 90^\circ$ □	Solving equation

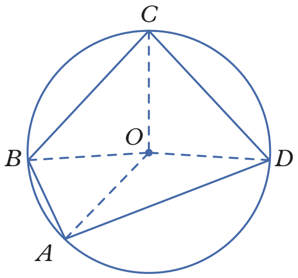
Circle theorem 3



Statements	Reasons
$\angle WOZ = 2\angle WXZ$	Circle theorem 1
$\angle WOZ = 2\angle WYZ$	Circle theorem 1
$2\angle WXZ = 2\angle WYZ$	Transitive law
$\angle WXZ = \angle WYZ$ □	Simplify equation

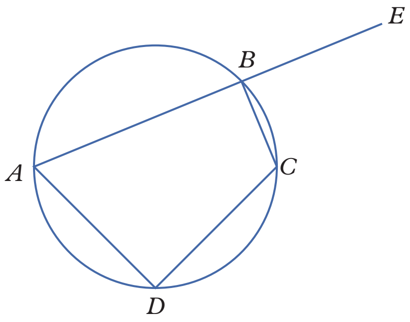
Circle theorem 4

The labels $\angle_O$ and $\angle_R$ are used to label obtuse and reflex angles, respectively.



Statements	Reasons
$\angle_R COA = 2\angle CBA$	Circle theorem 1
$\angle_O COA = 2\angle CDA$	Circle theorem 1
$\angle_R COA + \angle_O COA = 360^\circ$	Angles at a point add to 360°
$2\angle CBA + 2\angle CDA = 360^\circ$	Transitive law
$\angle CBA + \angle CDA = 180^\circ$	Simplify equation
$\angle_R BOD = 2\angle BAD$	Circle theorem 1
$\angle_O BOD = 2\angle BCD$	Circle theorem 1
$\angle_R BOD + \angle_O BOD = 360^\circ$	Angles at a point add to 360°
$2\angle BAD + 2\angle BCD = 360^\circ$	Transitive law
$\angle BAD + \angle BCD = 180^\circ$ □	Simplify equation

Circle theorem 5



Statements	Reasons
$\angle ABC + \angle ADC = 180^\circ$	Circle theorem 4
$\angle ABC + \angle CBE = 180^\circ$	Angles on a straight line add to 180°
$\angle ABC + \angle ADC =$ $\angle ABC + \angle CBE$	Transitive law
$\angle ADC = \angle CBE$ □	Subtraction property of equality

12 **a** $\angle DBC = 66^\circ$ **b** $\angle DEG = 19^\circ$

c $\angle ODC = 38^\circ$ **d** $\angle OAB = 62^\circ$

EX p376

7E Circle geometry: chords

1 **a** 101° **b** 38° **c** 58°

2 **a** 13 cm **b** 7.5 cm **c** 3.7 cm

3 **a** $x = 11$ cm

b $x = 4$ cm

c $x = 6$ cm, $y = 3.5$ cm

4 **a** 2.8 cm **b** 3.5 cm **c** 7.6 cm

5 **a** $x = 2$ cm, $y = 13$ cm **b** $x = 18$ cm

c $x = 2$ cm **d** $x = 4$ cm, $y = 21$ cm

e $x = 17^\circ$ **f** $x = 13.07$ cm

g $x = 4$ cm **h** $x = 2.75$ cm, $y = 4$ cm

i $x = 19^\circ$

6 **a** $x = 4.33$ cm, $y = 8$ cm

b $x = 2.67$ cm, $y = 4$ cm

c $x = 10^\circ$

d $x = 3$ cm, $y = 5$ cm

e $x = 35.5^\circ$

f $x = 2$ cm, $y = 4$ cm

7 **a** $x = 19^\circ$

b $x = 12^\circ$

c $x = 23.8^\circ$, $y = 15.25^\circ$

d $x = 5.66$ cm, $y = 4$ cm, $z = 1$ cm

e $x = 18^\circ$, $y = 18^\circ$

f x can take on any positive value, $y = 0.6$

8 **a** They are all equal in length.

b isosceles triangles

c $OA = OC$ (radii)
$OB = OD$ (radii)
$AB = CD$ (given)
So: $\triangle OAB \equiv \triangle OCD$ (using SSS)

d It follows that $\angle COD = \angle AOB$, because they are corresponding angles in congruent triangles.

9 **a** They are radii of the circle.

b Since $\angle OCA$ is given as 90°, $\angle OCB$ is also equal to 90°, because they are supplementary angles on the chord AB.

c $OA = OB$ (radii)

$\angle OCA = \angle OCB = 90°$ (supplementary angles)

OC is a side common to both triangles.

So: $\triangle ACO \equiv \triangle BCO$ (using RHS)

d AC and BC are corresponding sides in congruent triangles.

10 **a** They are all radii of the circle.

b Since OG is perpendicular to AB, it bisects the chord.

So: $AG = GB$.

Similarly, since OH is perpendicular to chord CD:

$CH = HD$

c If chords AB and CD are same length, and are both bisected by radii that are perpendicular to them, it follows that $AG = GB = CH = HD$.

d $OD = OB$ (radii)

$HD = GB$ (from **c**)

$\angle OHD = \angle OGB = 90°$

So: $\triangle OHD \equiv \triangle OGB$ (using RHS)

e $OH = OG$ because they are corresponding sides of congruent triangles.

f In a right-angled triangle, the longest side is the hypotenuse (in this case, the radius). The other two sides are shorter in length than that. Comparing any other line segment from the centre to the chord to the line segment which is perpendicular to the chord will always result in a right-angled triangle with the perpendicular line shorter than the other line.

11 **a** They are vertically opposite angles.

b They are angles at the circumference subtended by the same arc, DB.

c They are angles at the circumference subtended by the same arc, CA.

d The two triangles have three equal corresponding angles, AAA.

e In similar triangles, the ratio of the lengths of corresponding sides is constant:

$$\frac{AP}{CP} = \frac{BP}{DP}$$

Rearranging these ratios: $\frac{AP}{BP} = \frac{CP}{DP}$

or: $AP \times DP = CP \times BP$

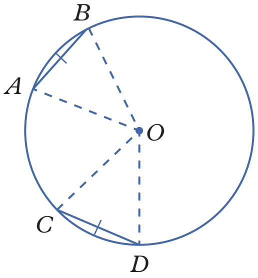
12 Since the quadrilateral is cyclic, opposite angles are supplementary.

Since the quadrilateral is a parallelogram, opposite angles are equal.

If two angles are equal and supplementary, they must each be 90°.

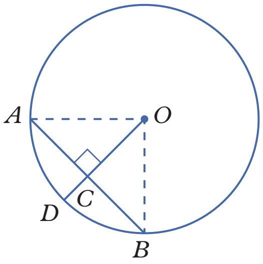
With all angles equal to 90°, the quadrilateral must be a rectangle.

13 Circle theorem 6



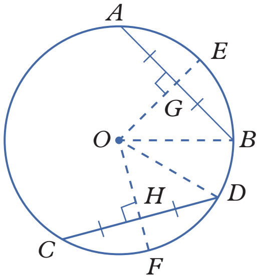
Statements	Reasons
$AB = CD$	Given
$OB = OD = OA = OC$	Radii of the circle
$\triangle AOB \equiv \triangle COD$	SSS (congruence)
$\angle BOA = \angle COD$ $\square$	Corresponding angles in congruent triangles are equal.

Circle theorem 7



Statements	Reasons
$OA = OB$	Radii of the circle
$\angle OCA = 90°$	Given
$\angle OCB + \angle OCA = 180°$	Supplementary angles add to 180°
$\angle OCB = 90°$	Substitution and solving equation
OC is common to $\triangle ACO$ and $\triangle BCO$	Common side
$\triangle ACO \equiv \triangle BCO$	RHS (congruence)
$AC = BC$ $\square$	Corresponding sides in congruent triangles are equal.

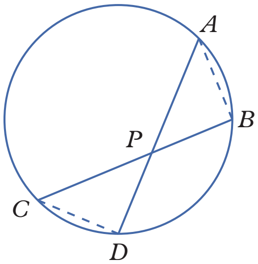
Circle theorem 8



Statements	Reasons
$OE = OF = OB = OD$	Radii of the circle
$\angle OGB = \angle OHD = 90°$	Shortest distance
$AG = BG$ and $CH = HD$	Circle theorem 7
$AB = AG + BG = 2AG$ $CD = CH + HD = 2CH$	Given and substitution
$AB = CD$	Given

Statements	Reasons
$2AG = 2CH$	Substitution
$BG = AG = CH = DH$	Simplification and transitivity
$\triangle AGO \equiv \triangle DHO$	RHS (congruence)
$OG = OH$ $\square$	Corresponding sides in congruent triangles are equal

Circle theorem 9



Statements	Reasons
$\angle APB = \angle CPD$	Vertically opposite angles are equal
$\angle BCD = \angle BAD$	Circle theorem 3
$\angle ABC = \angle ADC$	Circle theorem 3
$\triangle APB \sim \triangle UPV$	AAA (similar)
$\frac{AP}{PD} = \frac{BP}{PC}$	Scale factor between corresponding sides
$AP \times PC = BP \times PD$ $\square$	Rearranging the equation

14 **a** $\angle ACD = 90°$

b If a radius and a chord intersect perpendicularly, then the radius bisects the chord. Therefore, this radius goes through C (the midpoint of AB).

c $ACO = ACD = 90°$, therefore CO and CD are on the same line and O lies on the CD.

d Our finding from part **c** contradicts our assumption that O does not lie on the line segment CD, therefore our assumption was false, and the perpendicular bisector of a chord passes through the centre of a circle.

15 **a**

Statement	Reasons
A, B and C are non-collinear	Given
A, B and C form a triangle	Three points not in a straight line form a triangle
AC and BC are not parallel	No two sides of a triangle are parallel
DE and FG are not parallel	The perpendicular bisectors of two non-parallel lines are not parallel
DE and FG intersect at a unique point O	Two non-parallel straight lines only intersect at a unique point
$OA = OC$	Every point on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment
$OB = OC$	Every point on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment
$\therefore OA = OB = OC$	Transitive law
$\therefore$ A circle can be drawn with centre O and radius OA, which passes through the points A, B and C	Definition of a circle
All points equidistant from A and C lie on DE	A point which is equidistant from two distinct points lies on the perpendicular bisector of the two points
All points equidistant from B and C lie on FG	A point which is equidistant from two distinct points lies on the perpendicular bisector of the two points
The only point equidistant from A, B and C is O	DE and FG intersect at a unique point O
$\therefore$ The circle with O at the centre and OA, OB and OC as radii is the only circle that passes through A, B and C $\square$	

b Part **a** demonstrates that the perpendicular bisectors of all three sides of the triangle formed by three non-collinear points meet at a single point, and that this point is the centre of the unique circle that passes through these points.

16 (3, 3.5)

17

Statement	Reasons
PQ is a chord in the circle with centre M	Given
The perpendicular bisector of PQ passes through M	The perpendicular bisector of a chord passes through the centre of a circle
PQ is a chord in the circle with centre N	Given
The perpendicular bisector of PQ passes through N	The perpendicular bisector of a chord passes through the centre of a circle
The line passing through M and N is the perpendicular bisector of PQ $\square$	There is a unique line passing through any two distinct points

EX p383

7F Circle geometry: tangents and secants

1 **a** 90° **b** 45° **c** 69°

2 **a** $x = 6$ cm **b** $x = 2.75$ cm

c $x = 11.6$ cm, $y = 3$ cm

3 **a** 7.2 cm **b** 12.4 cm **c** 2 cm

4 **a** 14.1 cm **b** 2.7 cm **c** 4.5 cm

5 **a** 37° **b** 45° **c** 45°

6 **a** $x = 3$ cm (tangents to a circle from an external point are equal in length)

b $x = 72°$ (angle between radius and tangent is 90°, angle subtended by diameter is 90°, angle sum in a triangle is 180°)

c $x = 3$ cm, $y = 2$ cm (tangents to a circle from an external point are equal in length)

d $x = 1$ cm (product of the entire secant length by the external secant length is the same for each secant)

e $x = 2.5$ cm (product of entire secant length by external secant length is equal to square of tangent length)

f $x = 64.5°$ (angle sum in isosceles triangle, angle between radius and tangent is 90°)

7 a $x = 20.125$ cm, $y = 12.5$ cm (product of entire secant length by external secant length is equal to square of tangent length)

b $x = 24°$ (angles at a point, angle between tangent and radius is 90°, angle sum of quadrilateral)

c $x = 3.2$ cm, $y = 12.6$ cm (product of entire secant length by external secant length is equal to square of tangent length, tangents to a circle from an external point are equal in length)

8 a AB is a tangent to radius OB, and AD is a tangent to radius OD, so both angles are 90°.

b They are radii of the same circle.

c $OB = OD$ (radii of the same circle)

$\angle ABO = \angle ADO = 90°$

AO is a common side.

Therefore, $\triangle ABO \equiv \triangle ADO$ (RHS)

d AB and AD are corresponding sides in congruent triangles.

9 a $\angle BOE$ is the angle at the centre of the circle, while $\angle BAE$ is the angle at the circumference, standing on the same arc BDE.

b reflex $\angle BOE = 360^\circ - \text{obtuse } \angle BOE$

$= 360^\circ - 2\angle BAE$

c $\angle BDE$ and $\angle BAE$ are supplementary angles in a cyclic quadrilateral.

d $\angle BAE + \angle BDE = 180°$ (cyclic quadrilateral)

$\angle BDC + \angle BDE = 180°$ (angles on a straight line)

So $\angle BDC = \angle BAE$

e $\angle BCD$

f $\angle BDC = \angle BAE$

g If two pairs of angles in a triangle are equal to corresponding angles in another triangle, the third angles in the triangles must also be equal and the triangles are similar by AAA.

h As $\triangle ACE$ is similar to $\triangle DCB$, the ratio of corresponding sides is constant.

Therefore, $\frac{CA}{CD} = \frac{EC}{BC} = \frac{EA}{BD}$.

i $\frac{CA}{CD} = \frac{EC}{BC}$

Cross multiplying, $CA \times BC = CD \times EC$ or $AC \times BC = CE \times CD$

10 a $\angle BCD$

b $\angle BDO = \angle DBO$ (angles in an isosceles triangle opposite equal sides)

Therefore, $\angle BOD = 180 - 2\angle BDO$

Also, $\angle BOD = 2\angle BAD$ (angles subtended by the same arc BD)

Therefore, $2\angle BAD = 180° - 2\angle BDO$

$\angle BAD = 90° - \angle BDO$

As $\angle CDO = 90°$ (angle between radius and tangent)

$\angle CDB = 90° - \angle BDO$

Therefore, $\angle BAD = \angle CDB$ or $\angle CAD = \angle CDB$

c Take $\triangle BCD$ and $\triangle DCA$.

$\angle CDB = \angle CAD$ (proved previously)

$\angle DCB$ is common.

Therefore, the remaining angles in the two triangles are equal.

That is $\angle CBD = \angle CDA$

This means that $\triangle BCD$ is similar to $\triangle DCA$ (using AAA).

d In similar triangles, the ratio of corresponding sides is constant.

Therefore, $\frac{CB}{CD} = \frac{CD}{CA} = \frac{DB}{AD}$.

e Taking the first two ratios and cross multiplying,

$CB \times CA = CD \times CD$ or $CB \times CA = (CD)^2$

11 a $OB = OE$ (radii of the same circle)

$\triangle OBE$ is an isosceles triangle (definition of an isosceles triangle)

$\angle OBE = \angle OEB$ (sides opposite equal angles in an isosceles triangle)

b $\angle EOB + 2\angle OBE = 180°$ (angle sum in a triangle)

c A radius meets a tangent at 90°

d $\theta = 90° - \angle OBE$

e $\angle OBE = 90° - \theta$ (rearranging equation from part **d**)

$\angle EOB + 2(90° - \theta) = 180°$ (substituting into equation from part **b**)

$\angle EOB = 2\theta$

f The angle at the centre is twice the angle at the circumference standing on the same arc.

g Substituting equation from part **e** into equation from part **f**:

$2\theta = 2 \times \angle EDB$

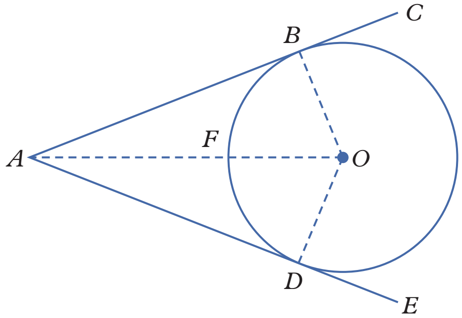
$\angle EDB = \theta$

12 a $x = 5.6$ cm

b $x = 5.7$ cm

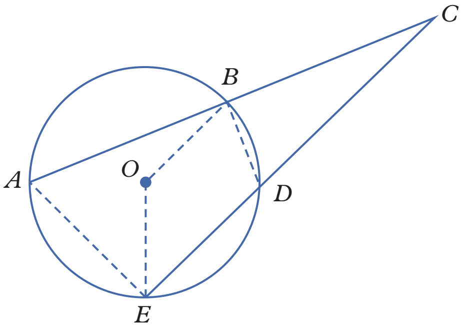
c $x = 17.7$ cm, $y = 4.5$ cm

13 Circle theorem 10



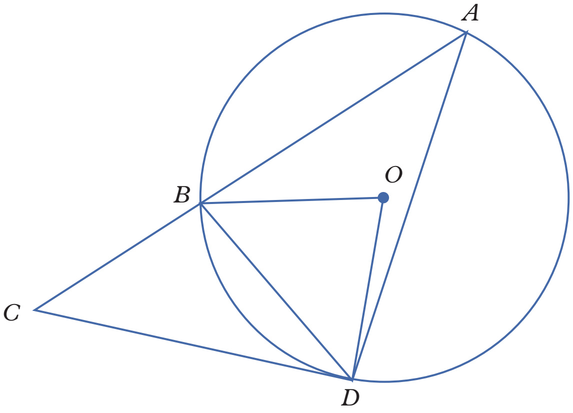
Statement	**Reasons**
$\angle ABO = \angle ADO = 90°$	Tangents meet radii on a circle at 90°
$OB = OD$	Radii of the same circle are equal
AO is common to both $\triangle ABO$ and $\triangle ADO$	Common side
$\therefore \triangle ABO \equiv \triangle ADO$ (RHS)	RHS (congruence)
$AB = AD$	Corresponding sides in congruent triangles are equal
∴ Tangents to a circle from an external point are equal in length □	

Circle theorem 11



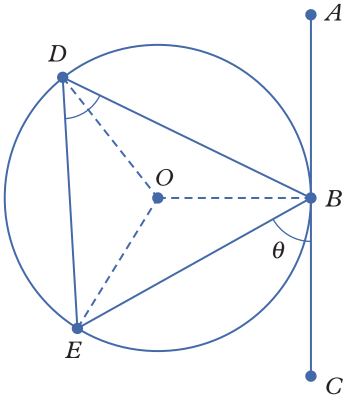
Statement	**Reasons**
$\angle BOE = 2\angle BAE$	The angle at the centre is twice the angle at the circumference standing on the same arc
reflex $\angle BOE = 360^{\circ} -$ obtuse $\angle BOE$	Angles at a point sum to 360°
reflex $\angle BOE = 360^{\circ} - 2\angle BAE$	Substitution
$\angle BAE + \angle BDE = 180°$	Opposite angles in a cyclic quadrilateral are supplementary
$\angle BDC + \angle BDE = 180°$	Angles on a straight line sum to 180°
$\angle BDC = \angle BAE$	Transitive law
$\angle BCD$ is common to both $\triangle DCB$ and $\triangle ACE$	Common angle
$\triangle DCB$ and $\triangle ACE$	AAA (similar)
$\frac{CA}{OD} = \frac{EC}{BC}$	The ratios of corresponding sides in similar triangles are constant
$AC \times BC = CE \times CD$	Simplifying equations
∴ From a point of intersection, the product of the entire secant length by the external secant length is the same for each secant □	

Circle theorem 12



Statement	**Reasons**
$\angle BCD$ is a common angle to $\triangle BCD$ and $\triangle DAC$	Common angle
$\angle BDO = \angle DBO$	Angles in an isosceles triangle opposite equal sides are equal
$\angle BOD = 180° - 2\angle BDO$	Angles in a triangle sum to 180°, substitution
$\angle BOD = 2\angle BAD$	The angle at the centre is twice the angle at the circumference standing on the same arc
$2\angle BAD = 180° - 2\angle BDO$	Substitution
$\angle BAD = 90° - \angle BDO$	Simplifying equation
$\angle CDO = 90°$	Tangents meet radii on a circle at 90°
$\angle CDB = 90° - \angle BDO$	Complementary angles
$\angle BAD = \angle CDB$	Substitution and simplifying equation
$\angle CAD = \angle CDB$	Equivalent angles
$\angle DCB$ is common to $\triangle BCD$ and $\triangle DCA$	Common angle
$\angle CBD = \angle CDA$	If two angles in two triangles are the same, the third angle will also be the same
$\therefore \triangle BCD \sim \triangle DCA$	AAA (Similarity)
$\frac{CB}{CD} = \frac{CD}{CA}$	The ratios of corresponding sides in similar triangles are constant
$CB \times CA = (CD)^2$	Simplifying equation
∴ The product of the entire secant length by the external secant length is equal to the square of the tangent length □	

Circle theorem 13



Statement	Reasons
$OB = OE$	Radii of the same circle are equal
$\triangle OBE$ is an isosceles triangle	A triangle with two equal sides is an isosceles triangle
$\angle OBE = \angle OEB$	Angles opposite the equal sides in an isosceles triangle are equal in size
$\angle EOB + 2\angle OBE = 180°$	Angle sum in a triangle is 180°
$\angle OBC = 90°$	Tangents meet radii on a circle at 90°
$\angle OBE = 90° - \theta$	Complementary angles
$\angle EOB + 2(90° - \theta) = 180°$	Substitution
$\angle EOB = 2\theta$	Simplifying equation
$\angle EOB = 2 \times \angle EDB$	The angle at the centre is twice the angle at the circumference standing on the same arc
$\angle EDB = \theta$	Substitution
∴ The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment □	

14

Statement	Reasons
$OB = OD$	Radii of the same circle are equal
$\angle ABO = \angle ADO = 90°$	Tangents meet radii on a circle at 90°
AO is a common side to $\triangle ABO$ and $\triangle ADO$	Common side
$\triangle ABO \equiv \triangle ADO$	RHS (congruence)
$\angle DAO = \angle BAO$	Corresponding angles in congruent triangles
$\angle BAD = \angle BAO + \angle DAO$	Angle addition postulate
∴ OA bisects $\angle BAD$ □	

15

Statement	Reasons
DE is a common tangent to both circles	Given
$\angle ABE = \angle CBE = 90°$	Tangents meet radii on a circle at 90°
$\angle ABC = \angle ABE + \angle CBE$	Angle addition postulate
$\angle ABC = 90° + 90°$	Substitution
$\angle ABC = 180°$	Simplifying equation
ABC is a straight line	Definition of a straight angle
∴ A, B and C are collinear □	

CHAPTER 7 review

Mathematical literacy review

1 All sides are equal in length and all interior angles are equal in size

2 The angle–side relationship states that the longest side of a triangle is always opposite the largest angle in a triangle. The right angle is the largest angle in a right-angled triangle, and the hypotenuse is the side opposite the right angle. Therefore, the hypotenuse is the longest side of a right-angled triangle.

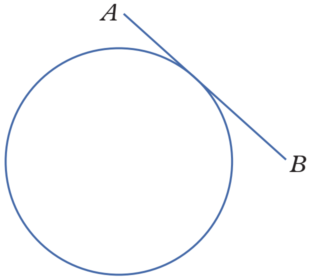
3 A definition is the minimum amount of information required to identify a plane shape and a property is a characteristic that applies to a plane shape.

4 **a** convex polygon

b chord

5 **a** The sum of the lengths of two sides of a triangle is always greater than the length of the third side. For example, if the lengths of two sides of a triangle are 4 cm and 6 cm, the length of the third side must be between 2 cm and 10 cm.

b A tangent is a line that touches the circumference of a circle at one point only. For example, AB is a tangent in the following diagram.



6 **a** supplementary

b theorem

Multiple choice

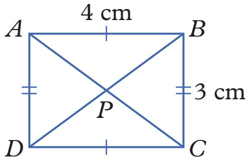
1 C **2** E **3** A **4** D

5 D **6** B **7** C **8** D

Short answer

1 **a** $a = 127°$, $b = 53°$ **b** $x = 105°$

2 **a**



b They are alternate angles between parallel lines AB and CD.

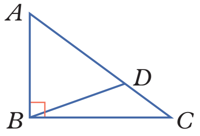
c AAS **d** 53°

3 **a** AAA (alternate and vertically opposite angles)

b $\frac{2}{5}$

c 7.5 cm

4 **a**



b

Statements	**Reasons**
$\angle ADB + \angle BDC = 180^\circ$	Angles on a straight line sum to 180°
$\angle DBC + \angle BCD + \angle BDC = 180^\circ$	Interior angle sum of a triangle is 180°
$\angle ADB + \angle BDC = \angle DBC + \angle BCD + \angle BDC$	Transitive law
$\angle ADB = \angle DBC + \angle BCD$ □	Subtraction property of equality

5 **a**

Statements	**Reasons**
$\angle AEC = \angle ADE + \angle DAE$	Exterior angle of a triangle is equal to the sum of the non-adjacent interior angles
$\angle ADE = \angle DAE$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle AEC = \angle ADE + \angle ADE$ $= 2\angle ADE$ □	Substitution and rearrangement

b

Statements	**Reasons**
$\angle ABC = \angle BCE = 90^\circ$	Given
$\angle ABC + \angle BCE = 180^\circ$	Substitution
$\therefore \angle AEC + \angle BAE = 180^\circ$	Interior angles of a quadrilateral sum to 360°
$\angle AEC = 2\angle ADE$	Given
$2\angle ADE + \angle BAE = 180^\circ$ □	Substitution

6

Statements	**Reasons**
$PQ = RS$	Given
$\angle PQT = \angle RST$	Alternate angles on parallel lines are equal
$\angle PTQ = \angle RTS$	Vertically opposite angles are equal
$\therefore \triangle PQT \equiv \triangle RST$	AAS (congruence)
$TQ = TS$	Corresponding sides in congruent triangles are equal
$TQ^2 + PT^2 = TS^2 + PT^2$	Transitive law
$PQ = PS$	Pythagoras' theorem
$\therefore PQRS$ is a rhombus □	A parallelogram with equal adjacent sides is a rhombus

7 **a** 54° **b** 27° **c** 63°

8

Statements	**Reasons**
$\angle RPQ = \angle RPO + \angle QPO$	Angle addition postulate
$\angle ROQ = 2\angle RPQ$	Circle theorem 1
$\therefore \angle ROQ = 2\angle RPO + 2\angle QPO$	Substitution
$\angle ROQ = 360^\circ - \angle POR - \angle POQ$	Angles at a point sum to 360°
$\therefore 2\angle RPO + 2\angle QPO = 360^\circ - \angle POR - \angle POQ$	Transitive law
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360^\circ$ □	Rearranging

9 **a** 46°

b

Statements	**Reasons**
$\angle DAC = \angle DBC$	Circle theorem 6
$\angle ACD = \angle ABD$	Circle theorem 6
$\angle DAC = \angle ACD = \angle ABD$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle DBC = \angle ABD$ □	Transitive law

10

Statement	**Reasons**
$AP = XP$	Tangents to a circle from an external point are equal in length
$PB = PY$	Tangents to a circle from an external point are equal in length
$AP + PB = XP + PY$	Equivalent equation
$\therefore AB = XY$ □	

Analysis

1 **a**

Statements	**Reasons**
$\angle DGC = 90^\circ$	Interior angles of quadrilateral $FDGC$ add to 360°
$\angle DFC = \angle DGC = 90^\circ$	Transitive law
$FC \parallel DG$	If co-interior angles are supplementary, the lines must be parallel
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal
CD is common to both $\triangle CFD$ and $\triangle CGD$	Common side
$\therefore \triangle CFD \equiv \triangle CGD$ □	AAS (congruence)

b

Statements	**Reasons**
$\angle CAB = \angle CFD = 90^\circ$	Given
$\angle CBA = \angle CDF$	The angle sum of triangles is 180°
$\angle ACB = \angle FCD$	Common to both triangles
$\therefore \triangle CAB \sim \triangle CFD$ □	AAA (similar)

c

Statements	**Reasons**
$\angle CAB = \angle DEB = 90^\circ$	Given
$\angle CBA = \angle DBE$	Vertically opposite angles are equal
$\angle ACB = \angle BDE$	The angle sum of triangles is 180°
$\therefore \triangle CAB \sim \triangle DEB$ □	AAA (similar)

d

Statements	**Reasons**
$AC = DE$	Given
$\angle ACB = \angle BDE$	Given
$\angle CBA = \angle DBE$	Given
$\therefore \triangle CAB \equiv \triangle DEB$ □	AAS (congruence)

Statements	Reasons
$CB = BD$	Corresponding sides in congruent triangles are equal
$\therefore B$ is the midpoint of CD □	Definition of midpoint (divides a line segment into two congruent pieces)

e

Statements	Reasons
$\frac{FC}{AC} = 3$	Given
$FC = FA + AC$	They make up the same line
$FC = 3AC$	Rearranging equation
$FA + AC = 3AC$	Transitive law
$FA = 2AC$	Rearranging the equation
$\angle AFD = \angle FDE =$ $\angle DEA = \angle EAF = 90^\circ$	Given
$\therefore AFDE$ is a rectangle	A quadrilateral with four right interior angles is a rectangle
$FA = DE$	The opposite sides of a rectangle are equal
$DE = 2AC$	Substitution
$\frac{DE}{AC} = 2$ □	Rearrangement

f

Statements	Reasons
$\angle DBE = 45^\circ$	Given
$\angle DBE + \angle BDE +$ $\angle DEB = 180^\circ$	The interior angle sum of a triangle is 180°
$\angle BDE = 45^\circ$	Substitution and solving equation
$\angle CDG = \angle BDE$	Given
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal
$\angle DCG = 45^\circ$	The interior angle sum of $\triangle CGD$ is 180°
$\triangle CGD$ is isosceles with $CG = DG$	If two angles in a triangle are equal, then the sides opposite the equal angles will be equal
$DG = CG$	Equal sides of an isosceles triangle
$CG = FD$	Opposite sides of rectangle $FDGC$
$FC = FD = DG = GC$	Sides of a triangle opposite equal angles are equal
$\therefore FDGC$ is a square □	A rectangle with two adjacent sides which are equal in length is a square

2 **a** The opposite angles are supplementary: $85^\circ + 95^\circ = 180^\circ$ and $116^\circ + 64^\circ = 180^\circ$.

b

Statements	Reasons
$ABCD$ is a rectangle	Given
$\angle ABC = \angle BCD =$ $\angle CDA = \angle DAB = 90^\circ$	Interior angles of a rectangle are 90°
$\angle ABC$ and $\angle CDA$ are supplementary	Supplementary angles sum to 180°
$\angle BCD$ and $\angle DAB$ are supplementary	Supplementary angles sum to 180°
$\therefore ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

c

Statements	Reasons
$AB \parallel DC$	Given
$\angle CDA = \angle BCD$	Given
$\angle ABC + \angle BCD = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle ABC + \angle CDA = 180^\circ$	Substitution
$\angle DAB + \angle CDA = 180^\circ$	Co-interior angles on parallel lines are supplementary
$\angle DAB + \angle BCD = 180^\circ$	Substitution
$\therefore ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

d Two of the edges of these quadrilaterals are radii of the circle, not chords.

e **i** Each pair of equal pronumerals are angles in the same segment subtended by the same chord.

ii $p = 41^\circ$, $q = 41^\circ$, $r = 62^\circ$, $t = 62^\circ$, $w = 23^\circ$, $y = 54^\circ$, $z = 23^\circ$

f $df = eg$

g

Statements	Reasons
$\angle BAD = \angle BCD = 90^\circ$	Given
$\angle ABC + \angle ADC +$ $\angle BAD + \angle BCD = 360^\circ$	Angle sum of a quadrilateral = 360°
$\angle ABC + \angle ADC + 180^\circ$ $= 360^\circ$	Substitution
$\angle ABC + \angle ADC = 180^\circ$	Simplifying
$ABCD$ is a cyclic quadrilateral □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

CHAPTER 8 Measurement

EX
p396

8A Area review

1 **a** 700 cm **b** 4.5 km **c** 8 400 000 m^2
d 0.29 cm^2 **e** 2.4 ha **f** 880 mm
g 0.003 42 km^2 **h** 1 022 000 cm **i** 300 ha

2 **a** V **b** I **c** IV
d II **e** III

3 **a** 42 m^2 **b** 272 cm^2 **c** 76 cm^2
d 348.5 cm^2 **e** 843.6 mm^2 **f** 164.4 cm^2

4 **a** 216 cm^2 **b** 242.1 cm^2 **c** 238.9 cm^2
d 375 mm^2 **e** 28.5 cm^2 **f** 409.5 cm^2

5 **a, b** chimney: trapezium, 1.8 m^2

roof: triangle, 6.5 m^2

front wall: rectangle, 15.54 m^2

front door: rectangle, 3.9 m^2

windows: square, 0.49 m^2 each window

c multiply by 4

d area to be painted = area of chimney + area of roof
+ area of front wall
– area of windows

e 21.88 m^2

f 1.5 L

6 **a** 59.4 cm^2 **b** 515.3 mm^2 **c** 2875.1 cm^2

7 **a** 168 cm^2 **b** 2 m^2 **c** 6.4 cm^2

d 348 cm^2 **e** 933.9 cm^2 **f** 219.5 cm^2

8 **a** Subtract the area of the inner circle from the area of the outer circle.

b 100.5 m^2

c **i** 115.5 m^2 **ii** 58.9 m^2 **iii** 125.7 m^2

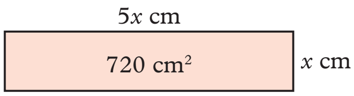
9 **a** $A = \frac{\pi r^2}{4}$

b $A = \frac{\pi r^2}{2}$

c A quadrant is one quarter of a circle, so its area is one quarter of the area of a circle. Similarly, a semicircle is half of a circle, so its area is half of the area of a circle.

d A sector with angle θ takes up $\frac{\theta}{360}$ of the area of a circle.

10 **a**



b The rectangle is 60 cm by 12 cm.

c 2880 cm^2

d 4 times the area

e 6480 cm^2, 9 times the area

f **i** When the dimensions are quadrupled, the area is 16 times as large.

ii When the dimensions are multiplied by a factor of n, the area becomes n^2 times as large.

11 **a** 75.4 cm^2 **b** 60°

c 12.6 cm^2 **d** 31.4 cm^2

12 **a** The radius of the full circle is twice the radius of the semi-circles.

b 904.8 cm^2

c The area of the white section is the same as that of the blue section: 904.8 cm^2.

13 108 cm^2

14 **a** 60π cm^2 **b** 40π cm^2 **c** 3 : 2

15 **a** $5\sqrt{3}$ cm **b** $25\sqrt{3}$ cm^2

c $150\sqrt{3}$ cm^2 **d** $\frac{3\sqrt{3}}{2}a^2$

16 **a**
$$A = \frac{1}{2}(a)(h) + \frac{1}{2}(b)(h)$$
$$= \frac{h}{2}(a+b)$$

b
$$A = ah + \frac{1}{2}(b-a)(h)$$
$$= \frac{ah}{2} + \frac{bh}{2}$$
$$= \frac{h}{2}(a+b)$$

c The two triangles are similar, and so $\frac{x}{x+h} = \frac{a}{b}$.
Solving for x gives:
$$xb = a(x+h)$$
$$xb = ax + ah$$
$$xb - ax = ah$$
$$x(b-a) = ah$$
$$x = \frac{ah}{b-a}$$
Therefore: $A = \frac{1}{2}(b)(h+x) - \frac{1}{2}(a)(x)$
$$= \frac{bh}{2} + \frac{bx}{2} - \frac{ax}{2}$$
$$= \frac{bh}{2} + \frac{x}{2}(b-a)$$
$$= \frac{bh}{2} + \frac{ah}{2(b-a)}(b-a)$$
$$= \frac{bh}{2} + \frac{ah}{2}$$
$$= \frac{h}{2}(a+b)$$

EX p404

8B Surface area review

1 **a** 262 cm^2 **b** 1350 cm^2 **c** 6144 mm^2

d 34.56 m^2 **e** 6156 mm^2 **f** 23.88 m^2

g 876 mm^2 **h** 332 cm^2 **i** 53.66 m^2

2 **a** 662.8 cm^2 **b** 2310.1 mm^2 **c** 117.4 cm^2

3 382.86 cm^2

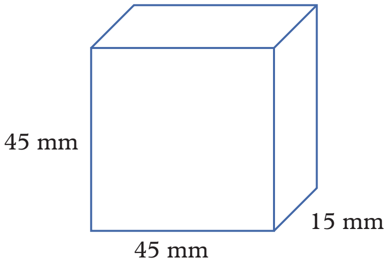
4 **a** 1809.6 cm^2 **b** 883.4 cm^2 **c** 11 309.7 mm^2

5 **a** 8224 cm^2 **b** 334.2 cm^2 **c** 470.4 cm^2

6 **a** 1350 mm^2

b 12 150 mm^2

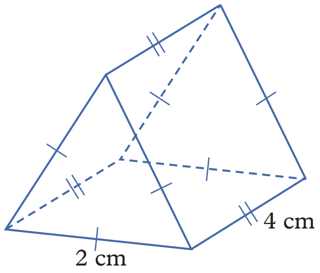
c



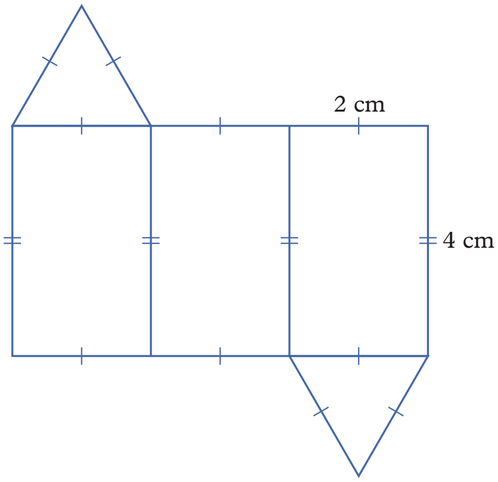
d 6750 mm^2

e 5400 mm^2; 24 surfaces of the dice are hidden within the structure (24 × 15^2 = 5400).

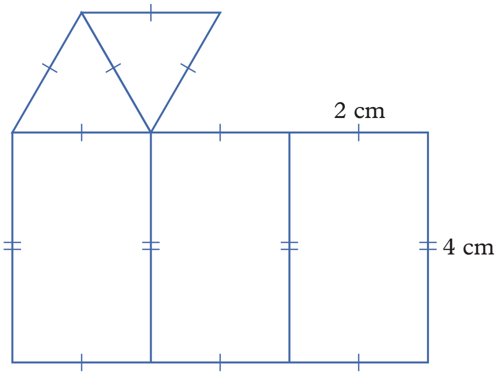
7 **a**



b Sample answer:



c No. For example, this diagram is not the net of the same triangular prism.



8 **a** 570.2 cm^2

b No. The pipe is open, so there is no top or bottom. This means $2\pi r^2$ is removed from formula.

9 279.9 cm^2

10 **a** 7191.3 cm^2

b No. The curved surface area will be double, but the one circular end, which needs painting, has the same area for both poles.

c 14 259.9 cm^2

$2 \times 7191.3 = 14\,382.6$ cm^2

This is 122.7 cm^2 larger than 14 259.9 cm^2.

This difference is the area of one end, which is not doubled.

This supports the answer to part **b**.

11 **a** 730.1 cm^2 **b** 650.8 cm^2

12 **a** 407.8 m^2 **b** 4815.8 m^2

13 1311 cm^2

14 208.9 cm^2

15 **a** 9 m^2 **b** 4 cans

EX p409

8C Volume review

1 **a** 12 500 000 cm^3 **b** 240 cm^3

c 34.2 m^3 **d** 550 000 000 mm^3

e 0.000 067 2 m^3 **f** 9000 mm^3

g 7.52 cm^3 **h** 8 740 000 cm^3

i 0.1429 m^3 **j** 0.073 m^3

2 **a** 2508 cm^3 **b** 16 128 mm^3

c 5.8 m^3 **d** 43 264 cm^3

e 689.1 cm^3 **f** 6.2 m^3

g 96 140 mm^3 or 96.1 cm^3 **h** 29 440 cm^3

i 1539 m^3 **j** 53.5 m^3

k 74 088 m^3 **l** 175 cm^3

3 **a** 1256.6 cm^3 **b** 172 240.2 mm^3

c 16 155.6 m^3

4 **a** 2508 mL **b** 16.1 mL

c 5800 L **d** 43 264 mL

e 689.1 mL **f** 6200 L

g 96.1 mL **h** 29 440 mL

i 1 539 000 L **j** 53 500 L

k 74 088 000 L **l** 175 mL

5 **a** 1102.5 cm^3 **b** 35 224 mm^3 **c** 15.58 m^3

6 **a** 12.48 m^3 **b** 10 560 mm^3 **c** 676.2 cm^3

d 9000 cm^3 **e** 17 090.3 mm^3 **f** 25 249.0 cm^3

g 356.7 mm^2 **h** 188.5 cm^2

i 255 254.4 cm^2

7 5193.4 cm^3

8 **a** rectangular prism, triangular prism, cylinder

b Find the total area of all surfaces showing

c 198 cm^2

d 97 cm^3

9 **a** 206.2 cm^3 **b** 360 cm^3 **c** 1687.5 cm^3

10 **a** 176 cm^3 **b** 0.176 kg **c** $3.70

11 **a** 45 000 cm^3 **b** 45 000 mL, 45 L **c** 39.6 L

12 14 cm

13 **a** 21 cm **b** 34 mm

14 **a** second tank **b** 1178 L

15 **a** 3015.9 cm^3

b **i** 754 cm^3 **ii** $\frac{1}{4}$ the volume

c **i** 12 063.7 cm^3 **ii** 4 times the volume

d **i** 1508 cm^3 **ii** $\frac{1}{2}$ the volume

e **i** 6031.9 cm^3 **ii** 2 times the volume

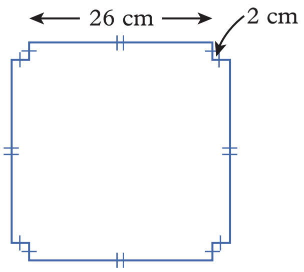
16 **a** 59 244 litres **b** 40 097 litres

17 **a** annulus, area = 84.8 cm^2

b 2121 cm^3

c Subtract the volume of the inner cylinder from the volume of the outer cylinder.

18 **a**



b length = 26 cm, width = 26 cm, height = 2 cm

c 1352 cm^3

d The box would be 24 cm × 24 cm × 3 cm, giving a volume of 1728 cm^3. The volume of the box would increase.

19 **a** circular nut: 107.7 mm^3
hexagonal nut: 144.7 mm^3

b hexagonal nut

EX
p413

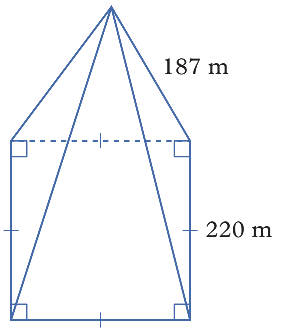
Checkpoint

1 **a** 4.5 cm **b** 4 m **c** 4530 mm
d 40 cm^2 **e** 5 200 000 m^2 **f** 6.8 m^2
2 13.125 m^2
3 **a** 219 cm^2 **b** 25 cm^2
4 306 cm^2
5 41.7 m^2
6 697.8 cm^2
7 2850 cm^2
8 7634.1 cm^3
9 **a** 4000 mm^3 **b** 4 cm^3
10 1292.655 cm^3

EX
p417

8D Surface area of pyramids and cones

1 **a** 2744 mm^2 **b** 118 cm^2 **c** 4856 mm^2
d 408 cm^2 **e** 3115 m^2 **f** 7605 cm^2
2 **a** 809.4 cm^2 **b** 1578.8 mm^2 **c** 9259.9 cm^2
3 1890 m^2
4 **a**



b 66 539.0 m^2
5 **a** 195.6 cm^2 **b** 914.3 mm^2 **c** 4875.0 cm^2
d 460.1 cm^2 **e** 619.2 mm^2 **f** 129.2 cm^2
6 **a** 9123.2 mm^2 **b** 2801.5 cm^2 **c** 624.4 cm^2
d 282.7 cm^2 **e** 628.3 cm^2 **f** 1582.2 mm^2
7 20 cm
8 slant height = 18.2 cm, SA = 1641.0 cm^2
9 **a** 27.7 cm^2 **b** surface area = $x^2\sqrt{3}$
10 339.3 cm^2
11 **a** 8
b 679.0 mm^2
c Each pyramid, including the base, has SA of 535.5 mm^2. So, SA of two pyramids is 1071.0 mm^2.
d The answers to parts **b** and **c** are not equal in this form. If you subtract the areas of the bases of the two pyramids (which are covered) the answers are the same. 1071.0 mm^2 – 2 × 196 mm^2 = 679 mm^2

12 **a** 162.7 cm^2 **b** 6579.8 mm^2 **c** 1847.3 cm^2
13 580.8 cm^2
14 **a** **i** 471.2 cm^2 **ii** 7.5 cm
b **i** $\theta = 180°$ **ii** $\theta = 120°$ **iii** $\frac{\theta}{360}r$
15 **a** 300 cm^2 **b** $\frac{2}{3}$
c **i** 16.2 cm **ii** 424.3 cm^2 **iii** 52.9%

EX
p422

8E Volume of pyramids and cones

1 **a** 1615 cm^3
b 37 762.7 mm^3
c 11.4 cm^3
d 6700.7 cm^3
e 28.9 cm^3 or 28 918.7 mm^3
f 38.2 m^3
g 3033.3 cm^3
h 93.6 cm^3
i 5118.8 mm^3
2 **a** 65 m^3 **b** 1239.3 cm^3 **c** 1760 cm^3
3 2 592 100 m^3
4 **a** 823 825.1 mm^3
b 1747.7 cm^3
c 119.7 cm^3 or 119 703.1 mm^3
d 5026.5 cm^3
e 5277.9 mm^3
f 53.1 m^3 or 53 136 105.4 cm^3
5 16 222.4 cm^3
6 15 519.2 mm^3
7 **a** 8.5 cm; 25.5 cm^2
b **i** 93.5 cm^3
ii 93 500 mm^3
c $V = \frac{1}{3} \times \frac{1}{2}bh_1h_2$, where h_1 and h_2 represent the height
8 **a** 7.0 cm **b** 2.9 cm
9 5.2 L
10 **a** 37 447.8 mm^2 **b** height of cone
c 566 755.9 mm^3
11 **a** 47.1 cm^3
b **i** 6 cm **ii** 188.5 cm^3
c 754.0 cm^3
d When the radius is doubled and the height remains the same, the volume increases by a factor of 4.
e 94.2 cm^3
f 188.5 cm^3
g 377 cm^3
h When the height is doubled and the radius remains the same, the volume is doubled.
i The formula for the volume of a cone involves r^2 so, when r is doubled, r^2 and the volume increase by a factor of 2^2 (or 4). The formula also involves h so, when h is doubled, this increases the volume by a factor of 2.

12 1.2%

13 **a** 6031.9 cm^3 **b** 7680 cm^3

c 1648.1 cm^3 **d** $4 - \pi : 4$

14 **a** 261.8 cm^3 **b** 7.9 cm

15 **a** Four faces are trapeziums; two faces are squares.

b 7291.7 cm^3 **c** 466.7 cm^3 **d** 6825 cm^3

16 3.0 m^3

17 **a** 225 723.4 mm^3 **b** 226 mL

18 One possible answer is shown here.

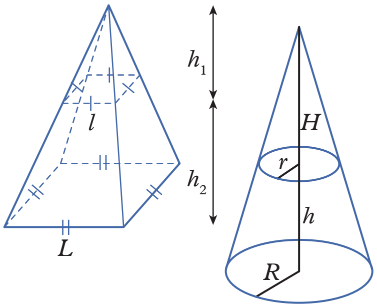
For a truncated square-based pyramid:

$$V = \frac{1}{3}L^2(h_1 + h_2) - \frac{1}{3}l^2h_1$$

For a truncated cone:

$$V = \frac{1}{3}\pi R^2(h + H) - \frac{1}{3}\pi r^2 H$$

or: $$V = \frac{\pi}{3}([h + H]R^2 - Hr^2)$$



EX p428

8F Surface area and volume of spheres

1 **a** 1256.6 mm^2 **b** 8494.9 cm^2

c 2290.2 cm^2 **d** 33 979.5 mm^2

2 **a** 3631.7 cm^2 **b** 7543.0 mm^2

c 10.2 m^2 **d** 8992.0 cm^2

3 **a** 4188.8 mm^3 **b** 73 622.2 cm^3

c 10 306.0 cm^3 **d** 588 977.4 mm^2

4 **a** 20 579.5 cm^3 **b** 61 600.9 mm^3

c 3.1 m^3 **d** 80 178.9 cm^3

5 11 cm

6 **a** 6 cm **b** 9 cm

c 13.0 cm **d** 16.0 cm

7 **a** 100π cm^2

b **i** 25π cm^2 **ii** 50π cm^2 **iii** 75π cm^2

8 **a** **i** 508.9 cm^2 **ii** 1526.8 cm^3

b **i** 3041.1 mm^2 **ii** 22 301.1 mm^3

c **i** 4561.6 mm^2 **ii** 22 301.1 mm^3

d **i** 1490.1 cm^2 **ii** 7649.3 cm^3

e **i** 82.0 cm^2 **ii** 53.8 cm^3

f **i** 9954.9 mm^2 **ii** 71 896.7 mm^3

9 **a** 18 145.8 cm^2 **b** 229 847.3 cm^3

10 **a** 33 cm **b** 6.7 cm

11 **a**

Planet	**Diameter (km)**	**Total surface area (km^2)**	**Volume (km^3)**
Mercury	4878	7.5×10^7	6.1×10^{10}
Venus	12 104	4.6×10^8	9.3×10^{11}
Earth	12 756	5.1×10^8	1.1×10^{12}
Mars	6787	1.4×10^8	1.6×10^{11}
Jupiter	142 800	6.4×10^{10}	1.5×10^{15}
Saturn	120 000	4.5×10^{10}	9.0×10^{14}
Uranus	51 118	8.2×10^9	7.0×10^{13}
Neptune	49 528	7.7×10^9	6.4×10^{13}

b The volume of Earth is about 18 times the volume of Mercury.

c The surface area of Jupiter is about 125 times the surface area of Earth.

d 362 942 012 km^2

12 **a** 8.6 cm **b** 75 cm^3

13 **a** 3591.4 cm^3 **b** 10 744.2 cm^3 **c** 10.7 L

d approximately 4.3 s

e It needs to be assumed that the rate of inflation is a constant rate; and that there is no time lost between filling one balloon and the next.

14 **a** 164.6 cm^3 **b** 658.5 cm^3

c dimensions of canister: diameter = 6.8 cm, height = 27.2 cm, volume = 987.8 cm^3

d 329.3 cm^3

e There would be more unused space because the cylinder would fit inside the rectangular prism.

15 132 mL

16 **a** 16 278.9 cm^3 **b** 4308.2 cm^3 **c** 738.2 cm^3

17 9714 cm^3

18 **a** 33 510.3 mm^3 or 33.5 cm^3 **b** 0.08 g/cm^3

19 **a** $V = \frac{1}{3}nAr$

b $nA = 4\pi r^2$

c The bases of all n pyramids cover the surface area of the sphere. Since nA is equal to n times the surface area of the base of one pyramid, it must be equal to the surface area of the sphere.

d It takes a large number of pyramids to approximate a sphere. For example, $n = 6$ only gives you a cube.

20 **a** **i** 20 cm **ii** 3811.2 cm^3

b **i** 8.7 cm **ii** 1720.7 cm^3

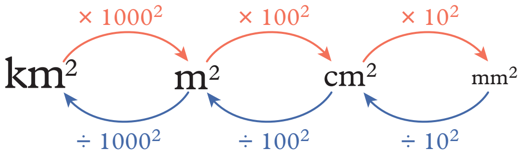
EX p433

CHAPTER 8 review

Mathematical literacy review

1 The cross-section of a prism is a polygon. Since a circle is not a polygon, a cylinder is not a prism.

2



3 **a** The surface area of a 3D object is found by summing the areas of each individual face.

b The volume of an object is how much space it takes up. The capacity is how much liquid an object can hold.

4 A 3D object has a uniform cross-section if, when cutting parallel through the object along its length, the same shape is always obtained. For example, all prisms and cylinders have uniform cross-sections and pyramids do not have a uniform cross-section.

5 **a** A regular tetrahedron is a triangle-based pyramid where all four faces are congruent equilateral triangles.

b A right cone is a 3D object with a circular base and a curved surface that tapers from the circular base to an apex with the apex directly above the centre of the circle.

c A sphere is a round 3D object where every point on its surface is equidistant from its centre.

Multiple choice

1 D **2** A **3** D **4** D **5** C
6 D **7** C **8** B **9** A **10** B

Short answer

1 3.9 cm^2

2 **a** 80.4 cm^2 **b** 63.3 cm^2

3 2664.2 mm^2

4 **a** 9600 cm^2, 38 400 cm^2, 153 600 cm^2

b 1 can

5 385 mL

6 64 000 cm^3, 512 000 cm^3, 4 096 000 cm^3

7 725 m^3

8 225 cm^2

9 341.9 cm^2

10 6363.96 cm^3

11 cylinder: 1024π cm^3

sphere: $\frac{2048\pi}{3}$ cm^3

12 **a** **i** 3769.9 cm^2 **ii** 9600 cm^2

b 10 856.6 cm^2

Analysis

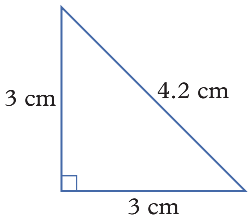
1 **a** cube, rectangular prism, square pyramid, triangular prism, cone, cylinder, sphere

b **i** 54 cm^2 **ii** 27 cm^3

c **i** The rectangular prism has a surface area of 90 cm^2, but the surface area of the cube is 54 cm^2. The rectangular prism is $1\frac{2}{3}$ times as large.

ii The volume of the rectangular prism is 54 cm^3. Its volume is twice the volume of the cube.

d **i**



ii 70.5 cm^2

iii No; the surface area of the triangular prism is more than half the SA of the rectangular prism because two new faces have been created by cutting the prism in half.

iv The volume of the triangular prism is half that of the rectangular prism, because it is the same size as the rectangular prism cut in half.

e **i** 70.7 cm^2 **ii** 42.4 cm^3

f **i**

Solid	Surface area
Cube	54 cm^2
Triangular prism	70.5 cm^2
Cylinder	70.7 cm^2
Rectangular prism	90 cm^2

ii

Solid	Volume
Cube and triangular prism	27 cm^3
Cylinder	42.4 cm^3
Rectangular prism	54 cm^3

g **i** 46.1 cm^2 **ii** 18 cm^3

h **i** 36.2 cm^2 **ii** 14.1 cm^3

i **i** 28.3 cm^2 **ii** 14.1 cm^3

j **i**

Solid	Surface area
Sphere	28.3 cm^2
Cone	36.2 cm^2
Square pyramid	46.1 cm^2
Cube	54 cm^2
Triangular prism	70.5 cm^2
Cylinder	70.7 cm^2
Rectangular prism	90 cm^2

ii

Solid	Volume
Sphere and cone	14.1 cm^3
Square pyramid	18 cm^3
Cube and triangular prism	27 cm^3
Cylinder	42.4 cm^3
Rectangular prism	54 cm^3

2 **a** $\frac{\sqrt{3}}{4}a^2$

b $5\sqrt{3}a^2$

c $\frac{5}{12}(3 + \sqrt{5})a^3 \approx 2.182a^3$

d $SA = 11.366a^2$, $V = 3.603a^3$

e 60.55%

f $8.067a^2$

g The truncated icosahedron is obtained by cutting off parts of the icosahedron, so it must have less volume.

h $2.048a^3$

i $SA = 8.574a^2$, $V = 2.361a^3$

j 86.74%

k By comparing the two percentages in parts **e** and **j**, we see truncated icosahedrons take up more of their circumscribed spheres than icosahedrons. This suggests that they are closer to being spherical and so are easier to make into spherical soccer balls.

CHAPTER 9 Pythagoras' theorem and trigonometry

EX p442

9A Pythagoras' theorem

1 **a** right-angled **b** acute-angled

c acute-angled **d** right-angled

e right-angled **f** obtuse-angled

2 **a** $5\sqrt{2}$ mm **b** $\sqrt{13}$ cm **c** 75 mm

d $2\sqrt{286}$ cm **e** $\sqrt{52.65}$ cm **f** $\sqrt{3.95}$ cm

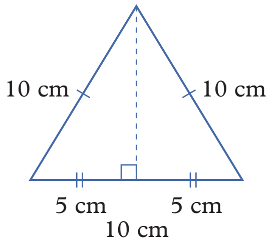
g $\sqrt{75.33}$ cm **h** $\sqrt{17.49}$ cm **i** $\sqrt{66.28}$ cm

3 **a** $2\sqrt{2}$ cm **b** $30\sqrt{2}$ mm **c** $13\sqrt{2}$ cm

d $35\sqrt{2}$ cm

4 $x\sqrt{2}$ cm

5 **a**



b $5\sqrt{3}$ cm

c $25\sqrt{3}$ cm²

6 $\frac{(x^2\sqrt{3})}{4}$

7 **a** 22.45 cm **b** 15.52 cm **c** 10.38 cm

d 41.23 cm **e** 36.66 cm **f** 18.03 mm

8 **a** acute **b** acute **c** obtuse

d acute **e** acute **f** acute

9 **a** 11.2 cm, 22.4 cm

b 7.9 cm, 23.7 cm

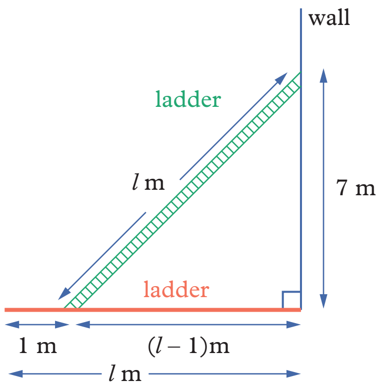
c 4.9 cm, 24.5 cm

10 38 m

11 **a** 6.4 m **b** 38.4 m **c** 74.6 m

d **i** 75.7 m **ii** 82 m

12 **a**



b 25 m

13 **a** $y = -\frac{1}{2}x$, $y = \frac{1}{3}x + \frac{5}{3}$, $y = 7x + 15$

b (–2, 1)

c $2\sqrt{5}$ units, $\sqrt{10}$, units, $5\sqrt{2}$ units

EX p447

9B Trigonometry

1 **a** $p = 5.2$ cm, $q = 4.3$ cm

b $t = 6.3$ mm, $s = 13.6$ mm

c $x = 15.1$ cm, $a = 16.0$ cm

d $k = 41.2$ cm, $m = 26.0$ cm

e $r = 8.6$ mm, $k = 9.2$ mm

f $t = 4.6$ cm, $s = 6.8$ cm

2 **a** 22°13′ **b** 32°20′ **c** 67°10′

d 50°17′ **e** 63°37′ **f** 0°34′

3 **a** $B = 66°$, $C = 24°$

b $T = 29°$, $P = 61°$

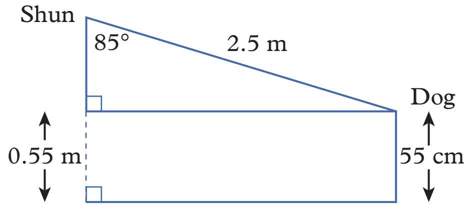
c $G = 32°$, $H = 58°$

d $K = 29°$, $M = 61°$

e $D = 27°$, $R = 63°$

f $D = 40°$, $F = 50°$

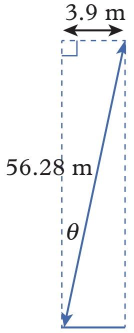
4 **a**



b 249 cm **c** 77 cm

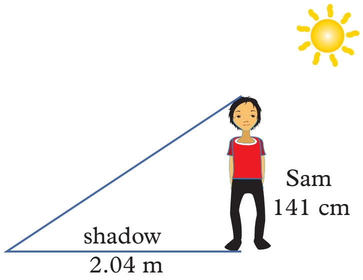
5 $l = 230$ mm, $w = 76$ mm

6 **a**



b $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ **c** 3°58′

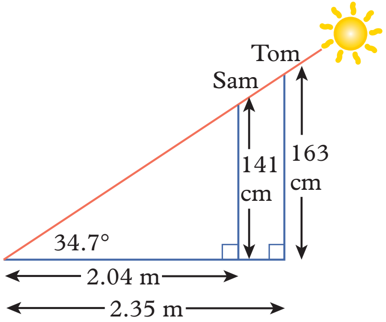
7 **a**



b Tom 34.7°, Sam 34.7°. The base angle represents the angle of elevation of the sun from the ground at the given time.

c **i** If the two triangles have the same base angle and a right angle then all three corresponding angles are equal, meaning the triangles are similar.

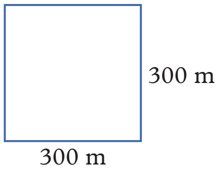
ii



8 **a** Ji-ho has written the trigonometric ratios in the wrong order, the numerator and denominator should be reversed

b When writing trigonometric ratios, make sure to follow the SOH CAH TOA rule

9 **a**



b 424.3 m

c 424.3 m

d The answers are identical.

10 **a** 1.7 m **b** $l = 100$ cm, $w = 80$ cm

11 **a** Two triangles are similar if each of the three angles in one triangle is equal to an angle in the other triangle.

b The angle whose size is unlabelled in both triangles must be $90° - \theta$ since the angle sum of a triangle is 180°. Then $\angle ABC = \angle XYZ = \theta$, $\angle ACB = \angle XZY = 90°$ and $\angle BAC = \angle YXZ = 90° - \theta$, so the triangles satisfy the AAA condition of similarity.

c Two triangles are similar if the lengths of all three pairs of corresponding sides are in the same ratio.

d **i** $\sin(\theta) = \frac{b}{c} = \frac{y}{z}$

ii $\cos(\theta) = \frac{a}{c} = \frac{x}{z}$

iii $\tan(\theta) = \frac{b}{a} = \frac{y}{x}$

e Given an acute angle θ, any two right-angled triangles with θ as an interior angle are similar, and so also have sides in the same ratio. These ratios include the three trigonometric ratios. This means that the values of the trigonometric ratios depend only on the value of θ.

12 **a** $x = 13.16$ cm **b** $x = 46.24°$

c $x = 19.66$ cm **d** $x = 145.32°$

13 20.78 cm

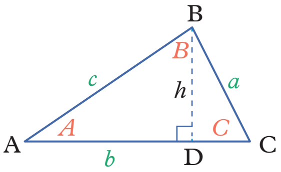
14 **a** **i** $x = 40$ cm, $y = 34.64$ cm

ii $x = 80.52$ cm, $y = 59.69$ cm

b **i** 692.8 cm^2

ii 2403.3 cm^2

15 **a, b**



c **i** $h = c\sin(A)$ **ii** $h = a\sin(C)$

iii $a\sin(C) = c\sin(A)$

$$a = \frac{c\sin(A)}{\sin(C)}$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

d $a\sin(B) = b\sin(A)$

$$a = \frac{b\sin(A)}{\sin(B)}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

So, with what was found in part **c**, we know that:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

16 **a** 67.5° **b** octagon **c** 80 cm

17 **a** **i** 0 **ii** It is 0 : 1.

b **i** a is equal in length to c. **ii** 1

c The sine increases from 0 : 1 at 0° to 1 : 1 at 90°.

d The cosine ratio has a value of 1 : 1 at 0°, becoming 0 : 1 at 90°.

e The tangent is 0 : 1 at 0°, becoming infinitely large as the angle approaches 90°.

18 The sine ratio has a minimum value of 0 and a maximum value of 1.

The cosine ratio has a minimum value of 0 and a maximum value of 1.

The tangent ratio has a minimum value of 0 and a maximum value of ∞.

EX p455

9C Applications of trigonometry

1 **a** $x = 117°, y = 63°$ **b** $x = 58°, y = 302°$

c $x = 45°, y = 45°$ **d** $x = 19°, y = 60°, z = 281°$

e $v = 153°, w = 57°, x = 150°, y = 30°, z = 282°$

f $v = 118°, w = 99°, x = 40.5°, y = 37°, z = 282.5°$

2 **a** $w = 95.7$ m **b** $x = 27.0$ m **c** $z = 148.2$ m

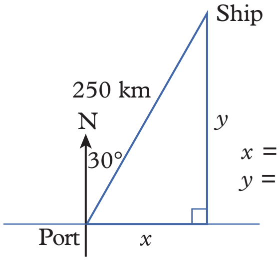
d $p = 121.7$ m **e** $s = 254.3$ m **f** $y = 235.2$ m

3 **a** 25 m **b** 144 m

4 **a** **i** 150°T **ii** 308°T **iii** 005°T

b **i** S20°W **ii** S80°E **iii** N60°W

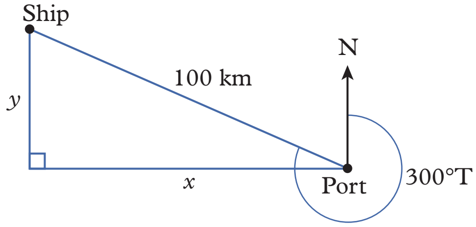
5 **a** **i**



x = distance travelled east
y = distance travelled north

ii 125 km east, 217 km north

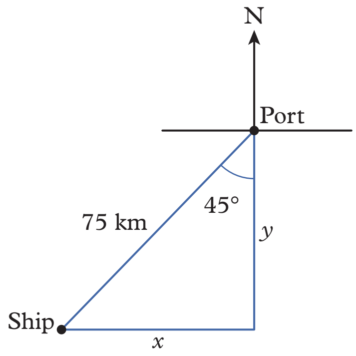
b **i**



x = distance travelled west
y = distance travelled north

ii 87 km west, 50 km north

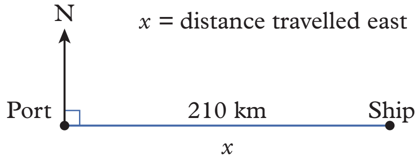
c **i**



x = distance travelled west
y = distance travelled south

ii 53 km west, 53 km south

d **i**



x = distance travelled east

ii 210 km east, 0 km north

6 **a** **i** 250 km on a bearing of S30°W

ii 250 km on a bearing of 210°T

b **i** 100 km on a bearing of S60°E

ii 100 km on a bearing of 120°T

c **i** 75 km on a bearing of N45°E

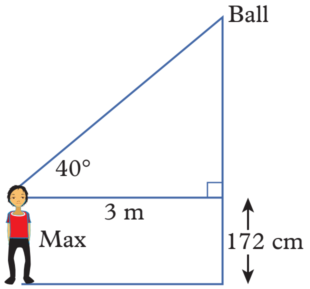
ii 75 km on a bearing of 045°T

d **i** 210 km on a bearing west

ii 210 km on a bearing of 270°T

7 **a** 10.39 m **b** 0.4 m/s

8 **a**



b 2.52 m (252 cm)

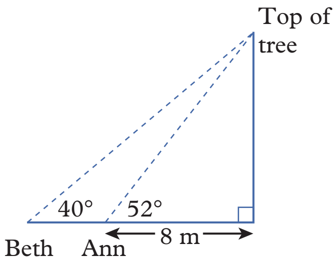
c 4.24 m (424 cm)

d The height of the ball is not significantly larger than the height of the eye level. If Max was measuring the height of a cliff, the height of the eye level may not be as important. If we do not consider the height of a person's eye in this situation, the error would be too big.

9 The angle of elevation and the angle of depression cannot be obtuse, because they are always measured from the line of sight pointing towards the other point. An obtuse angle would point backwards and therefore be measured facing the opposite direction.

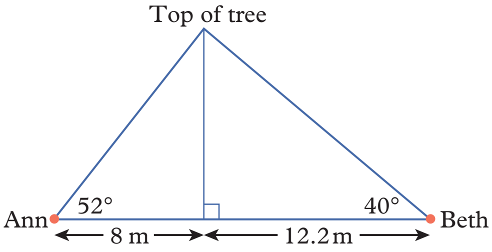
10 **a** 25.87 m **b** 0.55 m

11 **a**



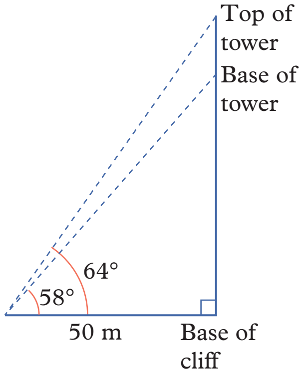
b 10.2 m **c** 12.2 m **d** 4.2 m

e **i**



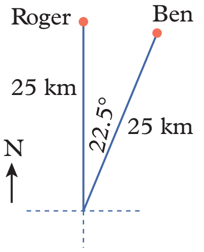
ii 20.2 m

12 **a**



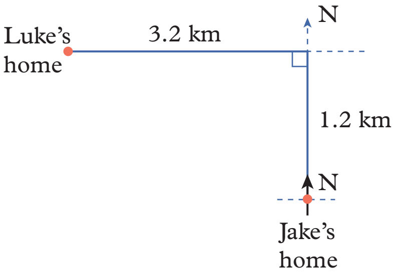
b 22.5 m

13 **a**



b 23.1 km **c** 9.6 km **d** 9.8 km

14 **a**



b 3.4 km **c** 291°T **d** 111°T

e The bearing and its back-bearing differ by 180°.

f **i** 3.4 km

ii The bearing from Jake's house to Luke's house is 291°T.

The bearing from Luke's house to Jake's house is 111°T.

g The answers are the same.

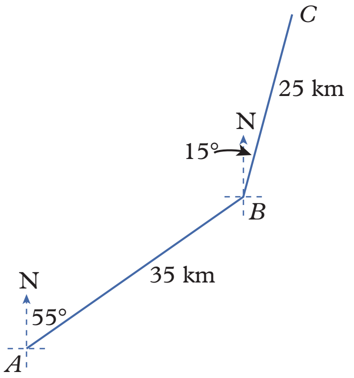
15 **a** 60° **b** 45°

c 40° **d** 10°

e **i** The angles in a triangle add to 180°; subtracting the acute measured angle and the right angle from 180° gives the angle of elevation. Alternatively, the acute measured angle and the angle of elevation are complementary as they are the non-right angles in a right-angled triangle.

ii The acute measured angle is vertically opposite the other non-right angle in the right-angled triangle. Alternatively, the obtuse measured angle forms a straight angle with the other non-right angle, so they are supplementary and must add to 180°. Then find the angle of depression by subtracting the acute measured angle and the right angle from 180° (or by finding the complement of the acute measured angle).

16 **a**



b **i** 20.1 km **ii** 24.1 km **iii** 44.2 km

c **i** 28.7 km **ii** 6.5 km **iii** 35.2 km

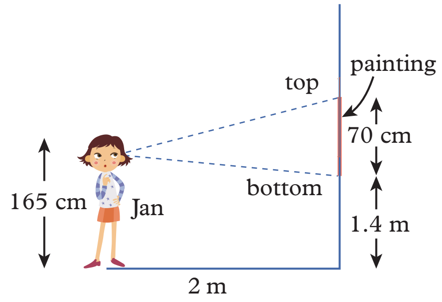
d **i** 56.5 km **ii** 039°T **iii** 219°T

e **i** 56 km, bearing of *C* from *A* is 039°T, bearing of *A* from *C* is 219°T

ii The answers are almost the same.

17 **a** 26.6 km **b** 062°T

18 **a**



b 12.7° **c** 7.1° **d** 13.3°

19 139 m

20 **a** 610 km

b **i** 26 km **ii** 12 km

c Yes, total distance would be 3552 km, total fuel was enough to fly 3560 km.

EX p461

9D Three-dimensional problems

1 **a** 21.21 cm **b** 25.98 cm **c** 35.26°

2 **a** diagonal of base (*d*):

$d = \sqrt{20^2 + 15^2}$

$d = \sqrt{625}$

$d = 25$ cm

diagonal of the box (*e*):

$e = \sqrt{10^2 + 25^2}$

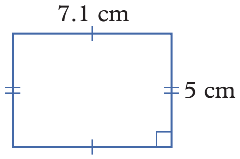
$e = \sqrt{725}$

$e = 26.93$ cm

The length of the rod is greater than the length of the diagonal of the box, so a 30 cm rod will not fit into the box.

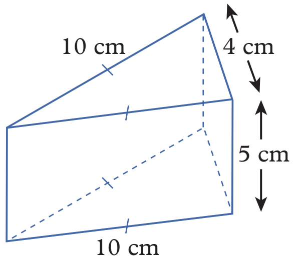
b 21.80°

3 **a** rectangle:

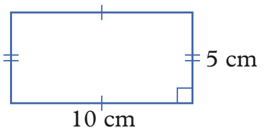


b Area of exposed face = 35.5 cm^2; area of cube face = 25 cm^2. So the area of the exposed face is 1.4 times greater than the area of one of the cube's faces.

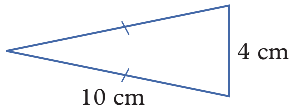
4 **a**



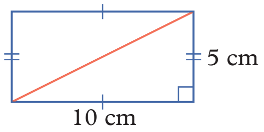
b **i**



ii



c



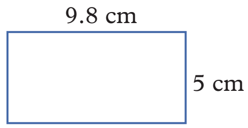
i 11.2 cm

ii 26.57°

iii The angles are complementary.
90° – 26.57° = 63.43°

d **i** 9.8 cm **ii** 19.6 cm^2

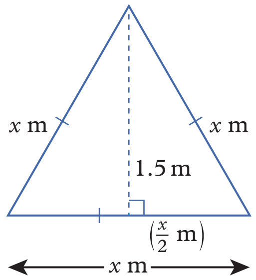
e **i**



ii Cut side: 9.8 × 5 = 49 cm^2
Other side: 10 × 5 = 50 cm^2

iii 11 cm

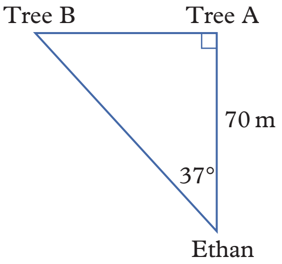
5 **a**



b 1.73 m

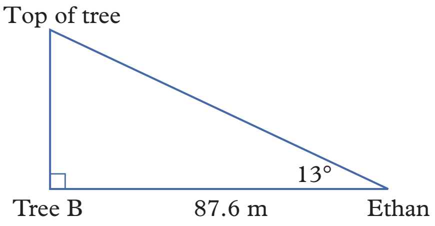
c Jack requires 2 × 1.73 = 3.46 m of material to cover the tent. 3.46 < 3.5, there is sufficient material.

6 **a**



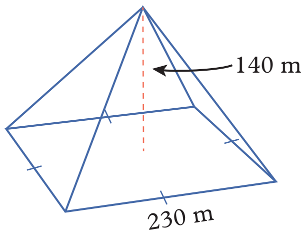
b 87.6 m

c



d 20.2 m

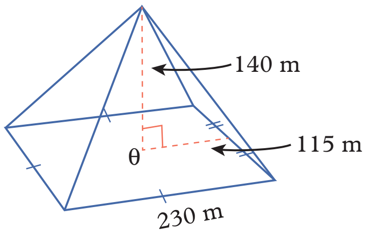
7 **a**



b 325.27 m

c 40.72°

d **i**



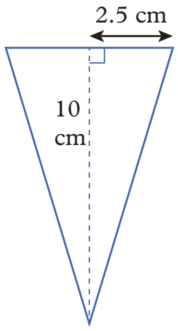
ii 50.60°

8 **a** 19.1 cm

b Internal diagonal = 22.6 cm. The spoon could fall in, because the internal diagonal of the saucepan is longer than the spoon.

9 **a** 2.5 cm

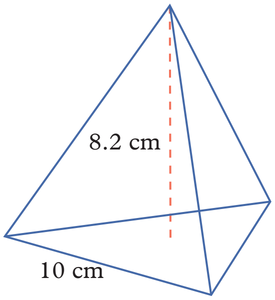
b



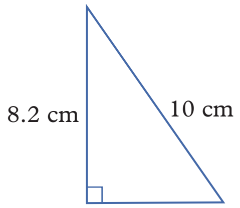
c 10.3 cm

10 9 streamers equalling the full slant length of the tree

11 **a, b**

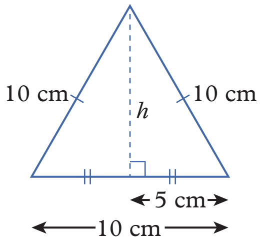


c **i**



ii 55°

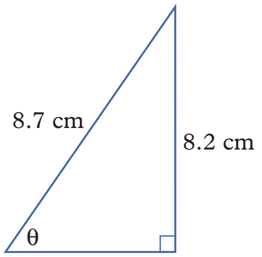
d



h = height of face

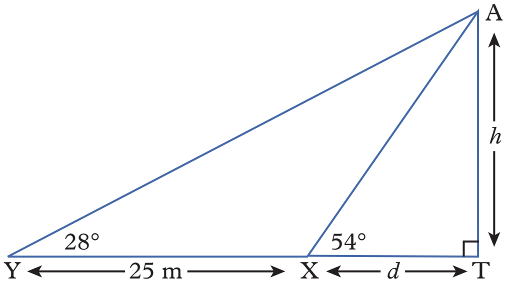
i 8.7 cm

ii



The angle (θ) between the base and a face of the tetrahedron is 71°.

12 a



b $h = d \times \tan(54°)$

c $h = (25 + d) \times \tan(28°)$

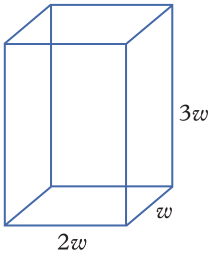
d 15.7 m **e** 21.7 m

13 a $\sqrt{w^2 + l^2}, \sqrt{l^2 + h^2}, \sqrt{w^2 + h^2}$

b $\sqrt{w^2 + l^2 + h^2}$

14 a $l = 2w$; $h = 3w$

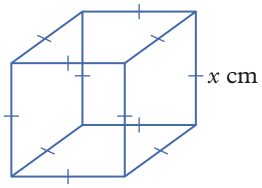
b



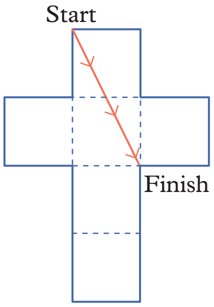
c $\sqrt{5}w$ **d** $\sqrt{14}w$ **e** 53°

15 3.6 cm

16 a



b i



ii $\sqrt{5}x$ cm

17 a 29.2 m **b** 36.8°

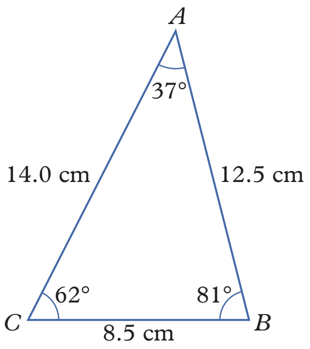
EX
p470

9E The sine and area rules

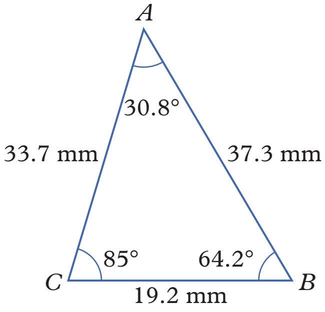
1 a 0.5 **b** 0.5 **c** 0.6428
d 0.6428 **e** 0.9848 **f** 0.9848
g 0.3746 **h** 0.3746

2 a 13.4 mm **b** 54.3° **c** 124.3°

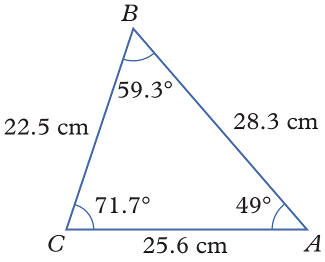
3 a $b = 14.0$ cm, $C = 62°$, $c = 12.5$ cm



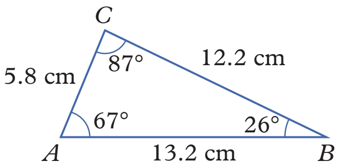
b $B = 64.2°$, $A = 30.8°$, $a = 19.2$ mm



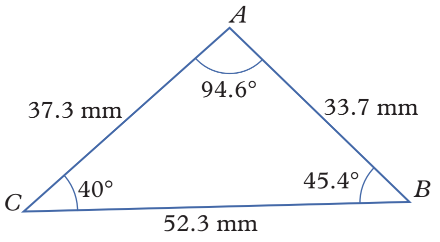
c $C = 71.7°$, $B = 59.3°$, $b = 25.6$ cm

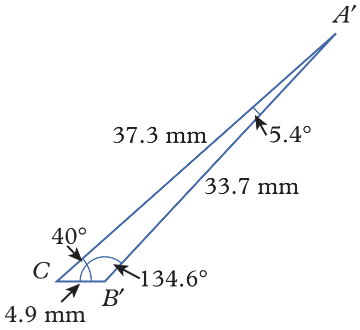


d $C = 87.0°$, $a = 12.2$ cm, $c = 13.2$ cm

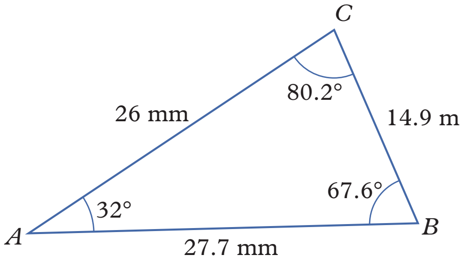


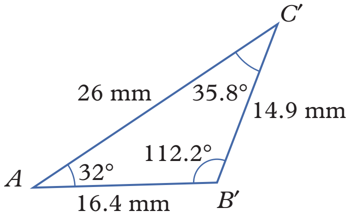
4 **a** $B = 45.4°$, $A = 94.6°$, $a = 52.3$ mm or $B = 134.6°$, $A = 5.4°$, $a = 4.9$ mm



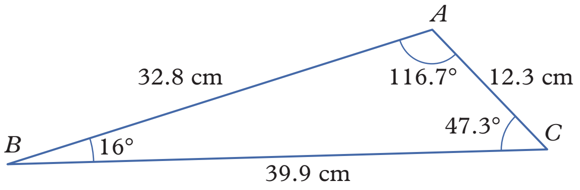


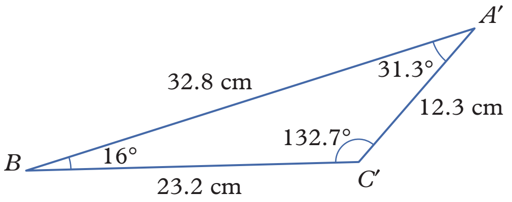
b $B = 67.6°$, $C = 80.4°$, $c = 27.7$ mm or $B' = 112.4°$, $C' = 35.6°$, $c = 16.4$ mm



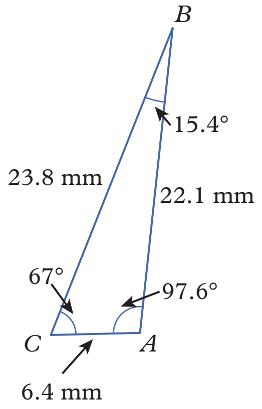


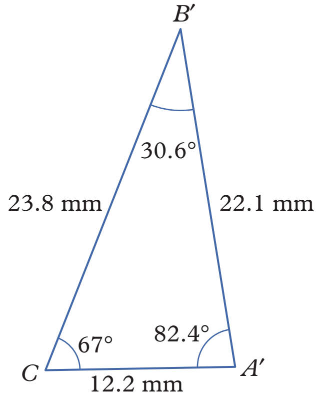
c $C = 47.3°$, $A = 116.7°$, $a = 39.9$ cm or $C' = 132.7°$, $A' = 31.3°$, $a = 23.2$ cm





d $A = 97.6°$, $B = 15.4°$, $b = 6.4$ mm or $A' = 82.4°$, $B' = 30.6°$, $b = 12.2$ mm





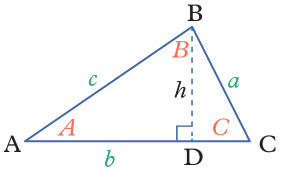
5 **a** 16.9 cm^2 **b** 12.4 mm^2 **c** 18.1 mm^2
d 39.2 cm^2

6 **a** 11.7 cm^2 **b** 9.7 m^2 **c** 130.4 cm^2
d 86.5 cm^2

7 **a** **i** Yes, an angle-side pair is known (*c*-*C*).
ii Yes, as the side opposite the angle, *c*, is shorter than the other length, *b*.
b **i** No, an angle-side pair is not known.
ii N/A
c **i** Yes, an angle-side pair is known (*b*-*B*).
ii No, two angles are given so it is not ambiguous.
d **i** Yes, an angle-side pair is known (*b*-*B*).
ii No, the side opposite the angle, *b*, is not shorter than the other length, *c*.

8 **a** 22.4 cm **b** 61.1 cm **c** 24.7 mm
d 31.5 cm **e** 64.3 mm **f** 53.7 mm

9 **a, b**



c **i** $h = c\sin(A)$
ii $h = a\sin(C)$
iii $a\sin(C) = c\sin(A)$

$$a = \frac{c\sin(A)}{\sin(C)}$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

d $a\sin(B) = b\sin(A)$

$$a = \frac{b\sin(A)}{\sin(B)}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

So, with what was found in part **c**, we know that:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

10 **a** 1

b $\sin(B) = \frac{b}{a}$, $\sin(C) = \frac{c}{a}$

c $a = a = a$

d In the right-angled triangle, the sine rule simplified to *a*, where *a* is the length of the hypotenuse. Yes, the sine rule holds for right-angled triangles.

e If the right angle was at ∠*ABC*, the answer would be *b*.

If the right angle was at ∠*BCA*, the answer would be *c*.

11 **a** $\sin(C) = \frac{c}{a}$ **b** $A = \frac{1}{2}bc$

c *A* represents the area, *b* represents the base of the triangle, *c* represents the height of the triangle. The formula from part **b** is written in the same form as $A = \frac{1}{2}bh$, the formula for the area of any right-angled triangle.

12 **a** **i** 133° **ii** 20°

b 19.9 m **c** 14.6 m **d** 13.6 m

e 28.6 m

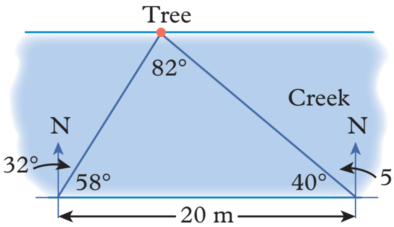
13 **a** 60° **b** 53° **c** 67°

d 28.6 km

e **i** N37°E **ii** S37°W **iii** west

f 310.0 km^2

14 **a**

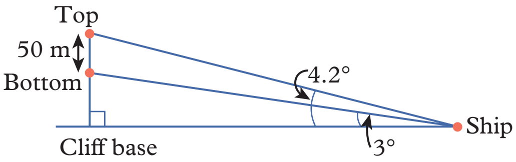


b **i** 13.0 m **ii** 17.1 m

c 11 m

15 693 cm^2

16 **a**



b **i** 2381.1 m **ii** 2384.2 m

c 2.4 km **d** 125 m

17 **a** 142°T and 218°T

b 13.1 km and 28.5 km

c 50.8 km

18 **a** The shorter known side is opposite the known angle.

b 100.1 cm, 39.9 cm

c 998.3 cm^2, 398.1 cm^2

d **i** 600.1 cm^2

ii $\frac{1}{2} \times 36.1 \times 36.1 \times \sin(112.9°)$

$= \frac{1}{2} \times 36.1^2 \times \sin(112.9°)$

19 **a** $\text{Area} = \frac{1}{2}ch$

b $h = b\sin(A)$, $\text{Area} = \frac{1}{2}bc\sin(A)$

c $h = a\sin(B)$, $\text{Area} = \frac{1}{2}ac\sin(B)$

d They are the same as the included angle area formula for the cases which include angles *A* and *B*. Doing this for one case is the main step in the proof of the formula.

20 **a** $\text{area} = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) = \frac{1}{2}ab\sin(C)$

b $\frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) = \frac{1}{2}ab\sin(C)$

$$\frac{bc\sin(A)}{abc} = \frac{ac\sin(B)}{abc} = \frac{ab\sin(C)}{abc}$$

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

c $(\sin(60°))^2 = \frac{3}{4}$

d $\sin(60°) = \frac{\sqrt{3}}{2}$

e $\text{area} = \frac{\sqrt{3}}{4}l^2$

21 **a** **i** 60° **ii** 6.3 cm **iii** 17.2 cm^2 **iv** 103.1 cm^2

b **i** 72° **ii** 11.8 cm **iii** 66.5 cm^2 **iv** 332.4 cm^2

c **i** 45° **ii** 9.1 cm **iii** 29.6 cm^2 **iv** 236.6 cm^2

d **i** 30° **ii** 7.0 cm **iii** 12.1 cm^2 **iv** 145.1 cm^2

e $$A = \frac{l^2 n\left(\sin\left(\frac{90°(n-2)}{n}\right)\right)^2}{2\sin\left(\frac{360°}{n}\right)}$$

22 **a** 28.40 cm

b **i** 233.80 cm^2 **ii** 633.25 cm^2 **iii** 399.45 cm^2

c **i** $x = 52.62$ cm, $a = 40.01°$, $b = 29.99°$, $y = 27.99$ cm

ii 1989.48 cm^2

23 **a** 2.29 cm

b 2.29 cm where *B* is at the right angle as the perpendicular distance from the ray to the point is the shortest distance to the ray *AB*.

c 4 cm, where one of the two positions for *B* is no longer creating a triangle.

d $x\sin(\theta) < BC < x$

EX
p475

Checkpoint

1 **a** yes **b** yes **c** no **d** yes

2 **a** **i** $\sqrt{41}$ cm **ii** 6.40 cm

b **i** $6\sqrt{14}$ mm **ii** 22.45 mm

c **i** $10\sqrt{5}$ mm **ii** 22.36 mm

d **i** $20\sqrt{2}$ cm **ii** 28.28 cm

3 **a** $x = 12.0$ mm, $y = 6.0$ mm, $\theta = 26.6°$

b $x = 9.3$ mm, $y = 6.3$ mm, $\theta = 42.7°$

c $x = 17.4$ mm, $y = 14.1$ mm, $\theta = 35.9°$

4 **a** $x = 63.8°$, $y = 26.2°$, $z = 20.1$ cm
b $x = 50.9°$, $y = 39.1°$, $z = 24.6$ mm

5 **a** 095° T **b** 145° T **c** 293.55° T

6 **a** 28.2 m **b** 26.6° **c** 15.8 m
d 8.6 m

7 **a** 78.7 m east and 216.1 m south **b** 056°T

8 **a** $x = 5$ cm, $y = 7.8$ cm
b $r = 5.3$ cm, $y = 9.8$ cm

9 **a** $x = 9.0$ mm, $z = 12.0$ mm
b $x = 63.4°$, $y = 22.4$ cm, $z = 12.6°$

10 **a** $x = 43°$, $y = 4.7$ cm, $z = 5.6$ cm
b $y = 23.5°$, $x = 37.5°$, $z = 11.0$ mm
c $x = 128°$, $y = 16.4$ cm, $z = 13.2$ cm
d $y = 30.2°$, $x = 84.8°$, $z = 8.9$ mm

11 **a** $y = 68.0°$, $x = 60.0°$, $z = 11.2$ mm or $y = 112.0°$, $x = 16.0°$, $z = 3.6$ mm
b $y = 85.5°$, $x = 65.5°$, $z = 16.7$ cm or $y = 94.5°$, $x = 56.5°$, $z = 15.3$ cm

12 **a** 126.61 mm^2 **b** 662.82 cm^2 **c** 253.03 cm^2
d 117.42 mm^2

EX p480

9F The cosine rule

1 **a** 0.5 **b** −0.5 **c** 0.7660
d −0.7660 **e** 0.1736 **f** −0.1736
g 0.9272 **h** −0.9272

2 **a** 3.6 cm **b** 8.8 cm **c** 22.5 cm
d 13.7 cm **e** 56.2 mm **f** 44.9 cm
g 55.4 mm **h** 14.7 m **i** 15.5 cm

3 **a** $A = 47.2°$, $B = 78.1°$, $C = 54.6°$
b $A = 81.8°$, $B = 66.0°$, $C = 32.2°$
c $A = 40.7°$, $B = 87.9°$, $C = 51.5°$
d $A = 56.9°$, $B = 84.2°$, $C = 38.9°$

4 **a** 109.2° **b** 48.6°
c 38.7° **d** 144.5°

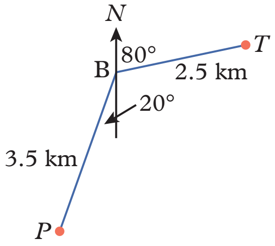
5 **a** $a = 18.5$ cm, $B = 94°$, $C = 38°$
b $e = 36.1$ mm, $D = 60°$, $F = 48°$
c $h = 48.2$ cm, $G = 53°$, $I = 68°$
d $j = 40.7$ m, $K = 50°$, $L = 67°$
e $n = 5.3$ m, $M = 41°$, $P = 60°$
f $s = 19.3$ mm, $Q = 51°$, $R = 60°$

6 **a** We would need to know the magnitude of a second angle, either B or C.
b 55.8 cm
c **i** Sample answer: use $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$ to find B. Subtract the sum of A and B from 180° to find C.
ii $B = 35°$, $C = 59°$
d $B = 35°$, $C = 59°$
e The answers are the same.

7 **a** $c^2 = a^2 + b^2 - 2ab\cos(C)$
b 0
c $c^2 = a^2 + b^2$; this is Pythagoras' Theorem.
d If C is an acute angle, then $0 < \cos(C) < 1$, so the adjustment term, $-2ab\cos(C)$, will be negative. Therefore, the adjustment term will decrease the sum of the squares of a and b, so $a^2 + b^2 - 2ab\cos(C) < a^2 + b^2$.
e If C is an obtuse angle, then $-1 < \cos(C) < 0$, so the adjustment term, $-2ab\cos(C)$, will be positive as the product of two negative numbers is positive. Therefore, the adjustment term will increase the sum of the squares of a and b, so: $a^2 + b^2 - 2ab\cos(C) > a^2 + b^2$.

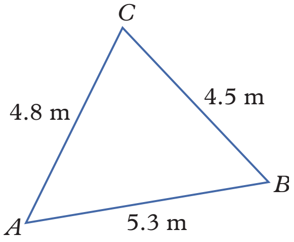
8 **a** $a^2 = h^2 + (b - x^2)$
b $h^2 = c^2 - x^2$
c
$$\begin{aligned} a^2 &= c^2 - x^2 + (b - x)^2 \\ &= c^2 - x^2 + b^2 - 2bx + x^2 \\ &= b^2 + c^2 - 2bx \end{aligned}$$
d $x = c\cos(A)$
f $b^2 = a^2 + c^2 - 2ac\cos(B)$
$c^2 = a^2 + b^2 - 2ab\cos(C)$
g **i** $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$
ii $\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$
iii $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$
h **i** two side lengths and the angle between them
ii three side lengths

9 **a**



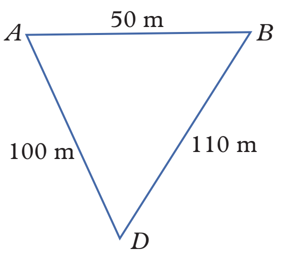
b 120° **c** 5.2 km **d** 25°
e N45°E **f** 35° **g** S45°W

10 **a**



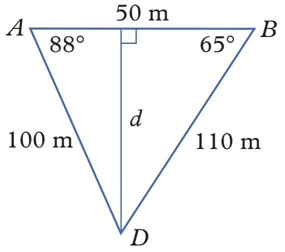
b C
c 69°
d The following formula can be used to calculate the area:
area $= \frac{1}{2}ab\sin(C)$.
$A = 10.1$ m^2

11 **a**



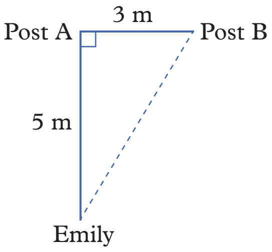
b 65° and 88°

c Use the sine ratio with either of the two right-angled triangles created.



d 99.9 m

12 **a**



b 5.8 m

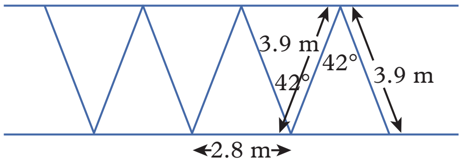
c 31°

d Emily has a greater chance of scoring a goal. She now has a larger angle of 35° within which she can score a goal.

13 **a** 3.9 m **b** 3.7 m

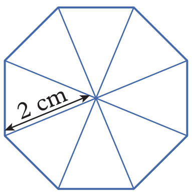
14 **a** 42°

b



c 3.6 m

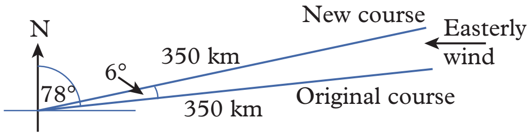
15 **a**



b **i** 45° **ii** 67.5°

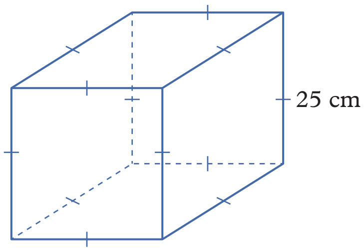
c 12.2 cm

16 **a**



b 36.6 km

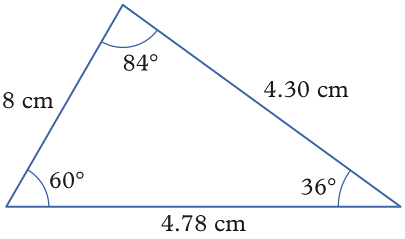
17 **a**



b $25\sqrt{2}$ cm **c** $25\sqrt{3}$ cm

18 **a** 36°, 60°, 84° **b** 4.78 cm

c



19 **a** $(38\sin(63°))^2 + (24 - 38\cos(63°)) = x^2$

$x = 34.5$ cm

b $(41\sin(116°))^2 + (17 - 41\cos(116°))^2 = x^2$

$x = 50.8$ cm

c $(b\sin(A))^2 + (c - b\cos(A))^2 = a^2$

$(b\sin(A))^2 + c^2 - 2c(b\cos(A)) + (b\cos(A))^2 = a^2$

$(b\sin(A))^2 + (b\cos(A))^2 + c^2 - 2bc\cos(A) = a^2$

$b^2 + c^2 - 2bc\cos(A)^2 = a^2$,

since $(b\sin(A))^2 + (b\cos(A))^2 = b^2$

20 **a** **i, ii** $r\sqrt{2}$ units

b Both methods give the same answer.

21 **a** $x^2 - 6\cos(64°)x - 16 = 0$

b $x = -2.9, x = 5.5$

c No, as one of the solutions, $x = -2.9$, is negative. Therefore, the unknown side length is 5.5 cm.

d $x = 2.3, x = 5.2$

e Yes, as both are positive values. This triangle has two possible lengths for the third side, 2.3 cm and 5.2 cm, as it is an ambiguous triangle as the shorter known side length is opposite the known angle.

f The sine rule would first need to determine the possible angles opposite the other known side length, then determine the corresponding third angle(s) in the triangle using the interior angle sum of a triangle, and then determine the possible lengths of the unknown side length separately. The cosine rule can directly find both possible lengths of the unknown side by solving a trinomial quadratic equation, which is a harder process than using the sine rule multiple times. The other angles in the triangle can then be determined if needed.

g $11^2 = x^2 + x^2 - 2xx\cos(103°)$

$11^2 = 2x^2 - 2x^2\cos(103°)$

$x = 7.0$ cm since x must be positive

22 Area = $5\frac{\sqrt{3}}{4} - \sqrt{3}\cos(\theta)$

23 **a** It is when the triangle inequality is not true.

b We proceed by contradiction. Suppose $a \geq b + c$. If a, b and c are the side lengths of a triangle, then $0 < a, b, c$ because distances are positive. This means that the square of the first inequality holds:

$a^2 \geq (b + c)^2 = b^2 + c^2 + 2bc$.

The cosine rule states that $a^2 = b^2 + c^2 - 2bc\cos(A)$. Combining with the previous inequality, we have:

$a^2 = b^2 + c^2 - 2bc\cos(A) \geq b^2 + c^2 + 2bc$.

Simplifying (since all values are positive), we have:

$-\cos(A) \geq 1$.

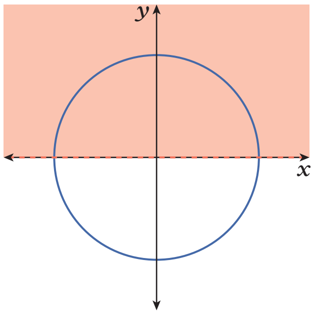
This implies that $\cos(A) = -1$, as $-1 \leq \cos(A) \leq 1$. But $\cos(A) = -1$ implies $A = 180°$. This is a contradiction because there can't be a 180° angle in a triangle as the interior angle sum of a triangle is 180° and there can't be 0°-angles in a triangle.

Hence, the original assumption that $a \geq b + c$ is false, which proves the triangle inequality.

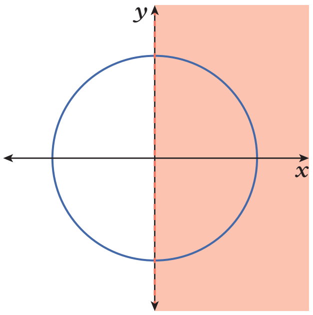
EX p490

9G The unit circle

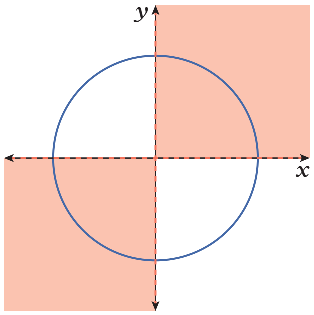
1 **a**



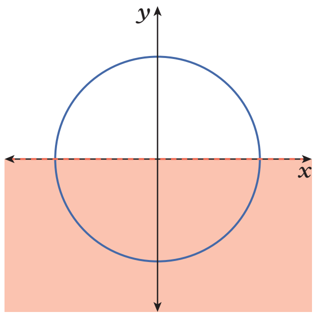
b



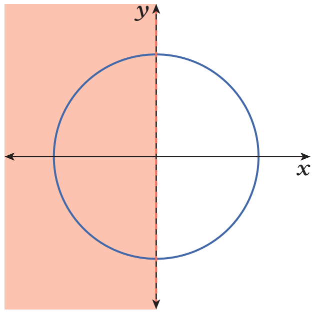
c



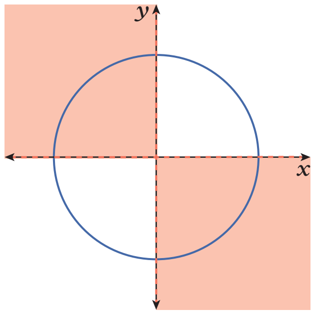
d



e



f



2 **a** quadrant 4 **b** quadrant 2 **c** quadrant 3
d quadrant 3 **e** quadrant 1 **f** quadrant 2

3 **a** positive **b** negative **c** negative
d positive **e** positive **f** negative
g negative **h** positive **i** negative
j negative

4 **a** 40° **b** 88° **c** 20°
d 10° **e** 58° **f** 45°
g 80° **h** 80° **i** 80°
j 30° **k** 15° **l** 10°
m 40° **n** 75° **o** 61°

5 **a** $-\sin(61°)$ **b** $\cos(56°)$ **c** $-\tan(85°)$
d $\sin(16°)$ **e** $-\cos(70°)$ **f** $-\cos(20°)$
g $\tan(20°)$ **h** $-\tan(2°)$ **i** $-\sin(75°)$
j $\sin(5°)$

6 **a** **i** $1 = \cos(0°)$ **ii** $0 = \sin(0°)$
b **i** $0 = \cos(90°)$ **ii** $1 = \sin(90°)$
c **i** $-1 = \cos(180°)$ **ii** $0 = \sin(180°)$
d **i** $0 = \cos(270°)$ **ii** $-1 = \sin(270°)$
e **i** $1 = \cos(360°)$ **ii** $0 = \sin(360°)$

7 **a** **i** $\tan(\theta) = \frac{y}{x}$ **ii** $\tan(\theta) = t$

b The gradient of the radius is its rise over its run, $\frac{y}{x}$. The tangent of θ is also equal to this ratio; therefore, the tangent of the angle is equal to the gradient of the radius.

c $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

d **i** $\tan(20°)$ **ii** $\tan(152°)$ **iii** $\frac{1}{\tan(218°)}$

iv $\tan(347°)$

e **i** 0 **ii** undefined **iii** 0

iv undefined **v** 0

8 **a** **i** $\sin(40°)$ **ii** $\sin(40°)$

b **i** $\cos(170°)$ **ii** $-\cos(10°)$

c **i** $\tan(80°)$ **ii** $\tan(80°)$

d **i** $\sin(270°)$ **ii** -1

e **i** $\cos(57°)$ **ii** $\cos(57°)$

f **i** $\cos(304°)$ **ii** $\cos(56°)$

g **i** $\tan(200°)$ **ii** $\tan(20°)$

h **i** $\cos(180°)$ **ii** -1

9 **a** **i** $\sin(315°)$ **ii** $-\sin(45°)$

b **i** $\cos(135°)$ **ii** $-\cos(45°)$

c **i** $\tan(225°)$ **ii** $\tan(45°)$

d **i** $\sin(45°)$ **ii** $\sin(45°)$

e **i** $\cos(212°)$ **ii** $-\cos(32°)$

f **i** $\tan(284°)$ **ii** $-\tan(76°)$

g **i** $\cos(90°)$ **ii** 0

h **i** $\sin(80°)$ **ii** $\sin(80°)$

10 **a** $\sin(158°) = \sin(22°) = -\sin(202°) = -\sin(338°)$

b $\cos(213°) = \cos(147°) = -\cos(33°) = -\cos(327°)$

c $\tan(285°) = \tan(105°) = -\tan(75°) = -\tan(255°)$

11 **a** $-0.8090... = -0.8090...$, $-0.587\,785... = -0.587\,785...$, $1.376\,38... = 1.376\,38...$

b 360°

c The reference angle, when the angle is in the fourth quadrant, is calculated using $360° - \theta°$. The same has been applied to 234° as $360° - 234° = 126°$. The RHS of each equation uses the sign of the trigonometric function in the fourth quadrant to ensure the expressions are equal.

12 **a** $\sin(\theta) = \frac{b}{c}$, $\cos(\theta) = \frac{a}{c}$, $\tan(\theta) = \frac{b}{a}$

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \tan(\theta)$$

b **i** $\theta' = 180° - \theta$, negative

ii $\tan(\theta) = -\tan(180° - \theta)$

iii $\sin(\theta) = \sin(180° - \theta)$, $\cos(\theta) = -\cos(180° - \theta)$

$$\begin{aligned}\frac{\sin(\theta)}{\cos(\theta)} &= \frac{\sin(180° - \theta)}{-\cos(180° - \theta)}\\ &= -\tan(180° - \theta) \text{ (by part } \mathbf{a}\text{)}\\ &= \tan(\theta)\end{aligned}$$

c **i** $\theta' = \theta - 180°$, positive

ii $\tan(\theta) = \tan(\theta - 180°)$

iii $\sin(\theta) = -\sin(\theta - 180°)$, $\cos(\theta) = -\cos(\theta - 180°)$

$$\begin{aligned}\frac{\sin(\theta)}{\cos(\theta)} &= \frac{-\sin(\theta - 180°)}{-\cos(\theta - 180°)}\\ &= \tan(\theta - 180°) \text{ (by part } \mathbf{a}\text{)}\\ &= \tan(\theta)\end{aligned}$$

d **i** $\theta' = 360° - \theta$, negative

ii $\tan(\theta) = -\tan(360° - \theta)$

iii $\sin(\theta) = -\sin(360° - \theta)$, $\cos(\theta) = \cos(360° - \theta)$

$$\begin{aligned}\frac{\sin(\theta)}{\cos(\theta)} &= \frac{-\sin(360° - \theta)}{\cos(360° - \theta)}\\ &= -\tan(360° - \theta) \text{ (by part } \mathbf{a}\text{)}\\ &= -\tan(\theta)\end{aligned}$$

e $\frac{\sin(0°)}{\cos(0°)} = \frac{0}{1} = 0 = \tan(0°)$

$\frac{\sin(90°)}{\cos(90°)} = \frac{1}{0} = \text{undefined} = \tan(90°)$

$\frac{\sin(180°)}{\cos(180°)} = \frac{0}{-1} = 0 = \tan(180°)$

$\frac{\sin(270°)}{\cos(270°)} = \frac{-1}{0} = \text{undefined} = \tan(270°)$

f Every angle θ outside $0° \leq \theta < 360°$ is coterminal to an angle inside $0° \leq \theta < 360°$. Parts **a** to **e** verify that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ for every angle inside this domain. Since any two coterminal angles have the same sine, cosine and tangent values in this equation, it must be true that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ for any value of θ.

13 **a** $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

b **i** $\sin(\theta) = \pm\sqrt{1 - (\cos(\theta))^2}$

ii $\cos(\theta) = \pm\sqrt{1 - (\sin(\theta))^2}$

c **i** $\cos(50°) = 0.6428$

ii $\sin(116°) = 0.8988$

iii $\cos(177°) = -0.9986$

iv $\sin(240°) = -0.8660$

14 **a** $\cos(60°)$ **b** $\cos(38°)$

c $\sin(66°)$ **d** $\sin(45°)$

e **i** $\cos(-68°)$ **ii** $\cos(292°)$ **iii** $\cos(68°)$

f **i** $\sin(-261°)$ **ii** $\sin(99°)$ **iii** $\sin(81°)$

g **i** $\cos(-121°)$ **ii** $\cos(239°)$ **iii** $-\cos(59°)$

h **i** $\cos(105°)$ **ii** $\cos(105°)$ **iii** $-\cos(75°)$

i **i** $\sin(-392°)$ **ii** $\sin(328°)$ **iii** $-\sin(32°)$

j **i** $\sin(645°)$ **ii** $\sin(285°)$ **iii** $-\sin(75°)$

EX p497

9H Exact values

1 **a** 30°, 150°, 210°, 330°

b 45°, 135°, 225°, 315°

c 60°, 120°, 240°, 300°

2

	θ	**sin(θ)**	**cos(θ)**	**tan(θ)**
a	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
b	300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
c	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
d	315°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
e	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
f	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

3 **a** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **b** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

d $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **e** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **f** $(0, -1)$

4 **a** $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **b** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

d $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **e** $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **f** $(-1, 0)$

5 **a** $-\sin(\theta)$ **b** $-\tan(\theta)$ **c** $\cos(\theta)$

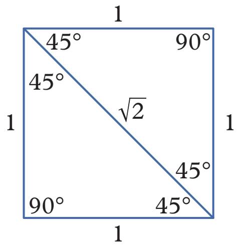
d $-\sin(\theta)$ **e** $-\sin(\theta)$ **f** $-\frac{1}{\tan(\theta)}$

6 **a** $\sin(\theta)$ **b** $-\cos(\theta)$ **c** $-\frac{1}{\tan(\theta)}$

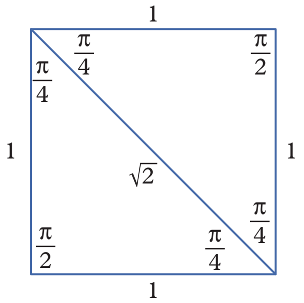
d $-\sin(\theta)$ **e** $-\cos(\theta)$ **f** $\frac{1}{\tan(\theta)}$

g $-\sin(\theta)$ **h** $\tan(\theta)$ **i** $-\sin(\theta)$

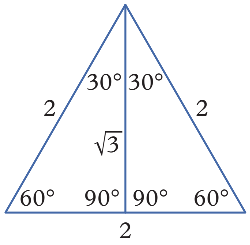
7 **a**



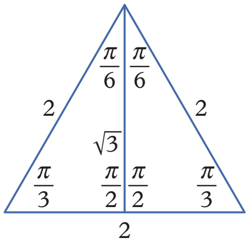
b



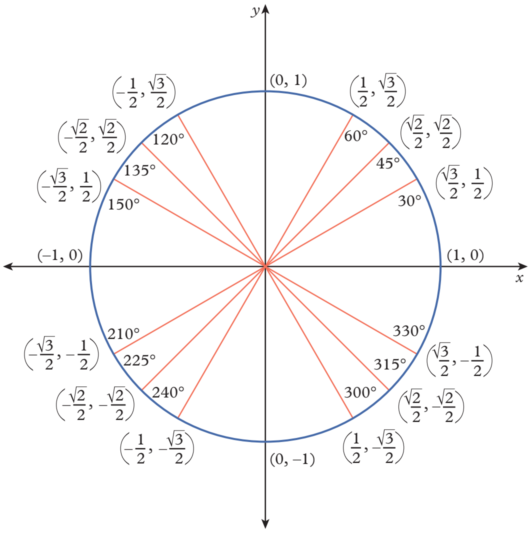
c



d



8 **a, b**



c **i** $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **ii** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

iii $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **iv** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

v $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **vi** $(0, 1)$

9 **a** $(-\sqrt{3}, -1)$ **b** $(\sqrt{2}, -\sqrt{2})$ **c** $(-5, 5\sqrt{3})$

d $(-3, -3\sqrt{3})$ **e** $\left(\frac{15\sqrt{2}}{2}, \frac{15\sqrt{2}}{2}\right)$ **f** $(-2\sqrt{3}, 2)$

10 **a** **i** $\sin(-60°)$ **ii** $-\sin(60°)$ **iii** $-\frac{\sqrt{3}}{2}$

b **i** $\cos(-45°)$ **ii** $\cos(45°)$ **iii** $\frac{\sqrt{2}}{2}$

c **i** $\tan(-30°)$ **ii** $-\tan(30°)$ **iii** $-\frac{\sqrt{3}}{3}$

d **i** $\cos(-90°)$ **ii** $\cos(90°)$ **iii** 0

e **i** $\sin(-30°)$ **ii** $-\sin(30°)$ **iii** $-\frac{1}{2}$

f **i** $\tan(-60°)$ **ii** $-\tan(60°)$ **iii** $-\sqrt{3}$

11 **a** **i** $\sin(60°) = \frac{\sqrt{3}}{2}$

ii $2\sin(30°)\cos(30°) = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

b **i** $\sin(90°) = 1$

ii $2\sin(45°)\cos(45°) = 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = 1$

c **i** $\sin(120°) = \frac{\sqrt{3}}{2}$

ii $2\sin(60°)\cos(60°) = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$

d **i** $\sin(180°) = 0$

ii $2\sin(90°)\cos(90°) = 2 \times 1 \times 0 = 0$

e **i** $\sin(240°) = -\frac{\sqrt{3}}{2}$

ii $2\sin(120°)\cos(120°) = 2 \times \frac{\sqrt{3}}{2} \times \left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$

f **i** $\sin(360°) = 0$

ii $2\sin(180°)\cos(180°) = 2 \times 0 \times 1 = 0$

12 a sin(30°)cos(30°) + cos(30°)sin(30°)

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \sin(60°)$

b sin(45°)cos(45°) + cos(45°)sin(45°)

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = 1 = \sin(90°)$

c sin(60°)cos(30°) + cos(60°)sin(30°)

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = 1 = \sin(90°)$

d sin(90°)cos(60°) + cos(90°)sin(60°)

$= 1 \times \frac{1}{2} + 0 \times \frac{\sqrt{3}}{2} = \frac{1}{2} = \sin(150°)$

e sin(90°)cos(30°) + cos(90°)sin(30°)

$= 1 \times \frac{\sqrt{3}}{2} + 0 \times \frac{1}{2} = \frac{\sqrt{3}}{2} = \sin(120°)$

f sin(150°)cos(60°) + cos(150°)sin(60°)

$= \frac{1}{2} \times \frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right) \times \frac{\sqrt{3}}{2} = -\frac{1}{2} = \sin(210°)$

g $\sin(2\theta) = \sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)$

$= \sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) = 2\sin(\theta)\cos(\theta)$

13 a i $\frac{\sqrt{2}(\sqrt{3} + 1)}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

ii $\frac{\sqrt{2}(\sqrt{3} + 1)}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

iii Since 105° = 180° − 75°, the identity $\sin(\theta) = \sin(180° - \theta)$ implies that sin(75°) = sin(105°).

b $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

c i $\frac{\sqrt{2}(\sqrt{3} - 1)}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$

ii $\frac{\sqrt{2}(\sqrt{3} + 1)}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

14 a (1 × cos(90°) − 0 × sin(90°), 1 × sin(90°) + 0 × cos(90°)) = (0, 1)

b i $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

ii $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

iii $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

iv $\left(-\frac{3\sqrt{3}}{2} - 1, \frac{3}{2} - \sqrt{3}\right)$

v $\left(\frac{\sqrt{3}}{2} - \frac{3}{2}, -\frac{3\sqrt{3}}{2} - \frac{1}{2}\right)$

vi $(\sqrt{3} + 2, 2\sqrt{3} - 1)$

c $(x\cos(\theta) + y\sin(\theta), y\cos(\theta) - x\sin(\theta))$

d i $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **ii** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

iii $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ **iv** $\left(\frac{3}{2} - \sqrt{3}, \frac{3\sqrt{3}}{2} + 1\right)$

v $\left(-\frac{3\sqrt{3}}{2} - \frac{1}{2}, \frac{3}{2} - \frac{\sqrt{3}}{2}\right)$ **vi** $(2\sqrt{3} - 1, -\sqrt{3} - 2)$

15 a $\left(2 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$ **b** $(\sqrt{2} - 3, \sqrt{2} + 2)$

c $\left(\frac{5\sqrt{3}}{2} - 2, -\frac{11}{2}\right)$ **d** $(3, 2\sqrt{3} + 3)$

16 a $x = 4\sqrt{3}$ cm **b** $x = \frac{23\sqrt{2}}{2}$ cm

17 a (1 × cos(α) − 0 × sin(α), 1 × sin(α) + 0 × cos(α)) = (cos(α), sin(α))

b (cos(α)cos(β) − sin(α)sin(β), cos(α)sin(β) + sin(α)cos(β))

c (cos(α + β), sin(α + β))

d $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$

e $\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$

$= \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta)$

$= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

18 a $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$

b $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

EX p505

9I Trigonometric graphs

1 a i decreased and approached −1

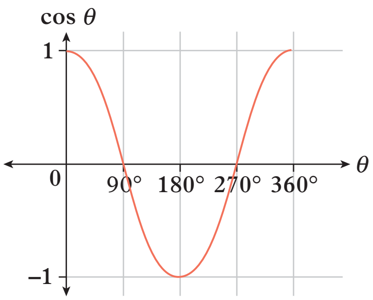
ii increased and approached 0

iii increased and approached 1

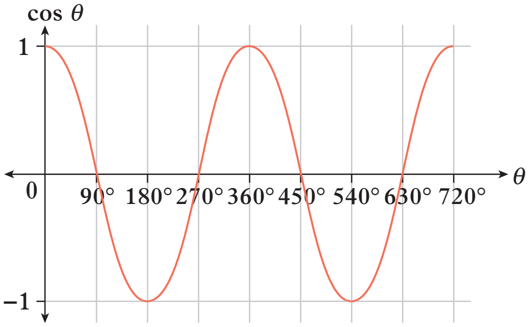
b

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos(θ)	1	0.9	0.5	0	–0.5	–0.9	–1	–0.9	–0.5	0	0.5	0.9	1

c



d



e 0°, 360°; the period of the cosine graph is 360°

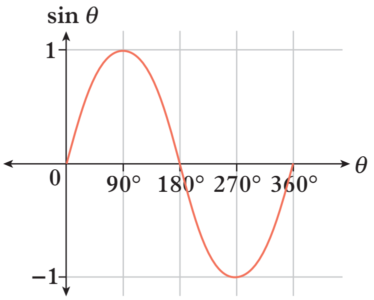
f amplitude = 1

g (0, 1)

2 a

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin (θ)	0	0.5	0.9	1	0.9	0.5	0	–0.5	–0.9	–1	–0.9	–0.5	0

b



c 360° **d** 1 **e** (0, 0)

3 **a** **i** 1 **ii** 0 **iii** 0
iv −1

b **i** 0.8 **ii** −0.2 **iii** 0.9
iv −0.9 **v** 0.2 **vi** −0.8

4 **a** 0°, 360° **b** 180° **c** 90°, 270°
d 70°, 290° **e** 130°, 230° **f** 40°, 320°

5 **a** **i** 0 **ii** 1 **iii** −1
iv 0

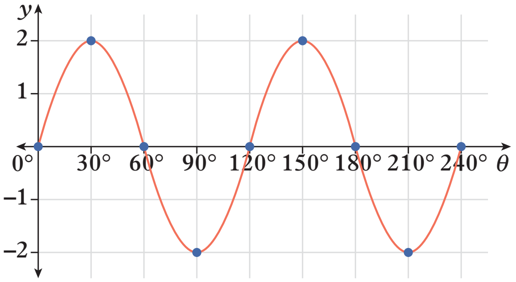
b **i** 0.6 **ii** 0.8 **iii** −0.2
iv −0.2 **v** −0.8 **vi** 0.6

6 **a** 90° **b** 270° **c** 0°, 180°, 360°
d 20°, 160° **e** 220°, 320° **f** 50°, 130°

7 **a**

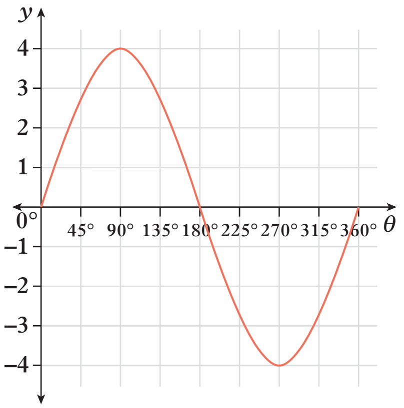
3θ	0°	90°	180°	270°	360°	450°	540°	630°	720°
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°
$\sin(3\theta)$	0	1	0	−1	0	1	0	−1	0
$y = 2\sin(3\theta)$	0	2	0	−2	0	2	0	−2	0

b, c

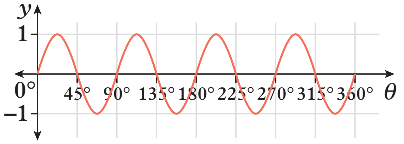


8 **a** amplitude = 3, period = 180°
b amplitude = 5, period = 120°
c amplitude = 4, period = 720°
d amplitude = 6, period = 2160°
e amplitude = $\frac{3}{4}$, period = 480°
f amplitude = $\frac{5}{3}$, period = 600°
g amplitude = $\frac{7}{8}$, period = 630°
h amplitude = $\frac{5}{12}$, period = 75°

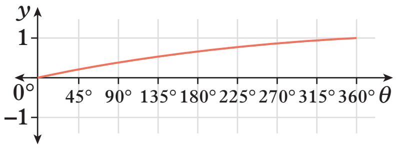
9 **a**



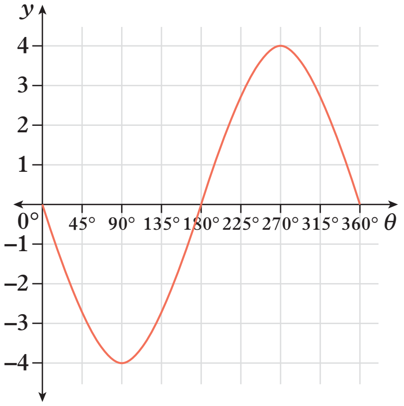
b



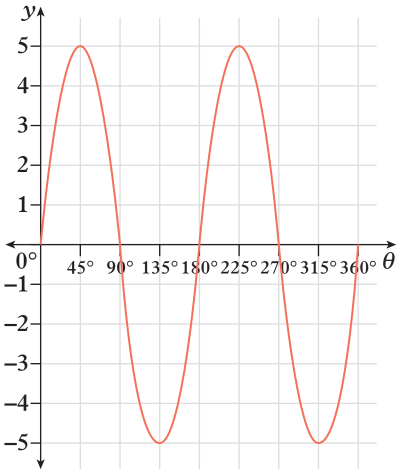
c



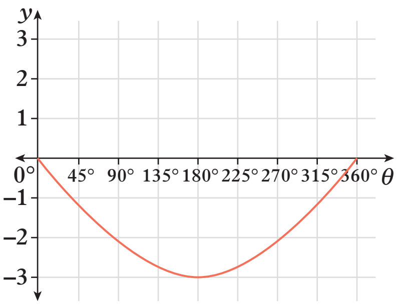
d



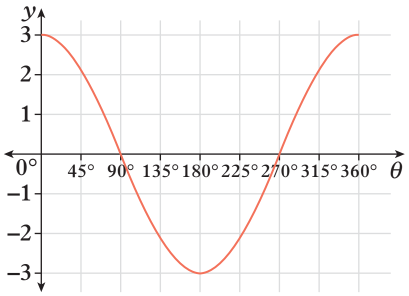
e



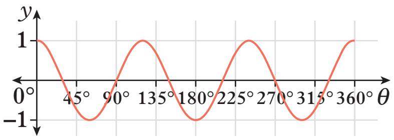
f



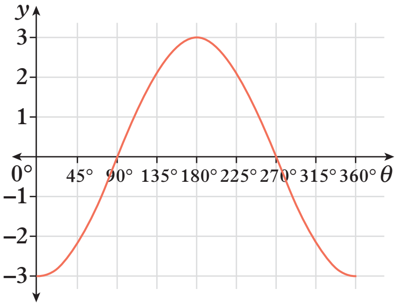
10 a



b



c



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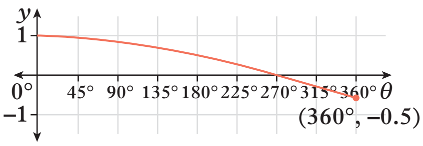
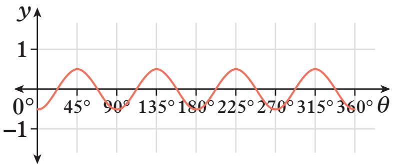
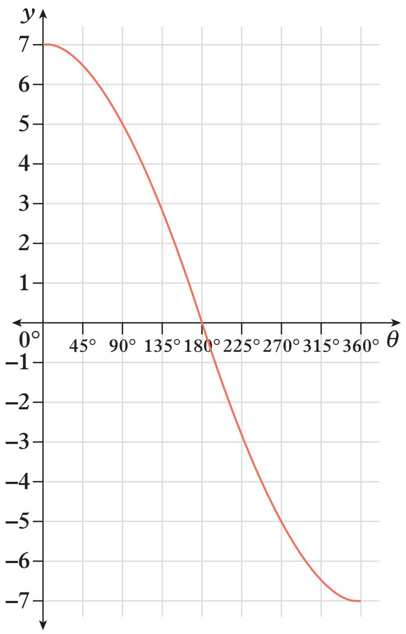


e



f



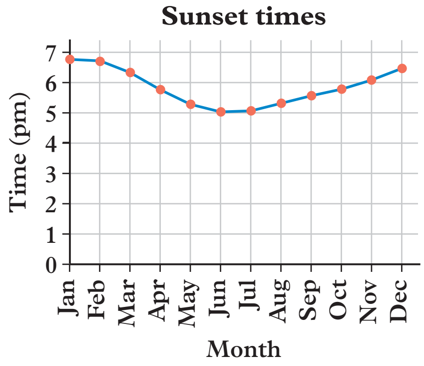
11 The sine graph starts at (0, 0) and the cosine graph starts at (0, 1).

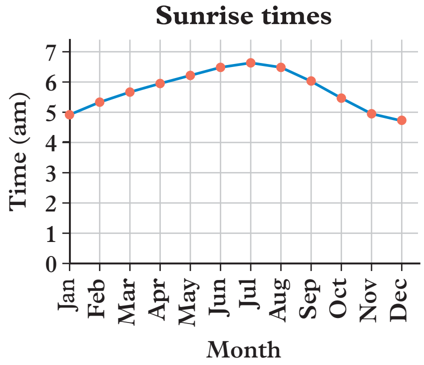
12 a A pendulum swings back and forth the same distance. Its motion is repetitive and regular, which means it is periodic.

b **i** 1.0 s **ii** 1.2 s **iii** 1.4 s

c The longer the pendulum, the longer the period of its motion.

13 a





b Answers may vary. Comments could include:
The graphs are approximately cyclic or periodic. The graphs have peaks and troughs.

c Similarities to sine and cosine graphs: The graphs have a repeating pattern. The graphs are cyclic or periodic. The graphs have peaks and troughs.

Differences from sine and cosine graphs: The graphs do not start at 1 or 0. The maximum point is not 1 and the minimum point is not –1. The graphs are completely above the x-axis. The shape of the graphs resembles a cosine graph more than a sine graph. The graphs are not perfect curves.

d Yes, the values on the graphs would repeat at regular intervals. In this case the interval is 1 year. But the repeated values would be approximately the same, not exactly the same.

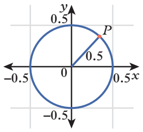
e Daylight hours are the hours which lie between the two graphs. These hours increase in summer and decrease in winter.

f Winter has the shortest days. These graphs indicate the least amount of daylight during the middle of the year. This is when winter occurs in the Southern Hemisphere.

14 a 3 **b** 11 **c** $\frac{1}{2}$ **d** b

e 4 **f** 90 **g** $\frac{3}{2}$ **h** b

15 a

	Graph	Circle	Radius	Revolution per 360°
i			10	4
ii			0.5	3
iii			3	$\frac{1}{2}$

b The amplitude is related to the length of the radius. The amplitude is equal to the length of the radius.

c The period is related to the speed the point travels around the circle. The period is the time it takes for the point to complete a revolution of the circle travelling at its constant speed. That is,

$$\text{period(time)} = \frac{\text{distance}(360°)}{\text{speed}}.$$

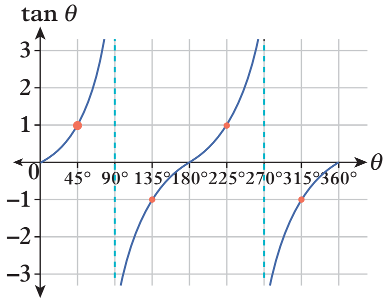
16 a

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
tan(θ)	0	0.6	1	1.7	⋆	–1.7	–1	–0.6	0

θ	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)	0.6	1	1.7	⋆	–1.7	–1	–0.6	0

⋆undefined

b



c i The graph is a curving line which extends from –∞ to +∞ with a point of inflection on the x-axis.

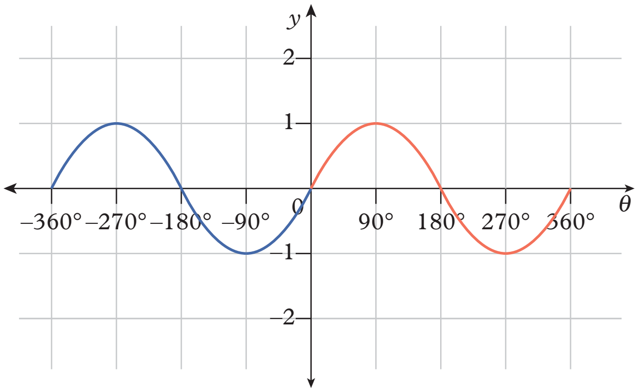
ii The graph line is not continuous; its cycle repeats after 180° (not 360°); it has values for which tan θ is undefined; the values approach ∞ and –∞.

iii At $\theta = 90°$ the value of tan θ is undefined so a vertical asymptote is shown at this point. An asymptote is a line which the graph approaches but never reaches. The next asymptote is at $\theta = 270°$.

iv The graph repeats after a certain period. Period = 180°.

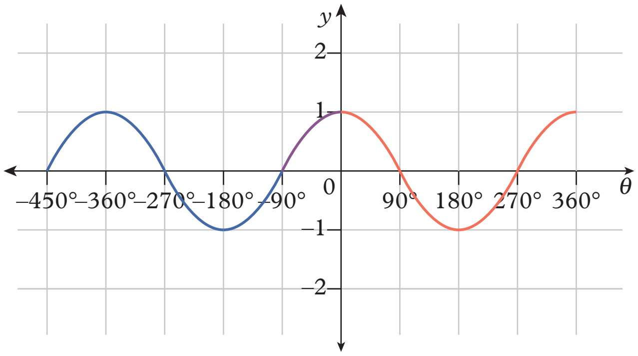
v The graph has no amplitude. The tan θ values approach ∞ and –∞.

17 a i, ii

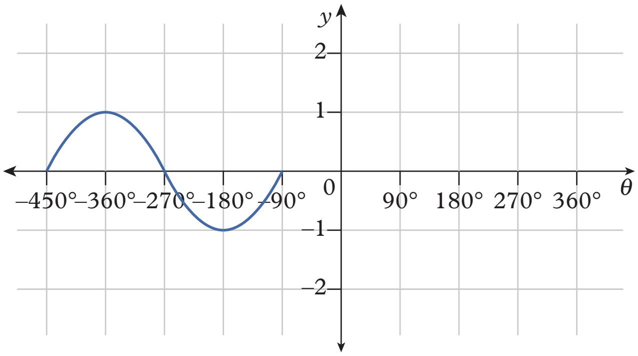


iii $\sin(-360°) = \sin(360°) = 0$

b i, ii

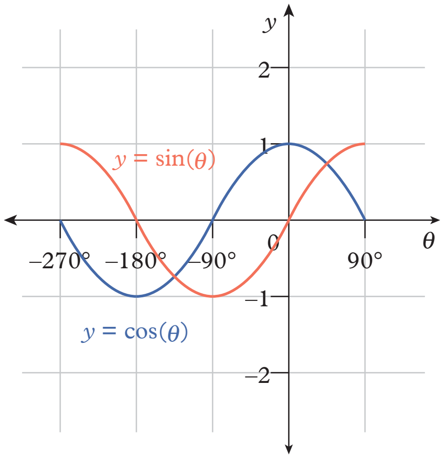


iii

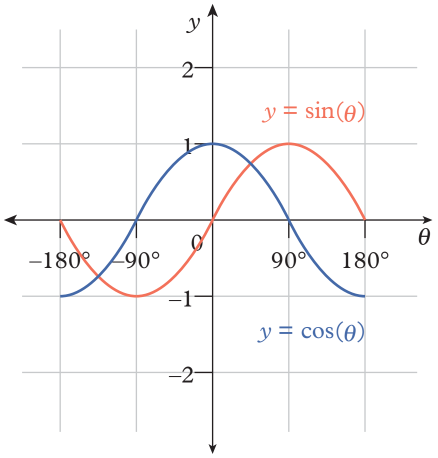


iv $\cos(-450°) = \cos(-90°) = 0$

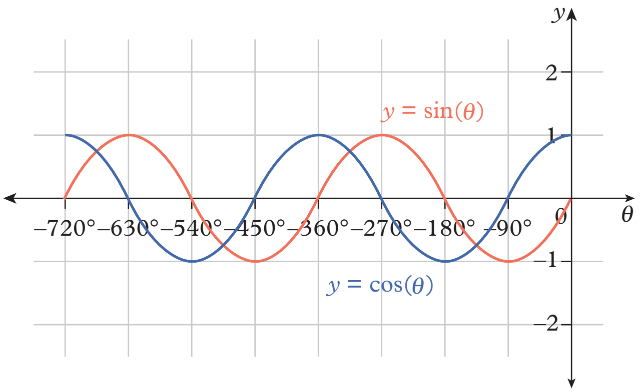
c **i**



ii

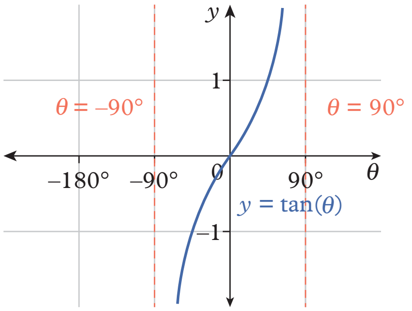


iii



18 **a** $\theta = -90^\circ$

b, c



d $\theta = -270^\circ$

e $\theta = 90^\circ + 180^\circ \times n$, for $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

EX p511

9J Solving trigonometric equations

1 **a** $\theta = 30^\circ, 330^\circ$ **b** $\theta = 225^\circ, 315^\circ$
c $\theta = 150^\circ, 330^\circ$ **d** $\theta = 240^\circ, 300^\circ$
e $\theta = 45^\circ, 315^\circ$ **f** $\theta = 45^\circ, 225^\circ$

2 **a** $\theta = 114^\circ, 246^\circ$ **b** $\theta = 197^\circ, 343^\circ$
c $\theta = 42^\circ, 222^\circ$ **d** $\theta = 37^\circ, 143^\circ$
e $\theta = 53^\circ, 307^\circ$ **f** $\theta = 98^\circ, 278^\circ$

3 **a** 90° **b** no
c $\theta = 90$ **d** $\theta = 90^\circ, 450^\circ$

4 **a** $\theta = 180^\circ$ **b** $\theta = 180^\circ, 540^\circ$

5 **a** 0°, 360° **b** 180°, 360°
c $\theta = 0^\circ, 180^\circ, 360^\circ$ **d** $\theta = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ$

6 **a** $\theta = 90^\circ, 270^\circ$ **b** $\theta = 0^\circ, 180^\circ, 360^\circ$
c $\theta = 270^\circ$ **d** $\theta = 0^\circ, 360^\circ$

7 **a** no domain within the stated range
b $90^\circ \leq \theta \leq 270^\circ$ **c** $60^\circ \leq \theta \leq 300^\circ$
d $60^\circ \leq \theta \leq 120^\circ$ **e** $45^\circ \leq \theta \leq 135^\circ$
f $150^\circ \leq \theta \leq 210^\circ$

8 **a** **i** $210^\circ \leq \theta \leq 330^\circ$ **ii** $210^\circ \leq \theta \leq 690^\circ$
b **i** $120^\circ \leq \theta \leq 240^\circ$ **ii** $120^\circ \leq \theta \leq 600^\circ$
c **i** $225^\circ \leq \theta \leq 315^\circ$ **ii** $225^\circ \leq \theta \leq 675^\circ$
d **i** $135^\circ \leq \theta \leq 225^\circ$ **ii** $135^\circ \leq \theta \leq 585^\circ$

9 **a–c** When each of these equations is simplified the sine or cosine value is greater than 1. Sine and cosine can only take values in the range –1 to 1.

10 **a** yes **b** yes **c** yes **d** yes
e No, when the equation is simplified the cosine value is less than –1.
f yes
g No, when the equation is simplified the cosine value is greater than 1.
h yes
i No, when the expression is simplified the sine value is less than –1.

11 **a** 1 **b** 0 **c** 1
d 1 **e** 1 **f** 1

12 **a** $\theta = 22.5^\circ, 67.5^\circ$
b $\theta = 45^\circ, 135^\circ$
c $\theta = 45^\circ, 105^\circ, 165^\circ$

13 **a** $\theta = 90^\circ$ **b** $\theta = 0^\circ$ **c** $\theta = 135^\circ, 225^\circ$
d $\theta = 240^\circ$ **e** $\theta = 225^\circ$ **f** $\theta = 150^\circ$

14 **a** $\theta = 420^\circ, 660^\circ$
b $\theta = 135^\circ, 945^\circ$
c $\theta = 300^\circ, 750^\circ$

15 Solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$ involves solutions that are multiples of 90°, whereas solving $\tan(\theta) = 1$ involves 45°. That is, the sine and cosine equations will only have one solution for $0^\circ \leq \theta \leq 180^\circ$, while the tangent equation will have two.

16 **a** 180° **b** 360° **c** 240°
d 540° **e** 360° **f** 480°

17 **a** $\theta = -300°, -240°$ **b** $\theta = -225°, -135°$

c $\theta = -300°, -120°$ **d** $\theta = -90°$

e $\theta = -90°, -270°$ **f** $\theta = 0°, -180°, -360°$

18 **a** $\theta = -180°, 0°, 180°$ **b** $\theta = \pm 30°$

c $\theta = -45°, 135°$ **d** $\theta = 45°, 135°$

e $\theta = \pm 180°$ **f** $\theta = -150°, 30°$

19 **a** $\dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{\cos(\theta)}{\cos(\theta)}$

$\tan(\theta) = 1$

b **i** $\theta = 45°, 225°$ **ii** $\theta = 60°, 240°$

iii $\theta = 150°, 330°$ **iv** $\theta = 30°, 210°$

20 **a** $\theta = 15°, 75°, 195°, 255°$

b $\theta = 360°$

c $\theta = 180°, 300°, 360°, 480°, 540°$

d $\theta = -900°, -540°, 540°, 900°, 1980°$

21 **a** $x = -\frac{1}{2}, 2$ **b** $\theta = 210°, 330°$

22 **a** $x = \frac{4}{3}, \frac{3}{2}$

b There are no solutions because $-1 \le \sin(\theta) \le 1$ and both solutions to part **a** are greater than 1.

c There are no solutions because $-1 \le \cos(\theta) \le 1$ and both solutions to part **a** are greater than 1.

d Infinitely many solutions (because $y = \tan(\theta)$ is periodic and its range is $\mathbb{R}$).

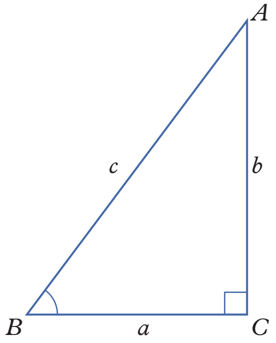
EX p515

CHAPTER 9 review

Mathematical literacy review

1 The square of the hypotenuse is equal to the sum of the squares of the other two sides.

2 $\sin(\theta) = \frac{a}{c}, \cos(\theta) = \frac{b}{c}, \tan(\theta) = \frac{b}{a}$

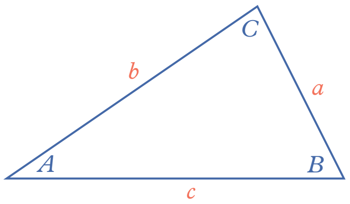


3 Compass bearings describe direction using angles between 0° and 90° within one of the four quadrants of a compass. They include a capital letter for the north–south direction, an angle and a capital letter for the east–west direction. For example, S50°E is a compass bearing. True bearings describe a direction using an angle between 0° and 360° clockwise from north. They are written with three digits for the angle followed by a capital T. The same direction as before as a true bearing is 230°T.

4 **a** The angle of depression is the angle measured down from the horizontal.

b The angle of elevation is the angle measured upwards from the horizontal.

5



6 **a** cosine rule, included angle area formula

b sine rule

7 **a** Two angles are coterminal if they differ by a multiple of 360°. For example, 90° and 450° are coterminal angles.

b A periodic function is a function which repeats in a set pattern as the value of *x* varies. For example, $y = \sin(x)$ is a periodic function.

8 **a** The amplitude of a trigonometric function is the distance from the centre of the curve to the maximum or minimum *y*-value.

b For a given angle, the reference angle is the acute angle between the *x*-axis and the line connecting the origin to the point on the unit circle.

Multiple choice

1	C	**2**	B	**3**	A	**4**	B	**5**	A
6	C	**7**	B	**8**	B	**9**	A	**10**	A
11	D	**12**	A	**13**	B				

Short answer

1 **a** $\frac{5\sqrt{3}}{2}$ cm **b** 4.3 cm

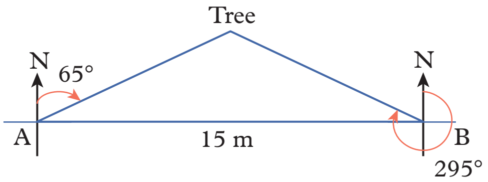
2 length = 13.50 cm, width = 8.21 cm

3 40 m

4 **a** 16.30 cm **b** 19.72 cm **c** 43.30 cm

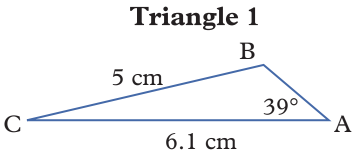
5 **a** 23° **b** 40° **c** 55°

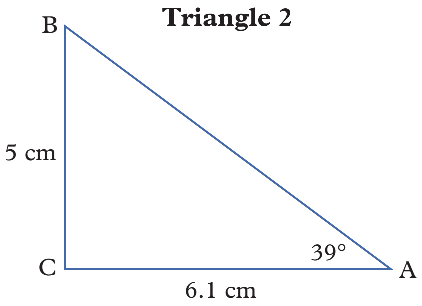
6 **a**



b 8.28 m in both positions

7 **a**



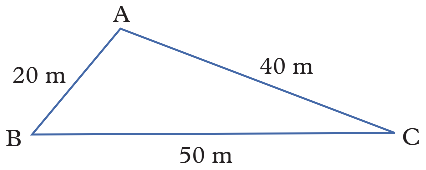


Because there are two sides and an angle given and the shorter side is opposite the angle it is possible to draw two distinct diagrams.

b Triangle 1: $B = 130^\circ$, $C = 11^\circ$, $c = 1.5$ cm

Triangle 2: $B = 50^\circ$, $C = 91^\circ$, $c = 7.9$ cm

8 a



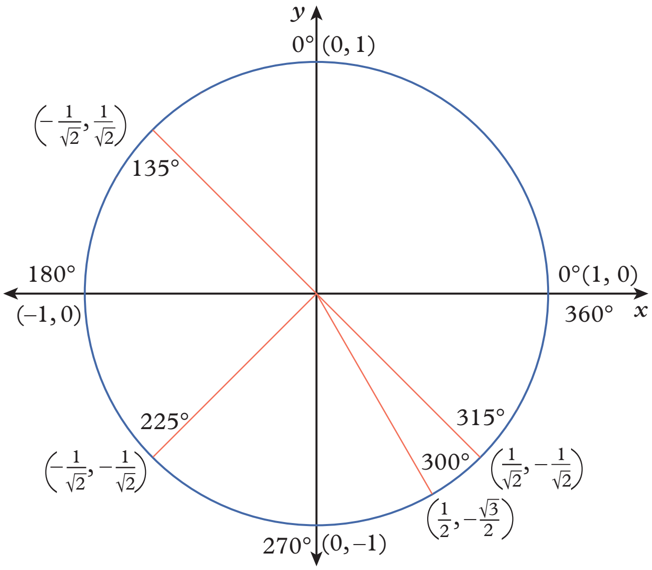
b $A = 108.2^\circ$, $B = 49.5^\circ$, $C = 22.3^\circ$, Area = 380 m²

c 15.20 m

9 a sin(60°) **b** −cos(30°)

c −tan(60°) **d** cos(45°)

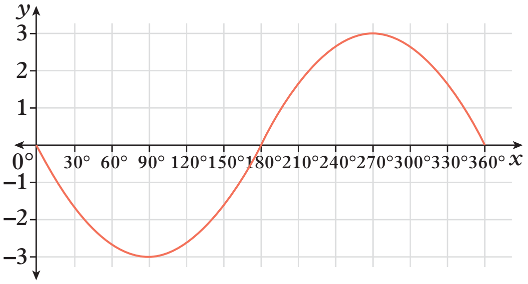
10



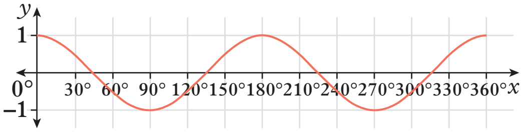
11 a $\cos(\theta)$ **b** $-\cos(\theta)$

c $-\tan(\theta)$ **d** $\cos(\theta)$

12 a

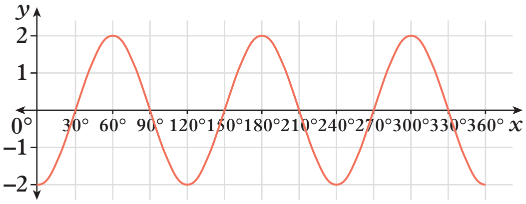


b



c

d



13 a $\theta = 180^\circ$ **b** $\theta = 0^\circ, 180^\circ, 360^\circ$

c $\theta = 30^\circ, 330^\circ$ **d** $\theta = 210^\circ, 330^\circ$

Analysis

1 a i $2R + G = 480$ mm; this is below the required range, so the staircase does not fall within building regulation guidelines.

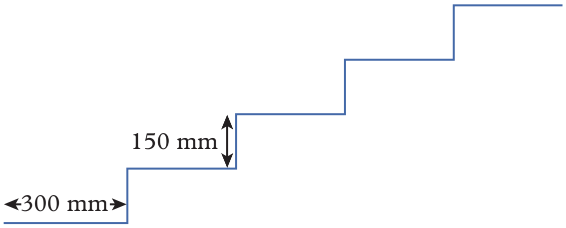
ii 25°

b i $2R + G = 735$ mm; this is above the required range, so the staircase does not fall within building regulation guidelines.

ii 28°

c i 115 mm to 172.5 mm **ii** 18° to 26°

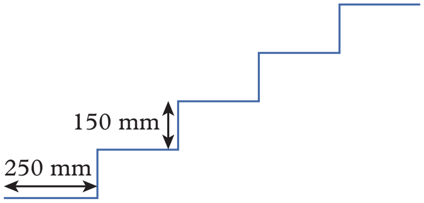
d



i 1500 mm **ii** 600 mm

iii 1615.5 mm **iv** 27°

e One possible solution is given.



115 mm ≤ 150 mm ≤ 190 mm

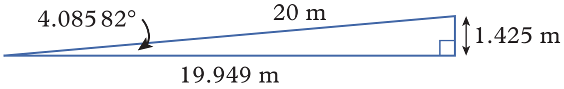
250 mm ≤ 250 mm ≤ 355 mm

$2R + G = 2 \times 150 + 250 = 300 + 250 = 550$ mm

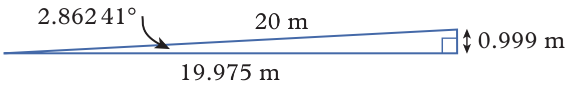
550 mm ≤ 550 mm ≤ 700 mm

f **i**

Internal ramp 1:14



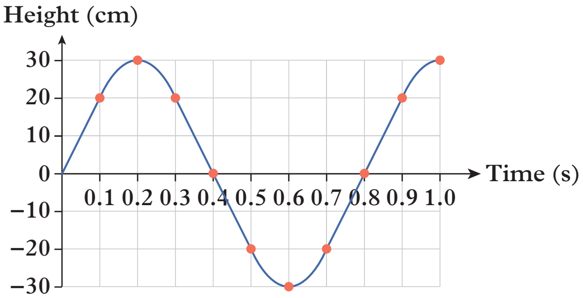
External ramp 1:20



ii Internal: rise = 1.425 m, going = 19.949 m

External: rise = 0.999 m, going = 19.975 m

2 **a**



b sine curve

c 0.8 seconds

d Starting position was 0 cm. It takes 0.1 seconds for the tag to go from 20 cm to 0 cm in height.

e 30 cm **f** 188.5 cm **g** 2.4 m/s

h After 10 seconds, the wheel will complete 12.5 rotations. If the tag's starting position was at 0 cm, then the tag will be at the same height of 0 cm but on the opposite side of the wheel.

3 **a** **i** 135° **ii** 3.2 km **iii** 2.2 km
iv 2.0 km **v** 116.9° **vi** 33.4°
vii 36.7° **viii** 123.9° **ix** 5.0 km
x 55.3° **xi** 3.2 km **xii** 4.1 km
xiii 6.0 km **xiv** 35.9° **xv** 23.1°

b AE = 9.9 km, 126°48′T

c Use more decimal places in calculations (for example)

CHAPTER 10 Networks

EX p524

10A Introduction to networks

1 **a** **i** 4 **ii** 3
b **i** 4 **ii** 4
c **i** 4 **ii** 2
d **i** 1 **ii** 3
e **i** 4 **ii** 6
f **i** 5 **ii** 4
g **i** 2 **ii** 4
h **i** 9 **ii** 12

2 **a**

Vertex	A	B	C	D
Degree	1	2	2	1

b

Vertex	A	B	C	D
Degree	2	2	2	2

c

Vertex	A	B	C	D
Degree	1	1	1	1

d

Vertex	A
Degree	6

e

Vertex	A	B	C	D
Degree	3	3	3	3

f

Vertex	A	B	C	D	E
Degree	4	1	1	1	1

g

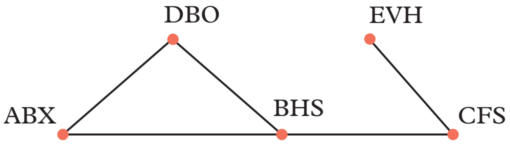
Vertex	A	B
Degree	3	5

h

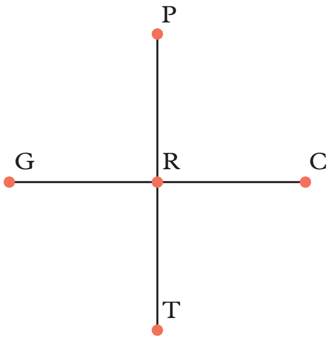
Vertex	A	B	C	D	E	F	G	H	I
Degree	2	2	6	2	4	2	2	2	2

3 **c**

4



5 **a**



b 4

6 **a** Sample answer: A, f, B, g, C, h, D (other walks from A to D exist)

b Sample answer: A, f, B, g, D (other walks from A to D exist)

c There is no walk from A to D.

7 **a**

Person	A	B	C	D
Number of hands shook	3	3	2	2

b 5 (the number of edges)

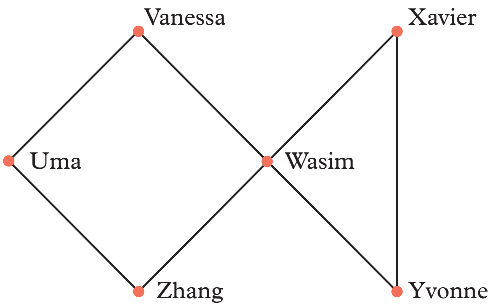
8 10 (twice the number of edges)

9 4 × 5 ÷ 2 = 10

10 Such a graph would have $\frac{3+2+2}{2} = 3.5$ edges, which is impossible.

11 A graph is defined by its vertices and edges, and both these diagrams contain the same vertices and edges, therefore they represent the same graph.

12 a



b

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree	2	2	4	2	2	2

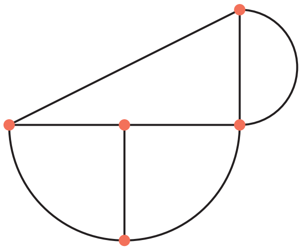
c Wasim (or Uma) **d** Uma

13 a Yes **b** Corner C

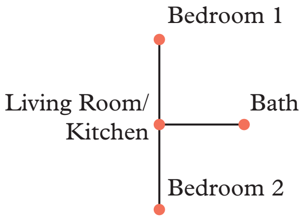
14 There are missing directions. East 42nd St does not connect to corner A.

15 Remove the edge between D and E.

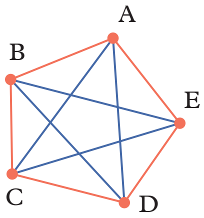
16



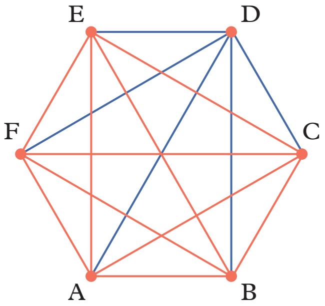
17



18 Sample answer:



19 a Sample answer:



b Sample answer: ACE.

c Question **18** shows you need to invite more than 5 people, since it contains no red or blue triangle. No matter how you answer question **19a** there is always a red or blue triangle, so you need 6 dinner party guests.

EX
p530

10B Planar graphs

1 a planar **b** planar
c planar **d** planar

2 a 2 **b** 4

c $e = f + v - 2$
$= 4 + 2 - 2$
$= 4$

3 a $v = 4, f = 2, e = 4; 4 + 2 = 4 + 2$

b $v = 1, f = 4, e = 3; 1 + 4 = 3 + 2$

c $v = 4, f = 3, e = 5; 4 + 3 = 5 + 2$

4 a The face inside the square is bordered by 4 edges so has degree 4. Similarly the outside face has degree 4. Hence, the sum of face degrees is $4 + 4 = 8 = 2e$.

b The inner circle and outer face have degree 1. The other two faces have degree 2. The sum of face degrees is $1 + 1 + 2 + 2 = 6 = 2e$.

c The face between A and B has degree 2. The other two faces have degree 4. The sum of face degrees is $4 + 4 + 2 = 10 = 2e$.

5 Graph **a** is simple.

6 4

7 Because it is not connected

8 A planar graph with 6 vertices has at most $3v - 6 = 12$ edges. This graph has 15 edges, so it cannot be planar.

9 a triangular face: 3, circular face: 1, outside face: 4

b $3 + 1 + 4 = 8$

c The sum of face degrees is twice the number of edges ($e = 4$).

10 a $e = 9, v = 6$

b yes (no loops or parallel edges)

c yes

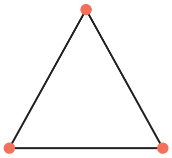
d $e = 9$ is less than $3v - 6 = 12$

11 Yes (see **10B** question **1a**)

12 No (see **10B** question **10**)

13 a If the graph is planar, then it has at most $3v - 6 = 3$ edges.

b



14 4

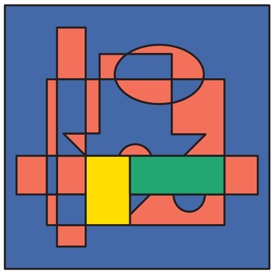
15 No. If such a graph did exist, then $e = f + v - 2 = 5 + 4 - 2 = 7$. This exceeds the limit of $3v - 6 = 6$ for planar graphs.

16 a a loop

b a pair of parallel edges

c 3

17 Sample answer:



EX p535

10C Platonic solids

1 6 edges

2 **a** true
  b false

3 **a** tetrahedron, octahedron, icosahedron
  b cube
  c dodecahedron

4 **a** tetrahedron, cube, dodecahedron
  b octahedron
  c icosahedron

5 d4: tetrahedron, d6: cube, d8: octahedron, d12: dodecahedron, d20: icosahedron

6 tetrahedron, octahedron or icosahedron

7 6

8 5

9 12 edges

10 30 edges

11 30 edges

12

Platonic solid	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices	4	8	6	20	12
Faces	4	6	8	12	20
Edges	6	12	12	30	30

13 **a** 3, 3, 3, 3, 3, 3, 3, 3
  b 4, 4, 4, 4, 4, 4
  c $8 \times 3 = 6 \times 4 = 24$

14 **a** 120 **b** 60 **c** 24

15 **a** 120 **b** 60 **c** 30

16 **a**

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3}+\frac{1}{3}-\frac{1}{2}=\frac{1}{6}$	$\frac{1}{4}+\frac{1}{3}-\frac{1}{2}=\frac{1}{12}$	$\frac{1}{5}+\frac{1}{3}-\frac{1}{2}=\frac{1}{30}$
$m = 4$	$\frac{1}{3}+\frac{1}{4}-\frac{1}{2}=\frac{1}{12}$	$\frac{1}{4}+\frac{1}{4}-\frac{1}{2}=0$	$\frac{1}{5}+\frac{1}{4}-\frac{1}{2}=-\frac{1}{20}$
$m = 5$	$\frac{1}{3}+\frac{1}{5}-\frac{1}{2}=\frac{1}{30}$	$\frac{1}{4}+\frac{1}{5}-\frac{1}{2}=-\frac{1}{20}$	$\frac{1}{5}+\frac{1}{5}-\frac{1}{2}=-\frac{1}{10}$

  b The five entries with positive values of $\frac{1}{e}$ correspond to the five platonic solids.
  c No, the number of edges must be zero or greater.
  d Yes, provided this table contains all the positive entries (see question **17**).

17 Suppose we want $\frac{1}{n}+\frac{1}{m}-\frac{1}{2}>0$. If m is large, then n must be small. Concretely, if $m \geq 6$ then $\frac{1}{n}>\frac{1}{2}-\frac{1}{m} \geq \frac{1}{2}-\frac{1}{6}=\frac{1}{3}$, so $3 > n$. This means $n = 2$ or $n = 1$.
By **10C** question **16**, we have parallel edges when $n = 2$ and loops when $n = 1$. Both cases are excluded because the graph is simple.

EX p538

Checkpoint

1

Vertex	A	B	C	D	E
Degree	3	4	4	3	4

2 yes

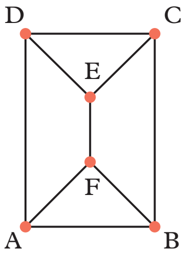
3 A network with vertex degrees 4, 3 and 2 has $\frac{(4+3+2)}{2} = 4.5$ edges, and we must have a whole number of edges.

4 yes (no parallel edges or loops)

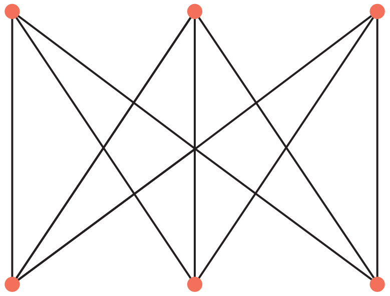
5 yes (can be redrawn without edges crossing)

6 **a** 15
  b No, a planar network with 6 vertices can have at most $3v - 6 = 12$ edges, so this network cannot be planar.

7 **a** Sample answer:



  b Sample answer:



8 6

9 **a** 8 **b** 2 **c** More likely to fail

10 **a** $2 \times 5 + 10 \times 3 = 40$
  b 20
  c 10

EX p541

10D Special types of walks

1 **a** A and C **b** No

2 **a** path **b** trail
  c invalid walk **d** walk

3 **a** A, B, C, D and A, B, C, D
  b No, because two different walks can have the same vertex sequence.

4 Closed walk

5 Sample answer: B, b, C, c, B

6 Sample answer: B, g, C, h, D, i, A, f, B

7 This network is not Eulerian because it has 10 vertices of odd degree.

8 This network is semi-Eulerian because it has 2 vertices of odd degree.

9 Sample answer: A, b, B, a, A, c, B, d, B

10 This network is Eulerian because no vertices have odd degree.

11 **a**

Vertex	N	K	S	L
Degree	3	5	3	3

b 4

c not Eulerian

d No, because the graph has too many vertices with odd degree.

12 **a** After adding one bridge, there will still be two vertices of odd degree.

b No, since the N vertex (and at least one other vertex) will still have odd degree.

c Yes, adding new bridges from N to S and N to L will make the graph semi-Eulerian, so there is a trail that crosses every bridge exactly once.

13 **a** semi-Eulerian **b** Eulerian

c semi-Eulerian

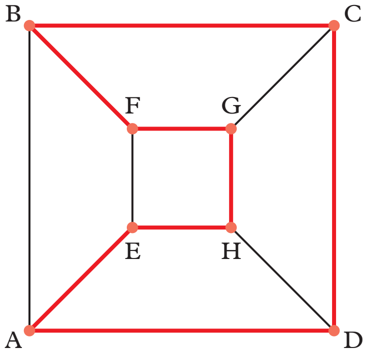
14 octahedron

15 **a** Sample answer:

b Not possible, since there are 4 intersections with odd degrees.

16 An Eulerian circuit exists because there are no vertices with odd degrees.

17 **a** Sample answer:



b There is no Hamiltonian cycle because to return to the starting vertex the central edge must be crossed twice.

EX
p546

10E Applications of networks

1 **a** D **b** H **c** C **d** E

2 **a** H **b** D **c** E **d** C

3

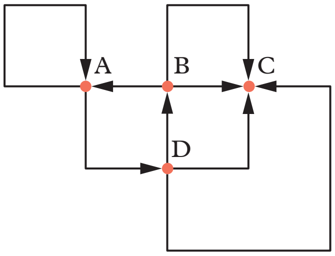
	Kin Group							
Father	A	B	C	D	E	F	G	H
Children	F	G	H	E	D	A	B	C

4 **a** A **b** G **c** G **d** A

5

	Kin group							
You	A	B	C	D	E	F	G	H
Cousins	A or G	B or H	C or E	D or F	E or C	F or D	G or A	H or B

6 Sample answer:



7 **a** 10 **b** 9

8 **a** 13

b Passing through Pride is shorter (12 stops)

c 15

9 **a** C **b** G **c** E **d** A

10 yes

11 no

12 also A

13 via Sydney and Melbourne ($1384 total)

14 **a**

	Kin group			
Mother	A	B	C	D
Children	B	D	A	C

b

	Kin group			
Father	A	B	C	D
Children	C	A	D	B

c dodecahedron

EX
p550

CHAPTER 10 review

Mathematical literacy review

1 A graph is connected if there is a walk between every pair of vertices.

2 tetrahedron, cube, octahedron, dodecahedron, icosahedron

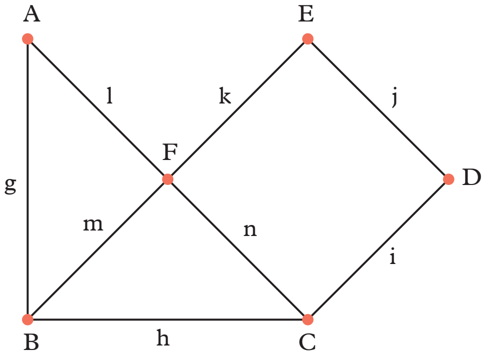
3 The degree of a face can be determined by counting the number of edges along its border.

4 A circuit is Eulerian if it contains every edge exactly once.

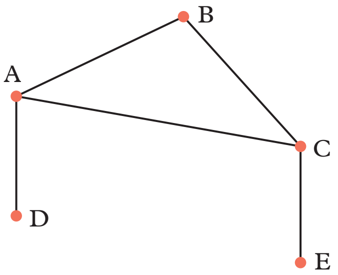
5 **a** path

b planar graph

6 **a** a walk in a graph without repeated edges or repeated vertices except for the start and end vertex which must be the same. For example, in the following graph F, n, C, i, D, j, E, k, F is a cycle.



b a graph without loops or parallel edges. For example:



7 **a** A graph is said to be connected if there is a walk between every pair of vertices.

b The handshaking lemma (faces version) states that in every planar graph the sum of face degrees is twice the number of edges.

Multiple choice

1 B **2** C **3** E **4** D **5** C
6 D **7** E **8** D **9** E **10** B

Short answer

1

Vertex	A	B	C	D	E	F
Degree	4	3	2	4	3	2

2 5

3 $E = F + V - 2 = 6 + 5 - 2 = 9$

4 48 $(6 \times 4 + 8 \times 3)$

5 All faces must be the same for a shape to be a platonic solid.

6 Because the number of edges would be $5 \times 3 \div 2 = 7.5$.

7

Solid	**Faces**	**Vertices**	**Edges**
Tetrahedron	4	4	6
Octahedron	8	6	12
Cube	6	8	12
Icosahedron	20	12	30
Dodecahedron	12	20	30

8 **a** 3

b 4

c Yes, because there are exactly 2 vertices with odd degrees.

9 Because there are 13 vertices of odd degree.

10 the small bedroom

11

	Kin Group							
You	A	B	C	D	E	F	G	H
Mother	B	C	D	A	H	E	F	G
Grandmother	C	D	A	B	G	H	E	F
Great-grandmother	D	A	B	C	F	G	H	E
Great-great-grandmother	A	B	C	D	E	F	G	H

Analysis

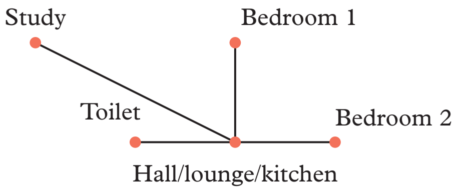
1 **a**

Shape	**Truncated cube**	**Truncated octahedron**	**Truncated icosahedron**
Face degrees	3 or 8	4 or 6	5 or 6

b **i** 4 **ii** 4 **iii** 36
iv 18 **v** 12

c **i** 72 **ii** 36 **iii** 24

2 **a** Sample answer (note: there is no door between the hall, lounge and kitchen):



b No, because there are 4 vertices with odd degree.

CHAPTER 11 Statistics and probability

EX p562

11A Bivariate data and scatter plots

1 **a** two variables

b two variables

c one variable

d one variable

2 **a** **i** a moderate to weak negative trend

ii With increasing height, the time taken to run 10 km decreases moderately.

b **i** weak positive trend

ii The number of visits to the doctor per year increases slightly as the number of colds per year increases.

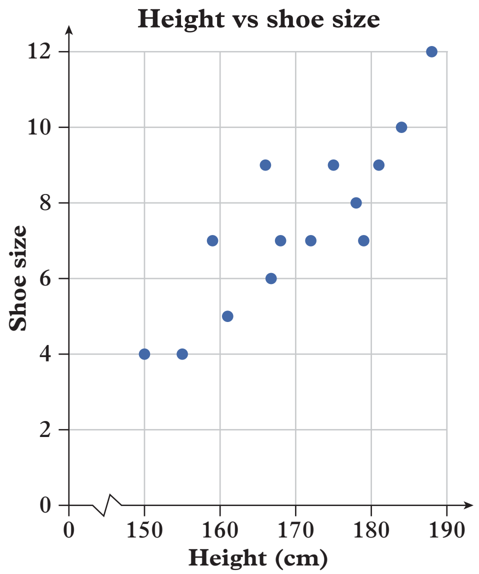
c **i** strong positive trend

ii The amount of petrol used increases greatly with an increase in the distance travelled.

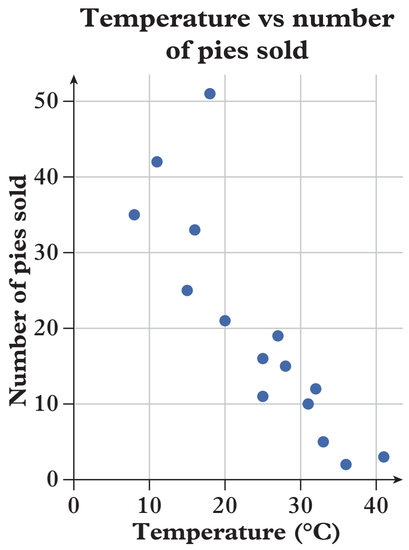
d **i** moderate negative trend

ii Exam marks decrease moderately with an increase in the hours of television watched per week.

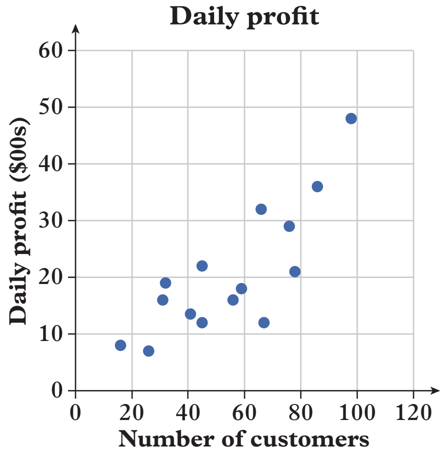
3 a



b



c

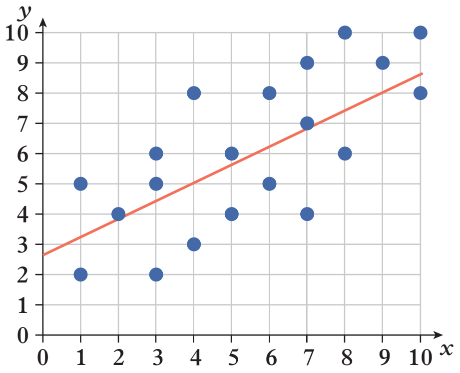


4 a There is a moderate increase in shoe size with an increase in people's height.

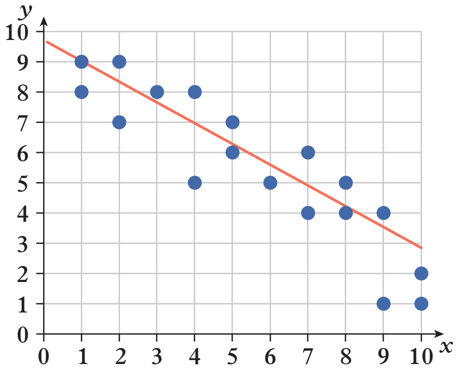
b There is a strong negative correlation, so the number of pies sold decreases greatly as temperature increases.

c There is a moderate increase in daily profit as the number of customers increases.

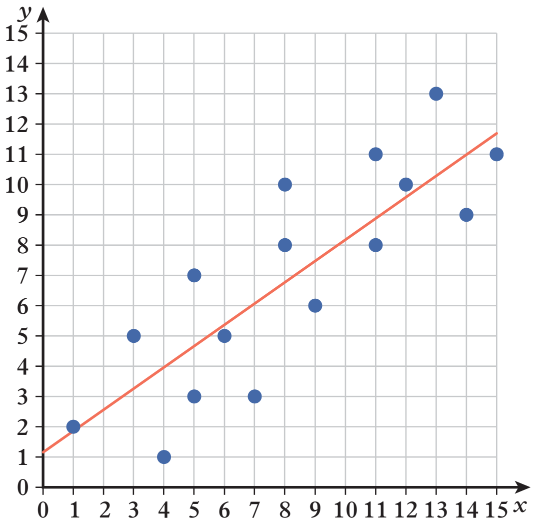
5 a



b

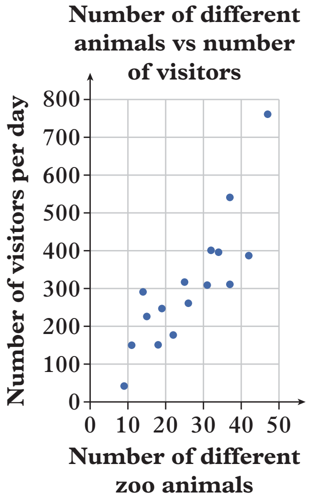


c



6 a moderate positive linear correlation

b



c There is a moderately strong linear increase in the number of visitors per day as the number of different zoo animals increases.

7 a The strong positive linear correlation between the hours studied and the marks achieved supports the view that more study time contributes to achieving a higher percentage result on a test. However, extra study time does not always guarantee a higher test score; the correlation is for a sample and for an individual the result can vary. Also, there may have been another underlying reason that explains the high marks and the extra time studying may have just been a coincidence.

b A high correlation does not imply that one variable 'caused' another to happen. There could be an underlying reason that is hidden and not known, and the correlation seen may actually just be a coincidence or due to choosing a particular sample.

c It is commonly thought if the relationship between two variables is strong, one must have a strong influence on the other. Sometimes this is true, but the particular circumstances need to be examined to determine this.

8 a The weight depends on height.

b As a person gets taller, he/she generally becomes heavier.

c i weight **ii** height

9 a i Up to a certain time (the age of the tree) there is a linear correlation.

ii Yes, as time passes, the tree grows taller.

iii The independent variable is time, the dependent variable is height.

iv The linear correlation is strong and positive in the early years of the tree's life.

b i Up to a certain age of car there is a linear correlation between the value of a car and its age. If the car becomes an antique, or a collectable item, this relationship changes.

ii As the car gets older, the price goes down.

iii The independent variable is the age of the car; the dependent variable is the value of the car.

iv The linear correlation is strong and negative, with the car losing value as it ages.

c There is no linear correlation between the weight of a person and the number of languages spoken.

d i There is a linear correlation between a person's shoe size and height.

ii As someone grows taller, their feet get larger, and they are likely to have a bigger shoe size.

iii The independent variable is height; the dependent variable is shoe size.

iv The linear correlation is strong and positive. Generally, as a person grows and gets taller, his/her shoe size increases.

e i There is a linear correlation between the number of girls in a park and the number of boys in a park.

ii There is no causal relationship.

iv There is a weak positive linear correlation.

f i There is an inverse correlation between the number of fans sold and the number of heaters sold.

ii There is no causal relationship.

10 a As temperature increases, the number of heaters purchased decreases strongly. There is a strong negative linear correlation between the temperature and the purchase of heaters, and there is a weak positive linear correlation between someone's earnings and their monthly phone bill. As the earnings increase, the phone bill does also tend to increase.

b about 4

c about $90

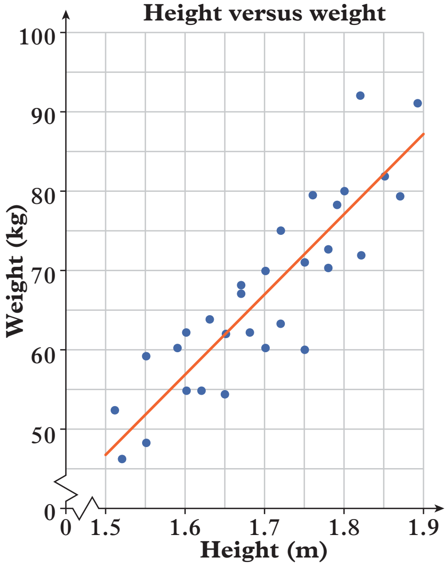
d The prediction from scatterplot A is not very reliable because there is not a strong relationship between the variables. For scatterplot B, the relationship is much stronger, so a prediction would be more reliable.

e around $270

f The prediction in part **e** is not as reliable as the prediction in part **c**, as the prediction in part **c** made within the original data range and the prediction in part **e** is outside the original range of data.

11 a The trend is linear, moderately strong and positive. As height increases, weight also increases.

b



c 82 kg (answers may vary)

d Quite confident because there is a strong correlation between height and weight.

e i interpolation

ii extrapolation

iii extrapolation

iv extrapolation

f 255 cm

g Making predictions outside the given data set can give unrealistic results. For example, in part **f**, a height of 255 cm is impossible. This means we cannot be confident about making a prediction outside this data set.

12 **a** no correlation

b moderate, positive linear correlation

13 Answers will vary.

14 While the previous examples have used only positive values for the dependent and independent variables, it is possible to plot data that uses negative values, such as temperature, and end up with a negative section on either axis.

15 To plot data values in a scatterplot you need to know the exact values of each variable. When dealing with grouped data, the exact values are no longer known, so you would not be able to accurately plot each point.

EX p568

11B Sampling methods

1 **a** convenience sampling

b stratified sampling

c self-selected sampling

d quota sampling

2 **a** Biased. People at an AFL football match are not a representative sample of the whole country.

b Fair. This should provide an answer representative of the views of all the people in school.

c Biased. The question is not fair and leads to only responses regarding people's attitude to dogs.

d Biased. A sample of 100 is too small to represent the population.

e Biased. People outside your community would probably not have any appreciation of particular community issues, so their views could skew the results.

f Biased. A small sample of one section of the population is not likely to be representative of the entire population.

3 **a** **i** The data collection method is not fair. Tess only surveyed people from her town when the population needing to be considered is the entire population of NSW.

ii Tess should have surveyed people across all of NSW rather than just in her town.

b **i** The sample is too small for any results to be reliable.

ii Survey a larger sample and watch out for outliers.

c **i** The question only asked if the people surveyed like Mac computers, but not if they prefer them to other types of computers. This is not a fair data collection method.

ii The question could have been, 'Which brand of computer do you prefer?'

d **i** The data collection method is fair.

e **i** The sample only consisted of people walking through the mall, so this could lead to biased results (especially when the survey is about time spent sitting down). It's also not clear who her population is.

ii Sienna should have specified her population and sampled people in various locations for her survey to be representative of the population.

f **i** Kane's survey method is biased because his neighbourhood contains only a small section of the target population and also includes people who are not part of the target population.

ii Kane should have surveyed a sample of people who will be attending the school dance.

4

Age group (years)	Percentage	Number to survey
0–4	6.179	618
5–9	6.382	638
10–14	6.133	613
15–19	5.923	592
20–24	6.935	694
25–29	7.525	753
30–34	7.458	746
35–39	7.021	702
40–44	6.291	629
45–49	6.616	662
50–54	6.051	605
55–59	6.091	609
60–64	5.473	547
65–69	4.829	483
70–74	4.171	417
75–79	2.895	290
80–84	1.993	199
85–89	1.234	123
90–94	0.605	61
95–99	0.174	17
100 and over	0.02	2

5

Zone	Frequency	Number to survey
Tibooburra	200 465	103
Broken Hill	1 197 474	614
Lower Darling	583 802	299
Cobar	102 480	53
Bourke	117 975	61
Narrabri	190 468	98
Coonabarabran	152 052	78
Griffith North	126 160	65
Griffith South	253 492	130

6 Teacher to check.

7 Teacher to check.

8 Sample answer: Not necessarily. The sample only includes people who watch the show and from those people it only includes those viewers who decide to call.

9 Sample answer: Both of these factors have made it more difficult to collect data from surveys over the phone and will have had an impact on the random sampling of phone surveys, as some demographic will be less likely to use landline phone and/or more likely to screen phone calls. Other surveying methods, such as online surveys, have become more prevalent during this time.

10 Sample answer: Ellie may have asked leading questions in her survey, which would make it more likely for people to answer in the way she wanted.

11 Teacher to check.

EX
p572

11C Analysing reports

1 **a** There is no indication of how people were chosen for the survey or exactly what they were asked.

b People shopping in an Apple store are probably Apple customers/supporters, so this is a biased sample.

c The sample is too small, and there is no indication of the gender or employment status of people making up the sample or whether it was their opinions or facts about their personal earnings that they were questioned about.

d Shareholders in a company have a vested interest in seeing that company's product succeed.

2 **a** If a phone is in stand-by mode, its battery could last up to 7 days. But, in normal use, a phone would not be in stand-by mode for 7 days!

b The 42 people surveyed could have been chosen for the survey because they were purchasing that brand of toilet paper at a supermarket. It is also a very small sample size that is not representative of all of Australia.

c This moisturising lotion was not compared with another moisturising lotion, but with soap.

3 **a** The scale on the *x*-axis is not uniform, and the scale on the *y*-axis does not start at zero, with no break indicated.

b **i** The revenue declined from $100 000 to $96 500, a loss of $3500, which is not a significant decline. The decrease from October to November is a decrease of $1000 over the course of a month. This is about 30% of the total loss over eleven months, which is not a slower rate.

ii The scale on the horizontal axis is not consistent, with there being a three-month interval between each of the first four labelled months, and then a one-month interval between the last two months. The spacing between the last two months is also significantly greater than that between the other months.

iii 3.5%

4 **a** This graph is misleading as the scale on the *y*-axis is not labelled so we do not know if it starts at zero or not or if the scale on the *y*-axis is uniform or not.

b **i** The graph is said to display the time taken before relief, therefore the company's brand takes longer before relief and is therefore less effective.

ii By not labelling the *y*-axis, the company can make the graph look like a bigger bar represents a more effective product as most people assume taller or bigger is better. The bar appears twice as large but with the lack of scaling we cannot verify if the value it actually represents is twice as large.

5 **a** This graph is misleading as the ages of 12 and 15 (at least) are missing.

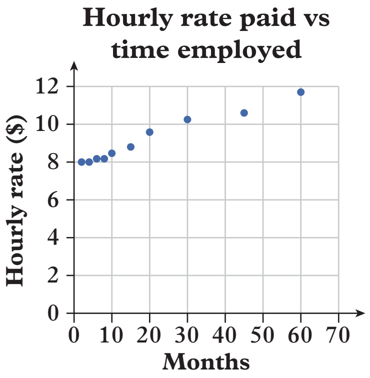
b **i** Since we do not know how many (if any) 12- and 15-year-old students joined the school band, we cannot accurately determine the shape of the distribution or know if there are students in all year levels.

ii By omitting the columns for 12- and 15-year-old students, the band conductor can make the data appear more symmetric with an average age of 16.

6 **a** Fatima's hourly rate of pay over 60 months of employment

b The scale on the *x*-axis is not uniform and the scale on the *y*-axis doesn't start at the origin.

c



7 **a** Only one person claimed 18 hours pain free. Most of the claims are far less than 18 hours but because one person said 18 hours, the claim of 'up to 18 hours' has been made.

b mean = 3, median = 2, mode = 2

c This claim is not too far from the truth. However, the median or the mode would be a better representative value.

8 How was the data collected? How big was the sample? Does the data represent a sample of the population? How was the questionnaire designed? Is the interpretation fair?

9 No. Success in school depends on more variables than just whether or not a student eats breakfast. Those who use the first statement may be trying to sell breakfast foods.

10 The refugee population in Australia as a percentage of the total Australian population in 2018 was approximately 0.23%. This is more than the refugee population as a percentage of the respective populations in New Zealand (0.03%), the United States (0.10%) and the United Kingdom (0.19%), but less than the refugee populations as a percentage of the respective total population in Canada (0.31%), Germany (1.28%), Turkey (4.44%), Pakistan (0.68%) and Jordan (7.18%).

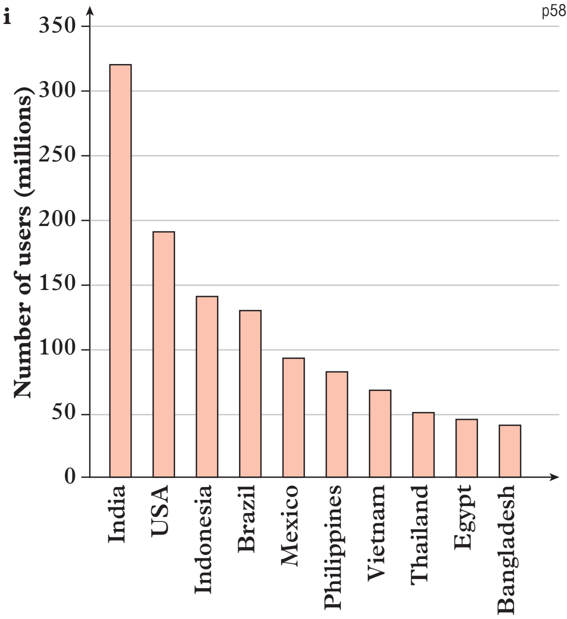
11 The source of information, the sample used to obtain the results and the source of funding for such surveys should be stated with any reported results.

12 Teacher to check.

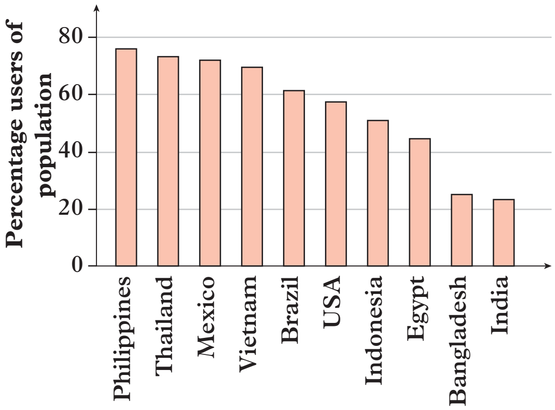
13 **a** 8th
  b Because 98 countries were assessed.
  c Other countries that were not included (e.g. China) may have handled the coronavirus pandemic better than Australia did.
  d Look up the Lowy Institute and find the data they used to rank the countries.

14 **a** incorrect, $\frac{18}{60} \times 100\% = 30\%$
  b correct, $\frac{4}{65} \times 100\% = 6\%$
  c correct, $\frac{36}{125} \times 100\% = 29\%$
  d incorrect (but close to correct), $\frac{31}{61} \times 100\% = 51\%$

15 **a** India: 23.2%, USA: 57.4%, Indonesia: 51.2%, Brazil: 61.1%, Mexico: 72.1%, Philippines: 75.7%, Vietnam: 69.9%, Thailand: 73.1%, Egypt: 44.0%, Bangladesh: 24.9%
  b **i** Philippines **ii** India
  c Facebook is banned in China, and China has its own social network (e.g. Weibo).
  d **i**



ii



  e **i** By ordering by number of users, it shows India and USA have the most number of users compared to the other countries.
  ii By ordering by the percentage of the population that are users of Facebook, India is at the bottom of this group of countries.

16 Sample answer: The operating profits of the mining industry were the most variable, with a big drop in the middle of the decade, followed by a big rise at the end of the decade. Manufacturing followed a similar pattern to mining, with a sharp initial drop in operating profit, followed by a steady rise to a new height at the end of the decade. After an initial drop in 2012–13, construction steadily increased in operating profit through the rest of the decade, while the operating of the retail industry gradually fell, and the profit of the media industry fell significantly after a slight boost in the middle of the decade.

17 Teacher to check.

18 Teacher to check.

EX
p580

11D Theoretical probability

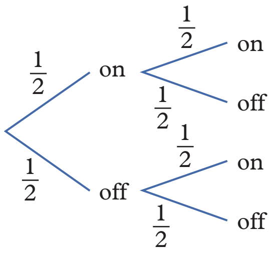
1 **a** {1, 2, 3, 4, 5, 6, 7, 8}
  b {A, U, S, T, R, L, I}
  c {red, white, blue}

2 **a** equally likely
  b not equally likely
  c not equally likely

3 **a** {M, A, T, H, E, I, C, S} **b** $\frac{2}{11}$

4 **a** 0.25 **b** 0.25 **c** 0.5

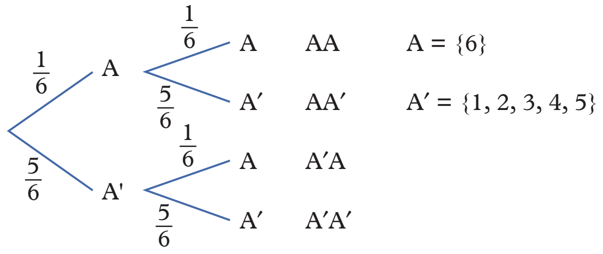
5 **a**



  b **i** 0.25 **ii** 0.25 **iii** 0.5

6 **a** not rolling a 6

b



c **i** $\frac{1}{36}$ **ii** $\frac{11}{36}$

7 **a** Each outcome now has an equal chance of occurring.

b 36

c **i** $\frac{1}{6}$ **ii** $\frac{1}{4}$

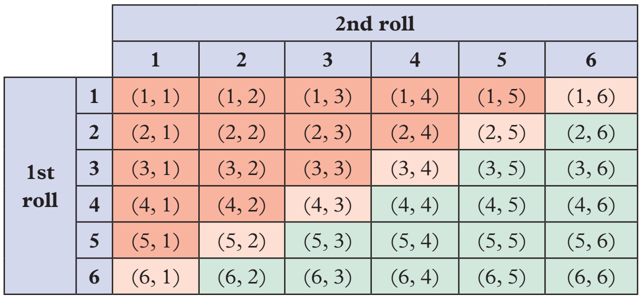
		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

8 **a** 0.125 **b** 0.75 **c** 0.5

9 **a, b** **i** Cells coloured red

ii Cells coloured orange

iii Cells coloured green



		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

c **i** There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

ii There are 6 favourable outcomes out of 36 possible outcomes; $\frac{6}{36} = \frac{1}{6}$.

iii There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

d The most successful strategy would be to predict a number greater than (or less than) 7.

10 **a** A = {2, 4, 6, 8, 10}, B = {4, 8}, C = {2, 3, 5, 7}

b B and C

c **i** $\frac{1}{2}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

11 **a** **i** 36

ii 6 outcomes are doubles; (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

iii $\frac{1}{6}$

b **i** 64

ii 8 outcomes are doubles, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)

iii $\frac{1}{8}$

c Pr(doubles) = $\frac{1}{n}$

12 10 marbles

13 **a** **i** $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{2}{9}$

b **i** For the outcomes from part **a**, an odd number is most likely.

ii For the outcomes from part **a**, a prime number is least likely.

14 **a** {1, 2, 3, 4, 5, 6, 8, 10, 12}

b Some scores can only be achieved one way, whereas others can be achieved two ways.

c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$

15 $\frac{11}{36}$

16 **a** 28

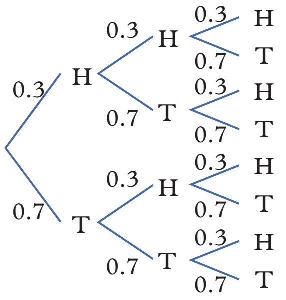
b The rows of tiles are grouped, in order from zero to six, according to the value shown on the top half of the tiles. Within each of these ordered groups, the values of the bottom half of the tiles have also been placed in order.

c 7

d **i** $\frac{1}{28}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{4}$

iv $\frac{3}{7}$ **v** $\frac{9}{14}$ **vi** $\frac{13}{28}$

17 **a**



Outcomes	**Probability**
HHH	0.027
HHT	0.063
HTH	0.063
HTT	0.147
THH	0.063
THT	0.147
TTH	0.147
TTT	0.343

b **i** TTT **ii** HHH

c **i** 0.441 **ii** 0.189 **iii** 0.216

18 The probabilities of the outcomes at the second and third steps of the experiment are not affected by the results of the previous steps of the experiment.

19 **a** 2

b 7

c **i** (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)

ii 6

20 5 red decorations

21 **a** The chance of landing on a red section is not the same as the chance of landing on a white section. There are more red bands than white and the combined area of the red bands appears larger than the combined area of the white bands.

b radii of circles from inner circle out: 1 cm, 3 cm, 5 cm, 7 cm, 9 cm

c bullseye = π cm^2, inner white band = 8π cm^2, middle red band = 16π cm^2, outer white band = 24π cm^2, outermost red band = 32π cm^2

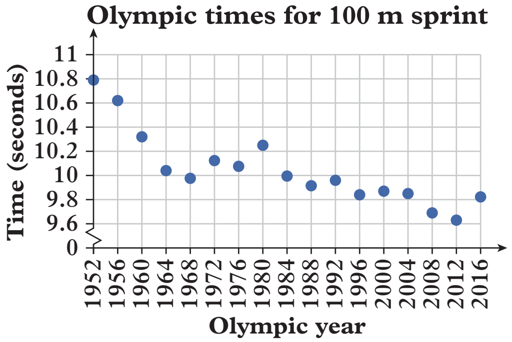
d **i** 0.012

ii 0.395

EX p584

Checkpoint

1 **a**



b Teacher to check.

c 2040 (answers may vary)

2 **a** weak **b** negative

3 **a** **i** systematic sampling

ii biased. The competitors at the swimming tournament are all likely to favour swimming over other sports.

b **i** convenience sampling

ii biased. The sample size is too small to reflect the population and the sample taken from the train station and does not represent all the people in NSW.

4 **a** While 51% is the majority, it is only just over half of the responses to a two-option question and there's no indication of the reliability of the result.

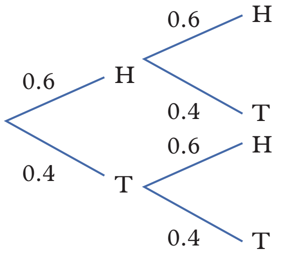
b The fraction '9 out of 10' sounds like it is a simplified fraction from a large sample. However, it is only 9 from a sample of 10, and those 9 were biased because they wanted their answer to positively reflect on their sponsor.

5 **a** {H, T} **b** {1, 2, 3, 4, 5, 6}

c {club, diamond, heart, spade} **d** {L, O, I, P}

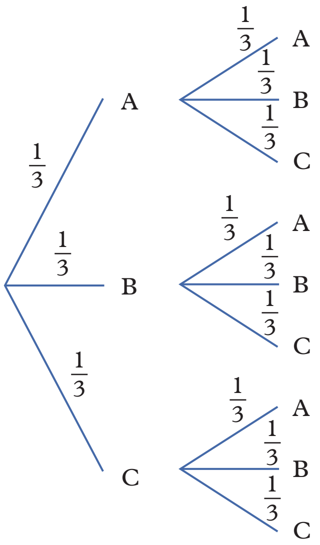
6 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

7 **a**



b **i** 0.36 **ii** 0.48

8 **a**

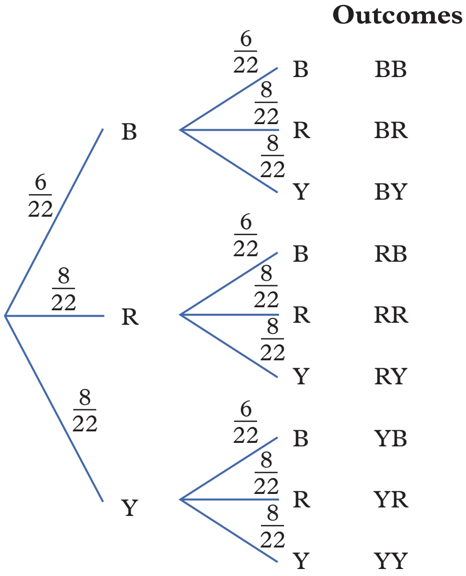


b $\frac{1}{9}$

EX p588

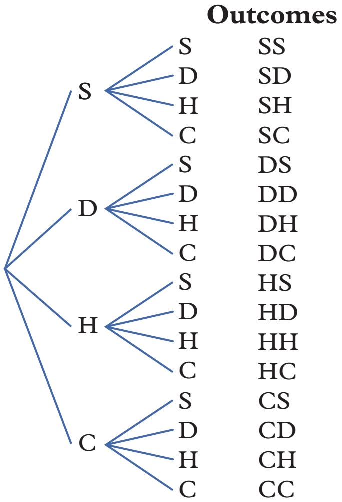
11E Experiments with and without replacement

1 **a**



b $\frac{41}{121}$

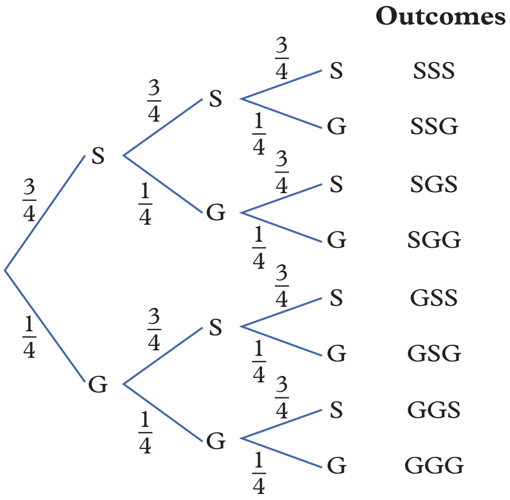
2 **a**



b $\frac{1}{16}$ **c** $\frac{1}{8}$ **d** $\frac{1}{4}$ **e** $\frac{3}{4}$

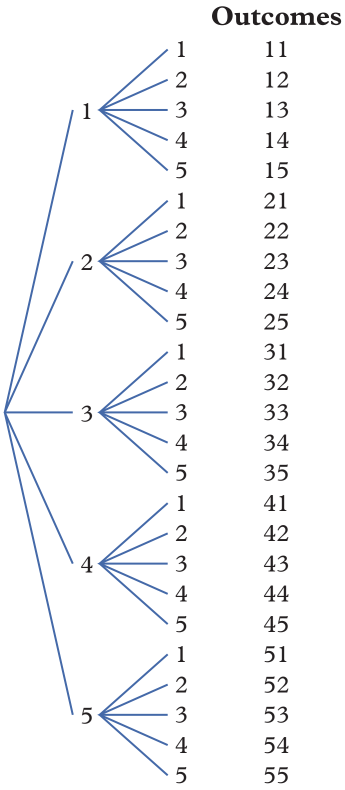
3 **a** **i** $\frac{3}{4}$ **ii** $\frac{1}{4}$

b



c **i** $\frac{27}{64}$ **ii** $\frac{1}{64}$ **iii** $\frac{9}{16}$

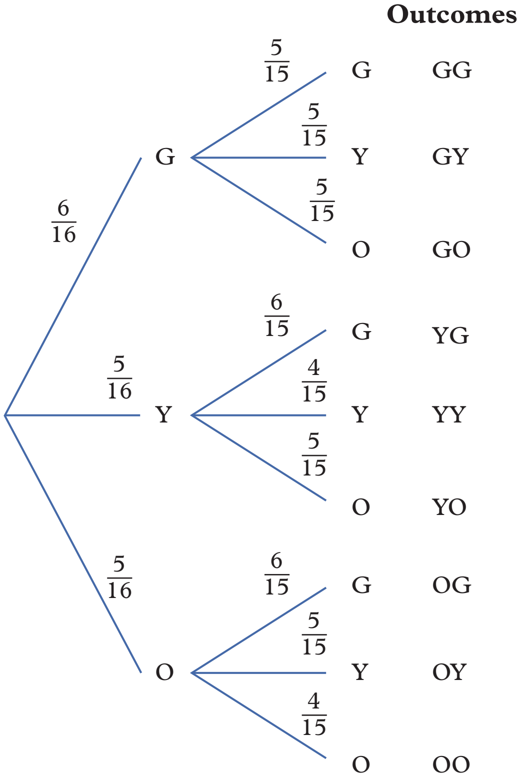
4 **a**



b 25 **c** 5

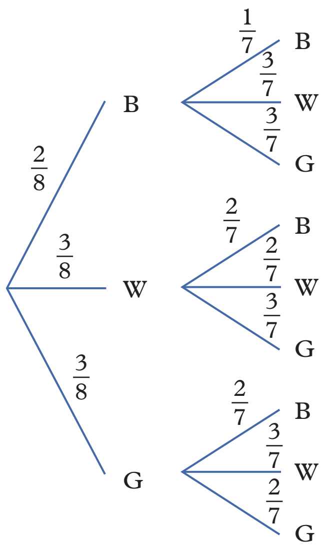
d **i** $\frac{4}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

5 **a**



b $\frac{7}{24}$

6 **a**

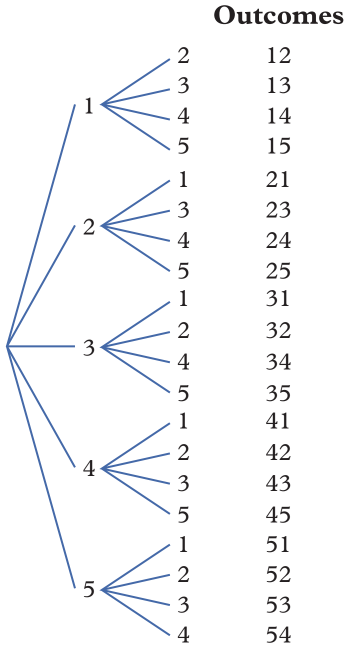


b $\frac{1}{4}$ **c** $\frac{3}{14}$

7 **a** $\frac{6}{55}$ **b** $\frac{28}{55}$

8 **a** $\frac{233}{338}$ **b** $\frac{44}{65}$

9 **a**



b 20

c **i** 0 **ii** 1 **iii** $\frac{1}{2}$

10 **a** $\frac{1}{4}$ **b** $\frac{23}{87}$

11 **a** $\frac{3}{8}$ **b** $\frac{25}{247}$ **c** $\frac{225}{988}$ **d** $\frac{42}{185}$

12 **a** **i** $\frac{9}{29}$ **ii** $\frac{2}{29}$ **iii** $\frac{3}{29}$

iv $\frac{3}{29}$ **v** $\frac{1}{435}$ **vi** $\frac{4}{435}$

b **i** $\frac{2}{87}$ **ii** $\frac{4}{435}$

c Undertaking the events in reverse order does not affect the overall probability of the events occurring.

13 **a** 0 **b** $\frac{27}{125}$

14 The probabilities of the events in a multi-step experiment with replacement are not affected by the results of the previous steps, whereas the probabilities of the events in a multi-step experiment without replacement are affected by the results of the previous steps.

15 **a** There are only two non-white balls in total.

b $\frac{1}{10}$

16 **a** **i** {2, 3, 4, 5, 6, 7, 8, 9, 10}

ii {3, 4, 5, 6, 7, 8, 9}

b **i** $\frac{3}{25}$ **ii** $\frac{1}{10}$

c **i** $\frac{1}{5}$ **ii** $\frac{1}{5}$

17 **a** **i** 125 **ii** 60

b **i** 729 **ii** 504

18 **a** 216 **b** 96

19 **a** **i** 10 **ii** 1 000 000

b **i** 17 576 000

ii You can form more possibilities with this system, because the alphabet has 26 letters, enabling more combinations than the 10 digits.

EX p594

11F Two-way tables

1 **a** 55 **b** 7 **c** $\frac{7}{55}$

2 **a** 75 **b** 11 **c** $\frac{58}{75}$

3 **a** $\frac{29}{85}$ **b** $\frac{37}{85}$ **c** $\frac{42}{85}$ **d** $\frac{23}{85}$

4 **a** $\frac{48}{125}$ **b** $\frac{19}{50}$ **c** $\frac{13}{125}$ **d** $\frac{31}{50}$

e $\frac{181}{250}$ **f** $\frac{27}{125}$

5

	Class A	Class B	Total
Prefer sweet food	0.23	0.32	0.55
Prefer savoury food	0.29	0.16	0.45
Total	0.52	0.48	1

6

	Reads books	Plays games	Plays sports	Total
Introvert	$\frac{7}{32}$	$\frac{11}{64}$	$\frac{1}{8}$	$\frac{33}{64}$
Extrovert	$\frac{3}{64}$	$\frac{11}{64}$	$\frac{17}{64}$	$\frac{31}{64}$
Total	$\frac{17}{64}$	$\frac{11}{32}$	$\frac{25}{64}$	1

7

	Light hair	Dark hair	Total
Prefer pop music	0.22	0.38	0.6
Prefer rock music	0.13	0.27	0.4
Total	0.35	0.65	1

8

	Spotify	No Spotify	Total
Netflix	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
No Netflix	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$
Total	$\frac{1}{4}$	$\frac{3}{4}$	1

9 **a**

	Cinema	Home	Total
Action	22	30	52
Comedy	19	14	33
Total	41	44	85

b **i** $\frac{41}{85}$ **ii** $\frac{14}{85}$ **iii** $\frac{52}{85}$

10

	Cat	No cat	Total
Purebred	29	75	104
Not purebred	84	12	96
Total	113	87	200

$$\text{Pr(non-cat purebred)} = \frac{75}{200} = \frac{3}{8}$$

11 **a**

	Women	Men	Total
Bath	0.33	0.09	0.42
Shower	0.22	0.36	0.58
Total	0.55	0.45	1

b **i** 0.36 **ii** 0.42 **iii** 0.22

c **i** 210 **ii** 275 **iii** 45

12 **a** $\frac{18}{83}$ **b** $\frac{67}{332}$ **c** $\frac{4}{83}$

d $\frac{125}{332}$ **e** $\frac{72}{83}$ **f** $\frac{69}{83}$

g $\frac{109}{166}$ **h** $\frac{47}{332}$ **i** $\frac{9}{83}$

13 **a** 60 **b** 160 **c** 980 **d** 120

14

	A	A'	Total
B	40	30	70
B'	20	30	50
Total	60	60	120

15 **a**

	A	A'	Total
B	40	35	75
B'	40	85	125
Total	80	120	200

	A	A'	Total
B	0.2	0.175	0.375
B'	0.2	0.425	0.625
Total	0.4	0.6	1

b

	C	C'	Total
D	4	5	9
D'	2	9	11
Total	6	14	20

	C	C'	Total
D	0.2	0.25	0.45
D'	0.1	0.45	0.55
Total	0.3	0.7	1

c

	E	E'	Total
F	18	4	22
F'	2	12	14
Total	20	16	36

	E	E'	Total
F	$\frac{1}{2}$	$\frac{1}{9}$	$\frac{11}{18}$
F'	$\frac{1}{18}$	$\frac{1}{3}$	$\frac{7}{18}$
Total	$\frac{5}{9}$	$\frac{4}{9}$	1

16 **a**

	Year 9	Year 10	Total
Maths	12	11	23
English	10	17	27
Total	22	28	50

b **i** $\frac{23}{50}$ **ii** $\frac{14}{25}$ **iii** $\frac{11}{50}$

c **i** 28 **ii** 11 **iii** $\frac{11}{28}$

d Part **c iii** says the student is in Year 10, hence the total number of possible outcomes is restricted to the total number of Year 10 students (28). In part **b iii** you are selecting from the total number of students (50), and of those you need to select a student in Year 10 who likes Maths.

e **i** likes English; 27

ii Year 9 students who like English

iii 10

iv $\frac{10}{27}$

17 **a** $\frac{13}{81}$ **b** $\frac{37}{96}$ **c** $\frac{1}{5}$

d $\frac{13}{43}$ **e** $\frac{71}{181}$ **f** $\frac{4}{5}$

EX p602

11G Venn diagrams

1 **a** 14 **b** 15 **c** 2 **d** 31

2 **a** 62 **b** 19 **c** 532 **d** 637

3 **a** $\frac{1}{5}$ **b** $\frac{3}{10}$ **c** $\frac{11}{20}$

d $\frac{7}{10}$ **e** $\frac{1}{10}$ **f** $\frac{1}{4}$

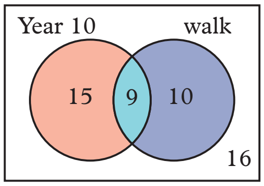
4 **a** $\frac{17}{45}$ **b** $\frac{1}{9}$ **c** $\frac{19}{45}$

d $\frac{1}{5}$ **e** $\frac{23}{45}$ **f** $\frac{31}{45}$

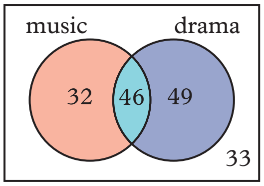
5 **a** $\frac{9}{41}$ **b** $\frac{21}{41}$ **c** $\frac{11}{41}$

d $\frac{8}{41}$ **e** $\frac{22}{41}$ **f** $\frac{33}{41}$

6



7



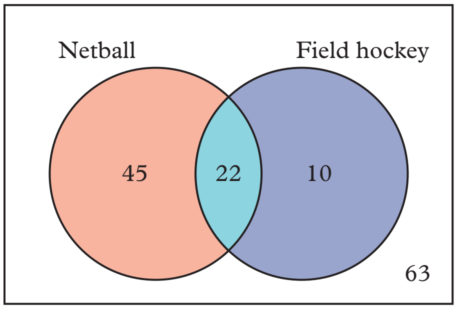
8 **a** $A \cap B$ **b** $A \cup B$

c $A' \cup B$ **d** $A \cap B'$

e $A \cup B'$ **f** $A' \cap B'$ or $(A \cup B)'$

g $A' \cap B$ **h** $A' \cup B'$ or $(A \cap B)'$

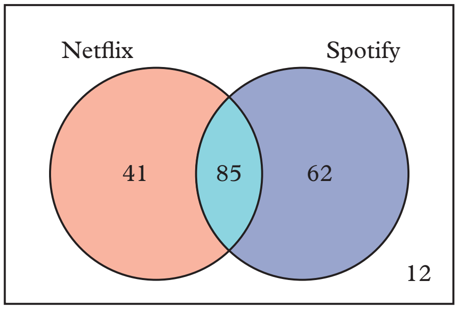
9



10 **a**

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)	41	12	53
Total	126	74	200

b



11

	A	A′	Total
B	15	7	22
B′	13	11	24
Total	28	18	46

12 For mutually exclusive events there are no elements in the intersection, so the circles can be drawn with no overlap. For non-mutually exclusive events there are elements in the intersection, so the circles need to be overlapping.

13 **a** 7

b

	Maths (M)	Not maths (M′)	Total
Science (S)	11	7	18
Not science (S′)	4	6	10
Total	15	13	28

14 **a** **i** 29 **ii** 43 **iii** 26

iv 10 **v** 50 **vi** 14

b **i** $\frac{12}{25}$ **ii** $\frac{21}{50}$ **iii** $\frac{1}{5}$

iv $\frac{7}{50}$ **v** $\frac{7}{50}$ **vi** $\frac{31}{50}$

15 **a** $(A \cap B)' = A' \cup B'$ and $(A' \cup B')' = A \cap B$

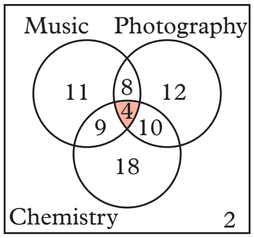
$(A' \cap B)' = A \cup B'$ and $(A \cup B')' = A' \cap B$

$(A \cap B')' = A' \cup B$ and $(A' \cup B)' = A \cap B'$

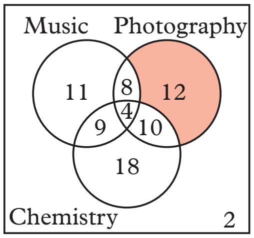
$(A' \cap B')' = A \cup B$ and $(A \cup B)' = A' \cap B'$

b The complement of an intersection is the union of the complements of the two events. The complement of a union is the intersection of the complements of the two events.

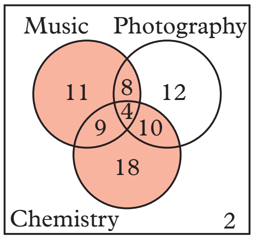
16 **a**



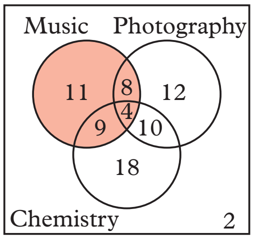
b



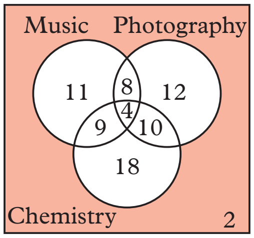
c



d



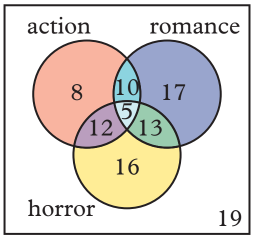
e



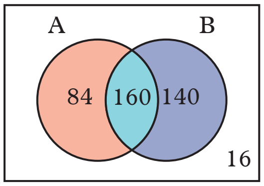
17 **a** $\frac{7}{34}$ **b** $\frac{4}{17}$ **c** $\frac{1}{17}$

d $\frac{5}{34}$ **e** $\frac{23}{34}$ **f** $\frac{7}{34}$

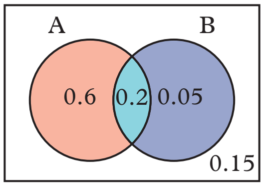
18 Sample answer:

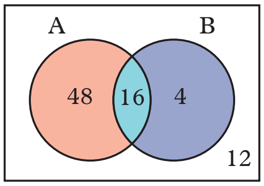


19



20 a





b **i** 0.8 **ii** 0.25

21 a **i** $\Pr(A \cup B)$ **ii** $\Pr(A \cap B)$

b $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

22 When sets A and B are mutually exclusive, $\Pr(A \cap B) = 0$.

23 a 0.7

b $\frac{5}{18}$

24 a $\frac{29}{61}$ **b** $\frac{29}{46}$ **c** $\frac{32}{61}$ **d** $\frac{9}{25}$

25 a $\frac{11}{30}$ **b** $\frac{11}{26}$ **c** $\frac{15}{26}$ **d** $\frac{2}{7}$

EX p610

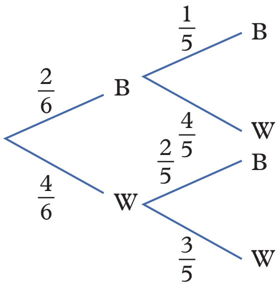
11H Conditional probability

1 a The probability $\frac{15}{20}$ means there are 15 soft centres out of a total of 20 chocolates. The probability $\frac{5}{20}$ means there are 5 hard centres out of the 20 chocolates.

b The branches on the tree indicate that if 1 soft centre is eaten, there are 14 soft centres and 5 hard centres left out of the remaining 19 chocolates. They also show that, if it had been 1 hard centre that had been eaten, there would have been 15 soft centres and 4 hard centres left out of the remaining 19 chocolates.

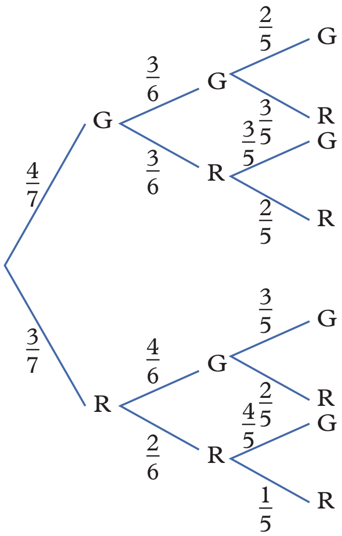
c $\frac{4}{19}$ **d** $\frac{5}{19}$ **e** $\frac{15}{19}$

2 a



b $\frac{3}{5}$ **c** $\frac{4}{5}$ **d** $\frac{2}{5}$

3 a



b $\frac{1}{2}$ **c** $\frac{2}{5}$ **d** $\frac{3}{5}$

4 a **i** 0.4 **ii** 0.5

b independent

5 a $\frac{1}{6}$ **b** $\frac{2}{3}$ **c** $\frac{1}{4}$

6 a $\frac{12}{13}$ **b** $\frac{12}{13}$

7 a 15 **b** 18 **c** 18 **d** 59

8 a $\frac{15}{41}$ **b** $\frac{9}{20}$ **c** $\frac{9}{50}$ **d** $\frac{59}{100}$ **e** $\frac{18}{59}$

9 a $\frac{4}{49}$ **b** $\frac{7}{12}$

10 a **i** $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{3}$ **iv** $\frac{4}{7}$ **v** $\frac{3}{11}$

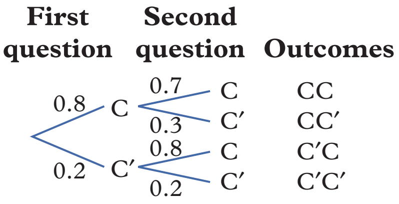
b $\frac{1}{2}$

11 a $\frac{1}{3}$ **b** 0.25

12 a dependent **b** independent

c independent **d** dependent

13 a



b **i** 0.7 **ii** 0.8 **iii** 0.3

14 a

	Hybrid	WFH	Total
Full-time employees	30	4	34
Part-time employees	5	11	16
Total	35	15	50

b **i** 4 **ii** 16 **iii** 35

c **i** $\frac{1}{7}$ **ii** $\frac{5}{16}$ **iii** $\frac{2}{17}$ **iv** $\frac{4}{15}$

d **i** In part **i**, the selection is made from hybrid workers (total of 35), whereas in part **ii** the selection is made from part-time employees (total of 16).

ii In part **iii**, the selection is made from full time employees (total of 34), whereas in part **iv** the selection is made from WFH employees (total of 15).

15 **a**

	Small	**Large**	**Total**
Silver	27	35	62
Coloured	18	10	28
Total	45	45	90

b **i** True; there are 45 of each.

ii True; there are 28 coloured paper clips but there are 62 silver paper clips.

iii False; there are 27 small silver paper clips out of a total of 45 small paper clips. $\frac{27}{45} = 60\%$

iv False; there are 10 large coloured paper clips out of a total of 28 coloured paper clips. $\frac{10}{28} \approx 36\%$

v True; there are 62 silver paper clips out of a total of 90 paper clips. $\frac{62}{90} \approx 69\%$

16 The probability of each roll of a die is an independent event and is not conditional on the results of the previous rolls.

17 Kaitlyn is correct. While the sample space is reduced, the number of favourable outcomes may also be reduced, so the probability can be either greater than or less than at the previous step.

18 0.78

19 **a** **i** 32 **ii** 41 **iii** 57

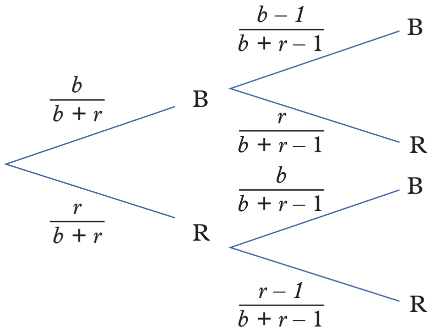
b **i** $\frac{11}{32}$ **ii** $\frac{11}{41}$ **iii** $\frac{35}{57}$

c They help to restrict the sample space, giving the appropriate denominator for the conditional probability.

20 **a** **i** 0.2 **ii** 0.5

b $\Pr(X) = 0.1$ and $\Pr(Y) = 0.25$

21 **a**



b $\frac{b}{b+r}$. This is equal to the probability that the first ball is blue. The denominator from the conditional probabilities in the second step cancels leaving this simplified fraction.

22 **a** 0.64 **b** $\frac{2}{3}$

23 **a** Start with $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$

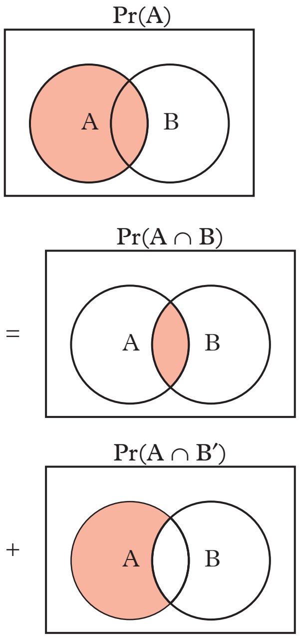
Multiplying through by Pr(A) gives
$\Pr(A \cap B) = \Pr(B|A) \times \Pr(A)$.
Substituting this result into

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ gives

$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$, which is Bayes' theorem.

b **i** 0.55 **ii** $\frac{7}{11}$

24 **a**



b $\Pr(A) \times \Pr(B') = \Pr(A) \times (1 - \Pr(B))$
$= \Pr(A) - \Pr(A) \times \Pr(B)$
$= \Pr(A) - \Pr(A \cap B)$
$= \Pr(A \cap B')$

EX
p615

CHAPTER 11 review

Mathematical literacy review

1 two

2 The intersection of two sets is the set of elements which appear in both sets, whereas the union of two sets is the set of elements which appear in either set.

3 If there is a positive relationship between two numerical variables, as one variable increases in value the other variable also increases in value. If there is a negative relationship between two numerical variables, as one variable increases in value the other variable decreases in value.

4 Mutually exclusive events are events that cannot both occur at the same time.

5 **a** systematic sampling **b** universal set

6 **a** a relationship between two variables where a change in one variable (the independent variable) directly causes a change in the other variable (the dependent variable)

b two events are independent if one event occurring has no effect on the probability of the other event occurring

7 **a** The complement of an event A is the event that A does not occur.

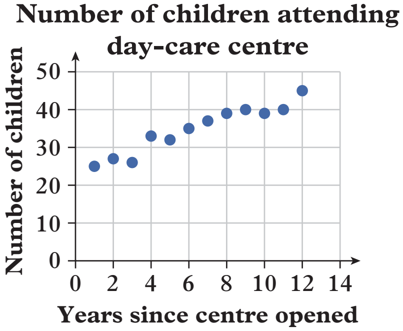
b Conditional probability is the probability of an event occurring, given the knowledge that another event has occurred.

Multiple choice

1 C	**2** B	**3** A	**4** B
5 D	**6** D	**7** C	**8** D
9 A	**10** D	**11** A	**12** D
13 C	**14** C	**15** E	**16** A

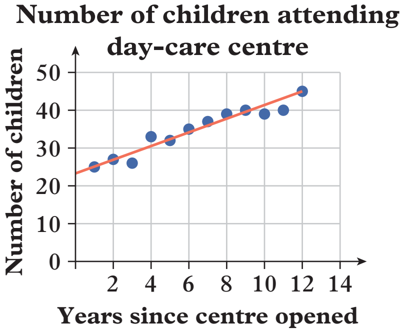
Short answer

1 **a**



b The number of children in the day care centre increased the more years the centre was open.

c



d 60 (answers may vary)

2 **a** systematic sampling

b stratified sampling

3 **a** This survey has used results from one gym to represent the entire population of Australia, meaning it is extremely biased.

b Two boys out of four does not represent a sufficient sample on which to base the claim. Information such as socio-economic backgrounds, ages, suburbs are not considered.

c If only five children were surveyed, the sample is not large enough to represent the larger population claimed. If the reference is to four-fifths of a larger population, more details are needed about what is considered a 'fair share' of chores and of whom that population was made up. Also, the children stated they 'believe they do' their fair share, but the report simply says they 'do' their fair share.

d The children were watching the carnival indoors, so there was no need for them to be wearing sunscreen. The report is misleading.

4 **a** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b **i** {1, 3, 5, 7, 9} **ii** {3, 6, 9}

iii {3, 9} **iv** {1, 3, 5, 6, 7, 9}

v {1, 5, 7}

5 **a** $\frac{9261}{17\,576}$ **b** $\frac{8315}{17\,576}$

c $\frac{9261}{17\,576}$ **d** $\frac{8315}{17\,576}$

6 **a** You can't use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number.

b 900 **c** $\frac{1}{900}$ **d** $\frac{1}{100}$

7 **a** You can't use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number. There also can't be any repetitions of digits.

b 648

c $\frac{1}{648}$ **d** 360

8 **a**

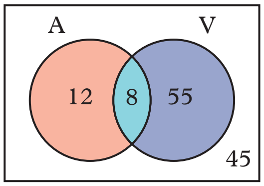
	Year 8	Year 9	Year 10	Total
Napoleon Perdis	32	25	45	102
Rimmel	22	10	22	54
Covergirl	17	31	13	61
Maybelline	49	29	15	93
Total	120	95	95	310

b **i** $\frac{3}{10}$ **ii** $\frac{11}{155}$ **iii** $\frac{5}{62}$

c **i** $\frac{5}{19}$ **ii** $\frac{12}{19}$

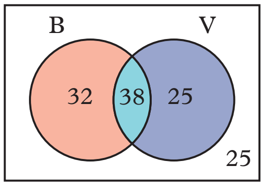
9 **a** 15 **b** 27 **c** 47 **d** 55

10 **a**



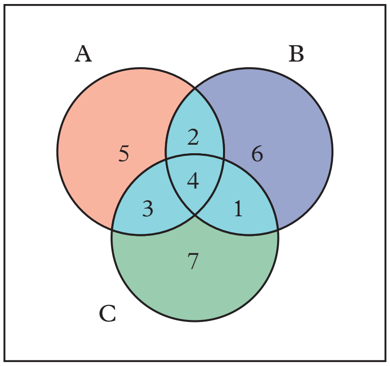
b **i** $\frac{1}{15}$ **ii** $\frac{5}{8}$ **iii** $\frac{3}{8}$ **iv** $\frac{21}{40}$

c



d **i** $\frac{19}{60}$ **ii** $\frac{11}{15}$ **iii** $\frac{4}{15}$ **iv** $\frac{7}{12}$

11



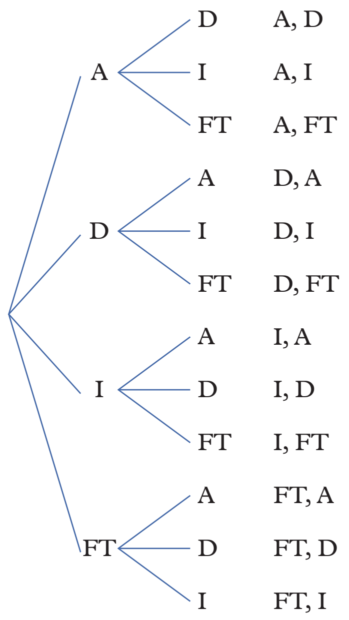
12 **a** $\frac{1}{5}$ **b** $\frac{1}{5}$ **c** $\frac{17}{55}$ **d** $\frac{38}{55}$

13 **a** $\frac{1}{11}$ **b** $\frac{2}{11}$ **c** $\frac{2}{5}$ **d** 1 **e** $\frac{3}{4}$

Analysis

1 **a** Without replacement, as the same subject cannot be selected twice.

b **i**



Key:
A = Art
D = Drama
I = Italian
FT = Food Technology

ii 12

c **i** $\frac{1}{6}$ **ii** $\frac{1}{6}$

d **i**

	Year 11	Year 12	Total
Art	0.24	0.12	0.36
Food technology	0.28	0.36	0.64
Total	0.52	0.48	1.00

ii $\frac{9}{25}$ **iii** $\frac{6}{25}$ **iv** $\frac{9}{25}$

v 72 **vi** 128

e **i** 24 **ii** 2 **iii** 68

iv $\frac{34}{75}$ **v** $\frac{28}{75}$

2 **a** **i** $\frac{1}{9}$ **ii** $\frac{1}{3}$ **iii** $\frac{5}{9}$

b {4, 5, 6, 7, 8, 10}

c **i** $\frac{1}{9}$ **ii** $\frac{2}{9}$ **iii** $\frac{1}{3}$

d **i** 9% **ii** 25% **iii** 24%

e **i** 30% **ii** 25% **iii** 86%

EX p622

SEMESTER 2 review

Short answer

1 **a** 8.48 cm² **b** 250.45 cm²

c 153.06 cm² **d** 100.53 cm²

e 127.31 cm² **f** 1086.40 cm²

2 **a** **i** 428.24 cm² **ii** 410.79 cm³

b **i** 154.40 cm² **ii** 96.00 cm³

c **i** 471.24 cm² **ii** 785.40 cm³

d **i** 502.38 cm² **ii** 695.70 cm³

3 **a** right-angled triangle **b** acute triangle

c right-angled triangle

4 **a** **i** 4 cm **ii** $a = 63°$, $b = 27°$

b **i** 13 cm **ii** $a = 23°$, $b = 67°$

5 **a** **i** test score = 32.5

ii test score = 38

iii hours on social media = 2.5

iv hours on social media = 24.5

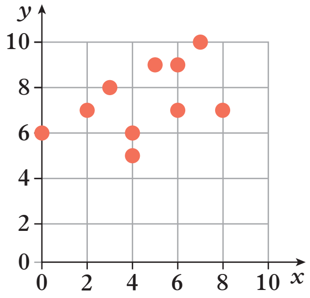
b **i** unreliable

ii reliable

iii reliable

iv unreliable

6



7 **a** moderate, negative, non-linear

b moderate, positive, linear

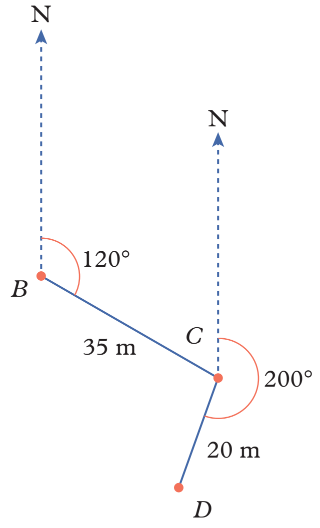
c weak, negative, linear

d no correlation

8 **a** 23.2° **b** 77 m

9 **a** **i** 334°T **ii** 154°T **iii** 2.8 km

b **i**



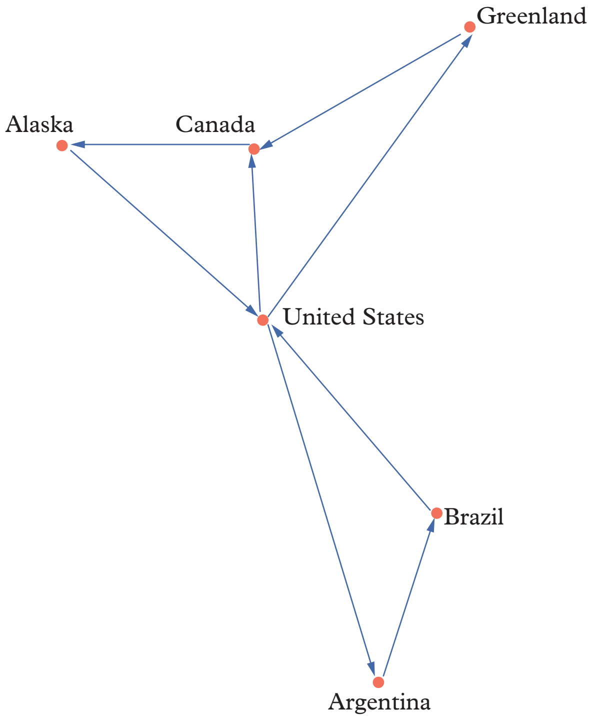
ii 17.5 m **iii** 30.3 m **iv** 18.8 m
v 6.8 m **vi** 327°T **vii** 43.2 m

10

Vertex	A	B	C	D	E
Degree	4	2	3	3	2

11 **a** closed walk **b** walk
c trail **d** invalid walk

12



13 **a** 228.25 cm^2 **b** 173.28 cm^2
c 127.31 cm^2 **d** 1086.40 cm^2
e 742.09 cm^2 **f** 2259.58 cm^2

14 **a** $a = 11.18$ cm, $b = 21.35$ cm, $c = 21.93$ cm
b $x = 13.97$ cm, $y = 26.16^\circ$, $z = 9.04$ cm

15 **a** $a = 65.16^\circ$, $b = 74.84^\circ$, $c = 25.53$ cm
or: $a = 114.84^\circ$, $b = 25.16^\circ$, $c = 11.24$ cm
b $a = 61.69^\circ$, $b = 70.06^\circ$
c $a = 49.07^\circ$, $b = 70.93^\circ$, $c = 51.29$ cm
d $a = 57.95^\circ$, $b = 68.05^\circ$, $c = 24.08$ cm
or: $a = 122.05^\circ$, $b = 3.95^\circ$, $c = 1.79$ cm
e $e = 30.35$ cm
f $f = 47.45$ cm
g $g = 145.14^\circ$
h $h = 35.46^\circ$

16 **a** **i** 384 cm^2 **ii** 384 cm^3
b **i** 84.82 cm^2 **ii** 56.55 cm^3
c **i** 197.92 cm^2 **ii** 183.80 cm^3
d **i** 452.39 cm^2 **ii** 904.78 cm^3
e **i** 131.48 cm^2 **ii** 73.09 cm^3
f **i** 392.21 cm^2 **ii** 636.70 cm^3

17 **a** Biased: the competitors at the tournament are all likely to favour the game they are playing in the tournament over other fighting games, and not enough people are being sampled.

b Fair: the sample size is large enough for the population; the students are questioned about what is being investigated.

c Biased: the sample size should be at least equal to the square root of the population so this sample size is too small.

d Biased: the sample of all Australians does not represent all the people in the Southern Hemisphere.

18 **a** **i** unfair
ii unfair
iii Peta should either sample people from a larger population than her local area or should have made a conclusion only about people in her local area who go to the local shopping centre in a five-hour period on Saturday.

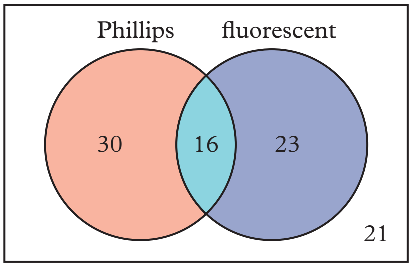
b **i** fair
ii unfair
iii Rob should have made his conclusions about basketball players in the NBA, not about all basketball players.

19 **a** $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ **b** $-\tan(45^\circ) = -1$
c $\cos(60^\circ) = \frac{1}{2}$

20 **a** $x = 210^\circ, 330^\circ$ **b** $x = 45^\circ, 315^\circ$
c $x = 60^\circ, 240^\circ$ **d** $x = 270^\circ$
e $x = 0^\circ, 360^\circ$ **f** $x = 135^\circ, 315^\circ$

21 **a** amplitude = 5, period = 90°
b amplitude = 3, period = 1080°

22 **a**



b **i** $\frac{8}{45}$ **ii** $\frac{17}{30}$ **iii** $\frac{22}{45}$ **iv** $\frac{2}{3}$

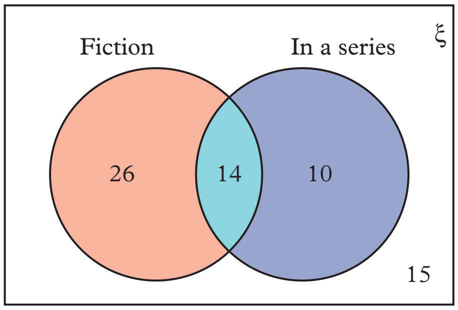
23 **a** **i** 60 **ii** 77 **iii** 52
iv 28 **v** 112 **vi** 17

b **i** $\frac{15}{28}$ **ii** $\frac{13}{28}$ **iii** $\frac{1}{4}$
iv $\frac{5}{16}$ **v** $\frac{5}{16}$ **vi** $\frac{95}{112}$

24 **a**

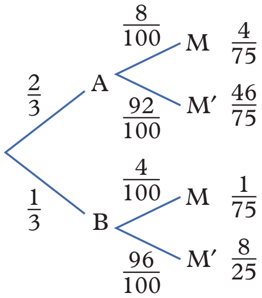
	Fiction	**Non-Fiction**	**Total**
In a series	14	10	24
Not in a series	26	15	41
Total	40	25	65

b



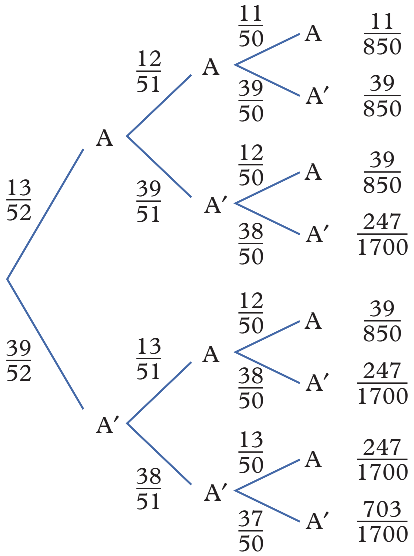
c **i** $\frac{2}{5}$ **ii** $\frac{2}{13}$
iii $\frac{7}{20}$ **iv** $\frac{15}{41}$

25 **a**



b **i** $\frac{4}{75}$ **ii** $\frac{8}{25}$
iii $\frac{2}{25}$ **iv** $\frac{4}{5}$

26 **a**

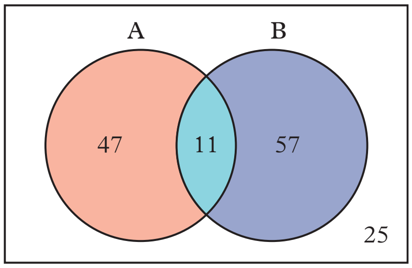


b **i** $\frac{11}{850}$ **ii** $\frac{117}{850}$
iii $\frac{247}{425}$ **iv** $\frac{13}{50}$

27 **a** $\frac{21}{40}$ **b** $\frac{1}{2}$ **c** $\frac{3}{20}$
d $\frac{13}{20}$ **e** $\frac{3}{10}$ **f** $\frac{14}{19}$

28 **a** dependent
b independent
c independent
d dependent

29 **a**



b

	A	**A′**	**Total**
B	5	21	26
B′	12	12	24
Total	17	33	50

30 **a**

	S	**S′**	**Total**
G	56	76	132
G′	42	38	80
Total	98	114	212

b **i** $\frac{57}{106}$ **ii** $\frac{33}{53}$ **iii** $\frac{19}{106}$
iv $\frac{59}{106}$

31 **a**

	A	**A′**	**Total**
B	$\frac{3}{5}$	$\frac{1}{25}$	$\frac{16}{25}$
B′	$\frac{7}{25}$	$\frac{2}{25}$	$\frac{9}{25}$
Total	$\frac{22}{25}$	$\frac{3}{25}$	1

b

	A	**A′**	**Total**
B	51	29	80
B′	15	25	40
Total	66	54	120

32 **e** **i** 1145.75 cm² **ii** 31.83 cm²
f **i** 9777.56 cm² **ii** 271.60 cm²

33 Proof that $\angle AOB = 2\angle ACB$:

Statements	**Reasons**
CD is a diameter	Given
O is the centre of the circle	Given
$OA = OB = OC$	All radii of the circle

Statements	**Reasons**
$\angle OAC = \angle OCA$	Base angles of an isosceles triangle
$\angle AOD = 180° - \angle AOC$	Angles on a straight line sum to 180°
$\angle AOC = 180° - \angle OAC - \angle OCA$	Interior angles of a triangle sum to 180°
$\angle AOD = 180° - (180° - \angle OAC - \angle OCA)$	Substitution of equal angles
$\angle AOD = \angle OAC + \angle OCA$	Simplifying
$\angle AOD = \angle OCA + \angle OCA$	Substitution of the equal angle
$\angle AOD = 2\angle OCA$	Grouping like angles
$\angle OBC = \angle OCB$	Base angles of an isosceles triangle
$\angle BOD = \angle OBC + \angle OCB$	The exterior angle of a triangle is equal to the sum of the two opposite interior angles
$\angle BOD = \angle OCB + \angle OCB$	Substitution of the equal angle
$\angle BOD = 2\angle OCB$	Grouping like angles
$\angle AOB = \angle AOD + \angle BOD$	
$\angle AOB = 2\angle OCA + 2\angle OCB$	Substitution of equal angle measures
$\angle AOB = 2(\angle OCA + \angle OCB)$	Factorising the RHS
but $\angle OCA + \angle OCB = \angle ACB$	Sum of adjacent angles
$\therefore \angle AOB = 2\angle ACB$ □	Substitution of the equal angle measure

34 **a** $j = 19°$, $k = 19°$

b $m = 51°$, $n = 81.5°$, $l = 55$ cm

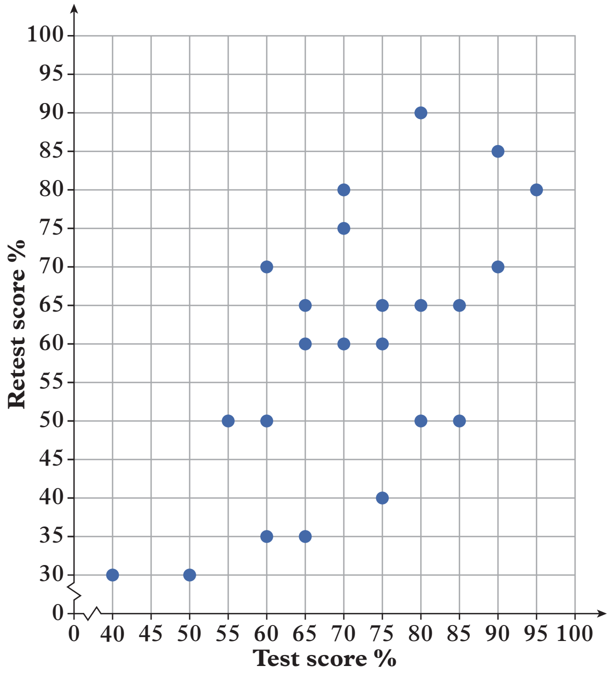
c $p = 36$ cm, $q = 80.91°$, $r = 99.09°$, $t = 43.86$ cm

d $w = 1.06$, $x = -0.85$, $y = 0.53$, $z = 0.62$

Analysis

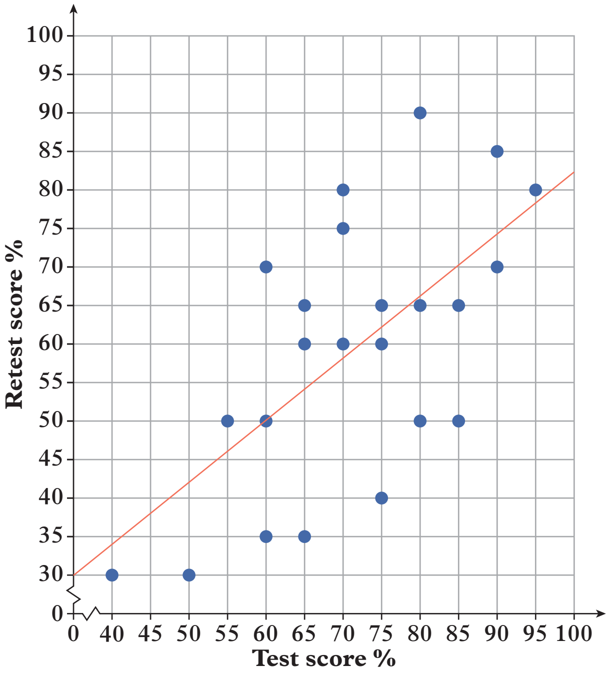
1 **a** test scores

b



c moderate, positive, linear

d

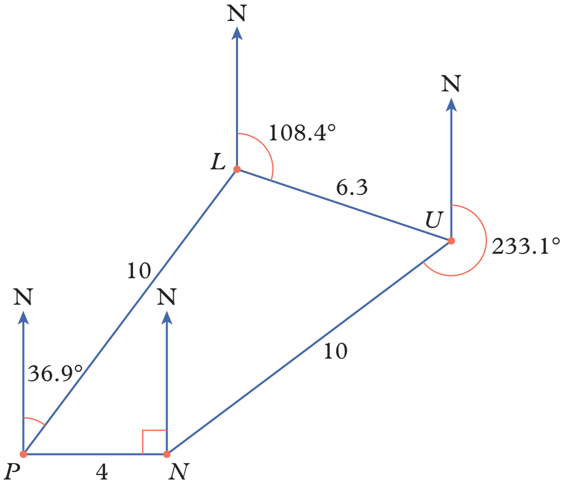


e **i** 58%

ii reliable

f There are 11 students who scored at or above 75% on the original test. Of these, 8 scored better on the retest than the median of those who scored less than 75% on the original test. The remaining 3 students of the 11 who scored at or above 75% on the original test received scores on the retest that were worse than a majority of those who scored less than 75% on the original test. Therefore, while the teacher's claim is somewhat accurate, the moderate correlation means that there are students whose results do not fit the teacher's claim.

2 **a**



b 30.3 km

c 5.05 km/h

d 243 m

e 13.4 km

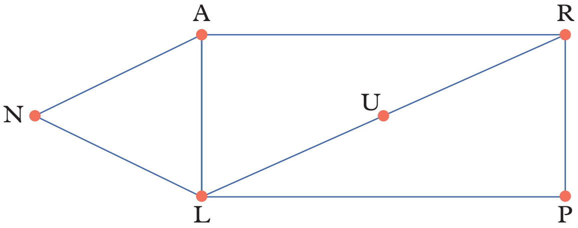
f 063°T

g 41.88 km^2

h **i** 2% **ii** 76%

iii 18% **iv** $\frac{2}{3}$

i



j **i** yes

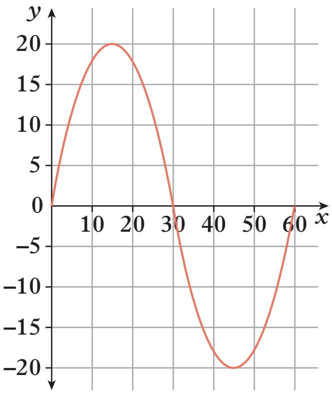
ii There are two vertices of odd degree in the graph. Yes, there is an Eulerian trail.

iii no

3 **a** amplitude = 20 m, period = 1 hour (60 minutes)

b $h = 20\sin\left(\frac{t\pi}{30}\right)$

c



d **i** $10\sqrt{3}$ m

ii 17.32 m

e 10 m

f 37.5 minutes and 52.5 minutes after boarding

g **i** $10\sqrt{2}$ m left and $10\sqrt{2}$ m above

ii 14.14 m left and 14.14 m above

h **i** 15π m

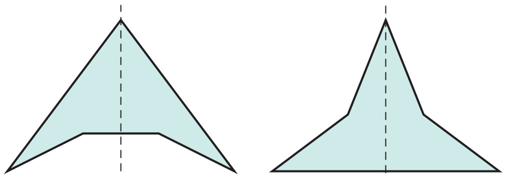
ii 47.12 m

EX p634

EXPLORATIONS 2

1 **a** None, so a thorn triangle is just an isosceles triangle.

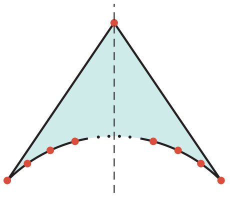
b



c Thorn heptagons have 4 reflex angles. Trying to keep them all separated would require at least 4 non-reflex angles.

d Thorn octagons have 5 reflex angles, leaving less than 180° to share among the remaining 3 angles.

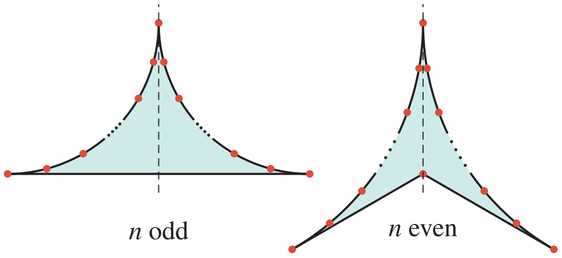
e 1) Place $n - 1$ equally spaced points along a circular arc, as shown. There are other strategies.



2) The angle sum of an n-gon is $180(n - 2)$, which is strictly less than the sum of $n - 2$ reflex angles.

f 9

g Consider equally spaced points along two identical circular arcs. Treat the odd and even cases separately.



2 **a** **i** $\frac{3}{8}$ **ii** $SU = \frac{1}{6}, SV = \frac{5}{24}, UV = \frac{1}{8}$ **iii** $\frac{3}{8}$

b $PQ = \frac{1}{7}$ or $\frac{3}{4}$

c $PQ = \frac{a}{b + c}$, so for 4 : 3 : 5, $PQ = \frac{1}{3}$

3 **a** **i** 48 cm^3

ii 48 cm, 60 cm, 64 cm

iii 88.78 cm^2, 118.10 cm^2, 153.21 cm^2

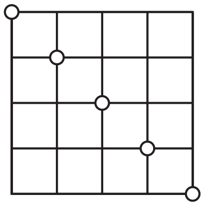
b It is enough to show the triangular faces have areas 15 555 cm^2, 23 925 cm^2 and 28 260 cm^2.

c Show that the perpendicular heights of the triangular faces are the integers c^3, $a(4b^2 + c^2)$ and $b(4a^2 + c^2)$.

4 **a** There are $2^4 = 16$ combinations of horizontal and vertical moves after 4 seconds. Four of these finish at P.

b $\frac{3}{16}$

c The counters can only meet when they are both halfway; that is, after 4 moves:



d $\frac{35}{128} \approx 0.273$

e Less likely, with probability $\frac{1120}{6561} \approx 0.171$

5 **a** **i** 9

ii week 19

b If there were k kirons, ℓ lirons and m mirons 5 weeks ago, then 1 week ago there were $\ell + m$ kirons and this week there are $k + \ell + m$ kirons. Yes, it is also true for lirons and mirons.

c 8 algae (4 kirons and 4 mirons)

d **i** week 13 **ii** 15 days

Glossary

AAA
the abbreviation for angle–angle–angle. Triangles are similar if all three angles are equal. The condition AAA does not necessarily mean the triangles are congruent

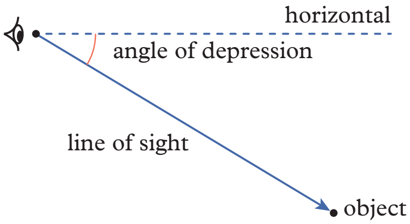
AAS
the abbreviation for angle–angle–side. Triangles are congruent if there are two pairs of corresponding equal angles and two corresponding sides are the same length

aim
the purpose of a statistical inquiry

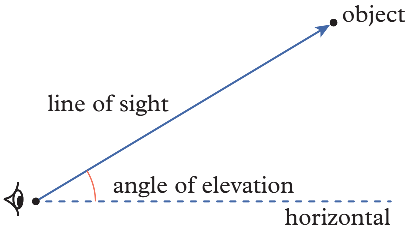
algebraic fractions
fractions that contain at least one pronumeral

amplitude
half the distance from the highest point of the curve to the lowest point of the curve in a trigonometric graph

angle of depression
the angle between a horizontal line and an observer's line of sight to an object that is below the horizontal



angle of elevation
the angle between a horizontal line and an observer's line of sight to an object that is above the horizontal



angle–side relationship
in a triangle, the longest side is opposite the largest angle

arc
part of a circle's circumference, which is also used to mark angles on diagrams

area
the amount of space enclosed by a two-dimensional (2D) shape

array
a collection of objects or numbers arranged in ordered rows and columns

asymptote
an imaginary line that a graph line approaches but never reaches

axis of symmetry
an imaginary line that divides a symmetrical shape or graph so that one side is a reflection of the other

base
a number that is repeatedly multiplied by itself; for example, in 2^4 the base is 2

base-ten
the traditional number system we use which uses the digits 0 to 9

bias
an inclination towards one person (group of people, things or ideas) over another

binomial
an algebraic expression for the sum or the difference of two terms

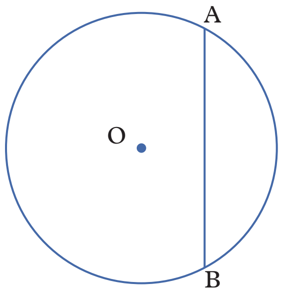
bivariate data
data consisting of two variables

bonus
an incentive or reward for good performance, which is given on top of a salary or wage

capacity
the maximum amount (usually of fluid or gas) that a three-dimensional object (container) can hold; usually measured in litres (L) or millilitres (mL)

causal relationship
a relationship between two variables where a change in one variable (the independent variable) directly causes a change in the other variable (the dependent variable)

chord
a line segment joining two points on the circumference of a circle. In the diagram below, AB is a chord



circuit
a trail in a graph/network with the same first and last vertices

circumference
the perimeter of a circle

closed walk
a walk in a graph/network with the same first and last vertices

coincident lines
two lines which lie on top of each other

commission
a payment which is a proportion of the total value of sales

compass bearing
direction describing one position from another by using the number of degrees from north or south towards east or west that the second position is from the first; for example, S43°E represents a direction that is 43° east of due south

complement
the complement of an event A is the event that A does not occur. The sum of the probabilities of an event and its complement is 1

complementary angles
angles that add to make a right angle (90°)

completing the square
the process of changing a quadratic expression from the general form, $ax^2 + bx + c$, to the form $a(x - h)^2 + k$, by applying the 'perfect square' rule

composite shape

a shape made up of more than one simple shape

compound event

an event with two or more favourable outcomes

compound interest

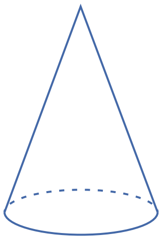
a type of interest whereby the amount of interest earned or charged is based on the present value of the loan or investment

conditional probability

the likelihood of an event occurring based on the outcome of a previous event

cone

a three-dimensional (3D) object with a curved surface tapering from its circular base to an apex



congruent

two figures are congruent if they are identical in shape and size, but not necessarily in position or orientation

conjugate

the conjugate of a binomial surd expression is formed by changing the operation sign between the two terms to the opposite operation sign (+ to – and – to +); for example, the conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$. When a binomial surd expression is multiplied by its conjugate, the product is a rational number

connected (graph)

a graph where a walk exists between every pair of vertices

constant of variation

if the relationship between two quantities x and y is $y = kx$ or $y = \frac{k}{x}$, k is the constant of variation (or proportion)

constant term

a term without any pronumeral component that has a fixed value

conversion factor

a multiplier for converting a quantity expressed in one set of units into another quantity expressed in a different set of units; for example, multiply by 10 to convert from cm to mm

convex polygon

a polygon in which all interior angles are less than 180°

correlation

the description of the strength (weak, moderate, strong) and direction (positive, negative) of the relationship between bivariate data displayed in a scatter plot

cosine (cos)

in trigonometry, the cosine of an angle in a right-angled triangle equals the ratio of the length of the adjacent side to the length of the hypotenuse

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

cosine rule

the relationship between three side lengths (for example, a, b and c) of a triangle ($\triangle ABC$) and one of the triangle's angles (A):
$a^2 = b^2 + c^2 - 2bc\cos(A)$.

Variations of this rule apply to other angles of the triangle:

$b^2 = a^2 + c^2 - 2ac\cos(B)$ and

$b^2 = a^2 + c^2 - 2ab\cos(C)$

coterminal angles

angles that differ by a multiple of 360°

cross-section

the shape of the surface revealed when we cut straight through an object. When the 'slice' is cut parallel to the base of a prism or cylinder, the cross-section has the same shape and size as that base

cubic equation

an equation that contains a pronumeral term or terms with a highest power of 3; for example $ax^3 = k$

cycle

a walk in a graph/network without repeated edges or repeated vertices and which starts and finishes at the same vertex

cylinder

a three-dimensional (3D) object with a uniform circular cross-section parallel to its circular base

definition

the minimum amount of information required to identify a plane shape

degree (face)

the number of edges along the border of a face in a graph/network

degree of a polynomial (*n*)

the highest power of any variable in a polynomial; for example, the polynomial $4x^3 - 2x^2 + 5x + 6$ has degree 3

degree (vertex)

the number of edges connected to a vertex in a graph/network

dependent variable

a variable whose value depends on the value of another variable in a relationship between two variables. The dependent variable is listed second in a table of values and in pairs of coordinates, and is shown on the vertical axis of the Cartesian plane

depreciating asset

an asset with a limited life expectancy that is expected to decline in value over the time it is used and which can be depreciated for a tax deduction

depreciation

the decline in the value of a depreciating asset over time

difference of two squares

when an expression is seen to be the difference of two perfect squares, $a^2 - b^2$, it can be factorised as:
$a^2 - b^2 = (a + b)(a - b)$

dilation
a transformation that can enlarge or compress the graph of a function in the x- or y-direction

diminishing-value depreciation
a type of depreciation whereby the depreciating asset is depreciated by a constant percentage each year

directed edge
an edge in a graph/network that has a direction indicated by an arrow

directed network
a network with directed edges

directly proportional
two variables are said to be directly proportional if $y = kx$, where k is the constant of variation. An increase in one quantity causes a corresponding increase in the other quantity, or a decrease in one quantity results in a corresponding decrease in the other quantity. Proportionality is denoted using the symbol $\propto$

discriminant
a value which is used in the quadratic formula to find the solution of quadratic equations

$\Delta = b^2 - 4ac$

distributive law
the result of first adding several numbers and then multiplying the sum by another number is the same as first multiplying each of the several numbers separately by that number and then adding the products: $a(b + c) = ab + ac$

dividend
the number or expression that is divided or distributed in a division problem

divisor
a number or expression that divides into another number or expression (dividend); for example, in $(4x^3 - 2x^2 + 5x + 6) \div 2x$, $2x$ is the divisor

domain
1 the complete set of possible values for the independent variable, shown on the horizontal axis of the Cartesian plane (the x- or θ-values), to produce a graph or define a rule

2 the set of input values of a function

edge (network)
a relation between two vertices in a graph/network

element
a member of a set; for example, 5 is an element of $\{1, 3, 5, 7, 9\}$

elimination method
the algebraic method used to solve simultaneous equations containing more than one variable by forming a new equation with all but one of the variables removed

ethics
the system of moral principles that distinguishes between right and wrong behaviour

equidistant
at equal distances from a point

Euler's formula
a formula that states that in a connected planar graph the sum of the number of faces and vertices is equal to two more than the number of edges; $f + v = e + 2$

Eulerian
a graph/network is Eulerian if it contains an Eulerian circuit

Eulerian circuit
a circuit containing every edge in a graph/network exactly once

Eulerian trail
a trail containing every edge in a graph/network exactly once

event
a subset of the sample space for a random experiment. It is one of the possible outcomes or sets of outcomes for the experiment; for example, for the experiment of rolling a die, a single-outcome event might be 'rolling a 2'; and an event that includes a set of outcomes might be 'rolling a 2 or a 4'

expansion of a perfect square
a pattern observable when expanding a factorised expression in which a factor is multiplied by itself, represented as:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + 2ab + b^2\end{aligned}$$

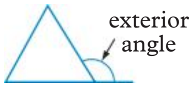
or: $$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - 2ab + b^2\end{aligned}$$

experiment
a repeatable procedure with a clearly defined set of possible results

exponent
see index

exponential relationship
a relationship between two variables, one of which is an index; for example, $y = 2^x$

exterior angle
an angle between any side of a shape and a line extended from the adjacent side



faces
the regions of a planar graph that are bordered by edges

function
a function f maps each element of one set S to precisely one element of a second set T

function notation
a way of expressing a relationship between two variables in the form $f(x)$ = [rule]. For example, $f(x) = x^2$ is the function which squares a number represented by the variable x

future value
the value of a loan or investment at a specified time in the future

general form
a form of the equation of a line; $ax + by + c = 0$

gradient (m)
the measure of the steepness of a straight line; or the numerical measure of the slope of a graph

$\text{gradient} = \frac{\text{rise}}{\text{run}}$ $m = \frac{y_2 - y_1}{x_2 - x_1}$

gradient–intercept form
a form of the equation of a line; $y = mx + c$, where m is the gradient and c is the y value of the y-intercept of the corresponding linear graph

graph
see network

handshaking lemma
propositions that state that in every graph/network the sum of vertex degrees is twice the total number of edges (edges version), and in every planar graph the sum of face degrees is twice the number of edges (faces version)

hypotenuse
the longest side in a right-angled triangle, appearing opposite the right angle

hypothesis
a proposed explanation that is being further investigated in a statistical inquiry

included angle area formula
the area of a triangle with two known side lengths b and c and an included angle A can be found using the formula

$\text{Area} = \frac{1}{2} bc \sin(A)$

income
money received over a set period of time in exchange for work or investment

income tax
the tax which is paid on an individual's taxable income, calculated by using a tax table or online tax calculator

independent events
two events in a probability experiment are considered independent when the outcome of one event does not affect the outcome of the other event

independent variable
the quantity in a relationship between variables which does not rely on any other variable for its value and is not affected by changes in another variable. The independent variable is the value listed first in a table of values and in pairs of coordinates, and the value shown along the horizontal axis on a Cartesian plane

index (plural: indices)
for a value expressed in index form, the index indicates the number of times the base is multiplied by itself as a repeated multiplication; for example, in 2^4 the index is 4 and $2^4 = 2 \times 2 \times 2 \times 2$

index form (index notation)
a shorter form of writing a repeated multiplication, in which a number is written with a base and an index; for example, 2^4 is written in index form and means $2 \times 2 \times 2 \times 2$

index laws
the set of laws that can be used to simplify calculations involving numbers in index form

index notation
see index form

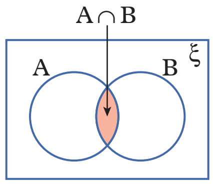
inference
conclusion drawn from evidence and reasoning

interest
a fee paid for an amount borrowed or invested, usually calculated as a percentage of the amount that has been borrowed or invested

interest rate
the rate, usually expressed as a percentage, at which interest is charged for a loan or paid for an investment

interior angle
an angle inside a plane shape

intersection
the intersection of two sets ($A \cap B$) is the set of elements that appear in both set A and set B



inversely proportional
if two variables are inversely proportional, then as one variable increases the other variable decreases at a proportional rate. The product of the two variables is constant

investment
money deposited into a bank, financial institution or other business for which interest is paid to the person depositing the money (the investor)

irrational number
a number that cannot be written in fraction form; irrational numbers are non-terminating decimals that are not recurring

leading coefficient
1. the coefficient of the squared variable in a quadratic trinomial; for example, in $ax^2 + bx + c$, a is the leading coefficient
2. the coefficient of the leading term in a polynomial; for example, the leading coefficient in the polynomial $4x^3 - 2x^2 + 5x + 6$ is 4

see also coefficient

see also leading term

leading term
the term with the highest power in a polynomial; for example, the leading term in the polynomial $4x^3 - 2x^2 + 5x + 6$ is $4x^3$

leave loading
an extra payment on top of annual leave that is usually calculated as 17.5% of normal pay for this period

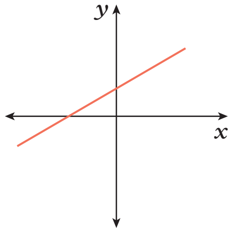
like surds
surd terms that have the same number or pronumeral/s under the root sign when in simplest form; for example, $2\sqrt{5}$, $-\sqrt{5}$ and $7\sqrt{5}$ are like surds

like terms
terms that contain the same pronumeral(s) raised to the same power(s)

line of best fit
a line that best represents the relationship between the two variables on a scatter plot. Lines of best fit are usually drawn as straight lines

linear equation
an equation that contains only a pronumeral term (or terms) for which the highest power is one; for example, $2x + 3 = 6$ is a linear equation

linear graph
the graph of a linear relationship. The graph is a straight line



linear inequality
an inequality containing pronumerals raised only to the power of 1 with no two pronumerals multiplied together; for example, $2x + 3 > 1$

linear relationship
a relationship between two variables that produces a linear graph

literal equation
an equation that contains mostly letters, such as $P = 2l + 2w$, where each variable represents a quantity 'literally'. Formulas are literal equations

loan
an amount of money lent by a bank or other institution to a person for a limited time (the term of the loan) for which the recipient (the borrower) must pay interest to the lender

logarithm (log)
the power (index) to which a given number (the base) must be raised in order to produce the required number; for example, in $2^3 = 8$, the logarithm is 3 and can be written as $\log_2(8) = 3$

logarithm of a power
the logarithm of a term that is expressed as a power. The related logarithm law is: $\log_a(x^n) = n \times \log_a(x)$

logarithm of a product
the logarithm of the product of two terms. The related logarithm law is: $\log_a(xy) = \log_a(x) + \log_a(y)$

logarithm of a quotient
the logarithm of the quotient of two terms. The related logarithm law is: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

logarithmic scale
a non-linear scale where the position of a value is marked using the logarithm of the value instead of the actual value

loop
an edge in a graph/network with the same vertex at both ends

maximum turning point
a point at which a parabola changes direction and at which it has its maximum y value

Medicare levy
a 2% tax on an individual's taxable income that helps to fund the health care system

midpoint
a point on a line segment or interval that divides the segment into two equal parts

minimum turning point
a point at which a parabola changes direction and at which it has its minimum y value

monic polynomial
a polynomial in which the leading coefficient is equal to one
see also leading coefficient

monic quadratic
a quadratic in which the leading coefficient is equal to one
see also leading coefficient

multiplicity
the number of times a factor is repeated in the factorisation of a polynomial into linear factors

mutually exclusive events
two or more events that cannot occur at the same time

negative index
an index with a negative value, indicating repeated division

negative relationship
(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable decreases in value

neighbours
the vertices on either side of an edge in a graph/network

net
a 2D plan that can be folded to form a 3D object

network
a collection of objects and the relations between them

non-linear relationship
a relationship between two variables where a change in one variable does not produce a uniform change in the other variable

non-monic polynomial
a polynomial in which the leading coefficient is not equal to one
see also leading coefficient

non-monic quadratic
a quadratic in which the leading coefficient is not equal to one
see also leading coefficient

non-planar graph
a graph/network that cannot be drawn in two dimensions without some edges crossing

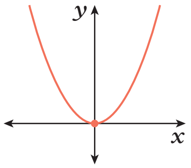
Null Factor Law
the law stating that, if the product of two factors is 0, one or both of the factors must be equal to 0

oblique pyramid
a pyramid with its apex (highest point) not directly above the centre of its base
see also pyramid

order of magnitude
the power of ten of a number in base ten

outcome
a possible result of a probability experiment

parabola
the approximately U-shaped graph of a quadratic relationship. A parabola can open up, down, left, right, or in any other direction, but is always symmetrical



parallel edges
two (or more) edges with the same endpoints

path
a walk in a graph/network without repeated edges or repeated vertices

penalty rates
higher rates of pay that are given to employees for working particular hours or days, such as weekends, public holidays or overtime

period
the horizontal distance covered by one full cycle of a periodic graph

periodic function
a function or relationship that repeats itself continuously in cycles;
for example, the sine, cosine and tangent functions are periodic functions

piece work
income which is paid for each item (or group of items) produced or action performed

planar graph
a graph/network that can be drawn without crossing edges

Platonic solid
a convex, three-dimensional (3D) shape where each face is an identical regular polygon

point-gradient form
a form of the equation of a line;
$y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) is a point on the line

point of inflection
point where a curve changes between being 'concave up' and 'concave down'

point of intersection
the point on a Cartesian plane where two graphs meet

polygon
a two-dimensional (2D) shape with straight edges

polynomial
an expression with terms containing one variable only. Each term in a polynomial contains the variable raised to a power that is a positive integer or zero; for example, $4x^3 - 2x^2 + 5x + 6$ is a polynomial $(4x^3 - 2x^2 + 5x + 6x^0)$

population
the complete set of individuals or things about which statistical information is being collected

positive relationship
(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable also increases in value

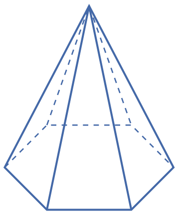
present value
the value of a loan or investment at the present time

principal
a sum of money invested, or an amount of money borrowed as a loan

prism
object with two ends that are identical polygons and joined by straight edges
see also polygon

property
a characteristic that applies to a plane shape

pyramid
a three-dimensional object whose base is a polygon that is joined to triangular faces, which all meet at a single point (the apex)



Pythagoras' theorem
the theorem stating that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two other sides:
$c^2 = a^2 + b^2$, where c is the length of the hypotenuse and a and b are lengths of the other two sides

quadrant
one of the four equal regions of a Cartesian plane indicated by the x- and y-axes and numbered anti-clockwise from the positive direction of the x-axis

quadratic equation
an equation that contains a pronumeral term or terms with a highest power of 2; the general form of a quadratic equation is $ax^2 + bx + c = 0$, for constants a, b and c with $a \neq 0$

quadratic formula
the formula used to solve equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. Solutions are found using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadratic trinomial
any expression of the form $ax^2 + bx + c$, where a, b and c are constants

quadrilateral
a polygon with four sides

quotient
the result of dividing one number or algebraic expression by another

range
1 the complete set of possible values for the dependent variable, shown on the vertical axis of the Cartesian plane (the y-values)
2 the set of output values of a function
3 a measure of spread determined by finding the difference between the highest value and lowest value in a numerical data set

rate of change
a rate that describes how one variable changes in relation to another variable

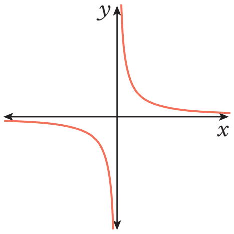
rational number
a number that can be written as a fraction with both the numerator and denominator being integers (whole numbers). The set of rational numbers includes whole numbers, fractions, terminating decimals and recurring decimals

real number
a number that can be represented by a point on a number line. Real numbers consist of all rational and irrational numbers

reciprocal
a value which, when multiplied by the given value, equals 1. For example, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$

rectangular hyperbola

an open curved graph line consisting of two branches whose ends continue to move apart from each other and begin to approximate, but never reach, two lines known as asymptotes; for example, the graph of $y = \frac{1}{x}$ is a hyperbola (see below)



reference angle

the acute angle between the x-axis and the line connecting the origin to the point on the unit circle

reflection

a transformation in which a shape or object is reflected in a mirror line to produce its exact mirror image (sometimes referred to as a ‘flip’)

regular polygon

a polygon with all sides equal in length and all interior angles equal in size

regular tetrahedron

a tetrahedron with four faces that are congruent equilateral triangles
see also tetrahedron

relation

a correspondence (map) between the elements of two sets; for example, ‘favourite colour’ maps from the set of students in a class (the domain) to the set of colours (the codomain)

remainder

the part ‘left over’ when dividing a number or expression into equal groups

RHS

1. abbreviation for right angle–hypotenuse–side. Right-angled triangles are congruent if their hypotenuses are the same length and another pair of side lengths are equal
2. abbreviation for right angle–hypotenuse–side. Right-angled triangles are similar if their hypotenuses are in the same ratio as another pair of sides

right cone

a cone with its apex (highest point) directly above the centre of the base
see cone

right pyramid

a pyramid with its apex (highest point) directly above the centre of the base
see pyramid

rise

the vertical distance between two points on a line

root

a solution to a polynomial equation

royalty

a payment made to a person or company for the use of their assets

run

the horizontal distance between two points on a line

salary

a fixed income which is paid to an employee on a regular basis

salvage value

the value of a depreciating asset after it has been depreciated

sample

a subset of a population. In statistical investigations a selection from a population is sometimes used to represent the whole population

sample space

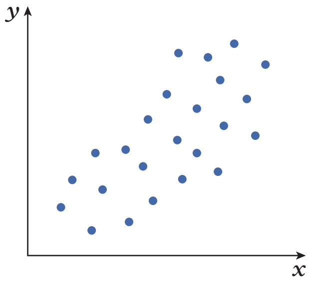
the list of all the different outcomes possible for a probability experiment; for example, the sample space when rolling a die is {1, 2, 3, 4, 5, 6}

SAS

1. abbreviation for side–angle–side. Triangles are congruent if two corresponding side lengths are equal and the angle between them is equal
2. abbreviation for side–angle–side. Triangles are similar if two corresponding side lengths are in the same ratio and angle between them is equal

scatter plot

a graph that displays a relationship between two variables in which the points are plotted but not joined



scientific notation

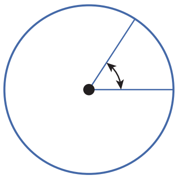
a value written in the form $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$ and m is an integer

secant

a line that cuts a circle twice

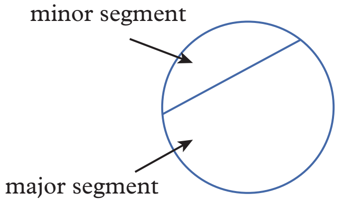
sector

a portion of a circle formed by two radii and part of the circumference



segment

the region enclosed in a circle by a chord and an arc
see also chord
see also arc



self-selected sampling

sample where respondents choose whether to take part in a survey when asked

semi-Eulerian

a graph is semi-Eulerian if it contains an Eulerian trail

set

a collection of distinct objects or elements
see also element

similar
two figures or objects are similar if they have exactly the same shape. Similar shapes do not need to be the same size

simple graph
a graph/network without loops or parallel edges

simple interest
a type of interest whereby the amount of interest earned or charged is based on the principal of the investment or loan

simultaneous linear equations
equations involving two or more unknowns that have the same values in those equations

sine (sin)
in trigonometry, the sine of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the hypotenuse

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

sine rule
the relationship between the side lengths (for example, a, b and c) and angles (A, B and C) of a triangle ($\triangle ABC$):

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

sphere
a three-dimensional (3D) object shaped like a ball where every point on its surface is equidistant from its centre

SSS
1 abbreviation for side–side–side. Triangles are congruent if all three side lengths are equal

2 abbreviation for side–side–side. Triangles are similar if all three side lengths are in the same ratio

straight-line depreciation
a type of depreciation whereby the depreciating asset is depreciated by a fixed amount each year

stratified sampling
a sample that divides the population into small groups known as strata and takes a random sample from each stratum

subject of an equation
the value that appears by itself on one side of an equation

substitution
replacement of a pronumeral with a number

substitution method
the algebraic method used to solve simultaneous equations containing more than one variable, by rearranging to make one of the variables the subject of an equation equivalent to the first equation and substituting the value of that variable from that equation into the second equation, resulting in an equation that has only one variable. This equation can then be solved and its solution substituted back into the first equation so it too can be solved

supplementary angles
angles that add to make a straight angle (180°)

surd
an irrational root of an integer that cannot be simplified to remove the root; for example, $\sqrt{2}$

surface area (SA)
the area of the surface of an object. The surface area of a prism is the sum of the areas of all the faces of that prism

survey
a list of questions designed to extract information about a population

systematic sampling
a sample that selects data at fixed intervals

tangent
1 the tangent (tan) of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the adjacent side

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

2 a straight line that touches but does not cross a curve

3 a line that touches the circumference of a circle at only one point

tax deductions
expenses that can be subtracted from a person's total income such as work-related expenses or gifts and donations

taxable income
an individual's total income minus any tax deductions, which is used to calculate the amount of income tax payable

test
a theorem that demonstrates a particular quality

tetrahedron
a pyramid with a triangular base
see pyramid

theorem
a statement or result that can be demonstrated to be true based on known facts

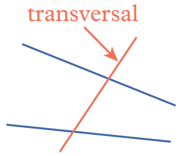
time period
the regular interval of time at which interest is calculated and added to the present value

trail
a walk in a graph/network without repeated edges

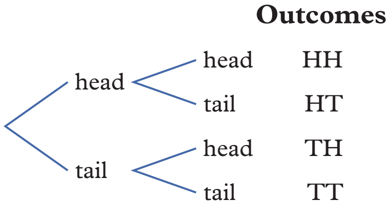
transformations
the general name for translations, reflections, rotations and dilations

translation
a transformation in which a shape, graph or object is moved in a straight line without turning or changing size. The movement is often described as a number of units up, down, left or right

transversal
a line that crosses (intersects) two or more lines



tree diagram
a display of outcomes for a multi-step experiment; for example, a tree diagram can be used to display the possible outcomes of flipping two coins (as shown below)



triangle

a polygon with three sides

triangle inequality

the sum of the lengths of two sides of a triangle is greater than the length of the third side

trigonometry

the study of the relationships between the angles and side lengths of a triangle

trinomial

an expression containing three terms

true bearing

the direction from one position to another, described by a number of degrees from north in a clockwise direction. The angle of a true bearing is written as three digits followed by T (or true); for example, 090°T is due east, 180°T is due south, and 270°T is due west

turning point

the point on a graph where a parabola changes direction

turning point form

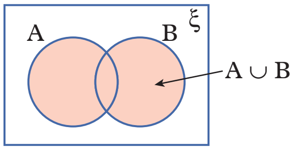
the general form of a quadratic equation, $y = a(x - h)^2 + k$, where a is the dilation factor, and the coordinates of the turning point are (h, k)

two-way table

a display of the outcomes of a survey or experiment that shows the relationship between two attributes

union

the union of two sets $(A \cup B)$ is the set of elements that appear in A or B or both



unit circle

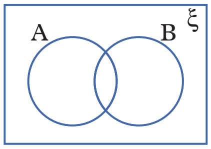
a circle drawn on the Cartesian plane with its centre at the origin and a radius of 1 unit. This circle is used to define trigonometric functions

universal set

the set containing all elements. Everything inside the rectangle of a Venn diagram belongs to the universal set. The symbol for the universal set is ξ

Venn diagram

a display of the relationship between different sets of data. A rectangle represents the universal set, ξ, and each set of data within the universal set is represented by a circle



vertex

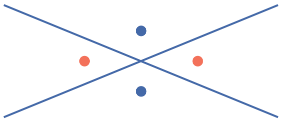
see turning point

vertical line test

a test that determines whether the graph of a relation is also a function. If a vertical line intersects or touches a graph at more than one point, then the graph is not a function

vertically opposite angles

a pair of equal angles that are not adjacent to (next to) each other, formed where two lines intersect. They are sometimes called ‘X angles’. There are two pairs of vertically opposite angles in the diagram below



vertices (network)

the objects in a network, represented by single points

volume

the quantity of space that a three-dimensional (3D) object occupies

wage

a fixed rate of income an employee receives for working a given time period; for example, $25 per hour

walk

a sequence of vertices and the edges between them in a graph/network

zero (of a polynomial)

a value a such that, for a polynomial $P(x)$, $P(a) = 0$

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