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MASTERING HSC MATHEMATICS

YEAR 12 EXTENSION 2 MATHEMATICS

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

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Errors:

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Features of this book

This book is suitable for all students studying the HSC Mathematics courses. It has been designed in a thoroughly organised manner to help students master each syllabus topic in the new Stage 6 HSC Mathematics Extension 2 course. This book will teach, consolidate, test and challenge students. It is an essential resource for all students and teachers.

In flavour with the new course, this book has the following features:

- Interpretation questions.
- Modelling and application problems.
- Verification questions.

Within each chapter, there are subsections divided as follows.

Fundamentals

The carefully constructed *fundamentals* section appears before the main body of questions. The purpose of this section is to

- test all key formulae, definitions, concepts and theory.
- test essential mathematical terms and language through cloze-passages.
- ensure that the student has knowledge of the essential prerequisites.
- provide a summary of basic requirements for the topic.

Questions

This is the main body of questions with the following features.

- Step-by-step questions to assist the student with more difficult problems.
- Carefully graded exercises.
- "Show"-type questions, both guides the student, and offers good exam preparation.
- Proofs and explanations to strengthen understanding and develop problem-solving skills.
- Application questions to demonstrate future uses of learned theory.
- Technology-based questions to teach and reinforce concepts.

Challenge

These are more difficult questions that provide

- a challenge for students wishing to test their mastery of the topic.
- rigour and higher-order thinking skills.
- extension and more in-depth treatment of the unit of work.

Chapter Review

This section appears at the end of every chapter, and offers the following.

- Revision and consolidation of the previous exercises.
- Questions that require a combination of ideas from previous exercises.

Investigations

These tasks are potential assignments and research projects. Teachers may use and adapt these to cover the new NESA requirements on investigative assessment tasks. This section provides for the student

- application and modelling scenarios.
- research tasks involving data collection and analysis.
- scaffolding of learning tasks.
- open-ended style problems for discussions.
- opportunity to use appropriate technology effectively in a range of contexts.
- opportunity for students to demonstrate critical thinking.

Answers

- Quick answers to questions.
- "Show" and "prove" answers can be found in the full worked solutions.

Full worked Solutions

- Can be found online for free, or a full-colour hard copy purchased for convenience.
- Provide complete worked solutions to all questions, except investigative tasks to maintain the open-ended nature of the tasks.
- Includes several alternative solutions to problems, where possible.

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CHAPTER 1

Exercise 1A

Language of Proof

Fundamentals

Fundamentals 1

(a) Expressions like 'The cat is black' or 'It will rain today' are called p__________. They are denoted by p and q.

(b) If we have

$$p \Rightarrow q$$

then it means that

'if p, then q'.

This is called a logical i__________ or a c__________ statement.

(c) The c________ of the statement is that

$$q \Rightarrow \underline{\quad}.$$

(d) If both directions are true, then we say that the statement is an 'iff' statement, which is short for __________________________.

Fundamentals 2

(a) The negation of a statement is the 'o________' of the statement.

(b) Complete the following negation.

Statement: All cats are cute.
Negation: ______ all cats are cute.

(c) The negation of a proposition p is denoted by ______.

Fundamentals 3

You can also have a negation of a conditional statement. The negation of a conditional statement is the directly contradictory statement that 'proves it wrong'.

Statement: If I study, I will do well in my exam.
Negation: I studied, but __________________________.

The negation of a conditional statement is important because that's how we disprove statements.

Question 1 Let p be the proposition

'Bob studies for his exams'

and q be the proposition

'Bob will get 99.95 ATAR'.

Write down the following in English. Whether or not the statements are true is a different matter!

(a) $p \to q$
(b) $q \to p$
(c) $\sim p \to q$
(d) $q \to \sim p$
(e) $\sim (p \to q)$
(f) $\sim p \to \sim q$

Question 2 Let p be the proposition

'It is cloudy'

and q be the proposition

'It will rain today'.

Write down the following in terms of p and q.

(a) If it is cloudy, then it will rain today.
(b) If it rains today, then it was cloudy.
(c) It was cloudy and it didn't rain.
(d) It didn't rain and it was cloudy.
(e) It wasn't cloudy and it didn't rain.
(f) If it doesn't rain today, then it was not cloudy.

Question 3 Write down the negation of the following statements.

(a) The cat is black.
(b) If it is a cat, it has a tail.
(c) If it doesn't have four legs, it is not cat.
(d) If I study, I will do well in my exam.

Question 4 [Negation of 'and' and 'or']

Write down the negation of the following statements.

(a) Bob likes swimming and running.
(b) Bob likes swimming or running.

Question 5 Write down the negation of the following statements.

(a) $a < b$
(b) $x \in [a, b]$
(c) $\forall n \in \mathbb{Z}$, $2n + 1$ is odd.
(d) 5 divides $2n + 1$ for some values of n.
(e) $\forall m \in \mathbb{Z}$, $\exists n \geq m$ such that $n = m^2$.
(f) $\exists x \in \mathbb{R}$ such that $\sin x = 0$.

Question 6 [Contrapositive]

Let p be the statement

'I am in Sydney'

and let q be the statement

'I am in NSW'.

(a) Write down $p \to q$.

(b) Write down $\sim q \to \sim p$.

(c) Comment on the truthfulness of the following.

$$(p \to q) \iff (\sim q \to \sim p)$$

Question 7 [Distinguishing 'negation' and 'converse']

Explain the difference between the *negation* of a statement $p \to q$ and the *converse* of a statement.

Question 8 Write down the following 'iff' statements as two separate 'if' statements.

(a) A number is divisible by 6 iff it is divisible by both 2 and 3.

(b) A quadratic has two real roots iff $\Delta > 0$.

(c) A polynomial $P(x)$ has $(x - a)$ being a factor iff $x = a$ is a zero.

Question 9 [Involving numbers]

For each of the following, write down the *converse* and state whether the converse is necessarily true.

(a) If a and b are odd, then $a + b$ is even.

(b) If a is odd, then a^2 is odd.

(c) If $a > b$ then $a^2 > b^2$.

(d) If $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$.

(e) If a and b are even, then ab is even.

(f) If a number ends with the digit 5, it is odd.

Question 10 [Geometry]

For each of the following, write down the *converse* and state whether the converse is necessarily true.

(a) If it is a rhombus, then it has perpendicular diagonals.

(b) If exactly two angles of a triangle are equal, then exactly two sides of the triangle will be equal.

(c) If the sides of a triangle are a, b, c and $a^2 + b^2 = c^2$, then the triangle is right-angled.

Question 11 [Graphs]

For each of the following, write down the *converse* and state whether the converse is necessarily true.

(a) If $f(x)$ has a vertical asymptote at $x = a$, then $f(a)$ does not exist.

(b) If $f(x)$ has a horizontal asymptote at $y = b$, then $f(x) = b$ has no solutions.

(c) If there is a point of inflexion at $x = a$, then $f''(a) = 0$.

(d) If $f'(a) = 0$, then there is a stationary point at $x = a$.

Question 12 Write down the following sentences using formal logic notation.

(a) For every complex value of z, there exists some complex conjugate $\bar{z}$ such that $z + \bar{z}$ is real.

(b) For every pair of distinct integers a, b, there exists some rational number r between a and b.

(c) For every real r, there exists a rational q that is at most d units away, where d is rational.

Question 13 Write down the following in English.

(a) $\forall m > n, \exists\, a \in \mathbb{R}$ such that $m - a = n$.

(b) If $P(x) = c_n x^n + \cdots + c_1 x + c_0$, where $c_k \in \mathbb{Z}$, and $\exists\, a \in \mathbb{Z}$ such that $P(a) = 0$, then $a \mid c_0$.

(c) If $n \neq k^2$, $k \in \mathbb{Z}$, then $\sqrt{n} \notin \mathbb{Q}$.

(d) If $z \in \mathbb{C}$, $z \neq 0$, then $\exists\, w \in \mathbb{C}$ such that $zw = 1$.

(e) If $a \in \mathbb{R}^+$, then $a + \dfrac{1}{a} \geq 2$.

Note from the author: Although it's nice to be able to read a string of logical symbols with fluency as though it's another language, keep in mind 'clarity before formality'. Sometimes, it's okay to forgo some mathematical precision for the sake of clarity. Try not to overuse notation for the sake of it.

Challenge Problems

Problem 1 Determine if the following are 'iff' statements.

(a) If α and β are the zeroes of a quadratic $P(x)$ and $\alpha^2 + \beta^2 > 0$, then the zeroes are real.

(b) If a and b are both irrational, then $a + b$ is irrational.

(c) If a and b are both irrational, then ab is irrational.

Problem 2 Suppose that $f(x)$ is a function that is continuous at $x = a$. Then $\lim_{x\to a} f(x) = L$ if $\forall\epsilon > 0$, $\exists\,\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for any $0 < |x - a| < \delta$. What does this mean in English?

Problem 3 Let $c_k \in \mathbb{R}$, where $k = 0, 1, 2, \cdots, n$. Define the polynomial

$$P(x) = \sum_{k=0}^{n} c_k x^k.$$

If $P(\alpha) = 0$, where $\alpha \in \mathbb{C}\backslash\mathbb{R}$, then $\exists\,\bar{\alpha}$ s.t $P(\bar{\alpha}) = 0$. What does this mean in English?

Exercise 1B

Direct Proof

Fundamentals

Fundamentals 1
Describe the general steps to prove that an integer n is

(a) even. (b) odd. (c) divisible by a.

Question 1 Prove that

(a) if m and n are even, then mn is even.
(b) if m and n are odd, then mn is odd.
(c) if m and n are even, then $m+n$ is even.
(d) if m and n are odd, then $m+n$ is even.

Question 2 Prove that

(a) if n is even, then n^2 is even.
(b) if n is even, then n^2+1 is odd.
(c) if n is odd, then n^2 is odd.
(d) if n is odd, then n^2+1 is even.

Question 3 Prove that if m is even and n is odd, then

(a) $m+n$ is odd.
(b) $m-n$ is odd.

Question 4

(a) Prove that n^2+n is always even.
(b) Prove that n^3-n is divisible by 6.
(c) Prove that n^2 is even if and only if n is even.
(d) Prove that n is divisible by 6 if and only if n is divisible by 2 and 3.

Question 5 Let n be odd. Prove that n^2 has a remainder of 1 when divided by 8.

Question 6 Prove that the sum of three consecutive integers is always divisible by 3.

Question 7 Prove that the sum of two consecutive positive powers of 4 is always a multiple of 20.

Question 8 Prove that $\frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6}$ is an integer $\forall n \in \mathbb{Z}$.

Question 9 Prove that every odd integer is the sum of two consecutive integers.

Question 10 [Application of binomial expansions]

Prove that $9^n - 1$ is divisible by 8.

Question 11 Prove that the difference between the squares of any two consecutive odd integers is always divisible by 8.

Question 12 Prove that if the sum of the digits of a 3-digit number is divisible by 3, then the number itself is divisible by 3.

Question 13 Prove that every odd integer is the difference between two consecutive perfect squares.

Challenge Problems

Problem 1 [Modular arithmetic properties]

(a) Prove that if a has a remainder of b when it is divided by n, then a^2 and b^2 will have the same remainder when they are divided by n.

(b) Prove that if a has a remainder of b when it is divided by n, then ac and bc will have the same remainder when they are divided by n.

Problem 2 Prove that $\forall n \in \mathbb{Z}^+$, $n \geq 3$, $\exists p$ prime such that $n < p < n!$

Problem 3 Prove that a number is divisible by 8 if and only if the last three digits themselves form a number that is divisible by 8.

Problem 4 Let p be a prime number and let q be some positive integer. Find the smallest value of q such that $p + q$ is never prime.

Problem 5 [Application of the Sophie Germain Identity]

(a) Show that $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.

(b) Hence, show that if $n > 1$, then $n^4 + 4^n$ is composite.

Problem 6 [Trivial proof]

Prove that no three positive integers a, b and c satisfy the equation

$$a^n + b^n = c^n$$

for any integer value of $n \geq 3$.

Exercise 1C

Contrapositive

Fundamentals

Fundamentals 1

Consider the statement $p \to q$.

(a) The contrapositive is of 'if p then q is

$$\text{If } \textbf{not } q\text{, then } \textbf{not } p$$

or using symbols, expressed as $\sim q \to$ ______ .

(b) The contrapositive is e__________ to the original statement, so

$$(p \to q) \iff (\sim q \to \sim p)$$

Fundamentals 2

(a) Sometimes, it is easier to prove statements indirectly by instead proving their c__________ .

(b) This is called proof by c____________ .

Question 1 Write down the contrapositive of each of the following statements.

(a) If I can see you, then you can see me.

(b) If it is raining, then Bob will drive to work.

(c) If Bob attends, then Mary will also attend.

(d) If I miss my train, I will be late to school.

(e) If it is cloudy, it will rain.

(f) If I have a cat, it has four legs.

(g) If Bob studies for his exam, he will do well.

(h) If you are in Sydney, you are in NSW.

Question 2 Write down the contrapositive of each of the following statements.

(a) If a number ends with the digit 5, it is odd.

(b) If a and b are odd, then $a + b$ is even.

(c) If a is positive, then $a + \dfrac{1}{a}$ is at least 2.

(d) If a and b are integers, then ab is an integer.

(e) If a quadrilateral is a square, it is also a rhombus.

(f) If all angles are equal in a triangle, then it has three equal sides.

Question 3 [Introduction to proof by contraposition]

Sometimes, we can prove statements indirectly by instead proving the contraposition of the statement.

(a) Prove that if n is even, then n^2 is even.

(b) Hence, it follows that if n^2 is o______, then n is ______.

Question 4 Prove the following statements indirectly by proving instead the contrapositive.

(a) If $3n + 7$ is even, then n is odd.

(b) If $n^2 - 6n + 5$ is even, then n is odd.

(c) If mn is even, then m or n are even.

(d) If mn is odd, then m and n are odd.

(e) If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

(f) If $n^3 - 1$ is even, then n is odd.

(g) If n is odd, then 8 divides $n^2 - 1$.

(h) If $n^2 + 1$ is even, then n is odd.

(i) If $n^2 - 2n + 7$ is even, then n is odd.

(j) If $n^2 - 6n + 5$ is even, then n is odd.

Question 5 Prove the following statements indirectly by instead proving the contrapositive.

(a) If $n^2 + 5n < 0$, then $n < 0$.

(b) If $a^3 + ab^2 \geq a^2b + b^3$, then $a \geq b$.

Question 6 Prove that if n^2 is divisible by 3, then n is divisible by 3.

Question 7 Prove that if ab is not divisible by 5, then neither a nor b are divisible by 5.

Question 8 Prove that n^2 is divisible by 5 if and only if n is divisible by 5.

Question 9 Prove that if $a^2 + b^2$ is even, then $a + b$ is even.

Challenge Problems

Problem 1 Let $a, b \in \mathbb{Z}$. Prove that if $a + b > 20$, then either $a > 10$ or $b > 10$.

Problem 2 Prove that if $2^n - 1$ is prime, then n is prime.

Exercise 1D

Proof by Contradiction

Fundamentals

Fundamentals 1

(a) Proof by contradiction is another technique to determine the truth value of a proposition p. We do this by first considering the n________ of p, denoted by $\sim p$.

(b) We prove that $\sim p$ is f______ by obtaining a contradiction.

(c) If $\sim p$ is false, then that means p is t______.

Question 1 Complete the following proof that $\sqrt{2}$ is irrational.

(a) Suppose that $\sqrt{2} = \dfrac{p}{q}$, where $p, q \in$ ___.

(b) Also, suppose that p and q have no common f______ so that $\dfrac{p}{q}$ cannot be further simplified.

(c) Squaring both sides, $p^2 =$ ___ so p^2 is e______, which implies p is also e______.

(d) Let $p = 2n$ for some $n \in$ ___.

(e) Hence, $q^2 =$ ___ so q^2 is e______, which implies q is also e______.

(f) Therefore, both p and q are e______ integers, which contradicts the statement in part ___.

(g) Hence, $\sqrt{2} \neq \dfrac{p}{q}$ for any $p, q \in$ ___, and therefore $\sqrt{2}$ is irrational.

Question 2 Use a similar proof to the previous question to prove that the following numbers are irrational.

(a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) $\sqrt{6}$ (d) $\sqrt{8}$

Question 3 Let p be any prime number. Prove that $\sqrt{p}$ is irrational.

Question 4 Let x be any irrational number. Prove that $\sqrt{x}$ is also irrational.

Question 5 Complete the following proof that $\log_2 3$ is irrational.

(a) Suppose that $\log_2 3 = \dfrac{p}{q}$, where $p, q \in$ ___.

(b) So $2^{\frac{p}{q}} =$ ___ and hence

$$2^p = \_\_\_$$

(c) The left-hand side is ___ whereas the right-hand side is ___, which is a contradiction. Hence, $\log_2 3$ is irrational.

Question 6 [Irrational proofs and logarithms]

Use a similar proof to the previous question to prove that the following are irrational.

(a) $\log_2 3$ (b) $\log_3 5$ (c) $\log_4 6$ (d) $\log_4 12$

Question 7 Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Question 8 Prove that there are no integers a and b such that $3a + 12b = 145$.

Question 9 Prove that $\sqrt[3]{2}$ is irrational.

Question 10 Prove that $n^2 - 3$ will never be divisible by 4.

Question 11 Let $a, b \in \mathbb{R}$. Prove that if $a + b$ is irrational, then at least one of a, b is irrational.

Challenge Problems

Problem 1 [Alternative way of stating an irrationality problem]

Prove that $4n + 3$ is never a square number for any $n \in \mathbb{Z}$.

Problem 2 Prove that if a and b are odd, then $a^2 + b^2$ is never a square number.

Problem 3 Define the polynomial

$$P(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Prove that $P(x) = 0$ cannot have any double-roots.

Problem 4 Define the polynomial

$$P(x) = x^3 + px + q$$

where $p,\ q \neq 0$. Prove that $P(x) = 0$ cannot have any triple-roots.

Problem 5 Let a be rational and let $\sqrt{b}$ be irrational. Prove that $a\sqrt{b}$ is irrational.

Problem 6 Prove that there are infinitely many prime numbers.

Exercise 1E

Examples and Counter-examples

Fundamentals

Fundamentals 1

A conditional statement has structure

$$p \to q$$

A counter-example to the statement is an element that satisfies p, but q does not follow. It is in other words the n________ of the conditional statement.

Fundamentals 2

A statement claims there exists some n such that a statement is true. How would you disprove the statement?

Fundamentals 3

A statement claims that a statement is true for all $n \in \mathcal{S}$, where $\mathcal{S}$ represents some set of numbers. How would you disprove the statement?

Question 1 Find a counter-example to show that the following statements are not always true, and give a possible modification to correct them.

(a) $\forall x \in \mathbb{R},\ 3x > 2x$

(b) $\forall x \in \mathbb{R},\ 2x \neq 2^x$

(c) $\forall x \in \mathbb{R},\ 3x < 2^x$

(d) $\forall x \in \mathbb{R},\ ax = bx \Rightarrow a = b$

(e) $\forall x \in \mathbb{R},\ \sqrt{x^2} = x$

(f) $\forall x \in \mathbb{R},\ \sqrt{1 - \sin^2 x} = \cos x$

(g) $\forall x, y \in \mathbb{R},\ |x| + |y| \leq |x + y|$

(h) $\forall x \in \mathbb{R},\ e^{\ln x} = x$

(i) $\forall x \in \mathbb{R},\ \sin\left(\sin^{-1} x\right) = x$

(j) $\forall x \in \mathbb{R},\ \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Question 2 Prove that the following statements are false by providing a counter-example.

(a) $\forall n \in \mathbb{Z}^+$, $7n + 4$ is prime.

(b) $\forall n \in \mathbb{Z}$, $n^2 + n + 1$ is prime.

(c) $\forall n \in \mathbb{Z}^+$, $n^2 + n + 41$ is prime.

(d) $\forall n \in \mathbb{Z}^+$, $n^2 + 1$ is never divisible by 25.

Question 3 Prove that the following statements are false by providing a counter-example.

(a) If $x^2 > 0$, then $x > 0$.

(b) If a and b are factors of n, then ab is also a factor of n.

(c) If n^2 is divisible by 4, then n is divisible by 4.

Question 4 Prove that the following statements are false by providing a counter-example.

(a) If x and y are non-zero integers, then $\frac{x}{y}$ is an integer.

(b) If x and y are irrational, then xy is irrational.

Question 5 Let $f(x)$ be any continuous smooth function with domain $x \in \mathbb{R}$. Prove that the following statements are false by providing a counter-example.

(a) $\forall a \in \mathbb{R}$, $f'(a) = 0$ implies a turning point at $x = a$.

(b) $\forall a \in \mathbb{R}$, $f''(a) = 0$ implies a horizontal point of inflection at $x = a$.

Question 6 Determine whether the following statements are true or false.

(a) If a cubic polynomial has roots α, β, γ, and $\alpha^2 + \beta^2 + \gamma^2 < 0$, then it has non-real roots.

(b) If m is a perfect square, then $\sqrt{mn}$ is irrational.

(c) Every odd number is either one more or one less than some multiple of 4.

(d) If a and b divide n, then ab divides n.

(e) If $f'(x) > 0$ and $f(0) = 0$, then $f(x) \geq 0$ for all $x > 0$.

Question 7 Determine if the following are 'iff' statements. If not, provide a counter-example.

(a) If a and b are odd, then $a + b$ is even.

(b) If a is odd, then a^2 is odd.

(c) If a and b are even, then ab is even.

(d) If a number ends with the digit 5, it is odd.

Question 8 [Polynomials]

Let $P(x), Q(x)$ be polynomials. Disprove the following statements by providing a counter-example.

(a) The sum of two quadratics is always a quadratic.

(b) If $P(x)$ is divisible by $(x - a)$ and $(x - b)$, then it is also divisible by $(x - a)(x - b)$.

(c) If $P(x)$ has an integer constant term and an integer root α, then α divides the constant term.

Question 9 [Geometry]

Determine if the following are 'iff' statements. If not, provide a counter-example.

(a) If a quadrilateral has perpendicular diagonals, then it is a rhombus.

(b) If two angles of a triangle are equal, then two sides of the triangle will be equal.

(c) If the sides of a triangle are a, b, c and $a^2 + b^2 = c^2$, then the triangle is right-angled.

Question 10 [Involving graphs]

Determine if the following are 'iff' statements. If not, provide a counter-example.

(a) If $f(x)$ has a vertical asymptote at $x = a$, then $f'(a)$ does not exist.

(b) If $f(x)$ has a horizontal asymptote at $y = b$, then $f(x) = b$ has no solutions.

(c) If $y'' = 0$ at $x = a$, then there is a point of inflexion at $x = a$.

(d) If $y' = 0$ at $x = a$, then there is a stationary point at $x = a$.

Question 11 Consider the following claim.

If n divides $(a + b)$ and n divides $(b + c)$, then n divides $(a + c)$.

Prove or disprove the claim.

Question 12 Consider the following claim.

Let n be a positive integer that is at least 2 digits long and divisible by 4. Let r be the remainder when n is divided by 100. Prove that r is also divisible by 4.

Prove or disprove the claim.

Challenge Problems

Problem 1 [Complex numbers]

State whether the following statements are true. If they are not true, provide a counter-example.

(a) If the roots of a quadratic $P(x)$ are α and β, and $\alpha^2 + \beta^2$ is negative, then $P(x)$ must have non-real roots.

(b) If a quadratic $P(x)$ has non-real roots, then $\alpha^2 + \beta^2$ will be negative.

Problem 2 [Further functions from Year 11 Extension 1]

Let $f(x)$ be any continuous smooth function with domain $x \in \mathbb{R}$. State whether the following statements are true. If they are not true, provide a counter-example.

(a) If $y = f(x)$ has a root at $x = a$, then $y = \sqrt{f(x)}$ has a vertical tangent at $x = a$.

(b) If $y = f(x)$ has a turning point at $x = a$, then $y = \dfrac{1}{f(x)}$ also has a turning point at $x = a$.

Problem 3 [Integration]

State whether the following statements are true. If they are not true, provide a counter-example.

(a) If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

(b) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for $x \in [a, b]$.

(c) If n is any integer, then $\int x^n\,dx = \dfrac{1}{n+1}x^{n+1} + C$.

Exercise 1F

Algebraic Inequalities

Fundamentals

Fundamentals 1

(a) If $a, b > 0$, then $a + b$ ___ 0.

(b) If $a, b > 0$, then ab ___ 0.

Fundamentals 2

(a) If $a \geq b$, then $a + c$ ___ $b + c$.

(b) If $a \leq b$, then $a + c$ ___ $b + c$.

(c) If $a \geq b$ and $c > 0$, then ac ___ bc.

(d) If $a \geq b$ and $c < 0$, then ac ___ bc.

Fundamentals 3

(a) If $a \geq b$ and $b \geq c$, then a ___ c.

(b) If $a \geq b$ and $c \geq d$, then $a + c$ ___ $b + d$.

(c) If $a \geq b > 0$ and $c \geq d > 0$, then ac ___ bd.

Fundamentals 4

If a and b are real, then $(a - b)^2$ ___ 0.

Fundamentals 5

If a and b are p______, then $\dfrac{a+b}{2} \geq \sqrt{ab}$.

Fundamentals 6

If x and y are real, then $|x| + |y| \geq |$______$|$.

Question 1 [Drill]

Prove the following statements for $a, b > 0$ by considering

$$(x - y)^2 \geq 0$$

for appropriate choices of x and y.

(a) $a^2 + b^2 \geq 2ab$

(b) $a^4 + b^4 \geq 2a^2b^2$

(c) $a + b \geq 2\sqrt{ab}$

(d) $a + \dfrac{1}{a} \geq 2$

(e) $\dfrac{1}{a^2} + \dfrac{1}{b^2} \geq \dfrac{2}{ab}$

(f) $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$

Question 2 [Demonstrating different proof techniques]

Consider the inequality

$$(a+b)^2 \geq 4ab$$

which is true for all $a, b \in \mathbb{R}$. Prove the inequality by

(a) starting from the left-hand side and working towards the right-hand side.

(b) investigating $(a+b)^2 - 4ab$ and proving that it is positive.

(c) assuming that $(a+b)^2 < 4ab$ and using a proof by contradiction.

Question 3 Suppose that $x, y > 1$. Prove that $x^2 + y^2 \geq x + y$.

Question 4 Let $a, b > 0$. Prove the following inequalities.

(a) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$

(b) $(a+b)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq 8$

Question 5 Let $a > b > 0$. Prove the following inequalities.

(a) $a^3 - b^3 \geq a^2b - ab^2$

(b) $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$

(c) $\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{8}{(a+b)^2}$

(d) $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$

Question 6 Prove that if $a^2 + b^2 = x^2 + y^2 = 1$, then $ax + by \leq 1$.

Question 7 Prove that if $a + b = 1$, then

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

Question 8 Prove that $(ab+cd)(ac+bd) \geq 4abcd$.

Question 9 Let $a, b > 0$.

(a) Show that $a^3 + b^3 \geq \frac{(a+b)^3}{4}$.

(b) Hence, show that $\sqrt[3]{\frac{a^3+b^3}{2}} \geq \frac{a+b}{2}$.

Question 10 [Three important means]

Prove the following inequalities for $0 < x < y$.

(a) $x < \frac{x+y}{2} < y$

(b) $x < \sqrt{xy} < y$

(c) $x < \frac{2}{\frac{1}{x}+\frac{1}{y}} < y$

Question 11 [HM/AM/GM-inequality]

Define the three expressions

$$A = \frac{x+y}{2}, \qquad G = \sqrt{xy}, \qquad H = \frac{2xy}{x+y}$$

Prove that $H \le G \le A$.

Question 12 Prove that

$$\sqrt{xy} \le \frac{x+y}{2} \le \sqrt{\frac{x^2+y^2}{2}}$$

for $x, y > 0$.

Question 13

(a) Prove that $(a^2 - b^2)(c^2 - d^2) \le (ac - bd)^2$.

(b) Hence, prove that $(a^2 - b^2)(a^4 - b^4) \le (a^3 - b^3)^2$.

Question 14 [Classic inequality and application]

Let $a, b, c > 0$.

(a) Prove that $a^2 + b^2 \ge 2ab$.

(b) Write down two more similar inequalities involving a, b and c.

(c) Hence, show that $a^2 + b^2 + c^2 \ge ab + bc + ac$.

(d) Deduce that for $x, y, z > 0$

$$x^2y^2 + x^2z^2 + y^2z^2 \ge xyz(x + y + z).$$

Question 15 Let $a, b, c > 0$.

(a) Prove that $a^2 + b^2 + c^2 \ge ab + bc + ac$.

(b) Prove that if $a + b + c = 1$, then $ab + bc + ac \le \frac{1}{3}$.

Question 16 Let $a, b, c > 0$.

(a) Prove that $a + b \ge 2\sqrt{ab}$.

(b) Write down two more similar inequalities involving a, b and c.

(c) Hence, show that

$$(a + b)(b + c)(a + c) \ge 8abc.$$

Question 17 [Proving AM/GM inequality for $n = 3$]

Let $a, b, c > 0$.

(a) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.

(b) Show that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$.

(c) Hence, show that $\dfrac{a^3 + b^3 + c^3}{3} \geq abc$.

(d) By making an appropriate substitution, deduce that

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}.$$

Question 18 Let $a, b, c > 0$.

(a) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.

(b) Write down two more similar inequalities involving a, b and c.

(c) Hence, show that $(a + b + c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$.

Question 19 Let $a, b, c > 0$. Prove the following inequalities.

(a) $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$

(b) $\dfrac{a + b}{c} + \dfrac{b + c}{a} + \dfrac{a + c}{b} \geq 6$

(c) $(a + b + c)^2 \left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}\right) \geq 27$

(d) $ab(a + b) + bc(b + c) + ac(a + c) \geq 6abc$

Question 20 [Proving AM/GM inequality for $n = 4$]

Let $p, q \geq 0$.

(a) Prove that $\dfrac{p + q}{2} \geq \sqrt{pq}$.

(b) Let $a, b, c, d > 0$. Deduce that

$$\frac{a + b + c + d}{4} \geq \sqrt[4]{abcd}.$$

Question 21 [Proving AM/GM inequality for $n = 3$]

By making an appropriate choice for d in the inequality in the previous question, prove that

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}.$$

Question 22 [Techniques from the Advanced course are useful!]

Prove that

$$\sum_{k=1}^{\infty} \frac{1}{\left(\log_a b + \log_b a\right)^k} \leq 1.$$

Question 23 Prove that if a, b, $c > 0$ and $(1+a)(1+b)(1+c) = 8$, then $abc \leq 1$.

Question 24 Let a, b and c be positive.

(a) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.

(b) Hence, prove that if x, y, $z > 0$, then

$$\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Question 25 [Bounding the factorial]

(a) Prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$, where $a \geq 0$ and $b \geq 0$.

(b) If $1 \leq x \leq y$, show that $x(y - x + 1) \geq y$.

(c) Let n and k be positive integers with $1 \leq k \leq n$. Prove that

$$\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}.$$

(d) For integers $n \geq 1$, prove that

$$\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$

Question 26 [Arithmetic-mean/Harmonic-mean inequality]

Let $a, b, c > 0$. Prove that

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Question 27 [Proving AM/GM inequality for $n = 3$]

Let $a, b, c > 0$.

(a) Show that $a^3 + b^3 \geq abc\left(\dfrac{a}{c} + \dfrac{b}{c}\right)$.

(b) Hence, show that $\dfrac{a^3 + b^3 + c^3}{3} \geq abc$.

(c) Deduce that

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ac.$$

Challenge Problems

Problem 1 Let a, b, $c \geq 0$ and $abc = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}.$$

Problem 2 Let a, b, $c > 0$. Prove that

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0.$$

Problem 3 [Combinatorics useful here]

Let $x_1, x_2, x_3, \ldots, x_n > 0$. Prove that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2.$$

Problem 4 [Nesbitt's inequality]

Let $a, b, c > 0$. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Problem 5 [Cauchy-Schwarz inequality]

Consider the quadratic polynomial with real coefficients

$$P(x) = \sum_{k=1}^{n} (a_k x - b_k)^2.$$

(a) Explain why $P(x)$ cannot have two distinct real roots.

(b) Deduce that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

(c) Let $S = x_1 + x_2 + x_3 + \cdots + x_n$ where $x_k > 0$ for all $1 \leq k \leq n$.

Show that

$$\frac{S}{S-x_1} + \frac{S}{S-x_2} + \frac{S}{S-x_3} + \cdots + \frac{S}{S-x_n} \geq \frac{n^2}{n-1}.$$

Exercise 1G

Mathematical Induction

Fundamentals

Fundamentals 1

(a) A first-order recurrence formula is a recurrence where the value of each term depends on the value of the term directly b________ it.

(b) In order to generate a sequence of numbers, an i________ value is needed, and then the other successive terms of the sequence can be generated by iterating the values back into the same recurrence formula. Since all other values come from this i________ value, it is sometimes called the *seed* value.

(c) A formula that instantly gives the n^{th} term is called the c______-form solution of the recurrence. For a given recurrence formula, different seed values will generate different sequences of numbers, and hence will result in different c______-form solutions.

(d) When we use induction, we are proving that the c______-form solution satisfies the recurrence. We are NOT attempting to prove the r________ formula, since it is given.

Fundamentals 2

For inequality induction problems, there are generally two types of problems.

(a) The first type is proving that

$$A \geq B$$

where A and B are single terms. In general, instead of proving that LHS $\geq$ RHS, it is a lot easier to prove that ________________ ≥ 0 instead.

(b) The second type is proving that

$$a_1 + a_2 + a_3 + \cdots + a_n \geq B$$

series-type problems. These types are a lot more difficult because you will often need to prove a 'smaller' inequality in order to prove the main inequality. These types can similarly be proven by showing that ________________ instead of ________________.

Fundamentals 3

For geometry induction problems, there is no single approach that 'always works'. However, you should always draw a d________ where possible.

Question 1 [Guided inequality induction problem]

This is a guided proof to show that

$$2^n \geq n^2$$

for all $n \geq 4$, $n \in \mathbb{Z}$, using mathematical induction.

(a) **Step 1:** Prove the statement is true for $n = 4$.

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$2^k \geq k^2,$$

for $k \geq 4$, $k \in \mathbb{Z}$.

Write down the statement required to prove true for $n = k + 1$.

$$\text{RTP} \qquad 2^{k+1} > \underline{\hspace{2cm}}$$

Note that equality was lost. This is because we already established equality at $k = 4$, so we need only to prove '>' part of the inequality.

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned}
\text{LHS} - \text{RHS} &= 2^{k+1} - (k+1)^2 \\
&= 2(2^k) - k^2 - 2k - 1 \\
&\geq 2(\underline{\hspace{0.5cm}}) - k^2 - 2k - 1 \quad \text{from assumption} \\
&= k^2 - 2k - 1 \\
&= (\underline{\hspace{1.5cm}})^2 - 2 \\
&\geq \underline{\hspace{0.5cm}}^2 - 2 \quad \text{since } k \geq \underline{\hspace{0.5cm}} \\
&> 0 \\
\therefore \text{LHS} &> \text{RHS}
\end{aligned}$$

Question 2 Use induction to prove that

(a) $5^n < n!$ for all $n \geq 12$.

(b) $n^2 \geq 2n + 3$ for all $n \geq 3$.

(c) $n^3 \geq 2n^2 - n$ for all $n \geq 1$.

(d) $2^n > 4n$ for all $n \geq 5$.

(e) $4^n > 10 \times 2^n$ for all $n \geq 4$.

(f) $3^n > 2n^2$ for all $n \geq 2$.

Question 3 [Guided first-order recurrence induction problem]

This is a guided proof to show that if

$$T_n = 3T_{n-1} + 2$$

with starting value $T_0 = 1$, then the closed form of the n^{th} term is

$$T_n = 2 \times 3^n - 1$$

for all $n \geq 0$, $n \in \mathbb{Z}$, using mathematical induction.

(a) **Step 1:** Prove the statement is true for $n = 0$.

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$T_k = 2 \times 3^k - 1,$$

for $k \geq 0$, $k \in \mathbb{Z}$.

Write down the statement required to prove true for $n = k + 1$.

RTP $\quad T_{k+1} =$ ____________________

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned}
\text{LHS} &= \underline{\hspace{4cm}} \quad \text{using the recurrence} \\
&= 3(2 \times 3^k - 1) + 2 \\
&= 6 \times 3^k - 1 \\
&= 2 \times 3 \times 3^k - 1 \\
&= 2 \times \underline{\hspace{1.5cm}} - 1 \\
&= \text{RHS}
\end{aligned}$$

Question 4 Consider the following first-order recurrence relations and initial values.

(a)
$$T_n = 2T_{n-1} - 1$$
$$T_1 = 3$$
Prove that $T_n = 2^n + 1$.

(b)
$$T_n = 1 + 2T_{n-1}$$
$$T_1 = 5$$
Prove that $T_n = 6 \times 2^{n-1} - 1$.

(c)
$$T_n = 8 + 3T_{n-1}$$
$$T_1 = 2$$
Prove that $T_n = 6 \times 3^{n-1} - 4$.

(d)
$$T_n = 2T_{n-1} + 2 - n$$
$$T_1 = 3$$
Prove that $T_n = 2^n + n$.

(e)
$$T_n = \frac{3T_{n-1} - 1}{4T_{n-1} - 1}$$
$$T_1 = 1$$
Prove that $T_n = \dfrac{n}{2n-1}$.

(f)
$$T_n = T_{n-1} + n$$
$$T_1 = 1$$
Prove that $T_n = \dfrac{n(n+1)}{2}$.

Question 5 [Proving the sum of an AP formula]

Let $T_n = T_{n-1} + d$ where $T_1 = a$ and $a,\ d \in \mathbb{R}$. Prove that

$$T_n = a + (n-1)d$$

for all $n \in \mathbb{Z}^+$.

Question 6 [Proving the sum of a GP formula]

Let $S_n - S_{n-1} = ar^{n-1}$ where $S_1 = a$ and $r \neq 1$. Prove that

$$S_n = \frac{a(r^n - 1)}{r-1}$$

for all $n \in \mathbb{Z}^+$.

Question 7 [Proving a general formula]

Let $T_n = pT_{n-1} + q$ where $T_1 = a$ and $p \neq 1$. Prove that

$$T_n = ap^{n-1} + q\left(\frac{p^{n-1} - 1}{p-1}\right)$$

for all $n \in \mathbb{Z}^+$.

Question 8 [Guided series-inequality induction proof]

This is a guided proof to show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\leq 2-\frac{1}{n}$$

for all $n \in \mathbb{Z}^+$, using mathematical induction.

(a) **Step 1:** Prove the statement is true for $n = 1$.

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}\leq 2-\frac{1}{k},$$

for $k \in \mathbb{Z}^+$.

Write down the statement required to prove true for $n = k + 1$.

$$\text{RTP} \qquad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < \underline{\qquad\qquad}$$

Note that equality was lost. This is because we already established equality at $k = 1$, so we need only to prove '$<$' part of the inequality.

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned}
\text{RHS} - \text{LHS} &= \left(2-\frac{1}{k+1}\right)-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\right)\\
&\geq \left(2-\frac{1}{k+1}\right)-\left(2-\frac{1}{k}+\frac{1}{(k+1)^2}\right) \quad \text{from assumption}\\
&= \frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}\\
&= \frac{?}{k(k+1)^2}\\
&> 0
\end{aligned}$$

$\therefore \text{RHS} > \text{LHS}$

Question 9 Use induction to prove the following for $n \in \mathbb{Z}^+$.

(a) $\sum_{k=1}^{n} \frac{1}{k} \le \frac{n+2}{2}$ (b) $\sum_{k=1}^{n} \frac{1}{k} \ge \frac{2n}{n+1}$ (c) $\sum_{k=1}^{n} (k-1)^3 \le \frac{n^4}{4}$

Question 10

(a) Prove that $k(4k+3)^2 < (k+1)(4k+1)^2$ for all $k \in \mathbb{Z}^+$.

(b) Prove that

$$\sum_{k=1}^{n} \sqrt{k} \le \frac{4n+3}{6}\sqrt{n}$$

for all $n \in \mathbb{Z}^+$.

Question 11

(a) Prove that $2k+3 > 2\sqrt{(k+1)(k+2)}$ for all integers $k \ge 0$.

(b) Use induction to prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2(\sqrt{n+1} - 1)$$

for all integers $n \ge 1$.

Question 12 Use induction to prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \ge \sqrt{n}$$

for all integers $n \ge 1$.

Question 13 Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1$$

for all integers $n \ge 2$.

Question 14 [The alternating partial harmonic series is never negative]

Prove that $\sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k} > 0$ for all $n \in \mathbb{Z}^+$.

Question 15 [Induction involving geometry]

Prove the following.

(a) A regular n-gon has $\frac{n(n-3)}{2}$ diagonals.

(b) n non-parallel lines, no three of which are concurrent, divide the plane into $\frac{1}{2}(n^2+n+2)$ regions.

(c) n non-parallel lines, no three of which are concurrent, have $\frac{n(n-1)}{2}$ points of intersection.

Question 16 [Induction involving calculus]

(a) Use induction to prove that

$$\frac{d^n}{d\theta^n}(\sin a\theta) = a^n \sin\left(a\theta + \frac{n\pi}{2}\right)$$

for all integers $n \geq 1$.

(b) Use induction to prove that

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$$

for all integers $n \geq 1$.

(c) Prove that

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$$

for all integers $n \geq 0$.

Question 17 [Induction involving combinatorics]

(a) Prove that the number of non-empty subsets of a set of n elements is $2^n - 1$.

(b) A room contains n people and each person shakes hands with everyone else exactly once. Prove that there are $\dfrac{n(n-1)}{2}$ handshakes in total.

Question 18 Use induction to prove that

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2}$$

for all integers $n \geq 1$.

Question 19

(a) Use induction to prove that

$$(2n)! > 2^n(n!)^2$$

for all integers $n \geq 2$.

(b) Deduce that

$$\binom{2n}{n} > 2^n$$

for all integers $n \geq 2$.

Question 20 [Bernoulli's Inequality]

Use induction to prove that

$$(1+x)^n \geq 1+nx$$

for all $n \in \mathbb{Z}^+$ and $x \geq -1$.

Question 21

(a) Prove that for all $n \in \mathbb{Z}^+$, there exist unique positive integers p_n and q_n such that

$$(1+\sqrt{2})^n = p_n + q_n\sqrt{2}.$$

(b) Write down a similar result for $(1-\sqrt{2})^n$.

(c) Hence, find the limit of $\dfrac{p_n}{q_n}$ as n gets large.

Question 22 Use mathematical induction to prove that

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) = \frac{1-x^{2^{n+1}}}{1-x}$$

for all $n \in \mathbb{Z}^+$ and $x > 1$.

Question 23 Use mathematical induction to prove that

$$e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

for all $n \in \mathbb{Z}^+$.

Question 24 Use mathematical induction to prove that

$$\tan\left(\frac{(2n+1)\pi}{4}\right) = (-1)^n$$

for all $n \in \mathbb{Z}^+$.

Question 25 [Not as simple as it looks!]

Use mathematical induction to prove that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \geq \frac{n}{2}$$

for all $n \in \mathbb{Z}^+$.

Challenge Problems

Problem 1 Define a sequence of angles $\theta_k \in \left[0, \dfrac{\pi}{2}\right]$ where $k = 1, 2, 3, \ldots, n$ such that

$$0 < \theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n < \frac{\pi}{2}$$

Use mathematical induction to prove that

$$\tan(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n) \geq \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \cdots + \tan\theta_n.$$

Problem 2 Prove that a $2^n \times 2^n$ square grid with exactly one tile removed can be completely tiled by L-shaped trominoes.

Problem 3 Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

using induction for all $n \in \mathbb{N}$.

Problem 4 Use induction to prove that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$$

for all integers $n \geq 1$.

Problem 5 [Generalised product rule]

Let $f^{(k)}(x)$ denote the k^{th} derivative of $f(x)$, or in other words $f^{(k)}(x) = \dfrac{d^k}{dx^k}(f(x))$. Use induction to prove that

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where u and v are functions in terms of x.

Problem 6 [Colouring problem]

The plane is divided by n lines into regions. Prove by induction that it is possible to colour all regions with 2 different colours such that no two neighbouring regions have the same colour.

Problem 7 Matthew flips a coin $(2n+1)$ times. Let $P(k)$ be the probability that Matthew flips exactly k heads.

(a) Show that $P(n) = P(n+1)$.

(b) Hence, prove by mathematical induction that Matthew has a probability of $\dfrac{1}{2}$ of flipping at least $(n+1)$ heads.

Exercise 1H

Inequalities using Differentiation

Fundamentals

Fundamentals 1

To prove that $f(x) \geq 0$ for $x \geq a$, we can prove two things.

- $f'(x) \geq 0$ for $x \geq a$
- $f(a) \geq 0$

It is important to note that both are required to safely conclude that $f(x) \geq 0$ for $x \geq a$.

(a) Explain what happens if we have only the first condition, but not the second condition.

(b) Explain what happens if we have only the second condition, but not the first condition.

Fundamentals 2

Suppose $f(x)$ has a global minimum stationary point (a, b). This means that $f(x) \geq$ ___ for all x in the domain of $f(x)$.

Question 1 Prove the following inequalities.

(a) $\forall x > 0,\ e^x > 1 + x$

(b) $\forall x > 0,\ \sin x < x$

Question 2 Prove the following inequalities.

(a) $\forall x > 0,\ x \geq \ln(1 + x)$

(b) $\forall x > 0,\ e^x > 1 + x + \dfrac{x^2}{2}$

(c) $\forall x > 0,\ \sin x > x - \dfrac{x^3}{6}$

(d) $\forall x > 0,\ x < \dfrac{e^x - e^{-x}}{2}$

(e) $\forall x \in \mathbb{R},\ \cos x \geq 1 - \dfrac{x^2}{2}$

(f) $\forall x \in \left[0, \dfrac{\pi}{2}\right],\ x < \ln(\sec x + \tan x)$

(g) $\forall x \in [0, 1],\ \sin^{-1} x \geq x$

(h) $\forall x > 0,\ \tan^{-1} x \geq x - x^3$

Question 3 [Not like the other ones!]

Prove that $\ln x \leq x - 1$ for all $x > 0$.

Question 4 Consider the curve $y = e^x\left(1 - \dfrac{x}{4}\right)^4$.

(a) Find the coordinates of the stationary points.

(b) Sketch the curve, labelling stationary points.

(c) Deduce that $e^x \le \dfrac{256}{(4-x)^4}$ for all $x < 4$.

(d) Hence, show that

$$\frac{625}{256} < e < \frac{256}{81}.$$

Question 5 Let $f(x) = e^x - x - 1$.

(a) Prove that $f(x) > 0$ for all $x \ne 0$.

(b) By finding a suitable value of x, prove that $e^\pi > \pi^e$.

Question 6 Let $f(x) = \dfrac{\ln x}{x}$.

(a) Find the stationary point of $y = f(x)$ and show that it is a maximum.

(b) Deduce that $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$.

Question 7 Let $f(x) = x^n + (2-x)^n$.

(a) Show that there is a minimum stationary point at $x = 1$.

(b) By letting $x = \dfrac{2a}{a+b}$, prove that

$$\left(\frac{a+b}{2}\right)^n \le \frac{a^n + b^n}{2}.$$

Question 8 Let $f(x) = x^n e^{-x}$.

(a) Show that there is a maximum turning point at $(n, n^n e^{-n})$.

(b) Sketch the graph for $x \ge 0$.

(c) Deduce that $\left(1 + \dfrac{1}{n}\right)^n < e$.

Challenge Problems

Problem 1 Let $f(x) = \ln x - x + 1$.

(a) Show that $\forall x > 0,\ f(x) \le 0$.

(b) Let $p_i > 0,\ \forall i = 1,\ 2,\ \ldots,\ n$ such that

$$p_1 + p_2 + \cdots + p_n = 1.$$

Prove that

$$\sum_{k=1}^{n} \ln(np_k) \le np_1 + np_2 + \cdots + np_n - n.$$

(c) Deduce that

$$0 < n^n p_1 p_2 p_3 \cdots p_n \le 1.$$

Problem 2 [Simple proof of the AM/GM inequality]

Let $a_k > 0$, where $k = 1,\ 2,\ \ldots,\ n$ be a set of n positive real numbers.

(a) Prove that $x \le e^{x-1}$ for all $x \in \mathbb{R}$.

(b) By finding an appropriate value of x to substitute, prove that

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 a_3 \ldots a_n}.$$

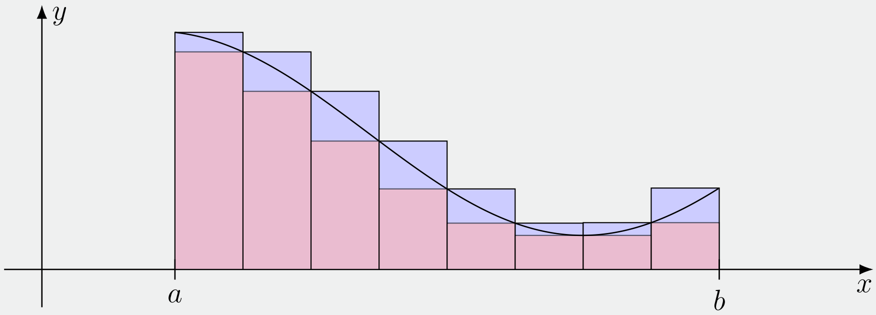
Exercise 1I

Inequalities using Integration

Fundamentals

Fundamentals 1

The diagram below shows a function $f(x)$ and a number of upper and lower-bound rectangles.

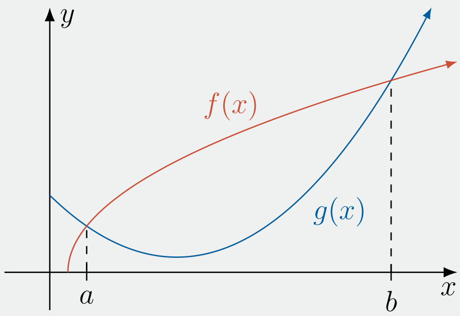


Let the total area of the upper and lower-bound rectangles be U and L respectively.

$$\_\_ < \int_a^b f(x)\,dx < \_\_$$

Fundamentals 2

The diagram below shows some function $f(x) \geq g(x)$ for $x \in [a, b]$.

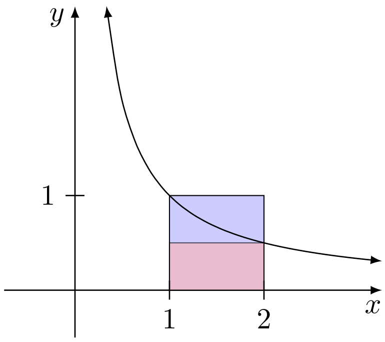


It follows that

$$\int_a^b f(x)\,dx > \int_?^? \_\_\_\_ \,dx$$

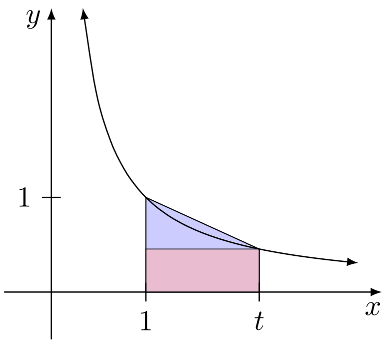
Equality is lost because although the functions were equal to each other originally at the point of intersections, their areas are often not equal and so their integrals are not necessarily equal.

Question 1 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the region $x \in [1, 2]$.



Use the diagram to prove that $\frac{1}{2} < \ln 2 < 1$.

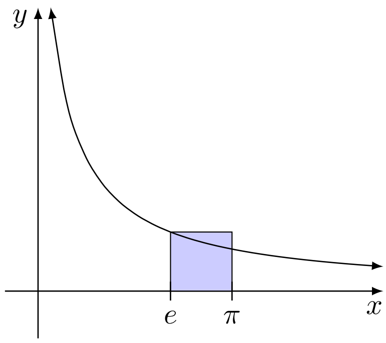
Question 2 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the region $x \in [1, t]$.



Use the diagram to prove that

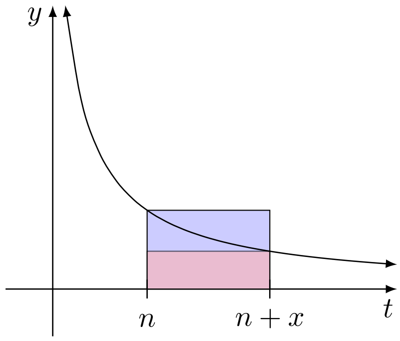
$$1 - \frac{1}{t} \leq \ln t \leq \frac{1}{2}\left(t - \frac{1}{t}\right).$$

Question 3 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the upper-bound rectangle in the domain $x \in [e, \pi]$.



Use the diagram to show that $e^\pi > \pi^e$.

Question 4 The diagram below shows a section of the graph of $y = \dfrac{1}{t}$.



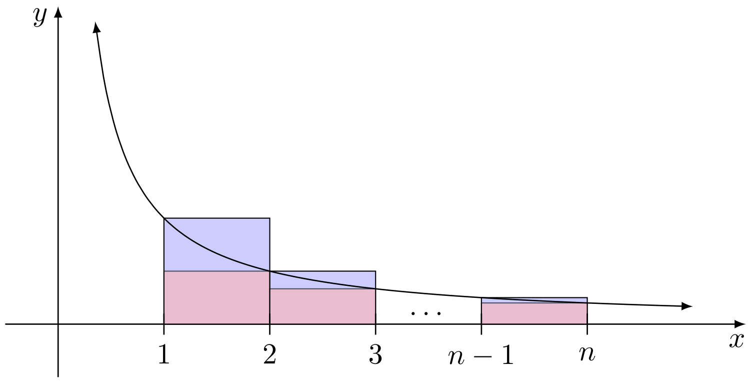
Consider the region $t \in [n, n+x]$, where $x > n$.

(a) Prove that $\dfrac{x}{1+\frac{x}{n}} < n \ln\left(1 + \dfrac{x}{n}\right) < x$.

(b) Hence, show that $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

Question 5 [Harmonic Series]

The diagram below shows the graph of $y = \dfrac{1}{x}$. Upper and lower-bound rectangles of unit width are constructed over the domain $x \in [1, n]$.



Define the series $H_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$.

(a) Show that

$$\frac{1}{n} + \ln n < H_n < 1 + \ln n.$$

(b) Hence, find two integers which are lower and upper bounds of the following sum.

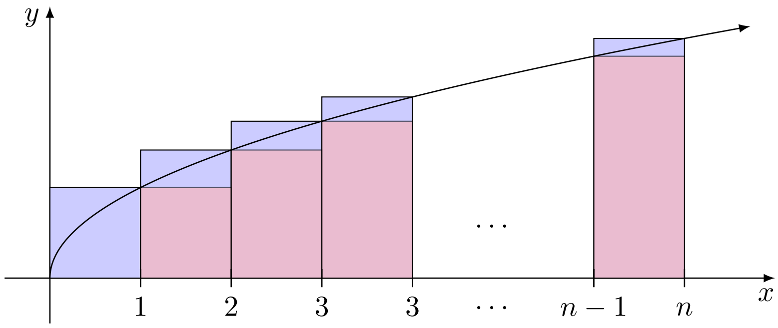
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2020}$$

(c) Does the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

have a finite limit?

Question 6 The diagram below shows the graph of $y = \sqrt{x}$. Upper and lower-bound rectangles of unit width are constructed over the domain $x \in [0, n]$.



Let $S_n = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$.

(a) Prove that

$$\frac{2n\sqrt{n}}{3} < S_n < \frac{(2n+3)\sqrt{n}}{3}.$$

(b) Hence, find two integers which are lower and upper bounds of the following sum.

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{2020}$$

Question 7 Let $0 \le t \le 1$.

(a) Prove that

$$\frac{1}{2} \le \frac{1}{1+t} \le 1.$$

(b) Deduce that for $0 \le x \le 1$

$$\frac{x}{2} \le \ln(1+x) \le x.$$

Question 8 Let $0 \le t \le 1$.

(a) Prove that

$$\frac{1}{2} \le \frac{1}{1+t^2} \le 1.$$

(b) Deduce that for $0 \le x \le 1$

$$\frac{x}{2} \le \tan^{-1}(x) \le x.$$

Question 9 [Proof of a well-known limit]

Let $x > 0$ and $n \in \mathbb{Z}^+$.

(a) Prove that $1 - x \leq \dfrac{1}{1+x} \leq 1$.

(b) Show that

$$1 - \frac{1}{2n} \leq n \ln\left(1 + \frac{1}{n}\right) \leq 1.$$

(c) Deduce that $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

Question 10 Define the integral for $n \geq 2$

$$I_n = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\, dx.$$

(a) Show that if $m > n$, then $I_m < I_n$.

(b) Hence, show that

$$\frac{1}{2} < I_n \leq \frac{\pi}{6}$$

for all $n \geq 2$.

Question 11 [Alternating harmonic series]

Consider the series

$$S_n = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1} t^{n-1}.$$

(a) Show that

$$S_n = \frac{1}{1+t} + (-1)^{n+1} \frac{t^n}{1+t}.$$

(b) Hence, show that

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n} = \ln 2 + (-1)^{n+1} \int_0^1 \frac{t^n}{1+t}\, dt.$$

(c) Hence, prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Question 12 [Gregory-Leibniz series]

(a) Show that

$$-x^{2n} \leq \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}\right) \leq x^{2n}.$$

(b) Hence, show that

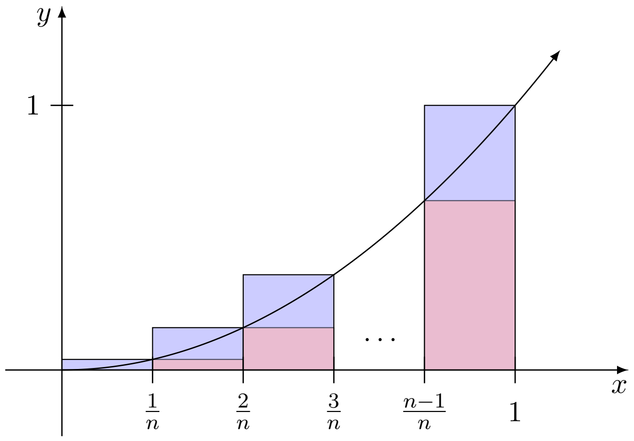
$$-\frac{1}{2n+1} \leq \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) \leq \frac{1}{2n+1}.$$

(c) Hence, write down the exact value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Question 13 [Integration by first principles]

The diagram below shows the graph of $f(x) = x^2$ in the domain $x \in [0, 1]$ that has been partitioned into n equal-width sub-intervals.



(a) Show that

$$\frac{1}{n^3}\sum_{k=0}^{n-1} k^2 < \int_0^1 f(x)\,dx < \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

(b) Use the fact that $\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$ to show that

$$\frac{(n-1)(2n-1)}{6n^2} < \int_0^1 f(x)\,dx < \frac{(n+1)(2n+1)}{6n^2}$$

(c) Find the limits of the expressions on either side of the inequality as $n \to \infty$, and state the significance of this result.

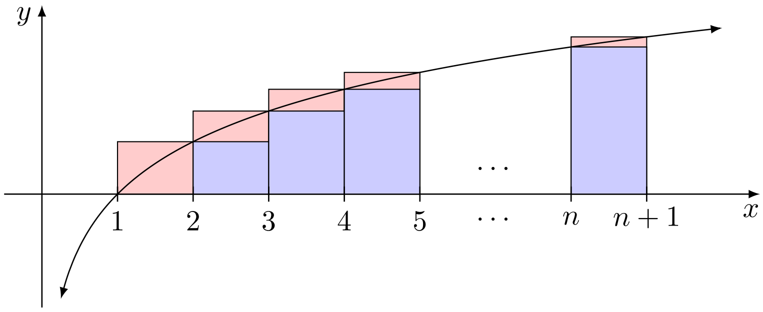
Challenge Problems

Problem 1 Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n^2}{n^2+k^2}.$$

Problem 2 [A way to approximate $\sqrt[n]{n!}$]

The diagram below shows the graph of $y = \ln x$. Upper and lower-bound rectangles of unit width are constructed over the domain $x \in [1, n+1]$.



(a) Prove that

$$\ln n! < \int_1^{n+1} \ln x \, dx < \ln(n+1)!$$

(b) Hence, show that

$$\frac{n+1}{e} < \sqrt[n]{n!} < \left(\frac{n+1}{e}\right) \sqrt[n]{n+1}.$$

(c) Enter $\sqrt[100]{100!}$ in your calculator. What is the output, and what might be causing it?

(d) Hence, find two integers which are lower and upper bounds of $\sqrt[100]{100!}$

Chapter 1 Review

The Nature of Proof

Review

Question 1 Consider the statement below.

If n is an odd integer, then $n^3 - n$ is a multiple of 12.

Either prove it, or disprove it.

Question 2 Consider the statement below.

For all $a, b \in \mathbb{Z}$, if a divides b and b divides a, then $a = b$.

Either prove it, or disprove it.

Question 3 Prove that

(a) the product of two odd numbers is odd.

(b) the sum of two even numbers is even.

(c) $n^2 - n$ is always even.

(d) $n^2 + n + 1$ is always odd.

(e) the product of three consecutive integers is always divisible by 6.

Question 4 Prove the following statements indirectly by instead proving the contrapositive.

(a) If n does not divide ab, then n does not divide a nor b.

(b) If ab is irrational, then either a or b must be irrational.

(c) If 5 is not a factor of mn, then 5 is neither a factor of m nor n.

(d) If a does not divide bc, then a does not divide b.

(e) If both ab and $a + b$ are even, then both a and b are even.

(f) If ab is divisible by 5, then a or b must be divisible by 5.

Question 5 Prove that $\sqrt{12}$ is irrational.

Question 6 Prove that $\log_4 8$ is irrational.

Question 7 Let n be a positive integer. Prove that $\sqrt{4n - 2}$ is irrational.

Question 8 Determine whether the following statements are true or false. Remember that a single counter-example is enough to render a statement false, even if it is *mostly* true.

(a) If $a > b$, then $a^2 > b^2$.

(b) If a^2 is not divisible by 4, then a is odd.

(c) If $n^2 + n$ is even, then n is even.

(d) If n is any integer, then $n^2 + n$ is even.

(e) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(f) If a and b are real, then $\frac{a}{b}$ is real.

(g) If $3x - 3y$ is even, then x and y are either both odd or both even.

(h) If $m^2 - n^2$ is divisible by 3, then $m - n$ is divisible by 3.

(i) If x and y are both irrational, then xy is irrational.

(j) If xy and $x + y$ are both even, then x and y must both be even.

(k) If a has a remainder of 1 when divided by 5, then a^2 has a remainder of 1 when divided by 5.

(l) If a has a remainder of b when divided by n, then a^3 will have a remainder of b^3 when divided by n.

Question 9 Prove that $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$.

Question 10 Prove that if $x_k, y_k > 0$ for all $k = 1, 2, 3, \ldots, n$ then $\sum_{k=1}^{n} \left(\frac{x_k}{y_k} + \frac{y_k}{x_k} \right) \geq 2n$.

Question 11 Prove that if $a \geq b \geq 1$, then $ab + b \geq a + b^2$.

Question 12 Let $a, b, c > 0$.

(a) Prove that $a^2 - ab + b^2 \geq ab$.

(b) Show that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

(c) Hence, show that $a^3 + b^3 + c^3 \geq ab\left(\frac{a+b}{2}\right) + bc\left(\frac{b+c}{2}\right) + ac\left(\frac{a+c}{2}\right)$.

Question 13 Let $a + b + c = 1$. Prove the following inequalities.

(a) $ab + bc + ac \leq \frac{1}{3}$

(b) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$

(c) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \geq 27$

(d) $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} \geq \frac{9}{2}$

(e) $(a + b)(b + c)(a + c) \leq \frac{8}{27}$

(f) $\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \geq 9$

Question 14 Use induction to prove that

(a) $3^n > n^3$ for all $n \geq 4$.

(b) $2^n < n!$ for all $n \geq 4$.

Question 15 Consider the following first-order recurrence relations and initial values.

(a)
$$T_{n+1} = 2T_{n-1} + 1$$
$$T_1 = 1$$
Prove that $T_n = 2^n - 1$.

(b)
$$T_{n+1} = 5T_n + 4$$
$$T_1 = 4$$
Prove that $T_n = 5^n - 1$.

Question 16 Use induction to prove that

$$\sum_{k=1}^{n} \frac{1}{k} < \sqrt{n}$$

for all integers $n \geq 7$.

Question 17 Use induction to prove that the interior angle sum of an n-gon is $(n-2)\pi$ for all integers $n \geq 3$.

Question 18 Prove that

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for $n \in \mathbb{Z}^+$ using induction.

Question 19 Let $x \geq 0$. Prove the following inequalities.

(a) $\tan x \geq x$

(b) $x \geq \ln\left(1 + x + \dfrac{x^2}{2}\right)$

Question 20 Define the integral for $n \geq 2$

$$I_n = \int_0^1 \frac{1}{1+x^n}\,dx.$$

(a) Show that if $m > n$, then $I_m < I_n$.

(b) Show that

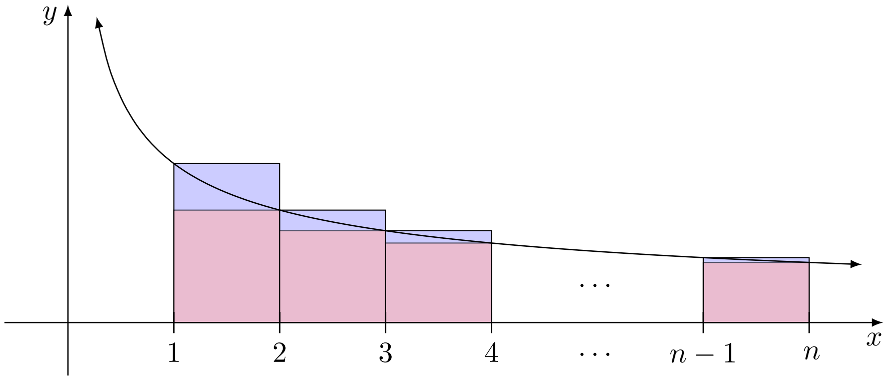
$$\frac{\pi}{4} \leq I_n < 1$$

for all $n \geq 2$.

Question 21 Prove that for $x > 0$

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Question 22 The diagram below shows the graph of $y = \dfrac{1}{\sqrt{x}}$. Upper and lower-bound rectangles of unit width are constructed over the domain $x \in [1, n]$.



Let $S_n = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$.

(a) Prove that

$$2\sqrt{n} + \frac{1}{\sqrt{n}} - 2 < S_n < 2\sqrt{n} - 1.$$

(b) Hence, find two integers which are lower and upper bounds of the following sum.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{2020}}$$

Investigation Task

Modular Arithmetic

In this chapter, we prove some basic number theory results using direct proof, proof by contraposition and proof by contradiction. However, modular arithmetic is a powerful tool that often does a quick job of many problems that would normally be time consuming.

Question 1 Explain briefly what modular arithmetic is.

Question 2 Complete the following modular arithmetic properties.

(a) If $a + b = c$, then $a \pmod{n} + b \pmod{n} \equiv \ldots$

(b) If $a \equiv b \pmod{n}$, then $a + k \equiv \ldots$

(c) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then ...

(d) If $a \equiv b \pmod{n}$, then $-a \equiv \ldots$

Question 3

(a) Write down the multiplication properties of modular arithmetic.

(b) Write down the exponentiation properties of modular arithmetic.

Question 4 Find $3^{2020} \pmod{4}$.

Question 5

(a) Explain how you could use modular arithmetic to calculate the last digit of a number.

(b) Find the last digit of 2020^{2020}.

Question 6

(a) Define the term *modular inverse.*

(b) Explain the relevance of the *Euclidean Algorithm* and *Bezout's Identity* in the context of the modular inverse.

(c) Find $17^{-1} \pmod{163}$.

(d) Explain why $69^{-1} \pmod{420}$ does not exist.

Question 7 Modular arithmetic is incredibly important in the field of *cryptography.* Write a one-page article with examples and explains how modular arithmetic is relevant in cryptography.

Investigation Task

Truth Tables

In the topic *Nature of Proof*, we lightly covered some areas of logic like implications, equivalent statements, counter-examples, contrapositive, and proof by contradiction. However, no study of logic is really complete without including truth tables.

Question 1 The truth table below contains three fundamental connectives between p and q.

p	q	$p \wedge q$	$p \vee q$	$p \to q$
T	T			
T	F			
F	T			
F	F			

(a) Complete the truth table.

(b) Explain how the truth values for $p \to q$ are obtained. Your answer should include an explanation of why two F's lead to a T.

Question 2 An implication containing three propositions P, Q and R is a theorem if it returns true for all truth combinations of P, Q and R. One of the following is a theorem and the other isn't. Determine which one is a theorem, and find all counter-examples for the one that isn't.

(a) $(p \vee r) \wedge (q \vee r) \to (p \wedge q) \vee r$

(b) $(p \vee r) \wedge (q \vee r) \to (p \vee q) \wedge r$

Question 3

(a) Research and define De Morgan's Laws for logical propositions.

(b) Give two 'word examples' for each law to demonstrate how they work.

(c) Prove De Morgan's Laws using truth tables.

Question 4 Use truth tables to verify that the contrapositive of $p \to q$ is logically equivalent.

Question 5 Use truth tables to verify that $p \to q$ is logically equivalent to $\sim p \vee q$.

2

COMPLEX NUMBERS

- Arithmetic of Complex Numbers
- Solving and Factorising Quadratics
- Polar Form and the Argand Diagram
- Vector Representation
- Locus
- De Moivre’s Theorem
- Applications of de Moivre’s Theorem
- Roots of Unity
- Applications of Roots of Unity
- Solving Polynomials
- Euler’s Formula

Exercise 2A

Arithmetic of Complex Numbers

Fundamentals

Fundamentals 1
If $z = a + ib$,

(a) a is called the r_____ component of z, and is denoted by _____.

(b) b is called the i_________ component of z, and is denoted by _____.

(c) If $a = 0$, then the number is purely i_________.

(d) If $b = 0$, then the number is purely r_____.

(e) The complex conjugate of z is $\overline{z} =$ _____.

(f) The modulus of z is $|z| =$ _____.

Fundamentals 2
Complete the following conjugate properties.

(a) $\overline{z + w} =$ _______ (b) $\overline{zw} =$ _____ (c) $\overline{\left(\dfrac{z}{w}\right)} =$ _____

Fundamentals 3
Simplify the following.

(a) $z + \overline{z}$ (b) $z - \overline{z}$ (c) $z\overline{z}$

Question 1 Simplify the following.

(a) i^2 (b) i^3 (c) i^4 (d) i^5

(e) i^{-5} (f) i^{2020} (g) i^{4n+1} (h) i^{2n}

Question 2 Let $z = 3 - 4i$ and $w = 2 + i$. Find the following.

(a) $z + w$ (b) $z - w$ (c) zw

(d) w^2 (e) $|z|$ (f) $|w|$

Question 3 Let $z = 2 + 3i$. Find the following.

(a) $\overline{z}$ (b) $z + \overline{z}$ (c) $z - \overline{z}$ (d) $z\overline{z}$

Question 4 Simplify the following.

(a) $\dfrac{1+i}{1-i}$ (b) $\dfrac{7-4i}{2+i}$ (c) $\dfrac{-7+24i}{3+4i}$

Question 5 Find x and y in each of the following.

(a) $3x + 4iy = 12 + 24i$ (b) $5x + iy = (2-i)(3+4i)$

(c) $\dfrac{10-5i}{x+iy} = 2+i$ (d) $\dfrac{x}{1+2i} + \dfrac{y}{2-i} = 2-i$

Question 6 Find z if $\dfrac{z}{1+z} = 1+i$.

Question 7 Prove the following conjugate properties.

(a) $\overline{z+w} = \overline{z} + \overline{w}$ (b) $\overline{zw} = (\bar{z})(\bar{w})$ (c) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{(\overline{z})}{(\overline{w})}$

Question 8 Prove the following conjugate properties.

(a) $z + \overline{z} = 2\text{Re}\,(z)$ (b) $z - \overline{z} = 2i\,\text{Im}\,(z)$ (c) $z\overline{z} = |z|^2$

Question 9 Prove that $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = \dfrac{2\text{Re}\,(z)}{|z|^2}$.

Question 10 Prove the following.

(a) $z^2 + (\overline{z})^2$ is purely real. (b) $z^2 - (\overline{z})^2$ is purely imaginary.

Question 11 Let $z = a + ib$. Express the real/imaginary parts of the following in terms of a and b. Hence, write them in the form $x + iy$, where x and y are in terms of a and b.

(a) z^2 (b) $\dfrac{1}{z}$ (c) $\dfrac{z+1}{z-1}$

Question 12 Let

$$z = \frac{a+bi}{a-bi},$$

where $a, b \in \mathbb{R}$. Show that $|z| = 1$.

Question 13 Find the possible value(s) of k if $\dfrac{k+i}{1+ki}$ is purely real.

Question 14 Prove that if z^2 is purely imaginary, then $|\text{Re}\,(z)| = |\text{Im}\,(z)|$.

Question 15 Prove that if $|z-1| = 1$, then $|z|^2 = 2\text{Re}\,(z)$.

Question 16 [Parallelogram Law]

Prove that if z and w are any complex numbers, then

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$

Challenge Problems

Problem 1

(a) Prove that if $\frac{z-1}{z+1}$ is purely real, then z is purely real too.

(b) Prove that if $|z|=1$, then $\frac{z-1}{z+1}$ is a purely imaginary number.

Problem 2 Prove that if $|2z-1|=|z-2|$, then $|z|=1$.

Problem 3 Let z and w be complex numbers.

(a) Show that $|z| \geq |\mathrm{Re}\,(z)\,|$.

(b) Hence, prove the triangle inequality $|z+w| \leq |z|+|w|$.

Problem 4 Show that if either $|z|=1$ or $|w|=1$, then $\left|\frac{z-w}{1-\bar{z}w}\right|=1$.

Problem 5 Let z_1, z_2, and z_3 be three complex numbers with modulus 1. Prove that

$$|z_1z_2+z_1z_3+z_2z_3|=|z_1+z_2+z_3|.$$

Problem 6 [A way to generate the vertices of the tangential quadrilateral]

Let α and β be any complex numbers. Let z be a complex number that satisfies both

$$z=\alpha(1+ki)$$
$$z=\beta(1-ki)$$

for some $k \in \mathbb{R}$.

(a) Show that $|\alpha|=|\beta|$.

(b) Show that $|z-\alpha|=|z-\beta|$.

(c) Interpret this result geometrically.

Problem 7 Consider the identity

$$\frac{1}{z}+\frac{1}{\bar{z}}=1.$$

(a) By letting $z=a+ib$, show that $(a-1)^2+b^2=1$.

(b) Find a possible non-real value of z.

(c) Show that $0<\mathrm{Re}\,(z)\leq 2$ and $-1\leq \mathrm{Im}\,(z)\leq 1$.

Exercise 2B

Solving and Factorising Quadratics

Fundamentals

Fundamentals 1
Expand and/or simplify the following.

(a) $\alpha + \bar{\alpha}$ (b) $\alpha\bar{\alpha}$ (c) $(z - \alpha)(z - \beta)$

Fundamentals 2
A monic quadratic polynomial $P(z)$ has complex roots α and $\bar{\alpha}$.

(a) The quadratic polynomial can be expressed as the product of complex linear factors.

$$P(z) = (z - \_\_\_)(z - \_\_\_)$$

(b) Expand $P(z)$.

$$P(z) = z^2 - \_\_\_\_\_ z + \_\_\_\_$$

Fundamentals 3

(a) To solve a quadratic with complex coefficients, we can use the q________ formula.

(b) The d___________ will most likely be a non-real number.

(c) Simplify the s_____ root term, and remember that plus/minus of plus/minus is overall just plus/minus.

(d) Once you obtain the two roots, you may like to use the s____/p________ of roots to verify that your roots are correct.

Question 1 Express each of the following in the form $x + iy$, where x and y are real.

(a) $\sqrt{3 + 4i}$ (b) $\sqrt{8 - 6i}$ (c) $\sqrt{5 - 12i}$

(d) $\sqrt{-2i}$ (e) $\sqrt{35 + 12i}$ (f) $\sqrt{9 - 40i}$

(g) $\sqrt{7 + 24i}$ (h) $\sqrt{-21 + 20i}$ (i) $\sqrt{-24 + 10i}$

Question 2 Solve the following quadratic equations and hence express them as the product of linear complex factors.

(a) $z^2 + 16 = 0$ (b) $z^2 + 2z + 5 = 0$ (c) $z^2 + z + 1 = 0$

Question 3 Write down the roots of the following quadratics.

(a) $(z - 1 + 2i)(z - 1 - 2i) = 0$ (b) $(z + 3 + 4i)(z + 3 - 4i) = 0$

Question 4 Solve the following quadratic equations.

(a) $z^2 - 3z + (3+i) = 0$

(b) $iz^2 - 2(1+i)z + 10 = 0$

Question 5 The quadratic equation $z^2 - (3+i)z + k = 0$ has a root $1 - 2i$. Find the value of k.

Challenge Problems

Problem 1 Express each of the following in the form $x + iy$, where x and y are real.

(a) $\sqrt{i}$

(b) $\sqrt{4+3i}$

(c) $\sqrt{-12+5i}$

Problem 2 Let $z = x + iy$ represent the square roots of $a + bi$. Prove the following.

$$x^2 = \frac{|z| + a}{2}$$

$$y^2 = \frac{|z| - a}{2}$$

Problem 3 Solve the polynomial equation $z^4 = -7 + 24i$.

Problem 4 [Useful result for roots of unity]

A monic quadratic equation has roots α and $\overline{\alpha}$. Show that the quadratic is of the form

$$z^2 - 2\mathrm{Re}\,(\alpha)\,z + |\alpha|^2.$$

Problem 5 [Conjugate root theorem for the quadratic]

The quadratic polynomial equation $az^2 + bz + c = 0$ has real coefficients a, b, c and a non-real root α. Prove that the other root is $\overline{\alpha}$.

Exercise 2C

Polar Form and the Argand Diagram

Fundamentals

Fundamentals 1

(a) Complex numbers have two components called the r___ and i________ components.

(b) Hence, they can be represented on a plane called the complex plane, otherwise known as the A______ diagram.

(c) On the complex plane, the r___ component of z is represented on the x-axis, and the i________ component of z is represented on the y-axis.

Fundamentals 2

Let $z = a + bi$ be a complex number.

(a) The distance of a complex number z from the origin is called the m________.

$$|z| =$$

(b) When connected to the origin, the angle that z makes with the positive real axis is called the a________, and it is given by solving

$$\tan\theta = \underline{\qquad}$$

where $\theta \in (\underline{\quad}, \underline{\quad}]$. Use quadrants to determine the correct argument.

(c) Combined together, we get the p______ form of z.

$$z = a + bi = r\big(\cos\theta + i\sin\theta\big) = r\underline{\qquad}$$

Fundamentals 3

If $z = A\operatorname{cis}\alpha$ and $w = B\operatorname{cis}\beta$, then

(a) $zw = \underline{\qquad\qquad}$

(b) $\dfrac{z}{w} = \underline{\qquad\qquad}$

Fundamentals 4

Complete the following properties.

(a) $|zw| = \underline{\quad}$

(b) $\left|\dfrac{z}{w}\right| = \underline{\quad}$

(c) $|kz| = \underline{\quad}$ if $k > 0$

(d) $\arg(zw) = \underline{\qquad}$

(e) $\arg\left(\dfrac{z}{w}\right) = \underline{\qquad}$

(f) $\arg(k) = \underline{\quad}$ if $k > 0$

Question 1 Convert the following to the form $r \operatorname{cis} \theta$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

(a) $1 + i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $1 - i\sqrt{3}$ (d) $-1 - i\sqrt{3}$

(e) $\sqrt{3} + i$ (f) $-\sqrt{3} + i$ (g) $\sqrt{3} - i$ (h) $-\sqrt{3} - i$

(i) $1 + i$ (j) $1 - i$ (k) $-1 + i$ (l) $-1 - i$

(m) $3i$ (n) $-4i$ (o) 5 (p) -2

Question 2 Convert the following to the form $r \operatorname{cis} \theta$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

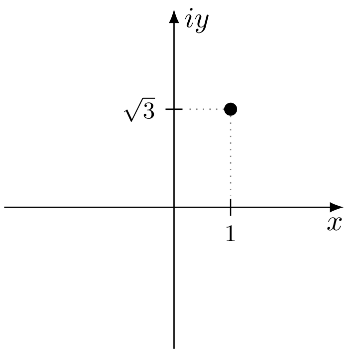
(a) $-2\sqrt{3} + 2i$ (b) $-\sqrt{2} - i\sqrt{6}$ (c) $i(1 - i)$

Question 3 Convert the following to Cartesian form.

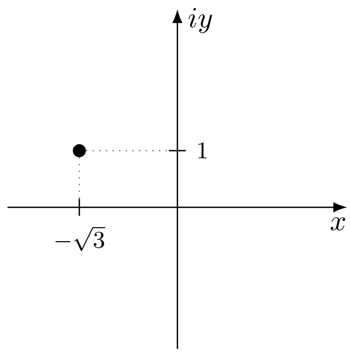
(a) $2 \operatorname{cis}\left(\frac{\pi}{3}\right)$ (b) $6 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ (c) $\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$

Question 4 The Argand diagrams below show a plot of z. Find the polar form of z.

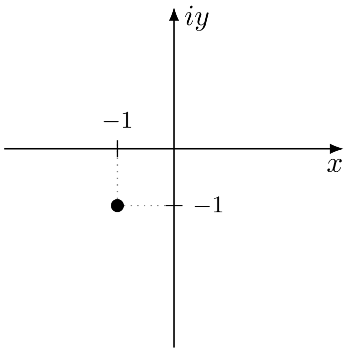
(a)



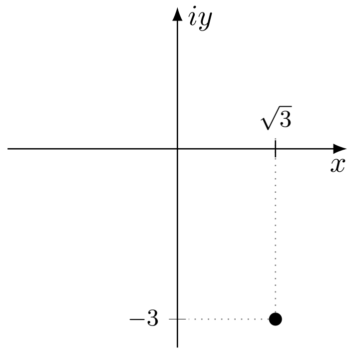
(b)



(c)



(d)



Question 5 Let $z = -1 + i$. Convert the following to polar form.

(a) $\bar{z}$ (b) z^{-1} (c) iz (d) z^2

Question 6 Express the following in the form $r \operatorname{cis} \theta$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

(a) $2 \operatorname{cis}\left(\frac{\pi}{6}\right) \times 4 \operatorname{cis}\left(\frac{\pi}{3}\right)$ (b) $4 \operatorname{cis}\left(\frac{2\pi}{3}\right) \times 3 \operatorname{cis}\left(\frac{-5\pi}{6}\right)$

(c) $6 \operatorname{cis}\left(\frac{7\pi}{6}\right) \div 2 \operatorname{cis}\left(\frac{\pi}{4}\right)$ (d) $4 \operatorname{cis}\left(\frac{3\pi}{4}\right) \div 2 \operatorname{cis}\left(\frac{-2\pi}{3}\right)$

Question 7 Simplify the following by first converting to polar form.

(a) $\dfrac{i}{1+\sqrt{3}i}$

(b) $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$

(c) $\dfrac{1-i}{\sqrt{3}-i}$

(d) $\dfrac{i(-\sqrt{3}+i)}{(-1+i)(1-i\sqrt{3})}$

Question 8

(a) Expand and simplify $(2+i)(3+i)$.

(b) Deduce that $\tan^{-1}\left(\dfrac{1}{2}\right)+\tan^{-1}\left(\dfrac{1}{3}\right)=\dfrac{\pi}{4}$.

Question 9 [To encourage use of the properties]

Let $z=-\sqrt{3}+3i$ and $w=\sqrt{2}-i\sqrt{2}$. Find the following.

(a) $|\bar{z}|$

(b) $|zw|$

(c) $\left|\dfrac{z}{w}\right|$

(d) $\left|\dfrac{\bar{z}}{z}\right|$

(e) $\arg\bar{z}$

(f) $\arg(z^2w)$

(g) $\arg\left(\dfrac{z}{w^2}\right)$

(h) $\arg(z\bar{z})$

Question 10 Let $z=1+i$ and $w=1+i\sqrt{3}$.

(a) Calculate zw and express in polar form.

(b) Deduce that $\sin\left(\dfrac{7\pi}{12}\right)=\dfrac{1+\sqrt{3}}{2\sqrt{2}}$ and $\cos\left(\dfrac{7\pi}{12}\right)=\dfrac{1-\sqrt{3}}{2\sqrt{2}}$.

Question 11 Prove the following by letting $z=r\operatorname{cis}\theta$.

(a) $z\bar{z}=|z|^2$

(b) If $|z|=1$, then $\bar{z}=z^{-1}$

Question 12 Let $z=\cos\theta+i\sin\theta$. Find z^2 using two different methods to prove the following.

(a) $\cos 2\theta=\cos^2\theta-\sin^2\theta$

(b) $\sin 2\theta=2\sin\theta\cos\theta$

Question 13 [Double-angle formulae can be used to simplify]

Show that

$$\frac{1}{1+\cos\theta+i\sin\theta}=\frac{1}{2}-\frac{1}{2}i\tan\left(\frac{\theta}{2}\right).$$

Challenge Problems

Problem 1 Let z be a complex number with argument θ and $\operatorname{Re}(z) \neq 0$. Show that

$$\frac{|z| - iz}{|z| + iz} = -i(\sec\theta + \tan\theta).$$

Problem 2 Let $z = \cos\theta + i\sin\theta$ and $w = \sin\theta + i\cos\theta$ where $0 < \theta < \frac{\pi}{2}$.

(a) Show that $|z + w| = 2\sin\left(\theta + \frac{\pi}{4}\right)$.

(b) What is the maximum value of $|z + w|$? For what value of θ does it occur?

Problem 3 Let $z = a + bi$ and $w = \frac{1}{a} + \frac{1}{b}i$, where a, $b > 0$.

(a) Calculate zw.

(b) Deduce that

$$\tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{2}.$$

Problem 4 Let $z = \cos\theta + i\sin\theta$. Show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$.

Problem 5 Prove that if $|z| = |w| = 1$, then

$$\arg(z + w) = \frac{1}{2}\left(\arg z + \arg w\right).$$

Problem 6 Prove that if $|z| = |w| = 1$ and $0 < \arg z < \arg w < \frac{\pi}{2}$, then

$$\arg(z - w) = \frac{1}{2}\left(\arg z + \arg w - \pi\right).$$

Problem 7 Define the polynomial

$$P(x) = x^2 + 2ax + b,$$

where $a^2 < b$. Let the zeroes of $P(x)$ be α and β, represented by A and B on the complex plane. Show that if $\angle AOB = \frac{\pi}{2}$, then $b = 2a^2$.

Exercise 2D

Vector Representation

Fundamentals

Fundamentals 1

(a) If the points A and B are represented by complex numbers a and b respectively, then

$$\overrightarrow{AB} = \_\_\_\_\_\_\_\_\_\_$$

(b) Similarly, the quantity $b - a$ represents the vector ______.

(c) When we say that a complex number represents a vector, what we mean is that the complex number represents the p__________ vector.

Fundamentals 2

Let the point $\overrightarrow{OP}$ be the position vector of the complex number z, and let $\alpha > 0$. Describe the effect on $\overrightarrow{OP}$ when we multiply z by

(a) 2 (b) $\operatorname{cis}\alpha$ (c) $\operatorname{cis}(-\alpha)$ (d) i

Note from the author: In this chapter, we will multiply vectors by $R\operatorname{cis}\theta$ to rotate the vector. This is not technically sound as we cannot multiply a *vector* by a *complex number*. What we really mean when we say $\overrightarrow{AB} \times R\operatorname{cis}\theta$ is that the *complex number* represented by $\overrightarrow{AB}$ is multiplied by $R\operatorname{cis}\theta$.

Question 1 [Practising using $\overrightarrow{AB} = b - a$]

Let $A = 3+2i$, $B = -5+i$ and $C = 4-3i$. Write down the complex number representing the position vectors of the following.

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BA}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{CB}$

Question 2 [Application of $\overrightarrow{AB} = b - a$]

Let $A = 3+2i$, $B = -5+i$ and $C = 4-3i$. Find the complex numbers representing P, Q and R if

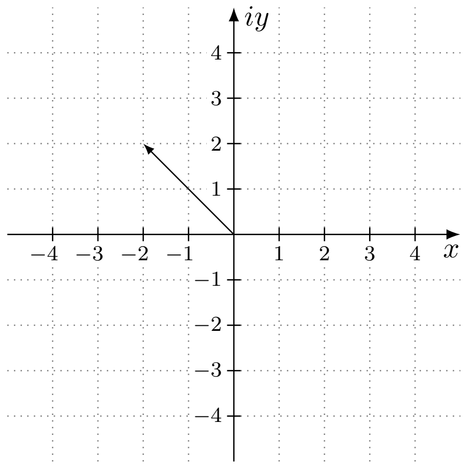
(a) $\overrightarrow{AP} = 8+4i$ (b) $\overrightarrow{QB} = 6-7i$ (c) $\overrightarrow{RC} = -3-5i$

Question 3 Let $z_1 = 3+2i$ and $z_2 = -1+4i$. Draw the following on separate Argand diagrams.

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $\bar{z}_1$

(d) $z_1 + \bar{z}_1$ (e) $z_1 - \bar{z}_1$ (f) $z_2 - \bar{z}_2$

Question 4 The diagram below displays a complex number z represented by the vector $\overrightarrow{OA}$. Roughly plot the following on the Argand diagram.



(a) $z \times 2\operatorname{cis}\left(\dfrac{\pi}{3}\right)$

(b) $z \times \dfrac{1}{2}\operatorname{cis}\left(-\dfrac{\pi}{3}\right)$

(c) $z \times i$

(d) $z \times (-i)$

Question 5 Let P be the point on the Argand diagram represented by a complex number z. What complex number should we multiply into z in order to

(a) double the length of $\overrightarrow{OP}$ and rotate it anti-clockwise by $\dfrac{5\pi}{6}$?

(b) halve the length of $\overrightarrow{OP}$ and rotate it clockwise by $\dfrac{\pi}{3}$?

(c) preserve the length of $\overrightarrow{OP}$ and rotate it clockwise by $\dfrac{\pi}{2}$?

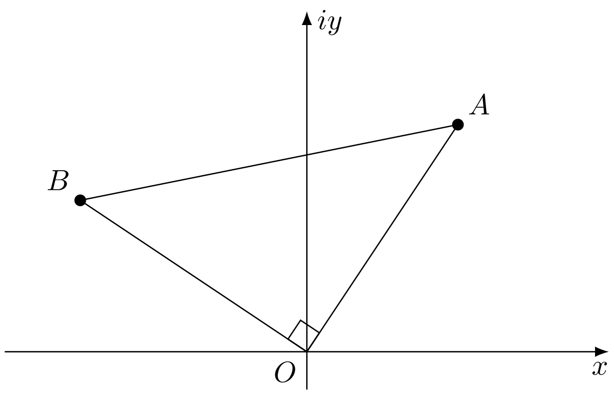
Question 6 State the effect of multiplying a vector by

(a) $1 + i\sqrt{3}$

(b) $\dfrac{1}{2\sqrt{2}} - \dfrac{1}{2\sqrt{2}}i$

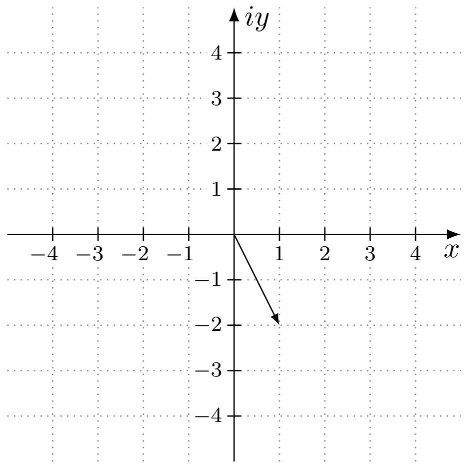
(c) $-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

Question 7 Suppose $a = 4+6i$ and b is some complex number such that $\triangle AOB$ forms a right-angled isosceles triangle and $\arg b > \arg a$.



Find the complex number represented by B.

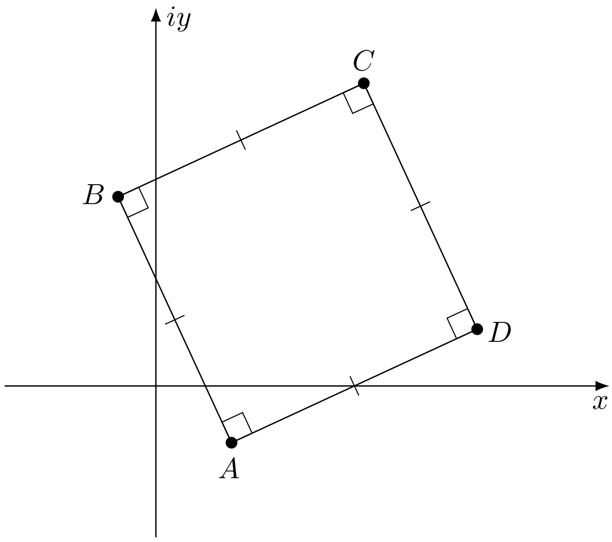
Question 8 The diagram below shows the vector represented by a complex number z.



Plot the position vectors represented by the following complex numbers.

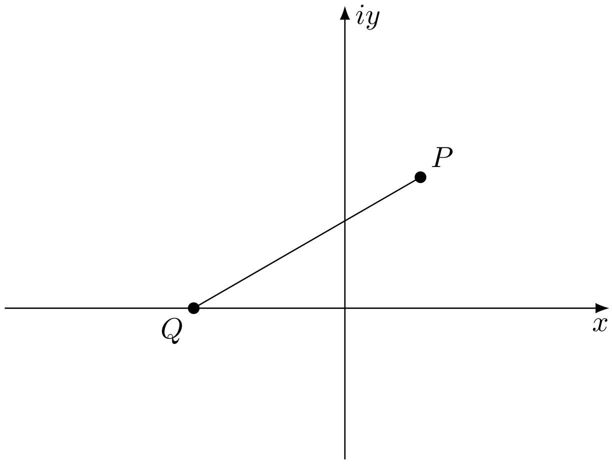
(a) iz (b) z^2 (c) $z\bar{z}$ (d) $(1-i)z$

Question 9 Let A and C be points representing $1-i$ and $3+4i$ respectively. Suppose that AC is a diagonal of a square $ABCD$.



Find the complex numbers represented by B and D.

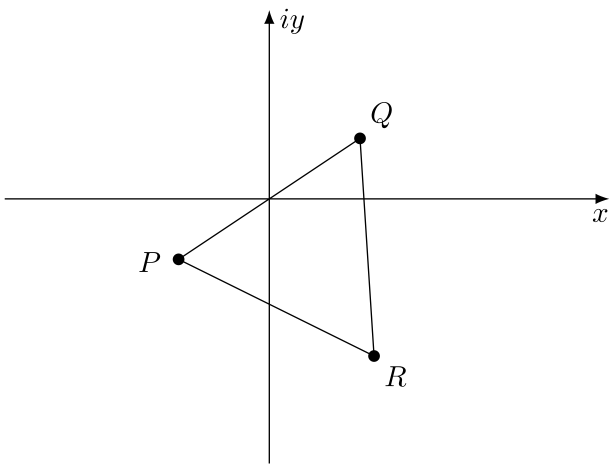
Question 10 Let $P = 1 + i\sqrt{3}$ and $Q = -2 + 0i$ be two points of an equilateral triangle $\triangle PQR$.



(a) Draw all possible positions of R.

(b) Find the complex number represented by R for all cases.

Question 11 Let $P = -6 - 4i$, $Q = 6 + 4i$ and R form an equilateral triangle. Let $PQRS$ be a rhombus where the labelling of the vertices is not necessarily in order.

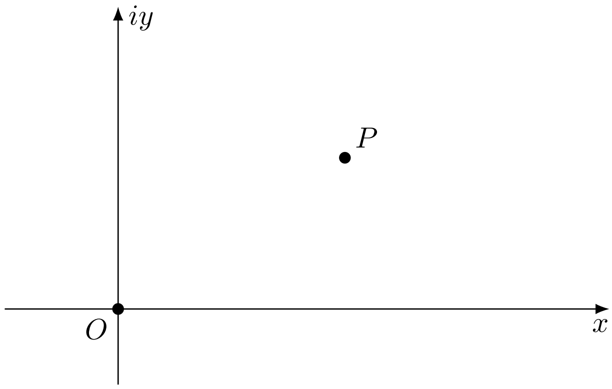


(a) Find the complex number represented by R in the diagram above.

(b) Draw all possible positions of S.

(c) Find the complex number represented by S for all cases.

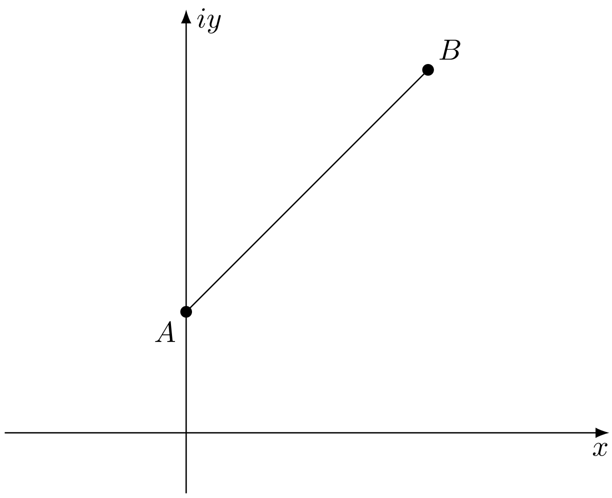
Question 12 Let $P = 6 + 4i$ be a point on a right-angled isosceles triangle $\triangle OPQ$.



(a) Draw all possible triangles $\triangle OQP$ if the right-angle occurs at Q, and find the coordinates of Q for each scenario.

(b) Draw all possible triangles $\triangle OPQ$ if the right-angle occurs at P, and find the coordinates of Q for each scenario.

Question 13 Let A and B be points representing $2i$ and $4 + 6i$ respectively.



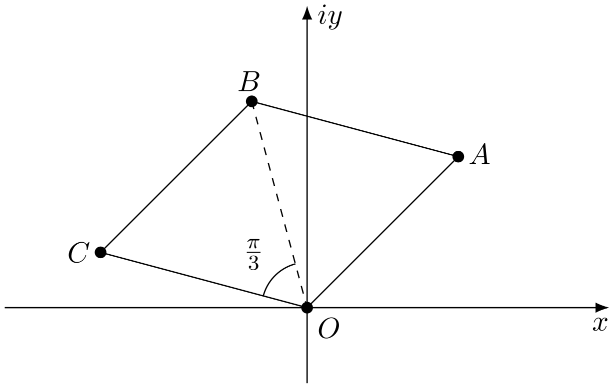
Suppose AB is one side of a square $ABCD$.

(a) Draw the two possible scenarios.

(b) Find the complex numbers representing C and D for both scenarios.

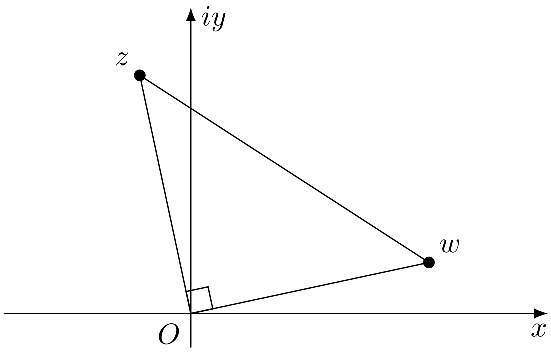
(c) Find the complex numbers representing C and D if AB was instead the *diagonal* of the square.

Question 14 Let $OABC$ represent a rhombus, where A represents the complex number $1+i$. Let B be in the second quadrant and $\angle BOC = \dfrac{\pi}{3}$.



Find the complex numbers that represent points B and C.

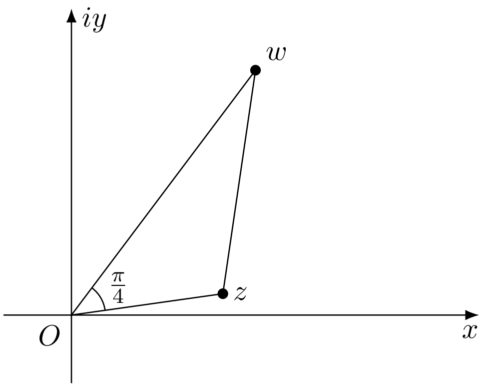
Question 15 The origin, w and z form a right-angled isosceles triangle such that $|z| = |w|$.



(a) Explain why either $z = iw$ or $w = iz$. Which one is shown in the diagram above?

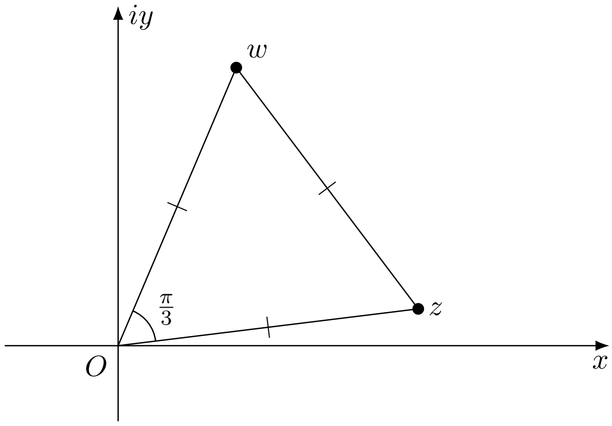
(b) Deduce that in either case $z^2 + w^2 = 0$.

Question 16 The diagram below shows the origin, w and z forming a triangle with angle $\dfrac{\pi}{4}$ at the origin such that $|w| = 2|z|$.



Prove that $w^4 + 16z^4 = 0$.

Question 17 The origin, w and z form an equilateral triangle.



(a) Prove that $z^3 + w^3 = 0$.

(b) Expand $(z + w)(z^2 - zw + w^2)$.

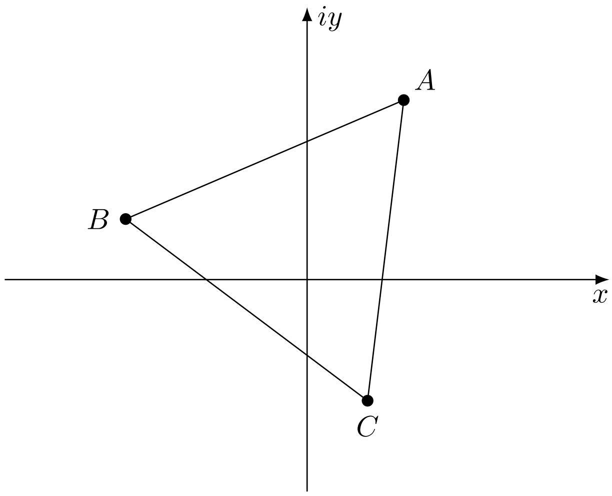
(c) Deduce that $z^2 + w^2 = zw$.

Question 18 Suppose that z_1, z_2 and z_3 are represented by A, B and C respectively on the complex plane, and $z_2 - z_1 = i(z_3 - z_1)$. What type of triangle is $\triangle ABC$?

Question 19 Show that if $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then z_1, z_2 and z_3 form an equilateral triangle on the unit circle.

Question 20 Suppose that z_1, z_2 and z_3 are represented by the points A, B and C respectively on the complex plane. Show that if A, B and C are collinear, then $\dfrac{z_3 - z_1}{z_2 - z_1}$ is a real number.

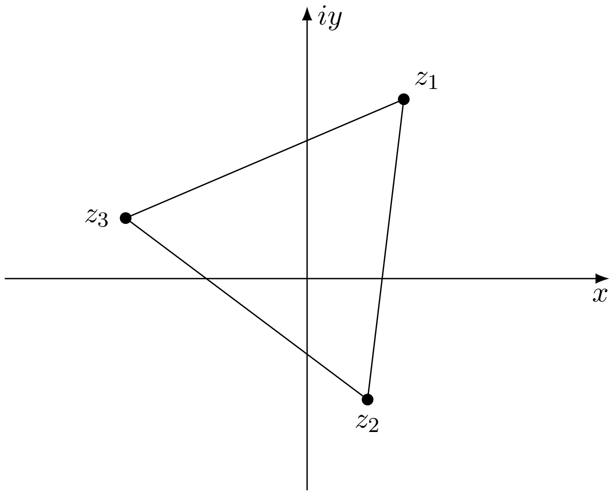
Question 21 Let a, b and c be complex numbers representing the vertices of $\triangle ABC$, as shown in the diagram below. Let $\omega = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$.



Show that $\triangle ABC$ is equilateral if and only if $a + b\omega + c\omega^2 = 0$.

Challenge Problems

Problem 1 Let z_1, z_2 and z_3 form an equilateral triangle as shown below.



(a) Show that $\dfrac{z_2 - z_1}{z_3 - z_1} = \text{cis}\left(\dfrac{\pi}{3}\right)$.

(b) Deduce that $z_1^2 + z_2^2 + z_3^2 = z_1z_3 + z_1z_2 + z_2z_3$.

Problem 2 If $|z| = |w|$, show that $\dfrac{z+w}{z-w}$ is purely imaginary.

Problem 3 Let z and w be non-real numbers such that $|z+w| = |z-w|$.

(a) What can we say about the origin, z, w and $z+w$?

(b) Hence, what is $\arg\left(\dfrac{z}{w}\right)$?

Problem 4 What can we say about z_1, z_2 and z_3 if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is a purely imaginary number?

Problem 5 [Cross-ratio]

Show that if

$$\frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

is real, then z_1, z_2, z_3 and z_4 are either concyclic or collinear.

Problem 6 [Application of circle geometry]

Prove that if the origin, z_1, z_2 and z_3 are concyclic, then $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear.

Exercise 2E

Locus

Fundamentals

Fundamentals 1

(a) Let z be some general complex number on the A________ diagram. If z has no restrictions, then it is free to be anywhere.

(b) However, if a condition is imposed upon z, then z must lie on a particular s___ of points. When we connect those points, it often forms some kind of l_____, c_____, or r_____.

(c) The set of all allowable points of z is called the l_____ of z.

Fundamentals 2

Describe the general shape of the locus of z if z is

(a) exactly r units away from a fixed point C.

(b) at most r units away from a fixed point C.

(c) at least r units away from a fixed point C.

(d) equidistant from two fixed points A and B.

(e) closer to A than it is to B.

(f) inclined at an angle of θ from the horizontal with respect to a point C.

Question 1 Sketch the locus of z defined by each condition.

(a) $|z| = 1$

(b) $|z - (2 + 3i)| = 2$

(c) $|z - 4 - 3i| = 3$

(d) $|z - 1 - i| = \sqrt{2}$

Question 2 Sketch the locus of z defined by each condition.

(a) $\arg(z) = \dfrac{\pi}{3}$

(b) $\arg(z) = -\dfrac{\pi}{4}$

(c) $\arg(z - 1) = \dfrac{2\pi}{3}$

(d) $\arg(z + 2) = -\dfrac{\pi}{6}$

(e) $\arg(z - (3 - 4i)) = \dfrac{\pi}{3}$

(f) $\arg(z + 4 - 3i) = \dfrac{\pi}{4}$

Question 3 Sketch the locus of z defined by each condition.

(a) $|z| > 1$

(b) $|z| \leq 1$

(c) $1 \leq |z - 2| \leq 2$

(d) $1 < |z + 1 - 2i| \leq 2$

Question 4 Sketch the locus of z defined by each condition.

(a) $\text{Re}(z) = 2$

(b) $-1 < \text{Im}(z) \leq 1$

(c) $\text{Re}(z) \geq -1$

(d) $\text{Re}(z) = \text{Im}(z)$

Question 5 Find the Cartesian equation and sketch the locus of z defined by each condition.

(a) $|z-2| = |z+4|$

(b) $|z+i| = |z-3i|$

(c) $|z-2i| = |z+2|$

(d) $|z-2+i| = |z-4-i|$

Question 6 Sketch the locus of z defined by each condition.

(a) $|z-4i| \geq |z-4|$

(b) $|z-2-i| < |z+1+4i|$

Question 7 Sketch the locus of z defined by each condition.

(a) $\dfrac{\pi}{3} \leq \arg z < \dfrac{2\pi}{3}$

(b) $0 \leq \arg(z-2) < \dfrac{2\pi}{3}$

(c) $0 \leq \arg(z-1+i) \leq \dfrac{3\pi}{4}$

(d) $-\dfrac{\pi}{3} \leq \arg(z+1-i\sqrt{3}) \leq \dfrac{\pi}{3}$

Question 8 Sketch the intersection of each pair of loci.

(a) $|z+1-2i| \leq 3$ and $-\dfrac{\pi}{3} \leq \arg z \leq \dfrac{\pi}{4}$

(b) $|z-1| \leq 2$ and $-\dfrac{\pi}{4} \leq \arg(z-1) \leq \dfrac{\pi}{4}$

(c) $|z+2| \geq 2$ and $|z-i| \leq 1$

(d) $1 \leq |z| \leq 2$ and $0 \leq z+\bar{z} \leq 3$

(e) $|z+\bar{z}| \leq 1$ and $|z-i| \leq 1$

(f) $|z-3+i| \leq 5$ and $|z+1| \leq |z-1|$

Question 9 Sketch the locus of z defined by each condition.

(a) $\arg(z-1) - \arg(z-i) = 0$

(b) $\arg(z-1) - \arg(z-i) = \pi$

Question 10 Find the Cartesian equation and sketch the locus of z defined by each condition.

(a) $|z|^2 = z+\bar{z}+1$

(b) $z^2 - \bar{z}^2 = 4i$

(c) $z+\bar{z} = \text{Im}(z)$

(d) $|z-i| = \text{Im}(z)+1$

(e) $\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z} = 2$

(f) $\dfrac{1}{z} + \dfrac{1}{\bar{z}} = 1$

Question 11 Consider the locus of z defined by $\arg(z-2) = \dfrac{\pi}{3}$.

(a) Find the range of values of $|z|$.

(b) Find the range of values of $\arg(z)$.

Question 12 Consider the locus of z defined by $|z-1-i| = 1$.

(a) Find the maximum value of $|z|$.

(b) Find the minimum value of $|z|$.

(c) Find the maximum value of $\arg(z)$.

(d) Find the minimum value of $\arg(z)$.

Question 13 Consider the locus of z defined by $|z - 4i| = 3$.

(a) Find the maximum value of $|z|$.

(b) Find the minimum value of $|z|$.

(c) Find the maximum value of $\arg(z)$.

(d) Find the minimum value of $\arg(z)$.

Question 14 Find the Cartesian equation and sketch the locus of z defined by each condition.

(a) $2|z - 1| = |z + 1|$

(b) $|2z - 1| = |z - 2|$

Question 15 Sketch the locus of z defined by each condition.

(a) $\dfrac{z-1}{z+i}$ is purely real.

(b) $\dfrac{1-z}{z}$ is purely imaginary.

(c) $\dfrac{z-i}{z+1}$ is purely real.

(d) $z - \dfrac{1}{z}$ is purely imaginary.

Question 16 [Circle Geometry]

Sketch the locus of z defined by each condition.

(a) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$

(b) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$

(c) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{2\pi}{3}$

Question 17 Find and sketch the locus of z if

$$\left|1 + \frac{1}{z}\right| \leq 1.$$

Question 18 Suppose $a, b \in \mathbb{R}$ and $a \neq b$. Let z be a complex number such that

$$|z - a|^2 - |z - b|^2 = |a^2 - b^2|.$$

Find and describe geometrically the locus of z.
Hint: There are two cases here!

Question 19 [A familiar result!]

Let a and b be complex numbers. Describe geometrically the locus of

$$|z - a|^2 + |z - b|^2 = |a - b|^2.$$

without explicitly letting $z = x + iy$.

Challenge Problems

Problem 1 [Möbius Transformations can map circles to circles]

Consider the expression

$$w = \frac{az+b}{z},$$

where a, b are real non-zero constants. Show that if z lies on the unit circle, then w will always lie on a circle centred at $(a,0)$ with radius b.

Problem 2 [Cool way to prove a classic problem!]

Let the origin, z_1, z_2 and z_3 be four points on a circle centred at c in the complex plane.

(a) Describe the locus of w satisfying $|w-k| = |w|$, where k is a fixed complex number.

(b) Deduce that $\frac{1}{z_1}$, $\frac{1}{z_2}$ and $\frac{1}{z_3}$ are collinear.

Problem 3 Show that the roots of

$$(z+1)^n + (z-1)^n = 0$$

are purely imaginary, by considering an appropriate locus.

Problem 4 [More on Möbius Transformations]

Find the locus of z if

$$\left|\frac{kz-1}{z-k}\right| = 1$$

where $k \in \mathbb{R}$ and $k \neq \pm 1$.

Problem 5 [Circle of Apollonius]

Let z_1 and z_2 be fixed complex numbers, and suppose $m > n > 0$. The locus of z given by

$$m|z-z_1| = n|z-z_2|$$

is a circle called the *Circle of Apollonius*.

(a) Show that the centre of the circle is $\dfrac{m^2 z_1 - n^2 z_2}{m^2 - n^2}$.

(b) Show that the radius is $\dfrac{mn}{m^2-n^2}|z_2 - z_1|$.

Exercise 2F

De Moivre's Theorem

Fundamentals

Fundamentals 1
Let $z = \cos\theta + i\sin\theta$. De Moivre's theorem states that for all $n \in \mathbb{Z}$

$$z^n = \_\_\_\_\_ + i\,\_\_\_\_\_$$

Fundamentals 2
The following results follow from de Moivre's theorem for all integer values of n.

(a) $|z^n| = \_\_\_\_\_$ (b) $\arg(z^n) = \_\_\_\_\_$ (c) $\overline{z^n} = \_\_\_\_\_$

Question 1 Simplify the following.

(a) $(\cos 2\theta + i\sin 2\theta)^3$
(b) $(\cos\theta - i\sin\theta)^5$
(c) $(\cos 3\theta + i\sin 3\theta)^{-2}$
(d) $(\cos 3\theta - i\sin 3\theta)^{-1}$

Question 2 Simplify the following.

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^4$
(b) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-4}$
(c) $\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$

Question 3 Let $z = 1 + i\sqrt{3}$.

(a) Find the modulus and argument of z.
(b) Express z in polar form.
(c) Hence, express z^4 in polar form.
(d) Express $\left(1 + i\sqrt{3}\right)^4$ in Cartesian form.

Question 4 Express the following in Cartesian form.

(a) $(1+i)^5$
(b) $(1+i\sqrt{3})^6$
(c) $(-\sqrt{3}+i)^3$
(d) $(\sqrt{3}-i)^7$
(e) $(1-i)^4$
(f) $(1-i\sqrt{3})^9$
(g) $(-\sqrt{3}-i)^6$
(h) $(-\sqrt{3}+3i)^3$
(i) $(-2+2i)^5$

Question 5 Show that

$$\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta.$$

Question 6 Show that for all $n \in \mathbb{Z}$

$$\big(\cos\theta - i\sin\theta\big)^n = \cos n\theta - i\sin n\theta.$$

Question 7 Let $z = \cos\theta + i\sin\theta$. Simplify the following.

(a) $\dfrac{z^n + \bar{z}^n}{2}$ (b) $\dfrac{z^n - \bar{z}^n}{2i}$

Question 8 [Proof of de Moivre's theorem]

(a) Use mathematical induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$.

(b) Deduce that de Moivre's theorem holds true for $n \in \mathbb{Z}$ in general.

Question 9 Simplify the following expression.

$$\frac{\big(\cos 3\theta + i\sin 3\theta\big)^2\big(\cos\theta - i\sin\theta\big)^4}{\big(\cos 2\theta - i\sin 2\theta\big)^{-3}}$$

Question 10 Find the values of n for which $(\sqrt{3} - i)^n$ is

(a) purely real. (b) purely imaginary.

Question 11 Find the values of n for which $(1 + i)^n$ is

(a) purely real. (b) purely imaginary.

Question 12 Find the values of n for which $(1 + i\sqrt{3})^n$ is

(a) purely real. (b) purely imaginary.

Question 13 Consider the complex number

$$z = (1 + i)^n + (1 - i)^n.$$

(a) Show that z is always real for $n \in \mathbb{Z}^+$.

(b) Can the same be said for $(1 + i)^n - (1 - i)^n$?

(c) Show that if n is a multiple of 4, then z is an integer.

Question 14 Suppose that z_1, z_2 and z_3 are complex numbers that all lie on the unit circle. Show that if $z_1 + z_2 + z_3 = 0$ then

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0.$$

Question 15

(a) Show that

$$(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right).$$

(b) Hence, show that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n$ is always an integer for $n \in \mathbb{Z}$.

Question 16 Simplify

$$\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n.$$

Question 17 Prove the following identity.

$$(1+\cos 2\theta + i\sin 2\theta)^n + (1+\cos 2\theta - i\sin 2\theta)^n = 2^{n+1}\cos^n\theta\cos n\theta$$

Question 18 Prove the following identity.

$$(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n\theta\cos n\theta$$

Question 19 Let $z = \cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)$. Show that

$$1+z+z^2+\cdots+z^{n-1} = 1+i\cot\left(\frac{\pi}{2n}\right).$$

Question 20 Let $z = \cos 2\theta + i\sin 2\theta$ and $w = \cos\theta + i\sin\theta$.

(a) Show that $1+z = 2w\mathrm{Re}\,(w)$.

(b) Hence, show that

$$\sum_{k=0}^{n}\binom{n}{k}\cos(2k\theta) = 2^n\cos^n\theta\cos n\theta.$$

Challenge Problems

Problem 1 [Dirichlet kernel]

Let $z = \cos\theta + i\sin\theta$. Define the series

$$S_n = 1 + z + z^2 + \cdots + z^n.$$

(a) Show that $S_n = \dfrac{\operatorname{cis}(n+1)\theta - 1}{\operatorname{cis}\theta - 1}$.

(b) Show that

$$S_n = \operatorname{cis}\left(\frac{n\theta}{2}\right) \times \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

(c) Deduce that

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin\left(\frac{n}{2}\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Problem 2 [Establishing one of the key steps for the Basel Problem]

Define the polynomial

$$P(x) = (x+i)^n - (x-i)^n,$$

where n is a positive integer.

(a) Show that

$$(\cot\theta + i)^n - (\cot\theta - i)^n = \frac{2i\sin n\theta}{\sin^n\theta}.$$

(b) Show that the zeroes of $P(x)$ are $x = \cot\left(\dfrac{k\pi}{n}\right)$ where $k = 1, 2, \cdots, n-1$.

(c) Hence, show that

$$\cot^2\left(\frac{\pi}{n}\right) + \cot^2\left(\frac{2\pi}{n}\right) + \cot^2\left(\frac{3\pi}{n}\right) + \cdots + \cot^2\left(\frac{(n-1)\pi}{n}\right) = \frac{(n-1)(n-2)}{3}.$$

(d) Deduce that

$$\csc^2\left(\frac{\pi}{n}\right) + \csc^2\left(\frac{2\pi}{n}\right) + \csc^2\left(\frac{3\pi}{n}\right) + \cdots + \csc^2\left(\frac{(n-1)\pi}{n}\right) = \frac{(n-1)(n+1)}{3}.$$

Exercise 2G

Applications of de Moivre's Theorem

Fundamentals

Fundamentals 1

Let α, β and γ be the roots of $ax^3 + bx^2 + cx + d = 0$. Write the following in terms of a, b, c, d.

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \alpha\gamma$ (c) $\alpha\beta\gamma$

Fundamentals 2

To express $\cos(n\theta)$ as a polynomial in terms of $\cos\theta$, follow the following steps.

(a) Define $z =$ ________ $+ i$ ________.

(b) Simplify z^n using ______________ theorem.

(c) Expand z^n manually using P______ triangle, or using the b________ expansion.

(d) Equate the real/imaginary (circle one) components of the two expressions for z^n.

(e) Turn all even powers of $\sin\theta$ into powers of ________ using the identity ______________.

Fundamentals 3

To express $\tan(n\theta)$ as a rational expression in terms of $\tan\theta$, follow the following steps.

(a) Obtain expressions for $\cos n\theta$ and ______ using the same steps outlined above. Keep the expressions as they are after equating real/imaginary components and do not modify them.

(b) D________ the two expressions to obtain $\tan n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(c) Divide the top and bottom by the highest power cosine/sine (circle one) term.

(d) Simplify and express everything in terms of ________

Question 1 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$

(b) $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

(c) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(d) $\cot 3\theta = \dfrac{3\cot^2\theta - 1}{\cot^3\theta - 3\cot\theta}$

Question 2 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Question 3 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(b) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$

(c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(d) $\cot 4\theta = \dfrac{4\cot^3\theta - 4\cot\theta}{\cot^4\theta - 6\cot^2\theta + 1}$

Question 4 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta$

(b) $\sin 5\theta = \sin^5\theta + 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta$

Question 5 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

(b) $\sin 6\theta = 6\sin\theta\cos^5\theta - 20\sin^3 x\cos^3\theta + 6\sin^5\theta\cos\theta$

Question 6 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$ for $n \in \mathbb{Z}^+$.

(b) Hence, show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$.

(c) Calculate $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$.

Question 7

(a) Use a similar technique to the previous question to prove that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(b) By finding a suitable substitution for θ, deduce that

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$

(c) Hence, show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$.

Question 8

(a) Show that

$$\cos^6\theta = \frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta).$$

(b) Find a similar result for $\sin^6\theta$.

Question 9 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Hence, solve $z^4 + 4z^3 + 2z^2 + 4z + 1 = 0$.

Question 10 [Guided question for a classic problem]

Define the following cubic polynomial.

$$P(x) = 8x^3 - 6x - 1$$

You may assume that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(a) Let $x = \cos\theta$. Show that solving $P(x) = 0$ is equivalent to solving $\cos 3\theta = \frac{1}{2}$.

(b) Solve the trigonometric equation to find 3 distinct values of θ.

(c) Hence, write down the three zeroes of $P(x)$.

(d) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{7\pi}{9}\right).$$

(e) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right).$$

(f) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right).$$

Question 11 You may assume that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(a) Find the zeroes of $P(x) = 8x^3 - 6x + 1$.

(b) Show that

$$\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = \cos\left(\frac{\pi}{9}\right).$$

(c) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right).$$

Question 12

(a) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

(b) Hence, solve $8x^4 - 8x^2 + 1 = 0$.

(c) Hence, find the exact values of $\cos\left(\frac{\pi}{8}\right)$ and $\cos\left(\frac{5\pi}{8}\right)$.

Question 13 You may assume that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(a) Solve the polynomial equation $t^3 - 3t^2 - 3t + 1 = 0$.

(b) Find the exact value of $\tan\left(\frac{\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$.

Question 14

(a) Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Hence, solve the polynomial equation $16x^4 - 20x^2 + 5 = 0$.

(c) Hence, show that

$$\cos\left(\frac{\pi}{10}\right) = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

(d) Write down the exact value of $\cos\left(\frac{3\pi}{10}\right)$.

Question 15

(a) Prove that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

(b) Show that $x = \sin\left(\frac{\pi}{10}\right)$ is a solution to the polynomial equation $16x^5 - 20x^3 + 5x - 1 = 0$.

(c) Find the polynomial $P(x)$ such that $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$.

(d) Find the value of a such that $P(x) = (4x^2 + ax - 1)^2$.

(e) Hence, find an exact value for $\sin\left(\frac{\pi}{10}\right)$.

Question 16

(a) Show that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

(b) Hence, find all the roots of the polynomial $32x^6 - 48x^4 + 18x^2 - 1 = 0$.

(c) Show that

$$\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) = \frac{1}{4}.$$

(d) Find the exact value of

$$\cos^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{5\pi}{12}\right).$$

(e) Hence, show that

$$\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}}{2}.$$

Question 17

(a) Show that

$$\cot 6\theta = \frac{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}.$$

(b) Find the exact value of $\tan\left(\frac{\pi}{12}\right)\tan\left(\frac{5\pi}{12}\right)$.

(c) Find the exact value of $\tan^2\left(\frac{\pi}{12}\right) + \tan^2\left(\frac{5\pi}{12}\right)$.

(d) Hence, find $\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{5\pi}{12}\right)$.

(e) Write down the equation of the quadratic polynomial that has roots $\tan\left(\frac{\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$.

(f) Hence, find the exact value of $\tan\left(\frac{\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$.

Question 18

(a) Prove that

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}.$$

(b) Hence, find the roots of the polynomial $t^4 - 10t^2 + 5 = 0$.

(c) Show that $\tan\left(\frac{\pi}{5}\right)\tan\left(\frac{2\pi}{5}\right) = \sqrt{5}$.

(d) Show that $\tan\left(\frac{\pi}{5}\right) + \tan\left(\frac{2\pi}{5}\right) = \sqrt{10 + 2\sqrt{5}}$.

(e) Hence, prove that $\tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$.

(f) Find the exact value of $\tan\left(\frac{2\pi}{5}\right)$ and justify your answer.

Challenge Problems

Problem 1 Show that if $-1 < r < 1$, then

$$1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}.$$

Problem 2 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^k + z^{-k} = 2\cos k\theta$, where $k \in \mathbb{Z}^+$.

(b) Let $n \in \mathbb{Z}^+$. Prove the following identity.

$$(2\cos\theta)^{2n} = 2\sum_{k=0}^{n}\binom{2n}{k}\cos(2n-2k)\theta.$$

(c) Hence, prove that

$$\int_0^{\frac{\pi}{2}} \cos^{2n}\theta\, d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

Problem 3 Let $z = \cos\theta + i\sin\theta$.

(a) Simplify $\left(z + \frac{1}{z}\right)^n z^n$.

(b) Show that

$$2^n\cos^n\theta\cos(n\theta) = \sum_{k=0}^{n}\binom{n}{k}\cos(2k\theta).$$

(c) Hence, show that

$$\int_{-\pi}^{\pi} \cos^n\theta\cos n\theta\, d\theta = \frac{\pi}{2^{n-1}}.$$

Exercise 2H

Roots of Unity

Fundamentals

Fundamentals 1

(a) The complex number ω is called a *root of unity* if $\omega^n =$ ___ for some $n \in \mathbb{Z}$, $n \neq 0$.

(b) The p________ root of unity is the root of unity with the smallest positive a________.

Fundamentals 2

Let ω be the principal cube root of unity.

(a) $\omega^3 =$ ___

(b) $1 + \omega + \omega^2 =$ ___

Fundamentals 3

Let ω be the principal n^{th} root of unity.

(a) $\omega^n =$ ___

(b) $1 + \omega + \omega^2 + \cdots + \omega^{n-1} =$ ___

Fundamentals 4

Follow the following steps to find the n^{th} root of any number a.

(a) Construct the polynomial equation $z^n =$ ___.

(b) Convert a to p_____ form.

(c) Add _____ to the argument, where $k \in \mathbb{Z}$. The polynomial equation is now

$$z^n = r\operatorname{cis}(\theta + 2k\pi).$$

(d) Raise both sides to the power of ___ to make z the subject. The solutions are

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

for $k = 0, \pm 1, \pm 2, \ldots$ until n values have been listed. If n is even then you will eventually need to decide whether to pick the positive or negative value of k. Choose the one that satisfies the p________ argument.

(e) List out all n solutions. If $a \in \mathbb{R}$ then the roots will occur in complex c________ pairs.

Question 1 Let ω be the principal cube root of unity.

(a) Write down the value of ω^3.

(b) Hence show that $1+\omega+\omega^2=0$.

Question 2 Let $P(z)=z^3-1$ and let ω be the principal root of $P(z)=0$.

(a) Find all the roots of $P(z)$.

(b) Plot them on the Argand diagram.

(c) Show that the other non-real root is ω^2.

(d) Prove that $\omega^2=\overline{\omega}$.

Question 3 Let ω be the principal cube root of unity. Simplify the following.

(a) $(1+\omega^{-1})(1+\omega^{-2})$

(b) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

(c) $\dfrac{1}{1+\omega}+\dfrac{1}{1+\omega^2}$

(d) $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$

Question 4 Let ω be the principal cube root of unity. Find the equation of the quadratic polynomial with zeroes $(1+\omega)$ and $(1+\omega^2)$.

Question 5 Sketch the roots of the following.

(a) $z^3=1$

(b) $z^4=1$

(c) $z^5=1$

(d) $z^6=1$

Question 6 Find all the roots of the following.

(a) $z^3+1=0$

(b) $z^3+27i=0$

(c) $z^3-4-4\sqrt{3}i=0$

Question 7 Find all the roots of the following.

(a) $z^4+1=0$

(b) $z^4-16i=0$

(c) $z^4+\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i=0$

Question 8 Find all the roots of the following.

(a) $z^5-1=0$

(b) $z^5+32i=0$

(c) $z^5-16\sqrt{2}+16\sqrt{2}i=0$

Question 9 Find all the roots of the following.

(a) $z^6-1=0$

(b) $z^6+64i=0$

(c) $z^6-32\sqrt{3}+32i=0$

Question 10 Let ω be the principal n^{th} root of unity.

(a) Prove that ω^k for $k\in\mathbb{Z}$ are also roots of unity.

(b) Deduce that $1+\omega+\omega^2+\cdots+\omega^{n-1}=0$.

Question 11 [Important exercise to match the roots of unity with their conjugate pairs]

Let ω be the principal n^{th} root of unity.

(a) Prove that $\overline{\omega} = \omega^{n-1}$.

(b) Prove that in general $\overline{\omega^k} = \omega^{n-k}$.

(c) Consider the 9 roots of $z^9 = 1$ and let ω be the principal root of unity. Write down all complex conjugate pairs. For example, the conjugate of ω is ω^8.

Question 12 Let ω be a non-real root of $z^5 = 1$.

(a) Show that $1 + \omega + \omega^2 + \ldots + \omega^4 = 0$.

(b) Hence, show that $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$.

Question 13 Let ω be a non-real root of $z^9 = 1$.

(a) Show that $1 + \omega + \omega^2 + \ldots + \omega^8 = 0$.

(b) Hence, show that $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = \cos\left(\frac{\pi}{9}\right)$.

Question 14 Let $P(z) = z^7 + 1$, and let ω be the principal root of unity.

(a) Find all the roots of $P(z) = 0$.

(b) Hence, show that $\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}$.

Question 15 Let ω be a seventh root of unity. Find the equation of the quadratic polynomial with roots $\omega + \omega^2 + \omega^4$ and $\omega^3 + \omega^5 + \omega^6$.

Question 16 Let ω be an n^{th} root of unity. Find the value of

$$\sum_{k=0}^{n} \left(\omega^k + \frac{1}{\omega^k}\right).$$

Question 17 Show that the roots of $(z+1)^8 - z^8 = 0$ are

$$z = -\frac{1}{2},\ -\frac{1}{2}\left(1 \pm i \cot\left(\frac{k\pi}{8}\right)\right),$$

where $k = 1,\ 2,\ 3$.

Challenge Problems

Problem 1 Let ω be a cube root of unity. Find all possible values of

$$1 + \omega^n + \omega^{2n}$$

and state when they occur.

Problem 2 Let ω be an n^{th} root of unity. Show that

$$(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1) = n.$$

Problem 3 Find all the roots of

$$(z+1)^n + (z-1)^n = 0$$

and show that they are purely imaginary.

Problem 4 Let ω be the $(n+1)^{\text{th}}$ root of unity. Show that

$$\sum_{k=1}^{n} \frac{1}{1-\omega^k} = \frac{n}{2}.$$

Problem 5 Find the value of $\displaystyle\sum_{k=0}^{n} \cos\left(\frac{2k\pi}{2n+1}\right)$.

Problem 6 Let $\alpha_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$ for $k = 1, 2, \ldots, n$.

(a) Show that

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1}) = n.$$

(b) Hence, show that

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\ldots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Exercise 2I

Applications of Roots of Unity

Fundamentals

Fundamentals 1

Let α be any complex number.

$$(z-\alpha)(z-\bar{\alpha}) = \rule{6cm}{0.4pt}$$

Fundamentals 2

(a) If n is any positive integer, then

$$z^n - 1 = (z-1)(\rule{7cm}{0.4pt}).$$

(b) If n is any positive *odd* integer, then

$$z^n + 1 = (z+1)(\rule{7cm}{0.4pt}).$$

Fundamentals 3

(a) If a polynomial $P(z)$ is factorised over $\mathbb{Z}$, it means that the factorisation contains only coefficients within the set of i________.

(b) If a polynomial $P(z)$ is factorised over $\mathbb{R}$, it means that the factorisation contains only coefficients within the set of r______ numbers.

Question 1 Expand the following using the identity

$$(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\mathrm{Re}\,(\alpha)\,z + |\alpha|^2.$$

(a) $(z-1+i)(z-1-i)$

(b) $(z-3+2i)(z-3-2i)$

Question 2 Express the following as the product of real quadratic factors

$$(z-\alpha)(z-\bar{\alpha})(z-\beta)(z-\bar{\beta}).$$

Question 3 [Useful identities for this topic]

Prove the following by expanding the right-hand side.

(a) $z^3+1 = (z+1)(z^2-z+1)$

(b) $z^3-1 = (z-1)(z^2+z+1)$

(c) $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$

(d) $z^5-1 = (z-1)(z^4+z^3+z^2+z+1)$

(e) $z^6+1 = (z^2+1)(z^4-z^2+1)$

(f) $z^6-1 = (z^2-1)(z^4+z^2+1)$

Question 4 Write down a factorisation of the following over $\mathbb{Z}$.

(a) z^7+1 (b) z^7-1 (c) z^8-1

Question 5 Let $P(z)=z^5-1$.

(a) Find the roots of $z^4+z^3+z^2+z+1=0$.

(b) Hence show that

$$z^4+z^3+z^2+z+1=\left(z^2-2z\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{5}\right)+1\right).$$

(c) Show that $\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$.

(d) Show that $\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{4}$.

(e) Construct a quadratic polynomial with roots $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$.

(f) Hence, find the exact values of $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$.

(g) Substitute $z=1$ into part (b) to prove that $\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}}{4}$.

Question 6 Define the polynomial

$$P_n(x)=(x+1)^{2n+1}+x^{n+2}$$

and let ω be a non-real cube root of unity. Prove that x^2+x+1 is a factor of $P_n(x)$.

Question 7 Let $z=\cos\theta+i\sin\theta$.

(a) Show that $z^n+z^{-n}=2\cos n\theta$ for $n\in\mathbb{Z}^+$.

(b) Show that $z^4+1=(z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1)$.

(c) Deduce that $\cos 2\theta=2\cos^2\theta-1$.

Question 8 Let $z=\cos\theta+i\sin\theta$.

(a) Show that $z^n+z^{-n}=2\cos n\theta$ for $n\in\mathbb{Z}^+$.

(b) Show that $z^6+1=(z^2+1)(z^2-\sqrt{3}z+1)(z^2+\sqrt{3}z+1)$.

(c) Deduce that

$$\cos 3\theta=4\cos\theta\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta-\cos\frac{5\pi}{6}\right).$$

Question 9 Let $P(z) = z^9 + 1$.

(a) Explain why the roots of $z^6 - z^3 + 1 = 0$ are also roots of $P(z)$.

(b) Find the roots of $z^6 - z^3 + 1 = 0$.

(c) Show that

$$z^6 - z^3 + 1 = \left(z^2 - 2z\cos\left(\frac{\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{5\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{7\pi}{9}\right) + 1\right).$$

(d) Deduce that

$$2\cos 3\theta - 1 = 8\cos\theta\left(\cos\theta - \cos\frac{\pi}{9}\right)\left(\cos\theta - \cos\frac{5\pi}{9}\right)\left(\cos\theta - \cos\frac{7\pi}{9}\right).$$

Question 10 Let $P(z) = z^9 - 1$.

(a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are also roots of $P(z)$.

(b) Find the roots of $z^6 + z^3 + 1 = 0$.

(c) Show that

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\left(\frac{2\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{8\pi}{9}\right) + 1\right).$$

(d) Deduce that

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = \frac{\sqrt{3}}{8}.$$

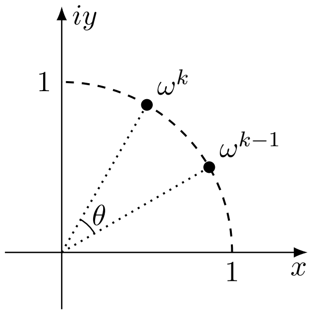
(e) Use a similar technique to prove that

$$\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)\cos\left(\frac{8\pi}{9}\right) = -\frac{1}{8}.$$

(f) Hence, find the exact value of $\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right)$.

Challenge Problems

Problem 1 Let ω be the principal n^{th} root of unity.



(a) Prove that for any two roots of unity ω^k and ω^{k-1} we have

$$|\omega^k - \omega^{k-1}| = \sqrt{2 - 2\cos\frac{2\pi}{n}}$$

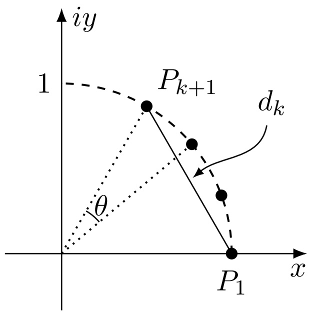
(b) The n roots of unity form a regular n-gon with some perimeter P_n. Explain why

$$P_n = n|\omega^k - \omega^{k-1}|.$$

(c) Hence, prove that as n gets large, $P_n \to 2\pi$.

(d) State the geometric significance of your result.

Problem 2 The diagram below shows an n-sided polygon centred at the origin with vertices P_i for $i = 1, 2, \ldots, n$, where P_1 is represented by $1 + 0i$.



Let $\theta = \dfrac{2\pi}{n}$, and let ω be the principal n^{th} root of unity. Define $d_k = |P_1P_{k+1}|$.

(a) Show that $d_k^2 = (1 - \cos k\theta)^2 + \sin^2 k\theta$.

(b) Show that $d_k^2 = 2 - \omega^k - \omega^{n-k}$.

(c) Deduce that $d_1^2 + d_2^2 + d_3^2 + \cdots + d_{n-1}^2 = 2n$.

Exercise 2J

Solving Polynomials

Fundamentals

Fundamentals 1

The Conjugate Root Theorem states that if the coefficients of a polynomial are r ______, then the non-real roots will occur in complex c __________ pairs.

Fundamentals 2

Write down the expansion of

$$(z-\alpha)(z-\bar{\alpha}) = \underline{\hspace{6cm}}.$$

Fundamentals 3

Let the roots of $ax^3+bx^2+cx+d=0$ be α, β and γ.

(a) $\alpha+\beta+\gamma = —$ (b) $\alpha\beta+\beta\gamma+\alpha\gamma = —$ (c) $\alpha\beta\gamma = —$

Fundamentals 4

Let the roots of $ax^4+bx^3+cx^2+dx+e=0$ be α, β, γ and δ.

(a) $\alpha+\beta+\gamma+\delta = —$

(b) $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = —$

(c) $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = —$

(d) $\alpha\beta\gamma\delta = —$

Question 1 [Guided question for solving cubic polynomials given a root]

Consider the polynomial equation

$$x^3 - x^2 - 7x + 15 = 0.$$

It is given that $x = 2+i$ is a root.

(a) Explain how we know that $x = 2-i$ is also a root.

(b) Use the sum of roots to find the third root.

(c) Hence fully factorise $P(x) = x^3 - x^2 - 7x + 15$.

Question 2 Use a similar technique to solve the following polynomial equations.

(a) $x^3 - x^2 + 2 = 0$, given that $x = 1+i$ is a root.

(b) $x^3 - 3x^2 + 4x - 12 = 0$, given that $x = 2i$ is a root.

Question 3 [Guided question for solving quartic polynomials given a root]

Consider the polynomial

$$P(x) = x^4 - x^3 - 12x^2 + 26x - 24.$$

It is given that $x = 1 + i$ is a root of $P(x) = 0$.

(a) Write down the other root and justify your answer.

(b) Construct a quadratic factor containing the above two roots.

(c) Divide out the quadratic factor from $P(x)$ to obtain another quadratic factor.

(d) Solve the other quadratic factor, and hence write down all the zeroes of $P(x)$.

Question 4 Use a similar technique to solve the following polynomial equations.

(a) $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$, given that $x = 1 - 2i$ is a root.

(b) $x^4 - 2x^3 + 6x^2 - 8x + 8 = 0$, given that $x = 2i$ is a root.

Question 5 Consider the quintic polynomial

$$P(x) = x^5 - 5x^4 + 12x^3 - 16x^2 + 12x - 4.$$

(a) Show that $x = 1 + i$ is a double-root.

(b) Hence, find all five zeroes of $P(x)$.

(c) Express $P(x)$ as the product of complex linear factors.

(d) Express $P(x)$ as the product of real linear and quadratic factors.

Question 6 Use polynomial long division to find the quotient and remainder when

$$P(x) = x^3 - 2x^2 + x + 1$$

is divided by $(x - i)$.

Question 7 [Remainder theorem and complex numbers]

(a) Prove the remainder theorem. That is, when $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$.

(b) Analyse your proof carefully and observe if at any point your proof relies on a being real. Determine whether the remainder theorem also works for complex numbers in general.

(c) Find the remainder when $P(x) = x^3 + x^2 - 2x + 4$ is divided by $(x + i)$.

Question 8 [Guided question for finding the remainder when dividing by a quadratic]

Consider the polynomial $P(x) = x^4 - x^3 + 5x^2 + 2x - 1$ and a divisor $x^2 + 1$.

(a) When $P(x)$ is divided by $x^2 + 1$, the remainder is $ax + b$ for some a, $b \in \mathbb{R}$. Explain why this is the case.

(b) Re-write $P(x)$ using the division transformation.

(c) Find $P(i)$ and hence find a and b.

(d) Write down the remainder when $P(x)$ is divided by $x^2 + 1$.

(e) Hence, find p and q such that $P(x) = x^4 - x^3 + 5x^2 + px + q$ is divisible by $x^2 + 1$.

Question 9 Find the remainder when the polynomial $P(x) = x^5 - 4x^3 + 2x + 1$ is divided by $x^2 + 1$.

Question 10 Find the remainder when the polynomial $P(x) = x^8 + x^4 - 1$ is divided by $x^2 + 2x + 2$.

Question 11 Find all the zeroes of the polynomial $P(x) = x^4 + 4x^3 + 11x^2 + 14x + 10$ given that the roots are in the form $a \pm bi$ and $a \pm 2bi$.

Question 12 Let the zeroes of $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be α, β, γ and δ.

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = a^2 - 2b$.

(b) Show that zeroes of $P(x) = x^4 - x^3 + 6x^2 + 3x + 1$ cannot all be real.

(c) Bob claims that the polynomial $P(x) = x^4 - 4x^3 + x^2 - 7x + 3$ has all real roots because $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$. Do you agree? Explain your answer.

Question 13 Consider the polynomial

$$P(x) = x^4 + ax^2 + b$$

where $a, b > 0$ and $a^2 - 4b \geq 0$.

(a) Prove that the zeroes are purely imaginary.

(b) Let two of the imaginary zeroes be αi and βi. Write down the other two zeroes.

(c) Hence, show that that $\alpha^2 + \beta^2 = a$.

Question 14 Define the general polynomial

$$P(x) = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0$$

where $c_k \in \mathbb{R}$. When $P(x)$ is divided by $(x - \alpha)$, where α is non-real, the remainder is β.

(a) Prove that when $P(x)$ is divided by $(x - \bar{\alpha})$, the remainder is $\bar{\beta}$.

(b) What familiar result follows on from this?

Question 15 [Palindromic coefficients lead to nice properties]

Let $P(x) = x^3 + kx^2 + kx + 1$, where $k \in \mathbb{R}$.

(a) Show that $x = -1$ is a zero of $P(x)$.

(b) Show that if α is a zero of $P(x)$, then $\dfrac{1}{\alpha}$ is also a zero of $P(x)$.

(c) Show that if α is a non-real zero of $P(x)$, then all the zeroes of $P(x)$ must lie on the unit circle in the complex plane.

Question 16 Consider the polynomial $P(x) = x^3 + px + q$ where $p,\, q > 0$.

(a) Prove that the polynomial cannot have all real zeroes.

(b) Prove that the real root γ must be negative.

(c) Let α be any non-real root. Show that $\operatorname{Re}(\alpha) > 0$.

(d) Hence, show that $|\alpha| > \sqrt{p}$.

Question 17 Define the polynomial

$$P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1,$$

where $k \in \mathbb{R}$. Let $\alpha = x + iy$ and it is given that α and $i\alpha$ are zeroes of $P(z)$, where $\bar{\alpha} \neq i\alpha$.

(a) Explain why $\bar{\alpha}$ and $-i\bar{\alpha}$ are zeroes of $P(z)$.

(b) Show that

$$P(z) = z^2(z-k)^2 + (kz-1)^2.$$

(c) Hence, show that if $P(z)$ has a real zero, then either

$$P(z) = (z^2+1)(z+1)^2$$

or

$$P(z) = (z^2+1)(z-1)^2$$

(d) Show that all the zeroes of $P(z)$ lie on the unit circle.

(e) Show that $k = x - y$.

(f) Deduce that $-\sqrt{2} \leq k \leq \sqrt{2}$.

Challenge Problems

Problem 1 [Cauchy's Bound]

Let α be a root of the complex monic polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let M be the value of the coefficient with largest magnitude, or in other words

$$M = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}.$$

(a) Show that

$$|\alpha|^n \leq M\left(|\alpha|^{n-1} + |\alpha|^{n-2} + \cdots + |\alpha| + 1\right).$$

(b) Deduce that all the zeroes of $P(z)$ satisfy

$$|\alpha| < 1 + M.$$

Problem 2 [Introduction to Root Bounding Polynomial Techniques]

Define the polynomial

$$P(z) = z^{n+1} - kz^n + kz - 1,$$

where $-1 \leq k \leq 1$ and $n \in \mathbb{Z}^+$.

(a) Show that if α is a zero of $P(z)$, then $\dfrac{1}{\alpha}$ is also a zero of $P(z)$.

(b) Suppose that there exists $n+1$ distinct zeroes of $P(z)$, none of which lie on the unit circle in the complex plane. Explain why there must exist at least one zero α *outside* of the unit circle.

(c) Show that $|\alpha|^{2n}|\alpha - k|^2 = |1 - k\alpha|^2$.

(d) Deduce that

$$\left(|\alpha|^2 - 1\right)\left(1 - k^2\right) < 0.$$

(e) Hence, prove that all the zeroes of $P(z)$ must lie on the unit circle.

Exercise 2K

Euler's Formula

Fundamentals

Fundamentals 1
Euler's identity states that

$$e^{i\theta} = \underline{\hspace{4cm}}.$$

This is called the e__________ form of a complex number.

Fundamentals 2
Let $z = \cos\theta + i\sin\theta$.

(a) $\dfrac{e^{i\theta} + e^{-i\theta}}{2} = \underline{\hspace{1.5cm}}$

(b) $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \underline{\hspace{1.5cm}}$

Question 1 Convert the following to exponential form.

(a) $1 + i\sqrt{3}$ (b) $2 - 2i$ (c) $-\sqrt{3} - i$

Question 2 Convert the following to Cartesian form.

(a) $4e^{\frac{i\pi}{3}}$ (b) $2e^{-\frac{5i\pi}{6}}$ (c) $\sqrt{2}e^{-\frac{i\pi}{4}}$

Question 3 Show that

(a) $\dfrac{e^{in\theta} + e^{-in\theta}}{2} = \cos n\theta$ (b) $\dfrac{e^{in\theta} - e^{-in\theta}}{2i} = \sin n\theta$

Question 4 Express the following in terms of $\cos n\theta$ and $\sin n\theta$, for some $n \in \mathbb{Z}^+$.

(a) $\left(e^{i\theta} - e^{-i\theta}\right)^2$ (b) $\left(e^{i\theta} + e^{-i\theta}\right)^2$

(c) $\left(e^{2i\theta} - e^{-2i\theta}\right)^3$ (d) $\left(e^{2i\theta} + e^{-2i\theta}\right)^3$

Question 5 Show that

(a) $e^{2i\theta} - 1 = 2i\sin\theta e^{i\theta}$ (b) $e^{2i\theta} + 1 = 2\cos\theta e^{i\theta}$

Question 6 Let $z = e^{i\theta}$. Simplify the following.

(a) $\dfrac{e^{2i\theta} - 1}{e^{2i\theta} + 1}$ (b) $\dfrac{e^{i\theta} - 1}{e^{2i\theta} + 1}$

Question 7

(a) Find an expression for $\tan\theta$ in complex exponential form.

(b) Hence, prove that

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Question 8

Use the complex exponential form to prove that

$$\cos^4\theta + \sin^4\theta = \frac{1}{4}\left(3+\cos 4\theta\right).$$

Question 9

[De Moivre's Theorem trivially follows from the exponential form]

Let $z = r\big(\cos\theta + i\sin\theta\big)$. Show that $z^n = r^n\big(\cos n\theta + i\sin n\theta\big)$.

Question 10

(a) Express $\cos\theta$ in complex exponential form.

(b) Hence, show that $\cos^3\theta = \dfrac{1}{4}\big(\cos 3\theta + 3\cos\theta\big)$.

Question 11

[The exponential form can be used to prove the double angle formulae]

Use the complex exponential form to prove that

(a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(b) $\sin 2\theta = 2\sin\theta\cos\theta$

Question 12

[The exponential form can be used to prove the compound angle formulae]

Use the complex exponential form to prove that

(a) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(b) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Challenge Problems

Problem 1

(a) Show that

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

(b) Hence, show that

$$e^{i\theta} + e^{i\phi} = 2\cos\left(\frac{\theta-\phi}{2}\right)e^{\frac{1}{2}i(\theta+\phi)}$$

Problem 2 [Proof of the Dirichlet kernel using Euler's Identity]

Let $z = e^{i\theta}$. Consider the expression

$$S_n = 1 + z + z^2 + \cdots + z^n.$$

(a) Show that

$$S_n = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

(b) Show that

$$S_n = e^{i\frac{n\theta}{2}} \times \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

(c) Deduce that

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin\left(\frac{n}{2}\theta\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Chapter 2 Review

Complex Numbers

Review

Question 1 Let $z = 10 - 5i$ and $w = 3 - 4i$. Find

(a) zw

(b) $\dfrac{z}{w}$

Question 2 Prove that if $\dfrac{z}{z-i}$ is purely real, then z is purely imaginary.

Question 3 Let

$$\frac{z+1}{z-1} = ki,$$

where $k \in \mathbb{R}$. Show that $|z| = 1$.

Question 4 Find $\sqrt{-24 + 70i}$.

Question 5 Solve the following quadratic equations.

(a) $z^2 - 6z + 13 = 0$

(b) $z^2 - (7 - i)z + (14 - 5i) = 0$

Question 6 Convert the following to Cartesian form.

(a) $2\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$

(b) $4e^{-\frac{i\pi}{6}}$

Question 7 Convert the following to polar and exponential form.

(a) $\sqrt{3} - 3i$

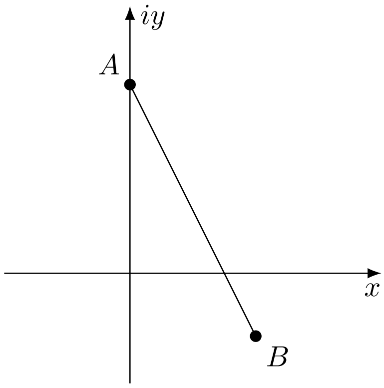
(b) $-2 - 2i$

Question 8 Simplify the following.

(a) $\dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$

(b) $\dfrac{1 + e^{2i\theta}}{1 - e^{2i\theta}}$

Question 9 Let A and B be points representing $3i$ and $2 - i$ respectively.

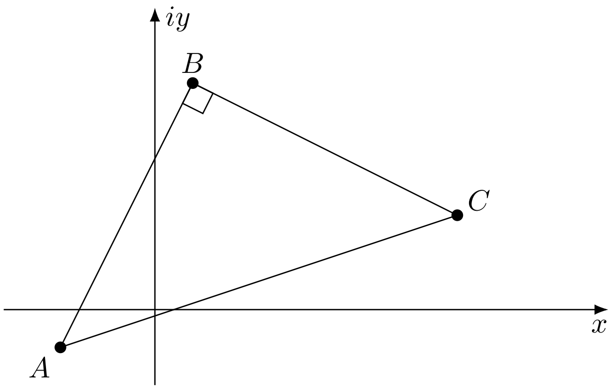


Suppose AB is one side of a square $ABCD$.

(a) Draw the two possible scenarios.

(b) Find C and D for each scenario.

Question 10 The diagram below shows vertices of a triangle ABC represented by z_1, z_2 and z_3 respectively.



The triangle is isosceles and right-angled at the point B.

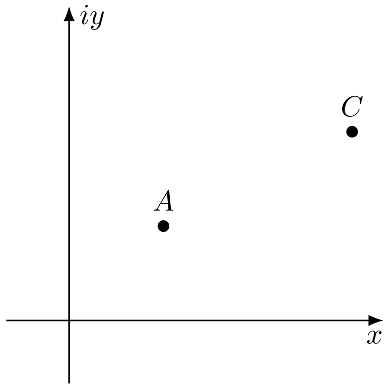
(a) Show that $(z_1 - z_2)^2 + (z_2 - z_3)^2 = 0$.

(b) The point D is chosen so that $ABCD$ is a square. Find the complex number z_4 that represents D in terms of z_1, z_2 and z_3.

(c) Hence, show that if $z_2 \neq 0$, then

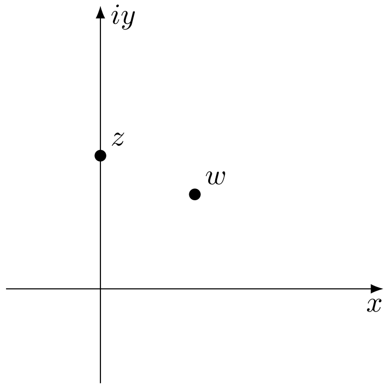
$$z_4 = \frac{1}{2}\left(\frac{z_1^2 + z_3^2}{z_2}\right).$$

Question 11 Let A and C be points representing $2 + 2i$ and $6 + 4i$ respectively.



Given AC is a diagonal of a square $ABCD$, find the complex numbers representing B and D.

Question 12 Let $z = i\sqrt{2}$ and $w = \dfrac{2}{1 - i}$.



(a) Express z and w in polar form.

(b) On the same Argand diagram, plot z, w and $z + w$.

(c) Show that $\arg(z + w) = \dfrac{3\pi}{8}$.

(d) Hence, find the exact value of $\tan\left(\dfrac{3\pi}{8}\right)$.

Question 13 Find the Cartesian equation of the following loci.

(a) $\operatorname{Im}(z - 1 + 3i) = 4$

(b) $\operatorname{Im}(z) = |z|$

(c) $z\overline{z} \leq 2(z + \overline{z})$

(d) $z + \overline{z} = \operatorname{Im}(z)$

(e) $z^2 - \overline{z}^2 = 8i$

(f) $z^2 + \overline{z}^2 = 4$

Question 14 On an Argand diagram, shade the region where the following conditions are both satisfied.

(a) $0 \leq \text{Re}(z) \leq 2$ and $|z - 1 + i| \leq 2$

(b) $|z| \leq |z - 2|$ and $-\dfrac{\pi}{4} \leq \arg z \leq \dfrac{\pi}{4}$

(c) $|z - \overline{z}| < 2$ and $|z - 1| \geq 1$

(d) $|z - 2 + i| \leq 2$ and $\text{Im}(z) \geq 0$

(e) $|z - 1| \leq 2$ and $-\dfrac{\pi}{4} \leq \arg(z - 1) \leq \dfrac{\pi}{4}$

(f) $|z - 1 - i| < 2$ and $0 < \arg(z - 1 - i) < \dfrac{\pi}{4}$

Question 15 Find the locus of z if $|2z + 1| = |z + 2|$.

Question 16 Consider the locus of z defined by $|z - 2 - 2i| = 1$.

(a) Find the maximum value of $|z|$.

(b) Find the minimum value of $|z|$.

(c) Find the maximum value of $\arg(z)$.

(d) Find the minimum value of $\arg(z)$.

Question 17 Find the integer values of n for which $\left(-\sqrt{3} + i\right)^n$ is purely real.

Question 18 Show that $\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Question 19 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Hence, solve $z^4 - 4z^3 + 2z^2 - 4z + 1 = 0$.

Question 20

(a) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

(b) Hence, solve $16x^4 - 16x^2 + 1 = 0$.

(c) Hence, find the exact values of $\cos\left(\dfrac{\pi}{12}\right)$ and $\cos\left(\dfrac{5\pi}{12}\right)$.

Question 21

(a) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Hence find the roots of $8x^3 - 6x + 1 = 0$.

(c) Deduce that $\sin\left(\dfrac{\pi}{18}\right) + \sin\left(\dfrac{5\pi}{18}\right) = \sin\left(\dfrac{7\pi}{18}\right)$.

Question 22 Find the roots of

(a) $z^5 + 1 = 0$

(b) $z^4 + 8 + 8\sqrt{3}i = 0$

Question 23 Let $\omega = e^{\frac{2\pi i}{7}}$.

(a) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

(b) Hence, show that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$

Question 24 Let ω be a non-trivial cube root of unity. Simplify the following.

(a) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

(b) $(1+2\omega+3\omega^2)(1+3\omega+2\omega^2)$

Question 25

(a) Show that

$$z^6 - 1 = (z^2-1)(z^2-z+1)(z^2+z+1).$$

(b) Deduce that

$$\sin 3\theta = \sin\theta(2\cos\theta+1)(2\cos\theta-1).$$

Question 26 Let $P(z) = z^5 + 1$.

(a) Express $P(z)$ as the product of real linear and quadratic factors.

(b) Factorise $P(z)$ over $\mathbb{Z}$.

(c) Show that $\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$.

(d) Show that $\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -\frac{1}{4}$.

(e) Write down the equation of the quadratic polynomial with roots $\cos\left(\frac{\pi}{5}\right)$ and $\cos\left(\frac{3\pi}{5}\right)$.

(f) Hence, find the exact value $\cos\left(\frac{\pi}{5}\right)$ and $\cos\left(\frac{3\pi}{5}\right)$.

Question 27 Find the roots of the following polynomial equations.

(a) $x^3 - 4x^2 + 14x - 20 = 0$, given that $x = 1 - 3i$.

(b) $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$, given that $x = 2 - i$.

Question 28 Find the remainder when $P(x) = x^5 - 6x^4 + x^3 + 2x - 1$ is divided by (x^2+1).

Question 29 Find all the zeroes of $P(x) = x^3 - 8x^2 + 29x - 52$.

(a) Show that $P(x)$ cannot have all real zeroes.

(b) Hence, find the zeroes of $P(x)$ given that two of the roots are in the form $a + bi$ and $2a$.

Question 30 Define the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

(a) Find all the zeroes of $P(x)$ given that two of the roots are in the form $a + bi$ and $a + 2bi$.

(b) Hence, express $P(x)$ as the product of real quadratic factors.

Question 31 Express the following in terms of $\cos n\theta$.

(a) $\left(e^{i\theta} + e^{-i\theta}\right)^3$

(b) $\left(e^{2i\theta} - e^{-2i\theta}\right)^4$

Question 32

(a) Express $\sin\theta$ in complex exponential form.

(b) Hence, show that $\sin^3\theta = \dfrac{1}{4}(3\sin x - \sin 3x)$.

Investigation Task

Failure of de Moivre's Theorem

In this chapter, we learn de Moivre's theorem and use it for integer powers. For example, we learn how to calculate

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^5$$

or even something like

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{-3}.$$

Without much further thought, it is tempting to do the following.

$$\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{\frac{1}{2}}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

However, this is incorrect. This investigation task allows the student to explore this further.

Question 1 Explain why de Moivre's theorem does not work for non-integer powers and what happens if you 'try' to do it anyway. What is the significance of the answer you initially obtain?

Question 2 Give a few examples that demonstrate failure of de Moivre's theorem for non-integer powers.

Question 3 The fact that de Moivre's theorem does not work for non-integer powers is not necessarily a bad thing, and in actual fact it is used for a particular topic in Complex Numbers. What is this topic, and why is it that we actually want de Moivre's theorem to not work?

Investigation Task

Complex Exponentiation

We know what it means to raise a real number to an integer power. We know what it means to raise a real number to a rational power. But what does it mean to raise a real number to anything beyond this? We can type things like $2^{\sqrt{2}}$ in our calculators and get answers, but upon deeper thought it's not entirely clear what it means to raise 2 to the power of something like $\sqrt{2}$, and even less so for the case of something like $e^{i\pi}$. Raising e to the power of $i\pi$ at first thought makes just about as much mathematical sense as e^{banana}.

Write an article that de-mystifies everything and explains intuitively what it means to raise a number to a complex number. Your answer should include a discussion of

- What does it mean to raise a real number to an *irrational* power?
- What does it mean to raise a real number to a *complex* power?
- Why is a number like i^i not well-defined?

Investigation Task

Fundamental Theorem of Algebra

When we study polynomials, we use the fact that if $P(a) > 0$ and $P(b) < 0$, then there is a real root $a < \alpha < b$. In other words, we have some way of 'guaranteeing' that there is a real root somewhere.

We also claim that if $P(x)$ is of degree n, then $P(x) = 0$ has n roots. This allows us to make conclusions like $P(x) = x^2 + 1$ has two zeroes, even though we physically cannot see two x-intercepts. But upon further thought, it's not obvious at all that all polynomials of degree n MUST have a root, let alone n of them. For example, consider the following polynomial equation.

$$P(x) = (37 - 29i)x^{2020} + (-17 + 13i)x^{2019} + \cdots$$

Is it really 'obvious' that $P(x)$ has 2020 values that we can plug in to get zero? This is the power of the Fundamental Theorem of Algebra, and this investigation task allows the student to explore this and appreciate that this theorem truly has earned the title of being a *fundamental* theorem.

Question 1 The idea that if $P(a) > 0$ and $P(b) < 0$ then there is a real root $a < \alpha < b$ is actually a theorem. What is the name of this theorem and what broader theorem is it actually a consequence of?

Question 2 Give the actual statement of the Fundamental Theorem of Algebra, and explain how this leads to the statement that a polynomial of degree n has n complex roots, including repeated roots.

Question 3 The full proof of the Fundamental Theorem of Algebra is not accessible to the majority of Extension 2 students. However, a rough outline of the proof and the intuition behind it is accessible. Produce a five to ten minute presentation that gives an outline of the proof of the theorem. Your presentation should address the following key questions.

- What is the brief history behind the theorem?
- What assumptions are made at certain steps?
- What was the intuition behind certain steps?

Investigation Task

Polynomial Root Bounds

Towards the end of **Exercise 2J**, there was a small collection of problems that explore the idea of polynomial root bounds. The theory behind this is rich and combines many aspects of polynomial theory and complex numbers. It also has many real-life applications in an assortment of fields of mathematics. This investigation task allows the student to get some hands-on experience with this theory and to allow them to explore how useful such theorems can be!

Question 1 Research and find two well-known root bounds that confine the roots of $P(x) = 0$ to a certain region. Your response should include

- any restrictions on the coefficients of the polynomial.
- an example of a polynomial and a visual indicator of the region of where the roots should reside within.

Question 2 Polynomials such as

$$P(z) = 9z^4 + 6z^3 + 5z^2 + 2z + 1$$

have strictly decreasing positive coefficients. Such polynomials have their zeroes satisfying a particularly nice condition that makes them easier to find.

(a) What is that 'nice' condition?

(b) Prove it.

Question 3 Produce a five minute presentation on the *Eneström-Kakeya Theorem*. Your presentation should include

- a statement of the theorem.
- any conditions on the coefficients.
- an example of a polynomial, and what the *Eneström-Kakeya Theorem* states about where the zeroes lie.
- any real-life applications or benefits of root-bounding theory in general.

3

3D VECTORS

- Introduction to 3D Vectors
- Proofs using 3D Vectors
- Vector Equation of a Line
- Parameterising 3D Curves
- Spheres and Circles

Exercise 3A

Introduction to 3D Vectors

Fundamentals

Fundamentals 1

Let $\underset{\sim}{u} = a_1\,\underset{\sim}{i} + a_2\,\underset{\sim}{j} + a_3\,\underset{\sim}{k}$ and $\underset{\sim}{u} = b_1\,\underset{\sim}{i} + b_2\,\underset{\sim}{j} + b_3\,\underset{\sim}{k}$. Write an expression for the following.

(a) $k\underset{\sim}{u}$ (b) $\underset{\sim}{u} + \underset{\sim}{v}$ (c) $\underset{\sim}{u} \cdot \underset{\sim}{v}$

(d) $\underset{\sim}{u} \cdot \underset{\sim}{u}$ (e) $|\underset{\sim}{u}|$ (f) $\hat{\underset{\sim}{u}}$

Fundamentals 2

Write down the formula for the following.

(a) $\text{scal}_{\mathbf{u}}\,\mathbf{v}$ (b) $\text{proj}_{\mathbf{u}}\,\mathbf{v}$

Question 1 Let P be the point $(3, -7, 4)$. Find the distance of P from the following.

(a) xy-plane (b) yz-plane (c) xz-plane

(d) x-axis (e) y-axis (f) z-axis

Question 2 Let $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}$. Calculate the following.

(a) $\underset{\sim}{u} - \underset{\sim}{v}$ (b) $\underset{\sim}{u} + 2\underset{\sim}{v}$ (c) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ (d) $\underset{\sim}{u} \cdot (\underset{\sim}{u} + \underset{\sim}{v})$

Question 3 Calculate the angle between the following vectors, correct to the nearest degree.

(a) $\underset{\sim}{u} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(b) $\underset{\sim}{u} = -2\,\underset{\sim}{i} + 5\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$
$\underset{\sim}{v} = 4\,\underset{\sim}{i} + \underset{\sim}{k}$

(c) $\underset{\sim}{u} = 3\,\mathbf{i} - 5\,\mathbf{j} - 2\,\mathbf{k}$
$\underset{\sim}{v} = -3\,\mathbf{j} + 3\,\mathbf{k}$

Question 4 Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{k}$. Find a unit vector that is perpendicular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Question 5 Let $\underset{\sim}{u} = \underset{\sim}{i} + 3\,\underset{\sim}{j} + 2\,\underset{\sim}{k}$, $\underset{\sim}{v} = 0\,\underset{\sim}{i} - 2\,\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{w} = 2\,\underset{\sim}{i} + \underset{\sim}{j} + 2\,\underset{\sim}{k}$. Find constants a, b, c such that

$$a\underset{\sim}{u} + b\underset{\sim}{v} + c\underset{\sim}{w} = 7\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + 10\,\underset{\sim}{k}.$$

Question 6 Calculate, correct to the nearest degree, the three angles in the triangle defined by $P(1,-3,4)$, $Q(-2,0,6)$ and $R(2,3,-1)$.

Question 7 Define $A(2,0,-3)$, $B(-1,4,2)$ and $C(0,6,-3)$. Find all possible points D such that $ABCD$ forms a parallelogram.

Question 8 Let $\mathbf{u} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\mathbf{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$. Find the following.

(a) $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ (b) $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ (c) $\text{scal}_{\mathbf{v}}\,\mathbf{u}$ (d) $\text{scal}_{\mathbf{u}}\,\mathbf{v}$

Question 9 What can we say about two vectors $\mathbf{u}$ and $\mathbf{v}$ if

(a) $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$? (b) $\text{scal}_{\mathbf{u}}\,\mathbf{v} = \text{scal}_{\mathbf{v}}\,\mathbf{u}$?

Question 10 Define $A(2,0,-3)$, $B(-1,4,2)$ and $C(0,6,-3)$. Let $\mathbf{u} = \overrightarrow{BA}$ and $\mathbf{v} = \overrightarrow{BC}$.

(a) Find $\text{scal}_{\mathbf{v}}\,\mathbf{u}$.

(b) Find $|\mathbf{u}|$.

(c) Hence, find the perpendicular distance of A from $\overrightarrow{BC}$.

(d) Find the area of $\triangle ABC$.

(e) Find the area of the same triangle but using an alternative method.

Question 11 Let $\underset{\sim}{u} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$. Find a vector with length 12 that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Question 12 Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be vectors in 3D space. Prove the following identity

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$$

Question 13 Consider the triangle formed by $A(1,-3,-2)$, $B(5,-1,2)$ and $C(-1,1,2)$.

(a) Prove that the triangle is isosceles.

(b) Show that it is not a right-angled isosceles triangle.

Question 14 Simplify the following by using the fact that $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$.

(a) $|\underset{\sim}{u} + \underset{\sim}{v}|^2$ (b) $|\underset{\sim}{u} - \underset{\sim}{v}|^2$

(c) $|\underset{\sim}{u} + \underset{\sim}{v}|^2 + |\underset{\sim}{u} - \underset{\sim}{v}|^2$ (d) $|\underset{\sim}{u} + \underset{\sim}{v}|^2 - |\underset{\sim}{u} - \underset{\sim}{v}|^2$

Question 15 [Deducing a well-known inequality from the Cauchy-Schwarz inequality]

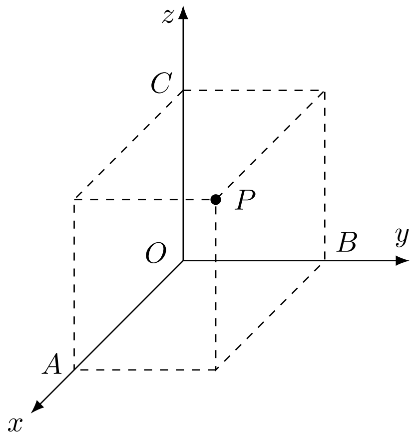
(a) Prove that $\underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}|$.

(b) Hence, prove that $|\underset{\sim}{u} + \underset{\sim}{v}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}|$.

(c) State the significance of this result.

Question 16 [Using algebraic symmetry to deduce similar results]

Let $P(a, b, c)$ be a point that forms a rectangular prism with the origin.



Let A, B and C be the points on the x, y and z axes representing their respective components of P.

(a) Find the size of $\angle OAP$.

(b) What can we deduce about $\angle OBP$ and $\angle OCP$ as well?

(c) Find an expression for $\angle APB$ and deduce similar expressions for $\angle BPC$ and $\angle APC$.

Challenge Problems

Problem 1 [The vector projection is a linear operator]

Let $P_{\mathbf{w}}(\mathbf{u})$ be the vector projection of $\mathbf{u}$ onto $\mathbf{w}$. Let $\mathbf{v}$ be any vector and $k \in \mathbb{R}$. Show that

$$P_{\mathbf{w}}(\mathbf{u} + k\mathbf{v}) = P_{\mathbf{w}}(\mathbf{u}) + kP_{\mathbf{w}}(\mathbf{v}).$$

Problem 2 Find the area of $\triangle OAB$ for points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Problem 3 [Cross product]

Find an expression for a vector that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$ if

$$\underset{\sim}{u} = a_1\,\underset{\sim}{i} + a_2\,\underset{\sim}{j} + a_3\,\underset{\sim}{k}$$
$$\underset{\sim}{v} = b_1\,\underset{\sim}{i} + b_2\,\underset{\sim}{j} + b_3\,\underset{\sim}{k}$$

Problem 4 [Normal vector]

Show that the vector $\underset{\sim}{n} = a\,\underset{\sim}{i} + b\,\underset{\sim}{j} + c\,\underset{\sim}{k}$ is perpendicular to the plane $ax + by + cz + d = 0$.

Exercise 3B

Proofs using 3D Vectors

Fundamentals

Fundamentals 1

Describe the main steps required to do the following.

(a) Prove that two vectors are perpendicular.

(b) Prove that two vectors are parallel.

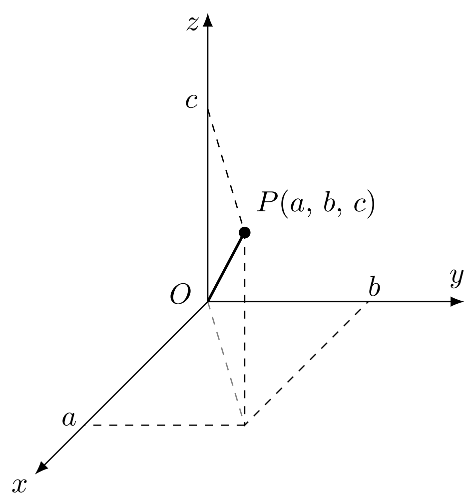
(c) Prove that three points are collinear.

(d) Find the angle between two vectors.

(e) Show that two intervals bisect each other.

Question 1 [Proving the distance formula]

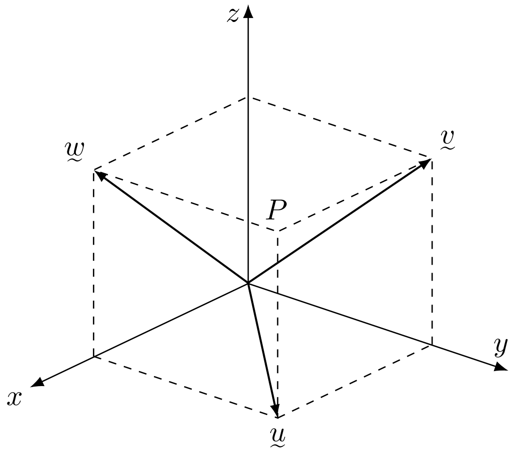
Define the point $P(a, b, c)$ as shown below.



Show that the distance of P from the origin is

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}.$$

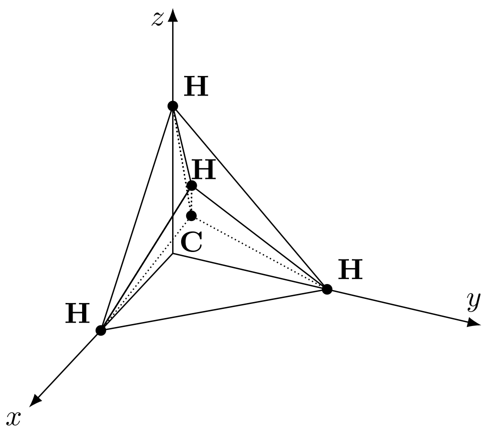
Question 2 The diagram below shows vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ in a rectangular prism produced by the origin and $P(a, b, c)$.



Show that if the angles between $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are all equal to $\frac{\pi}{3}$, then the rectangular prism is a cube.

Question 3 [Finding the bond angle of methane]

A molecule of methane $\mathbf{CH_4}$ is in the shape of a regular tetrahedron with a carbon atom at the centroid and four hydrogen atoms on each vertex of the tetrahedron.



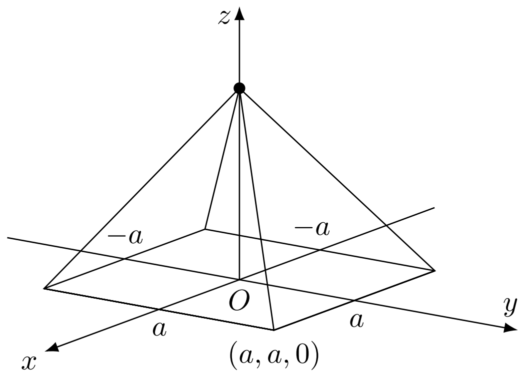
The *bond angle* is the angle formed by any $\mathbf{H} - \mathbf{C} - \mathbf{H}$ chain.

(a) Suppose that three of the hydrogen atoms are located at $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. What are the coordinates of the fourth hydrogen atom, if it is to be in the first octant?

(b) What are the coordinates of the carbon atom?

(c) Hence, show that the bond angle of methane is approximately $109.5°$.

Question 4 The diagram below shows a square pyramid where the side faces are equilateral triangles.

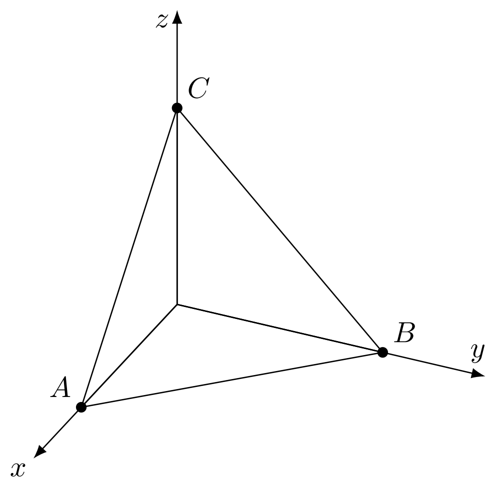


Let the coordinates of the square base be $(a, a, 0)$, $(a, -a, 0)$, $(-a, a, 0)$ and $(-a, -a, 0)$.

(a) Find the coordinates of the tip of the vertex in terms of a.

(b) Find the angle of inclination between any of the side faces and the square base.

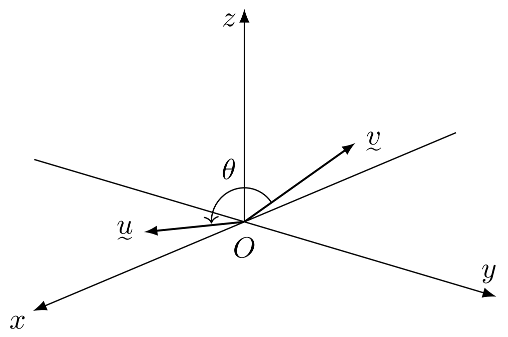
Question 5 Consider the tetrahedron defined by $A(t, 0, 0)$, $B(0, t, 0)$ and $C(0, 0, t)$.



Show that the angle between the plane ABC and the base is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Question 6 [Proving the dot product for 3D vectors]

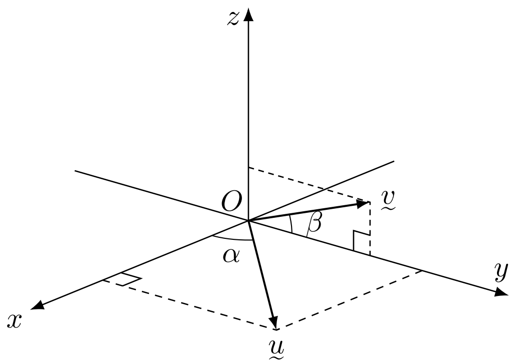
The diagram below shows two arbitrary vectors $\underset{\sim}{u} = u_1\underset{\sim}{i} + u_2\underset{\sim}{j} + u_3\underset{\sim}{k}$ and $\underset{\sim}{v} = v_1\underset{\sim}{i} + v_2\underset{\sim}{j} + v_3\underset{\sim}{k}$ and an angle θ between them.



(a) Write down two different expressions for the distance between the tips of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(b) Deduce that $\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Question 7 Let $\underset{\sim}{u}$ be a unit vector in the xy-plane, and let $\underset{\sim}{v}$ be a unit vector in the yz-plane. Let α be the angle between $\underset{\sim}{u}$ and $\underset{\sim}{i}$, and similarly let β be the angle between $\underset{\sim}{v}$ and $\underset{\sim}{j}$.

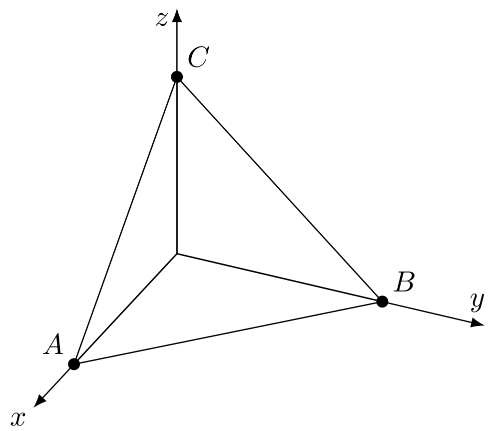


Let θ be the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Prove that

$$\cos\theta = \pm\sin\alpha\sin\beta.$$

Question 8 [Three-dimensional analogy of Pythagoras' Theorem]

Consider the tetrahedron defined by $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

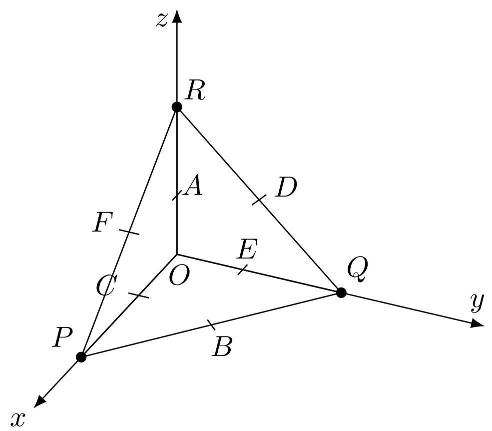


Show that if $|\triangle ABC|$ represents the area of $\triangle ABC$, then

$$|\triangle AOB|^2 + |\triangle AOC|^2 + |\triangle BOC|^2 = |\triangle ABC|^2.$$

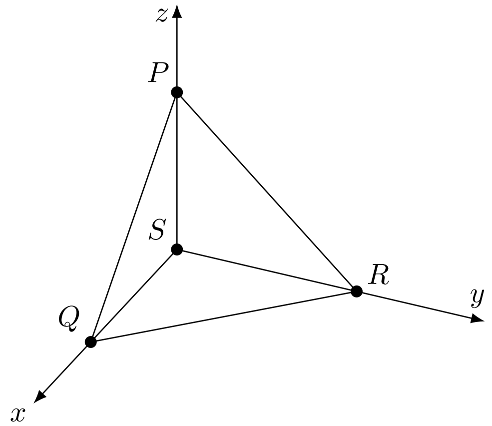
Question 9 [Three-dimensional analogy of the median property of triangles]

The diagram below shows a tetrahedron $OPQR$ where $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ are the position vectors of P, Q and R respectively. Let A, B, C, D, E and F be the midpoints of OR, PQ, OP, QR, OQ, and PR respectively.



Prove that AB, CD and EF are concurrent and bisect each other.

Question 10 The diagram below shows a tetrahedron $PQRS$ with vertices $P(0,0,p)$, $Q(q,0,0)$, $R(0,r,0)$ and $S(0,0,0)$. Let $\underset{\sim}{p}$ be the vector from P perpendicular to the face opposite point P with length being the area of that same face.



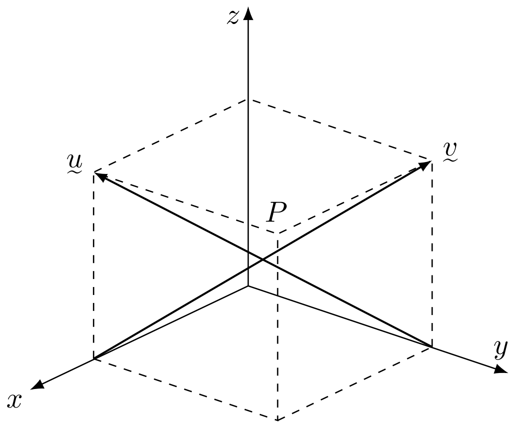
Define vectors $\underset{\sim}{q}$, $\underset{\sim}{r}$ and $\underset{\sim}{s}$ similarly.

(a) Show that $\underset{\sim}{p} = \left(-\frac{1}{2}qr\right)\underset{\sim}{k}$.

(b) Find similar expressions for $\underset{\sim}{q}$, $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

(c) Hence, show that $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} + \underset{\sim}{s} = 0$.

Question 11 The diagram below shows two main diagonals $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in a rectangular prism produced by the origin and the point $P(a,b,c)$.



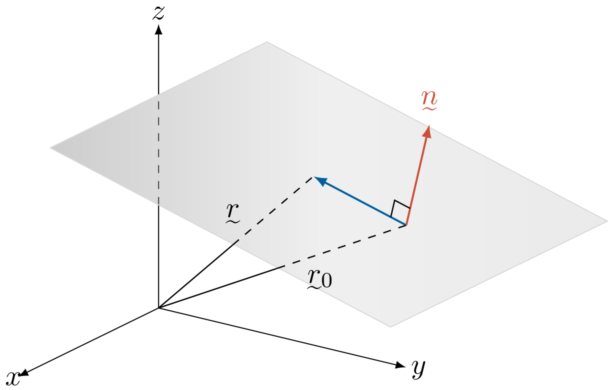
(a) Find expressions for $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in terms of a, b and c.

(b) Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular if and only if $a^2 + b^2 = c^2$.

(c) Find a possible point P that will have two perpendicular main diagonals.

Question 12 [Equation of a plane]

A plane in 3D space can be defined by a point on the plane and a vector perpendicular to the plane called the normal vector. This works similarly to how a line in 2D space can be defined by a point on the line and a gradient.



The diagram above shows two vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_0$ representing the position vectors of points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$ on a plane. Let $\underset{\sim}{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the vector perpendicular to the plane.

(a) Explain why

$$(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{n} = 0.$$

(b) Hence, show that the equation of the plane is

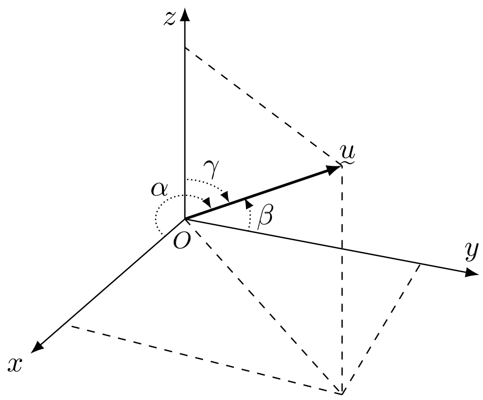
$$ax + by + cz = ax_0 + by_0 + cz_0.$$

(c) Find the equation of the plane that passes through $(2, -3, 1)$ and is perpendicular to the vector $-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

(d) What is the shortest possible distance between any point on the plane $ax + by + cz + d = 0$ and the origin?

Question 13 [Direction cosine]

The *direction cosines* of a vector $\underset{\sim}{u} = a\,\underset{\sim}{i} + b\,\underset{\sim}{j} + c\,\underset{\sim}{k}$ are the cosines of the angles from the vector to each of the coordinate axes. Let α, β and γ be the angles of vector $\underset{\sim}{u}$ from the x, y and z axes respectively.



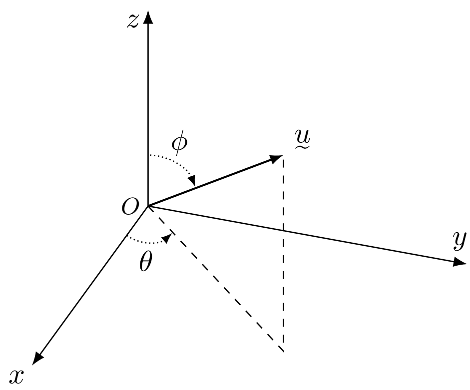
(a) Show that $\cos\alpha = \dfrac{a}{|\underset{\sim}{u}|}$.

(b) Deduce that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Question 14 [Connection between direction cosines and spherical coordinates]

Let θ and ϕ be the angles shown in the diagram below. Let the *direction cosines* be defined as in the previous question.



Show that the direction cosines of the vector $\underset{\sim}{u}$ can be expressed as the following.

$$\cos\alpha = \cos\phi\cos\theta$$
$$\cos\beta = \cos\phi\sin\theta$$
$$\cos\gamma = \sin\phi$$

Challenge Problems

Problem 1 [Special case of a regular polytopic distance]

Let P_1, P_2, P_3 and P_4 be the vertices of a regular tetrahedron circumscribed by the unit sphere. Let the position vectors of the vertices be $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$, $\underset{\sim}{p}_3$ and $\underset{\sim}{p}_4$ respectively.

(a) Show that $\underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3 + \underset{\sim}{p}_4 = 0$.

(b) Prove that $\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j = -\dfrac{1}{3}$ for all $i \neq j$.

(c) Let P be any arbitrary point on the sphere. Show that the expression

$$|\overrightarrow{PP_1}|^2 + |\overrightarrow{PP_2}|^2 + |\overrightarrow{PP_3}|^2 + |\overrightarrow{PP_4}|^2$$

is independent of the position of P.

Problem 2 Show that two non-zero vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are perpendicular *if and only if* their direction cosines satisfy

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0.$$

Problem 3 Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be any two non-zero vectors in the 3D plane. Define the following vector.

$$\underset{\sim}{w} = \underset{\sim}{u} - t\underset{\sim}{v}\,,$$

where $t \in \mathbb{R}$.

(a) Show that the value of t that minimises $|\underset{\sim}{w}|$ is $t = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}$.

(b) Interpret your result geometrically.

Problem 4 Let $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ be three mutually perpendicular vectors in the 3D plane. Define the following vector.

$$\underset{\sim}{v} = c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + c_3\underset{\sim}{v}_3$$

Show that the scalars c_1, c_2 and c_3 are given by

$$\frac{\underset{\sim}{v} \cdot \underset{\sim}{v}_i}{|\underset{\sim}{v}_i|^2}$$

for $i = 1, 2, 3$.

Exercise 3C

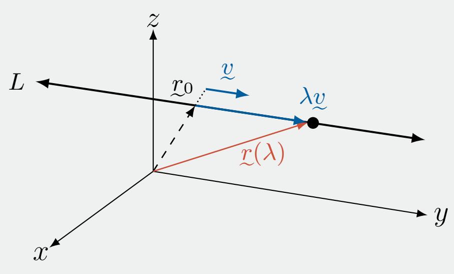
Vector Equation of a Line

Fundamentals

Fundamentals 1

The vector equation of the line ℓ that passes through $\underset{\sim}{r}_0$ in the direction of $\underset{\sim}{v}$ is

$$\underset{\sim}{r}(\lambda) = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}.$$



(a) The letter λ is called a p________.

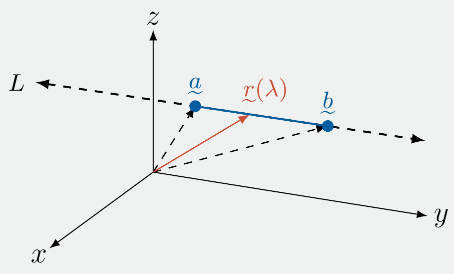
(b) It is called a p________ because it controls the position of a variable point P along the line. Each point P along the line has its own unique value of λ.

(c) It is important to note that $\underset{\sim}{r}(\lambda)$ is the p________ vector of the *set of points* along a line ℓ, but it is not the line ℓ itself.

Fundamentals 2

The line segment from $\underset{\sim}{a}$ pointing towards $\underset{\sim}{b}$ is represented by the vector equation

$$\underset{\sim}{r}(\lambda) = (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}.$$



(a) The domain of λ is $\lambda \in [\_\_\_, \_\_\_]$.

(b) When $\lambda = 0$, then $\underset{\sim}{r}(\lambda)$ points towards $\underset{\sim}{a}$ / $\underset{\sim}{b}$ (circle one).

(c) When $\lambda = 1$, then $\underset{\sim}{r}(\lambda)$ points towards $\underset{\sim}{a}$ / $\underset{\sim}{b}$ (circle one).

Question 1 Find the vector equation of the line that passes through the point P in the direction of $\underset{\sim}{v}$ for the following scenarios.

(a) $P(1,2,3)$ and $\underset{\sim}{v} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$

(b) $P(-3,1,5)$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$

(c) $P(2,1,-3)$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 3\underset{\sim}{k}$

(d) $P(3,-5,2)$ and $\underset{\sim}{v} = 2\underset{\sim}{j} + 7\underset{\sim}{k}$

Question 2

(a) Draw on the 2D plane the line ℓ represented by the following vector equation.

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(b) Plot on ℓ the two points corresponding to $\lambda = 1$ and $\lambda = 2$.

Question 3

(a) What is the gradient of the line with direction vector $\begin{bmatrix} a \\ b \end{bmatrix}$?

(b) Find a possible vector equation of $2x + 3y - 6 = 0$.

Question 4 Find the vector equation for the *line* AB if A and B are defined as such.

(a) $A(-7,3,4)$ and $B(2,6,-1)$

(b) $A(-4,1,5)$ and $B(6,-2,3)$

Question 5 Define points $A(3,-2,4)$ and $B(1,0,2)$. Find the vector equation for the

(a) interval AB.

(b) interval BA.

Question 6 Determine if the following points are collinear.

(a) $P(0,-5,5)$, $Q(1,-2,4)$, $R(3,4,2)$

(b) $P(2,4,2)$, $Q(3,7,-2)$, $R(1,3,3)$

Question 7 Determine whether the following points lie on the line represented by

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

(a) $(-3,-2,12)$

(b) $(-8,8,-9)$

(c) $(-4,0,8)$

Question 8 Let ℓ be the line with vector equation

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} -3 \\ 6 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

(a) Find the value of λ that corresponds to the point P where ℓ intersects the xy-axis.

(b) Hence, find the coordinates of P.

(c) Similarly, find the coordinates of the point Q where ℓ intersects the plane $y = 2$.

Question 9 Consider the line ℓ through $P(1, -2, 3)$ and parallel to $2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$.

(a) Find two points that lie on ℓ.

(b) Determine if the point $R(0, 1, -2)$ lies on ℓ.

(c) Determine if the point $R(7, 7, 0)$ lies on ℓ.

(d) Find where ℓ intersects the yz-plane.

Question 10 Find the vector equation of the line defined by the parametric equation

$$x = 3 + 2\lambda, \quad y = 1 - \lambda, \quad z = -2 + \lambda$$

where $\lambda \in \mathbb{R}$.

Question 11 Determine whether the lines represented by the vector equations

$$\underset{\sim}{r_1}(\lambda) = (\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} + \underset{\sim}{k})$$
$$\underset{\sim}{r_2}(\mu) = (3\underset{\sim}{i} + \underset{\sim}{j}) + \mu(-2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})$$

are parallel, intersect, or skew.

Question 12

(a) Find the vector equation of the line ℓ that passes through $P(-3, 6, 1)$ and $Q(1, 0, 3)$.

(b) Hence, find where ℓ intersects the plane $x + y - z + 2 = 0$.

Question 13 Find the point of intersection of the lines represented by the following vector equations.

(a) $$\underset{\sim}{r_1}(\lambda) = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$
$$\underset{\sim}{r_2}(\mu) = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(b) $$\underset{\sim}{r_1}(\lambda) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$\underset{\sim}{r_2}(\mu) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

Question 14 Consider a unit cube with all of the vertices in the first octant.

(a) Find the parametric vector equation of any two main diagonals.

(b) Hence, find the point of intersection of the main diagonals and verify that it is indeed halfway through the main diagonal.

Question 15 Let ℓ be the line represented by the vector equation

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Find the perpendicular distance between the point $P(2, 1, 3)$ and ℓ.

Question 16 Consider the two lines ℓ_1 and ℓ_2 represented by the vector equations

$$\underset{\sim}{r_1}(\lambda) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix}$$

$$\underset{\sim}{r_2}(\mu) = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

(a) Explain why ℓ_1 and ℓ_2 are parallel.

(b) Find the distance between ℓ_1 and ℓ_2.

Question 17 Let ℓ_1 be the line through $P(1, -2, 3)$ that is perpendicular to ℓ_2 represented by

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Find a vector representation of ℓ_2 if and intersects ℓ_1.

Challenge Problems

Problem 1 [Symmetric equations]

Consider the vector equation

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(a) Show that

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

(b) Interpret this result geometrically.

Problem 2 Consider two skew lines ℓ_1 and ℓ_2 defined by the vector equations

$$\underset{\sim}{r_1}(\lambda) = (\underset{\sim}{i} + 2\underset{\sim}{k}) + \lambda(-2\underset{\sim}{j} + \underset{\sim}{k})$$
$$\underset{\sim}{r_2}(\mu) = (-\underset{\sim}{j} + \underset{\sim}{k}) + \mu(\underset{\sim}{i} + \underset{\sim}{j})$$

(a) Find an expression for the vector that connects a point P on ℓ_1 to a point Q on ℓ_2. Express your answer in terms of μ and λ.

(b) Find the shortest distance between ℓ_1 and ℓ_2.

Problem 3 Let $\underset{\sim}{p}$ be the position vector of a point P on a sphere $\mathcal{S}$ with radius r and centred at C represented by $\underset{\sim}{c}$. You may use the fact that the position vector $\underset{\sim}{v}$ for the set of all points on $\mathcal{S}$ is

$$|\underset{\sim}{v} - \underset{\sim}{c}| = r.$$

(a) Consider a line ℓ that passes through P. If the direction vector of ℓ is $\underset{\sim}{b}$, write down a vector representation of ℓ.

(b) Show that if ℓ intersects $\mathcal{S}$, then $\lambda = 0$ or $\lambda = \dfrac{2\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{p})}{|\underset{\sim}{b}|^2}$.

(c) Deduce that ℓ is tangential to $\mathcal{S}$ if and only if

$$\underset{\sim}{b} \cdot (\underset{\sim}{p} - \underset{\sim}{c}) = 0$$

Exercise 3D

Parameterising 3D Curves

Fundamentals

Fundamentals 1

Similarly to straight lines, curves in 3D space are defined p__________ using

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

It is often difficult to draw the full curve and demonstrate all features properly by hand. So instead what we can do is draw the projections of the curve onto the ____, ____ and ____-planes to gain some insight as to what the curve may look like from different p__________.

Fundamentals 2

(a) The curve may have some restriction on the p__________. When this happens, it may restrict the curve so only a s________ of it is drawn.

(b) Be wary that the curve may also be restricted by other means. For example

$$x = t^2$$
$$y = \frac{1}{t^2}$$

is not simply the curve $y = 1/x$ because x and y are always p________ and so it only represents the first q________ of the curve.

Question 1 Find the Cartesian equation of the following, and state any restrictions where necessary.

(a) $\underset{\sim}{r}(t) = (t-1)\underset{\sim}{i} + (2t+3)\underset{\sim}{j}$

(b) $\underset{\sim}{r}(t) = (2t-1)\underset{\sim}{i} + (t^2)\underset{\sim}{j}$

(c) $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + (t^2+1)\underset{\sim}{j}$

(d) $\underset{\sim}{r}(t) = 4\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j}$

(e) $\underset{\sim}{r}(t) = 3\sec(t)\underset{\sim}{i} + 2\tan(t)\underset{\sim}{j}$

(f) $\underset{\sim}{r}(t) = (e^t)\underset{\sim}{i} + (e^{-t})\underset{\sim}{j}$

Question 2 Draw the curve defined by

$$\underset{\sim}{r}(t) = (t+1)\underset{\sim}{i} + (2t-1)\underset{\sim}{j} + (t^2)\underset{\sim}{k}$$

in the xy, xz and yz-planes for $t \in [0, 3]$.

Question 3

(a) Find the Cartesian equation of the curve defined parametrically by

$$\underset{\sim}{r}(t) = (\cos t + \sin t)\,\underset{\sim}{i} + (\cos t - \sin t)\,\underset{\sim}{j}.$$

(b) Find the Cartesian equation of the curve defined parametrically by

$$\underset{\sim}{r}(t) = \sqrt{2}\cos(t)\,\underset{\sim}{i} + \sqrt{2}\sin(t)\,\underset{\sim}{j}.$$

(c) Describe the difference in their position vectors as t varies.

Question 4 [Parametrisation of the ellipse]

Consider the ellipse defined parametrically by

$$\underset{\sim}{r}(t) = a\cos(t)\,\underset{\sim}{i} + b\sin(t)\,\underset{\sim}{j}$$

where $t \in [0, 2\pi]$.

(a) Find the Cartesian equation of the ellipse.

(b) Sketch the ellipse and indicate the orientation of the ellipse as t increases.

(c) How would you adjust the parametrisation such that the ellipse is the same, but the orientation is reversed?

(d) How would you adjust the parametrisation so that the starting point is instead on the positive y-axis?

Question 5 [Witch of Agnesi]

Consider the curve defined parametrically by

$$\underset{\sim}{r}(t) = 2a\tan(t)\,\underset{\sim}{i} + 2a\cos^2(t)\,\underset{\sim}{j}.$$

(a) Find the Cartesian equation.

(b) Sketch the curve.

Question 6 [Parametrisations are not unique!]

Consider the Cartesian equation $y = x^3$ and three possible parametrisations below.

$$\underset{\sim}{r}_1(t) = (2t)\,\underset{\sim}{i} + (8t^3)\,\underset{\sim}{j}$$
$$\underset{\sim}{r}_2(t) = (t^2)\,\underset{\sim}{i} + (t^6)\,\underset{\sim}{j}$$
$$\underset{\sim}{r}_3(t) = \left(\frac{1}{t}\right)\underset{\sim}{i} + \left(\frac{1}{t^3}\right)\underset{\sim}{j}$$

What is the difference between all three paths despite them having the same Cartesian equation?

Question 7 [Spiral behaviour]

All of the following parametrisations below represent spirals, but they differ in some way. Describe the behaviour of each spiral as t increases, given that $t \geq 0$.

(a) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (t)\,\underset{\sim}{k}$

(b) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (-t)\,\underset{\sim}{k}$

(c) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (t^2)\,\underset{\sim}{k}$

(d) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + \ln(t)\,\underset{\sim}{k}$

(e) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + e^{-t}\,\underset{\sim}{k}$

(f) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (1+\sin(t))\,\underset{\sim}{k}$

Question 8 Consider the parametrisation

$$\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (1-\sin(t))\,\underset{\sim}{k}.$$

(a) State what the curve looks like in the xy-plane.

(b) State what the curve looks like in the yz-plane.

(c) State what the curve looks like in the xz-plane.

(d) Describe geometrically what the parametrisation represents.

Question 9 Describe the shape of the curve defined by the following vector equations.

(a) $\underset{\sim}{r}(t) = (1-t)\,\underset{\sim}{i} + (t+2)\,\underset{\sim}{j} + (2t+3)\,\underset{\sim}{k}$

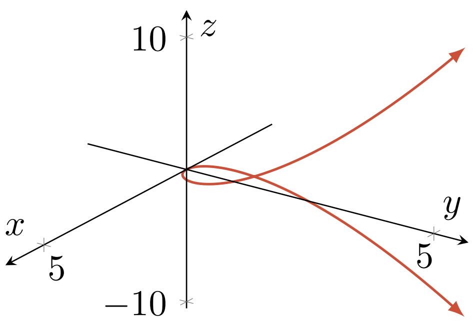
(b) $\underset{\sim}{r}(t) = (t)\,\underset{\sim}{i} + (t^2+2)\,\underset{\sim}{j} + (t)\,\underset{\sim}{k}$

(c) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + (t)\,\underset{\sim}{k}$

(d) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} + \sin(t)\,\underset{\sim}{k}$

Question 10 [Twisted Cubic]

The diagram below shows a *twisted cubic* for $t \in [-2, 2]$



and the parametrisation of the twisted cubic is

$$\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + t^2\,\underset{\sim}{j} + t^3\,\underset{\sim}{k}.$$

(a) Find the equation of the curve along the xy, xz and yz-axes.

(b) Sketch the curve along the xy, xz and yz-axes.

Question 11 [Parametrisations and motion]

Three different particles move in the xy-plane according to the trajectories defined parametrically by

$$\begin{aligned} \underset{\sim}{r}_1(t) &= \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} \\ \underset{\sim}{r}_2(t) &= \cos(2t)\,\underset{\sim}{i} + \sin(2t)\,\underset{\sim}{j} \\ \underset{\sim}{r}_3(t) &= \sin(t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} \end{aligned}$$

What is the difference in their motions?

Question 12 Two particles travel according to the trajectories defined by

$$\begin{aligned} \underset{\sim}{r}_1(t) &= (2t-1)\,\underset{\sim}{i} + (t^2)\,\underset{\sim}{j} + (t+1)\,\underset{\sim}{k} \\ \underset{\sim}{r}_2(t) &= (t^2)\,\underset{\sim}{i} + (3t-2)\,\underset{\sim}{j} + (3-t)\,\underset{\sim}{k} \end{aligned}$$

Determine if the particles will ever collide.

Question 13 A particle moves according to the path defined parametrically by

$$\underset{\sim}{r}(t) = (2t^3+t)\,\underset{\sim}{i} + (4-t^2)\,\underset{\sim}{j} + (3t+1)\,\underset{\sim}{k}$$

where t is time in seconds.

(a) Write down an expression for the velocity vector $\underset{\sim}{v}(t)$ and acceleration vector $\underset{\sim}{a}(t)$.

(b) Find the speed and acceleration of the particle after one second.

Question 14 Find a *possible* vector representation, with conditions if necessary, of the curve that satisfies each of the following descriptions. Note that there are multiple possible answers.

(a) A spiral section of radius 2 that wraps around the y-axis and begins at $(0, 0, 2)$.

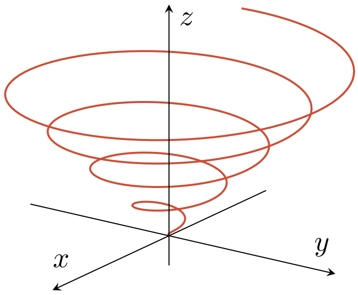
(b) A straight line that passes through $(3, -6, 1)$ and $(-2, 0, 4)$.

(c) A curve that is the parabola $y = x^2$ in the xy-plane but the linear function $z = 2y$ in the yz-plane.

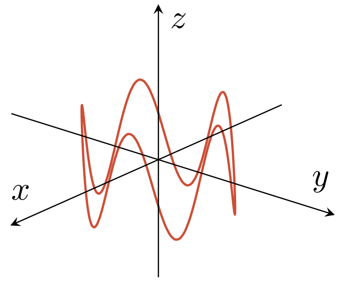
(d) An ellipse that in the xy-plane is the unit circle, but in the yz-plane forms the linear function $z = 2 - y$.

Question 15 The diagrams below show sketches of eight parametrically defined curves.

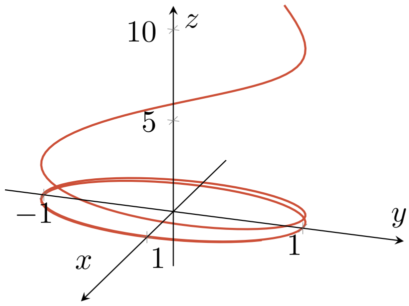
(i)



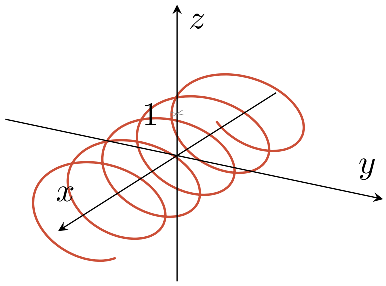
(ii)



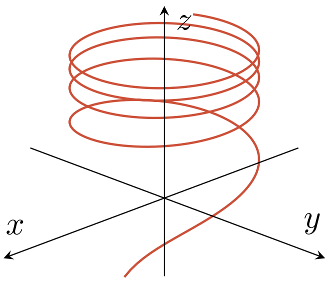
(iii)



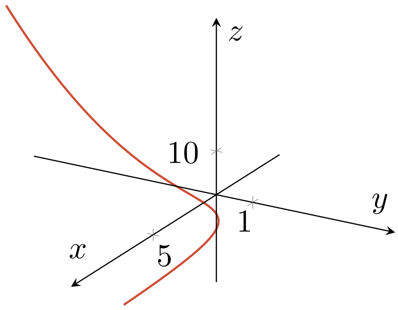
(iv)



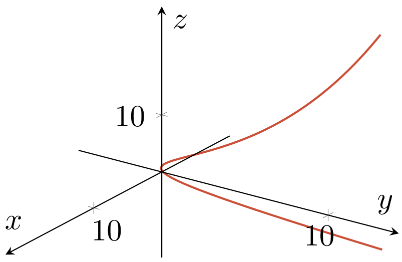
(v)



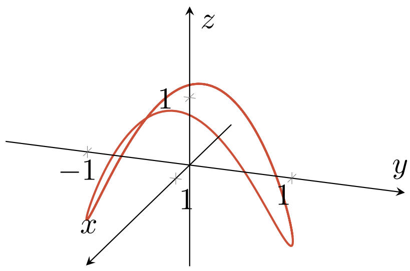
(vi)



(vii)



(viii)



Match the curves to the appropriate set of parametrisations below.

(a) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (e^{-0.5t})\underset{\sim}{k}$

(b) $\underset{\sim}{r}(t) = (t\cos 5t)\underset{\sim}{i} + (t\sin 5t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

(c) $\underset{\sim}{r}(t) = (t)\underset{\sim}{i} + (t^2)\underset{\sim}{j} + (t^3)\underset{\sim}{k}$

(d) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (\cos 2t)\underset{\sim}{k}$

(e) $\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + (\ln t)\underset{\sim}{k}$

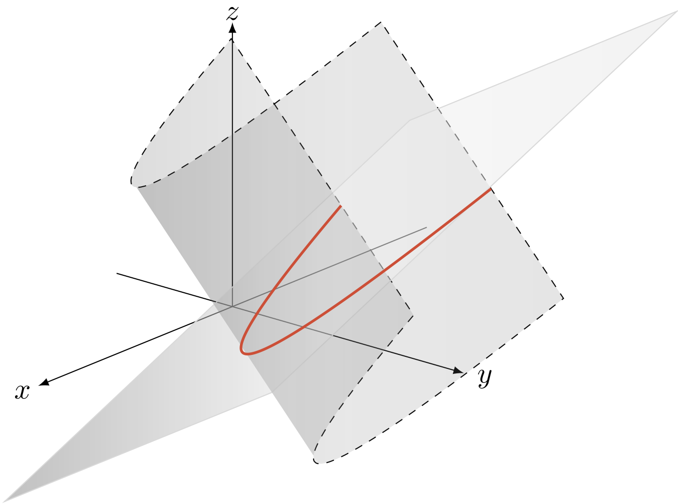
(f) $\underset{\sim}{r}(t) = (e^t)\underset{\sim}{i} + (t)\underset{\sim}{j} + (t^2)\underset{\sim}{k}$

(g) $\underset{\sim}{r}(t) = (t)\underset{\sim}{i} + (\cos 6t)\underset{\sim}{j} + (\sin 6t)\underset{\sim}{k}$

(h) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (\sin 5t)\underset{\sim}{k}$

Question 16 [Guided question to find the intersection of two surfaces]

The diagram below shows a parabolic cylinder $z = x^2 - y$ and a plane $z = y - 1$.



(a) Solve the surfaces simultaneously to show that $y = \frac{1}{2}\left(x^2 + 1\right)$. What is the geometric significance of this result?

(b) Set $x = t$ for some $t \in \mathbb{R}$ and hence show that the intersection of the two surfaces has parametric representation

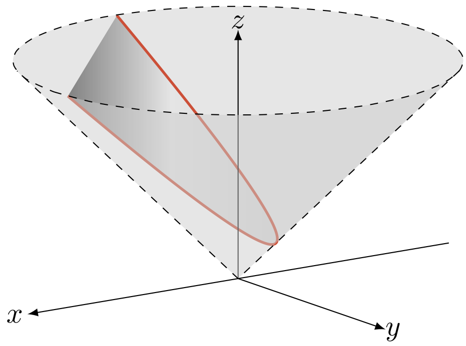
$$\underset{\sim}{r}(t) = (t)\underset{\sim}{i} + \frac{1}{2}\left(t^2 + 1\right)\underset{\sim}{j} + \frac{1}{2}\left(t^2 - 1\right)\underset{\sim}{k}.$$

(c) For the $\underset{\sim}{k}$-component above, the surface $z = y - 1$ was used. Is it incorrect to instead use the other surface $z = x^2 - y$?

(d) Suppose that the condition $z \leq 4$ were introduced. Find a corresponding restriction for t and hence find the endpoints of the curve of intersection.

Question 17 [A well-chosen initial parameter can result in a cleaner parametrisation]

The diagram below shows a cone $z = \sqrt{x^2 + y^2}$ and a plane $z = 1 + x$.



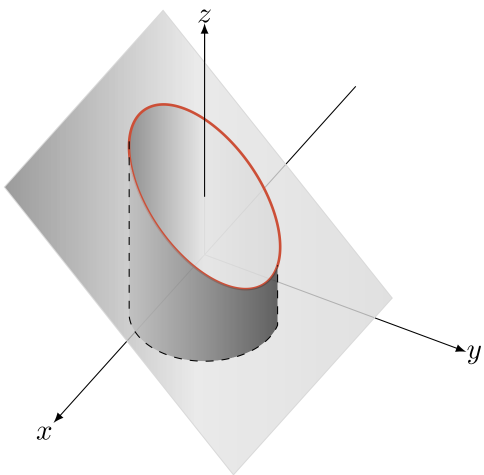
(a) Obtain a parametrisation of the curve of intersection by setting $x = t$.

(b) By setting $y = t$, show that a parametrisation of the intersection is

$$\underset{\sim}{r}(t) = \frac{1}{2}(t^2 - 1)\,\underset{\sim}{i} + t\,\underset{\sim}{j} + \frac{1}{2}(t^2 + 1)\,\underset{\sim}{k}.$$

(c) Why do you think the second method is better than the first even though it results in the same curve?

Question 18 The diagram below shows a cylinder $x^2 + y^2 = 1$ and a plane $z = 1 - y$.

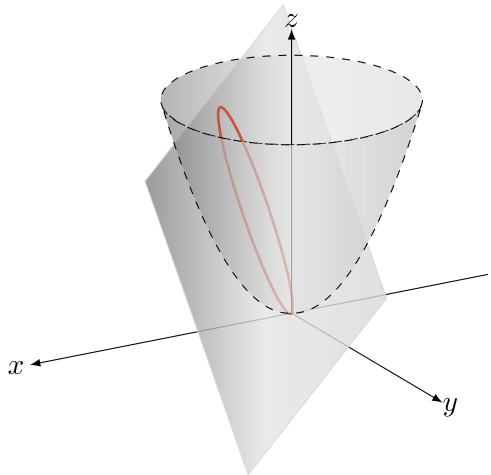


(a) Find a suitable parametrisation for the cylinder.

(b) Substitute this into the plane to obtain a parametrically defined curve.

(c) Describe what this parametrisation represents geometrically.

Question 19 The diagram below shows a paraboloid $z = x^2 + y^2$ and a plane $z = 2x - 2y$.



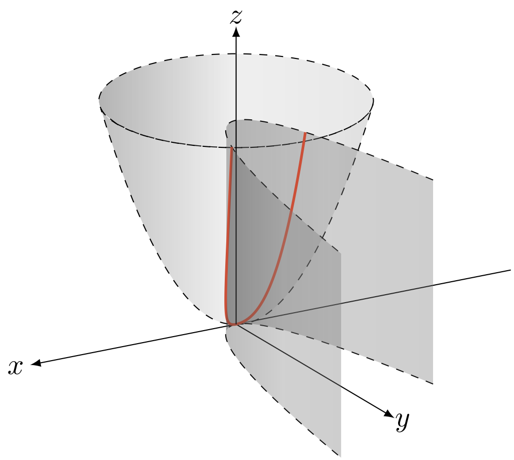
(a) Eliminate z to show that $(x-1)^2 + (y+1)^2 = 2$.

(b) Find a suitable parametrisation for the equation above.

(c) Hence show that the curve of intersection has parametric representation

$$\underset{\sim}{r}(t) = (1 + \sqrt{2}\cos t)\,\underset{\sim}{i} + (-1 + \sqrt{2}\sin t)\,\underset{\sim}{j} + \left(4 + 4\cos\left(t + \frac{\pi}{4}\right)\right)\underset{\sim}{k}.$$

Question 20 The diagram below shows a parabolic cylinder $y = x^2$ and a paraboloid $z = x^2 + y^2$ for $z \leq 12$.



Find a parametric representation for the curve of intersection, including any restrictions on the parameter t.

Challenge Problems

Problem 1 [A way to describe a familiar curve in 3D space]

Define two mutually perpendicular unit vectors in space as

$$\underset{\sim}{a} = a_1\,\underset{\sim}{i} + a_2\,\underset{\sim}{j} + a_3\,\underset{\sim}{k}$$
$$\underset{\sim}{b} = b_1\,\underset{\sim}{i} + b_2\,\underset{\sim}{j} + b_3\,\underset{\sim}{k}$$

Define the position vector
$$\underset{\sim}{p}(t) = (r\cos t)\underset{\sim}{a} + (r\sin t)\underset{\sim}{b}.$$

(a) Show that $\underset{\sim}{p}(t) \cdot \underset{\sim}{p}'(t) = 0$ for all $t \in \mathbb{R}$.

(b) Hence, what kind of curve does $\underset{\sim}{p}(t)$ trace out?

(c) Let $\underset{\sim}{c} = c_1\,\underset{\sim}{i} + c_2\,\underset{\sim}{j} + c_3\,\underset{\sim}{k}$ represent the position vector of some fixed point (c_1, c_2, c_3). Consider the curve defined by the following.

$$\underset{\sim}{p}(t) = \underset{\sim}{c} + (r\cos t)\underset{\sim}{a} + (r\sin t)\underset{\sim}{b}$$

What kind of curve does this define?

Problem 2 [The restriction can sometimes be hard to spot!]

Bob is given the parametric vector equation

$$\underset{\sim}{r}(t) = \begin{bmatrix} t + \dfrac{1}{t} \\ t^2 + \dfrac{1}{t^2} \end{bmatrix}$$

and asked to produce a sketch.

(a) Show that the Cartesian equation is $y = x^2 - 2$.

(b) Bob submits the graph of $y = x^2 - 2$ and his teacher says that he is wrong! Explain why.

Exercise 3E

Spheres and Circles

Fundamentals

Fundamentals 1
The sphere centred at (x_0, y_0, z_0) with radius r is

$$\rule{4cm}{0.4pt}$$

Fundamentals 2
The position vector $\underset{\sim}{v}$ of any point on a sphere is given by

$$|\underset{\sim}{v} - \underset{\sim}{c}| = r.$$

It represents a sphere centred at the position vector ___ with radius ___.

Fundamentals 3
Let $\mathcal{S}$ represent a sphere. Describe the main steps to

(a) determine if a point P lies inside $\mathcal{S}$, on the surface of $\mathcal{S}$, or outside of $\mathcal{S}$.

(b) determine where a line ℓ intersects $\mathcal{S}$.

(c) find the Cartesian equation of $\mathcal{S}$ given the centre C and a point P on the surface.

(d) find the Cartesian equation of $\mathcal{S}$ given two diametrically opposed foes A and B on $\mathcal{S}$.

Question 1

(a) Find the Cartesian equation of the sphere centred at $(2, -3, 5)$ with radius 6.

(b) Find the intersection of the sphere with each of the coordinate planes.

Question 2 Find the Cartesian equation of the sphere that passes through the origin O and has centre $C(-5, 3, 4)$.

Question 3 Find the centre and radius of the following spheres.

(a) $x^2 - 4x + y^2 + 2y + z^2 = 4$

(b) $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$

Question 4 Find the Cartesian equation of the sphere with centre $C(6, -2, 3)$ that touches the

(a) xy-plane.

(b) xz-plane.

(c) yz-plane.

Question 5 Find the Cartesian equation of the circles formed where the sphere

$$(x-2)^2 + (y-4)^2 + (z-3)^2 = 36$$

intersects the xy, yz, xz-planes.

Question 6 Find the Cartesian equation of the largest possible sphere centred at $C(3, 5, 2)$ but is still fully contained within the first octant.

Question 7 Define fixed points $A(1, -2, 0)$ and $B(1, 1, 3)$.

(a) Find the set of all points P where the distance of P to A is twice the distance of P to B, and show that it is a sphere.

(b) Find the centre and radius of the sphere.

Question 8 Find the Cartesian equation of the sphere that passes through $P(0, 3, 2)$ with centre $C(0, 1, 2)$.

Question 9 [Guided question for a line intersecting a sphere]

Consider the line represented by the vector equation

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

and the sphere centred at $(2, -1, 3)$ with radius 3.

(a) Write down the vector equation of the line in the form

$$\underset{\sim}{r}(\lambda) = f(\lambda)\,\underset{\sim}{i} + g(\lambda)\,\underset{\sim}{j} + h(\lambda)\,\underset{\sim}{k}.$$

(b) Write down the Cartesian equation of the sphere.

(c) Substitute the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components into the equation of the sphere to show that $7\lambda^2 - 8\lambda + 1 = 0$ and hence solve for λ.

(d) Hence, find the two points where the line intersects the sphere.

(e) What can we say if the quadratic in terms of λ has no real roots? How about one real root?

Question 10 [Demonstrating a 3D analogy of a familiar result]

Let ℓ be the line that passes through $(2, -1, 0)$ in the direction of $\underset{\sim}{b} = -\underset{\sim}{j} + \underset{\sim}{k}$. Let $\mathcal{S}$ be the sphere centred at $C(1, 0, -1)$ with radius r. Let the point of contact between $\mathcal{S}$ and ℓ be P.

(a) Find the value of r so that $\mathcal{S}$ touches ℓ.

(b) Substitute the vector representation of ℓ into the Cartesian equation of $\mathcal{S}$ and show that there is only one solution of λ. Was this to be expected?

(c) Find the coordinates of P.

(d) Find the angle between $\overrightarrow{CP}$ and ℓ. Explain briefly how this result was to be expected.

Question 11 Find the coordinates of the points where the line represented by

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

intersects the sphere centred at the origin with radius 3.

Question 12 Consider the two spheres

$$\mathcal{S}_1 : \quad (x-1)^2 + (y+1)^2 + (z-2)^2 = 16$$
$$\mathcal{S}_2 : \quad (x+1)^2 + (y-3)^2 + (z+2)^2 = 4$$

(a) Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are tangential to each other.

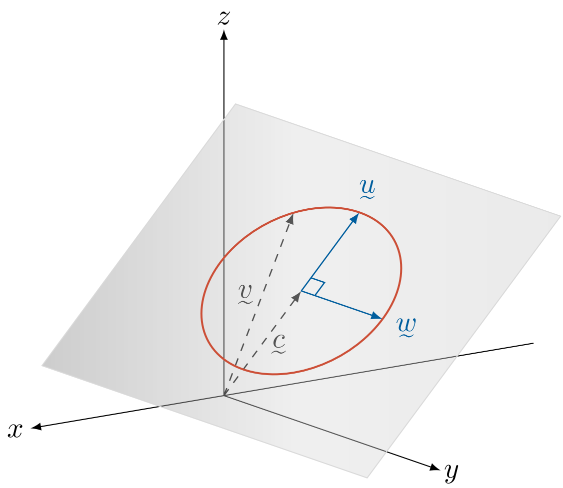
(b) Find the coordinates of the common point.

(c) Find another possible radius for $\mathcal{S}_2$ such that the spheres will be tangential.

(d) Find the coordinates of the common point when this occurs.

Question 13 [Parametric equation of a circle in 3D space]

Consider a circle with radius r on some plane $\mathcal{P}$ containing two perpendicular vectors $\underset{\sim}{u}$ and $\underset{\sim}{w}$ both with length r.



Let the centre of the circle have position vector $\underset{\sim}{c}$, and let $\underset{\sim}{v}$ represent the position vector of any point on the circle.

(a) Write down the standard parametrisation for the circle centred at the origin with radius r in the xy-plane.

(b) Use your previous answer to explain how

$$\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{u}\cos\theta + \underset{\sim}{w}\sin\theta$$

represents a circle in 3D space.

(c) Find the radius of the circle and verify that it is r.

Question 14 [Spherical coordinates]

Show that the set of equations

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

represent a sphere centred at the origin with radius r.

Challenge Problems

Problem 1 Consider the line ℓ defined parametrically by $\underset{\sim}{r}(\lambda) = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ and the sphere $\mathcal{S}$ defined by $|\underset{\sim}{v} - \underset{\sim}{c}| = r$.

(a) Suppose ℓ is a tangent to $\mathcal{S}$. Prove that this occurs when $\lambda = \underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a})$.

(b) Deduce that the tangent to the sphere is perpendicular to the radius at the point of contact.

Problem 2 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be the position vectors of two fixed points A and B in space, and let $\underset{\sim}{v}$ be a variable position vector that satisfies

$$(\underset{\sim}{v} - \underset{\sim}{a}) \cdot (\underset{\sim}{v} - \underset{\sim}{b}) = 0.$$

(a) Show that

$$\left|\underset{\sim}{v} - \frac{\underset{\sim}{a} + \underset{\sim}{b}}{2}\right| = \frac{1}{2}|\underset{\sim}{a} - \underset{\sim}{b}|.$$

(b) Geometrically, what does the position vector $\underset{\sim}{v}$ represent, and find the main features of it.

(c) What is the geometric significance of the above results?

Problem 3 Consider the two mutually tangential spheres

$$\mathcal{S}_1 : \quad |\underset{\sim}{v} - \underset{\sim}{c}_1| = r_1$$
$$\mathcal{S}_2 : \quad |\underset{\sim}{v} - \underset{\sim}{c}_2| = r_2$$

Show that the point of contact has position vector

$$\underset{\sim}{p} = \frac{r_2\underset{\sim}{c}_1 + r_1\underset{\sim}{c}_2}{r_1 + r_2}.$$

Chapter 3 Review

3D Vectors

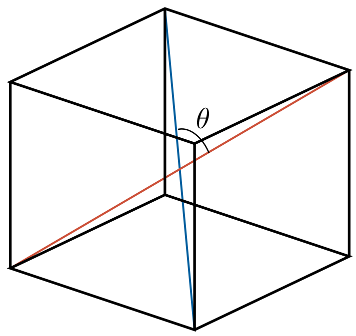
Review

Question 1 Find the angle between the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$.

Question 2 Define $A(1, 4, -1)$, $B(-2, 8, 4)$ and $C(-1, 10, -1)$. Find all possible points D such that $ABCD$ forms a parallelogram.

Question 3 Let $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{k}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$. Find a vector with length $\sqrt{129}$ that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Question 4 Consider a cube of side length a.



Prove that the acute angle between the two main diagonals as shown above is $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Question 5 Determine if the point $P(a, b, c)$ lies on the line represented by

$$\ell : \quad \underset{\sim}{r}(\lambda) = \underset{\sim}{a} + \lambda\underset{\sim}{b}.$$

Question 6 Find the vector equation for the *interval* AB if $A = (-7, 3, 4)$ and $B = (2, 6, -1)$.

Question 7 Define the lines

$$\ell_1 : \quad \underset{\sim}{r_1}(\lambda) = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$$

$$\ell_2 : \quad \underset{\sim}{r_2}(\mu) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -9 \\ 15 \\ 6 \end{bmatrix}$$

Determine whether ℓ_1 and ℓ_2 are parallel or not.

Question 8 Define the lines

$$\ell_1 : \quad \underset{\sim}{r_1}(\lambda) = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\ell_2 : \quad \underset{\sim}{r_2}(\mu) = \begin{bmatrix} 7 \\ 3 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$$

Determine whether ℓ_1 and ℓ_2 are perpendicular or not.

Question 9 Define the lines

$$\ell_1 : \quad \underset{\sim}{r_1}(\lambda) = \begin{bmatrix} -5 \\ 2 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

$$\ell_2 : \quad \underset{\sim}{r_2}(\mu) = \begin{bmatrix} 0 \\ -6 \\ -3 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$$

(a) Show that they intersect.

(b) Find the coordinates of the point of intersection.

Question 10 Define the line ℓ

$$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

Find where the line ℓ intersects the plane $2x - 3y + z + 2 = 0$.

Question 11 Define the lines ℓ_1 and ℓ_2 by

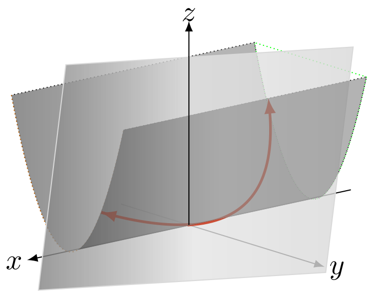
$$\ell_1 : \quad \underset{\sim}{r}(\lambda) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\ell_2 : \quad \underset{\sim}{r}(\mu) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(a) Prove that ℓ_1 and ℓ_2 are skew lines.

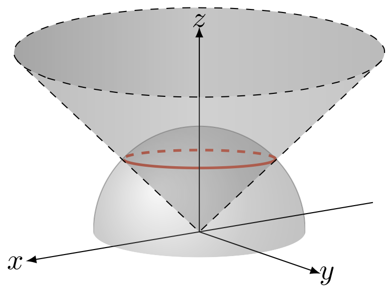
(b) Find the shortest distance between ℓ_1 and ℓ_2.

Question 12 The diagram below shows a parabolic cylinder $z = y^2$ and a plane $z = x + 2y$.



Find a parametric representation for the curve of intersection, including any restrictions on the parameter t.

Question 13 The diagram below shows a cone $z = \sqrt{x^2 + y^2}$ and the upper-half of the unit sphere $x^2 + y^2 + z^2 = 1$.



Find a parametric representation for the curve of intersection.

Question 14 Find the centre and radius of the sphere

$$x^2 + 4x + y^2 - 2y + z^2 - 6z - 2 = 0.$$

Question 15 Find the equation of the sphere that has centre $C(2, 3, 4)$ and touches the

(a) yz-plane. (b) x-axis.

Question 16 Find the coordinates of the points where the line passing through $P(3, -1, -2)$ and $Q(5, 3, -4)$ intersects the sphere $x^2 + y^2 + z^2 = 26$.

Question 17 Consider the two spheres

$$\mathcal{S}_1 : \quad (x-1)^2 + (y+1)^2 + (z-2)^2 = 64$$
$$\mathcal{S}_2 : \quad (x+1)^2 + (y-3)^2 + (z+2)^2 = 4$$

(a) Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are tangential.

(b) Find the coordinates of the point of contact.

Investigation Task

Lunar laser ranging retro-reflector

One of the ways of precisely measuring the distance of the Moon from the Earth is to use something called a *Laser Ranging Retro-reflector*, which shoots a laser from the Earth to a series of corner-mirrors on the moon that were planted during Apollo Programs 11, 14 and 15.

This investigation task will allow the student to see how the study of 3D vectors can be used in a practical physics scenario.

Write a two-page article that demonstrates how the mathematics behind 3D vectors was used in the laser ranging retro-reflector and in particular the idea behind corner mirrors. Your answer should include the following.

- A definition of an angle of incidence, an angle of reflection, and the relationship between the two angles.
- A derivation and/or proof of the *key result* about reflections that allows a corner-mirror to actually be useful.
- Relevant calculations that demonstrate an approximation of the distance of the Moon from the Earth.
- All relevant diagrams.

Investigation Task

Equation of a plane

In this chapter, we study lines in 3D space and work lightly with planes in 3D space. This investigation task will allow students to study planes more carefully, which will enrich their understanding of working in 3D space. This investigation task is also best accompanied by the investigation task on the *Cross Product* for a complete picture.

Question 1 In your study of linear functions, you needed a gradient and a point to uniquely define a linear function. What do you need to uniquely define a plane?

Question 2 Write a one-page article, complete with diagrams, that shows the equation of a plane and the derivation of that equation.

Question 3 Answer the following, and also provide an example to demonstrate your technique.

(a) How do you determine if two planes are parallel?

(b) If two planes intersect, how do you find the equation of the line of intersection?

(c) How do you find the angle between two planes?

(d) How do you find the distance between a point and a plane? What is the formula for it in general?

(e) How do you find the distance between two parallel planes?

(f) How do you find the plane that passes through a particular point and is perpendicular to a particular vector?

(g) How do you find the plane through a particular point that is parallel to a particular vector?

(h) How do you find the equation of a plane through three given points?

(i) How do you find the equation of a plane that contains two intersecting lines in space?

(j) How do you determine if two vectors are coplanar?

Investigation Task

Cross Product

Any standard course on linear algebra will include a section on the *cross product*, which is an incredibly useful tool when it comes to the study of 3D vectors. This investigation task, accompanied by the investigation task on the equation of the plane, will give a more complete study of 3D space and 3D vectors.

Question 1

(a) Define a *cross product* and give the formula for it.

(b) What is the output of a cross product?

(c) What does it mean when we say that a cross product is *non-commutative*?

(d) What is the *right-hand rule*, and why is it relevant to the cross product?

(e) What is *torque*, and why is it relevant to the cross product?

Question 2

(a) Give the formula for the *determinant* of a 2×2 matrix.

(b) How would you calculate the determinant of a 3×3 matrix?

(c) Give the definition of the cross product that involves the determinant of a 3×3 matrix.

(d) Make up two vectors, find their cross product, and show that the cross product is non-commutative.

Question 3

(a) What would you expect to happen if you cross product a vector with itself? Why?

(b) Prove that $\utilde{u} \times \utilde{v}$ is indeed perpendicular to both $\utilde{u}$ and $\utilde{v}$.

(c) For the dot product, we have the formula $\utilde{u} \cdot \utilde{v} = |\utilde{u}||\utilde{v}| \cos\theta$. There is a similar formula for the cross product. What is it, and explain how it can be used to find the area of a triangle.

(d) For the dot product, two vectors are perpendicular if and only if $\utilde{u} \cdot \utilde{v} = 0$. There is a similar property for the cross product. What is it, and provide derivations.

(e) What is the relationship between $\utilde{u} \times \utilde{v}$ and $\utilde{v} \times \utilde{u}$?

Question 4

(a) What is a *scalar triple product*? Prove that $\utilde{a} \cdot (\utilde{b} \times \utilde{c}) = (\utilde{a} \times \utilde{b}) \cdot \utilde{c}$.

(b) How is the scalar triple product used to calculate volume? Give full derivations.

(c) How can the scalar triple product be used to prove that three vectors are co-planar? Explain.

4

FURTHER INTEGRATION

- Integration by Substitution
- Trigonometric Integrals
- Trigonometric Substitutions
- Harder Standard Integrals
- Partial Fractions
- t-formula Substitutions
- Integration by Parts
- Reduction Formulae
- Further Substitutions

Exercise 4A

Integration by Substitution

Fundamentals

Fundamentals 1

Complete the following formulae.

(a) $\displaystyle\int f'(x)\left(f(x)\right)^n\,dx$ (b) $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ (c) $\displaystyle\int f'(x)e^{f(x)}\,dx$

Fundamentals 2

Complete the following formulae.

(a) $\displaystyle\int f'(x)\sin\left(f(x)\right)dx$ (b) $\displaystyle\int f'(x)\cos\left(f(x)\right)dx$ (c) $\displaystyle\int f'(x)\sec^2\left(f(x)\right)dx$

Fundamentals 3

Complete the following formulae.

(a) $\displaystyle\int \frac{f'(x)}{\sqrt{1-\left(f(x)\right)^2}}\,dx$ (b) $\displaystyle\int \frac{f'(x)}{1+\left(f(x)\right)^2}\,dx$

Note from the author: Although these formulae are on your reference sheet, it is useful to have them memorised to allow for quicker recognition and most importantly correct recognition of the correct form. It is easy to get the various forms mixed up, and many integrals can look very similar despite having entirely different answers.

For example, the integrand of

$$\int \frac{1}{x+\sqrt{x}}\,dx$$

looks similar to the integrand of

$$\int \frac{1}{x\sqrt{x}+\sqrt{x}}\,dx$$

but the first is a log integral and the second is an inverse tan integral. It is very difficult to see this if you rely too heavily on the reference sheet. Even worse, you may not even see it at all because you were not able to develop that ‘sixth sense’ that allows you to immediately spot the correct form.

Question 1 [Practising figuring out the substitution]

More often than not when doing integration by substitution, the substitution is of the form

$$u = \text{ the function inside another function}$$

For example, for the integral $\int xe^{x^2}\,dx$ we would let $u = x^2$ because x^2 is 'inside' the exponential function. State an appropriate substitution for each of the following integrals.

(a) $\int x^2 e^{x^3}\,dx$ (b) $\int x\sin(x^2)\,dx$ (c) $\int e^x\sqrt{1+e^x}\,dx$

(d) $\int \sin x(1+\cos x)^3\,dx$ (e) $\int \frac{x}{(1+x^2)^3}\,dx$ (f) $\int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

Question 2 Find the following integrals using an appropriate substitution or the formula

$$\int f'(x)\left(f(x)\right)^n\,dx = \frac{1}{n+1}\left(f(x)\right)^{n+1} + C.$$

(a) $\int \frac{x}{\sqrt{1+x^2}}\,dx$ (b) $\int \frac{\sin x - \cos x}{(\cos x + \sin x)^2}\,dx$ (c) $\int x\sqrt{x+1}\,dx$

(d) $\int \frac{x^3}{(x^2+1)^3}\,dx$ (e) $\int (x+1)(2-x)^5\,dx$ (f) $\int \frac{x}{\sqrt{1+x}}\,dx$

(g) $\int \frac{\sin x}{\cos^3 x}\,dx$ (h) $\int \frac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)^2}\,dx$ (i) $\int \frac{\ln x}{x}\,dx$

(j) $\int \frac{\sin x}{\sqrt{3-\cos x}}\,dx$ (k) $\int \sin x\cos^3 x\,dx$ (l) $\int \sec^2 x\tan^3 x\,dx$

Question 3 Find the following integrals using an appropriate substitution or the following formulae.

$$\int f'(x)\sin\left(f(x)\right)\,dx = -\cos\left(f(x)\right) + C$$
$$\int f'(x)\cos\left(f(x)\right)\,dx = \sin\left(f(x)\right) + C$$
$$\int f'(x)\sec^2\left(f(x)\right)\,dx = \tan\left(f(x)\right) + C$$

(a) $\int x\sin(x^2)\,dx$ (b) $\int e^x\cos\left(e^x\right)\,dx$ (c) $\int \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)\,dx$

(d) $\int \frac{1}{\sqrt{x}}\sin\left(\sqrt{x}\right)\,dx$ (e) $\int \frac{\cos(\ln x)}{x}\,dx$ (f) $\int \sec^2 x\sec^2\left(2\tan x\right)\,dx$

Question 4 Find the following integrals using an appropriate substitution or the formula

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C.$$

(a) $\displaystyle\int (2x+1)e^{x^2+x}\,dx$ (b) $\displaystyle\int e^{\cos x}\sin x\,dx$ (c) $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

(d) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ (e) $\displaystyle\int \frac{x}{e^{x^2}}\,dx$ (f) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

Question 5 Find the following integrals using an appropriate substitution or the formula

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

(a) $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx$ (b) $\displaystyle\int \frac{1}{x\ln x}\,dx$ (c) $\displaystyle\int \frac{1-\sec^2 x}{x-\tan x}\,dx$

(d) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$ (f) $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

Question 6 Find the following integrals using an appropriate substitution or the formula

$$\int \frac{f'(x)}{\sqrt{1-(f(x))^2}}\,dx = \sin^{-1}(f(x)) + C.$$

(a) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ (b) $\displaystyle\int \frac{\cos x}{\sqrt{25-4\sin^2 x}}\,dx$ (c) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

(d) $\displaystyle\int \frac{\sec^2 x}{\sqrt{16-25\tan^2 x}}\,dx$ (e) $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$ (f) $\displaystyle\int \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}\,dx$

Question 7 Find the following integrals using an appropriate substitution or the formula

$$\int \frac{f'(x)}{1+(f(x))^2}\,dx = \tan^{-1}(f(x)) + C.$$

(a) $\displaystyle\int \frac{x^2}{1+x^6}\,dx$ (b) $\displaystyle\int \frac{\cos x}{4+9\sin^2 x}\,dx$ (c) $\displaystyle\int \frac{e^{-x}}{1+e^{-2x}}\,dx$

(d) $\displaystyle\int \frac{\sec^2 x}{9+4\tan^2 x}\,dx$ (e) $\displaystyle\int \frac{1}{x\left(1+(\ln x)^2\right)}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

Question 8 Find the following integrals using an appropriate substitution.

(a) $\displaystyle\int e^x\tan(e^x)\,dx$ (b) $\displaystyle\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx$ (c) $\displaystyle\int \frac{1}{x^2}\tan\left(\frac{1}{x}\right)\,dx$

Question 9 [Useful way to handle square roots]

Find the following integrals using the substitution $x = u^2$.

(a) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ (b) $\displaystyle\int \frac{1}{1-\sqrt{x}}\,dx$ (c) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

Challenge Problems

Problem 1 [Euler substitution]

Find $\displaystyle\int \frac{1}{x\sqrt{x^2+4x-4}}\,dx$ using the substitution $u = \sqrt{x^2+4x-4} - x$.

Problem 2 [Verifying the mean of the normal distribution]

Recall that the probability density function of the normal distribution with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The formula for expected value in the discrete scenario is given by

$$E(X) = \sum_x xp(x).$$

The equivalent formula for the continuous case is

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx.$$

Verify that $E(X) = \mu$ for the normal distribution, using the formula above.

Problem 3 Find the following using appropriate manipulations and substitutions.

(a) $\displaystyle\int e^{e^x+x}\,dx$ (b) $\displaystyle\int \frac{e^{2x}+1}{e^{2x}-1}\,dx$ (c) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$

(d) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$ (e) $\displaystyle\int \frac{1}{x\sqrt{x+x^2}}\,dx$ (f) $\displaystyle\int \frac{x^4+1}{x^6+1}\,dx$

(g) $\displaystyle\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$ (h) $\displaystyle\int \frac{1}{x^5+x}\,dx$ (i) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

(j) $\displaystyle\int \frac{1}{\sqrt{x^3+x^2-x-1}}\,dx$ (k) $\displaystyle\int \frac{x^2+1}{x\sqrt{-x^4+3x^2-1}}\,dx$ (l) $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\,dx$

Exercise 4B

Trigonometric Integrals

Fundamentals

Fundamentals 1
The form below is the most important standard form when it comes to trigonometric integrals.

$$\int f'(x)\left(f(x)\right)^n\,dx$$

Write down the formula for it.

Question 1 Find the following.

(a) $\int \sin^2 x \cos x\,dx$ (b) $\int \cos^3 x \sin x\,dx$ (c) $\int \tan^4 x \sec^2 x\,dx$

(d) $\int \sec^5 x \tan x\,dx$ (e) $\int \sec^2 x \tan x\,dx$ (f) $\int \sec^3 x \tan x \sqrt{\sec x}\,dx$

Question 2 Find the following.

(a) $\int \sin^3 x\,dx$ (b) $\int \cos^3 x\,dx$ (c) $\int \tan^3 x\,dx$

(d) $\int \tan^2 x\,dx$ (e) $\int \sec^4 x\,dx$ (f) $\int \tan^4 x\,dx$

Question 3 Find the following.

(a) $\int \sin^5 x\,dx$ (b) $\int \cos^5 x\,dx$ (c) $\int \tan^5 x\,dx$

Question 4 [Double-angle formulae]

Find the following. For part (c), try it without using previous parts.

(a) $\int \sin^4 x\,dx$ (b) $\int \cos^4 x\,dx$ (c) $\int \sin^4 x + \cos^4 x\,dx$

Question 5 Find the following.

(a) $\int \tan x \sec^3 x\,dx$ (b) $\int \sec^2 x \tan^2 x\,dx$ (c) $\int \tan^3 x \sec x\,dx$

(d) $\int \sin^2 x \cos^3 x\,dx$ (e) $\int \sec^2 x \tan^3 x\,dx$ (f) $\int \sin^5 x \cos^2 x\,dx$

Question 6 Find the following.

(a) $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$ (b) $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$ (c) $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

(d) $\displaystyle\int \frac{\cos^3 x}{\sin x}\,dx$ (e) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$ (f) $\displaystyle\int \frac{\sec^4 x}{\tan^2 x}\,dx$

Question 7

(a) Find $\displaystyle\int \sin x \cos x\,dx$ using three different methods.

(b) Explain why you obtained three different answers, despite all three techniques being valid.

Question 8 [Products to sums]

Find the following.

(a) $\displaystyle\int \sin 2x \cos 3x\,dx$ (b) $\displaystyle\int \cos 5x \cos 2x\,dx$ (c) $\displaystyle\int \sin 7x \sin 5x\,dx$

Question 9 [Sums to products]

Find the following.

(a) $\displaystyle\int \frac{\sin 6x - \sin 2x}{\cos 6x + \cos 2x}\,dx$ (b) $\displaystyle\int \frac{\sin 2x - \sin x}{\cos 2x - \cos x}\,dx$

(c) $\displaystyle\int \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}\,dx$ (d) $\displaystyle\int \frac{\sin x + \sin 2x + \sin 3x}{1 + \cos x + \cos 2x}\,dx$

Challenge Problems

Problem 1 Find the following.

(a) $\displaystyle\int \frac{1}{a^2 + b^2 \sin^2 x}\,dx$ (b) $\displaystyle\int \frac{1}{a^2 + b^2 \cos^2 x}\,dx$

(c) $\displaystyle\int \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}\,dx$ (d) $\displaystyle\int \frac{1}{(a\cos x + b\sin x)^2}\,dx$

Problem 2 Evaluate $\displaystyle\int_0^{\pi} \frac{1}{a - b\cos x}\,dx$ using the substitution $(a-b\cos x)(a+b\cos y) = a^2-b^2$.

Problem 3 [Orthogonality relation for Fourier Analysis]

Prove that

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)\,dx = \begin{cases} 2L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Exercise 4C

Trigonometric Substitutions

Fundamentals

Fundamentals 1
Write down an appropriate trigonometric substitution for x to simplify the following.

(a) $\sqrt{a^2 - x^2}$ (b) $\sqrt{a^2 + x^2}$ (c) $\sqrt{x^2 - a^2}$

Question 1 Evaluate the following.

(a) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ (b) $\displaystyle\int_{1}^{2} \frac{\sqrt{4-x^2}}{x^2}\,dx$ (c) $\displaystyle\int_{0}^{\sqrt{3}} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$

Question 2 Find the following.

(a) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$ (c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

Question 3 Find the following.

(a) $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int \frac{1}{x\sqrt{4+x^2}}\,dx$ (c) $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

Question 4 Find the following.

(a) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ (b) $\displaystyle\int \frac{\sqrt{4+x^2}}{x^2}\,dx$ (c) $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Question 5 Find the following.

(a) $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$ (b) $\displaystyle\int \frac{\sqrt{4+x^2}}{x}\,dx$ (c) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

Question 6 Find the following.

(a) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\,dx$ (c) $\displaystyle\int \frac{x^2}{\sqrt{x^2-4}}\,dx$

Question 7 Find the following.

(a) $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$ (b) $\displaystyle\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$ (c) $\displaystyle\int \frac{1}{(x^2-4)^{\frac{3}{2}}}\,dx$

Question 8 Find the following.

(a) $\int \sqrt{a^2 - x^2}\, dx$ (b) $\int \sqrt{a^2 + x^2}\, dx$ (c) $\int \sqrt{x^2 - a^2}\, dx$

Question 9 Find the following.

(a) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ (b) $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$ (c) $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

Question 10 Find the following.

(a) $\int \frac{1}{(a^2 - x^2)^{\frac{3}{2}}}\, dx$ (b) $\int \frac{1}{(a^2 + x^2)^{\frac{3}{2}}}\, dx$ (c) $\int \frac{1}{(x^2 - a^2)^{\frac{3}{2}}}\, dx$

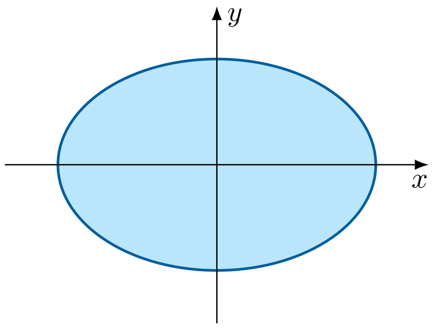
Challenge Problems

Problem 1 Evaluate the following.

(a) $\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-x}}\, dx$ (b) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ (c) $\int_0^1 \frac{\sqrt{x}}{1+x}\, dx$

(d) $\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{1-x}\, dx$ (e) $\int_0^1 \sqrt{2x - x^2}\, dx$ (f) $\int_1^3 \frac{x^2}{\sqrt{4x - x^2}}\, dx$

Problem 2 The diagram below shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the region inside it.



Prove that the area of the ellipse is πab by considering an appropriate integral and using a trigonometric substitution.

Problem 3 Let $b > a > 0$. Use the substitution $x = a\cos^2\theta + b\sin^2\theta$ to evaluate

$$\int_a^b \sqrt{\frac{x-a}{b-x}}\, dx.$$

Exercise 4D

Harder Standard Integrals

Fundamentals

Fundamentals 1

Consider an integral of the form

$$\int \frac{1}{ax^2 + bx + c}\, dx$$

and let Δ be the discriminant of the quadratic in the denominator. Describe the general technique to integrate the above if

(a) $\Delta = 0$ (b) $\Delta < 0$ (c) $\Delta > 0$

Fundamentals 2

When dealing with an integral of the form

$$\int \frac{px + q}{ax^2 + bx + c}\, dx$$

first force out the form $\int \frac{f'(x)}{f(x)}\, dx$ to handle the l_____ component of the numerator, and then it becomes the same as the above scenario.

Fundamentals 3

When dealing with an integral of the form

$$\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

complete the square and it will often be an i________ s_____ integral. Otherwise, a t__________ substitution can be used after completing the square.

Fundamentals 4

When dealing with an integral of the form

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx$$

first force out the form __________ and then it becomes the same as the above scenario.

Question 1 Find the following.

(a) $\displaystyle\int \frac{x}{2x-1}\,dx$ (b) $\displaystyle\int \frac{x^2}{x+1}\,dx$ (c) $\displaystyle\int \frac{2x+1}{x-1}\,dx$

(d) $\displaystyle\int \frac{x^2}{2x-1}\,dx$ (e) $\displaystyle\int \frac{x^2+4x-3}{x-1}\,dx$ (f) $\displaystyle\int \frac{x^4}{x-1}\,dx$

Question 2 Find the following.

(a) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ (b) $\displaystyle\int \frac{1}{4x^2+4x+5}\,dx$ (c) $\displaystyle\int \frac{1}{x^2+x+1}\,dx$

Question 3 Find the following.

(a) $\displaystyle\int \frac{x+2}{x^2+6x+10}\,dx$ (b) $\displaystyle\int \frac{6x+1}{9x^2+12x+4}\,dx$ (c) $\displaystyle\int \frac{2x+1}{x^2+6x+10}\,dx$

Question 4 Find the following.

(a) $\displaystyle\int \frac{x^2+1}{x^2+2x+2}\,dx$ (b) $\displaystyle\int \frac{x^2-x+1}{x^2+x+1}\,dx$ (c) $\displaystyle\int \frac{2x^2-1}{x^2+4x+5}\,dx$

Question 5 Find the following.

(a) $\displaystyle\int \frac{1}{\sqrt{6x-x^2}}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{-x^2+4x-3}}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{-2x-x^2}}\,dx$

Question 6 Find the following.

(a) $\displaystyle\int \frac{x+1}{\sqrt{x-x^2}}\,dx$ (b) $\displaystyle\int \frac{x}{\sqrt{2x-x^2}}\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{5+x-x^2}}\,dx$

Question 7 Find the following.

(a) $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$ (b) $\displaystyle\int \sqrt{\frac{x}{4-x}}\,dx$ (c) $\displaystyle\int \sqrt{\frac{x}{2-x}}\,dx$

Question 8 Suppose $b > a > 0$. Find the following.

(a) $\displaystyle\int \frac{1}{\sqrt{-a^2-2bx-x^2}}\,dx$ (b) $\displaystyle\int \frac{1}{x^2+2ax+b^2}\,dx$

Challenge Problems

Problem 1 [Alternating Harmonic Series]

Define the integral

$$I_n = \int_0^1 \frac{x^n}{1+x}\,dx$$

where n is a positive odd integer.

(a) Show that

$$I_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \ln 2.$$

(b) Prove that

$$0 < I_n < \frac{1}{n+1}.$$

(c) Hence, write down the exact value of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Problem 2 [Stretching $\pi \approx \dfrac{22}{7}$ to the limit]

Define the integral

$$I_n = \int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx$$

where n is a positive integer.

(a) Show that

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = P(x) + (-1)^n \frac{2^{2n}}{1+x^2}$$

where $P(x)$ is a polynomial of degree $8n-2$.

(b) Prove that $I_n \to 0$ as $n \to \infty$.

(c) Show that

$$I_n = 2^{2n-2}(-1)^n \pi + R_n$$

where R_n is some rational number.

(d) Hence, explain the significance of the number

$$\left| \frac{R_n}{2^{2n-2}} \right|$$

for large values of n.

Exercise 4E

Partial Fractions

Fundamentals

Fundamentals 1

There are three main categories of partial fraction problems. For each of them, write down the standard decomposition. A template for the first one has been done for you.

(a) Simple linear factors.

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \frac{?}{x+a} + \frac{?}{x+b}\,dx$$

(b) Irreducible quadratic factors.

$$\int \frac{1}{(x^2+a)(x+b)}\,dx = \cdots$$

(c) Repeated linear factors.

$$\int \frac{1}{(x+a)^2(x+b)}\,dx = \cdots$$

Question 1 [Simple linear factors]

Consider the partial fraction decomposition

$$\int \frac{x-1}{(x+1)(x-2)}\,dx = \int \frac{A}{x+1} + \frac{B}{x-2}\,dx.$$

(a) Show that

$$A(x-2) + B(x+1) \equiv x - 1.$$

(b) Equate the coefficient of x to find an equation in terms of A and B.

(c) Equate the constant term to find another equation in terms of A and B.

(d) Solve the expressions simultaneously to find A and B.

(e) Hence, find

$$\int \frac{x-1}{(x+1)(x-2)}\,dx.$$

Question 2 Find the following.

(a) $\displaystyle\int \frac{5}{(2x+1)(x+3)}\,dx$ (b) $\displaystyle\int \frac{1}{3x^2+18x+24}\,dx$ (c) $\displaystyle\int \frac{3x+2}{(x+1)(2x+1)}\,dx$

(d) $\displaystyle\int \frac{1}{x(x+1)(x-3)}\,dx$ (e) $\displaystyle\int_2^3 \frac{5}{(x-4)(1-x)}\,dx$ (f) $\displaystyle\int_1^2 \frac{x-8}{(2x-1)(x+2)}\,dx$

Question 3 [Useful for resisted motion and the logistic equation]

Show that

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

Question 4 [Irreducible quadratic factors]

Consider the partial fraction decomposition

$$\int \frac{x+1}{(x^2+1)(x-1)}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{C}{x-1}\,dx.$$

(a) Show that

$$(Ax+B)(x-1) + C(x^2+1) \equiv x+1.$$

(b) Substitute $x = 1$ to find the value of C.

(c) Equate the coefficient of x^2 to find A.

(d) Equate the constant term to find B.

(e) Hence, find

$$\int \frac{x+1}{(x^2+1)(x-1)}\,dx.$$

Question 5 Find the following.

(a) $\displaystyle\int \frac{x-2}{(x^2+4)(x+1)}\,dx$ (b) $\displaystyle\int \frac{4x+2}{(x^2+1)(x+3)}\,dx$ (c) $\displaystyle\int \frac{x^2+x+11}{(x+2)(x^2+9)}\,dx$

Question 6 [It's easier than you think!]

Find $\displaystyle\int \frac{x^2}{(x^2+1)(x^2-2)}\,dx.$

Question 7 [Repeated linear factors]

Consider the partial fraction decomposition

$$\int \frac{x}{(x-1)^2(x+1)}\,dx = \int \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}\,dx.$$

(a) Show that

$$A(x+1) + B(x-1)(x+1) + C(x-1)^2 \equiv x.$$

(b) Substitute $x = 1$ to find A.

(c) Substitute $x = -1$ to find C.

(d) Substitute any appropriate value of x to find B.

(e) Hence, find

$$\int \frac{x}{(x-1)^2(x+1)}\,dx.$$

Question 8 Find the following

(a) $\displaystyle\int \frac{x+9}{x(x-3)^2}\,dx$ (b) $\displaystyle\int \frac{x}{(x^2-1)(x+1)}\,dx$ (c) $\displaystyle\int \frac{x+1}{(x-1)(x+2)^2}\,dx$

Question 9 [Rationalising the integrand]

Use substitutions of the form $u = x^r$ to find the following.

(a) $\displaystyle\int \frac{\sqrt[3]{x}}{x-\sqrt[3]{x}}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x}}{1-x^3}\,dx$ (c) $\displaystyle\int \frac{1}{x\sqrt[3]{x}-1}\,dx$

Question 10

(a) Prove that $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$.

(b) Hence, evaluate

$$\int_0^\infty \frac{1}{x^4 + x^2 + 1}\,dx.$$

Question 11

(a) Prove that $x^4 - x^2 + 1 = (x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$.

(b) Hence, evaluate

$$\int_0^\infty \frac{1}{x^4 - x^2 + 1}\,dx.$$

Question 12 [Avoiding partial fractions]

Let $R > 1$. Define the integral

$$I = \int_{\frac{1}{R}}^{R} \frac{x}{x^3+1}\, dx.$$

(a) Prove that $x^3 + 1 = (x^2 - x + 1)(x + 1)$.

(b) Using the substitution $y = \frac{1}{x}$, show that

$$I = \int_{\frac{1}{R}}^{R} \frac{1}{y^3+1}\, dy.$$

(c) By considering the average of the two expressions for I, find I.

(d) Hence, evaluate

$$\int_0^{\infty} \frac{x}{x^3+1}\, dx.$$

Question 13

(a) Let $A(x)$ and $B(x)$ be of the form $p\cos x + q\sin x$ for $p, q \in \mathbb{R}$ in the following expression

$$\frac{12}{9\cos^2 x - 4\sin^2 x} = \frac{A(x)}{3\cos x - 2\sin x} + \frac{B(x)}{3\cos x + 2\sin x}.$$

Find $A(x)$ and $B(x)$.

(b) Hence, evaluate

$$\int_0^{\frac{\pi}{4}} \frac{12}{9\cos^2 x - 4\sin^2 x}\, dx.$$

Question 14

(a) Prove that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$.

(b) Find A and B such that

$$\frac{16}{x^4+4} = \frac{2x+A}{x^2+2x+2} - \frac{2x+B}{x^2-2x+2}.$$

(c) Hence, show that

$$\int_0^k \frac{16}{x^4+4}\, dx = \ln\left|\frac{k^2+2k+2}{k^2-2k+2}\right| + 2\tan^{-1}(k+1) + 2\tan^{-1}(k-1).$$

(d) Hence, evaluate

$$\int_0^{\infty} \frac{16}{x^4+4}\, dx.$$

Challenge Problems

Problem 1 [Residue method for partial fractions]

Let $P(x)$ be a monic polynomial of degree n with real distinct roots α_k, where $k = 1, 2, 3, \cdots, n$. Using partial fractions, $\dfrac{1}{P(x)}$ can be expressed in the form

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{c_k}{x - \alpha_k},$$

for some reals c_k.

(a) Show that

$$c_1 = \frac{x - \alpha_1}{P(x)} - \sum_{k=2}^{n} c_k \left(\frac{x - \alpha_1}{x - \alpha_k} \right).$$

(b) Hence, show that

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{1}{P'(\alpha_k)(x - \alpha_k)}.$$

Problem 2 [Generalised linear decomposition]

Let n be a positive integer and consider the decomposition

$$\frac{1}{x(x+1)(x+2)\cdots(x+n)} \equiv \frac{c_0}{x} + \frac{c_1}{x+1} + \frac{c_2}{x+2} + \cdots + \frac{c_n}{x+n}.$$

(a) Show that

$$c_k = \frac{(-1)^k}{n!} \binom{n}{k} \text{ for all } k = 0,\ 1,\ 2,\ \cdots,\ n.$$

(b) Hence, show that

$$\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{n}{k} = \frac{1}{n+1}.$$

Exercise 4F

t-formula Substitutions

Fundamentals

Fundamentals 1

If $t = \tan\left(\frac{\theta}{2}\right)$, write down the t-formula expansion of the following.

(a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$

Fundamentals 2

If $t = \tan\left(\frac{\theta}{2}\right)$, then write down an expression for $d\theta$.

Question 1 Find the following.

(a) $\displaystyle\int \frac{1}{\sin\theta + \tan\theta}\, d\theta$ (b) $\displaystyle\int \frac{1}{5 + 4\cos\theta}\, d\theta$ (c) $\displaystyle\int \frac{1}{1 + \cos\theta}\, d\theta$

(d) $\displaystyle\int \frac{1}{3 + 2\cos\theta}\, d\theta$ (e) $\displaystyle\int \frac{1}{2 - \cos\theta}\, d\theta$ (f) $\displaystyle\int \frac{1}{5 + 3\sin\theta - 4\cos\theta}\, d\theta$

Question 2 [Partial fractions involved]

Find the following.

(a) $\displaystyle\int \frac{5}{3\sin\theta + 4\cos\theta}\, d\theta$ (b) $\displaystyle\int \frac{1}{4 + 5\cos\theta}\, d\theta$ (c) $\displaystyle\int \frac{1}{3 + 5\cos\theta}\, d\theta$

(d) $\displaystyle\int \frac{1}{3 - 5\sin\theta}\, d\theta$ (e) $\displaystyle\int \frac{1}{12 + 13\cos\theta}\, d\theta$ (f) $\displaystyle\int \frac{1}{8 - 17\sin\theta}\, d\theta$

Question 3 Find the following.

(a) $\displaystyle\int \frac{1}{1 - \sin\theta + \cos\theta}\, d\theta$ (b) $\displaystyle\int \frac{1}{1 + \sin\theta - \cos\theta}\, d\theta$ (c) $\displaystyle\int \frac{1}{\sin\theta + \cos\theta}\, d\theta$

Question 4 Consider the integral

$$I = \int \sec\theta \, d\theta.$$

(a) Find I using t-formula substitutions.

(b) Find I using a different method, by first multiplying the top and bottom by $(\sec\theta + \tan\theta)$.

(c) Show that the two answers are equivalent.

Challenge Problems

Problem 1 Find

$$\int \frac{1}{1+\sin 2\theta}\, d\theta.$$

Problem 2 Find the following.

(a) $\displaystyle\int \frac{1}{1+4\cos^2\theta}\, d\theta$

(b) $\displaystyle\int \frac{1}{1+\sin\theta\cos\theta}\, d\theta$

(c) $\displaystyle\int \frac{1}{\sin^4\theta+\cos^4\theta}\, d\theta$

(d) $\displaystyle\int \frac{\sqrt{\tan\theta}}{\sin 2\theta}\, d\theta$

Problem 3

(a) Prove that $a^6+b^6=(a^4-a^2b^2+b^4)(a^2+b^2)$.

(b) Hence, find $\displaystyle\int \frac{1}{\sin^6\theta+\cos^6\theta}\, d\theta$.

Problem 4 Find

$$\int \frac{1}{a\sin\theta+b\cos\theta}\, d\theta.$$

Problem 5 Let $a>1$. Find the value of

$$\int_0^{\pi} \frac{1}{a-\cos\theta}\, d\theta.$$

Problem 6 [Differentiation Under The Integral Sign]

(a) Let $a>1$. Find the value of

$$\int_0^{\pi} \frac{1}{(a-\cos\theta)^2}\, d\theta.$$

(b) By differentiating the answer to the previous question, show that

$$\frac{d}{da}\int_0^{\pi} \frac{1}{a-\cos\theta}\, d\theta = \int_0^{\pi} \frac{d}{da}\left(\frac{1}{a-\cos\theta}\right) d\theta.$$

Exercise 4G

Integration by Parts

Fundamentals

Fundamentals 1

Complete the following formula for integration by parts.

$$\int uv'\,dx =$$

Fundamentals 2

Complete the following formula for integration by parts.

$$\int_a^b uv'\,dx =$$

Fundamentals 3

(a) When integrating an isolated function using integration by parts, it is often fruitful to set $v' =$ ___ to introduce an x term.

(b) When selecting what goes into the v' term, it is important to ensure that it will be easy to i________.

Question 1 Find the following using integration by parts.

(a) $\int xe^x\,dx$ (b) $\int x\ln x\,dx$ (c) $\int \sqrt{x}\ln x\,dx$

(d) $\int x\sin x\,dx$ (e) $\int x\sec^2 x\,dx$ (f) $\int x\sin^{-1}(x)\,dx$

(g) $\int x\tan^{-1}(x)\,dx$ (h) $\int \frac{\ln x}{x^2}\,dx$ (i) $\int x\sin x\cos x\,dx$

Question 2 [Integrating isolated functions]

Find the following.

(a) $\int \ln x\,dx$ (b) $\int \sin^{-1}(x)\,dx$ (c) $\int \tan^{-1}(x)\,dx$

(d) $\int \ln(x^2+1)\,dx$ (e) $\int e^{\sqrt{x}}\,dx$ (f) $\int \sin(\ln x)\,dx$

Question 3 [Two applications of integration by parts needed]

Find the following.

(a) $\int e^x\sin x\,dx$ (b) $\int x^2\sin x\,dx$ (c) $\int (\ln x)^2\,dx$

Question 4 [Prioritising the v' term]

Find the following.

(a) $\displaystyle\int x^5\sqrt{1+x^3}\,dx$ (b) $\displaystyle\int \frac{x^7}{\sqrt{1+x^4}}\,dx$ (c) $\displaystyle\int x^3\cos\left(x^2\right)\,dx$

(d) $\displaystyle\int x^3 e^{x^2}\,dx$ (e) $\displaystyle\int e^{6x}\sin\left(e^{3x}\right)\,dx$ (f) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

Question 5 Find $\displaystyle\int \sec^3 x\,dx$ using integration by parts.

Question 6 [Definite integrals using integration by parts]

Evaluate the following.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (b) $\displaystyle\int_1^{e} x\ln x\,dx$ (c) $\displaystyle\int_0^1 \tan^{-1}(x)\,dx$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$ (f) $\displaystyle\int_0^1 x^3\tan^{-1}(x)\,dx$

(g) $\displaystyle\int_0^{\infty} xe^{-x}\,dx$ (h) $\displaystyle\int_0^1 x^3e^{-x^2}\,dx$ (i) $\displaystyle\int_0^{\infty} \frac{\ln(1+e^x)}{e^x}\,dx$

Challenge Problems

Problem 1 Find the following.

(a) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ (b) $\displaystyle\int \sqrt{1-x^2}\,dx$ (c) $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$

(d) $\displaystyle\int \ln\left(x+\sqrt{x^2-a^2}\right)\,dx$ (e) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1+x}}\,dx$ (f) $\displaystyle\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{1+x}}\,dx$

Problem 2 [Application to the Laplace transform]

The *Laplace transform* is an advanced technique used to solve differential equations, usually taught in universities. It is an operation on $f(t)$ defined as

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt.$$

The output is a function in terms of s.

(a) Show that $\mathcal{L}(t) = \dfrac{1}{s^2}$. (b) Show that $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$.

Exercise 4H

Reduction Formulae

Fundamentals

Fundamentals 1

(a) A reduction formula is a recurrence formula where an integral I_n is expressed in terms of a similar integral, but with a lower i_____ such as I_{n-1} or I_{n-2}.

(b) Once the recurrence is obtained, the integral may be calculated by r_______ substituting the formula into itself until the integral reduces to either I_0 or I_1.

(c) Calculate I_0 or I_1 directly using the definition of _____. Do not use the reduction formula to try to calculate them because this will not work.

Fundamentals 2

Most reduction formula problems require the use of integration by p___, though there are some problems that do not require it. Be mindful that with some problems, you will need to use integration by p___ twice.

Fundamentals 3

The following identities are often useful for reduction formula problems.

(a) $\sin^2 x =$ _______.

(b) $\cos^2 x =$ _______.

(c) $\tan^2 x =$ _______.

(d) $\sec^2 x =$ _______.

(e) $x^n = 1 + ($_____$)$.

(f) $x^n = 1 - ($_____$)$.

Question 1 Define for $n \geq 0$

$$I_n = \int x^n e^x \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = x^n e^x - nI_{n-1}.$$

(b) Hence, find $\int x^2 e^x \, dx$.

Question 2 Define for $n \geq 0$

$$I_n = \int x(\ln x)^n \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}.$$

(b) Hence, find $\int x(\ln x)^2 \, dx$.

Question 3 Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx.$$

(a) For $n \geq 2$, show that

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}.$$

(b) Hence, find $\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$.

Question 4 [Integration by parts is not always needed]

Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx.$$

For $n \geq 2$, show that

$$I_n = \frac{1}{n-1} - I_{n-2}.$$

Question 5 Define for $n \geq 0$

$$I_n = \int_0^1 \frac{x^n}{1+x^2} \, dx.$$

For $n \geq 2$, show that

$$I_n = \frac{1}{n-1} - I_{n-2}.$$

Question 6 Define for $n \geq 0$

$$I_n = \int_0^1 (1-x^2)^n \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{2n}{2n+1} I_{n-1}.$$

(b) Hence, find $\int_0^1 (1-x^2)^3 \, dx$.

Question 7 Define for $n \geq 0$

$$I_n = \int_0^1 x(1-x^3)^n \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{3n}{3n+2} I_{n-2}.$$

(b) Hence, find $\int_0^1 x(1-x^3)^4 \, dx$.

Question 8 Define for $n \geq 0$

$$I_n = \int_0^1 x^n \sqrt{1-x} \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{2n}{2n+3} I_{n-1}.$$

(b) Hence, find $\int_0^1 x^3 \sqrt{1-x} \, dx$.

Question 9 Define for $n \geq 0$

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{2\sqrt{2}}{2n+1} - \frac{2n}{2n+1} I_{n-1}.$$

(b) Hence, find $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx$.

Question 10 Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx.$$

(a) For $n \geq 2$, show that

$$I_n = \frac{n-1}{n+2} I_{n-2}.$$

(b) Hence, find $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x\,dx$.

Question 11 Define for $n \geq 0$

$$I_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx.$$

(a) For $n \geq 2$, show that

$$I_n = -\frac{1}{n} x^{n-1}\sqrt{1-x^2} + \frac{n-1}{n} I_{n-2}.$$

(b) Hence, find $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$.

Question 12 Define for $n \geq 0$

$$I_n = \int \frac{1}{(1+x^2)^n}\,dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{1}{2(n-1)}\left(\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right).$$

(b) Hence, find $\displaystyle\int \frac{1}{(1+x^2)^3}\,dx$.

Question 13 Define for $n \geq 0$

$$I_n = \int_0^1 \frac{x^n}{(x+1)^2}\,dx.$$

(a) For $n \geq 2$, show that

$$I_n = \frac{1}{2(n-1)} - \frac{n}{n-1} I_{n-1}.$$

(b) Hence, find $\displaystyle\int_0^1 \frac{x^3}{(x+1)^2}\,dx$.

Question 14 Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx.$$

(a) For $n \geq 2$, show that $I_n = \dfrac{\sqrt{2}^{\,n-2}}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

(b) Hence, find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^6 x \, dx$.

Question 15 [Glimpse into the definition of the factorial for non-integer n]

Define for integer $n \geq 0$

$$I_n = \int_0^{\infty} x^{n-1} e^{-x} \, dx.$$

You may assume that for all real $n > 0$, $\displaystyle\lim_{x \to \infty} x^n e^{-x} = 0$.

(a) Show that $I_1 = 1$.

(b) Show that $I_{n+1} = nI_n$.

(c) Deduce that $I_{n+1} = n!$

(d) Let $J_n = \displaystyle\int_0^1 (\ln x)^n \, dx$. Prove that $\dfrac{I_n}{J_n} = \dfrac{(-1)^n}{n}$.

Question 16 [Wallis Product]

Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx.$$

(a) For $n \geq 2$, show that $I_n = \dfrac{n-1}{n} I_{n-2}$.

(b) Hence, show that

$$I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

and

$$I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1$$

(c) Explain why $I_k > I_{k+1}$.

(d) Deduce that

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 \times 4^2 \times \cdots \times (2n)^2}{1 \times 3^2 \times 5^2 \times \cdots \times (2n-1)^2(2n+1)} < \frac{\pi}{2}.$$

(e) Write down the value of the *Wallis Product*

$$\left(\frac{2}{1} \cdot \frac{2}{3}\right) \times \left(\frac{4}{3} \cdot \frac{4}{5}\right) \times \left(\frac{6}{5} \cdot \frac{6}{7}\right) \times \cdots$$

Challenge Problems

Problem 1 Define for $n \geq 0$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx.$$

(a) For $n \geq 1$, show that

$$I_n = \frac{2n-1}{2n} I_{n-1}.$$

(b) Hence, show that $I_n = \dfrac{\pi}{2^{2n+1}} \dbinom{2n}{n}$.

Problem 2 Define for $n \geq 0$

$$I_n = \int \frac{\sin nx}{\cos x} \, dx.$$

(a) For $n \geq 2$, show that

$$I_n = \frac{2}{1-n} \cos(n-1)x - I_{n-2}.$$

(b) Hence, find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5x \sin x}{\cos x} \, dx$.

Problem 3 Define for $n \geq 0$

$$I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} \, dx.$$

(a) For $n \geq 2$, show that $I_n = I_{n-2}$.

(b) Hence, find all possible values of I_n and state when they occur.

Problem 4 Define for $m, n \geq 0$

$$I(m,n) = \int_0^1 x^m (1-x)^n \, dx.$$

(a) For $n \geq 1$, show that

$$I(m,n) = \frac{n}{m+1} I(m+1, n-1).$$

(b) Hence, show that $I(m,n) = \dfrac{m!n!}{(m+n+1)!}$.

(c) Find the value of $\displaystyle\int_0^1 (x - x^2)^n \, dx$.

Problem 5 Define for $n \geq 0$

$$I_n = \int_0^{\pi} e^x \sin^n x \, dx.$$

For $n \geq 2$, show that $I_n = \dfrac{n^2 - n}{n^2 + 1} I_{n-2}$.

Exercise 4I

Further Substitutions

Fundamentals

Fundamentals 1
The variable in a definite integral is called the d________ variable, since the final answer does not contain that variable. Hence, what can we say about the following integrals?

$$\int_a^b f(x)\,dx \qquad \int_a^b f(u)\,du$$

Fundamentals 2
In general, let

$$u = \text{ expression inside the bracket}$$

when proving identities involving integration. Write down a suitable substitution for the following.

(a) $\int_a^b f(-x)\,dx$

(b) $\int_a^b f(ax)\,dx$

(c) $\int_a^b f(a-x)\,dx$

(d) $\int_a^b f(a+x)\,dx$

Question 1 Prove that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Question 2 Use the identity from the previous question to calculate the following.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

(c) $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$

(d) $\displaystyle\int_0^{\pi} x \sin^2 x\,dx$

(e) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

(f) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \sin 2x}\,dx$

Question 3 Prove that

$$\int_{-a}^a f(x)\,dx = \int_0^a f(x) + f(-x)\,dx.$$

Question 4 Use the identity from the previous question to calculate the following.

(a) $\displaystyle\int_{-1}^{1} \frac{x^2}{1 + e^x}\,dx$

(b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \sin^2 x}{1 + e^x}\,dx$

(c) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 + \sin x}\,dx$

Question 5 Show that

$$\int_0^1 x(1-x)^n\,dx = \frac{1}{(n+1)(n+2)}.$$

Question 6 Prove that

$$\int_1^{\frac{1}{a}} \frac{1}{x}\,dx = -\int_1^a \frac{1}{x}\,dx$$

and state what familiar logarithm law comes from this.

Question 7 Prove that

$$\int_0^1 f(ax)\,dx = \frac{1}{a}\int_0^a f(x)\,dx.$$

Question 8

(a) Prove that

$$\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx.$$

(b) Hence, evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx.$

Question 9

(a) Prove that $\displaystyle\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx.$

(b) A function $f(x)$ has the property that $f(x)+f(a-x)=f(a)$. Prove that

$$\int_0^a f(x)\,dx = \frac{a}{2}f(a).$$

Question 10

(a) Prove that $\displaystyle\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx.$

(b) Hence, evaluate

$$\int_0^a \ln\left(1+\tan a\tan x\right)\,dx$$

for $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$.

Question 11

(a) Prove that $\displaystyle\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx.$

(b) Prove that $\displaystyle\int_0^a f(a+x)\,dx = \int_a^{2a} f(x)\,dx.$

(c) Hence, evaluate

$$\int_0^{\frac{1}{2}} \sin^{-1}\left(\frac{1}{2}-x\right)+\sin^{-1}\left(\frac{1}{2}+x\right)\,dx.$$

Challenge Problems

Problem 1 Use the identities and techniques from this exercise to evaluate the following.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \ln\left(1+\tan x\right)\, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$

(c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(1+\tan^4 x)(1+\pi^x)}\, dx$

(d) $\displaystyle\int_0^{\pi} \frac{x}{a+b\cos^2 x}\, dx$ for $a,\, b>0$.

Problem 2 Let $0<\theta<\pi$ and $n\in\mathbb{R}$. Evaluate the following integral.

$$\int_0^{\infty} \frac{\tan^{-1}\left(x^n\right)}{x^2+2x\cos\theta+1}\, dx$$

Problem 3 Suppose $a>0$ is a constant.

(a) Prove that

$$\int_0^a \frac{f\left(\frac{a-x}{1+ax}\right)}{1+x^2}\, dx = \int_0^a \frac{f(x)}{1+x^2}\, dx.$$

(b) Hence, evaluate $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$.

Problem 4 Suppose that $f(x)$ is an even function and $a>0$ is a constant.

(a) Prove that

$$\int_{-a}^{a} \frac{f(x)}{1+e^x}\, dx = \int_0^a f(x)\, dx.$$

(b) Hence, evaluate $\displaystyle\int_0^1 \frac{(1-e^x)^2}{(1+e^x)^3}\, dx$.

Problem 5 Let φ be the positive solution to $x^2=x+1$, and n be a positive integer. Define the integral

$$I_n = \int_0^{n\pi} \frac{x}{\varphi-\cos^2 x}\, dx.$$

(a) Show that $I_n = \dfrac{n\pi}{2}\displaystyle\int_0^{n\pi} \frac{1}{\varphi-\cos^2 x}\, dx$.

(b) Show that $I_n = \dfrac{n^2\pi}{2} I_1$.

(c) Hence, evaluate I_n.

Chapter 4 Review

Further Integration

Review

Question 1 **Mixed problems (easy)**

(a) $\int \cos^3 x\,dx$ (b) $\int 2x\tan^{-1}(x)\;dx$ (c) $\int x\sqrt{1+x}\,dx$

(d) $\int \frac{x}{1-x^4}\,dx$ (e) $\int_0^{\frac{\pi}{2}} e^x\sin x\,dx$ (f) $\int \frac{1}{1-\sin x}\,dx$

(g) $\int \frac{1}{x^2-6x+13}\,dx$ (h) $\int \ln|x^2-1|\,dx$ (i) $\int \frac{x^2+x+3}{(x-1)(x^2+4)}\,dx$

Question 2 **Mixed problems (medium)**

(a) $\int \sin^4 x\,dx$ (b) $\int \sqrt{1+x^2}\,dx$ (c) $\int \cos\sqrt{x}\,dx$

(d) $\int \frac{\sin 2x}{1+\sin^2 x}\,dx$ (e) $\int \frac{1}{e^x+1}\,dx$ (f) $\int \frac{4x-3}{x^2+2x+5}\,dx$

(g) $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos\theta}\,d\theta$ (h) $\int \frac{1}{1-\sqrt{x}}\,dx$ (i) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Question 3 **Mixed problems (hard)**

(a) $\int \frac{1}{25\cos^2 x+4\sin^2 x}\,dx$ (b) $\int \frac{x^2}{(x^2+1)(x^2-2)}\,dx$ (c) $\int \frac{1}{\sin x-\cos x}\,dx$

(d) $\int \frac{1}{x+x^6}\,dx$ (e) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$ (f) $\int \frac{1}{e^{2x}-e^x-2}\,dx$

(g) $\int \sqrt{1+\sin x}\cot x\,dx$ (h) $\int \sqrt{\frac{x}{4-x}}\,dx$ (i) $\int \sqrt{1-\sqrt{x}}\,dx$

Question 4 **Mixed problems (very hard)**

(a) $\int \frac{1}{x\sqrt{x}+\sqrt{x}}\,dx$ (b) $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$ (c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\tan x+\cot x}\,dx$

(d) $\int \left(\frac{1+e^x}{1-e^x}\right)^2\,dx$ (e) $\int \frac{\ln(1+\ln x)}{x}\,dx$ (f) $\int \frac{x^2-1}{x^4+3x^2+1}\,dx$

(g) $\int_0^{\frac{\pi}{2}} \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$ (h) $\int \frac{1}{\cos x\sqrt{\sin 2x}}\,dx$ (i) $\int \ln\left(x+\sqrt{x^2-1}\right)\,dx$

Question 5 **Mixed problems (competition-level elementary techniques)**

(a) $\int \frac{1}{(1-x)\sqrt{1-x^2}}\,dx$ (b) $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx$ (c) $\int \frac{\sin x}{\sin x+\cos x}\,dx$

(d) $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$ (e) $\int_0^\infty \frac{x}{\sqrt{e^x-1}}\,dx$ (f) $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$

Question 6 Evaluate the following definite integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{x^2}{1+x^6}\,dx$ (b) $\displaystyle\int_{0}^{1} x^3\sqrt{1-x^2}\,dx$ (c) $\displaystyle\int_{0}^{\frac{1}{4}} \frac{1}{1-\sqrt{x}}\,dx$

(d) $\displaystyle\int_{0}^{\frac{\pi}{4}} \tan^4 x\,dx$ (e) $\displaystyle\int_{0}^{\frac{\pi}{2}} \cos^5 x\,dx$ (f) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^4 x\,dx$

(g) $\displaystyle\int_{0}^{1} \sqrt{4-x^2}\,dx$ (h) $\displaystyle\int_{0}^{2} \sqrt{\frac{x}{4-x}}\,dx$ (i) $\displaystyle\int_{0}^{1} \frac{x^2}{(1+x^2)^{\frac{3}{2}}}\,dx$

(j) $\displaystyle\int_{0}^{1} \frac{x}{x^2-x+1}\,dx$ (k) $\displaystyle\int_{-1}^{0} \frac{x+1}{x^2+x+1}\,dx$ (l) $\displaystyle\int_{2}^{4} \frac{x}{\sqrt{-8+6x-x^2}}\,dx$

(m) $\displaystyle\int_{1}^{2} \frac{1}{(2x+1)(x+2)}\,dx$ (n) $\displaystyle\int_{0}^{2} \frac{x^2-1}{(x+1)(x^2+4)}\,dx$ (o) $\displaystyle\int_{0}^{1} \frac{x^2+2x}{(x-2)^2(x+1)}\,dx$

(p) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{2+\cos x}\,dx$ (q) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{1}{1-\cos x+\sin x}\,dx$ (r) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{4\cos x+3\sin x}\,dx$

(s) $\displaystyle\int_{1}^{e} x^2\ln x\,dx$ (t) $\displaystyle\int_{0}^{\frac{1}{2}} \sin^{-1}(2x)\,dx$ (u) $\displaystyle\int_{0}^{1} x\tan^{-1}(x)\,dx$

Question 7

(a) Simplify $(x+1)(x^2-x+1)$. (b) Hence, evaluate $\displaystyle\int_{0}^{\infty} \frac{1}{x^3+1}\,dx$.

Question 8 Define for $n \geq 0$

$$I_n = \int (\ln x)^n\,dx.$$

(a) For $n \geq 2$, show that

$$I_n = x(\ln x)^n - nI_{n-1}.$$

(b) Hence, find I_3.

Question 9 Define for $n \geq 0$

$$I_n = \int \cos^n x\,dx.$$

(a) For $n \geq 2$, show that

$$I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}.$$

(b) Hence, find I_5.

Question 10 Define for $n \geq 0$

$$I_n = \int_{-1}^{0} x^n\sqrt{1+x}\,dx.$$

(a) For $n \geq 1$, show that

$$I_n = -\frac{2n}{2n+3}I_{n-1}.$$

(b) Hence, evaluate I_3.

Question 11 Define for $n \geq 0$

$$I_n = \int_0^1 x^n \left(1 - x^2\right)^5 \, dx.$$

(a) For $n \geq 2$, show that

$$I_n = \frac{n-1}{n+11} I_{n-2}.$$

(b) Hence, evaluate I_5

Question 12

(a) Prove that

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$$

(b) Hence, evaluate $\displaystyle\int_0^1 x^2 \sqrt{1-x}\, dx.$

Question 13

(a) Prove that

$$\int_{-a}^a f(x)\, dx = \int_0^a f(x) + f(-x)\, dx.$$

(b) Hence, evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos x}{1+e^x}\, dx.$

Question 14

(a) Prove that

$$\int_{\frac{1}{a}}^a \frac{f\left(\frac{1}{x}\right)}{x}\, dx = \int_{\frac{1}{a}}^a \frac{f(x)}{x}\, dx.$$

(b) Hence, evaluate $\displaystyle\int_{\frac{1}{a}}^a \frac{\sin\left(\frac{1}{x} - x\right)}{x}\, dx.$

Question 15

(a) Prove that

$$\int_0^\infty \frac{f\left(\frac{1}{x}\right)}{x^2+1}\, dx = \int_0^\infty \frac{f(x)}{x^2+1}\, dx.$$

(b) Hence, evaluate $\displaystyle\int_0^\infty \frac{1}{(x^2+1)^2}\, dx.$

Investigation Task

Further Applications of Integration

In the HSC course, we learn how to use integration to calculate area and volume. Another thing that can be calculated using integration is *arc length*. This investigation task will allow students to explore and get some hands-on experience with calculating arc length.

Question 1 Write a two-page article about *arc length*. Your response should include answers the following questions.

(a) State the formula for the arc length of a Cartesian equation, and prove it. Give an example.

(b) How do you find the arc length of *parametrically* defined curves? Give examples and include a verification that the circumference of a circle is πr^2.

(c) What assumptions are made in the derivation of the arc length formula?

Question 2 Write a two-page article about *polar curves*. Your response should include answers the following questions.

(a) How are such curves defined?

(b) Why do we need this method of defining curves?

(c) What are some examples of polar curves?

(d) What does the equation of a standard curve like a line or parabola look like when it is defined using polar coordinates instead of Cartesian coordinates?

(e) How do you find the arc length of such curves? Give examples including the cardioid.

(f) How do you find the area inside such curves? Give examples including the cardioid.

(g) What is a real-life example of arc length and/or area inside a curve defined using polar coordinates?

Question 3 Write a one-page article about *line integrals*. Your response should include answers the following questions.

(a) What is a line integral?

(b) How is it related to 'area under a curve' that we learn in the HSC course?

(c) Give an example of a calculation.

(d) Why does orientation matter?

(e) What is a real-life example or application of a line integral?

Investigation Task

Euler Substitution

In this chapter, we learn many integration techniques such as integration by parts and partial fractions. Another very useful technique that was mentioned in **Exercise 2A** Problem 1 is called the *Euler substitution*.

Write a three-page article on Euler substitutions. It should contain theory, proofs, a discussion on any special cases, and a number of questions and answers. Your response should also include a discussion of the following.

- What are the different substitutions and for what scenarios do you use which?
- What is the intuition behind using it?
- Give an example of an integral done using an Euler substitution, with the same integral done without it.

Your lesson should be sufficiently detailed so that a typical Extension 2 student will be able to independently perform an Euler substitution to solve a problem.

Investigation Task

Differentiating under the integral

A lesser-known trick in integration is to *differentiate under the integral.* Although not well-known, it is a very powerful tool that can be used to make quick work of integrals that are otherwise inaccessible using standard techniques.

Give a 5-10 minute presentation on this trick that is targeted towards an audience of Extension 2 students. Your answer should include the following.

- What is the difference between differentiating under the integral, and what is commonly referred to as *Feynman's Trick*?
- Worked examples using the trick, and the same problem without using it.
- How is the Fundamental Theorem of Calculus relevant to this trick?

Your presentation should be sufficiently detailed so that a typical Extension 2 student watching it will be able to solve simple problems independently.

Investigation Task

Bunch of Integrals

At the bottom of the first page of the chapter review is a section on 'competition-level elementary techniques' integral problems. Do them all and show full working out.

Question 1

(a) $\displaystyle\int \frac{1}{(1-x)\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx$

(c) $\displaystyle\int \frac{\sin x}{\sin x + \cos x}\,dx$

(d) $\displaystyle\int \frac{x^2}{(x\sin x + \cos x)^2}\,dx$

(e) $\displaystyle\int_0^{\infty} \frac{x}{\sqrt{e^x - 1}}\,dx$

(f) $\displaystyle\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx$

Many of these integrals have multiple solutions, so bonus points for alternative methods!

5

MECHANICS

- Velocity-Displacement Equations
- Simple Harmonic Motion
- Projectile Motion
- Resisted Horizontal Motion
- Resisted Vertical Motion
- Resisted Projectile Motion
- Inclined Planes and Pulleys

Exercise 5A

Velocity-Displacement Equations

Fundamentals

Fundamentals 1
Complete the following.

(a) $a = \frac{dv}{\quad}$

(b) $a = \frac{d}{\quad}\left(\underline{\qquad}\right)$

(c) $v = \frac{dx}{\quad}$

(d) $a = v\frac{dv}{\quad}$

Fundamentals 2
Prove that

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} = \frac{dv}{dt}$$

Fundamentals 3
Describe the steps to obtain an expression for x in terms of t given the following.

(a) $v = f(x)$

(b) $a = f(v)$

(c) $a = f(x)$

Note: Unless stated otherwise, all distance units are in metres and all time units are in seconds.

Question 1 A particle is initially at the origin and moves such that

$$v = e^{-x}.$$

Find the following as functions of time t.

(a) x

(b) v

(c) a

Question 2 A particle is initially at rest at $x = 1$ and moves such that

$$a = \frac{1}{2x+1}.$$

Find v^2 as a function of x.

Question 3 A particle moves with a constant acceleration of $8\,\text{m}\,\text{s}^{-2}$. When the particle is 5 metres to the right of the origin, it has a velocity of $12\,\text{m}\,\text{s}^{-1}$. Initially, the particle is at the origin.

(a) Find an expression for velocity in terms of displacement.

(b) Hence, find an expression for displacement in terms of time.

Question 4 The velocity of a particle moving in a straight line is given by

$$v = 3x + 1,$$

where x is the displacement from the origin.

(a) What is the acceleration of the particle when it is 2 metres to the right of the origin?

(b) If the particle is initially at the origin, when does it reach $x = 2$?

Question 5 A particle moves along the x-axis with velocity given by

$$v = -2x^2.$$

Initially, the particle is 2 metres to the right of the origin. Find an expression for

(a) a in terms of x.

(b) x in terms of t.

Question 6 The acceleration of a particle is given by

$$\ddot{x} = -\frac{9}{(x+2)^3}.$$

Initially, the particle is at the origin with a velocity of $v = \frac{3}{2}\,\mathrm{m\,s^{-1}}$.

(a) Express v in terms of x.

(b) Express t in terms of x.

Question 7 A particle moves in a straight line with acceleration

$$\ddot{x} = -\frac{x^3}{2}.$$

The particle is initially at rest at $x = 2$.

(a) Prove that $v^2 = \frac{1}{4}(16 - x^4)$.

(b) Find the value of the greatest speed, and where this occurs.

(c) Describe the motion of the particle.

Question 8 The acceleration of a particle is

$$\ddot{x} = 2x - 6.$$

Initially, the particle is at rest at $x = 4$.

(a) Show that $v^2 = 2(x^2 - 6x + 8)$.

(b) In which direction does the particle begin to move? Explain your answer.

(c) Show that the particle does not pass through the point $x = 3$.

(d) Sketch v^2 against x, and hence explain why $x \geq 4$.

(e) Describe the motion of the particle.

Question 9 A particle has velocity

$$v^2 = (4x^2 - 1)(x^2 - 9).$$

Initially, the particle is at the origin with a velocity of $3\,\mathrm{m\,s^{-1}}$. Show that at all times, the particle lies within the interval $|x| \leq \frac{1}{2}$.

Question 10 The acceleration of a particle is given by

$$\ddot{x} = -\frac{1}{2}e^{-4x}.$$

Initially, the particle is at the origin with a velocity of $v = \frac{1}{2}\,\mathrm{m\,s^{-1}}$.

(a) Show that $v = \frac{1}{2}e^{-2x}$.

(b) Find x as a function of t.

(c) What happens to x and v as $t \to \infty$?

(d) Describe the motion of the particle.

Question 11 A particle has an acceleration-displacement equation given by

$$a = e^{-x}.$$

It is initially at the origin and moving to the right with a speed of $4\,\mathrm{m\,s^{-1}}$.

(a) Find v as a function of x.

(b) Describe the particle's eventual behaviour.

Question 12 A particle moves with equation of motion given by

$$a = v^2 + 1,$$

and it is initially at rest at the origin.

(a) Show that $v = \tan t$.

(b) Show that $x = -\ln|\cos t|$.

(c) Show that $x = \frac{1}{2}\ln(1 + v^2)$.

(d) Show that $a = e^{2x}$.

Question 13 A particle moves with velocity-displacement equation

$$v = 2 - x,$$

and it is initially at the origin.

(a) In which direction does the particle travel initially?

(b) Find a as a function of x.

(c) Find x as a function of t.

(d) Hence, describe the behaviour of the particle as $t \to \infty$.

Question 14 A particle has displacement-time equation

$$x = 4 - e^{-2t}.$$

Show that $a = 4(x - 4)$ using two different methods.

Question 15 A particle has acceleration equation

$$a = 2(e^{2x} - e^{-2x}),$$

and is initially at the origin with a velocity of $2\sqrt{2}\ \mathrm{m\,s^{-1}}$.

(a) Show that $v = \sqrt{2}(e^{x} + e^{-x})$.

(b) Find x as a function of t.

Challenge Problems

Problem 1 [Escape velocity]

A particle is fired from the Earth's surface with initial velocity u, and the acceleration of the particle is given by

$$a = -\frac{k}{x^2},$$

where x is the distance of the particle from the centre of the Earth and $k \in \mathbb{R}^+$. Let R be the radius of the Earth.

(a) If acceleration due to gravity has a magnitude of $g\ \mathrm{m\,s^{-2}}$ on the Earth's surface, show that

$$a = -\frac{gR^2}{x^2}.$$

(b) Show that

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right).$$

(c) Show that if $u^2 = 2gR$, then $v \neq 0$.

(d) Explain the significance of this result.

Problem 2 [General formula for simple harmonic motion]

A particle is initially at the origin and moves to the right with acceleration equation

$$\ddot{x} = -n^2(x - x_0).$$

By solving the differential equation, prove that the displacement-time equation is

$$x = A\cos(nt + \alpha) + x_0,$$

where A, α, x_0 and n are constants.

Exercise 5B

Simple Harmonic Motion

Fundamentals

Fundamentals 1
The differential equation for simple harmonic motion centred about $x = x_0$ is

$$\ddot{x} = \underline{\hspace{3cm}}$$

Fundamentals 2
A particle moves according to the displacement-time equation

$$x = A\cos(nt + \alpha) + x_0.$$

Describe how each of the parameters A, n, α, and x_0 affect the motion of the particle.

Fundamentals 3

(a) A particle moving in simple harmonic motion about the origin satisfies

$$v^2 = \underline{\hspace{4cm}}.$$

(b) A particle moving in simple harmonic motion about the point $x = x_0$ satisfies

$$v^2 = \underline{\hspace{4.5cm}}.$$

Fundamentals 4
The formula for period is $T = \underline{\hspace{1.5cm}}$.

Fundamentals 5
A particle moves about the origin O with amplitude A. Write down the position(s) of the particle when it attains

(a) maximum speed.

(b) zero speed.

(c) maximum acceleration.

(d) zero acceleration.

Question 1 [Picking an appropriate model]

State whether you would use a sine or cosine equation to model the following scenarios. Also, state whether you would pick the positive or negative version of the function you chose. Assume that the centre of motion is the origin $x = 0$.

(a) A particle is initially at the origin and then moves to the left.

(b) A particle is initially at the origin and then moves to the right.

(c) A particle is initially at rest at $x = 5$.

(d) A particle is initially at rest at $x = -3$.

Question 2 A particle moves according to the displacement-time equation

$$x = 3\sin\left(2t + \frac{\pi}{3}\right) + 1.$$

Find the

(a) amplitude.

(b) period.

(c) centre of motion.

(d) range of motion.

(e) maximum speed.

(f) maximum acceleration.

(g) initial position.

(h) initial speed.

(i) initial acceleration.

Question 3 A particle moves according to the displacement-time equation

$$x = 2\cos\left(3t + \frac{2\pi}{3}\right) - 1.$$

Find the first two times when the particle

(a) passes through the origin.

(b) passes through the positive endpoint.

(c) has velocity $3\,\text{m}\,\text{s}^{-1}$.

(d) has maximum speed.

Question 4 A particle moves according to the displacement-time equation

$$x = 6 - 4\sin\left(2t - \frac{\pi}{6}\right).$$

Find the velocity of the particle when it passes through

(a) $x = 2$.

(b) $x = 10$.

(c) $x = 6$ for the first time.

(d) $x = 4$ for the second time.

Question 5 For each of the following, verify that the motion is simple harmonic motion by showing that they satisfy the differential equation

$$\ddot{x} = -n^2(x - x_0),$$

for constants n and x_0.

(a) $x = 2\cos 3t$

(b) $x = 2\sin 2t + 3\cos 2t - 1$

(c) $v^2 = 16 - 4x^2$

(d) $v^2 = 27 + 18x - 9x^2$

Question 6 A particle moves in simple harmonic motion about the origin with a period of 10 seconds. It is initially at the origin and moves to the right. In what direction is the particle moving

(a) 8 seconds later? (b) 15 seconds later? (c) 24 seconds later?

Question 7 [Be careful!]

Find the amplitude, period and centre of motion of the following particles.

(a) $x = \sin^2 t$ (b) $x = \cos^2 t$

(c) $x = 3 + 2\cos^2 t$ (d) $x = 2 - 3\sin^2 2t$

Question 8 [Using auxiliary-angle formulae]

Find the amplitude, period and centre of motion of the following particles.

(a) $x = 3\cos 2t + 4\sin 2t$ (b) $x = 4\sin 4t - 3\cos 4t$

(c) $x = 12\cos 3t + 5\sin 3t + 7$ (d) $x = \cos 6t - \sin 6t + 1$

Question 9 The motion of a particle is modelled by

$$x(t) = 3\cos\left(2t + \frac{\pi}{3}\right) - 3.$$

Sketch the following for $t \geq 0$.

(a) $x(t)$ (b) $\dot{x}(t)$ (c) $\ddot{x}(t)$

Question 10 [Tide problem]

The tide at a harbour can be modelled using simple harmonic motion. At the harbour, high tide is 12 metres and low tide is 2 metres. It takes 4 hours to go from low tide to high tide. Initially at 2am, it is at low tide. Let t be measured in hours.

(a) Show that $x = 7 - 5\cos\left(\frac{t\pi}{4}\right)$.

(b) A ship needs at least 5 metres of water in order to safely enter the harbour. Find the earliest time that the ship may enter the harbour.

(c) Find the latest time that the ship may leave the harbour after entering at the earliest time.

Question 11 A particle moves in simple harmonic motion with acceleration equation

$$\ddot{x} = -9x.$$

When the particle is at $x = 1$, it has a speed of $4\,\mathrm{m\,s^{-1}}$.

(a) Show that the velocity is given by

$$v^2 = 9\left(\frac{25}{9} - x^2\right).$$

(b) Find the speed of the particle when it is at the origin.

(c) Find the amplitude and period of the motion.

(d) Write down a possible displacement-time equation.

Question 12 A particle moves with displacement-time equation $x = 3\sin(2\pi t)$.

(a) Prove that the motion is simple harmonic motion.

(b) Find the initial velocity and acceleration of the particle.

(c) Find the maximum speed and acceleration of the particle.

(d) How far will the particle travel in the first 0.5 seconds of motion, and at what speed is it travelling at this point in time?

Question 13 A particle moves in simple harmonic motion. When it passes through the centre of motion $x = 0$, it has a speed of $3\sqrt{2}\,\mathrm{m\,s^{-1}}$.

(a) Find the amplitude in terms of n.

(b) Find the speed of the particle when it is halfway between the centre of motion and maximum displacement.

Question 14 A particle moves in a straight line so that the velocity is given by

$$v^2 = 4(5 - x)(x + 1).$$

Initially, the particle is at $x = 5$.

(a) Show that the motion is simple harmonic motion.

(b) Find the amplitude, period, and centre of motion.

(c) How far will the particle travel in the first $\frac{3\pi}{4}$ seconds of motion, and find its position then.

Question 15 A particle moves in a straight line so that the acceleration is given by $\ddot{x} = n^2(4 - x)$, where n is a constant. The particle initially moves from rest at $x = 0$.

(a) Show that $v^2 = n^2(8x - x^2)$.

(b) Hence, show that the particle never moves outside of a certain interval.

Question 16 A particle moves in simple harmonic motion. The velocity of the particle is respectively $3\,\text{m}\,\text{s}^{-1}$ and $\sqrt{8}\,\text{m}\,\text{s}^{-1}$ at distances $\sqrt{3}$ and 2 metres away from the centre of motion. Find the amplitude and period of motion.

Question 17 Two particles move in simple harmonic motion with displacement-time equations

$$x_1 = 3\cos 2t$$
$$x_2 = 5 + 3\sin 2t$$

Find the shortest distance between the two particles, and find the time when this first occurs.

Question 18 A particle moves in simple harmonic motion with endpoints $x = 2$ and $x = 14$. It has a maximum speed of $18\,\text{m}\,\text{s}^{-1}$. Find the amplitude and period of the motion.

Question 19 A particle moves in simple harmonic motion with period 0.5 seconds and maximum speed $4\,\text{m}\,\text{s}^{-1}$. Find the amplitude of the motion.

Question 20 A particle moves in simple harmonic motion with displacement-time equation

$$x = 8\left(\sin^6 t + \cos^6 t\right).$$

(a) Expand $(a+b)(a^2 - ab + b^2)$.

(b) Find the centre, amplitude and period of the particle.

Question 21 A particle satisfies the displacement time equation

$$x = A\cos^2\left(\frac{nt}{2}\right) + B\sin^2\left(\frac{nt}{2}\right),$$

where $0 < A < B$.

(a) Show that

$$x = \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\cos(nt).$$

(b) Hence, show that $A \le x \le B$.

Question 22 A particle moves in simple harmonic motion given by the model

$$x = A\sin nt.$$

Initially, the particle is at the origin and moves towards the positive extremity. Bob claims that when the particle has displacement $\frac{1}{k}$ of the positive extremity, the velocity is also $\frac{1}{k}$ of the maximum speed, where $k > 1$.

(a) Use a counter-example to demonstrate that this statement is not always true.

(b) Find the value of k for which the statement is true.

Challenge Problems

Problem 1 A particle moves in simple harmonic motion about the origin O. When it is p units from O, it has speed u. Similarly, when it is q units from O, it has speed w. Prove that

(a) the period is $T = 2\pi\sqrt{\dfrac{q^2 - p^2}{u^2 - w^2}}$.

(b) the amplitude is $A = \sqrt{\dfrac{u^2q^2 - w^2p^2}{u^2 - w^2}}$.

Problem 2 [Total mechanical energy]

The force equation for simple harmonic motion is

$$F_{\text{net}} = -kx(t),$$

where $k \in \mathbb{R}$ and $x(t)$ is the distance of the particle from the centre of motion at time t.

The *kinetic energy* of a particle is given by

$$K(t) = \frac{1}{2}mv^2(t),$$

where m is the mass of the object and $v(t)$ is the velocity of the object at time t.

The *potential energy* of a particle is given by

$$P(t) = \frac{1}{2}mx^2(t).$$

(a) Show that $x(t) = A\cos\left(\sqrt{\dfrac{k}{m}}\,t\right)$ satisfies the force equation for simple harmonic motion.

(b) Hence, show that the total mechanical energy

$$M(t) = K(t) + P(t)$$

depends only on the amplitude of the motion.

Problem 3 Two particles P and Q move in simple harmonic motion such that

$$x_P = A\cos(nt + \alpha)$$
$$x_Q = B\cos(nt + \beta)$$

A third particle moves with displacement-time equation $x = x_P + x_Q$. Prove that the third particle also moves in simple harmonic motion with amplitude R satisfying

$$R^2 = A^2 + B^2 + 2AB\cos(\alpha - \beta).$$

Exercise 5C

Projectile Motion

Fundamentals

Fundamentals 1

A particle is projected from the origin with initial velocity $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. Let $g\,\mathrm{m\,s^{-2}}$ be the acceleration due to gravity.

(a) Write down the standard time-equations of motion in vector form.

(b) Prove that the equation of the trajectory is

$$y = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta).$$

Fundamentals 2

Consider a particle projected from the origin with initial angle θ and initial speed $V\,\mathrm{m\,s^{-1}}$. Find the following in terms of V, θ and g.

(a) Time of flight.

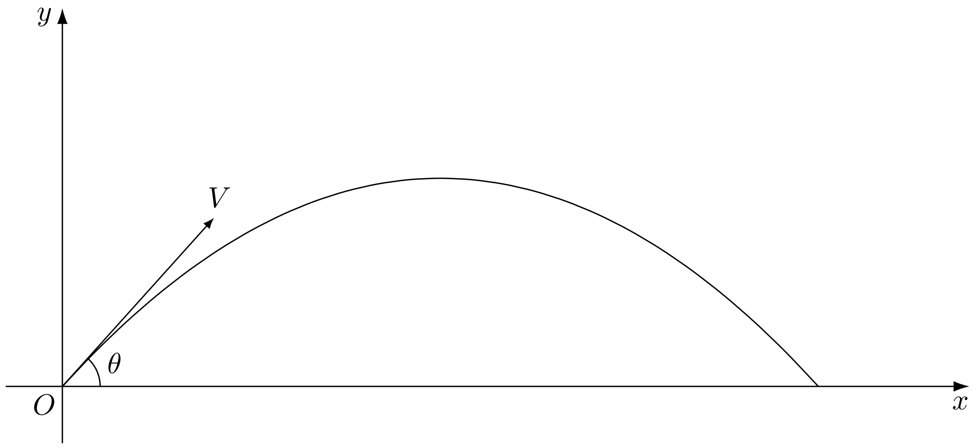
(b) Maximum height.

(c) Horizontal range.

(d) Maximum range.

Note from the author: Although the above results are **not** to be memorised, they are the bread-and-butter of projectile motion 'proof' questions, so their derivations should be practised heavily.

Question 1 A projectile is launched from the origin with fixed speed $V\,\mathrm{m\,s^{-1}}$ and some angle θ.



(a) Prove that the horizontal range of the particle is $\dfrac{V^2 \sin 2\theta}{g}$.

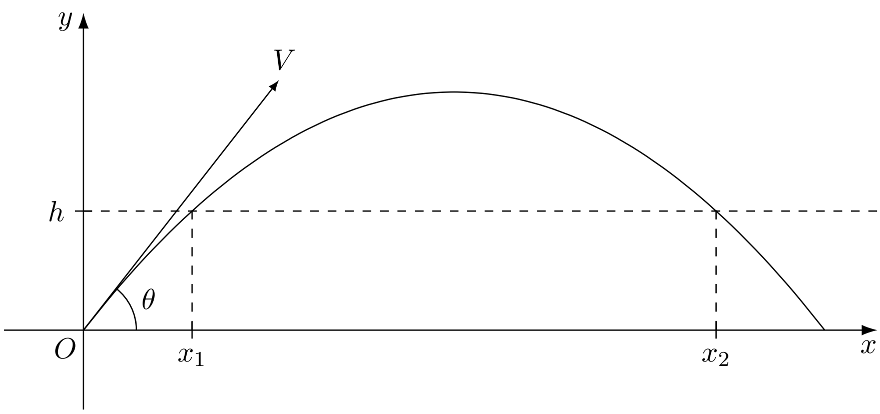
(b) When the particle is aimed at an angle of 15°, it travels a horizontal distance of 40 metres. Show that $V^2 = 80g$.

(c) Show that the equation of the trajectory is

$$y = x \tan\theta - \frac{x^2 \sec^2\theta}{160}.$$

(d) The particle is now aimed so that it hits an object 40 metres away horizontally at a height of 20 metres. Find the angles of projection required to hit the particle, rounded to the nearest degree.

Question 2 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. The particle has height h at $x = x_1$ and $x = x_2$.

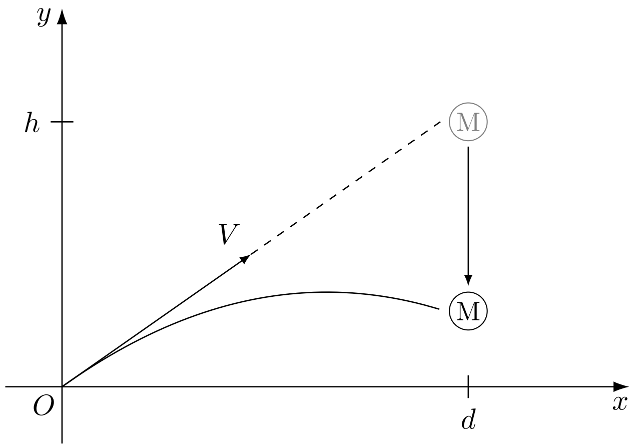


Show that

$$\tan\theta = h\left(\frac{1}{x_1} + \frac{1}{x_2}\right).$$

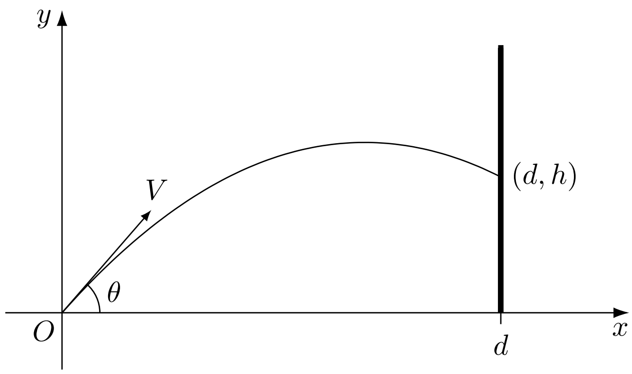
Question 3 [The famous 'Monkey and Hunter' problem]

A zoo-keeper aims a tranquilliser dart with some initial velocity $V\,\mathrm{m\,s^{-1}}$ at an escaped monkey that hangs from a tree d metres away horizontally and h metres high. The initial velocity is always enough so that the dart will at least make the tree. The moment the zoo-keeper fires the dart, the monkey lets go of the tree and falls vertically downwards.



Prove that regardless of V, the dart will always hit the monkey.

Question 4 A particle is projected with variable initial angle θ and fixed velocity $V\,\mathrm{m\,s^{-1}}$ at a pole d metres away horizontally. It hits the pole at a height of h metres.



You may assume that the equation of the trajectory is

$$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta.$$

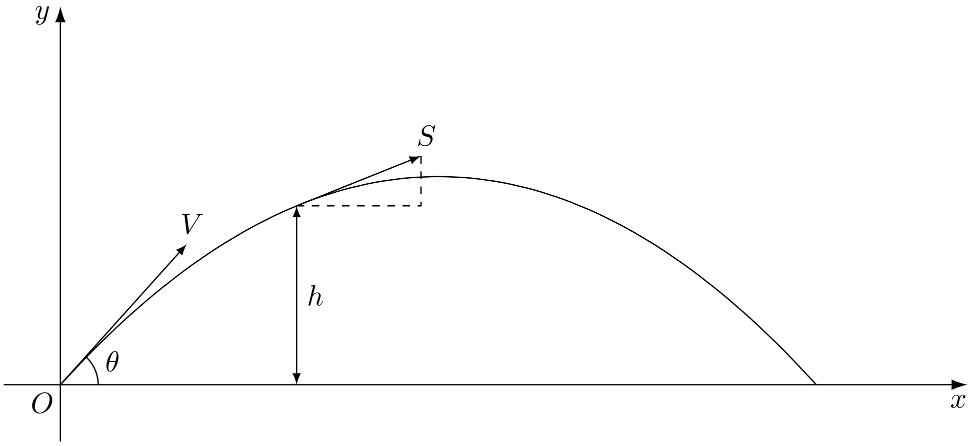
(a) Show that h is maximised when $\tan\theta = \dfrac{V^2}{gd}$.

(b) Prove that the maximum height at which the particle can hit the pole is given by

$$h = \frac{V^4 - g^2d^2}{2gV^2}.$$

(c) Deduce that $V^2 > gd$.

Question 5 A particle is fired from the ground with initial angle θ and initial velocity $V\,\mathrm{m\,s^{-1}}$.

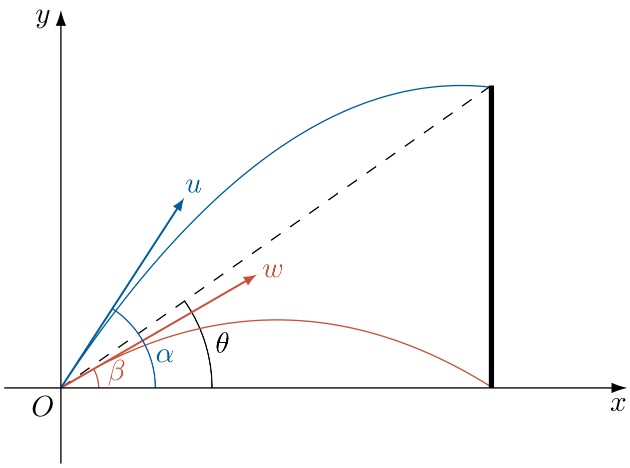


When the particle has height h, the speed of the particle is S.

(a) Prove that when the particle has height h, the vertical speed satisfies $\dot{y}^2 = V^2 \sin^2\theta - 2gh$.

(b) Deduce that $h = \dfrac{1}{2g}(V^2 - S^2)$.

Question 6 A pole has an angle of elevation of θ from the projection point of two cannons. Two projectiles are fired with initial angles of α and β, and initial speeds of u and w respectively.

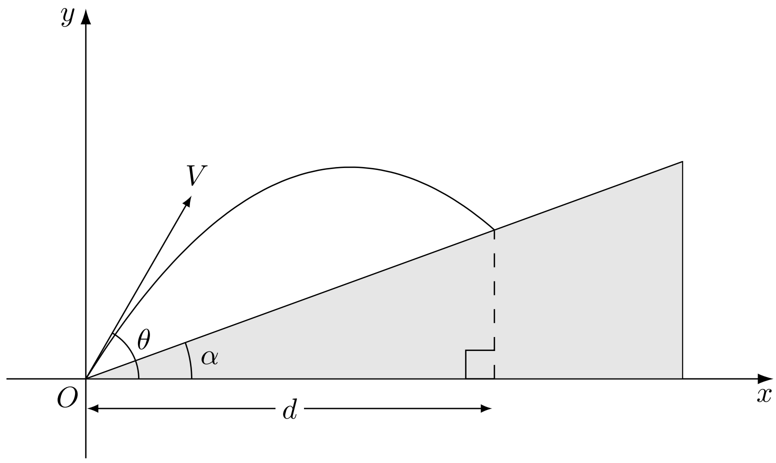


The projectiles hit the pole simultaneously at the base of the pole and at the top of the pole.

(a) Show that $u\cos\alpha = w\cos\beta$.

(b) Hence, prove that $\tan\theta = \tan\alpha - \tan\beta$.

Question 7 A particle is launched from the base O of a plane inclined at an angle of α from the horizontal plane.



Initially, the particle has a speed of $V\,\mathrm{m\,s^{-1}}$ and an angle of inclination of θ. You may assume the standard equations of motion in terms of time.

(a) Show that the equation of the trajectory is $y = x\tan\theta - \dfrac{gx^2}{2V^2}\sec^2\theta$.

(b) Show that when the particle hits the ramp, it has travelled a horizontal distance of

$$d = \frac{2V^2\cos\theta\sin(\theta-\alpha)}{g\cos\alpha}.$$

(c) Hence, show that the range of the particle up the inclined plane is

$$R = \frac{2V^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}.$$

(d) Prove that the range R up the ramp is maximised when the angle of projection is halfway between the vertical and the angle of the plane.

(e) Let T be the time of flight when this occurs. Show that $R = \dfrac{1}{2}gT^2$.

Question 8 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. The particle passes through the point $P(p,q)$, and has a horizontal range of R.

Show that

$$\tan\theta = \frac{qR}{p(R-p)}.$$

Question 9 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$.

(a) Show that the maximum possible height is $h = \dfrac{V^2}{2g}$.

(b) Show that the equation of the trajectory can be expressed as

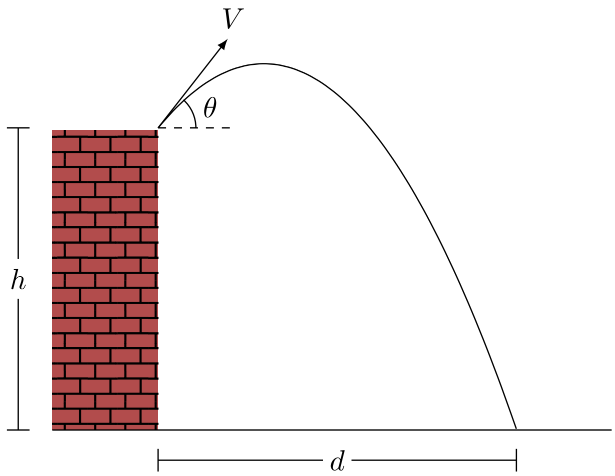
$$x^2 \tan^2\theta - 4hx\tan\theta + (4hy + x^2) = 0.$$

(c) Show that the point (X, Y) where $X \neq 0$ can be hit using two projection angles θ_1 and θ_2 if

$$X^2 < 4h(h - Y).$$

(d) Deduce that no point above the x-axis can be hit using two different projection angles θ_1 and θ_2 if both are less than $\dfrac{\pi}{4}$.

Question 10 A projectile is fired from the top of a h-metre tall building with a fixed muzzle velocity of $V\,\mathrm{m\,s^{-1}}$ and initial angle θ.



Let d be the horizontal distance of the particle from the building base when it lands on the ground.

(a) Show that

$$(g\sec^2\theta)d^2 - (2V^2\tan\theta)d - V^2h = 0.$$

(b) Let D be the maximum horizontal distance of the particle from the base of the building, and let α be the angle when this occurs. Show that $\tan\alpha = \dfrac{V^2}{gD}$.

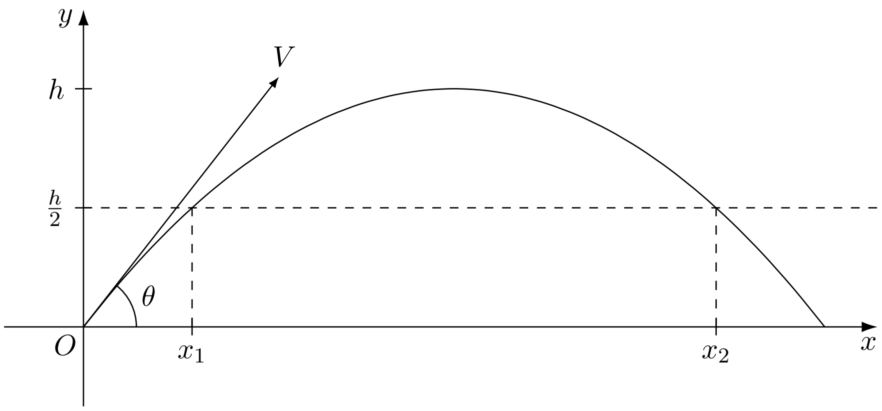
(c) Show that

$$D = \frac{V}{g}\sqrt{V^2 + 2gh}.$$

(d) Deduce that $\cot 2\alpha = \dfrac{h}{D}$.

Question 11 [Application of the sum and product of roots]

A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ.



There are two points x_1 and x_2 where the particle's vertical height is half of the maximum height h. Let $g\,\mathrm{m\,s^{-2}}$ be the acceleration due to gravity. You may assume that the Cartesian equation of the trajectory is

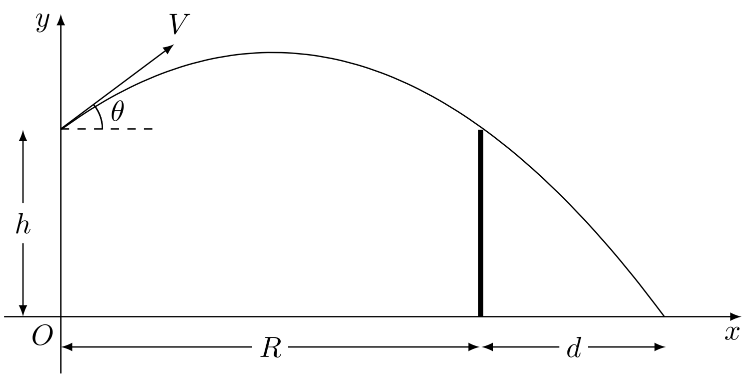
$$y = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta).$$

(a) Show that x_1 and x_2 are solutions of the quadratic

$$2g^2x^2(1+\tan^2\theta) - 4gxV^2\tan\theta + V^4\sin^2\theta = 0.$$

(b) Show that the horizontal distance between x_1 and x_2 is $\dfrac{R}{\sqrt{2}}$, where R is the horizontal range of the particle.

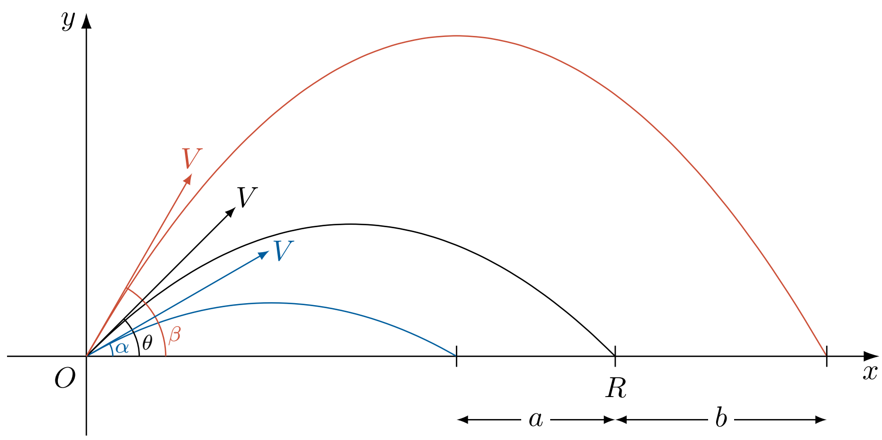
Question 12 A particle is projected from a point h metres above the ground with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$.



The particle clears a h-metre tall fence R metres away, and lands at a point d metres away from the base of the fence. Show that

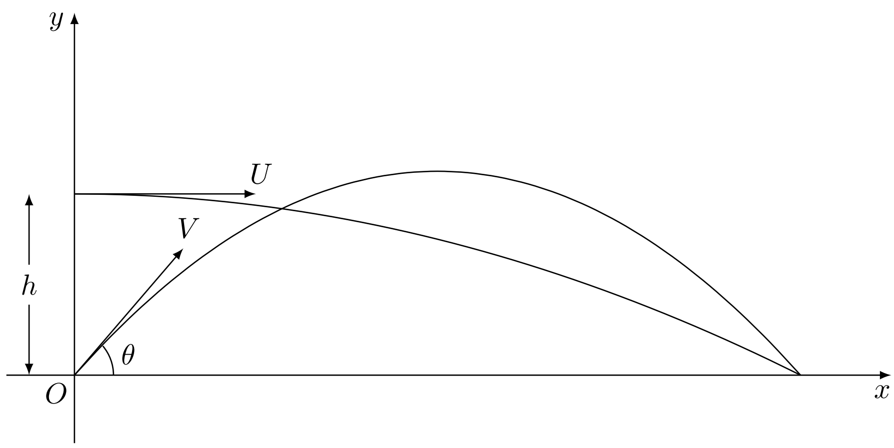
$$\tan\theta \geq \frac{hR}{d(d+R)}.$$

Question 13 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$. When the particle is projected with angle θ, it attains a horizontal range R. When it is projected with angle α, it falls short of R by a metres. Similarly, when it is projected with angle β, it overshoots R by b metres.



Show that $a+b=\dfrac{V^2}{g}(\sin 2\beta-\sin 2\alpha)$.

Question 14 A particle is projected horizontally from a point h metres above the ground with initial speed $U\,\mathrm{m\,s^{-1}}$. Simultaneously, another particle is projected from the ground with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ.

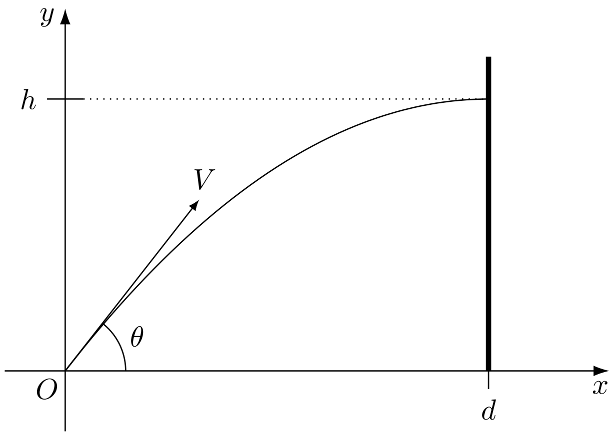


Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$.

(a) Show that if the particles collide, then $V>U$.

(b) Show that if the particles collide on the ground, then $V^2-U^2=\dfrac{1}{2}gh$.

Question 15 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$.

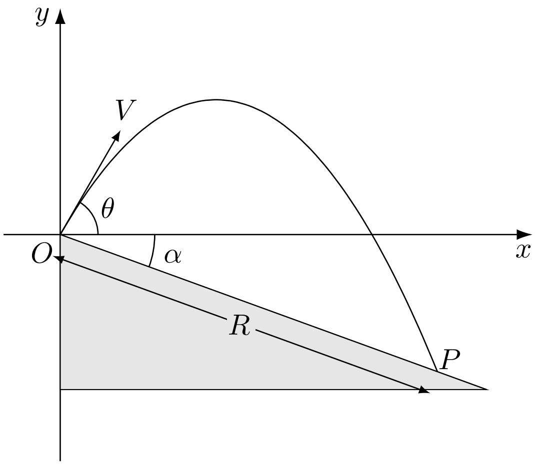


When the particle reaches maximum height, it hits a wall d metres away at height h.

(a) Show that $\theta = \tan^{-1}\left(\frac{2h}{d}\right)$.

(b) Show that $V^2 = \frac{g}{2h}(d^2 + 4h^2)$.

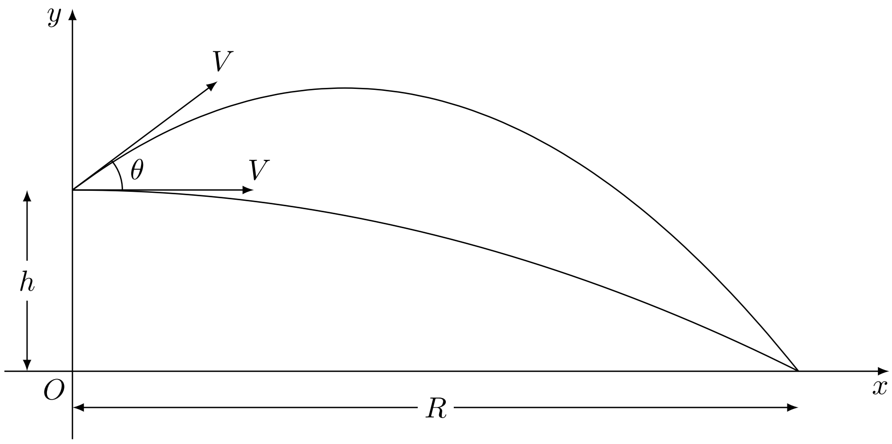
Question 16 A projectile is launched from a point O on a plane inclined downwards at an angle of α from the horizontal plane. Initially, the particle has speed $V\,\mathrm{m\,s^{-1}}$ and an angle of θ.



Prove that the range R down the ramp is maximised when the angle of projection is halfway between the vertical and the angle of the plane.

Challenge Problems

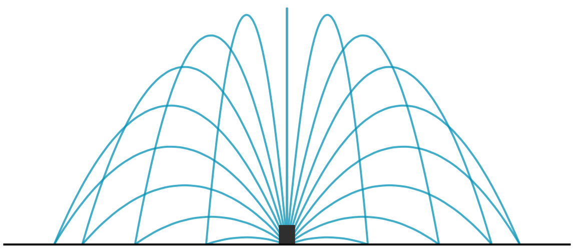
Problem 1 A particle is projected horizontally from a point h metres above the ground with initial speed $V\,\mathrm{m\,s^{-1}}$. It hits the ground R metres away from the base of the cliff after travelling for T seconds. The same point can also be hit using the same initial speed, but with angle of projection $\theta \neq 0$. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$.



Show that $R = \dfrac{1}{2}gT^2 \tan\theta$.

Problem 2 [Envelope of parabolas]

A hose sprays water with a fixed muzzle velocity of $V\,\mathrm{m\,s^{-1}}$.



The angle of inclination is allowed to vary from $-\pi \leq \theta \leq \pi$ so that anything inside the curve traced out by the stream can be hit by water.

(a) Prove that the horizontal range is given by $\dfrac{V^2 \sin 2\theta}{g}$.

(b) Show that the maximum horizontal range is given by $R = \dfrac{V^2}{g}$.

(c) Prove that anything inside the parabola

$$x^2 + 2Ry - R^2 = 0$$

can be hit by water.

Exercise 5D

Resisted Horizontal Motion

Fundamentals

Fundamentals 1
Write down at least three expressions for acceleration.

Fundamentals 2

(a) When a particle is projected horizontally through some medium, it experiences a resistive f_____ that is a function of the v_______ of the particle.

(b) For horizontal motion, we ignore the effects of g_______.

(c) Sometimes, the particle may have a limiting displacement and will never exceed this. The limiting displacement, if one exists, can be found by letting ___ approach infinity.

Question 1 A particle with mass 5kg is projected to the right with an initial velocity of $40\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $20v$, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

(a) Show that $a = -4v$.

(b) Show that $v = 40 - 4x$.

(c) Show that $v = 40e^{-4t}$.

(d) Combine parts (b) and (c) to show that $x = 10(1 - e^{-4t})$.

(e) Explain how you would obtain $x = 10(1 - e^{-4t})$ using either parts (b) or (c) only.

(f) Write down the limiting displacement of the particle.

Question 2 A particle with mass 5kg is projected to the right with an initial velocity of $10\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $20v^2$, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

(a) Show that $v = 10e^{-4x}$.

(b) Hence, show that $x = \ln(1 + 40t)$.

(c) Find how far the particle travels after 2 seconds.

(d) Find how long it takes for the particle to travel $\ln 2$ metres, and its speed at this time.

Question 3 A particle with mass 4kg is projected to the right with an initial velocity of $8\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of kv^2, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds. Find the value of k if

(a) after 4 seconds, the particle has a speed of $2\,\mathrm{m\,s^{-1}}$.

(b) the particle has a speed of $2\,\mathrm{m\,s^{-1}}$ when it has travelled 4 metres.

(c) after 2 seconds, the particle has travelled $8\ln 3$ metres.

Question 4 [Generalised linear drag]

A particle with mass m kg is projected to the right with an initial velocity of $u\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of mkv, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

(a) Show that $v = u - kx$.

(b) Show that $v = ue^{-kt}$.

(c) Show that $x = \dfrac{u}{k}(1 - e^{-kt})$.

(d) Hence state the limiting position of the particle.

(e) After how many seconds does the particle reach half of its limiting position?

(f) Use two different methods to show that when this occurs, the particle has half its initial velocity.

Question 5 A particle with mass 10kg is projected to the right with an initial velocity of $20\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of mkv, where v is the velocity of the particle after t seconds. The limiting displacement is 40 m. Let a be the acceleration of the particle after t seconds.

(a) Find the limiting displacement of the particle in terms of k.

(b) Hence, find the value of k.

Question 6 [Generalised quadratic drag]

A particle with mass m kg is projected to the right with an initial velocity of $u\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of mkv^2, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

(a) Show that $v = ue^{-kx}$.

(b) Show that $v = \dfrac{u}{1 + ukt}$.

(c) Show that $x = \dfrac{1}{k}\ln(1 + ukt)$.

(d) Does this particle have a limiting position?

Challenge Problems

Problem 1 A particle with mass m kg is projected to the right with an initial velocity of $u\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of mkv, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

Prove that when the particle is $r\,\%$ of the way to its limiting position, it will be moving at $(100 - r)\,\%$ of its initial velocity. For example, when the particle is 30% of the way to its limiting position, it will be moving at 70% of its initial velocity.

Problem 2 A particle with unit mass is projected to the right with an initial velocity of $u\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $v + v^3$, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

(a) Show that

$$x = \tan^{-1}\left(\frac{u - v}{1 + uv}\right).$$

(b) Show that

$$t = \frac{1}{2}\ln\left(\frac{u^2(1+v^2)}{v^2(1+u^2)}\right).$$

(c) State the limiting position of the particle.

(d) Show that the particle can never travel past $x = \dfrac{\pi}{2}$ with any initial speed.

(e) State the limiting speed of the particle.

Problem 3 Bob claims that if an object is projected horizontally through a resistive medium, then it must eventually stop, assuming that there are no further forces pushing the object forwards. Is Bob correct in saying this?

Problem 4 Suppose that two particles with differing mass are projected through the same resistive medium with linear drag, and with the same initial velocity. Bob claims that the lighter particle will travel further than the heavier one. Is Bob correct in saying this?

Exercise 5E

Resisted Vertical Motion

Fundamentals

Fundamentals 1
Resisted vertical motion could be going up or down. Separate equations will need to be produced for each direction of motion.

(a) When the particle goes up, weight force is considered positive/negative (circle one) and resistive force is considered positive/negative (circle one).

(b) When the particle goes down, weight force is considered positive/negative (circle one) and resistive force is considered positive/negative (circle one).

Fundamentals 2

(a) When an object is released in the air, it experiences a w________ force and a resistive force.

(b) For a particle falling, we consider the w________ force to be p________ since it pulls in the same direction as our motion. However, we consider the resistive force to be n________ because it opposes our motion.

Fundamentals 3

(a) When a particle falls vertically through a resistive medium, the speed will approach a limit called the t________ velocity.

(b) The t________ velocity can be found be letting either ___ or ___ approach infinity.

(c) However, the easiest way to find an expression for it is to let $a =$ ___ since this is when the particle has constant speed.

Question 1 A particle of mass 4kg is dropped from a tall building. It experiences a resistive force of $20v$, where v is the speed of the particle after t seconds, and a gravitational acceleration of $10\,\mathrm{m\,s^{-2}}$.

(a) Show that $a = 10 - 5v$.

(b) Show that $v = 2(1 - e^{-5t})$.

(c) Hence, find the terminal velocity of the particle using two different methods.

(d) Find the time that it takes for the particle to reach half of its terminal velocity.

(e) Find the displacement when this occurs.

Question 2 [Generalised upwards motion with linear drag]

A particle of mass m is projected vertically upwards through a resistive medium. It experiences a resistive force of mkv and a weight force mg. Suppose the particle has an initial velocity of $u\,\mathrm{m\,s^{-1}}$.

(a) Show that velocity is given by

$$v = \frac{1}{k}\left[(g+ku)e^{-kt} - g\right].$$

(b) Show that the time taken to reach maximum height is

$$T = \frac{1}{k}\ln\left(1 + \frac{ku}{g}\right).$$

(c) Show that displacement is given by

$$x = \frac{1}{k}\left[u - v - \frac{g}{k}\ln\left(\frac{g+ku}{g+kv}\right)\right].$$

(d) Show that the maximum height is

$$H = \frac{u}{k} - \frac{g}{k^2}\ln\left(1 + \frac{ku}{g}\right).$$

Question 3 [Generalised downwards motion with linear drag]

A particle of mass m falls vertically downwards through a resistive medium. It experiences a resistive force of mkv and a weight force mg. The particle has a terminal velocity of w.

(a) Show that $w = \dfrac{g}{k}$.

(b) Show that

$$x = -\frac{1}{k}\left[v + \frac{g}{k}\ln\left(1 - \frac{kv}{g}\right)\right].$$

(c) Hence, show that

$$x = \frac{1}{k}\left[w\ln\left(\frac{w}{w-v}\right) - v\right].$$

(d) Show that

$$t = \frac{1}{k}\ln\left(\frac{w}{w-v}\right).$$

Question 4 [Generalised upwards motion with quadratic drag]

A particle of mass m is projected vertically upwards through a resistive medium. It experiences a resistive force of mkv^2 and a weight force mg. Suppose the particle has an initial velocity of $u\,\mathrm{m\,s^{-1}}$.

(a) Show that

$$x = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right).$$

(b) Show that the maximum height is

$$H = \frac{1}{2k}\ln\left(1+\frac{ku^2}{g}\right).$$

(c) Show that

$$t = \frac{1}{\sqrt{gk}}\left[\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)\right].$$

(d) Show that the time taken to reach maximum height is

$$T = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right).$$

(e) Show that the particle will never take longer than $\dfrac{\pi}{2\sqrt{gk}}$ seconds to reach maximum height.

Question 5 A particle of mass m is projected vertically upwards through a resistive medium. It experiences a resistive force of mkv^2 and a weight force mg. Suppose the particle has an initial velocity of $u\,\mathrm{m\,s^{-1}}$. If the particle is dropped from rest instead, the terminal velocity of the particle is w.

(a) Show that $w^2 = \dfrac{g}{k}$.

(b) Show that

$$x = \frac{1}{2k}\ln\left(\frac{w^2+u^2}{w^2+v^2}\right).$$

(c) At some point in the upwards movement, the particle will have speed w. Show that when this occurs, the particle will have to travel $\dfrac{\ln 2}{2k}$ metres further before reaching maximum height.

Question 6 [Generalised downwards motion with quadratic drag]

A particle of mass m falls vertically downwards through a resistive medium. It experiences a resistive force of mkv^2 and a weight force mg. The particle has a terminal velocity of w.

(a) Show that

$$x = \frac{1}{2k} \ln\left(\frac{g}{g - kv^2}\right).$$

(b) Show that

$$v^2 = \frac{g}{k}(1 - e^{-2kx}).$$

(c) Find the terminal velocity of the particle using two different methods.

(d) Show that

$$x = \frac{1}{2k} \ln\left(\frac{w^2}{w^2 - v^2}\right).$$

(e) Show that

$$t = \frac{1}{2\sqrt{gk}} \ln\left|\frac{w + v}{w - v}\right|.$$

(f) Show that

$$v = w\left(\frac{e^{2t\sqrt{gk}} - 1}{e^{2t\sqrt{gk}} + 1}\right).$$

Question 7 A particle of mass m falls vertically downwards through a resistive medium. It experiences a resistive force mkv and a weight force mg. The particle has a terminal velocity of w.

(a) Show that the terminal velocity is $V = \dfrac{g}{k}$.

(b) Show that the speed at time t is $w(1 - e^{-kt})$.

(c) An identical particle is projected upwards from the same point of release as the first particle, but now with initial velocity u. Show that the second particle reaches maximum height when

$$t = \frac{1}{k} \ln\left(1 + \frac{u}{w}\right).$$

(d) Show that when the second particle reaches maximum height, the first particle has speed $\dfrac{uw}{u + w}$.

Question 8 A particle of mass m kg falls from rest and experiences an air resistance of mkv^2. Let the acceleration due to gravity be $g\,\mathrm{m\,s^{-2}}$ and let the terminal velocity of the particle be V.

(a) Show that the equation of motion is $\ddot{x} = g - kv^2$.

(b) Find V^2 in terms of k and g.

(c) The particle impacts the ground with velocity W. Show that the distance travelled is given by

$$D = \frac{1}{2k}\ln\left(\frac{V^2}{V^2 - W^2}\right).$$

(d) Prove that if the particle is projected from the ground with initial velocity U and air resistance is still mkv^2, then the maximum height reached is given by

$$H = \frac{1}{2k}\ln\left(\frac{U^2 + V^2}{V^2}\right).$$

(e) Show that $\dfrac{1}{W^2} = \dfrac{1}{U^2} + \dfrac{1}{V^2}$.

(f) Show that

$$W = \frac{UV}{\sqrt{U^2 + V^2}}.$$

(g) Hence, state whether the impact speed is lesser or greater than the initial projection speed.

Question 9 [What happens to simple harmonic motion when sufficiently damped]

A particle of mass m was originally intended to move in simple harmonic motion at time t according to the displacement equation $x = A\cos(nt)$ for positive constants A and n.

However, once the particle was released from rest it experienced a resistance force of $2mnv$, where v is the velocity of the particle.

(a) Explain why the acceleration equation of the particle with the resistance is

$$\ddot{x} = -n^2x - 2nv.$$

(b) Let $y = v + nx$. Show that $\dot{y} + ny = 0$.

(c) Hence, show that the displacement equation of the particle is given by

$$x = A(1 + nt)e^{-nt}.$$

Hint: Consider the fact that $\dfrac{d}{dx}\left(e^x f(x)\right) = e^x\left(f(x) + f'(x)\right)$.

(d) Sketch the displacement equation of the particle and describe its behaviour over time.

Challenge Problems

Problem 1 A particle A of unit mass is projected horizontally from the origin through a medium with initial speed u. It experiences a resistive force of kv^2, where v is the velocity of the particle after t seconds.

Simultaneously, a second particle B of unit mass is projected vertically through the same medium with the same initial speed u. It experiences a resistive force of kw^2, where w is the velocity of the particle after t seconds. Let $g\,\text{m}\,\text{s}^{-2}$ be the acceleration due to gravity.

(a) Let V be the velocity of particle A when particle B is at rest. Show that if v_T is the terminal velocity of particle B when allowed to fall indefinitely, then

$$\frac{1}{V} = \frac{1}{u} + \frac{1}{v_T}\tan^{-1}\left(\frac{u}{v_T}\right).$$

(b) Deduce that if u is sufficiently large, then V is approximately 64% of the terminal speed.

Problem 2 A particle of unit mass is projected vertically upwards through a resistive medium with initial speed u. It experiences a resistive force of kv and a weight force g. When the same particle lands on the ground, it has speed w. Prove that $u > w$.

Problem 3 A particle of unit mass is projected vertically upwards through a resistive medium with initial speed u. It experiences a resistive force of kv and a weight force g. When the same particle lands on the ground, it has speed w.

Show that the particle takes in total $T = \dfrac{u+w}{g}$ seconds to land back on the ground.

Problem 4 A particle A is dropped from a weather balloon. Its equation of motion is

$$\ddot{x} = g - kv_A,$$

where $g\,\text{m}\,\text{s}^{-2}$ is the acceleration due to gravity, k is a positive constant and v_A is the speed of particle A. After T seconds, an identical particle B is projected downwards from the same balloon with initial velocity $u\,\text{m}\,\text{s}^{-1}$. Its equation of motion is similarly

$$\ddot{x} = g - kv_B,$$

where v_B is the speed of particle B. Let w be the terminal velocity of both particles.

(a) Show that $T = \dfrac{1}{k}\ln\left(\dfrac{1}{w-u} \times \dfrac{w-v_A}{w-v_B}\right)$.

(b) Show that particle B's displacement is given by $x_B = \dfrac{1}{k}\left[u - v_B + w\ln\left(\dfrac{w-u}{g-v_B}\right)\right]$.

(c) Deduce that if particle B catches up with particle A, then particle B must have been projected no more than $\dfrac{u}{g}$ seconds after particle A was dropped.

Exercise 5F

Resisted Projectile Motion

Fundamentals

Fundamentals 1

For each of the following, express your answers in the form $x = x(t)$ and $y = y(t)$.

(a) Write down the acceleration equations for projectile motion assuming no air resistance.

(b) Write down the acceleration equations for projectile motion but now with linear drag, assuming that the drag constant is the same in both x and y directions.

Fundamentals 2

For each of the following, express your answers in the form $\ddot{\underset{\sim}{r}}(t)$, where $\underset{\sim}{r}(t)$ is the position vector of the particle at time t.

(a) Write down the acceleration equations for projectile motion assuming no air resistance.

(b) Write down the acceleration equations for projectile motion but now with linear drag, assuming that the drag constant is the same in both x and y directions.

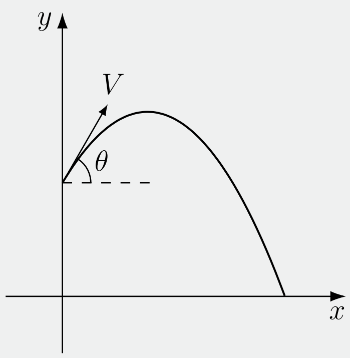
Fundamentals 3

When a particle undergoes projectile motion with air resistance, the motion is no longer p________. Instead, the particle will have a l________ horizontal displacement that can be found by letting t approach i________.

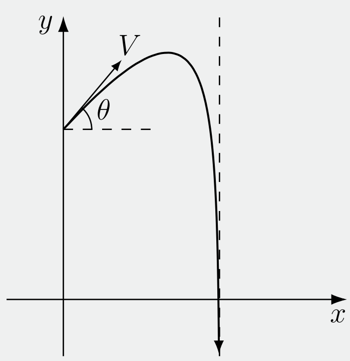
Fundamentals 4

The two diagrams below show projectile motion. Label the one that has no air resistance, and the one that has air resistance.

(a)



(b)



Question 1 [Standard derivations for linear drag]

A particle is projected from the origin with initial speed V with an angle of inclination of θ. The particle is subject to gravity and an air resistance proportional to the velocity such that the accelerations in the horizontal and vertical directions are given by

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

where k is a constant and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. Derive the following results.

(a) $\dot{x} = Ve^{-kt}\cos\theta$

(b) $\dot{y} = \left(\dfrac{g}{k} + V\sin\theta\right)e^{-kt} - \dfrac{g}{k}$

(c) $x = \dfrac{V\cos\theta}{k}(1 - e^{-kt})$

(d) $y = \dfrac{1}{k}\left(\dfrac{g}{k} + V\sin\theta\right)(1 - e^{-kt}) - \dfrac{g}{k}t$

Question 2 [Some key results for linear drag]

A particle is projected from the origin with initial speed V with an angle of inclination of θ. The particle is subject to gravity and an air resistance proportional to the velocity such that motion is modelled by

$$\ddot{\underset{\sim}{r}} = -k\dot{\underset{\sim}{r}} - g\,\underset{\sim}{j},$$

where k is a constant and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(a) Show that the limiting horizontal displacement of the particle is

$$x = \frac{V\cos\theta}{k}.$$

(b) Show that the time taken for the particle to reach maximum height is

$$T = \frac{1}{k}\ln\left(1 + \frac{kV}{g}\sin\theta\right).$$

(c) Show that when the particle reaches maximum height, the horizontal displacement is

$$D = \frac{V^2\sin 2\theta}{2(g + kV\sin\theta)}.$$

(d) Let α be the angle so that D is maximised. Show that

$$\sin\alpha = \frac{\sqrt{5} - 1}{2}.$$

Question 3 [Quadratic drag does not work in the way you may think it does]

A particle of unit mass is projected from the origin with initial speed V with an angle of inclination of θ. The particle is subject to gravity and an air resistance with a magnitude proportional to the *square* of the velocity, but acting in a direction opposite to the particle's motion.

Bob claims that the particle is modelled by the systems of equations

$$\ddot{x} = -k\dot{x}^2$$
$$\ddot{y} = -g - k\dot{y}^2$$

where $k \in \mathbb{R}^+$ for the *upwards journey* and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(a) Show that $x = \dfrac{1}{k}\ln\left(1 + ukt\cos\theta\right)$.

(b) Assume that gravity is absent. Write down a similar expression for y.

(c) Hence, show that the Cartesian equation of the trajectory is

$$y = \frac{1}{k}\ln\left|1 + \left(e^{kx} - 1\right)\tan\theta\right|$$

if the effect of gravity is to be ignored.

(d) Explain why this demonstrates that Bob's model does NOT accurately reflect projectile motion with quadratic drag.

Question 4 A particle of unit mass is projected horizontally from a building of height h with initial speed V. The particle is subject to gravity and an air resistance proportional to the velocity such that the accelerations in the horizontal and vertical directions are given by

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

where k is a constant and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

Let the terminal velocity in free fall be w, and let the limiting horizontal displacement be R.

(a) Show that $x = \dfrac{V}{k}\left(1 - e^{-kt}\right)$ and hence show that $k = \dfrac{V}{R}$.

(b) Show that $y = \dfrac{w}{k}\left(1 - e^{-kt}\right) - wt + h$.

(c) Hence, show that the equation of the trajectory is

$$y = h + \frac{w}{V}x + \frac{w}{k}\ln\left(1 - \frac{x}{R}\right).$$

(d) Show that if the particle hits the ground when it is at $x = \dfrac{R}{2}$, then

$$h = \frac{w}{2k}\left(2\ln 2 - 1\right).$$

Challenge Problems

Problem 1 [The correct model for projectile motion with quadratic drag]

(a) Explain why the correct model of projectile motion with quadratic drag is

$$\ddot{\underset{\sim}{r}} = -g\,\underset{\sim}{j} - k|\dot{\underset{\sim}{r}}|\dot{\underset{\sim}{r}}$$

where $k \in \mathbb{R}^+$ and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(b) Show that this leads to the systems of equations

$$\ddot{x} = -k\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$
$$\ddot{y} = -g - k\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$

Problem 2 [The Cartesian equation of the trajectory]

A particle of unit mass is projected from the origin with initial speed V with an angle of inclination of θ. The particle is subject to gravity and an air resistance proportional to the velocity such that the accelerations in the horizontal and vertical directions are given by

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

where k is a constant and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(a) Prove that the Cartesian equation of motion is

$$y = \left(\frac{g}{kV\cos\theta}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{V\cos\theta}\right).$$

(b) Show that the particle's limiting horizontal displacement is $R = \dfrac{V\cos\theta}{k}$.

(c) Show that the particle's terminal speed is $w = \dfrac{g}{k}$.

(d) Let X be the proportion of the particle's horizontal displacement with respect to R. Show that the Cartesian equation of the trajectory can be expressed as

$$y = \frac{w}{k}\Big(X + \ln(1 - X)\Big).$$

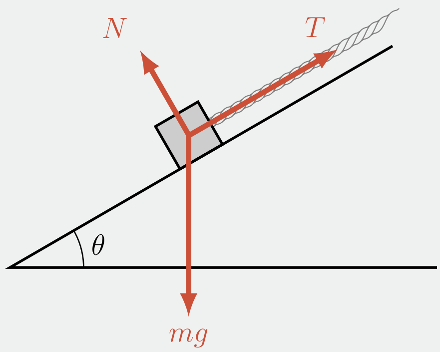
Exercise 5G

Inclined Planes and Pulleys

Fundamentals

Fundamentals 1

The following force diagram shows a normal force N, a tension force T and a weight force mg acting on an object.



(a) Resolve forces in the horizontal and vertical directions.

(b) Resolve forces in directions that are perpendicular and parallel to the tension force.

Fundamentals 2

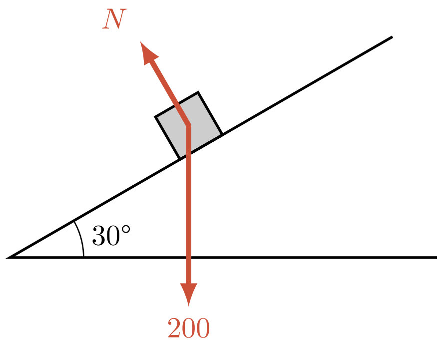
(a) If we want to find the acceleration equation of a particle that is travelling in a certain direction, we should resolve forces in directions p________ and p__________ to the direction of motion to make the working out easier.

(b) An i__________ of forces causes acceleration, and hence, movement, so if a particle is moving in a certain direction, it means there is a n___ force acting in that direction, so the acceleration in that direction can be studied.

Note from the author: For ramp problems in the following exercise, "Find the acceleration" is intended to mean "Find the acceleration of the object acting along the direction of the ramp".

Question 1 [Guided smooth ramp problem]

The following diagram shows an object of mass 20kg on a smooth ramp inclined at an angle of 30° from the horizontal.



The particle experiences a weight force and a normal force N. Let $g = 10\,\mathrm{m\,s^{-2}}$ be the acceleration due to gravity.

(a) Resolve forces in directions parallel and perpendicular to the plane.

(b) Show that there is a net force parallel to the plane with magnitude 100 Newtons.

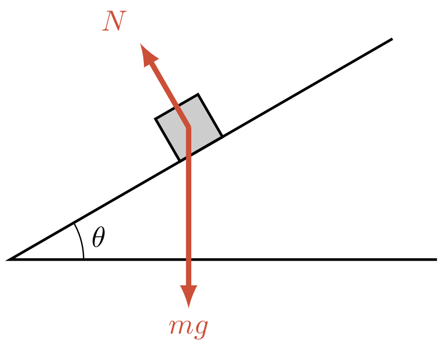
(c) Recall that $F_{\text{net}} = ma$. Show that the acceleration down the ramp is $a = 5\,\mathrm{m\,s^{-2}}$.

(d) Find the speed of the particle if it is released from rest and allowed to travel for 4 seconds.

(e) Find how far the particle slides down the ramp in 4 seconds.

Question 2 [Generalised smooth ramp problem]

The following diagram shows an object of mass m kg on a smooth ramp inclined at an angle of θ from the horizontal.



The particle experiences a weight force mg and a normal force N. Let a be the acceleration of the particle down the ramp. Show that after t seconds,

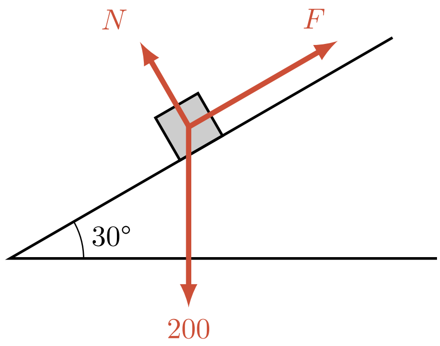
(a) $a = g\sin\theta$.

(b) $v = gt\sin\theta$.

(c) $x = \frac{1}{2}gt^2\sin\theta$.

Question 3 [Guided ramp problem with friction]

The following diagram shows an object of mass 20kg on a ramp inclined at an angle of 30° from the horizontal.



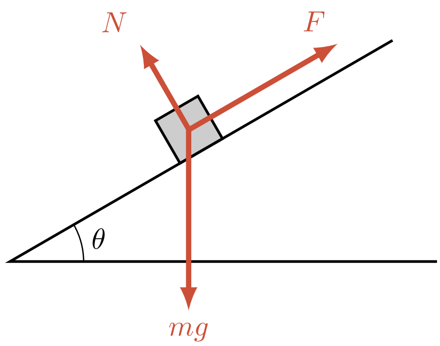
The particle experiences a weight force, a normal force N, and a friction force $F = 0.1N$. Let $g = 10\,\mathrm{m\,s^{-2}}$ be the acceleration due to gravity.

(a) Show that there is a net force down the ramp with magnitude $(100 - 10\sqrt{3})$ Newtons.

(b) Find the acceleration of the object.

Question 4 [Generalised ramp problem with friction]

The following diagram shows an object of mass m kg on a smooth ramp inclined at an angle of θ from the horizontal.



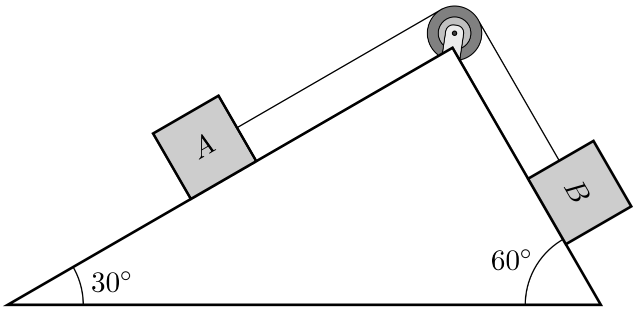
The particle experiences a weight force mg, a normal force N, and a friction force $F = \mu N$.

(a) Show that $a = \sin\theta - \mu\cos\theta$.

(b) Hence, show that if $\mu \geq \tan\theta$, then the particle will not slide down the ramp.

Question 5 [Guided double smooth ramp problem]

Two objects A and B with masses of 20kg and 10kg respectively are connected by a light inextensible string that runs through a smooth pulley. The objects lean on a double-sided smooth ramp inclined at angles of 30° and 60° from the horizontal, as shown below.



The system moves so that particle A slides down the ramp whilst particle B slides up the ramp.

(a) Explain briefly why the magnitude of tension is the same for both particles.

(b) Explain briefly why the magnitude of acceleration is the same for both particles.

(c) Resolve forces for particle A and show that

$$\ddot{x} = 5 - \frac{T}{20}.$$

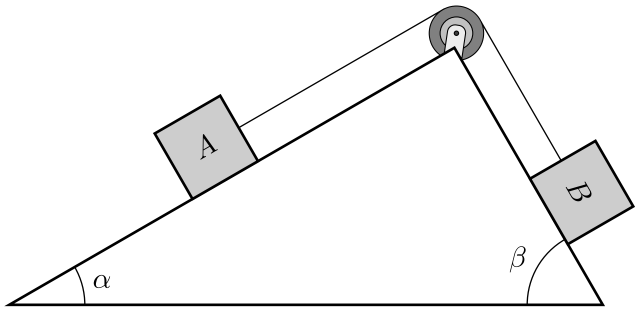
(d) Resolve forces for particle B and find a similar result for $\ddot{x}$.

(e) Hence, calculate the amount of tension in the string.

(f) Calculate the acceleration of particle A down the ramp.

Question 6 [Generalised double smooth ramp problem]

Two objects A and B with masses of $m_1$kg and $m_2$kg respectively are connected by a light inextensible string that runs through a smooth pulley. The objects lean on a double-sided smooth ramp inclined at angles of α and β from the horizontal, as shown below.



The system moves so that particle A slides down the ramp whilst particle B slides up the ramp.

(a) Show that

$$a = g\left(\frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2}\right).$$

(b) Hence, show that

$$T = g\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\sin\alpha + \sin\beta).$$

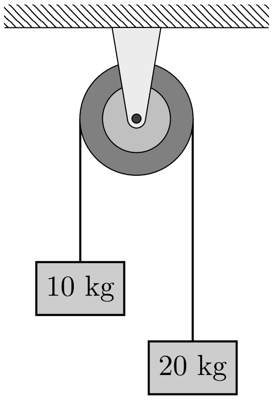
(c) Show that if

$$\frac{\sin\alpha}{\sin\beta} = \frac{m_2}{m_1},$$

then the system will remain at static equilibrium.

Question 7 [Guided smooth pulley problem]

The following diagram shows two objects with masses of 10kg and 20kg on either end of a light inextensible string that passes through a smooth pulley.



The heavier mass moves downwards whilst the lighter mass moves upwards.

(a) Show that for the heavier object $a = 10 - \dfrac{T}{20}$.

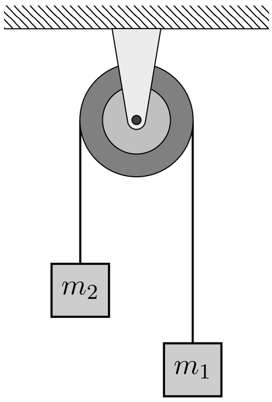
(b) Show that for the lighter object $a = -10 + \dfrac{T}{10}$.

(c) Explain why both particles have the same magnitude of acceleration at any time t.

(d) Hence, find the acceleration of both masses, and the amount of tension in the sting.

Question 8 [Generalised smooth pulley problem]

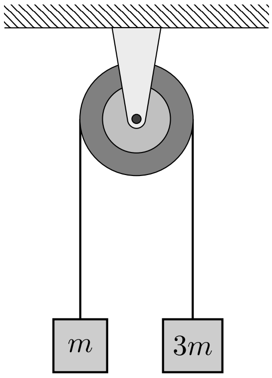
The following diagram shows two objects with masses of m_1 kg and m_2 kg, where $m_1 > m_2$, on either end of a light inextensible string that passes through a smooth pulley.



Show that the acceleration of the heavier particle is

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g.$$

Question 9 The following diagram shows two objects with masses of m kg and $3m$ kg on either end of a light inextensible string that passes through a smooth pulley. Both particles are released from rest simultaneously.



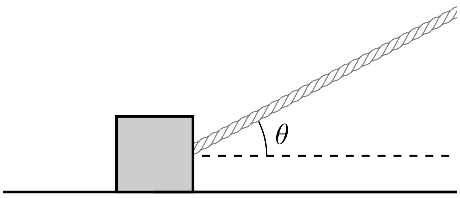
Let a be the acceleration of the heavier particle in the downwards direction. Let g be the acceleration due to gravity.

(a) Show that $a = \frac{g}{2}$.

(b) Hence, show that after 4 seconds, the heavier object travels $4g$ metres and has speed $2g\,\mathrm{m\,s^{-1}}$.

Question 10 [Minimal force problem on a flat surface]

An object of mass m rests on the surface of a table. It is attached to a rope inclined at an angle of θ from the horizontal that pulls it to the right.



The object experiences a friction force $F = \mu N$ that resists the motion of the object.

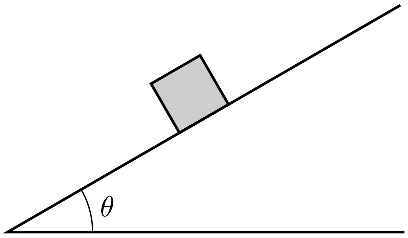
(a) Resolve forces in the vertical and horizontal directions.

(b) Hence, show that the amount of tension needed to overcome friction is

$$T = \frac{\mu m g}{\cos\theta + \mu \sin\theta}.$$

Question 11 [Minimal force problem involving a ramp]

An object of mass m sits on a ramp inclined at an angle of θ from the horizontal.



The object is pushed up the ramp and it experiences a friction force $F = \mu N$.

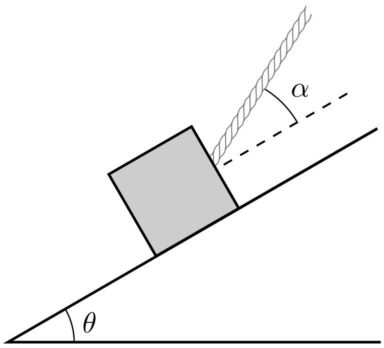
(a) Resolve forces parallel and perpendicular to the plane.

(b) Hence, show that the minimal amount of force needed to push the object up the ramp is

$$\mathbf{F}_{\text{push}} = mg(\sin\theta + \mu\cos\theta).$$

Question 12 [Minimal force problem involving a ramp and a rope]

An object of mass m sits on a ramp inclined at an angle of θ from the horizontal.



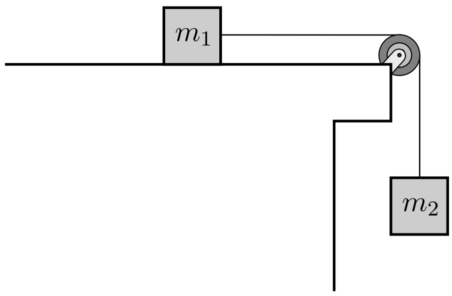
The object is dragged up the ramp by a rope that is angled α from the surface of the ramp, and it experiences a friction force $F = \mu N$.

(a) Resolve forces parallel and perpendicular to the plane.

(b) Hence, show that the minimal amount of force needed to pull the object up the ramp is

$$T = mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\alpha + \mu\sin\alpha}\right).$$

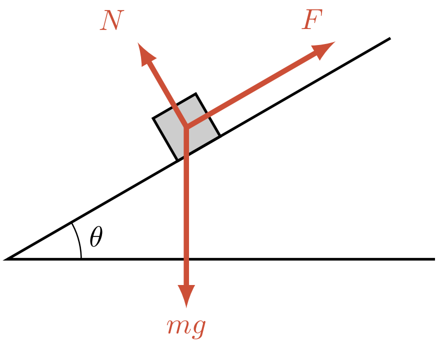
Question 13 The following diagram shows two objects with masses of m_1 kg and m_2 kg on either end of a light inextensible string that passes through a smooth pulley. The object with mass m_2 is hung off from the edge of a table whilst the object with mass m_1 rests on the surface of a table. The object on the table experiences a friction force $F = \mu N$ with the surface of the table.



Prove that if $\mu \geq \dfrac{m_1}{m_2}$, then there is sufficient friction to prevent the object on the table from moving.

Question 14 [Threshold problem]

An object rests on a plane inclined at a variable angle θ from the horizontal. It experiences a friction force F and a normal force N. Let α be the maximal angle that the plane can be tilted before the object begins to slide down the plane. When this occurs, the friction force is $F = \mu N$.



A horizontal force H is applied to the object so that if H is sufficiently large, then the particle will begin to slide *up* the plane and if H is sufficiently small, then the particle will still slide down the plane.

Show that for the object to remain motionless, H must satisfy

$$mg\tan(\theta - \alpha) \leq H \leq mg\tan(\theta + \alpha).$$

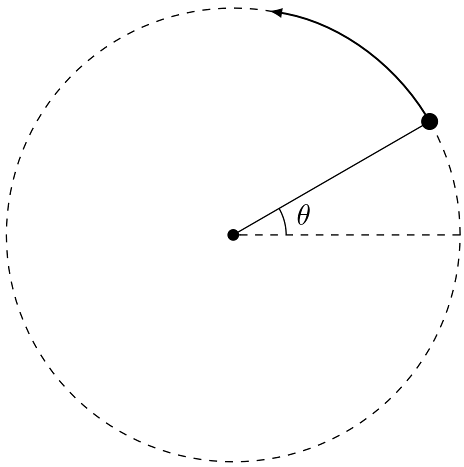
Question 15 [Glimpse into vector calculus]

Let $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{a} = \mathbf{a}(t)$ be the velocity and acceleration vectors respectively for a particle in motion. The speed of the particle at any time t is given by $|\mathbf{v}|$, and it is continuously changing as the particle moves around. Show that

$$\frac{d}{dt}(|\mathbf{v}|) = \frac{1}{|\mathbf{v}|}(\mathbf{v} \cdot \mathbf{a}).$$

Question 16 [Uniform circular motion]

The following diagram shows a particle travelling in a circle with constant angular velocity $\omega = \dfrac{d\theta}{dt}$, which is measured in radians per second.



The position of the particle at time t is given by the position vector

$$\underset{\sim}{r}(t) = (r\cos\omega t)\,\underset{\sim}{i} + (r\sin\omega t)\,\underset{\sim}{j}.$$

(a) Show that the velocity vector is given by

$$\dot{\underset{\sim}{r}}(t) = (-r\omega\sin\omega t)\,\underset{\sim}{i} + (r\omega\cos t)\,\underset{\sim}{j}.$$

(b) Show that the acceleration vector is given by

$$\ddot{\underset{\sim}{r}}(t) = (-r^2\omega\sin\omega t)\,\underset{\sim}{i} + (-r^2\omega\cos t)\,\underset{\sim}{j}.$$

(c) Show that at any time t, the linear speed of the particle is $v = r\omega$.

(d) Prove that the direction of the velocity vector $\dot{\underset{\sim}{r}}(t)$ is perpendicular to the direction of the position vector $\underset{\sim}{r}(t)$ at any time t.

(e) Show that at any time t, the linear acceleration of the particle is $r\omega^2$.

(f) Show that the direction of the acceleration vector $\ddot{\underset{\sim}{r}}(t)$ is opposite to the direction of the position vector $\underset{\sim}{r}(t)$.

(g) The *centripetal force* is the force that a particle in uniform circular motion continuously experiences in order to keep it in motion.

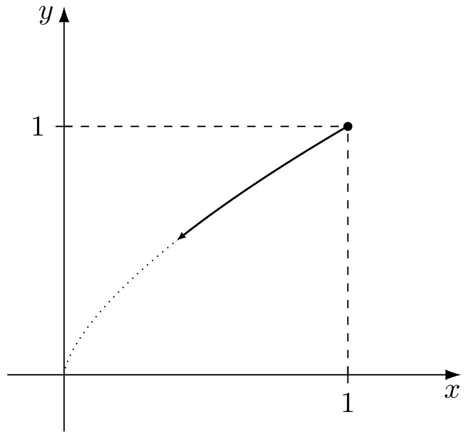
Show that the magnitude of centripetal force is given by $mr\omega^2$ or equivalently $\dfrac{mv^2}{r}$.

Challenge Problems

Problem 1 [Application to Kinetic Energy]

The following diagram shows a bead of mass m released from rest at $x = 1$ on the curve $y = \frac{3}{2}x^{\frac{2}{3}}$.

Let the position of the particle at time t be given by $\underset{\sim}{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.



The *kinetic energy* of a particle at time t is given by

$$E_k = \frac{1}{2}m|\dot{\underset{\sim}{r}}|^2.$$

The particle also experiences *gravitational potential energy*

$$E_p = mgh,$$

where h is the initial height of the particle from the ground.

(a) The *law of conservation of energy* states that the total energy of an isolated system remains constant. Use this to explain why

$$E = \frac{1}{2}m\left(|\dot{\underset{\sim}{r}}|^2 + 2gy\right)$$

is a constant function.

(b) Show that $\dot{x} = x^{\frac{1}{3}}\dot{y}$.

(c) Show that

$$\dot{y}^2 = 3g\left(\frac{3 - 2y}{3 + 2y}\right).$$

(d) Find how long it takes for the particle to fall to the ground.

Chapter 5 Review

Mechanics

Review

Question 1 A particle has acceleration equation

$$\ddot{x} = 4x + 2,$$

and is initially at the origin with velocity $1\,\mathrm{m\,s^{-1}}$.

(a) Show that the velocity is given by $\dot{x} = 2x + 1$.

(b) Find the time taken by the particle to attain a velocity of $16\,\mathrm{m\,s^{-1}}$.

Question 2 A particle has acceleration equation

$$a = x - 2,$$

and is initially at rest at $x = 3$.

(a) Show that the velocity is given by $v^2 = (x-1)(x-3)$.

(b) In what direction is the motion?

(c) Will the particle ever be at $x = 2$? Explain your answer.

(d) Find the position and acceleration of the particle when $v = 2\sqrt{2}$.

Question 3 A particle has acceleration equation

$$\ddot{x} = x - \frac{1}{x^3},$$

and is initially at $x = 1$ with velocity $v = 2\,\mathrm{m\,s^{-1}}$.

(a) Show that the velocity is $\dot{x} = x + \dfrac{1}{x}$.

(b) Hence, show that $x = \sqrt{2e^{2t} - 1}$.

Question 4 A particle has acceleration equation

$$\ddot{x} = -e^{-x} - e^{-2x},$$

and is initially at the origin with velocity $v = 2\,\mathrm{m\,s^{-1}}$.

(a) Show that the velocity is given by $\dot{x} = 1 + e^{-x}$.

(b) Hence, show that $x = \ln\left(2e^{t} - 1\right)$.

Question 5 Prove that the following particles are moving in simple harmonic motion.

(a) $x = 2\sin t$ (b) $x = 3\cos 2\pi t$ (c) $x = 1 - 2\sin 3t$

(d) $x = 4\sin\left(2t + \frac{\pi}{2}\right)$ (e) $x = 4\cos^2 t$ (f) $x = 3\cos 2t + 4\sin 2t$

(g) $x = 2 + 4\cos 3t + 6\sin 3t$ (h) $v^2 = 36 - 9x^2$ (i) $v^2 = 128 - 32x - 16x^2$

Question 6 A particle moves in simple harmonic motion with displacement-time equation

$$x = 4\cos\left(2t - \frac{\pi}{3}\right) + 2.$$

(a) Find the initial position of the particle.

(b) Find the first time when the particle passes through the origin.

(c) What is the speed of the particle when this occurs?

Question 7 A particle has velocity equation

$$\dot{x} = \sqrt{6x - x^2}.$$

(a) Show that the particle is moving in simple harmonic motion.

(b) Find the centre of motion, amplitude and period of the motion.

Question 8 A particle moves in simple harmonic motion and starts from rest at one of the end-points $x = 4$. The centre of motion is the origin. If $n = 3$, find

(a) when the particle first reaches $x = -2$.

(b) the speed when $x = -2$.

(c) how far the particle travels in the first $\frac{\pi}{6}$ seconds.

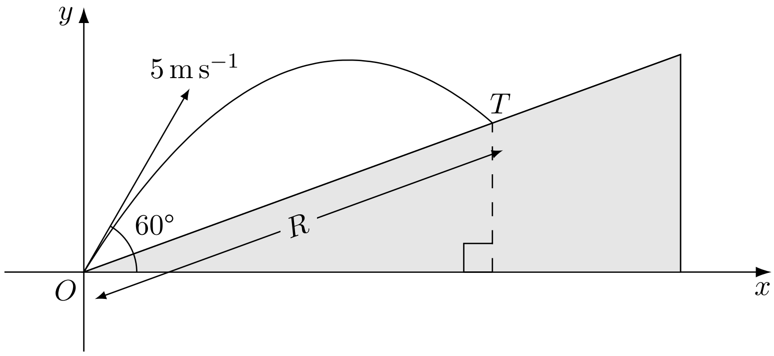
Question 9 The tides at a particular bay rise and fall in simple harmonic motion. Low tide is at 5am with a depth of 8 metres, while high tide is at 1pm with a depth of 12 metres. A ship needs at least 9 metres of water to pass through the bay safely. Let t be time in hours from 5am.

(a) Find the amplitude, period of motion, and the value of n.

(b) Write down an equation that models the above scenario.

(c) Hence, find the first range of times after 5am for when the ship can safely pass through the bay.

Question 10 A projectile is launched from the base O of a road inclined at an angle of α from the horizontal plane.



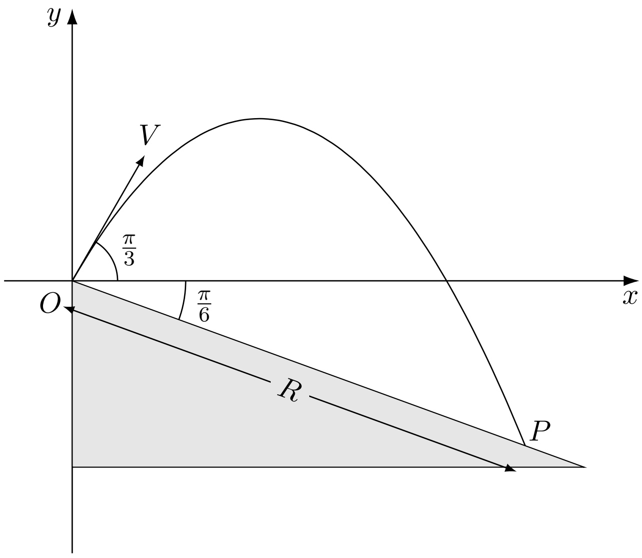
Initially, the particle has a speed of $5\,\text{m}\,\text{s}^{-1}$ and an angle of inclination of $60°$. Assume that acceleration due to gravity is $10\text{m}\,\text{s}^{-2}$.

(a) Show that the path of the trajectory is $y = x\sqrt{3} - \dfrac{4}{5}x^2$.

(b) Let $T(d, h)$ be the point on the road where the projectile lands. Show that $\tan\alpha = \dfrac{h}{d}$.

(c) Let R represent the distance of OT. Show that $R = \dfrac{5(\sqrt{3} - \tan\alpha)}{4\cos\alpha}$.

Question 11 A projectile is launched from the base O of a ramp that is inclined downwards at an angle of $\dfrac{\pi}{6}$ from the horizontal plane.



Initially, the particle has a speed of $10\,\text{m}\,\text{s}^{-1}$ and an angle of inclination of $\dfrac{\pi}{3}$. Assume that acceleration due to gravity is $10\text{m}\,\text{s}^{-2}$. Find the distance that the particle travels down the ramp.

Question 12 A particle is projected from the origin on level ground with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. Assume that acceleration due to gravity is $g\,\mathrm{m\,s^{-2}}$. Let R be the horizontal range attained by the particle when it lands.

(a) Show that $R = \dfrac{V^2 \sin 2\theta}{g}$.

(b) Hence, show that the Cartesian equation is $y = x\left(1 - \dfrac{x}{R}\right)\tan\theta$.

Question 13 A particle with unit mass is projected horizontally with initial velocity of $u\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $k(v+v^2)$, where v is the velocity of the particle after t seconds.

Show that the time when the particle has velocity v is given by $t = \dfrac{1}{k}\ln\left(\dfrac{v(1+u)}{u(1+v)}\right)$.

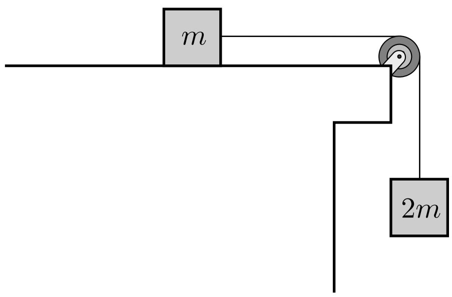
Question 14 A particle is projected vertically upwards with initial velocity u. When the particle falls back to the ground, it hits the original projection point with speed w. Let $g\,\mathrm{m\,s^{-2}}$ be the acceleration due to gravity.

(a) Prove that the maximum height of the particle is $H = \dfrac{1}{2k}\ln\left(1 + \dfrac{ku^2}{g}\right)$.

(b) Show that

$$(g + ku^2)(g - kw^2) = g^2.$$

Question 15 An object of mass m rests on a table and experiences a friction force $F = \mu N$. Another object of mass $2m$ is suspended off the edge of a table and is connected to the first object by a light inextensible string that runs through a smooth pulley.



(a) Prove that $T = \dfrac{2mg}{3}(1+\mu)$.

(b) Show that $a = \dfrac{g}{3}(2-\mu)$.

(c) Describe what happens physically if $\mu \geq 2$.

Investigation Task

Counter-intuitive results

Earlier in the chapter, we studied horizontal resisted motion for both cases when the resistance is proportional to velocity or proportional to the square of the velocity. This investigation task aims to shed some light on how mathematical modelling can be delicate, and how a poorly constructed mathematical model can cause confusion.

Question 1

(a) A particle of mass m kg is projected horizontally across two surfaces with initial speed $u\,\mathrm{m\,s^{-1}}$. One is made of ice, and the other is made of rubber. One of them has resistive force mkv and the other has resistive force mkv^2. Which surface do you think has which resistive force model? Explain your answer.

(b) Along which surface do you think the particle is more likely to travel further, given a sufficient amount of time?

(c) Show that for the surface with resistive force mkv, the displacement of the particle after t seconds is given by $x = \dfrac{u}{k}(1 - e^{-kt})$.

(d) Show that for the surface with resistive force mkv^2, the displacement of the particle after t seconds is given by $x = \dfrac{1}{k}\ln(1 + ukt)$.

Question 2 It appears that one of the surfaces allows for a limiting displacement, whereas the other one does not allow for that. Which surface did or did not allow for a limiting displacement to occur? Does this match your intuition?

Question 3 Justify your findings carefully, and de-mystify your findings.

Investigation Task

Uniform Circular Motion

When a particle moves in a circle at a constant angular speed, it is said to be moving in *uniform circular motion*. Many scenarios involving uniform circular motion can be analysed using techniques learned in this chapter. This investigation task allows students to explore Mechanics a little further for a scenario where the net force is non-trivial.

Question 1 Explain what centripetal force is, and provide a full derivation of the expression(s) for centripetal force for a particle travelling in uniform circular motion.

Question 2 What is providing the centripetal force in the following scenarios where uniform circular motion occurs?

(a) A car on a flat surface negotiating a bend.

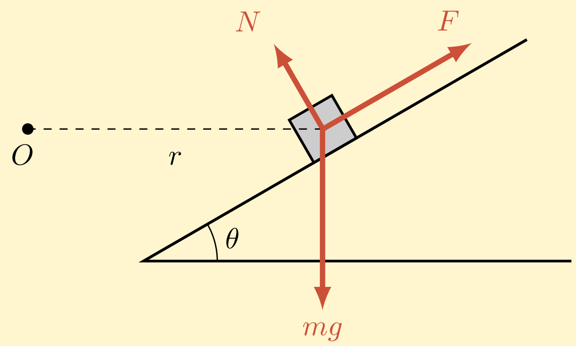
(b) A mass tied to the end of a string moving in a circle.

(c) A satellite orbiting the Earth.

(d) A car on a banked surface negotiating a bend.

(e) A mass tied to the top of a cone by a rope, and rotating across the surface of the cone.

Question 3 A car negotiates a circular bend with radius r on a surface inclined at an angle of θ from the horizontal.



It experiences a normal force, a friction force, and a weight force.

(a) What is the 'optimum speed' defined to be?

(b) Resolve forces vertically and horizontally.

(c) Resolve forces parallel and perpendicular to the plane.

(d) Prove that the optimum speed is given by $v^2 = gr\tan\theta$.

(e) Explain how the speed of the car is related to the direction of friction.

(f) Let $F = \mu N$. What happens if $\mu \geq \tan\theta$?

Investigation Task

How long does it take to fall?

In Problem 1 of Exercise 5G, a guided problem was given to allow students to calculate the amount of time that it takes for a particle to fall to the ground given a release point on a particular curve.

The task is to repeat the same problem, but with three different curves and release points of your choice. For example, you may like to use one quadratic function, one exponential function and one log function. More marks may be awarded for the following.

- Choosing curves and release points that result in neat values throughout the calculations.
- Choosing curves that require more advanced techniques from Mathematics Extension 2 Integration in order to solve for time.

1. The Nature of Proof

Exercise 1A
Language of Proof

F1
(a) propositions
(b) implication, conditional
(c) converse, p
(d) if and only if

F2
(a) opposite (b) Not (c) $\sim p$

F3
I did not do well in my exam.

Q1
(a) If Bob studies for his exam, he will get 99.95 ATAR.
(b) If Bob got 99.95 ATAR, then he studied for his exams.
(c) If Bob doesn't study for his exams, he will get 99.95 ATAR.
(d) If Bob got 99.95 ATAR, then he didn't study for his exams.
(e) Bob studied for his exam, but didn't get 99.95 ATAR.
(f) If Bob doesn't study for his exams, then Bob won't get 99.95 ATAR.

Q2
(a) $p \to q$ (b) $q \to p$
(c) $p \wedge \sim q$ (d) $\sim q \wedge p$
(e) $\sim p \wedge \sim q$ (f) $\sim q \to \sim p$

Q3
(a) The cat is not black.
(b) A cat that does not have a tail.
(c) It doesn't have four legs, but it is a cat.
(d) I studied, but did not do well.

Q4
(a) Bob dislikes either swimming or running.
(b) Bob likes neither swimming nor running.

Q5
(a) $a \geq b$
(b) $x \notin [a, b]$
(c) $\exists n \in \mathbb{Z}$, $2n + 1$ not odd.
(d) 5 divides $2n + 1$ for no values of n.
(e) $\exists m \in \mathbb{Z}$ such that $n \neq m^2$, $\forall n \geq m$.
(f) $\nexists x \in \mathbb{R}$ such that $\sin x = 0$.

Q6
(a) If I am in Sydney, then I am in NSW.
(b) If I am not in NSW, then I am not in Sydney.
(c) It is true!

Q7
The negation of a conditional statement is a counter-example. The converse of a statement is just reading it the other way around.

Q8
(a) If a number is divisible by 6, then it is divisible by both 2 and 3.

If a number is divisible by both 2 and 3, it is divisible by 6.
(b) If a quadratic has two real roots, then $\Delta > 0$.

If $\Delta > 0$, then the quadratic has two real roots.
(c) If $x = a$ is a zero, then $(x - a)$ is a factor.

If $(x - a)$ is a factor, then $x = a$ is a zero.

Q9
(a) If $a + b$ is even, then a and b are odd. Not true since a and b could both be even.
(b) If a^2 is odd, then a is odd. True.
(c) If $a^2 > b^2$, then $a > b$. Not true for $a = -2$, $b = 1$.
(d) If $a + b \in \mathbb{Q}$ then $a, b \in \mathbb{Q}$. Not true for $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$.
(e) If ab is even, then a and b are even. Not true since we only need at least one of them to be even, but not necessarily both.
(f) If a number is odd, then it ends with the digit 5. Not true. Consider 23.

Q10

(a) If it has perpendicular diagonals, then it is a rhombus. Not true because it could be a kite.

(b) If exactly two sides of the triangle are equal, then exactly two angles of the triangle are equal. True.

(c) If the triangle is right-angled, then $a^2 + b^2 = c^2$ where c is the length of the hypotenuse. True.

Q11

(a) If $f(a)$ does not exist, then $f(x)$ has a vertical asymptote at $x = a$. Not true since $x = a$ could be a 'hole' discontinuity.

(b) If $f(x) = b$ has no solutions, then $f(x)$ has a horizontal asymptote at $y = b$. Not true like how $y = x^2$ can never be equal to $y = -1$.

(c) If $f''(a) = 0$, then there is a point of inflexion at $x = a$. Not true for $f(x) = x^4$ at $x = 0$.

(d) If there is a stationary point at $x = a$, then $f'(a) = 0$. True.

Q12

(a) $\forall z \in \mathbb{C}, \exists \bar{z} \in \mathbb{C}$ such that $z + \bar{z} \in \mathbb{R}$.

(b) $\forall a, b \in \mathbb{Z}, a \neq b, \exists r \in \mathbb{Q}$ such that either $a < r < b$ or $b < r < a$.

(c) $\forall r \in \mathbb{R}, \exists q \in \mathbb{Q}$ such that $|r - q| \leq d, d \in \mathbb{Q}$.

Q13

(a) For all $m > n$, there exists some real number a such that $m - a = n$. Basically it means that if one number is larger than another, there exists a number that 'fills the gap' between them.

(b) If $P(x)$ is a polynomial with integer coefficients and there exists some integer zero a, then a is a factor of c_0. Basically this is the integer root theorem.

(c) If n is not a perfect square, $\sqrt{n}$ is not rational.

(d) If z is a non-zero complex number, then there exists some complex number w such that $zw = 1$.

(e) If a is a positive real number, then $a + \dfrac{1}{a}$ is at least 2.

P1

(a) The forward direction is false.
Let $\alpha = \bar{\beta} = 2 + i$. The converse is true.

(b) Neither direction is true.

(c) Neither direction is true.

P2

A function f approaches a limit L near $x = a$ if we can make f as close as we like to L by having x sufficiently close to a.

This is also known as the 'Epsilon-delta definition of the limit'. Almost every student studying their first mathematical analysis course at university will be confused by this for at least half of their degree.

P3

It's the complex conjugate root theorem.

Exercise 1B

Direct Proof

F1

(a) Prove that $n = 2m$ for $m \in \mathbb{Z}$.

(b) Prove that $n = 2m + 1$ or $n = 2m - 1$ for $m \in \mathbb{Z}$.

(c) $n = ma$ for $m \in \mathbb{Z}$.

Q1

(a) Let $m = 2a$, $n = 2b$.

(b) Let $m = 2a + 1$, $n = 2b + 1$.

(c) Let $m = 2a$, $n = 2b$.

(d) Let $m = 2a + 1$, $n = 2b + 1$.

Q2

(a) Let $n = 2m$.

(b) Let $n = 2m$.

(c) Let $n = 2m + 1$.

(d) Let $n = 2m + 1$.

Q3

For both parts, let $m = 2a$ and $n = 2b + 1$.

Q4

(a) Consider two cases where n is even or n is odd. Alternatively, recognise that $n^2 + n = n(n+1)$, which is the product of two consecutive integers.

(b) Recognise that $n^3 - n = (n-1)(n)(n+1)$, which is the product of three consecutive integers. This necessarily contains one multiple of 2 and one multiple of 3.

(c) Prove both directions separately.

(d) Prove both directions separately.

Q5

$n^2 = (2m+1)^2 = 4m(m+1)+1$ but $m(m+1)$ is even so $n^2 = 4(2a)+1 = 8a+1$.

Q6

$n + (n+1) + (n+2) = 3n + 3 = 3(n+1)$

Q7

$4^n + 4^{n+1} = 4^n(1+4) = 5 \times 4^n$

Q8

Note that

$$\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6} = \frac{n^3 + 3n^2 + 2n}{6}$$

So basically, we just need to prove that the numerator is divisible by 6. But $n^3 + 3n^2 + 2n = n(n+1)(n+2)$ which is the product of three consecutive integers.

Q9

Recognise that $2n + 1 = n + (n+1)$.

Q10

$9^n - 1 = (8+1)^n - 1$ then expand. When the 1 cancels out, the remaining terms are multiples of 8.

Q11

$(2m+1)^2 - (2m-1)^2 = 8m$.

Q12

Let the number be $n = 100a + 10b + c$ where $a, b, c \in \mathbb{Z}$. We are given that $a + b + c = 3m$ where $m \in \mathbb{Z}$. Substitute back into n to eliminate c and factorise out 3.

Q13

Consider $(n+1)^2 - n^2 = 2n + 1$. But this represents every odd integer.

P1

For both parts, let $a = kn + b$.

P2

$n!$ is divisible by all the integers from 1 to n inclusive, and so $n! - 1$ is not divisible by any of those. So $n! - 1$ is either itself prime, or it has some prime factor greater than or equal to $n+1$ since it's not divisible by $1, 2, 3, \ldots, n$. Hence either $n! - 1 = p$ or $p \geq n+1$. Note also p cannot exceed $n! - 1$ since it is a prime factor. And so

$$(n+1) \leq p \leq (n! - 1)$$

and hence

$$n < p < n!$$

P3

Let the number be $1000a + 100b + 10c + d$ and the last three digits form the number $100b + 10c + d$. Prove both directions.

P4

$q = 7$

P5

(a) Expand the RHS.

(b) If n is even, the result is trivial since everything is even. If n is odd, let $n = 2k+1$ for 4^n only and use the previous identity.

P6

Haha good luck!

Exercise 1C

Contrapositive

F1

(a) $\sim p$

(b) equivalent

F2

(a) contrapositive

(b) contraposition

Q1

(a) If you cannot see me, then I cannot see you.

(b) If Bob does not drive to work, then it is not raining.

(c) If Mary does not attend, then Bob is not attending.

(d) If I am not late to school, then I did not miss my train.

(e) If it does not rain, then it is not cloudy.

(f) If it does not have four legs, then it is not a cat.

(g) If Bob did not do well, then he did not study for his exam.

(h) If you are not in NSW, then you are not in Sydney.

Q2

(a) If a number is even, then it does not end with 5.

(b) If $a + b$ is odd, then either a or b are even.

(c) If $a + \frac{1}{a} < 2$ then a is negative.

(d) If ab is not an integer, then either a or b is not an integer.

(e) If a quadrilateral is not a rhombus, then it is not a square.

(f) If a triangle does not have three equal sides, then not all angles are equal.

Q3

(a) Let $n = 2m$.

(b) odd, odd

Q4

See full worked solutions.

Q5

(a) The contrapositive is that if $n > 0$, then $n^2 + 5n > 0$.

(b) The contrapositive is that if $a < b$, then $a^3 + ab^2 < a^2b + b^3$.

Q6

The contrapositive is that if n is not divisible by 3, then n^2 is not divisible by 3.

Q7

The contrapositive is that if either a or b are divisible by 5, then ab is divisible by 5.

Q8

Prove both directions. One can be done directly and the other can be done by contraposition.

Q9

The contrapositive is that if $a + b$ is odd, then $a^2 + b^2$ is odd.

P1

The contrapositive is that if neither a nor b are greater than 10, then $a + b < 20$.

P2

The contrapositive is that if n is composite, then $2^n - 1$ is composite.

Exercise 1D
Proof by Contradiction

F1

(a) negation (b) false (c) true

Q1

(a) $\mathbb{Z}$ (b) factors
(c) $2q^2$, even, even (d) $\mathbb{Z}$
(e) $2n^2$, even, even (f) even, (b)
(g) $\mathbb{Z}$

Q2

See full worked solutions.

Q3

See full worked solutions.

Q4

See full worked solutions.

Q5

(a) $\mathbb{Z}$ (b) 3, 3^q (c) even, odd

Q6

See full worked solutions.

Q7

See full worked solutions.

Q8

See full worked solutions.

Q9

See full worked solutions.

Q10

See full worked solutions.

Q11

See full worked solutions.

P1

See full worked solutions.

P2

Let $a = 2m + 1$ and $b = 2n + 1$ and suppose $a^2 + b^2 = k^2$. Upon expansion k must be even so let $k = 2p$ and then obtain a contradiction.

P3

See full worked solutions.

P4

See full worked solutions.

P5

See full worked solutions.

P6

See full worked solutions.

Exercise 1E

Examples and Counter-examples

F1

negation

F2

Show that there does not exist any n such that the statement is true.

F3

Find some $n \in \mathcal{S}$ where the statement is not true. In other words, find a counter-example.

Q1

(a) $x = -1$
$\forall x > 0,\ 3x > 2x$

(b) $x = 2$
$\forall x \in \mathbb{R}$ where $x \neq 1,\ 2,\ 2x \neq 2^x$

(c) $x = 2$
$\forall x > 4,\ 3x < 2^x$

(d) $x = 0$
$\forall x \neq 0,\ ax = bx \Rightarrow a = b$

(e) $x = -1$
$\forall x \geq 0,\ \sqrt{x^2} = x$

(f) $x = \dfrac{2\pi}{3}$
$\forall x$ where $\cos x \geq 0,\ \sqrt{1 - \sin^2 x} = \cos x$

(g) $x = 1,\ y = -2$
$\forall x,\ y \in \mathbb{R},\ |x| + |y| \geq |x + y|$

(h) $x = -1$
$\forall x > 0,\ e^{\ln x} = x$

(i) $x = 2$
$\forall x \in [-1, 1],\ \sin\left(\sin^{-1} x\right) = x$

(j) $x = -1$
$\forall x > 0,\ \tan^{-1}(x) + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$

Q2

(a) $n = 2$ (b) $n = 4$

(c) $n = 41$ (d) $n = 7$

Q3

(a) $x = -2$

(b) $a = 12,\ b = 6,\ n = 36$

(c) $n = 6$

Q4

(a) $x = 2,\ y = 3$ (b) $x = y = \sqrt{2}$

Q5

(a) $f(x) = (x - a)^3$ (b) $f(x) = (x - a)^4$

Q6

(a) True. $\alpha^2 + \beta^2 + \gamma^2 < 0$ is a sufficient condition for non-real roots.

(b) False. Let n be a perfect square.

(c) True. The odd integers have remainders 1 or 3 when divided by 4.

(d) False. Let $a = b = n$.

(e) True. The signed area bounded by the graph of $f'(x)$ is positive, as $f'(x)$ is positive.

Q7

(a) The converse is false. Let $a = b = 2$.

(b) True.

(c) The converse is false. Let $a = 1, b = 2$.

(d) The converse is false. Consider 1.

Q8

(a) Let $P(x) = x^2, Q(x) = -x^2$.

(b) Let $a = b,\ P(x) = (x - a)$.

(c) Let $P(x) = \dfrac{x}{2} - 1$.

Q9

(a) False. Consider a kite.

(b) True.

(c) True.

Q10

(a) True.

(b) False. Let $f(x) = e^{-x}\sin x, b = 0$. Then $f(x)$ intersects $b = 0$ infinitely many times.

(c) False. Let $y = (x-a)^4$.

(d) True. Definition of a stationary point.

Q11

The claim is false. Let $n = 3, a = c = 2, b = 4$.

Q12

The claim is true. Let $n = 100a + b$ where $0 \le b \le 99$.

P1

(a) True. The condition cannot be satisfied by real roots.

(b) False. Let $\alpha = \bar{\beta} = 2 + i$

P2

(a) False. Let $y = (x-a)^2$.

(b) False. Let $y = (x-a)^2$.

P3

(a) True.

(b) False. Let $a = 0,\ b = 3,\ f(x) = x,\ g(x) = 1$.

(c) False. Let $n = -1$.

Exercise 1F

Algebraic Inequalities

F1

(a) $>$ (b) $>$

F2

(a) $\ge$ (b) $\le$ (c) $\ge$ (d) $\le$

F3

(a) $\ge$ (b) $\ge$ (c) $\ge$

F4

$\ge$

F5

positive

F6

$|x+y|$

Q1

(a) $x = a,\ y = b$ (b) $x = a^2,\ y = b^2$

(c) $x = \sqrt{a},\ y = \sqrt{b}$ (d) $x = \sqrt{a},\ y = \dfrac{1}{\sqrt{a}}$

(e) $x = \dfrac{1}{a},\ y = \dfrac{1}{b}$ (f) $x = \sqrt{\dfrac{a}{b}},\ y = \sqrt{\dfrac{b}{a}}$

Q2

See full worked solutions.

Q3

$x(x-1) + y(y-1) \ge 0$

Q4

(a) Use $a + b \ge 2\sqrt{ab}$ and $\dfrac{1}{a} + \dfrac{1}{b} \ge \dfrac{2}{\sqrt{ab}}$ then multiply.

(b) Expand and use $\dfrac{x}{y} + \dfrac{y}{x} \ge 2$.

Q5

(a) Prove LHS $-$ RHS ≥ 0.

(b) Instead prove $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \ge 4$.

(c) Instead prove $(a+b)^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) \ge 8$.

(d) Prove LHS $-$ RHS ≥ 0.

Q6

$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{a^2+b^2}\sqrt{x^2+y^2}\cos\theta \le 1$. But $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax + by$.

Q7

Expand LHS.

Q8

Prove that LHS $-$ RHS ≥ 0.

Q9

(a) Instead prove that $4(a^3+b^3) - (a+b)^3 \ge 0$.

(b) Divide both sides of the previous inequality by 2, the cube root both sides.

Q10

See full worked solutions.

ANSWERS

Q11
See full worked solutions.

Q12
See full worked solutions.

Q13
(a) Prove that RHS − LHS ≥ 0.
(b) Replace c with a^2 and d with b^2 from the previous inequality.

Q14
(a) $(a-b)^2 \geq 0$
(b) $a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$
(c) Add the three inequalities and divide both sides by 2.
(d) Replace a with xy, b with xz and c with yz.

Q15
(a) Usual proof as done earlier.
(b) $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac)$

Q16
(a) $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$
(b) $a + c \geq 2\sqrt{ac}$

$b + c \geq 2\sqrt{bc}$
(c) Multiply all three inequalities.

Q17
(a) Standard proof as done earlier.
(b) Expand the RHS.
(c) The result in (b) is positive because of part (a).
(d) Replace a with $\sqrt[3]{x}$ and similarly for b and c.

Q18
(a) $\left(\sqrt{\dfrac{a}{b}} - \sqrt{\dfrac{b}{a}}\right)^2 \geq 0$
(b) $\dfrac{a}{c} + \dfrac{c}{a} \geq 2$

$\dfrac{b}{c} + \dfrac{c}{b} \geq 2$
(c) Multiply the LHS and apply (b).

Q19
(a) Prove that RHS − LHS ≥ 0.
(b) Split the numerator and use $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ repeatedly.
(c) Expand the LHS and use $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ repeatedly.
(d) $a(b-c)^2 + b(a-c)^2 + c(a-b)^2 \geq 0$

Q20
(a) $(\sqrt{p} - \sqrt{q})^2 \geq 0$
(b) Rewrite $\dfrac{a+b+c+d}{4}$ as $\dfrac{\frac{a+b}{2} + \frac{c+d}{2}}{2}$ and apply (a) repeatedly.

Q21
Let $d = \dfrac{a+b+c}{3}$ and raise both sides to the power of $\dfrac{4}{3}$.

Q22
Recall that $\log_b a = \dfrac{1}{\log_a b}$.

Q23
$1 + a \geq 2\sqrt{a}$ and similarly for b and c, then multiply.

Q24
(a) Standard proof as done earlier.
(b) Replace a with $\sqrt{\dfrac{x}{yz}}$ and similarly for the others.

Q25
See full worked solutions.

Q26
Instead prove that $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$, which was done earlier.

Q27
(a) Prove that LHS − RHS ≥ 0.
(b) Add three similar inequalities, factorise abc and then use $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ repeatedly.
(c) Suppose without loss of generality that a is largest.

$$\text{LHS} - \text{RHS} \geq \frac{S}{a}$$

where $S = a^3 + b^3 + c^3 - 3abc$.

P1

First prove that

$$a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ac}$$

and

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b} \geq a+b+c$$

and then combine the inequalities.

P2

Suppose without loss of generality that $a \geq b \geq c$. Observe that $b(b-a)(b-c)$ is the only negative term. Replace it with an 'even more' negative term $b(b-a)(a-c)$. Prove that the first two terms are positive. The third term is positive anyway.

P3

Expand the LHS and observe that you get n lots of '1' and a number of $\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}\right)$ pairs where $i \neq j$, each of which is at least 2. Precisely, there are $\binom{n}{2}$ such pairs so LHS $\geq 2 \times \binom{n}{2}+n=n^2$.

P4

Substitute $a=\frac{1}{2}(-x+y+z)$, $b=\frac{1}{2}(x-y+z)$ and $c=\frac{1}{2}(x+y-z)$ and use $\frac{p}{q}+\frac{q}{p} \geq 2$ repeatedly.

P5

(a) $P(x)$ is actually the sum of perfect squares, so geometrically it would either be a parabola on or above the x-axis i.e. it either has one real root or no real roots.

(b) $\Delta \leq 0$

(c) Let $a_k=\sqrt{\frac{S}{S-x_k}}$ and $b_k=\sqrt{\frac{S-x_k}{S}}$.

Exercise 1G

Mathematical Induction

F1

(a) before

(b) initial, initial

(c) closed, closed

(d) closed, recurrence

F2

(a) LHS – RHS

(b) LHS – RHS ≥ 0, LHS $\geq$ RHS

F3

diagram

Q1

(a) LHS $=2^4=16$

RHS $=4^2=16$

(b) $(k+1)^2$

(c) k^2, $k-1$, 3, 4

Q2

See full worked solutions.

Q3

(a) $T_0=2 \times 3^0-1=1$, as expected.

(b) $2 \times 3^{k+1}-1$

(c) $3T_k+2$, 3^{k+1}

Q4

See full worked solutions.

Q5

See full worked solutions.

Q6

See full worked solutions.

Q7

See full worked solutions.

Q8

(a) LHS $=\frac{1}{1^2}=1$

RHS $=2-\frac{1}{1}=1$

(b) $2-\frac{1}{k+1}$

(c) 1

Q9

See full worked solutions.

Q10

(a) Prove that RHS – LHS > 0.

(b) See full worked solutions.

Q11

(a) Prove that LHS^2 – RHS$^2 > 0$.

(b) See full worked solutions.

Q12
See full worked solutions.

Q13
See full worked solutions.

Q14
See full worked solutions.

Q15
See full worked solutions.

Q16
See full worked solutions.

Q17
See full worked solutions.

Q18
See full worked solutions.

Q19
(a) See full worked solutions.
(b) $\frac{(2n)!}{n!n!} = \binom{2n}{n}$

Q20
See full worked solutions.

Q21
(a) See full worked solutions.
(b) $p_n - q_n\sqrt{2}$
(c) $\sqrt{2}$

Q22
See full worked solutions.

Q23
See full worked solutions.

Q24
See full worked solutions.

Q25
See full worked solutions.

P1
See full worked solutions.

P2
See full worked solutions.

P3
See full worked solutions.

P4
See full worked solutions.

P5
See full worked solutions.

P6
See full worked solutions.

P7
(a) Calculate both using binomial probability and use the symmetry of the binomial coefficient.
(b) See full worked solutions.

Exercise 1H
Inequalities using Differentiation

F1
(a) The function is increasing, but it could start negative.
(b) The function could start positive, but then it decreases.

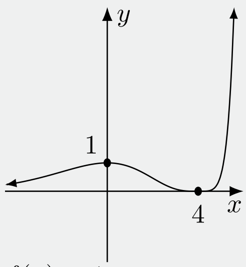
F2
b

Q1
(a) $f(x) = e^x - x - 1$
(b) $f(x) = x - \sin x$

Q2
See full worked solutions.

Q3
See full worked solutions.

Q4
(a) $(0, 1)$, $(4, 0)$
(b)



(c) $f(x) \leq 1$
(d) Substitute $x = 1$ and $x = -1$.

ANSWERS

Q5

(a) See full worked solutions.

(b) Let $x = \frac{\pi}{e} - 1$.

Q6

(a) $\left(e, \frac{1}{e}\right)$

(b) $f(\pi) < f(e)$

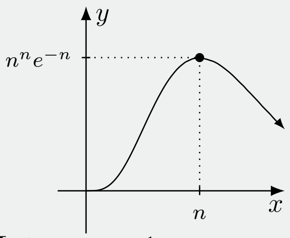
Q7

See full worked solutions.

Q8

(a) See full worked solutions.

(b)



(c) Let $x = n + 1$.

P1

(a) See full worked solutions.

(b) $\ln x \leq x - 1$ then substitute $x = np_k$ and then sum them.

(c) First note that $\sum_{k=1}^{n} \ln(np_k) \leq 0$, then use log laws and manipulate.

P2

(a) See full worked solutions.

(b) Let $x = \frac{a_k}{b}$ where $b = \frac{a_1 + a_2 + \cdots + a_n}{n}$.
Produce n such inequalities and multiply them.

Exercise 1I

Inequalities using Integration

F1

$L < \int_a^b f(x)\,dx < U$

F2

$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$

Q1

The area under the curve is between the rectangles.

Q2

The area under the curve is between the trapezium and rectangle.

Q3

The area under the curve is less than the area of the rectangle.

Q4

(a) The area under the curve is between the rectangles.

(b) See full worked solutions.

Q5

(a) The area under the curve is between the upper and lower rectangles.

(b) 7 and 9.

(c) No, since H_n grows asymptotically with $\ln n$ according to (a), and $\ln n$ diverges as $n \to \infty$.

Q6

(a) The area under the curve is between the upper and lower rectangles.

(b) 60525 and 60571.

Q7

(a) $1 \leq t + 1 \leq 2$ then flip everything.

(b) Integrate everything from 0 to x.

Q8

(a) $1 \leq 1 + t^2 \leq 2$ then flip everything.

(b) Integrate everything from 0 to x.

Q9

See full worked solutions.

Q10

See full worked solutions.

Q11

See full worked solutions.

Q12

See full worked solutions.

Q13

(a) The area under the curve is between the upper and lower rectangles.

(b) See full worked solutions.

(c) $\frac{1}{3}$

P1

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

P2

(a) The area under the curve is between the upper and lower rectangles. Then use log laws to form the factorials.

(b) Evaluate the integral in the double-inequality from the previous part, then algebraically manipulate the expressions.

(c) For most calculators you get `MATH ERROR` because the calculator tries to calculate the 100! first before taking the 100^{th} root.

(d) 37 and 39.

Chapter Review

R1

The statement is true.

R2

The statement is false for $a = 2$, $b = -2$.

R3

See full worked solutions.

R4

(a) The contrapositive is if n divides a or b, then n divides ab.

(b) The contrapositive is that if neither a nor b are irrational, then ab is rational.

(c) The contrapositive is that if 5 is either a factor of m or n, then 5 is a factor of mn.

(d) The contrapositive is that if a divides b, then a divides bc.

(e) The contrapositive is that if either a or b are odd, then either ab or $a + b$ are odd.

(f) The contrapositive is that if neither a nor b are divisible by 5, then ab is not divisible by 5.

R5

See full worked solutions.

R6

See full worked solutions.

R7

See full worked solutions.

R8

(a) False.	(b) True.	(c) False.
(d) True.	(e) False.	(f) False.
(g) True.	(h) False.	(i) False.
(j) True.	(k) True.	(l) False.

R9

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 \geq 0$$

R10

$\frac{x_k}{y_k} + \frac{y_k}{x_k} \geq 2$ and then sum n such inequalities.

R11

Prove that LHS $-$ RHS ≥ 0.

R12

(a) Prove that LHS $-$ RHS ≥ 0.

(b) Expand the RHS.

(c) Add three similar inequalities and use part the first inequality.

R13

See full worked solutions.

R14

See full worked solutions.

R15

See full worked solutions.

R16

See full worked solutions.

R17

See full worked solutions.

R18

See full worked solutions.

R19

(a) $f(x) = \tan x - x$

(b) $f(x) = x - \ln\left(1 + x + \frac{x^2}{2}\right)$

R20

See full worked solutions.

R21

First show that

$$1 - t \leq \frac{1}{1+t} \leq 1 - t + t^2$$

and then integrate each expression from 0 to x.

R22

(a) The area under the curve is between the upper and lower rectangles.

(b) 87 and 89.

2. Complex Numbers

Exercise 2A

Arithmetic of Complex Numbers

F1

(a) real, $\mathrm{Re}\,(z)$ (b) imaginary, $\mathrm{Im}\,(z)$

(c) imaginary (d) real

(e) $a - ib$ (f) $\sqrt{a^2 + b^2}$

F2

(a) $\overline{z} + \overline{w}$ (b) $(\overline{z})(\overline{w})$ (c) $\dfrac{(\overline{z})}{(\overline{w})}$

F3

(a) $2\mathrm{Re}\,(z)$ (b) $2i\,\mathrm{Im}\,(z)$ (c) $|z|^2$

Q1

(a) -1 (b) $-i$ (c) 1 (d) i

(e) $-i$ (f) 1 (g) i (h) $(-1)^n$

Q2

(a) $5 - 3i$ (b) $1 - 5i$ (c) $10 - 5i$

(d) $3 + 4i$ (e) 5 (f) $\sqrt{5}$

Q3

(a) $2 - 3i$ (b) 4

(c) $6i$ (d) 13

Q4

(a) i (b) $2 - 3i$ (c) $3 + 4i$

Q5

(a) $x = 4,\ y = 6$ (b) $x = 2,\ y = 5$

(c) $x = 3,\ y = -4$ (d) $x = 4,\ y = 3$

Q6

$z = -1 + i$

Q7

See full worked solutions.

Q8

Let $z = x + iy$ and expand/simplify the left-hand side.

Q9

Cross multiply and use the fact that $z + \overline{z} = 2\mathrm{Re}\,(z)$ and $z\overline{z} = |z|^2$.

Q10

Let $z = x + iy$ and expand/simplify.

Q11

(a) $a^2 - b^2 + 2abi$

(b) $\dfrac{a}{a^2 + b^2} - \dfrac{bi}{a^2 + b^2}$

(c) $\dfrac{a^2 + b^2 - 1}{(a-1)^2 + b^2} - \dfrac{2bi}{(a-1)^2 + b^2}$

Q12

See full worked solutions.

Q13

$k = \pm 1$

Q14

Let $\mathrm{Re}\,(z^2) = 0$ and show that $|x| = |y|$.

Q15

$(z-1)(\overline{z-1}) = 1$ and then use conjugate properties and expand.

Q16

$|z + w|^2 = (z + w)(\overline{z + w})$ and then use conjugate properties and expand. Similarly for $|z - w|^2$.

P1

(a) Let $z = x + iy$ and let $\mathrm{Im}\left(\dfrac{z-1}{z+1}\right) = 0$ to show that $y = 0$.

(b) Let $z = x + iy$ so $x^2 + y^2 = 1$. Show that $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right) = 0$.

P2

Square both sides, then use conjugate properties.

ANSWERS

P3

(a) $|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x|$

(b) Consider $|z + w|^2$, use conjugate properties and expand.

P4

See full worked solutions.

P5

Divide the LHS by $|z_1 z_2 z_3| = 1$, then recognise that since $|z_k| = 1$ then $\frac{1}{z_k} = \overline{z_k}$.

P6

(a) Take the modulus of both sides

(b) $|z - \alpha| = \left| z - \frac{z}{1 + ki} \right|$ and similarly for $|z - \beta|$

(c) Consider the circle centred at the origin in the complex plane that passes through α and β. The tangents drawn from α and β intersect at z.

P7

(a) See full worked solutions.

(b) $z = 1 \pm i$

(c) See full worked solutions.

Exercise 2B

Solving and Factorising Quadratics

F1

(a) $2\text{Re}\,(\alpha)$

(b) $|\alpha|^2$

(c) $(z - (\alpha + \beta)z + \alpha\beta)$

F2

(a) $(z - \alpha)(z - \bar{\alpha})$

(b) $P(z) = z^2 - 2\text{Re}\,(\alpha)\,z + |\alpha|^2$

F3

(a) quadratic (b) discriminant

(c) square (d) sum/product

Q1

(a) $\pm(2 + i)$ (b) $\pm(3 - i)$ (c) $\pm(3 - 2i)$

(d) $\pm(1 - i)$ (e) $\pm(6 + i)$ (f) $\pm(5 - 4i)$

(g) $\pm(4 + 3i)$ (h) $\pm(2 + 5i)$ (i) $\pm(1 + 5i)$

Q2

(a) $z = \pm 4i$
$z^2 + 16 = (z - 4i)(z + 4i)$

(b) $z = -1 \pm 2i$
$z^2 + 2z + 5 = (z + 1 - 2i)(z + 1 + 2i)$

(c) $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
$z^2 + z + 1 = \left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

Q3

(a) $z = 1 \pm 2i$ (b) $z = -3 \pm 4i$

Q4

(a) $z = 1 + i,\ 2 - i$ (b) $z = -1 - 3i,\ 3 + i$

Q5

$k = -8 + i$

P1

(a) $\pm\frac{1 + i}{\sqrt{2}}$ (b) $\pm\frac{3 + i}{\sqrt{2}}$ (c) $\pm\frac{1 + 5i}{\sqrt{2}}$

P2

See full worked solutions.

P3

$z = \pm(2 + i),\ \pm(1 - 2i)$

P4

Expand $(z - \alpha)(z - \bar{\alpha})$.

P5

$P(\alpha) = a\alpha^2 + b\alpha + c = 0$ then conjugate both sides to show that $P(\bar{\alpha}) = 0$.

Exercise 2C

Polar Form and the Argand Diagram

F1

(a) real, imaginary

(b) Argand

(c) real, imaginary

F2

(a) modulus, $a^2 + b^2$

(b) argument, $\frac{b}{a}$, $\theta \in (-\pi, \pi]$

(c) polar, $\text{cis}\,\theta$

F3

(a) $AB\,\text{cis}(\alpha+\beta)$ (b) $\frac{A}{B}\,\text{cis}(\alpha-\beta)$

F4

(a) $|z||w|$ (b) $\frac{|z|}{|w|}$

(c) $k|z|$ (d) $\arg z + \arg w$

(e) $\arg z - \arg w$ (f) 0

Q1

(a) $2\,\text{cis}\left(\frac{\pi}{3}\right)$ (b) $2\,\text{cis}\left(\frac{2\pi}{3}\right)$

(c) $2\,\text{cis}\left(-\frac{\pi}{3}\right)$ (d) $2\,\text{cis}\left(-\frac{2\pi}{3}\right)$

(e) $2\,\text{cis}\left(\frac{\pi}{6}\right)$ (f) $2\,\text{cis}\left(\frac{5\pi}{6}\right)$

(g) $2\,\text{cis}\left(-\frac{\pi}{6}\right)$ (h) $2\,\text{cis}\left(-\frac{5\pi}{6}\right)$

(i) $\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$ (j) $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

(k) $\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$ (l) $\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

(m) $3\,\text{cis}\left(\frac{\pi}{2}\right)$ (n) $4\,\text{cis}\left(-\frac{\pi}{2}\right)$

(o) $5\,\text{cis}(0)$ (p) $2\,\text{cis}(\pi)$

Q2

(a) $4\,\text{cis}\left(\frac{5\pi}{6}\right)$ (b) $2\sqrt{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)$

(c) $\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

Q3

(a) $1 + i\sqrt{3}$ (b) $-3 + 3\sqrt{3}i$

(c) $-1 - i$

Q4

(a) $2\,\text{cis}\left(\frac{\pi}{3}\right)$ (b) $2\,\text{cis}\left(\frac{5\pi}{6}\right)$

(c) $\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ (d) $2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$

Q5

(a) $\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ (b) $\frac{1}{\sqrt{2}}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

(c) $\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ (d) $2\,\text{cis}\left(\frac{3\pi}{2}\right)$

Q6

(a) $12\,\text{cis}\left(\frac{\pi}{2}\right)$ (b) $12\,\text{cis}\left(-\frac{\pi}{6}\right)$

(c) $3\,\text{cis}\left(\frac{11\pi}{12}\right)$ (d) $2\,\text{cis}\left(-\frac{7\pi}{12}\right)$

Q7

(a) $\frac{1}{2}\,\text{cis}\left(\frac{\pi}{6}\right)$ (b) $\text{cis}\left(\frac{\pi}{6}\right)$

(c) $\frac{1}{\sqrt{2}}\,\text{cis}\left(-\frac{\pi}{12}\right)$ (d) $\frac{1}{\sqrt{2}}\,\text{cis}\left(\frac{11\pi}{12}\right)$

Q8

(a) $5 + 5i$

(b) $\arg\big((2+i)(3+i)\big) = \arg(5+5i)$

Q9

(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) $\sqrt{3}$

(d) 1 (e) $-\frac{2\pi}{3}$ (f) $-\frac{11\pi}{12}$

(g) $-\frac{5\pi}{6}$ (h) 0

Q10

(a) $(1-\sqrt{3}) + i(1+\sqrt{3})$

(b) Equate real and imaginary components of both expressions for zw.

Q11

See full worked solutions.

Q12

See full worked solutions.

Q13

Use $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ and

$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)$

P1

Let $z = R\,\text{cis}\,\theta$

P2

(a) Add z and w, and then use the auxiliary angle formulae.

(b) 2, $\frac{\pi}{4}$

ANSWERS

P3

(a) $\left(\frac{a}{b}+\frac{b}{a}\right)i$

(b) Calculate $\arg(zw)$ using two different methods.

P4

Use $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$

P5

See full worked solutions.

P6

See full worked solutions.

P7

See full worked solutions.

Exercise 2D

Vector Representation

F1

(a) $b - a$

(b) $\overrightarrow{AB}$.

(c) position

F2

(a) Lengthen $\overrightarrow{OP}$ by a factor of 2.

(b) Rotate $\overrightarrow{OP}$ anti-clockwise by α.

(c) Rotate $\overrightarrow{OP}$ clockwise by α.

(d) Rotate $\overrightarrow{OP}$ anti-clockwise by $\frac{\pi}{2}$.

Q1

(a) $-8 - i$ (b) $8 + i$

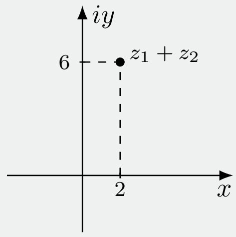
(c) $1 - 5i$ (d) $-9 + 4i$

Q2

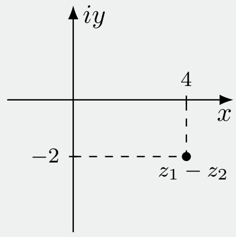
(a) $11 + 6i$ (b) $-11 + 8i$ (c) $7 + 2i$

Q3

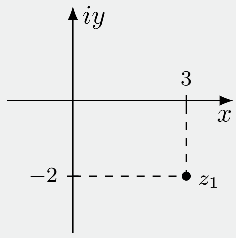
(a) $2 + 6i$



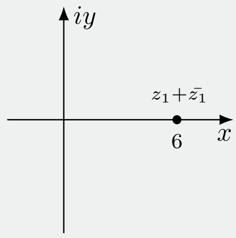
(b) $4 - 2i$



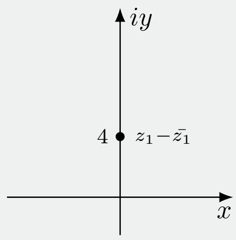
(c) $3 - 2i$



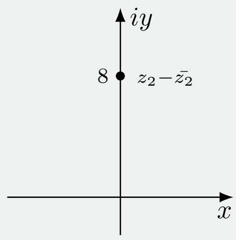
(d) 6



(e) $4i$

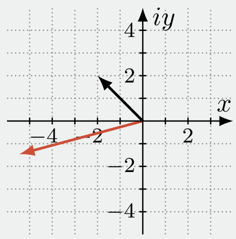


(f) $8i$

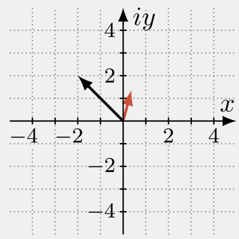


Q4

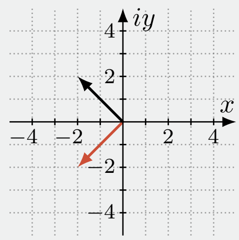
(a)



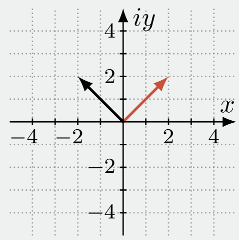
(b)



(c)



(d)



Q5

(a) $2\operatorname{cis}\left(\frac{5\pi}{6}\right)$ (b) $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$ (c) $\operatorname{cis}\left(-\frac{\pi}{2}\right)$

Q6

(a) Double the length, rotate anti-clockwise by $\frac{\pi}{3}$.

(b) Halve the length, rotate clockwise by $\frac{\pi}{4}$.

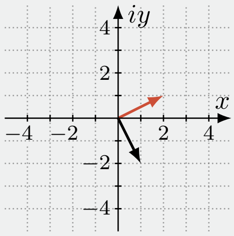
(c) Keep the length, rotate clockwise by $\frac{3\pi}{4}$.

Q7

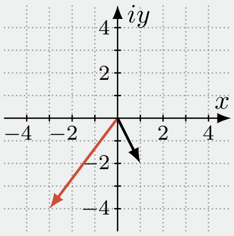
$-6+4i$

Q8

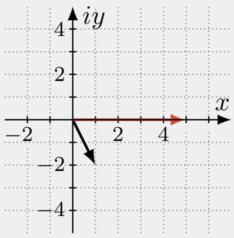
(a)



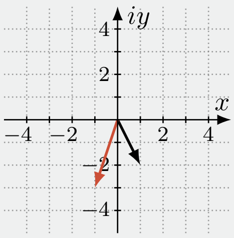
(b)



(c)



(d)

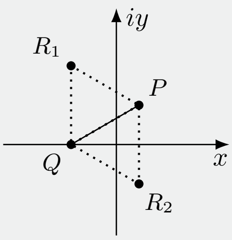


Q9

$B=-\frac{1}{2}+\frac{5}{2}i,\ D=\frac{9}{2}+\frac{1}{2}i$

Q10

(a)

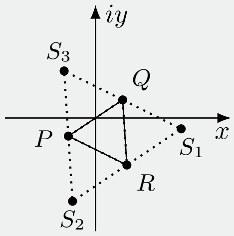


(b) $R_1=-2+2\sqrt{3}i$ or $R_2=1-i\sqrt{3}$

Q11

(a) $R=4\sqrt{3}-6\sqrt{3}i$

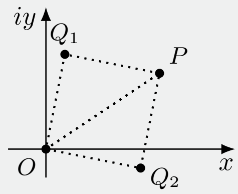
(b)



(c) $S_1=(12+4\sqrt{3})+i(8-6\sqrt{3})$
$S_2=(-12+4\sqrt{3})-i(8+6\sqrt{3})$
$S_3=-4\sqrt{3}+6\sqrt{3}i$

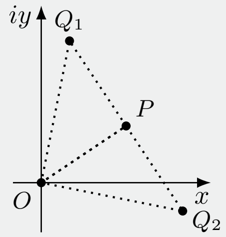
Q12

(a)



$Q_1=1+5i$
$Q_2=5-i$

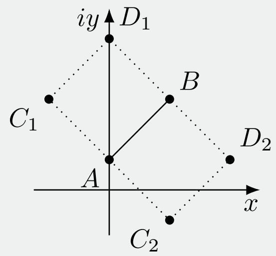
(b)



$Q_1=2+10i$
$Q_2=10-2i$

Q13

(a)



(b) $C_1=-4+6i,\ D_1=10i$
$C_2=4-2i,\ D_2=8+2i$

(c) $C=6i,\ D=4+2i$

Q14

$B=\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right),\ C=\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

Q15

(a) $\overrightarrow{OW} \times \text{cis}\left(\frac{\pi}{2}\right) = \overrightarrow{OZ}$ so $w = iz$. The other result is because z and w could be swapped. The diagram shows $z = iw$.

(b) Square both sides of either result, then re-arrange.

Q16

$w = z \times 2\,\text{cis}\left(\frac{\pi}{4}\right)$, then raise both sides to the fourth power.

Q17

(a) $z = w \times \text{cis}\left(\frac{\pi}{3}\right)$, then cube both sides and re-arrange.

(b) $z^3 + w^3$

(c) $z + w \neq 0$ so it follows that $z^2 - zw + w^2 = 0$.

Q18

$\overrightarrow{AB} = i\overrightarrow{AC}$ so it forms a right-angled isosceles triangle.

Q19

See full worked solutions.

Q20

Find $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$ and show that it is either 0 or π.

Q21

See full worked solutions.

P1

(a) Rotate one of the sides of the triangle to get another side of the triangle.

(b) Produce a similar result, equate the $\text{cis}\left(\pm\frac{\pi}{3}\right)$ part and then re-arrange.

P2

Draw out z, w, $z + w$ and $z - w$ and it forms a rhombus. Notice that

$$\arg\left(\frac{z+w}{z-w}\right) = \arg(z+w) - \arg(z-w) = \pm\frac{\pi}{2}$$

since the diagonals are perpendicular, and so it is purely imaginary.

P3

(a) They form a rectangle.

(b) $\pm\frac{\pi}{2}$

P4

$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \pm\frac{\pi}{2}$ so it forms a right-angled triangle.

P5

See full worked solutions.

P6

See full worked solutions.

Exercise 2E

Locus

F1

(a) Argand

(b) set, line, curve, region

(c) locus

F2

(a) A circle centred at C with radius r.

(b) The interior, including the circumference, of a circle centred at C with radius r.

(c) The exterior, including the circumference, of a circle centred at C with radius r.

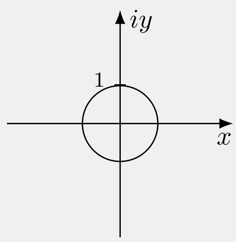
(d) The straight line that passes through the midpoint of AB and is perpendicular to AB.

(e) The half-plane containing A, bounded by the same straight line described in the previous part. The locus includes the boundary line itself.

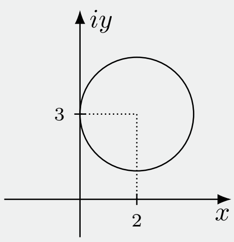
(f) The ray starting from C with an open circle, inclined at an angle of θ from the positive horizontal axis.

Q1

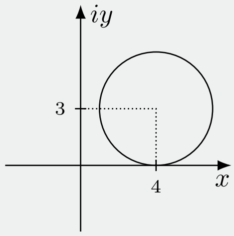
(a)



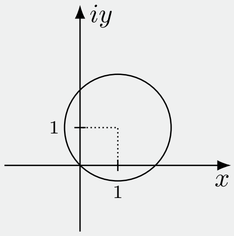
(b)



(c)

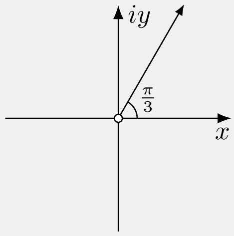


(d)

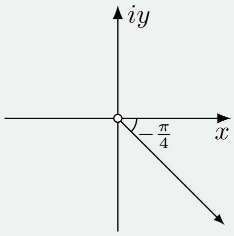


Q2

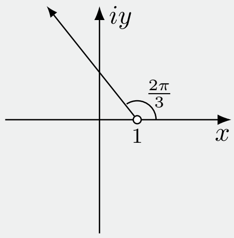
(a)



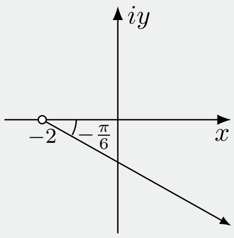
(b)



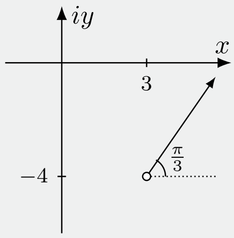
(c)



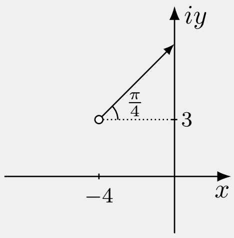
(d)



(e)

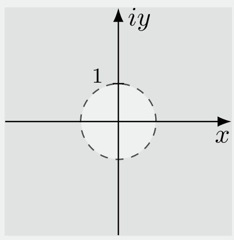


(f)

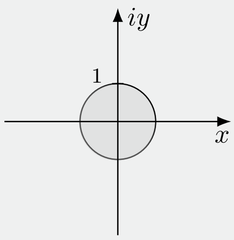


Q3

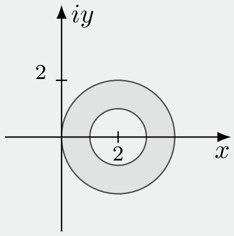
(a)



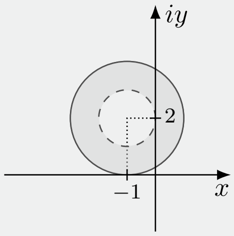
(b)



(c)

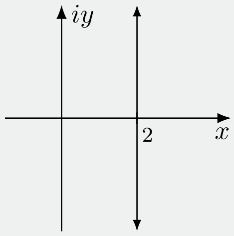


(d)

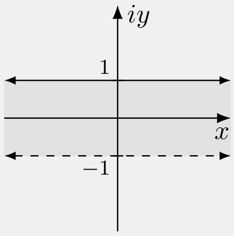


Q4

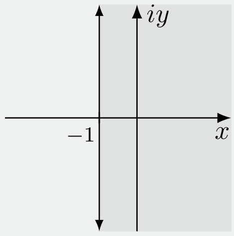
(a)



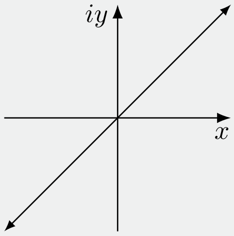
(b)



(c)

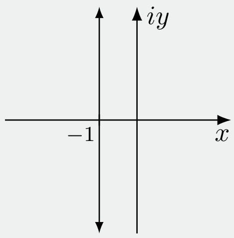


(d)

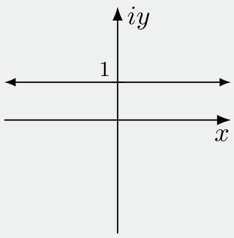


Q5

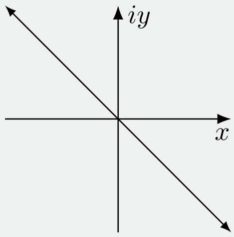
(a) $x = -1$



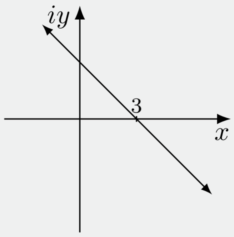
(b) $y = 1$



(c) $y = -x$

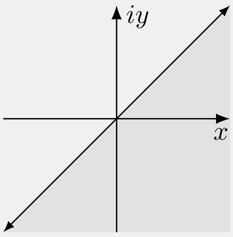


(d) $y = 3 - x$

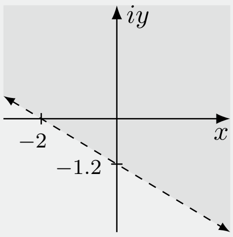


Q6

(a)

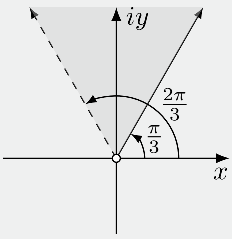


(b)

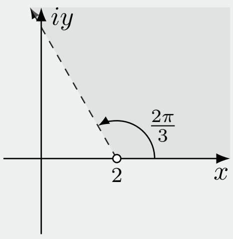


Q7

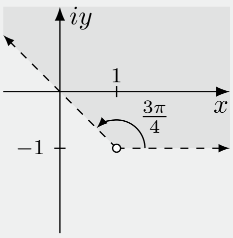
(a)



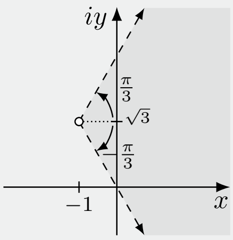
(b)



(c)

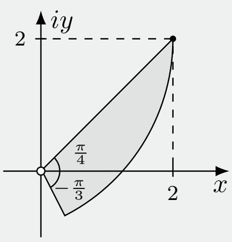


(d)

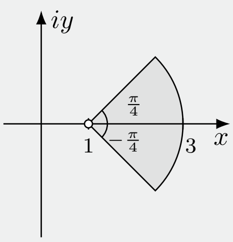


Q8

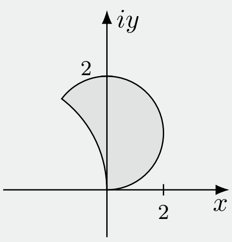
(a)



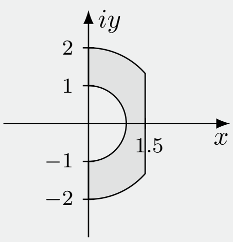
(b)



(c)



(d)



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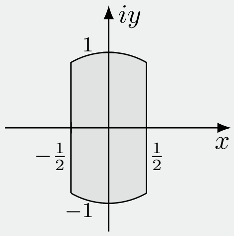
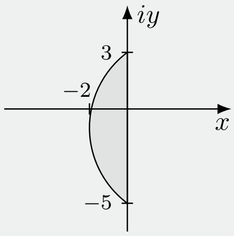
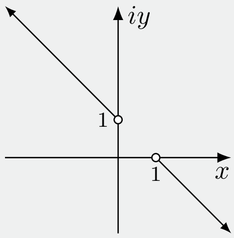


(f)

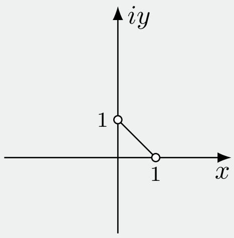


Q9

(a)

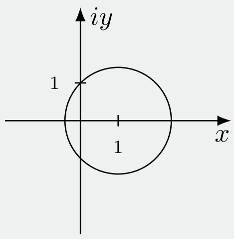


(b)

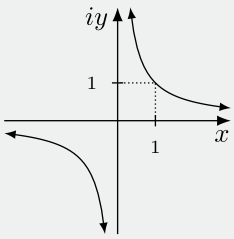


Q10

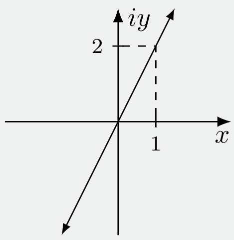
(a) $(x-1)^2 + y^2 = 2$



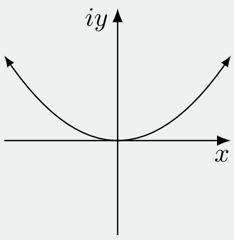
(b) $xy = 1$



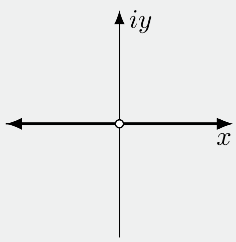
(c) $y = 2x$



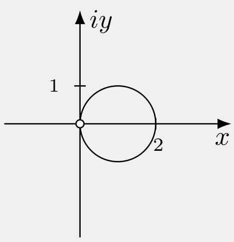
(d) $y = \frac{1}{4}x^2$



(e) $y = 0$ excluding $(0, 0)$.



(f) $(x-1)^2 + y^2 = 1$ excluding $(0, 0)$.



Q11

(a) $|z| > 2$

(b) $0 < \arg(z) < \dfrac{\pi}{3}$

Q12

(a) $1 + \sqrt{2}$

(b) $1 - \sqrt{2}$

(c) $\dfrac{\pi}{2}$

(d) 0

Q13

(a) 1

(b) 7

(c) $\dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{3}{4}\right)$

(d) $\dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{3}{4}\right)$

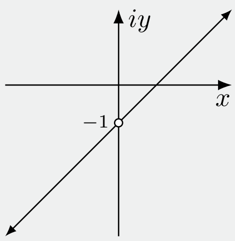
Q14

(a) $\left(x - \dfrac{5}{3}\right)^2 + y^2 = \dfrac{16}{9}$

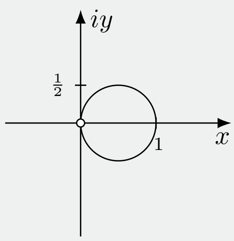
(b) $x^2 + y^2 = 1$

Q15

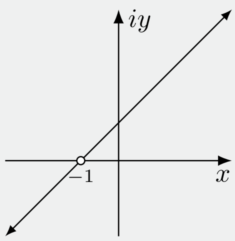
(a) $y = x - 1$ excluding $(0, -1)$



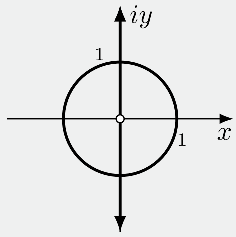
(b) $\left(x - \dfrac{1}{2}\right)^2 + y^2 = \dfrac{1}{4}$ excluding $(0, 0)$



(c) $y = x + 1$ excluding $(-1, 0)$

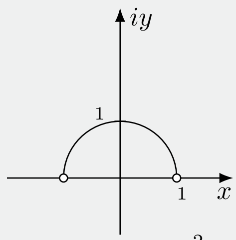


(d) $x = 0$ and $x^2 + y^2 = 1$ excluding $(0, 0)$

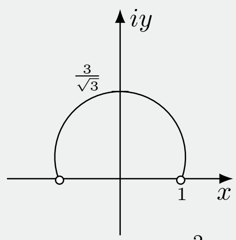


Q16

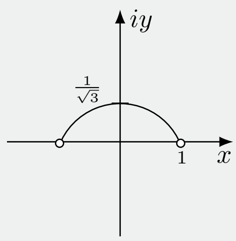
(a) $x^2 + y^2 = 1$, where $y > 0$.



(b) $x^2 + \left(y - \dfrac{1}{\sqrt{3}}\right)^2 = \dfrac{4}{3}$, where $y > 0$.



(c) $x^2 + \left(y + \dfrac{1}{\sqrt{3}}\right)^2 = \dfrac{4}{3}$, where $y > 0$.



Q17

$x \le -\dfrac{1}{2}$

Q18

$x = 0$ if $a > b$, and $x = a + b$ if $a < b$.

Q19

It is the circle with diameter being the interval connecting a to b.

P1

$w - a = \dfrac{b}{z}$ and take the modulus of both sides.

P2

(a) It is a straight line. More precisely, it is the perpendicular bisector of the interval from the origin to k on the complex plane.

(b) Let $w = \dfrac{1}{z_i}$, where $i = 1, 2, 3$.

P3

$(z+1)^n = -(z-1)^n$ then take the modulus of both sides. The locus is the imaginary axis and so the roots are imaginary.

P4

$x^2 + y^2 = 1$

P5

See full worked solutions.

Exercise 2F

De Moivre's Theorem

F1

$\cos n\theta$, $\sin n\theta$

F2

(a) $|z^n| = |z|^n$ (b) $\arg(z^n) = n \arg z$

(c) $\bar{z}^n$

Q1

(a) $\cos 6\theta + i \sin 6\theta$

(b) $\cos 5\theta - i \sin 5\theta$

(c) $\cos 6\theta - i \sin 6\theta$

(d) $\cos 3\theta + i \sin 3\theta$

Q2

(a) $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i$ (b) -1 (c) $-i$

Q3

(a) $|z| = 2$, $\arg z = \dfrac{\pi}{3}$

(b) $2 \operatorname{cis}\left(\dfrac{\pi}{3}\right)$

(c) $16 \operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$

(d) $-8 - 8\sqrt{3}i$

Q4

(a) $-4 - 4i$ (b) 64

(c) $8i$ (d) $-64\sqrt{3} + 64i$

(e) -4 (f) -512

(g) -64 (h) $24\sqrt{3}$

(i) $128 - 128i$

Q5

Multiply the top and bottom by $\cos\theta - i \sin\theta$

Q6

Use the fact that $\cos\theta - i\sin\theta = \cos(-\theta) + i \sin(-\theta)$

Q7

(a) $\cos n\theta$ (b) $\sin n\theta$

Q8

See full worked solutions.

Q9

$\cos 4\theta - i \sin 4\theta$

Q10

(a) $n = 6k$ where $k \in \mathbb{Z}$

(b) $n = 3 + 6k$ where $k \in \mathbb{Z}$

Q11

(a) $n = 4k$ where $k \in \mathbb{Z}$

(b) $n = 2 + 4k$ where $k \in \mathbb{Z}$

Q12

(a) $n = 3k$, where $k \in \mathbb{Z}$

(b) $n = \dfrac{3}{2} + 3k$, where $k \in \mathbb{Z}$

Q13

(a) Notice that z is in the form $z = \alpha^n + \bar{\alpha}^n = \alpha^n + \overline{\alpha^n} = 2\operatorname{Re}(\alpha^n)$

(b) No, because similarly it becomes $z = \alpha^n - \bar{\alpha}^n = \alpha^n - \overline{\alpha^n} = 2i\operatorname{Im}(\alpha^n)$ so it is actually purely imaginary.

(c) Use the fact that $z = 2^{\frac{n}{2}+1} \cos\left(\dfrac{n\pi}{4}\right)$

Q14

See full worked solutions.

Q15

(a) Turn everything to polar form then recognise that we are adding two complex conjugates.

(b) The cosine term is either $\pm\dfrac{1}{2}$, or ± 1.

Q16
$\cos n\theta + i \sin n\theta$

Q17

Use $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2 = 2\sin\theta\cos\theta$

Q18
Express each term with a common denominator and use de Moivre's theorem twice.

Q19
Use the sum of a geometric progression formula.

Q20
See full worked solutions.

P1
(a) Use the sum of a geometric progression formula.
(b) Factorise out $\operatorname{cis}\left(\frac{n\theta}{2}\right)$ and use the fact that $\alpha - \bar{\alpha} = 2i\text{Im}(\alpha)$
(c) Equate the imaginary components of the series expression for S_n, and the closed-form expression for S_n.

P2
(a) Use $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.
(b) Let $x = \cot\theta$ and solve $\sin n\theta = 0$.
(c) Use the sum of the squares of the zeroes of $P(x) = 0$ by first expanding using binomial expansions.
(d) Use $\cot^2\theta = \csc^2\theta - 1$

Exercise 2G
Applications of de Moivre's Theorem

F1
(a) $-\dfrac{b}{a}$ (b) $\dfrac{c}{a}$ (c) $-\dfrac{d}{a}$

F2
(a) $z = \cos\theta + i\sin\theta$
(b) de Moivre's
(c) Pascal's, binomial
(d) real
(e) $\cos\theta$, $\sin^2\theta = 1 - \cos^2\theta$

F3
(a) $\sin n\theta$ (b) divide
(c) cosine (d) $\tan\theta$

Q1
Find z^3 using two different methods and equate real/imaginary components.

Q2
Find z^3 using two different methods and equate real/imaginary components.

Q3
Find z^4 using two different methods and equate real/imaginary components.

Q4
Find z^5 using two different methods and equate real/imaginary components.

Q5
Find z^6 using two different methods and equate real/imaginary components.

Q6
(a) $z^n + z^{-n} = \operatorname{cis}(n\theta) + \operatorname{cis}(-n\theta) = 2\cos n\theta$
(b) Consider $\left(z + z^{-1}\right)^5$
(c) $\dfrac{8}{15}$

Q7
(a) Consider $\left(z + z^{-1}\right)^4$
(b) Replace θ with $\dfrac{\pi}{2} - \theta$ and use supplementary/complementary identities.
(c) Add the two previous results.

Q8
(a) Consider $\left(z + z^{-1}\right)^6$
(b) $\dfrac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right)$

Q9
(a) $z^n + z^{-n} = \operatorname{cis} n\theta + \operatorname{cis}(-n\theta) = 2\cos n\theta$
(b) $z = \pm i,\ -2 \pm \sqrt{3}$

ANSWERS

Q10

(a) See full worked solutions.

(b) $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

(c) $x = \cos\left(\frac{\pi}{9}\right), \cos\left(\frac{5\pi}{9}\right), \cos\left(\frac{7\pi}{9}\right)$

(d) 0

(e) $-\frac{4}{3}$

(f) $\frac{1}{8}$

Q11

(a) $x = \cos\left(\frac{2\pi}{9}\right), \cos\left(\frac{4\pi}{9}\right), \cos\left(\frac{8\pi}{9}\right) = -\cos\left(\frac{\pi}{9}\right)$

(b) Use the sum of roots.

(c) Use the product of roots.

Q12

(a) Consider $(\cos\theta + i\sin\theta)^4$

(b) $x = \pm\cos\left(\frac{\pi}{8}\right), \pm\cos\left(\frac{3\pi}{8}\right)$

(c) $\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$

$\cos\left(\frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$

Q13

(a) $t = -1, \tan\left(\frac{\pi}{12}\right), \tan\left(\frac{5\pi}{12}\right)$

(b) $\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}$

$\tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}$

Q14

(a) Consider $(\cos\theta + i\sin\theta)^5$

(b) $\cos\left(\frac{\pi}{10}\right), \cos\left(\frac{3\pi}{10}\right), \cos\left(\frac{7\pi}{10}\right), \cos\left(\frac{9\pi}{10}\right)$

(c) Solve the quartic in terms of x^2 by using the quadratic formula, and then take the square root again.

(d) $\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$

Q15

(a) Consider $(\cos\theta + i\sin\theta)^5$

(b) See full worked solutions.

(c) $P(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

(d) $a = 2$

(e) $\sin\left(\frac{\pi}{10}\right) = \frac{-1+\sqrt{5}}{4}$

Q16

(a) Consider $(\cos\theta + i\sin\theta)^6$

(b) $\pm\cos\left(\frac{\pi}{12}\right), \pm\cos\left(\frac{3\pi}{12}\right), \pm\cos\left(\frac{5\pi}{12}\right)$

(c) Use the product of roots.

(d) Use the sum of the squares of the roots.

(e) Use $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

Q17

(b) 1

(c) 14

(d) 4

(e) $x^2 - 4x + 1 = 0$

(f) $\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}$

$\tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}$

Q18

(a) Obtain $\sin 5\theta$ and $\cos 5\theta$ by usual means, then divide.

(b) $t = \pm\tan\left(\frac{\pi}{5}\right), \pm\tan\left(\frac{2\pi}{5}\right)$

(c) Use the product of the roots.

(d) Find the sum of the squares of the roots first.

(e) Construct a quadratic whose roots are $\tan\left(\frac{\pi}{5}\right)$ and $\tan\left(\frac{2\pi}{5}\right)$, then solve.

(f) $\sqrt{5+2\sqrt{5}}$

P1

Let $z = r\operatorname{cis}\theta$ and consider the limiting sum formula

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

then equate real components.

P2

(a) Use de Moivre's theorem.

(b) Expand $(z + z^{-1})^{2n}$.

(c) Integrate both sides. The right-hand side has all cosine terms except for the last term $\binom{2n}{n}$. The cosine terms integrate to zero but the last term survives.

P3

(a) $(z^2 + 1)^n$

(b) See full worked solutions.

(c) See full worked solutions.

Exercise 2H
Roots of Unity

F1

(a) 1

(b) principal, argument

F2

(a) 1 (b) 0

F3

(a) 1 (b) 0

F4

(a) a (b) polar

(c) $2k\pi$ (d) $\frac{1}{n}$, principal

(e) conjugate

Q1

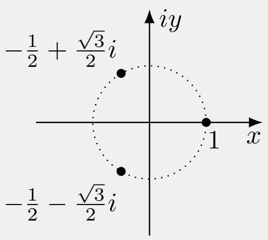
(a) 1

(b) $w^3 = 1$ so $w^3 - 1 = 0$ and then factorise.

Q2

(a) $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

(b)



(c) $P(\omega^2) = \omega^6 - 1 = 0$

(d) $\omega^3 = 1 = \omega\overline{\omega}$, then divide both sides by ω.

Q3

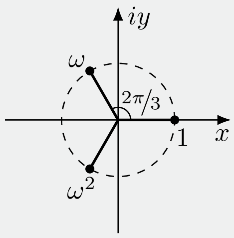
(a) 1 (b) 9 (c) 1 (d) ω

Q4

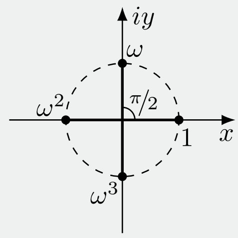
$P(x) = x^2 - x + 1$

Q5

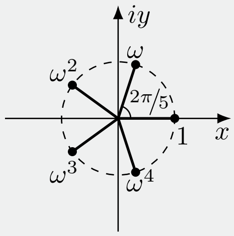
(a)



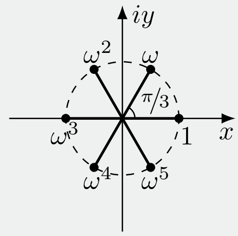
(b)



(c)



(d)



Q6

(a) $-1, \pm \operatorname{cis}\left(\frac{\pi}{3}\right)$

(b) $3i, 3\operatorname{cis}\left(-\frac{\pi}{6}\right), 3\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

(c) $2\operatorname{cis}\left(\frac{k\pi}{9}\right)$ where $k = -5, 1, 7$

Q7

(a) $\operatorname{cis}\left(\pm\frac{\pi}{4}\right), \operatorname{cis}\left(\pm\frac{3\pi}{4}\right)$

(b) $2\operatorname{cis}\left(\frac{k\pi}{8}\right)$ where $k = -7, -3, 1, 5$

(c) $\operatorname{cis}\left(\frac{k\pi}{24}\right)$ where $k = -19, -7, 5, 17$

Q8

(a) $1, \operatorname{cis}\left(\pm\frac{2\pi}{5}\right), \operatorname{cis}\left(\pm\frac{4\pi}{5}\right)$

(b) $-2i, 2\operatorname{cis}\left(\frac{k\pi}{10}\right)$ where $k = -9, -1, 3, 7$

(c) $2\,\text{cis}\left(\frac{k\pi}{20}\right)$ where $k = -17, -9, -1, 7, 15$

Q9

(a) $\pm 1, \text{cis}\left(\pm\frac{\pi}{3}\right), \text{cis}\left(\pm\frac{2\pi}{3}\right)$

(b) $\text{cis}\left(\frac{k\pi}{12}\right)$ where $k = -9, -5, -1, 3, 7, 11$

(c) $2\,\text{cis}\left(\frac{k\pi}{36}\right)$ where $k = -25, -13, -1, 11, 23, 35$

Q10

(a) ω is a zero of $P(x) = x^n - 1$. Show that $P(\omega^k) = 0$.

(b) Use the sum of the zeroes.

Q11

(a) $\omega^n = 1 = \omega\bar{\omega}$

(b) $\omega^n = 1 = \omega^k\overline{\omega^k}$

(c) $(\omega, \omega^8), (\omega^2, \omega^7), (\omega^3, \omega^6), (\omega^4, \omega^5)$

Q12

(a) $\omega^5 - 1 = (\omega - 1)(1 + \omega + \omega^2 + \ldots + \omega^4) = 0$

(b) Use the sum of the roots.

Q13

(a) $\omega^9 - 1 = (\omega - 1)(1 + \omega + \omega^2 + \ldots + \omega^8) = 0$

(b) Use the sum of the roots.

Q14

(a) $-1, \text{cis}\left(\pm\frac{\pi}{7}\right), \text{cis}\left(\pm\frac{3\pi}{7}\right), \text{cis}\left(\pm\frac{5\pi}{7}\right)$

(b) Use the sum of the zeroes.

Q15

$x^2 + x + 2 = 0$

Q16

2

Q17

Instead solve $\left(\frac{z+1}{z}\right)^8 = 1$

P1

$1 + \omega^n + \omega^{2n} = 3$ when n is a multiple of 3, otherwise it is equal to zero.

P2

Expand it and simplify.

P3

Re-arrange to instead solve $\left(\frac{z+1}{z-1}\right)^n = -1$

P4

Group them in pairs

$$\frac{1}{1-\omega^k} + \frac{1}{1-\omega^{n-k+1}} = 1$$

and recognise that we are just adding n such pairs.

P5

$\frac{1}{2}$

P6

See full worked solutions.

Exercise 2I

Applications of Roots of Unity

F1

$z^2 - 2\text{Re}(\alpha)\,z + |\alpha|^2$

F2

(a) $1 + z + z^2 + z^3 + \cdots + z^{n-1}$

(b) $1 - z + z^2 - z^3 + \cdots + z^{n-1}$

F3

(a) integers (b) real

Q1

(a) $z^2 - 2z + 2$ (b) $z^2 - 6z + 13$

Q2

$\left(z^2 - 2\text{Re}(\alpha)\,z + |\alpha|^2\right)\left(z^2 - 2\text{Re}(\beta)\,z + |\beta|^2\right)$

Q3

Expand the right-hand side and simplify.

Q4

(a) $(z+1)(1 - z + z^2 - \cdots + z^6)$

(b) $(z-1)(1 + z + z^2 + \cdots + z^6)$

(c) $(z-1)(z+1)(1 + z^2 + z^4 + z^6)$

Q5

(a) $\text{cis}\left(\pm\frac{2\pi}{5}\right)$, $\text{cis}\left(\pm\frac{4\pi}{5}\right)$

(b) Use $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$

(c) Equate the coefficient of z.

(d) Equate the coefficient of z^2.

(e) $4x^2 + 2x - 1 = 0$

(f) $\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$,
$\cos\left(\frac{4\pi}{5}\right) = \frac{-1-\sqrt{5}}{4}$

(g) Substitute in $z = 1$ and use double-angle formulae.

Q6

Show that $P_n(\omega) = P_n(\omega^2) = 0$ so $(x-\omega)(x-\omega^2)$ must be a factor, but this expands to become (x^2+x+1).

Q7

(a) Use de Moivre's theorem.

(b) Find all the roots of $z^4 + 1 = 0$ and then use $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$ two times.

(c) Divide both sides by z^2 and use the fact that if $z = \text{cis}\,\theta$ then $z^n + z^{-n} = 2\cos n\theta$

Q8

(a) Use de Moivre's theorem.

(b) Find all the roots of $z^6 + 1 = 0$ and then use $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$ three times.

(c) Divide both sides by z^3 and use the fact that if $z = \text{cis}\,\theta$ then $z^n + z^{-n} = 2\cos n\theta$

Q9

(a) $P(z) = (z^3+1)(z^6 - z^3 + 1)$, so the zeroes of $z^6 - z^3 + 1$ are a subset of the zeroes of $P(z)$.

(b) $\text{cis}\left(\pm\frac{\pi}{9}\right)$, $\text{cis}\left(\pm\frac{5\pi}{9}\right)$, $\text{cis}\left(\pm\frac{7\pi}{9}\right)$

(c) Use the result $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$ three times.

(d) Divide both sides by z^3 and use the fact that if $z = \text{cis}\,\theta$ then $z^n + z^{-n} = 2\cos n\theta$

Q10

(a) $P(z) = (z^3-1)(z^6 + z^3 + 1)$, so the zeroes of $z^6 + z^3 + 1$ are a subset of the zeroes of $P(z)$.

(b) $\text{cis}\left(\pm\frac{2\pi}{9}\right)$, $\text{cis}\left(\pm\frac{4\pi}{9}\right)$, $\text{cis}\left(\pm\frac{8\pi}{9}\right)$

(c) Use the result $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$ three times.

(d) Substitute $z = 1$ into the previous identity and use double-angle formulae.

(e) Substitute $z = i$ to the same identity.

(f) Divide the two previous results, and note that $\cos\left(\frac{8\pi}{9}\right) = -\cos\left(\frac{\pi}{9}\right)$

P1

(a) Find explicit expressions for ω^k and ω^{k-1} and calculate.

(b) The perimeter is n lots of the side length.

(c) See full worked solutions.

(d) The circumference of the unit circle is 2π.

P2

(a) Use the distance formula.

(b) First show that $d_k^2 = 2 - 2\cos k\theta$ and then use the fact that $\omega^k + \bar{\omega}^k = 2\cos k\theta$

(c) Sum the d_k^2 terms and use the fact that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

Exercise 2J
Solving Polynomials

F1

conjugate

F2

$z^2 - 2\text{Re}(\alpha)\, z + |\alpha|^2$

F3

(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$ (c) $-\frac{d}{a}$

F4

(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$ (c) $-\frac{d}{a}$ (d) $\frac{e}{a}$

ANSWERS

Q1

(a) The coefficients of the polynomial equation are real, so we can use the conjugate root theorem.

(b) $x = -3$

(c) $P(x) = (x+3)(x-2-i)(x-2+i)$

Q2

(a) $x = 1 \pm i,\ -1$ (b) $x = \pm 2i,\ 3$

Q3

(a) $x = 1 - i$, since the coefficients are real so we can use the conjugate root theorem.

(b) $x^2 - 2x + 2$

(c) $x^2 + x - 12$

(d) $x = 1 \pm i,\ -4,\ 3$

Q4

(a) $x = 1 \pm 2i,\ 2 \pm i$

(b) $x = 1 \pm i,\ \pm 2i$

Q5

(a) $P(1+i) = P'(1+i) = 0$. Note that the easiest way to calculate $P(1+i)$ is to first convert to polar form.

(b) $x = 1 \pm i,\ 1 \pm i,\ 1$

(c) $P(x) = (x-1+i)^2(x-1+i)^2(x-1)$

(d) $P(x) = (x^2 - 2x + 2)(x-1)$

Q6

$x^3 - 2x^2 + x + 1 = (x-i)\big(x^2 - (2-i)x - 2i\big) + 3$

Q7

(a) See full worked solutions.

(b) Yes, the remainder theorem also works for complex numbers.

(c) $P(-i) = 3 + 3i$

Q8

(a) The remainder is at most one degree less than the divisor. Since the divisor is quadratic, the remainder is at most linear i.e. $ax + b$.

(b) $P(x) = (x^2+1)Q(x) + ax + b$

(c) $P(i) = -5 + 3i$
$a = 3,\ b = -5$

(d) $3x - 5$

(e) $p = -1,\ q = 4$

Q9

$7x + 1$

Q10

11

Q11

$x = -1 \pm i,\ -1 \pm 2i$

Q12

(a) $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

(b) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 0$ which cannot happen if the roots are all real.

(c) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$ does not guarantee that all four roots are real. For example if $\alpha = i$, $\beta = -i$ and $\gamma = \delta = 5$ then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 > 0$ but the roots most certainly were not all real.

Q13

(a) Solve for x^2 and notice that it's always a negative number. Hence x is purely imaginary.

(b) $-\alpha i,\ -\beta i$

(c) $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

Q14

(a) $P(\alpha) = \beta$ and then use conjugate properties to eventually deduce that $P(\bar{\alpha}) = \bar{\beta}$.

(b) The conjugate root theorem is a special case of the property for when $\beta = 0$.

Q15

(a) $P(-1) = 0$

(b) $P\left(\dfrac{1}{\alpha}\right) = 0$ given $P(\alpha) = 0$.

(c) The two non-real zeroes are α and $\dfrac{1}{\alpha}$. But by the conjugate root theorem, this means that $\bar{\alpha} = \dfrac{1}{\alpha}$ and so $\alpha\bar{\alpha} = 1$. Hence all zeroes, including the real root, have modulus 1.

Q16

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 < 0$

(b) Use the product of the zeroes.

(c) Use the sum of the roots.

(d) Use the sum of the roots in pairs.

Q17

(a) They are the complex conjugates of α and $i\alpha$.

(b) Either re-arrange $P(z)$ or expand the given result.

(c) Note that $P(z)$ is the sum of perfect squares. So the only way it will be zero is if both parts are zero, so $k = \frac{1}{k}$ i.e. $k = \pm 1$.

(d) Consider two cases. One is if $P(z)$ has real zeroes, in which case actually find the zeroes and show that they are on the unit circle. The other case is if $P(z)$ does not have real zeroes, in which case use the product of the zeroes and use the fact that $|\alpha|^2 = \alpha\bar{\alpha}$.

(e) Use the sum of the zeroes.

(f) Since the zeroes lie on the unit circle, they are in the form $x + iy = \cos\theta + i\sin\theta$. So $k = x - y = \cos\theta - \sin\theta$ and use auxiliary-angle formulae.

P1

See full worked solutions.

P2

(a) $P\left(\frac{1}{\alpha}\right) = 0$ given $P(\alpha) = 0$.

(b) Suppose all zeroes are inside the unit circle. But from the previous part, $\frac{1}{\alpha}$ must also be a zero. But the modulus of $\frac{1}{\alpha}$ exceeds one and so at least one root is outside of the unit circle.

(c) Substitute $z = \alpha$ and re-arrange, then take the modulus of both sides.

(d) Use the fact that $|\alpha| > 1$. Substitute $\alpha = |\alpha| \operatorname{cis}\theta$ into the previous part and expand then simplify and factorise.

(e) $|\alpha| > 1$ and $-1 \le k \le 1$ so $\left(|\alpha|^2 - 1\right)\left(1 - k^2\right) \ge 0$. So any root lying outside of the unit circle will result in a contradiction. Hence, the only possible scenario left is that all the zeroes lie on the unit circle rather than inside or outside it.

Exercise 2K

Euler's Formula

F1

$\cos\theta + i\sin\theta$, exponential

F2

(a) $\cos\theta$ (b) $\sin\theta$

Q1

(a) $2e^{\frac{i\pi}{3}}$ (b) $2\sqrt{2}e^{-\frac{i\pi}{4}}$ (c) $2e^{-\frac{2i\pi}{3}}$

Q2

(a) $2 + 2i\sqrt{3}$ (b) $-\sqrt{3} - i$ (c) $1 - i$

Q3

(a) $\frac{1}{2}(\operatorname{cis}(n\theta) + \operatorname{cis}(-n\theta))$

(b) $\frac{1}{2i}(\operatorname{cis}(n\theta) - \operatorname{cis}(-n\theta))$

Q4

(a) $2\cos 2\theta - 2$ (b) $2\cos 2\theta + 2$

(c) $2i\sin 6\theta - 6i\sin 2\theta$ (d) $2\cos 6\theta + 6\cos 2\theta$

Q5

(a) Factorise out $e^{i\theta}$

(b) Factorise out $e^{i\theta}$

Q6

(a) $i\tan\theta$ (b) $e^{-\frac{i\theta}{2}} \times i\tan\theta$

Q7

(a) $\tan\theta = \frac{1}{i} \times \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$

(b) See full worked solutions.

Q8

Expand and simplify

$$\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^4$$

Q9

$z = re^{i\theta}$ so $z^n = r^n e^{ni\theta} = r^n\big(\cos n\theta + i\sin n\theta\big)$

Q10

(a) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

(b) Cube both sides and simplify.

ANSWERS

Q11

Start from the right-hand side, convert to exponential form, then expand.

Q12

(a) Find two different expressions for $\text{Re}\left(e^{i(\alpha+\beta)}\right)$

(b) Find two different expressions for $\text{Im}\left(e^{i(\alpha+\beta)}\right)$

P1

(a) Expand $\cos(A+B)+\cos(A-B)$ where $A=\dfrac{\theta+\phi}{2}$ and $B=\dfrac{\theta-\phi}{2}$. Similarly for the sine expression.

(b) $e^{i\theta}+e^{i\phi}=(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)$ then use the previous result.

P2

(a) Use the sum of a geometric progression.

(b) Factorise $e^{\frac{i(n+1)\theta}{2}}$ from the numerator and $e^{\frac{i\theta}{2}}$ from the denominator.

(c) Find two different expressions for $\text{Im}\,(S_n)$.

Chapter Review

R1

(a) $10-55i$ (b) $2+i$

R2

See full worked solutions.

R3

See full worked solutions.

R4

$\pm(5+7i)$

R5

(a) $z=3\pm 2i$ (b) $z=3-2i,\ 4+i$

R6

(a) $-1-i\sqrt{3}$ (b) $2\sqrt{3}-2i$

R7

(a) $2\sqrt{3}\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)=2\sqrt{3}e^{-\frac{2i\pi}{3}}$

(b) $2\sqrt{2}\operatorname{cis}\left(-\dfrac{3\pi}{4}\right)=2\sqrt{2}e^{-\frac{3i\pi}{4}}$

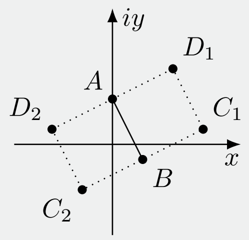
R8

(a) $\cos 2\theta - i\sin 2\theta$

(b) $i\cot\theta$

R9

(a)



(b) $C_1=6+i$, $D_1=4+5i$, $C_2=-2-3i$, $D_2=-4+i$

R10

(a) $\overrightarrow{BC}\times i=\overrightarrow{BA}$

(b) $z_4=z_1+z_3-z_2$

(c) See full worked solutions.

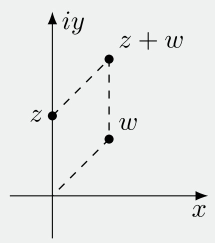
R11

$B=3+5i$, $D=5+i$

R12

(a) $\sqrt{2}\operatorname{cis}\left(\dfrac{\pi}{2}\right)$

(b)



(c) See full worked solutions.

(d) $1+\sqrt{2}$

R13

(a) $y=1$ (b) $x=0$

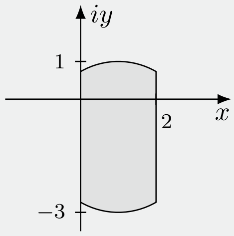
(c) $(x-2)^2+y^2=4$ (d) $y=2x$

(e) $xy=2$ (f) $x^2-y^2=2$

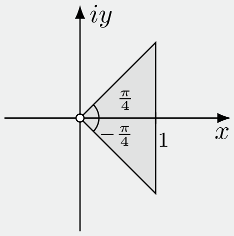
ANSWERS

R14

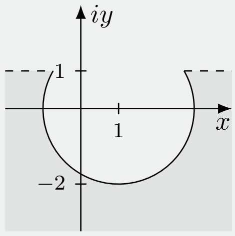
(a)



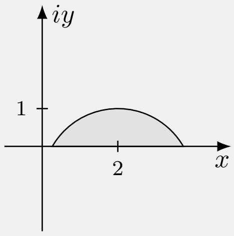
(b)



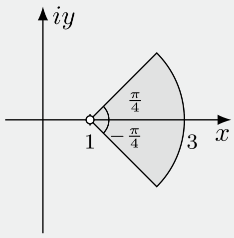
(c)



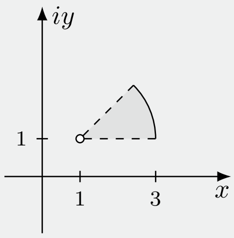
(d)



(e)



(f)



R15

$x^2 + y^2 = 1$

R16

(a) $2\sqrt{2} + 1$ (b) $2\sqrt{2} - 1$

(c) $\dfrac{\pi}{4} + \sin^{-1}\left(\dfrac{1}{2\sqrt{2}}\right)$ (d) $\dfrac{\pi}{4} - \sin^{-1}\left(\dfrac{1}{2\sqrt{2}}\right)$

R17

Multiples of 6

R18

Consider $\left(z + z^{-1}\right)^5$ to prove a result for $\cos^5\theta$, and then replace θ with $\dfrac{\pi}{2} - \theta$. Alternatively, consider $\left(z - z^{-1}\right)^5$. Note that the alternative method does not work for even powers of sine.

R19

(a) $z^n + z^{-n} = \operatorname{cis} n\theta + \operatorname{cis}(-n\theta) = 2\cos n\theta$

(b) $z = \pm i,\ 2 \pm \sqrt{3}$

R20

(a) Consider $(\cos\theta + i\sin\theta)^4$.

(b) $x = \pm\cos\left(\dfrac{\pi}{12}\right), \pm\cos\left(\dfrac{5\pi}{12}\right)$

(c) $\cos\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{3}+1}{2\sqrt{2}}, \cos\left(\dfrac{5\pi}{12}\right) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

R21

(a) Consider $(\cos\theta + i\sin\theta)^3$.

(b) $x = \cos\left(\dfrac{2\pi}{9}\right), \cos\left(\dfrac{4\pi}{9}\right), \cos\left(\dfrac{8\pi}{9}\right)$

(c) Use the sum of the roots, and then complementary angle identities.

R22

(a) $-1, \operatorname{cis}\left(\pm\dfrac{\pi}{5}\right), \operatorname{cis}\left(\pm\dfrac{3\pi}{5}\right)$

(b) $\operatorname{cis}\left(\dfrac{k\pi}{6}\right)$ where $k = -4, -1, 2, 5$

R23

(a) $\omega^7 = 1$ so $\omega^7 - 1 = 0$ then factorise.

(b) Group the identity into conjugate pairs and use the fact that $\alpha + \bar{\alpha} = 2\mathrm{Re}\,(\alpha)$.

R24

(a) 9 (b) 3

R25

(a) Find all the roots of $z^6 - 1 = 0$ and then use the result
$(z - \alpha)(z - \bar{\alpha}) = z^2 - 2\mathrm{Re}\,(\alpha)\,z + |\alpha|^2$ three times.

(b) Divide both sides by z^3 and then use the fact that $\alpha - \bar{\alpha} = 2i\mathrm{Im}\,(\alpha)$.

R26

(a) $(z+1)\left(z^2 - 2z\cos\left(\dfrac{\pi}{5}\right) + 1\right)$

$\left(z^2 - 2z\cos\left(\dfrac{3\pi}{5}\right) + 1\right)$

(b) $(z+1)(1 - z + z^2 - z^3 + z^4)$

(c) Equate the coefficient of z.

(d) Equate the coefficient of z^2.

(e) $4x^2 - 2x - 1 = 0$

(f) $\cos\left(\dfrac{\pi}{5}\right) = \dfrac{1+\sqrt{5}}{4}, \cos\left(\dfrac{3\pi}{5}\right) = \dfrac{1-\sqrt{5}}{4}$

R27

(a) $x = 1 \pm 3i,\ 2$ (b) $x = 2 \pm i,\ 1 \pm i$

R28

$2x - 7$

R29

(a) $\alpha^2 + \beta^2 + \gamma^2 = -52 < 0$, so all three roots cannot be real.

(b) $x = 2 \pm 3i,\ 4$

R30

(a) $x = 1 \pm i,\ 1 \pm 2i$

(b) $P(x) = (x^2 - 2x + 2)(x^2 - 2x + 5)$

R31

(a) $2\cos 3\theta + 6\cos\theta$

(b) $2\cos 8\theta - 8\cos 4\theta + 6$

R32

(a) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(b) Cube both sides and then convert to sines.

3. 3D Vectors

Exercise 3A

Introduction to 3D Vectors

F1

(a) $ka_1\underset{\sim}{i} + ka_2\underset{\sim}{j} + ka_3\underset{\sim}{k}$

(b) $(a_1 + b_1)\underset{\sim}{i} + (a_2 + b_2)\underset{\sim}{j} + (a_3 + b_3)\underset{\sim}{k}$

(c) $a_1b_1 + a_2b_2 + a_3b_3$

(d) $a_1^2 + a_2^2 + a_3^2$

(e) $\sqrt{a_1^2 + a_2^2 + a_3^2}$

(f) $\dfrac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right)$

F2

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}$ (b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$

Q1

(a) 4 (b) 3 (c) 7

(d) $\sqrt{65}$ (e) 5 (f) $\sqrt{58}$

Q2

(a) $6\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$ (b) $11\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

(c) -2 (d) 12

Q3

(a) 60° (b) 101° (c) 70°

Q4

$$\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

Q5

$a = 1,\ b = 2,\ c = 3$

Q6

82°, 33°, 65°

Q7

$(1, -2, 2),\ (-3, 10, 2),\ (3, 2, -8)$

Q8

(a) $\dfrac{1}{2}\begin{bmatrix}1\\-2\\3\end{bmatrix}$ (b) $\dfrac{1}{2}\begin{bmatrix}3\\1\\2\end{bmatrix}$

(c) $\sqrt{\dfrac{7}{2}}$ (d) $\sqrt{\dfrac{7}{2}}$

Q9

(a) $\mathbf{u} = \mathbf{v}$ (b) $|\mathbf{u}| = |\mathbf{v}|$

Q10

(a) $\dfrac{20}{\sqrt{30}}$

(b) $5\sqrt{2}$

(c) $\sqrt{\dfrac{110}{3}}$

(d) $\dfrac{10}{3}\sqrt{11}$

(e) Area $= \dfrac{1}{2}ab\sin\theta$

Q11

$$\underset{\sim}{w} = 4\sqrt{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Q12

See full worked solutions.

Q13

(a) $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 6$

(b) Show that $\overline{AB} \cdot \overline{AC} \neq 0$

Q14

(a) $|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 + 2\underset{\sim}{u} \cdot \underset{\sim}{v}$

(b) $|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}$

(c) $2|\underset{\sim}{u}|^2 + 2\underset{\sim}{v}|^2$

(d) $4\underset{\sim}{u} \cdot \underset{\sim}{v}$

Q15

(a) $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta \leq |\underset{\sim}{u}||\underset{\sim}{v}|$

(b) Prove instead the squared version of the required result.

(c) This is the triangle inequality.

Q16

(a) $\dfrac{\pi}{2}$

(b) Both are equal to $\dfrac{\pi}{2}$

(c) $\angle APB = \cos^{-1}\left(\dfrac{c^2}{\sqrt{b^2+c^2}\sqrt{a^2+c^2}}\right)$

$\angle BPC = \cos^{-1}\left(\dfrac{a^2}{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}\right)$

$\angle APC = \cos^{-1}\left(\dfrac{b^2}{\sqrt{a^2+b^2}\sqrt{b^2+c^2}}\right)$

P1

Use the formula for the vector projection.

P2

$$\frac{1}{2}\sqrt{(x_2y_1 - x_1y_2)^2 + (x_2z_1 - x_1z_2)^2 + (y_2z_1 - y_1z_2)^2}$$

P3

$$\begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

P4

Find two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ along the plane that are not parallel. Show that $\underset{\sim}{n}$ is perpendicular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Exercise 3B

Proofs using 3D Vectors

F1

(a) Show that their dot product is zero.

(b) Show that they are scalar multiples of each other.

(c) Show that the vectors that pass through the points are parallel, but using a common point.

(d) Use the formula $\cos\theta = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$

(e) Find the mid-point M of one interval. Show that the vector that makes the other interval also passes through M.

Q1

Use Pythagoras' Theorem twice.

Q2

Let all the angles be equal to $\dfrac{\pi}{3}$ and observe that $a = b = c$.

Q3

(a) $(1, 1, 1)$

(b) $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

(c) The exact angle is $\cos^{-1}\left(-\dfrac{1}{3}\right) \approx 109.5°$

Q4

(a) $(0, 0, a\sqrt{2})$

(b) $\tan^{-1}\left(\sqrt{2}\right)$

Q5

See full worked solutions.

Q6

(a) $|\underset{\sim}{u} - \underset{\sim}{v}| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

(b) Square both sides then use the fact that $|\underset{\sim}{u} - \underset{\sim}{v}|^2 = (\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v})$

Q7

First show that $\underset{\sim}{u} = \cos\alpha\,\underset{\sim}{i} \pm \sin\alpha\,\underset{\sim}{j}$ and $\underset{\sim}{v} = \pm\sin\beta\,\underset{\sim}{j} + \cos\beta\,\underset{\sim}{k}$

Q8

See full worked solutions.

Q9

See full worked solutions.

ANSWERS

Q10

(a) Since it is perpendicular to the face opposite $\triangle QRS$, it points vertically downwards. The area is just the area of the right-angled $\triangle QRS$.

(b) $q = \left(-\frac{1}{2}pr\right)\underset{\sim}{i}$, $r = \left(-\frac{1}{2}pq\right)\underset{\sim}{j}$,

$$s = \left(\tfrac{1}{2}pr\right)\underset{\sim}{i} + \left(\tfrac{1}{2}pq\right)\underset{\sim}{j} + \left(\tfrac{1}{2}qr\right)\underset{\sim}{k}$$

(c) Add the previous expressions and everything cancels.

Q11

(a) $\underset{\sim}{u} = a\,\underset{\sim}{i} - b\,\underset{\sim}{j} - c\,\underset{\sim}{k}$
$\underset{\sim}{v} = a\,\underset{\sim}{i} - b\,\underset{\sim}{j} + c\,\underset{\sim}{k}$

(b) $\underset{\sim}{u} \cdot \underset{\sim}{v} = a^2 + b^2 - c^2$, which is zero iff the diagonals are perpendicular.

(c) $P(3, 4, 5)$

Q12

(a) $(\underset{\sim}{r} - \underset{\sim}{r}_0)$ lies on the plane and $\underset{\sim}{n}$ is perpendicular to the plane, which means that it is perpendicular to any vector on the plane. So their dot product is zero.

(b) Expand $(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{n} = 0$

(c) $3x - 2y + z = 13$

(d) $\sqrt{a^2 + b^2 + c^2}$

Q13

See full worked solutions.

Q14

See full worked solutions.

P1

(a) See full worked solutions.

(b) $(\underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3 + \underset{\sim}{p}_4) \cdot (\underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3 + \underset{\sim}{p}_4) = 0$

(c) Find a general expression for $|\overrightarrow{PP_k}|^2 = \overrightarrow{PP_k} \cdot \overrightarrow{PP_k}$ and then sum them and use the previous result.

P2

Show that the given expression is equivalent to $\dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$

P3

(a) Instead, minimise $|\underset{\sim}{w}|^2 = |\underset{\sim}{u} - t\underset{\sim}{v}|^2$

(b) This proves that the minimum distance between a line ℓ and a point P is the perpendicular distance between them.

P4

Show that $\underset{\sim}{v} \cdot \underset{\sim}{v}_i = c_i(\underset{\sim}{v}_i \cdot \underset{\sim}{v}_i)$ and re-arrange.

Exercise 3C

Vector Equation of a Line

F1

(a) parameter (b) parameter (c) position

F2

(a) $0, 1$ (b) $\underset{\sim}{a}$ (c) $\underset{\sim}{b}$

Q1

(a) $\underset{\sim}{r}(\lambda) = \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1+4\lambda\\2+5\lambda\\3+6\lambda\end{bmatrix}$

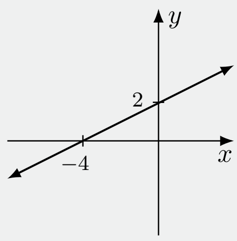
(b) $\underset{\sim}{r}(\lambda) = \begin{bmatrix}-3\\1\\5\end{bmatrix} + \lambda\begin{bmatrix}-2\\4\\1\end{bmatrix} = \begin{bmatrix}-3-2\lambda\\1+4\lambda\\5+\lambda\end{bmatrix}$

(c) $\underset{\sim}{r}(\lambda) = \begin{bmatrix}2\\1\\-3\end{bmatrix} + \lambda\begin{bmatrix}4\\0\\-3\end{bmatrix} = \begin{bmatrix}2+4\lambda\\1\\-3-3\lambda\end{bmatrix}$

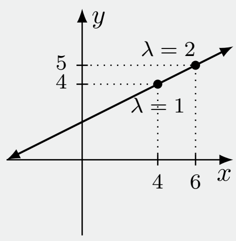
(d) $\underset{\sim}{r}(\lambda) = \begin{bmatrix}3\\-5\\2\end{bmatrix} + \lambda\begin{bmatrix}0\\2\\7\end{bmatrix} = \begin{bmatrix}3\\-5+2\lambda\\2+7\lambda\end{bmatrix}$

Q2

(a)



(b)



Q3

(a) $m = \dfrac{b}{a}$

(b) $\underset{\sim}{r}(\lambda) = \begin{bmatrix}0\\2\end{bmatrix} + \lambda\begin{bmatrix}3\\-2\end{bmatrix}$

ANSWERS

Q4

(a) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} -7 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 9 \\ 3 \\ -5 \end{bmatrix}$

(b) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 10 \\ -3 \\ -2 \end{bmatrix}$

Q5

(a) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ where $\lambda \in [0, 1]$

(b) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ where $\lambda \in [0, 1]$

Q6

(a) Collinear.

(b) Not collinear.

Q7

(a) Yes (b) No (c) Yes

Q8

(a) $\lambda = -1$

(b) $(-4, 8, 0)$

(c) $(-1, 2, 12)$

Q9

(a) $(1, -2, 3)$, $(3, 1, 2)$

(b) Does not lie on ℓ.

(c) Lies on ℓ.

(d) $(0, -3.5, 3.5)$

Q10

$\underset{\sim}{r}(\lambda) = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Q11

Skew.

Q12

(a) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}$

(b) $(1, 0, 3)$

Q13

(a) $(4, 4, 2)$

(b) $(1, 0, 2)$

Q14

(a) $\underset{\sim}{r}(\lambda) = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}(\mu) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(b) Show that $\lambda = \mu$ and $\lambda + \mu = 1$, hence

$\lambda = \mu = \frac{1}{2}$

Q15

$\sqrt{\frac{77}{13}}$

Q16

(a) Their direction vectors are scalar multiples of each other, and hence are parallel.

(b) $\sqrt{\frac{121}{7}}$

Q17

$\underset{\sim}{r}(\mu) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} -\frac{20}{3} \\ 2 \\ 1 \end{bmatrix}$

P1

(a) Obtain $x = x_0 + \lambda a$ and similarly for y and z. Equate their λ expressions.

(b) This gives the Cartesian equation of the line. More specifically, the xy, yz and xz-projections can be obtained from this by excluding one of the expressions.

P2

(a) $\overrightarrow{PQ} = \begin{bmatrix} \mu - 1 \\ \mu + 2\lambda - 1 \\ -\lambda - 1 \end{bmatrix}$

(b) 1 unit.

P3

(a) $\underset{\sim}{r}(\lambda) = \underset{\sim}{p} + \lambda \underset{\sim}{b}$

(b) Let $\underset{\sim}{r} = \underset{\sim}{v}$ and use the fact that $\underset{\sim}{u} \cdot \underset{\sim}{u} = |\underset{\sim}{u}|^2$

(c) ℓ is tangential to $\mathcal{S}$ if and only if there is exactly one point of contact i.e. $\lambda = 0$ is the only solution. This occurs if and only if $\underset{\sim}{b} \cdot (\underset{\sim}{p} - \underset{\sim}{c}) = 0$

Exercise 3D
Parameterising 3D Curves

F1

parametrically, xy, xz, yz, perspectives

F2

(a) parameter, segment

(b) positive, quadrant

Q1

(a) $y = 2x + 5$

(b) $y = \frac{1}{4}(x+1)^2$

(c) $y = x + 1$ where $x \geq 0$

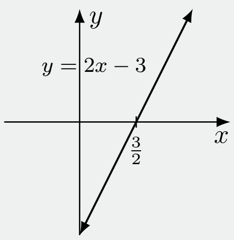
(d) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(e) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

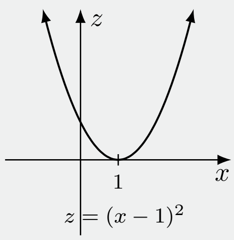
(f) $y = \frac{1}{x}$ where $x > 0$

Q2

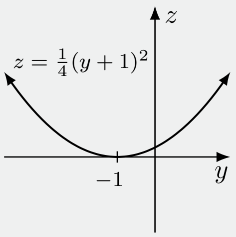
Plane xy



Plane xz



Plane yz



Q3

(a) $x^2 + y^2 = 2$

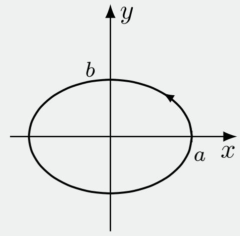
(b) $x^2 + y^2 = 2$

(c) The first one starts at $(1, 1)$ and goes clockwise as t increases. The second one starts at $(\sqrt{2}, 0)$ and goes anti-clockwise as t increases.

Q4

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(b)



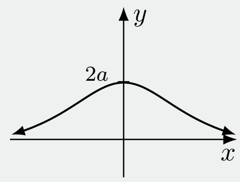
(c) Replace t with $-t$.

(d) Replace t with $t + \frac{\pi}{2}$.

Q5

(a) $y = \frac{8a^3}{x^2 + 4a^2}$

(b)



Q6

The first shows the full curve. The second only shows the first quadrant since $x \geq 0$ and the third excludes the origin since neither the x nor the y-coordinate can be zero.

Q7

(a) The spiral stretches out in the z-direction as t increases. Also it is only defined for $z \geq 0$.

(b) It is the same as the spiral from the previous part, except it spirals down the z-axis instead of up.

(c) The spiral stretches out even faster in the z-direction as t increases.

(d) The spiral gets 'compressed' in the z-direction as t increases. As $t \to 0$, the spiral has a 'tail' that shoots down to negative infinity.

(e) The spiral starts at $z = 1$ when $t = 0$, then spirals downwards indefinitely towards $z = 0$ but never touches the xy-plane.

(f) The 'spiral' oscillates up and down in the z-direction whilst still maintaining a circle in the xy-plane. So actually it is an ellipse that is slanted with respect to the xy-plane.

ANSWERS

Q8

(a) $x^2 + y^2 = 1$

(b) $z = 1 - y$

(c) $x^2 + (y-1)^2 = 1$

(d) It represents a slanted ellipse with respect to the xy-plane.

Q9

(a) This represents a straight line.

(b) This represents a slanted parabola.

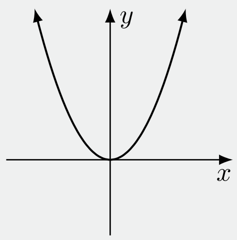
(c) This represents a spiral that wraps around the x-axis.

(d) This represents a slanted ellipse.

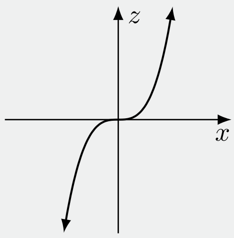
Q10

(a) $y = x^2$, $z = x^3$, $y = z^{\frac{2}{3}}$

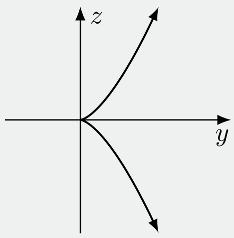
(b) Plane xy



Plane xz



Plane yz



Q11

The first particle begins at $(1, 0)$ and moves anti-clockwise at a certain speed. The second particle does the same thing, but traverses the circle at twice the speed. The third particle begins at $(0, 1)$ but travels clockwise at the same speed as the first particle.

Q12

The particles will collide when $t = 1$. this is found by equating any component, solving for t, and then testing the other components.

Q13

(a) $\underset{\sim}{v}(t) = (6t^2 + 1)\,\underset{\sim}{i} - (2t)\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$

$\underset{\sim}{a}(t) = (12t)\,\underset{\sim}{i} - 2\,\underset{\sim}{j}$

(b) $|\underset{\sim}{v}(1)| = \sqrt{62}$

$|\underset{\sim}{a}(1)| = 2\sqrt{145}$

Q14

(a) $\underset{\sim}{r}(t) = (2\sin t)\,\underset{\sim}{i} + t\,\underset{\sim}{j} + (2\cos t)\,\underset{\sim}{k}$

(b) $\underset{\sim}{r}(t) = (3 - 5t)\,\underset{\sim}{i} + (6t - 6)\,\underset{\sim}{j} + (3t + 1)\,\underset{\sim}{k}$

(c) $\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + (t^2)\,\underset{\sim}{j} + (2t^2)\,\underset{\sim}{k}$

(d) $\underset{\sim}{r}(t) = (\cos t)\,\underset{\sim}{i} + (\sin t)\,\underset{\sim}{j} + (2 - \sin t)\,\underset{\sim}{k}$

Q15

(a) (iii) (b) (i) (c) (vii) (d) (viii)

(e) (v) (f) (vi) (g) (iv) (h) (ii)

Q16

(a) Equate z. The parabola $y = \frac{1}{2}\left(x^2 + 1\right)$ is the projection of the curve of intersection onto the xy-plane.

(b) See full worked solutions.

(c) No, it is not wrong to use the other surface. We choose $z = y - 1$ because the algebra will be simpler.

(d) $-3 \le t \le 3$. Intersections at $(-3, 5, 4)$ and $(3, 5, 4)$.

Q17

(a) $\underset{\sim}{r}(t) = (t)\,\underset{\sim}{i} \pm \sqrt{1 + 2t}\,\underset{\sim}{j} + (1 + t)\,\underset{\sim}{k}$

(b) See full worked solutions.

(c) The first parametrisation works, but it has a plus/minus in it. So it can only give one branch of the parabolic intersection at a time. However, the first parametrisation gives the entire parabola in one go.

Q18

(a) $\left(\cos\theta\right)\underset{\sim}{i} + \left(\sin\theta\right)\underset{\sim}{j}$

(b) $\left(\cos\theta\right)\underset{\sim}{i} + \left(\sin\theta\right)\underset{\sim}{j} + \left(1 - \sin\theta\right)\underset{\sim}{k}$

(c) It represents the curve of intersection between the two surfaces, which is an ellipse. From the birds-eye view it is the unit circle. In the yz-plane, it is the straight line $z = 1 - y$.

Q19

(a) See full worked solutions.

(b) $\left(1 + \sqrt{2}\cos t\right)\underset{\sim}{i} + \left(-1 + \sqrt{2}\sin t\right)\underset{\sim}{j}$

(c) See full worked solutions.

Q20

$\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + t^2\,\underset{\sim}{j} + \left(t^2 + t^4\right)\underset{\sim}{k}$

$-\sqrt{3} \leq t \leq \sqrt{3}$

P1

(a) See full worked solutions.

(b) It traces out a circle.

(c) It defines the same circle as before, but now centred at the fixed point represented by $\underset{\sim}{c}$.

P2

(a) $x^2 = t^2 + \dfrac{1}{t^2} + 2 = y + 2$

(b) Recall the inequality $t + \dfrac{1}{t} \geq 2$ for $t > 0$. So the curve is actually the parabola but for $x \geq 2$ or $x \leq -2$.

Exercise 3E

Spheres and Circles

F1

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

F2

$\underset{\sim}{c}$, r

F3

(a) Find the distance between P and the centre of $\mathcal{S}$ and compare it to the radius.

(b) Find the vector equation of ℓ and substitute it into the Cartesian equation of the sphere. Solve the resulting quadratic for the parameter λ and substitute the solutions back into ℓ to find the point of intersection.

(c) Find the distance from C to P to find the radius, and the Cartesian equation can now be obtained since the centre and radius of $\mathcal{S}$ are known.

(d) The length of AB is the diameter and so the radius can be found. The midpoint of the interval is the centre of $\mathcal{S}$. Find the Cartesian equation from here.

Q1

(a) $(x - 2)^2 + (y + 3)^2 + (z - 5)^2 = 36$

(b) $(x - 2)^2 + (y + 3)^2 = 11$
$(x - 2)^2 + (z - 5)^2 = 27$
$(y + 3)^2 + (z - 5)^2 = 32$

Q2

$(x + 5)^2 + (y - 3)^2 + (z - 4)^4 = 50$

Q3

(a) $C(2, -1, 0)$, $r = 3$

(b) $C(-1, 2, 3)$, $r = 4$

Q4

(a) $(x - 6)^2 + (y + 2)^2 + (z - 3)^2 = 9$

(b) $(x - 6)^2 + (y + 2)^2 + (z - 3)^2 = 4$

(c) $(x - 6)^2 + (y + 2)^2 + (z - 3)^2 = 36$

Q5

$(x - 2)^2 + (y - 4)^2 = 27$
$(y - 4)^2 + (z - 3)^2 = 32$
$(x - 2)^2 + (z - 3)^2 = 20$

Q6

$(x - 3)^2 + (y - 5)^2 + (z - 2)^2 = 4$

Q7

(a) $(x - 1)^2 + (y - 2)^2 + (z - 4)^2 = 4$

(b) $C(1, 2, 4)$, $r = 2\sqrt{2}$

Q8

$x^2 + (y - 1)^2 + (z - 2)^2 = 4$

Q9

(a) $\underset{\sim}{r}(\lambda) = \begin{bmatrix} 1 + 3\lambda \\ -2 + 2\lambda \\ \lambda \end{bmatrix}$

(b) $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 9$

(c) $\lambda = 1, \dfrac{1}{7}$

(d) $(4, 0, 1)$ and $\left(\dfrac{10}{7}, -\dfrac{12}{7}, \dfrac{1}{7}\right)$

(e) If the quadratic had no real roots, then no real value of λ exists i.e. the line does not intersect the sphere. If there was only one real root, then it means the line is tangential to the sphere and hence intersects it at exactly one point.

Q10

(a) $r = 1$

(b) $\lambda = -1$. This was to be expected since ℓ is tangential to $\mathcal{S}$, so there should only be one point of intersection.

(c) $P(2, 0, -1)$

(d) The dot product is zero, so the two lines are perpendicular. This is to be expected since this is just the 3D analogy of the property "The tangent and radius are perpendicular at the point of contact" from circle geometry.

Q11

$(-2, -1, 2)$ and $(2, 1, 2)$

Q12

(a) Show that the distance between their centres is equal to the sum of their radii.

(b) $\left(-\frac{1}{3}, \frac{5}{3}, -\frac{2}{3}\right)$

(c) 10

(d) $\left(\frac{7}{3}, -\frac{11}{3}, \frac{14}{3}\right)$

Q13

(a) $\underset{\sim}{v} = (r\cos\theta)\,\underset{\sim}{i} + (r\sin\theta)\,\underset{\sim}{j}$

(b) The parametrisation above effectively consists of two perpendicular vectors of length r and combines $\cos\theta$ 'amount' of one of them with $\sin\theta$ 'amount' of the other one. The parametrisation at hand here does the same thing, except the basis vectors are now $\underset{\sim}{u}$ and $\underset{\sim}{w}$ instead of $\underset{\sim}{i}$ and $\underset{\sim}{j}$. The $\underset{\sim}{c}$ vector just shifts the whole thing to be centred at $\underset{\sim}{c}$.

(c) Show that $|\underset{\sim}{u}\cos\theta + \underset{\sim}{w}\sin\theta|^2 = r^2$

Q14

Show that the coordinates satisfy $x^2 + y^2 + z^2 = r^2$.

P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) It describes a circle with diameter AB.

(c) It proves that the set of all points that form perpendicular vectors with respect to two fixed points is a circle. In other words it proves the familiar 'angle in a semi-circle' property otherwise known as *Thales' Theorem.*

P3

See full worked solutions.

Chapter Review

R1

$\frac{\pi}{2}$

R2

$(0, 2, 4)$, $(-4, 14, 4)$, $(2, 6, -6)$

R3

$\underset{\sim}{w} = \begin{bmatrix} 4 \\ -7 \\ 8 \end{bmatrix}$

R4

Find expressions for the vectors and find the angle between the vectors.

R5

$(2, -4, 12)$

R6

$\underset{\sim}{r}(\lambda) = \begin{bmatrix} -7 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 9 \\ 3 \\ -5 \end{bmatrix}$ where $\lambda \in [0, 1]$

R7

Parallel.

R8

Perpendicular.

R9

(a) Equate two components and solve simultaneously for μ and λ. Test the solutions with the third component to check that it works.

(b) $(-2, -4, -1)$

R10

$(4, 1, -9)$

R11

(a) Equate two components and solve simultaneously for μ and λ. Test the solutions with the third component and show that it does not work.

(b) $\frac{3}{2}\sqrt{2}$

R12

$\underset{\sim}{r}(t) = (t^2 - 2t)\,\underset{\sim}{i} + t\,\underset{\sim}{j} + t^2\,\underset{\sim}{k}$

R13

$\underset{\sim}{r}(t) = \left(\frac{1}{\sqrt{2}}\cos t\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}\sin t\right)\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$

R14

$C(-2,1,3)$, $r=4$

R15

(a) $(x-2)^2+(y-3)^2+(z-4)^2=4$

(b) $(x-2)^2+(y-3)^2+(z-4)^2=25$

R16

$(4,1,-3)$ and $(1,-5,0)$

R17

(a) Note that the centre of $\mathcal{S}_2$ lies inside $\mathcal{S}_1$. Show that $d+r_2=r_1$, where d is the distance between their centres.

(b) $\left(-\frac{5}{3},\frac{13}{3},-\frac{10}{3}\right)$

4. Further Integration

Exercise 4A

Integration by Substitution

F1

(a) $\frac{1}{n+1}\big(f(x)\big)^{n+1}+C$

(b) $\ln|f(x)|+C$

(c) $e^{f(x)}+C$

F2

(a) $-\cos\big(f(x)\big)+C$

(b) $\sin\big(f(x)\big)+C$

(c) $\tan\big(f(x)\big)+C$

F3

(a) $\sin^{-1}\big(f(x)\big)+C$ (b) $\tan^{-1}\big(f(x)\big)+C$

Q1

(a) $u=x^3$ (b) $u=x^2$

(c) $u=1+e^x$ (d) $u=1+\cos x$

(e) $u=1+x^2$ (f) $u=1+\tan x$

Q2

(a) $\sqrt{1+x^2}+C$

(b) $\frac{1}{\cos x+\sin x}+C$

(c) $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$

(d) $\frac{1}{4(x^2+1)^2}-\frac{1}{2(x^2+1)}+C$

(e) $-\frac{1}{7}(2-x)^7-\frac{1}{2}(2-x)^6+C$

(f) $\frac{2}{3}(x+1)\sqrt{x+1}-2\sqrt{x+1}+C$

(g) $\frac{1}{2\cos^2 x}+C$

(h) $\frac{-1}{e^x+e^{-x}}+C$

(i) $\frac{1}{2}(\ln x)^2+C$

(j) $2\sqrt{3-\cos x}+C$

(k) $-\frac{1}{4}\cos^4 x+C$

(l) $\frac{1}{4}\tan^4 x+C$

Q3

(a) $-\frac{1}{2}\cos(x^2)+C$ (b) $\sin\left(e^x\right)+C$

(c) $-\tan\left(\frac{1}{x}\right)+C$ (d) $-2\cos(\sqrt{x})+C$

(e) $\sin\big(\ln x\big)+C$ (f) $\frac{1}{2}\tan\big(2\tan x\big)+C$

Q4

(a) $e^{x^2+x}+C$ (b) $-e^{\cos x}+C$

(c) $-e^{\frac{1}{x}}+C$ (d) $2e^{\sqrt{x}}+C$

(e) $-\frac{1}{2e^{x^2}}+C$ (f) $e^{\tan x}+C$

Q5

(a) $-\ln|1+\cos x|+C$ (b) $\ln|\ln x|+C$

(c) $\ln|x-\tan x|+C$ (d) $\ln|e^x+e^{-x}|+C$

(e) $2\ln|1+\sqrt{x}|+C$ (f) $-\ln|1+\cos^2 x|+C$

Q6

(a) $\frac{1}{2}\sin^{-1}(x^2)+C$

(b) $\frac{1}{2}\sin^{-1}\left(\frac{2\sin x}{5}\right)+C$

(c) $\sin^{-1}(e^x)+C$

(d) $\frac{1}{5}\sin^{-1}\left(\frac{5\tan x}{4}\right)+C$

(e) $\sin^{-1}(\ln x)+C$

(f) $-\sin^{-1}\left(\frac{1}{x}\right)+C$

Q7

(a) $\frac{1}{3}\tan^{-1}(x^3)+C$

(b) $\frac{1}{6}\tan^{-1}\left(\frac{3\sin x}{2}\right)+C$

(c) $-\tan^{-1}(e^{-x})+C$

(d) $\frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right)+C$

(e) $\tan^{-1}(\ln x)+C$

(f) $2\tan^{-1}(\sqrt{x})+C$

Q8

(a) $-\ln|\cos(e^x)|+C$ (b) $-2\ln|\cos(\sqrt{x})|+C$

(c) $\ln\left|\cos\left(\frac{1}{x}\right)\right|+C$

Q9

(a) $2\sqrt{x}-2\ln|1+\sqrt{x}|+C$

(b) $-2\sqrt{x}-2\ln|1-\sqrt{x}|+C$

(c) $x-2\sqrt{x}+2\ln|1+\sqrt{x}|+C$

P1

$\tan^{-1}\left(\frac{\sqrt{x^2+4x-4}-x}{2}\right)+C$

P2

See full worked solutions.

P3

(a) $e^{e^x}+C$

(b) $\ln|e^x-e^{-x}|+C$

(c) $2\ln|1+\sqrt{x}|+C$

(d) $-\sin^{-1}\left(\frac{1}{x}\right)+C$

(e) $-2\sqrt{\frac{1+x}{x}}+C$

(f) $\tan^{-1}(x)+\frac{1}{3}\tan^{-1}(x^3)+C$

(g) $\frac{1}{2}\ln\left|\frac{1-\sqrt{1-e^{2x}}}{1+\sqrt{1-e^{2x}}}\right|+C$

(h) $-\frac{1}{4}\ln|x^{-4}+1|+C$

(i) $2\sqrt{x}-2\tan^{-1}(\sqrt{x})+C$

(j) $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{x-1}}{\sqrt{2}}\right)+C$

(k) $\sin^{-1}\left(x-\frac{1}{x}\right)+C$

(l) $\frac{2}{3}\sin^{-1}(x\sqrt{x})+C$

Exercise 4B
Trigonometric Integrals

F1

$\frac{1}{n+1}(f(x))^{n+1}+C$

Q1

(a) $\frac{1}{3}\sin^3 x+C$ (b) $-\frac{1}{4}\cos^4 x+C$

(c) $\frac{1}{5}\tan^5 x+C$ (d) $\frac{1}{5}\sec^5 x+C$

(e) $\frac{1}{2}\tan^2 x+C$ (f) $\frac{2}{7}\sec^3 x\sqrt{\sec x}+C$

Q2

(a) $\frac{1}{3}\cos^3 x-\cos x+C$

(b) $\sin x-\frac{1}{3}\sin^3 x+C$

(c) $\frac{1}{2}\tan^2 x+\ln|\cos x|+C$

(d) $\tan x - x + C$

(e) $\tan x + \frac{1}{3}\tan^3 x + C$

(f) $x - \tan x + \frac{1}{3}\tan^3 x + C$

Q3

(a) $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

(b) $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

(c) $\frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$

Q4

(a) $\frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C$

(b) $\frac{1}{32}(12x + 8\sin 2x + \sin 4x) + C$

(c) $\frac{1}{16}(12x + \sin 4x) + C$

Q5

(a) $\frac{1}{3}\sec^3 x + C$

(b) $\frac{1}{3}\tan^3 x + C$

(c) $\frac{1}{3}\sec^3 x - \sec x + C$

(d) $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

(e) $\frac{1}{4}\tan^4 x + C$

(f) $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

Q6

(a) $\ln|\tan x| + C$

(b) $\frac{1}{2\cos^2 x} + C$

(c) $\frac{\cos^2 x}{2} + \ln|\sec x| + C$

(d) $\ln|\sin x| - \frac{\sin^2 x}{2} + C$

(e) $\left(\frac{2}{5}\cos^2 x - 2\right)\sqrt{\cos x} + C$

(f) $\tan x - \cot x + C$

Q7

(a) $\frac{\sin^2 x}{2} + C, -\frac{\cos^2 x}{2} + C, -\frac{\cos 2x}{4} + C$

(b) The three ‘different’ answers differ by constants, which can be proven using the double angle identities.

Q8

(a) $\frac{\cos x}{2} - \frac{\cos 5x}{10} + C$

(b) $\frac{\sin 3x}{6} + \frac{\sin 7x}{14} + C$

(c) $\frac{\sin 2x}{4} - \frac{\sin 12x}{24} + C$

Q9

(a) $-\frac{1}{2}\ln|\cos 2x| + C$

(b) $-\frac{2}{5}\ln\left|\sin\left(\frac{5}{2}x\right)\right| + C$

(c) $\ln|\sec x| + C$

(d) $-2\cos x + C$

P1

(a) $\frac{1}{a\sqrt{a^2+b^2}}\tan^{-1}\left(\frac{\sqrt{a^2+b^2}}{a}\tan x\right) + C$

(b) $\frac{1}{a\sqrt{a^2+b^2}}\tan^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\tan x\right) + C$

(c) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + C$

(d) $-\frac{\cos x}{b(a\cos x + b\sin x)} + C$

P2

$\frac{\pi}{\sqrt{a^2-b^2}}$

P3

See full worked solutions.

Exercise 4C

Trigonometric Substitutions

F1

(a) $a\sin\theta$ (b) $a\tan\theta$ (c) $a\sec\theta$

Q1

(a) $2-\sqrt{2}$ (b) $\sqrt{3}-\frac{\pi}{3}$ (c) $\frac{\sqrt{3}}{2}$

Q2

(a) $-\frac{\sqrt{4-x^2}}{4x}+C$ (b) $-\frac{\sqrt{x^2+4}}{4x}+C$

(c) $\frac{\sqrt{x^2-4}}{4x}+C$

Q3

(a) $-\frac{1}{2}\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right|+C$

(b) $-\frac{1}{2}\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right|+C$

(c) $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+C$

Q4

(a) $-\frac{\sqrt{4-x^2}}{x}-\sin^{-1}\left(\frac{x}{2}\right)+C$

(b) $\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right|+C$

(c) $\frac{\sqrt{x^2-4}}{2}+\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|+C$

Q5

(a) $\sqrt{4-x^2}-2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right|+C$

(b) $\sqrt{4+x^2}-2\ln\left|\frac{2+\sqrt{4+x^2}}{x}\right|+C$

(c) $\sqrt{x^2-4}-2\sec^{-1}\left(\frac{x}{2}\right)$

Q6

(a) $2\sin^{-1}\left(\frac{x}{2}\right)-\frac{x\sqrt{4-x^2}}{2}+C$

(b) $\frac{x\sqrt{4+x^2}}{2}-2\ln\left|x+\sqrt{x^2+4}\right|+C$

(c) $2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|+\frac{x\sqrt{x^2-4}}{2}+C$

Q7

(a) $\frac{x}{4\sqrt{4-x^2}}+C$ (b) $\frac{x}{4\sqrt{4+x^2}}+C$

(c) $-\frac{x}{4\sqrt{x^2-4}}+C$

Q8

(a) $\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

(b) $\frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|\right)+C$

(c) $\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|\right)+C$

Q9

(a) $\sin^{-1}\left(\frac{x}{a}\right)+C$

(b) $\ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|+C$

(c) $\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C$

Q10

(a) $\frac{x}{a^2\sqrt{a^2-x^2}}+C$ (b) $\frac{x}{a^2\sqrt{a^2+x^2}}+C$

(c) $-\frac{x}{a^2\sqrt{x^2-a^2}}+C$

P1

(a) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ (b) $\frac{\pi}{2}-1$

(c) $2-\frac{\pi}{2}$ (d) $2\ln(1+\sqrt{2})-\sqrt{2}$

(e) $\frac{\pi}{4}$ (f) $2\pi-\sqrt{3}$

P2

See full worked solutions.

P3

$\frac{\pi}{2}(b-a)$

ANSWERS

Exercise 4D

Harder Standard Integrals

F1

(a) Express as a perfect square and use the reference sheet formula for $\int f'(x)\big(f(x)\big)^n\,dx$.

(b) Complete the square and it is an inverse tan integral.

(c) Use partial fractions

F2

logarithmic

F3

inverse sine, trigonometric

F4

$\int f'(x)\big(f(x)\big)^n\,dx$

Q1

(a) $\frac{1}{4}\left(2x - 1 + \ln|2x-1| + C\right)$

(b) $\frac{x^2}{2} - x + \ln|x+1| + C$

(c) $2x + 3\ln|x-1| + C$

(d) $\frac{x^2+x}{4} + \frac{\ln|2x-1|}{8} + C$

(e) $\frac{x^2}{2} + 5x + 2\ln|x-1| + C$

(f) $\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$

Q2

(a) $\tan^{-1}(x+1) + C$

(b) $\frac{1}{4}\tan^{-1}\left(x + \frac{1}{2}\right) + C$

(c) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

Q3

(a) $\frac{1}{2}\ln|x^2+6x+10| - \tan^{-1}(x+3) + C$

(b) $\frac{1}{3x+2} + \frac{2}{3}\ln|3x+2| + C$

(c) $\ln|x^2+6x+10| - 5\tan^{-1}(x+3) + C$

Q4

(a) $x - \ln|x^2+2x+2| + \tan^{-1}(x+1) + C$

(b) $x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \ln|x^2+x+1|$

(c) $2x + 5\tan^{-1}(x+2) - 4\ln|x^2+4x+5| + C$

Q5

(a) $\sin^{-1}\left(\frac{x-3}{3}\right) + C$

(b) $\sin^{-1}(x-2) + C$

(c) $\sin^{-1}(x+1) + C$

Q6

(a) $\frac{3}{2}\sin^{-1}(2x-1) - \sqrt{x-x^2} + C$

(b) $\sin^{-1}(x-1) - \sqrt{2x-x^2} + C$

(c) $\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{21}}\right) - \sqrt{-x^2+x+5} + C$

Q7

(a) $\sin^{-1}(x) + \sqrt{1-x^2} + C$

(b) $2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2} + C$

(c) $\sin^{-1}(x-1) - \sqrt{2x-x^2} + C$

Q8

(a) $\sin^{-1}\left(\frac{x+b}{\sqrt{b^2-a^2}}\right) + C$

(b) $\frac{1}{\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{x+a}{\sqrt{b^2-a^2}}\right) + C$

P1

(a) Use the fact that $\frac{x^n}{1+x} = \frac{1+x^n-1}{1+x}$

(b) $\frac{x^n}{1+x} < x^n$ for $x \in [0,1]$.

(c) $\ln 2$

P2

(a) Use the same technique as in Chapter 2, Exercise 2J, Question 8.

(b) Notice that $0 \leq x(1-x) \leq \frac{1}{4}$.

(c) When $P(x)$ is integrated from 0 to 1, it yields a rational number since it has integer coefficients.

(d) It is a rational approximation of π that increases in accuracy as n gets larger.

Exercise 4E
Partial Fractions

F1

(a) $\int \frac{A}{x+a} + \frac{B}{x+b}\, dx$

(b) $\int \frac{Ax+B}{x^2+a} + \frac{C}{x+b}\, dx$

(c) $\int \frac{A}{(x+a)^2} + \frac{B}{x+a} + \frac{C}{x+b}\, dx$

Q1

(a)

$$\frac{x-1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

then multiply both sides by $(x+1)(x-2)$

(b) $A + B = 1$

(c) $-2A + B = -1$

(d) $A = \frac{2}{3}$, $B = \frac{1}{3}$

(e) $\frac{2}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + C$

Q2

(a) $\ln|2x+1| - \ln|x+3| + C$

(b) $\frac{1}{6}(\ln|x+2| - \ln|x+4|) + C$

(c) $\ln|x+1| + \frac{1}{2}\ln|2x+1| + C$

(d) $\frac{1}{4}\ln|x+1| - \frac{1}{3}\ln|x| + \frac{1}{12}\ln|x-3| + C$

(e) $\frac{10}{3}\ln 2$

(f) $4\ln 2 - \frac{7}{2}\ln 3$

Q3

Factorise the denominator and use partial fractions.

Q4

(a)

$$\frac{x+1}{(x^2+1)(x-1)} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

then multiply both sides by $(x^2+1)(x-1)$

(b) $C = 1$

(c) $A = -1$

(d) $B = 0$

(e) $\ln|x-1| - \frac{1}{2}\ln|x^2+1| + C$

Q5

(a) $\frac{1}{5}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3}{5}\ln|x+1| + \frac{3}{10}\ln|x^2+4| + C$

(b) $\tan^{-1}(x) - \ln|x+3| + \frac{1}{2}\ln|x^2+1| + C$

(c) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \ln|x+2| + C$

Q6

$\frac{1}{3}\tan^{-1}(x) + \frac{1}{3\sqrt{2}}(\ln|x-\sqrt{2}| - \ln|x+\sqrt{2}|) + C$

Q7

(a)

$$A(x+1) + B(x-1)(x+1) + C(x-1)^2 \equiv x$$

then multiply both sides by $(x+1)(x-1)^2$

(b) $A = \frac{1}{2}$

(c) $C = -\frac{1}{4}$

(d) Substituting $x = 0$ is easy and yields $B = \frac{1}{4}$

(e) $-\frac{1}{2(x-1)} + \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + C$

Q8

(a) $\ln|x| - \ln|x-3| - \frac{4}{x-3} + C$

(b) $\frac{1}{4}\left(\ln|x-1| - \ln|x+1| - \frac{2}{x-1}\right) + C$

(c) $\frac{2}{9}(\ln|x-1| - \ln|x+2|) - \frac{1}{3(x+2)} + C$

Q9

(a) $3\sqrt[3]{x}+\frac{3}{2}\ln\left|\frac{\sqrt[3]{x}-1}{\sqrt[3]{x}+1}\right|+C$

(b) $\ln\left|\frac{x\sqrt{x}+1}{x\sqrt{x}-1}\right|+C$

(c) $\frac{3}{4}\left(\ln\left|\frac{\sqrt[3]{x}-1}{\sqrt[3]{x}+1}\right|+2\tan^{-1}\left(\sqrt[3]{x}\right)\right)+C$

Q10

(a) Expand the right-hand side.

(b) $\frac{\pi}{2\sqrt{3}}$

Q11

(a) Expand the right-hand side.

(b) $\frac{\pi}{2}$

Q12

(a) Expand the right-hand side.

(b) See full worked solutions.

(c) $\frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{2R-1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{\frac{2}{R}-1}{\sqrt{3}}\right)\right)$

(d) $\frac{2\pi}{3\sqrt{3}}$

Q13

(a) $A(x)=2\cos x+3\sin x, B(x)=2\cos x-3\sin x$

(b) $\ln 5$

Q14

(a) Expand the right-hand side.

(b) $A=4,\ B=-4$

(d) 2π

P1

(a) Multiply both sides by $x-\alpha_1$ and isolate c_1.

(b) Take the limit as $x\to\alpha_1$ and recall the definition of the first derivative.

P2

(a) Multiply both sides by $x+k$ and isolate c_1. Then let $x=-k$ and simplify.

(b) Let $x=1$ and multiply both sides by $n!$

Exercise 4F

t-formula Substitutions

F1

(a) $\frac{2t}{1+t^2}$ (b) $\frac{1-t^2}{1+t^2}$ (c) $\frac{2t}{1-t^2}$

F2

$d\theta=\frac{2}{1+t^2}dt$

Q1

(a) $\frac{1}{2}\ln\left|\tan\left(\frac{\theta}{2}\right)\right|-\frac{1}{4}\tan^2\left(\frac{\theta}{2}\right)+C$

(b) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{\theta}{2}\right)\right)+C$

(c) $\tan\left(\frac{\theta}{2}\right)+C$

(d) $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\left(\frac{\theta}{2}\right)\right)+C$

(e) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\left(\frac{\theta}{2}\right)\right)$

(f) $-\frac{2}{3\left(3\tan\left(\frac{\theta}{2}\right)+1\right)}+C$

Q2

(a) $\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)-2}{2\tan\left(\frac{\theta}{2}\right)+1}\right|+C$

(b) $\frac{1}{3}\ln\left|\frac{3+\tan\left(\frac{\theta}{2}\right)}{3-\tan\left(\frac{\theta}{2}\right)}\right|+C$

(c) $\frac{1}{4}\ln\left|\frac{2+\tan\left(\frac{\theta}{2}\right)}{2-\tan\left(\frac{\theta}{2}\right)}\right|+C$

(d) $\frac{1}{4}\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)-3}{3\tan\left(\frac{\theta}{2}\right)-1}\right|+C$

(e) $\frac{1}{5}\ln\left|\frac{5+\tan\left(\frac{\theta}{2}\right)}{5-\tan\left(\frac{\theta}{2}\right)}\right|+C$

(f) $\frac{1}{15}\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)-4}{4\tan\left(\frac{\theta}{2}\right)-1}\right|+C$

Q3

(a) $-\ln\left|\tan\left(\frac{\theta}{2}\right)-1\right|+C$

(b) $\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)+1}\right|+C$

(c) $\frac{1}{\sqrt{2}}\ln\left|\frac{1-\sqrt{2}-\tan\left(\frac{\theta}{2}\right)}{1+\sqrt{2}-\tan\left(\frac{\theta}{2}\right)}\right|$

Q4

(a) $\ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right|+C_1$

(b) $\ln|\sec\theta+\tan\theta|+C_2$

(c) See full worked solutions.

P1

$-\frac{1}{1+\tan\theta}+C$

P2

(a) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{5}}\right)+C$

(b) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta+1}{\sqrt{3}}\right)+C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan 2\theta}{\sqrt{2}}\right)+C$

(d) $\sqrt{\tan\theta}+C$

P3

(a) Expand the right-hand side.

(b) $\tan^{-1}\left(\frac{\tan 2\theta}{2}\right)+C$

P4

$\frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{\sqrt{a^2+b^2}-a+b\tan\left(\frac{\theta}{2}\right)}{\sqrt{a^2+b^2}+a-b\tan\left(\frac{\theta}{2}\right)}\right|+C$

P5

$\frac{\pi}{\sqrt{a^2-1}}$

P6

(a) $\frac{\pi a}{(a^2-1)\sqrt{a^2-1}}$

(b) See full worked solutions.

Exercise 4G
Integration by Parts

F1

$uv-\int u'v\,dx$

F2

$uv\big|_a^b-\int_a^b u'v\,dx$

F3

(a) 1

(b) integrated

Q1

(a) $(x-1)e^x+C$

(b) $\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+C$

(c) $\frac{2x\sqrt{x}}{3}\left(\ln x-\frac{2}{3}\right)+C$

(d) $\sin x-x\cos x+C$

(e) $x\tan x+\ln|\cos x|+C$

(f) $\frac{1}{4}\left((2x^2-1)\sin^{-1}(x)+x\sqrt{1-x^2}\right)+C$

(g) $\frac{1}{2}\left((x^2+1)\tan^{-1}(x)-x\right)+C$

(h) $-\frac{1+\ln x}{x}+C$

(i) $\frac{1}{8}(\sin 2x-2x\cos 2x)+C$

Q2

(a) $x(\ln x-1)+C$

(b) $x\sin^{-1}(x)+\sqrt{1-x^2}+C$

(c) $x\tan^{-1}(x)+\frac{1}{2}\ln|x^2+1|+C$

(d) $x\ln(x^2+1)+2\tan^{-1}(x)-2x+C$

(e) $2(\sqrt{x}-1)e^{\sqrt{x}}+C$

(f) $\frac{x}{2}(\sin(\ln x)-\cos(\ln x))+C$

Q3

(a) $\frac{e^x}{2}(\sin x-\cos x)+C$

(b) $2x\sin x+(2-x^2)\cos x+C$

(c) $x\left((\ln x)^2-2\ln x+2\right)+C$

Q4

(a) $\dfrac{6x^3-4}{45}(1+x^3)\sqrt{1+x^3}+C$

(b) $\dfrac{x^4-2}{6}\sqrt{1+x^4}+C$

(c) $\dfrac{x^2\sin(x^2)+\cos(x^2)}{2}+C$

(d) $\dfrac{x^2-1}{2}e^{x^2}+C$

(e) $\dfrac{1}{3}\left(\sin(e^{3x})-e^{3x}\cos(e^{3x})\right)+C$

(f) $-\dfrac{3x^2+2}{15}(1-x^2)\sqrt{1-x^2}+C$

Q5

$\dfrac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+C$

Q6

(a) $\dfrac{\pi-2}{2}$

(b) $\dfrac{e^2+1}{4}$

(c) $\dfrac{\pi-2\ln 2}{4}$

(d) $\dfrac{1+e^{-\frac{\pi}{2}}}{2}$

(e) $\dfrac{\pi}{4}-\dfrac{\pi^2}{32}-\dfrac{\ln 2}{2}$

(f) $\dfrac{1}{6}$

(g) 1

(h) $\dfrac{1}{2}-\dfrac{1}{e}$

(i) $2\ln 2$

P1

(a) $-\dfrac{\sqrt{4-x^2}}{x}-\sin^{-1}\left(\dfrac{x}{2}\right)+C$

(b) $\dfrac{1}{2}x\sqrt{1-x^2}+\dfrac{1}{2}\sin^{-1}(x)+C$

(c) $\dfrac{x}{1+\ln x}+C$

(d) $x\ln\left(x+\sqrt{x^2-a^2}\right)-\sqrt{x^2-a^2}+C$

(e) $2\sqrt{1+x}\sin^{-1}(x)+4\sqrt{1-x}+C$

(f) $2\sqrt{1+x}\tan^{-1}(\sqrt{x})-2\ln\left|\sqrt{x}+\sqrt{1+x}\right|+C$

P2

See full worked solutions.

Exercise 4H

Reduction Formulae

F1

(a) index (b) repeatedly (c) I_n

F2

parts, parts

F3

(a) $1-\cos^2 x$ (b) $1-\sin^2 x$ (c) $\sec^2 x-1$

(d) $1+\tan^2 x$ (e) x^n-1 (f) $1-x^n$

Q1

(a) See full worked solutions.

(b) $(x^2-2x+2)e^x+C$

Q2

(a) See full worked solutions.

(b) $\dfrac{1}{2}x^2(\ln x)^2-\dfrac{1}{2}x^2\ln x+\dfrac{1}{4}x^2+C$

Q3

(a) See full worked solutions.

(b) $\dfrac{\pi^2}{4}-2$

Q4

See full worked solutions.

Q5

See full worked solutions.

Q6

(a) See full worked solutions.

(b) $\dfrac{16}{35}$

Q7

(a) See full worked solutions.

(b) $\dfrac{243}{1540}$

Q8

(a) See full worked solutions.

(b) $\dfrac{32}{315}$

Q9

(a) See full worked solutions.

(b) $\dfrac{14\sqrt{2}-16}{15}$

Q10

(a) See full worked solutions.

(b) $\dfrac{8}{105}$

Q11

(a) See full worked solutions.

(b) $-\dfrac{x^2+2}{3}\sqrt{1-x^2}+C$

Q12

(a) See full worked solutions.

(b) $\dfrac{x}{4\left(x^2+1\right)^2}+\dfrac{3x}{8\left(x^2+1\right)}+\dfrac{3}{8}\tan^{-1}(x)$

Q13

(a) See full worked solutions.

(b) $3\ln 2-2$

Q14

(a) See full worked solutions.

(b) $\dfrac{28}{15}$

Q15

See full worked solutions.

Q16

(a) See full worked solutions.

(b) Use the reduction formula recursively.

(c) The integrand of I_{k+1} is lower than the integrand of I_k since $0 \leq \sin x \leq 1$ in the domain of integration.

(d) See full worked solutions.

(e) $\dfrac{\pi}{2}$

P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) $\dfrac{23}{15}$

P3

(a) See full worked solutions.

(b) $I_n=\pi$ if n is odd, and $I_n=0$ if n is even.

P4

(a) See full worked solutions.

(b) See full worked solutions.

(c) $\dfrac{1}{n+1}\times\dfrac{1}{\dbinom{2n+1}{n}}$

P5

See full worked solutions.

Exercise 4I

Further Substitutions

F1

They are equal.

F2

(a) $u=-x$ (b) $u=ax$

(c) $u=a-x$ (d) $u=a+x$

Q1

See full worked solutions.

Q2

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi^2}{4}$

(d) $\dfrac{\pi^2}{4}$ (e) $\dfrac{\pi}{4}$ (f) $1-\dfrac{\pi}{4}$

Q3

See full worked solutions.

Q4

(a) $\dfrac{1}{3}$ (b) $\dfrac{\pi}{4}$ (c) 2

Q5

See full worked solutions.

Q6

See full worked solutions.

Q7

See full worked solutions.

Q8

(a) See full worked solutions.

(b) $\dfrac{\pi}{12}$

Q9

See full worked solutions.

Q10

(a) See full worked solutions.

(b) $a\ln(\sec a)$

Q11

(a) See full worked solutions.

(b) See full worked solutions.

(c) $\dfrac{\pi}{2}-1$

P1

(a) $\dfrac{\pi\ln 2}{8}$

(b) $-\dfrac{\pi\ln 2}{2}$

(c) $\dfrac{\pi}{4}$

(d) $\dfrac{\pi^2}{2\sqrt{a(a+b)}}$

P2

$\dfrac{\pi\theta}{4\sin\theta}$

P3

(a) See full worked solutions.

(b) $\dfrac{2}{1+e}-1$

P4

(a) See full worked solutions.

(b) $\dfrac{2}{1+e}-1$

P5

(a) Use the substitution $u = n\pi - x$.

(b) Consider $\cos^2 x = \cos^2(x + k\pi)$ for $k \in \mathbb{Z}$.

(c) $\dfrac{n^2\pi^2}{2}$

Chapter Review

R1

(a) $\sin x - \dfrac{\sin^3 x}{3} + C$

(b) $(x^2+1)\tan^{-1}(x) - x + C$

(c) $\dfrac{2}{5}(x+1)^{\frac{5}{2}} - \dfrac{2}{3}(x+1)^{\frac{3}{2}} + C$

(d) $\dfrac{1}{4}\ln\left|\dfrac{1+x^2}{1-x^2}\right| + C$

(e) $\dfrac{1+e^{\frac{\pi}{2}}}{2}$

(f) $\dfrac{2}{\cot\left(\frac{x}{2}\right)-1} + C$

(g) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right) + C$

(h) $x\ln|x^2-1| + \ln\left|\dfrac{1+x}{1-x}\right| - 2x + C$

(i) $\ln|x-1| + \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

R2

(a) $\dfrac{3x}{8} - \dfrac{\sin 2x}{4} + \dfrac{\sin 4x}{32} + C$

(b) $\dfrac{1}{2}\left(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|\right) + C$

(c) $2\cos\sqrt{x} + 2\sqrt{x}\sin\sqrt{x} + C$

(d) $\ln|1+\sin^2 x| + C$

(e) $-\ln|1+e^{-x}| + C$

(f) $2\ln|x^2+2x+5| - \dfrac{7}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right) + C$

(g) $\dfrac{\ln 3}{4}$

(h) $-2\sqrt{x} - 2\ln|1-\sqrt{x}| + C$

(i) $\dfrac{1}{2}\left(9\sin^{-1}\left(\dfrac{x}{3}\right) - x\sqrt{9-x^2}\right)$

R3

(a) $\frac{1}{10}\tan^{-1}\left(\frac{2\tan x}{5}\right)$

(b) $\frac{1}{3}\left(\tan^{-1}(x)+\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}-x}{\sqrt{2}+x}\right|\right)$

(c) $\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}-1-\tan\left(\frac{\theta}{2}\right)}{\sqrt{2}+1+\tan\left(\frac{\theta}{2}\right)}\right|$

(d) $-\frac{\ln|1+x^{-5}|}{5}$

(e) $6\sqrt[6]{x}-3\sqrt[3]{x}+2\sqrt{x}-6\ln|1+\sqrt[6]{x}|$

(f) $\frac{1}{6}\ln|1-2e^{-x}|+\frac{1}{3}\ln|1+e^{-x}|+C$

(g) $2\sqrt{1+\sin x}+\ln\left|\frac{1-\sqrt{1+\sin x}}{1+\sqrt{1+\sin x}}\right|+C$

(h) $4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)-\sqrt{x(4-x)}+C$

(i) $\frac{4}{5}(1-\sqrt{x})^{\frac{5}{2}}-\frac{4}{3}(1-\sqrt{x})^{\frac{3}{2}}+C$

R4

(a) $2\tan^{-1}(\sqrt{x})+C$

(b) $\frac{\pi\ln 2}{8}$

(c) $\frac{1}{4}$

(d) $x+\frac{4}{1-e^{x}}+C$

(e) $(1+\ln x)\ln(1+\ln x)-\ln x+C$

(f) $\tan^{-1}\left(x+\frac{1}{x}\right)+C$

(g) $\frac{1}{b}$

(h) $\frac{\sqrt{\sin 2x}}{\cos x}+C$

(i) $x\ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}+C$

R5

(a) $\sqrt{\frac{1+x}{1-x}}$

(b) $\frac{\pi}{\sqrt{2}}$

(c) $\frac{1}{2}(x-\ln(\cos x+\sin x))$

(d) $\frac{\sin x-x\cos x}{x\sin x+\cos x}$

(e) $2\pi\ln 2$

(f) 0

R6

(a) $\frac{\pi}{6}$

(b) $\frac{2}{15}$

(c) $2\ln 2-1$

(d) $\frac{\pi}{4}-\frac{2}{3}$

(e) $\frac{8}{15}$

(f) $\frac{3\pi}{16}$

(g) $\frac{\sqrt{3}}{2}+\frac{\pi}{3}$

(h) $\pi-2$

(i) $\ln(1+\sqrt{2})-\frac{1}{\sqrt{2}}$

(j) $\frac{\pi}{3\sqrt{3}}$

(k) $\frac{\pi}{3\sqrt{3}}$

(l) 3π

(m) $\frac{1}{3}\ln\left(\frac{5}{4}\right)$

(n) $\frac{1}{2}\ln 2-\frac{\pi}{8}$

(o) $\frac{4}{3}-\frac{11}{9}\ln 2$

(p) $\frac{\pi}{3\sqrt{3}}$

(q) $\ln 2$

(r) $\frac{1}{5}\ln 6$

(s) $\frac{1}{9}(2e^3+1)$

(t) $\frac{1}{4}(\pi-2)$

(u) $\frac{1}{4}(\pi-2)$

R7

(a) x^3+1

(b) $\frac{2\pi}{3\sqrt{3}}$

R8

(a) See full worked solutions.

(b) $x\left(\ln x\right)^3-3x\left(\ln x\right)^2+6x\ln x-6x+C$

R9

(a) See full worked solutions.

(b) $\frac{1}{5}\cos^4 x\sin x+\frac{4}{15}\cos^2 x\sin x+\frac{8}{15}\sin x$

ANSWERS

R10

(a) See full worked solutions.

(b) $-\dfrac{32}{315}$

R11

(a) See full worked solutions.

(b) $\dfrac{1}{336}$

R12

(a) See full worked solutions.

(b) $\dfrac{16}{105}$

R13

(a) See full worked solutions.

(b) 1

R14

(a) See full worked solutions.

(b) 0

R15

(a) See full worked solutions.

(b) $\dfrac{\pi}{4}$

5. Mechanics

Exercise 5A

Velocity-Displacement Equations

F1

(a) $\dfrac{dv}{dt}$ (b) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

(c) $\dfrac{dx}{dt}$ (d) $v\dfrac{dv}{dx}$

F2

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\times\frac{dv}{dx}\\ &= v\frac{dv}{dx}\\ &= \frac{dx}{dt}\times\frac{dv}{dx}\\ &= \frac{dv}{dt}\\ &= a\end{aligned}$$

F3

(a) Turn v into $\dfrac{dx}{dt}$ then solve the differential equation.

(b) Turn a into $\dfrac{dv}{dt}$ then solve the differential equation to obtain $v = f(t)$, then integrate again with respect to t.

(c) Turn a into $v\dfrac{dv}{dx}$ and solve the differential equation and obtain $v = f(x)$, then follow the same steps as above.

Q1

(a) $\ln t$ (b) $\dfrac{1}{t}$ (c) $\dfrac{1}{2t^2}$

Q2

$v^2 = \ln\left|\dfrac{2x+1}{3}\right|$

Q3

(a) $v = 4\sqrt{x+4}$ (b) $t = \dfrac{\sqrt{x+4}}{2} - 1$

Q4

(a) $21\,\mathrm{m\,s^{-2}}$ (b) $t = \dfrac{1}{3}\ln 7$ seconds

Q5

(a) $a = 8x^3$ (b) $x = \dfrac{2}{4t+1}$

Q6

(a) $v = \dfrac{3}{x+2}$ (b) $t = \dfrac{1}{6}(x^2+4x)$

Q7

(a) $v^2 = \dfrac{1}{4}(16 - x^4)$

(b) $2\,\mathrm{m\,s^{-1}}$ when $x = 0$

(c) Initially, the particle is at rest at $x = 2$. Since acceleration is then negative, the particle moves to the left. The particle then oscillates back and forth between $x = \pm - 2$. Speed is maximised at the origin. Acceleration is maximised at the end-points $x = \pm 2$.

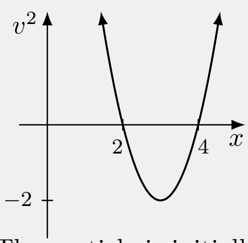
Q8

(a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 6$

(b) Right, since $\ddot{x} > 0$ when $x = 4$ initially.

(c) When $x = 3$, $v^2 < 0$

(d)



(e) The particle is initially at $x = 4$ and moves to the right indefinitely.

Q9

$v^2 \geq 0$ so either $|x| \leq \frac{1}{2}$ or $|x| \geq 3$. But initially the particle is at $x = 0$ so it must satisfy $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Q10

(a) Use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(b) $x = \frac{1}{2}\ln(1+t)$

(c) $x \to \infty$ and $v \to 0$

(d) Initially the particle is at the origin and moving to the right with speed $\frac{1}{2}\,\mathrm{m\,s^{-1}}$. The particle then moves to the right indefinitely.

Q11

(a) $v = \sqrt{18 - 2e^{-x}}$

(b) The particle moves to the right indefinitely with speed approaching $3\sqrt{2}\,\mathrm{m\,s^{-1}}$

Q12

See full worked solutions.

Q13

(a) To the right.

(b) $a = x - 2$

(c) $x = 2 - 2e^{-t}$

(d) The particle will approach $x = 2$ with decreasing speed indefinitely.

Q14

See full worked solutions.

Q15

(a) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) $x = \ln\left(\tan\left(t\sqrt{2} + \frac{\pi}{4}\right)\right)$

P1

(a) When $x = R$, $a = -g$.

(b) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(c) Substitute $u^2 = 2gR$.

(d) When the particle is projected with speed at least $\sqrt{2gR}$, then the particle will never return to Earth.

P2

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and solve the differential equation.

Exercise 5B

Simple Harmonic Motion

F1

$-n^2(x - x_0)$

F2

A affects how far back and forth the particle moves.
n affects the period of movement.
x_0 affects the centre of motion
α affects the initial position.

F3

(a) $v^2 = n^2(A^2 - x^2)$

(b) $v^2 = n^2(A^2 - (x - x_0)^2)$

F4

$T = \frac{2\pi}{n}$

F5

(a) $x = 0$ (b) $x = \pm A$

(c) $x = -A$ (d) $x = 0$

Q1

(a) Negative sine. (b) Positive sine.

(c) Positive cosine. (d) Negative cosine.

ANSWERS

Q2

(a) $A = 3$ (b) $T = \pi$

(c) $x_0 = 1$ (d) $-2 \le x \le 4$

(e) $v = 6$ (f) $a = 12$

(g) $x = 1 + \dfrac{3\sqrt{3}}{2}$ (h) $v = 3$

(i) $a = -6\sqrt{3}$

Q3

(a) $t = \dfrac{\pi}{3}, \dfrac{5\pi}{9}$ (b) $t = \dfrac{4\pi}{9}, \dfrac{10\pi}{9}$

(c) $t = \dfrac{\pi}{6}, \dfrac{7\pi}{18}$ (d) $t = \dfrac{5\pi}{18}, \dfrac{11\pi}{18}$

Q4

(a) $v = 0$ (b) $v = 0$

(c) $v = -8$ (d) $v = 4\sqrt{3}$

Q5

(a) $\ddot{x} = -9x$ (b) $\ddot{x} = -4(x+1)$

(c) $\ddot{x} = -4x$ (d) $\ddot{x} = -9(x-1)$

Q6

(a) Moving to the right towards O.

(b) At the origin and moving left.

(c) Moving left towards O.

Q7

(a) $A = \dfrac{1}{2}, T = \pi, x_0 = \dfrac{1}{2}$

(b) $A = \dfrac{1}{2}, T = \pi, x_0 = \dfrac{1}{2}$

(c) $A = 1, T = \pi, x_0 = 4$

(d) $A = \dfrac{3}{2}, T = \dfrac{\pi}{2}, x_0 = \dfrac{1}{2}$

Q8

(a) $A = 5, T = \pi, x_0 = 0$

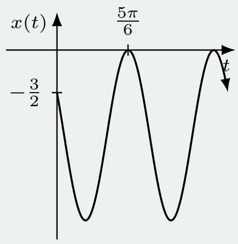
(b) $A = 5, T = \dfrac{\pi}{2}, x_0 = 0$

(c) $A = 13, T = \dfrac{2\pi}{3}, x_0 = 7$

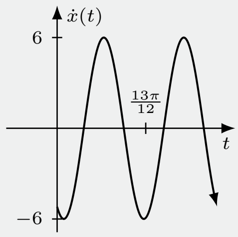
(d) $A = \sqrt{2}, T = \dfrac{\pi}{3}, x_0 = 1$

Q9

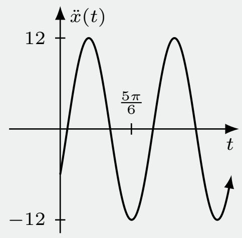
(a)



(b)



(c)



Q10

(a) $x_0 = \dfrac{12+2}{2} = 7$

$T = 8$ so $n = \dfrac{\pi}{4}$

$A = \dfrac{12-2}{2} = 5$

(b) 3:29am

(c) 8:31am

Q11

(a) $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

(b) 5

(c) $A = \dfrac{5}{3}, T = \dfrac{2\pi}{3}$

(d) $x = \dfrac{5}{3}\cos 3t$

Q12

(a) $a = -4\pi^2 x$ (b) $v = 6\pi,\ a = 0$

(c) $v = 6\pi,\ a = 12\pi^2$ (d) $6,\ v = -6\pi$

Q13

(a) $A = \dfrac{3\sqrt{2}}{n}$ (b) $3\sqrt{\dfrac{3}{2}}$

Q14

(a) $a = -4(x-2)$

(b) $A = 3,\ T = \pi,\ x_0 = 2$

(c) The particle travels 9 metres and is at $x = 2$.

Q15

(a) The centre of motion is $x = 4$ and when $x = 0,\ v = 0$

(b) $v^2 \geq 0$ so $8x - x^2 \geq 0$ and hence $0 \leq x \leq 8$

Q16

$A = 2\sqrt{3},\ n = 1$

Q17

Minimum distance is $5 - 3\sqrt{2}$ metres, which occurs when $t = \frac{7\pi}{8}$.

Q18

$A = 6,\ T = \frac{2\pi}{3}$

Q19

$A = \frac{1}{\pi}$

Q20

(a) $a^3 + b^3$

(b) $x_0 = 5,\ A = 3,\ T = \frac{\pi}{2}$

Q21

See full worked solutions.

Q22

(a) Consider $x = 2\sin t$, which has maximum speed $2\,\mathrm{m\,s^{-1}}$. According to the claim, when the particle is at $x = 1$ it should have speed $1\,\mathrm{m\,s^{-1}}$. But actually, the particle has speed $\sqrt{3}\,\mathrm{m\,s^{-1}}$.

(b) $k = \sqrt{2}$

P1

See full worked solutions.

P2

(a) Show that this satisfies $\ddot{x} = -\frac{k}{m}x$

(b) $M(t) = \frac{1}{2}kA^2$

P3

Let $x_P = A\cos(nt + \alpha)$ and let $x_Q = B\cos(nt + \beta)$.

Exercise 5C

Projectile Motion

F1

(a) $\mathbf{a}(t) = -g\,\mathbf{j}$

$\mathbf{v}(t) = V\cos\theta\,\mathbf{i} + \big(-gt + V\sin\theta\big)\,\mathbf{j}$

$\mathbf{r}(t) = Vt\cos\theta\,\mathbf{i} + \big(-\frac{1}{2}gt^2 + Vt\sin\theta\big)\,\mathbf{j}$

(b) Substitute $t = \dfrac{x}{V\cos\theta}$ into

$y = -\frac{1}{2}gt^2 + Vt\sin\theta.$

$$\begin{aligned}
y &= -\frac{1}{2}gt^2 + Vt\sin\theta \\
&= -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\sin\theta\left(\frac{x}{V\cos\theta}\right) \\
&= -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta \\
&= x\tan\theta - \frac{gx^2}{2V^2}\left(1 + \tan^2\theta\right)
\end{aligned}$$

F2

(a) $T = \dfrac{V\sin\theta}{g}$

(b) $H = \dfrac{V^2\sin^2\theta}{2g}$

(c) $R = \dfrac{V^2\sin 2\theta}{g}$

(d) $R_{\max} = \dfrac{V^2}{g}$

Q1

(a) Let $y = 0$, solve for t, then substitute into x.

(b) Substitute $\theta = 15°$, $x = 40$ into the previous result.

(c) Usual Cartesian equation derivation, but this time using $V^2 = 80g$.

(d) $\theta = 45°,\ 72°$

Q2

Substitute (x_1, h) and (x_2, h) into the Cartesian equation. Eliminate the $-\dfrac{g}{2V^2}\sec^2\theta$ parts by manipulating and dividing the two equations to get

$$\frac{h - x_1\tan\theta}{h - x_2\tan\theta} = \frac{x_1^2}{x_2^2}$$

ANSWERS

Re-arrange to make $\tan\theta$ the subject. Alternatively, substitute $y = h$ into the Cartesian equation and obtain a quadratic in terms of x, which has roots x_1 and x_2. Find $x_1 + x_2$ and x_1x_2 using the sum/product of roots and divide them.

Q3

Equate the y-equations of the trajectory of the monkey $y = h - \frac{1}{2}gt^2$ and the dart $y = -\frac{1}{2}gt^2 + Vt\sin\theta$ to get $t = \frac{h}{V\sin\theta}$. Substitute this into the x-equation of the dart $x = Vt\cos\theta$ to get $x = h\cot\theta$. But $\cot\theta = \frac{d}{h}$ and so $x = d$ at that same point in time.

Q4

(a) $h = -\frac{gd^2}{2V^2}\sec^2\theta + d\tan\theta$. This is a quadratic in terms of $\tan\theta$. Let $\frac{dh}{d\theta} = 0$ and solve for $\tan\theta$.

(b) Substitute the previous result into the Cartesian equation.

(c) Use the fact that $h > 0$.

Q5

(a) First square both sides of $\dot{y}$ to get

$$(\dot{y})^2 = g^2t^2 - 2Vgt\sin\theta + V^2\sin^2\theta$$

then expand and substitute in $h = -\frac{1}{2}gt^2 + Vt\sin\theta$

(b) $S^2 = (\dot{x})^2 + (\dot{y})^2$. Use the previous result and $\dot{x} = V\cos\theta$.

Q6

(a) $x = ut\cos\alpha$ and $x = wt\cos\beta$. When $t = T$, $x = d$ and then equate.

(b) Do a similar thing for y to get $h = T(u\sin\alpha - w\sin\beta)$. Divide with $d = uT\cos\alpha$ and use the fact that $\frac{h}{d} = \tan\theta$, and the previous result.

Q7

(a) Usual Cartesian equation proof.

(b) Solve the equation of the trajectory simultaneously with the equation of the ramp, which is $y = x\tan\alpha$.

(c) Use right-angled triangles and note that $\cos\alpha = \frac{d}{R}$.

(d) Let $\frac{dR}{d\theta} = 0$ and solve to get $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$, which is precisely halfway between the vertical and the angle of the plane.

(e) Find R and T for $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$.

Q8

Substitute $P(p, q)$ into the Cartesian equation and manipulate.

Q9

(a) $H = \frac{V^2\sin^2\theta}{2g}$ and the max of $\sin^2\theta$ is 1.

(b) Standard Cartesian equation and substitute in $h = \frac{V^2}{2g}$.

(c) Part (b) is a quadratic in terms of $\tan\theta$. The discriminant is positive.

(d) The product of roots is greater than 1, which cannot occur if $\tan\theta_1 < 1$ and $\tan\theta_2 < 1$.

Q10

(a) Substitute $(d, 0)$ into the Cartesian equation $y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta + h$

(b) The previous result is a quadratic in terms of d. When $\theta = \alpha$, the derivative with respect to θ is zero.

(c) Substitute the previous result and $d = D$ back into part (a) and re-arrange.

(d) Use the double angle formula for $\cot 2\alpha$.

Q11

(a) Show that $H = \frac{V^2\sin^2\theta}{2g}$ by usual means, then substitute $y = \frac{H}{2}$ into the Cartesian equation and re-arrange to express it as a quadratic in terms of x.

(b) The horizontal distance is

$$|x_2 - x_1| = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$$

then use the sum and product of roots to find $x_1 + x_2$ and x_1x_2.

Q12

First show that $V^2 \geq \frac{gR}{2\sin\theta\cos\theta}$ since it clears the fence, then substitute into the Cartesian equation.

Q13

Show that $R(\theta) = \dfrac{V^2 \sin 2\theta}{g}$ and calculate $R - a$ and $R + b$, then subtract the results.

Q14

(a) Equate the x components and use the fact that $0 < \cos\theta < 1$

(b) Equate y to find the time to collide. Substitute this value into the time equation for the particle being projected horizontally.

Q15

(a) The time of flight is $T = \dfrac{V \sin\theta}{g}$. When $t = T$, $x = d$ and $y = h$. Obtain two expressions and combine them.

(b) Square one of the previous results, use the fact that $\cos^2\theta = 1 - \sin^2\theta$, and substitute the other result in to eliminate θ.

Q16

Similar set of steps to Question 7.

P1

See full worked solutions.

P2

See full worked solutions.

Exercise 5D

Resisted Horizontal Motion

F1

$$a = \ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

F2

(a) force, velocity

(b) gravity

(c) t

Q1

(a) $5a = -20v$

(b) $a = v\dfrac{dv}{dx}$

(c) $a = \dfrac{dv}{dt}$

(d) Equate v and re-arrange.

(e) For (b), use $v = \dfrac{dx}{dt}$ then re-arrange and solve. For (c), integrate directly with respect to t.

(f) $x = 10$

Q2

(a) $a = -4v^2$ then use $a = v\dfrac{dv}{dx}$

(b) $v = \dfrac{dx}{dt}$ then re-arrange and solve for x.

(c) $x = 4\ln 3$

(d) $t = \dfrac{3}{8}$, $v = \dfrac{5}{8}$

Q3

(a) $k = \dfrac{3}{8}$ (b) $k = 2\ln 2$ (c) $k = 2$

Q4

(a) Use $a = v\dfrac{dv}{dx}$

(b) Use $a = \dfrac{dv}{dt}$

(c) Combine the previous two results.

(d) $x = \dfrac{u}{2k}$

(e) $t = \dfrac{1}{k}\ln 2$

(f) Substitute $x = \dfrac{u}{2k}$ into $v = u - kx$, or substitute $t = \dfrac{1}{k}\ln 2$ into $v = ue^{-kt}$.

Q5

(a) $x = \dfrac{20}{k}$ (b) $k = \dfrac{1}{2}$

Q6

(a) Use $a = v\dfrac{dv}{dx}$

(b) Use $a = \dfrac{dv}{dt}$

(c) Combine the previous two results or obtain it directly from either one.

(d) As $t \to \infty$, $x \to \infty$ so no it does not have a limiting position.

P1

Find the limiting position $\frac{u}{k}$ and form the relationship $v = f(x)$. Substitute $x = r\% \times \frac{u}{k}$ into and show that $v = (100 - r)\% \times u$

P2

(a) $a = -(v + v^3)$ and use $a = v\frac{dv}{dx}$

(b) $a = \frac{dv}{dt}$

(c) $x = \tan^{-1}(u)$

(d) As $u \to \infty$, the limiting position approaches $\frac{\pi}{2}$.

(e) $v = 0$

P3

No. A counter-example is $a = -v^2$ which leads to $x = \ln(1 + ut)$ where u is the initial speed. This particle moves to the right indefinitely.

P4

No. In fact, the heavier particle travels further.

Exercise 5E

Resisted Vertical Motion

F1

(a) negative, negative

(b) positive, negative

F2

(a) weight

(b) weight, positive, negative

F3

(a) terminal

(b) terminal, t, x

(c) 0

Q1

(a) $4a = 40 - 20v$

(b) $a = \frac{dv}{dt}$

(c) One is to let $a = 0$, and the other is to let $t \to \infty$. Either method yields $v_T = 2$

(d) $\frac{1}{5}\ln 2$

(e) $x = \frac{1}{5}\left(2\ln 2 - 3\right)$

Q2

(a) $a = \frac{dv}{dt}$

(b) Let $v = 0$.

(c) $a = v\frac{dv}{dx}$

(d) Let $v = 0$.

Q3

(a) $a = g - kv$ then let $a = 0$.

(b) $a = v\frac{dv}{dx}$

(c) Substitute in $w = \frac{g}{k}$ into the previous result.

(d) $a = \frac{dv}{dt}$ then repeat as above.

Q4

(a) $a = v\frac{dv}{dx}$

(b) Let $v = 0$ in the above result.

(c) $a = \frac{dv}{dt}$

(d) Let $v = 0$ in the above result.

(e) Take the limit of T as $u \to \infty$.

Q5

(a) $a = g - kv^2$ and let $a = 0$

(b) Form the acceleration equation for upwards motion

$$a = -g - kv^2 = -k\left(\frac{g}{k} + v^2\right) = -k(w^2 + v^2)$$

then use $a = v\frac{dv}{dx}$.

(c) Let $v = w$ to find the displacement. Then find the difference between that displacement and the maximum height.

ANSWERS

Q6

(a) $a = v\dfrac{dv}{dx}$

(b) Re-arrange the previous result and make v^2 the subject.

(c) Either let $a = 0$ or take the limit as $x \to \infty$ to get $w = \sqrt{\dfrac{g}{k}}$

(d) Divide the numerator and denominator of the term in the bracket by k then substitute in $w^2 = \dfrac{g}{k}$.

(e) $a = \dfrac{dv}{dt}$ and use partial fractions.

(f) Re-arrange the previous result.

Q7

(a) $a = g - kv$ and let $a = 0$

(b) $a = \dfrac{dv}{dt}$

(c) Produce a new acceleration equation $a = -g - kv$ and use $a = \dfrac{dv}{dt}$.

(d) Substitute the result from (c) into (b).

Q8

(a) $ma = mg - mkv^2$

(b) $V = \dfrac{g}{k}$

(c) Derive x in terms of v, and substitute $x = D$, $v = W$.

(d) Produce a new acceleration equation $a = -g - kv^2$ and use $a = v\dfrac{dv}{dx}$. Find x in terms of v and substitute $x = H$, $v = 0$.

(e) Observe that $H = D$ since the particle falls the same amount that it travels up.

(f) Make W the subject from the previous result.

(g) The impact speed is lesser.

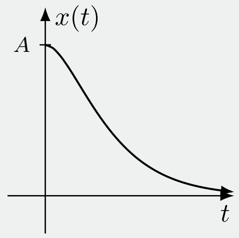
Q9

(a) Start with our standard acceleration for simple harmonic motion $\ddot{x} = -n^2x$. Then add the resistive acceleration which is $-2nv$.

(b) Calculate the left-hand side and show it is equivalent to (a).

(c) Solve the first-order differential equation for y, then substitute back into (b) and solve again for $x(t)$. Alternatively just differentiate the given result and show that it satisfies the differential equation.

(d) The particle approaches $x = 0$ from the right, but never crosses it. So the particle does not even make it to the origin from the starting point at $x = A$.



P1

(a) See full worked solutions.

(b) As $u \to \infty$, $V \to \dfrac{2}{\pi}v_T$

P2

See full worked solutions.

P3

Find the time for the upwards and downwards journey separately, then add them.

P4

See full worked solutions.

Exercise 5F
Resisted Projectile Motion

F1

(a) $\ddot{x} = 0$
$\ddot{y} = -g$

(b) $\ddot{x} = -k\dot{x}$
$\ddot{y} = -k\dot{y} - g$

F2

(a) $\ddot{\underset{\sim}{r}}(t) = -g\underset{\sim}{j}$

(b) $\ddot{\underset{\sim}{r}}(t) = -k\dot{\underset{\sim}{r}}(t) - g\underset{\sim}{j}$

F3

parabolic, limiting, infinity

F4

(a) No air resistance.

(b) With air resistance.

Q1

See full worked solutions.

Q2

(a) Show that

$$x = \frac{V\cos\theta}{k}\left(1 - e^{-kt}\right)$$

and then take the limit of x as $t \to \infty$

(b) Show that

$$\dot{y} = \left(\frac{g}{k} + V\sin\theta\right)e^{-kt} - \frac{g}{k}$$

and then let $\dot{y} = 0$

(c) Substitute the time value from (b) into

$$y = \frac{1}{k}\left(\frac{g}{k} + V\sin\theta\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$$

(d) Let $\frac{dD}{d\theta} = 0$.

Q3

(a) $\ddot{x} = \frac{d\dot{x}}{dt}$

(b) $x = \frac{1}{k}\ln\left(1 + ukt\sin\theta\right)$

(c) Make t the subject from the x-equation, and substitute into the y-equation.

(d) If Bob's model was correct, then it should show a linear trajectory since gravity is not present. However, the trajectory is clearly not linear.

Q4

See full worked solutions.

P1

(a) We want the magnitude of drag to be proportional to the square of the speed of the particle, which $k|\dot{\underset{\sim}{r}}|\dot{\underset{\sim}{r}}$ satisfies. We also want drag to point against the direction of the particle at any time t, which is why $\dot{\underset{\sim}{r}}$ has been left in vector form.

(b) Note that $|\dot{\underset{\sim}{r}}| = \sqrt{\dot{x}^2 + \dot{y}^2}$.

P2

See full worked solutions.

Exercise 5G

Inclined Planes and Pulleys

F1

(a) $N\cos\theta + T\sin\theta = mg$
$N\sin\theta - T\cos\theta = 0$

(b) $N = mg\cos\theta$
$T = mg\sin\theta$

F2

(a) parallel, perpendicular

(b) imbalance, net

Q1

(a) $0 = N - mg\cos 30°$
$ma = mg\sin 30°$

(b) $ma = mg\sin 30° = 20 \times 10 \times \frac{1}{2} = 100$ Newtons.

(c) $ma = 100$ and $m = 20$, so $a = 5\,\text{m}\,\text{s}^{-2}$.

(d) $20\,\text{m}\,\text{s}^{-1}$

(e) $40\,\text{m}$

Q2

(a) Resolve parallel to the plane.

(b) Integrate the previous result.

(c) Integrate the previous result.

Q3

(a) Resolve parallel to the plane.

(b) $a = \left(5 - \frac{\sqrt{3}}{2}\right)\,\text{m}\,\text{s}^{-2}$

Q4

(a) See full worked solutions.

(b) Set $a \leq 0$ for the particle to not slide down the ramp.

Q5

(a) The ramp and pulley are frictionless, so the pulley only re-directs the tension force without affecting their magnitude.

(b) The string is inextensible so if we pull the string by some amount, the other side of the string gets pulled by the exact same amount.

(c) Resolve parallel to the plane.

(d) $\ddot{x} = \frac{T}{10} - 50\sqrt{3}$

(e) $T = \frac{100}{\sqrt{3}}(1+\sqrt{3})$ Newtons

(f) $\ddot{x} = \frac{5}{3}(2-\sqrt{3})\,\mathrm{m\,s^{-2}}$

Q6

(a) See full worked solutions.

(b) See full worked solutions.

(c) Set $a = 0$ in part (a).

Q7

(a) $ma = mg - T$

(b) $ma = -mg + T$

(c) The string is inextensible and taut so both particles accelerate by the same amount. It is similar to when a person drags a mass tied to a rope at a speed of 1 metre per second. The mass moves at the same speed as the person as long as the string is taut.

(d) $T = \frac{40}{3}$ Newtons, $a = \frac{28}{3}\,\mathrm{m\,s^{-2}}$

Q8

See full worked solutions.

Q9

See full worked solutions.

Q10

(a) $N + T\sin\theta - mg = 0$
$T\cos\theta - \mu N = ma$

(b) See full worked solutions.

Q11

(a) $N - mg\cos\theta = 0$
$F_{\text{push}} - \mu N - mg\sin\theta = ma$

(b) See full worked solutions.

Q12

(a) $N + T\sin\alpha - mg\cos\theta = 0$
$T\cos\alpha - mg\sin\theta - \mu N = ma$

(b) See full worked solutions.

Q13

See full worked solutions.

Q14

See full worked solutions.

Q15

See full worked solutions.

Q16

See full worked solutions.

P1

(a) $E = E_k + E_p$ and recognise that $h = y$.

(b) Differentiate both sides of $y = \frac{3}{2}x^{\frac{2}{3}}$ with respect to time, and use the chain rule.

(c) Substitute (b) into (a) and also find the actual value of E by recognising that E is constant and so we can just use the initial gravitational potential energy $E_p = mgh$ where $h = \frac{3}{2}$.

(d) $1 + \frac{3\pi}{4}$

Chapter Review

R1

(a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) $t = 2\ln 2$ seconds

R2

(a) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) To the right.

(c) No, because $v^2 \geq 0$ so $x \geq 3$ or $x \leq -1$, so it will never be at $x = 2$.

(d) $x = 5$, $a = 3$

R3

(a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) $\dot{x} = \frac{dx}{dt}$ and solve for x.

R4

(a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) $\dot{x} = \frac{dx}{dt}$ then solve for x.

R5

(a) $\ddot{x} = -x$

(b) $\ddot{x} = -4\pi^2 x$

(c) $\ddot{x} = -9(x-1)$

(d) $\ddot{x} = -4x$

(e) $\ddot{x} = -4(x-2)$

(f) $\ddot{x} = -4x$

(g) $\ddot{x} = -9(x-2)$

(h) $\ddot{x} = -18x$

(i) $\ddot{x} = -32(x+1)$

R6

(a) $x = 4$ (b) $x = \frac{\pi}{2}$ (c) $4\sqrt{3}\,\text{m s}^{-1}$

R7

(a) $\ddot{x} = -(x - 3)$

(b) $x_0 = 3,\ A = 3,\ T = 2\pi$

R8

(a) $t = \frac{2\pi}{9}$ (b) $6\sqrt{3}\,\text{m s}^{-1}$ (c) 4 metres

R9

(a) $A = 2,\ T = 16,\ n = \frac{\pi}{8}$

(b) $x = 10 - 2\cos\left(\frac{\pi}{8}t\right)$

(c) 7:40am till 6:20pm

R10

See full worked solutions.

R11

$R = \frac{40}{3}$

R12

See full worked solutions.

R13

See full worked solutions.

R14

See full worked solutions.

R15

(a) See full worked solutions.

(b) See full worked solutions.

(c) There is enough friction so that the object on the table will not slide.